Calculus I

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Limits

Limit of a function: $\lim_{x \to a} f(x) = L$ if f(x) is defined near f(a) approaching from both sides and both the left and right limits must equal each other. $\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = \lim_{x \to a} f(x) = L$.
f(a) does not have to be defined.

Infinite Limits: $\lim_{x \to a} f(x) = \infty$ or $-\infty$ where f(x) is defined on both sides of a except at a.

Ex: $\lim_{x \to 0} \frac{1}{x^2} = \infty$

On a graph there would be vertical asymptotes there.

Limit Laws: (do not apply to limits at infinity technically)

Sum/Difference: $\lim_{x \to a} f(x) \pm g(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

Constant: $\lim_{x \to a} \mathbf{c}f(x) = \mathbf{c} \lim_{x \to a} f(x)$

Product/Quotient: $\lim_{x \to a} f(x) \bullet g(x) = \lim_{x \to a} f(x) \bullet \lim_{x \to a} g(x)$ or $\lim_{x \to a} f(x)/g(x) = \lim_{x \to a} f(x) / \lim_{x \to a} g(x)$

Power: $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$

Root: $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$

Direction Substitution: $\lim_{x \to a} f(x) = f(a)$

Theorem: $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$ if $f(x) \leq g(x)$ when x is near a except at a, and the limits of f(x) and g(x) both exist as x approaches a.

Squeeze theorem: if $f(x) \leq g(x) \leq h(x)$ for $x \approx a$ except at a and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x)$ then $\lim_{x \to a} g(x) = L$

Precise definition of a limit: if $|x - a| < \delta$ then $|f(x) - L| < \varepsilon$. This means if x is within δ of a, then f(x) will be within ε of L

- example: prove $\lim_{x \to 3} f(x)\ 4x-5 = 7$

  if $|x - 3| < \delta$ then $|4x - 5| < \varepsilon$ $\quad$ $|4x - 5 - 7| < \varepsilon$.
  $|4x - 5 - 7| = |4x - 12| = 4|x - 3| ... 4|x - 3| < \varepsilon$ so $|x - 3| <$ ¾ so…
  $\delta = \varepsilon/4$ or $4\delta = \varepsilon$
  if $|x - 3| < \delta$ then $|4x - 5| < \varepsilon$
  $4|x - 3| < \varepsilon$, $\varepsilon = 4\delta$, $4|x - 3|/4 < 4\delta/4 … |x - 3| < \delta$

Definition of limit at ∞

$\lim_{x \to \infty} f(x)$= L, then there will be a horizontal asymptote at y = L (same for - ∞)

- When doing these problems with rational functions, divide the numerator and denominator by the highest power of x in the denominator.
- There will be horizontal asymptotes at y = L

Precise Definition of a limit at infinity

If $\lim_{x \to \infty} f(x)$= L, then there is an N such that if x > N, then |f(x) - L| < ε

- this means f(x) gets within ε of L as x gets larger than N, where x depends on N

If $\lim_{x \to -\infty} f(x)$= L, then there is an N such that if x < N, then |f(x) - L| < ε

- this means f(x) gets within ε of L as x gets larger than N, where x depends on N

Example: show $\lim_{x \to \infty} \frac{1}{x} = 0$

1) if x > N, then $\left|\frac{1}{x} - 0\right| < \varepsilon$ 2) if x > N then $\left|\frac{1}{x}\right| < \varepsilon$

3) x→∞ so x > 0, so if x > N then $\frac{1}{x} < \varepsilon$ and x > $\frac{1}{\varepsilon}$

4) so since x > N, then $N = \frac{1}{\varepsilon}$ so $x > \frac{1}{\varepsilon}$

5) $x > \frac{1}{\varepsilon}$ so εx > 1 so $\frac{1}{x} < \varepsilon$

Special Limits:

$\lim_{x \to 0} \frac{sinx}{x}$= 1 $\lim_{x \to 0} \frac{cosx-1}{x}$= 0

Continuity

Definition: f(x) is continuous at x = a if **1)** f(a) is defined and **2)** $\lim_{x \to a} f(x)$ is defined (see the limits section for conditions necessary for this) and **3)** $\lim_{x \to a} f(x)$ = f(a)

- f(x) is continuous on an interval if it is continuous at every number in the interval.
- In simpler terms, a function is continuous if it's graph can be drawn on a piece of paper without ever lifting the pencil off of the paper.

Theorem: if f(x) and g(x) are continuous at a, and *c* is a constant, then the following are also continuous:

- f(a) $\pm$ g(a)
- *c*f(a), *c*g(a)
- f(x)g(x)
- f(x) / g(x), g ≠ 0

Theorem: 1) A polynomial is continuous everywhere. **2)** A rational function is continuous in its domain. Root functions, trig, inverse trig, exponential, and log functions are also continuous in their domain.

Theorem: if f(x) is continuous at b, and $\lim_{x \to a} g(x)$= b, then $\lim_{x \to a} f(g(x))$= f(b)

- more simply, $\lim_{x \to a} f(g(x))$= f($\lim_{x \to a} g(x)$)

Theorem: if g(x) is continuous at a, and f(x) is continuous at g(a), then f(g(x)) is continuous at a

Intermediate value theorem: if f(x) is continuous on [a,b] and f(a) < N < f(b) where f(a) ≠ f(b) then there exists a number *c* between [a,b] such that f(*c*) = N.

- A continuous function on its domain takes on every intermediate value in its domain between the ends of the domain.
- Example problem: show there is a root of $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2.
    - say why f(x) is continuous (polynomial). Find an x value *c* such that f(0) = 0. Set [a,b] = [1,2]. Find f(1) and f(2). f(1) < 0 and f(2) > 0 so by IVT there is a *c* such that f(*c*) = 0

Derivatives

Precise Definition of a Derivative:

f ′(x) at f(a) = $\lim_{x \to h} \frac{f(a+h) - f(a)}{h}$ or $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

Fact: In rational functions, when determining whether an x value is **point discontinuity** or **vertical asymptote**, point discontinuities occur when terms cancel, and vertical asymptotes are terms that don't cancel.

- f(x) = $\frac{x^2 - 4}{x - 2}$ has a point discontinuity at x = 2 because $\frac{x^2 - 4}{x - 2} = \frac{(x-2)(x+2)}{x - 2}$ and the (x - 2) terms cancel.
- f(x) = $\frac{x + 2}{x - 2}$ has a vertical asymptote at x = 2 because no terms cancel.

Theorem: f(x) is differentiable at a if f ′(a) exists. It is differentiable on an open interval if it is differentiable at every number in that interval. So, if a function is differentiable at a point it is continuous there also. However, just because a function is continuous at a point does not mean it's differentiable there. This is the case in corners (absolute value functions).

- In general, a function is not differentiable at **corners**, **point discontinuities, jump discontinuities,** and **vertical asymptotes**.

Mean value theorem: if f(x) is continuous on [a,b] and differentiable on (a,b) then there is a *c* in (a,b) such that f ′(*c*) = $\frac{f(b) - f(a)}{b - a}$

Rolle's Theorem: if f(x) is continuous on [a,b] and differentiable on (a,b) and f(a) = f(b) then there is a *c* in (a,b) such that f ′(*c*) = 0

Linearization: L(x) = f(x) + f ′(x)(x - a)

Differentials: for y = f(x) $\frac{dy}{dx} = f'(x)$ dy = f'(x)dx

Exponential growth: Ae^{kx}

Continuous Compounding: $\lim_{n \to \infty} (1 + \frac{r}{n})^n$… say r was 10%, then r would be 1/10 and the limit would equal $100e^{1/10}$

- for interest, $A_0(1 + \frac{r}{n})^{nt}$ where A_0 is initial amount, r is rate, n is # times being compounded within one interval of time t

Definition of e: $\lim_{x \to 0} (1 + x)^{\frac{1}{x}}$

Related Rates: Use volume of sphere as an example.

- V = $\frac{4}{3}\pi r^3$... $\frac{dV}{dt} = \frac{dV}{dr} \bullet \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$

Implicit Differentiation:

- Think of y as a function of x so use chain rule. This means do the normal derivatives for x and y (for example, x^2 becomes 2x, y^2 becomes 2y) but then also add on the dy/dx to the y terms (so y^2 ultimately becomes 2ydy/dx)

Logarithmic Differentiation:

- Take the natural logarithm of each side and use laws of logarithms to simplify. Then use implicit differentiation. Example: y = $x^{\sqrt{x}}$

$\ln y = \ln x^{\sqrt{x}}$ $\quad \ln y = \sqrt{x}\ln x$ $\quad \frac{d}{dx}(\ln y = \sqrt{x}\ln x)$ $\quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{2\sqrt{x}}\ln x + \frac{\sqrt{x}}{x}$ $\quad \frac{dy}{dx} = y(\frac{1}{2\sqrt{x}}\ln x + \frac{\sqrt{x}}{x})$

Absolute Maximum: f(c) where f(c) $\geq$ f(x) for all x in the domain.
Absolute Minimum: f(c) where f(c) $\leq$ f(x) for all x in the domain.
Local Maximum: f(c) where f(c) $\geq$ f(x) for all x near c.
Local Minimum: f(c) where f(c) $\leq$ f(x) for all x near c.
Extreme Value Theorem: if f(x) is continuous on [a,b] then there must be a *c* and *d* such that f(*c*) $\geq$ all f(x) and f(*d*) $\leq$ all f(x) …. there must be an absolute maximum *c* and absolute minimum *d*.
Critical number: An x value *c* such that f'(*c*) = 0 or DNE.
Critical point: the (x, y) coordinate pair where x is the critical number.
Inflection point: The points where f''(x) = 0.

- f''(x) > 0 means concave up (slope is increasing)
- f''(x) < 0 means concave down (slope is decreasing)

e exponents: $e^{\ln x}$ = x …. x can be term like "x + 2"

Functions and inverses:

- given a function y = f(x), solve for x in terms of y. Then swap the x's and y's. This is $f^{-1}(x)$.
- $f(f^{-1}(x)) = f^{-1}(f(x)) = x$
- The range of $f^{-1}(x)$ is the domain of f(x).