COMPSCI 3MI3

Assignment 1

1 Solution Set

1.1 Q1

Define the relation R' as follows:

$$R' = R \cup \{(s,s) \mid s \in S\}$$

To show R' is the *reflexive closure* of R we must show (1) $R \subseteq R'$, (2) R' is reflexive and (3) R' is the smallest reflexive relation such that $R \subseteq R'$.

(1) Prove $R \subseteq R'$

$$\begin{aligned} & R' \\ = \; & R \cup \{(s,s) \mid s \in S\} && \langle \text{def. of } R' \rangle \\ \supseteq \; & R && \langle A \subseteq A \cup B \; \rangle \end{aligned}$$

(2) Show R' is reflexive.
Since the relation R' contains the set of all reflexive pairs in S $\{(s,s) \mid s \in S\}$ we know that R' is reflexive by definition.

(3) Show R' is the smallest reflexive relation possible such that $R \subseteq R'$.
We must show R' is the smallest possible set that satisfies the above two properties. Suppose K is a reflexive relation on S extending R. By reflexivity of K, we know $\{(s,s) \in S\} \subseteq K$. Since we also know K extends R then $R \subseteq K$, it follows that $R \cup \{(s,s) \in S\} \subseteq K$, which is $R' \subseteq K$. Thus, R' must be the smallest possible reflexive relation extending R.

We have shown that R' is the reflexive closure of R.

1.2 Q2

We suppose that R is some binary relation on S and P is a predicate on S that preserves R. Let P^* be the *relexive transitive* closure of R. We will show that P also preserves P^*.

If P preserves P^* then for any $s, t \in S$ the following should hold:

$$P(s) \land s\ P^*\ t \rightarrow P(t)$$

This is equivalent to saying that for all pairs $(s,t) \in P^*$ if $P(s)$ holds then $P(t)$ holds. We can then define P^* as:

$$P^* = R \cup K \cup T$$

where K contains all reflexive pairs (s,s) in S and T contains all pairs that together make P^* transitive. Thus, to prove P preserves P^* we show that P preserves R, K, T such that for any pair $(s,t) \in R \cup K \cup T$ we have $P(s) \rightarrow P(t)$.

(1) (P preserves R) This is trivial since we know that P preserves R as given in the question, thus for any pair $(s,t) \in R$ we know $P(s) \rightarrow P(t)$.

(2) (P preserves K) Since all pairs in K are reflexive they all share the form (s,s) where $s \in S$. Thus for any $(s,s) \in K$ we have $P(s) \rightarrow P(s)$, which is true by reflexivity of implication. Therefore, P preserves K.

(3) (P preserves T) Since T contains all pairs that together make P^* transitive we know that for any pair $(s,t) \in T$ there must exist two pairs (s,r),$(r,t) \in R \cup T$. We can prove by induction that for any pair $(s,t) \in T$ we have $P(s) \rightarrow P(t)$. For any pair $(s,t) \in T$ if $(s,r), (r,t) \in R$ then the pair can be obtained in "one step of transitivity" and from (1) we know such that $P(s) \rightarrow P(r)$ and $P(r) \rightarrow P(t)$. By transitivity of implication we have that $P(s) \rightarrow P(t)$. Similarly, all other pairs $(s,t) \in T$ follow this principle since they are formed in "one step of transitvity" from two pairs (s,r) and (r,t) that already follow the principle. Thus P preserves T.

Therefore, we have shown that P preserves P^*.

1.3 Q3

Define the relation R^+ as given in the question:

$$R_0 = R$$

$$R_{i+1} = R_i \cup \{(s,u) \mid \text{for some}\, t, (s,t) \in R_i \text{ and } (t,u) \in R_i\}$$

$$R^+ = \bigcup_i R_i$$

To show R^+ is the *transitive closure* of R we must show (1) $R \subseteq R^+$, (2) R^+ is transitive, (3) R^+ is the smallest transitive relation such that $R \subseteq R^+$.

(1) Prove $R \subseteq R^+$

$$
\begin{aligned}
& R+ \\
&= \bigcup_i R_i && \langle \text{def. of } R^+ \rangle \\
&\supseteq R_0 && \langle \text{def. of } \bigcup,\ A \subseteq A \cup B \rangle \\
&= R && \langle R_0 = R \rangle
\end{aligned}
$$

(2) Show R^+ is transitive.
Suppose some $a, b, c \in S$ with $a\ R_i\ b$ and $b\ R_i\ c$ for some i. By definition of R^+ the pair (a, c) will appear in R_{i+1} such that $a\ R_{i+1}\ c$. Since R^+ is the union of all R_i the pairs (a, b),(b, c),$(a, c) \in R^+$, thus R^+ is transitive by definition.

(3) Show R^+ is the smallest transitive relation possible such that $R \subseteq R^+$.
We must show R^+ is the smallest possible set that satisfies the above two properties. Suppose T is a transitive relation on S that extends R. We know T extends R so $R \subseteq T$. Additionally, since T is transitive for any $a, b, c \in S$ such that $a\ R\ b$ and $b\ R\ c$ we know $a\ T\ c$, which implies $R_1 \subseteq T$. We can then inductively prove that for any $a, b, c \in S$ with $a\ R_i\ b$ and $b\ R_i\ c$ we will have $a\ T\ c$ thus $R_{i+1} \subseteq T$, or simply $R_i \subseteq T$ for all i. It then follows that $\bigcup_i R_i \subseteq T$ which is $R^+ \subseteq T$. Therefore, R^+ must be the smallest transitive relation extending R.

1.4 Q4

Define the function $\mathsf{fib} : \mathbb{N} \to \mathbb{N}$ such that $\mathsf{fib}(n)$ returns the n^{th} number of the Fibbonaci sequence (0-indexed). fib is defined by:

$$
\mathsf{fib}(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ \mathsf{fib}(n-1) + \mathsf{fib}(n-2) & \text{if } n \geq 2 \end{cases}
$$

We will prove that each element in the Fibbonaci sequence above the 2^{nd} is greater than the preceeding number. Formally, we prove that $P(n) \equiv \mathsf{fib}(n) > \mathsf{fib}(n-1)$ for all $n \in \mathbb{N}$ with $n > 2$ by ordinary induction on n.

Base case: $n = 3$. Show $P(3)$ holds.

$$\begin{aligned}
&\mathsf{fib}(3) && \langle\text{RHS}\rangle\\
&= \mathsf{fib}(3-1) + \mathsf{fib}(3-2) && \langle\text{def. of } \mathsf{fib}.3\rangle\\
&= \mathsf{fib}(2) + \mathsf{fib}(1) && \langle\text{arithmetic}\rangle\\
&= \mathsf{fib}(2) + 1 && \langle\text{def. of } \mathsf{fib}.2\rangle\\
&> \mathsf{fib}(2) && \langle b > 0 \to a + b > a;\ \text{LHS}\rangle
\end{aligned}$$

So $P(3)$ holds.

Induction step: $n > 3$. Assume $P(n)$ holds. We must show $P(n+1)$ holds.

$$\begin{aligned}
&\mathsf{fib}(n+1) && \langle\text{RHS}\rangle\\
&= \mathsf{fib}(n+1-1) + \mathsf{fib}(n+1-2) && \langle\text{def. of } \mathsf{fib}.3\rangle\\
&= \mathsf{fib}(n) + \mathsf{fib}(n-1) && \langle\text{arithmetic}\rangle\\
&> \mathsf{fib}(n) && \langle b > 0 \to a + b > a\rangle\\
&= \mathsf{fib}(n+1-1) && \langle\text{arithmetic; LHS}\rangle
\end{aligned}$$

Thus, $P(n+1)$ holds.

Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by ordinary induction. We have shown that each element in the Fibbonaci sequence above the 2^{nd} is greater than the preceeding number.

1.5 Q5

Using the same definition of fib from Q4 we will prove that each element in the Fibbonaci sequence above the 2^{nd} is greater than the preceeding number. Formally, we prove that $P(n) \equiv \mathsf{fib}(n) > \mathsf{fib}(n-1)$ for all $n \in \mathbb{N}$ with $n > 2$ by complete induction on n.
Base case: $n = 3$. Show $P(3)$ holds.

$$\begin{aligned}
&\mathsf{fib}(3) && \langle\text{RHS}\rangle\\
&= \mathsf{fib}(3-1) + \mathsf{fib}(3-2) && \langle\text{def. of } \mathsf{fib}.3\rangle\\
&= \mathsf{fib}(2) + \mathsf{fib}(1) && \langle\text{arithmetic}\rangle\\
&= \mathsf{fib}(2) + 1 && \langle\text{def. of } \mathsf{fib}.2\rangle\\
&> \mathsf{fib}(2) && \langle b > 0 \to a + b > a;\ \text{LHS}\rangle
\end{aligned}$$

So $P(3)$ holds.

Base case: $n = 4$. Show $P(4)$ holds.

$$\begin{aligned}
&\mathsf{fib}(4) && \langle\text{RHS}\rangle \\
&= \mathsf{fib}(4-1) + \mathsf{fib}(4-2) && \langle\text{def. of } \mathsf{fib}.3\rangle \\
&= \mathsf{fib}(3) + \mathsf{fib}(2) && \langle\text{arithmetic}\rangle \\
&= \mathsf{fib}(3) + \mathsf{fib}(1) + \mathsf{fib}(0) && \langle\text{def. of } \mathsf{fib}\text{; arithmetic}\rangle \\
&= \mathsf{fib}(3) + 1 && \langle\text{def. of } \mathsf{fib}\text{; arithmetic}\rangle \\
&> \mathsf{fib}(3) && \langle b > 0 \to a + b > a\text{; LHS}\rangle
\end{aligned}$$

So $P(4)$ holds.

Induction step: $n > 4$. Assume $P(m)$ holds for all $2 < m < n$. We must show $P(n)$ holds.

$$\begin{aligned}
&\mathsf{fib}(n) && \langle\text{RHS}\rangle \\
&= \mathsf{fib}(n-1) + \mathsf{fib}(n-2) && \langle\text{def. of } \mathsf{fib}\rangle \\
&> \mathsf{fib}(n-2) + \mathsf{fib}(n-2) && \langle\text{Ind. hyp. } P(n-1)\text{; Monotonicity of } +\rangle \\
&> \mathsf{fib}(n-2) + \mathsf{fib}(n-3) && \langle\text{Ind. hyp. } P(n-2)\text{; Monotonicity of } +\rangle \\
&= \mathsf{fib}(n-1) && \langle\text{def. of } \mathsf{fib}\text{; LHS}\rangle
\end{aligned}$$

Thus $P(n)$ holds for $n > 4$.

Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by complete induction. We have shown that each element in the Fibbonaci sequence above the 2^{nd} is greater than the preceeding number.

1.6 Q6

Define a binary tree data structure $\mathsf{BinTree}$ representative of binary search trees over an arbitrary data type a for which the properties of binary search trees are respected. $\mathsf{BinTree}$ is defined by pattern matching with the following constructors:

$$\mathsf{Leaf} : \mathsf{a} \to \mathsf{BinTree}$$

$$\mathsf{Fork} : \mathsf{BinTree} \times \mathsf{a} \times \mathsf{BinTree} \to \mathsf{BinTree}$$

We define $P(t) \equiv$ search operations over the binary search tree t modeled as a $\mathsf{BinTree}$ only need to search one branch of t. We will prove $P(t)$ for all $t \in \mathsf{BinTree}$ by structural induction on t.

Base case: $t = \mathsf{Leaf}(n)$. Prove $P(t)$ holds.

It is trivially obvious that the search algorithm over t can only check one element, the leaf of the tree. Thus, there is only one branch to search: the leaf. So $P(t)$ holds.

Inductive step: $t = \mathsf{Fork}(b_1, n, b_2)$. Assume $P(b_1)$ and $P(b_2)$ hold. Prove $P(t)$ holds.

According to the properties of binary search trees we know that t follows the binary search property: that all leaves containing values less than or equal to n are stored in b_1, and any leaves with values greater than or equal to n are stored in b_2. Thus, the search operation algorithm can compare the search value x with n and choose which sub-tree to search in.

We then have 3 cases.

- If $x = n$ the search operation has only had to check one value or one branch of possible values so $P(t)$ holds.
- If $x < n$ the search operation can choose the left branch b_1 to continue its search. Since $P(b_1)$ holds we know the search operation will only search one branch in the left sub-tree as well. Together, the search operation will only have to search one branch overall of t. Therefore, $P(t)$ holds for $x < n$.
- Similarly, we can follow the same logic for $x > n$ with the right branch b_2 since $P(b_2)$ holds. Thus, $P(t)$ holds for all cases.

We have shown that $P(t)$ holds for all binary trees modeled by $\mathsf{BinTree}$, thus search operations need only to search one branch of binary search trees.