

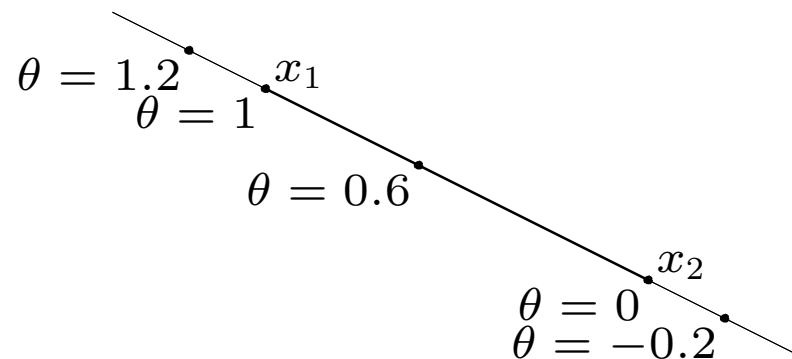
2. Convex sets

- affine and convex sets
- some important examples
- operations that preserve convexity
- generalized inequalities
- separating and supporting hyperplanes
- dual cones and generalized inequalities

Affine set

line through x_1, x_2 : all points

$$x = \theta x_1 + (1 - \theta)x_2 \quad (\theta \in \mathbf{R})$$



affine set: contains the line through any two distinct points in the set

example: solution set of linear equations $\{x \mid Ax = b\}$

(conversely, every affine set can be expressed as solution set of system of linear equations)

Convex set

line segment between x_1 and x_2 : all points

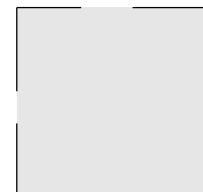
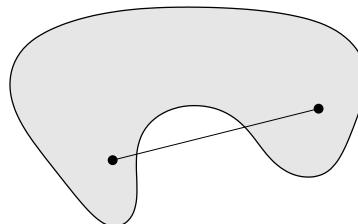
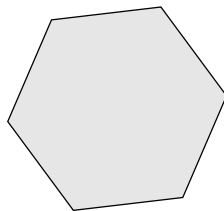
$$x = \theta x_1 + (1 - \theta)x_2$$

with $0 \leq \theta \leq 1$

convex set: contains line segment between any two points in the set

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \implies \quad \theta x_1 + (1 - \theta)x_2 \in C$$

examples (one convex, two nonconvex sets)



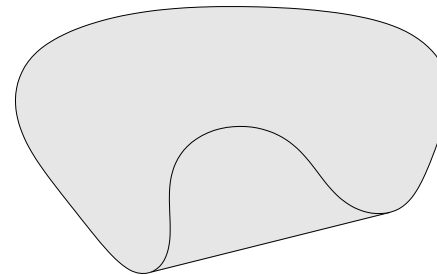
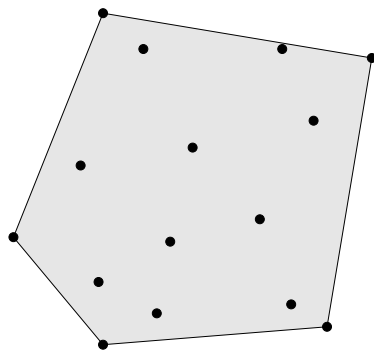
Convex combination and convex hull

convex combination of x_1, \dots, x_k : any point x of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \dots + \theta_k x_k$$

with $\theta_1 + \dots + \theta_k = 1$, $\theta_i \geq 0$

convex hull $\text{conv } S$: set of all convex combinations of points in S

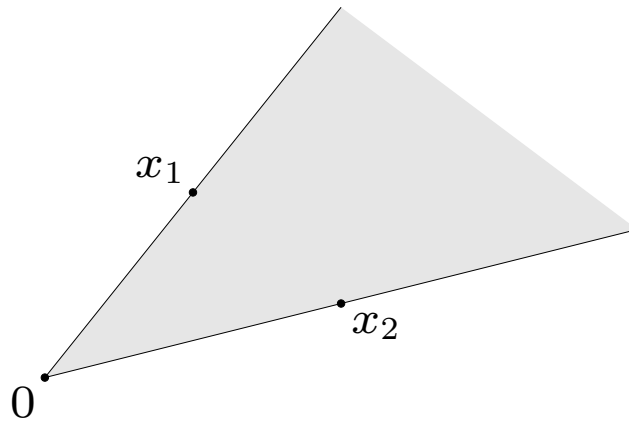


Convex cone

conic (nonnegative) combination of x_1 and x_2 : any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2$$

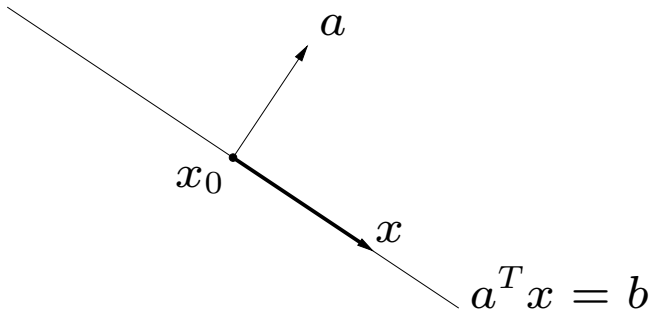
with $\theta_1 \geq 0$, $\theta_2 \geq 0$



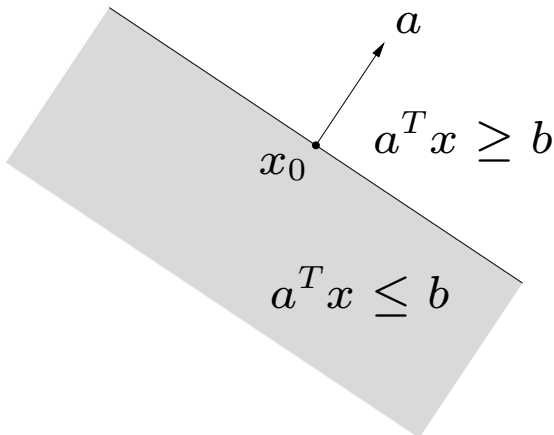
convex cone: set that contains all conic combinations of points in the set

Hyperplanes and halfspaces

hyperplane: set of the form $\{x \mid a^T x = b\}$ ($a \neq 0$)



halfspace: set of the form $\{x \mid a^T x \leq b\}$ ($a \neq 0$)



- a is the normal vector
- hyperplanes are affine and convex; halfspaces are convex

Euclidean balls and ellipsoids

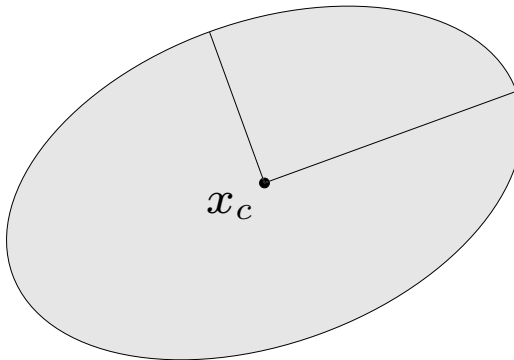
(Euclidean) ball with center x_c and radius r :

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\} = \{x_c + ru \mid \|u\|_2 \leq 1\}$$

ellipsoid: set of the form

$$\{x \mid (x - x_c)^T P^{-1} (x - x_c) \leq 1\}$$

with $P \in \mathbf{S}_{++}^n$ (*i.e.*, P symmetric positive definite)



other representation: $\{x_c + Au \mid \|u\|_2 \leq 1\}$ with A square and nonsingular

Norm balls and norm cones

norm: a function $\| \cdot \|$ that satisfies

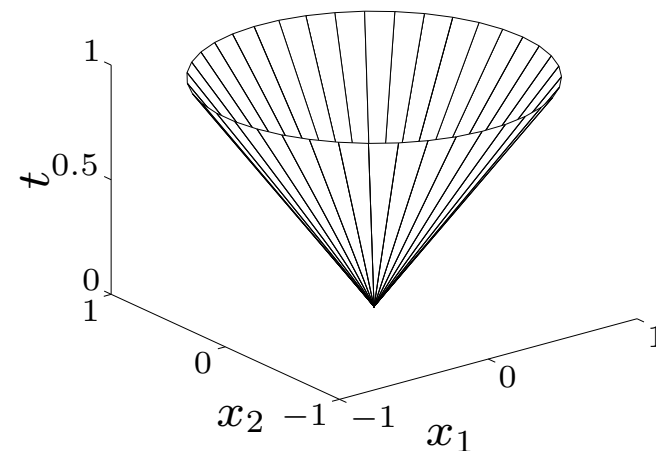
- $\|x\| \geq 0$; $\|x\| = 0$ if and only if $x = 0$
- $\|tx\| = |t| \|x\|$ for $t \in \mathbf{R}$
- $\|x + y\| \leq \|x\| + \|y\|$

notation: $\| \cdot \|$ is general (unspecified) norm; $\| \cdot \|_{\text{symb}}$ is particular norm

norm ball with center x_c and radius r : $\{x \mid \|x - x_c\| \leq r\}$

norm cone: $\{(x, t) \mid \|x\| \leq t\}$

Euclidean norm cone is called second-order cone



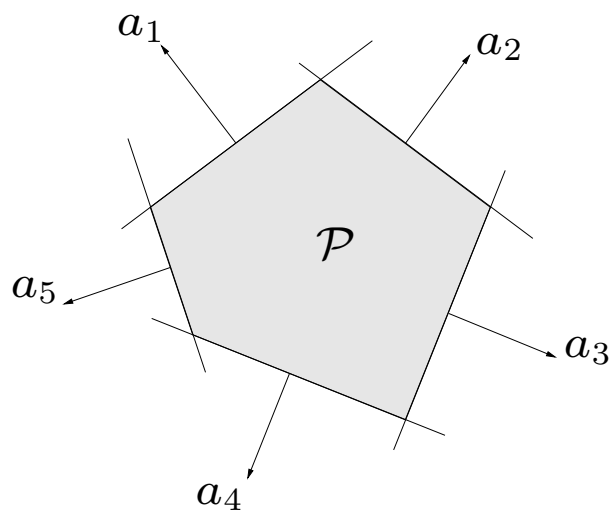
norm balls and cones are convex

Polyhedra

solution set of finitely many linear inequalities and equalities

$$Ax \preceq b, \quad Cx = d$$

($A \in \mathbf{R}^{m \times n}$, $C \in \mathbf{R}^{p \times n}$, \preceq is componentwise inequality)



polyhedron is intersection of finite number of halfspaces and hyperplanes

Positive semidefinite cone

notation:

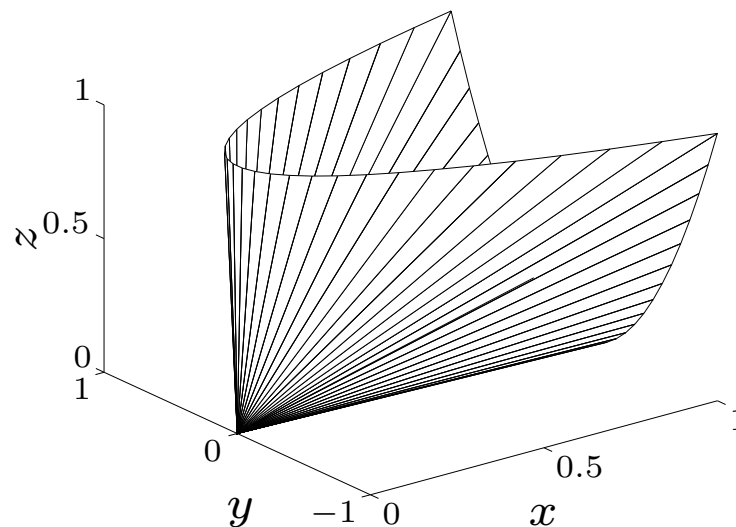
- \mathbf{S}^n is set of symmetric $n \times n$ matrices
- $\mathbf{S}_+^n = \{X \in \mathbf{S}^n \mid X \succeq 0\}$: positive semidefinite $n \times n$ matrices

$$X \in \mathbf{S}_+^n \iff z^T X z \geq 0 \text{ for all } z$$

\mathbf{S}_+^n is a convex cone

- $\mathbf{S}_{++}^n = \{X \in \mathbf{S}^n \mid X \succ 0\}$: positive definite $n \times n$ matrices

example: $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}_+^2$



Operations that preserve convexity

practical methods for establishing convexity of a set C

1. apply definition

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \implies \quad \theta x_1 + (1 - \theta)x_2 \in C$$

2. show that C is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, . . .) by operations that preserve convexity

- intersection
- affine functions
- perspective function
- linear-fractional functions

Intersection

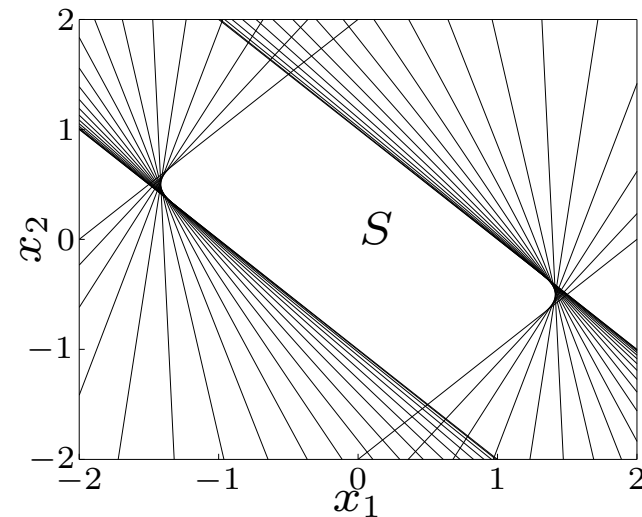
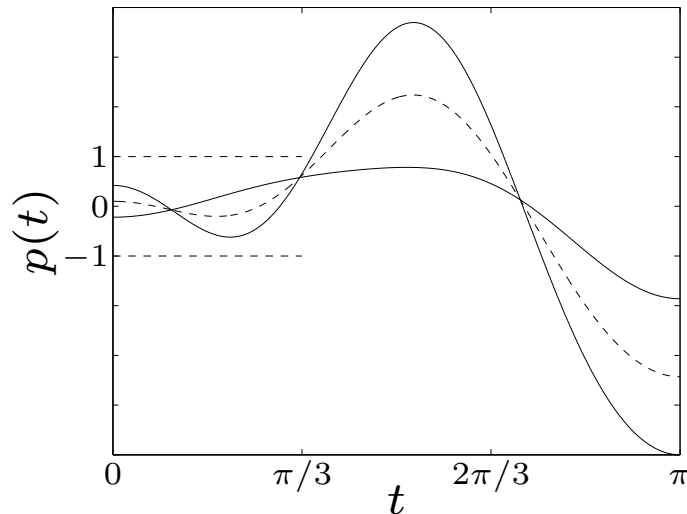
the intersection of (any number of) convex sets is convex

example:

$$S = \{x \in \mathbf{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \pi/3\}$$

where $p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt$

for $m = 2$:



Affine function

suppose $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is affine ($f(x) = Ax + b$ with $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$)

- the image of a convex set under f is convex

$$S \subseteq \mathbf{R}^n \text{ convex} \implies f(S) = \{f(x) \mid x \in S\} \text{ convex}$$

- the inverse image $f^{-1}(C)$ of a convex set under f is convex

$$C \subseteq \mathbf{R}^m \text{ convex} \implies f^{-1}(C) = \{x \in \mathbf{R}^n \mid f(x) \in C\} \text{ convex}$$

examples

- scaling, translation, projection
- solution set of linear matrix inequality $\{x \mid x_1 A_1 + \cdots + x_m A_m \preceq B\}$
(with $A_i, B \in \mathbf{S}^p$)
- hyperbolic cone $\{x \mid x^T P x \leq (c^T x)^2, c^T x \geq 0\}$ (with $P \in \mathbf{S}_+^n$)

Perspective and linear-fractional function

perspective function $P : \mathbf{R}^{n+1} \rightarrow \mathbf{R}^n$:

$$P(x, t) = x/t, \quad \text{dom } P = \{(x, t) \mid t > 0\}$$

images and inverse images of convex sets under perspective are convex

linear-fractional function $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$:

$$f(x) = \frac{Ax + b}{c^T x + d}, \quad \text{dom } f = \{x \mid c^T x + d > 0\}$$

images and inverse images of convex sets under linear-fractional functions are convex

example of a linear-fractional function

$$f(x) = \frac{1}{x_1 + x_2 + 1}x$$

