13.8 Series

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Problem 1

A geometric series is one having the form $a + ar + ar^2 + ar^3 + \ldots$, where $a, r \in \mathbb{R}$. (The first term in the sum is a, and beyond that the kth terms i r times the previous term.) Prove that if $|r| < 1$, then the series converges to $\frac{a}{1-r}$. Also, if $a \neq 0$ and $|r| \geq 1$, then the series diverges.

First we need to prove the radius of convergence. Use the root test:

$$\alpha = \limsup |ar^n|^{1/n} = |r| \limsup |a|^{1/n} = |r|$$

We have $\limsup |a|^{1/n} = 1$ for any nonzero a because taking higher roots of a nonzero number drives it to 1. The root test tells us that when $\alpha < 1$ the series converges. Since $\alpha = |r|$, we equivalently say the series converges when $|r| < 1$. Likewise if $|r| > 1$ then the series diverges.

If $|r| = 1$ then and $a \neq 0$ then the series simplifies to

$$a + a(1) + a(1^2) + a(1^3) = \ldots = a + a + a + \ldots$$

which will diverge to positive or negative infinity, depending on the sign of a.

When the series does converge, prove it converges to $\frac{a}{1-r}$, let $s = \sum ar^k$. In longhand:

$$s = a + ar + ar^2 + \ldots$$

If we multiply this by r we get

$$sr = ar + ar^2 + ar^3 + \ldots$$

which is nearly the same as s but omitting the first term. Therefore we can write

$$s - sr = a \implies s = \frac{a}{1-r}$$

Problem 2

Prove the comparison test: Suppose $\sum a_k$ and $\sum b_k$ are series. If $0 \leq a_k \leq b_k$ for each k, and $\sum b_k$ converges, then $\sum a_k$ converges. Also, if $0 \leq b_k \leq a_k$ for k, and $\sum b_k$ diverges, then $\sum a_k$ diverges.

This benefits from the **Cauchy criterion** for series: a series converges if and only if for all $\epsilon > 0$ there exists an N such that $m, n > N$ imply

$$\left| \sum_{k=m}^{n} a_k \right| < \epsilon$$

which is to say that eventually, the series' tail gets arbitrarily small. From here we apply this to the series b_k, and since a_k is termwise less than b_k and bounded below by 0, we get the following:

For any $\epsilon > 0$ let N be so large that $m, n > N$ imply that $|\sum_{k=m}^{n} b_k| < \epsilon$. Then

$$\left| \sum_{k=m}^{n} a_k \right| \leq \left| \sum_{k=m}^{n} b_k \right| < \epsilon$$

therefore a_k satisfies the Cauchy criterion and it converges by comparison.

To prove the divergence statement, start with b_k being divergent. Then it fails the Cauchy criterion and its tail cannot become arbitrarily small. The a_k series is termwise greater than b_k which is bounded below by 0, therefore the tail of a_k cannot be less than the b_k tail and a_k must also fail the Cauchy criterion.

Problem 3

Prove the limit comparison test: Suppose $\sum a_k$ and $\sum b_k$ are series for which $a_k, b_k > 0$ for each k. If $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges.

Intuitively, this means the denominator is growing faster than the numerator, causing the ratio to shrink to 0. More formally, we're given that $\frac{a_k}{b_k} \to 0$. So for any $\epsilon > 0$ there exists an N so large that any $n > N$ implies $\frac{a_n}{b_n} < \epsilon$. This means $a_n < \epsilon b_n$. More generally, no matter how small an ϵ we choose, b_n will eventually be larger than a_n termwise. By the comparison test, we have $a_n < b_n$ (for $\epsilon < 1$) and $\sum b_n$ converges, therefore $\sum a_n$ converges.

Problem 4

Prove the absolute convergence test: Let $\sum a_k$ be a series. If $\sum |a_k|$ converges, then $\sum a_k$ converges.

Apply the Cauchy criterion: since $\sum |a_k|$ converges, it must satisfy the Cauchy criterion: for any $\epsilon > 0$ there is an N large enough that $m, n > N$ imply:

$$\sum_{k=m}^{n} |a_k| < \epsilon$$

By the triangle inequality, this is larger than summing the terms first then taking the absolute value:

$$\left| \sum_{k=m}^{n} a_k \right| \leq \sum_{k=m}^{n} |a_k| < \epsilon$$

Therefore $\sum a_k$ also satisfies the Cauchy criterion and converges.

Problem 5

Prove the ratio test: Given a series $\sum a_k$ with each a_k positive, if $\lim_{n\to\infty} \frac{a_{k+1}}{a_k} = L < 1$, then $\sum a_k$ converges. Also, if $L > 1$, then $\sum a_k$ diverges.

We're given that the limit of $\frac{a_{n+1}}{a_n} = L < 1$. This means that eventually the ratio gets arbitrarily close to L which is less than 1; so the ratio will eventually be less than 1 and stay less than 1. For all terms in the tail, $a_{n+1} < a_n$.

More formally, for any $\epsilon > 0$ there exists an N so large that $n > N$ implies $|a_{n+1}/a_n - L| < \epsilon$. Viewed another way:

$$\frac{a_{n+1}}{a_n} \in (L - \epsilon, L + \epsilon) \implies \frac{a_{n+1}}{a_n} < L + \epsilon$$

which we can write:

$$a_{n+1} < (L + \epsilon)a_n$$

Now let ϵ be so small that $L + \epsilon < 1$, and say $L + \epsilon = r$. For all subsequent terms in the tail we have $a_{n+1} < ra_n$ and applying this repeatedly we get:

$$a_{n+2} < ra_{n+1} < r^2 a_n$$

$$a_{n+k} < r^k a_n$$

And since $|r| < 1$ this is a convergent geometric series as $k \to \infty$. Therefore the tail satisfies the Cauchy criterion and the series converges. Alternatively we can also assert the series converges because we have a convergent geometric series $\sum\limits_{k=N+1}^{\infty} a_N r^{k-N}$ preceded by a finite series $\sum\limits_{k=1}^{N} a_k$, which is simply a sum of two finite terms.