

# Chapter 1

Notation: I use  $\mathbf{a}_i$  to denote the  $i$ th column of matrix  $A$ , and  $\mathbf{a}_j$  to denote the  $j$ th row.

1.7 Find a formula for  $\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^n$ , and prove it by induction.

**Discussion.** Call this matrix  $A$ , and consider what  $A$  does to  $B$  in the product  $BA$ :  $A$  places  $B$ 's first column in the product's first column,  $B$ 's column 1 +  $B$ 's column 2 in the product's second column, and the sum of  $B$ 's 3 columns in the product's third column.

$$BA = \begin{bmatrix} | & | & | \\ \mathbf{b}_1 & \mathbf{b}_1 + \mathbf{b}_2 & \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3 \\ | & | & | \end{bmatrix}$$

$$A^2 = AA = \begin{bmatrix} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_1 + \mathbf{a}_2 & \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 \\ | & | & | \end{bmatrix}$$

If the left matrix is itself  $A$ , we can guess that repeated application of  $A$  on the right will leave column 1 unchanged, so we suppose the first column of  $A^n = (1, 0, 0)$ . If this is correct, each right-multiplication by  $A$  adds  $(1, 0, 0)$  to  $(1, 1, 0)$ , making column 2 of  $A^n = (n, 1, 0)$ . For column 3 of  $A^2$  we add  $(1, 0, 0) + (1, 1, 0) + (1, 1, 1) = (3, 2, 1)$ . For column 3 of  $A^3$  we add  $(1, 0, 0) + (2, 1, 0) + (3, 2, 1)$ , and we suspect that in  $A^n$ 's 3rd column, the first entry is the sum  $1 + 2 + \dots + n$ . So we guess the third column is  $(\sum_{k=1}^n k, n, 1)$ . The sum of the first  $k$  natural numbers has a well-known closed form,  $n(n+1)/2$ , so we can now claim:

$$A^n = \begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ & 1 & n \\ & & 1 \end{bmatrix}$$

**Proof by induction.**

For the base case we may use  $n = 1$ . Then  $\frac{n(n+1)}{2} = 1(2)/2 = 1$  and the base case holds. Assuming this holds for some  $n$ , for the inductive step we would have:

$$\begin{aligned} A^{n+1} &= A^n A && \text{associativity} \\ &= \begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ & 1 & n \\ & & 1 \end{bmatrix} A \\ &= \begin{bmatrix} 1 & n+1 & \frac{n(n+1)}{2} + n+1 \\ & 1 & n+1 \\ & & 1 \end{bmatrix} \end{aligned}$$

The diagonal and superdiagonal are as we claimed. The top-right entry can be rewritten as:

$$\frac{n(n+1)}{2} + n + 1 = \frac{n(n+1) + 2n + 2}{2} = \frac{n^2 + 3n + 2}{2} = \frac{(n+1)(n+2)}{2}$$

which is also as claimed. Thus the proof is complete.

1.14 Find infinitely many matrices  $B$  such that  $BA = I_2$  when

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$$

and prove there is no matrix  $C$  such that  $AC = I_3$ .

**Discussion.** To find infinitely many solutions, we can find a vector that when added to  $B$  doesn't change the product  $BA$ . By the row picture of matrix multiplication, the rows of  $B$  specify linear combinations of  $A$ 's rows to make the product  $BA$ . So we'll look for a linear combination of  $A$ 's rows that sum to  $\mathbf{0}$ . Then any multiple of this linear combination will also be  $\mathbf{0}$ , and can be added to  $B$  without changing  $BA$ . Technically, we seek a vector in  $A$ 's left null space.

Consider the row space of  $A$ : there are three vectors in a 2-dimensional space so they must be linearly dependent. In fact we might see that the first row is the sum of the other two, and so  $\mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3 = \mathbf{0}$ . Therefore the row vector  $[1 \ -1 \ -1]$  is in  $A$ 's left null space:

$$\lambda [1 \ -1 \ -1] A = \mathbf{0}$$

Let  $N = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$ , stacking this vector. Then:

$$NA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } (B + \lambda N)A = BA + \lambda NA = BA + \mathbf{0} = BA$$

Next we seek a matrix  $\tilde{B}$  such that  $\tilde{B}A = I_2$ . We could ignore  $A$ 's third row and invert the 2-by-2 matrix  $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ , its inverse is  $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ . To make it the right size we append a column of zeros to get:

$$\tilde{B} = \begin{bmatrix} 2 & -3 & 0 \\ -1 & 2 & 0 \end{bmatrix}, \quad \tilde{B}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Now we can let  $B = \tilde{B} + \lambda N$  for any choice of scalar  $\lambda$ , and we'll have:

$$BA = (\tilde{B} + \lambda N)A = \tilde{B}A = I_2$$

To prove that there *is* a right-inverse  $C$ , observe that  $A$  has 2 linearly independent columns and a 3-dimensional codomain. By rank-nullity,  $A$  is not full-rank and cannot produce  $I_3$ .

1.15 With  $A$  arbitrary, determine the products  $e_{ij}A$ ,  $Ae_{ij}$ ,  $e_jAe_k$ ,  $e_{ii}Ae_{jj}$ , and  $e_{ij}Ae_{k\ell}$

**Discussion.** Note that Artin defines  $e_{ij}$  as a ‘unit matrix’, a matrix with a 1 at the  $(i, j)$  coordinate and 0 elsewhere.

- $e_{ij}A$ : since  $e_{ij}$  is multiplying  $A$  on the left, it is making linear combinations of  $A$ ’s rows. Since it only has a 1 in column  $j$  it extracts the  $j$ th row of  $A$  and puts it in the  $i$ th row of the product, which is 0 everywhere else.

$$e_{ij}A = \begin{matrix} & j \\ i & \begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix} \end{matrix} \begin{bmatrix} - & \mathbf{a}_1 & - \\ & \vdots & \\ - & \mathbf{a}_j & - \\ & \vdots & \\ - & \mathbf{a}_n & - \end{bmatrix} = \begin{matrix} & & \\ i & \begin{bmatrix} - & \mathbf{a}_j & - \end{bmatrix} \end{matrix}$$

- $Ae_{ij}$ : Note that the  $j$ th column of  $e_{ij}$  is also the  $i$ th basis vector  $e_i$ . Multiplying  $A$  by a basis vector  $e_i$  simply extracts the  $i$ th column from  $A$ :  $Ae_i = \mathbf{a}_i$ . Next, consider that in  $e_{ij}$  there are  $j - 1$  columns of  $\mathbf{0}$  before the  $\mathbf{e}_j$  column. So  $Ae_{ij}$  is a matrix with  $\mathbf{a}_i$  in the  $j$ th column and 0’s everywhere else.

$$Ae_j = \begin{bmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_n \\ | & & | \end{bmatrix} \begin{bmatrix} | \\ e_i \\ | \end{bmatrix}$$

$$Ae_{ij} = \begin{matrix} & j \\ \begin{bmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_n \\ | & & | \end{bmatrix} & \begin{bmatrix} \dots & \mathbf{0} & e_i & \mathbf{0} & \dots \\ | & | & | & | & | \end{bmatrix} \end{matrix}$$

Alternatively, we can write  $Ae_{ij} = (e_{ij}^\top A^\top)^\top = (e_{ji}A^\top)^\top$  and use the previous result:  $e_{ji}A^\top$  places the  $i$ th row of  $A^\top$  in the  $j$ th row of the product, then the transpose  $(e_{ji}A^\top)^\top$  has the  $i$ th row of  $A$  in the  $j$ th column (and zeros everywhere else).

- $e_j A e_k$ : the product  $e_j A$  is not defined for an arbitrary  $A$ . Suppose  $e_j$  is  $m \times 1$ , then for  $e_j A$  to be defined  $A$  must have the shape  $1 \times n$ . Then  $e_k$  must be  $n \times 1$  for the product  $e_j A e_k$  to be defined. It would be very unconventional for  $e_j$  and  $e_k$  to have different shapes, but nevertheless  $A e_k$  would be the single entry matrix  $[a_{1k}]$ , which then multiplies  $e_j$  like a scalar to produce  $a_{1k} e_j$ .  
In case Artin intended the product to read  $e_j^\top A e_k$ , the answer would be that  $e_j^\top A$  extracts the  $j$ th row of  $A$ , then multiplying by  $e_k$  extracts the  $k$ th component of that row:  $e_j^\top A e_k = a_{jk}$ .
- $e_{ii} A e_{jj}$ : using previous results,  $A e_{jj}$  has the effect of zeroing out  $A$  except for the  $j$ th column. Multiplying this by  $e_{ii}$  on the left zeros out everything but the  $i$ th row, which leaves only  $a_{ij}$  and zeros everywhere else:  $e_{ii} A e_{jj} = a_{ij} e_{ij}$ .
- $e_{ij} A e_{k\ell}$ : the product  $A e_{k\ell}$  is a matrix with  $a_{k\ell}$  in the  $\ell$ th column and zeros everywhere else. Multiplying this by  $e_{ij}$  on the left takes the  $j$ th component of this column,  $a_{jk}$ , and places it in the  $i$ th row of the product, with zeros everywhere else. Therefore we have only  $a_{jk}$  in the  $(i, \ell)$  entry, or  $a_{jk} e_{i\ell}$ .

## Exercises from 1st edition

**1.1.16** A square matrix  $A$  is called *nilpotent* if  $A^k = 0$  for some  $k > 0$ . Prove that if  $A$  is nilpotent then  $I + A$  is invertible.

### Discussion

If  $I + A$  is invertible then its inverse is some matrix  $(I + A)^{-1}$ . If we consider this a function of  $A$  then we find that it is infinitely differentiable and we might attempt to evaluate its Maclaurin series. Note the resemblance to a geometric series:

$$f(A) = (I + A)^{-1} = I - A + A^2 - A^3 + \dots$$

Since  $A$  is nilpotent all powers of  $A$  from  $k$  onwards are 0, so this is really a finite sum:

$$(I + A)^{-1} = I - A + A^2 - \dots (-1)^{k-1} A^{k-1}$$

We'll assume Artin meant that  $k$  is the *lowest* integer such that  $A^k = 0$ .

### Proof

We check if it inverts  $I + A$ , and indeed it does:

$$\begin{aligned} & (I + A)(I - A + A^2 - \dots (-1)^{k-1} A^{k-1}) \\ &= (I(I - A + A^2 - \dots (-1)^{k-1} A^{k-1})) + (A(I - A + A^2 - \dots (-1)^{k-1} A^{k-1})). \end{aligned}$$

$$\begin{aligned}
&= I + A - A + A^2 - A^2 + \dots (-1)^{k-1} A^{k-1} + (-1)^{k-2} A^{k-1} + 0 \\
&= I
\end{aligned}$$

which confirms that  $I + A - A^2 + \dots (-1)^{k-1} A^{k-1}$  is indeed the inverse of  $I + A$ .