THE PRESCRIBED CROSS CURVATURE PROBLEM ON THREE-DIMENSIONAL UNIMODULAR LIE GROUPS

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ABSTRACT. Let G be a three-dimensional unimodular Lie group and let T be a left-invariant $(0,2)$-tensor field on G. This report studies the necessary and sufficient conditions for the existence of a left-invariant metric g that satisfies the equation $X(g) = T$ on three-dimensional unimodular Lie groups, where $X(g)$ is the cross curvature of g. We then look at the uniqueness of the solutions to the above equation. In particular, we study the groups $E(2), E(1,1), \mathrm{H}_3$ and $\mathbb{R}^3$.

1. INTRODUCTION

A class of problems in Riemannian Geometry is that of prescribing curvature on a given manifold. One of the most widely studied of these is the problem of prescribing Ricci curvature. Let M be a Riemannian manifold, we are interested in the existence and uniqueness of a Riemannian metric g such that

$$\mathrm{Ric}(g) = T \tag{1.1}$$

for a given a $(0,2)$-tensor field T. This problem was studied in [Ham84] where Hamilton proved the existence of a left-invariant metric solving (1.1) specifically for $M = SU(2)$. This result was extended to five other Lie groups by Buttsworth in [But19]. Both of these papers and many more, rather than solving (1.1), look at solving the equation with a scalar multiple of T. This was motivated by a result on $\mathbb{S}^2$ presented by Hamilton and DeTurck in [Ham84] and [DeT16] respectively, where they gave a restriction on the solvability of (1.1). The work on the existence and uniqueness of solutions to (1.1) has ties to research in Ricci flow, a main area of work in Riemannian geometry. This gives motivation for us to study a similar problem for the cross curvature.

The cross curvature, initially introduced by Chow and Hamilton in [CH04] in the context of uniformising three-manifolds, is a $(0,2)$-tensor field $X(g)$. The prescribed cross curvature problem on a Riemannian manifold M states, given a $(0,2)$-tensor field T does there exist a Riemannian metric g such that,

$$X(g) = T, \tag{1.2}$$

and if so is the solution unique? The problem is most interesting in three dimensions yet studying the global solvability and uniqness of solutions to (1.2) is difficult. Some results however, have been obtained on $\mathbb{S}^3$ in [BP21] and [Gki08]. In this report, we focus on the homogeneous case: prescribing cross curvature of left-invariant metrics on three-dimensional unimodular Lie groups. Note that (1.2) has no scaling factor. This is since the cross curvature is not scaling invariant while the Ricci curvature is.

To define the cross curvature tensor, we first introduce the Einstein tensor. For further resources on the basic notations of Riemannian geometry, we direct the reader to [dC92] and [Lee18].

Definition 1.1 (Einstein Tensor Field). Let M be a three-dimensional Riemannian manifold with metric g. Define the $(0,2)$-tensor field

$$Ein(g) := \mathrm{Ric}(g) - g\frac{S}{2},$$

where S is the scalar curvature of M.

We find it useful to define the $(1,1)$-tensor

$$\mathcal{E}(g) = \left(\text{Ric}(g) - g\frac{S}{2}\right)^{\#}.$$

Definition 1.2 (Cross Curvature). We define the cross curvature as

$$X(g)(\cdot, \cdot) := \det(\mathcal{E}) \, g\left(\mathcal{E}^{-1}\cdot, \cdot\right), \tag{1.3}$$

here $X(g)$ is the cross curvature tensor.

Gkigkitzis in [Gki08] defines an analogous problem to (1.2) for the cross transformation, a $(1,2)$-tensor, and proves a result on the existence and uniqueness of left-invariant metrics on $SU(2)$. Pulemotov and Buttsworth show that the two problems are equivalent in [BP21].

Let G be a three-dimensional unimodular Lie group with Lie algebra $\mathfrak{g}$. According to [HL09], for any left-invariant metric g on G, there exists a basis $\{V_1, V_2, V_3\}$ of $\mathfrak{g}$ that diagonalises the metric and gives us the Lie bracket relations:

$$[V_i, V_j] = \sum_{k=1}^{3} \epsilon_{ijk}\lambda_k V_k \tag{1.4}$$

for some coefficients λ_k. The structure coefficients λ can be scaled and the basis of $\mathfrak{g}$ can be oriented to uniquely describe each of the 6 unimodular three-dimensional Lie groups. In [But19], it was shown to be sufficient to consider values of λ_i in $\{-2, 0, 2\}$. We focus on the four groups in which at least one of the λ_i values is 0. Our results are encapsulated by the following theorem:

Theorem 1.1. *Let T be a left-invariant $(0,2)$-tensor field on the Lie group $E(2), E(1,1)$ or H_3. Then, there exists a left-invariant Riemannian metric g solving (1.2) if and only if the following are satisfied:*

> *(i) T is diagonalisable in a basis $\{V_i\}_{i=1}^3$ in which (1.4) holds, and*
> *(ii) The diagonal elements $T(V_i, V_i) =: T_i$ have signature as in theorems 4.1, 4.2 and 4.3 when looking at the groups $E(2), E(1,1)$ and H_3 respectively.*

In the same basis, if none of the diagonal elements T_i are zero, the solution g is unique. Otherwise there are infinitely many solutions.

2. The Cross Curvature Tensor on a Unimodular Lie Group

In this section we compute the cross curvature tensor for a general three-dimensional unimodular Lie group G. We present this in the following lemma.

Lemma 2.1. *The cross curvature tensor for a left-invariant metric $g = x_1 \, d\theta_1^2 + x_2 \, d\theta_2^2 + x_3 \, d\theta_3^2$ for $\{\theta_i\}_{i=1}^3$ a basis for $\mathfrak{g}^*$, dual to $\{V_i\}_{i=1}^3$, on a three-dimensional unimodular Lie group has*

$$X(g)(V_i, V_j) = \delta_{ij}\frac{(\mathbf{x_i}\mathbf{x_{i+2}} - \mathbf{x_{i+1}}\mathbf{x_{i+2}} - \mathbf{x_i}\mathbf{x_{i+1}})(\mathbf{x_i}\mathbf{x_{i+1}} - \mathbf{x_i}\mathbf{x_{i+2}} - \mathbf{x_{i+1}}\mathbf{x_{i+2}})}{x_i x_{i+1}^2 x_{i+2}^2} \tag{2.1}$$

where $\mathbf{x_i} := \frac{\lambda_{i+2}x_{i+2} + \lambda_{i+1}x_{i+1} - \lambda_i x_i}{2}$ for $i = 1, 2, 3$.

Proof. Fix a basis $\{V_i\}_{i=1}^3$ in which (1.4) holds and g is diagonal. We first compute the covariant derivatives required to find the Riemann curvature tensor by using the Koszul formula:

$$g(\nabla_X Y, Z) = \frac{1}{2}\left(g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X)\right),$$

for vector fields X, Y, Z. In our basis $\{V_i\}_{i=1}^3$, recal that λ_i for $i = 1, 2, 3$ are the coefficients in the Lie bracket relation (1.4). We compute

$$
\begin{aligned}
g(\nabla_{V_i} V_{i+1}, V_{i+2}) &= \frac{1}{2}\left(g([V_i, V_{i+1}], V_{i+2}) - g([V_i, V_{i+2}], V_{i+1}) - g([V_{i+1}, V_{i+2}], V_i)\right), \\
&= \frac{1}{2}\left(g(\lambda_{i+2}V_{i+2}, V_{i+2}) - g(-\lambda_{i+1}V_{i+1}, V_{i+1}) - g(\lambda_i V_i, V_i)\right), \\
&= \frac{\lambda_{i+2}x_{i+2} + \lambda_{i+1}x_{i+1} - \lambda_i x_i}{2}.
\end{aligned}
$$

This gives us that

$$
\nabla_{V_i} V_{i+1} = \frac{\lambda_{i+2}x_{i+2} + \lambda_{i+1}x_{i+1} - \lambda_i x_i}{2x_{i+2}}V_{i+2}.
$$

The Riemann curvature tensor is given by

$$
R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]}Z,
$$

and is symmetric. Thus, we only need to compute $R(V_i, V_{i+1}, V_i, V_{i+1})$ for $i = 1, 2, 3$. We use the fact that $R(X, Y, X, Y) = g(R(X, Y)Y, X)$ and hence compute:

$$
R(V_i, V_{i+1}, V_i, V_{i+1}) = \lambda_{i+2}\left(\mathbf{x}_{i+2} - \frac{\mathbf{x}_i \mathbf{x}_{i+1}}{\mathbf{x}_i + \mathbf{x}_{i+1}}\right).
$$

Now we compute the Ricci tensor elements, $\mathrm{Ric}_{bd} = g^{ac}R_{abcd}$. It is important to note that $R_{abcd} = R_{badc}$. Then,

$$
\mathrm{Ric}(V_i, V_j) = \delta_{ij}\frac{2\mathbf{x}_{i+1}\mathbf{x}_{i+2}}{x_{i+1}x_{i+2}}.
$$

The scalar curvature is given by

$$
\begin{aligned}
S &= g^{ij}\,\mathrm{Ric}(V_i, V_j), \\
&= \frac{1}{x_1}\frac{2\mathbf{x}_2\mathbf{x}_3}{x_2 x_3} + \frac{1}{x_2}\frac{2\mathbf{x}_1\mathbf{x}_3}{x_1 x_3} + \frac{1}{x_3}\frac{2\mathbf{x}_1\mathbf{x}_2}{x_1 x_2}, \\
&= \frac{2}{x_1 x_2 x_3}\left(\mathbf{x}_2\mathbf{x}_3 + \mathbf{x}_1\mathbf{x}_3 + \mathbf{x}_1\mathbf{x}_2\right).
\end{aligned}
$$

We now compute the $(1,1)$-tensor field $\mathcal{E}(g)$ elements,

$$
\mathcal{E}_j^i = g^{ik}\,\mathrm{Ric}_{jk} - \frac{S}{2}\delta_j^i,
$$

where δ_j^i is the Kronecker symbol. We see that the off diagonal elements vanish and so,

$$
\begin{aligned}
\mathcal{E}_i^i &= \frac{1}{x_i}\frac{2\mathbf{x}_{i+1}\mathbf{x}_{i+2}}{x_{i+1}x_{i+2}} - \frac{\frac{2}{x_1 x_2 x_3}\left(\mathbf{x}_{i+1}\mathbf{x}_{i+2} + \mathbf{x}_i\mathbf{x}_{i+2} + \mathbf{x}_i\mathbf{x}_{i+1}\right)}{2}, \\
&= \frac{\mathbf{x}_{i+1}\mathbf{x}_{i+2} - \mathbf{x}_i\mathbf{x}_{i+2} - \mathbf{x}_i\mathbf{x}_{i+1}}{x_i^2 x_{i+1}x_{i+2}}.
\end{aligned}
$$

Using (1.3) we compute the cross curvature tensor,

$$
X(g)(V_i, V_j) = \delta_{ij}\frac{\left(\mathbf{x}_i\mathbf{x}_{i+2} - \mathbf{x}_{i+1}\mathbf{x}_{i+2} - \mathbf{x}_i\mathbf{x}_{i+1}\right)\left(\mathbf{x}_i\mathbf{x}_{i+1} - \mathbf{x}_i\mathbf{x}_{i+2} - \mathbf{x}_{i+1}\mathbf{x}_{i+2}\right)}{x_i x_{i+1}^2 x_{i+2}^2} \tag{2.2}
$$

$\square$

Given a basis in which T is diagonal with $T(V_i, V_j) = \delta_{ij}T_i$, we see that solving (1.2) is equivalent to solving the following system of equations:

$$\begin{cases} \frac{(\mathbf{x_1x_3}-\mathbf{x_2x_3}-\mathbf{x_1x_2})(\mathbf{x_1x_2}-\mathbf{x_1x_3}-\mathbf{x_2x_3})}{x_1x_2^2x_3^2} = T_1, \\ \frac{(\mathbf{x_2x_3}-\mathbf{x_1x_3}-\mathbf{x_1x_2})(\mathbf{x_1x_2}-\mathbf{x_1x_3}-\mathbf{x_2x_3})}{x_1^2x_2x_3^2} = T_2, \\ \frac{(\mathbf{x_2x_3}-\mathbf{x_1x_3}-\mathbf{x_1x_2})(\mathbf{x_1x_3}-\mathbf{x_2x_3}-\mathbf{x_1x_2})}{x_1^2x_2^2x_3} = T_3. \end{cases} \tag{2.3}$$

Recall here $\mathbf{x}_i = \frac{\lambda_{i+2}x_{i+2}+\lambda_{i+1}x_{i+1}-\lambda_i x_i}{2}$ for $i = 1, 2, 3$.

3. Existence and Uniqueness Argument

We present two lemmas that are analogs of results proven in [But19].

Lemma 3.1. *Let T be a left-invariant $(0, 2)$-tensor field on a three-dimensional unimodular Lie group G. There exists a left-invariant metric g solving (1.2) if and only if the following conditions hold:*

(i) *There exists a basis $\{V_i\}_{i=1}^3$ of $\mathfrak{g}$ satisfying (1.4) in which T is diagonal.*
(ii) *There is a solution (x_1, x_2, x_3) of (2.3).*

Lemma 3.2. *Let T be a non-zero left-invariant $(0, 2)$-tensor field on G and let g be a left-invariant metric such that $X(g) = T$. The metric g is unique if the following conditions hold:*

(i) *g is diagonal in any basis $\{V_i\}_{i=1}^3$ such that T is diagonal and (2.3) holds, and*
(ii) *The solution (x_1, x_2, x_3) of (2.3) is unique whenever T_1, T_2, T_3 are the diagonal components of T in a basis $\{V_i\}_{i=1}^3$ satisfying (1.4).*

By lemma 3.1, a solution (x_1, x_2, x_3) to (2.3) gives the existence of a metric solving (1.2). Claiming this metric is unique requires a more subtle argument than solving (1.2) with unique values for x_1, x_2 and x_3. Suppose we have a solution, g, to (1.2) in a basis where both g and T are diagonal. Now, fix a new basis in which T is still diagonal but g is not. This could lead us to another solution for $X(g) = T$. So, given all bases where T is diagonal, we must determine whether our solution g is also diagonal. The following lemma from [But19] is utilised in conjunction with lemma 3.2 to give conditions on the uniqueness of solutions to (1.2).

Lemma 3.3. *For $E(2), E(1, 1)$ and H_3, let $U_i = \sum_{j=1}^3 a_{ji}V_j$ be the change of basis from $\{V_i\}_{i=1}^3$ that satisfies (1.4). If $\{U_i\}_{i=1}^3$ satisfies (1.4), the following constraints must be satisfied for a given λ_2 value:*

(i) *$\lambda_2 = 2$: $\quad a_{31} = a_{32} = 0, \quad a_{33} = \pm 1, \quad a_{21} = \mp a_{12}$ and $a_{11} = \pm a_{22}$.*
(ii) *$\lambda_2 = -2$: $\quad a_{31} = a_{32} = 0, \quad a_{33} = \pm 1, \quad a_{21} = \pm a_{12}$ and $a_{11} = \pm a_{22}$.*
(iii) *$\lambda_2 = 0$: $\quad a_{11} = a_{22}a_{33} - a_{32}a_{23}$ and $a_{21} = a_{31} = 0$.*

To verify condition (i) of lemma 3.2, we need only check that if the matrix representation of T remains diagonal under change of basis, the matrix representation of the metric also remains diagonal. As both are diagonal to begin with, this amounts to showing that the change of basis maps diagonal matrices to diagonal matrices. We compute the change of basis for the matrix of T to be

$$A^T \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{pmatrix} A.$$

That is,

$$\begin{pmatrix} a_{11}^2 T_1 + a_{21}^2 T_2 + a_{31}^2 T_3 & a_{11}a_{12}T_1 + a_{21}a_{22}T_2 + a_{31}a_{32}T_3 & a_{11}a_{13}T_1 + a_{21}a_{23}T_2 + a_{31}a_{33}T_3 \\ a_{11}a_{12}T_1 + a_{21}a_{22}T_2 + a_{31}a_{32}T_3 & a_{12}^2 T_1 + a_{22}^2 T_2 + a_{32}^2 T_3 & a_{12}a_{13}T_1 + a_{22}a_{23}T_2 + a_{32}a_{33}T_3 \\ a_{11}a_{13}T_1 + a_{21}a_{23}T_2 + a_{31}a_{33}T_3 & a_{12}a_{13}T_1 + a_{22}a_{23}T_2 + a_{32}a_{33}T_3 & a_{13}^2 T_1 + a_{23}^2 T_2 + a_{33}^2 T_3 \end{pmatrix}$$

It can be seen that if $T_i \neq 0$ for all $i = 1, 2, 3$, the above being diagonal implies that

$$A^T \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix} A$$

is also diagonal in all three Lie groups.

4. RESULTS ON $E(2)$, $E(1,1)$ AND H_3

4.1. The Euclidean Group, $E(2)$.

Theorem 4.1. *Let T be a left-invariant $(0,2)$ tensor field on $E(2)$. There exists a left-invariant metric g such that $X(g) = T$ if and only if T is diagonalisable in a basis $\{V_i\}_{i=1}^3$ satisfying (1.4) with $\lambda_1 = \lambda_2 = \lambda_3 = 2$ and exactly one of the following conditions is satisfied:*

(i) $T_1 = T_2 = T_3$,
(ii) $T_1 > 0$ and $T_2, T_3 < 0$,
(iii) $T_1, T_3 < 0$ and $T_2 > 0$.

In (i), there are infinitely many solutions to (1.2) while in (ii) and (iii), the solution g is unique.

Proof. Fix a basis $\{V_i\}_{i=1}^3$ of $\mathfrak{g}$ in which T is diagonal and (1.4) holds. We compute the variables $\mathbf{x}_1 = x_2 - x_1, \mathbf{x}_2 = x_1 - x_2, \mathbf{x}_3 = x_1 + x_2$. This simplifies (2.3) to

$$\begin{cases} T_1 = \frac{(x_1 + 3x_2)(x_1 - x_2)^3}{x_1 x_2^2 x_3^2} \\ T_2 = \frac{-(3x_1 + x_2)(x_1 - x_2)^3}{x_1^2 x_2 x_3^2} \\ T_3 = \frac{-(x_2 + 3x_1)(3x_2 + x_1)(x_1 - x_2)^2}{x_1^2 x_2^2 x_3}. \end{cases} \tag{4.1}$$

We treat the conditions by cases:

Condition (i): As $x_1, x_2, x_3 > 0$, for $T_1 = 0$ we require $x_1 = x_2$. This ensures $T_2 = T_3 = 0$. As there is no dependence on x_3, there are infinite solutions g to (1.2).

Condition (ii): From the first equation we have $x_3^2 = \frac{(x_1 + 3x_2)(x_1 - x_2)^3}{x_1 x_2^2 T_1}$. Substituting this into the second equation we find, $0 = x_1^2 T_2 + 3x_1 x_2 (T_1 + T_2) + x_2^2 T_1$. This gives an expression for x_1,

$$x_1 = x_2 \frac{-3(T_1 + T_2) \pm (9T_1^2 + 14T_1 T_2 + 9T_2^2)^{\frac{1}{2}}}{2T_2}.$$

As $x_1 > 0$, we see $-3(T_1 + T_2) \pm (9T_1^2 + 14T_1 T_2 + 9T_2^2) < 0$. Since $-3(T_1 + T_2) < (9T_1^2 + 14T_1 T_2 + 9T_2^2)$, we may disregard the positive and solve uniquely for (x_1, x_2, x_3).

Condition (iii): This case is similar to the previous however as $T_2 > 0$, we disregard the negative of the square root in the expression for x_1. Again we obtain a unique solution of (x_1, x_2, x_3). This solves the existence of our metric.

In cases (ii) and (iii), we have that all $T_i \neq 0$ for $i = 1, 2, 3$. Thus, by the discussion in § 3, we apply lemma 3.2 to obtain uniqueness of the metric. □

4.2. **The Minkowski Group,** $E(1,1)$.

Theorem 4.2. *Let T be a left-invariant $(0,2)$ tensor field on $E(1,1)$. There exists a left-invariant Riemannian metric g such that $X(g) = T$ if and only if T is diagonalisable in a basis $\{V_i\}_{i=1}^3$ satisfying (1.4) with $\lambda_1 = 2, \lambda_2 = -2, \lambda_3 = 0$ and one of the following conditions is satisfied:*

  (i) $T_1 < 0, , T_2 = T_3 = 0$,
  (ii) $T_1 = T_3 = 0, T_2 < 0$,
  (iii) $T_1 > 0, T_2, T_3 < 0$,
  (iv) $T_1 < 0, T_2, T_3 > 0$, *and*
  (v) $T_1, T_2, T_3 < 0$.

Here we denote $T_i = T(V_i, V_i)$. In (i) and (ii) there are infinitely many solutions g to (1.2) while in (iii), (iv) and (v), the solution g is unique.

Proof. Fix a basis $\{V_i\}_{i=1}^3$ of $\mathfrak{g}$ in which T is diagonal and (1.4) holds. We compute the variables $\mathbf{x}_1 = -(x_2 + x_1), \mathbf{x}_2 = x_2 + x_1, \mathbf{x}_3 = x_1 - x_2$. This simplifies (2.3) to

$$\begin{cases} T_1 = \frac{(x_1 - 3x_2)(x_1 + x_2)^3}{x_1 x_2^2 x_3^2} \\ T_2 = \frac{-(3x_1 - x_2)(x_1 + x_2)^3}{x_1^2 x_2 x_3^2} \\ T_3 = \frac{-(x_2 - 3x_1)(3x_2 - x_1)(x_1 + x_2)^2}{x_1^2 x_2^2 x_3}. \end{cases} \tag{4.2}$$

Condition (i): Given $T_2, T_3 = 0$ we conclude that $3x_1 = x_2$ as $x_1, x_2, x_3 > 0$. This simplifies the first equation to be $T_1 = \frac{-512x_1}{9x_3^2}$. This has infinitely many solutions.

Condition (ii): Similar to above, given $T_1, T_3 = 0$ we find $T_2 = \frac{-512x_2}{9x_3^2}$ which has infinitely many solutions.

For the next three conditions, we solve for x_3^2 and substitute the expression into the second equation, solving for x_1. We find, $x_3^2 = \frac{(x_1 - 3x_2)(x_1 + x_2)^3}{x_1 x_2^2 T_1}$ and hence $x_1 = x_2 \frac{-3(T_1 - T_2) \pm \left(9T_1^2 - 14T_1 T_2 + 9T_2^2\right)^{\frac{1}{2}}}{2T_2}$.

Condition (iii): As $T_2, T_3 < 0$ and $T_1 > 0$, we require $x_1 \geq 3x_2$. This implies $3 \leq \frac{-3(T_1 - T_2) \pm \left(9T_1^2 - 14T_1 T_2 + 9T_2^2\right)^{\frac{1}{2}}}{2T_2}$ and so $3T_2 + 3T_1 \geq \pm \left(9T_1^2 - 14T_1 T_2 + 9T_2^2\right)^{\frac{1}{2}}$. This gives us two expressions, the first of which cannote hold since $T_2 < 0$. Thus we take the negative of the square root, giving us a single expression for x_1 in terms of x_2. Substituting this into our expression for x_3 and finally into the third equation, we solve a unique expression for x_2. We then recover expressions for x_1 and x_3, giving us a unique solution.

Condition (iv): This case is similar to the previous condition however we find $\frac{1}{3}x_2 \leq x_1 \leq 3x_2$ and so $\frac{1}{3} \leq \frac{-3(T_1 - T_2) \pm \left(9T_1^2 - 14T_1 T_2 + 9T_2^2\right)^{\frac{1}{2}}}{2T_2}$. Again this gives us two expressions,

$$\frac{49}{9}T_2^2 - 14T_1 T_2 + 9T_1^2 < 9T_2^2 - 14T_1 T_2 + 9T_1^2$$

$$\frac{49}{9}T_2^2 - 14T_1 T_2 + 9T_1^2 > 9T_2^2 - 14T_1 T_2 + 9T_1^2$$

We disregard the first and hence solve uniquely for x_2, recovering x_1 and x_3 in the process.

Condition (v): Here we find that $x_1 \leq \frac{1}{3}x_2$ and so $\frac{1}{3} \geq \frac{-3(T_1-T_2)\pm\left(9T_1^2-14T_1T_2+9T_2^2\right)^{\frac{1}{2}}}{2T_2}$. Similarly to the two previous cases we have two inequalities:

$$\frac{49}{9}T_2^2 - 14T_1T_2 + 9T_1^2 < 9T_2^2 - 14T_1T_2 + 9T_1^2$$

$$\frac{49}{9}T_2^2 - 14T_1T_2 + 9T_1^2 > 9T_2^2 - 14T_1T_2 + 9T_1^2$$

The second inequality cannot hold and so we disregard it. This allows us to conclude a unique solution (x_1, x_2, x_3) to (4.2). This concludes the existence of a metric solving $X(g) = T$ in this basis. For uniqueness, we note that in cases $(iii), (iv)$ and (v), all $T_i \neq 0$ for $i = 1, 2, 3$. Thus, we may apply lemma 3.2 to conclude that our solution g solving (1.2) is unique. $\qquad \square$

4.3. **The Heisenberg Group,** H_3.

Theorem 4.3. *Let T be a left-invariant $(0, 2)$ tensor field on H_3. There exists a unique left-invariant Riemannian metric g such that $X(g) = T$ if and only if T is diagonalisable in a basis $\{V_i\}_{i=1}^3$ satisfying (1.4) with $\lambda_1 = 2, \lambda_2 = \lambda_3 = 0$ and $T_1 > 0, T_2, T_3 < 0$. Here we denote $T(V_i, V_i) = T_i$.*

Proof. Fix a basis $\{V_i\}_{i=1}^3$ of the Lie algebra $\mathfrak{g}$ in which T is diagonal and (1.4) holds. We compute the variables $\mathbf{x}_1 = -x_1, \mathbf{x}_2 = x_1, \mathbf{x}_3 = x_1$. This simplifies (2.3) to

$$\begin{cases} T_1 = \frac{x_1^3}{x_2^2 x_3^2} \\ T_2 = \frac{-3x_1^2}{x_2 x_3^2} \\ T_3 = \frac{-3x_1^2}{x_2^2 x_3} \end{cases} \tag{4.3}$$

We rearrange the second equation, resulting in $x_2 = \frac{-3x_1^2}{x_3^2 T_2}$. Substituting this into the first equation we find an expression, $x_1 = \frac{x_3^2 T_2^2}{9T_1}$. Finally we substitute both expressions into the third equation to find $x_3 = \frac{-27T_1^2}{T_2^2 T_3}$. Now we compute $x_1 = \frac{81T_1^3}{T_2^2 T_3^2}$ and $x_2 = \frac{-27T_1^2}{T_2 T_3^2}$. As these three expressions depend only on T, there must exist a single metric g solving $X(g) = T$ in our fixed basis. By lemma 3.1, this proves the existence portion of the theorem. Since all $T_i \neq 0$ for $i = 1, 2, 3$, as per § 3, we may apply lemma 3.2 giving us uniqueness of g solving (1.2). $\qquad \square$

4.4. **Euclidean Space,** $\mathbb{R}^3$.
The fourth Lie group with at least 1 structure coefficient being zero is Euclidean space. Indeed, all structure coefficients λ_i are 0. $\mathbb{R}^3$ is Ricci-flat and so it has Einstein tensor 0. Thus $\mathcal{E}$ is not well defined and so the cross curvature is not well defined for $\mathbb{R}^3$.

This concludes the analysis on three-dimensional unimodular Lie groups.

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