Invariants and Coinvariants of $\mathrm{Sym}(sl_2(k))$

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January 6, 2024

1 Introduction to the problem

For the remainder of this section, G is an arbitrary group and V is an arbitrary vector space over an arbitrary field k. In later sections we will restrict ourselves to the case where G is finite and V is finite-dimensional, but the results in this section will hold in general.

Definition 1.1. A **representation** of a group H on a vector space W over a field F is a group homomorphism

$$\varphi : H \to GL(W)$$

where $GL(W)$ is seen as a group under composition of maps.

Using this definition, the last assumption that we will need for this section is that we have some representation of G on V over k,

$$\varphi : G \to GL(V).$$

Definition 1.2. For any group H and any ring R, we will denote by $R[H]$ the **group ring of H over R**. Elements of $R[H]$ can be thought of as formal linear combinations of elements of H with coefficients in R. That is,

$$R[H] = \left\{ \sum_{h \in H} \alpha_h h : \alpha_h \in R \right\}.$$

We define addition on $R[H]$ group-element-wise: that is, we define the sum

$$\sum_{h \in H} \alpha_h h + \sum_{h \in H} \beta_h h = \sum_{h \in H} (\alpha_h + \beta_h) h.$$

We will then define multiplication as follows:

$$\left(\sum_{g \in H} \alpha_g g \right) \left(\sum_{h \in H} \beta_h h \right) = \sum_{g \in H} \sum_{h \in H} (\alpha_g \beta_h)(gh)$$

where $\alpha_g \beta_h$ are our coefficients in R and gh are elements of the group H. Equipped with these operations, $R[H]$ has the structure of a ring.

From this definition and our representation of G on V, we are able to conclude that V has the structure of a $k[G]$-module. That is, we can think of elements of $k[G]$ as acting on elements of V in such a way that the properties of a module are satisfied.

Now that we have an action of G on V, we can extend to an action of G on Sym$V$ (or equivalently, we will extend to a representation of G on Sym$V$). To define this, we will define a representation of G on $\bigotimes_{i=1}^n V$ for every positive integer n. The algebra $\bigotimes_{i=0}^\infty V$ is then said to be **filtered**, and this is the corresponding sequence of subspaces. Once we've defined a representation of G on $\bigotimes_{i=0}^\infty V$, on can obtain the representation on Sym$V$ very easily. But first, we'll define the representation on this filtration.

Definition 1.3. A **filtered algebra** over a field k is an algebra $(A, \cdot)$ over k such that there exists an increasing sequence

$$\{0\} \subset F_0 \subset F_1 \subset \ldots \subset A$$

of subspaces F_i of A such that

$$A = \bigcup_{i=0}^{\infty} F_i$$

and that these subspaces are compatible with multiplication in the sense that for all $n, m \geq 0$, $F_n \cdot F_m \subset F_{n+m}$.

Given that we have a representation φ of G on V where $g \mapsto \varphi_g$ for all $g \in G$, we will define the representation of G on $\bigotimes_{i=0}^{n} V$ by

$$\rho : G \to GL\left(\bigotimes_{i=0}^{n} V\right)$$

$$g \mapsto \rho_g$$

$$\rho_g(v_1 \otimes v_2 \otimes \ldots \otimes v_n) = \varphi_g(v_1) \otimes \varphi_g(v_2) \otimes \ldots \otimes \varphi_g(v_n)$$

for all $g \in G$ and all $v_1 \otimes v_2 \otimes \ldots \otimes v_n \in \bigotimes_{i=0}^{n} V$. Note that because $\varphi_g(v) \in V$ for all $g \in G$ and all $v \in V$, each ρ_g is indeed mapping elements of $\bigotimes_{i=0}^{n} V$ to other elements in $\bigotimes_{i=0}^{n} V$.

Definition 1.4. Let W be a vector space over a field F. For all $m \geq 0$, we define the $\boldsymbol{m}^{\mathbf{th}}$ **tensor power** of W to be the tensor product of W with itself m times, that is

$$\bigotimes_{i=1}^{m} W =: T^m W.$$

By convention we define $T^0 W$ to simply be the base field F.

We then define the **tensor algebra** of W, $T(W)$, to be the direct sum of all of these tensor powers, that is

$$T(W) = \bigoplus_{m=0}^{\infty} T^m W = F \oplus \bigoplus_{m=1}^{\infty} \left(\bigotimes_{i=1}^{m} W\right)$$

Definition 1.5. Given a vector space W over a field F, we can construct the **symmetric algebra** $\mathrm{Sym}W$ of W over F as follows:

$$\mathrm{Sym}W = T(W)/\langle v \otimes w - w \otimes v : v, w \in T(W)\rangle$$

That is, we quotient the tensor algebra of W by the two-sided ideal generated by all commutators. Given the quotient, it is clear that $\mathrm{Sym}W$ is a commutative algebra containing W. However, the symmetric algebra also satisfies the following universal property which in some sense make $\mathrm{Sym}W$ the smallest algebra satisfying these properties.

For every linear map L from W to a commutative algebra A, there is a unique algebra homomorphism $g : \mathrm{Sym}W \to A$ such that $f = g \circ i$, where i is the inclusion map of W into $\mathrm{Sym}W$.

Recall that we have defined a representation of G on $\bigotimes_{i=0}^{n} V$, for every $n \geq 0$. We would like to extend this to a representation of G on the tensor algebra $T(V)$. To do this, recall that

$$T(V) = \bigoplus_{m=0}^{\infty} T^m V$$

Therefore, an arbitrary element of $T(V)$ can be written as a necessarily finite direct sum of elements of tensor powers of V. Concretely, if $v \in T(V)$ then we can write

$$v = v_{m_1} \oplus v_{m_2} \oplus \ldots \oplus v_{m_N}$$

where these m_is are distinct nonnegative integers and each v_{m_i} is an element of the m_i^{th} tensor power of V, $T^{m_i}V$. We then can define a representation of G on $T(V)$ (or rather, extend our representation) by

$$\alpha : G \to GL\left(T(V)\right)$$

$$g \mapsto \alpha_g$$

$$\alpha_g(v_{m_1} \oplus v_{m_2} \oplus \ldots \oplus v_{m_N}) = \rho_g(v_{m_1}) \oplus \rho_g(v_{m_2}) \oplus \ldots \oplus \rho_g(v_{m_N})$$

Thus, to compute the action on an element of $T(V)$, we break apart the direct sum and compute the action on each tensor power, and then direct sum the results back together. Here we use a bit of an abuse of notation, with ρ taking the place of a family of representations, one for each tensor power. In particular, when we compute $\rho_g(v_{m_1})$ we're using the representation

$$\rho : G \to GL\left(\bigotimes_{i=1}^{m_1} V\right)$$

but when we compute $\rho_g(v_{m_2})$ we're using the representation

$$\rho : G \to GL\left(\bigotimes_{i=1}^{m_2} V\right)$$

To simplify notation, we will accept this ambiguity as it should be apparent which representation is being used by what the input is.

As a last remark on this representation α, note that because ρ is a representation, each $\rho_g(v_{m_i})$ is an element of the m_i^{th} tensor power of V. Thus, since the components of the direct sum get mapped back into their own tensor powers, the resulting direct sum is certainly an element of $T(V)$.

Finally, we will extend this to a representation of G on $\text{Sym}V$. Indeed, as $\text{Sym}V$ is a quotient of $T(V)$ this map is easy to define - an element of $\text{Sym}V$ is an equivalence class of $T(V)$, so if we let C by the ideal of commutators,

$$C = \langle v \otimes w - w \otimes v : v, w \in T(V) \rangle$$

then any element of $\text{Sym}V$ can be written as $u + C$, where $u \in T(V)$. We then define our representation

$$\beta : G \to GL(\text{Sym}V)$$

$$g \mapsto \beta_g$$

$$\beta_g(u + C) = \alpha_g(u) + C$$

as one might expect. What's left is to verify that this action is well-defined, and to do so we simply need to verify that the ideal C of $T(V)$ is stable under the representation α of G on $T(V)$.

Proposition 1.6. *The ideal C of $T(V)$ is stable under the representation α of G on $T(V)$.*

Proof. Fix $g \in G$. We will argue that $\alpha_g(v) \in C$ for all $v \in C$. Indeed, suppose that $v \otimes w - w \otimes v$ is an arbitrary element of C (so $v, w \in T(V)$). Then we have that

$$\begin{aligned}\alpha_g(v \otimes w - w \otimes v) &= \alpha_g(v \otimes w) - \alpha_g(w \otimes v) \\ &= \alpha_g(v) \otimes \alpha_g(w) - \alpha_g(w) \otimes \alpha_g(v)\end{aligned}$$

which is an element of C because both $\alpha_g(v)$ and $\alpha_g(w)$ are elements of $T(V)$. The fact that α_g is compatible with the tensor product in this way is a consequence of how it was defined - in particular refer back to the definition of ρ_g. □

Because the ideal C is stable under the representation α, the representation β is well-defined, and we obtain the desired representation of G on Sym$V$. This is the action which we will be concerned with in the remainder of this paper.

2 Algebras of Invariants and Coinvariants

Again, we assume that G is any group, that V is any vector space (so, in particular, V may be a symmetric algebra, as it will be in later sections), and that we have a representation $\rho : G \to GL(V)$. Throughout this section we will periodically use group action notation to tidy up the work being done. That is, we will write $g \cdot v$ to signify $\rho(g)(v)$.

Definition 2.1. We define the **space of invariants** of this representation as

$$V^G := \{x \in V : \rho(g)(x) = x \text{ for all } g \in G\}$$

That is, V^G is the fixed point set of this action.

Proposition 2.2. *If V is an algebra, then the space of invariants V^G is a subalgebra of V. If V is not an algebra, then V^G is a subspace of V.*

Proof. Suppose that $A, B \in V^G$ and that $\alpha \in k$. We will argue that $A - B$, αA, and AB are elements of V^G as well and conclude that V^G is an algebra. If V is not an algebra, then showing that $A - B$ and αA are in V will suffice to show that V^G is a subspace of V. To this end, suppose that $g \in G$. Then we see that

$$\begin{aligned} g \cdot (A - B) =& (g \cdot A) - (g \cdot B) \quad G \text{ acts linearly} \\ =& A - B \\ g \cdot (\alpha A) =& \alpha(g \cdot A) \quad G \text{ acts linearly} \\ =& \alpha A \\ g \cdot (AB) =& (g \cdot A)(g \cdot B) \quad \text{by construction of our action on Sym}V \\ =& AB \end{aligned}$$

Thus, $(\text{Sym}V)^G$ is a subalgebra of $\text{Sym}V$. ☐

Definition 2.3. We also define the **space of coinvariants** of this action as

$$V_G := V/I$$

where I is the ideal

$$I := \langle X - g \cdot X : X \in V, g \in G \rangle.$$

Note 2.4. Suppose that W is any vector space and that H is a group acting linearly on W via $*$. Let W' be a subspace of W. If $H * W' \subset W'$ (or in other words, if W' is an invariant subspace of W under the linear transformations $v \mapsto h * v$ for all $h \in H$), then we can define an induced action of H on the quotient space W/W' by $h * (w + W') = h * w + W'$, and this action will be well-defined.

Remark 2.5. Using the above note, let's show that we can define an induced action of G upon V_G:

$$\rho : G \to GL(V_G)$$

$$g \mapsto \rho_g$$

$$\rho_g(X + I) = \rho_g(X) + I$$

To do so, it suffices to show that I is an invariant subspace of V under the action of G on V. To do this, suppose that $g \in G$ and that $X - h \cdot X \in I$. Observe then that

$$\begin{aligned} g \cdot (X - h \cdot X) &= g \cdot X - g \cdot (h \cdot X) \\ &= g \cdot X - (gh) \cdot X \\ &= g \cdot X - (ghg^{-1}g) \cdot X \\ &= (g \cdot X) - (ghg^{-1}) \cdot (g \cdot X) \end{aligned}$$

which is an element of I because $g \cdot X \in V$ and $ghg^{-1} \in G$. Thus, the coinvariant ideal is an invariant subspace under this action, so we can conclude that the induced action of G on V_G is well-defined.

Proposition 2.6. *Suppose that $\rho : G \to GL(V)$ is a representation of G on V. Then the following properties are satisfied by the spaces of invariants and coinvariants:*

1. *V^G is the largest subspace of V upon which the action of G is trivial. That is, if W is any subspace of V upon which the action of G is trivial, then it must be that $W \subset V^G$.*

2. *V_G is the largest quotient of V upon which the action of G is trivial. That is, if V/W is any quotient of V which is invariant under the induced quotient action, then there exists a surjective homomorphism $V_G \to V/W$.*

Proof. Suppose that $\rho : G \to GL(V)$ is a representation of G on V.

1. Suppose that W is any subspace of V upon which the action of G is trivial - that is, $\rho(g)(w) = w$ for all $w \in W$ (and for all $g \in G$). But we defined V^G to be the set of elements of V which satisfy this condition (that is, V^G is the set of elements of V which are fixed under the action), so each element of W is therefore an element of V^G, and we obtain that $W \subset V^G$. Hence, V^G is the largest subspace of V upon which the action of G is trivial.

2. First, let's show that the induced action of G on V_G is trivial. To this end, suppose that $g \in G$ and that $v + I \in V_G$. Observe then that

   $$\begin{aligned} g \cdot (v + I) &= g \cdot v + I \\ &= g \cdot v + (v - g \cdot v) + I \quad \text{as } v - g \cdot v \in I \\ &= v + I \end{aligned}$$

   Thus, the induced action of G on V_G is indeed trivial.

   Next, suppose that V/W is a quotient of V which is invariant under the induced quotient action. First, note that in order for the induced quotient action to even be well-defined, it must be that $G \cdot W \subset W$ (W is an invariant subspace under the action of G).

   In order to obtain the desired result, we will first argue that $I \subset W$. Recall that I is the ideal generated by elements of the form $X - g \cdot X$, $g \in G$, $X \in V$. Thus, because I is generated by such elements, to show that $I \subset W$ it will suffice to show that each of these generators is an element of W. To this end, suppose that $g \in G$ and $X \in V$. Because V/W is invariant under the induced quotient action, we have that

$$\begin{aligned} g \cdot (X+W) = X+W &\implies g \cdot X + W = X + W \\ &\implies X - g \cdot X + W = W \\ &\implies X - g \cdot X \in W \end{aligned}$$

Thus, as W contains all of these generators, we conclude that $I \subset W$. As we've shown that I is a subspace of W, it is straightforward to find a surjective homomorphism from V_G to V/W. Indeed, define the map

$$V_G \xrightarrow{\varphi} V/W$$

$$v + I \mapsto v + W$$

The fact that φ is well-defined is a direct consequence of the fact that $I \subset W$. Indeed, suppose that $v + I = u + I$. Observe then that

$$\begin{aligned} v + I = u + I &\implies v - u \in I \\ &\implies v - u \in W \\ &\implies v + W = u + W \\ &\implies \varphi(v+I) = \varphi(u+I) \end{aligned}$$

Thus, φ is well-defined. The fact that φ is surjective is clear, if $v + W$ is any element of V/W then we have that $\varphi(v+I) = v + W$.

$\square$

Thus, the invariant space V^G can be thought of as the largest subspace of V upon which the action of G is trivial, and V_G can be thought of as the largest quotient of V upon which the action of G is trivial.

3 Theorems which will be used

Suppose that G is any connected, simply connected, semisimple algebraic group over $\mathbb{C}$, and fix a maximal torus H of G. Then let $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$. Finally, let Δ be the root system of G with respect to H, and fix a positive root system Δ^+.

Theorem 3.1. *Chevalley's Restriction*

The restriction of the natural map $\mathbb{C}[\mathfrak{g}] \to \mathbb{C}[\mathfrak{h}]$ to $\mathbb{C}[\mathfrak{g}]^G$, that is the map

$$\textit{rest: } \mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{h}],$$

is injective, and its image coincides with the invariant polynomial ring $\mathbb{C}[\mathfrak{h}]^W$ with respect to the action of the Weyl group W on $\mathfrak{h}$ from our root system.

A proof of this theorem can be found on page 260 of Hotta et al.[3]. Thus, this theorem gives us that $\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\mathfrak{h}]^W$.

Theorem 3.2. *Bourbaki*

$\mathbb{C}[\mathfrak{h}]^W$ is generated by ℓ $(:= \dim \mathfrak{h})$ algebraically independent homogeneous polynomials over $\mathbb{C}$. In particular, $\mathbb{C}[\mathfrak{h}]^W$ is isomorphic to a polynomial ring of ℓ variables.

A proof of this theorem can be found in the cited portion of Bourbaki[5]. Putting these two results together, we have that $\mathbb{C}[\mathfrak{g}]^G$ is isomorphic to a polynomial ring in ℓ variables.

The degrees of the generators described in the theorem above can be described explicitly using the following procedure:

Set $c := s_{\alpha_1} \cdots s_{\alpha_\ell} \in W$, where $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ is the set of simple roots of our root system and s_{α_i} is the reflection in W associated with the root α_i. We call c the **Coxeter transformation**.

The conjugacy class of c in W will not depend on the choice of Π or on the numbering of the α_i (we'll use this fact without justification, but more on this can be found on page 261 of Hotta et al.[3]).

First, assume that the root system Δ is irreducible. Then we set h to be the order of c in W - h here is called the **Coxeter number**. Then, write the eigenvalues of the operator c on $\mathfrak{h}$ as

$$\exp(2\pi i m_1/h), \ldots, \exp(2\pi i m_\ell/h)$$

$$0 \leq m_1 \leq m_2 \leq \ldots \leq m_\ell < h.$$

It is know that

$$0 < m_1 = 1 < m_2 \leq \ldots \leq m_\ell < h$$

$$m_i + m_{\ell-i+1} = h$$

$$\sum_{i=1}^{\ell} m_i = |\Delta^+|.$$

We'll call these m_i the **exponents** of the irreducible root system Δ. If Δ is not irreducible, then we define the exponents of Δ to be the union of those exponents in its irreducible components.

Theorem 3.3. *Let $m_1, \ldots, m_\ell$ be the exponents of our root system Δ. Then the degrees of the ℓ algebraically independent homogeneous polynomials described in Theorem 2.2 are $m_1 + 1, \ldots, m_\ell + 1$.*

A proof of this theorem can be in the cited portion of Bourbaki[5].

4 The Problem in Characteristic Zero

Problem
Let $\mathfrak{g} = sl_2(\mathbb{C})$ and $G = SL_2(\mathbb{C})$. Of interest to us is the structure of the algebras $\mathbb{C}[\mathfrak{g}]^G$ and $\mathbb{C}[\mathfrak{g}]_G$. In particular, we'd like to show that these two algebras are isomorphic to some polynomial algebras which are to be determined.

Definition 4.1. A **Cartan Subalgebra** $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a nilpotent subalgebra of $\mathfrak{g}$ that is self-normalising.

A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is said to be self-normalising if whenever $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$, then $Y \in \mathfrak{h}$.

Definition 4.2. Given a root system Φ in a Euclidean vector space E, the group of isometries of E generated by reflections through hyperplanes associated to the roots of Φ is called that **Weyl group** of Φ.

The process being followed and the construction here are inspired by Hotta et al. [3]

Consider the subalgebra $\mathfrak{h} = \{d(a_1, a_2) : \sum a_i = 0\}$ of $\mathfrak{g}$, where we're using the notation

$$d(a_1, \dots, a_n) = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}.$$

For $i = 1, 2$, define $\lambda_i \in \mathfrak{h}^*$ (where $\mathfrak{h}^*$ denotes the dual space of $\mathfrak{h}$) by

$$\lambda_i(d(a_1, a_2)) = a_i$$

From the fact that elements of $\mathfrak{h}$ are diagonal matrices with zero trace, it's clear that

$$\mathbb{C}[\mathfrak{h}] \simeq \mathbb{C}[\lambda_1, \lambda_2]/(\lambda_1 + \lambda_2)$$

I claim that this resulting algebra is isomorphic to a polynomial algebra in one variable.

Proposition 4.3.

$$\mathbb{C}[\mathfrak{h}] \simeq \mathbb{C}[x],$$

where x is a formal variable.

Proof. Indeed, by definition of $\mathfrak{h}$ we have that

$$\mathbb{C}[\mathfrak{h}] \simeq \mathbb{C}[\lambda_1, \lambda_2]/(\lambda_1 + \lambda_2)$$

But given an element of $\mathbb{C}[\lambda_1, \lambda_2]/(\lambda_1 + \lambda_2)$ (an element of this algebra will be an equivalence class, but it will have a representative which is a polynomial in λ_1 and λ_2), it can be written as a polynomial in only λ_1 via the map $\lambda_2 \mapsto -\lambda_1$. Indeed,

$$\begin{aligned} \lambda_1 + \lambda_2 + (\lambda_1 + \lambda_2) &= (\lambda_1 + \lambda_2) \\ \implies \lambda_2 + (\lambda_1 + \lambda_2) &= -\lambda_1 + (\lambda_1 + \lambda_2) \end{aligned}$$

Thus, we can obtain an isomorphism from $\mathbb{C}[\mathfrak{h}]$ to $\mathbb{C}[x]$ by substituting $-\lambda_1$ for λ_2 in order to get our polynomials in a single variable, and then simply mapping $\lambda_1 \mapsto x$. $\square$

Next, we need to find the Weyl group associated with $\mathfrak{h}$.

Remark 4.4. Suppose that $\mathfrak{g}$ is any complex semisimple Lie algebra and that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ (this is the case for $sl_2(\mathbb{C})$ and its subalgebra $\mathfrak{h}$ described earlier). Then a root system can be constructed as follows:

We will say that $\alpha \in \mathfrak{h}^*$ is a root of $\mathfrak{g}$ relative to $\mathfrak{h}$ if $\alpha \neq 0$ and there exists some $X \neq 0 \in \mathfrak{g}$ such that

$$[H, X] = \alpha(H)X$$

for all $H \in \mathfrak{h}$.

Proposition 4.5. *The Weyl group of the root system constructed from $\mathfrak{g}$ relative to its Cartan subalgebra $\mathfrak{h}$ is isomorphic to S_2.*

Proof. I claim that the root system constructed in this manner will have two roots, namely α and $-\alpha$ (elements of $\mathfrak{h}^*$), where

$$\alpha \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} = h_1 - h_2$$

Indeed, consider

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and let

$$H = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \in \mathfrak{h}$$

Then we have that

$$\begin{aligned}
[H, X] &= HX - XH \\
&= \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \\
&= \begin{pmatrix} 0 & h_1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & h_2 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & h_1 - h_2 \\ 0 & 0 \end{pmatrix} \\
&= (h_1 - h_2) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= \alpha(H)X
\end{aligned}$$

Thus, α is a root of $\mathfrak{g}$ relative to $\mathfrak{h}$. Then, of course, $-\alpha$ must also be a root - indeed, consider now the element $-X$ of $\mathfrak{g}$. We then have that

$$\begin{aligned}[H, -X] &= H(-X) - (-X)H \\ &= -(HX - XH) \\ &= -[H, X] \\ &= -(\alpha(H)X) \\ &= (-\alpha)(H)X\end{aligned}$$

for any $H \in \mathfrak{h}$. Thus, $-\alpha$ is also a root of $\mathfrak{g}$ relative to $\mathfrak{h}$.

It is straightforward but tedious to verify that these are the only two possible roots of $\mathfrak{g}$ relative to $\mathfrak{h}$.

Thus, the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$ is $\{\alpha, -\alpha\}$. The Weyl group corresponding to this root system is the isometry group generated by reflections through the hyperplanes associated with these roots. In this case, as our two roots are scalar multiples of each other, this group is generated by a single reflection, and in fact it is simply isomorphic to the group S_2 (indeed, this is a tad excessive - there is only one root system of rank 1, which is the root system consisting of some nonzero vector and its negation. Any root system of rank 1 has the same Weyl group, namely S_2). $\square$

Writing $S_2 = \{e, \sigma\}$, the action of the Weyl group S_2 on $\mathfrak{h}$ is then given by $\sigma \cdot \lambda_i = \lambda_{\sigma(i)}$, and this extends naturally to an action of S_2 on $\mathbb{C}[\mathfrak{h}]$. Using the fact that $\mathbb{C}[\mathfrak{h}] \simeq \mathbb{C}[x]$, we see that this action is equivalent to an action of S_2 on $\mathbb{C}[x]$. In particular, it's given by

$$\sigma \cdot x = -x$$

which is clear as our isomorphism between $\mathbb{C}[\mathfrak{h}]$ and $\mathbb{C}[x]$ mapped $\lambda_1 \mapsto x$, $\lambda_2 \mapsto -x$. We are interested in $\mathbb{C}[\mathfrak{h}]^W \simeq \mathbb{C}[x]^W$, the set of elements invariant under the action of σ.

As the action of σ is that of negating x, the polynomials which will be invariant under the action of σ will be those in which all terms have even degree (σ will have the effect of negating all polynomial terms of odd degree). Thus, we have that

$$\mathbb{C}[\mathfrak{h}]^W \simeq \mathbb{C}[x]^W = \mathbb{C} \oplus \langle x^2 \rangle$$

Calculating the Coxeter transformation in this case, we obtain that

$$c = (1\,2) \in W = S_2$$

The single eigenvalue of this transformation is then -1, the Coxeter number is 2, and we have the single exponent 1. This serves as a sanity check, the degree of x^2 is 2, which agrees with Theorem 3.3.

By Chevalley's Restriction Theorem (Theorem 3.1), we have that $\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\mathfrak{h}]^W$. Thus, we obtain

$$\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C} \oplus \langle x^2 \rangle$$

Finally, we would like to determine the structure of $\mathbb{C}[\mathfrak{g}]_G$.

Theorem 4.6. $\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\mathfrak{g}]_G$.

Proof. Let $\mathrm{Sym}_n \mathfrak{g}$ denote the n^{th} step of the natural filtration of $\mathrm{Sym}\mathfrak{g} = \mathbb{C}[\mathfrak{g}]$. To argue that $\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\mathfrak{g}]_G$, we will argue that $\mathrm{Sym}_n \mathfrak{g}^G$ and $\mathrm{Sym}_n \mathfrak{g}_G$ are isomorphic for the adjoint action of $G = SL_2(\mathbb{C})$ on its Lie algebra $\mathfrak{g}$ for every $n \geq 0$.

By Weyl's theorem on complete reducibility, $\mathrm{Sym}_n\mathfrak{g}$ (viewed as a G-representation) can be written as the direct sum of irreducible G-representations. That is, we can write

$$\mathrm{Sym}_n\mathfrak{g} = \bigoplus_{i=1}^{N} I_i$$

where each I_i is an irreducible G-representation. G will act trivially on a certain number of these subspaces, so we can write

$$\mathrm{Sym}_n\mathfrak{g} = \bigoplus_{i=1}^{N_1} J_i \oplus \bigoplus_{j=1}^{N_2} K_j$$

where G acts trivially on each subspace J_i, and G does not act trivially on any subspace K_j. It is then clear that $(\mathrm{Sym}_n\mathfrak{g})^G$, the largest subspace of $\mathrm{Sym}_n\mathfrak{g}$ upon which G acts trivially, will simply be the direct sum of all of these components upon which G acts trivially. That is, we have

$$(\mathrm{Sym}_n\mathfrak{g})^G = \bigoplus_{i=1}^{N_1} J_i$$

Now, since we've written $\mathrm{Sym}_n\mathfrak{g}$ as the direct sum of these irreducible components, we can also take the quotient of $\mathrm{Sym}_n\mathfrak{g}$ by $\bigoplus_{j=1}^{N_2} K_j$ to obtain that

$$\mathrm{Sym}_n\mathfrak{g}/\bigoplus_{j=1}^{N_2} K_j = \left(\bigoplus_{i=1}^{N_1} J_i \oplus \bigoplus_{j=1}^{N_2} K_j\right)/\bigoplus_{j=1}^{N_2} K_j \simeq \bigoplus_{i=1}^{N_1} J_i$$

Now, I claim that this quotient is $\mathrm{Sym}_n\mathfrak{g}_G$. That is, I claim that this is the largest quotient of $\mathrm{Sym}_n\mathfrak{g}$ that is invariant under the induced action. First, note that this quotient is indeed invariant under the induced action as it's isomorphic to an invariant subspace.

Suppose that we were to quotient by a smaller number of subspaces in an attempt to create a larger invariant quotient space - say

$$\mathrm{Sym}_n\mathfrak{g}/\bigoplus_{j=1, j\neq M}^{N_2} K_j =: \mathrm{Sym}_n\mathfrak{g}/A$$

thus, we leave out K_M from our quotient. Can this quotient space be invariant under the induced action?

The answer, of course, is no. Indeed, because G does not act trivially on K_M, there must exist some $k \in K_M$ and some $g \in G$ such that $g \cdot k \neq k$. We then consider the action

$$g \cdot (k + A) = g \cdot k + A$$

If this new quotient we've created were invariant under the induced action, then we would have that $g \cdot k - k \in A$. Since g does not act trivially on k, this difference is not the identity. Now, k is certainly not in A. But if $g \cdot k$ were in A, then this would imply that $k \in A$. Thus, it must be the case that $g \cdot k$ is in $\bigoplus_{i=1}^{N_2} K_j$. But then we have that $g \cdot k - k \in \bigoplus_{i=1}^{N_2} K_j$, which contradicts the direct sum. Hence, we've reached a contradiction - if we remove any K_i from our quotient then the resulting quotient space cannot be invariant under the induced action. Therefore, we conclude that this is indeed the largest quotient which is invariant under the induced action, and therefore we have that

$$(\mathrm{Sym}_n\mathfrak{g})_G \simeq \bigoplus_{i=1}^{N_1} J_i = (\mathrm{Sym}_n\mathfrak{g})^G$$

Since this isomorphism holds at every step of the filtration, we have that

$$(\mathrm{Sym}\mathfrak{g})^G \simeq (\mathrm{Sym}\mathfrak{g})_G$$

or, using the notation we've been using throughout this section,

$$\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\mathfrak{g}]_G$$

$\square$

Hence, putting Theorem 4.5 together with the characterization of $\mathbb{C}[\mathfrak{g}]^G$ derived earlier, we obtain that

$$\mathbb{C}[\mathfrak{g}]_G \simeq \mathbb{C} \oplus < x^2 > .$$

5 The Problem in General

Problem
Let $\mathfrak{g} = sl_2(k)$ and $G = SL_2(k)$ for field k of positive characteristic. Of interest to us is the structure of the algebras $k[\mathfrak{g}]^G$ and $k[\mathfrak{g}]_G$. In particular, we'd like to show that these two algebras are isomorphic to some polynomial algebras which are to be determined.

We'll take the same approach towards this problem as we did in the complex case, modifying the process where necessary.

Consider the subalgebra $\mathfrak{h} = \{d(a_1, a_2) : \sum a_i = 0\}$ of $\mathfrak{g}$. We will invoke a modified Chevalley's restriction theorem to again conclude that $k[\mathfrak{g}]^G \simeq k[\mathfrak{h}]^W$ (where W is the Weyl group associated with the roots of $\mathfrak{g}$ relative to $\mathfrak{h}$). But first, let us determine the structure of $k[\mathfrak{h}]^W$. Again define $\lambda_i \in \mathfrak{h}^*$ by

$$\lambda_i(d(a_1, a_2)) = a_i$$

for $i = 1, 2$. From the fact that elements of $\mathfrak{h}$ are diagonal matrices with zero trace, it's clear that

$$k[\mathfrak{h}] \simeq k[\lambda_1, \lambda_2]/(\lambda_1 + \lambda_2) \simeq k[x]$$

where x is a formal variable. The proof of Proposition 4.1 can be modified to show the second isomorphism above (in fact, the proof given will work for an arbitrary field).

Next, we need to find the Weyl group associated with $\mathfrak{h}$. In fact, the Weyl group structure will be exactly as it was in the complex case. Indeed, all the work we did in proving Proposition 4.5 holds over an arbitrary field of any characteristic. Therefore, using the exact same process, we see that the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$ is $\{\alpha, -\alpha\}$ where

$$\alpha \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} = h_1 - h_2.$$

The Weyl group corresponding to his root system is again $S_2 = \{e, \sigma\}$, and the action of $W := S_2$ on $\mathfrak{h}$ is given by

$$\sigma \cdot \lambda_i = \lambda_{\sigma(i)}$$

which extends to an action of W on $k[\mathfrak{h}]$. Using the fact that $k[\mathfrak{h}] \simeq k[x]$, we see that this action is equivalent to the action of W on $k[x]$ given by

$$\sigma \cdot x = -x.$$

We are interested in $k[\mathfrak{h}]^W \simeq k[x]^W$, the set of polynomials invariant under the action of σ. Working in positive characteristic does not at all change which polynomials will be invariant under this action. It's clear that the invariant polynomials here will still simply be those in which all terms have even degree. Therefore, we have that

$$k[\mathfrak{h}]^W \simeq k[x]^W = k \oplus \langle x^2 \rangle.$$

Next we would like to show that $k[\mathfrak{g}]^G \simeq k[\mathfrak{h}]^W$. Consider the natural map

$$k[\mathfrak{g}] \to k[\mathfrak{h}]$$

$$f \mapsto f|_{\mathfrak{h}}$$

As we're viewing elements of these algebras as polynomial functions, this map is essentially simply restricting the domain of any polynomial function on $\mathfrak{g}$. We then consider the restriction of this map to $k[\mathfrak{g}]^G$, that is, the map

$$rest : k[\mathfrak{g}]^G \to k[\mathfrak{h}]$$

mapping G-invariant polynomial functions on $\mathfrak{g}$ to their corresponding polynomial functions on $\mathfrak{h}$. We will argue that this map is injective and that its image is $k[\mathfrak{h}]^W$. This restriction map is in fact injective in general - a proof of this over $\mathbb{C}$ can be found on page 260 of Hotta et al.[3], and this proof can be modified for the case of positive characteristic.

What's left to show is that the image of $rest$ is $k[\mathfrak{h}]^W$.

Thus, putting this result together with our characterization of $k[\mathfrak{h}]^W$, we have that

$$k[\mathfrak{g}]^G \simeq k[\mathfrak{h}]^W \simeq k \oplus \langle x^2 \rangle$$

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