4 Linear Algebra II

- Let's review matrix multiplication once before moving forward:

$$\boldsymbol{A} = \underset{2\times 3}{\begin{bmatrix} 1 & 2 & -3 \\ 4 & 0 & -2 \end{bmatrix}}, \quad \boldsymbol{B} = \underset{3\times 2}{\begin{bmatrix} 3 & 1 \\ 2 & 4 \\ -1 & 5 \end{bmatrix}}$$

- The dimensions of the product of these two matrices is the outside dimensions:

$$\boldsymbol{AB} = (\mathbf{2} \times 3) \cdot (3 \times \mathbf{2}) = 2 \times 2$$
$$\boldsymbol{BA} = (\mathbf{3} \times 2) \cdot (2 \times \mathbf{3}) = 3 \times 3$$

- We know we can multiply the two together in either direction because the inner dimensions match in either order.

$$\begin{bmatrix} 1 & 2 & -3 \\ 4 & 0 & -2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 4 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 3+4+3 & 1+8-15 \\ 12+0+2 & 4+0-10 \end{bmatrix} = \begin{bmatrix} 10 & -6 \\ 14 & -6 \end{bmatrix}$$

- One more example:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 7 & 8 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4+14+1 & 3+16+2 \\ 0+7+1 & 0+8+2 \end{bmatrix} = \begin{bmatrix} 19 & 21 \\ 8 & 10 \end{bmatrix}$$

4.1 Matrix Representation of Systems of Equations

- Take this system of equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

- Or in matrix form:

$$\underset{m\times n}{\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}} \underset{n\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} = \underset{m\times 1}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}$$

- Ex:

$$
\begin{aligned}
2x_1 + 3x_2 - 1x_3 &= 7 \\
4x_1 + 5x_2 + 6x_3 &= 8 \\
-1x_1 + 2x_2 + 1x_3 &= 9
\end{aligned}
$$

$$\Downarrow$$

$$
\underbrace{\begin{bmatrix} 2 & 3 & -1 \\ 4 & 5 & 6 \\ -1 & 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}}_{b}
$$

4.2 Linear Dependence and Independence

- The **span** of a vector is all of its linear combinations. I.e. $c \begin{bmatrix} 2 \\ 3 \end{bmatrix} \forall c$.
- If one vector falls in the span of another vector then the two are **linearly dependent**. There is no new information in the second vector.
  - $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$ are linearly dependent. The second is the first times 2.
- If one vector *does not* fall in the span of another vector, then the two are **linearly independent**.
  - $\begin{bmatrix} 7 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ are linearly independent. One cannot be represented as a linear combination of the other.
- If we were to draw each item in the vector set, then we are interested in whether or not the overall span of a set changes when a vector is added or removed.

4.3 Properties of Matrix Operators

- Matrix Addition
  - If $\boldsymbol{A}$ and $\boldsymbol{B}$ are both the same size $(m \times n)$:
    1. $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$
    2. $\boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}$
    3. Additive Inverse: $\boldsymbol{A} + (-\boldsymbol{A}) = 0_{m \times n}$
- Matrix Multiplication:
  - If $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ are conformable:
    1. $\boldsymbol{A}(\boldsymbol{B}\boldsymbol{C}) = (\boldsymbol{A}\boldsymbol{B})\boldsymbol{C}$

2. $\boldsymbol{IA} = \boldsymbol{AI} = \boldsymbol{A}$
   - Where $\boldsymbol{I}$ is an identity matrix, e.g. $\boldsymbol{I}_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
3. $\boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{AB} + \boldsymbol{AC} \neq \boldsymbol{BA} + \boldsymbol{CA}$ because order matters

- Matrix Exponents
  1. $\boldsymbol{A}^p \cdot \boldsymbol{A}^q = \boldsymbol{A}^{p+q}$
  2. $(\boldsymbol{A}^p)^q = \boldsymbol{A}^{pq}$
  3. $(\boldsymbol{AB})^p \neq \boldsymbol{A}^p\boldsymbol{B}^p$ unless $\boldsymbol{AB} = \boldsymbol{BA}$
- Scalar Multiplication; $\boldsymbol{A}, \boldsymbol{B}$ are matrices and r, s are scalars:
  1. $r(s\boldsymbol{A}) = (rs\boldsymbol{A})$
  2. $(r + s)\boldsymbol{A} = r\boldsymbol{A} + s\boldsymbol{A}$
  3. $r(\boldsymbol{A} + \boldsymbol{B}) = r\boldsymbol{A} + r\boldsymbol{B}$
  4. $A(r\boldsymbol{B}) = r\boldsymbol{AB} = \boldsymbol{AB}r$
- Matrix Transposition
  1. $(\boldsymbol{A}^T)^T = \boldsymbol{A}$
  2. $(\boldsymbol{A} + \boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$
  3. $(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$
  4. $(rA)^T = r(A^T)$
- Symmetric Matrix: A square matrix is symmetric if $\boldsymbol{A}^T = \boldsymbol{A}$
  - Ex: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 1 & 5 \end{bmatrix}$

4.4 Idempotent Matrices

- A matrix, $\boldsymbol{A}$, is idempotent if $\boldsymbol{AA} = \boldsymbol{A}$. Multiplying the matrix by itself returns the original matrix.

4.5 Reduced Row/Row Echelon Form and Solving Linear Systems of Equations Gauss-Jordan Reduction/Elimination

- We can use a matrix to represent a system of equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &= b_m \end{aligned}$$

- Rewritten as an **augmented matrix**:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & ... & a_{1n} & b_1 \\ a_{21} & a_{22} & ... & a_{2n} & b_2 \\ \vdots & & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} & b_m \end{array}\right]$$

- To solve for a system of equations, we often want to translate a matrix into **row echelon** or **reduced row echelon** form. What conditions describe a matrix in row echelon and reduced row echelon form. The conditions are:

  1. If any rows are zeros, then they are below nonzero rows (include at least one nonzero element):
  $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right]$
  2. The first non-zero entry in any row is 1 (except all zero row).
  $\left[\begin{array}{ccc|c} 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right]$
  3. For each non-zero row the leading 1 is to the right of each 1 in the row above.
  $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$
  4. **Reduced row-echelon form**: makes the solution to a system of equations obvious. Below $x_1 = 7, x_2 = 0, x_3 = 2$
  $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$

- Transforming a matrix into row echelon and reduced row echelon form is referred to as **Gauss-Jordan elimination**. We do so through **elementary row operators**:

- Multiplying a row by a constant
  - Remember, the matrix is a system of equations. So we're just multiplying both sides of an equation.
- Adding/subtracting rows
- Interchanging rows

- Example:

$$\begin{aligned} 1x_1 + 2x_2 + 4x_3 &= 3 \\ 2x_1 + 1x_2 + 3x_3 &= 2 \\ 1x_1 - 2x_2 + 2x_3 &= 3 \end{aligned}$$

  - As an augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 2 & 1 & 3 & 2 \\ 1 & -2 & 2 & 3 \end{array}\right]$$

  - Add $-2r1$ to $r2$ and $-1r1$ to $r3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & -3 & -5 & -4 \\ 0 & -4 & -2 & 0 \end{array}\right]$$

  - $r3 \times -\frac{1}{4}$ and interchange $r2$ and $r3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 1/2 & 0 \\ 0 & -3 & -5 & -4 \end{array}\right]$$

  - Add $3r2$ to $r3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & -7/2 & -4 \end{array}\right]$$

  - Multiply $r3$ by $-\frac{2}{7}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 1 & 8/7 \end{array}\right]$$

  - Subtract $4r3$ from $r1$ and subtract $\frac{1}{2}r3$ from $r2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -11/7 \\ 0 & 1 & 0 & -4/7 \\ 0 & 0 & 1 & 8/7 \end{array}\right]$$

- Subtract $2r2$ from $r1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3/7 \\ 0 & 1 & 0 & -4/7 \\ 0 & 0 & 1 & 8/7 \end{array}\right]$$

- $x_1 = -\frac{3}{7}, x_2 = -\frac{4}{7}, x_3 = \frac{8}{7}$
- Or: $\mathrm{I}_3 X = \begin{bmatrix} -3/7 \\ -4/7 \\ 8/7 \end{bmatrix}$