Constructive Approximation of Functions

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Abstract. These are notes from the Spring 2024 Brown University APMA DRP on the constructive approximation of functions, specifically polynomials and rational functions of one variable, under the direction of Dr. Wenjun Zhao. We closely followed the text *Approximation Theory and Approximation Practice* by Trefethen [1] and additionally engaged in material on universal approximation as it relates to neural networks out of interest.

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1 Week 1: 2/22/2024

Chebyshev interpolants have a very strong approximation properties, as opposed to uniformly spaced points. They are the points that correspond to the real part of equispaced points on the unit circle in the complex plane. That is, the Chebyshev points are

$$x_j = \cos\left(\frac{j\pi}{n}\right)$$

and a key property is that they "collect" near the ends of the interval in higher density. Namely, this key property is that each point is, on average, the same distance away from every other point. For the most part, we deal with approximating functions on the interval $[-1, 1]$, which any function on any interval $[a, b]$ can be scaled to.

There are connections that can be drawn between the Chebyshev, Fourier, and Laurent settings, with each being used in numerical, complex, and real analysis heavily, respectively. In the Chebyshev settings, we approximate functions $f(x), x \in [-1, 1]$ with the form

$$f(x) \approx \sum_{k=0}^{n} a_k T_k(x)$$

while using $z \in S^1 \subset \mathbb{C}$ equispaced points on the complex plane gives us the Laurent setting with Laurent polynomials

$$F(z) = F(z^{-1}) = \frac{1}{2}\sum_{k=0}^{n} a_k(z^k + z^{-k})$$

and finally using the angle $\theta \in [-\pi, \pi]$ to define $\mathcal{F}(\theta) = F(e^{i\theta}) = f(\cos(\theta))$ gives us Fourier series as

$$\mathcal{F}(\theta) \approx \frac{1}{2}\sum_{k=0}^{n} a_k(e^{ik\theta} + e^{-ik\theta})$$

Their corresponding canonical grid systems are as follow:

Chebyshev points	$x_j = \cos\left(\frac{j\pi}{n}\right)$,	$0 \le j \le n$
Roots of unity (Laurent)	$z_j = e^{\frac{j\pi}{n}}$,	$-n+1 \le j \le n$
Equispaced points (Fourier)	$\theta_j = \frac{j\pi}{n}$,	$-n+1 \le j \le n$

1.1 Chebyshev Series and Polynomials

Definition 1.1 (k-th Chebyshev polynomial)**.** The k-th Chebyshev polynomial is the real part of the function z^k on the unit circle; i.e.

$$T_k(x) = \Re(z^k) = \frac{1}{2}(z^k + z^{-k}) = \cos(k\theta)$$

Theorem 1.2 (Existence of Chebyshev Series)**.** Suppose that f is Lipschitz on $[-1, 1]$, i.e. that there exists $C \in \mathbb{R}$ such that $|f(x) - f(y)| \le C|x - y|$ for any $x, y \in \mathbb{R}$. Then f admits a unique representation as a Chebyshev series

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x)$$

which is absolutely and uniformly convergent. The coefficients a_k are given by

$$a_k = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x) T_k(x)}{\sqrt{1-x^2}} dx$$

for $k \geq 1$, and for $k = 0$ by the same formula with a $\frac{1}{\pi}$ factor instead.

2 Week 2: 2/29/2024

2.1 Interpolants and projections

3 Bibliography

[1] Trefethen, L. N. (2012). *Approximation Theory and Approximation Practice (Other Titles in Applied Mathematics)*. Society for Industrial and Applied Mathematics, USA.