# Chapter 1

Notation: I use $\mathbf{a}_i$ to denote the $i$th column of matrix $A$, and $\mathbf{a}_j$ to denote the $j$th row.

1.7 Find a formula for $\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^n$, and prove it by induction.

**Discussion.** Call this matrix $A$, and consider what $A$ does to $B$ in the product $BA$: $A$ places $B$'s first column in the product's first column, $B$'s column 1 + $B$'s column 2 in the product's second column, and the sum of $B$'s 3 columns in the product's third column.

$$BA = \begin{bmatrix} | & | & | \\ \mathbf{b}_1 & \mathbf{b}_1 + \mathbf{b}_2 & \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3 \\ | & | & | \end{bmatrix}$$

$$A^2 = AA = \begin{bmatrix} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_1 + \mathbf{a}_2 & \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 \\ | & | & | \end{bmatrix}$$

If the left matrix is itself $A$, we can guess that repeated application of $A$ on the right will leave column 1 unchanged, so we suppose the first column of $A^n = (1,0,0)$. If this is correct, each right-multiplication by $A$ adds $(1,0,0)$ to $(1,1,0)$, making column 2 of $A^n = (n,1,0)$. For column 3 of $A^2$ we add $(1,0,0)+(1,1,0)+(1,1,1) = (3,2,1)$. For column 3 of $A^3$ we add $(1,0,0)+(2,1,0)+(3,2,1)$, and we suspect that in $A^n$'s 3rd column, the first entry is the sum $1+2+\ldots+n$. So we guess the third column is $(\sum_{k=1}^{n} k, n, 1)$. The sum of the first k natural numbers has a well-known closed form, $n(n+1)/2$, so we can now claim:

$$A^n = \begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ & 1 & n \\ & & 1 \end{bmatrix}$$

**Proof by induction.**

For the base case we may use $n = 1$. Then $\frac{n(n+1)}{2} = 1(2)/2 = 1$ and the base case holds. Assuming this holds for some $n$, for the inductive step we would have:

$$\begin{aligned} A^{n+1} &= A^n A && \text{associativity} \\ &= \begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ & 1 & n \\ & & 1 \end{bmatrix} A \\ &= \begin{bmatrix} 1 & n+1 & \frac{n(n+1)}{2} + n + 1 \\ & 1 & n+1 \\ & & 1 \end{bmatrix} \end{aligned}$$

The diagonal and superdiagonal are as we claimed. The top-right entry can be rewritten as:

$$\frac{n(n+1)}{2}+n+1=\frac{n(n+1)+2n+2}{2}=\frac{n^2+3n+2}{2}=\frac{(n+1)(n+2)}{2}$$

which is also as claimed. Thus the proof is complete.

1.14 Find infinitely many matrices $B$ such that $BA = I_2$ when

$$A=\begin{bmatrix}2&3\\1&2\\1&1\end{bmatrix}$$

and prove there is no matrix $C$ such that $AC = I_3$.

**Discussion.** To find infinitely many solutions, we can find a vector that when added to $B$ doesn't change the product $BA$. By the row picture of matrix multiplication, the rows of $B$ specify linear combinations $A$ rows to make the product $BA$. So we'll look for a linear combination of $A$ rows that sum to $\mathbf{0}$. Then any multiple of this linear combination will also be $\mathbf{0}$, and can be added to $B$ without changing $BA$. Technically, we seek a vector in $A$'s left null space.

Consider the row space of $A$: there are three vectors in a 2-dimensional space so they must be linearly dependent. In fact we might see that the first row is the sum of the other two, and so $\mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3 = \mathbf{0}$. Therefore the row vector $[1\ -1\ -1]$ is in $A$'s left null space:

$$\lambda\begin{bmatrix}1&-1&-1\end{bmatrix}A=\mathbf{0}$$

Let $N=\begin{bmatrix}1&-1&-1\\1&-1&-1\end{bmatrix}$, stacking this vector. Then:

$$NA=\begin{bmatrix}0&0\\0&0\end{bmatrix},\ \text{and } (B+\lambda N)A = BA+\lambda NA = BA+0 = BA$$

Next we seek a matrix $\tilde{B}$ such that $\tilde{B}A = I_2$. We could ignore $A$'s third row and invert the 2-by-2 matrix $\begin{bmatrix}2&3\\1&2\end{bmatrix}$, its inverse is $\begin{bmatrix}2&-3\\-1&2\end{bmatrix}$. To make it the right size we append a column of zeros to get:

$$\tilde{B}=\begin{bmatrix}2&-3&0\\-1&2&0\end{bmatrix},\quad \tilde{B}A=\begin{bmatrix}1&0\\0&1\end{bmatrix}=I_2$$

Now we can let $B = \tilde{B} + \lambda N$ for any choice of scalar $\lambda$, and we'll have:

$$BA = (\tilde{B}+\lambda N)A = \tilde{B}A = I_2$$

To prove that there $A$ has no right-inverse $C$, observe that $A$ has 2 linearly independent columns and a 3-dimensional codomain. By rank-nullity, $A$ is not full-rank and cannot produce $I_3$.

1.15 With $A$ arbitrary, determine the products $e_{ij}A, Ae_{ij}, e_jAe_k, e_{ii}Ae_{jj}$, and $e_{ij}Ae_{k\ell}$

**Discussion.** Note that Artin defines $e_{ij}$ as a 'unit matrix', a matrix with a 1 at the $(i, j)$ coordinate and 0 elsewhere.

- $e_{ij}A$: since $e_{ij}$ is multiplying $A$ on the left, it is making linear combinations of $A$'s rows. Since it only has a 1 in column $j$ it extracts the $j$th row of $A$ and puts in in the $i$th row of the product, which is 0 everywhere else.

$$e_{ij}A = \begin{array}{cc} & \begin{array}{ccc} & j & \phantom{0} \end{array} \\ i & \begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix} \end{array} \begin{bmatrix} — & \mathbf{a}_1 & — \\ & \vdots & \\ — & \mathbf{a}_j & — \\ & \vdots & \\ — & \mathbf{a}_n & — \end{bmatrix} = \begin{array}{cc} i & \begin{bmatrix} — & \mathbf{a}_j & — \\ & & \end{bmatrix} \end{array}$$

- $Ae_{ij}$: Note that the $j$th column of $e_{ij}$ is also the $i$th basis vector $e_i$. Multiplying $A$ by a basis vector $e_i$ simply extracts the $i$th column from $A$: $Ae_i = \mathbf{a}_i$. Next, consider that in $e_{ij}$ there are $j - 1$ columns of $\mathbf{0}$ before the $\mathbf{e}_j$ column. So $Ae_{ij}$ is a matrix with $\mathbf{a}_i$ in the $j$th column and 0's everywhere else.

$$Ae_j = \begin{array}{cc} & \begin{bmatrix} | \\ e_i \\ | \end{bmatrix} \\ \begin{bmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_n \\ | & & | \end{bmatrix} & \begin{bmatrix} | \\ \mathbf{a}_i \\ | \end{bmatrix} \end{array}$$

$$Ae_{ij} = \begin{array}{cc} & \begin{array}{c} \begin{array}{ccccc} & & j & & \end{array} \\ \begin{bmatrix} & | & | & | & \\ \dots & \mathbf{0} & e_i & \mathbf{0} & \dots \\ & | & | & | & \end{bmatrix} \end{array} \\ \begin{bmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_n \\ | & & | \end{bmatrix} & \begin{bmatrix} & | & | & | & \\ \dots & \mathbf{0} & \mathbf{a}_i & \mathbf{0} & \dots \\ & | & | & | & \end{bmatrix} \end{array}$$

Alternatively, we can write $Ae_{ij} = (e_{ij}^\top A^\top)^\top = (e_{ji}A^\top)^\top$ and use the previous result: $e_{ji}A^\top$ places the $i$th row of $A^\top$ in the $j$th row of the product, then the transpose $(e_{ji}A^\top)^\top$ has the $i$th *row* of $A$ in the $j$th *column* (and zeros everywhere else).

- $e_j A e_k$: the product $e_j A$ is not defined for an arbitrary $A$. Suppose $e_j$ is $m \times 1$, then for $e_j A$ to be defined $A$ must have the shape $1 \times n$. Then $e_k$ must be $n \times 1$ for the product $e_j A e_k$ to be defined. It would be very unconventional for $e_j$ and $e_k$ to have different shapes, but nevertheless $Ae_k$ would be the single entry matrix $[a_{1k}]$, which then multiplies $e_j$ like a scalar to produce $a_{1k} e_j$.

  In case Artin intended the product to read $e_j^\top A e_k$, the answer would be that $e_j^\top A$ extracts the $j$th row of $A$, then multiplying by $e_k$ extracts the $k$th component of that row: $e_j^\top A e_k = a_{jk}$
- $e_{ii} A e_{jj}$: using previous results, $Ae_{jj}$ has the effect of zeroing out $A$ except for the $j$th column. Multiplying this by $e_{ii}$ on the left zeros out everything but the $i$th row, which leaves only $a_{ij}$ and zeros everywhere else: $e_{ii} A e_{jj} = a_{ij} e_{ij}$ .
- $e_{ij} A e_{k\ell}$: the product $Ae_{k\ell}$ is a matrix with $\mathbf{a}_k$ in the $\ell$th column and zeros everywhere else. Multiplying this by $e_{ij}$ on the left takes the $j$th component of this column, $a_{jk}$, and places it in the $i$th row of the product, with zeros everywhere else. Therefore we have only $a_{jk}$ in the $(i, \ell)$ entry, or $a_{jk} e_{i\ell}$.

## Exercises from 1st edition

**1.1.16** A square matrix $A$ is called *nilpotent* if $A^k = 0$ for some $k > 0$. Prove that if $A$ is nilpotent then $I + A$ is invertible.

### Discussion

If $I + A$ is invertible then its inverse is some matrix $(I + A)^{-1}$. If we consider this a function of $A$ then we find that it is infinitely differentiable and we might attempt to evaluate its Maclaurin series. Note the resemblance to a geometric series:

$$f(A) = (I + A)^{-1} = I - A + A^2 - A^3 + \dots$$

Since $A$ is nilpotent all powers of $A$ from $k$ onwards are 0, so this is really a finite sum:

$$(I + A)^{-1} = I - A + A^2 - \dots (-1)^{k-1} A^{k-1}$$

We'll assume Artin meant that $k$ is the *lowest* integer such that $A^k = \mathbf{0}$.

### Proof

We check if it inverts $I + A$, and indeed it does:

$$(I + A)(I - A + A^2 - \dots (-1)^{k-1} A^{k-1})$$

$$= \left(I(I - A + A^2 - \dots (-1)^{k-1} A^{k-1})\right) + \left(A(I - A + A^2 - \dots (-1)^{k-1} A^{k-1})\right).$$

$$= I + A - A + A^2 - A^2 + \dots (-1)^{k-1}A^{k-1} + (-1)^{k-2}A^{k-1} + 0$$
$$= I$$

which confirms that $I + A - A^2 + \dots (-1)^{k-1}A^{k-1}$ is indeed the inverse of $I + A$.