

Subfiles package example

Overleaf

1 Introduction

Chapter 1

1.1 a

1.2 a

1.3 b

1.4 c

1.5 d

1.6 e

1.7 Find a formula for $\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^n$, and prove it by induction.

Discussion. Call this matrix A . The action of multiplying any matrix B by A on the right will yield a product whose first column is the first column of B , second column is the sum of the first two columns, and whose third column is the sum of all three columns:

$$BA = \left[\begin{array}{c|c|c} | & | & | \\ \mathbf{b}_1 & \mathbf{b}_1 + \mathbf{b}_2 & \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3 \\ | & | & | \end{array} \right]$$

We can guess that repeated application of A on the right of A will have the effect of leaving column 1 unchanged. Since column 1 will always be $(1, 0, 0)$, each right-multiplication of A adds $(1, 0, 0)$ to column 2 so we can guess that column 2 of $A^n = (n, 1, 0)$. Column 3 of A^n we can guess is the sum of the first n numbers, n , then 1: $(\sum_{k=1}^n k, n, 1)$. The sum of the first k natural numbers has a well-known closed form, $n(n+1)/2$, so we can now claim:

$$A^n = \begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ & 1 & n \\ & & 1 \end{bmatrix}$$

Proof by induction.

For the base case we may use $n = 1$. Then $\frac{n(n+1)}{2} = 1(2)/2 = 1$ and the base case holds. Assuming this holds for some n , for the inductive step we would have:

$$\begin{aligned} A^{n+1} &= A^n A && \text{associativity} \\ &= \begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ & 1 & n \\ & & 1 \end{bmatrix} A \\ &= \begin{bmatrix} 1 & n+1 & \frac{n(n+1)}{2} + n+1 \\ & 1 & n+1 \\ & & 1 \end{bmatrix} \end{aligned}$$

The diagonal and superdiagonal are as we claimed. The top-right entry can be rewritten as:

$$\frac{n(n+1)}{2} + n+1 = \frac{n(n+1) + 2n+2}{2} = \frac{n^2 + 3n + 2}{2} = \frac{(n+1)(n+2)}{2}$$

which is also as claimed. Thus the proof is complete.

1.8 x

1.9 x

1.10 x

1.11 x

1.12 x

1.13 x

1.14 Find infinitely many matrices B such that $BA = I_2$ when

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$$

and prove there is no matrix C such that $AC = I_3$.

Discussion. To find infinitely many solutions, we'll probably need to find a vector that when added to B doesn't change the product AB . By the row picture of matrix multiplication, the rows of B denote linear combinations of the rows of A to form the product BA . So we'll look for linear combination of A rows that sum to $\mathbf{0}$, then any multiple of this linear combination will also be $\mathbf{0}$, and can be added to B without changing BA . Technically, we seek a vector in A 's left null space.

Consider the row space of A : there are three vectors in a 2-dimensional space so they must be linearly dependent. In fact we might see that the first row is the sum of the other two, and so $\mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3 = \mathbf{0}$. Therefore the row vector $[1 \ -1 \ -1]$ is in A 's left null space:

$$\lambda [1 \ -1 \ -1] A = \mathbf{0}$$

Let $N = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$, stacking this vector. Then:

$$NA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and by linearity, $(B + \lambda N)A = BA$ for any λ .

Next we seek a matrix \tilde{B} such that $\tilde{B}A = I_2$. We could ignore A 's third row and invert the 2-by-2 matrix $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, its inverse is $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$. To make it the right size we append a column of zeros to get:

$$\tilde{B} = \begin{bmatrix} 2 & -3 & 0 \\ -1 & 2 & 0 \end{bmatrix}, \quad \tilde{B}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Now we can let $B = \tilde{B} + \lambda N$ for any choice of scalar λ , and we'll have:

$$BA = (\tilde{B} + \lambda N)A = \tilde{B}A = I_2$$

To prove that there A has no right-inverse C , observe that A has 2 linearly independent columns and a 3-dimensional codomain. By rank-nullity, A is not full-rank and cannot produce I_3 .

1.15 With A arbitrary, determine the products $e_{ij}A$, Ae_{ij} , e_jAe_k , $e_{ii}Ae_{jj}$, and $e_{ij}Ae_{kl}$

Discussion. Note that Artin defines e_{ij} as a 'unit matrix', a matrix with a 1 at the (i, j) coordinate and 0 elsewhere.

- $e_{ij}A$: since e_{ij} is multiplying A on the left, it is making linear combinations of A 's rows. Since it only has a 1 in column j it extracts the j th row of A and puts in in the i th row of the product, which is 0 everywhere else.

$$\text{Mat}_{\varphi \text{ to } M} = c_1c_2c_3c_4c_5r_11111r_201001r_300101r_400011r_500001$$

$$e_{ij}A = \begin{bmatrix} & 1 & \\ & & \end{bmatrix} \begin{bmatrix} - & \mathbf{a}_1 & - \\ & \vdots & \\ - & \mathbf{a}_j & - \\ & \vdots & \\ - & \mathbf{a}_n & - \end{bmatrix}$$