Nonlinear instability in long time calculations of a partial difference equation*

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The stability of long time ($t \rightarrow \infty$) calculations of *nonlinear* partial difference equations is examined. The model chosen is a discretisection in space and time of the Korteweg de Vries equation. The standard *linearized* von Neumann stability analysis, although necessary, is not sufficient, since it ignores the quadratic nature of the nonlinearity in the problem. Nonlinear stability is analysed by perturbing a constant solution with a period 3 Fourier mode. This leads to a nonlinear system of ordinary differential equations, the stability of which is examined by phase plane analysis. The results obtained contrast strongly with those obtained from the linear analysis. A numerical illustration shows how a single Fourier mode initial condition through side band growth, eventually causes blow-up in a long time calculation.

INTRODUCTION

Partial difference equation models of nonlinear time dependent problems, either obtained by finite difference or finite element methods from differential equation, or as models in their own right, are of use only if they are *stable*. There is no accepted difinition of stability for nonlinear partial difference equations particularly in *long time* ($t \rightarrow \infty$) calculations with a *fixed time step*, where lack of blow-up of the solution is necessary for stability but not sufficient. In linear problems there is no such difficulty and many stability theorems exist based on Fourier or energy methods.

If Fourier modes are used to assess the stability of a nonlinear difference formula, where linearization has taken place as in the von Neumann method, any Fourier mode with an amplification factor greater than one is enough to make the method unstable. On the other hand, satisfaction of the von Neumann condition, although necessary for stability is not sufficient and *nonlinear* effects can still render the difference formula unstable. Alternatively, if the difference formula retains its stability, despite the presence of nonlinear effects, the solution of the nonlinear problem may bifurcate into a new stable solution which exists for grid parameter values beyond the linearized stability limits[1]. These can be illustrated by studying discretizations of two entirely different time dependent partial differential equations involving the nonlinear term u^2, viz. the Korteweg de Vries equation[2]

$$\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial}{\partial x}(u^2)+\varepsilon\frac{\partial^3 u}{\partial x^3}=0, \qquad \varepsilon>0 \tag{1}$$

with dispersive wave solutions, and Fisher's[3] equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+u-u^2 \tag{2}$$

a model for reaction diffusion problems in mathematical biology. In the latter case, standard space discretization of equation (2) followed by Euler's method in time, leads to the linearized stability condition

$$r\leqslant\frac{1}{2}-\frac{1}{4}k \tag{3}$$

where $r=k/h^2$, h and k being the step lengths in space and time respectively. It has been shown that stable periodic solutions of the nonlinear partial difference equation exist for values of r and k in the range $r>\frac{1}{2}-\frac{1}{4}k$ (Ref. 4).

Our main objective, however, is the study of *nonlinear instability* and so for the remainder of the paper, we shall concentrate on equation (1). When $\varepsilon=0$ and equation (1) becomes the inviscid Burgers equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0 \tag{4}$$

several important results exist for nonlinear instability of difference models. These are gathered together in a clearly written paper by Trefethen[5], where reference is made to fundamental papers on nonlinear instability by Phillips[6], Fornberg[7], Briggs *et al.*[8], and others.

THE KORTEWEG DE VRIES EQUATION

Useful analytical properties of the KdV equation are:

(i) the initial value problem with compact support initial data and solutions which tend to zero rapidly as $|x|$ tends to infinity is completely integrable by inverse scattering, and
(ii) the equation has an infinite number of conservation laws. Consequently it might be thought that numerical solutions of the KdV equation will present few problems, in contrast to the case when $\varepsilon=0$, when

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considerable difficulties are known to arise. This is true in some cases e.g. solutions emerging from compact support, positive initial data, but not in others.

The problem we actually consider involves equation (1) in the x-range $[0, L]$ with *periodic* boundary conditions and an initial condition $u(x, 0) = f(x)$, $0 \leqslant x \leqslant L$. Initially we discretize equation (1) in space only to give

$$\dot{U}_j = -\frac{\theta}{4h}(U_{j+1}^2 - U_{j-1}^2) - \frac{1-\theta}{2h}U_j(U_{j+1} - U_{j-1})$$
$$-\frac{\varepsilon}{2h^3}(U_{j+2} - 2U_{j+1} + 2U_{j-1} - U_{j-2})$$
$$0 \leqslant j \leqslant J-1 \tag{5}$$

where $x = jh$, $Jh = L$, a dot denotes differentiation with respect to time, and θ is a parameter in the range $0 \leqslant \theta \leqslant 1$. This conserves $\sum_{j=0}^{J-1} U_j^2$ exactly if $\theta = 2/3$.

For the purpose of comparing linear and nonlinear stability, we consider *leap frog* discretization in time of equation (5) giving

$$U_j^{n+1} = U_j^{n-1} - r(U_{j+1}^n - U_{j-1}^n)$$
$$\times\left[\frac{1}{2}\theta(U_{j+1}^n + U_{j-1}^n) + (1-\theta)U_j^n\right]$$
$$-\frac{\varepsilon r}{h^2}(U_{j+2}^n - 2U_{j+1}^n + 2U_{j-1}^n - U_{j-2}^n)$$
$$0 \leqslant j \leqslant J-1, \qquad n = 1, 2, \ldots \tag{6}$$

where $t = nk$ and $r = k/h$. The extra starting condition required by equation (6) is obtained from equation (5), with Euler discretization in time. The nonlinear partial difference formula equation (6) is second order accurate in space and time, and for $\theta = 2/3$ conserves $\sum_{j=0}^{J-1} (U_j^n)^2$ to order k^2.

LINEAR STABILITY

We now linearize equation (6) by making the substitution

$$U_j^n \sim C + \delta_j^n$$

where δ is a small perturbation about the constant solution C. This gives

$$\delta_j^{n+1} = \delta_j^{n-1} - Cr(\delta_{j+1}^n - \delta_{j-1}^n)$$
$$-\frac{\varepsilon r}{h^2}(\delta_{j+2}^n - 2\delta_{j+1}^n + 2\delta_{j-1}^n - \delta_{j-2}^n) \tag{7}$$

a result *independent of θ and the initial condition*, and where $u_{\min} \leqslant C \leqslant u_{\max}$.

Standard von Neumann analysis with

$$\delta_j^n \sim e^{\alpha nk} e^{i\beta jh}, \qquad \alpha, \beta \text{ real}$$

leads to *neutral* stability ($|e^{\alpha k}| = 1$) for equation (7) if

$$\max_{\phi}\left|r \sin\theta\left(C - \frac{4\varepsilon}{h^2}\sin^2\frac{\phi}{2}\right)\right| \leqslant 1 \tag{8}$$

where $\phi = \beta h$. If it is assumed that $u_{\min} = 0$, and so the constant about which the linearization is performed is never negative, Vliegenhart[9] derives from equation (8) the stability condition

$$r \leqslant \left(u_{\max} + \frac{4\varepsilon}{h^2}\right)^{-1}$$

and Sanz Serna[10], the more realistic linearized condition

$$r \leqslant \frac{2}{3\sqrt{3}}\frac{h^2}{\varepsilon} \tag{9}$$

provided $u_{\max} \leqslant 1/r$.

NONLINEAR STABILITY OF THE SEMI-DISCRETE SYSTEM

Nonlinear stability is analysed by considering a solution of equation (5) in the form

$$U_j(t) = A_1(t)\, e^{i(2\pi/3)j} + (*) + C, \qquad j = 0, 1, 2 \tag{10}$$

where A_1 is complex and $(*)$ denotes complex conjugate. This represents a period 3 perturbation about the constant solution C. The nonlinear terms in equation (5) acting on the $2\pi/3$ Fourier mode give

$$e^{i(2\pi/3)j} \cdot e^{i(2\pi/3)j} = e^{i(4\pi/3)j} = e^{-i(2\pi/3)j}$$

and

$$e^{-i(2\pi/3)j} \cdot e^{-i(2\pi/3)j} = e^{-i(4\pi/3)j} = e^{i(2\pi/3)j}$$

respectively, and so the quadratic interactions are closed. Hence no new Fourier modes are created. This phenomenon is known as *aliasing*.

If we substitute (10) into (5) and put

$$A_1(t) = V_1(t) + iV_2(t), \qquad V_1, V_2 \in \mathbf{R}$$

we obtain the system of ordinary differential equations

$$\dot{V}_1 = 2\lambda V_1 V_2 - \beta V_2$$
$$\dot{V}_2 = \lambda(V_1^2 - V_2^2) + \beta V_1$$
$$\dot{C} = 0 \tag{11}$$

with $\lambda = \sqrt{3}/4h\ (2 - 3\theta)$ and $\beta = \sqrt{3}/2h\ (3\varepsilon/h^2 - 1/2(2 - 3\theta)C)$. Phase plane analysis in the $(V_1 V_2)$ plane reveals four critical points with $(0, 0)$ stable, provided β and λ are not zero. Special cases are

(i) $\lambda = 0$, $(\theta = 2/3)$ when the system is linearized and integrates to give the conservation law

$$V_1^2 + V_2^2 = \text{constant} \tag{12}$$

and

(ii) $\beta = 0$, giving $C = 6\varepsilon/(2 - 3\theta)h^2$ $(\theta \neq 2/3)$, when the system reduces to

$$\dot{V}_1 = 2\lambda V_1 V_2$$
$$\dot{V}_2 = \lambda(V_1^2 - V_2^2) \tag{13}$$

This system is also obtained for perturbations about the zero solution when $\varepsilon=0$ and $\theta\neq 2/3$ (Ref. 11), where it is shown that the only critical point is a centre at (0, 0) and the integral curves satisfy

$$V_1(V_1^2-3V_2^2)=\text{constant} \tag{14}$$

The initial condition determines the solution curve all of which eventually lead to infinity. Thus the system equation (13) is completely unstable.

In a similar manner, a period 4 perturbation about the constant solution C given by

$$U_j(t)=A(t)\,e^{i(\pi/2)j}+(*)+B(t)\cos\pi j+C(t), \qquad j=0,1,2,3 \tag{15}$$

results in the system

$$h\dot{V}_1=[(1-2\theta)B+\beta]V_2$$

$$h\dot{V}_2=[(1-2\theta)B-\beta]V_1$$

$$h\dot{B}=4(1-\theta)V_1V_2$$

$$\dot{C}=0 \tag{16}$$

where $\beta=(C-26/h^2)$. Special cases of equation (16) are

(i) $\theta=1$. The system is linearized and integrates to give

$$V_1^2-\gamma V_2^2=\text{constant} \tag{17}$$

where $\gamma=(B-\beta)(B+\beta)^{-1}$. Hence depending on the sign of γ, the orbits in the phase plane are either ellipses or hyperbolae. In the latter case, the system is unstable.

(ii) $\theta=2/3$. Here the system integrates to give the energy conservation

$$V_1^2+V_2^2+\frac{1}{2}B^2=\text{constant} \tag{18}$$

and so the solution cannot blow-up.

We conclude from the two cases $J=3,4$ that nonlinear analysis of the semi-discrete system is very dependent on θ, in sharp contrast to the linearized analysis for the *fully* discretized system, where the stability condition is independent of θ.

LEAP FROG DISCRETIZATION IN TIME AND A NUMERICAL EXPERIMENT

We now return to equation (6) which is the leap frog discretization in time of equation (5) and put $\theta=2/3$. This results in the partial difference formula

$$U_j^{n+1}=U_j^{n-1}-\frac{1}{3}r(U_{j+1}^n-U_{j-1}^n)(U_{j+1}^n+U_j^n+U_{j-1}^n)$$

$$-\frac{\varepsilon r}{h^2}(U_{j+2}^n-2U_{j+1}^n+2U_{j-1}^n-U_{j-2}^n)$$

$$0\leqslant j\leqslant J-1, \qquad n=1,2,\ldots \tag{19}$$

which was introduced first by Zabusky and Kruskal[12] as a means of solving the KdV equation.

A numerical experiment is now set up in order to study the growth of nonlinear instabilities. It is based on a long time calculation of equation (19) with a fixed time step. The initial condition is chosen from equation (10) to be

$$U_j(0)=2\sigma\left(\cos\frac{2\pi}{3}j-\sin\frac{2\pi}{3}j\right)+C, \qquad j=0,1,2 \tag{20}$$

and since periodic boundary conditions require J to be a multiple of 3, we choose $J=120$. The additional starting values U_j^1, $j=0,1,\ldots,N-1$, are calculated using Euler's rule. Although the quantity $\sum(U_j^n)^2$ is conserved by equation (5) with $\theta=2/3$, leap frog discretization reduces the conservation to order k^2, leaving blow-up a possibility using equation (19). The other parameters in equations (19) and (20) are $r=1$, $h=1/120$, $\varepsilon=0.2315\times10^{-4}$ and $C=1$, values which satisfy the linearized stability condition equation (9) but marginally violate the upper limit condition on $u_{\max}$. It is shown that for this choice of parameters, round-off errors in the numerical calculation induce nonlinear instability, during which the relative spread and growth of Fourier modes can be studied prior to blow-up.

We define the energy in the perturbation term of the initial condition equation (20) by

$$E=h\sum_{j=0}^{J-1}(U_j^0-C)^2=4\sigma^2 \tag{21}$$

and blow up time by the first value of n for which

$$\max_j|U_j^n|>10^5$$

In fact in our numerical experiment, with $E=0.01$, blow-up occurred after 16 774 time steps. The distribution of Fourier modes in the solution as time increases is best seen in Fourier space where the Fourier transform of U_j^n, $j=0,1,\ldots,J-1$ is obtained from the formula

$$\hat{U}_k=\frac{1}{\sqrt{J}}\sum_{j=0}^{J-1}U_j^n\,e^{2\pi ijk/J}, \qquad k=0,1,\ldots,J-1 \tag{22}$$

and the initial condition

$$|\hat{U}_k^0|=\begin{cases}0 & k\neq 40\\ \sqrt{\dfrac{JE}{2}} & k=40\end{cases} \tag{23}$$

In x-space for the initial stages the solution is $U_j^n=U_j^0$. As the calculation proceeds in time, the first sign of instability is a modulation of the constant envelope and then the solution begins to focus at local patches on the grid. Blowup occurs within approximately one hundred steps after focusing. A fuller account of the numerical experiment can be found in Herring[13].

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