For convenience let $\bar{L}(k) = \bigcup_{i \leq k} L(i)$.

Proposition 1. *$\bar{L}(k)//A(n-k)$ is a free module over $A(n)//A(n-k)$, for any $n \geq k$.*

In particular, we will show that $\bar{L}(k)//A(n-k)$ has $A(n)//A(n-k)$-basis

$$B = \{\mathrm{Sq}^I \mid I = (2^{n+1}j_1, 2^n j_2, \ldots, 2^{n-k+2}j_k), j_1 \geq j_2 \geq \cdots \geq j_k \geq 0\}.$$

First, let us show that the module generated by B is free. Here we follow a similar argument of Welcher. Let t be the top class of $A(n)//A(n-k)$. Then I claim that for any $A(n)//A(n-k)$-module M, the submodule generated by $m \in M$ is free if $tm \neq 0$.

Then it suffices to show that tB is linearly independent. To do this, we consider only the top terms in the left lexicographic ordering, and show that no two are the same. Following Welcher, we let $\equiv$ mean congruent modulo both lower terms and $A(\mathrm{Sq}^i)_{i \leq n-k}$. In particular, we will show that if $K = (1 + n2^{n+1}, 1 + (n-1)2^n, \ldots, 1 + (n-k+1)2^{n-k+2})$, and $\mathrm{Sq}^I \in B$, then $t\mathrm{Sq}^I \equiv \mathrm{Sq}^{I+K}$. Clearly none of these are the same, so tB will then be linearly independent.

To show this, recall that in the Wood Z-basis, the top class t is represented by the element

$$\mathrm{Sq}^{2^{n+1}-2^n} \cdots \mathrm{Sq}^{2^{n+1}-1} \mathrm{Sq}^{2^n - 2^{n-1}} \cdots \mathrm{Sq}^{2^{n-k+2}-2^{n-k+1}} \cdots \mathrm{Sq}^{2^{n-k+2}-1}.$$

Then we get