

Notes on Naive Set Theory by Halmos

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1 Axiom of Extension

Sets are equal if and only if they contain exactly the same members.

2 Axiom of Specification

2.1 The Axiom of Specification

Sets can be filtered by logical conditions to determine subsets.

2.2 Russell's Paradox

There cannot be a 'universal' set that contains everything.

Let $B = \{x \in A : x \notin x\}$; the elements of A that don't contain themselves. While B is a perfectly valid set, there is no set A that it can belong to.

Proof Assume toward a contradiction that $B \in A$ for some set A . We cannot have $B \in B$ since that contradicts the specification for B membership. But if $B \notin B$ then B qualifies for B membership, which means it *is* in B . Since this is a contradiction, there cannot be a set A that has B as a member.

3 Unordered Pairs

3.1 The Empty Set

The empty set \emptyset has no members. It's considered a subset of every set since, for any set A , \emptyset has no members that are not in A .

3.2 The Axiom of Pairing

For any two sets there is a set they both belong to.

3.3 Unordered Pairs

We can use specification to make an unordered pair $\{a, b\}$ out of any distinct a and b , a set that contains them and nothing else.

3.4 Specifying sets with only a proposition

There are some special sets that can be specified by only a proposition $S(x)$, such as $\{x : x \neq x\} = \emptyset$ or $\{x : x = a\} = \{a\}$. But in general, the axiom of specification requires a proposition $S(x)$ be joined with a set A as in $\{x \in A : S(x)\}$.

4 Unions and Intersections

4.1 Axiom of unions

A set can be formed from the members of each set in a collection of sets.

4.2 Union and Intersection are associative and commutative

Unions and intersections can be thought of as specific logical propositions involving *or* and *and*. Since logical *or* and logical *and* are associative and commutative in Boolean algebra, unions and intersections are as well.

4.3 Intersections with the empty set

For a collection of sets \mathcal{C} define

$$\bigcup \mathcal{C} = \{x : x \in X \text{ and } X \in \mathcal{C}\}$$

If \mathcal{C} is empty, then no x can satisfy the specification and the intersection is empty. Halmos makes a technical point about requiring \mathcal{C} to be non-empty. But this is an artifact of the way he phrased intersecting a collection, in order to avoid x 's vacuously satisfying the intersection condition. This is not needed if the specification above is used.

5 Complements and Powers

5.1 Complements

The set difference $A - B$ are the elements of A that are not also in B . This is the *relative complement* of B in A . We can write this as B' when the set A is clear from context.

5.2 Axiom of powers

For each set A there is a set $\mathcal{P}(A)$ whose members are all possible subsets of A .

6 Ordered Pairs

6.1 Ordered Pairs from Sets

To encode an ordering of elements, make a collection where each element to be ordered is replaced by a set containing that element and all its predecessors.

$$X = \{a, b, c\}, \text{ order desired: } (c, b, a)$$

$$(c, b, a) \mapsto \{\{c\}, \{c, b\}, \{c, b, a\}\}$$

6.2 Equality of Ordered Pairs

With ordered pairs if $(a, b) = (x, y)$ then each coordinate must be equal: $a = x, b = y$. By the encoding above, each ordered pair has exactly one singleton and one unordered pair, so there is no ambiguity which is which.

7 Relations

Using sets, we can encode relations as ordered pairs: (x, y) . A set of ordered pairs can then collect all the elements x, y such that x has the given relation to y .

8 Functions

8.1 Defining functions

On $X \times Y$, a function pairs every element of X with exactly one element of Y . Compare to a relation, which does not require every $x \in X$ to be in a pair, and doesn't require a unique y to be paired with any x .

8.2 The Canonical Map

The canonical map is the function that maps $x \in X$ to its equivalence class in X/R .

8.3 Characteristic Function

For $A \subset X$, the characteristic function $\chi_A(x) = 1$ if $x \in A$ and 0 if $x \in X - A$. Note that A is a subset of X , so $A \in \mathcal{P}(X)$, but χ_A is a function, so it is a member of 2^X . However, we can naturally associate a subset to its characteristic function with the correspondence $A \rightarrow \chi_A$.

9 Families

9.1 Index sets and families

It may be convenient to give certain elements of X a label such as x_i or x_α . To do so we establish a function $I \longrightarrow X$ from the *index set* I (the set of labels) to X . Like any function this pairs an $i \in I$ with a specific $x \in X$. Thinking of X being *indexed* by I , we call this function the *family* $\{x_i\}$.