6. Show that if x is independent of y then their covariance is 0.

$$cov[x,y] = \mathbb{E}_{x,y}[xy] - \mathbb{E}[x]\mathbb{E}[y] = \mathbb{E}[x]\mathbb{E}[y] - \mathbb{E}[x]\mathbb{E}[y] = 0$$

where $\mathbb{E}_{x,y}[xy] = \mathbb{E}[x]\mathbb{E}[y]$ since x is independent of y.

7. We know the normalizing constant for the normal distribution is $1/\sqrt{2\pi\sigma^2}$ and this is why. Consider the integral of the density function

$$I = \int_{\mathcal{R}} \exp\left(-\frac{1}{2\sigma^2}x^2\right)\,dx$$

Not an easy integral. However, we can square it like so:

$$I^2 = \left(\int_{\mathcal{R}} \exp\left(-\frac{1}{2\sigma^2}x^2\right)\,dx\right)\left(\int_{\mathcal{R}} \exp\left(-\frac{1}{2\sigma^2}y^2\right)\,dy\right) = \int_{\mathcal{R}^2} \exp\left(-\frac{1}{2\sigma^2}(x^2+y^2)\right)\,dx\,dy$$

So instead of integrating over the real line we integrate over the plane. Now we can switch to polar coordinates, $r^2 = x^2 + y^2$, which will induce a Jacobean of r in the integrand:

$$I^2 = \int_0^{2\pi}\int_0^{\infty} \exp\left(-\frac{1}{2\sigma^2}r^2\right) r\;dr\,d\theta$$

Now we'll do a change of variable:

$$u = \frac{1}{2\sigma^2}r^2, \qquad \sigma^2 du = r\,dr$$

Checking the limits of integration, we have $u \to 0$ as $r \to 0$, and $u \to \infty$ as $r \to \infty$. So the integral over dr needn't change limits.

$$\implies I^2 = \sigma^2 \int_0^{2\pi}\int_0^{\infty} e^{-u}\,du\,d\theta$$

$$= \sigma^2 \int_0^{2\pi} d\theta \int_0^{\infty} e^{-u}du = \sigma^2(2\pi)\left[-e^{-u}\right]_0^{\infty} = 2\pi\sigma^2.$$

Knowing the density function is everywhere positive we can then take the principal square root of I^2:

$$I = \sqrt{I^2} = \sqrt{2\pi\sigma^2}$$