Your Document Title

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December 3, 2024

Problem 1

Given two or more functions $f_1, f_2, \ldots, f_n$ suppose that $\lim_{x\to c} f_i(x)$ exists for each $1 \le i \le n$. Prove that $\lim_{x\to c}(f_1(x) + f_2(x) + \ldots + f_n(x)) = f_1(c) + f_2(c) + \ldots + f_n(c)$. Use induction on n, with Theorem 13.5 (sum rule for limits) as the base case.

The base case is proven in Theorem 13.5. Now assume this holds for n functions. Let $g(x) = f_1(x) + f_2(x) + \ldots + f_n(x)$, which is just another function. Then we can apply the base case again to $g(x) + f_{n+1}(x)$ and the result holds.

Problem 2

Given two or more functions $f_1, f_2, \ldots, f_n$, suppose that $\lim_{x\to c} f_i(x)$ exists for each $1 \le i \le n$. Prove that $\lim_{x\to c}(f_1(x) \cdot f_2(x) \cdot \ldots \cdot f_n(x)) = f_1(c) \cdot f_2(c) \cdot \ldots \cdot f_n(c)$. Use induction on n, with Theorem 13.7 (product rule for limits) as the base case.

Following much the same as before, the theorem proves our base case for the product of two functions. Now suppose the result holds for the product of n functions. Let $g(x) = f_1(x) \cdot \ldots \cdot f_n(x)$, and apply the base case to $g(x) \cdot f_{n+1}(x)$.

Problem 3

Use the previous two exercises adn the constant multiple rule to prove that if $f(x)$ is a polynomial, then $\lim_{x\to c} f(x) = f(c)$ for any $c \in \mathbb{R}$.

In other words, we're asked to prove that all real-variable polynomials are everwhere continuous. We can prove this by induction on the degree of a polynomial, since polynomials by definition have a finite degree but it can be arbitrarily large. For a base case assume the degree is 0 and we have a constant function. This is continuous everywhere. Now assume the result holds for a polynomial of degree n. If we add an $n+1$ degree term it takes the form $g(x) = ax^{n+1}$ for some nonzero coefficient a. With the exponent, this term is equivalent to $a \cdot \underbrace{x \cdot x \cdot \ldots \cdot x}_{n+1 \text{ times}}$. By the constant multiple rule and the product rule we have

$$\lim_{x\to c} g(x) = \lim_{x\to c} ax \cdot x \ldots \cdot x = a \lim_{x\to c} x \cdot x \ldots \cdot x = ac \cdot c \ldots \cdot c = ac^{n+1} = g(c)$$

By the inductive hypothesis the polynomial's first n terms' sum is continuous, so we can sum the lower-degree terms in the limit as well and find that the $n+1$ degree polynomial is continuous.

Problem 4

Use Exercise 3 with a limit law to prove that if $\frac{f(x)}{g(x)}$ is a rational function (a polynomial divided by a polynomial), and $g(c) \neq 0$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}$.

To apply the division rule for limits it requires that the limits exist individually for the numerator and denominator. Since f and g are polynomials they are continuous everywhere and the limits exist at c for them individually. Then by the division rule, as long as $g(c) \neq 0$, the limit divides here as well.

Problem 5

Use Definition 13.2 (epsilon-delta definition of the limit) to prove that limits are unique in the sense that if $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} f(x) = M$, then $L = M$.

Suppose at least one limit exists and there are two symbols L and M for limits of f as x approaches c.

ATAC that $L \neq M$. Then there is some distance between them, call it ϵ. Since the limit of f exists as x approaches c, this requires for $\epsilon/2$ there is some δ_1 that $|x - c| < \delta_2$ implies that $|f(x) - L| < \epsilon/2$. In other words, $f(x) \in (L - \frac{\epsilon}{2}, L + \frac{\epsilon}{2})$.

But we can say the same thing about M: that for $\epsilon/2$ there is some δ_2 where $|x-c| < \delta_2 \implies |f(x)-M| < \epsilon/2$. So $f(x) \in (M - \frac{\epsilon}{2}, M + \frac{\epsilon}{2})$.

In fact if we set the challenge at $\epsilon/2$ and let $\delta = \min\{\delta_1, \delta_2\}$, then for any x within δ of c requires $f(x)$ to be in both intervals. But this is impossible because L and M are ϵ apart as we stated earlier, so the intervals

$$\left(L - \frac{\epsilon}{2}, L + \frac{\epsilon}{2}\right) \quad \text{and} \quad \left(M - \frac{\epsilon}{2}, M + \frac{\epsilon}{2}\right)$$

have no intersection and $f(x)$ can't be in both. Therefore $L = M$.

Problem 6

Prove the *squeeze theorem*: Suppose $g(x) \leq f(x) \leq h(x)$ for all $x \in \mathbb{R}$ satisfying $0 < |x - c| < \delta$ for some $\delta > 0$. If $\lim_{x \to c} g(x) = L = \lim_{x \to c} h(x)$, then $\lim_{x \to c} f(x) = L$.

ATAC that g and h converge to L as $x \to c$ but f does not. It could be that f diverges to infinity or it could be f converges to some limit M unequal to L. Both, however would lead at a contradiction.

First consider the case that f diverges to $\pm\infty$ as x gets closer to c. This means that for any magnitude M, there is a small enough δ where $|x - c| < \delta$ implies $|f(x)| > M$. But we also know that $f(x) \leq h(x)$ and $h(x)$ is converging to L. So we could set the magnitude to beat be L itself; then there's a δ_1 that constrains $|f(x)| > L$ and a δ_2 that constrains $|h(x) - L| < \epsilon$ for any $\epsilon > 0$. This is a contradiction because then for $\delta = \min\{\delta_1, \delta_2\}$ we have $f(x) > L$ and $h(x) < L$ for $|x - c| < \delta$, contradicting $f(x) \leq h(x)$ (a similar argument is made if f diverges to $-\infty$, using the fact that $f(x) \geq g(x)$).

Now consider the case that f converges to some M unequal to L. Then there is a distance between the two limits, call it ϵ. Then for any $\epsilon > 0$ let δ_1 be small enough that $|g(x) - L| < \epsilon/2$ and $|h(x) - L| < \epsilon/2$, and let δ_2 be small enough that $|f(x) - M| < \epsilon/2$. Now setting $\delta = \min\{\delta_1, \delta_2\}$, we have

$$g(x) \in \left(L - \frac{\epsilon}{2}, L + \frac{\epsilon}{2}\right), h(x) \in \left(L - \frac{\epsilon}{2}, L + \frac{\epsilon}{2}\right), f(x) \in \left(M - \frac{\epsilon}{2}, M + \frac{\epsilon}{2}\right)$$

This is a contradiction because it is violating the inequalities we started with. If $M > L$ then $f(x) > h(x)$, and if $M < L$ then $f(x) < g(x)$.