Abstract

The Steenrod Algebra $\mathcal{A}$ is the algebra of stable natural endomorphisms of the $\mathbb{Z}/2$-cohomology functor; it is generated by elements Sq^{2^i}. Let $\mathcal{A}(k)$ be the subalgebra generated by the Sq^{2^i} for $i \leq k$. Consider the modules $L(k)$ spanned by sequences of Steenrod operations of length k. Welcher proved that $L(k)$ is a free module over $\mathcal{A}(k-1)$. We are interested in finding the structure of $L(k)$ as an $\mathcal{A}(r)$-module for any r. We conjecture that $L(k)$ is built as an $\mathcal{A}(r)$-module out of $\mathcal{A}(r)//\mathcal{A}(r-k)$, in the sense that it has an increasing filtration with quotients isomorphic to $\mathcal{A}(r)//\mathcal{A}(r-k)$, and present partial results towards that claim. In addition, we prove some interesting commutation relations in the Steenrod algebra relating to representations of Steenrod Algebra elements in Wood's Z-basis.

1 Introduction

Definition 1. *Following Welcher [**?**], we let $L(k)$ be the subspace of $\mathcal{A}$ spanned by admissible monomials of length k.*

The vector space $L(k)$ may be considered as a (left or right) $\mathcal{A}$-module, by simply doing the multiplication in $\mathcal{A}$, and considering any terms which are not of length k to be zero. Similarly, we may consider $L(k)$ as a module over any of the subalgebras $\mathcal{A}(k)$. $L(k)$ has a nice periodic structure, which we consider in more detail in the next section.

2 Definitions and Background

2.1 The Adem Relations

Recall that the Steenrod squares satisfy the Adem relations:

$$\mathrm{Sq}^i\mathrm{Sq}^j = \sum_{m=0}^{m/2} \binom{j-m-1}{i-2m} \mathrm{Sq}^{i+j-k}\mathrm{Sq}^k.$$

The binomial coefficient is taken modulo 2, and it is a standard result in combinatorics that it is nonzero if and only if for each bit of the binary representation of $i-2m$ that is 1, the corresponding bit of $j-m-1$ must also be 1. We will use this fact extensively.

2.2 The Periodicity of $L(k)$

If we consider $L(k)$ as a module over $\mathcal{A}(n)$, we find from the Adem relations that it has a nice periodic structure. In particular, the left action of Sq^{2^i} commutes with the operation of decreasing each monomial Sq^I componentwise by the tuple $(d_1, d_2, \ldots, d_k)$ where d_j is divisible by 2^{i+2-j} if $j < i+2$, removing the

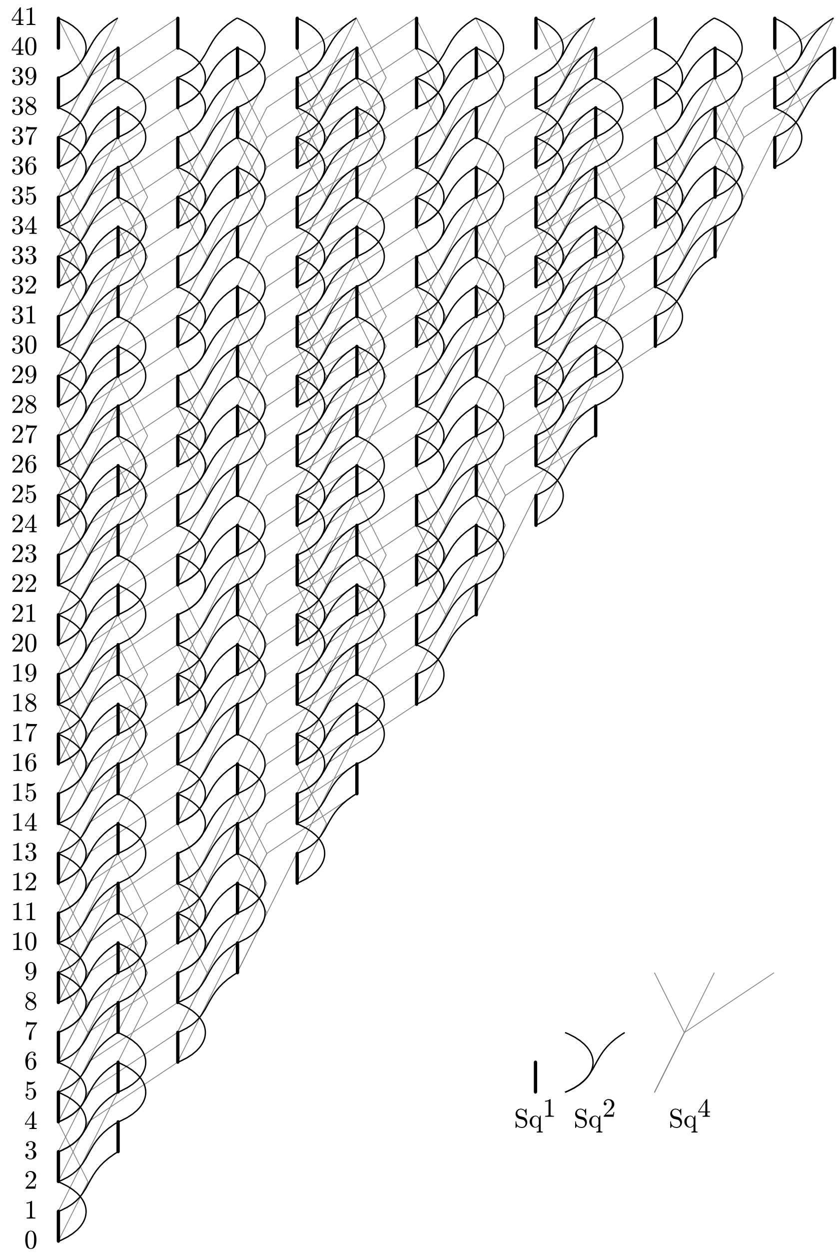

Figure 1: The structure of $\bar{L}(2)$ as an $\mathcal{A}(2)$-module. Each point is an element of the Adem basis for $\bar{L}(2)$; for example the bottommost point in the first column is Sq^0, while the bottommost point in the second column is $\mathrm{Sq}^2\mathrm{Sq}^1$. The lines show the action of $\mathcal{A}(2)$; for instance the light gray lines connect each monomial Sq^I to the terms of $\mathrm{Sq}^4\mathrm{Sq}^I$. Shifting the diagram down by 8 rows or right by 4 columns shows the periodic structure. $L(2)$ consists of all but the first column; $L(k)_{-\infty}$ extends infinitely far to the left.

monomial if it is no longer admissible. This fact follows trivially from the Adem relations; in particular adding a large power of two to the exponent of the second term does not affect the binomial coefficients.

In many cases, that periodic structure becomes somewhat easier to consider if we include all admissible monomials of length *at most* k.

Definition 2. *Let* $\bar{L}(k) = \bigcup_{i \le k} L(i)$ *include all admissible monomials of length* at most k.

For example, the structure of $\bar{L}(2)$ as an $\mathcal{A}(2)$-module is shown in Figure 1; it is periodic modulo $(8, 4)$.

In some cases it is simpler to dispense with the edge cases of the periodic structure entirely, and simply extend $L(k)$ back to negative infinity (or equivalently, to the left in Figure 1). Formally, we let $L(k)_{-\infty}$ consist of admissible monomials composed of factors Sq^i, where i may be negative. (Admissible monomials are still defined as those in which each term is at least twice the previous one, so that $\mathrm{Sq}^3\mathrm{Sq}^{-2}$ and $\mathrm{Sq}^3\mathrm{Sq}^{-1}$ are both admissible, but $\mathrm{Sq}^{-3}\mathrm{Sq}^{-1}$ is not.) The module structure of $L(k)_{-\infty}$ over $\mathcal{A}(k)$ may be defined by extending the periodic structure of $L(k)$ backwards, so that $\mathrm{Sq}^{2^i}\mathrm{Sq}^I$ is defined by adding some sufficiently large admissible tuple $J = (2^{i+1}m_1, 2^i m_2, \ldots)$ to I so that it becomes positive, performing the multiplication, and then subtracting the tuple J from the exponent of each term of the result, discarding any terms which are no longer admissible.

2.3 The Wood Z basis

In addition to the Adem basis, the Z basis described by Wood [?] will be important to our work. Let

$$X_n = \mathrm{Sq}^{2^{n+1}-2^n}\mathrm{Sq}^{2^{n+1}-2^{n-1}} \cdots \mathrm{Sq}^{2^{n+1}-1},$$

and let $Z_n = X_n X_{n-1} \cdots X_0$. Then Wood showed that Z_n is the top element of $\mathcal{A}(n)$, and furthermore that the following is a basis for $\mathcal{A}(n)$.

Definition 3. *The Wood Z basis for* $\mathcal{A}(n)$ *consists of the set of monomials in Steenrod squares produced by multiplying any subset of the Steenrod squares in* Z_n *in order.*

The Steenrod Algebra, then, consists of all finite subproducts of the infinite product

$$\cdots \mathrm{Sq}^8\mathrm{Sq}^{12}\mathrm{Sq}^{14}\mathrm{Sq}^{15}\mathrm{Sq}^4\mathrm{Sq}^6\mathrm{Sq}^7\mathrm{Sq}^2\mathrm{Sq}^3\mathrm{Sq}^1.$$

For example,

$$Z_2 = \mathrm{Sq}^4\mathrm{Sq}^6\mathrm{Sq}^7\mathrm{Sq}^2\mathrm{Sq}^3\mathrm{Sq}^1 = \mathrm{Sq}^17\mathrm{Sq}^5\mathrm{Sq}^1$$

is the top class of $\mathcal{A}(2)$. Furthermore, the following set forms a basis of $\mathcal{A}(2)$:

$$\begin{array}{c}
\mathrm{Sq}^4\mathrm{Sq}^6\mathrm{Sq}^7\mathrm{Sq}^2\mathrm{Sq}^3\mathrm{Sq}^1 \\
\mathrm{Sq}^4\mathrm{Sq}^6\mathrm{Sq}^7\mathrm{Sq}^2\mathrm{Sq}^3 \quad \mathrm{Sq}^4\mathrm{Sq}^6\mathrm{Sq}^7\mathrm{Sq}^2\mathrm{Sq}^1 \quad \mathrm{Sq}^4\mathrm{Sq}^6\mathrm{Sq}^7\mathrm{Sq}^3\mathrm{Sq}^1 \\
\mathrm{Sq}^4\mathrm{Sq}^6\mathrm{Sq}^2\mathrm{Sq}^3\mathrm{Sq}^1 \quad \mathrm{Sq}^4\mathrm{Sq}^7\mathrm{Sq}^2\mathrm{Sq}^3\mathrm{Sq}^1 \quad \mathrm{Sq}^6\mathrm{Sq}^7\mathrm{Sq}^2\mathrm{Sq}^3\mathrm{Sq}^1 \\
\vdots \\
\mathrm{Sq}^4 \quad \mathrm{Sq}^6 \quad \mathrm{Sq}^7 \quad \mathrm{Sq}^2 \quad \mathrm{Sq}^3 \quad \mathrm{Sq}^1 \\
1
\end{array}$$

Note that the Z basis builds larger $\mathcal{A}(n)$ from right to left: in particular, $Z_n = X_n Z_{n-1}$, and therefore each element of the Z basis for $\mathcal{A}(n)$ is formed by multiplying some subproduct of X_n by an element of the Z basis for $\mathcal{A}(n-1)$. This means that the algebras $\mathcal{A}(n)/\!/\mathcal{A}(n-k)$ have a particularly nice representation in the Z-basis.

Proposition 1. *The set of monomials produced by multiplying any subset of the Steenrod squares in $X_n X_{n-1} \cdots X_{n-k+1}$ in any order forms a set of representatives for a basis for the algebra $\mathcal{A}(n)/\!/\mathcal{A}(n-k)$. In particular, $X_n X_{n-1} \cdots X_{n-k+1}$ is a representative of its top class.*

Proof. Consider some element z of the Z basis for $\mathcal{A}(n)$, which must be a subproduct of $X_n X_{n-1} \cdots X_{n-k+1} Z_{n-k}$. If it contains any Steenrod squares from Z_{n-k}, then it can be written $z = xy$ where $x \in \mathcal{A}(n)$ and $y \in \mathcal{A}(n-k)$. Then in $\mathcal{A}(n)/\!/\mathcal{A}(n-k) = \mathcal{A}(n) \otimes_{\mathcal{A}(n-1)} \mathbb{F}_2$, z is equal to zero. Otherwise, it is in the proposed basis. So since a basis of $\mathcal{A}(n)$ must span $\mathcal{A}(n)/\!/\mathcal{A}(n-k)$, the proposed basis must span. Furthermore, $\mathcal{A}(n)$ has dimension $2^{2^{n+1}-1}$, and therefore $\mathcal{A}(n)/\!/\mathcal{A}(n-k)$ has dimension $2^{2^{n+1}-2^{n-k+1}}$, so since $X_n X_{n-1} \cdots X_{n-k+1}$ is a product of $2^{n+1} - 2^{n-k+1}$ Steenrod squares, there are the right number of subproducts, so they must form a basis. The element $X_n X_{n-1} \cdots X_{n-k+1}$ is the only element in the right degree to be a representative of the top class, so it must in fact be. $\square$

For example,

$$\mathrm{Sq}^4\mathrm{Sq}^6\mathrm{Sq}^7\mathrm{Sq}^2\mathrm{Sq}^3$$

is a representative of the top class of $\mathcal{A}(2)/\!/\mathcal{A}(0)$, and the following set forms a basis of $\mathcal{A}(2)/\!/\mathcal{A}(0)$:

$$\begin{array}{c}
\mathrm{Sq}^4\mathrm{Sq}^6\mathrm{Sq}^7\mathrm{Sq}^2\mathrm{Sq}^3 \\
\mathrm{Sq}^4\mathrm{Sq}^6\mathrm{Sq}^7\mathrm{Sq}^2 \quad \mathrm{Sq}^4\mathrm{Sq}^6\mathrm{Sq}^7\mathrm{Sq}^3 \\
\mathrm{Sq}^4\mathrm{Sq}^6\mathrm{Sq}^2\mathrm{Sq}^3 \quad \mathrm{Sq}^4\mathrm{Sq}^7\mathrm{Sq}^2\mathrm{Sq}^3 \quad \mathrm{Sq}^6\mathrm{Sq}^7\mathrm{Sq}^2\mathrm{Sq}^3 \\
\vdots \\
\mathrm{Sq}^4 \quad \mathrm{Sq}^6 \quad \mathrm{Sq}^7 \quad \mathrm{Sq}^2 \quad \mathrm{Sq}^3 \\
1
\end{array}$$

3 The Commutation Relation

We now prove a commutation relation involving the X_n which is interesting in its own right, and will prove useful in the remainder of the paper.

Lemma 1. *For any positive integers m and n,*

$$X_n X_{n-1} \mathrm{Sq}^{2^{n+1}m} = X_n \mathrm{Sq}^{2^{n+1}m} X_{n-1}.$$

Proof. First, we show the following identities by application of the Adem relations:

Proposition 2. *For any positive integers m and n,*

$$\mathrm{Sq}^{2^n-1}\mathrm{Sq}^{2^{n+1}m} = \mathrm{Sq}^{2^{n+1}m+2^n-1}.$$

Proposition 3. *For any positive integers k, m, and n, with $k \leq n$,*

$$\mathrm{Sq}^{2^n-2^k}\mathrm{Sq}^{2^n-2^{k-1}+2^{n+1}m} = \mathrm{Sq}^{2^n-2^k+2^{n+1}m}\mathrm{Sq}^{2^n-2^{k-1}} + \mathrm{Sq}^{2^{n+1}-2^k}\mathrm{Sq}^{2^{n+1}m-2^{k-1}}.$$

Proof of Proposition 2. The Adem relations give us that

$$\mathrm{Sq}^{2^n-1}\mathrm{Sq}^{2^{n+1}m} = \sum_{j=0}^{2^{n-1}-1} \binom{2^{n+1}m-1-j}{2^n-1-2j} \mathrm{Sq}^{2^{n+1}m+2^n-1-j}\mathrm{Sq}^j. \tag{1}$$

Recall that in $\mathbb{F}_2$, $\binom{p}{q}$ is 1 if and only if p has a one in every position of its binary expansion that q has a 1. Let $2^n > j > 0$ and let $2^{\omega(j)}$ be the largest power of 2 dividing j. Then the binary expansion of $2^{n+1}m - 1 - j$ ends in $\omega(j)$ ones preceded by a zero, while that of $2^n - 1 - 2j$ ends in a zero and then $\omega(j) + 1$ ones, so the binomial coefficient in the Adem relation is zero. Then the only term of the Adem relation in (1) that is nonzero is that when $j = 0$, where $2^{n+1}m - 1$ ends with $n + 1$ ones, while $2^n - 1$ consists of n ones. The result follows immediately. $\square$

Proof of Proposition 3. We expand the first and last terms using the Adem relations, and show that they are identical except for the middle term. For convenience, let $d = 2^{n+1}(m + 1) - 2^k - 2^{k-1}$, the degree of the elements.

$$\mathrm{Sq}^{2^n-2^k}\mathrm{Sq}^{2^n-2^{k-1}+2^{n+1}m} = \sum_{j=0}^{2^{n-1}-2^{k-1}} \binom{2^n-2^{k-1}+2^{n+1}m-1-j}{2^n-2^k-2j} \mathrm{Sq}^{d-j}\mathrm{Sq}^j$$

$$\mathrm{Sq}^{2^{n+1}-2^k}\mathrm{Sq}^{2^{n+1}m-2^{k-1}} = \sum_{j=0}^{2^n-2^{k-1}} \binom{2^{n+1}m-2^{k-1}-1-j}{2^{n+1}-2^k-2j} \mathrm{Sq}^{d-j}\mathrm{Sq}^j$$

First, let $0 \leq j \leq 2^{n-1} - 2^{k-1}$; we will show the two binomial coefficients are equal. We again consider the binary expansions. The expansion of $2^n - 2^k - 2j$ is the same as that of $2^{n+1}-2^k-2j$, except that the leading one has been removed. The corresponding bits of $2^n - 2^{k-1} + 2^{n+1}m - 1 - j$ and $2^{n+1}m - 2^{k-1} - 1 - j$ are also zero and one, respectively, so the two binomial coefficients are the same.

Now let $2^{n-1} - 2^{k-1} < j < 2^n - 2^{k-1}$; we will show that the latter binomial coefficient

$$\binom{2^{n+1}m - 2^{k-1} - 1 - j}{2^{n+1} - 2^k - 2j}$$

is zero. Let 2^q be the next power of 2 strictly larger than $2^n - 2^{k-1} + j$, so $q \leq n-1$. Then the qth position of the binary expansion will not satisfy the condition to make the binomial coefficient one, so it will be zero.

Finally, if $j = 2^n - 2^{k-1}$, the binomial coefficient in the second formula has a zero on the bottom, thus is one, so we get $\mathrm{Sq}^{2^n-2^k+2^{n+1}m}\mathrm{Sq}^{2^n-2^{k-1}}$, the middle term of the claimed identity, as the only leftover term, and the identity holds. $\square$

Then in particular, by repeated application of the above identities, we may write

$$\begin{aligned}
X_{n-1}\mathrm{Sq}^{2^{n+1}m} &= \mathrm{Sq}^{2^n-2^{n-1}}\mathrm{Sq}^{2^n-2^{n-2}}\cdots\mathrm{Sq}^{2^n-2}\mathrm{Sq}^{2^{n+1}m+2^n-1} \\
&= \mathrm{Sq}^{2^n-2^{n-1}}\cdots\mathrm{Sq}^{2^n-4}\mathrm{Sq}^{2^{n+1}m+2^n-2}\mathrm{Sq}^{2^n-1} \\
&\quad + \mathrm{Sq}^{2^n-2^{n-1}}\cdots\mathrm{Sq}^{2^n-4}\mathrm{Sq}^{2^{n+1}-2}\mathrm{Sq}^{2^{n+1}m-1} \\
&\vdots \\
&= \mathrm{Sq}^{2^{n+1}m}X_{n-1} \\
&\quad + \mathrm{Sq}^{2^n-2^{n-1}}\cdots\mathrm{Sq}^{2^n-4}\mathrm{Sq}^{2^{n+1}-2}\mathrm{Sq}^{2^{n+1}m-1} \\
&\quad + \cdots + \mathrm{Sq}^{2^n}\mathrm{Sq}^{2^{n+1}m-2^{n-1}}\mathrm{Sq}^{2^n-2^{n-2}}\cdots\mathrm{Sq}^{2^n-1}.
\end{aligned}$$

Now all of the extra terms are products of (from left to right) an element of $\mathcal{A}(n-1)$, then an element of the form $\mathrm{Sq}^{2^{n+1}-2^k}$ for $k < n+1$, then some element of $\mathcal{A}$. In particular, they are zero in $\mathcal{A}(n-1) \otimes_{\mathcal{A}(n)} \mathcal{A}$. Then since X_n is the top class of $\mathcal{A}(n) \otimes_{\mathcal{A}(n-1)} \mathbb{F}_2$, thus annihilates any element of $\mathcal{A}(n)$ that is not in $\mathcal{A}(n-1)$, $1 \otimes x \mapsto X_n x$ is well-defined as a map from $\mathcal{A}(n-1) \otimes_{\mathcal{A}(n)} \mathcal{A}$ to $\mathcal{A}$, so the two sides are the same when multiplied by X_n:

$$X_n X_{n-1}\mathrm{Sq}^{2^{n+1}m} = X_n\mathrm{Sq}^{2^{n+1}m}X_{n-1}.$$

$\square$

4 Representatives for $\mathcal{A}(n)//\mathcal{A}(n-k)$

Lemma 2. *Any element t of $\mathcal{A}(n)$ may be written as $x + yz$ where $x \in \bar{L}(k)$, $y \in \mathcal{A}$, and $z \in \mathcal{A}(n-k) \setminus \mathbb{F}_2$ for any $k \geq 0$. Equivalently, $\mathcal{A}(n) \subset \bar{L}(k) + \mathcal{A}(\mathrm{Sq}^{2^i})_{i \leq n-k}$, where we take $\mathcal{A}(\mathrm{Sq}^{2^i})_{i \leq n-k}$ to be the left ideal of $\mathcal{A}$ generated by $\{\mathrm{Sq}^{2^i}\}_{i \leq n-k}$, for any $k \geq 0$.*

Proposition 2 essentially says that there exists a set of representatives for a basis of $\mathcal{A}(n)//\mathcal{A}(n-k)$, all of which lie in $\bar{L}(k)$.

Proof. It suffices to prove the proposition on some basis for $\mathcal{A}(n)$; we use the Wood Z basis. We induct on the number of squares selected to form the basis element. The empty product is 1, which is in $\bar{L}(k)$, so we are done. So now suppose the proposition holds on all elements of the Wood Z-basis of length at most $l-1$, and let t be an element of length l. Then, for some $m = 2^j - 2^i$, $i < j \leq n+1$, we may write $t = \mathrm{Sq}^m(x + yz)$ where x, y, and z are as in the statement of the proposition.

Now $\mathrm{Sq}^m yz$ is clearly of the prescribed form, since $\mathrm{Sq}^m y$ is in $\mathcal{A}$. If $x \in \bar{L}(k-1)$, then $\mathrm{Sq}^m x \in \bar{L}(k)$ (since application of the Adem relations cannot increase the length of a monomial), so we may suppose $x \in L(k)$.

Write x in admissible form as $\mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2}\cdots\mathrm{Sq}^{i_k}$. If $\mathrm{Sq}^m\mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2}\cdots\mathrm{Sq}^{i_k}$ is already in admissible form, then we have that $2^k i_k \leq 2^{k-1}i_{k-1} \leq \cdots \leq 2i_1 \leq m < 2^{n+1}$, so in particular, $i_k < 2^{n-k+1}$, so $\mathrm{Sq}^{i_k} \in \mathcal{A}(\mathrm{Sq}^{2^i})_{i \leq n-k}$, so in fact the entire product $\mathrm{Sq}^m x$ is in $\mathcal{A}(\mathrm{Sq}^{2^i})_{i\leq n-k}$. Then we may apply the Adem relations at least once, to get

$$\begin{aligned}\mathrm{Sq}^m\mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2}\cdots\mathrm{Sq}^{i_k} = a_0\mathrm{Sq}^{m+i_1}\mathrm{Sq}^{i_2}\cdots\mathrm{Sq}^{i_k}\\ + \sum_{1\leq j\leq \frac{m}{2}} a_j\mathrm{Sq}^{m+i_1-j}\mathrm{Sq}^j\mathrm{Sq}^{i_2}\cdots\mathrm{Sq}^{i_k};\end{aligned}$$

the first term we know is in $L(k)$. For each of the remaining terms, if we may repeat the same process, ignoring the first factor; either the product is in $\mathcal{A}(\mathrm{Sq}^{2^i})_{i\leq n-k}$, or it may be further reduced by the Adem relations. Finally, if we apply the Adem relations k times, the rightmost factor must be Sq^i for $i \leq \frac{m}{2^k} < 2^{n-k+1}$, so similarly the product lies in the ideal $\mathcal{A}(\mathrm{Sq}^{2^i})_{i\leq n-k}$. Then $\mathrm{Sq}^m x$ may be written in the desired form, so we are done. □

5 Linear Independence

Lemma 3. *Taken as elements of $L(k)_{-\infty}$ as a vector space over $\mathbb{F}_2$, the set*

$$\{X_nX_{n-1}\cdots X_{n-k+1}\mathrm{Sq}^I \mid I = (2^{n+1}m_1, 2^n m_2, \ldots, 2^{n-k+2}m_k), m_1 \geq \cdots \geq m_k\}$$

is linearly independent.

Proof. Note that by the commutation relation we have shown, the element $s = X_nX_{n-1}\cdots X_{n-k+1}\mathrm{Sq}^I$ may be rewritten as

$$X_n\mathrm{Sq}^{2^{n+1}m_1}X_{n-1}\mathrm{Sq}^{2^n m_2}\cdots X_{n-k+1}\mathrm{Sq}^{2^{n-k+2}m_k}.$$

Let $M_{k,n}$ be the subspace of $\mathcal{A}$ which is spanned by monomials of the form yz where $y \in L(k)$ and $z \in \mathcal{A}(n)$ with the degree of z at least 1. If $b \in \mathcal{A}$, we will write $b + M_{k,n}$ for the subset of $\mathcal{A}$ consisting of elements of the form $b + m$ for $m \in M_{k,n}$.

Proposition 4. *The element*

$$X_n \mathrm{Sq}^{2^{n+1}m_n} X_{n-1} \mathrm{Sq}^{2^n m_{n-1}} \cdots X_{n-k+1} \mathrm{Sq}^{2^{n-k+2} m_{n-k+1}}$$

is in

$$\mathrm{Sq}^{2^{n+1}(m_n+n)+1} \mathrm{Sq}^{2^n(m_{n-1}+n-1)+1} \cdots \mathrm{Sq}^{2^{n-k+2}(m_{n-k+1}+n-k+1)+1} + M_{k,n-k}.$$

Proof. We proceed by induction on k for fixed n. If $k = 1$, we know by Lemma 2 that $X_n \in M_{1,n-1} + \mathrm{Sq}^{n2^{n+1}+1}$; the latter term is the only element of $\bar{L}(1)$ in the right dimension, so we know it must be the value of x from the lemma. Then since, as an $\mathcal{A}(n)$-module, $L(1)$ is periodic in degrees modulo 2^{n+1}, $X_n \mathrm{Sq}^{2^{n+1}m} \in M_{1,n-1} + \mathrm{Sq}^{2^{n+1}(n+m)+1}$.

Now suppose the proposition holds for some k; we will show it for $k+1$. Let

$$\begin{aligned} t &= X_n \mathrm{Sq}^{2^{n+1}m_n} X_{n-1} \mathrm{Sq}^{2^n m_{n-1}} \cdots X_{n-k+1} \mathrm{Sq}^{2^{n-k+2} m_{n-k+1}} \\ t' &= X_{n-k} \mathrm{Sq}^{2^{n-k+1} m_{n-k}} \\ b &= \mathrm{Sq}^{2^{n+1}(m_n+n)+1} \mathrm{Sq}^{2^n(m_{n-1}+n-1)+1} \cdots \mathrm{Sq}^{2^{n-k+2}(m_{n-k+1}+n-k+1)+1} \\ b' &= \mathrm{Sq}^{2^{n-k+1}(m_{n-k}+n-k)+1}. \end{aligned}$$

Then by inductive hypothesis we know that $t \in b + M_{k,n-k}$ and by the base case we know that $t' \in b' + M_{1,n-k-1}$; we wish to show that $tt' \in bb' + M_{k+1,n-k-1}$. To do this, it suffices to show that $bM_{1,n-k-1}$ and $M_{k,n-k}t'$ are contained in $M_{k+1,n-k-1}$; for simplicity we will show that this is true of the spanning sets we have constructed.

First, let $y \in L(1)$, $z \in \mathcal{A}(n-k-1)$ with z not in degree 0, so that $yz \in M_{1,n-k-1}$. Then by looking at degrees, by is already in admissible form, so it is in $L(k+1)$, so $byz \in M_{k+1,n-k-1}$ as desired. Since elements of the form yz span $M_{1,n-k-1}$, we get that $bM_{1,n-k-1} \subset M_{k+1,n-k-1}$.

Second, let $y \in L(k)$, $z \in \mathcal{A}(n-k)$ with z not in degree 0, so that $yz \in M_{k,n-k}$. If $z \notin \mathcal{A}(n-k-1)$, then zX_{n-k} and thus yzt' must be zero in $\mathcal{A}(n-k)$. Then we may assume $z \in \mathcal{A}(n-k-1)$; we will show that both yzb' and $yzM_{1,n-k-1}$ are contained in $M_{k+1,n-k-1}$. Now if we write zb' in admissible form, multiplying by y will give an element already in admissible form by looking at degrees, so we need only show that $zb' \in M_{1,n-k-1}$. If we multiply out using the Adem relations, this is clearly true, modulo showing that zb' contains no terms of length 1. This is easy to show; by the Milnor basis, z cannot end in a Steenrod square with degree a multiple of 2^{n-k}, but b' is a Steenrod square in a degree which is 1 modulo 2^{n-k}, so on multiplying the two, the length 1 term must vanish, as desired.

Finally, then, we must show that $yzM_{1,n-k-1} \subset M_{k+1,n-k-1}$; let $y' \in L(1)$ and $z' \in \mathcal{A}(n-k-1)$, so we must show that $yzy'z' \in M_{k+1,n-k-1}$. Again looking at the Adem relations, we may write zy' in the form $y''z''$ where $y'' \in L(1)$ and $z'' \in \mathcal{A}(n-k-1)$. Then $z''z' \in \mathcal{A}(n-k-1)$ and is not in degree zero, and $yy'' \in L(k+1)$, so $yy''z''z' \in M_{k+1,n-k-1}$.

Then the induction is complete; all terms of $tt' - bb'$ are in $M_{k+1,n-k-1}$ as desired. □

Note that the terms

$$\mathrm{Sq}^{2^{n+1}(m_n+n)+1}\mathrm{Sq}^{2^n(m_{n-1}+n-1)+1}\cdots\mathrm{Sq}^{2^{n-k+2}(m_{n-k+1}+n-k+1)+1} \qquad (2)$$

are outside of $M_{k,n-k}$ since all elements of $\mathcal{A}(n-k)$ have degree smaller than the last factor of (2). Then since the elements in question may be written in the given form, we know that each has a unique term outside of $M_{k,n-k}$. Those terms are distinct, so the terms are linearly independent. □

6 Filtrations of $L(k)$

6.1 The General Case

Lemma 3 suggests that the set

$$B_{n,k} = \{Sq^I \mid I = (2^{n+1}m_1, 2^n m_2, \ldots, 2^{n-k+2}m_k), m_1 \geq \cdots \geq m_k\}$$

may allow us to construct a filtration of $L(k)_{-\infty}$ with copies of $\mathcal{A}(n)//\mathcal{A}(n-k)$ as the filtration quotients, through the following theorem.

Theorem 1. *The set $B_{n,k}$ is linearly independent in $L(k)_{-\infty}$ with respect to coefficients taken from $\mathcal{A}(n)//\mathcal{A}(n-k)$.*

Proof. Suppose it were not: suppose that there were some relation $\sum_i x_i b_i = 0$, with $x_i \in \mathcal{A}(n)//\mathcal{A}(n-k)$ and $b_i \in B_{n,k}$. Let z be the top class of $\mathcal{A}(n)//\mathcal{A}(n-k)$. We now have two cases.

First, consider the case where not all b_i are in the same degree; let b_j be one of the ones in the highest possible degree. Since $\mathcal{A}(n)//\mathcal{A}(n-k)$ is a Poincaré algebra, there exists some y_j in $\mathcal{A}(n)//\mathcal{A}(n-k)$ such that $y_j x_j = z$. Then for any $b_{j'}$ in a lower degree than b_j, the corresponding $x_{j'}$ will be in a higher degree than x_j. Then $y_j x_j = 0$, so multiplying the entire relation by y_j leaves a new relation amongst only the b_i in the highest possible degree.

So now we consider the case where all b_i are in the same degree. As before, let y be such that $yx_1 = z$. Then in the relation $\sum_i yx_i b_i = 0$, the first coefficient is z, and all of the later ones are either z or zero. But each zb_i is an element of the linearly independent set from Lemma 3, so there can be no relation among them. Then we have a contradiction: there is no relation among the b_i over $\mathcal{A}(n)//\mathcal{A}(n-k)$, so the set $B_{n,k}$ is linearly independent. □

Given this, it remains only to show that in some sense the "basis" $B_{n,k}$ has the right size. Unfortunately, this is more difficult than it sounds. If we take the full $B_{n,k}$, the vector space of elements in each degree of $L(k)_{-\infty}$ is infinite-dimensional, so the argument cannot proceed. If we consider only the intersection of $B_{n,k}$ with $\bar{L}(k)$, this difficulty is removed, but another one presents itself:

some classes are not spanned by the $B_{n,k}$. While these are irrelevant to the desired final result, in that the periodic structure is still entirely encapsulated, with the exception of some initial conditions, a dimension-counting argument would require a careful accounting of these initial exceptions, which we found intractible in the general case.

Thus we were unable to show the desired result in the general case. We do believe that a proof could proceed along these lines, we have just been unable to complete the dimension-counting portion of the argument, and hope that future research will yield fruits in the area.

6.2 The case of small n and k

Fortunately, in small cases the dimension-counting is tractable. We begin by considering $L(1)_{-\infty}$ as a module over some $\mathcal{A}(n)$. The space $L(1)_{-\infty}$ is particularly simple, since it is one-dimensional in each integral degree.

Theorem 2. *As an $\mathcal{A}(n)$-module, $L(1)_{-\infty}$ has a filtration with each filtration quotient consisting of a copy of $\mathcal{A}(n)//\mathcal{A}(n-1)$.*

Proof. We begin by showing that $\bar{L}(1)$ has such a filtration, with a single finite-dimensional base space of the filtration not isomorphic to $\mathcal{A}(n)//\mathcal{A}(n-1)$. The filtration quotients, in particular, are each a copy of $\mathcal{A}(n)//\mathcal{A}(n-1)$ built on each element of $B_{n,1} \cap \bar{L}(1)$.

These quotients are linearly independent by Theorem 1. In addition, I claim they contain one class in each degree above $(n-1)2^{n+1}$. In particular, consider some degree $d = m2^{n+1} - d'$, where $d' < 2^{n+1}$ and $m \geq n$. Suppose that the binary representation of d' has ones in the bits $k_1, k_2, \ldots, k_j$. Then the class

$$\mathrm{Sq}^{2^{n+1}-2^{k_1}}\mathrm{Sq}^{2^{n+1}-2^{k_2}} \cdots \mathrm{Sq}^{2^{n+1}-2^{k_j}}\mathrm{Sq}^{(m-j)2^{n+1}}$$

has is in the correct degree and is in the proposed filtration. So the proposed filtration indeed spans all of $\bar{L}(1)$ with all quotients copies of $\mathcal{A}(n)//\mathcal{A}(n-1)$ except possibly for some finite-dimensional base space containing classes in small degrees.

Since the structure of $L(1)_{-\infty}$ is periodic, this finite-dimensional base disappears if we extend the filtration to $L(1)_{-\infty}$, so it has a filtration with all quotients consisting of a copy of $\mathcal{A}(n)//\mathcal{A}(n-1)$ as desired. $\square$

Next we consider the case of $\bar{L}(2)$. Welcher's result shows that $\bar{L}(2)$ is free as an $\mathcal{A}(1)$-module. As an $\mathcal{A}(2)$-module, we wish to show that $L(2)_{-\infty}$ admits a filtration with quotients $\mathcal{A}(2)//\mathcal{A}(0)$. Again, only the dimension-counting remains.

Theorem 3. *As an $\mathcal{A}(2)$-module, $L(2)_{-\infty}$ has a filtration with each filtration quotient consisting of a copy of $\mathcal{A}(n)//\mathcal{A}(n-1)$.*

Proof. Again we begin by showing that $\bar{L}(2)$ has a filtration, this time with a single base space which we construct, and which is finite in each degree, and each other filtration quotient is isomorphic to $\mathcal{A}(2)/\!/\mathcal{A}(0)$, again built on elements of $B_{2,2} \cap \bar{L}(2)$.

These quotients are linearly independent by Theorem 1. In addition, I claim that none of the filtration quotients built on this basis span any of the following basis elements of $\bar{L}(2)$:

$$\mathrm{Sq}^1, \mathrm{Sq}^3\mathrm{Sq}^1, \mathrm{Sq}^4\mathrm{Sq}^1, \mathrm{Sq}^5\mathrm{Sq}^1, \ldots. \tag{3}$$

This is easy to see from the Adem relations: clearly the only elements from $B_{2,2} \cap \bar{L}(2)$ that could be multiplied by elements of $\mathcal{A}(2)$ to get anything ending in Sq^1 are the Sq^{8i}, and it is easy to see from the Adem relations that any product which could create a term in (3) and no others must involve multiplying by something ending in Sq^1, i.e., by something which is zero in $\mathcal{A}(2)/\!/\mathcal{A}(0)$. (Note that this is the point where using $\bar{L}(2)$ rather than $L(2)$ makes things much simpler: the set of excluded elements would be much more complicated in $L(2)$.)

We now use Poincaré series to show that after adding those elements from (3), the claimed filtration quotients add up to have the right dimensionality in each degree, showing that the claimed filtration exists. Recall that the Poincaré series of a graded vector space is the infinite series $v(x) = \sum_{i=0}^{\infty} x^i \dim V_i$. By definition, the Poincaré series of $\bar{L}(2)$ is

$$l(x) = 1+x+x^2+2x^3+2x^4+2x^5+3x^6+\cdots = (1+x+x^2+\cdots)(1+x^3+x^6+\cdots).$$

Furthermore, from looking at the Z basis, we can see that the Poincaré series of $\mathcal{A}(2)/\!/\mathcal{A}(0)$ is

$$a(x) = (1+x^4)(1+x^6)(1+x^7)(1+x^2)(1+x^3).$$

The Poincaré series of the set $B_{2,2} \cap \bar{L}(2)$ is

$$b(x) = (1+x^8+x^{16}+\cdots)(1+x^{12}+x^{24}+\cdots).$$

Finally, the Poincaré series of the extra elements from (3) is

$$c(x) = x + x^3 + x^4 + x^5 + \cdots.$$

We wish to show that $l(x) = a(x)b(x) + c(x)$. This is just a matter of algebra:

$$\begin{aligned}
a(x)b(x)+c(x) &= (1+x^7)(1+x^2+x^4+\cdots)(1+x^3+x^6+\cdots)+c(x)\\
&= (1-x+x^2-\cdots+x^6)(1+x+x^2+\cdots)(1+x^3+x^6+\cdots)\\
&\qquad +x+x^3(1+x+x^2+\cdots)\\
&= (1+x+x^2+\cdots)(1-x+x^2+x^3+x^6+x^9+\cdots)+x\\
&= l(x)+x(x-1)(1+x+x^2+\cdots)+x\\
&= l(x)-x+x\\
a(x)b(x)+c(x) &= l(x)
\end{aligned}$$

Then since this filtration has linearly independent filtration quotients, and has the right total dimensionality, it must span the entirety of $\bar{L}(2)$. As we continue backwards to $L(2)_{-\infty}$, the excluded elements are not a part of the periodic structure, so they get filled in, and we get that $L(2)_{-\infty}$ has a filtration consisting only of copies of $\mathcal{A}(2)//\mathcal{A}(0)$. $\square$