# Notes on Naive Set Theory by Halmos

Benjamin Basseri

## 1 Axiom of Extension

Sets are equal if and only if they contain exactly the same members.

## 2 Axiom of Specification

### 2.1 The Axiom of Specification

Sets can be filtered by logical conditions to determine subsets.

### 2.2 Russell's Paradox

There cannot be a 'universal' set that contains everything.

Let $B = \{x \in A : x \notin x\}$; the elements of $A$ that don't contain themselves. While $B$ is a perfectly valid set, there is no set $A$ that it can belong to.

**Proof** Assume toward a contradiction that $B \in A$ for some set $A$. We cannot have $B \in B$ since that contradicts the specification for $B$ membership. But if $B \notin B$ then $B$ qualifies for $B$ membership, which means it *is* in $B$. Since this is a contradiction, there cannot be a set $A$ that has $B$ as a member.

## 3 Unordered Pairs

### 3.1 The Empty Set

The empty set $\varnothing$ has no members. It's considered a subset of every set since, for any set $A$, $\varnothing$ has no members that are not in $A$.

### 3.2 The Axiom of Pairing

For any two sets there is a set they both belong to.

### 3.3 Unordered Pairs

We can use specification to make an unordered pair $\{a, b\}$ out of any distinct $a$ and $b$, a set that contains them and nothing else.

### 3.4 Specifying sets with only a proposition

There are some special sets that can be specified by only a proposition $S(x)$, such as $\{x : x \neq x\} = \varnothing$ or $\{x : x = a\} = \{a\}$. But in general, the axiom of specification requires a proposition $S(x)$ be joined with a set $A$ as in $\{x \in A : S(x)\}$.

## 4 Unions and Intersections

### 4.1 Axiom of unions

A set can be formed from the members of each set in a collection of sets.

### 4.2 Union and Intersection are associative and commutative

Unions and intersections can be thought of as specific logical propositions involving *or* and *and*. Since logical *or* and logical *and* are associative and commutative in Boolean algebra, unions and intersections are as well.

### 4.3 Intersections with the empty set

For a collection of sets $\mathcal{C}$ define

$$\bigcup \mathcal{C} = \{x : x \in X \text{ and } X \in \mathcal{C}\}$$

If $\mathcal{C}$ is empty, then no $x$ can satisfy the specification and the intersection is empty. Halmos makes a technical point about requiring $\mathcal{C}$ to be non-empty. But this is an artifact of the way he phrased intersecting a collection, in order to avoid $x$'s vacuously satisfying the intersection condition. This is not needed if the specification above is used.

## 5 Complements and Powers

### 5.1 Complements

The set difference $A - B$ are the elements of $A$ that are not also in $B$. This is the *relative complement* of $B$ in $A$. We can write this as $B'$ when the set $A$ is clear from context.

### 5.2 Axiom of powers

For each set $A$ there is a set $\mathcal{P}(A)$ whose members are all possible subsets of $A$.

# 6 Ordered Pairs

## 6.1 Ordered Pairs from Sets

To encode an ordering of elements, make a collection where each element to be ordered is replaced by a set containing that element and all its predecessors.

$$X = \{a, b, c\}, \text{ order desired: } (c, b, a)$$

$$(c, b, a) \longmapsto \{\{c\}, \{c, b\}, \{c, b, a\}\}$$

## 6.2 Equality of Ordered Pairs

With ordered pairs if $(a, b) = (x, y)$ then each coordinate must be equal: $a = x, b = y$. By the encoding above, each ordered pair has exactly one singleton and one unordered pair, so there is no ambiguity which is which.

# 7 Relations

Using sets, we can encode relations as ordered pairs: $(x, y)$. A set of ordered pairs can then collect all the elements $x, y$ such that $x$ has the given relation to $y$.

# 8 Functions

## 8.1 Defining functions

On $X \times Y$, a function pairs every element of $X$ with exactly one element of $Y$. Compare to a relation, which does not require every $x \in X$ to be in a pair, and doesn't require a unique $y$ to be paired with any $x$.

## 8.2 The Canonical Map

The canonical map is the function that maps $x \in X$ to its equivalence class in ${}^{X}\!/_{R}$.

## 8.3 Characteristic Function

For $A \subset X$, the characteristic function $\chi_A(x) = 1$ if $x \in A$ and 0 if $x \in X - A$. Note that $A$ is a subset of $X$, so $A \in \mathcal{P}(X)$, but $\chi_A$ is a function, so it is a member of $2^X$. However, we can naturally associate a subset to its characteristic function with the correspondence $A \longrightarrow \chi_A$.

# 9 Families

## 9.1 Index sets and families

It may be convenient to give certain elements of $X$ a label such as $x_i$ or $x_\alpha$. To do so we establish a function $I \longrightarrow X$ from the *index set* $I$ (the set of labels) to $X$. Like any function this pairs an $i \in I$ with a specific $x \in X$. Thinking of $X$ being *indexed* by $I$, we call this function the *family* $\{x_i\}$.

## 9.2 Unioning a family

For a family $\{A_i\}$, the notation $\bigcup_{i \in I} A_i$ or $\bigcup_i A_i$ indicates the union of the range of the family

# 10 Inverses and Composites

## 10.1 Image subset maps

For a function $f : X \longrightarrow Y$, and $A \subset X$, we let $f(A)$ denote the set of $Y$ elements mapped to by some element in $A$. Technically, this is a function on $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ and we *overload* the meaning of $f$.

## 10.2 Inverse maps

Let $f^{-1}(B)$, where $B \subset Y$, denote the elements in $X$ that map into $B$. Rather than implying that $f$ is invertible, this says the inverse map generally is from $\mathcal{P}(Y)$ to $\mathcal{P}(X)$. If $f$ is injective then $f^{-1}$ will map singletons to singletons, and we might call the function $f$ invertible.

## 10.3 Relation inverses

Inverse relations can be phrased as such: if $x$ has some relation $R$ to $y$, then $y$ has the relation $R^{-1}$ to $x$.