Voronoï Percolation in the Hyperbolic Plane

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Joint work with Tobias Müller

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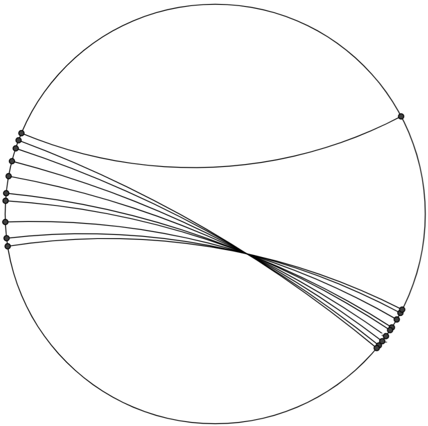
Mark Kac Seminar

Hyperbolic Plane $\mathbb{H}^2$

Hyperbolic Plane

- A surface in a high dimensional space
- Every point is a saddle point (Gaussian curvature is negative)
- Fails Euclid's fifth axiom



Poincaré Disk
Equip unit disk with the right metric

$$d_{\mathbb{H}^2}(x, y) = 2\,\mathrm{arcsinh}\left(\frac{||x - y||}{\sqrt{(1 - ||x||^2)(1 - ||y||^2)}}\right)$$

$$\mathrm{Area}_{\mathbb{H}^2}(B) = \int_B \frac{4}{(1 - x^2 - y^2)^2}\,dydx$$

Poisson Point Process on $\mathbb{H}^2$

A Poisson point process on $\mathbb{H}^2$ is a random countable collection of points $\mathcal{Z}$ in $\mathbb{H}^2$.

(Homogenous) Poisson point process: $\lambda > 0$

- Number of points in region B is Poisson with expectation $\lambda \cdot \text{Area}(B)$.
- Disjoint regions: the number of points in each region are independent.

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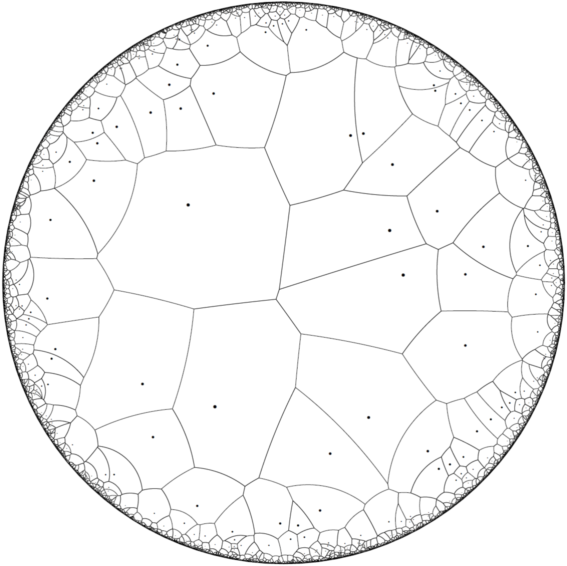
PPP (intensity λ) on the Poincaré disk is inhomogeneous PPP on unit disk

- Expected number of points in B:

$$\int_B \frac{4\lambda}{(1 - x^2 - y^2)^2} dxdy$$

Voronoï Cells in $\mathbb{H}^2$

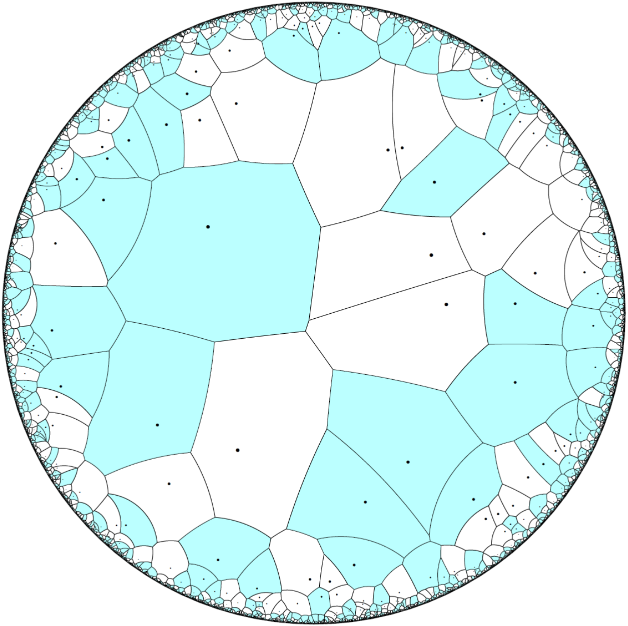
The hyperbolic metric assigns every $x \in \mathbb{H}^2$ to the "closest" element of $\mathcal{Z}$ to form the hyperbolic cells.



Percolation on $\mathbb{H}^2$

Colour cells black independently with probability p.

- In the picture, $p = 1/2$ and $\lambda = 1$.



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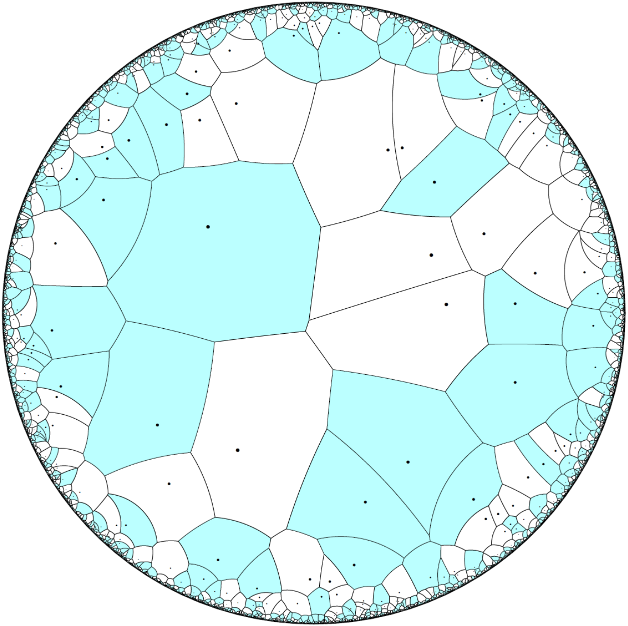

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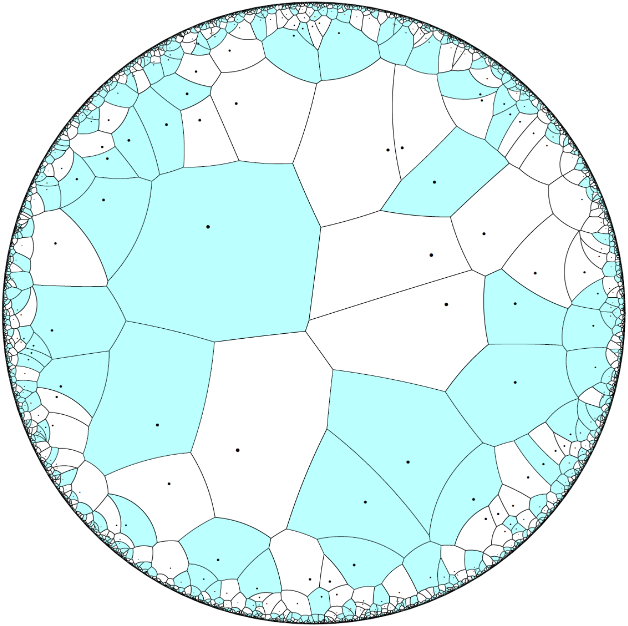

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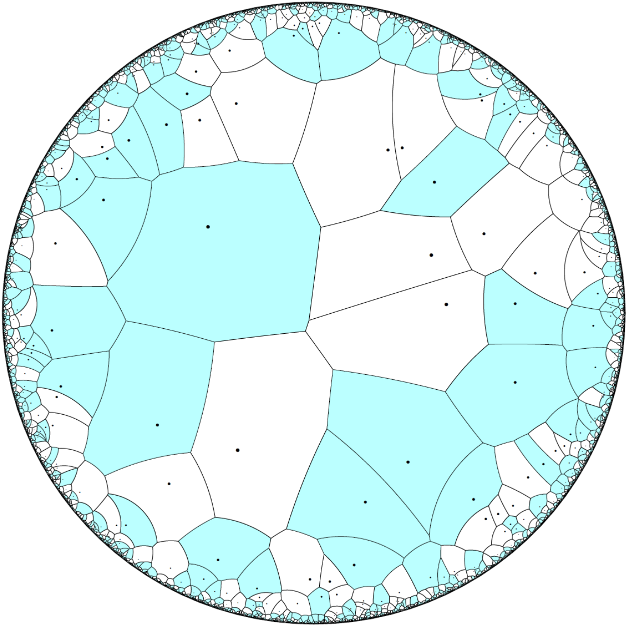

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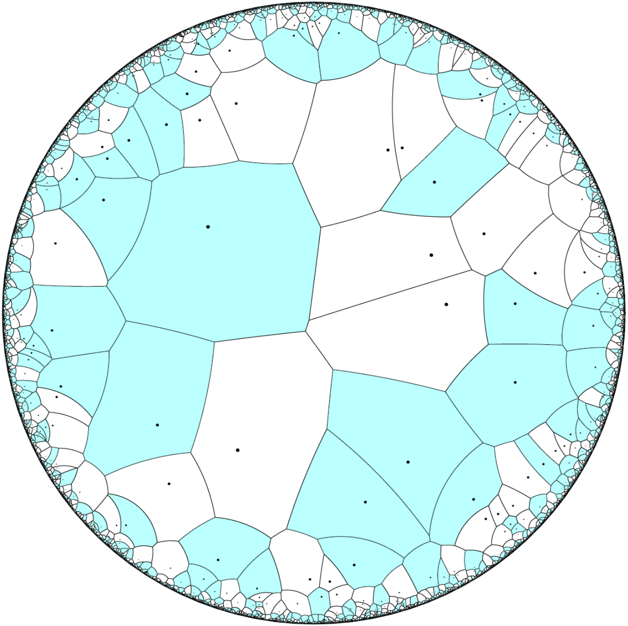
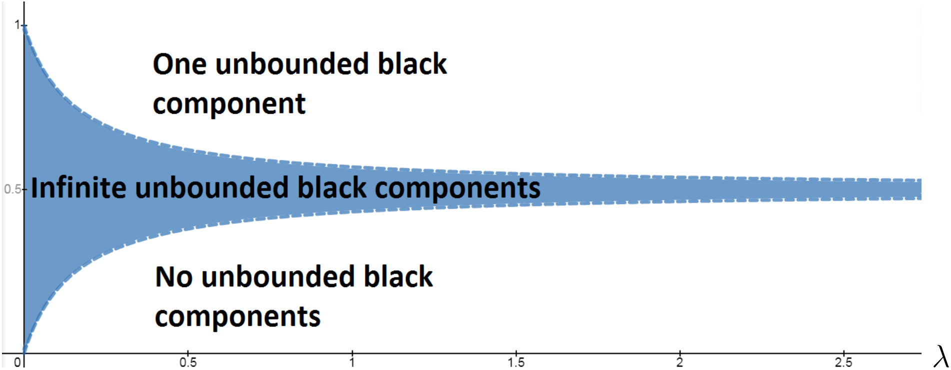
- Two sources of randomness
- Both p_c and p_u depend on λ



Phase Diagram (Benjamini + Schramm 2001)

$0 < p_c(\lambda) \leq \frac{1}{2} - \frac{1}{4\pi\lambda+2}$ and $p_u(\lambda) = 1 - p_c(\lambda)$,



Also p_c is continuous, a.s. no unbounded black component for $(\lambda, p_c(\lambda))$.

Conjecture and Theorem

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Theorem (H + Müller 2020+)

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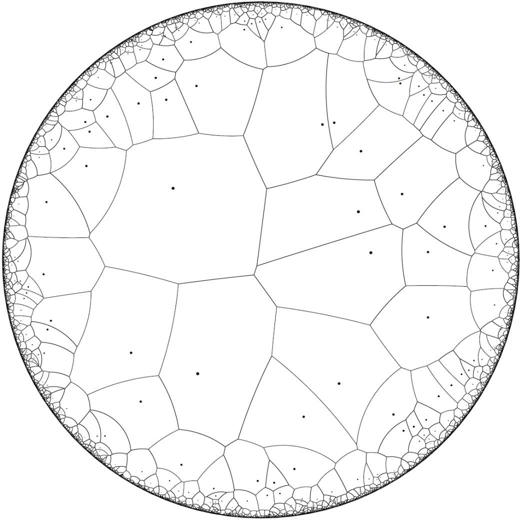
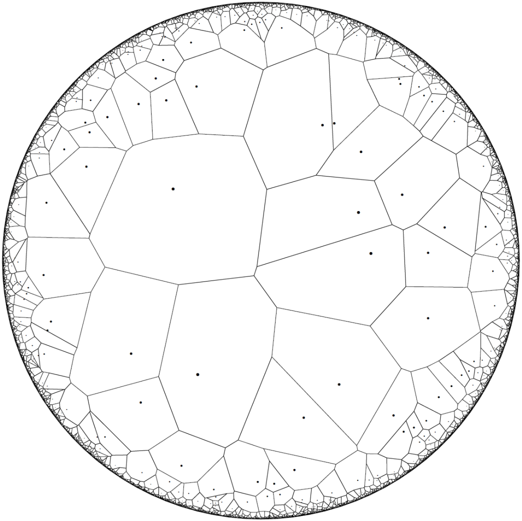
Adjacency Graphs are Isomorphic

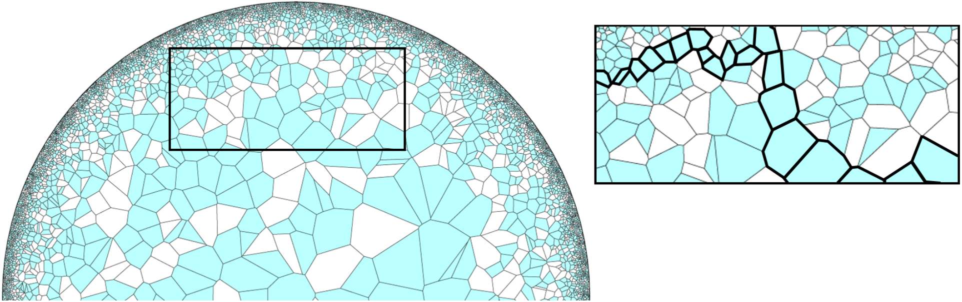
Hyperbolic Voronoï cells are adjacent iff Euclidean Voronoï cells are adjacent, a.s.





Crossing Events

A rectangle R has a long, black crossing if there is a curve $\gamma \subseteq R$ using Euclidean Voronoï cells from one short side of R to the opposite side such that all points of γ are black. Denote this event as $\text{cross}(R)$.

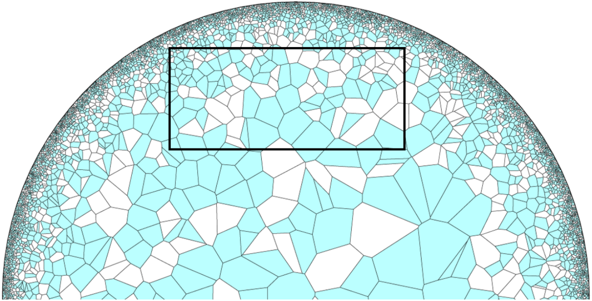


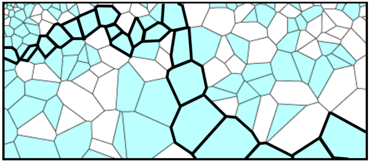
Main Lemma

Lemma (H + Müller 2020+)

For $p > 1/2$ and each rectangle $R \subseteq D$,

$$\lim_{\lambda \to \infty} \mathbb{P}_{p,\lambda}(\mathrm{cross}(R)) = 1.$$





Main Idea for the Lemma

Use a consequence of Bollobás + Riordan [2006]:

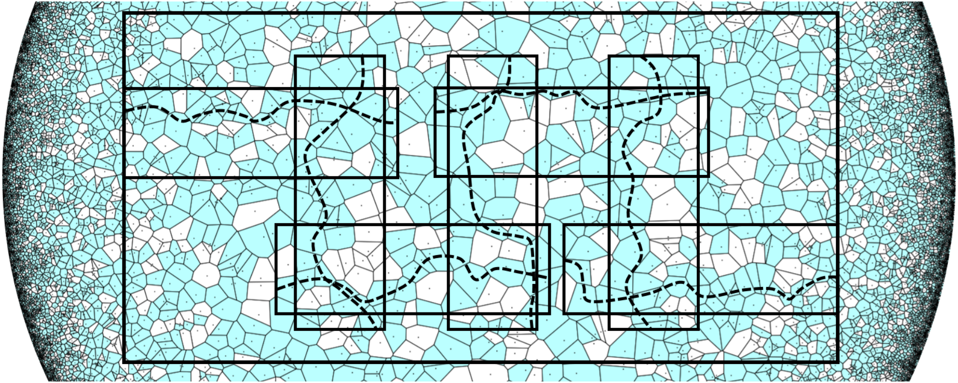
Lemma

For $p > 1/2$, for Euclidean Voronoï percolation and any rectangle R,

$$\lim_{\lambda \to \infty} \mathbb{P}(\mathrm{cross}(R)) = 1$$

For small rectangles, $\mathcal{Z}$ and a homogeneous Poisson point process can be coupled so that cross using $\mathcal{Z}$ is easier.

For Large Rectangles

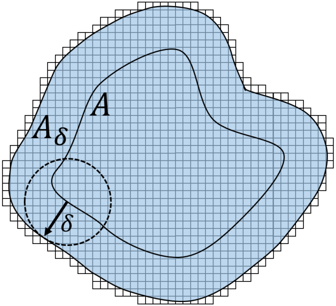


Colouring is Local and Cells are Small

Let $A \subseteq D$ and $\delta > 0$.

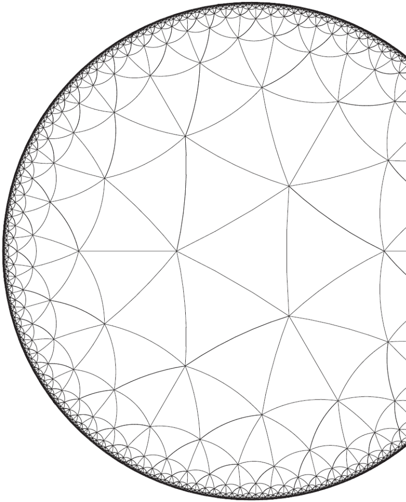
Let nearby(A, δ) be the event that the colouring of A is determined by $\mathcal{Z}$ in A_δ under both metrics and all cells that intersect A have Euclidean diameter at most δ.

$$\lim_{\lambda \to \infty} \mathbb{P}_{p,\lambda}(\text{nearby}(A, \delta)) = 1.$$



Dependent Percolation on the 7 Regular Triangulation

Start with a triangle centred at the origin. Use the following mechanism to colour triangles. Let $p < 1/2$,



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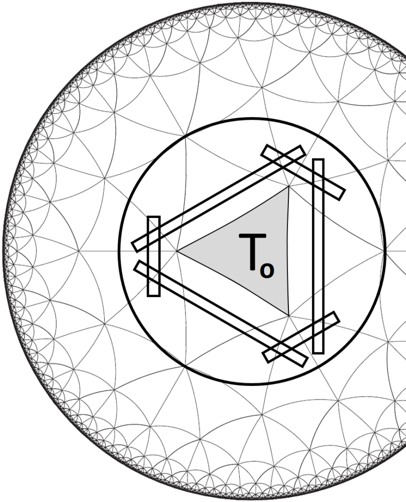

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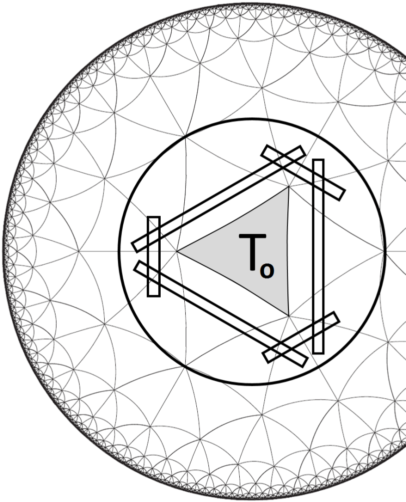

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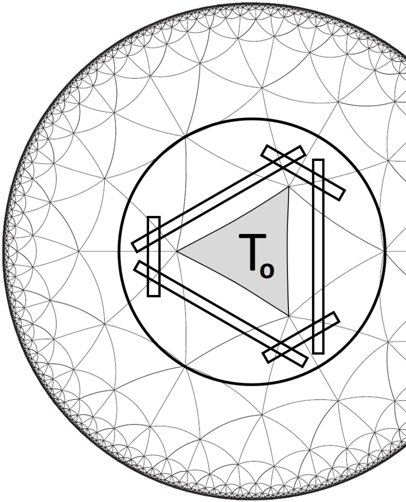

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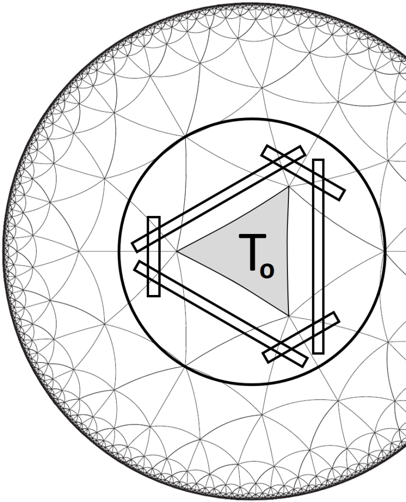

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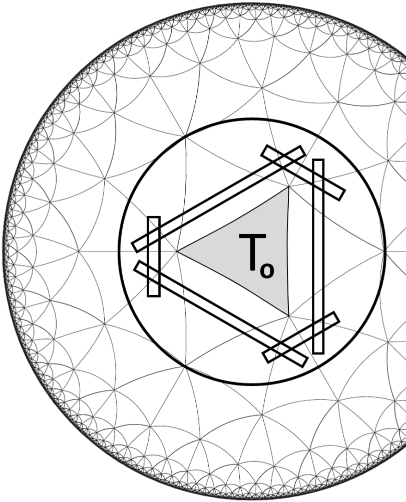

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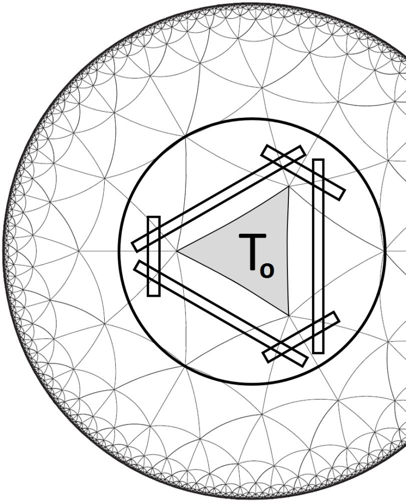
- closed(T) can be defined for all faces. Colour T white if closed(T) holds. Otherwise, black.



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- $p < 1/2$ and λ large $\Rightarrow$ The black clusters of triangles must be finite a.s.

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Thank you! Questions?