

# Exercises from Naive Set Theory by Halmos

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## 3 Unordered Pairs

### 3.1

Consider the sets  $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}$ , etc.; consider the pairs, such as  $\{\emptyset, \{\emptyset\}\}$ , formed by any two of them; consider the pairs formed by any two such pairs, or else the mixed pairs formed by any singleton and any pair; and proceed so on ad infinitum. Are all the sets obtained in this way distinct from one another?

**Discussion** Halmos asks if there are two different ways of pairing these sets together that somehow end up constructing the same set. The key fact here is that all sets made this way are unordered pairs so they have the form  $\{a, b\}$ . The members  $a$  and  $b$  are themselves pairs or singletons. The axiom of extension requires that there is only one  $\{a, b\}$  and anything equal to it must have members  $a$  and  $b$ , and nothing else. So the set  $\{a, b\}$  is distinct from any pair with alternative choices of members, which means each set constructed as above is distinct.

**Proof** Suppose you have a set  $E$  that can be constructed from this process by pairing  $A$  and  $B$  or by pairing  $C$  and  $D$ . Then  $E$  is the unordered pair  $\{A, B\}$  and at the same time  $E = \{C, D\}$ , so  $\{A, B\} = \{C, D\}$ . By the axiom of extension, it must be that these two pairs have the same members and so either  $A = C$  and  $B = D$  or  $A = D$  and  $B = C$ . Now,  $A$  and  $B$  are either singletons or pairs, and the same reasoning can be applied to them, ‘ad infinitum’. So there are not two distinct ways to construct the same set in this manner, and all sets so constructed must be distinct.

## 4 Unions and Intersections

A necessary and sufficient condition that  $(A \cap B) \cup C = A \cap (B \cup C)$  is that  $C \subset A$ . Observe that the condition has nothing to do with the set  $B$ .

**Discussion** Recall that ‘necessary and sufficient’ is jargon for an ‘if and only if’ proof, i.e. a biconditional. So we split up the biconditional into the forward and reverse implications.

**Forward implication.** Assume  $(A \cap B) \cup C = A \cap (B \cup C)$ . Any  $x \in C$  must be in the union  $(A \cap B) \cup C$ . By assumption this equals  $A \cap (B \cup C)$ , so  $x$

is in this intersection as well. But this requires  $x$  to be in each set intersected, so  $x \in A$ . Therefore  $x \in C$  implies  $x \in A$  as well:  $C \subset A$ .

**Reverse implication.** Assume  $C \subset A$  and note that  $A \cup C = A$ . By the distributive law:

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C) = A \cap (B \cup C)$$

## 5 Complements and Powers

### 5.1

Prove that  $\mathcal{P}(E) \cap \mathcal{P}(F) = \mathcal{P}(E \cap F)$  and  $\mathcal{P}(E) \cup \mathcal{P}(F) = \mathcal{P}(E \cup F)$ .

**Proof.**  $A \in \mathcal{P}(E) \cap \mathcal{P}(F)$  if and only if  $A$  is a subset of  $E$  and a subset of  $F$ . This occurs if and only if every member of  $A$  is in  $E \cap F$ , which means  $A$  is a subset of  $E \cap F$ . Hence  $A$  must be in  $\mathcal{P}(E \cap F)$ .

If  $A \in \mathcal{P}(E) \cup \mathcal{P}(F)$  then  $A$  is a subset of  $E$  or  $F$ . The set  $E \cup F$  contains all members of both  $E$  and  $F$ , so  $A$  must be a subset of  $E \cup F$  as well, and hence a member of  $\mathcal{P}(E \cup F)$ .

### 5.2

These assertions can be generalized to

$$\bigcap_{X \in \mathcal{C}} \mathcal{P}(X) = \mathcal{P}\left(\bigcap_{X \in \mathcal{C}} X\right)$$

$$\bigcup_{X \in \mathcal{C}} \mathcal{P}(X) \subset \mathcal{P}\left(\bigcup_{X \in \mathcal{C}} X\right);$$

find a reasonable interpretation of the notation in which these generalizations were here expressed and then prove them.

**Interpretation and proof.** What's remains in the left intersection are the subsets that appear in each  $X$ 's power set. Since these subsets are in every  $X$ , they are subsets in  $\bigcap X$ , the intersection on the right. The left union collects every subset of every  $X$  in  $\mathcal{C}$ . Any such subset of an  $X$  is certainly still a subset of  $X$  unioned with anything else, since unioning can only add members. Therefore all the subsets making up the left union must also be in the right union.

### 5.3

Show that  $E$  is always equal to  $\bigcap_{X \in \mathcal{P}(E)} X$  (that is,  $E = \bigcup \mathcal{P}(E)$ ), but that the result of applying  $\mathcal{P}$  and  $\bigcup$  to  $E$  in the other order is a set that includes  $E$  as a subset, typically a proper subset.

**Proof.** Every element in  $E$  appears in some subset of  $E$ , and any element *not* in  $E$  will not appear in any such subset. Therefore if we union all the subsets

of  $E$  together, which is exactly what the notation indicates, we reassemble the set  $E$ .

To prove  $E \subset \mathcal{P}(\bigcup E)$  consider that  $E$  is itself a collection of sets and recall that  $\mathcal{P}(\bigcup E)$  is just alternative notation for  $\mathcal{P}(\bigcup_{X \in E} X)$ . Now any set  $X$  will be an element of  $\mathcal{P}(X)$ , so unioning  $\mathcal{P}(X)$  for all  $X \in E$  places each  $X \in E$  in union. Therefore  $E$ , which is just the collection of the  $X$ 's, must be a subset of  $\mathcal{P}(\bigcup_{X \in E} X)$ .

Let's explore Halmos' comment that  $E$  is typically a proper subset of  $\mathcal{P}(\bigcup E)$ . Consider what it would take for  $E$  to *not* be a proper subset. Since  $E$  must be contained in  $\mathcal{P}(\bigcup E)$ , to be improper means  $E$  must be all of  $\mathcal{P}(\bigcup E)$ . If  $E = \mathcal{P}(\bigcup E)$  then this says that  $E$  is the power set of the set made from all  $E$ 's members' elements. In other words,  $E$  must itself have the structure of a power set for this equality to hold. So  $E$  is a proper subset of this union unless  $E = \mathcal{P}(A)$  for some set  $A$ .

## 6 Unordered Pairs

### 6.1

If  $A, B, X$ , and  $Y$  are sets, then

$$(i) \quad (A \cup B) \times X = (A \times X) \cup (B \times X)$$

**Proof** The set  $(A \cup B) \times X$  are all possible ordered paris  $(c, x)$  with  $c \in A \cup B$  and  $x \in X$ . So equivalently, we could take the set of pairs  $A \times X$  which contains all pairs including an  $A$  element, and  $B \times X$  which has the rest, and union them.

$$(ii) \quad (A \cap B) \times (X \cap Y) = (A \times X) \cap (B \times Y)$$

**Proof.** The set on the left are all pairs with first coordinate in both  $A$  and  $B$ , and second coordinate in both  $X$  and  $Y$ . Then the set on the right starts with  $A \times X$  but the intersection operation removes from it any pairs not found in  $B \times Y$ . What's left are precisely those pairs whose first coordinate is in both  $A$  and  $B$ , and whose second coordinate is in both  $X$  and  $Y$ .

$$\begin{aligned} (a, x) \in (A \cap B) \times (X \cap Y) &\iff a \in (A \cap B) \text{ and } x \in (X \cap Y) \\ &\iff a \in A, a \in B, x \in X, x \in Y \\ &\iff (a, x) \in (A \cap X) \text{ and } (a, x) \in (B \cap Y) \\ &\iff (a, x) \in (A \cap X) \cap (B \cap Y) \end{aligned}$$

$$(iii) \quad (A - B) \times X = (A \times X) - (B \times X)$$

**Proof.** The set on the left are all pairs  $(a, x)$  with  $a \in A, x \in X$  but having removed any pairs where the  $A$  coordinate is also a member of  $B$ , which is precisely what the set on the right specifies.

## 7 Relations

### 7.1

For each of the three possible properties (reflexive, symmetric, transitive) find a relation that does not have that property but does have the other two.

**Reflexive** A sibling relation may be thought of as reflexive if we specify that no one is their own sibling. Then if  $x$  is a sibling of  $y$ ,  $y$  is a sibling of  $x$  (symmetric) and if  $x$  is a sibling of  $y$  and  $y$  is a sibling of  $z$ , then  $x$  is a sibling of  $z$  (transitive).

**Symmetric** The  $\leq$  relation on real numbers is reflexive since  $x \leq x$  always, and it's transitive since  $x \leq y$  and  $y \leq z$  implies  $x \leq z$ . But it is not symmetric since if  $x \leq y$  it is not required that  $y \leq x$ .

**Transitive** Take movie stars and let the relation be 'has been in a movie with'. Everyone has been in a movie with themselves (reflexive), and if star  $a$  has been in a movie with star  $b$  then  $b$  has been in that same movie with  $a$  (symmetric). But if  $a$  and  $b$  have made a movie together, and  $b$  and  $c$ , it doesn't mean  $a$  and  $c$  have been in a movie together (not transitive).

### 7.2

Show that  $X/R$  is indeed a set by exhibiting a condition that specifies exactly the subset  $X/R$  of the power set  $\mathcal{P}(X)$

**Proof.** Each  $x \in X$  belongs to equivalence class in  $X/R$  specified as  $\{y \in X : xRy\}$ . Since the construction is limited to elements of  $X$ , the class must be a subset of  $X$  and therefore a member of  $\mathcal{P}(X)$ . So  $X/R$ , the set of all equivalence classes, is a set and more specifically it's a subset of  $\mathcal{P}(X)$ .

### 7.3

Relation composition is always associative.

**Proof** Suppose we have the relation  $U = T \circ S \circ R$  between sets  $A, B, C, D$ :

$$A \xrightarrow{R} B \xrightarrow{S} C \xrightarrow{T} D$$

We say  $a \in A$  has the relation  $U$  to  $d \in D$  when there's a  $b \in B$  and a  $c \in C$  where  $aRb$ ,  $bSc$ , and  $cTd$ . If one of those relations is missing then we don't have  $aUd$ . The question is only about what pairs exist in each relation set  $R, S, T$ : we may phrase the question as  $T \circ (S \circ R)$  or  $(T \circ S) \circ R$  or  $T \circ S \circ R$  but the choice won't affect the membership of  $R, S$ , or  $T$ . Hence we could evaluate in any grouping and get the same outcome, and relation composition is associative.

### 7.4

Relation composition is always connected with inversion via the equation  $(SR)^{-1} = R^{-1}S^{-1}$

**Proof.** Suppose  $xRy$  and  $ySz$ . Then by definition, the inverse relations  $yR^{-1}x$  and  $zS^{-1}y$  hold. Since we have  $zS^{-1}y$  and  $yR^{-1}x$  they compose as  $zS^{-1}R^{-1}x$ .

## 8 Functions

### 8.1

In what special cases are projection maps one-to-one?

**Solution.** Suppose  $f : X \times Y \longrightarrow X$  is a projection onto  $X$ . For  $f$  to be injective (one-to-one) then for any  $x \in X$  that  $f$  maps to, it must map to  $x$  only once. This cannot happen if there is more than one  $y \in Y$ , since if there were two distinct  $y, y' \in Y$ , then  $f(x, y) = x$  and  $f(x, y') = x$  as well, mapping to  $x$  twice. So  $f$  can only be injective if  $Y$  is a singleton. If  $Y$  is empty, then  $X \times Y$  is empty as well as the projection  $f$ , which makes it vacuously injective.

### 8.2

- (i)  $Y^\varnothing$  has exactly one element, namely  $\varnothing$ , whether  $Y$  is empty or not.

**Proof.**  $Y^\varnothing$  is the set of all possible functions from the empty set to  $Y$ . Any such function is a subset of  $\varnothing \times Y$ , which is empty, so the set of possible functions  $Y^\varnothing$  is empty as well.

- (ii) if  $X$  is not empty, then  $\varnothing^X$  is empty.

**Proof.**  $\varnothing^X$  is the set of all possible functions from  $X$  to the empty set, and must be a subset of  $X \times \varnothing$ , which is empty.

## 9 Families

### 9.1

Formulate and prove a generalized version of the commutative law for unions.

**Proof** Consider unioning some number of sets:  $A \cup B \cup \dots \cup X$ . The commutative law states that the order in which we place the sets doesn't change the result of the union. By the axiom of specification, the only condition for an element to be in the union is that it appears in at least one of the sets being unioned: first set, last set, or any other. Hence, order doesn't matter for set unions.

### 9.2

If both  $\{A_i\}$  and  $\{B_i\}$  are families of sets, then

$$\left( \bigcup_i A_i \right) \cap \left( \bigcup_j B_j \right) = \bigcup_{i,j} (A_i \cap B_j),$$

$$\left(\bigcap_i A_i\right) \cup \left(\bigcap_j B_j\right) = \bigcap_{i,j} (A_i \cup B_j)$$

**Discussion.** Suppose  $c \in \left(\bigcup_i A_i\right) \cap \left(\bigcup_j B_j\right)$ . Then there is some  $i$  where  $A_i$  contains  $c$  and there is some  $j$  where  $B_j$  contains  $c$ . This means  $c \in A_i \cap B_j$ , which means  $c \in \bigcup_{i,j} A_i \cap B_j$ .

For the second equality, the left hand side specifies elements that are in every  $A_i$  or in every  $B_j$ . So for an element  $c$  to *not* be in this set, there would have to be an  $A_i$  and a  $B_j$  that didn't contain it. For that  $(i, j)$  pair,  $c \notin (A_i \cup B_j)$  and will be cut out of the intersection  $\bigcap_{i,j} (A_i \cup B_j)$ . So the right side contains the same elements as the left side, and no more.

### 9.3

Prove that  $\left(\bigcup_i A_i\right) \times \left(\bigcup_j B_j\right) = \bigcup_{i,j} (A_i \times B_j)$ , and that the same equation holds for intersections (provided that the domains of the families involved are not empty).

**Proof.** An  $(a, b)$  is in the product on the left if and only if there is an  $i$  and  $j$  where  $a \in A_i$  and  $b \in B_j$  which means  $(a, b) \in A_i \times B_j$ . This  $(i, j)$  pair is one of the indices in  $\bigcup_{i,j}$  so  $A_i \times B_j$  will be in the union on the right.

### 9.4

Prove (with appropriate provisos about empty families) that  $\bigcap_i X_i \subset X_j \subset \bigcup_i X_i$  for each index  $j$  and that intersection and union can in fact be characterized as the extreme solutions of these inclusions.

**Proof.** Intersecting a set can only remove elements from the result and unioning sets can only add elements to the result. So anything in  $\bigcap_i X_i$  must be in  $X_j$  and anything in  $X_j$  must be in  $\bigcup_i X_i$ .

## 10 Inverses and Composites

### 10.1

If  $\{A_i\}$  is a family of subsets of  $X$ , then  $f(\bigcup_i A_i) = \bigcup_i f(A_i)$ .

**Discussion.** Suppose  $f$  maps  $X$  to some set  $Y$ . Then the notation  $f(A_i)$  refers to the subset of  $Y$  that is mapped to any element in  $A_i$ . So  $y \in f(\bigcup_i A_i)$  if and only if there is an  $A_i \in \{A_i\}$  with a member  $a$  that maps to  $y$  on  $f$ . This  $y$  will then also be in the set  $\bigcup_i f(A_i)$ , since this is unioning the image of all  $A_i$ 's. Likewise, if an  $A_i$  contains an  $a \xrightarrow{f} y$  then  $a \in \bigcup_i A_i$  and  $y \in f(\bigcup_i A_i)$ .

### 10.2

A necessary and sufficient condition that  $f$  map  $X$  onto  $Y$  is that the inverse image under  $f$  of each non-empty subset of  $Y$  be a non-empty subset of  $X$ .

**Discussion.** If  $f$  is surjective then by definition, any  $y \in Y$  has at least one  $x \in X$  such that  $f(x) = y$ . Then if  $B$  is any non-empty subset of  $Y$ ,  $B$  has  $y$ -elements in it, and there are  $x$ -elements that map to them. So the inverse image  $f^{-1}(B)$  is non-empty.

### 10.3

Assume in each case that  $f$  is a function from  $X$  to  $Y$ .

- (i) If  $g$  is a function from  $Y$  to  $X$  such that  $gf$  is the identity on  $X$ , then  $f$  is one-to-one and  $g$  maps  $Y$  onto  $X$ .

**Proof.** If  $f$  wasn't injective there'd be two distinct elements  $x, x' \in X$  that  $f$  maps to the same  $y$ . Now  $g(y)$  must map back to the original element in  $X$  so that  $gf$  is the identity map, but if  $g(y) = x$  then  $gf$  is not the identity for  $x'$  and if  $g(y) = x'$  then  $gf$  isn't the identity on  $x$ . So  $f$  must be injective. Since  $gf$  must be the identity on  $X$ , every  $x \in X$  must have a  $y \in Y$  such that  $g(y) = x$ , which means  $g$  must be surjective.

- (ii) A necessary and sufficient condition that  $f(A \cap B) = f(A) \cap f(B)$  for all subsets  $A$  and  $B$  of  $X$  is that  $f$  be one-to-one.

**Forward implication:** If  $f(A \cap B) = f(A) \cap f(B)$  for all  $A, B \subset X$  then it must hold when  $A$  and  $B$  are distinct singletons. Then this equality requires  $f(A) \cap f(B) = f(\emptyset) = \emptyset$ , which means  $f$  must map each  $x \in X$  to a distinct  $y \in Y$ , so that  $f$  is injective.

**Reverse implication:** For each  $y \in f(A) \cap f(B)$  there is an  $x \in X$  such that  $f(x) = y$ . Since  $f$  is injective, there is only one such  $x$ , which must be in both  $A$  and  $B$ . Hence,  $x \in A \cap B$  and  $f(x) \in f(A \cap B)$ , which shows that  $f(A) \cap f(B) \subset f(A \cap B)$ . Reversing these steps shows  $f(A \cap B) \subset f(A) \cap f(B)$ , which shows equality.

- (iii) A necessary and sufficient condition that  $f(X - A) \subset Y - f(A)$  for all subsets  $A$  of  $X$  is that  $f$  be one-to-one.

**Forward implication:** Assume toward a contradiction that  $f$  isn't injective. Then there is an  $f(x) = f(x') = y$  for distinct  $x, x' \in X$ . Now let  $A$  be a subset that contains  $x'$  but not  $x$ . Then  $x \in X - A$  so  $y \in f(X - A)$ . However, since  $x' \in A$  then  $y \in f(A)$ , which means  $y$  is removed from  $Y - f(A)$ . Then  $f(X - A)$  has  $y$  but  $Y - f(A)$  does not, which contradicts  $f(X - A) \subset Y - f(A)$ .

**Reverse implication:** If  $f$  is injective then the elements removed from  $X$  in  $X - A$  are the only ones that can map to  $f(A)$ , therefore  $f(A)$  is certainly removed from the image  $f(X - A)$ . Then, since  $f(X)$  must be contained in  $Y$ , we have  $f(X - A) \subset Y - f(A)$ .

- (iv) A necessary and sufficient condition that  $Y - f(A) \subset f(X - A)$  for all subsets  $A$  of  $X$  is that  $f$  map  $X$  onto  $Y$ .

**Proof.** Since this must hold for  $A$  being any subset of  $X$ , consider what happens when  $A = \emptyset$ . Then  $Y - f(A) \subset f(X - A)$  means  $Y \subset f(X)$ . For  $f(X)$  to contain  $Y$  it must be all of  $Y$  itself, meaning  $f$  is surjective.