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Problem 1

Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.

Proof: direct, using the topology axioms and given information We're given for each $x \in A$ there is an open set U where $x \in U \subset A$. We have arbitrary unions of open sets are open from the topology axioms. So for each x call the containing open set U_x that is itself contained in A, then A is an arbitrary union of open sets, which is open:

$$A = \bigcup_{x \in A} U_x$$

Problem 2

Consider the nine topologies on the set $X = \{a, b, c\}$ indicated in Example 1 of §12. Compare them; that is, for each pair of topologies, determine whether they are comparable, and if so, which is the finer.

The first topology is $\mathscr{T}_{\text{trivial}} = \{\varnothing, X\}$, or the 'one big clump' topology. Since all topologies will contain $\varnothing$ and X, this topology will be comparable and coarser than all others. Likewise, the last topology is the discrete topology, comparable and finer than all others.

The other topologies are:

$$\mathscr{T}_{\text{nested-}a} = \{\varnothing, \{a\}, \{a, b\}, X\}$$

$$\mathscr{T}_{\text{nested}-b} = \{\varnothing, \{b\}, \{a, b\}, \{b, c\}, X\}$$

$$\mathscr{T}_{\text{isolated}-b} = \{\varnothing, \{b\}, X\}$$

$$\mathscr{T}_{\text{separated}} = \{\varnothing, \{a\}, \{b, c\}, X\}$$

$$\mathscr{T}_{\text{split-}b} = \{\varnothing, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\}$$

$$\mathscr{T}_{\text{cut-}c} = \{\varnothing, \{a, b\}, X\}$$

$$\mathscr{T}_{\text{closed-}c} = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$$

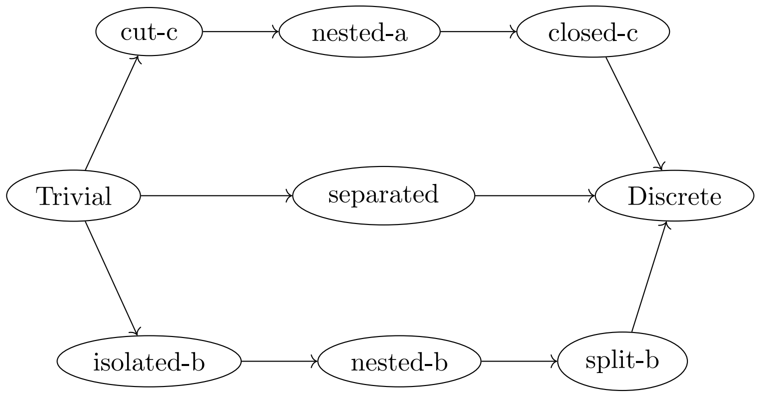
For $\mathscr{T}_{\text{nested-}a}$ it is only comparable to $\mathscr{T}_{\text{cut-}c}$ and $\mathscr{T}_{\text{closed-}c}$. For the other topologies above it either has $\{a\}$ or $\{a, b\}$ open and the others do not so it is not contained by them, and the others have $\{b\}$ or $\{b, c\}$ open and $\mathscr{T}_{\text{nested-}a}$, so it cannot contain them. This means

$$\mathscr{T}_{\text{trivial}} \subset \mathscr{T}_{\text{cut-}c} \subset \mathscr{T}_{\text{nested-}a} \subset \mathscr{T}_{\text{closed-}c} \subset \mathscr{T}_{\text{discrete}}$$

For the other topologies

$$\mathscr{T}_{\text{trivial}} \subset \mathscr{T}_{\text{isolated}-b} \subset \mathscr{T}_{\text{nested-}b} \subset \mathscr{T}_{\text{split-}b} \subset \mathscr{T}_{\text{discrete}}$$

$$\mathscr{T}_{\text{trivial}} \subset \mathscr{T}_{\text{separated}} \subset \mathscr{T}_{\text{discrete}}$$


cut-c
nested-a
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Trivial
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Discrete
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