13.6 Limits at Infinity

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Problem 1

$\lim\limits_{x\to\infty} \frac{1}{x^n} = 0$ if $n \in \mathbb{N}$.

To prove this for all $n \in \mathbb{N}$ use induction. For the base case $n = 1$ we want to show $\lim\limits_{x\to\infty} \frac{1}{x} = 0$. For any $\epsilon > 0$ let $N > 1/\epsilon$. Then for $x > N$ we have

$$\begin{aligned} x > N &> 1/\epsilon \\ 1/x &< \epsilon \end{aligned}$$

Now for the inductive step suppose the result holds up to an arbitrary n. Then using the limit law for products, take the inductive step:

$$\lim_{x\to\infty} \frac{1}{x^{n+1}} = \lim_{x\to\infty} \frac{1}{x^n} \cdot \frac{1}{x} = \left(\lim_{x\to\infty} \frac{1}{x^n}\right)\left(\lim_{x\to\infty} \frac{1}{x}\right) = 0 \cdot 0 = 0$$

Problem 2

$\lim\limits_{x\to\infty} \frac{3x+2}{2x-1} = \frac{3}{2}$.

We can prove this using limit laws by multiplying the function by $\frac{1/x}{1/x}$:

$$\lim_{x\to\infty} \frac{3x+2}{2x-1} = \lim_{x\to\infty} \frac{3+2/x}{2-1/x} = \frac{\lim_{x\to\infty} 3+2/x}{\lim_{x\to\infty} 2-1/x} = \frac{3+0}{2-0} = \frac{3}{2}$$

Alternatively we can do an $\epsilon - \delta$ proof starting with scratchwork:

$$\begin{aligned} \left|\frac{3x+2}{2x-1} - \frac{3}{2}\right| &< \epsilon \\ \left|\frac{6x+4-6x+3}{4x-2}\right| &< \epsilon \\ \left|\frac{7}{4x-2}\right| &< \epsilon \\ \frac{7}{2}\left|\frac{1}{2x-1}\right| &< \epsilon \\ \frac{1}{2x-1} &< \frac{2\epsilon}{7} \\ 2x-1 &> \frac{7}{2\epsilon} \\ 2x &> \frac{7}{2\epsilon} + 1 \\ x &> \frac{7}{4\epsilon} + \frac{1}{2} \end{aligned}$$

Note that we can drop the absolute value bars since for sufficiently large x the fraction $\frac{1}{2x-1}$ will be positive. Now for any $\epsilon > 0$, let $x > \frac{7}{4\epsilon} + \frac{1}{2}$. Then follow the scratchwork above to arrive at the limit.

Problem 3

If $a \in \mathbb{R}$, then $\lim_{x\to\infty} a = a$.

This says that constant functions have limits at infinity. To formalize this let $f(x) = a$ be the constant a function. Then $|f(x) - a| = |a - a| = 0$ which is necessarily less than any $\epsilon > 0$, regardless of a delta or the location of x. So for any domain value c, any $\epsilon > 0$ let $\delta = \epsilon$ and if $|x - c| < \delta$ then $|f(x) - a| < \epsilon$ (and we don't even need the 'if' part).

Problem 4

If $\lim_{x\to\infty} f(x)$ exists, and $a \in \mathbb{R}$, then $\lim_{x\to\infty} a(f(x)) = a \lim_{x\to\infty} f(x)$.

This says that constants factor out of limits. Prove this by applying the limit law for products:

$$\lim_{x\to\infty} af(x) = \left(\lim_{x\to\infty} a\right)\left(\lim_{x\to\infty} f(x)\right) = a \lim_{x\to\infty} f(x)$$

Problem 5

If both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ exist, then $\lim_{x\to\infty} (f(x) + g(x)) = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$.

This is the sum rule for limits. Prove this by using the $\epsilon/2$ trick. Call

$$\lim_{x\to\infty} f(x) = L, \lim_{x\to\infty} g(x) = M$$

For any $\epsilon > 0$, choose x sufficiently large that $|f(x) - L| < \epsilon/2$ and $|g(x) - M| < \epsilon/2$. Then

$$|f(x) + g(x) - (L + M)| \leq |f(x) - L| + |g(x) - M| < \epsilon/2 + \epsilon/2 = \epsilon$$

Problem 6

If both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ exist, then $\lim_{x\to\infty} (f(x) \cdot g(x)) = \lim_{x\to\infty} f(x) \cdot \lim_{x\to\infty} g(x)$.

This is the product rule for limits. Prove this by adding and subtracting a form of zero in the scratchwork:

$$\begin{aligned}
|f(x)g(x) - L \cdot M| &< \epsilon \\
|f(x)g(x) - f(x)M + f(x)M - L \cdot M| &< \epsilon \\
|f(x)(g(x) - M) + M(f(x) - L)| &< \epsilon \\
|f(x)(g(x) - M)| + |M(f(x) - L)| &< \epsilon
\end{aligned}$$

From here we need to use a little trickery to account for cases where L, M might be negative or zero. But first notice that $f(x)$ is still in the inequality. Since $f(x) \to L$ we can bound this value by $|L| + 1$, i.e. $f(x) < |L| + 1$ for sufficiently large x. This with Cauchy-Schwarz gives us

$$|f(x)(g(x) - M)| + |M(f(x) - L)| < (|L| + 1)|g(x) - M| + |M||f(x) - L|$$

Make $g(x) - M$ and $f(x) - L$ so small that even with their multipliers, the term will still come out smaller than $\epsilon/2$. For $|f(x) - L|$ that would be $\epsilon/2|M|$ except that M might be zero. To avoid this we can add 1 to the denominator so that $|f(x) - L| < \epsilon/2(|M| + 1)$.

Now we can state the formal proof: for any $\epsilon > 0$ choose x sufficiently large so that $|f(x) - L| < \epsilon/2(|M| + 1)$ and $|g(x) - M| < \epsilon/2(|L| + 1)$. Then the scratchwork above shows that $|f(x)g(x) - LM| < \epsilon$.

Problem 7

If both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ exist, then $\lim_{x\to\infty}(f(x)-g(x)) = \lim_{x\to\infty} f(x) - \lim_{x\to\infty} g(x)$.

This statement is equivalent to the limit rule for sums but where the function is $-g(x)$. By the previously proven results for sums and constant multipliers:

$$\lim_{x\to\infty}(f(x)-g(x)) = \lim_{x\to\infty} f(x) + (-g(x)) = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} -g(x) = \lim_{x\to\infty} f(x) - \lim_{x\to\infty} g(x)$$

Problem 8

If both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ exist, and $\lim_{x\to\infty} g(x) \neq 0$, then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)}$.

If we can prove that $\lim_{x\to\infty} 1/g(x) = 1/\lim_{x\to\infty} g(x)$ then we can apply the product rule for limits to the function $f(x) \cdot 1/g(x)$. For scratchwork:

$$\begin{aligned}
&\left|\frac{1}{g(x)} - \frac{1}{M}\right| < \epsilon \\
&\left|\frac{g(x)-M}{Mg(x)}\right| < \epsilon \\
&|g(x)-M| < \epsilon|Mg(x)| \\
&|g(x)-M| < \epsilon|M||M+1|
\end{aligned}$$

At the end there, we bind $|g(x)| < |M+1|$ since for sufficiently large x, $g(x)$ gets arbitrarily close to M. So we can choose x large enough that $|g(x)-M| < \epsilon|M||M+1|$ and follow the scratchwork to prove that $\lim_{x\to\infty} 1/g(x) = 1/\lim_{x\to\infty} g(x)$. This proves the quotient rule for limits to infinity.

Problem 9

This was already proven as Theorem 13.9.

Problem 10

Prove that $\lim_{x\to\infty} \sin(x)$ does not exist.

Of course the sine function oscillates forever and does not converge to any limit as $x \to \infty$. To formalize this, derive a contradiction. Suppose $\lim_{x\to\infty} \sin(x) = L$. Whatever value L is there of course will exist some increment $x + y$ such that $\sin(x+y) \neq L$, which can show the contradiction. To formalize, we can note that $\sin(x+\pi) = -\sin(x) = -L$. If $L = 0$ then x is a multiple of π and then $\sin(x+\frac{\pi}{2}) = \pm 1$, which cannot pass the $\epsilon - \delta$ test. If $L \neq 0$ then there is an interval of length $2|L|$ between L and $-L$ and we can use that to derive a contradiction as well. Let $\epsilon < |L|/2$. Then supposedly there is an N sufficiently large that $x > N$ ensures $|\sin(x) - L| < |L|/2$. However, for such an x consider $\sin(x+\pi) = -L$. Then $|\sin(x+\pi) - L| = 2|L| > |L|$, which is a contradiction.