

# 11.5

Benjamin Basseri

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## Problem 1

Write the addition and multiplication tables for  $\mathbb{Z}_2$ .

+	0	1
0	0	1
1	1	1

, 

×	0	1
0	0	0
1	0	1

## Problem 2

Write the addition and multiplication tables for  $\mathbb{Z}_3$ .

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

, 

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

## Problem 3

Write the addition and multiplication tables for  $\mathbb{Z}_4$ .

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

, 

×	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

## Problem 4

Write the addition and multiplication tables for  $\mathbb{Z}_6$ .

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

, 

×	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

## Problem 5

Suppose  $[a], [b] \in \mathbb{Z}_5$  and  $[a] \cdot [b] = [0]$ . Is it necessarily true that either  $[a] = [0]$  or  $[b] = [0]$ ?

Yes because we can see from its multiplication chart that the only products becoming  $[0]$  come from factors of  $[0]$ . More generally any member of  $[5] = 5x$  for some integer  $x$ . In the prime factorization of  $5x$ , the prime 5 has at least a power of 1. And if  $5x = a \cdot b$  for some integers  $a$  and  $b$ , it would require that either  $a$  or  $b$  have a factor of 5 which would require  $[a] = 0$  or  $[b] = 0$ .

#### Problem 6

Suppose  $[a], [b] \in \mathbb{Z}_6$  and  $[a] \cdot [b] = [0]$ . Is it necessarily true that either  $[a] = [0]$  or  $[b] = [0]$ ? What if  $[a], [b] \in \mathbb{Z}_7$ ?

No, we saw from its multiplication table that nonzero classes could multiply to  $[0]$  such as  $[2] \cdot [3] = [0]$ . More generally, the prime factorization of any element in  $[0]$  looks like  $2 \cdot 3 \cdot x$  for some integer  $x$ . This means if  $2 \cdot 3 \cdot x = a \cdot b$  for some integers  $a$  and  $b$ , it could be that  $a$  provides the factor of 2 and  $b$  provides the factor of 3. For example,  $a = 2, b = 3, x = 1$ .

If we're in  $\mathbb{Z}_7$  we might observe that we're in a field which would require that  $[a] = [0]$  or  $[b] = [0]$ . If we don't make that observation again we could argue using prime decomposition. The prime factorization of any number in  $[0] = 7n$  for some integer  $n$ . If  $7n = a \cdot b$  for integers  $a$  and  $b$ , then either  $a$  or  $b$  must have a factor of 7, since 7 is prime and cannot be the product of smaller factors. And if  $a$  or  $b$  has a factor of 7 then it is a multiple of 7 and belongs to  $[0]$ .

#### Problem 7

Do the following calculations in  $\mathbb{Z}_9$ , in each case expressing your answer as  $[a]$  with  $0 \leq a \leq 8$ .

1.  $[8] + [8]$
2.  $[24] + [11]$
3.  $[21] \cdot [15]$
4.  $[8] \cdot [8]$

1.  $[8] + [8] = [16] = [7]$
2.  $[24] + [11] = [35] = [8]$
3.  $[21] \cdot [15] = [3] \cdot [6] = [18] = [0]$
4.  $[8] \cdot [8] = [64] = [1]$

#### Problem 8

Suppose  $[a], [b] \in \mathbb{Z}_n$  and  $[a] = [a']$  and  $[b] = [b']$ . Alice adds  $[a]$  and  $[b]$  as  $[a] + [b] = [a + b]$ . Bob adds them as  $[a'] + [b'] = [a' + b']$ . Show that their answers  $[a + b]$  and  $[a' + b']$  are the same.

Since  $a \in [a]$  we can write it as  $a = xn + c$  where  $c$  is the remainder. Likewise  $a' \in [a]$  so  $a' = x'n + c$ . Also write  $b = yn + d, b' = y'n + d$ .

Now  $a + b = (x + y)n + (c + d)$  so it is in the equivalence class  $[c + d]$ . Likewise  $a' + b' = (x' + y')n + (c + d)$  so it is in the equivalence class  $[c + d]$ . Therefore Alice and Bob's answers are the same.