# Notes on Naive Set Theory by Halmos

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## 1 Axiom of Extension

### 1.1 Sets

Sets, which will not be formally defined, are collections of objects. This text is more concerned with what we can *do* with sets rather than what sets *are*.

### 1.2 The Axiom of Extension

Sets are equal if and only if they contain exactly the same members. This means we consider two sets to be identical when they have the same elements: the members of the set define the set.

### 1.3 Inclusion both ways

One technique to prove sets $A$ and $B$ are equal is to prove that $A \subset B$ and $B \subset A$. If both inclusions hold it means that every element of $A$ is also in $B$ but every element of $B$ is also in $A$. Then $A$ and $B$ have all the same members, and $A = B$ by extension.

### 1.4 The belonging relation

Objects can 'belong' to sets, e.g. $x \in A$. So 'belonging' is a relationship between $x$ and $A$. Halmos notes that the Axiom of Extension not only defines equality for sets, but it puts conditions on the 'belonging' relation as well. For instance, belonging is not symmetric: if $x \in A$ it does not mean that $A \in x$.

## 2 Axiom of Specification

### 2.1 The Axiom of Specification

Sets can be filtered by logical conditions to determine subsets.

### 2.2 Russell's Paradox

There cannot be a 'universal' set that contains everything.

Let $B = \{x \in A : x \notin x\}$; the elements of $A$ that don't contain themselves. While $B$ is a perfectly valid set, there is no set $A$ that it can belong to.

**Proof** Assume toward a contradiction that $B \in A$ for some set $A$. We cannot have $B \in B$ since that contradicts the specification for $B$ membership. But if $B \notin B$ then $B$ qualifies for $B$ membership, which means it *is* in $B$. Since this is a contradiction, there cannot be a set $A$ that has $B$ as a member.

# 3 Unordered Pairs

## 3.1 The Empty Set

The empty set $\varnothing$ has no members. It's considered a subset of every set since, for any set $A$, $\varnothing$ has no members that are not in $A$.

## 3.2 The Axiom of Pairing

For any two sets there is a set they both belong to.

## 3.3 Unordered Pairs

We can use specification to make an unordered pair $\{a, b\}$ out of any distinct $a$ and $b$: a set that contains them and nothing else.

## 3.4 Specifying sets with only a proposition

There are some special sets that can be specified by only a proposition $S(x)$, such as $\{x : x \neq x\} = \varnothing$ or $\{x : x = a\} = \{a\}$. But in general, the axiom of specification requires a proposition $S(x)$ be joined with a set $A$ as in $\{x \in A : S(x)\}$.

# 4 Unions and Intersections

## 4.1 Axiom of unions

A set can be formed by collecting the members of multiple sets.

## 4.2 Union and Intersection are associative and commutative

Unions and intersections can be thought of as specific logical propositions involving *or* and *and*. Since logical *or* and logical *and* are associative and commutative in Boolean algebra, unions and intersections are as well.

### 4.3 Intersections with the empty set

For a collection of sets $\mathcal{C}$ define

$$\bigcup \mathcal{C} = \{x : x \in X \text{ and } X \in \mathcal{C}\}$$

If $\mathcal{C}$ is empty, then no $x$ can satisfy the specification and the union is empty.

Halmos makes a technical point about requiring $\mathcal{C}$ to be non-empty if we consider $\bigcap \mathcal{C}$. But this is an artifact of the way he phrased intersecting a collection, in order to avoid $x$'s vacuously satisfying the intersection condition. We can simply define the intersection of the empty set with anything to be the empty set.

## 5 Complements and Powers

### 5.1 Complements

The set difference $A - B$ are the elements of $A$ that are not also in $B$. This is the *relative complement* of $B$ in $A$. We can write this as $B'$ when the set $A$ is clear from context.

### 5.2 Axiom of powers

For each set $A$ there is a set $\mathcal{P}(A)$ whose members are all possible subsets of $A$.

## 6 Ordered Pairs

### 6.1 Ordered Pairs from Sets

To encode an ordering of elements, make a collection where each element to be ordered is replaced by a set containing that element and all its predecessors.

$$X = \{a, b, c\}, \text{ order desired: } (c, b, a)$$

$$(c, b, a) \longmapsto \{\{c\}, \{c, b\}, \{c, b, a\}\}$$

### 6.2 Equality of Ordered Pairs

With ordered pairs if $(a, b) = (x, y)$ then each coordinate must be equal: $a = x, b = y$. By the encoding above, each ordered pair has exactly one singleton and one unordered pair, so there is no ambiguity which is which.

## 7 Relations

Using sets, we can encode relations as ordered pairs: $(x, y)$. A set of ordered pairs can then collect all the elements $x, y$ such that $x$ has the given relation to $y$.

# 8 Functions

## 8.1 Defining functions

On $X \times Y$, a function pairs every element of $X$ with exactly one element of $Y$. Compare to a relation, which does not require every $x \in X$ to be in a pair, and doesn't require a unique $y$ to be paired with any $x$.

## 8.2 The Canonical Map

The canonical map is the function that maps $x \in X$ to its equivalence class in $X/R$.

## 8.3 Encoding $\mathbb{N}$ with sets.

We can encode the natural numbers with nothing but the ordered set enoding:

$$0 = \varnothing, 1 = \{\varnothing\}, 2 = \{0, 1\}, \ldots$$

## 8.4 Characteristic Function

For $A \subset X$, the characteristic function $\chi_A(x) = 1$ if $x \in A$ and 0 otherwise. Note that $A$ is a subset of $X$, so $A \in \mathcal{P}(X)$, but $\chi_A$ is a function, so it is a member of $2^X$. However, we can naturally associate a subset to its characteristic function with the correspondence $A \longrightarrow \chi_A$.

# 9 Families

## 9.1 Index sets and families

It may be convenient to give certain elements of $X$ a label such as $x_i$ or $x_\alpha$. To do so we establish a function $I \longrightarrow X$ from the *index set* $I$ (the set of labels) to $X$. Like any function this pairs an $i \in I$ with a specific $x \in X$. Thinking of $X$ being *indexed* by $I$, we call this function the *family* $\{x_i\}$.

## 9.2 Unioning a family

For a family $\{A_i\}$, the notation $\bigcup_{i \in I} A_i$ or $\bigcup_i A_i$ indicates the union of the range of the family

# 10 Inverses and Composites

## 10.1 Image subset maps

For a function $f : X \longrightarrow Y$, and $A \subset X$, we let $f(A)$ denote the set of $Y$ elements mapped to by some element in $A$. Technically, this is a function on $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ and we *overload* the meaning of $f$.

## 10.2 Inverse maps

Let $f^{-1}(B)$, where $B \subset Y$, denote the elements in $X$ that map into $B$. Rather than implying that $f$ is invertible, this says the inverse map generally is from $\mathcal{P}(Y)$ to $\mathcal{P}(X)$. If $f$ is injective then $f^{-1}$ will map singletons to singletons, and we might call the function $f$ invertible.

## 10.3 Relation inverses

Inverse relations can be phrased as such: if $x$ has some relation $R$ to $y$, then $y$ has the relation $R^{-1}$ to $x$.