THE PRESCRIBED CROSS CURVATURE PROBLEM ON THREE-DIMENSIONAL UNIMODULAR LIE GROUPS

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Abstract. Let G be a three-dimensional unimodular Lie group and let T be a left-invariant $(0,2)$-tensor field on G. We study the necessary and sufficient conditions for the existence of a left-invariant metric g that satisfies the equation $X(g) = T$ on three-dimensional unimodular Lie groups, where $X(g)$ is the cross curvature of g. We discuss uniqueness of solutions for the groups $\mathsf{E}(2), \mathsf{E}(1,1), \mathsf{H}_3$ and $\mathbb{R}^3$.

1. Introduction

The problem of prescribing curvature on a given manifold is a fundamental topic in Riemannian geometry. A particularly well-studied instance of this is the problem of prescribing Ricci curvature. Let M be a Riemannian manifold, and let T be a given $(0,2)$-tensor field. One is interested in the existence and uniqueness of a Riemannian metric g such that

$$\mathrm{Ric}(g) = T. \tag{1.1}$$

Hamilton proved existence in the case where $M = \mathsf{SU}(2)$ [Ham84]. Buttsworth later extended the result to $\mathbb{S}^2$, $\mathsf{E}(2), \mathsf{E}(1,1), \mathsf{H}_3$ and $\mathbb{R}^3$. Motivated by a restriction on the solvability of (1.1) on $\mathbb{S}^2$ [Ham84], [DeT16], and the scale invariance of the Ricci curvature, work on the prescribed Ricci curvature problem often considers (1.1) with a scalar multiple of T. In this paper, we study the analogue of this problem for the cross curvature tensor.

The cross curvature, introduced by Chow and Hamilton [CH04] in the context of uniformising three-manifolds, is a $(0,2)$-tensor field $X(g)$. Let M be a Riemannian manifold. The prescribed cross curvature problem asks for a given $(0,2)$-tensor field T, does there exist a Riemannian metric g such that,

$$X(g) = T. \tag{1.2}$$

Some existence and uniqueness results have been obtained when $M = \mathbb{S}^3$ [BP21], [Gki08]. We consider the particular problem of prescribing cross curvature of left-invariant metrics on three-dimensional unimodular Lie groups. Note that the cross curvature is not scaling invariant and hence we drop the scaling factor in (1.2).

Definition 1.1 (Einstein Tensor Field)**.** Let (M, g) be a three-dimensional Riemannian manifold. Define the $(0,2)$-tensor field

$$\mathrm{Ein}(g) := \mathrm{Ric}(g) - g\frac{S}{2},$$

where S is the scalar curvature of M.

We find it useful to define the $(1,1)$-tensor

$$\mathcal{E}(g) = \left(\mathrm{Ric}(g) - g\frac{S}{2}\right)^{\#}.$$

Definition 1.2 (Cross Curvature)**.** The cross curvature tensor is

$$X(g)(\cdot,\cdot) := \det(\mathcal{E})\; g\left(\mathcal{E}^{-1}\cdot,\cdot\right). \tag{1.3}$$

Gkigkitzis defines an analogous problem to (1.2) for the cross transformation, a $(1,2)$-tensor, and proves a result on the existence and uniqueness of left-invariant metrics on $SU(2)$ [Gki08]. Pulemotov and Buttsworth show that the two problems are equivalent [BP21].

Let G be a three-dimensional unimodular Lie group with Lie algebra $\mathfrak{g}$. For any left-invariant metric g, there exists a basis $\{V_1, V_2, V_3\}$ of $\mathfrak{g}$ that diagonalises the metric and gives us the Lie bracket relations:

$$[V_i, V_j] = \sum_{k=1}^{3} \epsilon_{ijk}\lambda_k V_k \tag{1.4}$$

for some coefficients $\lambda_k \in \mathbb{R}$ [HL09]. The structure coefficients λ_k can be scaled and the basis of $\mathfrak{g}$ can be oriented to uniquely describe each of the 6 unimodular three-dimensional Lie groups. In [But19], it was shown to be sufficient to consider values of λ_k in $\{-2,0,2\}$. We focus on the four groups in which at least one of the λ_k values is 0 and obtain the following result:

Theorem 1.3. Let T be a left-invariant $(0,2)$-tensor field on $\mathsf{E}(2), \mathsf{E}(1,1)$ or H_3. Then, there exists a left-invariant Riemannian metric g solving (1.2) if and only if:

(i) T is diagonalisable in a basis $\{V_i\}_{i=1}^3$ in which (1.4) holds, and
(ii) The diagonal elements $T(V_i,V_i) =: T_i$ have signature as in theorems 4.1, 4.2 and 4.3 $\mathsf{E}(2), \mathsf{E}(1,1)$ and H_3 respectively.

In the same basis, if none of the diagonal elements T_i are zero, the solution g is unique.

2. The Cross Curvature Tensor on a Unimodular Lie Group

In this section we compute the cross curvature tensor for a three-dimensional unimodular Lie group G:

Lemma 2.1. Let (G, g) be a three-dimensional unimodular Lie group with Lie algebra $\mathfrak{g}$. Let $\{V_i\}_{i=1}^3$ be a basis of $\mathfrak{g}$ that diagonalises g. That is, $g = x_1 \mathrm{d}\theta_1 + x_2 \mathrm{d}\theta_2 + x_3 \mathrm{d}\theta_3$ where $\{\theta_i\}_{i=1}^3$ is the basis of $\mathfrak{g}^*$. Then the cross curvature tensor is:

$$X(g)(V_i,V_j) = \delta_{ij}\frac{\left(\mathbf{x}_i\mathbf{x}_{i+2} - \mathbf{x}_{i+1}\mathbf{x}_{i+2} - \mathbf{x}_i\mathbf{x}_{i+1}\right)\left(\mathbf{x}_i\mathbf{x}_{i+1} - \mathbf{x}_i\mathbf{x}_{i+2} - \mathbf{x}_{i+1}\mathbf{x}_{i+2}\right)}{x_i x_{i+1}^2 x_{i+2}^2} \tag{2.1}$$

where $\mathbf{x}_i := \frac{\lambda_{i+2}x_{i+2} + \lambda_{i+1}x_{i+1} - \lambda_i x_i}{2}$ for $i = 1,2,3$.

Proof. Fix a basis $\{V_i\}_{i=1}^3$ in which (1.4) holds and g is diagonal. We first compute

$$\begin{aligned} g(\nabla_{V_i}V_{i+1}, V_{i+2}) &= \frac{1}{2}\left(g([V_i, V_{i+1}], V_{i+2}) - g([V_i, V_{i+2}], V_{i+1}) - g([V_{i+1}, V_{i+2}], V_i)\right), \\ &= \frac{1}{2}\left(g(\lambda_{i+2}V_{i+2}, V_{i+2}) - g(-\lambda_{i+1}V_{i+1}, V_{i+1}) - g(\lambda_i V_i, V_i)\right), \\ &= \frac{\lambda_{i+2}x_{i+2} + \lambda_{i+1}x_{i+1} - \lambda_i x_i}{2} \end{aligned}$$

by the Koszul formula:

$$g(\nabla_X Y, Z) = \frac{1}{2}\left(g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X)\right).$$

This gives us that

$$\nabla_{V_i}V_{i+1} = \frac{\lambda_{i+2}x_{i+2} + \lambda_{i+1}x_{i+1} - \lambda_i x_i}{2x_{i+2}} V_{i+2}.$$

Since the Riemann curvature tensor is symmetric, we only need to compute $R(V_i, V_{i+1}, V_i, V_{i+1})$ for $i = 1, 2, 3$. We use the fact that $R(X, Y, X, Y) = g(R(X, Y)Y, X)$ to compute:

$$R(V_i, V_{i+1}, V_i, V_{i+1}) = \lambda_{i+2}\left(\mathbf{x}_{i+2} - \frac{\mathbf{x}_i\mathbf{x}_{i+1}}{\mathbf{x}_i + \mathbf{x}_{i+1}}\right)$$

and subsequenty

$$\mathrm{Ric}(V_i, V_j) = \delta_{ij}\frac{2\mathbf{x}_{i+1}\mathbf{x}_{i+2}}{x_{i+1}x_{i+2}}.$$

The scalar curvature is then given by

$$\begin{aligned} S &= g^{ij}\,\mathrm{Ric}(V_i, V_j), \\ &= \frac{1}{x_1}\frac{2\mathbf{x}_2\mathbf{x}_3}{x_2x_3} + \frac{1}{x_2}\frac{2\mathbf{x}_1\mathbf{x}_3}{x_1x_3} + \frac{1}{x_3}\frac{2\mathbf{x}_1\mathbf{x}_2}{x_1x_2}, \\ &= \frac{2}{x_1x_2x_3}\left(\mathbf{x}_2\mathbf{x}_3 + \mathbf{x}_1\mathbf{x}_3 + \mathbf{x}_1\mathbf{x}_2\right). \end{aligned}$$

We now have all the elements required to compute the $(1,1)$-tensor field $\mathcal{E}(g)$:

$$\mathcal{E}^i_j = g^{ik}\,\mathrm{Ric}_{jk} - \frac{S}{2}\delta^i_j.$$

The off diagonal elements vanish and we are left with

$$\begin{aligned} \mathcal{E}^i_i &= \frac{1}{x_i}\frac{2\mathbf{x}_{i+1}\mathbf{x}_{i+2}}{x_{i+1}x_{i+2}} - \frac{\frac{2}{x_1x_2x_3}\left(\mathbf{x}_{i+1}\mathbf{x}_{i+2} + \mathbf{x}_i\mathbf{x}_{i+2} + \mathbf{x}_i\mathbf{x}_{i+1}\right)}{2}, \\ &= \frac{\mathbf{x}_{i+1}\mathbf{x}_{i+2} - \mathbf{x}_i\mathbf{x}_{i+2} - \mathbf{x}_i\mathbf{x}_{i+1}}{x_i^2x_{i+1}x_{i+2}}. \end{aligned}$$

Using (1.3) we finally obtain

$$X(g)(V_i, V_j) = \delta_{ij}\frac{\left(\mathbf{x}_i\mathbf{x}_{i+2} - \mathbf{x}_{i+1}\mathbf{x}_{i+2} - \mathbf{x}_i\mathbf{x}_{i+1}\right)\left(\mathbf{x}_i\mathbf{x}_{i+1} - \mathbf{x}_i\mathbf{x}_{i+2} - \mathbf{x}_{i+1}\mathbf{x}_{i+2}\right)}{x_ix_{i+1}^2x_{i+2}^2} \tag{2.2}$$

□

Given a basis in which T is diagonal with $T(V_i, V_j) = \delta_{ij} T_i$, we see that solving (1.2) is equivalent to solving the following system of equations:

$$\begin{cases} \frac{(\mathbf{x}_1\mathbf{x}_3 - \mathbf{x}_2\mathbf{x}_3 - \mathbf{x}_1\mathbf{x}_2)(\mathbf{x}_1\mathbf{x}_2 - \mathbf{x}_1\mathbf{x}_3 - \mathbf{x}_2\mathbf{x}_3)}{x_1 x_2^2 x_3^2} = T_1, \\ \frac{(\mathbf{x}_2\mathbf{x}_3 - \mathbf{x}_1\mathbf{x}_3 - \mathbf{x}_1\mathbf{x}_2)(\mathbf{x}_1\mathbf{x}_2 - \mathbf{x}_1\mathbf{x}_3 - \mathbf{x}_2\mathbf{x}_3)}{x_1^2 x_2 x_3^2} = T_2, \\ \frac{(\mathbf{x}_2\mathbf{x}_3 - \mathbf{x}_1\mathbf{x}_3 - \mathbf{x}_1\mathbf{x}_2)(\mathbf{x}_1\mathbf{x}_3 - \mathbf{x}_2\mathbf{x}_3 - \mathbf{x}_1\mathbf{x}_2)}{x_1^2 x_2^2 x_3} = T_3. \end{cases} \tag{2.3}$$

Recall here $\mathbf{x}_i = \frac{\lambda_{i+2}x_{i+2} + \lambda_{i+1}x_{i+1} - \lambda_i x_i}{2}$ for $i = 1, 2, 3$.

3. Existence and Uniqueness

We present two lemmas analogous to Lemmas 3.1 and 3.2 in [But19].

Lemma 3.1. Let T be a left-invariant $(0, 2)$-tensor field on a three-dimensional unimodular Lie group G. There exists a left-invariant metric g solving (1.2) if and only if the following conditions hold:

(i) There exists a basis $\{V_i\}_{i=1}^3$ of $\mathfrak{g}$ satisfying (1.4) in which T is diagonal.
(ii) There is a solution (x_1, x_2, x_3) of (2.3).

Lemma 3.2. Let T be a non-zero left-invariant $(0, 2)$-tensor field on G and let g be a left-invariant metric such that $X(g) = T$. The metric g is unique if the following hold:

(i) Given a basis $\{V_i\}_{i=1}^3$ such that T is diagonal and (2.3) holds, then g is diagonal;
(ii) The solution (x_1, x_2, x_3) of (2.3) is unique whenever T_1, T_2, T_3 are the diagonal components of T in a basis $\{V_i\}_{i=1}^3$ satisfying (1.4).

The following lemma from [But19] is utilised in conjunction with lemma 3.2 to give conditions on the uniqueness of solutions to (1.2).

Lemma 3.3. For $\mathsf{E}(2), \mathsf{E}(1, 1)$, and H_3, let $U_i = \sum_{j=1}^3 a_{ji} V_j$ be the change of basis from $\{V_i\}_{i=1}^3$ that satisfies (1.4). If $\{U_i\}_{i=1}^3$ satisfies (1.4), the following constraints must be satisfied for a given value of λ_2:

(i) $\lambda_2 = 2$:

$$a_{31} = a_{32} = 0, \qquad a_{33} = \pm 1,$$
$$a_{21} = \mp a_{12}, \qquad a_{11} = \pm a_{22}.$$

(ii) $\lambda_2 = -2$:

$$a_{31} = a_{32} = 0, \qquad a_{33} = \pm 1,$$
$$a_{21} = \pm a_{12}, \qquad a_{11} = \pm a_{22}.$$

(iii) $\lambda_2 = 0$:

$$a_{11} = a_{22}a_{33} - a_{32}a_{23},$$
$$a_{21} = a_{31} = 0.$$

Condition (i) of lemma 3.2 amounts to showing that the change of basis maps diagonal matrices to diagonal matrices. We compute the change of basis for the matrix of T to be

$$A^T \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{pmatrix} A.$$

That is,

$$\begin{pmatrix} a_{11}^2T_1+a_{21}^2T_2+a_{31}^2T_3 & a_{11}a_{12}T_1+a_{21}a_{22}T_2+a_{31}a_{32}T_3 & a_{11}a_{13}T_1+a_{21}a_{23}T_2+a_{31}a_{33}T_3 \\ a_{11}a_{12}T_1+a_{21}a_{22}T_2+a_{31}a_{32}T_3 & a_{12}^2T_1+a_{22}^2T_2+a_{32}^2T_3 & a_{12}a_{13}T_1+a_{22}a_{23}T_2+a_{32}a_{33}T_3 \\ a_{11}a_{13}T_1+a_{21}a_{23}T_2+a_{31}a_{33}T_3 & a_{12}a_{13}T_1+a_{22}a_{23}T_2+a_{32}a_{33}T_3 & a_{13}^2T_1+a_{23}^2T_2+a_{33}^2T_3 \end{pmatrix}$$

It is not hard to verify that if the above is diagonal and all $T_i \neq 0$, then

$$A^T \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix} A$$

is also diagonal.

4. Results for $\mathsf{E}(2)$, $\mathsf{E}(1,1)$ and H_3

4.1. **The Euclidean Group, $\mathsf{E}(2)$.**

Theorem 4.1. Let T be a left-invariant $(0,2)$ tensor field on $\mathsf{E}(2)$. There exists a left-invariant metric g such that $X(g) = T$ if and only if T is diagonalisable in a basis $\{V_i\}_{i=1}^3$ satisfying (1.4) with $\lambda_1 = \lambda_2 = \lambda_3 = 2$ and exactly one of the following conditions is satisfied:

(i) $T_1 = T_2 = T_3$,
(ii) $T_1 > 0$ and $T_2, T_3 < 0$,
(iii) $T_1, T_3 < 0$ and $T_2 > 0$.

In *(i)*, there are infinitely many solutions. In *(ii)* and *(iii)*, the solution g is unique.

Proof. Fix a basis $\{V_i\}_{i=1}^3$ of $\mathfrak{g}$ in which T is diagonal and (1.4) holds. We compute the variables $\mathbf{x}_1 = x_2 - x_1,\ \mathbf{x}_2 = x_1 - x_2,$ and $\mathbf{x}_3 = x_1 + x_2$. This simplifies (2.3) to

$$\begin{cases} T_1 = \frac{(x_1+3x_2)(x_1-x_2)^3}{x_1x_2^2x_3^2} \\ T_2 = \frac{-(3x_1+x_2)(x_1-x_2)^3}{x_1^2x_2x_3^2} \\ T_3 = \frac{-(x_2+3x_1)(3x_2+x_1)(x_1-x_2)^2}{x_1^2x_2^2x_3}. \end{cases} \tag{4.1}$$

By the positive definiteness of g, we know that $T_3 \leq 0$ and T_2 is opposite in sign to T_1. We proceed with three cases:

(i) Assume $T_1 = 0$. As $x_1, x_2, x_3 > 0$ we know $x_1 = x_2$. This ensures $T_2 = T_3 = 0$. As there is no dependence on x_3, there are infinite solutions g to (1.2).
(ii) Assume $T_1 > 0$. From the first equation we have $x_3^2 = \frac{(x_1+3x_2)(x_1-x_2)^3}{x_1x_2^2T_1}$. Substituting this into the second equation we find, $0 = x_1^2T_2 + 3x_1x_2(T_1+T_2) + x_2^2T_1$. This gives an expression for x_1,

$$x_1 = x_2\frac{-3(T_1+T_2) \pm (9T_1^2 + 14T_1T_2 + 9T_2^2)^{\frac{1}{2}}}{2T_2}.$$

As $x_1 > 0$, we see $-3(T_1+T_2) \pm (9T_1^2 + 14T_1T_2 + 9T_2^2) < 0$. Since $-3(T_1+T_2) < (9T_1^2 + 14T_1T_2 + 9T_2^2)$, we may disregard the positive and solve uniquely for (x_1, x_2, x_3).
(iii) Assume $T_1 < 0$. This case is similar to the previous however as $T_2 > 0$, we disregard the negative of the square root in the expression for x_1. Again we obtain a unique solution of (x_1, x_2, x_3). This solves the existence of our metric.

In cases (ii) and (iii), we have that all $T_i \neq 0$ for $i = 1, 2, 3$. Thus, by the discussion in § 3, we apply lemma 3.2 to obtain uniqueness of the metric. $\square$

4.2. **The Minkowski Group, $\mathsf{E}(1,1)$.**

Theorem 4.2. Let T be a left-invariant $(0,2)$ tensor field on $\mathsf{E}(1,1)$. There exists a left-invariant Riemannian metric g such that $X(g) = T$ if and only if T is diagonalisable in a basis $\{V_i\}_{i=1}^3$ satisfying (1.4) with $\lambda_1 = 2, \lambda_2 = -2, \lambda_3 = 0$ and one of the following conditions is satisfied:

(i) $T_1 < 0, , T_2 = T_3 = 0,$
(ii) $T_1 = T_3 = 0, T_2 < 0,$
(iii) $T_1 > 0, T_2, T_3 < 0,$
(iv) $T_1 < 0, T_2, T_3 > 0$, and
(v) $T_1, T_2, T_3 < 0.$

Here we denote $T_i = T(V_i, V_i)$. In (i) and (ii) there are infinitely many solutions g to (1.2). In (iii), (iv) and (v), the solution g is unique.

Proof. Fix a basis $\{V_i\}_{i=1}^3$ of $\mathfrak{g}$ in which T is diagonal and (1.4) holds. We compute the variables $\mathbf{x}_1 = -(x_2 + x_1), \mathbf{x}_2 = x_2 + x_1, \mathbf{x}_3 = x_1 - x_2$. This simplifies (2.3) to

$$\begin{cases} T_1 = \frac{(x_1-3x_2)(x_1+x_2)^3}{x_1 x_2^2 x_3^2} \\ T_2 = \frac{-(3x_1-x_2)(x_1+x_2)^3}{x_1^2 x_2 x_3^2} \\ T_3 = \frac{-(x_2-3x_1)(3x_2-x_1)(x_1+x_2)^2}{x_1^2 x_2^2 x_3}. \end{cases} \tag{4.2}$$

It is clear that the five conditions on the signature T_i listed are exhaustive so we consider existence and uniqueness in each. For the last three conditions, it is useful to solve for x_3^2 and substitute the expression into the second equation, solving for x_1. We find

$$x_3^2 = \frac{(x_1 - 3x_2)(x_1 + x_2)^3}{x_1 x_2^2 T_1},$$

and hence

$$x_1 = x_2 \frac{-3(T_1 - T_2) \pm (9T_1^2 - 14T_1T_2 + 9T_2^2)^{\frac{1}{2}}}{2T_2}.$$

(i) Given $T_2 = T_3 = 0$ we conclude that $3x_1 = x_2$ as $x_1, x_2, x_3 > 0$. This simplifies the first equation to $T_1 = \frac{-512x_1}{9x_3^2}$. This has infinitely many solutions.
(ii) Given $T_1, T_3 = 0$, we find $T_2 = \frac{-512x_2}{9x_3^2}$. This has infinitely many solutions.
(iii) Given $T_2, T_3 < 0$ and $T_1 > 0$, we require $x_1 \geq 3x_2$. This implies $3 \leq \frac{-3(T_1-T_2)\pm\left(9T_1^2-14T_1T_2+9T_2^2\right)^{\frac{1}{2}}}{2T_2}$ and so $3T_2 + 3T_1 \geq \pm(9T_1^2 - 14T_1T_2 + 9T_2^2)^{\frac{1}{2}}$. This gives us two expressions, the first of which cannot hold since $T_2 < 0$. Thus we take the negative of the square root, giving us a single expression for x_1 in terms of x_2. Substituting this into our expression for x_3 and finally into the third equation, we solve a unique expression for x_2. We then recover expressions for x_1 and x_3, giving us a unique solution.

(iv) This case is similar to the previous, however we find $\frac{1}{3}x_2 \leq x_1 \leq 3x_2$ and so $\frac{1}{3} \leq \frac{-3(T_1-T_2)\pm\left(9T_1^2-14T_1T_2+9T_2^2\right)^{\frac{1}{2}}}{2T_2}$. Again this gives us two expressions,

$$\frac{49}{9}T_2^2 - 14T_1T_2 + 9T_1^2 < 9T_2^2 - 14T_1T_2 + 9T_1^2$$
$$\frac{49}{9}T_2^2 - 14T_1T_2 + 9T_1^2 > 9T_2^2 - 14T_1T_2 + 9T_1^2$$

We disregard the first and solve uniquely for x_2, recovering x_1 and x_3 in the process.

(v) Here we find that $x_1 \leq \frac{1}{3}x_2$ and so $\frac{1}{3} \geq \frac{-3(T_1-T_2)\pm\left(9T_1^2-14T_1T_2+9T_2^2\right)^{\frac{1}{2}}}{2T_2}$. Similarly to the two previous cases, we have two inequalities:

$$\frac{49}{9}T_2^2 - 14T_1T_2 + 9T_1^2 < 9T_2^2 - 14T_1T_2 + 9T_1^2$$
$$\frac{49}{9}T_2^2 - 14T_1T_2 + 9T_1^2 > 9T_2^2 - 14T_1T_2 + 9T_1^2$$

The second inequality cannot hold and so we disregard it. This allows us to conclude a unique solution (x_1, x_2, x_3) to (4.2).

Note that in cases $(iii), (iv)$ and (v), all $T_i \neq 0$ for $i = 1, 2, 3$. Thus, we may apply lemma 3.2 to conclude that the solution is unique. □

4.3. The Heisenberg Group, H_3.

Theorem 4.3. Let T be a left-invariant $(0,2)$-tensor field on H_3. There exists a unique left-invariant Riemannian metric g such that $X(g) = T$ if and only if T is diagonalisable in a basis $\{V_i\}_{i=1}^3$ satisfying (1.4) with $\lambda_1 = 2, \lambda_2 = \lambda_3 = 0$ and $T_1 > 0, T_2, T_3 < 0$..

Proof. Fix a basis $\{V_i\}_{i=1}^3$ of the Lie algebra $\mathfrak{g}$ in which T is diagonal and (1.4) holds. We compute the variables $\mathbf{x}_1 = -x_1, \mathbf{x}_2 = x_1, \mathbf{x}_3 = x_1$. This simplifies (2.3) to

$$\begin{cases} T_1 = \frac{x_1^3}{x_2^2x_3^2} \\ T_2 = \frac{-3x_1^2}{x_2x_3^2} \\ T_3 = \frac{-3x_1^2}{x_2^2x_3} \end{cases} \tag{4.3}$$

The second equation becomes $x_2 = \frac{-3x_1^2}{x_3^2T_2}$. Substituting this into the first gives, $x_1 = \frac{x_3^2T_2^2}{9T_1}$. Finally we substitute both expressions into the third equation to find $x_3 = \frac{-27T_1^2}{T_2^2T_3}$. We compute $x_1 = \frac{81T_1^2}{T_2^2T_3^2}$ and $x_2 = \frac{-27T_1^2}{T_2T_3^2}$. As these three expressions depend only on T, there must exist a single metric g solving $X(g) = T$ in our fixed basis. By lemma 3.1, this proves the existence portion of the theoerm. Since all $T_i \neq 0$ for $i = 1, 2, 3$, as per § 3, we may apply lemma 3.2 giving us uniqueness of g solving (1.2). □

4.4. Euclidean Space, $\mathbb{R}^3$.

Euclidean Space is Ricci-flat and so it has Einstein tensor 0. Thus $\mathcal{E}$, and subsequently the cross curvature, is not well defined.

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