

# Linear Algebra and Learning from Data

## Chapter 1

Selected solutions by Benjamin Basseri

1. Give an example where a combination of three nonzero vectors in  $\mathbb{R}^4$  is the zero vector. Then write your example in the form  $Ax = 0$ . What are the shapes of  $A$ ,  $x$  and  $0$ ?

We can choose two vectors then make the third vector a linear combination of the first two:

$$\mathbf{a} + \mathbf{b} + (\lambda_1 \mathbf{a} + \lambda_2 \mathbf{b})$$

In this case we may choose:

$$a = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, c = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Then  $a + b - c = 0$ . This linear combination of  $a, b, c$  can be expressed in matrix form as:

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \mathbf{0}$$

where  $A$  is  $4 \times 3$ ,  $x$  is  $3 \times 1$ , and  $0$  is  $4 \times 1$ .

2. Suppose a combination of the columns of  $A$  equals a different combination of those columns. Write that as  $Ax = Ay$ . Find two combinations of the columns of  $A$  that equal the zero vector (in matrix language, find two solutions to  $Az = 0$ ).

$$Ax = Ay \implies Ax - Ay = 0 \implies A(x - y) = 0 \implies x - y \in \ker A$$

This also implies  $y - x$  is in the null space of  $A$  since we could subtract  $Ax$  from both sides in the first step just as well.

3. The vectors  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$  are in  $m$ -dimensional space  $\mathbb{R}^m$ , and a combination  $c_1 \mathbf{a}_1 + \dots + c_n \mathbf{a}_n$  is the zero vector. That statement is at the vector level.

- a) Write that statement at the matrix level. Use the matrix  $A$  with the  $\mathbf{a}$ 's in its columns and use the column vector  $\mathbf{c} = (c_1, \dots, c_n)$ .

One of Strang's pictures of matrix multiplication describes  $A\mathbf{c}$  as a linear combination of the columns of  $A$ , with the combination given by the components of  $\mathbf{c}$ . Thus

$$A\mathbf{c} = \begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = 0$$

$$\iff c_1\mathbf{a}_1 + \dots c_n\mathbf{a}_n = 0$$

- b) Write that statement at the scalar level. Use subscripts and sigma notation to add up numbers. The column vector  $\mathbf{a}_j$  has components  $a_{1j}, a_{2j}, \dots, a_{mj}$ .

From the matrix picture we can write out in summation for the  $j$ th component of  $A\mathbf{c}$ :

$$(A\mathbf{c})_j = \sum_{i=1}^m c_i a_{ji} = 0$$

$$\text{altogether: } A\mathbf{c} = \sum_{j=1}^n \sum_{i=1}^m c_i a_{ij} \mathbf{e}_j = \mathbf{0}$$

4. Suppose  $A$  is the 3 by 3 matrix of all ones. Find two independent vectors  $x$  and  $y$  that solve  $Ax = 0$  and  $Ay = 0$ . Write that first equation  $Ax = 0$  (with numbers) as a combination of the columns of  $A$ . Why don't I ask for a third independent vector with  $Az = 0$ ?

We can choose  $x = (1, -1, 0)$ ,  $y = (0, 1, -1)$ . These are independent since clearly there is no scalar  $\lambda$  such that  $\lambda x = y$ : the scalar  $\lambda$  would have to map  $x_1 \mapsto 0$  which implies  $\lambda = 0$ , but then  $x_2 \mapsto 1$  which implies  $\lambda \neq 0$ .

Let  $\mathbf{1}$  be the vector of 1's, which in this case is each column of  $A$ . Then

$$Ax = \sum_{i=1}^3 x_i \mathbf{1} = x_1 \mathbf{1} + x_2 \mathbf{1} + x_3 \mathbf{1} = 1 \cdot \mathbf{1} - 1 \cdot \mathbf{1} + 0 = 0$$

There is no 3rd independent vector in the null space of  $A$ . This follows from rank-nullity:  $A$  has a rank of 1, its column space is the subspace spanned by  $\mathbf{1}$  (a line). Since we're in  $\mathbb{R}^3$ , that means  $N(A)$  has dimension 2.

5. The linear combinations of  $v = (1, 1, 0)$  and  $w = (0, 1, 1)$  fill a plane in  $\mathbb{R}^3$ .
- a) Find a vector  $z$  that is perpendicular to  $Wv$  and  $w$ .

Such a vector  $z$  must have a 0 dot product with  $v$  and  $w$ . By inspection we see that if  $z = (1, -1, 1)$  then the dot product with either  $v$  or  $w$  will get a 1 from the first or third component and a -1 from the second:

$$z \cdot v = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$z \cdot w = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0$$

b) Find a vector  $u$  that is not on the plane. Check that  $u^\top z \neq 0$

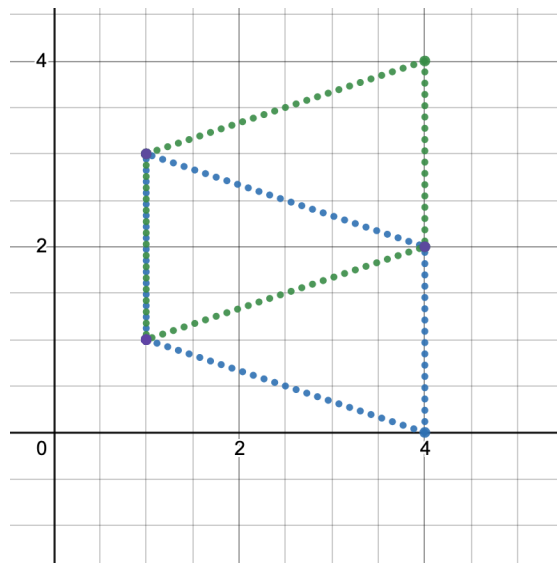
Of course  $z$  is not in the plane and since  $z \neq 0$ , we have  $z^\top z \neq 0$ .

6. If three corners of a parallelogram are  $(1,1)$ ,  $(4,2)$ , and  $(1,3)$ , what are all three of the possible fourth corners? Draw two of them.

By the parallelogram rule of vector addition, we can take the vector represented by a point  $x$  and add the other two points' *differences* to  $x$  to get the fourth point of the parallelogram. Starting at  $(1,1)$ , we have differences to  $(1,3)$  and  $(4,2)$  as follows:

$$(1,3) - (1,1) = (0,2), \quad (4,2) - (1,1) = (3,1)$$

so a point that makes a parallelogram would be  $(1,1) + (0,2) + (3,1) = (4,4)$ .



7. Describe the column space of  $A = [v \ w \ v + 2w]$ . Describe the nullspace of  $A$ : all vectors  $x = (x_1, x_2, x_3)$  that solve  $Ax = 0$ . Add the “dimensions” of that plane (the column space of  $A$  and that line (the nullspace of  $A$ )).

Let  $C(A)$  denote the column space of  $A$ . Since the 3rd column is a linear combination of the first 2,  $C(A) = C([v, w]) = \text{span}\{v, w\}$ . Assuming  $w \neq \lambda v$  the null

space  $N(A) = \{(1, 2, -1)\}$ , since for any  $x \in N(A) = \lambda(1, 2, -1)$  for some  $\lambda$ , then

$$Ax = \lambda A(1, 2, -1) = v + 2w - (v + 2w) = 0$$

$A$  spans a 2-dimensional plane and the nullspace is 1-dimensional, in keeping with rank-nullity.

8.  $A = CR$  is a representation of the columns of  $A$  in the basis formed by the columns of  $C$  with coefficients in  $R$ . If  $A_{ij} = j^2$  is 3 by 3, write down  $A$  and  $C$  and  $R$ .

Given the recipe  $A_{ij} = j^2$  we have

$$A = \begin{bmatrix} 1 & 4 & 9 \\ 1 & 4 & 9 \\ 1 & 4 & 9 \end{bmatrix}$$

The columns of  $A$  are linearly dependent, we may keep the first column for  $C$  and then  $R$  is the matrix necessary to make  $CR = A$ :

$$C = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 4 & 9 \end{bmatrix}$$

9. Suppose the column space of an  $m \times n$  matrix is all of  $\mathbb{R}^3$ . What can you say about  $m$ ? What can you say about  $n$ ? What can you say about the rank  $r$ ?

Given the span of a matrix is  $\mathbb{R}^3$  it must be  $m = 3$  and  $n \geq 3$ . If  $m$ , the number of rows, was any less than 3 it would not be able to span  $\mathbb{R}^3$  (we omit the possibility that  $m > 3$  and it spans a 3-dimensional subspace isomorphic to  $\mathbb{R}^3$ ). Likewise if the number of columns  $n$  is less than 3 then the matrix could not have 3 linearly independent vectors, which are necessary to form a minimally spanning set of  $\mathbb{R}^3$ . Altogether this means the rank  $r$  of the matrix is 3.

10. Find the matrices  $C_1$  and  $C_2$  containing independent columns of  $A_1$  and  $A_2$ :

$$A_1 = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ 2 & 6 & -4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

For  $A_1$  we see its second and third columns are multiples of the first. So  $C_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ .

For  $A_2 = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]$  we see the third column is a linear combination of the other two:  $\mathbf{a}_3 = 2\mathbf{a}_2 - \mathbf{a}_1$ . The first two columns are linearly independent which

makes  $C_2 = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}$

11. Factor each of those matrices into  $A = CR$ . The matrix  $R$  will contain the numbers that multiply the columns of  $C$  to recover columns of  $A$ .

Having identified which columns are linearly columns of which, we encode that information into  $R$ :

$$C_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 1 & 3 & -2 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

12. Produce a basis for the column spaces of  $A_1$  and  $A_2$ . What are the dimensions of those columns spaces – the number of independent vectors? What are the ranks of  $A_1$  and  $A_2$ ? How many independent rows in  $A_1$  and  $A_2$ ?

By construction the columns of  $C_1$  and  $C_2$  form bases for  $A_1$  and  $A_2$  since they are sets of linearly independent vectors that span the column space of the matrix. So we may choose for bases  $B_1$  and  $B_2$ :

$$B_1 = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \right\}, \quad B_2 = \left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \right\}$$

The dimension of each column space is the number of linearly independent vectors in each basis. Hence  $\dim C(A_1) = 1$ ,  $\dim C(A_2) = 2$ . Equivalently, these are the ranks of matrices  $A_1$  and  $A_2$ . Lastly, the rank of  $A$  is equal to the rank of  $A^T$  for a square  $A$ , so the matrices have the same number of independent rows as they do columns.

13. Create a 4 by 4 matrix of rank 2. What shapes are  $C$  and  $R$ ?

Before construction, we know the shape of  $C$  will be  $4 \times 2$ . It must have 4 rows since it must match the original matrix. It will have 2 columns since we're constructing the original matrix to have 2 linearly independent columns.

Let us construct the matrix:

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first two columns of  $A$  are independent and the second two are linear combinations of the others. This gives  $C$  and  $R$ :

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$