7 Calculus II

7.1 The Chain Rule

Let's revisit the chain rule in more detail. Another way to write it out is:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

So we first specify one of the functions as u and then calculate $\frac{dy}{du}$. Then we calculate $\frac{du}{dx}$, which is the change in the function u, given a change in x. Last, we write out the product of the two derivatives. This is an incredibly common and powerful trick. One more example:

$$\begin{aligned} y &= \left[\frac{(x^2+1)(3x+9)}{(8x^2-9)}\right]^3 \\ u &= \frac{(x^2+1)(3x+9)}{(8x^2-9)} \\ \frac{dx}{dy} &= 3[u]^2 \cdot u' \end{aligned}$$

Where we solve for u' with the product and quotient rules.

7.2 Implicit Differentiation

Sometimes we cannot fully x and y when finding $\frac{dy}{dx}$. This means we cannot solve for y solely in terms of x. We use implicit differentiation for these situations. You'll notice that our answer to the following example includes both x and y. The trick is to write out all parts of our function in terms of $\frac{dy}{dx}$. Then we solve for $\frac{dy}{dx}$.

$$\begin{aligned} f(x,y) = y^5 + xy + x^2 \text{ at } f(x,y) = 3 \\ \frac{d}{dx} f(x,y) &= \frac{d}{dx}(3) \\ \frac{d}{dx}(y^5 + xy + x^2) &= \frac{d}{dx}(3) \\ \frac{d}{dy}y^5 + \frac{d}{dx}xy + \frac{d}{dx}x^2 &= 0 \\ 5y^4 + \frac{dx}{dx}y + x\frac{dy}{dx} + 2x &= 0 \\ 5y^4 + y + \frac{dy}{dx}x + 2x &= 0 \\ \frac{dy}{dx} &= \frac{-5y^4 - y - 2x}{x} \end{aligned}$$

Now, we have a quantity for $\frac{dy}{dx}$, it just depends upon the values of $x\ and\ y$.

7.3 Partial Derivatives

What if we have a multivariate function (multiple variables are changing) and are only concerned with the change in y, given the change in *one* variable? The trick is to treat other variables as a constant, because we are only concerned with variation in one variable. So:

$$\begin{aligned} f(x,y) &= 5x^2y^3 \\ \frac{df}{dx} &= 10xy^3 \\ \frac{df}{dy} &= 15x^2y^2 \end{aligned}$$

Some additional notation: $\frac{df}{dx} = f_x$ or $\frac{df}{dy} = f_y$
Examples:

$$\begin{aligned} f(x,y) &= (x+4)(3x+2y) \\ f_x &= 1\cdot(3x+2y)+(x+4)(3) \\ f_y &= 0(3x+2y)+(x+4)2 \\ &= 2(x+4) \end{aligned}$$

But:

$$\begin{aligned} f(x,y,z) &= (x+4)(3x+2y)(3z) \\ f_z &= 3(x+4)(3x+2y) \end{aligned}$$

This last example is simpler because the other parts of the function don't include z, so we treat them like constants and then derive $3z$.

7.4 Gradient

A vector with all of a function's patial derivatives is the **gradient**. Take a function $f(x)$, with n variables. The gradient – or $\nabla f(x)$ – is:

$$\nabla f(x) = \left[\frac{df(x)}{dx_1}, \frac{df(x)}{dx_2}, \ldots, \frac{df(x)}{dx_n}\right]$$

For a function $f(x,y) = \left[\frac{df}{dx}, \frac{df}{dy}\right] = [f_x, f_y]$

7.5 Second Derivatives and the Hessian

We can take higher-order derivatives, which are just the rate of change of the previous derivative. I.e. $f(x) = x^3$, $f'(x) = 3x^2$, $f''(x) = 6x$, and so on...
In your statistics courses you will encounter the **Hessian**, which is a matrix of all combinations of second derivatives:

$$\begin{bmatrix} \frac{d^2f}{dx_1dx_1} & \cdots & \frac{d^2f}{dx_ndx_1} \\ \frac{d^2f}{dx_2dx_1} & \frac{d^2f}{dx_2dx_2} & \vdots \\ \vdots & & \ddots \\ \frac{d^2f}{dx_ndx_1} & \cdots & \frac{d^2f}{dx_ndx_n} \end{bmatrix}$$

This gives us the change of the curvature of a function in all directions.

7.6 More on limits

We can also take 'one-sided' limits, which come from different directions. Consider $f(x) = \frac{1}{x}$:

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty$$
$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$

We also sometimes need to simplify a function to find the limit. In this example, plugging 1 in gives us 0/0. But simplifying gives us 2.

$$\lim_{x\to 1} \frac{1-x^2}{1-x} = \frac{(1-x)(1+x)}{1-x} = 1 + x = 2$$

7.7 L'Hôpital's Rule

One more trick for limits: If $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $= \pm\infty$, *and* $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists, then:

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

This is useful for undefined limits. Ex:

$$
\begin{aligned}
\lim_{x\to 1} \frac{x^2-1}{x^2+3x-4} = \frac{1^2-1}{1^2+3(1)-4} = \frac{0}{0} \\
&= \lim_{x\to 1} \frac{2x}{2x+3} \\
&= \frac{2(1)}{2(1)+3} \\
&= \frac{2}{5}
\end{aligned}
$$