5 Linear Algebra III

5.1 Matrix Inversion

- For a scalar a, the inverse: $a^{-1} = \frac{1}{a}$, where $a^{-1}a = 1$.
- We can also invert a matrix A: $A^{-1}A = AA^{-1} = \mathrm{I}$
  - Note: if A^{-1} does not exist, then A is singular/not invertible.
  - $A^T \neq A^{-1}$
- For a 2×2 matrix, we use the following steps. We'll cover larger matrices after, for which we'll use a lengthier, but more intuitive process.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Where $ad - bc$ is the *determinant*, which we discuss more later. An example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
$$A^{-1} = \frac{1}{(1 \cdot 4) - (3 \cdot 2)} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$
$$= \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -\frac{1}{2} \end{bmatrix}$$

- How to check if $A^{-1}A = \mathrm{I}$?

$$\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

5.2 Properties of Matrix Inversion

Singular vs nonsingular:

- Invertible = Nonsingular
- Noninvertible = Singular
- If A is nonsingular, then A^{-1} is nonsingular
- If A & B are nonsingular, then $(AB)^{-1} = B^{-1}A^{-1}$

- If A is nonsingular, then $(A^T)^{-1} = (A^{-1})^T$

Some conditions for nonsingularity (we can find A^{-1}, there are often more conditions listed, but they are technically covered by these three):

1. Rows and columns are linearly independent
   - No rows and/or columns add up to each other. Generally speaking if rows and columns are not linearly independent, then the matrix is invertible.
2. Matrix A is row equivalent to I (can we use row operators to turn A into the identity matrix?)
3. The determinant (we cover below) is not 0.

One intuitive and practical procedure for finding A^{-1}, regardless of the size:

1. Find $[A|\mathrm{I}]$ where both A and I are both $n \times n$
2. Find the reduced row echelon form for A (left side)
3. If step 2 gives us $[\mathrm{I}|C]$, then $C = A^{-1}$

A note on rank: If A is $m \times n$, then the rank(A) =

$$\min \begin{cases} \text{max \# of linearly ind. rows} \\ \text{max \# of linearly ind. cols} \end{cases}$$

- E.g.: $\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{1} & 2 & 3 \\ \mathbf{2} & 4 & 6 \end{bmatrix}$ Only the bold rows and columns are linearly ind. So, rank$(A) = \min(1,1) = 1$

- **Examples:**

$$\boldsymbol{A}\mathrm{I} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{array}\right]$$

  - Subtract 1/2 r2 from r1, divide r2 by 2, subtract 5r1 from r3

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -1/2 & 1 & -1/2 & 0 \\ 0 & 1 & 3/2 & 0 & 1/2 & 0 \\ 0 & 0 & 4 & -5 & 0 & 1 \end{array}\right]$$

  - Subtract 1/8 r3 from r1, add 3/8 r3 to r2, divide r3 by -4

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 13/8 & -1/2 & -1/8 \\ 0 & 1 & 0 & 15/8 & 1/2 & 3/8 \\ 0 & 0 & 1 & 5/4 & 0 & -1/4 \end{array}\right]$$

  - Where the right-hand side matrix is A^{-1}

- Another example, but A^{-1} doesn't exist:

  $$\left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 5 & -2 & -3 & 0 & 0 & 1 \end{array}\right]$$

  - Subtract r1 from r2, subtract 5r1 from r3

  $$\begin{bmatrix} 1 & 2 & 3 & \dots & \dots & \dots \\ 0 & -4 & 4 & \dots & \dots & \dots \\ 0 & -12 & 12 & \dots & \dots & \dots \end{bmatrix}$$

  - We aren't concerned with the right-hand side matrix because the rows of the left matrix are not independent. r3 is a linear combination of r2, meaning the matrix is singular...

5.3 Determinant

- Determinants convert a matrix into a scalar but can only be defined for a square matrix. Determinants are useful for checking if a matrix is invertible. They also can play a role in solving for systems of equations. The formula is straightforward for a 2×2 matrix, but less so for larger matrices.

- Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

  - The determinant of A is the two diagonal products differenced:

    $|A| = (a_{11} \cdot a_{22}) - (a_{21} \cdot a_{12})$

- Examples:

  - $\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} \Rightarrow (2 \cdot 5) - (3 \cdot 1) = 7$
  - $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \Rightarrow 2 - 2 = 0$

- For a 3×3 matrix we sum the products of all elements in any row or column, alternating signs, and the determinants of a specific 2×2 submatrix. An element's submatrix is the remaining elements when the elements from the relevant row and column are removed. I.e., for:

  $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

  The submatrix for a_{23} is $\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$

Taking the first column, the determinant of A is:

$a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$

- We can use any row or column. But it is best to use one with zeros, if available.
- The **minor** of an element is the determinant of its submatrix.
  - $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = (a_{21} \cdot a_{33}) - (a_{31} \cdot a_{23})$

5.3.1 Determinants via Cofactor Expansion

- The **cofactor** of any element i, j: $C_{ij} = (-1)^{i+j} M_{ij}$, which is used for calculating the determinants of $n \times n$ matrices where $n > 2$. i is rows, j is columns.
  - $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$
  - Ex: $C_{11} = (-1)^{1+1}\ M_{11} = 1 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$
- The determinant of a $n \times n$ matrix where $n > 2$ is the sum of the products of each element and its cofactor for any row or column. We just choose a single row or a single column – ideally one with as many zeros as possible.
  - Given row i:
  $$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}$$
  - Or row j:
  $$\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij}$$
- Let's show what this means:
- Choose the row or column with the most zeros:
  - Ex: $\begin{bmatrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 0 & 0 & 0 & -3 \\ 2 & 0 & -2 & 3 \end{bmatrix}$
  - So let's take all elements in row 3, multiply each times its cofactor, and add it all together. Because of the zeros, we only have to find one cofactor!

$$
\begin{aligned}
|A| = \sum_{j=1}^{4} a_{3j}C_{3j} &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} + a_{34}C_{34} \\
a_{34}C_{34} &= 0 + 0 + 0 + a_{34}C_{34} \\
a_{34}C_{34} &= -3\left[(-1)^{3+4}\begin{vmatrix} 1 & 2 & -3 \\ -4 & 2 & 1 \\ 2 & 0 & 2 \end{vmatrix}\right] \\
&= -3 \cdot -1\left[2\begin{vmatrix} 2 & -3 \\ 2 & 1 \end{vmatrix} - 0\begin{bmatrix} 1 & -3 \\ -4 & 1 \end{bmatrix} + (-2)\begin{vmatrix} 1 & 2 \\ -4 & 2 \end{vmatrix}\right] \\
&= 3\left[2(2+6) - 2(2+8)\right] \\
&= 3[16 - 20] = 3(-4) = -12
\end{aligned}
$$

- To wrap up, some useful properties of determinants:
  1. $|A| = |A^T|$
  2. If a row or column of A is a linear combination of other rows or columns, then $\det(A) = 0$
  3. If A is diagonal, then $|A|$ is the product of the diagonals.
  4. $|AB| = |A| \cdot |B|$
  5. If A is non-singular, then $|A^{-1}| = \frac{1}{|A|}$

5.4 Adjoint Matrix

- **Adjoint matrix**: the transpose of a matrix of the cofactors of each element:

$$
\operatorname{adj}(A) = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & \dots & \dots & c_{2n} \\ \vdots & & & \\ c_{n1} & \dots & \dots & c_{nn} \end{bmatrix}^T
$$

Where c is an element's cofactor. For example:

$$
A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \Rightarrow C = \begin{bmatrix} +\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} & -\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} & +\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} \\ +\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix}
$$

Where the last matrix is the adjoint matrix.

This information gives us another way to calculate the inverse of A with the following formula:

$$A^{-1} = \frac{\text{adj}(A)}{|A|}$$

5.5 Trace

Let's close with something simpler. The **trace** of a $n \times n$ matrix is just the sum of the diagonal elements.

$$\text{tr}(A) = \Sigma_{i=1}^{n} a_{11} = a_{11} + a_{22} + ... + a_{nn}$$

This is less frequently used, but worth being aware of.