Percolation Theory

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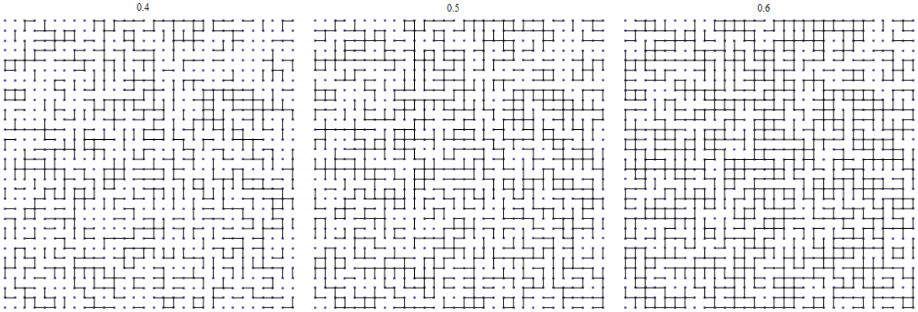
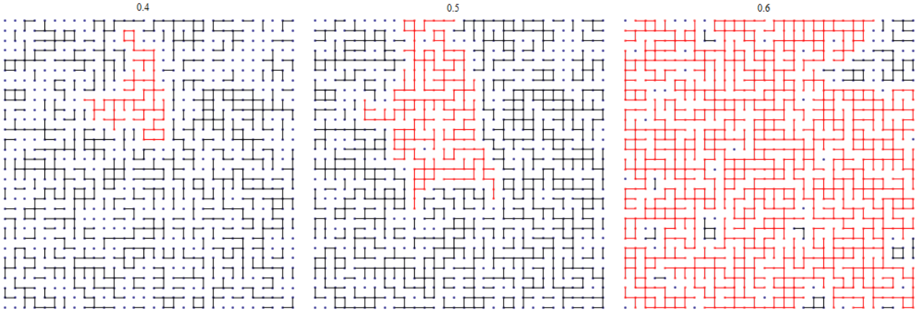
Differences Among These Graphs?



With Largest Cluster Highlighted



Outline

1. History and Overview
2. Unique Boundary of a Finite Open Cluster
3. Bounding the Critical Probability: p_H
   - Proof of Lower Bound
   - Proof of Upper Bound

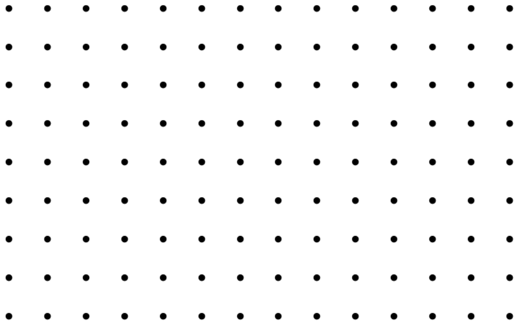
History and Overview

What is Percolation?

- Broadbent (1954): Gas Masks
- Fluid vs Medium
- Diffusion vs Percolation
- Broadbent and Hammersley: Monte Carlo Simulations
- Murray Hill (1961) Program's Running Time: 39 Hours

Definitions

- Square Lattice, $\mathbb{Z}^2$
- Sites and Bonds
- Open and Closed
- Dual Lattice, $\mathbb{Z}^{2*}$
- Bond Configuration, p



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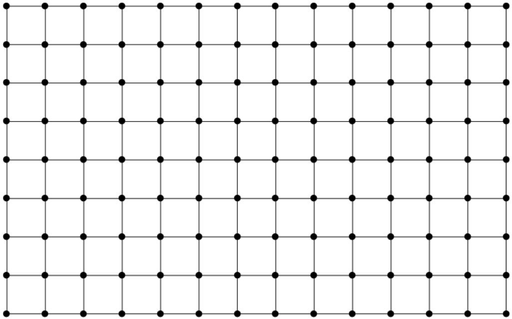

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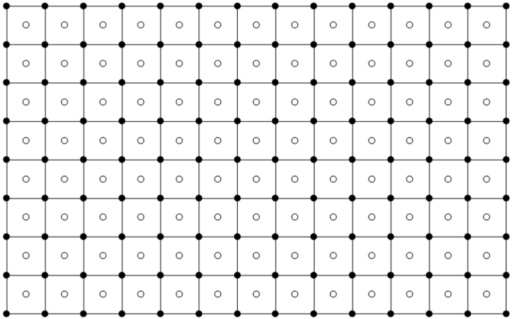

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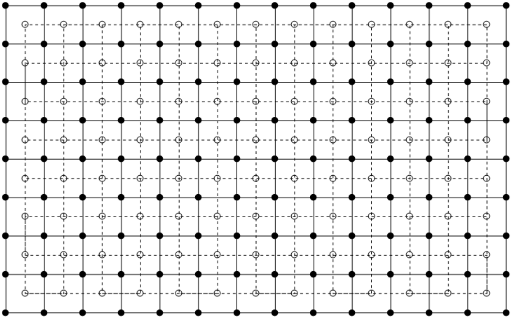

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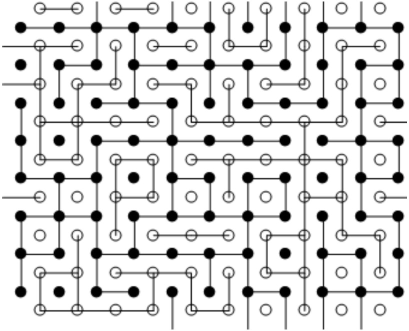
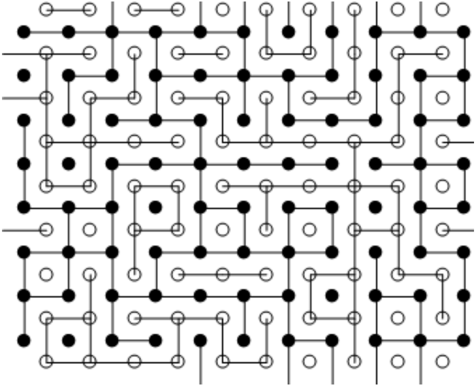


More Definitions

- Walk (Open Walk)
- Path (Open Path)
- μ_d: The number of paths of length d
- Cycle (Open Cycle)
- Open Cluster
- Infinite Cluster



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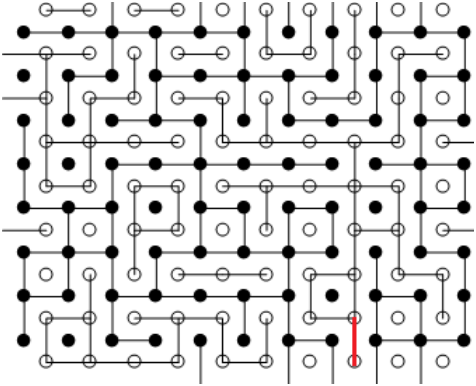

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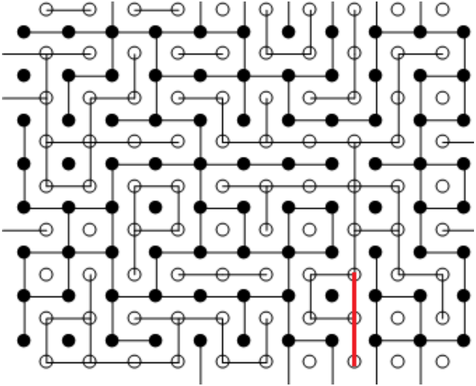

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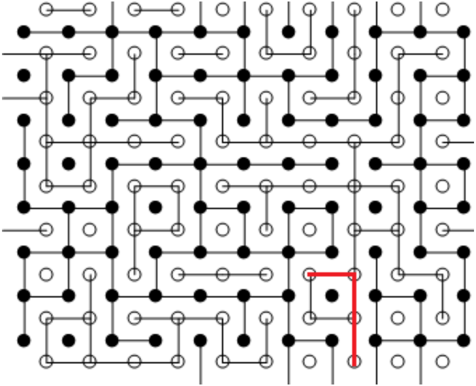

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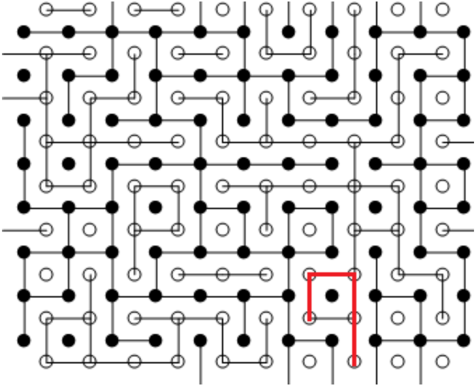

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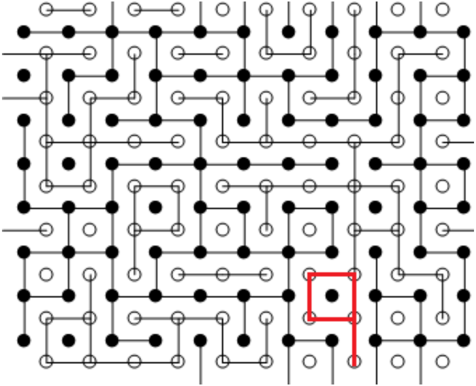

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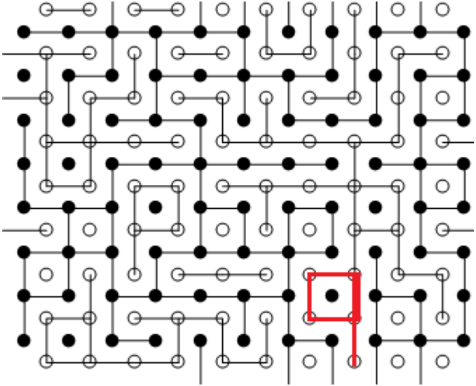

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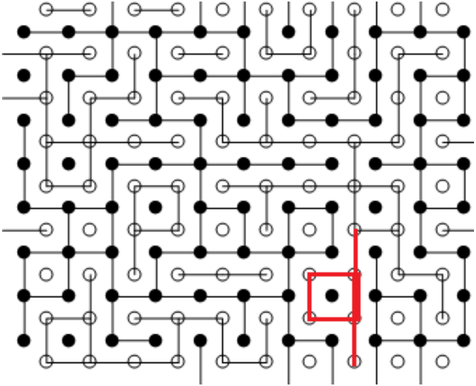

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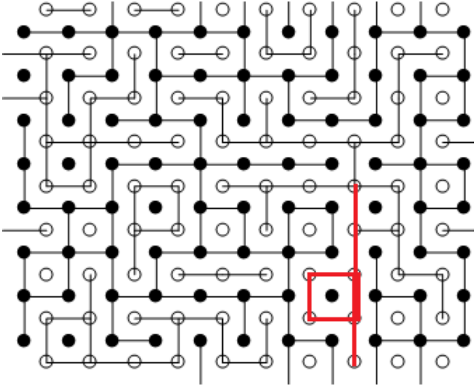

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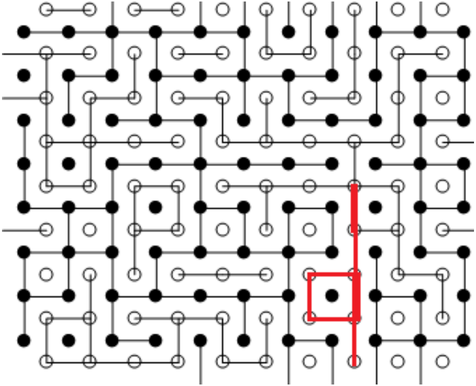

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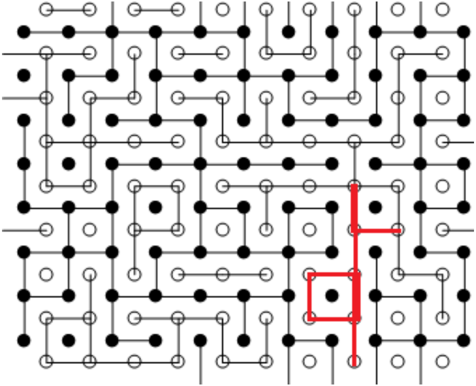

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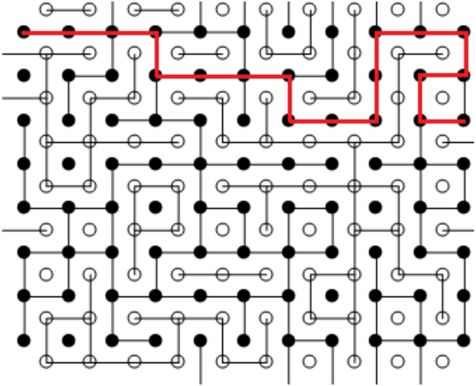

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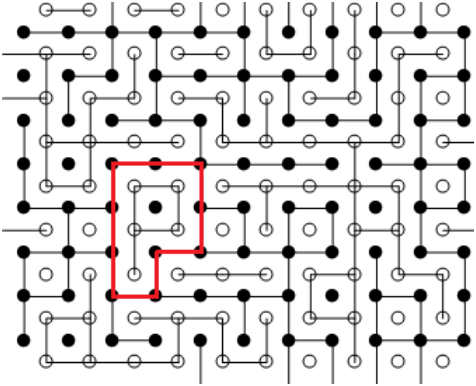

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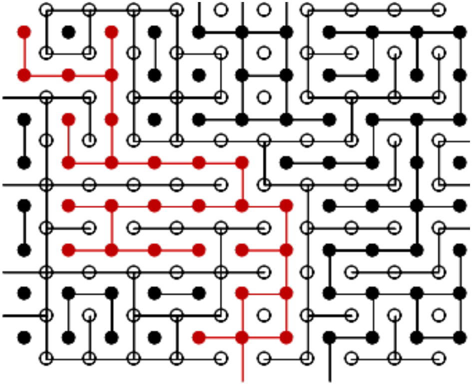

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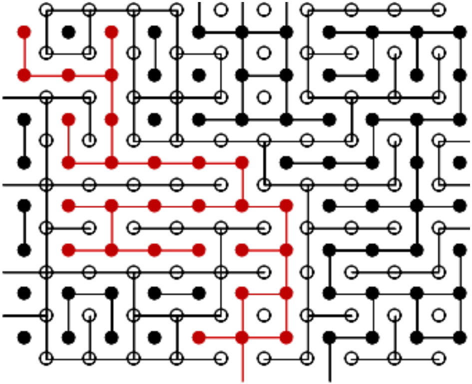
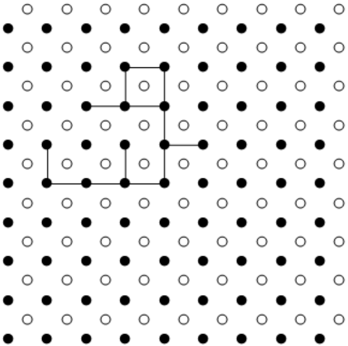


Boundary of a Finite Open Cluster

Theorem:

We can draw an open boundary in the dual lattice around every finite open cluster.



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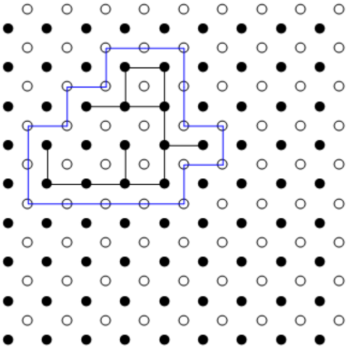

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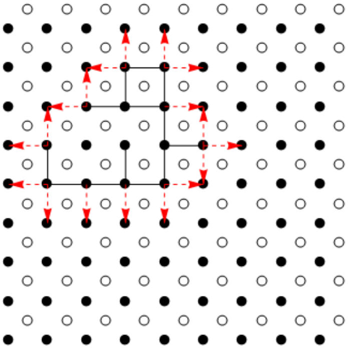

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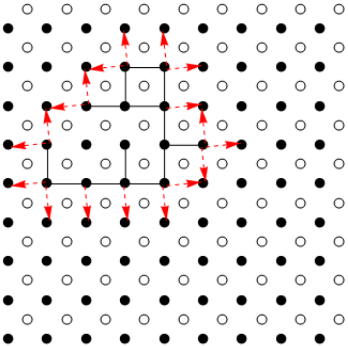

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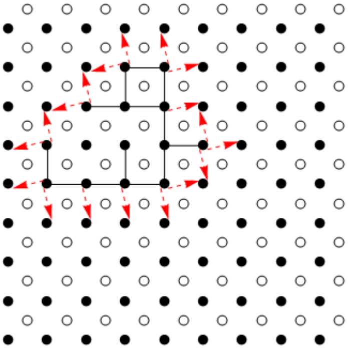

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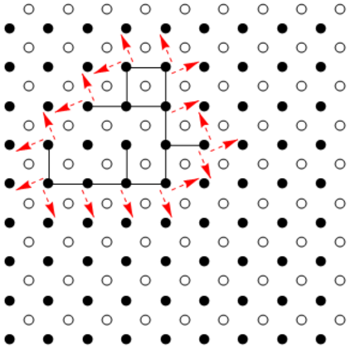

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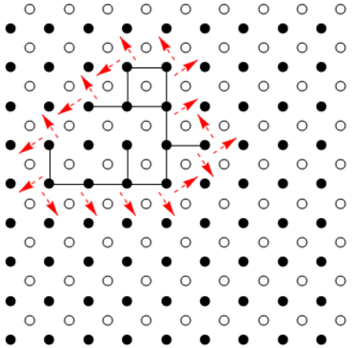

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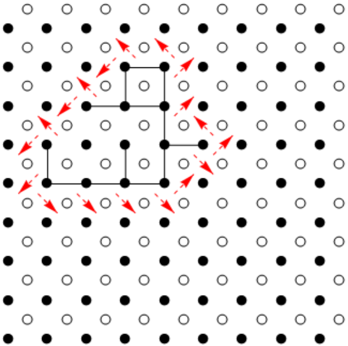

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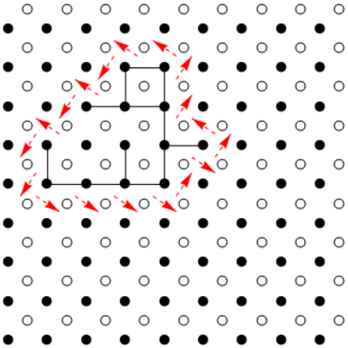

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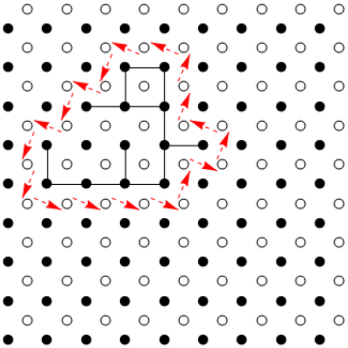

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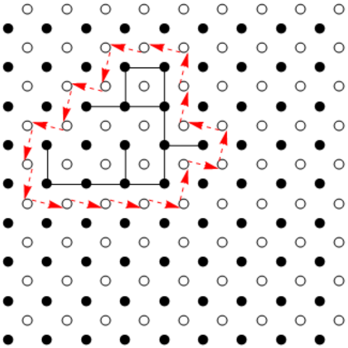

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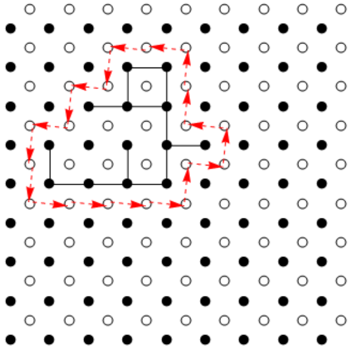

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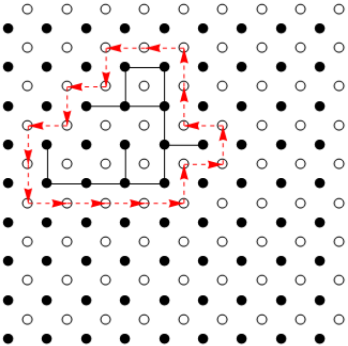


Bounding the Critical Probability: p_H

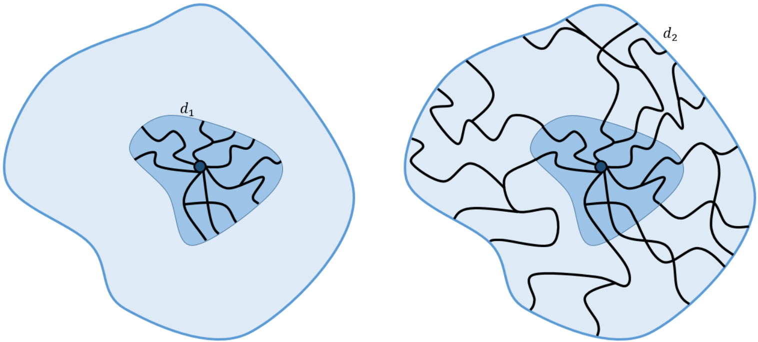
Let p_H be the value for which if $p > p_H$, the probability the origin is part of an infinite cluster, $\theta(p)$, is greater than zero and if $p < p_H$, then $\theta(p) = 0$.

We will show: $1/3 \leq p_H \leq 2/3$.

Proof of Lower Bound: $1/3 \leq p_H$

Two battling forces: As we increase d,

- The number of paths, μ_d, increases
- The probability that a path is open decreases.



Who Wins?

Let $N(d)$ be the number of open paths of length d. As d increases, for $p < 1/3$:

$$\begin{aligned}
\theta(p) &\leq \mathbb{P}(N(d) \geq 1) && \text{Infinite cluster} \Rightarrow \text{path of length } d \\
&\leq \mathbb{E}(N(d)) && \sum p(x) \leq \sum p(x)x \text{ for } x \geq 1 \\
&= \mu_d p^d && \text{Expectation is linear} \\
&\leq 4(3^{d-1})p^d && \mu_d \leq 4(3^{d-1}) \\
&= \frac{4}{3}(3p)^d \to 0
\end{aligned}$$

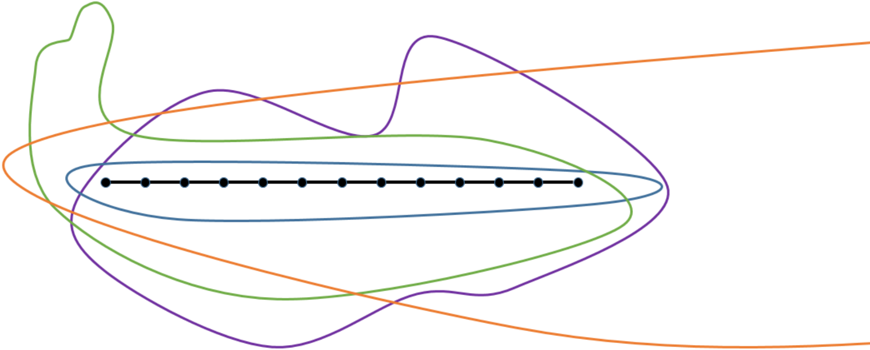
Probability a path isn't open beats the number of paths. Hence $1/3 \leq p_H$.

Proof of Upper Bound: $p_H \leq 2/3$

If $p > 2/3$, it's easy to have open paths in $\mathbb{Z}^2$ and hard to paths in the dual.
Assume we have a open line of length m.

Proof of Upper Bound: $p_H \leq 2/3$

Count the expected number of dual cycles around our line, $\mathbb{E}_p(Y_m)$.
Counting dual cycles is hard! Counting paths is easier!



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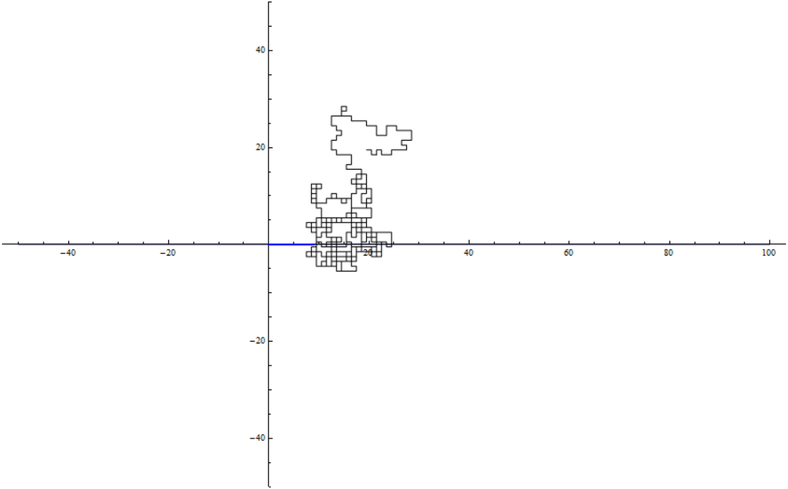
Some of the things we will end up including:



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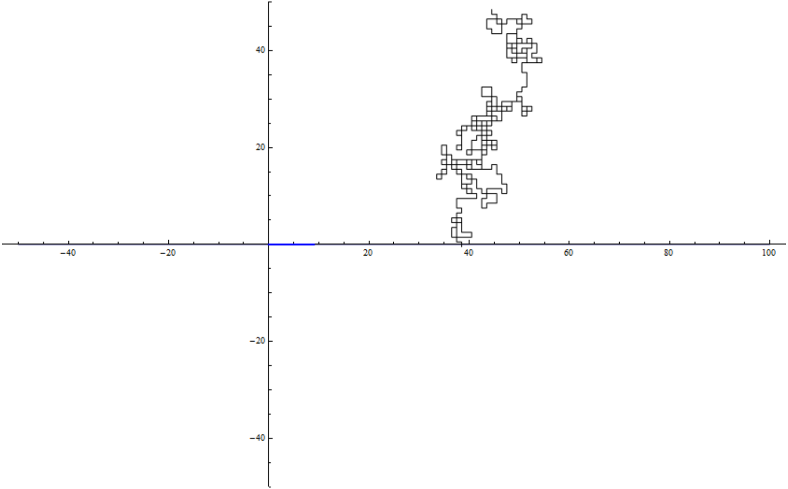

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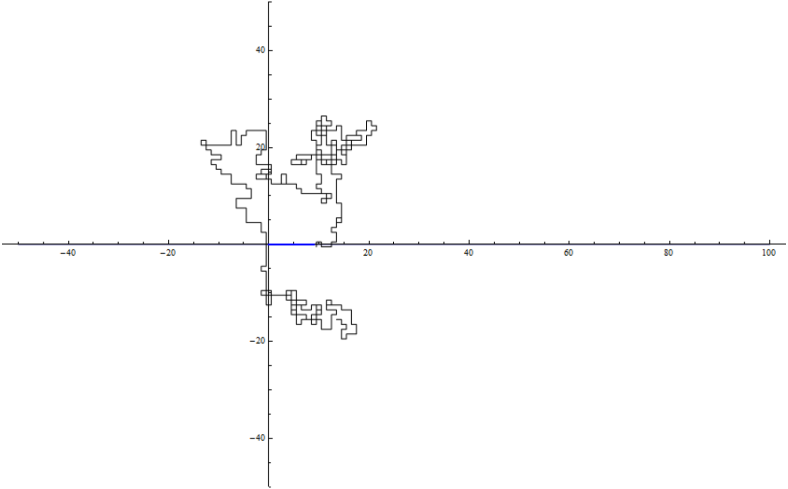

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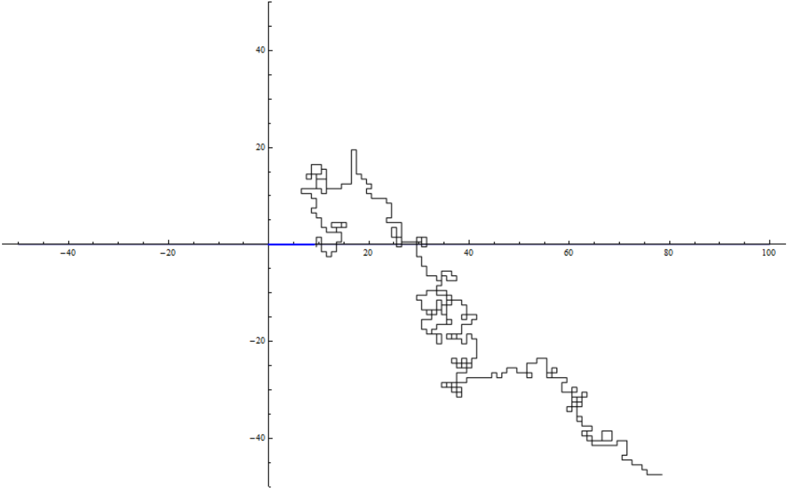
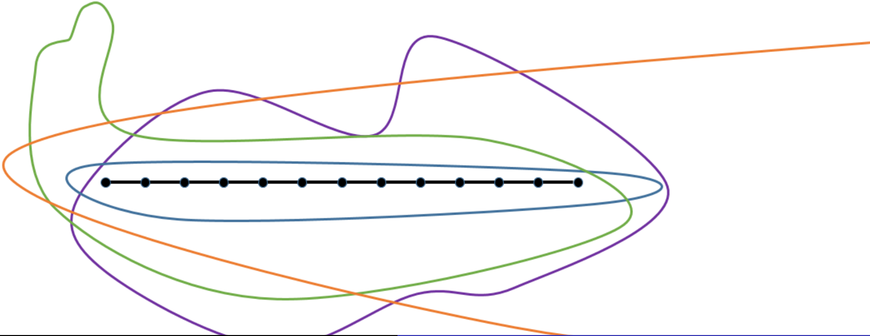


Expected Number of Open Dual Cycles

Every length, $2d$, of a cycle has at most d choices for a starting location. For one length d:
μ_d x(# of starting locations)x(probability path is open)

$$\mathbb{E}_p(Y_m) \leq \sum_{d=m}^{\infty} \mu_{2d} d(1-p)^{2d} \leq \frac{4}{3} \sum_{d=m}^{\infty} d[3(1-p)]^{2d} < \infty$$

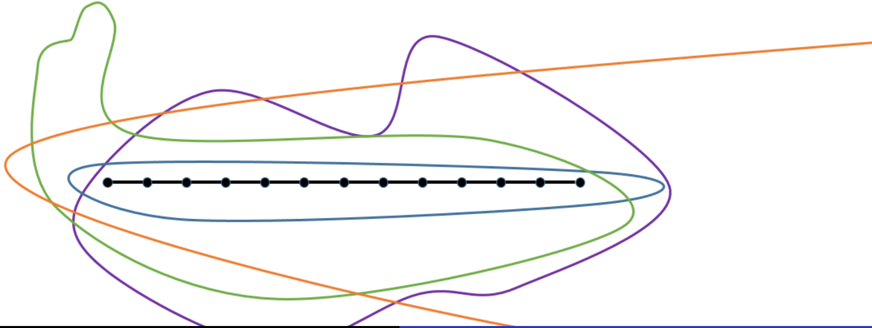


Proof of Upper Bound: $p_H \leq 2/3$

Crank m up!

$$\mathbb{E}_p(Y_m) \leq \frac{4}{3} \sum_{d=m}^{\infty} d[3(1-p)]^{2d} < 1$$

For long enough line, the expected number of dual cycles is less than one! $\mathbb{E}_p(Y) < 1 \Rightarrow \mathbb{P}(\text{No Dual Cycles!}) > 0$.



Positive Probability of No Dual Cycles

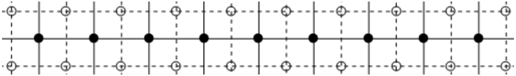
No dual cycles $\Rightarrow$ no boundary.
No boundary $\Rightarrow$ our open cluster is infinite!

Hold the phone, assuming the **bonds in that line are open.**

Proof of Upper Bound: $p_H \leq 2/3$

If our path is of length m, then our line will be open with probability $p^m > 0$.
Two Events:

- No dual cycles
- The line is open



These events are independent!
So we have some chance of having an infinite cluster. So $p_H \leq 2/3$.
Hence $1/3 \leq p_H \leq 2/3$.

Summary

Percolation

- Randomness Due to the Medium
- A Rapid Transitions in a Random System
- Really, Really Big Networks

Applications

- Chemistry and Material Science
- Epidemiology
- Impressing Your Barista

Any Questions?

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- Hammersley, J.M. *Origins of Percolation Theory.* Annals of the Israel Physical Society, Vol. 5. 1983