12.5 Inverse Functions

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January 9, 2025

Problem 1

Check that $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 6 - n$ is bijective. Then compute f^{-1}.

To show that it is bijective, show that it is injective and surjective. Injectivity: suppose $f(n) = f(n)$. Then $6 - m = 6 - n$ and $m = n$. Surjectivity: for any $y \in Z$ you want to reach, we solve $y = 6 - n$ for n, obtaining $n = 6 - y$. To check, $f(6 - y) = 6 - (6 - y) = y$. Since any y can be reached on f, the function is surjective.

To find the inverse, set $y = 6 - n$ and solve for n. We just did that, so $f^{-1}(y) = 6 - y$. To check:

$$f^{-1}(f(n)) = f^{-1}(6 - n) = 6 - (6 - n) = n$$

Since $f^{-1} \circ f$ is the identity, this confirms f^{-1} is the correct inverse function.

Problem 2

In Exercise 9 of Section 12.2 you proved that $f : \mathbb{R} - \{2\} \to \mathbb{R} - \{5\}$ defined by $f(x) = \frac{5x+1}{x-2}$ is bijective. Now find its inverse.

Write $f(x) = y$ and solve for y:

$$\begin{aligned}
\frac{5x+1}{x-2} &= y \\
5x + 1 &= xy - 2y \\
2y + 1 &= xy - 5x \\
2y + 1 &= x(y - 5) \\
\frac{2y+1}{y-5} &= x
\end{aligned}$$

Now we can say $f^{-1}(x) = \frac{2x+1}{x-5}$. To check:

$$\begin{aligned}
f(f^{-1}(x)) &= f\left(\frac{2x+1}{x-5}\right) \\
&= \frac{5\left(\frac{2x+1}{x-5}\right) + 1}{\frac{2x+1}{x-5} - 2} \\
&= \frac{11x}{11} \\
&= x
\end{aligned}$$

Problem 3

Let $B = \{2^n : n \in \mathbb{Z}\} = \{\ldots, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, \ldots\}$. Show that the function $f : \mathbb{Z} \to B$ defined as $f(n) = 2^n$ is bijective. Then find f^{-1}.

To show bijectivity we could show the inverse, but let's show it's injective and surjective. Injectivity: suppose $f(m) = f(n)$. Then $2^m = 2^n$ and m must equal n. The function is surjective by the construction of B: the set is all integer powers of 2 and the domain of f is all integers, so any integer power of 2 is in the image of f.

The inverse function must map 2^n back to n, and the $\log_2$ function does this:

$$\log_2 2^n = n \log_2 2 = n \cdot 1 = n$$

Problem 4

The function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by the formula $f(m, n) = (5m + 4n, 4m + 3n)$ is bijective. Find its inverse.

We can solve a system of equations:

$$\begin{aligned} 5m + 4n &= x \\ 4m + 3n &= y \end{aligned}$$

Solving the system for m, n gives $m = 4y - 3x, n = 4x - 5y$. Checking we see

$$\begin{aligned} f(f^{-1}(m, n)) &= f(4y - 3x, 4x - 5y) \\ &= (5(4y - 3x) + 4(4x - 5y), 4(4y - 3x) + 3(4x - 5y)) \\ &= (20y - 15x + 16x - 20y, 16y - 12x + 12x - 15y) \\ &= (x, y) \end{aligned}$$

Problem 5

Show that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the formula $f(x, y) = ((x^2 + 1)y, x^3)$ is bijective. Then find its inverse.

To show bijectivity, we can show the inverse, but let's show injectivity and surjectivity.

Injectivity: suppose $f(a, b) = f(x, y)$. This means $((a^2 + 1)b, a^3) = ((x^2 + 1)y, x^3)$. But since the coordinates must match this gives $a^3 = x^3$. Since the cubing function is bijective, this means $a = x$. Then the first coordinates being equal means $(a^2 + 1)b = (x^2 + 1)y$. Since $a = x$ we can write this as $(x^2 + 1)b = (x^2 + 1)y$, and by the cancelation law $b = y$.

For surjectivity, the second coordinate can reach any real value by choosing the corresponding cube root for x. For the first coordinate, if it is 0 let $y = 0$. Otherwise $(x^2 + 1)y \neq$ which means $x^2 + 1 \neq 0$. Then for any desired value z on the first coordinate, let $y = \frac{z}{x^2+1}$. So any value in $\mathbb{R}^2$ can be reached by f.

To find the inverse where $f(x, y) = (a, b)$, we must have $x^3 = b$ or $x = b^{1/3}$. To solve for y we have $(x^2 + 1)y = a$, or by substition $(b^{2/3} + 1)y = a$ so that $y = \frac{a}{b^{2/3}+1}$. Checking we see

$$\begin{aligned} f(f^{-1}(a, b)) &= f(b^{1/3}, \frac{a}{b^{2/3} + 1}) \\ &= ((b^{2/3} + 1)\frac{a}{b^{2/3} + 1}, b) \\ &= (a, b) \end{aligned}$$

Problem 6

Is the function $\theta : \mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{Z})$ defined as $\theta(X) = \overline{X}$ bijective? If so, find its inverse.

It is bijective because it has an inverse function: itself. The complement of a complement is the original set. Therefore $\theta(\theta(X)) = X$ and θ composed with θ is the identity function.

Problem 7

Consider the function $f : \mathbb{R} \times \mathbb{N} \to \mathbb{N} \times \mathbb{R}$ defined as $f(x, y) = (y, 3xy)$. Check that this is bijective; find its inverse.

Injectivity: suppose $f(a, b) = f(x, y)$. Then $(b, 3ab) = (y, 3xy)$. This means $b = y$ and $3ab = 3xb$, so $a = x$. Surjectivity: for any $y \in \mathbb{N}$ needed in the codomain's first coordinate, choose that y as the second input. Then for any $z \in \mathbb{R}$ needed in the codomain's second coordinate, z must equal $3xy$. Then $x = z/3y$ (assuming $0 \notin \mathbb{N}$). Therefore the function's image is all of $\mathbb{N} \times \mathbb{R}$.

This also give us the inverse function $f^{-1}(x, y) = (y/3x, x)$. Checking:

$$f(f^{-1}(x, y)) = f(y/3x, x) = (x, 3(y/3x)) = (x, y)$$

Problem 8

Consider $f : \mathbb{N} \to \mathbb{Z}$ defined as $f(n) = \frac{(-1)^n(2n-1)+1}{4}$. Find its inverse.

Evaluating the first few function values we get:

$$\begin{aligned} f(1) &= 0 \\ f(2) &= 1 \\ f(3) &= -1 \\ f(4) &= 2 \\ f(5) &= -2 \end{aligned}$$

So we see that negative integers come from odd naturals and positive integers come from even ones. From this we can form a piecewise function:

$$f^{-1}(x) = \begin{cases} 2x, & x > 0 \\ 1 - 2x, & x \leq 0 \end{cases}$$