COMPSCI 3MI3

Assignment 6

1 Solution Set

1.1 Q1

We assume the syntax and semantics for TAE as defined in the slides for topic 4 and 7. We define the syntax for a *logical and* operator, and *arithmetic addition* operator with the following syntax in EBNF:

```
t ::= ...
    | and t t
    | plus t t

v ::= ...
nv ::= ...
T ::= ...
```

Where `t` defines terms, `v` values, `nv` numerical values and `T` the set of types for TAE as given in the slides. We add our logical and operator **`and`**, as well as our arithmetic addition operator `plus` to the definition of `t`.
We use the same set of types `T` as defined in slide set 7 and assume any previously defined typing rules hold.

(a) We define the following typing rules for our logical and operator **`and`**:

$$\frac{t_1 : \texttt{Bool} \qquad t_2 : \texttt{Bool}}{\texttt{and}\ t_1\ t_2 : \texttt{Bool}} \tag{T-And}$$

(b) We define the following typing rules for our arithmetic addition operator `plus`:

$$\frac{t_1 : \texttt{Nat} \qquad t_2 : \texttt{Nat}}{\texttt{plus}\ t_1\ t_2 : \texttt{Nat}} \tag{T-Plus}$$

1.2 Q2

We will prove the property that every subterm of a well-typed term is also well-typed, for all terms in TAE. Formally, if t_1 is a subterm of term $t : \alpha$, then we have $t_1 : \beta$ with $\alpha, \beta \in \mathtt{T}$, $\mathtt{T}$ the set of types in TAE. We will prove this property holds by induction on the typing derivation $t : \alpha$. We consider the typing derivations defined in the slides for topic 7.

Base case. The base case denotes typing derivations on terms for which there does not exists any sub-typing derivations. The possible typing derivations are of the following:

Case: T-True, T-False, T-Zero
If T-True is the final typing derivation we know $t = \mathtt{true}$ and $\mathtt{T} = \mathtt{Bool}$. Since `true` does not have any subterms, the left hand side of our implication is false, thus our property is vacuously true. We can apply the same argument for T-False and T-Zero.
Thus, our base case holds.

Induction step. We assume that for all typing sub-derivations the above property holds. We will prove that for the final typing derivation our property holds. The typing derivation must be one of the following:

Case: T-If
If T-If was the last typing derivation we know $t = \mathtt{if}\ t_1\ \mathtt{then}\ t_2\ \mathtt{else}\ t_3 : \mathtt{T}$, and have the premises that $t_1 : \mathtt{Bool}$, $t_2 : \mathtt{T}$ and $t_3 : \mathtt{T}$. By our induction hypothesis we know that if t_1 is well typed then all of its subterms are well-typed. Since $t_1 : \mathtt{Bool}$ we have that all subterms of t_1 are well-typed. We can apply the same argument for t_2 and t_3. Thus, all subterms of t_1, t_2 and t_3 are well-typed. Therefore for our term $t : \mathtt{T}$ we've shown t_1, t_2 and t_3 are well-typed by our premise, and all subterms of t_1, t_2 and t_3 are well-typed, thus all subterms of t must be well-typed. Therefore our property holds for the T-If case.

Case: T-Pred
If T-Pred was the last typing derivation we know $t = \mathtt{pred}\ t_1 : \mathtt{Nat}$, and have the premise that $t_1 : \mathtt{Nat}$. By our induction hypothesis and since we know that t_1 is well-typed because of our premise, we know that all subterms of t_1 are also well-typed. Thus, for our term $t : \mathtt{Nat}$ we've show that t_1 is well-typed and all subterms of t_1 are well-typed, therefore all all subterms of t are well typed. Therefore our property holds for the T-Pred case.

Case: T-Succ
We can prove the T-Succ case similar to T-Pred. If T-Succ is the last typing derivation we know $t = \mathtt{succ}\ t_1 : \mathtt{Nat}$, and we have the premise that $t_1 : \mathtt{Nat}$. By supplying our induction hypothesis with our premise that t_1 is well-typed we have that all subterms of t_1 are also well-typed. Thus, for our term $t : \mathtt{Nat}$ we've show that t_1 is well-typed and all subterms of t_1 are well-typed, therefore all all subterms of t are well typed. Therefore our property holds for the T-Succ case.

Case: T-IsZero
We can prove the T-IsZero case similar to T-Pred. If T-IsZero was the last typing derivation we know $t = \texttt{iszero}\ t_1 : \texttt{Bool}$, and have the premise that $t_1 : \texttt{Nat}$. By our induction hypothesis and since we know that t_1 is well-typed because of our premise, we know that all subterms of t_1 are also well-typed. Thus, for our term $t : \texttt{Bool}$ we've show that t_1 is well-typed and all subterms of t_1 are well-typed, therefore all all subterms of t are well typed. Therefore our property holds for the T-IsZero case.

We have shown that for all possible typing derivations over t over property holds. Therefore our induction step holds.

Thus, we've shown that every subterm of a well-typed term is also well-typed, for all terms in TAE. □

1.3 Q3

We will prove the property of preservation holds for all terms in TAE. Formally, for terms t, t' and type $\texttt{T}$ in TAE the following holds:

$$t : \texttt{T} \wedge t \to t' \implies t' : \texttt{T}$$

We will prove the property of preservation holds by induction on the derivation $t \to t'$.

Base case. The base case denotes the final derivation where $t \to t'$ will yield a value instead of a term evaluatable by a sub-derivation. The possible derivations that yield a value are of the following:

Case: E-IfTrue, E-IfFalse
If E-IfTrue was the last derivation and yields a value, we know $t = \texttt{if}\,\texttt{true}\,\texttt{then}\,t_2\,\texttt{else}\,t_3$ and $t' = t_2$ with t_2 being some value. The only typing rule we can apply to t is T-If, thus we know $t : \texttt{T}$, $t_2 : \texttt{T}$ and $t_3 : \texttt{T}$ with $\texttt{T}$ a type in TAE. Since $t : \texttt{T}$ and $t_2 = t' : \texttt{T}$ for the derivation $t \to t'$ of E-IfTrue our, property holds. We can prove E-IfFalse using the same argument, swapping $t' = t_3$. Thus, our property of preservation holds for E-IfTrue and E-IfFalse.

Case: E-PredZero
If E-PredZero was the last derivation we immediately know $t = \texttt{pred}\,0$ and $t' = 0$, with t' a value. By T-Pred we immediately know $t : \texttt{Nat}$, and by T-Zero we know $t' : \texttt{Nat}$. Thus, our property of preservation holds for the E-PredZero case.

Case: E-PredSucc
If E-PredSucc was the last derivation we know $t = \texttt{pred}\,(\texttt{succ}\,t_2)$ and $t' = t_2$, where t_2 is a numerical value. The only typing rule we can apply to t is T-Pred, thus we know $t : \texttt{Nat}$

and $\texttt{succ}\, t_2 : \texttt{Nat}$. By the inversion lemma we know $\texttt{succ}\, t_2 : R \implies R = \texttt{Nat} \wedge t_2 : \texttt{Nat}$. Therefore we know $t : \texttt{Nat}$ and $t_2 = t : \texttt{Nat}$ for the derivation $t \to t'$ of E-PredSucc. Thus, our property holds.

Case: E-IsZeroZero
If E-IsZeroZero was the last derivation we immediately know $t = \texttt{iszero}\, 0$ and $t' = \texttt{true}$, with t' a value. We can only apply T-IsZero to t, thus we know $t : \texttt{Bool}$, and by T-True we know $t' : \texttt{Bool}$ as well. Thus, our property of preservation holds for the E-IsZeroZero case.

Case: E-IsZeroSucc
If E-IsZeroSucc was the last derivation we know $t = \texttt{iszero}\,(\texttt{succ}\, t_2)$ with t_2 a numerical value, and we know $t' = \texttt{false}$. The only typing rule we can apply to t is T-IsZero, thus we know $t : \texttt{Bool}$, and by T-False we know $t' : \texttt{Bool}$ as well. Therefore we know $t : \texttt{Bool}$ and $t' : \texttt{Bool}$ for the derivation $t \to t'$ of E-IsZeroSucc. Thus, our property holds.

Thus, our base case holds.

Induction step. We assume that for all sub-derivations of $t \to t'$ the property of preservation holds. We will prove that for all possible derivations $t \to t'$ the property holds. The derivation $t \to t'$ must be one of the following:

Case: E-IfTrue, E-IfFalse
We can prove the cases of E-IfTrue and E-IfFalse in exactly the same manner as for the base case, this time assuming that the derivation $t \to t'$ for E-IfTrue and E-IfFalse do not yield a value, thus t' is not a value. Thus, our property holds for the case E-IfTrue and E-IfFalse

Case: E-If
If E-IF was the last derivation, we know $t = \texttt{if}\, t_1\, \texttt{then}\, t_2\, \texttt{else}\, t_3$, $t' = \texttt{if}\, t_1'\, \texttt{then}\, t_2\, \texttt{else}\, t_3$ and we have the premise that $t_1 \to t_1'$. The only typing rule we can apply to t is T-If. From T-If we know $t_1 : \texttt{Bool}$, $t_2 : \texttt{T}$, $t_3 : \texttt{T}$ and $t : \texttt{T}$ with $\texttt{T}$ a type in TAE. By our induction hypothesis we have $t_1 : \texttt{T} \wedge t_1 \to t_1' \implies t_1' : \texttt{T}$. Since we have the premises $t_1 : \texttt{Bool}$ and $t_1 \to t_1'$ we can conclude $t_1' : \texttt{Bool}$. Since $t' = \texttt{if}\, t_1'\, \texttt{then}\, t_2\, \texttt{else}\, t_3$, with $t_1' : \texttt{Bool}$, $t_2 : \texttt{T}$ and $t_3 : \texttt{T}$, by T-If we can show $t' : \texttt{T}$. Thus, we've shown the property of preservation holds if the last derivation is E-If.

Case: E-Succ
If E-Succ was the last derivation, we know $t = \texttt{succ}\, t_1$, $t' = \texttt{succ}\, t_1'$ and have the premise that $t_1 \to t_1'$. Since $t = \texttt{succ}\, t_1$, we can only apply the typing derivation T-Succ. From T-Succ we know $t : \texttt{Nat}$ and have the premise $t_1 : \texttt{Nat}$. By our induction hypothesis we have $t_1 : \texttt{T} \wedge t_1 \to t_1' \implies t_1' : \texttt{T}$. Since we have the premises $t_1 : \texttt{Nat}$ and $t_1 \to t_1'$ we can conclude $t_1' : \texttt{Nat}$. We now know $t_1 : \texttt{Nat}$ and $t_1' = \texttt{succ}\, t_1'$, then by T-Succ we can

show $t' : \texttt{Nat}$. Thus, we've shown the property of preservation holds if the last derivation is E-Succ.

Case: E-Pred
We proceed in the same manner as with E-Succ. If If E-Pred was the last derivation, we know $t = \texttt{pred}\, t_1$, $t' = \texttt{pred}\, t_1'$ and have the premise that $t_1 \to t_1'$. Since $t = \texttt{succ}\, t_1$, we can only apply the typing derivation T-Pred. From T-Pred we know $t : \texttt{Nat}$ and have the premise $t_1 : \texttt{Nat}$. By our induction hypothesis we have $t_1 : \texttt{T} \wedge t_1 \to t_1' \implies t_1' : \texttt{T}$. Since we have the premises $t_1 : \texttt{Nat}$ and $t_1 \to t_1'$ we can conclude $t_1' : \texttt{Nat}$. We now know $t_1 : \texttt{Nat}$ and $t_1' = \texttt{pred}\, t_1'$, then by T-Pred we can show $t' : \texttt{Nat}$. Thus, we've shown the property of preservation holds if the last derivation is E-Pred.

Case: E-IsZero
We can prove the E-IsZero case in similar fashion to the E-Succ case. If E-IsZero was the last derivation, we know $t = \texttt{iszero}\, t_1$, $t' = \texttt{iszero}\, t_1'$ and have the premise that $t_1 \to t_1'$. Since $t = \texttt{iszero}\, t_1$, we can only apply the typing derivation T-IsZero. From T-IsZero we know $t : \texttt{Bool}$ and have the premise $t_1 : \texttt{Nat}$. By our induction hypothesis we have $t_1 : \texttt{T} \wedge t_1 \to t_1' \implies t_1' : \texttt{T}$. Since we have the premises $t_1 : \texttt{Nat}$ and $t_1 \to t_1'$ we can conclude $t_1' : \texttt{Nat}$. We now know $t_1 : \texttt{Nat}$ and $t_1' = \texttt{pred}\, t_1'$, then by T-IsZero we can show $t' : \texttt{Bool}$. Thus, we've shown the property of preservation holds if the last derivation is E-IsZero.

We've shown our property holds over all cases. Therefore our induction step holds.
We have shown that for all possible derivations $t \to t'$ our property of preservation holds, thus we've shown the property of preservation holds for all terms in TAE. $\square$