8 Proofs Involving Sets

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Problem 1

Prove that $\{12n : n \in \mathbb{Z}\} \subseteq \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$.

Proof: use direct proof Any multiple of 12 can be written $12n = 2 \cdot 3 \cdot 4n$, which means it is a multiple of both 2 and 3, and therefore in the intersection of the two sets on the right.

Problem 2

Prove that $\{6n : n \in \mathbb{Z}\} = \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$.

Any $n \in \{6n : n \in \mathbb{Z}\}$ can be written as $6k = 2 \cdot 3k$ for some $k \in \mathbb{Z}$. Therefore n is divisible by both 2 and 3, making it a member of both $\{2n : n \in \mathbb{Z}\}$ and $\{3n : n \in \mathbb{Z}\}$.

Problem 3

If $k \in \mathbb{Z}$, then $\{n \in \mathbb{Z} : n \mid k\} \subseteq \{n \in \mathbb{Z} : n \mid k^2\}$.

Suppose n divides k. Then n divides $k \cdot k = k^2$. Therefore $n \in \{n \in \mathbb{Z} : n \mid k^2\}$.

Problem 4

If $m, n \in \mathbb{Z}$, then $\{x \in \mathbb{Z} : mn|x\} \subseteq \{x \in \mathbb{Z} : m|x\} \cap \{x \in \mathbb{Z} : n|x\}$.

Proof: use direct proof and a relevant theorem Recall the theorem that if $a \mid b$ and $b \mid c$ then $a \mid c$. Then for any $x \in \mathbb{Z}$ such that $mn \mid x$, it must be that $m \mid x$ and $n \mid x$ by this theorem. Therefore x is a member of both sets $\{x \in \mathbb{Z} : m|x\}$ and $\{x \in \mathbb{Z} : n|x\}$, which means it is in their intersection.

Problem 5

If p and q are positive integers, then $\{pn : n \in \mathbb{N}\} \cap \{qn : n \in \mathbb{N}\} \neq \varnothing$.

Proof: directly show the intersection is nonemtpy Since p and q are positive integers they are both in $\mathbb{N}$. Therefore pq is both in $\{pn : n \in \mathbb{N}\}$ and $\{qn : n \in \mathbb{N}\}$, so the intersection is nonempty.

Problem 6

Suppose A, B, and C are sets. Prove that if $A \subseteq B$, then $A - C \subseteq B - C$.

Proof: use direct proof Suppose $a \in A - C$, then $a \in A$ and $a \notin C$. Given $A \subseteq B$, we know $a \in B$ as well. And since $a \notin C$, a will be in the set difference $B - C$. Therefore $A - C \subseteq B - C$.

Problem 7

Suppose A, B, and C are sets. Prove that if $B \subseteq C$, then $A \times B \subset A \times C$.

Proof: use direct proof Suppose $(a, b) \in A \times B$. Then $a \in A$ and $b \in B$. Given $B \subseteq C$, we know $b \in C$ as well. Therefore $(a, b) \in A \times C$.

Problem 8

If A, B and C are sets, then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof: use logic laws

$$\begin{aligned} A \cup (B \cap C) &= \{x \in A \vee (x \in B \wedge x \in C)\} \\ &= \{x \in A \vee x \in B \wedge x \in A \vee x \in C\} \\ &= \{x \in A \vee x \in B\} \wedge \{x \in A \vee x \in C\} \\ &= (A \cup B) \cap (A \cup C) \end{aligned}$$

Problem 9

If A, B and C are sets, then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof: use logic laws

$$\begin{aligned} A \cap (B \cup C) &= \{x \in A \wedge (x \in B \vee x \in C)\} \\ &= \{x \in A \wedge x \in B \vee x \in A \wedge x \in C\} \\ &= \{x \in A \wedge x \in B\} \vee \{x \in A \wedge x \in C\} \\ &= (A \cap B) \cup (A \cap C) \end{aligned}$$

Problem 10

If A, and B are sets in a universal set U, then $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Proof: show inclusion both ways Although using pure logic laws might be more concise, let's prove this by mutual inclusion.

Consider $x \in \overline{A \cap B}$. Then x does not belong to both A and B. This means x does not belong to A or it does not belong to B (or both): $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.

Now consider $x \in \overline{A} \cup \overline{B}$. If $x \notin A$ then x cannot belong to the intersection $A \cap B$, and $x \in \overline{A \cap B}$. Similarly, if $x \notin B$ then $x \in \overline{A \cap B}$. Therefore $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.

Problem 11

If A, B are sets in a universal set U, then $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Proof: use logic laws
Apply De Morgan's laws on sets: for any $x \in U$, let $\alpha(x)$ be the proposition that $x \in A$ and $\beta(x)$ be the proposition that $x \in B$.

$$\begin{aligned} \overline{A \cup B} &= \{x \in U : x \notin A \cup B\} \\ &= \{x \in U : \neg(\alpha(x) \vee \beta(x))\} \\ &= \{x \in U : \neg\alpha(x) \wedge \neg\beta(x)\} \\ &= \{x \in U : x \notin A\} \wedge \{x \in U : x \notin B\} \\ &= \overline{A} \cap \overline{B} \end{aligned}$$

Problem 12

If A, B, and C are sets, then $A - (B \cap C) = (A - B) \cup (A - C)$.

Proof: use direct proof

A logic or mutual inclusion proof might be more convincing but for variety here is a direct proof that demonstrates both sets specify the same membership criteria:

Suppose $a \in (A - B) \cup (A - C)$. Any element a in this set must belong to A. If it belongs to B but not C then a gets removed from $A - B$ but will remain in $A - C$, and then will remain in any union. Likewise if a belongs to C but not B, it will remain in $A - B$. Only if it a belongs to both B and C will it be removed from $(A - B) \cup (A - C)$. Therefore the members of $(A - B) \cup (A - C)$ are exactly the members of A with elements belongong to both B and C removed, which is the same as $A - (B \cap C)$.

Problem 13

If A, B, and C are sets then $A - (B \cup C) = (A - B) \cap (A - C)$.

Proof: use logic and De Morgan's laws

$$\begin{aligned}
A - (B \cup C) &= \{x \in A : x \notin B \cup C\} && \text{Given} \\
&= \{x \in A : \neg(x \in B \vee x \in C)\} && \text{Rewrite in logic} \\
&= \{x \in A : \neg x \in B \wedge \neg x \in C\} && \text{De Morgan} \\
&= \{x \in A \wedge x \notin B \wedge x \in A \wedge x \notin C\} && x \in A = (x \in A) \wedge (x \in A) \\
&= \{x \in A \wedge x \notin B\} \cap \{x \in A : x \notin C\} && \text{Def. intersection} \\
&= (A - B) \cap (A - C)
\end{aligned}$$

Problem 14

If A, B, and C are sets then $(A \cup B) - C = (A - C) \cap (B - C)$.

Proof: use direct proof

For variety let's use a direct proof. The set $(A \cup B) - C$ contains all elements belonging to either A or B, but not belonging to C. This is the same as taking the C members out of A, then taking the C members out of B, then putting the results together (i.e. $(A - C) \cap (B - C)$).

Problem 15

If A, B, and C are sets then $(A \cap B) - C = (A - C) \cap (B - C)$

Proof: use direct proof to show logical equivalence

The set $(A \cap B) - C$ first collects elements belonging to both A and B, then removes C members. Equivalently, you can first remove C members from A to make $A - C$, then remove C members from B to make $B - C$, then form the set of elements belonging to both $A - C$ and $B - C$, which is the intersection $(A - C) \cap (B - C)$).

Problem 16

If A, B, and C are sets then $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Proof: use logic laws This is an application of logic's distribution laws. For any $(a, b) \in A \times (B \cup C)$ you have $a \in A$ and $b \in B \cup C$. Logically:

$$(a \in A) \wedge (b \in B \vee b \in C)$$

By the distributive properties, rewrite this as:

$$((a \in A) \wedge (b \in B)) \vee ((a \in A) \wedge (b \in C))$$

which specifies the set $(A \times B) \cup (A \times C)$.

Problem 17

If A, B, and C are sets then $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Proof: use logic laws This is another application of a logical distribution law. For any $(a, b) \in A \times (B \cap C)$ you have $a \in A$ and $b \in B \cap C$. Logically:

$$(a \in A) \wedge (b \in B \wedge b \in C)$$

which equals

$$((a \in A) \wedge (b \in B)) \vee ((a \in A) \wedge (b \in C))$$

Problem 18

If A, B, and C are sets then $A \times (B - C) = (A \times B) - (A \times C)$.

Proof: use direct proof The set $A \times (B - C)$ contains all pairs (a, b) where $a \in A$ and $b \in B$ but not in C. The set $A \times B$ contains all pairs (a, b) for any $b \in B$. But subtracting the set $A \times C$ removes any pair (a, c) where $c \in C$ and $c \in B$ as well. Therefore $(A \times B) - (A \times C)$ contains all pairs (a, b) where $a \in A$ and $b \in B$ but not in C.

Problem 19

Prove that $\{9^n : n \in \mathbb{Z}\} \subseteq \{3^n : n \in \mathbb{Z}\}$, but $\{9^n : n \in \mathbb{Z}\} \neq \{3^n : n \in \mathbb{Z}\}$.

Proof: use direct proof.

These expressions say that every power of 9 is also a power of 3, but not every power of 3 is a power of 9.

Using exponent rules, $9^n = (3^2)^n = 3^{2n}$ which proves $\{9^n : n \in \mathbb{Z}\} \subseteq \{3^n : n \in \mathbb{Z}\}$. However, $3^1 = 3$ is not a power of 9. Since there is an element in the second set that is not in the first, the sets are not equal.

Problem 20

Prove that $\{9^n : n \in \mathbb{Q}\} = \{3^n : n \in \mathbb{Q}\}$.

Proof: show inclusion both ways.

Any $9^n = (3^2)^n = 3^{2n}$. Since n is rational, so is $2n$ and $3^{2n} \in \{3^n : n \in \mathbb{Q}\}$. Therefore $\{9^n : n \in \mathbb{Q}\} \subseteq \{3^n : n \in \mathbb{Q}\}$. Likewise, any 3^n can be written as $(9^{1/2})^n = 9^{n/2}$, which is also rational. Therefore $\{3^n : n \in \mathbb{Q}\} \subseteq \{9^n : n \in \mathbb{Q}\}$.

Problem 21

Suppose A and B are sets. Prove $A \subseteq B$ if and only if $A - B = \varnothing$.

Proof: chain biconditionals B contains A if and only if every element of a belongs to B as well. This occurs if and only if removing every element of B from the set A would leave A empty, i.e. $A - B = \varnothing$.

Problem 22

Let A and B be sets. Prove that $A \subseteq B$ if and only if $A \cap B = A$.

Proof: prove implication both ways

Forward direction: Suppose $A \subseteq B$. Then any $a \in A$ belongs to B as well, and $a \in A \cap B$ which means $A \subseteq A \cap B$. For any intersection we have $A \cap B \subset A$. Since we have mutual inclusion, $A \cap B = A$.

Reverse direction: Suppose $A \cap B = A$. Then any $a \in A$ must be in B as well for it to survive the intersection. Therefore $A \subseteq B$.

Problem 23

For each set $a \in \mathbb{R}$, let $A_a = \{(x, a(x^2 - 1)) \in \mathbb{R}^2 : x \in \mathbb{R}\}$. Prove that $\bigcap_{a \in \mathbb{R}} A_a = \{(-1, 0), (1, 0)\}$.

Proof: show inclusion both ways

Forward direction: use the contrapositive statement. Let's prove that if (x, y) does not equal (1, 0) or (-1, 0) then it cannot be in every A_a. Suppose $x \neq \pm 1$ and consider the set A_a, A_b, where $b \neq a$. Then the only pair with x on the first coordinate in A_a is $(x, a(x^2 - 1))$, but the only pair with x on the first coordinate in B_b is $(x, b(x^2 - 1))$. Since x is not 1 or -1, $(x^2 - 1) \neq 0$. And since $a \neq b$, it cannot be that $a(x^2 - 1) = b(x^2 - 1)$. Therefore if $x \neq \pm 1$ then $(x, a(x^2 - 1))$ cannot be in every A.

Reverse direction: use direct proof. Since $a(x^2 - 1)$ factors to $a(x + 1)(x - 1)$, its roots are ± 1 regardless of the factor a. Therefore (1, 0) and (-1, 0) satisfy $a(x^2 - 1)$ for any a, and appear in every A_a.

Problem 24

Prove that $\bigcap_{x \in \mathbb{R}} [3 - x^2, 5 + x^2] = [3, 5]$.

Proof: show inclusion both ways

Forward direction: direct proof. Note that x^2 is everywhere at least 0. So $3 - x^2$ is at most 3 and decreases as x gets further from 0. Similarly, $5 + x^2$ is at least 5 and increases as x gets further from 0. So at $x = 0$ the interval is at [3, 5] exactly. For any nonzero x, the lower boundary decreases and the upper boundary increases. Therefore all intervals in the intersection contain [3, 5]. For any wider interval $[3 - a^2, 5 + a^2]$ there is some b closer 0 than a (i.e. $|b| < |a|$) so that the wider interval gets trimmed to the shorter one during intersection. Since this is true of any nonzero number, the only sub-interval common to all intervals is $[3, 5]$.

Problem 25

Suppose A, B, C and D are sets. Prove that $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

Proof: use direct proof

For any $(a, b) \in (A \times B) \cup (C \times D)$, it must be that $(a, b) \in A \times B$ or $(a, b) \in C \times D$. So a belongs to A or C, and b belongs to B or D, and $a \in A \cup C, b \in B \cup D$. Therefore $(a, b) \in (A \cup C) \times (B \cup D)$.

Problem 26

Prove that $\{4k + 5 : k \in \mathbb{Z}\} = \{4k + 1 : k \in \mathbb{Z}\}$.

Proof: show inclusion both ways

Suppose $x = 4k + 5$ for some $k \in \mathbb{Z}$. Then $x = 4k + 4 + 1 = 4(k + 1) + 1$ implying $x \in \{4k + 1 : k \in \mathbb{Z}\}$. If

$x = 4k + 1$ for some $k \in \mathbb{Z}$ then $x = 4(k-1) + 5$ implying $x \in \{4k + 5 : k \in \mathbb{Z}\}$.

Problem 27

Prove that $\{12a + 4b : a, b \in \mathbb{Z}\} = \{4c : c \in \mathbb{Z}\}$

Proof: show inclusion both ways. $12a + 4b = 4(3a + b)$. Since a, b are integers, $3a + b$ is in integer as well. Therefore any element in the left set also belongs to the right set. Conversely, for any $4c$ let $a = 0$ so $4c = 12(0) + 4c$, which matches the criteria for the left set.

Problem 28

Prove that $\{12a + 25b : a, b \in \mathbb{Z}\} = \mathbb{Z}$.

Proof: show inclusion both ways. For the forward direction, the result of $12a + 4b$ for any integers a, b results in an integer, so the left set is contained in the integers. For the reverse direction, any integer n can be written as $12(-2n) + 25n$, therefore the left set contains all integers.

Problem 29

Suppose $A \neq \varnothing$. Prove that $A \times B \subset A \times C$ if and only if $B \subset C$.

Proof: show implication both ways Suppose that $A \times B \subseteq A \times C$. Then $(a, b) \in A \times B$ implies $(a, b) \in A \times C$ and therefore $b \in C$. This means that $B \subseteq C$. Now suppose $B \subseteq C$. Then for any $b \in B$ we have $b \in C$ as well. If any pair $(a, b) \in A \times B$ it must be that $b \in B$, which implies $b \in C$ as well and therefore $(a, b) \in A \times C$.

Problem 30

Prove that $(\mathbb{Z} \times \mathbb{N}) \cap (\mathbb{N} \times \mathbb{Z}) = \mathbb{N} \times \mathbb{N}$.

Proof: show inclusion both ways
In forming the intersection, both the first and second coordinates must be numbers commong to both $\mathbb{N}$ and $\mathbb{Z}$. There are just the naturals $\mathbb{N}$, therefore $(\mathbb{Z} \times \mathbb{N}) \cap (\mathbb{N} \times \mathbb{Z})$ is contained in $\mathbb{N} \times \mathbb{N}$. Conversely, for pair of naturals $(a, b) \in \mathbb{N} \times \mathbb{N}$ the numbers a and b are both naturals and integers, so $(a, b) \in \mathbb{Z} \times \mathbb{N}$ and $(a, b) \in \mathbb{N} \times \mathbb{Z}$, therefore $(a, b) \in (\mathbb{Z} \times \mathbb{N}) \cap (\mathbb{N} \times \mathbb{Z})$.

Problem 31

Suppose $B \neq \varnothing$ and $A \times B \subseteq B \times C$. Prove that $A \subseteq C$.

Proof: use direct proof with a bit of contradition. Since $A \times B \subseteq B \times C$ we must have $A \subseteq B$ and $B \subseteq C$ and inclusion is transitive therefore $A \subseteq C$. If we did not have $A \subseteq B$ there could be an $a \in A \setminus B$ and then the pair (a, b) could not belong to $B \times C$. Likewise if B was not contained in C then it would have some b for which (a, b) would belong to $A \times B$ but not $B \times C$.