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Theoretical and Computational Fluid Dynamics

A New Class of Equations for Rotationally Constrained Flows[1]

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Abstract. The incompressible Navier–Stokes equation is considered in the limit of rapid rotation (small Ekman number). The analysis is limited to horizontal scales small enough so that both horizontal and vertical velocities are comparable, but the horizontal velocity components are still in geostrophic balance. Asymptotic analysis leads to a pair of nonlinear equations for the vertical velocity and vertical vorticity coupled by vertical stretching. Statistically stationary states are maintained against viscous dissipation by boundary forcing or energy injection at larger scales. For thermal forcing direct numerical simulation of the reduced equations reveals the presence of intense vortical structures spanning the layer depth, in excellent agreement with simulations of the Boussinesq equations for rotating convection by Julien *et al.* (1996).

1. Introduction

The last few years have seen increased interest in the dynamics of rapidly rotating fluids, with particular emphasis on geophysical and astrophysical applications. Rapidly rotating flows are generally believed to be simpler to understand because the strong rotation reduces the effective dimensionality of the system. The classical statement of this tendency is provided by the Taylor–Proudman theorem (Taylor, 1923) which asserts that, under appropriate conditions, rapid rotation confines any motion to two-dimensional planes orthogonal to the rotation axis (hereafter the z axis). Nevertheless, rotating flows in nature (and in the

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laboratory) are seldom purely two-dimensional, particularly if they are forced. For example, both thermal forcing (present in rotating convection) and surface shear stresses (resulting from no-slip boundaries) violate the basic assumptions of the theorem, and generate flow parallel to the rotation axis via buoyancy or Ekman pumping (Ekman, 1905). In both of these cases the boundary conditions prevent strict two-dimensional motion and generate deviations from the z-independent flow predicted by the theorem. Such deviations may concentrate in thin "boundary layers" or permeate the entire fluid layer, and are a consequence of viscous or baroclinic effects ignored by the theorem. Despite these complications the Taylor–Proudman theorem indicates that motion along the rotation axis will be inhibited for sufficiently rapid rotation. Consequently, the range of vertical wave numbers capable of participating in mode-coupling as turbulence develops will be limited. Recent work of Babin *et al.* (1995) exploits this property to obtain a rigorous description of what might be termed ($2+\varepsilon$)–dimensional turbulence, using averaging in the "Schrödinger representation" of the fluid equations. This procedure leads to a reduced mode-coupling theory in wave number space, and can be extended to stratified hydrodynamics (Babin *et al.*, 1997). Related work by Embid and Majda (1997) and Majda and Embid (1998), employing rigorous averaging for partial differential equations, yields a description of strongly stratified rapidly rotating flows in terms of reduced partial differential equations. It is hoped, in general, that such reduced descriptions will prove more tractable, both analytically and numerically, than fully three-dimensional turbulence, and some steps along these lines have already been taken (Mahalov and Zhou, 1996).

In this paper we describe a complementary approach to problems of this type. Our methodology is also based on the rapidly rotating limit of the fluid equations. However, we focus on horizontal scales small enough so that both the horizontal and vertical velocities are comparable. On these scales the horizontal velocity components are still in geostrophic balance at leading order, but in contrast to the usual treatments valid for large horizontal scales, the vertical velocity becomes a dynamic variable responsible for strong stretching in the vertical. This stretching can lead to intensification of the vertical vorticity which can, in turn, sustain the vertical velocity. This process is described by a pair of nonlinear equations for the vertical velocity and vertical vorticity. In our formulation (statistically) stationary states are maintained against viscous dissipation by boundary forcing or energy injection at larger scales. The resulting equations are relevant to laboratory experiments as well as to the planetary boundary layer. Application to the ocean mixed layer is also envisaged. Since the theory is fully nonlinear it offers scope for going beyond the usual linear Ekman-type analysis employed in most models of rotating boundary layers. Only recently has the need for a nonlinear treatment of such boundary layers become apparent (Hart, 1995, 1996); however, in contrast to Hart's treatment our approach avoids modeling and is therefore, in principle, exact. In particular it should be possible to use the equations derived here to evaluate existing models of rotating boundary layers by investigating the collective influence of smaller-scale motions which are usually neglected (or filtered out) in large-scale descriptions.

In contrast to the work of Babin *et al.* (1995), which is in effect an expansion in powers of the Rossby number (the ratio of the shear rate on the scale of the energy-containing eddies to the rotation rate), we employ a multiple scale expansion in both time *and* space. Specifically, we define the Ekman number $E \equiv \nu/2\Omega d^2$, where ν is the kinematic viscosity, d is a typical length scale in the problem (imposed by the boundaries), and $\mathbf{\Omega} \equiv \Omega\hat{\mathbf{z}}$ is the rotation vector, and treat E as a small parameter. We scale time in units of $(2\Omega)^{-1}$, distances in units of d, and velocities in units of $2\Omega d$. In the absence of stratification the incompressible Navier–Stokes equation then becomes

$$\frac{D\mathbf{u}}{Dt} + \hat{\mathbf{\Omega}} \times \mathbf{u} = -\nabla\pi + E\nabla^2\mathbf{u} + \mathbf{f}, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2}$$

where $\mathbf{f}$ is an as yet unspecified body force. In Section 5 we present results for the specific case of rotating convection for which $\mathbf{f} = (Ra/\sigma)E^2T\hat{\mathbf{z}}$ and (1) and (2) are supplemented with the energy equation

$$\sigma\frac{DT}{Dt} = E\nabla^2T. \tag{3}$$

Here T is the temperature, *Ra* is the Rayleigh number, and $\sigma = \nu/\kappa$ is the Prandtl number; κ is the thermal diffusivity. For most of the paper such a specific choice of $\mathbf{f}$ is unnecessary, however.

In the following we assume that the rotation rate is large in the sense that $E \ll 1$. Although this requirement says nothing about the local Rossby number Ro on the scales of interest ($Ro \equiv U/2\Omega L$, where U and L are, respectively, the characteristic velocity and length scales), we focus on scales L for which this number is small. However, in contrast to standard geophysical applications in which the smallness of the Rossby number is due to the large scales considered, we consider small scales with correspondingly slow velocities, and the Rossby number is small because the rotation rate is large. Such a characterization is typical for turbulent rotating convection (Julien *et al.*, 1996) in which both long-lived thermal and vortical coherent structures are observed (Figure 1). These structures, whose horizontal and vertical scales are close to the dissipation scale and the layer depth, respectively, cannot be captured by reduced prescriptions such as the (2+ε)–dimensional models discussed above. A complementary approach such as the one presented in this paper is therefore worth pursuing.

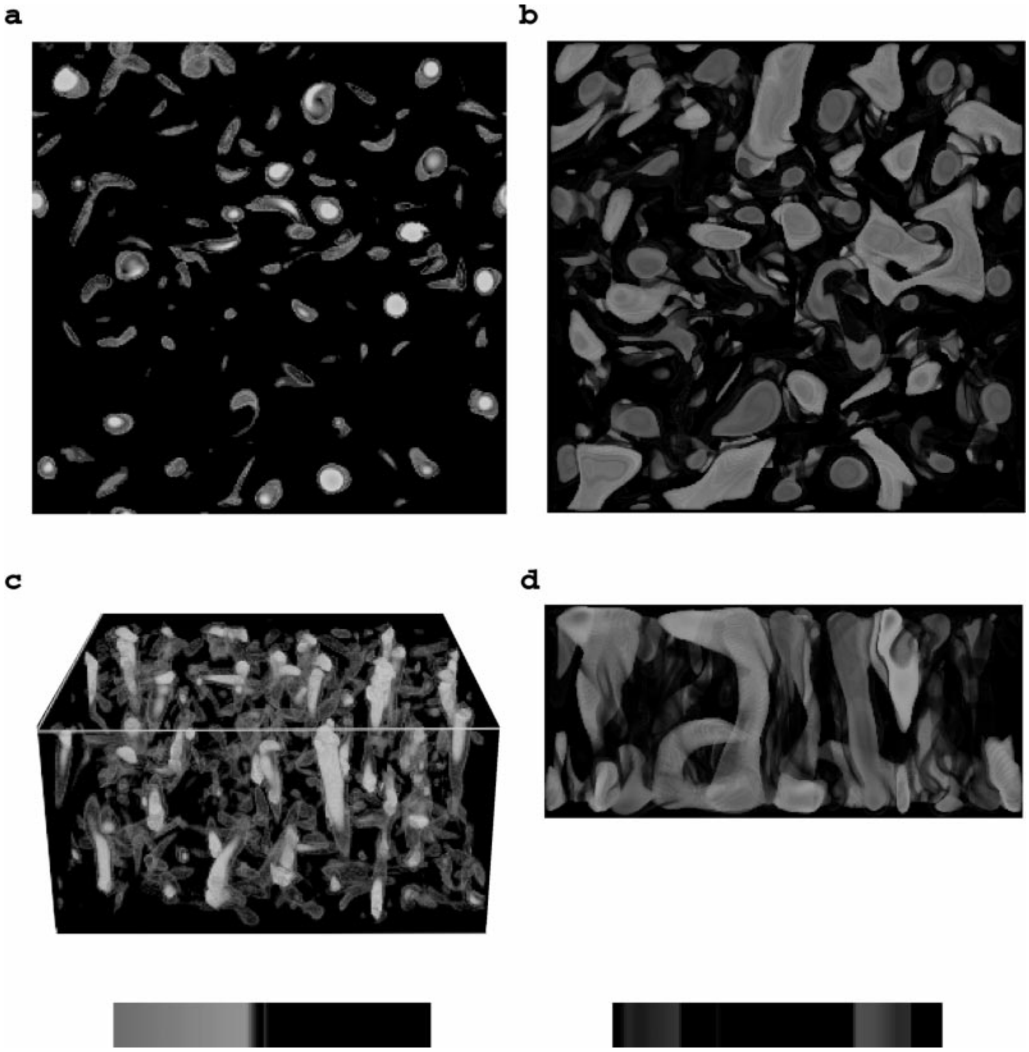

Figure 1. Results of direct numerical simulations of three-dimensional Boussinesq convection in the rapidly rotating regime (Julien *et al.*, 1996). The visualization of vortical structures in (a) and (c) is in terms of the quantity λ_2 suggested by Jeong and Hussain (1995), where λ_2 is the intermediate eigenvalue of $R^2 + S^2$, where R and S are, respectively, the rotation and strain matrices of the velocity field. The fluctuating temperature field θ is shown in (b) and (d). Plots (a), (b) are top views, (c) is the perspective, and (d) is the side view. The parameter values are $Ra = 1.0 \times 10^7$, $E = 9.4 \times 10^{-5}$, $Ro = 0.3$; the aspect ratio is $2 \times 2 \times 1$ with periodic boundary conditions in the horizontal. The simulation reveals the presence of slender coherent structures that span the depth of the layer. These structures are noticeably thinner in the λ_2 visualization than in the temperature field.

2. Streamfunction Formulation

In order to satisfy the incompressibility requirement (2) we employ the streamfunction formulation, and write $\mathbf{u}$ in the form

$$\mathbf{u} := (u, v, w) = \nabla \times \varphi\hat{\mathbf{z}} + \nabla \times \nabla \times \psi\hat{\mathbf{z}}, \tag{4}$$

or, more explicitly,

$$\mathbf{u} = \begin{pmatrix} \partial_y\varphi + \partial_x\partial_z\psi \\ -\partial_x\varphi + \partial_y\partial_z\psi \\ -\nabla_\perp^2\psi \end{pmatrix}, \qquad \boldsymbol{\omega} = \begin{pmatrix} \partial_x\partial_z\varphi - \nabla^2\partial_y\psi \\ \partial_y\partial_z\varphi + \nabla^2\partial_x\psi \\ -\nabla_\perp^2\varphi \end{pmatrix}. \tag{5}$$

Here $\boldsymbol{\omega} \equiv \nabla\times\mathbf{u}$ is the vorticity, and partials with subscripts denote differentiation with respect to the subscript variable, e.g., $\partial_x \equiv \partial/\partial x$; $\nabla_\perp^2 \equiv \partial_{xx} + \partial_{yy}$ is the horizontal Laplacian. Horizontal mean flows $\overline{u}(z)$, $\overline{v}(z)$ (if present) require that the streamfunctions φ, ψ be nonperiodic in (x, y). However, in the expansion that follows such flows only come in at higher order in E and hence are determined *self-consistently* by the leading-order contributions to φ and ψ. Consequently, in the following we take both φ and ψ to be periodic in the horizontal and calculate the associated mean flows only *a posteriori* (see Julien and Knobloch, 1998a).

The evolution equations for φ and ψ follow on taking $(\hat{\mathbf{z}}\cdot\nabla\times)$ and $(\hat{\mathbf{z}}\cdot\nabla\times\nabla\times)$ of the momentum equation (1). In the absence of the body force $\mathbf{f}$ these are

$$\partial_t\nabla_\perp^2\varphi - (\hat{\boldsymbol{\Omega}}\cdot\nabla)\nabla_\perp^2\psi + N_\varphi(\varphi,\psi) = E\nabla^2\nabla_\perp^2\varphi, \tag{6}$$

$$\partial_t\nabla^2\nabla_\perp^2\psi + (\hat{\boldsymbol{\Omega}}\cdot\nabla)\nabla_\perp^2\varphi + N_\psi(\varphi,\psi) = E\nabla^4\nabla_\perp^2\psi, \tag{7}$$

where

$$N_\varphi(\varphi,\psi) \equiv (\boldsymbol{\omega}\cdot\nabla)w - (\mathbf{u}\cdot\nabla)\zeta, \tag{8}$$

$$N_\psi(\varphi,\psi) \equiv \hat{\mathbf{z}}\cdot\nabla\times\nabla\times(\boldsymbol{\omega}\times\mathbf{u}). \tag{9}$$

Here ζ denotes the vertical component of the vorticity ω. In the streamfunction representation these terms become

$$\begin{aligned} N_\varphi = & -J[\varphi,\nabla_\perp^2\varphi] - J[\nabla^2\psi,\nabla_\perp^2\psi] + \nabla_\perp(\nabla_\perp^2\varphi)\cdot\nabla_\perp(\partial_z\psi) \\ & -\nabla_\perp(\partial_z\varphi)\cdot\nabla_\perp(\nabla_\perp^2\psi) - \nabla_\perp^2\psi\nabla_\perp^2(\partial_z\varphi) + \nabla_\perp^2\varphi\nabla_\perp^2(\partial_z\psi), \end{aligned} \tag{10}$$

and

$$\begin{aligned} N_\psi = & -\nabla^2\Big\{J[\varphi,\nabla^2\psi] + J[\partial_z\varphi,\partial_z\psi] - \nabla_\perp\varphi\cdot\nabla_\perp(\partial_z\varphi) - \nabla_\perp(\partial_z\psi)\cdot\nabla_\perp(\nabla^2\psi)\Big\} \\ & - \partial_z\Big\{J[\partial_z\psi,\nabla^2\varphi] - J[\varphi,\nabla^2\partial_z\psi] - 2J[\partial_z\varphi,\nabla^2\psi] + \nabla_\perp\varphi\cdot\nabla_\perp(\nabla^2\varphi) \\ & + \nabla_\perp(\partial_z\psi)\cdot\nabla_\perp(\nabla^2\partial_z\psi) + \nabla_\perp^2\psi\nabla^2(\nabla_\perp^2\psi) + |\nabla_\perp(\partial_z\varphi)|^2 + |\nabla_\perp(\nabla^2\psi)|^2 + (\nabla_\perp^2\varphi)^2\Big\}, \end{aligned} \tag{11}$$

with the horizontal Jacobian operator given by $J[f,g] \equiv \partial_x f\,\partial_y g - \partial_y f\partial_x g$. Appropriate boundary conditions must be supplied with these equations.

3. Reduced Interior Equations

In the following we focus on horizontal scales of order $E^{1/3}d$; scales of this order are the dominant ones selected by linear stability theory if the motion is forced by uniform heating from below (Chandrasekhar, 1961), but we do not wish to prejudge their origin. The analysis that follows applies equally to scales of this size "fed" by an inertial range energy cascade, i.e., the theory that follows describes a particular *range* of scales in the energy spectrum of rapidly rotating turbulence. However, in the vertical direction the scales of interest will be of order d: the spatial scales are thus anisotropic. This anisotropy is an inevitable consequence of the Taylor–Proudman constraint which requires slow variation in z relative to the variation in (x, y). We therefore introduce "fast" horizontal variables $x' \equiv E^{-1/3}x$, $y' \equiv E^{-1/3}y$ and use the notation $D \equiv \partial_z$ to denote derivatives with respect to the “slow” variable z. Since the motion on the horizontal scales will be predominantly in geostrophic balance, we also introduce a slow time $t' \equiv E^{1/3}t$:

$$\partial_x, \partial_y = E^{-1/3}(\partial_{x'}, \partial_{y'}), \qquad \partial_t = E^{1/3}\partial_{t'}, \qquad \partial_z = D. \tag{12}$$

Finally, we scale the streamfunctions as

$$\varphi = E\varphi', \qquad \psi = E^{4/3}\psi'. \tag{13}$$

This rescaling implies that on the length scales of interest the vertical and horizontal velocities are of the same order, both $\mathcal{O}(E^{2/3})\Omega d$ in dimensional units. The local Rossby number is thus $\mathcal{O}(E^{1/3})$ and hence is small; i.e., even though the scales of interest are small, the flow is still rotation dominated. Moreover, with this scaling the horizontal velocity components are in geostrophic balance at leading order, as can be verified from (5).

Scaling (6) and (7) according to (12) and (13) (and dropping all primes) now leads to the following reduced system:

$$\partial_t \nabla_\perp^2 \varphi - J[\varphi, \nabla_\perp^2 \varphi] - D\nabla_\perp^2 \psi = \nabla_\perp^4 \varphi + \mathcal{O}(E^{1/3}), \tag{14}$$

$$\partial_t \nabla_\perp^2 \psi - J[\varphi, \nabla_\perp^2 \psi] + D\varphi = \nabla_\perp^4 \psi + \mathcal{O}(E^{1/3}), \tag{15}$$

i.e., a pair of equations for the vertical velocity $w \equiv -\nabla_\perp^2 \psi$ and the vertical vorticity $\zeta \equiv -\nabla_\perp^2 \varphi$, coupled via the vertical derivatives D, i.e., by vertical stretching. Note, in particular, that the anisotropic scaling in the horizontal and vertical directions resulted in a substantial reduction in order of the vertical derivative. Henceforth we refer to (14) and (15) as the reduced (3-ε–dimensional!) equations for rapidly rotating turbulence.

3.1. Relation to the Equations of Geophysical Fluid Dynamics

Equations (14) and (15) are closely related to those used in geophysical fluid dynamics. The standard (shallow water) scaling (Pedlosky, 1979) yields at leading order an equation of the form

$$\partial_t \nabla_\perp^2 \varphi - J[\varphi, \nabla_\perp^2 \varphi] + Dw_1 = Re^{-1}\nabla_\perp^4 \varphi. \tag{16}$$

In this equation Re (= Ud/ν) denotes the Reynolds number of the interior flow, i.e., the ratio of the characteristic shear rate to the rate of viscous dissipation, and $Ro\; w_1$ is the leading-order vertical velocity. Since the Rossby number $Ro \ll 1$, this velocity is much smaller than the $\mathcal{O}(1)$ horizontal velocities. Despite this difference from our scaling, (16) is similar to our (14). However, with this standard scaling, (16) is not coupled to an equation of the form (15). Instead, w_1 must be specified in order to close the system. This is done by integrating (16) over $0 \leq z \leq 1$ using the fact that, at leading order, φ (and hence ζ) is independent of z. This is a consequence of the large horizontal scales assumed by the theory, which in turn imply that at leading order the fluid is in hydrostatic balance (in addition to the geostrophic balance in the horizontal). The vertical velocity at $z = 0, 1$ (assumed rigid) is determined by matching to viscous boundary layers. The resulting Ekman suction generates nonzero w_1 at both boundaries, which can also be written in terms of ζ. In this way one obtains the closed (potential) vorticity equation

$$\partial_t \nabla_\perp^2 \varphi - J[\varphi, \nabla_\perp^2 \varphi] = -r\nabla_\perp^2 \varphi + Re^{-1}\nabla_\perp^4 \varphi. \tag{17}$$

Here, the Rayleigh friction coefficient r measures the viscous effect of the horizontal Ekman boundary layers on the bulk flow, while, for turbulent conditions, the higher-order dissipation term involving the Reynolds number need only be retained in the presence of no-slip side boundaries; elsewhere this term is negligible because of the large value of the Reynolds number on the scales of interest. Since (17) does not include the forcing term **f**, all its solutions ultimately decay; there is no possibility of sustained nontrivial dynamics. Other closure schemes based on the shallow-water scaling such as those of quasi-geostrophy (Pedlosky, 1979) and balanced models (Gent and McWilliams, 1983) lead to equations similar to the potential vorticity equation (17) even though they relax some of the shallow-water assumptions.

It is now a simple matter to identify the origin of the differences between (14), (15), and (17). Specifically, with our chosen scaling, both the vertical and horizontal velocities are comparable. Consequently, these velocities are dynamically coupled and both depend on z. More importantly, for this scaling viscous dissipation in the bulk comes in at leading order. Despite this, Ekman pumping is still an important factor. This is because (14) and (15) are of second order in z; consequently at horizontal boundaries only impermeability can be imposed and Ekman boundary layers are required to satisfy no-slip boundary conditions. However, in contrast to the case discussed by Pedlosky (1979), in our case there is no simple relation between the Ekman velocity $\hat{w}_0$ and the vertical vorticity. Thus both approaches contain a parameter related to Ekman friction: in the shallow-water approach this is the parameter r while in the present approach it is the Ekman velocity $\hat{w}_0$. There is, however, an important difference between these two cases. In the former the parameter r really is a free parameter, since it specifies the magnitude of the viscosity which is otherwise absent from the problem for infinite Re (at least when sidewalls are absent). In contrast in our approach the viscosity is specified by the Ekman number used in the expansion. As a result $\hat{w}_0$ is *not* an independent parameter, but has to be determined self-consistently by the requirement that the Ekman suction responsible for the solution is in turn generated by the solution it creates. However, since a no-slip boundary, e.g., at $z = 0$, generates an Ekman velocity $\hat{w}_0$ that is proportional to $E^{1/6} D\psi(0)$ (Julien and Knobloch, 1998a) this procedure suggests that ψ must in fact be $\mathcal{O}(E^{1/6})$ in the boundary layer and not $\mathcal{O}(1)$ as assumed for the bulk. Evidently, the Ekman forcing is too weak in the rapid rotation limit to sustain the flow in the absence of large-scale forcing such as might arise from an energy cascade down to the scales of interest.

4. Elementary Properties of the Reduced Equations

In this section we summarize a few of the more elementary properties of (14) and (15). A detailed discussion of the equations together with numerical solutions will appear elsewhere. However, preliminary numerical results for rotating convection are included in Section 5 below.

Special Cases

The simplest case of (14) and (15) arises when $D \equiv 0$. This case corresponds to strictly two-dimensional flow, and equations for the vertical velocity and vertical vorticity decouple. Both equations are of advection–diffusion type in the horizontal. Consequently, in the absence of forcing, both quantities are homogenized in the horizontal (Weiss, 1966; Rhines and Young, 1982, 1983; Lingevitch and Bernoff, 1994). In the absence of boundaries, w = const., ζ = const. is a general solution of this type.

Two-Dimensional Equilibria

There are a number of nontrivial, z-independent equilibria (φ_0, w_0) satisfying the nonlinear equations

$$J(\varphi_0, \nabla_\perp^2 \varphi_0) = -\nabla_\perp^4 \varphi_0, \qquad J(\varphi_0, w_0) = -\nabla_\perp^2 w_0, \tag{18}$$

subject to appropriate boundary conditions. These boundary conditions must provide boundary forcing that can sustain the flow against viscous decay in the presence of Ekman boundary layers. Although it is possible to write down explicit solutions of these equations in certain cases, their stability properties have not been investigated. It should be noted that these solutions are the result of vorticity diffusion into the body of the fluid and its subsequent amplification by vertical shear; in equilibrium this amplification must be balanced by viscous dissipation. This observation raises the possibility that sustained turbulence can be produced by *finite* amplitude perturbations, even though infinitesimal amplitude perturbations of the trivial state always decay (in the absence of boundary forcing) as described next.

Spatially Periodic Equilibria

Equations (14) and (15) have the trivial solution $w = \zeta = 0$. It is a straightforward exercise to show that infinitesimal perturbations about this solution decay in the absence of boundary forcing. Note, however, that for fixed horizontal planforms the linearized equations take the form of a (damped) *wave* equation. Specifically, if $\nabla_\perp^2 \psi + k^2\psi = 0$ and dissipation is neglected, one obtains the equation

$$\partial_{tt}\varphi - k^{-2}D^2\varphi = 0. \tag{19}$$

This propagative aspect of the dynamics is a consequence of the strong vertical stretching and may well be important for understanding the resulting turbulence.

When boundary forcing is present the basic equilibrium becomes nontrivial. We illustrate what happens with the Ekman-like forcing (see Pedlosky, 1979)

$$w_0 = \mp\hat{w}_0\, f(x,y) \qquad \text{on} \quad z = 0, 1, \tag{20}$$

where $\overline{f} = 0$ in order to satisfy incompressibility; the overbar indicates horizontal averaging. We search for solutions of the form $\varphi = \Phi(z)f(x,y)$, $\psi = \Psi(z)f(x,y)$, and suppose that $f(x,y)$ satisfies the planform equation $\nabla_\perp^2 f + k^2 f = 0$. The resulting solutions

$$\varphi = -\frac{\hat{w}}{k}\frac{\cosh k^3(z-\frac{1}{2})}{\sinh(k^3/2)}\, f(x,y), \tag{21}$$

$$\psi = -\frac{\hat{w}}{k^2}\frac{\sinh k^3(z-\frac{1}{2})}{\sinh(k^3/2)}\, f(x,y), \tag{22}$$

satisfy the *nonlinear* equations (14) and (15). The amplitude of this state increases with increasing $\hat{w}$ and it is possible that such a state will lose stability at a critical value of $\hat{w}$. Such an instability could be detected by linearizing (14) and (15) about the equilibrium (21), (22) and solving the resulting stability problem.

Exact solutions with asymmetry with respect to the midplane $z = \frac{1}{2}$ can be found by similar methods.

Integral Relations

In the following we assume periodic boundary conditions in the two horizontal directions, and derive several useful integral relations describing balance between stretching and dissipation in the time-stationary regime. We multiply (14) by ψ and (15) by φ, and average in the horizontal. The following relations follow on adding and subtracting the results:

$$\partial_t(\overline{\nabla_\perp\psi\cdot\nabla_\perp\varphi}) + \overline{\nabla_\perp^2\varphi J[\psi,\varphi]} + 2\overline{\nabla_\perp^2\psi\nabla_\perp^2\varphi} = \tfrac{1}{2}D\overline{|\nabla_\perp\psi|^2} + \tfrac{1}{2}D\overline{\varphi^2}, \tag{23}$$

$$\overline{\nabla_\perp^2\psi\,\partial_t\varphi - \nabla_\perp^2\varphi\,\partial_t\psi} - \overline{\nabla_\perp^2\varphi J[\psi,\varphi]} + \tfrac{1}{2}D\overline{|\nabla_\perp\psi|^2} - \tfrac{1}{2}D\overline{\varphi^2} = 0. \tag{24}$$

In writing these expressions we have used the general result that $\overline{J[f,g]} \equiv 0$. Hence for steady or time-periodic solutions

$$D\overline{\varphi^2} = 2\overline{\nabla_\perp^2\psi\nabla_\perp^2\varphi} \quad \text{and} \quad D(\overline{|\nabla_\perp\psi|^2 - \varphi^2}) = 2\overline{\nabla_\perp^2\varphi J[\psi,\varphi]}. \tag{25}$$

Similarly, multiplying (14) by φ, (15) by $\nabla_\perp^2\psi$, averaging, and subtracting the results yields

$$\tfrac{1}{2}\partial_t(\overline{(\nabla_\perp^2\psi)^2 + |\nabla_\perp\varphi|^2}) - D\overline{\nabla_\perp\varphi\cdot\nabla_\perp\psi} = -\overline{|\nabla_\perp\nabla_\perp^2\psi|^2} - \overline{(\nabla_\perp^2\varphi)^2}. \tag{26}$$

We can thus identify the energy

$$\mathcal{E} = \tfrac{1}{2}\overline{(\nabla_\perp^2\psi)^2 + |\nabla_\perp\varphi|^2}, \tag{27}$$

the flux

$$\mathcal{F} = -\overline{\nabla_\perp\varphi\cdot\nabla_\perp\psi}, \tag{28}$$

and the dissipation

$$\varepsilon = \overline{|\nabla_\perp\nabla_\perp^2\psi|^2} + \overline{(\nabla_\perp^2\varphi)^2}; \tag{29}$$

we can check that $\mathcal{E}$ is the leading-order contribution to the kinetic energy, and that $\mathcal{F}$ is the corresponding approximation to the energy flux. Note that in contrast to fluxes obtained in isotropic turbulence, which possess a cubic term, this flux is quadratic. Its time evolution is given by (23). The integral of the energy equation (26) over $0 \le z \le 1$ yields the simple result that the net flux $\mathcal{F}^- - \mathcal{F}^+$ must balance the total dissipation; here $\mathcal{F}^+$ (resp. $\mathcal{F}^-$) is the flux obtained via vorticity diffusion through the top (resp. bottom) of the layer. Such a flux can be sustained against viscous decay through forcing at the boundaries. For example, in convection, heating at the lower surface creates a buoyancy flux characterized by the Rayleigh number.

An alternative way to sustain the flow is to input energy by "stirring" at a particular (horizontal) scale ℓ; this is equivalent to replacing boundary forcing with an explicit body force. If we scale the horizontal lengths by ℓ while retaining the vertical scale on which the boundary conditions are imposed, and scale time and the fields φ and ψ with ℓ^{-1}, ℓ^3, and ℓ^4, respectively, we then obtain the scaled equations

$$\partial_t \nabla_\perp^2 \varphi - J[\varphi, \nabla_\perp^2 \varphi] - D\nabla_\perp^2 \psi = \ell^{-3}\nabla_\perp^4 \varphi, \tag{30}$$

$$\partial_t \nabla_\perp^2 \psi - J[\varphi, \nabla_\perp^2 \psi] + D\varphi = \ell^{-3}\nabla_\perp^4 \psi. \tag{31}$$

Thus ℓ^3 plays the role of the Reynolds number, with $\ell \gg 1$ implying a large separation between the scales at which energy is input and those of interest. Interestingly, in the absence of boundary conditions in z the additional freedom to rescale lengths in this direction yields a one-parameter family of equations of the form of (30) and (31) with $Re \equiv \ell^\alpha$, $\alpha > 0$. The introduction of such an effective Reynolds number into (14) and (15) is extremely useful for their analysis.

5. Rapidly Rotating Convection

As mentioned already in Section 1, the present approach allows us to describe rapidly rotating convection by introducing the buoyancy force $\mathbf{f} = (Ra/\sigma)E^2 T\hat{\mathbf{z}}$ into (1). Here T is the temperature obeying (3), which in the streamfunction formulation takes the form

$$\sigma(\partial_t T + N_T(\varphi, \psi, T)) = E\nabla^2 T, \tag{32}$$

where

$$N_T \equiv -J[\varphi, T] + \nabla_\perp \partial_z \psi \cdot \nabla_\perp T - \nabla_\perp^2 \psi \partial_z T. \tag{33}$$

In the following we introduce the scaled Rayleigh number $Ra' \equiv E^{4/3}Ra$ and split the temperature into its mean and fluctuating parts: $T(x,y,z,t) \equiv \overline{T}(z) + E^{1/3}\theta(x,y,z,t)$. Here, the overbar denotes a spatial average in the horizontal *and* a time-average. The time-averaging is an essential aspect of our decomposition, and allows us to close the problem (Julien and Knobloch, 1998a). Specifically, to leading order,

$$D^2\overline{T} = -\sigma D(\overline{\nabla_\perp^2 \psi\, \theta}). \tag{34}$$

This equation is integrable, i.e., $D\overline{T} = \text{const.} - \sigma\overline{\nabla_\perp^2 \psi\, \theta}$, where $-\overline{\nabla_\perp^2 \psi\, \theta} \equiv \overline{w\,\theta}$ is the horizontally averaged *and* time-averaged vertical convective flux.

Noting that the scaled temperature difference across the layer is one, the final equations may be summarized as follows:

$$\partial_t \nabla_\perp^2 \varphi - J[\varphi, \nabla_\perp^2 \varphi] - D\nabla_\perp^2 \psi = \nabla_\perp^4 \varphi + \mathcal{O}(E^{1/3}), \tag{35}$$

$$\partial_t \nabla_\perp^2 \psi - J[\varphi, \nabla_\perp^2 \psi] + D\varphi = -\frac{Ra}{\sigma}\theta + \nabla_\perp^4 \psi + \mathcal{O}(E^{1/3}), \tag{36}$$

$$\sigma(\partial_t \theta - J[\varphi, \theta] - \nabla_\perp^2 \psi D\overline{T}) = \nabla_\perp^2 \theta + \mathcal{O}(E^{1/3}), \tag{37}$$

$$D\overline{T} = -1 - \sigma\overline{(\nabla_\perp^2 \psi\, \theta - \langle \nabla_\perp^2 \psi\, \theta\rangle)} + \mathcal{O}(E^{1/3}). \tag{38}$$

Here the angular brackets denote a vertical average, while Ra is the *scaled* Rayleigh number. These equations describe the dynamics of an interior solution, exclusive of any momentum (Ekman) boundary layers required by horizontal boundaries. We reiterate that such boundary layers only contribute at higher order and are therefore passive (Julien and Knobloch, 1998a).

Although these equations are closed they are slightly awkward to use for initial value problems because of the implied time-averaging. To solve them we therefore proceed as follows. We start with the *unaveraged* (in time) mean temperature equation

$$\sigma\big(\partial_t \overline{T} - E^{2/3} D(\overline{\nabla_\perp^2 \psi\, \theta})\big) = E^{2/3} D^2 \overline{T} + \mathcal{O}(E), \tag{39}$$

and note that $\partial_t \overline{T} = \mathcal{O}(E^{2/3})$. Consequently the time steps required to advance (39) are $\mathcal{O}(E^{-\frac{2}{3}})$ larger than those required for (35)–(37) and one can set $\partial_t \overline{T} = 0$ during the transient response to the initial conditions. Once the mean temperature reaches a statistically stationary state a running average may be accumulated so that (38) is employed correctly. In practice we find that using a large enough box (as in the calculations reported below) effectively eliminates the need to carry out the above time-averaging since the horizontal average of the rising and falling plumes becomes equivalent to a time-average if the domain is sufficiently large.

Figures 2–4 show fully nonlinear numerical solutions of (35)–(38) using only spatial averaging for the mean terms. After a very short initial transient, vortical buoyant plumes emerge and mutually advect one

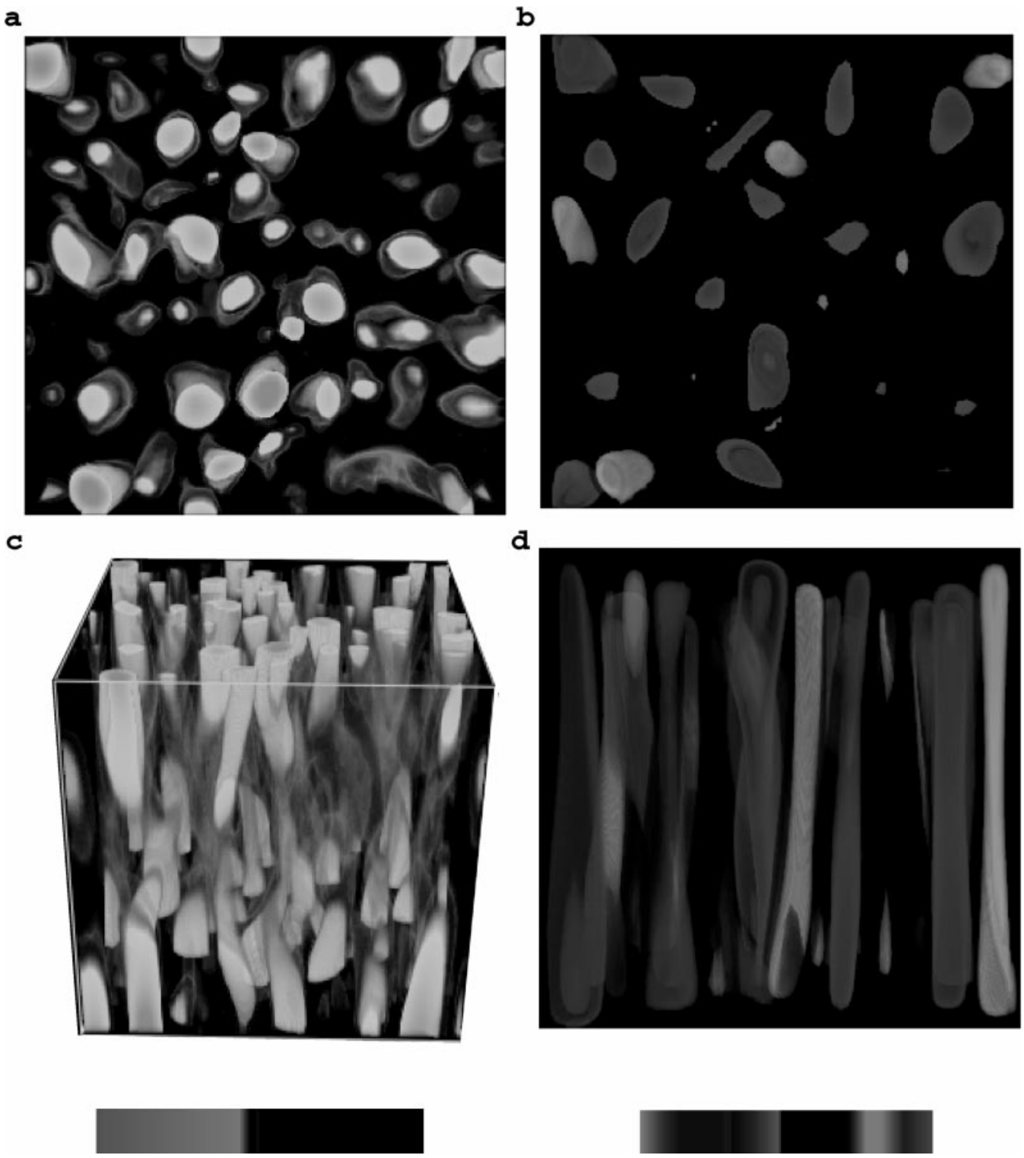

Figure 2. Visualization in λ_2 (a) and (c) and θ (b) and (d) for the reduced equations (35)–(38) with $Ra = 20E^{-4/3}$; the aspect ratio is $6E^{-1/3} \times 6E^{-1/3} \times 1$ (not to scale). The flow characterization captures the coherent structures first observed in simulations of the full three-dimensional Boussinesq equations (Figure 1).

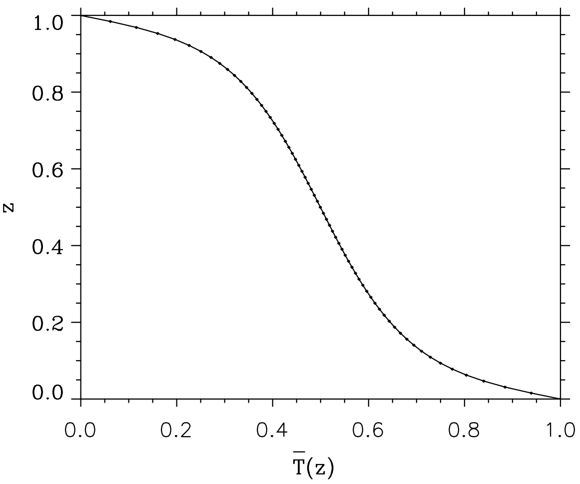

Figure 3. Mean temperature profile $\overline{T}(z)$ at Ra = $20E^{-4/3}$.

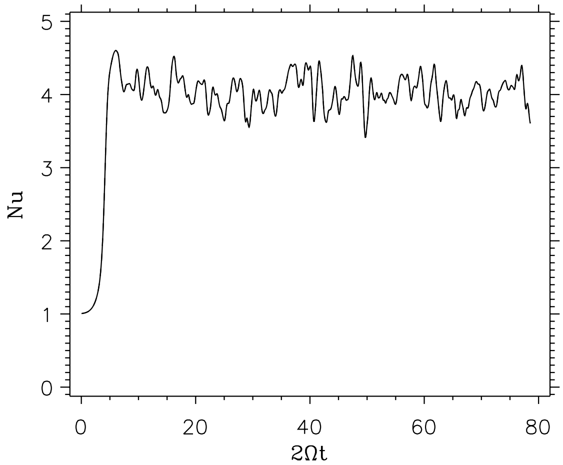

Figure 4. The unaveraged Nusselt number $Nu \equiv -D\overline{T}$ from (39) as a function of $2\Omega t$, showing rapid relaxation to a statistically stationary state.

another laterally. The plumes are columnar, spanning the layer depth, as one expects given the Taylor–Proudman constraint. Very near the boundaries, however, sharp temperature gradients appear, as anticipated from the thermal boundary conditions. Such boundary layers are evident in $\overline{T}(z)$ in Figure 3 for $z < 0.1$ or $z > 0.9$ where $D\overline{T}$ approaches -4.0; in comparison, at midlayer, $D\overline{T}$ is a factor of 10 smaller ($D\overline{T} \approx -0.4$). Semianalytical investigations by Julien and Knobloch (1998a,b) indicate that this ratio continues to grow monotonically with increased thermal forcing. These features resemble closely those found in numerical solutions of the full three-dimensional Boussinesq convection equations at large rotation rates (Julien *et al.*, 1996) (Figure 1). Differences between full Boussinesq simulations (Figure 1) and the reduced equations presented here (Figure 2) are most notable near the boundaries. This is to be expected since the reduced system describes only the leading-order dynamics and so is insensitive to the details of the mechanical boundary conditions. Such a system cannot capture the structure of the resulting velocity boundary layers. Finally, Figure 4 shows the Nusselt number (or nondimensional heat transfer) of the time-evolving flow, and indicates that the flow develops rapidly to a near-stationary state. These solutions are representative of the possible dynamics contained within the new class of equations derived here. A detailed discussion of these equations and of the physical significance of their solutions will be given elsewhere.

The numerical solutions were computed with a pseudospectral Petrov–Galerkin method in which field variables are represented with sines or cosines in the vertical and periodic Fourier modes in the horizontal. Time stepping is via a mixed implicit/explicit third-order Runge–Kutta scheme developed by Spalart *et al.* (1991). We treat diffusion and forcing terms implicitly and nonlinear and stretching terms explicitly. We set Ra' = 20 (2.2 times critical), σ = 1, and the domain width to six times the most unstable linear wavelength ($\lambda_\perp \approx$ 4.8154) in both horizontal directions. Top and bottom boundary conditions are impenetrable/fixed temperature and side boundaries are periodic. The calculations were conducted with 64^3 spectral modes and were de-aliased in all spatial directions at each Runge–Kutta sub-time-step.

6. Conclusion

We have presented a new class of equations (14) and (15) for rotationally constrained flows based on the relaxation of the Taylor–Proudman constraint due to viscous and small-scale baroclinic effects. The asymptotic theory leading to these equations is valid for appropriately chosen horizontal scales in the limit of large rotation rates. These scales are defined by a balance between advection and dissipation. The resulting horizontal velocity field is in geostrophic balance at leading order but the vertical motions are strongly nonhydrostatic. This is in contrast to the geostrophic *and* hydrostatic balance that holds on larger scales (Pedlosky, 1979; Gent and McWilliams, 1983). We believe this theory to be relevant to various laboratory flows and in particular to rotating convection, in which the velocity field is essentially isotropic and coherent structures exist close to the dissipation scale (Figure 1). The resulting equations bear some semblance to the equations describing excitable media: small perturbations of the trivial state decay but the equations exhibit finite amplitude equilibria with nonzero vertical vorticity (the excited state). It appears likely that stable excited states can be found within our reduced equations, although this speculation is still subject to verification. Such states would be characterized by intense vertical stretching that feeds energy into the motion statistically balanced by viscous dissipation in the layer. Our simulations of the equations with thermal forcing strongly suggest that such states do indeed exist. Although the work reported here remains preliminary, all indications are that the equations derived here merit further study, particularly in connection with modeling efforts usually employed for boundary-layer turbulence or laboratory experiments.

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