13.3 Limits That Do Not Exist

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Prove that the following limits do not exist.

Problem 1

$\lim_{x \to 0} \log_{10} |x|$

Looking at the graph, this diverges to $-\infty$ as $x \to 0$. Using Ross' sequence definition of the limit, we can show that the sequence $(1/n)$ goes to 0 as $n \to \infty$. However

$$\log_{10} \frac{1}{n} = \log_{10} n^{-1} = -\log_{10} n$$

which decreases without bound as $n \to \infty$.

Using Hammack's approach:

Assume towards a contradiction that a limit L exists, so for $\epsilon = 1$ there exists $\delta > 0$ such that $|x| < \delta$ implies $|\log_{10} |x| - L| < 1$. Knowing that the function decreases without bound we can derive a contradiction by making x even closer to 0. We'll have a contradiction if we also show that $|\log_{10} |x| - L| \geq 1$. If we set up

$$|\log_{10} |x| - L| > 1$$

Some scratch work:

$$\begin{aligned} |\log_{10} |x| - L| &> 1 \\ \log_{10} |x| - L &< -1 \text{ (perhaps)} \\ \log_{10} |x| &< L - 1 \\ |x| &< 10^{L-1} \end{aligned}$$

So if we set $|x| < \min\{\delta, 10^{L-1}\}$, then $|x| < 10^{L-1}$. Then $|\log_{10} |x| - L| > 1$ following the steps above in reverse. Only this time, we don't have to worry about the assumption on line 2 since we are free to make $|x|$ as small as we like. Therefore we have a contradiction.

Problem 2

$\lim_{x \to 0} \frac{|x|}{x}$

This limit doesn't exist because of the discontinuity at $x = 0$; on the right hand side the function is a constant 1 and on the left side it's a constant -1. This is easy to prove using the sequence definition of the limit but we'll use a proof by contradiction. Let $f(x) = \frac{|x|}{x}$ and assume that a limit L exists. Then for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|x| < \delta$ means $|f(x) - L| < \epsilon$. But this also means $|-x| < \delta$ implies $|f(-x) - L| < \epsilon$. Consider that for any $x \neq 0$ the distance between $f(x)$ and $f(-x)$ will always be 2:

$$|f(x) - f(-x)| = 2$$

since one of them evaluates to 1 and the other to -1. Now choose an $\epsilon < 1$ and for $|x| < \delta$, by adding a form of 0 and the triangle inequality we can write:

$$2 = |f(x) - f(-x)| = |f(x) - L + L - f(-x)| \leq |f(x) - L| + |L - f(-x)| < 2\epsilon$$

This is a contradition because 2 cannot be less than 2 times any number less than 1.

Problem 3

$$\lim_{x \to 0} \frac{1}{x^2}$$

Assume towards a contradiction that a limit L exists. Then for any $\epsilon > 0$ there exists a $\delta > 0$ so that $|x| < \delta$ implies $|f(x) - L| < \epsilon$. Set $\epsilon = 1$, so there is some δ that makes the implication true. Notice if $|x| < \delta$ then $|\frac{x}{2}| < \delta$ as well. So we have:

$$|f(x) - L| < 1, \qquad |f(x/2) - L| < 1$$

Combining these we have:

$$|f(x) - L| + |L - f(x/2)| < 2$$

Apply the triangle inequality:

$$|f(x) - f(x/2)| = |f(x) - L + L - f(x/2)| \leq |f(x) - L| + |L - f(x/2)| < 2$$

Substitute the definition of f for the first term:

$$|f(x) - f(x/2)| = \left| \frac{1}{x^2} - \frac{1}{(x/2)^2} \right| = \left| \frac{1}{x^2} - \frac{4}{x^2} \right| = \left| \frac{3}{x^2} \right|$$

Use this to derive a contradiction: we have required that $|x| < \delta$ for some unknown δ, but for the implication to hold it must be true for $|x|$ equal to every value less than δ, including when $|x| < 1$ (or more precisely, when $|x| < \min\{1, \delta\}$). So when $|x| < 1$ we'd have $\left|\frac{3}{x^2}\right| > 3$, and this would form our contradiction:

$$3 < |f(x) - f(x/2)| < 2$$

Problem 4

$$\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$$

Of course, the function will oscillate increasingly rapidly as $x \to 0$, and we'll use this to derive the contradiction. ATAC that the limit L exists. Choose $\epsilon = 1/2$ and for any $\delta > 0$, if $|x| < \delta$ then $|cos(1/x) - L| < \epsilon$. There exists a $2k$ for $k \in \mathbb{N}$ large enough that $\frac{1}{2k\pi} < \delta$, and therefore the implication should hold that $|\cos(k\pi) - L| < 1/2$. Since $\cos(2k) = 1$ this means that L must be in the interval $(1/2, 3/2)$. However, consider $\frac{1}{(2k+1)\pi}$ is also less than δ so the implication should hold, but $\cos((2k+1)\pi) = -1$, which means L is in the interval $(-3/2, -1/2)$, contradicting the earlier interval we found.

Problem 5

$$\lim_{x \to 0} x \cot\left(\frac{1}{x}\right)$$

Notice that we can make the cotangent 1 by choosing $x = \frac{1}{\frac{\pi}{4} + 2k\pi}$ and make k sufficiently large. Then as k increases x goes to 0, making the limit apparently 0.

However as $x \to 0$ there are an infinite number of values $y \in (0, x)$ such that $\cot(1/y)$ blows up to infinity. More precisly, for any value M there is a $y \in (0, x)$ where $\cot(1/y) > M$. So it cannot be that for all values in $(0, \delta)$ that the function is within an epsilon band of L.

Problem 6

$$\lim_{x\to 1} \frac{1}{x^2 - 2x + 1}$$

We can factor the fraction to get

$$\frac{1}{(x-1)^2}$$

Let's rephrase the limit as:

$$\lim_{x\to j} \frac{1}{(x-j)^2}$$

We're given $j = 1$, but the limit will be the same for any value of j since that is just horizontal translation in plane, the y values remain the same, symmetric about j. Therefore this limit is equivalent to

$$\lim_{x\to 0} \frac{1}{(x-0)^2} = \lim_{x\to 0} \frac{1}{x^2}$$

which we have previously proven diverges to infinity.