Abstract

The Steenrod Algebra $\mathcal{A}$ is the algebra of stable natural endomorphisms of the $\mathbb{Z}/2$-cohomology functor; it is generated by elements Sq^{2^i}. Let $\mathcal{A}(k)$ be the subalgebra generated by the Sq^{2^i} for $i \leq k$. Consider the modules $L(k)$ spanned by sequences of Steenrod operations of length k. Welcher proved that $L(k)$ is a free module over $\mathcal{A}(k-1)$. We are interested in finding the structure of $L(k)$ as an $\mathcal{A}(r)$-module for any r. We conjecture that $L(k)$ is built as an $\mathcal{A}(r)$-module out of $\mathcal{A}(r)/\!/\mathcal{A}(r-k)$, in the sense that it has an increasing filtration with quotients isomorphic to $\mathcal{A}(r)/\!/\mathcal{A}(r-k)$, and present partial results towards that claim. In addition, we prove some interesting commutation relations in the Steenrod algebra relating to representations of Steenrod Algebra elements in Wood's Z-basis.

1 The Commutation Relation

Recall that, in the Wood Z-basis, the top class of $A(n) \otimes_{A(n-1)} \mathbb{F}_2$ may be written as $\mathrm{Sq}^{2^{n+1}-2^n}\mathrm{Sq}^{2^{n+1}-2^{n-1}} \cdots \mathrm{Sq}^{2^{n+1}-1}$; denote this by X_n.

Lemma 1. *For any positive integers m and n,*

$$X_n X_{n-1} \mathrm{Sq}^{2^{n+1}m} = X_n \mathrm{Sq}^{2^{n+1}m} X_{n-1}.$$

Proof. First, we show the following identities by application of the Adem relations:

Proposition 1. *For any positive integers m and n,*

$$\mathrm{Sq}^{2^n-1}\mathrm{Sq}^{2^{n+1}m} = \mathrm{Sq}^{2^{n+1}m+2^n-1}.$$

Proposition 2. *For any positive integers k, m, and n, with $k \leq n$,*

$$\mathrm{Sq}^{2^n-2^k}\mathrm{Sq}^{2^n-2^{k-1}+2^{n+1}m} = \mathrm{Sq}^{2^n-2^k+2^{n+1}m}\mathrm{Sq}^{2^n-2^{k-1}} + \mathrm{Sq}^{2^{n+1}-2^k}\mathrm{Sq}^{2^{n+1}m-2^{k-1}}.$$

Proof of Proposition 1. The Adem relations give us that

$$\mathrm{Sq}^{2^n-1}\mathrm{Sq}^{2^{n+1}m} = \sum_{j=0}^{2^{n-1}-1} \binom{2^{n+1}m-1-j}{2^n-1-2j} \mathrm{Sq}^{2^{n+1}m+2^n-1-j}\mathrm{Sq}^j.$$

Recall that in $\mathbb{F}_2$, $\binom{p}{q}$ is 1 if and only if p has a one in every position of its binary expansion that q has a 1. Let $2^n > j > 0$ and let $2^{\omega(j)}$ be the largest power of 2 dividing j. Then the binary expansion of $2^{n+1}m - 1 - j$ ends in a zero and then $\omega(j)$ ones, while that of $2^n - 1 - 2j$ ends in a zero and then $\omega(j)+1$ ones, so the binomial coefficient in the Adem relation is zero. Then the only term of the Adem relation that is nonzero is that when $j = 0$, where $2^{n+1}m - 1$ ends with $n+1$ ones, while $2^n - 1$ consists of n ones. $\square$

Proof of Proposition 2. We expand the first and last terms using the Adem relations, and show that they are identical except for the middle term. For convenience, let $d = 2^{n+1}(m+1) - 2^k - 2^{k-1}$, the degree of the elements.

$$\mathrm{Sq}^{2^n-2^k}\mathrm{Sq}^{2^n-2^{k-1}+2^{n+1}m} = \sum_{j=0}^{2^{n-1}-2^{k-1}} \binom{2^n - 2^{k-1} + 2^{n+1}m - 1 - j}{2^n - 2^k - 2j}\mathrm{Sq}^{d-j}\mathrm{Sq}^j$$

$$\mathrm{Sq}^{2^{n+1}-2^k}\mathrm{Sq}^{2^{n+1}m-2^{k-1}} = \sum_{j=0}^{2^n-2^{k-1}} \binom{2^{n+1}m - 2^{k-1} - 1 - j}{2^{n+1} - 2^k - 2j}\mathrm{Sq}^{d-j}\mathrm{Sq}^j$$

First, let $0 \le j \le 2^{n-1} - 2^{k-1}$; we will show the two binomial coefficients are equal. We again consider the binary expansions. The expansion of $2^n - 2^k - 2j$ is the same as that of $2^{n+1} - 2^k - 2j$, except that the leading one has been removed. The corresponding bits of $2^n - 2^{k-1} + 2^{n+1}m - 1 - j$ and $2^{n+1}m - 2^{k-1} - 1 - j$ are also zero and one, respectively, so the two binomial coefficients are the same.

Now let $2^{n-1} - 2^{k-1} < j < 2^n - 2^{k-1}$; we will show that the binomial coefficient

$$\binom{2^{n+1}m - 2^{k-1} - 1 - j}{2^{n+1} - 2^k - 2j}$$

is zero. Let 2^q be the next power of 2 strictly larger than $2^n - 2^{k-1} + j$, so $q \le n-1$. Then the qth position of the binary expansion will not satisfy the condition to make the binomial coefficient one, so it will be zero.

Finally, if $j = 2^n - 2^{k-1}$, the binomial coefficient in the second formula has a zero on the bottom, thus is one, so we get the middle term of the claimed identity. □

Then in particular, by repeated application of the above identities, we may write

$$\begin{aligned}
X_{n-1}\mathrm{Sq}^{2^{n+1}m} &= \mathrm{Sq}^{2^n-2^{n-1}}\mathrm{Sq}^{2^n-2^{n-2}}\cdots\mathrm{Sq}^{2^n-2}\mathrm{Sq}^{2^{n+1}m+2^n-1} \\
&= \mathrm{Sq}^{2^n-2^{n-1}}\cdots\mathrm{Sq}^{2^n-4}\mathrm{Sq}^{2^{n+1}m+2^n-2}\mathrm{Sq}^{2^n-1} \\
&\quad + \mathrm{Sq}^{2^n-2^{n-1}}\cdots\mathrm{Sq}^{2^n-4}\mathrm{Sq}^{2^{n+1}-2}\mathrm{Sq}^{2^{n+1}m-1} \\
&\vdots \\
&= \mathrm{Sq}^{2^{n+1}m}X_{n-1} \\
&\quad + \mathrm{Sq}^{2^n-2^{n-1}}\cdots\mathrm{Sq}^{2^n-4}\mathrm{Sq}^{2^{n+1}-2}\mathrm{Sq}^{2^{n+1}m-1} \\
&\quad + \cdots + \mathrm{Sq}^{2^n}\mathrm{Sq}^{2^{n+1}m-2^{n-1}}\mathrm{Sq}^{2^n-2^{n-2}}\cdots\mathrm{Sq}^{2^n-1}.
\end{aligned}$$

Now all of the extra terms are products of (from left to right) an element of $\mathcal{A}(n-1)$, then an element of the form $\mathrm{Sq}^{2^{n+1}-2^k}$ for $k < n+1$, then some element of $\mathcal{A}$. In particular, they are zero in $\mathcal{A}(n-1) \otimes_{\mathcal{A}(n)} \mathcal{A}$. Then since X_n

is the top class of $\mathcal{A}(n) \otimes_{\mathcal{A}(n-1)} \mathbb{F}_2$, thus annihilates any element of $\mathcal{A}(n)$ that is not in $\mathcal{A}(n-1)$, $1 \otimes x \mapsto X_n x$ is well-defined as a map from $\mathcal{A}(n-1) \otimes_{\mathcal{A}(n)} \mathcal{A}$ to $\mathcal{A}$, so the two sides are the same when multiplied by X_n:

$$X_n X_{n-1} \mathrm{Sq}^{2^{n+1}m} = X_n \mathrm{Sq}^{2^{n+1}m} X_{n-1}.$$

□

2 Representatives for $\mathcal{A}(n)//\mathcal{A}(n-k)$

For convenience let $\bar{L}(k) = \bigcup_{i \leq k} L(i)$.

Proposition 3. *Any element t of $\mathcal{A}(n)$ may be written as $x + yz$ where $x \in \bar{L}(k)$, $y \in \mathcal{A}$, and $z \in \mathcal{A}(n-k) \setminus \mathbb{F}_2$ for any $k \geq 0$. Equivalently, $\mathcal{A}(n) \subset \bar{L}(k) + \mathcal{A}(\mathrm{Sq}^i)_{i \leq k}$, where we take $\mathcal{A}(\mathrm{Sq}^i)_{i \leq k}$ to be the left ideal of $\mathcal{A}$ generated by $\{\mathrm{Sq}^i\}_{i \leq k}$, for any $k \geq 0$.*

Proof. It suffices to prove the proposition on some basis for $\mathcal{A}(n)$. Recall that in Wood's Z-basis of the Steenrod algebra, a basis for $\mathcal{A}(n)$ consists of all strings of squares selected from (1) and multiplied in the given order:

$$\mathrm{Sq}^{2^{n+1}-2^n} \mathrm{Sq}^{2^{n+1}-2^{n-1}} \cdots \mathrm{Sq}^{2^{n+1}-1} \mathrm{Sq}^{2^n-2^{n-1}} \cdots \mathrm{Sq}^{2^n-1} \cdots \mathrm{Sq}^2 \mathrm{Sq}^3 \mathrm{Sq}^1 \quad (1)$$

We induct on the number of squares selected. The empty product is 1, which is in $\bar{L}(k)$, so we are done. So now suppose the proposition holds on all elements of the Wood Z-basis of length at most $l-1$, and let t be an element of length l. Then, for some $m = 2^j - 2^i$, $i < j \leq n+1$, we may write $t = \mathrm{Sq}^m (x + yz)$ where x, y, and z are as in the statement of the proposition.

Now $\mathrm{Sq}^m yz$ is clearly of the prescribed form, since $\mathrm{Sq}^m y$ is in $\mathcal{A}$. If $x \in \bar{L}(k-1)$, then $\mathrm{Sq}^m x \in \bar{L}(k)$ (since application of the Adem relations cannot increase the length of a monomial), so we may suppose $x \in L(k)$.

Write x in admissible form as $\mathrm{Sq}^{i_1} \mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_k}$. If $\mathrm{Sq}^m \mathrm{Sq}^{i_1} \mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_k}$ is already in admissible form, then we have that $2^k i_k \leq 2^{k-1} i_{k-1} \leq \cdots \leq 2 i_1 \leq m < 2^{n+1}$, so in particular, $i_k < 2^{n-k+1}$, so $i_k \in \mathcal{A}(\mathrm{Sq}^i)_{i \leq k}$, so in fact the entire product $\mathrm{Sq}^m x$ is in $\mathcal{A}(\mathrm{Sq}^i)_{i \leq k}$. Then we may apply the Adem relations at least once, to get

$$\mathrm{Sq}^m \mathrm{Sq}^{i_1} \mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_k} = a_0 \mathrm{Sq}^{m+i_1} \mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_k} + \sum_{1 \leq j \leq \frac{m}{2}} a_j \mathrm{Sq}^{m+i_1-j} \mathrm{Sq}^j \mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_k};$$

the first term we know is in $L(k)$. For each of the remaining terms, if we may repeat the same process, ignoring the first factor; either the product is in $\mathcal{A}(\mathrm{Sq}^i)_{i \leq k}$, or it may be further reduced by the Adem relations. Finally, if we apply the Adem relations k times, the rightmost factor must be Sq^i for $i \leq \frac{m}{2^k} < 2^{n-k+1}$, so similarly the product lies in the ideal $\mathcal{A}(\mathrm{Sq}^i)_{i \leq k}$. Then $\mathrm{Sq}^m x$ may be written in the desired form, so we are done. □

3 Linear Independence

Lemma 2. *Taken as elements of $L(k)_{-\infty}$ as a module over $\mathcal{A}(n) \otimes_{\mathcal{A}(n-k)} \mathbb{F}_2$, the set*

$$\{X_n X_{n-1} \cdots X_{n-k+1} \mathrm{Sq}^I \mid I = (2^{n+1}m_1, 2^n m_2, \ldots, 2^{n-k+2}m_k), m_1 \geq m_2 \geq \cdots \geq m_{n-k}\}$$

is linearly independent.

Proof. Note that by the commutation relation we have shown, the element $s = X_n X_{n-1} \cdots X_{n-k+1} \mathrm{Sq}^I$ may be rewritten as

$$X_n \mathrm{Sq}^{2^{n+1}m_1} X_{n-1} \mathrm{Sq}^{2^n m_2} \cdots X_{n-k+1} \mathrm{Sq}^{2^{n-k+2}m_k}.$$

Let $M_{k,n}$ be the subspace of $\mathcal{A}$ which is spanned by elements of the form yz where $y \in L(k)$ and $z \in \mathcal{A}(n)$ with the degree of z at least 1. If $b \in \mathcal{A}$, we will write $b + M_{k,n}$ for the subset of $\mathcal{A}$ consisting of elements of the form $b + m$ for $m \in M_{k,n}$.

Proposition 4. *The element*

$$X_n \mathrm{Sq}^{2^{n+1}m_n} X_{n-1} \mathrm{Sq}^{2^n m_{n-1}} \cdots X_{n-k+1} \mathrm{Sq}^{2^{n-k+2}m_{n-k+1}}$$

is in

$$\mathrm{Sq}^{2^{n+1}(m_n+n)+1} \mathrm{Sq}^{2^n(m_{n-1}+n-1)+1} \cdots \mathrm{Sq}^{2^{n-k+2}(m_{n-k+1}+n-k+1)+1} + M_{k,n-k}.$$

Proof. We proceed by induction on k for fixed n. If $k = 1$, we know that $X_n \in M_{1,n-1} + \mathrm{Sq}^{n2^{n+1}+1}$, since the latter term is the only element of $L(1)$ in the right dimension. Then since, as an $\mathcal{A}(n)$-module, $L(1)$ is periodic in degrees modulo 2^{n+1}, $X_n \mathrm{Sq}^{2^{n+1}m} \in M_{1,n-1} + \mathrm{Sq}^{2^{n+1}(n+m)+1}$.

Now suppose the proposition holds for some k; we will show it for $k+1$. Let

$$t = X_n \mathrm{Sq}^{2^{n+1}m_n} X_{n-1} \mathrm{Sq}^{2^n m_{n-1}} \cdots X_{n-k+1} \mathrm{Sq}^{2^{n-k+2}m_{n-k+1}},$$

let $t' = X_{n-k} \mathrm{Sq}^{2^{n-k+1}m_{n-k}}$, let

$$b = \mathrm{Sq}^{2^{n+1}(m_n+n)+1} \mathrm{Sq}^{2^n(m_{n-1}+n-1)+1} \cdots \mathrm{Sq}^{2^{n-k+2}(m_{n-k+1}+n-k+1)+1},$$

and let $b' = \mathrm{Sq}^{2^{n-k+1}(m_{n-k}+n-k)+1}$. Then by inductive hypothesis we know that $t \in b + M_{k,n-k}$ and by the base case we know that $t' \in b' + M_{1,n-k-1}$; we wish to show that $tt' \in bb' + M_{k+1,n-k-1}$. To do this, it suffices to show that $bM_{1,n-k-1}$ and $M_{k,n-k}t'$ are contained in $M_{k+1,n-k-1}$; for simplicity we will show that this is true of the spanning sets we have constructed.

First, let $y \in L(1)$, $z \in \mathcal{A}(n-k-1)$ with z not in degree 0, so that $yz \in M_{1,n-k-1}$. Then by looking at degrees, by is already in admissible form, so it is in $L(k+1)$, so $byz \in M_{k+1,n-k-1}$ as desired. Since elements of the form yz span $M_{1,n-k-1}$, we get that $bM_{1,n-k-1} \subset M_{k+1,n-k-1}$.

Second, let $y \in L(k)$, $z \in \mathcal{A}(n-k)$ with z not in degree 0, so that $yz \in M_{k,n-k}$. If $z \notin \mathcal{A}(n-k-1)$, then zX_{n-k} must be zero in $\mathcal{A}(n-k)$. Then we may assume $z \in \mathcal{A}(n-k-1)$; we will show that both yzb' and $yzM_{1,n-k-1}$ are contained in $M_{k+1,n-k-1}$. Now if we write zb' in admissible form, multiplying by y will give an element already in admissible form by looking at degrees, so we need only show that $zb' \in M_{1,n-k-1}$. If we multiply out using the Adem relations, this is clearly true, modulo showing that zb' contains no terms of length 1. This is easy to show; by the Milnor basis, z cannot end in a Steenrod square with degree a multiple of 2^{n-k}, but b' is a Steenrod square in a degree which is 1 modulo 2^{n-k}, so on multiplying the two, the length 1 term must vanish, as desired.

Finally, then, we must show that $yzM_{1,n-k-1} \subset M_{k+1,n-k-1}$; let $y' \in L(1)$ and $z' \in \mathcal{A}(n-k-1)$, so we must show that $yzy'z' \in M_{k+1,n-k-1}$. Again looking at the Adem relations, we may write zy' in the form $y''z''$ where $y'' \in L(1)$ and $z'' \in \mathcal{A}(n-k-1)$. Then $z''z' \in \mathcal{A}(n-k-1)$ and is not in degree zero, and $yy'' \in L(k+1)$, so $yy''z''z' \in M_{k+1,n-k-1}$.

Then the induction is complete; all terms of $tt' - bb'$ are in $M_{k+1,n-k-1}$ as desired. □

Then since the elements in question may be written in the given form, their shortest terms are unique and distinct, so the terms are linearly independent. □