For convenience let $\bar{L}(k) = \bigcup_{i \le k} L(i)$.

Proposition 1. *Any element t of $\mathcal{A}(n)$ may be written as $x + yz$ where $x \in \bar{L}(k)$, $y \in \mathcal{A}$, and $z \in \mathcal{A}(n-k) \setminus \mathbb{F}_2$ for any $k \ge 0$. Equivalently, $\mathcal{A}(n) \subset \bar{L}(k) + \mathcal{A}(\mathrm{Sq}^i)_{i \le k}$, where we take $\mathcal{A}(\mathrm{Sq}^i)_{i \le k}$ to be the left ideal of $\mathcal{A}$ generated by $\{\mathrm{Sq}^i\}_{i \le k}$, for any $k \ge 0$.*

Proof. It suffices to prove the proposition on some basis for $\mathcal{A}(n)$. Recall that in Wood's Z-basis of the Steenrod algebra, a basis for $\mathcal{A}(n)$ consists of all strings of squares selected from (1) and multiplied in the given order:

$$\mathrm{Sq}^{2^{n+1}-2^n}\mathrm{Sq}^{2^{n+1}-2^{n-1}} \cdots \mathrm{Sq}^{2^{n+1}-1}\mathrm{Sq}^{2^n-2^{n-1}} \cdots \mathrm{Sq}^{2^n-1} \cdots \mathrm{Sq}^2\mathrm{Sq}^3\mathrm{Sq}^1 \tag{1}$$

We induct on the number of squares selected. The empty product is 1, which is in $\bar{L}(k)$, so we are done. So now suppose the proposition holds on all elements of the Wood Z-basis of length at most $l-1$, and let t be an element of length l. Then, for some $m = 2^j - 2^i$, $i < j \le n+1$, we may write $t = \mathrm{Sq}^m(x + yz)$ where x, y, and z are as in the statement of the proposition.

Now $\mathrm{Sq}^m yz$ is clearly of the prescribed form, since $\mathrm{Sq}^m y$ is in $\mathcal{A}$. If $x \in \bar{L}(k-1)$, then $\mathrm{Sq}^m x \in \bar{L}(k)$ (since application of the Adem relations cannot increase the length of a monomial), so we may suppose $x \in L(k)$.

Write x in admissible form as $\mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_k}$. If $\mathrm{Sq}^m\mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_k}$ is already in admissible form, then we have that $2^k i_k \le 2^{k-1} i_{k-1} \le \cdots \le 2i_1 \le m < 2^{n+1}$, so in particular, $i_k < 2^{n-k+1}$, so $i_k \in \mathcal{A}(\mathrm{Sq}^i)_{i \le k}$, so in fact the entire product $\mathrm{Sq}^m x$ is in $\mathcal{A}(\mathrm{Sq}^i)_{i \le k}$. Then we may apply the Adem relations at least once, to get

$$\mathrm{Sq}^m\mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_k} = a_0\mathrm{Sq}^{m+i_1}\mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_k} + \sum_{1 \le j \le \frac{m}{2}} a_j \mathrm{Sq}^{m+i_1-j}\mathrm{Sq}^j\mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_k};$$

the first term we know is in $L(k)$. For each of the remaining terms, if we may repeat the same process, ignoring the first factor; either the product is in $\mathcal{A}(\mathrm{Sq}^i)_{i \le k}$, or it may be further reduced by the Adem relations. Finally, if we apply the Adem relations k times, the rightmost factor must be Sq^i for $i \le \frac{m}{2^k} < 2^{n-k+1}$, so similarly the product lies in the ideal $\mathcal{A}(\mathrm{Sq}^i)_{i \le k}$. Then $\mathrm{Sq}^m x$ may be written in the desired form, so we are done. ∎