Linear Algebra and Learning from Data

Chapter 1

Selected solutions by Benjamin Basseri

1. Give an example where a combination of three nonzero vectors in $\mathbb{R}^4$ is the zero vector. Then write your example in the form $Ax = 0$. What are the shapes of A, x and 0?

   We can choose two vectors then make the third vector a linear combination of the first two:
   $$\mathbf{a} + \mathbf{b} + (\lambda_1 \mathbf{a} + \lambda_2 \mathbf{b})$$

   In this case we may choose:
   $$a = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, c = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

   Then $a + b - c = 0$. This linear combination of a, b, c can be expressed in matrix form as:
   $$[a\ b\ c] \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \mathbf{0}$$

   where A is 4×3, x is 3×1, and 0 is 3×1.

2. Suppose a combination of the columns of A equals a different combination of those columns. Write that as $Ax = Ay$. Find two combinations of the columns of A that equal the zero vector (in matrix language, find two solutions to $Az = 0$).

   $$Ax = Ay \implies Ax - Ay = 0 \implies A(x - y) = 0 \implies x - y \in \ker A$$

   This also implies $y - x$ is in the null space of A since we could subtract Ax from both sides in the first step just as well.

3. The vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are in m-dimensional space $\mathbb{R}^m$, and a combination $c_1 \mathbf{a}_1 + \ldots + c_n \mathbf{a}_n$ is the zero vector. That statement is at the vector level.

a) Write that statement at the matrix level. Use the matrix A with the $\mathbf{a}$'s in its columns and use the column vector $\mathbf{c} = (c_1, \ldots, c_n)$.

One of Strang's pictures of matrix multiplication describes $A\mathbf{c}$ as a linear combination of the columns of A, with the combination given by the components of $\mathbf{c}$. Thus

$$A\mathbf{c} = \begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = 0$$

$$\iff c_1\mathbf{a}_1 + \ldots c_n\mathbf{a}_n = 0$$

b) Write that statement at the scalar level. Use subscripts and sigma notation to add up numbers. The column vector $\mathbf{a}_j$ has components $a_{1j}, a_{2j}, \ldots, a_{mj}$.

From the matrix picture we can write out in summation for the jth component of $A\mathbf{c}$:

$$(A\mathbf{c})_j = \sum_{i=0}^{m} c_i a_{ji} = 0$$

$$\text{altogether: } A\mathbf{c} = \sum_{j=1}^{n}\sum_{i=1}^{m} c_i a_{ij}\mathbf{e}_j = \mathbf{0}$$

4. Suppose A is the 3 by 3 matrix of all ones. Find two independent vectors x and y that solve $Ax = 0$ and $Ay = 0$. Write that first equation $Ax = 0$ (with numbers) as a combination of the columns of A. Why don't I ask fo ra third independent vector with $Az = 0$?

   We can choose $x = (1, -1, 0), y = (0, 1, -1)$. These are independent since clearly there is no scalar λ such that $\lambda x = y$: the scalar λ would have to map $x_1 \mapsto 0$ which implies $\lambda = 0$, but then $x_2 \mapsto 1$ which implies $\lambda \neq 0$.

   Let $\mathbf{1}$ be the vector of 1's, which in this case is each column of A. Then

$$Ax = \sum_{i=1}^{3} x_i\mathbf{1} = x_1\mathbf{1} + x_2\mathbf{1} + x_3\mathbf{1} = 1\cdot\mathbf{1} - 1\cdot\mathbf{1} + 0 = 0$$

   There is no 3rd independent vector in the null space of A. This follows from rank-nullity: A has a rank of 1, its column space is the subspace spanned by $\mathbf{1}$ (a line). Since we're in $\mathbb{R}^3$, that means $N(A)$ has dimension 2.

5. The linear combinations of $v = (1, 1, 0)$ and $w = (0, 1, 1)$ fill a plane in $\mathbb{R}^3$.

   a) Find a vector z that is perpendicular to Wv and w.

   Such a vector z must have a 0 dot product with v and w. By inspection we see that if $z = (1, -1, 1)$ then the dot product with either v or w will get a 1 from the first or third component and a -1 from the second:

$$z \cdot v = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$z \cdot w = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0$$

b) Find a vector u that is not on the plane. Check that $u^\top z \neq 0$

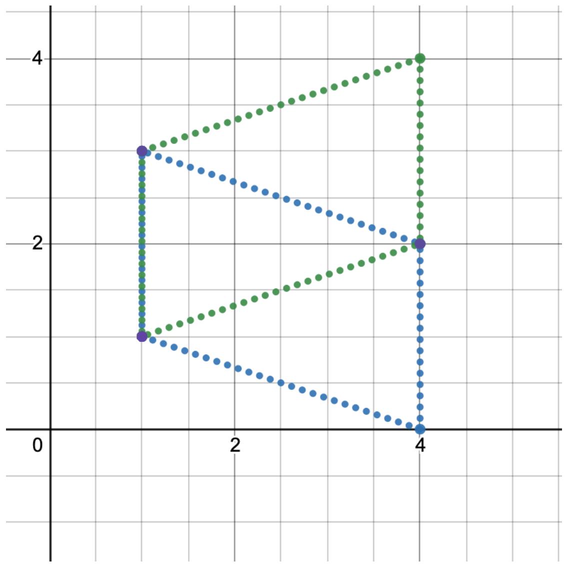
Of course z is not in the plane and since $z \neq 0$, we have $z^\top z \neq 0$.

6. If three corners of a parallelogram are (1,1), (4,2), and (1,3), what are all three of the possible fourth corners? Draw two of them.

By the parallelogram rule of vector addition, we can take the vector represented by a point x and add the other two points' *differences* to x to get the fourth point of the parallelogram. Starting at $(1,1)$, we have differences to $(1,3)$ and $(4,2)$ as follows:

$$(1,3) - (1,1) = (0,2), \quad (4,2) - (1,1) = (3,1)$$

so a point that makes a parallelogram would be (1,1) + (0, 2) + (3,1) = (4, 4).



7. Describe the column space of $A = [v \; w \; v + 2w]$. Describe the nullspace of A: all vectors $x = (x_1, x_2, x_3)$ that solve $Ax = 0$. Add the "dimensions" of that plane (the column space of A and that line (the nullspace of A).

Let $C(A)$ denote the column space of A. Since the 3rd column is a linear combination of the first 2, $C(A) = C([v, w]) = \text{span}\{v, w\}$. Assuming $w \neq \lambda v$ the null space $N(A) = \{(1, 2, -1)\}$, since for any $x \in N(A) = \lambda(1, 2, -1)$ for some λ, then

$$Ax = \lambda A(1, 2, -1) = v + 2w - (v + 2w) = 0$$

A spans a 2-dimensional plane and the nullspace is 1-dimensional, in keeping with rank-nullity.

8. $A = CR$ is a representation of the columns of A in the basis formed by the columns of C with coefficients in R. If $A_{ij} = j^2$ is 3 by 3, write down A and C and R.

   Given the recipe $A_{ij} = j^2$ we have

$$A = \begin{bmatrix} 1 & 4 & 9 \\ 1 & 4 & 9 \\ 1 & 4 & 9 \end{bmatrix}$$

   The columns of A are linearly dependent, we may keep the first column for C and then R is the matrix necessary to make $CR = A$:

$$C = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 4 & 9 \end{bmatrix}$$

9. Suppose the column space of an $m \times n$ matrix is all of $\mathbb{R}^3$. What can you say about m? What can you say about n? What can you say about the rank r?

   Given the span of a matrix is $\mathbb{R}^3$ it must be $m = 3$ and $n \geq 3$. If m, the number of rows, was any less than 3 it would not be able to span $\mathbb{R}^3$ (we omit the possibility that $m > 3$ and it spans a 3-dimensional subspace isomorphic to $\mathbb{R}^3$). Likewise if the number of columns n is less than 3 then the matrix could not have 3 linearly independent vectors, which are necessary to form a minimally spanning set of $\mathbb{R}^3$. Altogether this means the rank r of the matrix is 3.

10. Find the matrices C_1 and C_2 containing independent columns of A_1 and A_2:

$$A_1 = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ 2 & 6 & -4 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

   For A_1 we see its second and third columns are multiples of the first. So $C_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$.

   For $A_2 = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ we see the third column is a linear combination of the other two: $\mathbf{a}_3 = 2\mathbf{a}_2 - \mathbf{a}_1$. The first two columns are linearly independent which makes $C_2 = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}$

11. Factor each of those matrices into $A = CR$. The matrix R will contain the numbers that multiply the columns of C to recover columns of A.

    Having identified which columns are linearly columns of which, we encode that information into R:

$$C_1 = \begin{bmatrix}1\\3\\2\end{bmatrix}, \qquad R_1 = \begin{bmatrix}1 & 3 & -2\end{bmatrix}$$

$$C_2 = \begin{bmatrix}1 & 2\\4 & 5\\7 & 8\end{bmatrix}, \qquad R_2 = \begin{bmatrix}1 & 0 & -1\\0 & 1 & 2\end{bmatrix}$$

$$\begin{array}{cc} & \begin{bmatrix}1 & 0 & -1\\0 & 1 & 2\end{bmatrix} \\ \begin{bmatrix}1 & 2\\4 & 5\\7 & 8\end{bmatrix} & \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}\end{array}$$

12. Produce a basis for the column spaces of A_1 and A_2. What are the dimensions of those columns spaces – the number of independent vectors? What are the ranks of A_1 and A_2? How many independent rows in A_1 and A_2?

    By construction the columns of C_1 and C_2 form bases for A_1 and A_2 since they are sets of linearly independent vectors that span the column space of the matrix. So we may choose for bases B_1 and B_2:

$$B_1 = \left\{ \begin{bmatrix}1\\3\\2\end{bmatrix} \right\}, \qquad B_2 = \left\{ \begin{bmatrix}1\\4\\7\end{bmatrix}, \begin{bmatrix}2\\5\\8\end{bmatrix} \right\}$$

    The dimension of each column space is the number of linearly independent vectors in each basis. Hence $\dim C(A_1) = 1, \dim C(A_2) = 2$. Equivalently, these are the ranks of matrices A_1 and A_2. Lastly, the rank of A is equal to the rank of $A^\top$ for a square A, so the matrices have the same number of independent rows as they do columns.

13. Create a 4 by 4 matrix of rank 2. What shapes are C and R?

    Before construction, we know the shape of C will be 4×2. It must have 4 rows since it much match the original matrix. It will have 2 columns since we're constructing the original matrix to have 2 linearly independent columns.

    Let us construct the matrix:

$$A = \begin{bmatrix}1 & 0 & 1 & -1\\0 & 1 & 1 & -1\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

The first two columns of A are independent and the second two are linear combinations of the others. This gives C and R:

$$\begin{array}{cc} & \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

14. Suppose two matrices A and B have the same column space.

    a) Show that their row spaces can be different

    Consider the following matrices:

    $$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

    The column space of A is $\{(1,0,0),(0,1,0)\}$, or the xy plane. In B, the third column is a linear combination of the first two, so its column space is also the subspace generated by $\{(1,0,0),(0,1,0)\}$, the xy plane once again.

    However, the row spaces are different. Row space $R(M) = C(M^\top)$ for any matrix M. A is a symmetric matrix so $A^\top = A$, which implies $R(A) = C(A)$. Hence the row space of A is also the xy plane. But the row space of B is given by the column space of $B^\top$:

    $$B^\top = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

    The row space of B is then the subspace generated by the columns of $B^\top$, that is $\{(1,1,0),(0,1,1)\}$. This plane is clearly not the xy plane. As proof, consider that $(0,0,1)$ is orthogonal to the xy plane (the row space of A) but not orthogonal to the row space of B. So $R(A) \neq R(B)$ even though $C(A) = C(B)$.

    b) Show that the matrices C (basic columns) can be different.

    Take the same A and B as before. The algorithm for constructing the basic column matrices C_A and C_B would extract the first two columns of A for C_A and the first two columns of B for C_B, yielding:

    $$C_A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C_B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

So even though the column spaces of A and B are equal, their basic column matrices are different.

c) What number will be the same for A and B?

A and B have the same column space which means they both span the same linear subspace, and A and B have the same *rank*. By the rank nullity theorem the dimension of their null space will also be the same.

15. If $A = CR$, the first row of A is a combination of the rows of R. Which part of which matrix holds the coefficients in that combination – the numbers that multiply the rows of R to produce row 1 of A?

The first row of C gives the coefficients for the linear combination of R rows that yield the first row of A.

$$\begin{array}{cc} & \begin{bmatrix} — & r_1 & — \\ & \vdots & \\ — & r_m & — \end{bmatrix} \\ CR = \begin{bmatrix} c_{11} & \dots & c_{1m} \\ \vdots & \ddots & \\ c_{n1} & \dots & c_{nm} \end{bmatrix} & \begin{bmatrix} — & \sum c_{1i}\mathbf{r}_i & — \\ & & \\ & & \end{bmatrix} \end{array}$$

16. The rows of R are a basis for the row space of A. *What does that sentence mean?*

A basis a minimal set of linearly independent vectors that span a subspace. Taking each row of R to be a vector, the statement means that the R rows will be linearly independent and will span the row space of the matrix A.

17. For these matrices with square blocks, find $A = CR$. What ranks?

$$A_1 = \begin{bmatrix} \text{zeros} & \text{ones} \\ \text{ones} & \text{ones} \end{bmatrix}_{4\times 4}, \quad A_2 = \begin{bmatrix} A_1 \\ A_1 \end{bmatrix}_{8\times 4}, \quad A_3 = \begin{bmatrix} A_1 & A_1 \\ A_1 & A_1 \end{bmatrix}_{8\times 8}$$

Writing out A_1 explicitly, we see that columns 1 and 3 are linearly independent:

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Thus its C matrix will be $\begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$. The row matrix R necessary to make $CR = A$ is then $R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

For A_2, we see this is two copies of A_1 stacked vertically. Again its first and third columns are linearly independent, the second column is a copy of the first and the fourth is a copy of the third. So it has the same row matrix R as before but the C matrix is now:

$$C = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

For A_3, we see it is two copies of A_2 stacked horizontally. As such it will have the same column space as A_2 and the same C matrix will be factored as A_2. The row matrix R becomes two copies of the previous row matrix stacked horizontally: $[R\ R]$.

18. "Elimination" subtracts a number ℓ_{ij} times row j from row i: a "row operation." Show how those steps can reduce the matrix A in Example 4 to R (except that this row echelon form R has a row of zeros). The rank won't change!

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix} \to \to R = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{rref}(A)$$

One possible rref sequence is:

$$\begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix} \stackrel{1_{21}}{\to} \begin{bmatrix} 1 & 3 & 8 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{bmatrix} \stackrel{-2_{22}}{\to} \begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \stackrel{1_{32}}{\to} \begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\stackrel{3_{21}}{\to} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$