Sets with Equal Cardinalities

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A. Show that the two given sets have equal cardinality by describing a bijection from one to the other. Describe your bijection with a formula (not as a table).

Problem 1

$\mathbb{R}$ and $(0, \infty)$

Look for a function that takes any real number and outputs any positive real number. Many such functions exist, such as $f(x) = e^x$. As x gets very negative, e^x approaches 0 and e^x can grow without bound as x increases. And since e^x is invertible on the positive reals (using log), this is a bijection.

Problem 2

$\mathbb{R}$ and $(\sqrt{2}, \infty)$

The target set $(\sqrt{2}, \infty)$ only differs from $(0, \infty)$ by the lower endpoint. Take the function $f(x) = e^x$ and follow it with the function $g(x) = x + \sqrt{2}$. Put together:

$$h(x) = g(f(x)) = e^x + \sqrt{2}$$

The function h is bijective since it's the composition of bijective functions: e^x on a positive set and $g(x)$ inverse would simply be to subtract $\sqrt{2}$.

Problem 3

$\mathbb{R}$ and $(0, 1)$

In the text they prove this by showing transitively $|\mathbb{R}| = |(0, \infty)| = |(0, 1)|$. We could compose the two functions used to get a bijection:

$$h : \mathbb{R} \longrightarrow (0, 1)$$
$$x \longmapsto \frac{2^x}{2^x + 1}$$

Taking $f(x) = \frac{x}{x+1}$ (a bijection between $(0, \infty)$ and $(0, 1)$) and $g(x) = 2^x$ (a bijection between $\mathbb{R}$ and $(0, \infty)$), we compose them to make $h = g \circ f$. As a composition of bijections, h is bijective as well.

Problem 4

The set of even integers and the set of odd integers.

Let $f(x) = x + 1$ mapping even integers to odd integers. Its inverse function is the 'minus 1' function, therefore f is a bijection between evens and odds.

Problem 5

$A = \{3k : k \in \mathbb{Z}\}$ and $B = \{7k : k \in \mathbb{Z}\}$.

Both sets A and B are in a sense generated from $\mathbb{Z}$. In fact, if we divide each $a \in A$ by 3 we'll return the the set of integers $\mathbb{Z}$, which we can then multiply by 7 to convert to the set B. So we may suspect that the map $f(x) = \frac{7}{3}x$ will be our bijection. It maps A members to B members as just described, and f is invertible by multiplying by $3/7$. Therefore f is a bijection between A adn B and $|A| = |B|$.

Problem 6

$\mathbb{N}$ and $S = \{\frac{\sqrt{2}}{n} : n \in \mathbb{N}\}$

Similar to the previous problem, we see that S is generated by $\mathbb{N}$ and in fact to transform $n \in N$ to an $s \in S$ we take $n \mapsto \sqrt{2}/n$. This map is surjective because this is how all elements in S are specified. It is also injective since if $n \neq m$ then $1/n \neq 1/m$ and $\sqrt{2}/n \neq \sqrt{2}/m$. Therefore the function $f(n) = \sqrt{2}n$ is a bijection between these sets and $|\mathbb{N}| = |S|$.

Problem 7

$\mathbb{Z}$ and $S = \{\ldots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16, \ldots\}$

The set S seems to be all integer powers of 2. So the simplest map from $\mathbb{Z}$ to S would be $f(x) = 2^x$. This is invertible by the $\log_2$ function, so a bijection exists and the sets have equal cardinality.

Problem 8

$\mathbb{Z}$ and $S = \{x \in \mathbb{R} : \sin x = 1\}$

We know that $\sin x = 1$ when $x = \pi/2$ or a multiple of 2π plus $\pi/2$:

$$\sin x = 1 \implies x = \frac{\pi}{2} + 2\pi k, k \in \mathbb{Z}$$

So we could rewrite S as

$$S = \{x \in \mathbb{R} : x = \pi/2 + 2\pi k, k \in \mathbb{Z}\}$$

So the function $f(k) = \pi/2 + 2\pi k$ maps integers to S elements. This function is also invertible:

$$f^{-1}(y) = \frac{y - \pi/2}{2\pi}$$

So the bijection exists between $\mathbb{Z}$ and S and $|\mathbb{Z}| = |S|$.

Problem 9

$\{0,1\} \times \mathbb{N}$ and $\mathbb{N}$

Similar to the table that shows $|\mathbb{N}| = |\mathbb{Z}|$ we can write a table:

$\mathbb{N}$	1	2	3	4	5	$\cdots$
$\{0,1\} \times \mathbb{N}$	$(0,1)$	$(1,1)$	$(0,2)$	$(1,2)$	$(0,3)$	$\cdots$

however the problem asks us to specify an actual formula. We do see that when $n \in \mathbb{N}$ is odd the first coordinate in $\{0,1\} \times \mathbb{N}$ is 0, and it's 1 otherwise. The second coordinate is $(n+1)//2$ (floor division). Using these coordinate functions we can compose a bijection:

$$f(n) : \mathbb{N} \longrightarrow \{0,1\} \times \mathbb{N}$$

$$n \longmapsto (n+1 \pmod 2), (n+1)//2)$$

To show that f is a bijection, we'll construct the inverse function.

$$g(a,b) = 2b - (1-a)$$

If $a = 0$ then n was odd, and the result of $(n+1)/2 = b$ since we can drop the floor division and use regular division for odd n. Solving for n gives us $n = 2b - 1$, which matches the formula when $a = 0$.

If $a = 1$ then n was even and $n/2 = (n+1)//2 = b$. Solving for n we have $n = 2b$, which matches our formula as well for when $a = 1$.

Problem 10

$\{0,1\} \times \mathbb{N}$ and $\mathbb{Z}$

If we use $\{0,1\}$ to encode the sign we'd have an easy time of it:

$\{0,1\} \times \mathbb{N}$	$(0,1)$	$(1,1)$	$(0,2)$	$(1,2)$	$\cdots$
$\mathbb{N}$	-1	1	-2	2	$\ldots$

However we have to include 0 in the mapping, so try shifting everything to the right:

$\{0,1\} \times \mathbb{N}$	$(0,1)$	$(1,1)$	$(0,2)$	$(1,2)$	$(0,3)$	$\cdots$
$\mathbb{N}$	0	-1	1	-2	2	$\ldots$

Call the tuple (a,b). We can encode the sign as $(-1)^a$ so when $a = 1$ we have a -1 coefficient, and it simplifies to 1 when $a = 0$. For the magnitude, we see it's equal to b when a is 1 or 'switched on', and it's $b-1$ when a is 'off'. So we can use a one-hot switch to encode that: $b-(1-a)$. Putting this together we have

$$f(a,b) = (-1)^a(b-(1-a))$$

Prove this is a bijection by showing the inverse function. First, if $x \in Z = 0$ then $f^{-1}(0) = (0,1)$.

For any other $x \in \mathbb{Z}$, we used a to encode its sign:

$$a = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases}$$

And we used b to encode its magnitude:

$$b = \begin{cases} |x|, & x < 0 \\ |x|+1, & x > 0 \end{cases}$$

Put together we have:

$$f^{-1}(x) = \begin{cases} (1, |x|), & x < 0 \\ (0, |x|+1), & x > 0 \end{cases}$$

To show this is indeed the inverse, suppose $x = (-1)^a(b-(1-a))$. If $x > 0$ then it must be $a = 0$ and $|x| = |b-1|$. However since $x > 0$ and $b - 1 \geq 0$ we can drop the absolute value bars and state $x = b-1$ or $x+1 = b$. So we have $(a,b) = (0, x+1)$ which equals $(0, |x|+1)$, matching f^{-1}. If $x < 0$ then $a = 1$ and $|x| = |b|$. Since $b > 0$ we have $|b| = b$ and more simply, $(a,b) = (1, |x|)$, again as f^{-1} predicts. Therefore f^{-1} in all cases takes an integer x and returns the (a,b) pair the mapped to it on f.

Problem 11

$[0,1]$ and $(0,1)$.

Here take inspiration from Hammack's proof but note there is a mistake in his solution: his function $g : [0, 1) \to (0, 1)$ actually has codomain $[0, 1)$ since as defined, $g(0) = 0$. Thus it doesn't achieve the goal of mapping into the open interval.

The trick here is to take the unwanted endpoints, 0 and 1, and map them into the interval's interior in a way that won't overlap with other mappings. For instance, we can get rid of 1 by sending the natural reciprocals, $\frac{1}{n}$ for $n \in \mathbb{N}$ to $\frac{1}{n+1}$ and leaving everything else alone.

Formally: let $X = \{1/n : n \in \mathbb{N}\}$ and define f:

$$f(x) = \begin{cases} x & x \in [0, 1] - X \\ \frac{1}{x+1} & \text{otherwise} \end{cases}$$

We see that $f(1) = 1/2$ and anything less than 1 must map into $[0, 1)$. The function f is invertible: on the first branch it's the identity function so nothing happened to the input, and on the second branch we mapped a natural reciprocal to the next natural reciprocal (e.g. $1/3 \mapsto 1/4 \mapsto 1/5 \mapsto \ldots$) so we just have to map it back:

$$f^{-1}(x) = \begin{cases} x & x \in [0, 1] - X \\ \frac{1}{x-1} & x \in X \end{cases}$$

So we have a bijection between $[0, 1]$ and $[0, 1)$. We can do something similar to shave off the left endpoint 0. Before, we sent $1 \mapsto 1/2 \mapsto 1/3 \mapsto 1/4 \mapsto \ldots$. Now we need a similar chain but starting with 0. So if we want $0 \mapsto 1/2 \mapsto 1/3 \mapsto 1/4 \mapsto \ldots$ we can define the branch $n \mapsto \frac{1}{n+2}$ for $n \in \mathbb{N} \cup \{0\}$. So let $Y = \{\frac{1}{n+2} : n \in \mathbb{N} \cup \{0\}\}$ and our second function is

$$g(x) = \begin{cases} x, & x \in [0, 1) - Y \\ \frac{1}{n+2}, & x \in Y \end{cases}$$

The g function is also bijective since any natural reciprocal follows from second branch, and $\frac{1}{x} \mapsto \frac{1}{x-1}$ unless $x = 2$, then $g^{-1}(1/2) = 0$. Otherwise, $g(x)$ is on the identity branch and g^{-1} is the identity. Now we have a bijective function $g : [0, 1) \to (0, 1)$.

And we can have $g \circ f$ which bijectively maps from $[0, 1]$ to $(0, 1)$, proving they have equal cardinality.

Problem 12

$\mathbb{N}$ and $\mathbb{Z}$

As proven in section 12.2, one such bijection is

$$f : \mathbb{N} \longrightarrow \mathbb{Z}$$

$$n \longmapsto \frac{(-1)^n(2n-1)+1}{4}$$

Problem 13

$\mathcal{P}(\mathbb{N})$ and $\mathcal{P}(\mathbb{Z})$

Since $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality it makes sense that their power sets have the same cardinality as well. Since $|\mathbb{N}| = |\mathbb{Z}|$ there exists a bijection $f : \mathbb{N} \to \mathbb{Z}$. To define a bijection between their powersets we'll use f again. Define

$$F : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{Z})$$

$$A \subseteq \mathbb{N} \longmapsto B = \{f(a) : a \in A\}$$

So F is basically the set-valued version of f. In any $A \subseteq \mathbb{N}$, the f function uniquely maps the members to integers encoding a subset B in $\mathbb{Z}$. Since f is bijective we could take the integers in B and map them back to A on f^{-1}.

Problem 14

$\mathbb{N} \times \mathbb{N}$ and $\{(n, m) \in \mathbb{N} \times \mathbb{N} : n \leq m\}$

Although we need a formula, not a table, we can use a table for the second set for inspiration.

$n \backslash m$	1	2	3	4
1	$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$
2		$(2,2)$	$(2,3)$	$(2,4)$
3			$(3,3)$	$(3,4)$
4				$(4,4)$

Notice that the table for the second set is the upper triangle of the table for $\mathbb{N} \times \mathbb{N}$. One bijection might be starting with $\mathbb{N} \times \mathbb{N}$ and shifting the pairs to the right, since this seems to happen in a simple pattern. So the map would be somthing like $f(n, m) = (n, m + x)$. We can see that the second coordinate controls how far to the right something gets shifted, it is second coordinate minus one. The bijection then is

$$f(n, m) = (n, n + m - 1)$$

To show that f is a bijection we can show its inverse: on the first coordinate f is the identity, on the second coordinate we have $m \mapsto n + m - 1$. Therefore for any (p, q) in the second set we can undo f by subtracting the first coordinate p and adding back 1:

$$f^{-1}(p, q) = (p, q - p + 1)$$

B. Answer the following questions concerning bijections from this section.

Problem 15

Find a formula for the bijection f in Example 14.2, $|\mathbb{N}| = |\mathbb{Z}|$

The table given looks like:

n	1	2	3	4	5	6	7
$f(n)$	0	1	-1	2	-2	3	-3

Apart from $f(1) = 0$ the parity of $n \in \mathbb{N}$ determines the sign of $m \in \mathbb{Z}$ with even naturals mapping to positive integers and odd naturals mappping to negative integers. We also see that if n is even it maps to $n/2$, but if n is odd it maps to the negative of $(n - 1)/2$. So the function can be written as:

$$f(n) = \begin{cases} n/2 & n \text{ even} \\ -(n-1)/2 & n \text{ odd} \end{cases}$$

Problem 16

Verify that the function f in Example 14.3 is a bijection.

This example claims that $f(x) = \frac{x}{x+1}$ is a bijection from $(0, \infty)$ to $(0, 1)$. We can solve for its inverse function to show it is a bijection:

$$y = \frac{x}{x+1} \implies x = \frac{y}{y-1}$$