COMMENTS ON THE FAVORABLE-BET THEOREM

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Professors Taub and Bodenhorn[1] have demonstrated the useful theorem that, if your marginal utilities are well-defined at a point of no risk, you can judge among some small bets or investments by pretending that your marginal utilities are constant at the no-risk level. Although they cite a 1963 item from my *Collected Scientific Papers*, there are many items there that utilize various Taylor-expansion approximations to the expected utility function.[2] Thus, a closely related result is that an investor, who holds cash or a safe security and faces the option of an investment whose mean return per dollar exceeds that of the safe security, will necessarily benefit himself from investing some positive amount in that security.[3] Or to rescue mean-variance theory from its dubious dependence on quadratic utility, or on normal-Gaussian probability distributions, or on the confused view that any two-parameter family of distributions can have those parameters defined in terms of the first two moments alone (an irrelevancy when one realizes that a portfolio of two two-parameter distributions no longer is a 2-parameter family), recourse has been made[4] to the concept of "compact" probability distributions whose expected utility value can be asymptotically approximated by one-, two-, or any specified number of statistical moments.

I offer a few comments here that relate to corners in the utility function, to 2-way as distinct from 1-way unlimited-scale options, to problems of nonconvergence of the expected value of utility, to objective as against subjective definitions of "fair" and "favorable" investment or betting op-

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1. A.J. Taub and D. Bodenhorn, "Risk and the Scale of a Bet," *Economic Inquiry* (Sept. 1974), 12, pp.

2. P A. Samuelson, "Risk and Uncertainty: A Fallacy of Large Numbers," *Scientia*, 6th Series, 57 (April-May, 1963), pp. 1-6, reproduced in my *Collected Scientific Papers*, II (MIT Press, Cambridge, Mass., 1965), Ch. 16, pp. 153-158. Similar considerations are utilized in Chs. 12, 15 and in *CSP*, III (1972), Chs. 200, 201, 202, 203, 204. In particular they lie at the heart of my works on portfolio decision making, since the betting or investing options discussed here can be easily modified to be relevant for alternative security investments of a given initial wealth.

3. P. A. Samuelson, "General Proof that Diversification Pays," *Journal of Financial and Quantitative Analysis*, 2 (March, 1967), pp. 1-13, reproduced as Chapter 201 in P. A. Samuelson *Collected Scientific Papers*, III (MIT Press, Cambridge, Mass., 1972), pp. 848-859. See particularly Corollary I, p. 6 or 853.

4. P. A. Samuelson, "The Fundamental Approximation Theorem of Portfolio Analysis in Terms of Means, Variances and Higher Moments," *Review of Economic Studies*, 37, (October, 1970), pp. 537-542, reproduced as Chapter 203 in *CSP*, III, pp. 877-890.

tions ("bets" for short), and finally to uncontrollable variations in the states of the world that go on whatever the investors decisions and which preclude him from having open to him a no-risk base and which have a bearing on the "subjectivity" of defining "fair."

1. If a person has a corner in his utility function at the no-risk base point, then, as noted in 1963, the argument fails. To be sure, almost nowhere can a continuous-increasing function have corners. But suppose (as in the classic Domar-Musgrave analysis) the decision maker feels differently about gains than about losses. Then, so to speak, a corner follows such a "regretter" around to wherever he may land (just as a myopic Friedman-Savage bettor on long shots finds an inflection point waiting for him at some higher income than wherever he happens to land). When such corners occur, as Taub and Bodenhorn make clear, the usual conclusions have to be qualified. Example: at a corner a risk-averter will definitely shun favorable bets that are not "sufficiently" favorable.

2. It makes a difference whether a scale-free stochastic option is considered to be a 2-way rather than 1-way option. Thus, if eH and $-eK$ are the authors' vectors of gain and loss at probabilities p and $1-p$, we have a 2-way option only when the scale parameter that the investor is free to choose, e, can be negative as well as positive or zero.

If the bet is a two way option and is not originally a "fair" bet, one can always by picking the appropriate algebraic sign for e contrive a "favorable" bet. (Instead of "buying" an unfavorable option, one reverses its direction by "selling" it.)

The general theorem for 2-way options becomes this.

> Theorem 1. If a 2-way option is not a "fair" bet, there must always exist an (algebraic) scale for it which is better than staying in the no-risk base situation. This is independent of whether the maximizer of *expected* (smooth) utility is a risk-averter (with strictly decreasing marginal utilities and concave utility), or is a risk-lover (with strictly convex utility), or is of mixed allegiance (as in the Friedman-Savage *curiosum*). But is is only in the first two of these cases that we can be assured that the final optimal decision, e^*, involves investing in the direction of the favorable bet. The risk-lover must plunge without limit into any favorable bet (a result incompatible with his remaining small enough to be realistically confronted by scale-free or unlimited scale investing options). The risk-averter can usually be expected to take only a finite optimal scale, e^*, for a given favorable bet, but examples can be easily found in which the bet will be so favorable compared to the rate at which his marginal utility decreases that he too will be an unlimited plunger (a result not possible however if his marginal utility of wealth

satisfies "Inada" conditions of becoming infinite at zero wealth and zero at infinite wealth). For the mixed allegiance man, we cannot be sure that his final optimal e^* has the algebraic sign that it would have to have if he were confined to small-scale betting and thereby led in the direction of the favorable bet.

All "fair" bets will be rejected by risk-averters; be plunged into without limit by risk-lovers; but cannot have any fixed rule laid down for them in the case of mixed types.

The proof of this general theorem can be given for any case where you begin with a no-risk base situation, X, and face an option with random algebraic outcomes, Y, whose probability distribution is defined as

$$\text{(1)} \qquad Probability\ [Y \leqslant Z] = P(Z)$$

Then the expected value of any function of Y can be computed as an integral. In particular the expected value of utility of outcomes for any scale of bet e, which is the scalar quantity that is to be maximized with respect to e, is expressible as the following Stieltjes integral

$$\text{(2)} \qquad Expected\ u = v(e) = \int_{-\infty}^{\infty} u[X+eZ]\ P(dZ)$$

This provides convenient notation that can handle either discrete probabilities or continuous probability densities. Moreover, (2) may involve a multiple rather than single integral if we wish to let X, Y, and Z go beyond being merely the scalar magnitude of money wealth. Instead our notation handles for nothing, so to speak, the case where X is a row vector of goods and Y is a random vector of algebraic gains of those goods: Y = (tea gained, . . . , salt gained), etc. In the case of scalar money wealth we denote by $u_X[X+A]$ the marginal utility of money wealth at the point $X+A$. But to handle the vector of goods case, we need merely interpret $u_X[X+A]$ to mean the column vector of marginal utilities of the respective goods or the gradient of $u[x]$.

Example: For the authors' case where the bet involves a gain of $(k_1, \ldots, k_n)$ with probability p and a loss of $(h_1, \ldots, h_n)$ with probability $1\text{-}p$, our (2) would read, $pu[X+eK] + (1\text{-}p)u[X\text{-}eH]$.

To find the optimal e^* that maximizes $v(e)$, we differentiate (2) and look for an e root that makes the following first-derivative vanish

$$\text{(3)} \qquad v'(e) = \int_{-\infty}^{\infty} Zu_X[X+eZ]\ P(dZ) = 0$$

For risk-averters any e root will be sufficient to provide the maximizing e^*. For risk-lovers any possible e root would provide a minimum that is to be shunned; and hence we know that risk-lovers never have a proper finite-e^* optimum but instead must be plungers. For mixed types the necessary conditions of (3) are not enough to indicate where e^* will be and must be supplemented by secondary-condition investigations.

We are now in a position to give an exact definition of what constitutes a "fair," "favorable," or "unfavorable" bet to the person in question. Our criterion is to be the algebraic sign of $v'(e)$ at $e = 0$, or $v'(0)$: namely,

$$(4) \qquad v'(0) = \int_{-\infty}^{\infty} ZP(dZ)u_X[X] = \text{Expected } Yu_X[X]$$

These inner products are merely the expected values of the physical goods—tea, . . . , salt—summed and weighted by their respective marginal utilities in the no-risk situation. By definition a favorable bet involves $v'(0)$ positive; an unfavorable bet involves $v'(0)$ negative; and a fair bet involves $v'(0)$ zero. For Y a money scalar, this boils down to the original mathematicians' definition of fair and favorable. But in the vector case, unless every element of the vector involves a favorable bet in the physical commodity alone, we need to balance the favorable and unfavorable physical results by their respective marginal utility weights to arrive at an unambiguous definition.

Now suppose $v'(0)$ does not vanish. Then by picking a small e of its algebraic sign one will get a resulting $v(e)$ that must be better than $v(0)$. This is independent of the higher derivatives of $v(e)$ and hence establishes the first part of the theorem.

The second part of the theorem, that e^* must end up the same sign as $v'(0)$ for risk-averters follows immediately from the fact that if $u[X]$ is strictly concave so must be $v(e)$ for all non-trivial bets. If e^* were to be of opposite sign to $v'(0)$ for a strictly concave function, that would involve the contradiction that for such a function there would have to be a minimum e between e^* and 0, a patent absurdity for a concave function. When we examine the case of risk-lovers, we realize that the optimal e^* cannot be finite and that things do indeed get indefinitely better as one proceeds away from zero in the direction of a favorable bet. That e^* can be of opposite sign to $v'(0)$ in the mixed case is shown by the simple example of the Friedman-Savage guinea pig who faces an unfavorable long-shot bet. It would be better for him to reverse roles and take the function of a bookie in selling a small amount of such a bet as against standing pat; but so great is the marginal utility to him of being really wealthy that his optimal e^* is seen to be positive as he takes the bet in defiance of its negative $v'(0)$.

The final part of the theorem that I have added, which deals with fair bets, is proved as follows. For a risk-averter a fair bet provides a maximum at $e^* = 0$. A strictly concave function can have but one maximum. For a risk-lover $e^* = 0$ provides a minimum, from which he will certainly want to depart, gaining the more he does so. Finally, a mixed-allegiance person might or might not have distant reversals of his curvature that were strong enough to induce him into taking a given fair bet.

Note that where we have only a 1-way option, and e must be non negative, we can easily use Kuhn-Tucker programming to maximize $v(e)$, replacing the vanishing of $v'(e^*)$ by its being non-positive but being capable of being negative only if e^* vanishes. As the authors' suggest, a risk-lover confronted by an unfavorable 1-way option will shun sufficiently small bets and (perhaps more surprisingly) may shun *all* unfavorable bets at any scale if they are not "sufficiently not too unfavorable." This last paradox can be shown to arise when the rising marginal utility is confined between upper and lower bounds.

3. The above discussion of large-scale bets alerts us to a warning that is needed anyway: the expected value of utility, $v(e)$, may not be well-defined for all e scales; and it may particularly not be defined for negative e. Thus, suppose that money Y has the log-normal distribution so beloved of finance theorists. Then the largest gain is infinite. But now suppose that Y is an unfavorable bet. As a 2-way option, selling Y short produces a favorable bet. But will one wish to choose a negative e, as indicated in Theorem 1? Certainly not if one's marginal utility goes to infinity at zero wealth, as seems reasonable and is implied by the Bernoulli utility function $log\ X$ or by any member of the constant-relative-risk aversion family $u[X] = X^a/a,\ a < 1$. For such cases $v(e)$ is defined only for non-negative e and the 2-way option will never be relevant even if it is available. Risk averters will shun such a 1-way unfavorable or fair bet. Risk-lovers must plunge into fair 1-way bets but, as noted, may (if their strong convexity attenuates at high wealths) find it never pays to embrace at all highly unfavorable bets.

4. All that has been said so far hinges on the crucially important postulate that the decision maker has a no-risk base from which he can operate. Life need not be like that. Whatever we do there may be so many uncontrollable states of the world that could occur that we stand to face a variable outcome even if we stand pat. In that case the simple Theorem 1 has no fulcrum from which to operate.

Let me illustrate with a simple dollar-wealth case where X and Y are scalars. I begin at X of money wealth. I face the scale-free option eY. But no matter what my choice of e, I also am faced with uncontrollable dollar increments, S, that must be added to X—giving me in the end as final chance wealth outcomes $X + S + eY$, where I face a joint probability distribution for S and Y.

$$\text{(5)} \qquad Prob\left\{S \leqslant T \text{ and } Y \leqslant Z\right\} = P(T, Z)$$

Now I act to maximize

$$\text{(6)} \qquad v(e) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u[X + T + eZ] P(dT, dZ)$$

and

$$\text{(7)} \qquad v'(e) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Zu_X[X + T + eZ] P(dT, dZ)$$

And now the new fairness criterion would seem, as before, to hang on the algebraic sign of $v'(0)$ in

$$\text{(8)} \qquad v'(0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Zu_X[X + T] P(dT, dZ)$$

Note that this will by no means have to agree with the traditional dollar criterion of a fair gamble, since (8)'s new $v'(0)$ need *not* have the algebraic sign of

$$\text{(9)} \qquad Expected\ Y = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Z\, P(dT, dZ)$$

As shown in Samuelson-Merton,[5] we must replace the objective utility of $P(T, Z)$ by the "util prob" metric, $Q(T, Z)$, where by definition

$$\text{(10)} \qquad Q(dT, dZ) \equiv u_X[X + T] P(dT, dZ)/c$$

$$c = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_X[X + T] P(dT, dZ)$$

There is, however, the special case where the betting options outcomes happen to be independently distributed from the S uncontrollable variations. The independence case involves

$$\text{(11)} \qquad P(T, Z) \equiv R(T) P(Z)$$

With independence, our dollar fairness criterion does agree with the conventional definition, since now

$$\text{(12)} \qquad v'(0) = \int_{-\infty}^{\infty} Z\, P(dZ) \quad u_X[X + T] R(dT)$$

$$= (Expected\ Y)\ (Expected\ u_X)$$

With marginal utility always positive, the second factor in this last product

5. P. A. Samuelson and R. C. Merton, "A Complete Model of Warrant Pricing that Maximizes Utility," *Industrial Management Review*, 10 (Winter, 1969), pp. 17-46, reproduced as Chapter 200 in *CSP*, pp. 818-847. See particularly p. 19 or 820 and Appendix A's useful lemmas provided by Prof. David T. Scheffman, now of the University of Western Ontario. Robert Solow reminds me that (8) can be interpreted as "the expected value of 'Z' times 'the conditioned expected value of U_X, conditional on Z'" or *Expected* $[ZE\{U_X|Z\}]$ from which (12) follows intuitively in the independence case.

must be positive, giving $v'(0)$ the same algebraic sign as the first factor. Independence thus gives the same result with objective probability or with personal util-prob. These formulas hold also for X, S, and Y commodity vectors.

The independence case permits us to generalize Theorem 1. Now, both in the scalar dollar-wealth case and the vector many-commodity case, we can make the same inferences about "favorable" and fair scale-free 2-way options, provided only we carefully define "fair" bets so that, in the many-commodity case the expected values of the physical commodities are combined not with no-risk marginal utilities as weights, as in (4)'s $u_X[X]$, but rather with (12)'s expected marginal utilities as weights (the expectations being averaged over the $e = 0$ outcomes).

Theorem 2. Scale-free 2-way investing options whose outcomes are independently distributed from uncontrollable variations are subject to *all* the results of Theorem 1, provided our definition of fairness uses as weightings of physical-commodities' expected values (not the no-risk marginal utilities but rather) the marginal utilities' expected values in the zero-bet situation. Thus, a bet that is not fair will at some small scale be better than an $e = 0$ decision . . . ; risk-averters will shun fair bets, just as risk-lovers will shun small unfavorable 1-way bets; . . . ; etc.

5. Actually, if we are willing to work with definitions of fairness that are completely subjective, *being different for each different person* (even in the scalar dollar-wealth case!), we can generalize both these theorems as follows.

Theorem 3. Applying as the criterion of fairness for a 2-way option $v'(0)$ in (8), we can validly state *all* the conclusions of the two earlier theorems on how risk-averters, risk-lovers and mixed-type persons will behave. In addition, new conclusions are valid such as the following: A risk-averter will definitely want to accept a bet that is "fair in *objective* money (or every-commodity) terms" provided it is distributed in probability in *negative dependence*[6] on the no-bet S outcomes, in the sense that

$$\text{(13)} \qquad \textit{Conditional probability } [Y < Z \textit{ when } S = T] = P(Z|T)$$
$$\text{with } P(Z|T_1) > P(Z|T_2) \textit{ if } T_1 > T_2 .$$

6. For more on this see P. A. Samuelson, "General Proof that Diversification Pays," cited in footnote 3. Theorem III is particularly relevant. Note: a printer's error on p. 8 (or p. 855) prevents $\partial P(\underline{x}_i | \underline{x}_j)/\partial x_j > 0$ appearing as required here. See also the more difficult developments of this coal-and-ice company theorem in D. Scheffman, *Two Essays in Economic Theory*, MIT Doctoral Dissertation (1971), parts of which should be published in journal articles.

6. This lack of objectivity in the definition of a fair bet must at first seem distressing. Thus, Jones may rationally utilize Theorem 3 to make a dollar bet whose objective expected value he believes to be negative, even though Jones is a risk-averter. At the same time, risk-averter Smith may rationally bet in the opposite direction. (Recall Friedman-Savage's example.) Since economists often can only observe how people bet and not what their probability estimates are, doesn't Theorem 3 come close to saying, "The bets risk-averters make must be 'favorable' or they wouldn't have made them." "People do what they do" constitutes only an empty truism that rises to the level of a fatuity.

As the author's analysis makes clear, we were *already* in Theorem 1 in the morass of subjectivity when dealing with many-commodity vectors. Even if Smith and Jones agreed that tea's expected value in the bet was positive and salt's expected value was negative, they might rationally as risk-averters choose to be on opposite sides of the bet—provided that their relative marginal utilities for salt and tea differed in the no-risk situation.

However, one might have hoped—superficially as I shall show—to get rid of subjectivity as follows.

Suppose in the no risk situation, Pareto-optimality has somehow *already* been achieved. Then the $u_X[X]$ marginal-utilities vectors in (4) of two different persons will *already* have become proprotional to each other. Hence, speaking loosely, we can say that everybody "in the same market" will have the *same* "fair" bet criterion. (We speak loosely because Pareto-optimality need not come about from use of market pricing.)

Moreover, pushing along this same tempting logical path, we might try to bring Theorem 2 also into the *objective* camp by the following trick. Arrow-Debreu contingency-security markets, it is known,[7] can hope to bring about Pareto-optimality in stochastic situations. So let us hope that the random S that is added to each person's X—say $X^k + S^k$ for the k-th person—will already because of Arrow-Debreu contracts be arranged so that we have proportionality for all persons, k or otherwise, to the same weighting (column) vector *Expected* $u^k[X^k + S^k]$ in (8). If this hope can come true, Theorem 2 would seem to operate in terms of *objective* definitions of a "fair" or "favorable" bet.

7. However, before pushing our luck to trying to bring Theorem 3's general case on to the *terra firma* of objectivity, notice the fallacy of the hoped-for line of escape. Why should the *no-risk* situation's marginal utilities be made Pareto-optimal when in fact Theorem 1 says we shall all be

7. Cf. K. J. Arrow, *Essays in the Theory of Risk Bearing* (Markham Publishing Co., Chicago, 1971), especially Ch. 4, a 1963 English translation of Arrow's classic Paris paper of 1952; G. Debreu, *Theory of Value* (John Wiley and Sons, New York, 1959), final chapter.

departing from the no-risk bases? Arrow-Debreu Pareto-optimization ought to apply to the non-zero e^* states of the world, not to the $e = 0$ pre-bet states of the world.

Theorem 3 alerts us to the absolute need for subjective differences in "fairness," for it is just those differences that make it possible for Smith and Jones to both want to bet with each other. Some examples can illustrate.

Let Y be a fair bet in dollar terms. But let Y's gains tend to occur when risk-averter Smith's S shows losses and when risk-averter Jones' S shows gains. Then, as already noted, an arm's-length bet of Smith at positive e with Jones at negative e makes sense for *both* of them. They are mutually reinsuring in the Arrow-Debreu fashion! Mark Twain said that it is differences of opinion [about probabilities] that make horse races. We can add: Even with the same opinions about probabilities, it is differences in people's situations that make zero-sum bets possible.

8. I conclude by defining "a person's subjective marginal 'fair' bet, which is to be fair (not necessarily at the $e = 0$ level, but rather) at any specified scale level, e, and is to involve a zero $v'(e)$ there." The optimal scale for a 2-way bet is in the general case defined by e^*, at the root of e for which

$$(14) \qquad 0 = v'(e^*) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Z\, u_X[X + T + e^*Z]\, P(dT, dZ)$$

Hence, using the algebraic sign of $v'(e)$ as the criterion of "marginal favorability or fairness" at the scale e, we have the circular theorem.

> Theorem 4. At the optimal scale for a bet, e^*, the bet must be *marginally* fair (with zero expected value under the util-prob there). All different securities one holds have the same "marginally fair" property with Samuelson-Merton util-prob expected values equal.

In a trivial sense, everyone in his e^* equilibrium experiences a marginal fair bet, but one man's $v'(e^*)$ vanishes at an e^* that is different from the e^* of the man he bets with.

9. I must not leave the impression that bets or investment gambles have to be zero-sum, with as much economic weight on one side as on the other. For each of us—and even Robinson Crusoe alone—can also make bets with Mother Nature, as for example when we plant seeds in the ground and reap a random harvest. Unlimited-scale bets, with other people or with Nature, are usually possible only when (a) each one of us is so small as to be a "bet *taker*" too unimportant to affect the odds; (b) and/or Nature's stochastic production function has *expected values* of outputs that are homogeneous-first degree functions of all inputs, (c) or, more strongly, has vector random outputs that are proportional to input

in the following sense—if H represents a vector of inputs and R a random vector of outputs, then

$$(15) \qquad Prob \quad R \leqslant Q|H = H \quad = \pi(Q|H^\circ) \equiv \pi(mQ/mH^\circ),\ m > 0$$

Does man have *limited liability* when it comes to bets with Nature? Often we have nothing to lose but our positive production inputs. But when we sow too much U-235, we may reap the whirlwind!

Axiom systems, like that of Arrow,[8] often assume utility to be bounded both above and below. And I do agree that many (perhaps most) people will, on introspection, feel that *their* utility is indeed bounded above; in the same way, even more people will feel that their utility is not linear in money. Sophisticated people in the latter group do not need to contemplate the famous St. Petersburg Paradox to learn that their marginal utility is decreasing; but some others may be made aware of this fact by such a dramatic example. Similarly, a super-St. Petersburg paradox of the Karl Menger type (as referred to by Arrow) may alert some subset of bounded-utility people to awareness of this fact. So much is not in dispute. But nothing said yet deprives a person from having linear utility, if on introspection he finds he really does. It's a free country! Two people in the universe with linear (or even convex) utility will gamble with each other inveterately even if an infinite-stakes game is feasible only in one-sided thought experiments. On the other hand, in a universe where everyone had a strictly-concave utility that is unbounded above, there would be no super-St. Petersburg game that can actually find a Peter to offer its option to the willing Pauls. So I fail to see the Menger theorem's compelling relevance. Nor do I think I should accept the bounded-utility axiom merely out of the fear that without it there may exist probability options that I am unable to rank by their first moments of utility—since their expected values may fail to exist. Am I ever likely to be confronted with choices among such option in the real world? I wonder.

As we saw with short-sales of IBM or other stocks, failure of convergence is most to be feared at low or zero levels of wealth. Bankruptcy or Death provides difficulties for any theory of finance or stochastic decision making. "One might as well be hung for a sheep as a lamb; or go bankrupt owing a million as a cent," are slogans that can cost one's fellowmen or co-passengers dearly—which is why every viable society tries to protect itself from *après-moi-le-deluge* rashness by placing crucial decision-making in staggered-age committees and family councils. These help defend the utility function from a relevant lower bound that destroys its strong con-

8. K. J. Arrow, *op. cit.*, Ch. 2, especially pp. 64-69.

cavity property on which the nice theorems about risk-averters must depend.[9]

Even if the utility function need only be defined for wealth outcomes that are non-negative, convergence difficulties can arise if marginal utility becomes infinite at zero wealth. This is because, in real life, no matter what one does, one may well face a finite probability of "ruin." (Like Socrates, all men are mortal.) In that case, no situation has a defined expected utility.

What to do? A case can be then made, I think, for the following lexicographic ordering that involves a minimax strategy.

1. Act always, between decision A and decision B, so as to select the option with lowest probability of literal ruin.

2. As between all options that share the same (lowest) probability of ruin, choose the one with the highest expected utility over *positive* outcomes.

Such a minimax (or maximin) strategy seems to be free of the usual minimaxer's paranoia which pretends that Nature is an omnipotent and malevolent adversary who will construe the worst.

9. The behavior of utility at lowest levels offers, I think, the more perplexing problems. I must confess I don't understand why anyone would permit me to make a short sale if (a) my wealth is finite, and (b) we both believe that the stocks' price is log-normally distributed with expected value not less than current price. But, as a thought experiment, I can still be permitted to shun any offered short sale on the ground that, if it results in a negative T outcome that makes my final outcome, $T+X$, negative, the dishonor of bankruptcy is deemed by me so infinitely bad as to make me prefer to the short sale any alternative that, with probability greater than $\epsilon > 0$, leaves me with no less than $0 + \eta > 0$, where ϵ and η are arbitrarily small. This does mean that my relevant $v(e)$ is not defined for negative e (or has to be defined as "minus infinity"). Whether $u[X+T]$ is termed concave for non-positive arguments is a semantic question of not much interest: certainly $u[-w^2]/u[w^2+a^2]$ is effectively zero for *all* w and for any positive a^2, however small.