Chapter 1

1.1 a

1.2 a

1.3 b

1.4 c

1.5 d

1.6 e

1.7 Find a formula for $\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^n$, and prove it by induction.

Discussion. Call this matrix A. The action of multiplying any matrix B by A on the right will yield a product whose first column is the first column of B, second column is the sum of the first two columns, and whose third column is the sum of all three columns:

$$BA = \begin{bmatrix} | & | & | \\ \mathbf{b}_1 & \mathbf{b}_1 + \mathbf{b}_2 & \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3 \\ | & | & | \end{bmatrix}$$

We can guess that repeated application of A on the right of A will have the effect of leaving column 1 unchanged. Since column 1 will always be $(1, 0, 0)$, each right-multiplication of A adds $(1, 0, 0)$ to column 2 so we can guess that column 2 of $A^n = (n, 1, 0)$. Column 3 of A^n we can guess is the sum of the first n numbers, n, then 1: $(\sum_{k=1}^{n} k, n, 1)$. The sum of the first k natural numbers has a well-known closed form, $n(n+1)/2$, so we can now claim:

$$A^n = \begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ & 1 & n \\ & & 1 \end{bmatrix}$$

Proof by induction.

For the base case we may use $n = 1$. Then $\frac{n(n+1)}{2} = 1(2)/2 = 1$ and the base case holds. Assuming this holds for some n, for the inductive step we would have:

$$\begin{aligned} A^{n+1} &= A^n A && \text{associativity} \\ &= \begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ & 1 & n \\ & & 1 \end{bmatrix} A \\ &= \begin{bmatrix} 1 & n+1 & \frac{n(n+1)}{2} + n + 1 \\ & 1 & n+1 \\ & & 1 \end{bmatrix} \end{aligned}$$

The diagonal and superdiagonal are as we claimed. The top-right entry can be rewritten as:

$$\frac{n(n+1)}{2} + n + 1 = \frac{n(n+1) + 2n + 2}{2} = \frac{n^2 + 3n + 2}{2} = \frac{(n+1)(n+2)}{2}$$

which is also as claimed. Thus the proof is complete.

1.8 x

1.9 x

1.10 x

1.11 x

1.12 x

1.13 x

1.14 Find infinitely many matrices B such that $BA = I_2$ when

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$$

and prove there is no matrix C such that $AC = I_3$.

Discussion. To find infinitely many solutions, we'll probably need to find a vector that when added to B doesn't change the product AB. By the row picture of matrix multiplication, the rows of B denote linear combinations of the rows of A to form the product BA. So we'll look for linear combination of A rows that sum to $\mathbf{0}$, then any multiple of this linear combination will also be $\mathbf{0}$, and can be added to B without changing BA. Technically, we seek a vector in A's left null space.

Consider the row space of A: there are three vectors in a 2-dimensional space so they must be linearly dependent. In fact we might see that the first row is the sum of the other two, and so $\mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3 = \mathbf{0}$. Therefore the row vector $[1\ -1\ -1]$ is in A's left null space:

$$\lambda \begin{bmatrix} 1 & -1 & -1 \end{bmatrix} A = \mathbf{0}$$

Let $N = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$, stacking this vector. Then:

$$NA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and by linearity, $(B + \lambda N)A = BA$ for any λ.

Next we seek a matrix $\tilde{B}$ such that $\tilde{B}A = I_2$. We could ignore A's third row and invert the 2-by-2 matrix $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, its inverse is $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$. To make it the right size we append a column of zeros to get:

$$\tilde{B} = \begin{bmatrix} 2 & -3 & 0 \\ -1 & 2 & 0 \end{bmatrix}, \quad \tilde{B}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Now we can let $B = \tilde{B} + \lambda N$ for any choice of scalar λ, and we'll have:

$$BA = (\tilde{B} + \lambda N)A = \tilde{B}A = I_2$$

To prove that there A has no right-inverse C, observe that A has 2 linearly independent columns and a 3-dimensional codomain. By rank-nullity, A is not full-rank and cannot produce I_3.

1.15 With A arbitrary, determine the products $e_{ij}A, Ae_{ij}, e_jAe_k, e_{ii}Ae_{jj}$, and $e_{ij}Ae_{k\ell}$

Discussion. Note that Artin defines e_{ij} as a 'unit matrix', a matrix with a 1 at the (i, j) coordinate and 0 elsewhere.

- $e_{ij}A$: since e_{ij} is multiplying A on the left, it is making linear combinations of A's rows. Since it only has a 1 in column j it extracts the jth row of A and puts in in the ith row of the product, which is 0 everywhere else.

$$e_{ij}A = \begin{array}{cc} & \begin{array}{ccc} & j & \end{array} \\ i & \begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix} \end{array} \begin{bmatrix} \text{—} & \mathbf{a}_1 & \text{—} \\ & \vdots & \\ \text{—} & \mathbf{a}_j & \text{—} \\ & \vdots & \\ \text{—} & \mathbf{a}_n & \text{—} \end{bmatrix} = \begin{array}{cc} i & \begin{bmatrix} \text{—} & \mathbf{a}_j & \text{—} \end{bmatrix} \end{array}$$

- Ae_{ij}: Note that the jth column of e_{ij} is also the jth basis vector $\mathbf{e}_j$, with a 1 on the jth component and 0's everywhere else. $A\mathbf{e}_j$ returns $\mathbf{a}_j$, the jth column of A. Then consider that in e_{ij} there are $i-1$ columns of $\mathbf{0}$ before the $\mathbf{e}_j$ column. So Ae_{ij} is a matrix with $\mathbf{a}_j$ in the ith column and 0's everywhere else.
  Alternatively, we can write $Ae_{ij} = (e_{ij}^\top A^\top)^\top = (e_{ji}A^\top)^\top$, and using the previous result we have $e_{ji}A^\top$ is the jth row of $A^\top$ in the ith row of the product, then transposing this again we get the product is the jth column of A in the ith column (and zeros everywhere else).
- e_jAe_k: the product e_jA is not defined since e_j is an $n \times 1$ column vector and A is an $n \times n$ matrix. Likewise, it does no good to try to compute Ae_k first asince the result is another $n \times 1$ column vector.

- $e_{ii}Ae_jj$: from our previous result we know Ae_{jj} is a matrix with $mathbfa_j$ in place and zeros everywhere else. Multiplying this by e_{ii} takes the ith component of $mathbfa_j$ and puts in in the ith row and column of the product. We get a matrix with a_{ij} in place and zeros everywhere else, i.e. $a_{ij}e_{ij}$.
- $e_{ij}Ae_{k\ell}$: the product $Ae_{k\ell}$ is a matrix with $\mathbf{a}_k$ in the ℓth column and zeros everywhere else. Multiplying this by e_{ij} take the jth component of this column, a_{jk}, and places it in the ith row of the product, with zeros everywhere else. Therefore we have only a_{jk} in the (i, ℓ) entry, or $a_{jk}e_{i\ell}$.