Chapter 1

N: natural numbers (0, 1, 2, 3…)
Z: integers (...-3, -2, -1, 0, 1, 2, 3…)
Q: rational numbers (can be put into a fraction)
R: real numbers
C: complex numbers

Cardinality: |A| means the cardinality of A...the number of elements in set A
Subset: $A \subseteq B$ A is a subset of mean. Every element of A is in B. The NULL set is a subset of every set.
Equality: A = B sets are equal
If $A \subseteq A$ and $Ø \subseteq A$ and A is non-empty, then A is an **improper subset** of itself. All other subsets of A, including Ø are **proper subsets**.

Basic Operations

Intersection: $A \cap B$ set of elements in both A and B
Union: $A \cup B$ set of elements in A or B
Disjoint: A and B are disjoint if they share no elements ($A \cap B = Ø$)
Universe: set of all possible elements in a given problem. Could be all real numbers for example, or indicated by U = {1, 2, 3...10}
Complement:

- B - A the complement of A relative to B, the set of elements in B but not also in A
- A^C is U - A, all elements in the universe not in A

Symmetric Difference: $A \oplus B$ set of all elements in A or B but not both. The corollary to $A \cap B$
Cartesian Product: A x B = {(a, b) | $a \in A$ and $b \in B$}

- set or ordered pairs using sets A and B
- cardinality of A x B is the product of the individual cardinalities. |A x B| = (|A|)(|B|)
- A x B x C would be (a, b, c) for example (up to n sets)
- A x A = A^2, A x A x A = A^3 etc.

Power Set: the set of all subsets including the Ø set. Denoted P(S)--power set of set S

- *cardinality of power sets:* given set S, $|P(S)| = 2^{|S|}$--the cardinality of a power set is 2 raised to the cardinality of the original set.

Binary

General format: 0 1 1 0 0 0 1 0
2^7 2^6 2^5 2^4 2^3 2^2 2^1 2^0

Decimal to binary conversion

1) start with empty list of bits 2) assign k = n 3) while k > 0, do the following
   a) k / 2, get quotient (k / 2) and remainder (k % 2)
   b) attach remainder to right hand side of bits
   c) k = quotient

example: 41
41 / 2 = 20 r 1 1
20 / 2 = 10 r 0 01

10 / 2 = 5 r 0	001
5 / 2 = 2 r 1	1001
2 / 2 = 1 r 0	01001
1 / 2 = 0 r 1	101001
0 / 2 stop	

Summation Notation

$\sum_{i=0}^{5} k = 0 + 1 + 2 + 3 + 4 + 5$ $\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots A_n$ $\bigcup_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots$

A_n

$\prod_{i=1}^{n} k = (1)(2(3)\ldots(n)$

Chapter 2

Rule of Products: if there are p_1 ways to do operation 1, p_2 ways to do operation 2 etc. then there are $(p_1)(p_2)\ldots(p_n)$ ways to perform an operation

Combinatorics

For set A, if |A| = n then there are n! permutations i.e. if A = {1, 2, 3, 4} then a permutation could be {1, 2, 4, 3} or {2, 3, 4, 1}

Permutation (official): ordered arrangement of k elements selected from a set of n elements where no two elements are the same is denoted P(n, k) = n(n-1)(n-2)...(n-k+1).

- example: how many ways can 10 runners place 1st, 2nd, and 3rd? P(10, 3) = (10)(9)(8) = 720. Note how order DOES matter meaning {runner 1, runner 2, runner 7} is different than {runner 2, runner 1, runner 7}. Even if a permutation has the same elements, it’s still a unique permutation since the actual position of each element matters. This is not always the case.
- P(n, k) also = n!/(n-k)

Binomial coefficient: this is for when order doesn't matter. For example, how many ways can three prizes of $1000 be distributed to 10 people? Since each prize is the same amount, {Person A, Person B, Person C} is equivalent to {Person C, Person A, Person B}

$\binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}$ for k elements from set of size n.

- special cases/rewriting the coefficient

$\binom{n}{k} = \binom{n}{n-k}$ $\binom{n}{0} = 1$ $\binom{n}{1} = n$

Binomial Coefficient

$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ x and y can be anything ($2x$, $4y^2$ for example)

The coefficient of $x^{n-k}y^k$ is $\binom{n}{k}$. If it's 2x-3y, then you also have to include the 2 and -3 so it's $\binom{n}{k}$ $(2)^{n-k}(-3)^k$

Partitions/Law of Addition

Partition: set of one or more nonempty subsets of A such that every element of A is in exactly one set. So, $A_1 \cup A_2 \cup A_3 \ldots = A$.

- There can be no repeats of an element or an element missing. So, if $i \neq j$, then $A_i \cap A_j = \varnothing$
- each subset is called a **block**, and # of blocks is $\leq$ A for a finite set.
- $|A| = |A_1| + |A_2| + \ldots |A_n|$. The cardinality of A equals the sum of the cardinalities of all the blocks in its partition.

Law of Inclusion/Exclusion:

$|A \cup B| = |A| + |B| - |A \cap B|$ for two finite sets.

$|A \cup B \cup C| = |A| + |B| + |C| - (|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C|$ for 3 finite sets

- these give you the total number of terms among 2 or 3 sets

Chapter 4

Laws of Set Theory

Distributive:	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Commutative:	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associative:	$A \cup (B \cup C) = (A \cup B) \cup C$	$A \cap (B \cap C) = (A \cap B) \cap C$
DeMorgan:	$(A \cup B)^c = A^c \cap B^c$	$(A \cap B)^c = A^c \cup B^c$
Identity:	$A \cup \varnothing = A$	$A \cap U = A$
Complement:	$A \cup A^c = U$	$A \cap A^c = \varnothing$
Idempotent:	$A \cup A = A$	$A \cap A = A$
Null:	$A \cup U = U$	$A \cap \varnothing = \varnothing$
Absorption:	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
Involution:	$(A^c)^c = A$	

Chapter 3

Proposition: a sentence that's either true or false

Conjunction: $p \wedge q$ p and q

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

Disjunction: $p \vee q$ p or q

p	q	$p \wedge q$
0	0	0
0	1	1
1	0	1
1	1	1

Negation: $\neg p$ not p

p	$\neg p$
0	1
1	0

Conditional: $p \rightarrow q$ if p then q

p	q	$p \rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

Converse: $q \rightarrow p$, converse of $p \rightarrow q$

p	q	$q \rightarrow p$
0	0	1
0	1	0
1	0	1
1	1	1

Contrapositive: $\neg q \rightarrow \neg p$ if not q the not p

- logically equivalent to condition. If the conditional is true, then the contrapositive is true

p	q	$\neg q \rightarrow \neg p$
0	0	1
0	1	1
1	0	0
1	1	1

Biconditional Proposition: $p \leftrightarrow q$ p if and only if q

p	q	$p \leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

<u>if p then q</u> vs.	<u>p if and only if</u>
p implies q	p is necessary and sufficient for q
q follows from p	p is equivalent to q
p, only if q	if p then q and if q then p
q, if p	if p, then q and conversely
p is sufficient for q	
q is necessary for p	

Truth Tables/Propositions Generated by a Set

- Let S be any set of propositions. A proposition generated by set S is any valid combination of propositions in S with conjunction (and), disjunction (or), and negation (opposite).
  - i.e. if $p \in S$ then p is a proposition generated by S.
    if x and y are propositions generated by S then so are x, $\neg x$, $x \wedge y$, and $x \vee y$
  - conditional/biconditional are generated from conjunction and disjunction and negation
  - *propositional hierarchy:* negation, conjunction, disjunction, conditional operation, biconditional operation. Parentheses at any place can override these. Work left to right.

Tautology: something is true in all cases
Contradiction: something is false in all cases
Equivalence: $r \Leftrightarrow s$
- let r and s be propositions generated over a set S. r and s are equivalent ($r \Leftrightarrow s$) if and only if $r \leftrightarrow s$ is a tautology (all biconditional propositions are true)

Implication: $r \Rightarrow s$
- r implies s if $r \rightarrow s$ is a tautology (all conditional propositions are true)

All tautologies are equivalent to each other. All contradictions are equivalent to each other

Laws of Logic

Basic

Commutative: $p \vee q \Leftrightarrow q \vee p$ $\quad$ $p \wedge q \Leftrightarrow q \wedge p$
Associative: $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$ $\quad$ $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$
Distributive: $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ $\quad$ $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
Demorgan's: $\neg(p \vee q) \Leftrightarrow (\neg p) \wedge (\neg q)$ $\quad$ $\neg(p \wedge q) \Leftrightarrow (\neg p) \vee (\neg q)$
Absorption: $p \wedge (p \vee q) \Leftrightarrow p$ $\quad$ $p \vee (p \wedge q) \Leftrightarrow p$
Involution: $\neg(\neg p) \Leftrightarrow p$
Identity: $p \vee 0 \Leftrightarrow p$ $\quad$ $p \wedge 1 \Leftrightarrow p$
Negation: $p \wedge \neg p \Leftrightarrow 0$ $\quad$ $p \vee \neg p \Leftrightarrow 1$
Idempotent: $p \vee p \Leftrightarrow p$ $\quad$ $p \wedge p \Leftrightarrow p$
Null: $p \wedge 0 \Leftrightarrow 0$ $\quad$ $p \vee 1 \Leftrightarrow 1$

Common Implications and Equivalences

Detachment: $(p \rightarrow q) \wedge p \Rightarrow q$
Chain: $(p \rightarrow q) \wedge (q \rightarrow r) \Rightarrow (p \rightarrow r)$
Disjunctive Addition: $p \Rightarrow (p \vee q)$ $\quad$ $q \Rightarrow (p \vee q)$
Indirect Reasoning: $(p \rightarrow q) \wedge \neg q \Rightarrow \neg p$
Conjunctive Simplification: $(p \wedge q) \Rightarrow p$ $\quad$ $(p \wedge q) \Rightarrow q$
Disjunctive Simplification: $(p \vee q) \wedge \neg p \Rightarrow q$ $\quad$ $(p \vee q) \wedge \neg q \Rightarrow p$
Contrapositive: $(p \rightarrow q) \Leftrightarrow (\neg q \rightarrow \neg p)$
Conditional Equivalence: $p \rightarrow q \Leftrightarrow \neg p \vee q$
Biconditional Equivalence: $(p \leftrightarrow q) \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p) \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$

Universes/Truth Sets

- A proposition over U is a sentence that contains a variable that can take on any value in U and that has a definite truth value as a result of any such substitution.
  ex: universe = integers
  $4x^2 - 3x = 0 \qquad 0 \leq n \leq 5$
- all 3.4 laws of logic are valid for propositions over a universe

truth set: if p is a proposition over U, the truth set os p is $T_p = \{a \in U \mid p(a) \text{ is true}\}$

- a proposition p over U is a tautology if its truth set is U. Contradiction if its truth set is empty
  $T_{p \wedge q} = T_p \cap T_q \qquad T_{p \vee q} = T_p \cup T_q \qquad T_{p \leftrightarrow q} = (T_p \cap T_q) \cup (T_p \cup T_q)$
  $T_{p \rightarrow q} = T_p^c \cup T_q \qquad T_{\neg p} = T_p^c$
- $p \Leftrightarrow q$ if $T_p = T_q$ (p is equivalent to q if the truth sets are equal to each other).
- $p \Rightarrow q$ if $T_p \subseteq T_q$ (p implies q when truth set of p is a subset of the truth set of q)

Quantifiers

$\exists$**:** existential quantifier, “there exists” $(\exists_X)_Z\,(q(x))$: there exists an integer x such that q(x) is true

A: there does not exist

$\forall$**:** universal quantifier, “for all” or “for every”

Negating Quantifiers

$\neg((\forall_n)_U(p(n))) \Leftrightarrow (\exists_n)_U(\neg p(n))$

$\neg((\exists_n)_U(p(n))) \Leftrightarrow (\forall_n)_U(\neg p(n))$

Propositions with more than one variable

$(\forall_a)_Q((\forall_b)_R)(q(x, y)))$ For all a in Q and all b in R, proposition p(x, y) is true.

- you can commute two $\forall$’s or two $\exists$’s but not a combo of both

Mathematical Proofs

Proving if p then q

1) Direct proof: assume p is true, then deduce q must be true using laws of logic
2) Indirect proof: suppose p is true then assume q is false. With these assumptions, deduce a series of statements to get a contradiction to prove q is true

Proving p if and only if q

- prove if p then q, and then prove if q then p

Other methods of if p then q

- prove the contrapositive.
- case analysis: break into cases

Mathematical Induction

- p(n) is a proposition over P. Prove p(n) is true for all $n \in P$

1) Say “we will use proof by induction”.
2) Step 1: basis of induction. Prove p(1) is true
3) Step 2: Induction step: let $k \in P$. Assume p(k) is true. Show if p(k) is true, then p(k+1) is true.

Chapter 5: Matrices

Matrix: rectangular array of numbers called entries.

- m x n: m rows, n columns (order/dimension)

Notations:

- a_{ij} is the (i, j)th entry (ith row, jth column)
- the set of all m x n entries with entries from the set S is $M_{m \times n}(S)$
  i.e. A = 1 3 5 written as $A \in M_{2 \times 3}(Z)$
  0 2 8

- You can add/subtract only matrices of the same dimensions.
- For scalar multiplication (like 3A), just multiply every term in the matrix by the number (3).
- When multiplying 2 matrices, the number of columns in the first matrix must match the number of rows in the second matrix
  i.e. (2 x 3) x (3 x 4) is valid but (2 x 3) x (2 x 4) is not valid.

determinant: for a 2x2 matrix A = $\left[\frac{a\ b}{c\ d}\right]$ the determinant **detA** = ad - bc. Alternatively denoted |A|, not to be confused with the cardinality of A

Inverse: A has an inverse (A^{-1}) only if the determinant does not equal 0. If it exists, then the inverse matrix is $\frac{1}{detA}\left[\frac{d\ -b}{-c\ a}\right]$

Multiplicative Identity: I is the multiplicative identity matrix because AI = A for every n x n matrix and $AA^{-1} = A^{-1}A = I$ for n x n matrix whose inverse exists.

Matrix Properties

- AB does not always equal BA
- $(A + B)^2 = A^2 + 2AB + B^2$ not always true
- AB = 0 does not always mean A or B = 0. However, if AB = 0 and A^{-1} exists, then B must = 0
  $A^{-1}AB = A^{-1}0 = IB$
- If AB = AC then B = C not always true. However, ff AB = AC and A^{-1} exists, then B = C
  AB = AC $A^{-1}AB = A^{-1}AC$ IB = IC B = C

$0_{m \times n}$ is m x n matrix with all zeroes

n x n is a square matrix. With these you can take powers like A^2, A^3 etc.

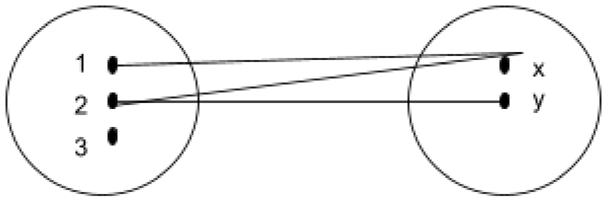
Diagonal matrix: all non-diagonal entries are zero 2 0 0
0 3 0
0 0 1

Identity Matrix: n x n diagonal matrix whose diagonal entries all equal 1. Denoted I_n or I

- AI = A or IA = A
- if A is an n x n matrix and AB = I and BA = I, then B is an inverse of A.
- There is only one inverse of a matrix

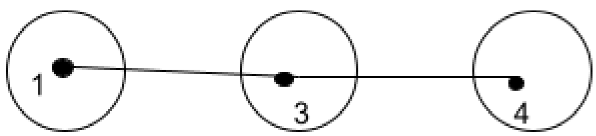
Chapter 6

Relation: let A and B be sets. A relation from A to B is a subset of A x B.
- i.e. A = {1, 2, 3} and B = {x, y}. Then, r = {(1, x), (2, y), (2, x)} is a potential relation
- the number of subsets of A x B is $2^{|A \times B|}$



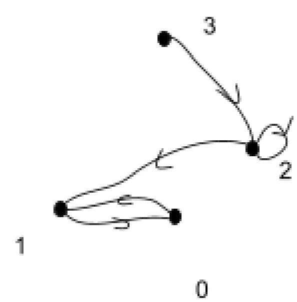
A **relation on** set A is a relation from A to itself.
- ex: a relation on Z could be r = {(x, y) ∈ ZZ | y = x^2} and (-2, 4) and (3, 9) would both be ∈ r

Notation
- arb or (a,b) ∈ r both mean a and b are an element of relation r
- if r is a relation from A into B and s is a relation from B into C then rs is a relation from A into C
- find (a,c) ∈ rs by using the b value from (a, b) from relation r and then using that b value to find all (b, c) elements in relation s. For example, if (1, 3) ∈ r and (3, 4) ∈ s then (1, 4) ∈ rs



- if r is a relation on A, the r^2 = rr, r^3 = rrr etc.
  ex: r = {(1, 1), {2, 4), (3, 9), (4, 16)...} then…
  rr = r^2 = {(1, 1), (2, 16), (3, 81)...}

Directed graph/digraph: for r a relation on set A
- **vertices:** points represent each element of A
- **directed edges:** the lines
- ex: r = {(0, 1), (1, 0), (2, 1), (2, 2), (3, 2)}



Relation Properties

Let r be a relation on set A

Reflexive: r is reflexive if $(a, a) \in r$ for every $a \in A$. In a digraph, there would be a loop at every vertex as seen above with 2.

- ex: $\leq$ is a reflexive relation on Z since for any integer n, $n \leq n$

Symmetric: r is symmetric if for any $a, b \in A$ if $(a, b) \in r$ then $(b, a) \in r$. On digraph, this means if there's an arrow going from one element to another, there's another arrow going back as seen above with 0 and 1.

Transitive: r is transitive if for all $a, b, c \in A$, if $(a, b) \in r$ and $(b, c) \in r$ then $(a, c) \in r$. In a digraph, if a is connected to b, then b is connected to c

Antisymmetric: r is antisymmetric if the following is true for all $a, b, c \in A$: if $(a, b) \in r$ and $a \neq b$, then $(b, a) \notin r$ i.e. $\leq$ is antisymmetric since $a \leq b$ but b not $\leq a$.

equivalence relation: a relation r is an equivalence relation if it's reflexive, symmetric, and transitive.

- ex: = is an equivalence relation on z

equivalence class: let r be an equivalence relation on set A. Then the equivalence class of a $c \in A$ is $c(a) = \{b \in A \mid arb\}$ (the set of all elements of A that are related to a). The set of all equivalence classes from a partition of the set A if r is any equivalence relation of A

partial ordering: r is a partial ordering on A if its reflexive, transitive, and antisymmetric. A set with a partial ordering is called a **partially ordered set/poset.**

- ex: $\leq$ is a partial ordering on Z
- ex: $\subseteq$ is a partial ordering on P(A) (subset is a partial ordering of the power set)
- $\preccurlyeq$ is the generic notation for partial ordering (looks like a less than or equal to but the equal to part is slanted down)
- common partial ordering: a | b...a divides b (b is a multiple of a) on Z, the set of integers

Matrices of Relations

r is a relation from $A = \{a_1, a_2...a_m\}$ and to $B = \{b_1, b_2...b_3\}$. The adjacency matrix of r is the matrix $R = (R_{ij})$ so that $R_{ij} = 1$ if $(a_i, b_j) \in r$

0 otherwise

0's and 1's are counted with boolean arithmetic

- 1 + 1 = 1 (everything else is the same)...like 1 or 1
- 1 + 0 = 1...like 1 or 0
- 1 x 1 = 1...like 1 and 1
- 1 x 0 = 0...like 1 and 0
- 0 x 0 = 0...like 0 and 0

example: let s = {(4, x), (4, y), (5, y), (5, z)} be a relation from R = {(4, 5)} to C = {(x,y, z)}.

let r = {(1, 4), (3, 4), (3, 5), (5, 5)} be a relation from A = {1, 3, 5} and R = {4, 5} (same R)

Adjacency matrix S — Adjacency Matrix R

$$\begin{array}{c|ccc} & x & y & z \\ \hline 4 & 1 & 1 & 0 \\ 5 & 0 & 1 & 1 \end{array} \qquad \begin{array}{c|cc} & 4 & 5 \\ \hline 1 & 1 & 0 \\ 3 & 1 & 1 \\ 5 & 0 & 1 \end{array}$$

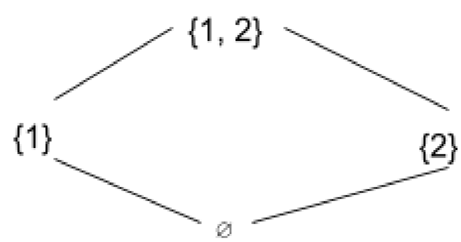
Adjacency Matrix RS: (for rs)

$$\underset{R}{\begin{array}{c|ccc} & x & y & z \\ \hline 4 & 1 & 1 & 0 \\ 5 & 0 & 1 & 1 \end{array}} \quad X \quad \underset{S}{\begin{array}{c|cc} & 4 & 5 \\ \hline 1 & 1 & 0 \\ 3 & 1 & 1 \\ 5 & 0 & 1 \end{array}} \quad = \quad \underset{RS}{\begin{array}{c|ccc} & x & y & z \\ \hline 1 & 1 & 1 & 0 \\ 3 & 1 & 1 & 1 \\ 5 & 0 & 1 & 1 \end{array}}$$

Alternatively, you could first calculate rs = {(1, x), (1, y), (3, x), (3, y), (3, z), (5, y), (5, z)}

Hasse Diagram: for a poset A with partial ordering
- represent each element of A with a matrix
- put matrix a between b if a ≤ b
- Draw line form a to b if a ≤ b and there's no c ∈ A's so that a ⩽ c ⩽ b

ex: consider P({1, 2}) with partial ordering on the subsets (recall P is the power set of {1, 2})
- so, {1} ⊆ {1, 2} and ∅ ⊆ {2} etc.
- the Hasse diagram would be



Chapter 7: Functions

Function definition (formal): a function f from set A to set B is a relation from A to B so that for each a $\in$ A, there's actually one b $\in$ B so that (a, b) $\in$ f.

- **informal definition:** A function from A into B is a way of assigning each element of A to an element of B

input	function	output
a $\in$ A	f	b $\in$ B

- if a $\in$ A is assigned to b $\in$ B, this can be encoded as an ordered pair (a, b)
- The function f corresponds to a set of ordered pairs A x B

Notation:

- f: A $\longrightarrow$ B
  - **domain:** A (input)
  - **codomain:** B (output)
- if (a, b) $\in$ f then f(a) = b

Ways of defining a function

let A = {1, 2, 4} and B = {1, 3, 5, 7} then.

- f = {(1, 5), (2, 3), (4, 7)} is a function from A to B (just one example)
- define f: A $\longrightarrow$ B by f(1) = 5, f(2) = 3, f(4) = 7
- define: f: A $\longrightarrow$ B by f(x) = 2x - 3

Determining if something is a function

- every element of the domain must correspond to an element in the codomain
- every element in the domain must correspond to <u>one</u> element in the codomain, not multiple.
- not every element in the codomain has to be included.
- different elements of the domain can correspond to the same elements in the codomain

image: if f(a) = b, then f(a) is the image of a under f

range: the range of f is the set of all images. Notation: f(A) = {f(a) | a $\in$ A}

- ex: A = {2, 4, 5} and B = {1, 2...10}. Define f: A$\longrightarrow$ B by f(2) = 1, f(4) = 5, f(5) = 7. The range is f(A) = {1, 5, 7}

Calculating Possible functions:

- using above example, there are 1000 different potential functions since there are 10 ways to assign an element of A to an element of B for each element of A so 10 x 10 x 10 = 1000

Equal functions: functions f: A $\longrightarrow$ B and g: A $\longrightarrow$ B are equal if f(x) = g(x) for all x $\in$ A

Properties of Functions

define f: A $\longrightarrow$ B

injective:

- if for any a, b $\in$ A, if a $\neq$ b then f(a) $\neq$ f(b). Another way of saying this…
- for any a, b $\in$ A, if f(a) = f(b) then a = b
- no two elements have the same output. If f(2) = 3 then f(3) cannot = 3 also

surjective:

- for every $b \in B$ there exists an $a \in A$ so that $f(a) = B$ i.e. the range of f equals the codomain B, every element of B is the image of an element in A

bijective: a function is both injective and surjective

- every element of the domain has a single, unique output. Not two elements can have the same output
- the range of the function equals the codomain.

Compositions

Composition: define $f: A \longrightarrow B$ and $g: B \longrightarrow C$. The composition $g \circ f\, A \longrightarrow C$ is defined by $(g \circ f)(x) = g(f(x))$ for all $x \in A$

In set of real numbers

- **multiplicative identity:** $(a)(1) = (1)(a) = a$
- **multiplicative inverse:** for $a \neq 0$, $aa^{-1} = a^{-1}a = 1$

Identity function: let A be a set. Then the identity function $i_A: A \longrightarrow A$ is defined by $i_A(x) = x$ for all $x \in A$.

- notation: $i: A \longrightarrow A$
- *fact:* $i \circ f = f \circ i = f$ for any $f: A \longrightarrow A$

  $(f \circ x)(x) = f(x)$ for all $x \in A$.

Alternate Definition of Inverse Function: let $f: A \longrightarrow B$ be a bijection. The inverse of f is the function $f^{-1}: B \longrightarrow A$ defined by if $f(a) = b$, then $f^{-1}(b) = a$

- *fact:* $f \circ f^{-1} = i_B$ and $f^{-1} \circ f = i_A$

Chapter 8: Sequences/Recurrence Relations

Sequence: infinite list of numbers called terms denoted S_1, $S_2 \ldots S_n$. Notation is $\{S_k\}^{\infty}_{k=0}$

- same as sequences in calculus

recurrence relation: a sequence can be defined using a recurrence relation. Define $\{S_k\}$ by using previous terms in the sequence

- ex: let $\{S_k\}$ defined by $S_k = 2S_{k-1} + S_{k-2}$ with initial condition $S_0 = 3$ and $S_1 = -1$
- ultimate goal is to find a closed formula for $\{S_k\}$ in terms of k. This can be accomplished by calculating terms and looking for a pattern. Some sequences can be found in the following way

Linear Homogeneous Recurrence Relations of order n: recurrences that can be written as follows

$$S_k + C_1S_{k-1} - C_2S_{k-2} + \ldots C_nS_{k-n} = 0$$

Use $\circledast$ to refer to linear homogeneous recurrence relation of order n so you don't have to write it out every time

- ex: $S_k - 3S_{k-1} + 2S_{k-2} - S_{k-3} = 0$ is of order 3 since there are 4 terms.

Solving these $\circledast$ relations involves the following steps

Given: $S_k = 5S_{k-1} - 6S_{k-2}$

1) find the **characteristic equation:**
    a) rewrite in form $\circledast$: $S_k - 5S_{k-1} + 6S_{k-2} = 0$ (order 2 b/c 3 terms)
    b) characteristic equation: $a^2 - 5a + 6$
2) find the **general solution:** of the form $S_k = b_1a_1^{\,k} + b_2a^k + b_na_n^{\,k}$ for some constants b_1, $b_2 \ldots b_n$ and n distinct roots
    a) $a^2 - 5a + 6 = (a - 2)(a - 3)$ so there are 2 roots: 2 and 3
    b) **general solution:** $S_k = b_13^k + b_22^k$
3) Then you'll be given some initial conditions and you use those to solve for b_1 and b_2