1 Lecture 1: Introduction, Notation, Definitions, and Basic Mathematics

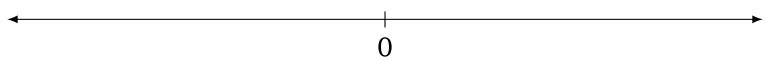
1.1 About this Document

Although not all of the following material is drawn from Moore and Siegel (2013), the overlap is sizeable and intentional. Many thanks to the authors. Some of the examples are also drawn from Jason Morgan's lectures (2015) with additions by Drew Rosenberg and Daniel Kent. This document was originally compiled by Daniel Kent.

1.2 Variables and Constants

- **Variable**: A concept or a measure that takes different values in a given set.
  - E.g., GDP, polity score, party identification, etc.
- **Constant**: A concept or a measure that has a single value for a given set.

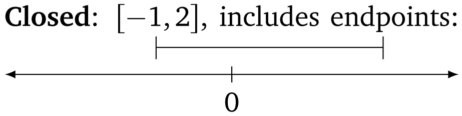
1.3 Number Types and Notation



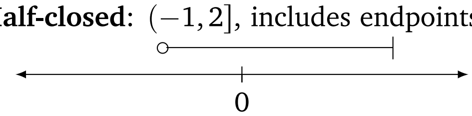
- **Real numbers**: ($\mathbb{R}$), can be placed anywhere on the line
- **Natural numbers**: ($\mathbb{N}$), positive with no decimal
- **Integer**: ($\mathbb{Z}$), non-decimal, both positive and negative
- **Rational number**: can be expressed as a ratio or fraction
- **Irrational number**: cannot be expressed as a fraction: π, e
- $\mathbb{R}^k$ is a k-dimensional space

1.4 Intervals

- **Open**: $(-1, 2)$, does not include endpoints, so greater than -1 and less than 2.
- **Closed**: $[-1, 2]$, includes endpoints: greater than or equal to -1 and less than or equal to 2.



- **Half-closed**: $(-1, 2]$, includes endpoints: greater than or equal to -1 and less than or equal to



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1.5 Levels of measurement

- **Nominal**: No mathematical relationship among the values
  - E.g., race, gender, country, party
- **Ordinal**: Ranking, but cannot do arithmetic because the distance between values is not equal.
  - E.g., K-12, age cohort, ideology (left-to-right)

- **Interval**: Fixed differences, but zero is arbitrary
  - E.g., temperature, date/time
- **Ratio**: Fixed differences with a true zero
  - E.g., age, length, income, votes

1.6 Sets

- A **set**, $S = \{\}$, is a collection of elements.
  - Order does not matter: $S = \{x, y, z\} = \{z, y, x\}$
    - $x \in S$, meaning x is an element of S
    - $a \notin S$, meaning a is not an element of S
- Other examples:
  - $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ or $\{x|x \text{ is an integer}\}$
  - $T = \{x|8 \geqslant x \geqslant 7\}$ or $T = [7, 8]$
- Set notation:
  - $S = \{x, y, z\}$, $T = \{a, b, x\}$, $N = \{a, b\}$
    - Intersection: $T \cap S = \{x\}$
      - $N \cap S = \varnothing$: "empty set"
    - Union: $N \cup S = \{a, b, x, y, z\}$
    - Complement (not in the set): $(T \cap S)^c = \{a, b, y, z\}$
  - $\mathbb{U}$: universal set, all possible values
  - $A = \{x, y\}$
    - $A \subset S$: "A is a subset of S"
    - If $\not\subset$, then not a subset of. All subsets of A: $\{x\}$, $\{y\}$, $\{\varnothing\}$, $\{x, y\}$
- Transitivity:
  - If $Z \in Q$ and $Q \in R$, then $Z \in R$
- Disjoint:
  - No elements in common, more formally, two sets are disjoint if the intercept of sets is the null set: $N \cap S = \{\varnothing\}$

1.7 Independent and Dependent Variables

Let $y = f(x)$, where y is the outcome and x the input.

- **Independent variable**: the input – x
- **Dependent variable**: outcome – y

In a linear model, i.e. $y = \alpha + \beta x$, the dependent variable is y and the independent variable is x.

1.8 Functions

- $f(x) = y$
- **Constant function**: a function whose outcome is the same no matter the input. If $f(x) = c$, then no matter x the output is c.
- **Polynomial function**: $y = a + bx + cx^2$
- **Rational function**: can be defined by a fraction (ratio):

$$f(x, y) = \frac{a + bx^2}{1 + y}$$

1.9 Inequalities and Absolute Values

$$|x| = \sqrt{x^2} = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Useful properties:

- $|m| + |n| \geqslant |m + n|$
- $|m| \times |n| = |m \times n|$

1.10 Exponent Rules

- $x^n = x \cdot x \cdot x \cdot x \cdot \ldots \cdot x$ n times; $2^2 = 2 \cdot 2 = 4$
- $x^0 = 1$
- $x^m \cdot x^n = x^{m+n}$
- $\frac{x^m}{x^n} = x^{m-n}$
- $\frac{1}{x^m} = x^{-m}$
- $x^1 = x$
- $x^{\frac{1}{m}} = \sqrt[m]{x}$
- $(x^m)^n = x^{m \cdot n}$
- $x^m \cdot y^m = (xy)^m$

But:

- $x^m + y^n \neq (x + y)^{m+n}$

1.11 Commutative, Associative, and Distributive Laws

- **Associative property**: rewriting the parentheses does not change the outcome
  - $(a+b)+c = a+(b+c)$
  - $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- **Commutative property**: order of operands (inputs) does not change the outcome
  - $a+b = b+a$
  - $a \cdot b = b \cdot a$
- **Distributive property**:
  - $a(b+c) = ab+ac$

Also relevant:

- **Inverse property**:
  - For any x, there exists a $-x$ such that $-x+x=0$.
    - Formally[1]: $\exists(-x)$ s.t. $-x+x=0$
  - For any x, there exists a x^{-1} such that $x \cdot x^{-1} = 1$.
    - Formally: $\exists(x^{-1})$ s.t. $x^{-1} \cdot x = 1$
- **Identity property**:
  - $\exists(0)$ s.t. $x+0=x$
  - $\exists(1)$ s.t. $x \cdot 1 = x$
  - Commonly, we see: $\mathrm{I}(x) = x$

[1] $\exists$ = 'there exists'