Chapter 1

1.1 a

1.2 a

1.3 b

1.4 c

1.5 d

1.6 e

1.7 Find a formula for $\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^n$, and prove it by induction.

Discussion. Call this matrix A. The action of multiplying any matrix B by A on the right will yield a product whose first column is the first column of B, second column is the sum of the first two columns, and whose third column is the sum of all three columns:

$$BA = \begin{bmatrix} | & | & | \\ \mathbf{b}_1 & \mathbf{b}_1 + \mathbf{b}_2 & \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3 \\ | & | & | \end{bmatrix}$$

We can guess that repeated application of A on the right of A will have the effect of leaving column 1 unchanged. Since column 1 will always be $(1, 0, 0)$, each right-multiplication of A adds $(1, 0, 0)$ to column 2 so we can guess that column 2 of $A^n = (n, 1, 0)$. Column 3 of A^n we can guess is the sum of the first n numbers, n, then 1: $(\sum_{k=1}^{n} k, n, 1)$. The sum of the first k natural numbers has a well-known closed form, $n(n+1)/2$, so we can now claim:

$$A^n = \begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ & 1 & n \\ & & 1 \end{bmatrix}$$

Proof by induction.

For the base case we may use $n = 1$. Then $\frac{n(n+1)}{2} = 1(2)/2 = 1$ and the base case holds. Assuming this holds for some n, for the inductive step we would have:

$$\begin{aligned} A^{n+1} &= A^n A && \text{associativity} \\ &= \begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ & 1 & n \\ & & 1 \end{bmatrix} A \\ &= \begin{bmatrix} 1 & n+1 & \frac{n(n+1)}{2} + n + 1 \\ & 1 & n+1 \\ & & 1 \end{bmatrix} \end{aligned}$$

The diagonal and superdiagonal are as we claimed. The top-right entry can be rewritten as:

$$\frac{n(n+1)}{2} + n + 1 = \frac{n(n+1) + 2n + 2}{2} = \frac{n^2 + 3n + 2}{2} = \frac{(n+1)(n+2)}{2}$$

which is also as claimed. Thus the proof is complete.

1.8 x

1.9 x

1.10 x

1.11 x

1.12 x

1.13 x

1.14 Find infinitely many matrices B such that $BA = I_2$ when

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$$

and prove there is no matrix C such that $AC = I_3$.

Discussion. To find infinitely many solutions, we'll probably need to find a vector that when added to B doesn't change the product AB. By the row picture of matrix multiplication, the rows of B denote linear combinations of the rows of A to form the product BA. So we'll look for linear combination of A rows that sum to $\mathbf{0}$, then any multiple of this linear combination will also be $\mathbf{0}$, and can be added to B without changing BA. Technically, we seek a vector in A's left null space.

Consider the row space of A: there are three vectors in a 2-dimensional space so they must be linearly dependent. In fact we might see that the first row is the sum of the other two, and so $\mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3 = \mathbf{0}$. Therefore the row vector $[1\ -1\ -1]$ is in A's left null space:

$$\lambda \begin{bmatrix} 1 & -1 & -1 \end{bmatrix} A = \mathbf{0}$$

Let $N = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$, stacking this vector. Then:

$$NA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and by linearity, $(B + \lambda N)A = BA$ for any λ.

Next we seek a matrix $\tilde{B}$ such that $\tilde{B}A = I_2$. We could ignore A's third row and invert the 2-by-2 matrix $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, its inverse is $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$. To make it the right size we append a column of zeros to get:

$$\tilde{B} = \begin{bmatrix} 2 & -3 & 0 \\ -1 & 2 & 0 \end{bmatrix}, \quad \tilde{B}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Now we can let $B = \tilde{B} + \lambda N$ for any choice of scalar λ, and we'll have:

$$BA = (\tilde{B} + \lambda N)A = \tilde{B}A = I_2$$

To prove that there A has no right-inverse C, observe that A has 2 linearly independent columns and a 3-dimensional codomain. By rank-nullity, A is not full-rank and cannot produce I_3.

1.15 With A arbitrary, determine the products $e_{ij}A, Ae_{ij}, e_jAe_k, e_{ii}Ae_{jj}$, and $e_{ij}Ae_{k\ell}$

Discussion. Note that Artin defines e_{ij} as a 'unit matrix', a matrix with a 1 at the (i, j) coordinate and 0 elsewhere.

- $e_{ij}A$: since e_{ij} is multiplying A on the left, it is making linear combinations of A's rows. Since it only has a 1 in column j it extracts the jth row of A and puts in in the ith row of the product, which is 0 everywhere else.

$$e_{ij}A = \begin{array}{cc} & \begin{array}{ccc} & j & \end{array} \\ i & \begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix} \end{array} \begin{bmatrix} — & \mathbf{a}_1 & — \\ & \vdots & \\ — & \mathbf{a}_j & — \\ & \vdots & \\ — & \mathbf{a}_n & — \end{bmatrix} = \begin{array}{cc} i & \begin{bmatrix} — & \mathbf{a}_j & — \\ & & \end{bmatrix} \end{array}$$

- Ae_{ij}: Note that the jth column of e_{ij} is also the ith basis vector e_i. Multiplying A by a basis vector e_i simply extracts the ith column from A: $Ae_i = \mathbf{a}_i$. Next, consider that in e_{ij} there are $j-1$ columns of $\mathbf{0}$ before the $\mathbf{e}_j$ column. So Ae_{ij} is a matrix with $\mathbf{a}_i$ in the jth column and 0's everywhere else.

$$\begin{array}{ccc} & & \begin{bmatrix} | \\ e_i \\ | \end{bmatrix} \\ Ae_j = & \begin{bmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_n \\ | & & | \end{bmatrix} & \begin{bmatrix} | \\ \mathbf{a}_i \\ | \end{bmatrix} \end{array}$$

$$
Ae_{ij} = \begin{array}{cc}
 & \begin{array}{ccccc} & & j & & \\ \end{array} \\
 & \begin{bmatrix} & | & | & | & \\ \dots & \mathbf{0} & e_i & \mathbf{0} & \dots \\ & | & | & | & \end{bmatrix} \\
\begin{bmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_n \\ | & & | \end{bmatrix} & \begin{bmatrix} & | & | & | & \\ \dots & \mathbf{0} & \mathbf{a}_i & \mathbf{0} & \dots \\ & | & | & | & \end{bmatrix}
\end{array}
$$

Alternatively, we can write $Ae_{ij} = (e_{ij}^\top A^\top)^\top = (e_{ji}A^\top)^\top$ and use the previous result: $e_{ji}A^\top$ places the ith row of $A^\top$ in the jth row of the product, then the transpose $(e_{ji}A^\top)^\top$ has the ith *row* of A in the jth *column* (and zeros everywhere else).

- e_jAe_k: the product e_jA is not defined for an arbitrary A. Suppose e_j is $m \times 1$, then for e_jA to be defined A must have the shape $1 \times n$. Then e_k must be $n \times 1$ for the product e_jAe_k to be defined. It would be very unconventional for e_j and e_k to have different shapes, but nevertheless Ae_k would be the single entry matrix $[a_{1k}]$, which then multiplies e_j like a scalar to produce $a_{1k}e_j$.

  In case Artin intended the product to read $e_j^\top Ae_k$, the answer would be that $e_j^\top A$ extracts the jth row of A, then multiplying by e_k extracts the kth component of that row: $e_j^\top Ae_k = a_{jk}$
- $e_{ii}Ae_{jj}$: using previous results, Ae_{jj} has the effect of zeroing out A except for the jth column. Multiplying this by e_{ii} on the left zeros out everything but the ith row, which leaves only a_{ij} and zeros everywhere else: $e_{ii}Ae_{jj} = a_{ij}e_{ij}$.
- $e_{ij}Ae_{k\ell}$: the product $Ae_{k\ell}$ is a matrix with $\mathbf{a}_k$ in the ℓth column and zeros everywhere else. Multiplying this by e_{ij} on the left takes the jth component of this column, a_{jk}, and places it in the ith row of the product, with zeros everywhere else. Therefore we have only a_{jk} in the (i, ℓ) entry, or $a_{jk}e_{i\ell}$.

Exercises from 1st edition

1.1.16 A square matarix A is called *nilpotent* if $A^k = 0$ for some $k > 0$. Prove that if A is nilpotent then $I + A$ is invertible.

Discussion. If $I + A$ is invertible then its inverse is some matrix B such that $(I + A)B = I$. We're tempted to just divide both sides by $I + A$ and get $B = \frac{I}{I+A}$. Although we don't really have division by matrices, we can note that $I/(I + A)$ looks similar to a geometric sum. We might guess:

$$B \stackrel{?}{=} \frac{I}{I + A} = \frac{I}{I - (-A)} \stackrel{?}{=} I - A + A^2 - A^3 + \dots$$

Since A^k is 0, the infinite sum actually does end:

$$I - A + A^2 - A^3 + \ldots = I - A + A^2 - \ldots (-1)^{k-1}A^{k-1}$$

Try this for B to see if it inverts $I + A$, and indeed it does.
Proof.

$$\begin{aligned}
(I + A)B &= (I + A)(I - A + A^2 - \ldots (-1)^{k-1}A^{k-1}) \\
&= (I - A + A^2 - \ldots (-1)^{k-1}A^{k-1}) + A(I - A + A^2 - \ldots (-1)^{k-1}A^{k-1}) \\
&= (I - A + A^2 - \ldots (-1)^{k-1}A^{k-1}) + (A - A^2 + A^3 - \ldots (-1)^{k-1}A^{k}) \\
&= I + (A - A) + (A^2 - A^2) - \ldots \big((-1)^{k-1}A^{k-1} + (-1)^{k-2}A^{k-1}\big) + 0 \\
&= I
\end{aligned}$$

Here we see that after distributing, all the A powers cancel out leaving only I, therefore $B = I - A + A^2 - \ldots (-1)^{k-1}A^{k-1}$ is an inverse of $I + A$.