12.1 Functions

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Problem 1

Suppose $A = \{0, 1, 2, 3, 4\}, B = \{2, 3, 4, 5\}$ and $f = \{(0, 3), (1, 3), (2, 4), (3, 2), (4, 2)\}$. State the domain and range of f. Find $f(2)$ and $f(1)$.

The domain of f is the set of all first coordinates in f's set representation, which is A. The range of f is another term for the image, meaning the set of all second coordinates in f's set representation, which is $\{2, 3, 4\}$. Scanning through the set f we find $f(2) = 4, f(1) = 3$.

Problem 2

Suppose $A = \{a, b, c, d\}, B = \{2, 3, 4, 5, 6\}$ and $f = \{(a, 2), (b, 3), (c, 4), (d, 5)\}$. State the domain and range of f. Find $f(b)$ and $f(d)$.

The domain is A, the set of all first coordinates. The range is $\{2, 3, 4, 5\}$ which is not quite B but a subset of B. Scanning the set f we see $f(b) = 3, f(d) = 5$.

Problem 3

There are four different functions $f : \{a, b\} \to \{0, 1\}$. List them

$$\begin{array}{c|c} x & f(x) \\ \hline a & 0 \\ b & 0 \end{array}, \quad \begin{array}{c|c} x & f(x) \\ \hline a & 1 \\ b & 1 \end{array}, \quad \begin{array}{c|c} x & f(x) \\ \hline a & 0 \\ b & 1 \end{array}, \quad \begin{array}{c|c} x & f(x) \\ \hline a & 1 \\ b & 0 \end{array}$$

Problem 4

There are eight different functions $f : \{a, b, c\} \to \{0, 1\}$. List them

Constant 0, constant 1, three functions where exactly one element in the domain maps to 0 while the others map to 1, and three functions where exactly two elements in the domain map to 0 and exactly one element maps to 1.

Problem 5

Give an example of a relation from $\{a, b, c, d\}$ to $\{d, e\}$ that is not a function.

For a relation to *not* be a function it must fail the 'vertical line test', which in the set view means we can't have two ordered pairs with the same first coordinate. So for example:

$$R = \{(a, d), (a, e)\}$$

is a relation from the first set to the second but a relates to two different elements. As a function this would not be 'well-defined'.

Problem 6

Suppose $f : \mathbb{Z} \to \mathbb{Z}$ is defined as $f = \{(x, 4x + 5) : x \in \mathbb{Z}\}$. State the domain, codomain, and range of f. Find $f(10)$.

The domain is $\mathbb{Z}$ since the first coordinate of all pairs in f come directly from $\mathbb{Z}$. The codomain is also $\mathbb{Z}$ since for any integer x, $4x+5$ will also be an integer. The range is $\{\ldots, -3, 1, 5, 9, \ldots\}$ or $\{4x + 5 : x \in \mathbb{Z}\}$. By plugging in 10 we get $f(10) = 45$.

Problem 7

Consider the set W$f = \{(x, y) \in Z \times \mathbb{Z} : 3x + y = 4\}$. Is this a function from $\mathbb{Z}$ to $\mathbb{Z}$? Explain.

It's certainly a relation from $\mathbb{Z}$ to $\mathbb{Z}$ since $f \subseteq \mathbb{Z} \times \mathbb{Z}$. For it to be a function, f must have a unique y for any x in a pair (x, y). I claim this is true for f. To prove this we'll use the 'set up two things and prove they're equal' technique.

Suppose $(x, y) \in f$. Then $3x+y = 4$ or in terms of y we have $y = 4-3x$. Now if there is another element $(x, z) \in f$ then $z = 4 - 3x$ which means $z = y$.

Problem 8

Consider the set $f = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x + 3y = 4\}$. Is this a function from $\mathbb{Z}$ to $\mathbb{Z}$? Explain.

Yes, for precisely the same reasons as before. Set f is a subset of $\mathbb{Z} \times \mathbb{Z}$ so it's a relation. By the same equality argument as before we can show that f is well-defined. Therefore it's a function.

Problem 9

Consider the set $f = \{(x^2, x) : x \in \mathbb{R}\}$. Is this a function from $\mathbb{R}$ to $\mathbb{R}$? Explain.

No, because it's not well-defined. For example $(4, 2)$ and $(4, -2)$ satisfy the specification for f and so are both members.

Problem 10

Consider the set $f = \{(x^3, x) : x \in \mathbb{R}\}$. Is this a function from $\mathbb{R}$ to $\mathbb{R}$? Explain.

Yes, this is a function. Another way of looking at the relation is that the second coordinate is the cube root of the first. Since every $x \in \mathbb{R}$ is a cube of some number, the first coordinate (domain) is all $\mathbb{R}$. And since every number has a unique cube root, there is only one possible x for any x^3, making f well-defined.

Problem 11

Is the set $\theta = \{(X, |X|) : X \subseteq \mathbb{Z}_5\}$ a function? If so, what is its domain and range?

If we consider $\mathbb{Z}_5 = \{[0], [1], [2], [3], [4]\}$ then $\mathbb{Z}_5$ is finite (even though its members aren't) and the subsets of $\mathbb{Z}_5$ are finite with well-defined size. So θ is a valid function. It's domain is the equivalence classes modulo 5 and its range is $\{0, 1, 2, 3, 4, 5\}$.

Problem 12

Is the set $\theta = \{((x, y), (3y, 2x, x + y)) : x, y \in \mathbb{R}\}$ a function? If so, what is its domain and range? What can be said about the codomain?

We can see that θ associates points in $\mathbb{R}^2$ to points in $\mathbb{R}^3$, and each second coordinate results from three linear component functions, therefore f is a subset of $\mathbb{R}^2 \times \mathbb{R}^3$ and is well-defined, so it's a function.