**COMPSCI 3MI3**

# Assignment 7

## 1 Solution Set

### 1.1 Q1

As described in the slides for topics 7 and 8, when defining a typing relation over the set of terms $\mathcal{T}$, the typing relation is not always total. Thus $dom(:) \subseteq \mathcal{T}$, with the set of terms that are syntactically correct, but semantically incorrect $\mathcal{W} \subseteq \mathcal{T} \setminus dom(:)$. This is to say that all terms that are well-typeable must be evaluatable under our semantics to a value. While $\Omega$ is a syntactically correct, semantically it represents an incorrect term in lambda calculus, being unable to evaluate to a value. Additionally, we cannot type these recursive forms since pure simply-typed lambda caluclus only allows us to assign a base type to values. Since $\Omega$ will never evaluate to a value we cannot assign a base type to build up a well-typed definition. Therefore, $\Omega$ cannot be well-typed.

### 1.2 Q2

We can show the following expressions have the indicated types by deriving their types formally.

(a) Show $f : Bool \Rightarrow Bool \vdash f\ (\texttt{if false then true else false}) : Bool$

| | | |
|---|---|---|
| (1) | $\boxed{f : Bool \Rightarrow Bool \vdash f\ (\texttt{if false then true else false})}$ | Assume |
| (2) | $\boxed{f : Bool \Rightarrow Bool \vdash f}$ | Assume |
| (3) | $f : Bool \Rightarrow Bool \vdash f : Bool \Rightarrow Bool$ | T-Var on 2 |
| (4) | $\boxed{f : Bool \Rightarrow Bool \vdash \texttt{if false then true else false}}$ | Assume |
| (5) | $\boxed{f : Bool \Rightarrow Bool \vdash \texttt{false}}$ | Assume |
| (6) | $Bool$ | T-False on (5) |
| (7) | $f : Bool \Rightarrow Bool \vdash \texttt{false} : Bool$ | T-Intro on (5), (6) |

| | | |
|---|---|---|
| (8) | $\boxed{f : Bool \Rightarrow Bool \vdash \texttt{true}}$ | Assume |
| (9) | $Bool$ | T-True on (8) |
| (10) | $f : Bool \Rightarrow Bool \vdash \texttt{true} : Bool$ | T-Intro on (8), (9) |
| (11) | $\boxed{f : Bool \Rightarrow Bool \vdash \texttt{false}}$ | Assume |
| (12) | $Bool$ | T-False on (11) |
| (13) | $f : Bool \Rightarrow Bool \vdash \texttt{false} : Bool$ | T-Intro on (11), (12) |
| (14) | $Bool$ | T-If on (4), (7), (10), (13) |
| (15) | $f : Bool \Rightarrow Bool \vdash \texttt{if false then true else false} : Bool$ | T-Intro on (4), (14) |
| (16) | $f : Bool \Rightarrow Bool \vdash f\ (\texttt{if false then true else false}) : Bool$ | T-App on (3), (15) |

(b) Show $f : Bool \Rightarrow Bool \vdash \lambda x : Bool.\ f\ (\texttt{if}\ x\ \texttt{then false else}\ x) : Bool \Rightarrow Bool$

| | | |
|---|---|---|
| (1) | $\boxed{f : Bool \Rightarrow Bool \vdash \lambda x : Bool.\ f\ (\texttt{if}\ x\ \texttt{then false else}\ x)}$ | Assume |
| (2) | $\boxed{f : Bool \Rightarrow Bool, x : Bool \vdash f\ (\texttt{if}\ x\ \texttt{then false else}\ x)}$ | Assume |
| (3) | $\boxed{f : Bool \Rightarrow Bool, x : Bool \vdash f}$ | Assume |
| (4) | $f : Bool \Rightarrow Bool, x : Bool \vdash f : Bool \Rightarrow Bool$ | T-Var on 3 |
| (5) | $\boxed{f : Bool \Rightarrow Bool, x : Bool \vdash \texttt{if}\ x\ \texttt{then false else}\ x}$ | Assume |
| (6) | $\boxed{f : Bool \Rightarrow Bool, x : Bool \vdash x}$ | Assume |
| (7) | $f : Bool \Rightarrow Bool, x : Bool \vdash x : Bool$ | T-Var on (6) |
| (8) | $\boxed{f : Bool \Rightarrow Bool, x : Bool \vdash \texttt{false}}$ | Assume |
| (9) | $Bool$ | T-False on (8) |
| (10) | $f : Bool \Rightarrow Bool, x : Bool \vdash \texttt{false} : Bool$ | T-Intro on (8), (9) |
| (11) | $\boxed{f : Bool \Rightarrow Bool, x : Bool \vdash x}$ | Assume |
| (12) | $f : Bool \Rightarrow Bool, x : Bool \vdash x : Bool$ | T-Var on (11) |
| (13) | $Bool$ | T-If on 5, 7, 10, 12 |

$$\begin{array}{lll}
(14) & \quad f : Bool \Rightarrow Bool, x : Bool \vdash \texttt{if}\ x\ \texttt{then false else}\ x : Bool & \text{T-Intro on (5), (13)} \\
(15) & \quad f : Bool \Rightarrow Bool, x : Bool \vdash f\ (\texttt{if}\ x\ \texttt{then false else}\ x) : Bool & \text{T-App on (4), (14)} \\
(16) & f : Bool \Rightarrow Bool \vdash \lambda x : Bool.\ f\ (\texttt{if}\ x\ \texttt{then false else}\ x) : Bool \Rightarrow Bool & \text{T-Abs on (15)}
\end{array}$$

### 1.3 Q3

We will prove the property that if $\Gamma \vdash t : T$ and $\Delta$ is a permutation of $\Gamma$, then $\Delta \vdash t : T$ and $\texttt{depth}(\Gamma \vdash t : T) = \texttt{depth}(\Delta \vdash t : T)$, with depth as defined below. We will prove this property holds by induction over typing derivations $\Gamma \vdash t : T$. We consider the definition of simply-typed pure lambda calculus and its typing derivations presented in the slides for topic 7 and 8.

We define a permutation of elements in a set to be the rearrangement of those elements in a linear order different from the set's original linear ordering. A permutation of a set of elements does not alter the elements the set contains.

We define the *depth* of a typing derivation to be the the number of typing "steps" to fully type a well-typeable term $t$. We define depth by cases as follows:

$\texttt{depth}(\Gamma \vdash \texttt{true} : \texttt{Bool}) = 1$

$\texttt{depth}(\Gamma \vdash \texttt{false} : \texttt{Bool}) = 1$

$\texttt{depth}(\Gamma \vdash \texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3 : T) = 1 +$
$\max(\texttt{depth}(\Gamma \vdash t_1 : \texttt{Bool}), \texttt{depth}(\Gamma \vdash t_2 : T), \texttt{depth}(\Gamma \vdash t_3 : T))$

$\texttt{depth}(\Gamma \vdash x : T) = 1$

$\texttt{depth}(\Gamma \vdash \lambda x : T_1.\ t_2 : T_1 \Rightarrow T_2) = 1 + \texttt{depth}(\Gamma, x : T_1 \vdash t_2 : T_2)$

$\texttt{depth}(\Gamma \vdash t_1\ \ t_2 : T_2) = 1 + \max(\texttt{depth}(\Gamma \vdash t_1 : T_1 \Rightarrow T_2),\ \texttt{depth}(\Gamma \vdash t_2 : T_1))$

*Base case.* The base case denotes typing derivations on terms for which there does not exist any sub-typing derivations. The possible typing derivations are of the following:

**Case**: T-True, T-False
If T-True is the final typing derivation we know t = `true`, T = `Bool`. To type `true` by T-True we need no typing information so we can ignore $\Gamma$, and subsequently $\Delta$. By T-True we know `true` : `Bool`, thus we can attach our irrelevant permutated typing context to obtain $\Delta \vdash \texttt{true} : \texttt{Bool}$. Therefore we have $\Delta \vdash t : T$.
By the defintion of depth we know $\texttt{depth}(\Gamma \vdash \texttt{true} : \texttt{Bool}) = 1 = \texttt{depth}(\Delta \vdash \texttt{true} : \texttt{Bool})$ thus we know both derivations have the same depth.
Therefore our property holds if T-True is the final typing derivation. We can prove the T-False case in the same way.

**Case**: T-Var
If T-Var is the final typing derivation, we have $t = x$, $T = T$, and have the premise that $x : T \in \Gamma$. Since we know $\Delta$ is a permutation of $\Gamma$ and permutations of a set contain all elements of the original set, we know $x : T \in \Delta$. Since we have $x : T \in \Delta$, by T-Var we know $\Delta \vdash x : T$, which is $\Delta \vdash t : T$.
By the defintion of depth we know $\texttt{depth}(\Gamma \vdash x : T) = 1 = \texttt{depth}(\Delta \vdash x : T)$, thus we know both derivations have the same depth.
Therefore our property holds for the T-Var case.

Thus our base case holds.

*Induction step.* We assume for all typing sub-derivations our property holds. We will prove for the reminaing typing derivations that our property holds. The typing derivation must be one of the following:

**Case**: T-If
If T-If was the final typing derivation we know $t = \texttt{if}\, t_1\, \texttt{then}\, t_2\, \texttt{else}\, t_3$ and have the premises that in the typing context $\Gamma$ we know $t_1 : \texttt{Bool}$, $t_2 : T$ and $t_3 : T$. Since we know $\Delta$ is a permutation of $\Gamma$ and have the premise $\Gamma \vdash t_1 : \texttt{Bool}$, by our induction hypothesis on $t_1$ we know $\Delta \vdash t_1 : \texttt{Bool}$. By applying the induction hypothesis on $t_2$ and $t_3$ we similarly know $\Delta \vdash t_2 : T$ and $\Delta \vdash t_3 : T$. Since we have the previous premises by T-If we can conclude $\Delta \vdash \texttt{if}\, t_1\, \texttt{then}\, t_2\, \texttt{else}\, t_3 : T$.
By the definition of depth we know $\texttt{depth}(\Gamma \vdash \texttt{if} t_1 \texttt{then} t_2 \texttt{else} t_3 : T) = 1 + \max(\texttt{depth}(\Gamma \vdash t_1 : \texttt{Bool}), \texttt{depth}(\Gamma \vdash t_2 : T), \texttt{depth}(\Gamma \vdash t_3 : T))$. By our induction hypothesis on $t_1$ we know $\texttt{depth}(\Gamma \vdash t_1 : \texttt{Bool}) = \texttt{depth}(\Delta \vdash t_1 : \texttt{Bool})$, and can prove the derivations for $t_2$ and $t_3$ have the same depth similarly. Thus we know $\max(\texttt{depth}(\Gamma \vdash t_1 : \texttt{Bool}), \texttt{depth}(\Gamma \vdash t_2 : T), \texttt{depth}(\Gamma \vdash t_3 : T)) = \max(\texttt{depth}(\Delta \vdash t_1 : \texttt{Bool}), \texttt{depth}(\Delta \vdash t_2 : T), \texttt{depth}(\Delta \vdash t_3 : T))$, thus it follows that $(\Gamma \vdash \texttt{if}\, t_1\, \texttt{then}\, t_2\, \texttt{else}\, t_3 : T) = (\Delta \vdash \texttt{if}\, t_1\, \texttt{then}\, t_2\, \texttt{else}\, t_3 : T)$, both derivations have the same depth.
Therefore our property holds for the T-If case.

**Case**: T-Abs
If T-Abs was the final typing derivation we have $t = \lambda x : T_1.\, t_2$ and $T = T_1 \Rightarrow T_2$ with the premise $\Gamma, x : T_1 \vdash t_2 : T_2$. Since $\Delta$ is a permutation of $\Gamma$, we know $\Delta, x : T_1$ is a permutation of $\Gamma, x : T_1$. Therefore by our induction hypothesis on $t_2$ with our premise $\Gamma, x : T_1 \vdash t_2 : T_2$ and $\Delta, x : T_1$ a permutation of $\Gamma x : T_1$ we know $\Delta, x : T_1 \vdash t_2 : T_2$. Thus, by T-Abs we can show $\Delta \vdash \lambda x : T_1.\, t_2 : T_1 \Rightarrow T_2$. Therefore we have $\Delta \vdash t : T$.
By the definition of depth we know $\texttt{depth}(\Gamma \vdash \lambda x : T_1 . t_2 : T_1 \Rightarrow T_2) = 1 + \texttt{depth}(\Gamma, x : T_1 \vdash t_2 : T_2)$. By our induction hypothesis on $t_2$, and since we know $\Delta, x : T_1$ is a permutation of $\Gamma, x : T_1$ we can conclude $\texttt{depth}(\Gamma, x : T_1 \vdash t_2 : T_2) = \texttt{depth}(\Delta, x : T_1 \vdash t_2 : T_2)$. Therefore by the defintion of depth we know $\texttt{depth}(\Gamma \vdash \lambda x : T_1.\, t_2 : T_1 \Rightarrow T_2) = \texttt{depth}(\Delta \vdash \lambda x : T_1.\, t_2 : T_1 \Rightarrow T_2)$, the two derivations have the same depth.

Therefore our property holds for T-Abs.

**Case**: T-App
If T-App was the final typing derivation we have $t = t_1\ t_2$ and $T = T_2$, with the premises $\Gamma \vdash t_1 : T_1 \Rightarrow T_2$ and $\Gamma \vdash t_2 : T_1$. Since we know $\Delta$ is a permutation of $\Gamma$ and have the premise $\Gamma \vdash t_1 : T_1 \Rightarrow T_2$, by our induction hypothesis on $t_1$ we know $\Delta \vdash t_1 : T_1 \Rightarrow T_2$. Similarly since we have the premise $\Gamma \vdash t_2 : T_1$, by our induction hypothesis on $t_2$ we know $\Delta \vdash t_2 : T_1$. Thus, by T-App we can show $\Delta \vdash t_1\ t_2 : T_2$. Therefore we have $\Delta \vdash t : T$.
By the definition of depth we know $\texttt{depth}(\Gamma \vdash t_1\ t_2 : T_2) = 1 + \max(\texttt{depth}(\Gamma \vdash t_1 : T_1 \Rightarrow T_2),\ \texttt{depth}(\Gamma \vdash t_2 : T_1))$. Since we know $\Delta$ is a permutation of $\Gamma$, by our induction hypothesis on $t_1$ we know $\texttt{depth}(\Gamma \vdash t_1 : T_1 \Rightarrow T_2) = \texttt{depth}(\Delta \vdash t_1 : T_1 \Rightarrow T_2)$. Similarly by our induction hypothesis on $t_2$ we know $\texttt{depth}(\Gamma \vdash t_2 : T_1) = \texttt{depth}(\Delta \vdash t_2 : T_1)$. Thus their maximum will be the same, $\max(\texttt{depth}(\Gamma \vdash t_1 : T_1 \Rightarrow T_2),\ \texttt{depth}(\Gamma \vdash t_2 : T_1)) = \max(\texttt{depth}(\Delta \vdash t_1 : T_1 \Rightarrow T_2),\ \texttt{depth}(\Delta \vdash t_2 : T_1))$. Then, by the definition of depth we know both derivations have the same depth with $\texttt{depth}(\Gamma \vdash t_1\ t_2 : T_2) = \texttt{depth}(\Delta \vdash t_1\ t_2 : T_2)$.
Therefore our property holds for T-App.
We have shown that for all possible typing derivations our property holds. Therefore our induction step holds.
Thus we've shown that if $\Gamma \vdash t : T$ and $\Delta$ is a permutation of $\Gamma$, then $\Delta \vdash t : T$ and the depth of both derivations is the same, i.e. $\texttt{depth}(\Gamma \vdash t : T) = \texttt{depth}(\Delta \vdash t : T)$.

$\square$

## 1.4 Q4

We will prove the property that if $\Gamma \vdash t : T$ and $x \notin dom(\Gamma)$, then $\Gamma, x : S \vdash t : T$ and the latter derivation has the same depth as the former, or formally $\texttt{depth}(\Gamma \vdash t : T) = \texttt{depth}(\Gamma, x : S \vdash t : T)$, with the definition of depth defined in Q3. We will prove this property holds by induction over typing derivations $\Gamma \vdash t : T$. We consider the definition of simply-typed pure lambda calculus and its typing derivations presented in the slides for topic 7 and 8.

*Base case.* The base case denotes typing derivations on terms for which there does not exist any sub-typing derivations. The possible typing derivations are of the following:

**Case**: T-True, T-False
If T-True is the final typing derivation we know t = `true`, T = `Bool`. T-True requires no typing context to type `true` as `Bool`, thus we have $\Gamma, x : S \vdash \texttt{true} : \texttt{Bool}$ by T-True. Therefore we've proven $\Gamma, x : S \vdash t : T$.
By the defintion of depth we know $\texttt{depth}(\Gamma \vdash \texttt{true} : \texttt{Bool}) = 1 = \texttt{depth}(\Gamma, x : S \vdash \texttt{true} : \texttt{Bool})$ thus we know both derivations have the same depth.

Therefore our property holds if T-True is the final typing derivation. We can prove the T-False case in the same way.

**Case**: T-Var
If T-Var is the final typing derivation, we have $t = y$, $T = T$, and have the premise that $y : T \in \Gamma$. If $y = x$, then by our premise $x : T \in \Gamma$ we know $x \notin \mathit{dom}(\Gamma)$ is false. Thus, the left side of our implication is false and our property is vacuously true. If $y \neq x$, then we know $y : T \in \Gamma$ and $y : T \in \Gamma, x : S$, since growing our typing context will not remove $y$ from $\mathit{dom}(\Gamma)$. By T-Var we then have $\Gamma, x : S \vdash y : T$, which is $\Gamma, x : S \vdash t : T$.
By the defintion of depth we know $\texttt{depth}(\Gamma \vdash y : T) = 1 = \texttt{depth}(\Gamma, x : S \vdash y : T)$, thus we know both derivations have the same depth.
Therefore our property holds for the T-Var case.

Thus, our base case holds.

*Induction step.* We assume for all typing sub-derivations our property holds. We will prove for the reminaing typing derivations that our property holds. The typing derivation must be one of the following:

**Case**: T-If
If T-If was the final typing derivation we know $t = \texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3$ and have the premises $\Gamma \vdash t_1 : \texttt{Bool}$, $\Gamma \vdash t_2 : T$ and $\Gamma \vdash t_3 : T$. Since we know $x \notin \mathit{dom}(\Gamma)$ and have the premise $\Gamma \vdash t_1 : \texttt{Bool}$, by our induction hypothesis on $t_1$ we know $\Gamma, x : S \vdash t_1 : \texttt{Bool}$. By applying the induction hypothesis on $t_2$ and $t_3$ we similarly know $\Gamma, x : S \vdash t_2 : T$ and $\Gamma, x : S \vdash t_3 : T$. Since we have the previous premises by T-If we can conclude $\Gamma, x : S \vdash \texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3 : T$.
By the definition of depth we know $\texttt{depth}(\Gamma \vdash \texttt{if} t_1 \texttt{then} t_2 \texttt{else} t_3 : T) = 1 + \max(\texttt{depth}(\Gamma \vdash t_1 : \texttt{Bool}), \texttt{depth}(\Gamma \vdash t_2 : T), \texttt{depth}(\Gamma \vdash t_3 : T))$. By our induction hypothesis on $t_1$ we know $\texttt{depth}(\Gamma \vdash t_1 : \texttt{Bool}) = \texttt{depth}(\Gamma, x : S \vdash t_1 : \texttt{Bool})$, and can prove the derivations for $t_2$ and $t_3$ have the same depth similarly. Thus we know $\max(\texttt{depth}(\Gamma \vdash t_1 : \texttt{Bool}), \texttt{depth}(\Gamma \vdash t_2 : T), \texttt{depth}(\Gamma \vdash t_3 : T)) = \max(\texttt{depth}(\Gamma, x : S \vdash t_1 : \texttt{Bool}), \texttt{depth}(\Gamma, x : S \vdash t_2 : T), \texttt{depth}(\Gamma, x : S \vdash t_3 : T))$, thus it follows that $(\Gamma \vdash \texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3 : T) = (\Gamma, x : S \vdash \texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3 : T)$, both derivations have the same depth.
Therefore our property holds for the T-If case.

**Case**: T-Abs
If T-Abs was the final typing derivation we have $t = \lambda y : T_1.\, t_2$ and $T = T_1 \Rightarrow T_2$ with the premise $\Gamma, y : T_1 \vdash t_2 : T_2$. Since we know $x \notin \mathit{dom}(\Gamma)$, by our induction hypothesis on $t_2$ we know $\Gamma, y : T_1, x : S \vdash t_2 : T_2$. By the lemma of permutation we proved in Q3, we know typing derivations do not change if we permute the typing context. Therefore $\Gamma, y : T_1, x : S \vdash t_2 : T_2$ is equivalent to $\Gamma, x : S, y : T_1 \vdash t_2 : T_2$, swapping the ordering of $x$ and $y$. By T-Abs with the premise $\Gamma, x : S, y : T_1 \vdash t_2 : T_2$ we know $\Gamma, x : S \vdash \lambda y : T_1.\, t_2 : T_1 \Rightarrow T_2$. Therefore we have $\Gamma, x : S \vdash t : T$.

By the definition of depth we know $\texttt{depth}(\Gamma \vdash \lambda y : T_1.\, t_2 : T_1 \Rightarrow T_2) = 1 + \texttt{depth}(\Gamma, y : T_1 \vdash t_2 : T_2)$. By our induction hypothesis on $t_2$ we know $\texttt{depth}(\Gamma, y : T_1 \vdash t_2 : T_2) = \texttt{depth}(\Gamma, y : T_1, x : S \vdash t_2 : T_2)$ and by our permutation lemma we know $\texttt{depth}(\Gamma, y : T_1, x : S \vdash t_2 : T_2) = \texttt{depth}(\Gamma, x : S, y : T_1 \vdash t_2 : T_2)$. Therefore by the defintion of depth we know $\texttt{depth}(\Gamma \vdash \lambda y : T_1.\, t_2 : T_1 \Rightarrow T_2) = \texttt{depth}(\Gamma, x : S \vdash \lambda y : T_1.\, t_2 : T_1 \Rightarrow T_2)$, the two derivations have the same depth.
Therefore our property holds for the T-Abs case.

**Case**: T-App
If T-App was the final typing derivation we have $t = t_1\ t_2$ and $T = T_2$, with the premises $\Gamma \vdash t_1 : T_1 \Rightarrow T_2$ and $\Gamma \vdash t_2 : T_1$. Since we know $x \notin \mathit{dom}(\Gamma)$, by our induction hypothesis on $t_1$ we know $\Gamma, x : S \vdash t_1 : T_1 \Rightarrow T_2$. Similarly by our induction hypothesis on $t_2$ we know $\Gamma, x : S \vdash t_2 : T_1$. By the premises $\Gamma, x : S \vdash t_1 : T_1 \Rightarrow T_2$ and $\Gamma, x : S \vdash t_2 : T_1$, by T-App we then derive $\Gamma, x : S \vdash t_1\ t_2 : T_2$. Therefore we have $\Gamma, x : S \vdash t : T$.
By the definition of depth we know $\texttt{depth}(\Gamma \vdash t_1\ t_2 : T_2) = 1 + \max(\texttt{depth}(\Gamma \vdash t_1 : T_1 \Rightarrow T_2),\ \texttt{depth}(\Gamma \vdash t_2 : T_1))$. Since we know $x \notin \mathit{dom}(\Gamma)$, by our induction hypothesis on $t_1$ we know $\texttt{depth}(\Gamma \vdash t_1 : T_1 \Rightarrow T_2) = \texttt{depth}(\Gamma, x : S \vdash t_1 : T_1 \Rightarrow T_2)$. Similarly by our induction hypothesis on $t_2$ we know $\texttt{depth}(\Gamma \vdash t_2 : T_1) = \texttt{depth}(\Gamma, x : S \vdash t_2 : T_1)$. Thus their maximum will be the same, $\max(\texttt{depth}(\Gamma \vdash t_1 : T_1 \Rightarrow T_2),\ \texttt{depth}(\Gamma \vdash t_2 : T_1)) = \max(\texttt{depth}(\Gamma, x : S \vdash t_1 : T_1 \Rightarrow T_2),\ \texttt{depth}(\Gamma, x : S \vdash t_2 : T_1))$. Then, by the definition of depth we know both derivations have the same depth with $\texttt{depth}(\Gamma \vdash t_1\ t_2 : T_2) = \texttt{depth}(\Gamma, x : S \vdash t_1\ t_2 : T_2)$.
Therefore our property holds for T-App.
We have shown that for all possible typing derivations our property holds. Therefore our induction step holds.
Thus we've shown that if $\Gamma \vdash t : T$ and $x \notin \mathit{dom}(\Gamma)$, then $\Gamma, x : S \vdash t : T$ and $\texttt{depth}(\Gamma \vdash t : T) = \texttt{depth}(\Gamma, x : S \vdash t : T)$.

$\square$