6 Calculus I

Before we begin introducing mathematical formula, what is a derivative? A derivative calculates the rate of change of a function at any given point. In high school when we learned about the slope of a line – $y = mx + b$, where m is the slope – that slope is a derivative. In that context, the derivative is the same at all points because the line is straight. But we often encounter functions that are not a straight line and we care about calculating the slope across values of that function. We can use calculus for this.

6.1 Sequences and Limits

A sequence is an ordered list of numbers, e.g.

$$\{x_n\} = \{x_1, x_2, ..., x_n\}$$

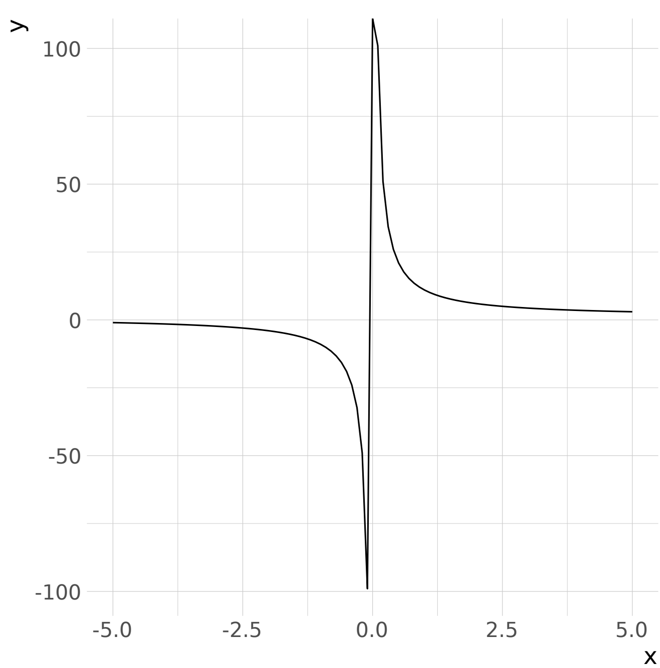
Where x_n is a real number that extends from x_1 to x_n. We usually encounter n extending to ∞. Another way to write the series is:

$$\{x_n\}_{n=1}^{\infty}$$

Central to calculus is the notion of a sequence "converging to a limit", generally where $n \to \infty$ or $n \to 0$. This is written as:

$$\lim_{n\to\infty} y_n = L$$

where L is the limit. Let's visualize for an arbitrary function, $f(x) = \frac{x+10}{x}$:



We can see that as $x \to \infty$ the value of $f(x)$ stabilizes. Indeed:

$$\lim_{x\to\infty} \frac{x+10}{x} = \lim_{x\to\infty} \left(1 + \frac{10}{x}\right) = \underbrace{\lim_{x\to\infty} 1}_{1} + \underbrace{\lim_{x\to\infty} \frac{10}{x}}_{0} = 1$$

This gives us a limit of 1.

6.2 Derivatives and the Difference Quotient

The derivative is the rate of change of $f(x)$ at any x. For a straight line, i.e $y = mx + b$, the derivative is constant at all points. But for a nonlinear function, i.e. $y = 2x^4$, the rate of change varies across x.

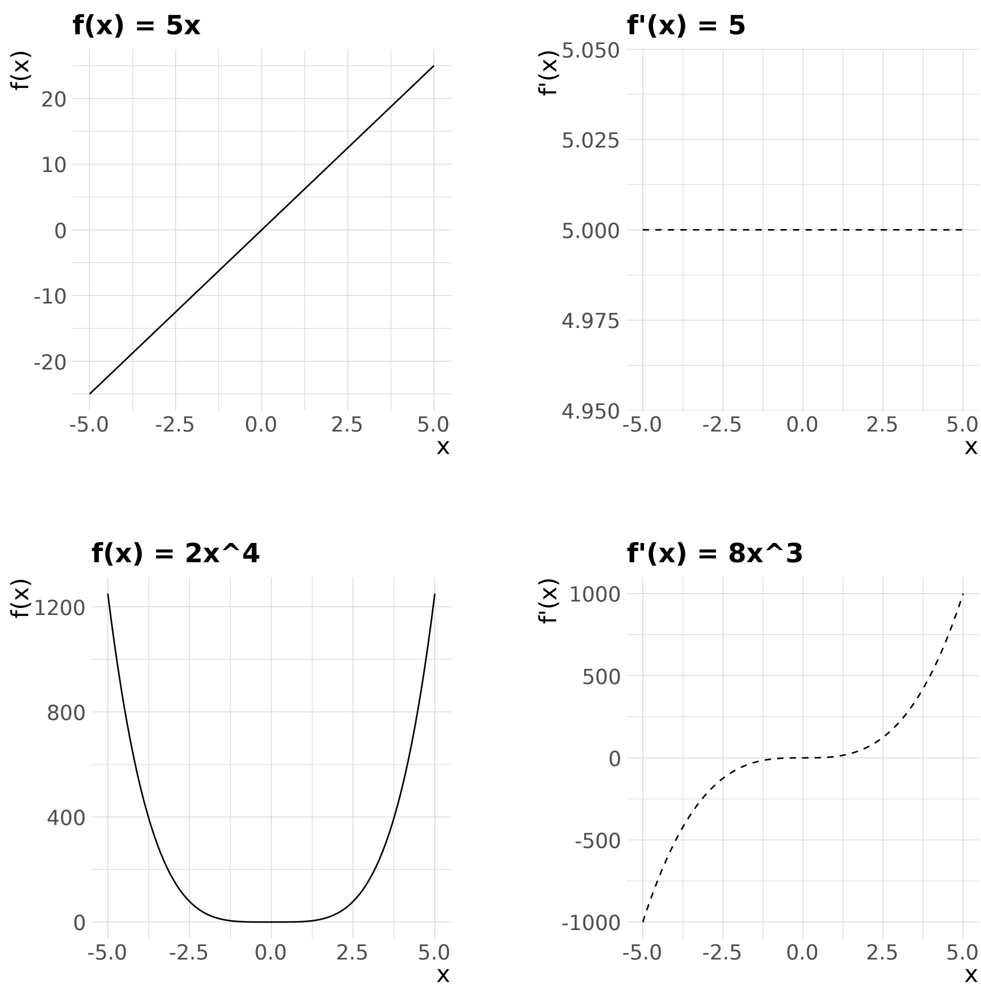


Figure 1: Functions and their derivatives

Now, let's introduce how we produce a derivative. Let f be a function with an open interval that contains x. Let h be the interval where $f(x)$ changes. Below we are simply calculating rise over run at each specified interval.

$$\frac{d}{dx}f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

The numerator represents the change in $f(x)$ as Δx approaches 0 and the denominator is the change in Δx – or the change in x. We generally see two notations for derivatives:

- Lagrange: $f'(x)$
- Leibniz: $\frac{d}{dx}y = \frac{dy}{dx}$
  - The change in y, given change in x.

Side-note, on the problem set, when we are asked about the difference quotient as a function of $x+\Delta x$, think about how a specified function would look if it were plugged in for $f(x)$ and $f(x+\Delta x)$.

6.3 Rules for Derivatives

- Power rule:
  - $y = f(x) = ax^n$, $f'(x) = nax^{n-1}$
- Constant multiplier rule:
  - $f(x) = ax$, $f'(x) = a$
- Constant rule:
  - $f(x) = a$, $f'(x) = 0$
- Summation rule:
  - $f(x) = g(x) + h(x)$, $f'(x) = g'(x)h'(x)$
- Product rule:
  - $f(x) = g(x)h(x)$
  - $f'(x) = g'(x)h(x) + g(x)h'(x)$
- Quotient rule:
  - $f(x) = g(x)/h(x)$
  - $f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{[h(x)]^2}$
- Chain rule:
  - $f(x) = h[g(x)]$
  - $f'(x) = h'[g(x)] \cdot g'(x)$
- Exponent rule:
  - $f(x) = a^x$

  - $f'(x) = a^x(ln(a))$
  - $f(x) = e^x$, $f'(x) = e^x$
- Logarithm rule:
  - $f(x) = \log_a x$
  - $f'(x) = \frac{1}{x \ln a}$
  - Natural log: $f(x) = ln(x)$, $f'(x) = \frac{1}{x}$

6.4 Derivative Examples

1. $f(x) = 40x^{400}$
   - $f'(x) = 400 \cdot 40x^{399} = 16000x^{399}$
2. $f(x) = 16x$
   - $f'(x) = 16$
3. $f(x) = 1000$
   - $f'(x) = 0$
4. $f(x) = 3x^{100} + 5x^2$
   - $f'(x) = 300x^{99} + 10x$
5. $f(x) = (3x+5)(9x+2)$
   - $f'(x) = 3(9x+2) + 9(3x+5)$
6. $f(x) = \frac{3x+5}{9x+2}$
   - $f'(x) = \frac{3(9x+2) - 9(3x+5)}{(9x+2)^2}$
7. $f(x) = 20(x+3)^{10}$
   - $f'(x) = 200(x+3)^9 \cdot 1$
8. $f(x) = 20(x^3+3x)^{10}$
   - $f'(x) = 200(x^3+3x)^9 \cdot (3x^2+3)$
9. $f(x) = 20[(x+3)^3 + 4x]^{10}$
   - $f'(x) = 200[(x+3)^3 + 4x]^9 \cdot g'(x)$
   - $g'(x) = 3(x+3)^2 \cdot 1 + 4$
   - $f'(x) = 200[(x+3)^3 + 4x]^9 \cdot 3(x+3)^2 + 4$

10. $f(x) = e^{\sqrt{x}}$, $f'(x) = e^{x^{\frac{1}{2}}} \cdot (\frac{1}{2}x^{-\frac{1}{2}})$

11. $f(x) = a^{\sqrt{x}}$, $f'(x) = a^{\sqrt{x}} ln(a) \cdot (\frac{1}{2}x^{-\frac{1}{2}})$

    - Keep in mind for the problem set.

12. $f(x) = ln(3x^2 + 3x + 3)$, $f'(x) = \frac{1}{3x^2+3x+3} \cdot 6x + 3 = \frac{6x+3}{3x^2+3x+3}$