14.4 The Cantor-Bernstein Schröder Theorem

Benjamin Basseri

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Problem 1

Show that if $A \subseteq B$ and there is an injection $g : B \to A$, then $|A| = |B|$.

Use Schroeder-Bernstein.

If we show an injection $f : A \to B$ exists then we'll have injections both ways and can invoke Schroder-Bernstein. Since $A \subseteq B$, we can define f as the inclusion map $f(a) = a$ in B. This map is injective since if $f(a) = f(a')$ then $a = a'$. Since we have injections both ways, there exists a bijection between A and B and they have equal cardinality.

Problem 2

Show that $|\mathbb{R}^2| = |\mathbb{R}|$.

Transform to easier sets and use Schroeder-Bernstein.

We can construct an injection from $\mathbb{R} \to \mathbb{R}^2$ by $x \mapsto (x, 0)$ or any number of simple maps like that. Going the other way seems trickier. However, we know from previous exercises that $|\mathbb{R}| = |(0,1)|$. To construct an injection from $(0,1) \times (0,1)$ to $(0,1)$ might be a bit more straightforward: any real number in this interval is 0 followed by an infinite decimal string. For $(0,1) \times (0,1)$ we can interleave the decimals to make one real in number in $(0,1)$. For instance, for $(x, y) \in (0,1)^2$ let $x = 0.x_1x_2x_3\ldots$ and $y = 0.y_1y_2y_3\ldots$. Then let $f(x, y) = 0.x_1y_1x_2y_2x_3y_3\ldots$.

I claim that f is injective. If $f(x, y) = f(z, w)$ then $x_1 = z_1, y_1 = w_1, x_2 = z_2, \ldots$ and so on for all decimal places. This means that z matches x on all decimal places as does y match w, therefore $z = x, y = w$ and f is injective.

For the other injection we can just map $x \in (0,1)$ to $(x, 1/2)$. Having injections both ways allows us to invoke Schroeder-Bernstein, and the sets have equal cardinality.

Problem 3

Let $\mathscr{F}$ be the set of all function $\mathbb{N} \to \{0,1\}$. Show that $|\mathbb{R}| = |\mathscr{F}|$.

Notice that $\mathscr{F}$ is the set of all binary sequences. If we allow the terms of a binary sequence to represent the decimals in base-2, then the binary sequence represents a real number between 0 and 1 inclusive. Therefore we have a bijection between $\mathscr{F}$ and $[0,1]$ and they have equal cardinality. We have from previous results that the closed interval has equal cardinality to the open interval, which in turn has equal cardinality to $\mathbb{R}$. So altogether:

$$|\mathscr{F}| = |[0,1]| = |(0,1)| = |\mathbb{R}|$$

Problem 4

Let $\mathscr{F}$ be the set of all function $\mathbb{R} \to \{0,1\}$. Show that $|\mathbb{R}| < |\mathscr{F}|$.

Solution: recognize the similarity to another setting.

Another way to notate $\mathscr{F}$ is $2^{\mathbb{R}}$, meaning it is the set of all possible functions from $\mathbb{R}$ to a set size 2. This is equivalent to the size of its power set, which we know is strictly greater than $\mathbb{R}$:

$$|\mathscr{F}| = |2^{\mathbb{R}}| = |\mathscr{P}(\mathbb{R})| > |\mathbb{R}|$$

Problem 5

Consider the subset $B = \{(x, y) : x^2 + y^2 \leq 1\} \subseteq \mathbb{R}^2$. Show that $|B| = |\mathbb{R}^|$.

Solution: convert to easier sets of equal cardinality.

Let's rewrite B using polar coordinates: it's the same as (r, θ) where $r \in [0, 1]$ and $\theta = [0, 2\pi)$. The set $[0, 2\pi)$ has equal cardinality to $[0, 1)$ since the map $f(x) = x/2\pi$ is bijective. And we know from previous results that both $[0, 1]$ and $[0, 1)$ have equal cardinality to $\mathbb{R}$. Therefore:

$$|B| = |[0, 1] \times [0, 1)| = |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}^2|$$

Problem 6

Show that $|\mathscr{P}(\mathbb{N} \times \mathbb{N})| = |\mathscr{P}(\mathbb{N})|$.

We might be tempted to write:

$$|\mathscr{P}(\mathbb{N} \times \mathbb{N})| = 2^{|\mathbb{N} \times \mathbb{N}|} = 2^{|\mathbb{N}|} = |\mathscr{P}(\mathbb{N})|$$

however this isn't really rigorous since it assumes without proof that if $|A| = |B|$ then the powerset of A has the same cardinality as the powerset of B. To be more careful, we'll construct injections both ways and invoke Schroeder-Bernstein.

For an injection $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$ we can just map $n \in \mathbb{N}$ to $(n, 1) \in \mathbb{N} \times \mathbb{N}$. Defining this function on powersets, we get:

$$f : \mathscr{P}(\mathbb{N}) \longrightarrow \mathscr{P}(\mathbb{N} \times \mathbb{N})$$

$$\{n_1, n_2, \ldots\} \longmapsto \{(n_1, 1), (n_2, 1), \ldots\}$$

For an injection from $\mathscr{P}(\mathbb{N} \times \mathbb{N})$ back to $\mathscr{P}(\mathbb{N})$, consider that a subset of $\mathbb{N} \times \mathbb{N}$ is a collection of tuples $\{(a, b), (c, d), \ldots\}$. We want to map this to a set of natural numbers $\{n_1, n_2, \ldots\}$ injectively. One approach would be to leverage the uniqueness of every natural number's prime decomposition. For every tuple $(n, m) \in X \in \mathscr{P}(\mathbb{N} \times \mathbb{N})$, map it to the nth prime with exponent m. I claim this map ϕ is injective. If $\phi(X) = \phi(Y)$ then both values are subsets of $\mathbb{N}$ containing the exact same natural numbers which are primes to some exponent:

$$\phi(X) = \phi(Y) = \{(p_{n_1})^{m_1}, (p_{n_2})^{m_2}, \ldots\}$$

Since each element of this set is a prime to a power $(p_i)^{m_i}$ there is only one tuple that could have mapped to it, (n_i, m_i). Therefore $X = \{(n_1, m_1), (n_2, m_2), \ldots\} = Y$ and ϕ is injective.

Problem 7

Prove or disprove: If there is an injection $f : A \to B$ and a surjection $g : A \to B$, then there is a bijection $h : A \to B$.

This is not in general true since smaller cardinality sets will have injections to strictly larger cardinality sets, and the larger sets have surjections to the smaller sets; it doesn't mean they have the same cardinality. For example, there are injections $\mathbb{N} \to \mathbb{R}$ and surjections $\mathbb{R} \to \mathbb{N}$ but $|\mathbb{N}| \neq |\mathbb{R}|$.

One injection $\mathbb{N} \to \mathbb{R}$ is the inclusion map. For a surjection from $\mathbb{R} \to \mathbb{N}$ consider $f(x) = \lfloor |x| + 1 \rfloor$.