Voronoï Percolation in the Hyperbolic Plane

Benjamin Hansen
University of Groningen

Joint work with Tobias Müller

July 15, 2019

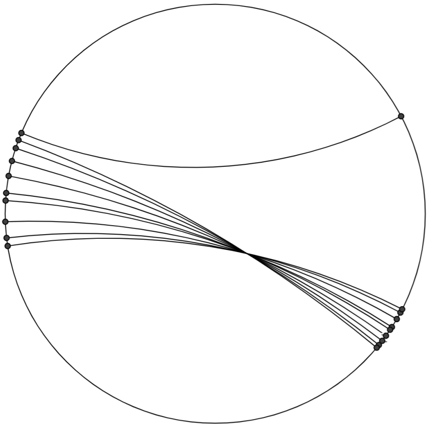
Hyperbolic Plane $\mathbb{H}^2$

Hyperbolic Plane

- A surface in a high dimensional space
- Every point is a saddle point (Gaussian curvature is negative)
- Fails Euclid's fifth axiom

Poincaré Disk

Equip unit disk with the right metric



$$d_{\mathbb{H}}(x,y) = 2\,\text{arcsinh}\left(\frac{||x-y||}{\sqrt{(1-||x||^2)(1-||y||^2)}}\right)$$

$$\text{Area}(B) = \int_B \frac{4}{(1-x^2-y^2)^2}\,dydx$$

Poisson Point Process on $\mathbb{H}^2$

A Poisson point process on $\mathbb{H}^2$ is a random countable collection of points $\mathcal{Z}$ in $\mathbb{H}^2$.

For a (homogenous) Poisson point process with intensity $\lambda > 0$, the number of points falling in a region B is Poisson random variable with expectation $\lambda \cdot \text{Area}(B)$.

Additionally, for disjoint regions the number of points falling in each region are independent.

For inhomogeneous case replace $\lambda \cdot \text{Area}(B)$ with $\int_B f dA$ for some $f \geq 0$.

Poisson Point Process on $\mathbb{H}^2$

A Poisson point process on $\mathbb{H}^2$ is a random countable collection of points $\mathcal{Z}$ in $\mathbb{H}^2$.

For a (homogenous) Poisson point process with intensity $\lambda > 0$, the number of points falling in a region B is Poisson random variable with expectation $\lambda \cdot \text{Area}(B)$.

Additionally, for disjoint regions the number of points falling in each region are independent.

For inhomogeneous case replace $\lambda \cdot \text{Area}(B)$ with $\int_B f dA$ for some $f \geq 0$.

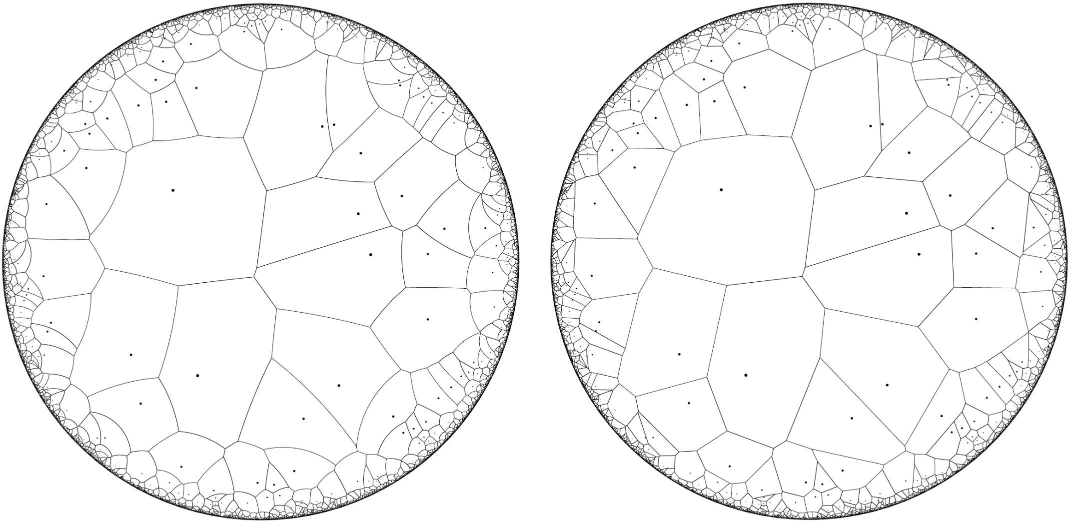
A homogeneous PPP with intensity λ on the Poincaré disk can be seen as an inhomogeneous PPP on the unit disk D where the expected number of points falling in B is

$$\int_B \frac{4\lambda}{(1-x^2-y^2)^2} dxdy.$$

Voronoï cells on $\mathbb{H}^2$

For the two metrics, $d_{\mathbb{H}}$ and d_E, the hyperbolic and Euclidean metrics, assign every $x \in D$ to the "closest" element of $\mathcal{Z}$.

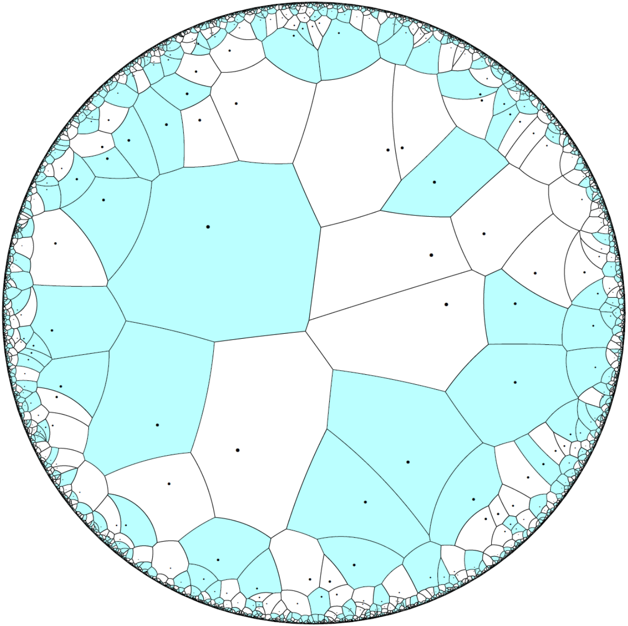
Two Voronoï tessellations for D : $C_{\mathbb{H}}(\mathcal{Z})$ and $C_E(\mathcal{Z})$



Percolation on $\mathbb{H}^2$

For $C_{\mathbb{H}}(\mathcal{Z})$, colour cells black independently with probability p.

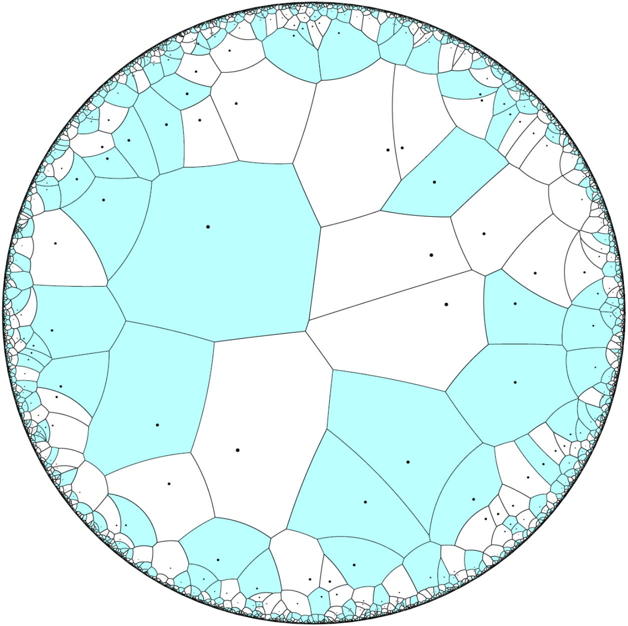
- In the picture, $p = 1/2$ and $\lambda = 1$.



Percolation on $\mathbb{H}^2$

For $C_{\mathbb{H}}(\mathcal{Z})$, colour cells black independently with probability p.

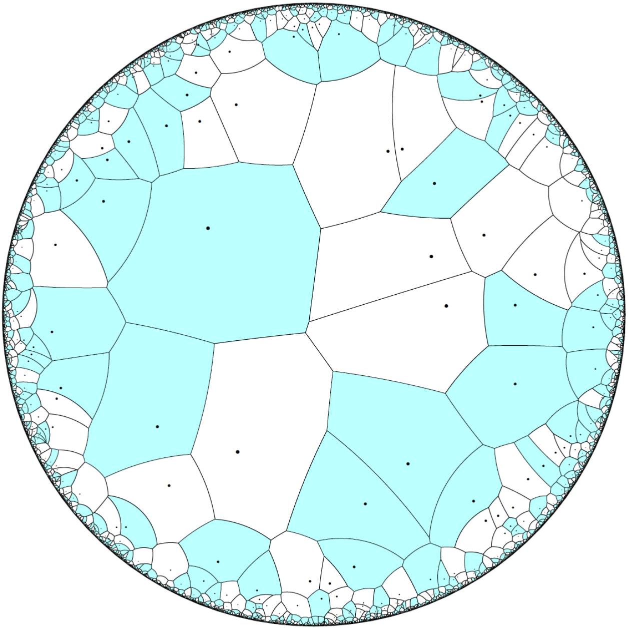
- In the picture, $p = 1/2$ and $\lambda = 1$.
- Is there an unbounded black component? (critical value p_c)



Percolation on $\mathbb{H}^2$

For $C_{\mathbb{H}}(\mathcal{Z})$, colour cells black independently with probability p.

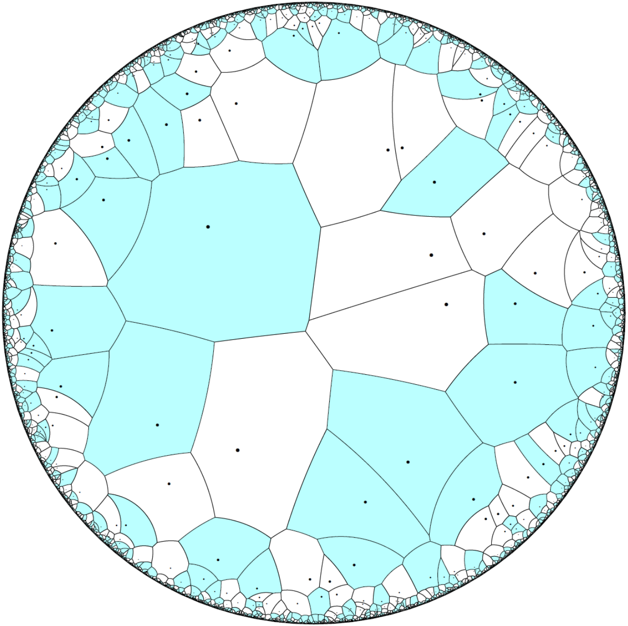
- In the picture, $p = 1/2$ and $\lambda = 1$.
- Is there an unbounded black component? (critical value p_c)
- A unique unbounded black component? (critical value p_u)



Percolation on $\mathbb{H}^2$

For $C_{\mathbb{H}}(\mathcal{Z})$, colour cells black independently with probability p.

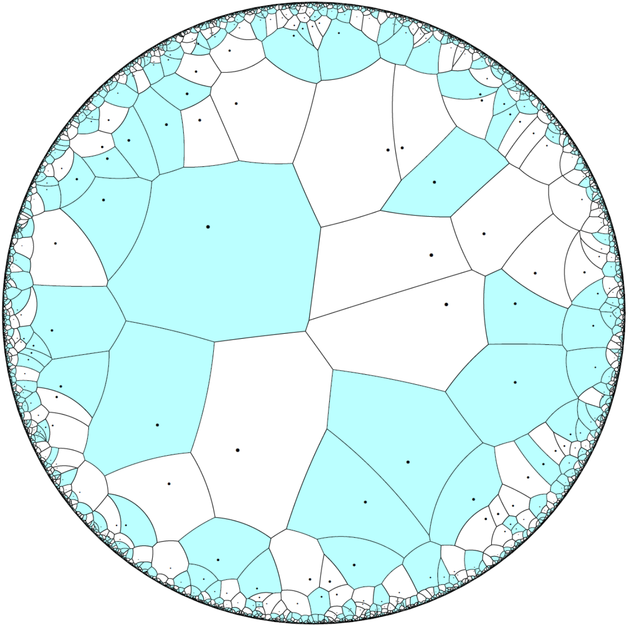
- In the picture, $p = 1/2$ and $\lambda = 1$.
- Is there an unbounded black component? (critical value p_c)
- A unique unbounded black component? (critical value p_u)
- Two sources of randomness



Percolation on $\mathbb{H}^2$

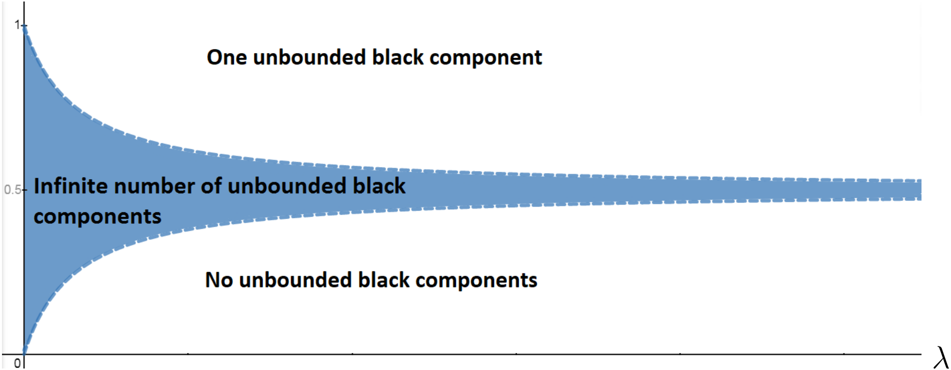
For $C_{\mathbb{H}}(\mathcal{Z})$, colour cells black independently with probability p.

- In the picture, $p = 1/2$ and $\lambda = 1$.
- Is there an unbounded black component? (critical value p_c)
- A unique unbounded black component? (critical value p_u)
- Two sources of randomness
- Both p_c and p_u depend on λ



Phase Diagram (Benjamini + Schramm 2001)

$0 < p_c(\lambda) \leq \frac{1}{2} - \frac{1}{4\pi\lambda+2}$ and $p_u(\lambda) = 1 - p_c(\lambda)$,



Also p_c is continuous, a.s. no unbounded black component for $(\lambda, p_c(\lambda))$, and conjectured $\lim_{\lambda\to\infty} p_c(\lambda) = 1/2$, the value of p_c in the plane for a homogenous PPP under the Euclidean metric (as shown by Bollobás + Riordan in 2006).

Statements

Theorem (H + Müller 2019+)

$\lim_{\lambda\to\infty} p_c(\lambda) = 1/2$.

Lemma (H + Müller 2019+)

In the Poincaré disk, consider any region R that when viewed as a subset of $\mathbb{R}^2$ is a rectangle. Denote two opposing sides as Left *and* Right*. If $p > 1/2$, then, under the intensity function $\frac{4\lambda}{(1-x^2-y^2)^2}$ and hyperbolic metric $d_{\mathbb{H}}$,*

$$\lim_{\lambda\to\infty} \mathbb{P}_\lambda(\exists \textit{ path in } R \textit{ from } \text{Left} \textit{ to } \text{Right} \textit{ using only black cells}) = 1.$$

Crossing "Rectangles": Changing Metric

For $\mathcal{Z}$ a PPP on the Poincaré disk, almost surely two cells are adjacent in $C_{\mathbb{H}}(\mathcal{Z})$ if and only if they are adjacent in $C_E(\mathcal{Z})$. (The circles which determine adjacency under both metrics share the same boundary.)

Crossing “Rectangles”: Changing Metric

For $\mathcal{Z}$ a PPP on the Poincaré disk, almost surely two cells are adjacent in $C_{\mathbb{H}}(\mathcal{Z})$ if and only if they are adjacent in $C_E(\mathcal{Z})$. (The circles which determine adjacency under both metrics share the same boundary.)

$+$ When the intensity is large, the boundaries of cells do not change significantly when changing the metric

Crossing “Rectangles”: Changing Metric

For $\mathcal{Z}$ a PPP on the Poincaré disk, almost surely two cells are adjacent in $C_{\mathbb{H}}(\mathcal{Z})$ if and only if they are adjacent in $C_E(\mathcal{Z})$. (The circles which determine adjacency under both metrics share the same boundary.)

$+$ When the intensity is large, the boundaries of cells do not change significantly when changing the metric

$\Rightarrow$ Hence if a “rectangle” is crossed under one metric, the same cells can be used to cross it in the other metric.

Use the Euclidean metric instead of the hyperbolic metric.

Crossing Rectangles: Inhomogeneous Poisson Process

The area element $\frac{4}{(1-x^2-y^2)^2}$ is a continuous function. So if we stay away from the boundary of the unit disk, the area element is uniformly continuous.

Implies for small rectangles R away from the boundary, the inhomogeneous (hyperbolic) PPP $\mathcal{Z}$ coloured black with probability $p > 1/2$ can be coupled with a homogenous PPP on $\mathbb{R}^2$ denoted $\mathcal{Z}_R$ coloured black with probability $p_{new} > 1/2$.

On R, every black point of $\mathcal{Z}_R$ is a black point in $\mathcal{Z}$ and every white point of $\mathcal{Z}$ is a white point in $\mathcal{Z}_R$.

By monotonicity, crossings of R are harder for $\mathcal{Z}_R$ then $\mathcal{Z}$.

Critical Ingredient (Bollobás + Riordan 2006)

Let $a, b, t > 0$, and rectangle R_t have side lengths at and bt. Let $\mathcal{Z}$ be a PPP with constant density on $\mathbb{R}^2$ and colour cells black independently with probability $p > 1/2$. Then

$$\lim_{t \to \infty} \mathbb{P}(\text{Horizontal black crossing of } R_t) = 1.$$

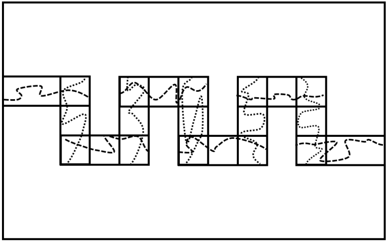
Crossing $\mathcal{Z}_R$ is Easy

How hard is it for R to be crossed using black cells of $\mathcal{Z}_R$?

The intensity of $\mathcal{Z}_R$ is at least 4λ.

So rescale the previous R and $\mathcal{Z}_R$ by the square root of intensity to get a constant density and increasing R.

By Bollobás + Riordan 2006, crossing R is easy with $\mathcal{Z}_R$ and so it is easy with the inhomogeneous $\mathcal{Z}$ (as $\lambda \to \infty$).

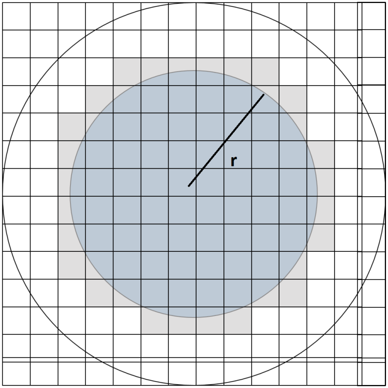


Crossing hyperbolic "rectangles" is easy when $p > 1/2$.

Fill Event

Cover the unit disk with squares of width $q > 0$ and fix a disk D_r centered at the origin with radius $r > 0$.

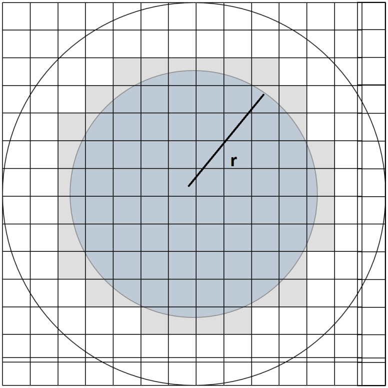
- If a point from the PPP falls in every square that intersects D_r, the event Fill occurs.



Fill Event

Cover the unit disk with squares of width $q > 0$ and fix a disk D_r centered at the origin with radius $r > 0$.

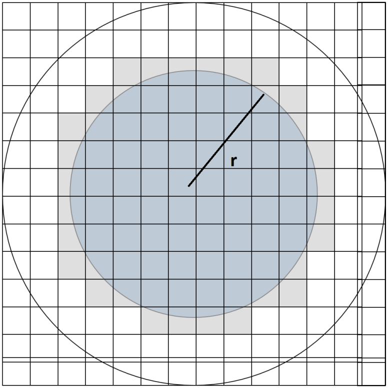
- If a point from the PPP falls in every square that intersects D_r, the event Fill occurs.
- For event A depending on the colouring of a sufficiently smaller disk inside D_r, $A \cap \mathit{Fill}$ is independent of $\mathcal{Z} \cap D_r^c$.



Fill Event

Cover the unit disk with squares of width $q > 0$ and fix a disk D_r centered at the origin with radius $r > 0$.

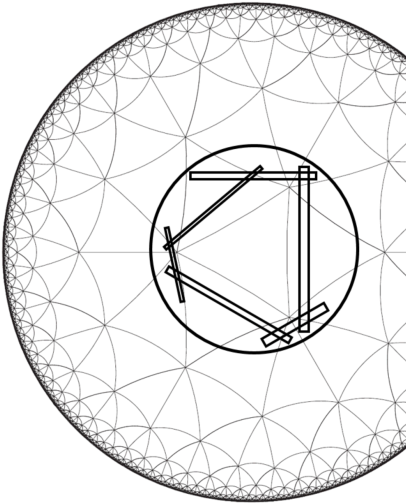
- If a point from the PPP falls in every square that intersects D_r, the event Fill occurs.
- For event A depending on the colouring of a sufficiently smaller disk inside D_r, $A \cap \mathit{Fill}$ is independent of $\mathcal{Z} \cap D_r^c$.
- Finite $\#$ squares $\implies$ as $\lambda \to \infty$, Fill holds a.a.s.



Dependent Face Percolation on the $\{3,7\}$ Tiling

Start with a face centered at the origin. Use the following mechanism to colour faces of the triangles. Let $p < 1/2$,

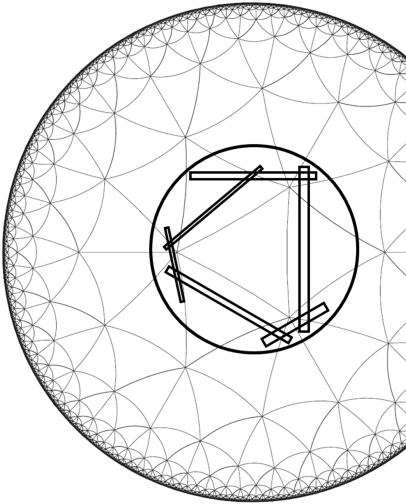
- Fix a set of “rectangles” around this triangle.



Dependent Face Percolation on the {3,7} Tiling

Start with a face centered at the origin. Use the following mechanism to colour faces of the triangles. Let $p < 1/2$,

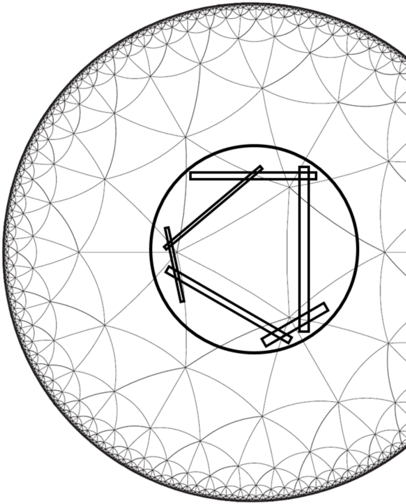
- Fix a set of "rectangles" around this triangle.
- Generate $C_{\mathbb{H}}(\mathcal{Z})$.



Dependent Face Percolation on the $\{3,7\}$ Tiling

Start with a face centered at the origin. Use the following mechanism to colour faces of the triangles. Let $p < 1/2$,

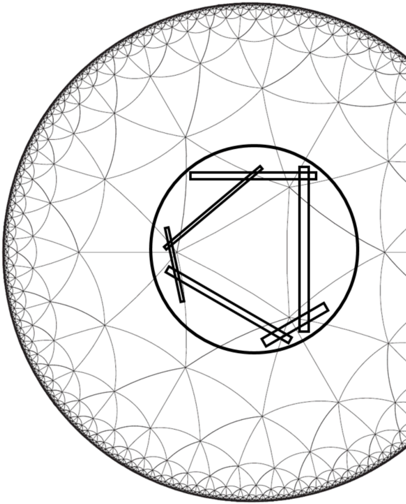
- Fix a set of "rectangles" around this triangle.
- Generate $C_{\mathbb{H}}(\mathcal{Z})$.
- All "rectangles" having long white crossings $\implies$ $\exists$ white circuit in a small ball with the triangle on the interior.



Dependent Face Percolation on the $\{3,7\}$ Tiling

Start with a face centered at the origin. Use the following mechanism to colour faces of the triangles. Let $p < 1/2$,

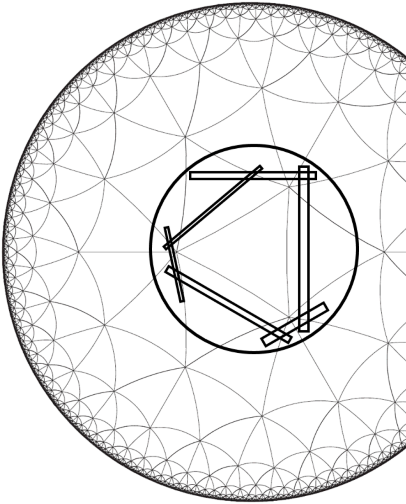
- Fix a set of "rectangles" around this triangle.
- Generate $C_{\mathbb{H}}(\mathcal{Z})$.
- All "rectangles" having long white crossings $\implies$ $\exists$ white circuit in a small ball with the triangle on the interior.
- The event Fill on a slightly larger ball B makes Fill $\cap$ "white circuit" independent of $\mathcal{Z}$ outside B.



Dependent Face Percolation on the $\{3,7\}$ Tiling

Start with a face centered at the origin. Use the following mechanism to colour faces of the triangles. Let $p < 1/2$,

- Fix a set of "rectangles" around this triangle.
- Generate $C_{\mathbb{H}}(\mathcal{Z})$.
- All "rectangles" having long white crossings $\implies \exists$ white circuit in a small ball with the triangle on the interior.
- The event Fill on a slightly larger ball B makes Fill $\cap$ "white circuit" independent of $\mathcal{Z}$ outside B.
- For each face, isometrically map the tiling and $C_{\mathbb{H}}(\mathcal{Z})$ so that the face takes on the role of the first face. If Fill $\cap$ "white circuit" occurs, colour that face white. Otherwise, black.



No Infinite Black Clusters in the $\{3, 7\}$ Tiling

- Dependent!

No Infinite Black Clusters in the $\{3, 7\}$ Tiling

- Dependent!
- But not that dependent. (k independent)

No Infinite Black Clusters in the $\{3,7\}$ Tiling

- Dependent!
- But not that dependent. (k independent)
- $\lim_{\lambda\to\infty} \mathbb{P}_\lambda(\text{Face is white}) = 1$ (Duality with black for $p > 1/2$)

No Infinite Black Clusters in the $\{3, 7\}$ Tiling

- Dependent!
- But not that dependent. (k independent)
- $\lim_{\lambda \to \infty} \mathbb{P}_\lambda(\text{Face is white}) = 1$ (Duality with black for $p > 1/2$)
- For $p < 1/2$ and λ large $\Rightarrow$ The black clusters of faces must be finite a.s. [Liggett + Schonmann + Stacey 1997]

But what if..

- If there is also an unbounded cluster of black cells in $C_{\mathbb{H}}(\mathcal{Z})$

But what if..

- If there is also an unbounded cluster of black cells in $C_{\mathbb{H}}(\mathcal{Z})$
- Then there will be a black path tending toward the boundary of the disk.

But what if..

- If there is also an unbounded cluster of black cells in $C_{\mathbb{H}}(\mathcal{Z})$
- Then there will be a black path tending toward the boundary of the disk.
- The "white circuit" cannot hold for any triangles this path intersects and the path will intersect adjacent triangles $\Rightarrow$

But what if..

- If there is also an unbounded cluster of black cells in $C_{\mathbb{H}}(\mathcal{Z})$
- Then there will be a black path tending toward the boundary of the disk.
- The "white circuit" cannot hold for any triangles this path intersects and the path will intersect adjacent triangles $\Rightarrow$
- $\exists$ Black cluster of faces that is infinite.

But what if..

- If there is also an unbounded cluster of black cells in $C_{\mathbb{H}}(\mathcal{Z})$
- Then there will be a black path tending toward the boundary of the disk.
- The "white circuit" cannot hold for any triangles this path intersects and the path will intersect adjacent triangles $\Rightarrow$
- $\exists$ Black cluster of faces that is infinite.
- For $p < 1/2$ and large enough λ, almost surely there are no unbounded black clusters in $C_{\mathbb{H}}(\mathcal{Z})$.

But what if..

- If there is also an unbounded cluster of black cells in $C_{\mathbb{H}}(\mathcal{Z})$
- Then there will be a black path tending toward the boundary of the disk.
- The "white circuit" cannot hold for any triangles this path intersects and the path will intersect adjacent triangles $\Rightarrow$
- $\exists$ Black cluster of faces that is infinite.
- For $p < 1/2$ and large enough λ, almost surely there are no unbounded black clusters in $C_{\mathbb{H}}(\mathcal{Z})$.
- $\lim_{\lambda \to \infty} p_c(\lambda) = 1/2$.

But what if..

- If there is also an unbounded cluster of black cells in $C_{\mathbb{H}}(\mathcal{Z})$
- Then there will be a black path tending toward the boundary of the disk.
- The "white circuit" cannot hold for any triangles this path intersects and the path will intersect adjacent triangles $\Rightarrow$
- $\exists$ Black cluster of faces that is infinite.
- For $p < 1/2$ and large enough λ, almost surely there are no unbounded black clusters in $C_{\mathbb{H}}(\mathcal{Z})$.
- $\lim_{\lambda \to \infty} p_c(\lambda) = 1/2$.

Thank you! Questions?