3 Lecture 3: Linear Algebra I

Most, if not all, algebra learned throughout K-12 educations deals with *scalar* algebra. Each variable only represents a single number. But, a variable can represent more than one element. Instead:

- *Scalar*: one element, x
- *Vector*: n elements
  - $\boldsymbol{x}$ includes the set of scalars $x_1, x_2, ..., x_i, ..., x_n$
- *Matrix*: $n \times m$ elements
  - $\boldsymbol{X}$ include the set of scalars $x_{11}, ..., x_{1m}, x_{21}, ..., x_{2m}, ..., x_{n1}, ..., x_{nm}$

There are considerable gains to be made through this notation. Beyond a more efficient notation, we're able to easily carry out all sorts of useful manipulations. We'll work up to these benefits in this lecture through various concepts.

3.1 Systems of equations

- *System of equations*: two or more equations with the same variables
  - To find a unique solution, we need as many equations as variables. E.g.,

$$\begin{aligned} 6x_1 - 3x_2 + 4x_3 &= -13 \\ 6x_1 &= -13 + 3x_2 - 4x_3 \\ x_1 &= -\frac{13}{6} + \frac{1}{2}x_2 - \frac{2}{3}x_3 \end{aligned}$$

  - We can use this single equation and solution to create a series of solutions, i.e. if $x_2 = 2$ and $x_3 = 3$, then $x_1 = -\frac{13}{6}$ and so on.

- Consider the following system of (linear) equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + ... + a_{mn}x_n &= b_m \end{aligned}$$

  Drawing on yesterday's lecture, we can also write this all as: $\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_j = b_i$ Or, bringing matrix notation in, as: $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{A}$ is an $n \times m$ matrix.

3.1.1 Solving for a system of equation

Consider the following system of equations. Throughout r references row number. By setting all variables but one per equation, we are solving by *elimination*. We'll later introduce Gauss-Jordan elimination, which is the same process, but with an augmented matrix.

- First, the equations:

$$1x_1 + 1x_2 = 100,000$$
$$0.05x_1 + 0.09x_2 = 7,800$$

- Add $-0.05r1$ to $r2$

$$1x_1 + 1x_2 = 100,000$$
$$0x_1 + 0.04x_2 = 2,800$$

- $25r2$

$$1x_1 + 1x_2 = 100,000$$
$$0x_1 + 1x_2 = 70,000$$

- Subtract r2 from r1

$$1x_1 + 0x_2 = 30,000$$
$$0x_1 + 1x_2 = 70,000$$

- This gives us: $x_1 = 30,000$ and $x_2 = 70,000$

Now, let's try a lengthier example. Again, we're not substituting equations into each variable. We're eliminating all variables but one from each equation.

- First, the equations:

$$1x_1 + 2x_2 + 3x_3 = 6$$
$$2x_1 - 3x_2 + 2x_3 = 14$$
$$3x_1 + 1x_2 - 1x_3 = -2$$

- Subtract $-2r1$ from $r2$ and subtract $-3r1$ from $r3$

$$\begin{aligned} 1x_1 + 2x_2 + 3x_3 &= 6 \\ 0x_1 - 7x_2 - 4x_3 &= 2 \\ 0x_1 - 5x_2 - 10x_3 &= -20 \end{aligned}$$

- Multiply $r3$ by $-\frac{1}{5}$ and flip $r2$ and $r3$

$$\begin{aligned} 1x_1 + 2x_2 + 3x_3 &= 6 \\ 0x_1 + 1x_2 + 2x_3 &= 4 \\ 0x_1 - 7x_2 - 4x_3 &= 2 \end{aligned}$$

- Subtract $2r2$ from $r1$ and add $7r2$ to $r3$

$$\begin{aligned} 1x_1 + 0x_2 - 1x_3 &= -2 \\ 0x_1 + 1x_2 + 2x_3 &= 4 \\ 0x_1 + 0x_2 + 10x_3 &= 30 \end{aligned}$$

- Divide $r3$ by 10

$$\begin{aligned} 1x_1 + 0x_2 - 1x_3 &= -2 \\ 0x_1 + 1x_2 + 2x_3 &= 4 \\ 0x_1 + 0x_2 + 1x_3 &= 3 \end{aligned}$$

- Add $r3$ to $r1$ and subtract $2r3$ from $r2$

$$\begin{aligned} 1x_1 + 0x_2 - 0x_3 &= 1 \\ 0x_1 + 1x_2 + 0x_3 &= -2 \\ 0x_1 + 0x_2 + 1x_3 &= 3 \end{aligned}$$

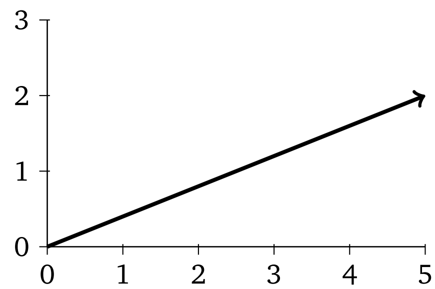
- $x_1 = 1, x_2 = -2, x_3 = 3$

Writing all of the $x's$ gets tedious though. So let's introduce vectors.

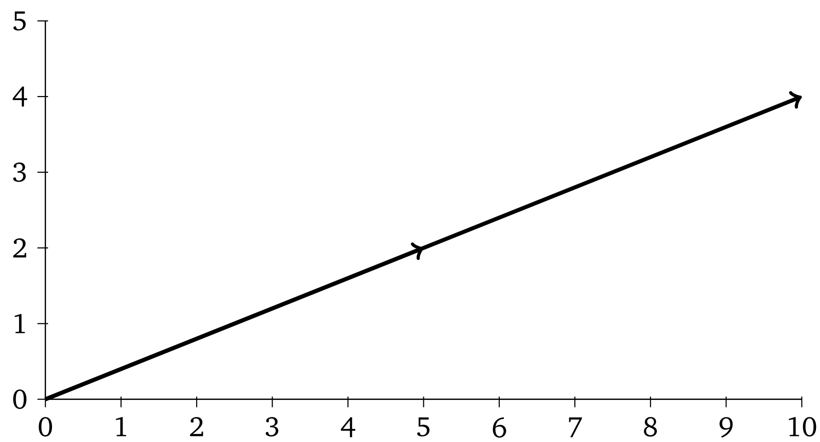
3.2 Vectors

- Examples:
  - $\boldsymbol{x} = [x_1, x_2, x_3, ..., x_n]$

- $\boldsymbol{x} = (5, 2)$
  - If, like in this case, there are two dimensions (number of components), then we can visually understand the vector as:



- If we multiply by a scalar: $a = 2$, $aX = [10, 4]$



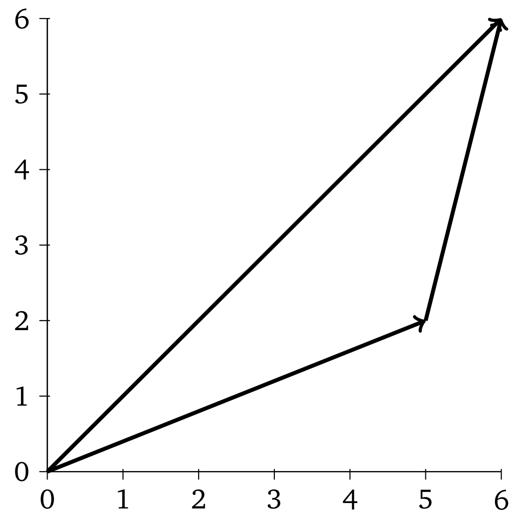
3.2.1 Vector length

- Also called the *norm*, length is not the same as *dimensions*. The formula is:

$$||x|| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$$

- Considering the visuals above, in two dimensions we are just using the Pythagorean theorem. But the formula is extendable to k-dimensions. For $\boldsymbol{x} = (5, 2)$, $||\boldsymbol{x}|| = \sqrt{25 + 4} = \sqrt{29}$.

- Another example
  - $x = (5, 2), y = (1, 4)$
    - $x + y = (6, 6)$
    - $||x + y|| = \sqrt{36 + 36} = \sqrt{72}$



3.2.2 Vector multiplication

- If c is a *scalar* and we multiply $a(x_1, x_2, ..., x_n)$, then we get $(ax_1, ax_2, ..., ax_n)$. Dividing by a scalar works the same way.
- But what about multiplying one vector by another vector? We use the **dot product**: $\boldsymbol{a} \cdot \boldsymbol{b}$. Another name for this operation is the **inner product**.[2]
  - If $\boldsymbol{a}$ and $\boldsymbol{b}$ are both n-dimensional, then $\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + ... + a_n b_n = \sum_{i=1}^{n} a_i b_i$

$$\text{Ex: } \begin{bmatrix} 5 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 \end{bmatrix} = (5 \cdot 1) + (2 \cdot 4) = 5 + 8 = 13$$

  - Note: the result of the dot product of vectors is a *scalar*.
- The **outer product** of two vectors instead produces a matrix:

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cdot \begin{bmatrix} b_1 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{bmatrix}$$

  - The dimensions of this matrix are the two outer dimensions of the vectors multiplied together:

$$\underset{3\times 1}{\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}} \underset{1\times 3}{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}} = \underset{3\times 3}{\begin{bmatrix} 3 & 6 & 9 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}}$$

[2]For a nice review of vector manipulation, see https://people.cs.clemson.edu/~dhouse/courses/401/notes/vectors.pdf

- ○ *But* the inner dimensions must match up. See 1 and 1 above. If the first matrix's number of columns is not equal to the second matrix's number of rows, then cannot multiply.

3.3 Matrices

- A **matrix** is a rectangular table of numbers or variables arranged in a specific order in rows and columns. We express dimensions by rows, n, and columns, m. The dimensions of a matrix $A_{n\times m}$ are pronounced 'n by m'.

$$\boldsymbol{X} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \in \mathbb{R}^{nm} = \mathbb{R}^{3\times 3}$$

- If $m = n$, then the matrix is square.
- **Types of matrices**:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{zero matrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \text{diagonal matrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{identity matrix}$$

3.4 Matrix operators

- **Addition**
  - ○ Must have the same number of elements

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \text{can't do}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & 4 \\ 6 & 8 \end{bmatrix}$$

- **Transposition**
  - ○ Rotate so that the first column becomes the first row:

$$\boldsymbol{X} = \underset{3\times 2}{\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}}, \quad \boldsymbol{X}^T = \underset{2\times 3}{\begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \end{bmatrix}}$$

- **Multiplication**

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 7 & 0 & -1 \\ 1 & 3 & 1 \end{bmatrix}$$

  - Because $\boldsymbol{A}$ is 2×2 and $\boldsymbol{B}$ is 2×3, we can multiply. But $\boldsymbol{A}\boldsymbol{B}^T$ is undefined because $\boldsymbol{B}$ is 3×2.

$$\begin{aligned}
\boldsymbol{AB} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 7 & 0 & -1 \\ 1 & 3 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{A}_{11}\boldsymbol{B}_{11} + \boldsymbol{A}_{12}\boldsymbol{B}_{21} & \boldsymbol{A}_{11}\boldsymbol{B}_{12} + \boldsymbol{A}_{12}\boldsymbol{B}_{22} & \boldsymbol{A}_{11}\boldsymbol{B}_{13} + \boldsymbol{A}_{12}\boldsymbol{B}_{23} \\ \boldsymbol{A}_{21}\boldsymbol{B}_{11} + \boldsymbol{A}_{222}\boldsymbol{B}_{21} & \boldsymbol{A}_{21}\boldsymbol{B}_{12} + \boldsymbol{A}_{22}\boldsymbol{B}_{22} & \boldsymbol{A}_{21}\boldsymbol{B}_{13} + \boldsymbol{A}_{22}\boldsymbol{B}_{23} \end{bmatrix} \\
&= \begin{bmatrix} (1 \cdot 7) + (2 \cdot 1) & (1 \cdot 0) + (2 \cdot 3) & (1 \cdot -1) + (2 \cdot 1) \\ (3 \cdot 7) + (4 \cdot 1) & (3 \cdot 0) + (4 \cdot 3) & (3 \cdot -1) + (4 \cdot 1) \end{bmatrix} = \begin{bmatrix} 9 & 6 & 1 \\ 25 & 12 & 1 \end{bmatrix}
\end{aligned}$$

  - Ex with identity matrix:

$$\begin{aligned}
\boldsymbol{I}_{2\times 2}\boldsymbol{X} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\
\text{b/c} &\begin{bmatrix} (1 \cdot 1) + (0 \cdot 3) & (0 \cdot 1) + (1 \cdot 2) \\ (0 \cdot 1) + (1 \cdot 3) & (0 \cdot 3) + (1 \cdot 4) \end{bmatrix}
\end{aligned}$$