12.4 Composition

Benjamin Basseri

January 9, 2025

Problem 1

Suppose $A = \{5, 6, 8\}$, $B = \{0, 1\}$, $C = \{1, 2, 3\}$. Let $f : A \to B$ be the function $f = \{(5, 1), (6, 0), (8, 1)\}$, and $g : B \to C$ be $g = \{(0, 1), (1, 1)\}$. Find $g \circ f$.

We can go through the domain of f and its mappings through g or notice that g is the constant 1 function, so the set presentation of f is:

$$f = \{(5, 1), (6, 1), (8, 1)\}$$

Problem 2

Suppose $A = \{1, 2, 3, 4\}$, $B = \{0, 1, 2\}$, $C = \{1, 2, 3\}$. Let $f : A \to B$ be $f = \{(1, 0), (2, 1), (3, 2), (4, 0)\}$, and $g : B \to C$ be $g = \{(0, 1), (1, 1), (2, 3)\}$. Find $g \circ f$.

$$f(1) = 0, g(0) = 1 \implies g(f(1)) = 1$$
$$f(2) = 1, g(1) = 1 \implies g(f(2)) = 1$$
$$f(3) = 2, g(2) = 3 \implies g(f(3)) = 3$$
$$f(4) = 0, g(0) = 1 \implies g(f(4)) = 1$$

$$g \circ f = \{(1, 1), (2, 1), (3, 3), (4, 1)\}$$

Problem 3

Suppose $A = \{1, 2, 3\}$. Let $f : A \to A$ be the function $f = \{(1, 2), (2, 2), (3, 1)\}$, and let $g : A \to A$ be the function $g = \{(1, 3), (2, 1), (3, 2)\}$. Find $g \circ f$ and $f \circ g$.

$$f(1) = 2, g(2) = 1 \implies g(f(1)) = 1$$
$$f(2) = 2, g(2) = 1 \implies g(f(2)) = 1$$
$$f(3) = 1, g(1) = 3 \implies g(f(3)) = 3$$
$$g \circ f = \{(1, 1), (2, 1), (3, 3)\}$$

$$g(1) = 3, f(3) = 1 \implies f(g(1)) = 1$$
$$g(2) = 1, f(1) = 2 \implies f(g(2)) = 2$$
$$g(3) = 2, f(2) = 2 \implies f(g(3)) = 2$$
$$f \circ g = \{(1, 1), (2, 2), (3, 2)\}$$

Problem 4

Suppose $A = \{a, b, c\}$. Let $f : A \to A$ be the function $f = \{(a, c), (b, c), (c, c)\}$, and let $g : A \to A$ be the function $g = \{(a, a), (b, b), (c, a)\}$. Find $g \circ f$ and $f \circ g$.

$$f(a) = c, g(c) = a \implies g(f(a)) = a$$
$$f(b) = c, g(c) = a \implies g(f(b)) = a$$
$$f(c) = c, g(c) = a \implies g(f(c)) = a$$
$$g \circ f = \{(a, a), (b, a), (c, a)\}$$

For $f \circ g$ we only need notice that f is the constant c function, so any input g provides, $f \circ g$ also maps to c and is the contstant c function.

Problem 5

Consider the function $f, g : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \sqrt[3]{x+1}$ and $g(x) = x^3$. Find the formulas for $g \circ f$ and $f \circ g$.

$$(g \circ f)(x) = g(f(x)) = g(\sqrt[3]{x+1}) = \left(\sqrt[3]{x+1}\right)^3 = x + 1$$
$$(f \circ g)(x) = f(g(x)) = f(x^3) = \sqrt[3]{x^3+1}$$

Problem 6

Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \frac{1}{x^2+1}$ and $g(x) = 3x + 2$. Find the formulas for $g \circ f$ and $f \circ g$.

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{1}{x^2+1}\right) = \frac{3}{x^2+1} + 2$$
$$(f \circ g)(x) = f(g(x)) = f(3x+2) = \frac{1}{(3x+2)^2+1} = \frac{1}{9x^2+12x+5}$$

Problem 7

Consider the functions $f, g : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$ defined as $f(m, n) = (mn, m^2)$ and $g(m, n) = (m+1, m+n)$. Find the formulas for $g \circ f$ and $f \circ g$.

$$(g \circ f)(m, n) = g(f(m, n)) = g(mn, m^2) = (mn + 1, mn + m^2)$$
$$(f \circ g)(m, n) = f(g(m, n)) = f(m+1, m+n) = ((m+1)(m+n), (m+1)^2) = (m^2+mn+m+n, m^2+2m+1)$$

Problem 8

Consider the functions $f, g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined as $f(m, n) = (3m - 4n, m2m + n)$ and $g(m, n) = (5m + n, m)$. Find the formulas for $g \circ f$ and $f \circ g$.

$$(g \circ f)(m, n) = g(f(m, n)) = g(3m-4n, 2m+n) = (5(3m-4n)+(2m+n), 3m-4n) = (17m-19n, 3m-4n)$$
$$(f \circ g)(m, n) = f(g(m, n)) = f(5m + n, m) = (3(5m + n) - 4m, 2(5m + n) + m) = (11m + 3n, 11m + 2n)$$

Problem 9

Consider the functions $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined as $f(m,n) = m + n$ and $g : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined as $g(m) = (m,m)$. Find the formulas for $g \circ f$ and $f \circ g$

$$(g \circ f)(m,n) = g(f(m,n)) = g(m+n) = (m+n, m+n)$$
$$(f \circ g)(m) = f(g(m)) = f(m,m) = (2m, 2m)$$

Problem 10

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the formula $f(x,y) = (xy, x^3)$. Find a formula for $f \circ f$.

$$(f \circ f)(x,y) = f(xy, x^3) = ((xy)x^3, (xy)^3) = (x^4 y, x^3 y^3)$$