13.7 Sequences

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Problem 1

Prove that $\left\{\frac{2^n}{n!}\right\}$ converges to 0.

Prove this using the product law for limits. View 2^n as $2 \times 2 \times \ldots \times 2$, which has n factors. So does $n!$. Write the general term as

$$\frac{2^n}{n!} = \frac{2}{1} \cdot \frac{2}{2} \cdot \frac{2}{3} \cdot \ldots \cdot \frac{2}{n-1} \cdot \frac{2}{n}$$

See how after the first two factors, the denominator will always be larger than the numerator. Using the limit law for products, we get

$$\lim_{n\to\infty} \frac{2^n}{n!} = \left(\lim_{n\to\infty} \frac{2}{1}\right) \cdot \left(\lim_{n\to\infty} \frac{2}{2}\right) \cdot \ldots \cdot \left(\lim_{n\to\infty} \frac{2}{n-1}\right) \cdot \left(\lim_{n\to\infty} \frac{2}{n}\right) = 2 \cdot 0 = 0$$

Problem 2

Prove that $\left\{5 + \frac{2}{n^2}\right\}$ converges to 5.

Apply the limit sum rule to the general term:

$$\lim_{n\to\infty} 5 + \frac{2}{n^2} = \lim_{n\to\infty} 5 + \lim_{n\to\infty} \frac{2}{n^2} = 5 + 0 = 5$$

Problem 3

Prove that $\left\{\frac{2n^2+1}{3n-1}\right\}$ diverges to ∞.

Simplify the general term by multiplying by $\frac{1/n}{1/n}$ and applying limit laws:

$$\lim_{n\to\infty} \frac{2n^2+1}{3n-1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{2n - \frac{1}{n}}{3 - \frac{1}{n}} = \frac{\lim_{n\to\infty} 2n - \frac{1}{n}}{\lim_{n\to\infty} 3 - \frac{1}{n}} = \frac{1}{3} \lim_{n\to\infty} 2n = +\infty$$

Problem 4

Prove that $\left\{1 - \frac{1}{2^n}\right\}$ converges to 1.

Apply the limit sum rule to the general term:

$$\lim_{n\to\infty} 1 - \frac{1}{2^n} = \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{2^n} = 1 - 0 = 1$$

Problem 5

Prove that $\left\{\frac{2n+1}{3n-1}\right\}$ converges to $\frac{2}{3}$.

Simplify the general term by multiplying by $\frac{1/n}{1/n}$ and applying limit laws:

$$\lim_{n\to\infty} \frac{2n+1}{3n-1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{2+\frac{1}{n}}{3-\frac{1}{n}} = \frac{\lim\limits_{n\to\infty} 2+\frac{1}{n}}{\lim\limits_{n\to\infty} 3-\frac{1}{n}} = \frac{2}{3}$$

Problem 6

Prove that $\left\{\frac{5n^2+n+1}{4n^2+2}\right\}$ converges to $\frac{5}{4}$.

Simplify the general term by multiplying by $\frac{1/n^2}{1/n^2}$ and applying limit laws:

$$\lim_{n\to\infty} \frac{5n^2+n+1}{4n^2+2} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{5+\frac{1}{n}+\frac{1}{n^2}}{4+\frac{2}{n^2}} = \frac{\lim\limits_{n\to\infty} 5+\frac{1}{n}+\frac{1}{n^2}}{\lim\limits_{n\to\infty} 4+\frac{2}{n^2}} = \frac{5}{4}$$

Problem 7

Prove that if a sequence diverges to infinity, the it diverges.

This is a tautology because ‘diverge to infinity’ is contained in the definition of divergence. But to formalize it, suppose a_n diverges to infinity. Then the sequence can obtain arbitrarily large values. ATAC that sequence does converge to some limit L. By assumption, for any $\epsilon > 0$ there exists an N sufficiently large that $n > N$ implies $|a_n - L| < \epsilon$. But since the series diverges to infinity there is an M sufficiently large that $n > M$ implies $a_n > L + \epsilon$. Then choose $P = \max(N, M)$ and any $n > P$ implies

$$|a_n - L| \geq a_n - L > \epsilon \qquad \text{and} \qquad |a_n - L| < \epsilon$$

which is a contradiction.

Problem 8

Prove that the **constant sequence** $c, c, c, c, \ldots$ converges to c for any $c \in \mathbb{R}$.

This can be done with an $\epsilon - \delta$ proof since viewing the sequence as a function, the difference between any function value and c will always be 0. So for any $\epsilon > 0$ and any $c \in \mathbb{R}$, let $\delta = \epsilon$. Then for any $n > 0$ we have $|a_n - c| = 0 < \epsilon$.

Problem 9

Prove that if $\{a_n\}$ converges to L, and $c \in \mathbb{R}$, then the sequence $\{ca_n\}$ converges to cL.

Apply the limit law for functions times constants:

$$\lim_{n\to\infty} c \cdot a(n) = c \lim_{n\to\infty} a(n) = c \cdot \lim_{n\to\infty} a_n = cL$$

Problem 10

Prove that if $\{a_n\}$ converges to L and $\{b_n\}$ converges to M then the sequence $\{a_n + b_n\}$ converges to $L + M$.

Apply the addition rule to the general term after viewing the sequences as functions: since $a_n \to L$ this means the function that $a(n)$ approaches L as n approaches infinity. Similarly $b(n)$ approaches M. Then

$$\lim_{n\to\infty} a(n) + b(n) = \lim_{n\to\infty} a(n) + \lim_{n\to\infty} b(n) = L + M$$

Problem 11

Prove that if $\{a_n\}$ converges to L and $\{b_n\}$ converges to M then the sequence $\{a_n \cdot b_n\}$ converges to $L \cdot M$.

Apply the product rule to the general term as in the previous problems. Since $a_n \to L$ this means that for any $\epsilon > 0$ there exists an N sufficiently large that for $n > N$, $|a(n) - L| < \epsilon$. Likewise for b_n there exists a threshold M such that $n > M$ implies $|b(n) - M| < \epsilon$. Then for $n > \max(N, M)$. Then for scratchwork:

$$\begin{aligned}
|a(n)b(n) - LM| &< \epsilon \\
|a(n)b(n) - Lb(n) + Lb(n) - LM| &< \epsilon \\
|b(n)(a(n) - L) + L(b(n) - M)| &< \epsilon \\
|b(n)||a(n) - L| + |L||b(n) - M| &< \epsilon
\end{aligned}$$

As in the previous section's proof, we can make $|a(n) - L|$ and $|b(n) - M|$ arbitrarily small, so each term ends up less than $\epsilon/2$. For the first term, we want to bound $|b(n)|$ so it no longer depends on n. Since $b(n) \to M$, we can say for sufficiently large n that $|b(n)| < |M|+1$. Then if we make $|a(n)-L| < \epsilon/2(|M|+1)$ and $|b(n) - M| < \epsilon/2(|L| + 1)$, then the sum of the two terms will be less than ϵ.

So the formal proof proceeds that for any $\epsilon > 0$, there exists an N so large that $n > N$ implies $|a(n)-L| < \epsilon/2(|M| + 1)$ and $|b(n) - M| < \epsilon/2(|L| + 1)$. Then for $n > N$ we have $|a(n)b(n) - LM| < \epsilon$ by following the scratchwork above.

Problem 12

Prove that if $\{a_n\}$ converges to L and $\{b_n\}$ converges to $M \neq 0$ then the sequence $\{a_n/b_n\}$ converges to L/M. (You may assum that $b_n \neq 0$ for each $n \in \mathbb{N}$.)

Apply the division rule to the general term:

$$\lim_{n\to\infty} \frac{a(n)}{b(n)} = \frac{\lim_{n\to\infty} a(n)}{\lim_{n\to\infty} b(n)} = \frac{L}{M}$$

Problem 13

For any sequence $\{a_n\}$, there is a corresponding sequence $\{|a_n|\}$. Prove that if $\{|a_n|\}$ converges to 0, then $\{a_n\}$ converges to 0. Give an example of a sequence $\{a_n\}$ for which $\{|a_n|\}$ converges to a number $L \neq 0$ but $\{a_n\}$ diverges.

To find a sequence where $\{|a_n|\}$ converges to a number $L \neq 0$ but $\{a_n\}$ diverges, we need to find one where taking the terms' absolute value somehow makes the sequence more well-behaved. One example would be the alternating sequence $-1, 1, -1, 1, \ldots$ or $a_n = (-1)^n$. Then $\{|a_n|\}$ is the constant 1 sequence which converges to 1, but $\{a_n\}$ diverges because it oscillates between two values forever.

Problem 14

Suppose that $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are sequences for which $a_n \leq b_n \leq c_n$ for all sufficiently large n. (That is, $a_n \leq b_n \leq c_n$ for all $n > M$ for some integer M.) Prove that if $\{a_n\}$ and $\{c_n\}$ converge to the same limit L, then $\{b_n\}$ also converges to L.

While this 'squeeze theorem' makes intuitive sense, let's prove it formally using $\epsilon - \delta$. For any $\epsilon > 0$ let N_1 be large enough that $n > N_1$ implies $|a_n - L| < \epsilon$ and N_2 be large enough that $n > N_2$ implies $|c_n - L| < \epsilon$. Choose $N = \max(N_1, N_2)$. Then for any $n > N$ we have

$$a_n \in (L - \epsilon, L + \epsilon) \quad \text{and} \quad c_n \in (L - \epsilon, L + \epsilon)$$

By the constraint that $a_n \leq b_n \leq c_n$, we must have c_n in the same interval $(L - \epsilon, L + \epsilon)$. Written with absolute values this means $|b_n - L| < \epsilon$. So for any $\epsilon > 0$ there exists an N such that $n > N$ implies $|b_n - L| < \epsilon$.