Multiple Imputation with Angular Covariates

Imputing Incomplete Angular Data with Projected Normal Regression

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- The projected normal distribution and multiple imputation

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- Complications with imputing angular data

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- Simulation results
- An illustrative analysis of US adult asthma rates' connection to pollution and wind

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Review of Multiple Imputation

- Let $\mathbf{X} = (X_1, \ldots, X_p, Y)$ be complete and $\Theta = (\theta, \cos\theta, \sin\theta)$ be partially observed, $\epsilon \sim N_n(0, \sigma^2 I_n)$, and

$$Y = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p + \beta_{p+1}\cos\theta + \beta_{p+2}\sin\theta + \epsilon$$

Multiple Imputation Procedure [Rubin, 1987] :

1. **Impute** - Use an imputation method g to impute Θ_{mis} by $\dot{\theta} = g(\mathbf{X}, \Theta_{obs})$ or $(\cos\dot{\theta}, \sin\dot{\theta})' = g(\mathbf{X}, \Theta_{obs})$ M times to create M completed data sets

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2. **Analyze** - For each completed data $(\mathbf{X}, \Theta_{obs}, \Theta_{mis}^{(m)})$ for $m = 1, \ldots, M$, estimate β using least squares to collect $\hat{\beta}^{(m)}$ and $U^{(m)} = se(\hat{\beta}^{(m)})^2$

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3. **Combine** - Apply **Rubin's rules** to get point estimate $\bar{Q}$ and total variance T

A Brief Intro to Directional Statistics

- Directional data consist of angles and points on the unit sphere

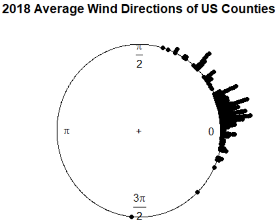

Figure 1: 2018 Average Wind Directions in US Counties

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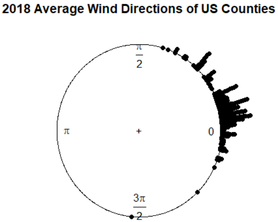

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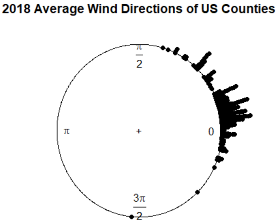

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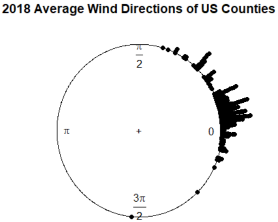
- $\theta \in [0, 2\pi)$ but $0 \equiv 2\pi$ and methods should be rotation invariant
- Distributions on the circle can be intrinsic (von Mises) or arise out of transformations (wrapping or projection)



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Incomplete Data Structure

Analysis model:

$$Y = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p + \beta_{p+1} \cos\theta + \beta_{p+2} \sin\theta + \epsilon$$

$D = (\mathbf{X}, \Theta)$

$\mathbf{X}$	θ	$\cos\theta$	$\sin\theta$
$\mathbf{x}_1$	θ_1	$\cos\theta_1$	$\sin\theta_1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{x}_{n_{obs}}$	$\theta_{n_{obs}}$	$\cos\theta_{n_{obs}}$	$\sin\theta_{n_{obs}}$
$\mathbf{x}_{n_{obs}+1}$	?	?	?
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{x}_n$	?	?	?

Θ_{obs}

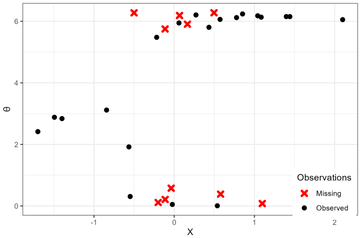
θ	$\cos\theta$	$\sin\theta$
θ_1	$\cos\theta_1$	$\sin\theta_1$
$\vdots$	$\vdots$	$\vdots$
$\theta_{n_{obs}}$	$\cos\theta_{n_{obs}}$	$\sin\theta_{n_{obs}}$

Θ_{mis}

θ	$\cos\theta$	$\sin\theta$
?	?	?
$\vdots$	$\vdots$	$\vdots$
?	?	?

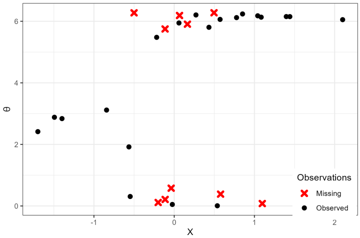
$\mathbf{X} = (X_1, \ldots, X_p, Y)$

Passive Imputation

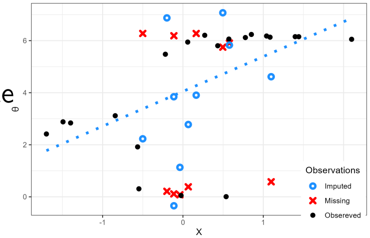


1. Impute by $\theta = \beta_0 + \beta_1 X + \beta_2 Y$
2. Transform by $\theta \mapsto (\cos\theta, \sin\theta)'$

Passive Imputation



Impute



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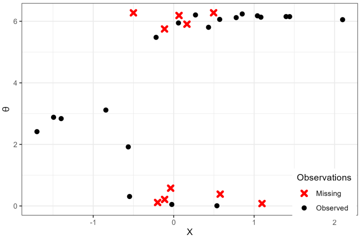

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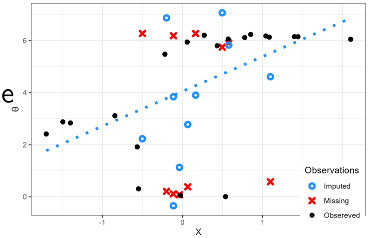
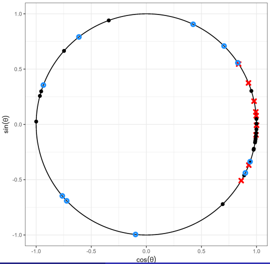


Transform

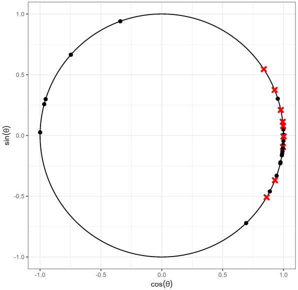


1. Impute by $\theta = \beta_0 + \beta_1 X + \beta_2 Y$
2. Transform by $\theta \mapsto (\cos\theta, \sin\theta)'$

Direct Imputation

- Impute cartesian coordinates instead by

$$\begin{pmatrix} \cos\theta_i \\ \sin\theta_i \end{pmatrix} = \begin{pmatrix} \beta_{10} + \beta_{11} X_i + \beta_{12} Y_i \\ \beta_{20} + \beta_{21} X_i + \beta_{22} Y_i \end{pmatrix}$$



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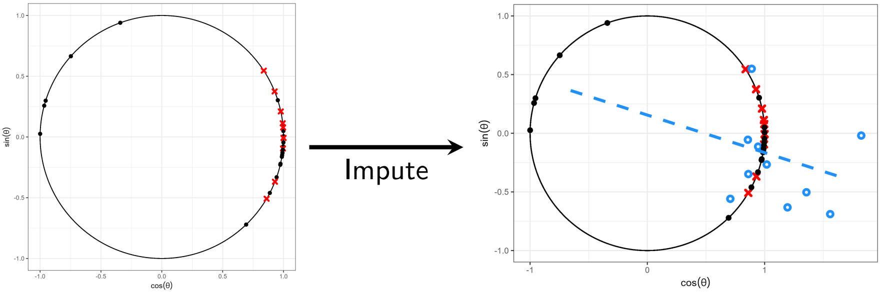


Congeniality

Congeniality The imputation should, at minimum, contain all of the variables that will be in the analysis model [Meng, 1994].

- With a circular model, imputing θ is equivalent to imputing $\mathbf{u}$ and any structure between θ and $Y, \mathbf{X}$ is in both the imputation and analysis models
- Imputing with an inline model, the imputation model is congenial if imputing on $\mathbf{u}$, but not if imputing on θ and then transforming to $\mathbf{u}$
  - Analysis model: $p(Y, X_1, \ldots, X_p, \cos\theta, \sin\theta)$
  - Uncongenial imputation model: $p_{Imp}(Y, X_1, \ldots, X_p, \theta)$ then transform $\theta \mapsto (\cos\theta, \sin\theta)'$
  - Congenial imputation model: $p_{Imp}(Y, X_1, \ldots, X_p, \cos\theta, \sin\theta)$

The Projected Normal Distribution I

Projected Normal Distribution

A multivariate normal vector $\mathbf{y}$ with $E(\mathbf{y}) = \mu$ and $Var(\mathbf{y}) = \Sigma$ normalized by its length results in a projected normal vector $\mathbf{u} = \mathbf{y}/||\mathbf{y}||$

- More flexible than the von Mises or wrapped family
- Equivalent to von Mises and wrapped normal when unimodal and symmetric for small and large concentrations
- Estimate latent lengths $l_i = ||\mathbf{y}_i||$ for each $\mathbf{u}_i \equiv \theta_i$ to work with $\tilde{\mathbf{y}}_i = \tilde{l}_i \mathbf{u}_i \dot{\sim} N_k(\mu, \Sigma)$ [Presnell et al., 1998, Nuñez-Antonio and Gutiérrez-Peña, 2005]

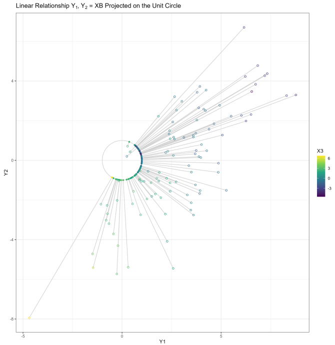
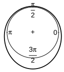


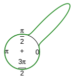
Figure 2: Projected normal regression with latent $\mathbf{y}$.

Projected Normal Examples

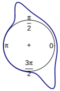
- If Σ is unconstrained, then the parameters are non-identifiable
  - Assume $\Sigma = I_k$ for symmetric and unimodal distribution
  - Assume some other constrained form to get bi-modal, skewed, or symmetric distributions [Hernandez-Stumpfhauser et al., 2017]



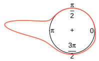
(a) Low concentration



(b) High concentration



(c) Bi-modal



(d) Skewed

Figure 3: Examples of projected normal distributions

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Standard Methods

- Let ω be one of $\{\theta, \cos\theta, \sin\theta\}$ depending on the imputation procedure with $\mathbf{X} = (1, Y, X_1, \ldots, X_p)$
- Assume only ω is incomplete

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Linear Regression

- Use the observed data to fit the model $\omega_i = \mathbf{x}_i'\gamma + \eta_i$ for $i = 1, \ldots, n_{obs}$; $\eta_i \overset{iid}{\sim} N(0, \tau^2)$
- Impute ω_i by drawing from the predictive posterior of the regression model given the predictors $\mathbf{x}_j$ for each incomplete observation, $j \notin \{1, \ldots, n_{obs}\}$

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Predictive Mean Matching

- Fit a Bayesian regression with a weakly informative prior
- Use draws from the predictive posterior to create a set of neighbors for drawing imputations

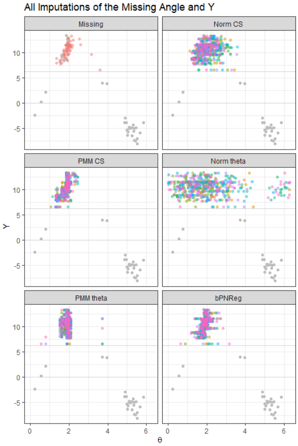
Projected Normal Regression I

- Let $\theta_i \sim PN_2(\mu_i, I_2)$ where $\mu_i = \mathbf{x}_i'\mathbf{B}$ and $\mathbf{B} = (\beta_1, \beta_2) \in \mathbb{R}^{(p+2)\times 2}$
- Normal prior on $\beta_{ij} \sim N(0, 10000)$
    - Implemented by `bpnreg` package in R [Cremers, 2021]
- Assumes a symmetric and unimodal distribution for θ_i given $\mathbf{X}_i$

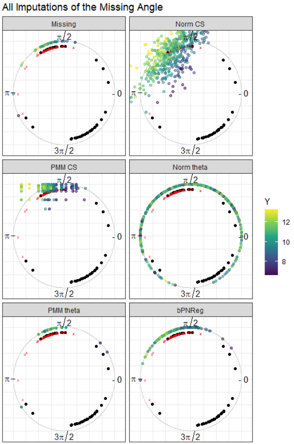
Projected Normal Imputation

- Fit the projected normal regression model using the observed data and obtain posterior draws $\dot{\mathbf{B}}$ of $\mathbf{B}$
- Sample $\dot{Y}_j \sim N_2(\mathbf{x}_j'\dot{\mathbf{B}}, I_2)$ and normalize to get $U_j = Y_j/||Y_j||^{-1} = (\cos\dot{\theta}_j, \sin\dot{\theta}_j)'$ for $j \notin \{1, \ldots, n_{obs}\}$
- Impute the missing θ_j with predictive posterior draws of $\dot{\theta}_j$

Projected Normal Regression II



(a) On the unit circle



(b) Coordinates in $\mathbb{R}^2$

Figure 4: Imputations with angular data generated by projected normal regression

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Simulation Overview

How well do the projected normal imputations perform under various settings?

1. Different Sample Sizes - $N = 50; 100; 500; 1,000$

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3. Different Data Generating Processes
   a. High Concentration Projected Normal
   b. Low Concentration Projected Normal
   c. Skewed Projected Normal
   d. Bi-modal Projected Normal
   e. Projected Normal Regression
   f. von Mises Regression
   $\theta \sim M(\mu_i, 15)$ with $\mu_i = 2\arctan(-3X_{i1} + 3X_{i2})$

Results I

- Projected normal imputation provides better inferences (nominal CI coverage with low average width; low bias and MSE) compared to imputations of just θ using linear regression or PMM
- Imputations on $(\cos\theta, \sin\theta)'$ also provide robust inferences despite not being angular
- All investigated imputation strategies perform similarly across differing sample sizes and missingness proportions; wider intervals/greater MSE with low N and high p_{miss}

Results II

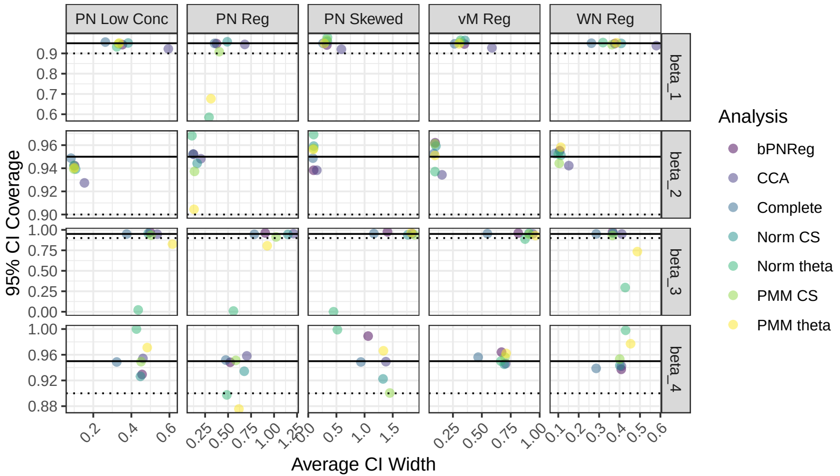

Figure 5: 95% Confidence interval coverages and average widths for each coefficient using the different analyses and data generating processes.

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Overview I

- Air pollution is strongly connected to asthma [Glad et al., 2012, Gleason et al., 2014, Lin et al., 2002, Weisel et al., 1995, Wilson et al., 2005]
- Model the impact of air pollution on asthma accounting for meteorological conditions
- Data collected at the US county-level and averaged over the year from CDC () and EPA (2021)
- Census data (2021) is used to weight the county observations by population

Overview II

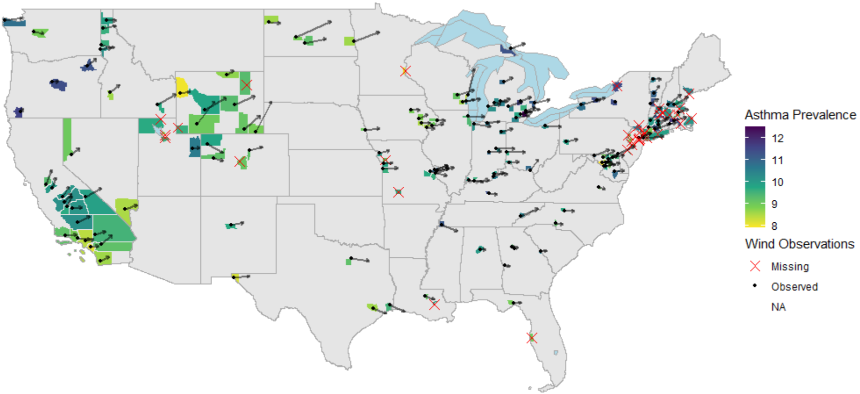

Figure 6: Map of US counties asthma prevalence and average wind directions and speed.

Analysis

- Impute using each of the previously discussed methods for the angular data
- Impute inline data with linear regression or predictive mean matching
- Include all variables collected for this step

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- Mixed effects linear model (centered and scaled predictors)
  - Fixed effects for the meteorological and air pollution variables
  - Random intercepts for the state-level

$$
\begin{aligned}
Asthma_i = {} & \beta_0 + \beta_1 AirPressure_i + \beta_2 RH_i + \beta_3 Temperature_i \\
& + \beta_4 WindSpeed_i + \beta_5 \cos WindAngle_i + \beta_6 \sin WindAngle_i \\
& + \beta_7 USGD_i + \beta_8 MedianAQI_i + \beta_9 90thPercentileAQI_i \\
& + \beta_{10} CODays_i + \beta_{11} NO2Days_i + \beta_{12} Ozone_i \\
& + \beta_{13} PM2.5_i + \beta_{14} PM10_i + \mathbf{Z}\eta + \epsilon_i
\end{aligned}
$$

Results I

- Imputation strategies are generally consistent and provide better estimates than complete case analysis (CCA)
- Projected normal regression and linear regression imputations on $(\cos\theta, \sin\theta)$ produce similar inferences
- Predictive mean matching with either θ or $(\cos\theta, \sin\theta)'$ are similar

Results II

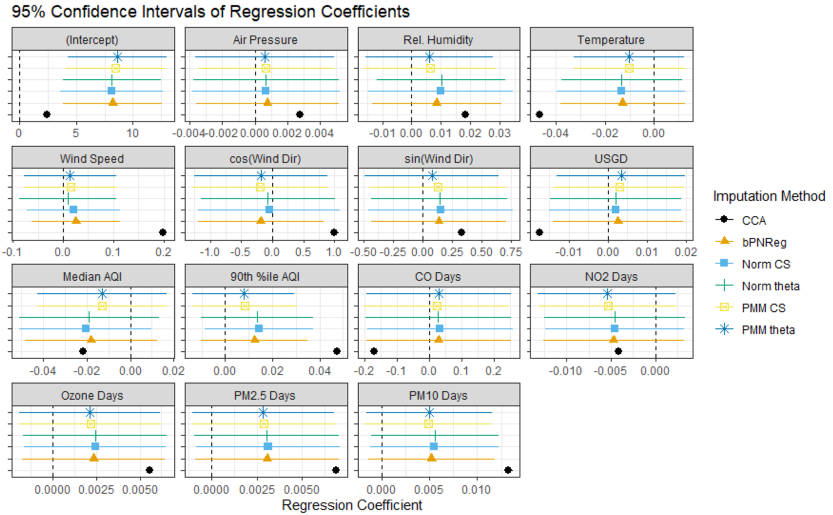

Figure 7: Bias of coefficient estimates using different imputation strategies.

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What's Next?

- Imputation for spherical data
- Applying angular imputation models when the response or outcome is angular
- Incomplete data analysis for angular data in temporal or spatial settings

Acknowledgements

- Research was supported by NSF AGEP-GRS supplement for Award #2015320.
- Thank you to my advisor Dr. Ofer Harel for support and assistance throughout the research process.

Questions?

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Appendix I

Rubin's Rules - Multivariate [Rubin, 1987, Harel and Zhou, 2007]

- $\bar{Q} = \frac{1}{M} \sum_{m=1}^{M} \hat{\beta}^{(m)}$ is the mean of the estimates
- $\bar{U} = \frac{1}{M} \sum_{m=1}^{M} U^{(m)}$ is the mean of the within variance
- $B = \frac{1}{M-1} \sum_{m=1}^{M} (\hat{\beta}^{(m)} - \bar{Q})(\hat{\beta}^{(m)} - \bar{Q})'$ is the between variance
- $T = \bar{U} + (1 + \frac{1}{M})B$ is the total variance
- Proportionality assumption: $B \propto \bar{U}$ so that $\tilde{T} = (1-t)\bar{U}$ where $t = (1 + \frac{1}{M}) tr(B\bar{U}^{-1})$

Appendix II

Simulated Data Generating Processes

Different Data Generating Processes parametrized by Φ_j

1. High Concentration Projected Normal - $\Phi_1 = ((10, 0)', I_2)'$
2. Low Concentration Projected Normal - $\Phi_2 = ((1, 0)', I_2)'$
3. Skewed Projected Normal - $\Phi_3 = \left(\begin{pmatrix} 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix} \right)'$
4. Bi-modal Projected Normal - $\Phi_4 = \left(\begin{pmatrix} 0.5 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ -2 & 10 \end{pmatrix} \right)'$
5. Projected Normal Regression - $\Phi_5 = (\mathbf{B}, I_2)'$ so that $\mu_i' = (1, X_{i1}, X_{i2})\mathbf{B}$ and $\mathbf{B} = (\beta_1, \beta_2)'$ where $\beta_1 = (0, 1, -0.5)'$ and $\beta_2 = (0, -3, 1)'$
6. von Mises Regression - $\Phi_6 = (0, -3, 3, 15)'$ so that $\theta \sim M(\mu_i, 15)$ with $\mu_i = 2\arctan(-3X_{i1} + 3X_{i2})$

Appendix III

Simulation Procedure

1. Create data set and impose missing values via logistic model
   a. $(X_1, X_2)' \sim N_2\left(\begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ -2 & 10 \end{pmatrix}\right)$
   b. $\theta \sim F(\theta | X_1, X_2, \Phi)$ where F is the projected normal or von Mises distributions/regressions parametrized by Φ
   c. $Y = \mathbf{X}\beta + \epsilon$ where $\epsilon \sim N(0, 1)$
2. Impute the missing angular data using each method M times
3. Fit the regression model $Y = \mathbf{X}\beta$ to each imputed data set, the complete data, and complete case analysis
4. Combine results for multiply imputed data sets using Rubin's rules
5. Repeat 1000 times

Appendix IV

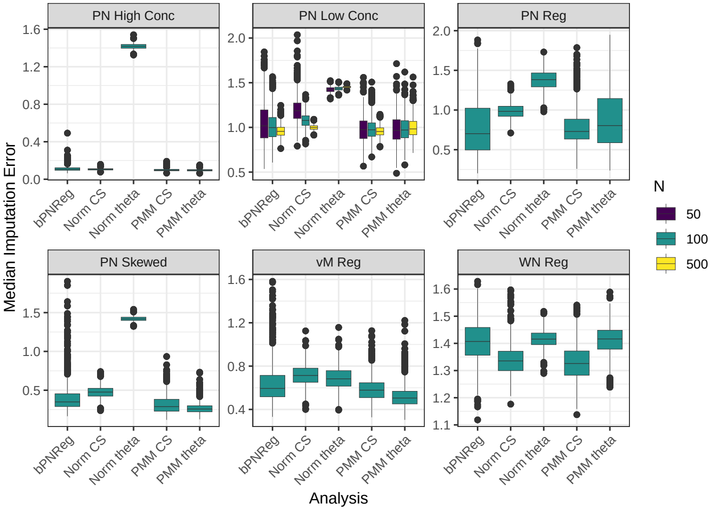


Figure 8: Median imputation error for p(miss) = 0.5.

Appendix V

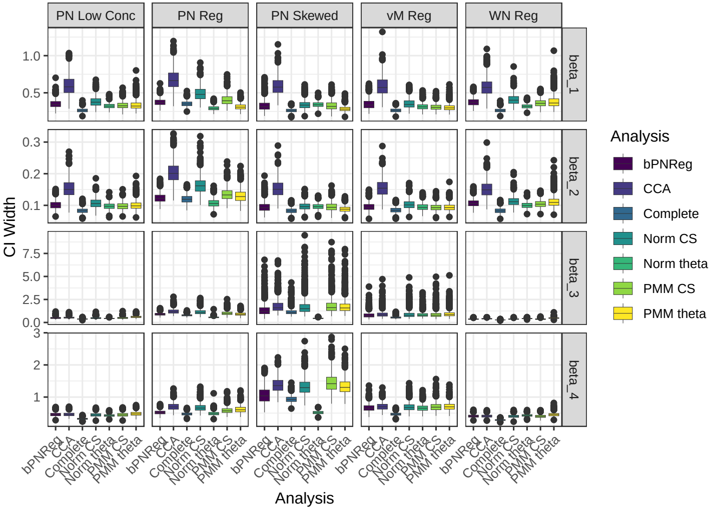
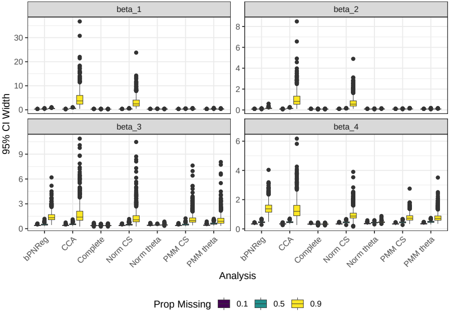


Figure 9: 95% Confidence interval widths for each coefficient using the different analyses and data generating processes.

Appendix VI



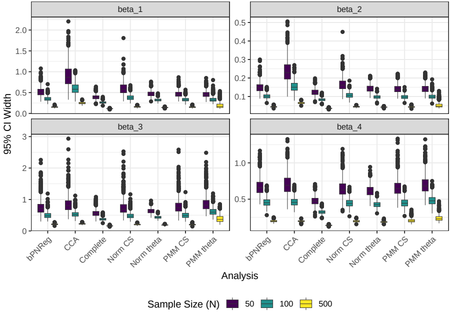


Figure 10: Varying p_{miss} with $N = 100$. Figure 11: Varying N with $p_{miss} = 0.5$.

95% Confidence interval widths for each coefficient with low concenctration projected normal angular data.