13.5 Continuity and Derivatives

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Problem 1

Prove that the function $f(x) = \sqrt{x}$ is continuous at any number $c > 0$. Deduce that $\lim_{x\to c} \sqrt{g(x)} = \sqrt{\lim_{x\to c} g(x)}$, provided $\lim_{x\to c} g(x)$ exists and is greater than 0.

First we show $\sqrt{x}$ is continuous using and $\epsilon - \delta$ proof. For scratchwork:

$$\begin{aligned}
|\sqrt{x} - \sqrt{c}| &< \epsilon \\
|(\sqrt{x} - \sqrt{c}) \cdot \frac{\sqrt{x} + \sqrt{c}}{\sqrt{x} + \sqrt{c}}| &< \epsilon \\
|\frac{x - c}{\sqrt{x} + \sqrt{c}}| &< \epsilon \\
|x - c| &< \epsilon(\sqrt{x} + \sqrt{c}) \\
|x - c| &< \epsilon\sqrt{c}
\end{aligned}$$

At the last line, we can drop the $\sqrt{x}$ term multiplying ϵ because we can make x as close to c as we like, and since the domain is positive numbers the roots are positive. Therefore $\epsilon\sqrt{c} < \epsilon(\sqrt{x} + \sqrt{c})$ for any $x > 0$. Now write the formal proof: for any $\epsilon > 0$ set $\delta < \epsilon\sqrt{c}$. Then for $0 < |x - c| < \delta$ we have

$$\begin{aligned}
|f(x) - f(c)| &= |\sqrt{x} - \sqrt{c}| \\
&= \frac{|x - c|}{\sqrt{x} + \sqrt{c}} \\
&< \frac{\delta}{\sqrt{c}} \\
&< \frac{\epsilon\sqrt{c}}{\sqrt{c}} \\
&= \epsilon
\end{aligned}$$

Now that we know the square root function is continuous at any positive real number, we can apply the theorem that

$$\lim_{x\to c} \sqrt{g(x)} = \sqrt{\lim_{x\to c} g(x)}$$

provided the root function is continuous at $g(c)$.

Problem 2

Show that the condition of continuity in Theorem 13.9 is necessary by finding functions f and Wg for which $\lim_{x\to c} g(x) = L$, and f is *not* continuous at $x = L$, and $\lim_{x\to c} f(g(x)) \neq f\left(\lim_{x\to c} g(x)\right)$.

Theorem 13.9 is that continuous functions compose:

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right)$$

The problem statement asks to show a function that is not continuous at L. One simple solution is to define a piecewise function with a point discontinuity:

$$f(x) = \begin{cases} x & x \neq 0 \\ \pi & x = 0 \end{cases}$$

This f is almost the identity function except at 0, it jumps to π. Combine that with $g(x) = x$, the actual identity function. Then

$$\lim_{x \to 0} f(g(x)) = \lim_{x \to 0} f(x) = 0$$

Since as x approaches 0 (but does not equal 0), $f(x)$ using the 'identity' branch, approaches 0 as well. However if we take the limit of g before sending to f we get $f(0) = \pi$.