Chapter 1

Notation: I use $\mathbf{a}_i$ to denote the ith column of matrix A, and $\mathbf{a}_j$ to denote the jth row.

1.7 Find a formula for $\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^n$, and prove it by induction.

Discussion. Call this matrix A, and consider what A does to B in the product BA: A places B's first column in the product's first column, B's column 1 + B's column 2 in the product's second column, and the sum of B's 3 columns in the product's third column.

$$BA = \begin{bmatrix} | & | & | \\ \mathbf{b}_1 & \mathbf{b}_1 + \mathbf{b}_2 & \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3 \\ | & | & | \end{bmatrix}$$

$$A^2 = AA = \begin{bmatrix} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_1 + \mathbf{a}_2 & \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 \\ | & | & | \end{bmatrix}$$

If the left matrix is itself A, we can guess that repeated application of A on the right will leave column 1 unchanged, so we suppose the first column of $A^n = (1, 0, 0)$. If this is correct, each right-multiplication by A adds $(1, 0, 0)$ to $(1, 1, 0)$, making column 2 of $A^n = (n, 1, 0)$. For column 3 of A^2 we add $(1, 0, 0) + (1, 1, 0) + (1, 1, 1) = (3, 2, 1)$. For column 3 of A^3 we add $(1, 0, 0) + (2, 1, 0) + (3, 2, 1)$, and we suspect that in A^n's 3rd column, the first entry is the sum $1 + 2 + \ldots + n$. So we guess the third column is $(\sum_{k=1}^n k, n, 1)$. The sum of the first k natural numbers has a well-known closed form, $n(n+1)/2$, so we can now claim:

$$A^n = \begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ & 1 & n \\ & & 1 \end{bmatrix}$$

Proof by induction.

For the base case we may use $n = 1$. Then $\frac{n(n+1)}{2} = 1(2)/2 = 1$ and the base case holds. Assuming this holds for some n, for the inductive step we would have:

$$\begin{aligned} A^{n+1} &= A^n A && \text{associativity} \\ &= \begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ & 1 & n \\ & & 1 \end{bmatrix} A \\ &= \begin{bmatrix} 1 & n+1 & \frac{n(n+1)}{2} + n + 1 \\ & 1 & n+1 \\ & & 1 \end{bmatrix} \end{aligned}$$

The diagonal and superdiagonal are as we claimed. The top-right entry can be rewritten as:

$$\frac{n(n+1)}{2} + n + 1 = \frac{n(n+1) + 2n + 2}{2} = \frac{n^2 + 3n + 2}{2} = \frac{(n+1)(n+2)}{2}$$

which is also as claimed. Thus the proof is complete.

1.14 Find infinitely many matrices B such that $BA = I_2$ when

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$$

and prove there is no matrix C such that $AC = I_3$.

Discussion. To find infinitely many solutions, we can find a vector that when added to B doesn't change the product BA. By the row picture of matrix multiplication, the rows of B specify linear combinations A rows to make the product BA. So we'll look for a linear combination of A rows that sum to $\mathbf{0}$. Then any multiple of this linear combination will also be $\mathbf{0}$, and can be added to B without changing BA. Technically, we seek a vector in A's left null space.

Consider the row space of A: there are three vectors in a 2-dimensional space so they must be linearly dependent. In fact we might see that the first row is the sum of the other two, and so $\mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3 = \mathbf{0}$. Therefore the row vector $[1 \ -1 \ -1]$ is in A's left null space:

$$\lambda \begin{bmatrix} 1 & -1 & -1 \end{bmatrix} A = \mathbf{0}$$

Let $N = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$, stacking this vector. Then:

$$NA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } (B + \lambda N)A = BA + \lambda NA = BA + 0 = BA$$

Next we seek a matrix $\tilde{B}$ such that $\tilde{B}A = I_2$. We could ignore A's third row and invert the 2-by-2 matrix $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, its inverse is $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$. To make it the right size we append a column of zeros to get:

$$\tilde{B} = \begin{bmatrix} 2 & -3 & 0 \\ -1 & 2 & 0 \end{bmatrix}, \quad \tilde{B}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Now we can let $B = \tilde{B} + \lambda N$ for any choice of scalar λ, and we'll have:

$$BA = (\tilde{B} + \lambda N)A = \tilde{B}A = I_2$$

To prove that there A has no right-inverse C, observe that A has 2 linearly independent columns and a 3-dimensional codomain. By rank-nullity, A is not full-rank and cannot produce I_3.

1.15 With A arbitrary, determine the products $e_{ij}A, Ae_{ij}, e_jAe_k, e_{ii}Ae_{jj}$, and $e_{ij}Ae_{k\ell}$

Discussion. Note that Artin defines e_{ij} as a 'unit matrix', a matrix with a 1 at the (i, j) coordinate and 0 elsewhere.

- $e_{ij}A$: since e_{ij} is multiplying A on the left, it is making linear combinations of A's rows. Since it only has a 1 in column j it extracts the jth row of A and puts in in the ith row of the product, which is 0 everywhere else.

$$e_{ij}A = \begin{array}{cc} & \begin{array}{ccc} & j & \end{array} \\ i & \begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix} \end{array} \begin{bmatrix} — & \mathbf{a}_1 & — \\ & \vdots & \\ — & \mathbf{a}_j & — \\ & \vdots & \\ — & \mathbf{a}_n & — \end{bmatrix} = \begin{array}{cc} i & \begin{bmatrix} & & \\ — & \mathbf{a}_j & — \\ & & \end{bmatrix} \end{array}$$

- Ae_{ij}: Note that the jth column of e_{ij} is also the ith basis vector e_i. Multiplying A by a basis vector e_i simply extracts the ith column from A: $Ae_i = \mathbf{a}_i$. Next, consider that in e_{ij} there are $j-1$ columns of $\mathbf{0}$ before the $\mathbf{e}_j$ column. So Ae_{ij} is a matrix with $\mathbf{a}_i$ in the jth column and 0's everywhere else.

$$Ae_j = \begin{array}{cc} & \begin{bmatrix} | \\ e_i \\ | \end{bmatrix} \\ \begin{bmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_n \\ | & & | \end{bmatrix} & \begin{bmatrix} | \\ \mathbf{a}_i \\ | \end{bmatrix} \end{array}$$

$$Ae_{ij} = \begin{array}{cc} & \begin{array}{ccccc} & & j & & \end{array} \\ & \begin{bmatrix} & | & | & | & \\ \dots & \mathbf{0} & e_i & \mathbf{0} & \dots \\ & | & | & | & \end{bmatrix} \\ \begin{bmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_n \\ | & & | \end{bmatrix} & \begin{bmatrix} & | & | & | & \\ \dots & \mathbf{0} & \mathbf{a}_i & \mathbf{0} & \dots \\ & | & | & | & \end{bmatrix} \end{array}$$

Alternatively, we can write $Ae_{ij} = (e_{ij}^\top A^\top)^\top = (e_{ji}A^\top)^\top$ and use the previous result: $e_{ji}A^\top$ places the ith row of $A^\top$ in the jth row of the product, then the transpose $(e_{ji}A^\top)^\top$ has the ith *row* of A in the jth *column* (and zeros everywhere else).

- $e_j A e_k$: the product $e_j A$ is not defined for an arbitrary A. Suppose e_j is $m \times 1$, then for $e_j A$ to be defined A must have the shape $1 \times n$. Then e_k must be $n \times 1$ for the product $e_j A e_k$ to be defined. It would be very unconventional for e_j and e_k to have different shapes, but nevertheless Ae_k would be the single entry matrix $[a_{1k}]$, which then multiplies e_j like a scalar to produce $a_{1k} e_j$.

  In case Artin intended the product to read $e_j^\top A e_k$, the answer would be that $e_j^\top A$ extracts the jth row of A, then multiplying by e_k extracts the kth component of that row: $e_j^\top A e_k = a_{jk}$
- $e_{ii} A e_{jj}$: using previous results, Ae_{jj} has the effect of zeroing out A except for the jth column. Multiplying this by e_{ii} on the left zeros out everything but the ith row, which leaves only a_{ij} and zeros everywhere else: $e_{ii} A e_{jj} = a_{ij} e_{ij}$.
- $e_{ij} A e_{k\ell}$: the product $Ae_{k\ell}$ is a matrix with $\mathbf{a}_k$ in the ℓth column and zeros everywhere else. Multiplying this by e_{ij} on the left takes the jth component of this column, a_{jk}, and places it in the ith row of the product, with zeros everywhere else. Therefore we have only a_{jk} in the (i, ℓ) entry, or $a_{jk} e_{i\ell}$.

Exercises from 1st edition

1.1.16 A square matrix A is called *nilpotent* if $A^k = 0$ for some $k > 0$. Prove that if A is nilpotent then $I + A$ is invertible.

Discussion

If $I + A$ is invertible then its inverse is some matrix $(I + A)^{-1}$. If we consider this a function of A then we find that it is infinitely differentiable and we might attempt to evaluate its Maclaurin series. Note the resemblance to a geometric series:

$$f(A) = (I + A)^{-1} = I - A + A^2 - A^3 + \dots$$

Since A is nilpotent all powers of A from k onwards are 0, so this is really a finite sum:

$$(I + A)^{-1} = I - A + A^2 - \dots (-1)^{k-1} A^{k-1}$$

We'll assume Artin meant that k is the *lowest* integer such that $A^k = \mathbf{0}$.

Proof

We check if it inverts $I + A$, and indeed it does:

$$(I + A)(I - A + A^2 - \dots (-1)^{k-1} A^{k-1})$$

$$= \left(I(I - A + A^2 - \dots (-1)^{k-1} A^{k-1})\right) + \left(A(I - A + A^2 - \dots (-1)^{k-1} A^{k-1})\right).$$

By distributing $I + A$ over $(I + A)^{-1}$, we get a 'copy' of $(I + A)^{-1}$ added to a copy of $(I + A)^{-1}$ with its terms raised one power (considering I as A^0):

$$(I - A + A^2 - \dots (-1)^{k-1} A^{k-1}) + (A - A^2 + \dots (-1)^{k-1} A^k)$$

This means for $A, \dots, A^{k-1}$ in the first term we have $A, \dots, A^{k-1}$ in the second term with the opposite sign. Therefore, everything cancels out except I and A^k, which is 0. So the result of this product is I, which confirms that $I + A - A^2 + \dots (-1)^{k-1} A^{k-1}$ is indeed the inverse of $I + A$.