Recall that, in the Wood Z-basis, the top class of $A(n) \otimes_{A(n-1)} \mathbb{F}_2$ may be written as $\mathrm{Sq}^{2^{n+1}-2^n}\mathrm{Sq}^{2^{n+1}-2^{n-1}} \cdots \mathrm{Sq}^{2^{n+1}-1}$; denote this by X_n.

Lemma 1. *For any positive integers m and n,*

$$X_n X_{n-1} \mathrm{Sq}^{2^{n+1}m} = X_n \mathrm{Sq}^{2^{n+1}m} X_{n-1}.$$

Proof. First, we show the following identities by application of the Adem relations:

Proposition 1. *For any positive integers m and n,*

$$\mathrm{Sq}^{2^n-1}\mathrm{Sq}^{2^{n+1}m} = \mathrm{Sq}^{2^{n+1}m+2^n-1}.$$

Proposition 2. *For any positive integers k, m, and n, with $k \leq n$,*

$$\mathrm{Sq}^{2^n-2^k}\mathrm{Sq}^{2^n-2^{k-1}+2^{n+1}m} = \mathrm{Sq}^{2^n-2^k+2^{n+1}m}\mathrm{Sq}^{2^n-2^{k-1}} + \mathrm{Sq}^{2^{n+1}-2^k}\mathrm{Sq}^{2^{n+1}m-2^{k-1}}.$$

Proof of Proposition 1. The Adem relations give us that

$$\mathrm{Sq}^{2^n-1}\mathrm{Sq}^{2^{n+1}m} = \sum_{j=0}^{2^{n-1}-1} \binom{2^{n+1}m-1-j}{2^n-1-2j} \mathrm{Sq}^{2^{n+1}m+2^n-1-j}\mathrm{Sq}^j.$$

Recall that in $\mathbb{F}_2$, $\binom{p}{q}$ is 1 if and only if p has a one in every position of its binary expansion that q has a 1. Let $2^n > j > 0$ and let $2^{\omega(j)}$ be the largest power of 2 dividing j. Then the binary expansion of $2^{n+1}m - 1 - j$ ends in a zero and then $\omega(j)$ ones, while that of $2^n - 1 - 2j$ ends in a zero and then $\omega(j) + 1$ ones, so the binomial coefficient in the Adem relation is zero. Then the only term of the Adem relation that is nonzero is that when $j = 0$, where $2^{n+1}m - 1$ ends with $n + 1$ ones, while $2^n - 1$ consists of n ones. $\square$

Proof of Proposition 2. We expand the first and last terms using the Adem relations, and show that they are identical except for the middle term. For convenience, let $d = 2^{n+1}(m + 1) - 2^k - 2^{k-1}$, the degree of the elements.

$$\mathrm{Sq}^{2^n-2^k}\mathrm{Sq}^{2^n-2^{k-1}+2^{n+1}m} = \sum_{j=0}^{2^{n-1}-2^{k-1}} \binom{2^n-2^{k-1}+2^{n+1}m-1-j}{2^n-2^k-2j} \mathrm{Sq}^{d-j}\mathrm{Sq}^j$$

$$\mathrm{Sq}^{2^{n+1}-2^k}\mathrm{Sq}^{2^{n+1}m-2^{k-1}} = \sum_{j=0}^{2^n-2^{k-1}} \binom{2^{n+1}m-2^{k-1}-1-j}{2^{n+1}-2^k-2j} \mathrm{Sq}^{d-j}\mathrm{Sq}^j$$

First, let $0 \leq j \leq 2^{n-1} - 2^{k-1}$; we will show the two binomial coefficients are equal. We again consider the binary expansions. The expansion of $2^n - 2^k - 2j$ is the same as that of $2^{n+1} - 2^k - 2j$, except that the leading one has been removed.

The corresponding bits of $2^n - 2^{k-1} + 2^{n+1}m - 1 - j$ and $2^{n+1}m - 2^{k-1} - 1 - j$ are also zero and one, respectively, so the two binomial coefficients are the same.

Now let $2^{n-1} - 2^{k-1} < j < 2^n - 2^{k-1}$; we will show that the binomial coefficient

$$\binom{2^{n+1}m - 2^{k-1} - 1 - j}{2^{n+1} - 2^k - 2j}$$

is zero. Let 2^q be the next power of 2 strictly larger than $2^n - 2^{k-1} + j$, so $q \leq n-1$. Then the qth position of the binary expansion will not satisfy the condition to make the binomial coefficient one, so it will be zero.

Finally, if $j = 2^n - 2^{k-1}$, the binomial coefficient in the second formula has a zero on the bottom, thus is one, so we get the middle term of the claimed identity. □

Then in particular, by repeated application of the above identities, we may write

$$\begin{aligned}
X_{n-1}\mathrm{Sq}^{2^{n+1}m} &= \mathrm{Sq}^{2^n-2^{n-1}}\mathrm{Sq}^{2^n-2^{n-2}} \cdots \mathrm{Sq}^{2^n-2}\mathrm{Sq}^{2^{n+1}m+2^n-1} \\
&= \mathrm{Sq}^{2^n-2^{n-1}} \cdots \mathrm{Sq}^{2^n-4}\mathrm{Sq}^{2^{n+1}m+2^n-2}\mathrm{Sq}^{2^n-1} \\
&\quad + \mathrm{Sq}^{2^n-2^{n-1}} \cdots \mathrm{Sq}^{2^n-4}\mathrm{Sq}^{2^{n+1}-2}\mathrm{Sq}^{2^{n+1}m-1} \\
&\vdots \\
&= \mathrm{Sq}^{2^{n+1}m}X_{n-1} \\
&\quad + \mathrm{Sq}^{2^n-2^{n-1}} \cdots \mathrm{Sq}^{2^n-4}\mathrm{Sq}^{2^{n+1}-2}\mathrm{Sq}^{2^{n+1}m-1} \\
&\quad + \cdots + \mathrm{Sq}^{2^n}\mathrm{Sq}^{2^{n+1}m-2^{n-1}}\mathrm{Sq}^{2^n-2^{n-2}} \cdots \mathrm{Sq}^{2^n-1}.
\end{aligned}$$

Now all of the extra terms are products of (from left to right) an element of $\mathcal{A}(n-1)$, then an element of the form $\mathrm{Sq}^{2^{n+1}-2^k}$ for $k < n+1$, then some element of $\mathcal{A}$. In particular, they are zero in $\mathcal{A}(n-1) \otimes_{\mathcal{A}(n)} \mathcal{A}$. Then since X_n is the top class of $\mathcal{A}(n) \otimes_{\mathcal{A}(n-1)} \mathbb{F}_2$, thus annihilates any element of $\mathcal{A}(n)$ that is not in $\mathcal{A}(n-1)$, $1 \otimes x \mapsto X_n x$ is well-defined as a map from $\mathcal{A}(n-1) \otimes_{\mathcal{A}(n)} \mathcal{A}$ to $\mathcal{A}$, so the two sides are the same when multiplied by X_n:

$$X_n X_{n-1}\mathrm{Sq}^{2^{n+1}m} = X_n \mathrm{Sq}^{2^{n+1}m} X_{n-1}.$$

□