

Assignment 1

October 29, 2021

1 Solution Set

1.1 Q1

Define the relation R' as follows:

$$R' = R \cup \{(s, s) \mid s \in S\}$$

To show R' is the *reflexive closure* of R we must show (1) $R \subseteq R'$, (2) R' is reflexive and (3) R' is the smallest reflexive relation such that $R \subseteq R'$.

(1) Prove $R \subseteq R'$

$$\begin{aligned} R' &= R \cup \{(s, s) \mid s \in S\} && \langle \text{def. of } R' \rangle \\ &\supseteq R && \langle A \subseteq A \cup B \rangle \end{aligned}$$

(2) Show R' is reflexive.

Since the relation R' contains the set of all reflexive pairs in S $\{(s, s) \mid s \in S\}$ we know that R' is reflexive by definition.

(3) Show R' is the smallest reflexive relation possible such that $R \subseteq R'$.

We must show R' is the smallest possible set that satisfies the above two properties. Suppose K is a reflexive relation on S extending R . By reflexivity of K , we know $\{(s, s) \in S\} \subseteq K$. Since we also know K extends R then $R \subseteq K$, it follows that $R \cup \{(s, s) \in S\} \subseteq K$, which is $R' \subseteq K$. Thus, R' must be the smallest possible reflexive relation extending R .

We have shown that R' is the reflexive closure of R .

1.2 Q2

We suppose that R is some binary relation on S and P is a predicate on S that preserves R . Let P^* be the *reflexive transitive* closure of R . We will show that P also preserves P^* .

If P preserves P^* then for any $s, t \in S$ the following should hold:

$$P(s) \wedge s P^* t \rightarrow P(t)$$

This is equivalent to saying that for all pairs $(s, t) \in P^*$ if $P(s)$ holds then $P(t)$ holds. We can then define P^* as:

$$P^* = R \cup K \cup T$$

where K contains all reflexive pairs (s, s) in S and T contains all pairs that together make P^* transitive. Thus, to prove P preserves P^* we show that P preserves R, K, T such that for any pair $(s, t) \in R \cup K \cup T$ we have $P(s) \rightarrow P(t)$.

- (1) (P preserves R) This is trivial since we know that P preserves R as given in the question, thus for any pair $(s, t) \in R$ we know $P(s) \rightarrow P(t)$.
- (2) (P preserves K) Since all pairs in K are reflexive they all share the form (s, s) where $s \in S$. Thus for any $(s, s) \in K$ we have $P(s) \rightarrow P(s)$, which is true by reflexivity of implication. Therefore, P preserves K .
- (3) (P preserves T) Since T contains all pairs that together make P^* transitive we know that for any pair $(s, t) \in T$ there must exist two pairs $(s, r), (r, t) \in R \cup T$. We can prove by induction that for any pair $(s, t) \in T$ we have $P(s) \rightarrow P(t)$. For any pair $(s, t) \in T$ if $(s, r), (r, t) \in R$ then the pair can be obtained in “one step of transitivity” and from (1) we know such that $P(s) \rightarrow P(r)$ and $P(r) \rightarrow P(t)$. By transitivity of implication we have that $P(s) \rightarrow P(t)$. Similarly, all other pairs $(s, t) \in T$ follow this principle since they are formed in “one step of transitivity” from two pairs (s, r) and (r, t) that already follow the principle. Thus P preserves T .

Therefore, we have shown that P preserves P^* .

1.3 Q3

Define the relation R^+ as given in the question:

$$R_0 = R$$

$$R_{i+1} = R_i \cup \{(s, u) \mid \text{for some } t, (s, t) \in R_i \text{ and } (t, u) \in R_i\}$$

$$R^+ = \bigcup_i R_i$$

To show R^+ is the *transitive closure* of R we must show (1) $R \subseteq R^+$, (2) R^+ is transitive, (3) R^+ is the smallest transitive relation such that $R \subseteq R^+$.

(1) Prove $R \subseteq R^+$

$$\begin{aligned} & R^+ \\ = & \bigcup_i R_i && \langle \text{def. of } R^+ \rangle \\ \supseteq & R_0 && \langle \text{def. of } \bigcup, A \subseteq A \cup B \rangle \\ = & R && \langle R_0 = R \rangle \end{aligned}$$

(2) Show R^+ is transitive.

Suppose some $a, b, c \in S$ with $a R_i b$ and $b R_i c$ for some i . By definition of R^+ the pair (a, c) will appear in R_{i+1} such that $a R_{i+1} c$. Since R^+ is the union of all R_i the pairs $(a, b), (b, c), (a, c) \in R^+$, thus R^+ is transitive by definition.

(3) Show R^+ is the smallest transitive relation possible such that $R \subseteq R^+$.

We must show R^+ is the smallest possible set that satisfies the above two properties. Suppose T is a transitive relation on S that extends R . We know T extends R so $R \subseteq T$. Additionally, since T is transitive for any $a, b, c \in S$ such that $a R b$ and $b R c$ we know $a T c$, which implies $R_1 \subseteq T$. We can then inductively prove that for any $a, b, c \in S$ with $a R_i b$ and $b R_i c$ we will have $a T c$ thus $R_{i+1} \subseteq T$, or simply $R_i \subseteq T$ for all i . It then follows that $\bigcup_i R_i \subseteq T$ which is $R^+ \subseteq T$. Therefore, R^+ must be the smallest transitive relation extending R .

1.4 Q4

Define the function $\text{fib} : \mathbb{N} \rightarrow \mathbb{N}$ such that $\text{fib}(n)$ returns the n^{th} number of the Fibonacci sequence (0-indexed). fib is defined by:

$$\text{fib}(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ \text{fib}(n-1) + \text{fib}(n-2) & \text{if } n \geq 2 \end{cases}$$

We will prove that each element in the Fibonacci sequence above the 2nd is greater than the preceding number. Formally, we prove that $P(n) \equiv \text{fib}(n) > \text{fib}(n-1)$ for all $n \in \mathbb{N}$ with $n > 2$ by ordinary induction on n .

Base case: $n = 3$. Show $P(3)$ holds.

$$\begin{array}{ll}
 \text{fib}(3) & \langle \text{RHS} \rangle \\
 = \text{fib}(3-1) + \text{fib}(3-2) & \langle \text{def. of fib.3} \rangle \\
 = \text{fib}(2) + \text{fib}(1) & \langle \text{arithmetic} \rangle \\
 = \text{fib}(2) + 1 & \langle \text{def. of fib.2} \rangle \\
 > \text{fib}(2) & \langle b > 0 \rightarrow a + b > a; \text{LHS} \rangle
 \end{array}$$

So $P(3)$ holds.

Induction step: $n > 3$. Assume $P(n)$ holds. We must show $P(n+1)$ holds.

$$\begin{array}{ll}
 \text{fib}(n+1) & \langle \text{RHS} \rangle \\
 = \text{fib}(n+1-1) + \text{fib}(n+1-2) & \langle \text{def. of fib.3} \rangle \\
 = \text{fib}(n) + \text{fib}(n-1) & \langle \text{arithmetic} \rangle \\
 > \text{fib}(n) & \langle b > 0 \rightarrow a + b > a \rangle \\
 = \text{fib}(n+1-1) & \langle \text{arithmetic; LHS} \rangle
 \end{array}$$

Thus, $P(n+1)$ holds.

Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by ordinary induction. We have shown that each element in the Fibonacci sequence above the 2nd is greater than the preceding number.

1.5 Q5

Using the same definition of fib from Q4 we will prove that each element in the Fibonacci sequence above the 2nd is greater than the preceding number. Formally, we prove that $P(n) \equiv \text{fib}(n) > \text{fib}(n-1)$ for all $n \in \mathbb{N}$ with $n > 2$ by complete induction on n .

Base case: $n = 3$. Show $P(3)$ holds.

$$\begin{array}{ll}
 \text{fib}(3) & \langle \text{RHS} \rangle \\
 = \text{fib}(3-1) + \text{fib}(3-2) & \langle \text{def. of fib.3} \rangle \\
 = \text{fib}(2) + \text{fib}(1) & \langle \text{arithmetic} \rangle \\
 = \text{fib}(2) + 1 & \langle \text{def. of fib.2} \rangle \\
 > \text{fib}(2) & \langle b > 0 \rightarrow a + b > a; \text{LHS} \rangle
 \end{array}$$

So $P(3)$ holds.

Base case: $n = 4$. Show $P(4)$ holds.

$$\begin{array}{ll}
\text{fib}(4) & \langle \text{RHS} \rangle \\
= \text{fib}(4 - 1) + \text{fib}(4 - 2) & \langle \text{def. of fib.3} \rangle \\
= \text{fib}(3) + \text{fib}(2) & \langle \text{arithmetic} \rangle \\
= \text{fib}(3) + \text{fib}(1) + \text{fib}(0) & \langle \text{def. of fib; arithmetic} \rangle \\
= \text{fib}(3) + 1 & \langle \text{def. of fib; arithmetic} \rangle \\
> \text{fib}(3) & \langle b > 0 \rightarrow a + b > a; \text{LHS} \rangle
\end{array}$$

So $P(4)$ holds.

Induction step: $n > 4$. Assume $P(m)$ holds for all $2 < m < n$. We must show $P(n)$ holds.

$$\begin{array}{ll}
\text{fib}(n) & \langle \text{RHS} \rangle \\
= \text{fib}(n - 1) + \text{fib}(n - 2) & \langle \text{def. of fib} \rangle \\
> \text{fib}(n - 2) + \text{fib}(n - 2) & \langle \text{Ind. hyp. } P(n - 1); \text{Monotonicity of } + \rangle \\
> \text{fib}(n - 2) + \text{fib}(n - 3) & \langle \text{Ind. hyp. } P(n - 2); \text{Monotonicity of } + \rangle \\
= \text{fib}(n - 1) & \langle \text{def. of fib; LHS} \rangle
\end{array}$$

Thus $P(n)$ holds for $n > 4$.

Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by complete induction. We have shown that each element in the Fibonacci sequence above the 2nd is greater than the preceeding number.

1.6 Q6

Define a binary tree data structure **BinTree** representative of binary search trees over an arbitrary data type **a** for which the properties of binary search trees are respected. **BinTree** is defined by pattern matching with the following constructors:

Leaf : $\mathbf{a} \rightarrow \mathbf{BinTree}$

Fork : $\mathbf{BinTree} \times \mathbf{a} \times \mathbf{BinTree} \rightarrow \mathbf{BinTree}$

We define $P(t) \equiv$ search operations over the binary search tree t modeled as a **BinTree** only need to search one branch of t . We will prove $P(t)$ for all $t \in \mathbf{BinTree}$ by structural induction on t .

Base case: $t = \text{Leaf}(n)$. Prove $P(t)$ holds.

It is trivially obvious that the search algorithm over t can only check one element, the leaf of the tree. Thus, there is only one branch to search: the leaf. So $P(t)$ holds.

Inductive step: $t = \text{Fork}(b_1, n, b_2)$. Assume $P(b_1)$ and $P(b_2)$ hold. Prove $P(t)$ holds.

According to the properties of binary search trees we know that t follows the binary search property: that all leaves containing values less than or equal to n are stored in b_1 , and any leaves with values greater than or equal to n are stored in b_2 . Thus, the search operation algorithm can compare the search value x with n and choose which sub-tree to search in.

We then have 3 cases.

- If $x = n$ the search operation has only had to check one value or one branch of possible values so $P(t)$ holds.
- If $x < n$ the search operation can choose the left branch b_1 to continue its search. Since $P(b_1)$ holds we know the search operation will only search one branch in the left sub-tree as well. Together, the search operation will only have to search one branch overall of t . Therefore, $P(t)$ holds for $x < n$.
- Similarly, we can follow the same logic for $x > n$ with the right branch b_2 since $P(b_2)$ holds. Thus, $P(t)$ holds for all cases.

We have shown that $P(t)$ holds for all binary trees modeled by **BinTree**, thus search operations need only to search one branch of binary search trees.