Determining the Structure of Length-k Steenrod Operations as $A(r)$-Modules

The Steenrod Algebra $\mathcal{A}$ is the algebra of stable natural endomorphisms of the $\mathbb{Z}/2$-cohomology functor; it is generated by elements Sq^{2^i}. Let $\mathcal{A}(k)$ be the subalgebra generated by the Sq^{2^i} for $i \leq k$. Consider the modules $L(k)$ spanned by sequences of Steenrod operations of length k. Welcher proved that $L(k)$ is a free module over $\mathcal{A}(k-1)$. We are interested in finding the structure of $L(k)$ as an $\mathcal{A}(r)$-module for any r. We conjecture that $L(k)$ is built as an $\mathcal{A}(r)$-module out of $\mathcal{A}(r)//\mathcal{A}(r-k)$, in the sense that it has an increasing filtration with quotients isomorphic to $\mathcal{A}(r)//\mathcal{A}(r-k)$, and present partial results towards that claim. In addition, we prove some interesting commutation relations in the Steenrod algebra relating to representations of Steenrod Algebra elements in Wood's Z-basis.