Linear Algebra Done Right

Section 3A

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1. Suppose $b, c \in \mathbb{R}$. Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz)$$

Show that T is linear if and only if $b = c = 0$.

Solution. For the forward proof, assume T is linear. Suppose we have two vectors in $\mathbb{R}^3$, $\mathbf{x} = (x, y, z)$ and $\mathbf{x}' = (x', y', z')$. Then by definition of T and vector addition:

$$T(\mathbf{x} + \mathbf{x}') = T(x + x', y + y', z + z')$$
$$= (2(x + x') - 4(y + y') + 3(z + z') + b, 6(x + x') + c(x + x')(y + y')(z + z'))$$

Since T is linear, it is additive and we can also write $T(\mathbf{x} + \mathbf{x}')$ as:

$$\begin{aligned}
T(\mathbf{x} + \mathbf{x}') &= T(\mathbf{x}) + T(\mathbf{x}') && \text{linearity}\\
&= (2x - 4y + 3z + b, 6x + cxyz)\\
&\quad + (2x' - 4y' + 3z' + b, 6x' + cx'y'z')\\
&= \big(2(x + x') - 4(y + y') + 3(z + z') + 2b, 6(x + x') + c(xyz + x'y'z')\big)
\end{aligned}$$

By matching coordinates, we have $2b = b \implies b = 0$. We must also have $c(x + x')(y + y')(z + z') = c(xyz + x'y'z')$ for all $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^3$, which implies $c = 0$.

Since T is linear it has homogeneity, so we can write:

$$\lambda T(\mathbf{x}) = \lambda(2x - 4y + 3z + b, 6x + cxyz) = T(\lambda \mathbf{x})$$

by distributivity of multiplication.

For the reverse direction, assume $b = c = 0$. Then from the expansion of $T(\mathbf{x} + \mathbf{x}')$ above we see that the surviving terms in both coordinates satisfy linearity.