14.3 Comparing Cardinalities

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Problem 1

Suppose B is an uncountable set and A is a set. Given that ther is a surjective function $f : A \to B$, what can be said about the cardinality of A?

Solution: recall key facts about surjections and injections.

Since there is a surjection from A to B there exists an injection from B to A. Construct the injection using f: for any $b \in B$ there exists an $a \in A$ such that $f(a) = b$. There may be multiple such A members though, but since f is surjective there's guaranteed to be at least one. So for every $b \in B$ we can define a map back to A by choosing some $a \in f^{-1}(b)$, the preimage of b. This map is injective because if some $a \in f^{-1}(b)$ it cannot be in the inverse image of any b' distinct from b or f would not be a well-defined function and $f(a)$ would somehow equal both b and b'.

Since there's an injection from B to A, and we don't know about a bijection, we can say that $|B| \leq |A|$. Thus a surjection from A to B gives us the relation $|B| \leq |A|$.

Problem 2

Prove that the set $\mathbb{C}$ of complex numbers is uncountable.

Solution: use a theorem to simplify the problem. We have a theorem that if A is uncountable and $A \subseteq B$, then B must be uncountable as well. In this case we have $\mathbb{R} \subseteq \mathbb{C}$ and $\mathbb{R}$ is uncountable, which implies $\mathbb{C}$ is uncountable as well.

Problem 3

Prove or disprove: If A is uncountable, then $|A| = |R|$

This is not true. Even though we don't yet know about cardinalities beyond uncountably infinite, we know that there are infinite cardinalities since $|A| < |\mathscr{P}(A)|$. This applies to $\mathbb{R}$ as well, and $|\mathbb{R}| < |\mathscr{P}(\mathbb{R})|$. So if $A = \mathscr{P}(\mathbb{R})$ then it would not be true that A has the same cardinality as $\mathbb{R}$.

Problem 4

Prove or disprove: if $A \subseteq B \subseteq C$ and A and C are countably infinite, then B is countably infinite.

Solution: direct proof.

Claim that B must be at least countably infinite. If it was less than countably infinite then it would be finite, and then B could not contain A. Therefore B must be infinite, countable or uncountable. Next, observe that B is contained in a countably infinite set. An uncountable set cannot be contained in a countable set, therefore, B is at most uncountably infinite.

Problem 5

Prove or disprove: the set $\{0, 1\} \times \mathbb{R}$ is uncountable.

Solution: prove by comparing to a subset.

We have $\mathbb{R}$ as basically a subset of this: take the subset $P = \{(0, x), x \in \mathbb{R}\}$, this is in bijection with $\mathbb{R}$ by just taking its second coordinate. Therefore $|P| = |\mathbb{R}|$ and is uncountable. Since P is a subset of our set in question, the parent set must be uncountable as well.

Problem 6

Prove or disprove: Every infinite set is a subset of a countably infinite set.

Solution: apply a handy theorem.

An infinite set can mean countable or uncountable. We have a theorem that demonstrates an uncountable set cannot be contained in a countable set. Therefore the statement is false.

Problem 7

Prove or disprove: if $A \subseteq B$ and A is countably infinite and B is uncountable, then $B - A$ is uncountable.

The statement asks if we take a countably infinite set out of an uncountably infinite set, is the result countable or uncountable? The result is still uncountable.

ATAC that $B - A$ is countable. Then there is a bijective function $f : \mathbb{N} to B - A$. We also know that A is countable so there is a bijection $g : \mathbb{N} to A$. Now we could interleave f and g to construct a bijective function $h : \mathbb{N} to B$:

$$h(n) = \begin{cases} f(\frac{n+1}{2}) & , n \text{ even} \\ g(\frac{n}{2}) & , n \text{ odd} \end{cases}$$

which implies B is countable, contradicting the statement that it is uncountable.

Problem 8

Prove or disprove: the set $\{(a_1, a_2, a_3, \ldots) : a_i \in \mathbb{Z}\}$ of infinite sequences of integers is countably infinite.

This is uncountably infinite. Suppose for contradiction that the set is countably infinite. Then we can enumerate the sequences as $s_1, s_2, \ldots$ and package the sequences as rows in a table. Now we are in precisely the same situation as in Cantor's diagonalization argument on the uncountability of the real numbers.

Problem 9

Prove that if A and B are finite sets with $|A| = |B|$, then any infection $f : A \to B$ is also a surjection. Show this is not necessarily true if A and B are not finite.

If A and B are both finite and $|A| = |B|$ then the pigeonhole principle states that any injection is also a surjection. Prove by induction on the size of the sets. If $|A| = |B| = 1$, then there is just one map from A to B and it is both injective and surjective.

Now assume the property holds up to some size n and consider sets A and B size $n+1$. If we remove one element from each set (an $a \in A$ and $b \in B$) then they are both size n and the inductive hypothesis gives us an injection f from A to B. We can then extend f to include mapping a to b (the removed elements). So having an injection on sets size n leads to an injection on sets size $n + 1$ that is also a surjection.

However if the sets are infinite, an injection doesn't necessarily mean there is a surjection. To find a counterexample, let's look for sets between which we know there cannot be a surjection. For instance, $\mathbb{N} \to \mathbb{R}$. We know there cannot be a surjection from $\mathbb{N}$ to $\mathbb{R}$ however there are many injections, such as the inclusion map $f(n) = n$.

Problem 10

Prove that if A and B are finite sets with $|A| = |B|$, then any surjection $f : A \to B$ is also an injection. Show this is not necessarily true if A and B are not finite.

Proof: induction on set size.

Proceed by induction on the size of the sets. For sets size 1 there is only one map between them which is a bijection, and hence both injective and surjective. Assume this holds for sets up to size n and consider $|A| = |B| = n + 1$. Take some $a \in A, b \in B$ and remove them from the sets: $A - \{a\}$ and $B - \{b\}$ are both sets size n, so any surjection f from $A - \{a\}$ to $B - \{b\}$ is also an injection. Now we can extend f to f_A, mapping from A to B by defining $f_A(a) = b$ (and $f_A = f$ otherwise). The extension f_A is injective and surjective.

This property doesn't generally hold for infinite sets. Like before, there is no injection from $\mathbb{R}$ to $\mathbb{N}$ even though there are any number of surjections, such as $g(x) = x$ if $x \in \mathbb{N}$ and 1 otherwise.