Incomplete Angular Time Series Imputation with a Projected Normal Autoregressive Process and Exogenous Predictors

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Hartford PM2.5 Concentration Data

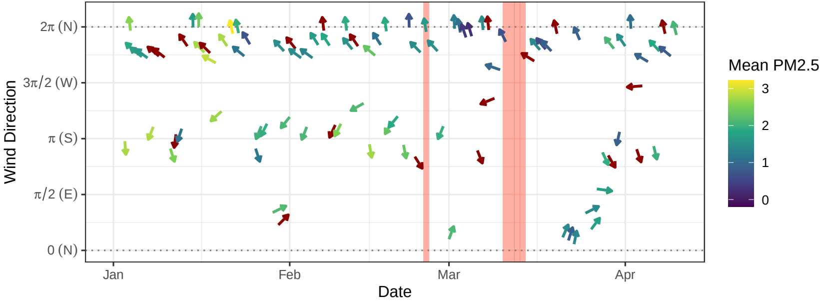


Figure 1: The first 100 days (January 1, 2018 to April 10, 2018) of wind directions. Red bars are where observations are missing.

- Daily data were obtained from the EPA's Air Quality System (AQS) [1] for the period from January 1, 2018 to December 31, 2018.

The Problem

Q: How can we analyze an outcome time series Y with incomplete angular predictors Θ, e.g. air pollution data with meteorological predictors?

Our Proposed Solution

We propose using **multiple imputation** with incomplete angular data imputed by a **projected normal autoregressive process** (PN AR(1)).

Projected Normal Distribution I

Definition

- Let $\mathbf{w}_t \sim N_2(\mu, \Sigma)$ be a latent vector with length $l_t = ||\mathbf{w}_t||$.
- $\mathbf{u}_t = \mathbf{w}_t / l_t = (\cos\theta_t, \sin\theta_t)'$ is a unit vector corresponding to θ_t.
- Then $\theta_t \sim PN(\mu, \Sigma)$.

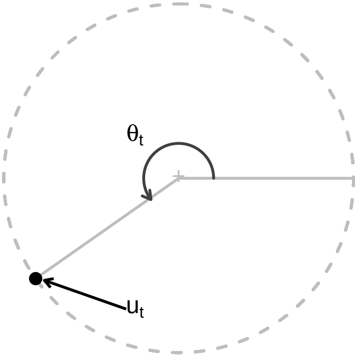


Figure 2: The projected normal angle $\theta_t \sim PN(\mu, \Sigma)$ is equivalent to the unit vector $\mathbf{u}_t$.

Fitting Projected Normal Models

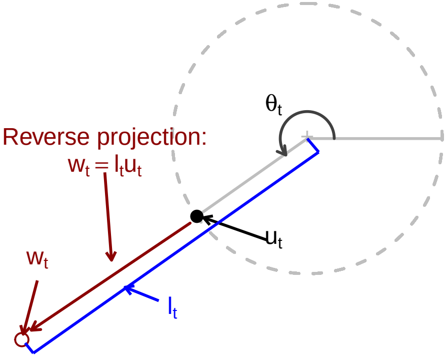


Figure 3: We estimate the latent length l_t to reverse the projection and obtain the latent vector $\mathbf{w}_t = l_t \mathbf{u}_t \sim N_2(\mu, \Sigma)$. Repeat with the whole data set to estimate μ and Σ using $\mathbf{W} = (\mathbf{w}_1, \ldots, \mathbf{w}_T)'$ where $\mathbf{w}_t \overset{iid}{\sim} N_2(\mu, \Sigma)$.

Multiple Imputation Procedure [2, 3]

1. Multiply Impute M times with $\dot{\theta}_{imp}^{(m)} = g(\mathbf{X}, \Theta_{obs})$.

Incomplete Data: $(\Theta_{obs}, \Theta_{mis}, \mathbf{Z})$

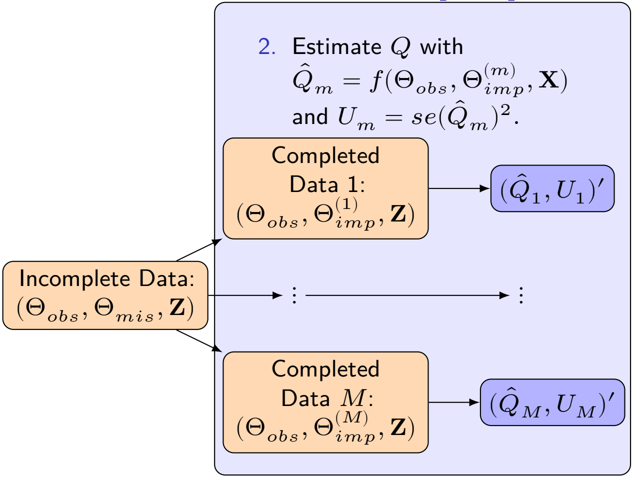
Completed Data 1: $(\Theta_{obs}, \Theta_{imp}^{(1)}, \mathbf{Z})$

$\vdots$

Completed Data M: $(\Theta_{obs}, \Theta_{imp}^{(M)}, \mathbf{Z})$

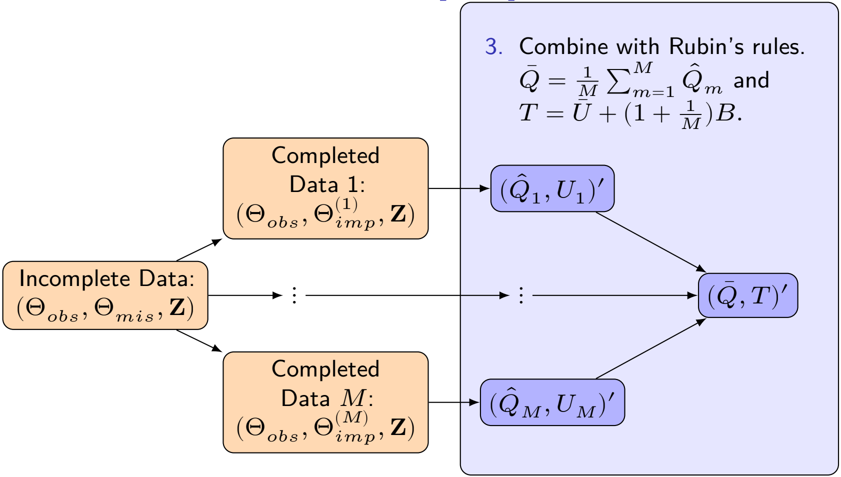
$\Theta = (\theta_0, \dots, \theta_T)'$ and $\mathbf{Z} = (Y, \mathbf{X})'$

Multiple Imputation Procedure [2, 3]



Multiple Imputation Procedure [2, 3]

3. Combine with Rubin's rules.
$\bar{Q} = \frac{1}{M}\sum_{m=1}^{M} \hat{Q}_m$ and
$T = \bar{U} + (1 + \frac{1}{M})B$.



$$B = \frac{1}{M-1}\sum_{m=1}^{M}(\hat{Q}_m - \bar{Q})^2 \text{ and } \bar{U} = \frac{1}{M}\sum_{m=1}^{M} U_m.$$

Projected Normal Autoregressive Process

PN AR(1) Process

- Let $\theta_t| \ . \ \sim PN(\mu_t, \Sigma)$ where

$$\mu_t = \mu_0 + \Phi l_t \mathbf{u}_{t-1} + \mathbf{B}\mathbf{x}_t$$

- l_t are latent lengths such that $W_t = l_t(\cos\theta_t, \sin\theta_t)' \sim N_2(\mu_t, I_2)$.
- Weakly informative normal priors for $l_t \sim N_+(0, 10^2)$, $\beta_{ik} \sim N(0, 100^2)$ and ϕ_{ij} $N_{[-1,1]}(0, 100^2)$.

- The PN AR(1) process [4] is implemented in the Bayesian setting to draw imputations from the posterior predictive distribution.

Projected Normal-based Imputation

Imputation Procedure

1. **Fit** the PN AR(1) model using the observed data $(\Theta_{obs}, \mathbf{X}, Y)$ and obtain posterior draws for the model parameters $(\mathbf{B}^{(q)}, \Phi^{(q)})'$ for the last MCMC iteration q.

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2. For each missing observation, **sample** $\dot{\mathbf{w}}_t \sim N_2(\dot{\mu}_t^{(q)}, I_2)$ and **project** onto the unit circle to get $\dot{\mathbf{u}}_t = \dot{\mathbf{w}}_t / ||\dot{\mathbf{w}}_t||$.

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3. **Convert** $\dot{\mathbf{u}}_t$ to $\dot{\theta}_t$ and use these posterior predictive draws to **impute** the missing θ_t.

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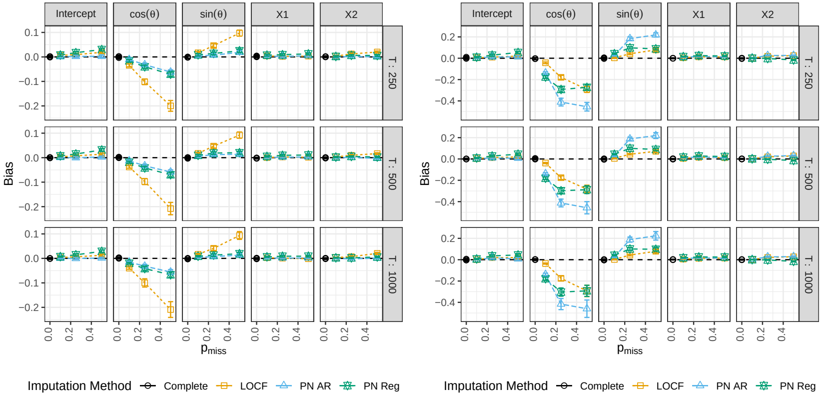
4. **Repeat** M times, refitting the model each time.

Set-Up

- **Aims** - Evaluate the LOCF [5], PN Regression [6, 7], and PN AR(1) imputation models with an ARX analysis model.
- **Data Generating Mechanism** - Simulate response variable from a ARX model and angular data from PN AR(1) model.
  - Simulate with *high* or *low* autocorrelation, varied sample sizes, and varied proportions of missingness.
- **Estimand** - The regression coefficient vector β.
  - $Y_t = \beta_0 + \beta_1 X_{t1} + \beta_2 X_{t2} + \beta_3 \cos\theta_t + \beta_4 \sin\theta_t + \phi_y Y_{t-1} + \epsilon_t$
  - $\epsilon_t \overset{iid}{\sim} N(0, \sigma_y^2)$
- **Performance Measure** - Bias and 95% CI Coverage[1]

[1] The code for the simulations is available at Github: https://github.com/benjamin-stockton/ch2-ts-mi-sim.

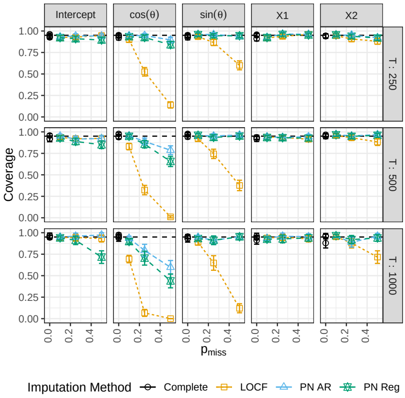
Results



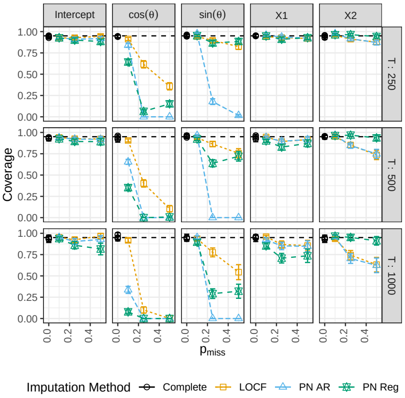
(a) Low Autocorrelation

(b) High Autocorrelation

Figure 4: Regression coefficient bias.



(a) Low Autocorrelation



(b) High Autocorrelation

Figure 5: 95% confidence interval coverage. The horizontal dashed line is at 95% coverage.

Hartford PM2.5 Concentration Data

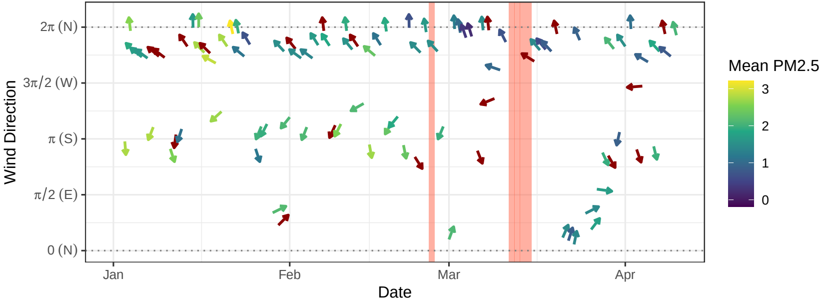


Figure 6: The first 100 days (January 1, 2018 to April 10, 2018) of wind directions. Red bars are where observations are missing.

Imputed Wind Directions

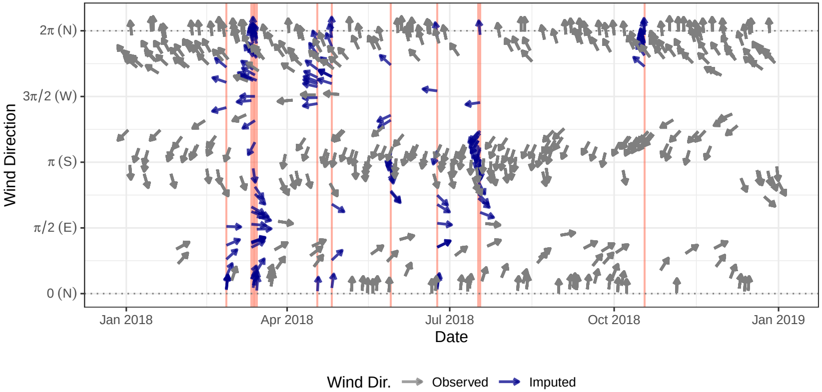


Figure 7: Ten (out of 25 total) completed data sets imputed with PN AR(1) and predictive mean matching using MICE [8]. 3% of wind observations and 13% of PM2.5 observations are missing.

Results

Table 1: Parameter estimates from the multiply imputed PM2.5 and wind data.

Variable	Est.	95% LB	95% UB
Intercept	6.55	5.82	7.29
Max WindSpeed	-1.05	-1.62	-0.49
$\cos\theta$	0.14	-0.41	0.68
$\sin\theta$	-0.73	-1.73	0.29
Max WindSpeed $\times \cos\theta$	0.47	-0.15	1.10
Max WindSpeed $\times \sin\theta$	-0.46	-1.69	0.77
$\cos\theta \times \sin\theta$	-0.22	-1.64	1.19
Max WindSpeed $\times \cos\theta \times \sin\theta$	1.46	-0.32	3.27
ϕ_1	0.49	0.37	0.59
σ_Y	3.33	3.07	3.63

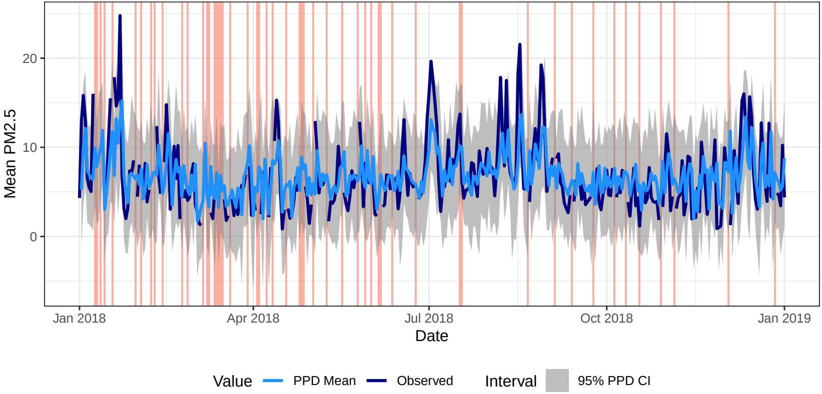


Figure 8: Posterior predictive mean PM2.5 concentration with 95% CI and observed data.

Conclusion

- We have developed a projected normal autoregressive model for imputing angular time series with MICE [8].[2]
- The proposed method works best under low-to-moderate autocorrelation settings. LOCF may be a viable alternative with high autocorrelation.
- Multiply imputed regression analysis of daily PM2.5 concentrations in Hartford, CT showed no to weak associations between lagged maximum daily wind speed and direction and the PM2.5 concentration.

[2]Available at: https://github.com/benjamin-stockton/pnregstan and https://github.com/benjamin-stockton/imputeangles

Acknowledgements

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Appendix

Table 2: Summary of missing data proportions for each variable.

Variable	n_{miss}	p_{miss}
PM2.5	47	0.128
Pressure	8	0.022
Relative Humidity	58	0.158
Temperature	10	0.027
Wind Direction	12	0.033
Wind Speed	12	0.033