Frequency Analysis of Entanglement Dynamics in Coupled Quantum Oscillators

Bin Shi[1], Chao Xu[1]

[1]College of Control Science and Engineering, Zhejiang University
Interdisciplinary Center for Quantum Information

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Outline

Introduction

In quantum engineering periodic signals have been applied to quantum systems and lots of interesting phenomena occur.

- suppressing the tunneling effect [Grossman, 1991]
- changing the squeezing factor of quantum states [Brown, 1987]
- creation and survival of entanglement of quantum oscillators [Galve,2010]

Periodic parameter variation makes possible an entangled nonequilibrium state at high temperatures in some models such as **quantum parametric oscillators**. It builds the bridge between quantum oscillators and the classical ones. Meanwhile, mathematical development and tools enable us to investigate more complicated models such as QPOs with quasi-periodic couplings and multiple driving terms.

Outline

Mathematical Model

Hamiltonian of N quantum oscillators: $H = H_{System} + H_{Base} + H_{Interaction}$

$$
\begin{aligned}
H_{System} &= \sum_{i=1}^{N}\left[\frac{P_i^2}{2M_i} + \frac{M_i\Omega_i^2(t)X_i^2}{2}\right] + \sum_{1\le i<j\le N} C_{ij}(t)X_iX_j \\
H_{Base} &= \sum_{k=1}^{\infty}\left[\frac{p_k^2}{2m_k} + \frac{m_k\omega_k^2x_k^2}{2}\right] \\
H_{Interaction} &= \sum_{k=1}^{\infty}\left[-\sqrt{2}\frac{c_k}{\sqrt{N}}x_k\left(\sum_{i=1}^{N}X_i\right) + \frac{c_k^2}{2Nm_k\omega_k^2}\left(\sum_{i=1}^{N}X_i\right)^2\right]
\end{aligned} \tag{1}
$$

Assume all oscillators have the same parameters:

$$M_i = M, \Omega_i = \Omega, C_{ij}(t) = C(t), \forall i = 1, ..., N$$

where coupling terms of oscillators $C(t)$ are usually **periodic** or **quasi-periodic**.

Mathematical Model

There exists a proper $N \times N$ orthogonal matrix R that transforms H in $\{\mathbf{X}, \mathbf{P}\}$ into the sum of N commutable Hamiltonians under $\{\bar{\mathbf{X}}, \bar{\mathbf{P}}\}$.

$$\{\mathbf{X}, \mathbf{P}\} \xrightarrow[\bar{\mathbf{P}}=R\mathbf{P}]{\bar{\mathbf{X}}=R\mathbf{X}} \{\bar{\mathbf{X}}, \bar{\mathbf{P}}\}$$

$$\sigma\{\mathbf{X}, \mathbf{P}\} \xrightarrow[f(\sigma,R)]{} \bar{\sigma}\{\bar{\mathbf{X}}, \bar{\mathbf{P}}\} = diag(\sigma_{1+}, \sigma_{2-}, \cdots \sigma_{N-})$$

where transformation matrix R can be derived from an initial matrix R_{ori} by Schmidt orthogonalization. For example, when $N = 4$:

$$R_{ori} = \begin{bmatrix} \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \cdots & \frac{1}{\sqrt{N}} \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \xrightarrow[orth.]{Schmidt} R_4 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2\sqrt{3}} & \frac{3}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Mathematical Model

The system is divided into N independent subsystems after transformation:

$$
\begin{aligned}
H &= \sum_{i=1}^{N} \bar{H}_i \\
\bar{H}_1 &= \frac{\bar{P}_1^2}{2M} + \frac{M\Omega_+^2(t)\bar{X}_1^2}{2} + \sum_{k=1}^{\infty}\left[\frac{p_k^2}{2m_k} + \frac{m_k\omega_k^2}{2}\left(x_k - \frac{\sqrt{2}c_k}{m_k\omega_k^2}\bar{X}_1\right)^2\right] \\
\bar{H}_i &= \frac{\bar{P}_i^2}{2M} + \frac{M\Omega_-^2(t)\bar{X}_i^2}{2}, i = 2, 3, \cdots, N
\end{aligned} \tag{2}
$$

H_1 describes a **dissipative parametric oscillator** coupled to a common heat bath; H_i describe $N-1$ **free parametric oscillators** with the same resonance parameters.

$$
\begin{aligned}
\Omega_+^2 &= \Omega(t)^2 + (N-1)C(t)/M \\
\Omega_-^2 &= \Omega(t)^2 - C(t)/M
\end{aligned}
$$

Classical parametric oscillator (CPO)

CPO can be described by Mathieu equation with damping:

$$\ddot{x} + \gamma\dot{x} + (\delta + \varepsilon\cos t)x = 0 \tag{3}$$

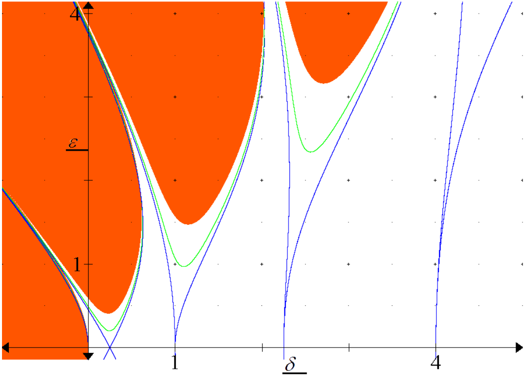


Figure 1: Ince-Strutt instability diagram with different damping rates. $\gamma = 0$ [blue], $\gamma = 0.2$ [green],$\gamma = 0.4$ [orange]. White areas are stable zones.

Quantum parametric oscillator (QPO)

Taking for example two QPOs cases, it shows the matchup between classical instability and numerical divergence of elements of covariance.

- **Free Parametric Oscillator**

$$\begin{aligned}\sigma_{X\_X\_} &= \sigma^0_{P\_P\_}\phi_1^2 + \sigma^0_{X\_X\_}\phi_2^2 + \sigma^0_{X\_P\_}\phi_1\phi_2 \\ \sigma_{P\_P\_} &= \sigma^0_{P\_P\_}\dot{\phi}_1^2 + \sigma^0_{X\_X\_}\dot{\phi}_2^2 + \sigma^0_{X\_P\_}\dot{\phi}_1\dot{\phi}_2 \\ \sigma_{X\_P\_} &= \sigma^0_{P\_P\_}\phi_1\dot{\phi}_1 + \sigma^0_{X\_X\_}\phi_2\dot{\phi}_2 + \sigma^0_{X\_P\_}(\phi_1\dot{\phi}_2 + \dot{\phi}_1\phi_2)\end{aligned} \tag{4}$$

  $\sigma^0_{(\cdot)}$ are initial values and ϕ_1, ϕ_2 are the solutions of: $\ddot{x} + \Omega_-^2(t)x = 0$ parameters of Ω_-^2 in unstable zones lead to numerical divergence of σ and quantum entanglement.

- **Dissipative Parametric Oscillator**

$$\sigma_+ = f(\sigma_+^0, \gamma, T) \tag{5}$$

  Calculation of dissipative covariances is complicated. It's partly related to solution of $\ddot{x} + \gamma\dot{x} + \Omega_+^2(t)x = 0$.
  In room temperature, σ_+ is generally convergent.

Quantum covariances calculation

- **Covariance Matrix**: in $Q = (X_1, P_1, X_2, P_2, \cdots, X_N, P_N)$

$$\sigma_{A_i B_j} = \langle A_i B_j + B_j A_i \rangle /2 - \langle A_i \rangle \langle B_j \rangle$$

where $A, B = X \; or \; P, \; i, j = 1, 2, \cdots, N$

- **Initial States**: Gaussian states, not squeezed states

$$\sigma_{XX}^{0} = 1/2 r_s \sqrt{\Omega_0^2}, \; \sigma_{XP}^{0} = 0$$
$$\sigma_{PP}^{0} = r_s \sqrt{\Omega_0^2}/2, \;\; \sigma_{PX}^{0} = (\sigma_{XP}^{0})^T, \;\; r_s = 1$$

- **Entanglement Dynamics**: the degree of entanglement can be measured by the logarithmic negativity

$$E_n = \begin{cases} 0, & \text{if} \quad v_- \geq 1/2 \\ -\log(2v_-), \text{if} & v_- < 1/2 \end{cases} \tag{6}$$

where $v_- = f(Det(\sigma_{1+}), Det(\sigma_{i-}))$

Quantum covariance calculation

Periodic parametric variations may drive QPOs to entangle states. When driving amplitude is not large, E_n grows approximately, **linearly** over time. Define τ_r, r respectively, as the **initial entangled time** and the average **growth rate** of the entanglement. Small vibrations are called **ripples**.

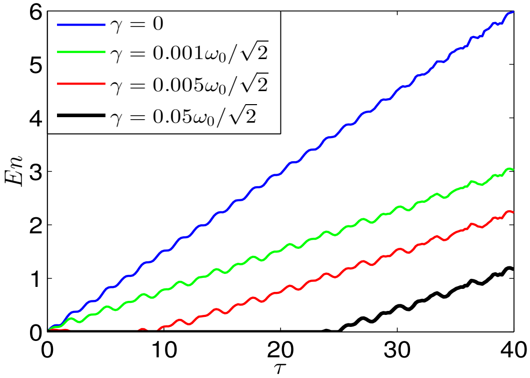


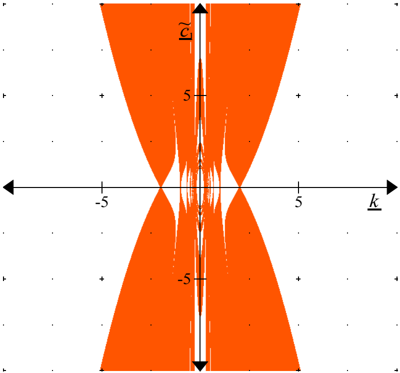
Figure 2: Example of E_n grows over time at different damping rates.

Outline

Role of relative frequencies in instability diagram

Considering periodic coupling: $\Omega_-^2 = \Omega(t)^2 - C(t)/M = \omega_0^2 + c_1 \cos(\omega_D t)$

$$\frac{d^2x}{dt^2} + (\omega_0^2 + c_1 \cos(\omega_D t))x = 0 \xrightarrow[k=\omega_D/\omega_0]{\tau=\omega_D t} \frac{d^2x}{d\tau^2} + \frac{1}{k^2}(1 + \tilde{c}_1 \cos(\tau))x = 0 \qquad (7)$$



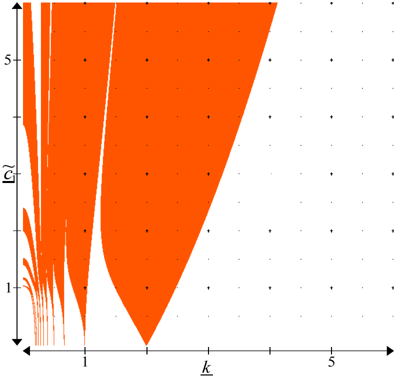


Figure 3: The Ince-Strutt diagram of relative driving frequency to relative frequency $\tilde{c}_1 \sim k$, white zones represent stability regions.

Role of relative frequencies in instability diagram

Periodic driving produces an entangled nonequilibrium state at high temperatures $k_B T > E_{typ}$ and helps sustain quantun coherence.

$$T = \tilde{T} T_0, \tilde{T} = 100, \omega_0 = 1, T_0 = \hbar\omega_0 / k_B, \tilde{c}_1 = 0.3, \gamma = 0.005\omega_0/\sqrt{2}$$

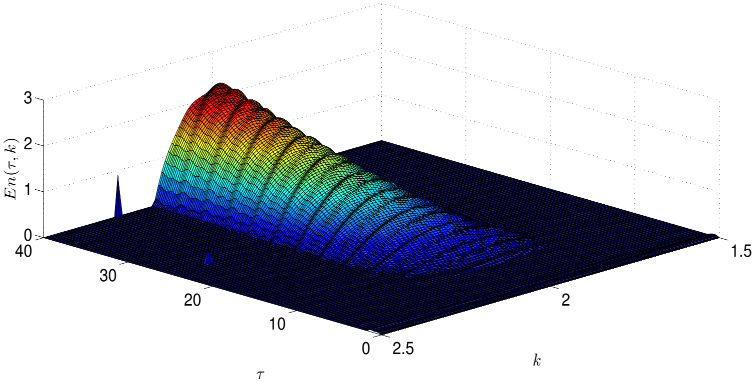


Figure 4: Surface of logarithm negativity $E_n \sim (\tau, k)$ with maxima at $k = 2$.

Role of relative frequencies in instability diagram

Entanglement only survives near $k = 2$ with minimal initial entangled time τ_r and maximal growth rate r.

$$T = \tilde{T}\,T_0,\ \tilde{T} = 100,\ T_0 = \hbar\omega_0/k_B, \omega_0 = 1, k = 2, \tilde{c}_1 = 0.3$$

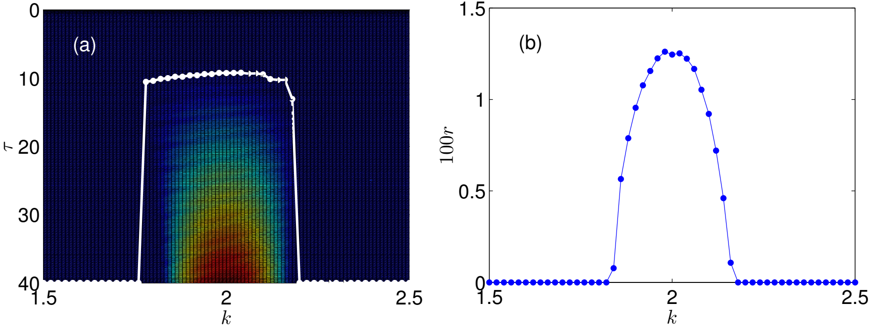


Figure 5: The influence of relative frequency on characteristic parameters of entanglement dynamics. (a)$\tau_r \sim k$; (b)$r \sim k$.

Role of absolute frequencies in quantum entanglement

Not only relative frequencies k, but also absolute frequencies ω_0 influence characteristic parameters of entanglement dynamics.

$$T = \tilde{T}T_0, \tilde{T} = 100, T_0 = \hbar\omega_0/k_B, k = 2, \tilde{c}_1 = 0.3, \gamma = 0.005\omega_0/\sqrt{2}$$

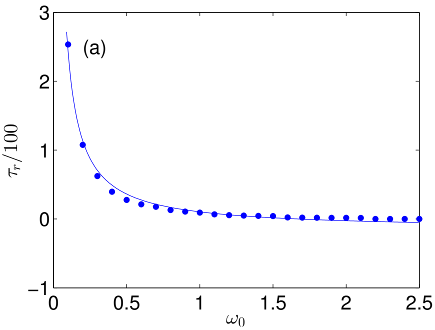


Figure 6: (a)τ_r suggests a linear fit to ω_0^{-1}: $\tau_r = 25.85\omega_0^{-1} - 15.53$. (b) r suggests a linear fit to ω_0: $r = 0.7297\omega_0 + 0.0593$.

Frequency analysis of ripples

In the case of only one periodic driving, the ripple and its frequency characteristics are exhibited below.

$$T = \tilde{T}T_0,\ \tilde{T} = 100,\ T_0 = \hbar\omega_0/k_B, k = 2, \omega_0 = 1, \tilde{c}_1 = 0.3, \gamma = 0.005\omega_0/\sqrt{2}$$

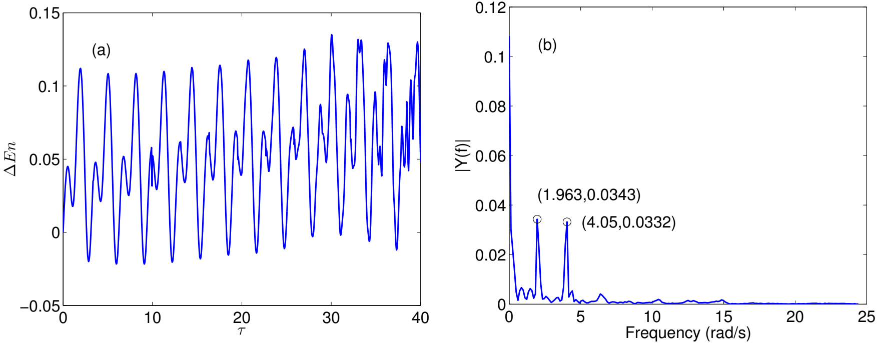


Figure 7: (a)Time evolution of the ripple. (b)Frequency spectrum obtained by Fast Fourier Transform. The sample period is 0.05.

Frequency analysis of ripples

The properties of ripples are closely related to the driving terms.

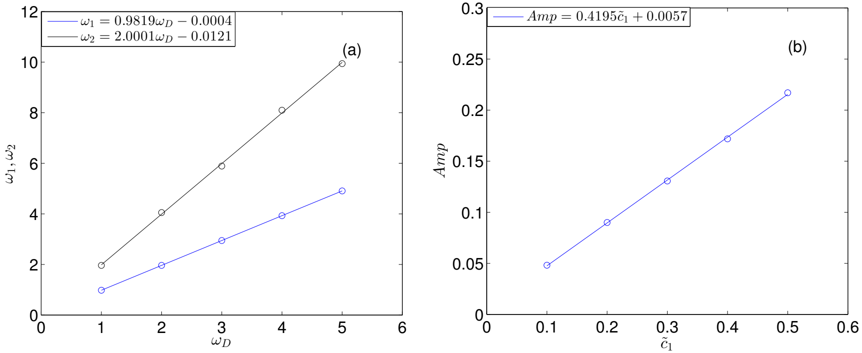


Figure 8: (a)When $\tilde{c}_1 = 0.3$, the frequencies of ripples have a linear fit to the driving frequency, $\omega_1 = 0.9819\omega_D - 0.0004$, $\omega_2 = 2.0001\omega_D - 0.0121$; (b)When $\omega_D = 1$, the amplitude of ripples has a linear fit to the driving amplitude, $Amp = 0.4195\tilde{c}_1 + 0.0057$.

Outline

The augmented Mathieu equation

Considering QPOs with multiple driving terms

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + (\omega_0^2 + \sum_{n=1}^{\infty} c_n\cos(\omega_n t))x = 0$$

- **Augmented Mathieu Eq.**: same critical stable nodes, different tongue structure.

$$\frac{d^2x}{d\tau^2} + \frac{1}{\omega_D^2}(\omega_0^2 + \sum_{n=1}^{\infty} c_n\cos(n\tau))x = 0$$

- **Quasi-periodic Mathieu Eq.**: taking two driving cases for example, ω is irrational and stability chart change a lot.

$$\ddot{x} + (\delta + \varepsilon(\cos t + \cos\omega t))x = 0$$

Transition curve can be derived from harmonic balance method and Hill determinant of A according to Diophantine condition.

$$x_{even}(t) = \sum_{a=0}^{\infty}\sum_{b=-\infty}^{\infty} A_{ab}\cos\left(\frac{a+b\omega}{2}t\right),\ x_{odd}(t) = \sum_{a=0}^{\infty}\sum_{b=-\infty}^{\infty} B_{ab}\sin\left(\frac{a+b\omega}{2}t\right)$$

The quasi-periodic Mathieu equation

The relationships between $\delta, \varepsilon, \omega$ and stability chart:

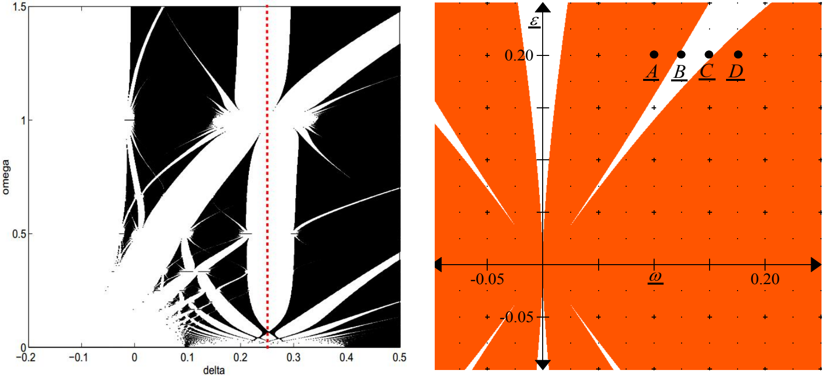


Figure 9: (a)[Zounes, 1998]* Stability chart of $\delta - \omega$ when $\varepsilon = 0.1$. Black zones represent stability regions. (b)Stability chart of $\varepsilon - \omega$ when $\delta = 0.25$(*or* $k = 2$). White zones represent stability regions.

* Zounes, 1998, SIAM Journal on Applied Mathematics.

The quasi-periodic Mathieu equation

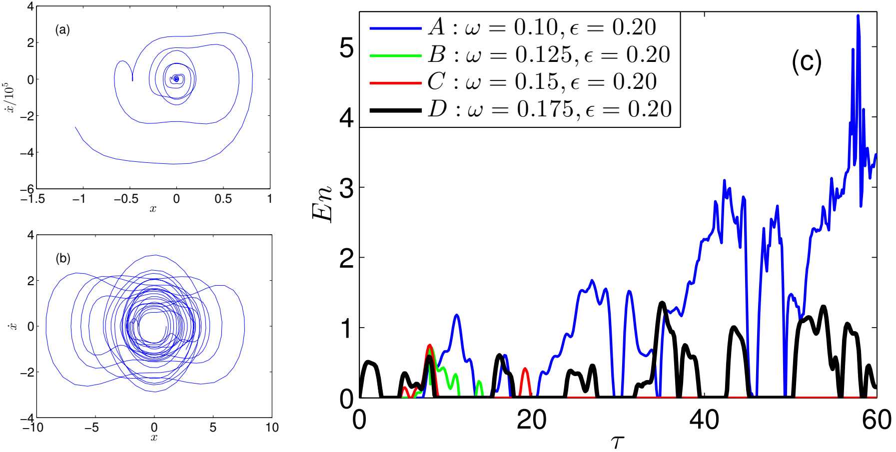


Figure 10: (a)Phase plane plot at A diverges. Initial value is $(1,0)$. (b)Phase plane plot at C converges. (c) Time evolution of entanglement when the parameter pairs locate at A,B,C,D in the parameter plane. A [blue], D [bold black] in unstable regions. B [green], C [red], are in the stable regions.

Frequency analysis

Abundant frequency information is found and dynamics of quantum oscillators with quasi-periodic couplings is much more complicated than the periodic one.

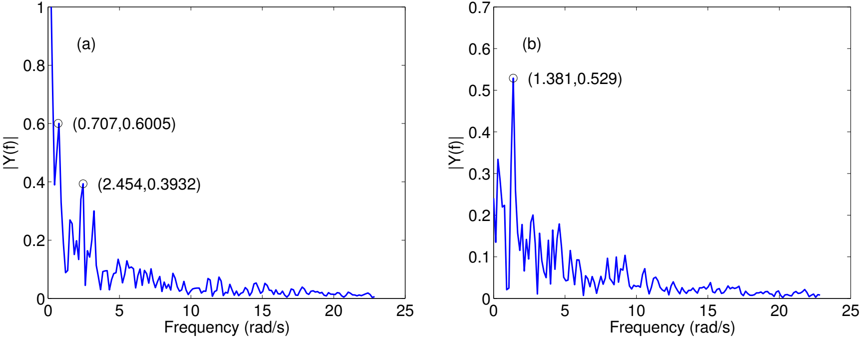


Figure 11: Frequency analysis of ripples. (a)When $\omega = 0.10, \varepsilon = 0.2$. (b) When $\omega = 0.175, \varepsilon = 0.2$

Control landscape

The parameter landscape for controlling entanglement. Quasi-periodic driving produces an entangled nonequilibrium state at high temperatures.

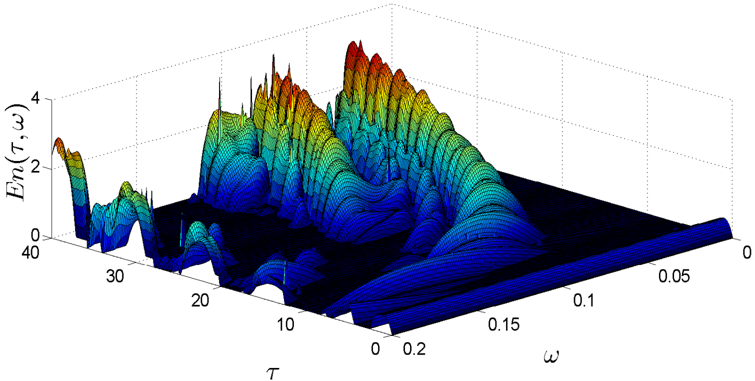


Figure 12: Surface of logarithm negativity $E_n \sim (\tau, \omega)$ when $k = 2$, and the regions where entanglement exists coincides with Fig.(9).

Outline

Conclusion

- **Modeling**: Transformation matrix for multiple oscillators is given.
- **Peirodic Coupled QPO**: Roles of relative frequency, absolute frequency on characteristic parameters of entanglement dynamics are analyzed. In one driving term case, frequencies and amplitudes of ripples are closely related to the driving term.
- **Quasi-Peirodic Coupled QPO**: Stability Chart of quasi-periodic couplings is given. Briefly analyze its characteristics in frequency domain and parameter landscape for controlling entanglement dynamics.

- **Future Work**
  - Stability chart with multiple driving terms.
  - Application in quantum engineering.

THANK

YOU!