13.2 The Definition of a Limit

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Problem 1

Prove that $\lim_{x \to 5} (8x - 3) = 37$.

For scratch work:

$$\begin{aligned} |8x - 3 - 37| &< \epsilon \\ |8x - 40| &< \epsilon \\ 8|x - 5| &< \epsilon \\ |x - 5| &< \frac{\epsilon}{8} \end{aligned}$$

Then for any $\epsilon > 0$ let $\delta < \frac{\epsilon}{8}$. Then if $|x - 5| < \delta < \epsilon/8$ we have $8|x - 5| < \epsilon$ or $|8x - 3 - 37| < \epsilon$.

Problem 2

Prove that $\lim_{x \to -1} (4x + 6) = 2$.

For scratch work:

$$\begin{aligned} |4x + 6 - 2| &< \epsilon \\ |4x + 4| &< \epsilon \\ 4|x + 1| &< \epsilon \\ 4|x - (-1)| &< \epsilon \\ |x - (-1)| &< \frac{\epsilon}{4} \end{aligned}$$

Then for any $\epsilon > 0$ use $\delta < \epsilon/4$. If $|x - (-1)| < \delta < \epsilon/4$ then $4|x + 1| < \epsilon$ or $|4x + 6 - 2| < \epsilon$.

Problem 3

Prove that $\lim_{x \to 0} (x + 2) = 2$.

For scratch work:

$$\begin{aligned} |x + 2 - 2| &< \epsilon \\ |x| &< \epsilon \end{aligned}$$

So for any $\epsilon > 0$ use $|x| < \epsilon$. Then $|x + 2 - 2| < \epsilon$

Problem 4

Prove that $\lim_{x \to 8}(2x - 7) = 9$.

Scratch work:

$$\begin{aligned} |2x - 7 - 9| &< \epsilon \\ |2x - 16| &< \epsilon \\ 2|x - 8| &< \epsilon \\ |x - 8| &< \frac{\epsilon}{2} \end{aligned}$$

So for any $\epsilon > 0$ use $\delta < \frac{\epsilon}{2}$. Then $|x - 8| < \delta$ implies $|x - 8| < \frac{\epsilon}{2}$ and $2|x - 8| = |2x - 7 - 9| < \epsilon$.

Problem 5

Prove that $\lim_{x \to 3}(x^2 - 2) = 7$.

Scratch work:

$$\begin{aligned} |x^2 - 2 - 7| &< \epsilon \\ |x^2 - 9| &< \epsilon \\ |x - 3||x + 3| &< \epsilon \\ |x - 3||x + 3| &< \epsilon \end{aligned}$$

Now as x approaches 3, the $x + 3$ term approaches 6. We can constrain $|x - 3| < 1$ so that x is within 1 unit of 3. Since x is bounded above by 4, $x + 3$ must be less than 7. So we can continue the scratch work:

$$\begin{aligned} |x - 3||x + 3| &< |x - 3|7 && < \epsilon \\ |x - 3| &< \frac{\epsilon}{7} \end{aligned}$$

So for any $\epsilon > 0$ use $\delta < \min\{1, \epsilon/7\}$. Then

$$|x - 3| < \delta \implies |x - 3| < \min\{1, \epsilon/7\}$$

If $|x - 3| < \delta$ then x is within 1 unit of 3 so $|x + 3| < 7$ and $7|x - 3| < \epsilon$, so $|x + 3||x - 3| < \epsilon$, or that $|x^{-}2 - 7| < \epsilon$.

Problem 6

Prove that $\lim_{x \to 1}(4x^2 + 1) = 5$.

Scratch work:

$$\begin{aligned} |4x^2 + 1 - 5| &< \epsilon \\ 4|x^2 - 1| &< \epsilon \\ |x - 1||x + 1| &< \frac{\epsilon}{4} \end{aligned}$$

As x approaches 1 $x + 1$ approaches 2, so we can force x to be close enough to 1 that $x + 1$ is bounded above by 3, making

$$|x-1||x+1| < 3|x-1| < \frac{\epsilon}{4}$$
$$\implies |x-1| < \frac{\epsilon}{12}$$

Then for any $\epsilon > 0$ choose $\delta < \min\{1, \epsilon/12\}$. If $|x-1| < \delta$ then x is within 1 unit of 1, making $|x+1| < 3$ and $3|x-1| < \epsilon/4$. Then $|x+1||x-1| < \epsilon$ and $|4x^2 + 1 - 5| < \epsilon$.