

Chapter 1 Notation: I use \mathbf{a}_i to denote the i th column of matrix A , and \mathbf{a}_j to denote the j th row.

1.1 a

1.2 a

1.3 b

1.4 c

1.5 d

1.6 \mathbf{a}_j

1.7 Find a formula for $\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^n$, and prove it by induction.

Discussion. Call this matrix A , and consider what A does to B in the product BA : A places B 's first column in the product's first column, B 's column 1 + B 's column 2 in the product's second column, and the sum of B 's 3 columns in the product's third column.

$$BA = \begin{bmatrix} | & | & | \\ \mathbf{b}_1 & \mathbf{b}_1 + \mathbf{b}_2 & \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3 \\ | & | & | \end{bmatrix}$$

$$A^2 = AA = \begin{bmatrix} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_1 + \mathbf{a}_2 & \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 \\ | & | & | \end{bmatrix}$$

If the left matrix is itself A , we can guess that repeated application of A on the right will leave column 1 unchanged, so we suppose the first column of $A^n = (1, 0, 0)$. If this is correct, each right-multiplication by A adds $(1, 0, 0)$ to $(1, 1, 0)$, making column 2 of $A^n = (n, 1, 0)$. For column 3 of A^2 we add $(1, 0, 0) + (1, 1, 0) + (1, 1, 1) = (3, 2, 1)$. For column 3 of A^3 we add $(1, 0, 0) + (2, 1, 0) + (3, 2, 1)$, and we suspect that in A^n 's 3rd column, the first entry is the sum $1 + 2 + \dots + n$. So we guess the third column is $(\sum_{k=1}^n k, n, 1)$. The sum of the first k natural numbers has a well-known closed form, $n(n+1)/2$, so we can now claim:

$$A^n = \begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ & 1 & n \\ & & 1 \end{bmatrix}$$

Proof by induction.

For the base case we may use $n = 1$. Then $\frac{n(n+1)}{2} = 1(2)/2 = 1$ and the base case holds. Assuming this holds for some n , for the inductive step we would have:

$$\begin{aligned} A^{n+1} &= A^n A && \text{associativity} \\ &= \begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ & 1 & n \\ & & 1 \end{bmatrix} A \\ &= \begin{bmatrix} 1 & n+1 & \frac{n(n+1)}{2} + n+1 \\ & 1 & n+1 \\ & & 1 \end{bmatrix} \end{aligned}$$

The diagonal and superdiagonal are as we claimed. The top-right entry can be rewritten as:

$$\frac{n(n+1)}{2} + n+1 = \frac{n(n+1) + 2n+2}{2} = \frac{n^2 + 3n + 2}{2} = \frac{(n+1)(n+2)}{2}$$

which is also as claimed. Thus the proof is complete.

1.8 x

1.9 x

1.10 x

1.11 x

1.12 x

1.13 x

1.14 Find infinitely many matrices B such that $BA = I_2$ when

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$$

and prove there is no matrix C such that $AC = I_3$.

Discussion. To find infinitely many solutions, we can find a vector that when added to B doesn't change the product BA . By the row picture of matrix multiplication, the rows of B specify linear combinations of rows of A to make the product BA . So we'll look for a linear combination of rows of A that sum to $\mathbf{0}$. Then any multiple of this linear combination will also be $\mathbf{0}$, and can be added to B without changing BA . Technically, we seek a vector in A 's left null space.

Consider the row space of A : there are three vectors in a 2-dimensional space so they must be linearly dependent. In fact we might see that the

first row is the sum of the other two, and so $\mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3 = \mathbf{0}$. Therefore the row vector $[1 \ -1 \ -1]$ is in A 's left null space:

$$\lambda [1 \ -1 \ -1] A = \mathbf{0}$$

Let $N = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$, stacking this vector. Then:

$$NA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } (B + \lambda N)A = BA + \lambda NA = BA + 0 = BA$$

Next we seek a matrix \tilde{B} such that $\tilde{B}A = I_2$. We could ignore A 's third row and invert the 2-by-2 matrix $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, its inverse is $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$. To make it the right size we append a column of zeros to get:

$$\tilde{B} = \begin{bmatrix} 2 & -3 & 0 \\ -1 & 2 & 0 \end{bmatrix}, \quad \tilde{B}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Now we can let $B = \tilde{B} + \lambda N$ for any choice of scalar λ , and we'll have:

$$BA = (\tilde{B} + \lambda N)A = \tilde{B}A = I_2$$

To prove that there A has no right-inverse C , observe that A has 2 linearly independent columns and a 3-dimensional codomain. By rank-nullity, A is not full-rank and cannot produce I_3 .

1.15 With A arbitrary, determine the products $e_{ij}A$, Ae_{ij} , e_jAe_k , $e_{ii}Ae_{jj}$, and $e_{ij}Ae_{kl}$

Discussion. Note that Artin defines e_{ij} as a 'unit matrix', a matrix with a 1 at the (i, j) coordinate and 0 elsewhere.

- $e_{ij}A$: since e_{ij} is multiplying A on the left, it is making linear combinations of A 's rows. Since it only has a 1 in column j it extracts the j th row of A and puts in in the i th row of the product, which is 0 everywhere else.

$$e_{ij}A = \begin{matrix} & & j \\ i & \begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix} \end{matrix} \begin{bmatrix} - & \mathbf{a}_1 & - \\ & \vdots & \\ - & \mathbf{a}_j & - \\ & \vdots & \\ - & \mathbf{a}_n & - \end{bmatrix} = \begin{matrix} & & \\ i & \begin{bmatrix} - & \mathbf{a}_j & - \end{bmatrix} \end{matrix}$$

- Ae_{ij} : Note that the j th column of e_{ij} is also the i th basis vector e_i . Multiplying A by a basis vector e_i simply extracts the i th column from A : $Ae_i = \mathbf{a}_i$. Next, consider that in e_{ij} there are $j - 1$ columns of $\mathbf{0}$ before the \mathbf{e}_j column. So Ae_{ij} is a matrix with \mathbf{a}_i in the j th column and 0's everywhere else.

$$\begin{aligned}
 Ae_j &= \begin{bmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_n \\ | & & | \end{bmatrix} \begin{bmatrix} | \\ e_i \\ | \end{bmatrix} \\
 Ae_{ij} &= \begin{bmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_n \\ | & & | \end{bmatrix} \begin{bmatrix} | & & | \\ \dots & \mathbf{0} & e_i & \mathbf{0} & \dots \\ | & & | & & | \end{bmatrix}
 \end{aligned}$$

Alternatively, we can write $Ae_{ij} = (e_{ij}^\top A^\top)^\top = (e_{ji} A^\top)^\top$ and use the previous result: $e_{ji} A^\top$ places the i th row of A^\top in the j th row of the product, then the transpose $(e_{ji} A^\top)^\top$ has the i th row of A in the j th column (and zeros everywhere else).

- $e_j A e_k$: the product $e_j A$ is not defined for an arbitrary A . Suppose e_j is $m \times 1$, then for $e_j A$ to be defined A must have the shape $1 \times n$. Then e_k must be $n \times 1$ for the product $e_j A e_k$ to be defined. It would be very unconventional for e_j and e_k to have different shapes, but nevertheless $A e_k$ would be the single entry matrix $[a_{1k}]$, which then multiplies e_j like a scalar to produce $a_{1k} e_j$.

In case Artin intended the product to read $e_j^\top A e_k$, the answer would be that $e_j^\top A$ extracts the j th row of A , then multiplying by e_k extracts the k th component of that row: $e_j^\top A e_k = a_{jk}$

- $e_{ii} A e_{jj}$: using previous results, $A e_{jj}$ has the effect of zeroing out A except for the j th column. Multiplying this by e_{ii} on the left zeros out everything but the i th row, which leaves only a_{ij} and zeros everywhere else: $e_{ii} A e_{jj} = a_{ij} e_{ij}$.
- $e_{ij} A e_{k\ell}$: the product $A e_{k\ell}$ is a matrix with \mathbf{a}_k in the ℓ th column and zeros everywhere else. Multiplying this by e_{ij} on the left takes the j th component of this column, a_{jk} , and places it in the i th row of the product, with zeros everywhere else. Therefore we have only a_{jk} in the (i, ℓ) entry, or $a_{jk} e_{i\ell}$.

Exercises from 1st edition

1.1.16 A square matrix A is called *nilpotent* if $A^k = 0$ for some $k > 0$. Prove that if A is nilpotent then $I + A$ is invertible.

Discussion. If $I + A$ is invertible then its inverse is some matrix $(I + A)^{-1}$. Since I and A commute with each other and are differentiable, we can rewrite $(I + A)^{-1}$ using the generalized binomial theorem (proof of the theorem depends on the exponent being real-valued, which it is, and $I + A$ being infinitely differentiable, which it is). In fact the result resembles a geometric series:

$$(I + A)^{-1} = I - A + A^2 - A^3 + \dots$$

Since A is nilpotent all powers of A from k onwards are 0, so this is really a finite sum:

$$(I + A)^{-1} = I - A + A^2 - \dots (-1)^{k-1} A^{k-1}$$

We'll assume Artin meant that k is the *lowest* integer such that $A^k = \mathbf{0}$. We check if it inverts $I + A$, and indeed it does.

Proof.

$$\begin{aligned} (I + A)(I + A)^{-1} &= (I + A)(I - A + A^2 - \dots (-1)^{k-1} A^{k-1}) \\ &= (I - A + A^2 - \dots (-1)^{k-1} A^{k-1}) + A(I - A + A^2 - \dots (-1)^{k-1} A^{k-1}) \\ &= (I - A + A^2 - \dots (-1)^{k-1} A^{k-1}) + (A - A^2 + A^3 - \dots (-1)^{k-1} A^k) \\ &= I + (A - A) + (A^2 - A^2) - \dots ((-1)^{k-1} A^{k-1} + (-1)^{k-2} A^{k-1}) + 0 \\ &= I \end{aligned}$$

Here we see that after distributing, all the A powers cancel out leaving only I , therefore $B = I - A + A^2 - \dots (-1)^{k-1} A^{k-1}$ is an inverse of $I + A$.