# *Bubble-Driven* Business Cycles

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**Abstract:** Pronounced and persistent fluctuations in aggregate wealth and real activity – boom-bust episodes – have become more prevalent in recent history. In this paper I provide a quantitative explanation for such boom-bust episodes that is based on rational bubbles. To this end, I set up an overlapping generations model with many generations, financial frictions, aggregate uncertainty and rational bubbles. The calibrated model generates empirically plausible *bubble-driven* business cycles. I decompose the macroeconomic effect of rational bubbles into several different channels and use the calibrated model to asses their relative strength. The decomposition shows that one particular channel that operates through the creation of bubbles is necessary for plausible bubbles to exist. I then apply the model to replicate the observed series of real output and aggregate wealth during the two recent US boom-bust episodes between 1990 and 2010. By decomposing the model-implied series for aggregate wealth I show that on average one third of the deviations of aggregate wealth from its trend can be explained by fluctuations in an aggregate rational bubble.

**Key words:** Computable General Equilibrium, Bubble, Asset Price, Real Activity.
**JEL classification:** D58, E32, E44.

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# 1. Introduction

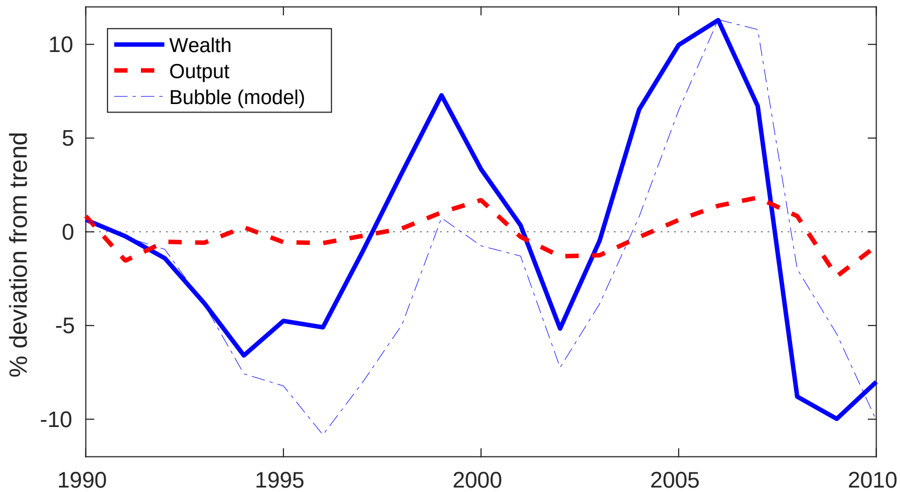
Pronounced and persistent fluctuations in aggregate wealth and real activity – boom-bust episodes – have become more prevalent in recent history.[1] Two such prominent episode are the US Dotcom and the housing boom-bust episodes depicted in figure I. After reaching its peak in 2000 aggregate real wealth declined by more than 10 percent within two years. During the subsequent housing boom-bust episode aggregate real wealth dropped by more than 20 percent between 2007 and 2009. This was almost entirely driven by price dynamics in stock and housing markets. Output co-moved with aggregate wealth, exhibiting much smaller fluctuations during the first boom-bust episode and a pronounced recession in the second bust. In this paper I provide a quantitative explanation for such boom-bust episodes with an overlapping generations model (OLG) featuring many generations, financial frictions, aggregate uncertainty and – most importantly – rational bubbles.

In the model rational bubbles can exert different effects on output when they exist in general equilibrium. At the core is the interplay between possibly borrowing constrained entrepreneurs that demand credit and savers that supply credit. Entrepreneurs invest in capital and run the production in the economy, but can borrow only up to a fraction of the collateral value of their firms (Kiyotaki and Moore, 1997). The effect of bubbles on real activity operates through different and partly opposing channels that are all connected to the credit market. The proposed model captures three prominent channels through which bubbles affect output: the *crowding-out channel* which goes back to the seminal contribution by Tirole (1985), the *liquidity channel* introduced by Farhi and Tirole (2012), and the *bubble-creation channel* put forward by Martin and Ventura (2012, 2016). I compare the relative strength of these channels in a calibrated model that is confronted with the data.

Bubbles raise the demand for credit in the economy, leading to a higher return on credit, which in turn reduces leverage and depresses investment through the *crowding-out channel*. This is the classical crowding-out effect of bubbles as the capital stock and output decline with the size of the bubble (Tirole, 1985). Since this implication is at odds with empirical observations – see figure I – alternative channels through which bubbles can raise output have been proposed by Farhi and Tirole (2012); Martin and Ventura (2012, 2016).[2] The relevant feature is the introduction of market incompleteness in the form of borrowing constraints. When a bubble exists and the return on credit increases, this might imply that current savers accumulate more wealth. When these savers become entrepreneurs in the future and face a borrowing constraint, then their liquidity or wealth is larger and they invest more, hence raising over-

[1]Based on a panel dataset of 17 developed economies Jordà *et al.* (2015, Table 2) show that the magnitude of boom-bust episodes is higher in the post-WW2 period than in the pre-WW2 period. Similarly, Carvalho *et al.* (2012, Figure 1) show that fluctuations in real US net-worth became more pronounced since the 1950s.

[2]There are many more contributions that study alternative channels through which rational bubbles affect output. See the recent survey article on the macroeconomics of rational bubbles by Martin and Ventura (2018).




*Source: Output corresponds to gross domestic product and wealth to domestic net wealth. Domestic net wealth is obtained by adding the net foreign asset position to national net wealth. National net wealth comprises all private and non-for profit assets minus liabilities. Both output and wealth are deflated with the CPI to obtain real series. All series are from the US NIPA, retrieved from FRED, `https://fred.stlouisfed.org/`, on January 2019. All time series are expressed in logarithms and have been detrended with the HP filter. Output is detrended with a HP parameter value of 6.25 as proposed by Ravn and Uhlig (2002), but for wealth a smoother filter of 100 is chosen. See figure VIII for details. The thin dash-dot line is computed with the model, as explained in section 5.*

**Figure I:** Two recent US boom-bust episodes

all output through the *liquidity channel*. When a saver becomes an entrepreneur she is also able to create new bubbles with her new firm. This raises the expected firm value, implying a larger collateral value and subsequently larger investment and output produced by these entrepreneurs – the *bubble-creation channel*. Bubbles can raise output through both the *liquidity* and *bubble-creation channel* if entrepreneurs are borrowing constrained. I use the calibrated model to study i) when the expansionary *liquidity* and *bubble-creation channels* dominate the contractionary *crowding-out channel*, and ii) whether the *liquidity* or *bubble-creation channel* is more relevant for this result. The calibrated model shows that the overall effect of bubbles on output is positive and that this effect is almost entirely driven by the *bubble-creation channel*. In an economy without the *bubble-creation channel*, bubbles only increase output when the annual depreciation rate is above 63 percent – an empirically implausible value. The *bubble-creation channel* by far outweighs the *liquidity channel* and I also show that this is robust to variations in other parameters.

I further compare two different aggregate shocks: standard total factor productivity (TFP) shocks and market sentiment shocks that directly affect the evolution of the stochastic bubbles. The numerically solved model is able to generate *bubble-driven* business cycles in addition to real business cycles that are driven by TFP shocks. Since it is challenging to explain the strong fluctuations in real wealth during the boom-bust episodes depicted in figure I as a result of strong negative fundamental shocks like TFP shocks, it seems warranted to provide a quantitative, model-based explanation of boom-bust episodes resulting from stochastic rational bubbles. This is underlined by the evidence from two empirical studies by Brunnermeier and Schnabel (2015) and Jordà *et al.* (2015). Both come to the conclusion that bursting bubbles can cause recessions. In order to test the model's ability to explain boom-bust episodes, I use the model to replicate the observed behavior of output and wealth during the two recent US boom-bust episodes. I then decompose the model-generated series of aggregate wealth into a bubble component and a fundamental component. The model implies that on average one third of the deviations of aggregate wealth from its trend is due to changes in the aggregate bubble, as can be seen by the dash-dotted line in figure I.

This paper provides also a methodological contribution. I apply the numerical solution approach proposed by Boppart *et al.* (2018) to an overlapping generations model with many generations and aggregate shocks. The authors apply the method to an infinitely lived heterogeneous agent model. It remains to be seen how the method performs in more non-linear settings. The application of the method from Boppart *et al.* (2018) to the present paper which features overlapping generations and financial frictions provides an important step towards better understanding when linear solution methods are viable and when global methods are necessary. I test whether the linearity assumption underlying the method from Boppart *et al.* (2018) is satisfied. Linearity seems to be very plausible for TFP shocks, but for very large sentiment shocks results are mixed, implying that the method from Boppart *et al.* (2018) might face its limitations with very large shocks.

**Related Literature** This paper is related to the large body of literature on boom-bust episodes in macroeconomic models. The literature can be divided in two branches. The first branch is centered around quantitative DSGE models with financial frictions and without bubbles. The second branch of the literature studies rational bubbles in general equilibrium, mainly in stylized, analytical macroeconomic models.

First, numerous papers study boom-bust cycles within quantitative DSGE models, often with heterogeneous agents. Some papers study how shocks emanating from the real sector are amplified through the financial sector, for example Christiano *et al.* (2015). A common approach to replicating observed aggregate price fluctuations is to introduce housing demand shocks into the utility function, see for example Iacoviello (2005); Iacoviello and Neri (2010), or Kaplan *et al.* (2017). The latter consider changes in expectations about housing preference shocks. Other contributions study shocks that emanate directly from the financial sector. Two examples are *valuation shocks* in Gertler and Karadi (2011) and *liquidity shocks* in Kiyotaki and Moore (2012). A few contributions within the infinitely-lived agent DSGE literature do also consider asset price bubbles. Bernanke and Gertler (1999) consider *irrational* asset price bubbles within a RBC model with financial frictions. As the authors state they "use the term “bubble” [...] loosely to denote temporary deviations of asset prices from fundamental values"

(p. 19). Similarly, Luik and Wesselbaum (2014) consider "near-rational" asset price bubbles. In contrast to *rational* bubbles these bubbles are not micro-founded.

Second, since the seminal contribution by Tirole (1985), which was predated by Samuelson (1958), numerous papers incorporated rational asset price bubbles in general equilibrium models. In the Tirole (1985) model bubbles always crowd-out capital and hence reduce output. Empirically it is not plausible that the capital stock and output decline during episodes of existing asset price bubbles, and increase when bubbles burst. The correlation should be exactly the opposite as can be inferred from figure I. Therefore, recent models extend the Tirole (1985) model – mostly by financial frictions – in order to derive equilibria where bubbles co-move with output. Some examples from this include Martin and Ventura (2011, 2012); Farhi and Tirole (2012); Galí (2014); Martin and Ventura (2016).[3] Most of these contributions consider stylized *two-* or *three-period* OLG models. *Two-period* OLG models are well suited for deriving insightful analytical results for the economic mechanisms at work, but their drawback is that they are not capable of replicating the short and medium run behavior of empirical time series by numerical solutions. The reason is that calibrating *two-period* OLG models implies that one period in the model corresponds to approximately 30 years in real time. Accordingly, the length of a recession would be at least 30 years and bubbles would exist also at least for 30 years – both implausibly long. I contribute to this literature by considering rational bubbles within a *large-scale* OLG model. As a result, one period in the model corresponds to one year in real time such that the model with bubbles can be brought to the data.

Some papers within the literature on rational bubbles in macroeconomic models follow a more quantitative approach. Miao *et al.* (2015) numerically solve a large DSGE model with several shocks and rational bubbles relying on local perturbation methods and focusing on a different mechanism that is based on Miao and Wang (2018). Galí (2017) studies nominal bubbles in a perpetual-youth New-Keynesian model without financial frictions. Domeij and Ellingsen (2018) show that rational bubbles can exist in an infinitely-lived agent incomplete-markets model with public debt. To the best of my knowledge, this is the first paper to study stochastic rational bubbles in a life-cycle RBC model – as first introduced by Ríos-Rull (1996) – that is extended by financial frictions.

**Outline** The rest of the paper is organized as follows: Section 2 describes the model and provides a first discussion of the existence properties of bubbles in this economy. In section 3 I discuss my calibration strategy and the resulting age profiles of earnings, consumption, and wealth for different household groups. Section 4 studies through which different transmission channels bubbles affect the economy, assessing the relative strength of each, and showing how the existence and properties of bubbles change with variations in different parameters. Finally, in section 5 I compare a TFP-driven business cycle with a *bubble-driven* business cycle and I

[3] See Martin and Ventura (2018) for a recent survey of the literature on the macroeconomics of rational bubbles.

apply the model to shed light on the two recent US boom-bust episodes before offering some concluding comments in section 6.

## 2. Model

The economy is inhabited by households belonging to different, overlapping generations. Some households are entrepreneurs while the others are savers. Entrepreneurs run the production in the economy, while savers provide labor and credit to entrepreneurs. Entrepreneurial borrowing is limited by the collateral value of their firms. The firm value consists of a fundamental and a bubble component, implying that changes in the bubble affect the entrepreneurs' borrowing capacity. By being part of the firms' collateral, bubbles affect credit demand, investment, and hence overall output in a non-trivial way, as will be studied below. The economy is exposed to two aggregate shocks, i) standard total factor productivity (TFP) shocks and ii) sentiment shocks that affect the realized rate of return to bubbles. The model is constructed such that it nests several prominent model classes as special cases. First, the model incorporates the OLG models with rational bubbles by Tirole (1985); Farhi and Tirole (2012); Martin and Ventura (2012, 2016). These contributions introduce different mechanisms through which rational bubbles affect the real economy. Incorporating these different mechanisms into one model makes it possible to asses the qualitative and quantitative relevance of the different channels for the first time. Second, in the absence of bubbles and financial frictions the model boils down to the frictionless life-cycle RBC model as first studied by Ríos-Rull (1996) because then savers and entrepreneurs are isomorph. Capturing these two model classes as benchmark cases allows to study the macroeconomic implications of rational bubbles in an economy with financial frictions.

### 2.1. Setup

**Demographics.** Time $t$ is discrete and the closed economy is populated by $J$ overlapping generations. Total population size is denoted by $N_t$, normalized to one in period $t = 0$, and growing at a constant factor $n$ such that $N_t = n^t$. Household age is denoted by the subindex $j \in \{1, 2, ..., J\}$, where $J \geq 2$ is the maximum number of periods a household stays in the economy. Let $N_{t,j}$ denote the mass of cohort $j$ in period $t$.[4] Aggregation implies $\sum_{j=1}^{J} N_{t,j} = N_t$. Lifetimes are uncertain. Let $\zeta_j$ denote the unconditional probability of surviving up to age $j$, with $\zeta_1 = 1$ and $\zeta_{J+1} = 0$. The probability of surviving up to age $j$ conditional on being alive at age $j-1$ is denoted by $\varrho_j \equiv \frac{\zeta_j}{\zeta_{j-1}}$. The mass of each cohort grows at the same constant factor

[4]For brevity, I use the term "cohort $j$" to refer to the group of all households of age $j$.

$n$, such that cohort shares in total population are constant over time.[5]

**Preferences.** Households have standard time-separable preferences over stochastic consumption and labor supply streams $\{c_j, l_j\}_{j=1}^{J}$ represented by[6]

$$\mathbb{E}\sum_{j=1}^{J}\beta^{j-1}\zeta_j u\left(c_j, l_j\right), \tag{1}$$

where $\beta$ is the time discount factor and $u\left(c_j, l_j\right)$ is the period utility function. The period utility function is given by GHH (Greenwood, Hercowitz and Huffman, 1988) preferences

$$u(c,l) = \frac{[c - G(l)]^{1-\sigma} - 1}{1-\sigma},$$

where $G(l)$ denotes disutility from labor and $\sigma > 0$ is the inverse of the intertemporal elasticity of substitution in the composite consumption good $c - G(l)$. The functional form of $G(l)$ is given by $G(l) = g^t\theta\frac{l^{1+\chi}}{1+\chi}$, where $g$ is a growth factor to be specified below, $\theta$ a coefficient measuring the disutility from work, and $\chi$ is the inverse of the Frisch elasticity of labor supply. Because the economy will exhibit long-run growth, it is necessary to include the growth factor $g$ in order for labor supply being bounded between zero and one. The assumption of GHH preferences serves two purposes. First, it rules out implausible labor supply responses to wealth shocks. With the common KPR (King *et al.*, 1988) preferences an exogenously triggered drop in wealth – which can, but does not need be the result of a bursting bubble – can lead to rising labor supply. Second, GHH preferences reduce the computational burden because individual and hence aggregate labor supply are only functions of the wage rate.[7]

**Earnings and pensions.** Households are endowed with one unit of time per period that can be allocated to leisure or to labor. The choice between labor and leisure depends on the exogenous, age-dependent labor productivity profile $\{e_j\}_{j=1}^{J}$. When a household of age $j$ chooses to spend $l_j$

[5]The mass of cohort $j$ in period $t$ can hence be expressed as

$$N_{t,j} = N_t \cdot N_{0,j} = n^t \cdot N_{0,j},$$

where $N_{0,j} = \zeta_j\left(\sum_{s=1}^{J}\zeta_s n^{j-s}\right)^{-1}$ is the size of cohort $j$ in period $t = 0$.

[6]I suppress the time and age indices whenever this implies no confusion.

[7]Another argument in favor of GHH preferences as compared to KRP preferences is that with the latter the age profile of labor supply might be monotonically decreasing, which is empirically implausible (see, e.g., Ríos-Rull (1996) for the empirical labor profile). An example where that happens is Heer and Scharrer (2018, figure 1), where age profiles of labor supply are monotonically decreasing. The reason is that with KPR preferences individuals accumulate wealth over roughly the first half of the life-cycle, consumption is increasing, and therefore labor supply declines as wealthier individuals consume also more leisure. This wealth effect is absent with GHH preferences and the labor supply profile is hence only a mirror image of the the efficiency profile. See also Ascari and Rankin (2007) for a very related argument on KPR vs GHH preferences in life-cycle economies.

units of her time endowment working, then she earns $(1-\tau)wl_j e_j$ units of the final output good, where $\tau$ are proportional labor taxes and $w$ is the wage rate. After working for $J^w \in \{1, ..., J-1\}$ periods retirement is mandatory. This is captured by setting labor productivity $e_j$ equal to zero for $j > J^w$. Households spend up to $J - J^w$ periods in retirement before leaving the economy. Retired households receive a pension $pen$ from the government during each period of retirement. The government finances a pay-as-you-go pension system by levying a proportional labor income tax at the rate $\tau$. I abstract from government debt by assuming a balanced budget in each period

$$\tau_t w_t L_t = \sum_{j=J^w+1}^{J} N_{t,j} pen_t = \frac{1}{1+\phi^{-1}} n^t pen_t, \tag{2}$$

where $L_t$ denotes aggregate labor supply and $\phi \equiv \left(\sum_{j=J^w+1}^{J} N_{t,j}\right) / \left(\sum_{j=1}^{J^w} N_{t,j}\right)$ is the constant old-age dependency ratio, i.e. the share of retired households over working households in the economy. Equation (2) states that government revenues, given by the left hand side, are equal to government expenditure as given by the right hand side.

**Savers and Entrepreneurs.** Within a cohort some households are entrepreneurs and some are savers. Every household enters the economy as a saver and becomes an entrepreneur in a given period $J^E \in \{1, 2, ..., J^w - 1\}$ with probability $p^E$. Once a household is an entrepreneur she remains an entrepreneur for the rest of her life. Hence, households face an irreversible one-time idiosyncratic shock at age $j = J^E$ of becoming an entrepreneur.

An entrepreneur is defined as follows. He possesses the necessary skills and abilities to run a firm. An entrepreneur of age $j$ faces the Cobb-Douglas production function

$$y_{t,j} = Z_t^Y k_{t,j}^{\alpha} \left(g^t h_{t,j}\right)^{1-\alpha},$$

where $0 < 1 - \alpha < 1$ is the labor income share, and $y_{t,j}, k_{t,j}, h_{t,j}$ are output, capital stock and labor demand by entrepreneur of age $j$ in period $t$, respectively. The production function features exogenous labor-augmenting technological growth with the constant growth factor $g \geq 0$. Aggregate total factor productivity (TFP) shocks are captured by $Z_t^Y$ which follows an AR(1) process in logs

$$\ln Z_{t+1}^Y = \rho^Y \ln Z_t^Y + \epsilon_{t+1}^Y,$$

where $\epsilon_t^Y \sim N(0, \sigma^Y)$ is the innovation term and $\rho^Y \in [0, 1)$ captures the persistence of TFP shocks.

In a given period $t$ the capital stock $k_{t,j}$ is predetermined and $Z_t^Y$ is known at the beginning of the period, such that it is possible to separate the static labor demand problem from the full dynamic optimization problem faced by entrepreneurs.[8] An entrepreneur hires labor on a

[8] This can be shown by including the labor demand problem into the dynamic problem and verifying that first

perfectly competitive labor market in order to maximize current period profits, taking the wage rate $w_t$ as given. The static labor demand problem then reads

$$\max_{h_{t,j}} \Pi_{t,j} = Z_t^Y k_{t,j}^{\alpha} \left(g^t h_{t,j}\right)^{1-\alpha} - w_t h_{t,j} - \delta k_{t,j}, \tag{3}$$

where $\delta \in (0,1]$ is the capital depreciation rate.[9] The first order condition reads

$$(1-\alpha) Z_t^Y (g^t)^{1-\alpha} \left(\frac{k_{t,j}}{h_{t,j}}\right)^{\alpha} = w_t = (1-\alpha) Z_t^Y (g^t)^{1-\alpha} \left(\frac{K_t}{L_t}\right)^{\alpha}, \tag{4}$$

where $K_t$ and $L_t$ are aggregate capital stock and labor demand. Equation (4) shows that every entrepreneur demands an amount of labor that results in the same capital to labor ratio. The wage rate then also equals the *aggregate* marginal product of labor. Since every entrepreneur chooses the same capital-labor ratio, aggregate output can be written as a function of aggregate capital and labor (the distribution of capital and labor across firms does not matter here)

$$Y_t = \sum_{j=J^E}^{J} p^E N_{t,j} y_{t,j} = Z_t^Y K_t^{\alpha} (g^t L_t)^{1-\alpha}. \tag{5}$$

When an entrepreneur chooses the capital stock in a given period, she anticipates her optimal labor demand in the subsequent period. This implies that the profits obtained by investing $k_{t,j}$ units of capital in $t-1$ are given by the profit function

$$\Pi_t(k_{t,s}) = (R_t - 1) k_{t,s},$$

where the (gross) rate of return on capital is

$$R_t = 1 + \alpha \frac{Y}{K} - \delta. \tag{6}$$

The last equation is obtained by inserting optimal labor demand (4) into the profit function (3) and applying (5).

Savers can transfer income into the future by either lending to entrepreneurs or by using an inferior production technology. The inferior production scale is linear in capital

$$y_{t,j,S} = (\gamma - 1 + \delta) k_{t,j,S},$$

where $\gamma \geq 0$ is a parameter that governs the relative profitability of the savers' production

order conditions are identical.

[9] I assume that capital depreciates independently of its utilization such that $k_{t,j}$ is no choice variable in period $t$. That implies that if profits happens to be negative even for the optimal $h_{t,j}$ (due to large negative TFP shocks or over-accumulation), entrepreneurs cannot escape negative profits.

function in comparison to the entrepreneurs' production function. When savers invest in this technology they earn the (gross) rate of return $\gamma$. This production function is inferior when the capital stock in the entrepreneurial sector is sufficiently low, or equivalently when $R > \gamma$.

The saver's production scale admits two alternative and equivalent interpretations. One, households have access to international credit markets where the exogenous interest rate is constant over time and given by $\gamma$. Two, savers have access to a storage technology that transforms one unit of current income into $\gamma$ units of next period's income.

**Credit market.** The economy is characterized by an imperfect credit market. In this market non-contingent one-period credit contracts are traded at the gross return $R^d_{t+1}$, which is predetermined in period $t$.[10] Entrepreneurs can borrow from savers in order to invest in their firms. Due to limited enforcement, however, they can only credibly pledge the resale value of their firm which is denoted by $W_{t,j}$. As a result, entrepreneurs face the following borrowing constraint

$$R^d_{t+1} d_{t+1,j+1} \leq \mathbb{E}_t W_{t+1,j+1}, \tag{7}$$

where $d_{t+1,j+1}$ is the amount of debt issued by entrepreneur $j$ in period $t$. The constraint states that the certain repayment of outstanding debt including interest has to be smaller or equal to the expected resale value of the firm.

**Firm value.** The fundamental value of the firm at the end of a period is given by $(1-\delta)k_{t,j}$. In equilibrium, however, the firm value can also contain a bubble component:

$$W_{t+1,j+1} = \underbrace{(1-\delta)k_{t+1,j+1}}_{\text{fundamental}} + \underbrace{Q_{t+1}\left[b_{t+1,j+1} + b^N_{t+1,j+1}\right]}_{\text{bubble}}. \tag{8}$$

The fundamental firm value consists of the capital stock remaining after production and depreciation took place. The bubble component consists of the gross rate of return on bubbles, $Q_{t+1}$, multiplied by the value of the existing bubble, $b_{t+1,j+1}$, and the value of newly created bubble $b^N_{t+1,j+1}$. The rate of return on bubbles, $Q_{t+1}$, can be stochastic, capturing the possibility that bubbles evolve differently than expected.

The creation of new bubbles is exogenous and only new entrepreneurs ($j = J^E$) create bubbles when starting a new firm. That follows the arguments put forward by Diba and Grossman (1987) according to which a bubble has to exist when an asset is created and cannot appear on an existing asset. To be precise, this implies that $b^N_{t+1,J^E+1} \geq 0$ while $b^N_{t+1,j+1} = b^N_t = 0$ for all $j \neq J^E$.

[10] I assume that credit is non-contingent for the following reasons. First, this follows many recent macro models with financial frictions, e.g., Iacoviello (2005). Second, this way the financial amplification mechanisms is present. Third, it simplifies the solution, because I would otherwise have to solve for all state-contingent returns, which might even be dependent on entrepreneurial age $j$.

Let $B_t^N = p^E N_{t,J^E} b_t^N$ denote the aggregate amount of new bubbles and $B_t$ the aggregate value of the existing bubble. The law of motion of aggregate bubbles then reads

$$B_{t+1} = Q_t(B_t + B_t^N).$$

A bubble bursts when $Q_{t+1} \to 0$. After a bubble did burst a new bubble can only emerge when bubble creation is strictly positive. This is why bubble creation is necessary for recurrent *bubble-driven* business cycles to exist.

There are two equivalent interpretations for the firm value in (8). First, it consists of the fundamental asset capital and a zero-dividend asset. If the value of the zero-dividend asset is strictly positive, then $b > 0$ and $b^N > 0$, and it is by definition a bubble. Entrepreneurs trade capital and existing bubbles, while new bubbles are created exogenously. Second, instead of trading two distinct assets, entrepreneurs trade portfolios of firm values on a stock market. The bubble can then be thought of as pertaining to the firm rather than as an isolate asset.

**Accidental bequests.** I abstract from the existence of perfect annuity markets that allow households to fully insure against the risk of longevity. This has two reasons. First, it is not necessarily incompatible, but more cumbersome together with the financial frictions and the working of the credit market in this economy. Second, as shown by Hansen and İmrohoroğlu (2008), with perfect annuities it would not be possible to replicate the empirically observed hump-shaped consumption age-profile (Fernández-Villaverde and Krueger, 2007) in this model framework. I therefore follow Hansen and İmrohoroğlu (2008) and assume that households leave accidental bequests that are redistributed lump sum across surviving households. This deficit-neutral redistribution scheme is run by the government. Let cash-on-hand (net worth) be denoted by $m_{t,j,i}$, where $i \in \{S, E\}$ indices savers and entrepreneurs, respectively. Bequests received by a surviving household are denoted by $beq_t$ and are given by

$$\begin{aligned}\sum_{j=1}^{J} N_{t,j} beq_t &= \sum_{j=2}^{J^E} (1-\varrho_j) N_{t-1,j-1} m_{t,j,S} + \sum_{j=J^E+1}^{J} (1-\varrho_j) N_{t-1,j-1} \left[ (1-p^E) m_{t,j,S} + p^E m_{t,j,E} \right] \\ \Leftrightarrow beq_t &= \sum_{j=2}^{J^E} \frac{1-\varrho_j}{\varrho_j} N_{0,j} m_{t,j,S} + \sum_{j=J^E+1}^{J} \frac{1-\varrho_j}{\varrho_j} N_{0,j} \left[ (1-p^E) m_{t,j,S} + p^E m_{t,j,E} \right].\end{aligned}$$

The first line equates all received bequests on the left hand side with all accidental bequest on the right hand side while the second line expresses bequests as a function of cash-on-hand.

**Savers' problem.** Savers can save by lending to entrepreneurs at the risk-free rate $R_{t+1}^d$ or by investing into their inferior production technology at the rate of return $\gamma$. Since both investments are risk-free it has to hold in equilibrium that $R_{t+1}^d \geq \gamma$. The inferior technology will only be used when both returns are equalized and then savers are indifferent between the two. It is therefore sufficient to consider only the savers' total wealth $a_{t,j}$ in the optimization problem instead of introducing two variables for lending and inferior capital. A saver maximizes intertemporal utility as given by (1) subject to the following constraints

$$\begin{aligned} c_{t,j} + a_{t+1,j+1} &= (1-\tau)w_t e_j l_{j,t} + R_t^d a_{t,j} + I_j^r pen_t + beq_t \\ a_{t,1} &= \frac{m_{t,1,S}}{R_t^d}, a_{t,j} \geq 0, \end{aligned} \tag{9}$$

where $I_j^r$ is an indicator that equals one 1 for retired agents and 0 otherwise.[11]

**Entrepreneurs' problem.** Entrepreneurs face a richer set of investment possibilities than savers. They can invest in capital $k$ yielding the return $R$, bubbles $b$ yielding the return $Q$, and take on debt $d$ at the rate $R^d$. Debt $d$ could also be negative, implying lending rather than borrowing. Let $m$ denote cash-on-hand or, equivalently, net worth as

$$m_{t,j,E} = \begin{cases} 0 & \text{if } j < J^E \\ R_t^d a_{t,j} & \text{if } j = J^E \\ R_t k_{t,j} - R_t^d d_{t,j} + Q_t(b_{t,j} + b_{t,j}^N) & \text{if } j > J^E. \end{cases} \tag{10}$$

Since individuals can only become entrepreneurs at age $J^E$, cash-on-hand is zero for $j < J^E$. Cash-on-hand of a household who just became an entrepreneur, $j = J^E$, consists of the gross-return on all bonds purchased when this household was a saver in the previous period. Existing entrepreneurs' ($j > J^E$) cash-on-hand comprises the entire portfolio return including capital, debt, and bubbles.

An entrepreneur maximizes intertemporal utility (1) subject to the following constraints

$$\begin{aligned} c_{t,j} + k_{t+1,j+1} + b_{t+1,j+1} - d_{t+1,j+1} &= (1-\tau)w_t e_j l_{j,t} + m_{t,j,E} + I_j^r pen_t + beq_t \\ R_{t+1}^d d_{t+1,j+1} &\leq (1-\delta)k_{t+1,j+1} + Q_{t+1}\left(b_{t+1,j+1} + b_{t+1,j+1}^N\right) \\ k_{t,J^E} = b_{t,J^E} = d_{t,J^E} &= 0, \quad k_{t,j} \geq 0. \end{aligned} \tag{11}$$

The first line is the budget constraint. Entrepreneurs spend their cash-on-hand, labor income, received bequests, and – in the case of retired entrepreneurs – also pension income for consumption $c$ and investment in capital $k$, bubbles $b$ and debt $d$. The second line is the borrowing constraint and the third line states that i) an entrepreneur starts with zero capital, bubbles, and debt, ii) capital cannot be negative, iii) wealth has to be positive after death, and iv) cash-on-hand is given by (10).

[11] The non-negativity restriction on lending results endogenously from the credit market imperfection in this economy because the savers' collateral is equal to zero. I assume that inferior capital does not serve as collateral.

The problems of savers and entrepreneurs differ with regard to their investment possibilities, but labor earnings, pensions, and bequests are identical. This implies that when no bubbles exist and when the borrowing constraint is not binding, both groups face the same rate of returns and hence the exactly same problems, such that the model reduces to the frictionless life-cycle RBC model (Ríos-Rull, 1996). The symmetry of savers and entrepreneurs regarding labor income, pensions, and received bequests implies also that whenever $R > R^d$ (and abstracting from risk), entrepreneurs are better off from being entrepreneurs than from remaining savers. If becoming an entrepreneur were an explicit choice that is available under positive realizations of the entrepreneur shock, then entrepreneurs would either be indifferent or voluntarily decide to actually become entrepreneurs. I could therefore equivalently assume that becoming an entrepreneur – given a positive realization of the idiosyncratic shock – is a choice as in Cagetti and De Nardi (2006) and all results would be exactly the same.

### 2.2. Equilibrium

A sequential competitive equilibrium consists of sequences of individual consumption and labor supply $\left\{c_{t,j}, l_{t,j}\right\}_{j=1}^{J}$ for both savers and entrepreneurs for $t \geq 0$, as well as of sequences of bubbles, bonds, capital, and debt $\{b_{t,j}, a_{t,j}, k_{t,j}, d_{t,j},\}_{j=1}^{J}$ for all $t \geq 0$ maximizing the households' intertemporal utility (1) subject to the constraints (9) and (11), a sequence of prices $\{w_t, R_t, R_t^d, Q_t\}_{t=0}^{\infty}$ satisfying (4), (6) and (13), a sequence of shocks $\left\{Z_t^Y\right\}_{t=1}^{\infty}$ drawn from its respective distribution and initial values $\left\{b_{0,j}, a_{0,j}, k_{0,j}, d_{0,j}\right\}_{j=1}^{J}, Z_0^Y, R_0^d$ such that

- the capital market clears[12]

$$K_t = \sum_{j=J^E+1}^{J} \frac{p^E N_{t,j}}{\varrho_j} k_{t,j},$$

- the market for bubbles clears

$$\underbrace{B_t}_{\text{supply of bubbles in } t-1} = \underbrace{\sum_{j=J^E+1}^{J} \frac{p^E N_{t,j}}{\varrho_j} b_{t,j}}_{\text{demand for bubbles in } t-1},$$

- the credit market clears

$$A_t \equiv \sum_{j=2}^{J^E} \frac{N_{t,j}}{\varrho_j} a_{t,j} + \sum_{s=J^E+1}^{J} \frac{(1-p^E) N_{t,j}}{\varrho_j} a_{t,j} \geq \sum_{j=J^E+1}^{J} \frac{p^E N_{t,j}}{\varrho_j} d_{t,j} \equiv D_t,$$

[12] In period $t-1$ entrepreneurs of age $j-1$ and mass $p^E N_{t-1,j-1}$ invest into capital stock $k_{t,j}$. I assume that production will take place in period $t$ also for the capital stock installed by entrepreneurs that pass away between periods $j$ and $j+1$. Hence, total capital stock of entrepreneurs aged $j-1$ in $t-1$ is given by $N_{t-1,j-1} k_{t,j} = \frac{p^E N_{t,j}}{\varrho_j} k_{t,j}$

- the labor market clears[13]

$$\underbrace{\sum_{j=1}^{J} N_{t,j} e_j l_{j,t}}_{\text{labor supply}} = \underbrace{\sum_{j=J^E+1}^{J} \frac{p^E N_{t,j}}{\varrho_j} h_{j,t}}_{\text{labor demand}} = L_t,$$

- the goods market clears[14]

$$Y_t + \gamma(A_t - D_t) = C_t + K_{t+1} - (1-\delta)K_t + (A_{t+1} - D_{t+1}),$$

- bubbles are freely disposable and the capital stock is positive

$$B_t \geq 0, K_t \geq 0.$$

If the credit market clearing condition holds with equality, then savers do not use their inferior production technology. Otherwise the total amount of assets used for the inferior production technology is given by the difference of credit demanded by entrepreneurs, $D$, and the savers' wealth, $A$. This also explains why total output consists of $Y + \gamma(A_t - D_t)$, where the second term is output generated with the inferior production technology, and investment into new inferior capital is equal to $(A_{t+1} - D_{t+1})$. The last two conditions on bubbles and capital determine whether bubbles can exist in equilibrium or not. First, bubbles have to be positive which is related to the notion of free disposal. If a bubble were negative, an entrepreneur could close the existing firm and transfer the capital stock into a new firm without a bubble, resulting in a higher firm value than before. Since establishing new firms or closing existing firms features not costs, a negative bubble can not exist in equilibrium. Second, bubbles cannot be too large. A bubble is too large when the capital stock becomes negative. Both conditions are standard in the literature on rational bubbles, see e.g. Martin and Ventura (2018). In the following sections I will show when these conditions are satisfied and when not.

### 2.3. Rates of return in the steady state

Before calibrating and solving the model numerically I study the relationship between the potentially different rates of return in steady state equilibria with and without bubbles. The non-stochastic steady state is defined as the steady state of an economy where TFP is equal to its

[13] Savers and entrepreneurs of the same age supply the same amount of labor, as shown in appendix A.1.

[14] Although it is not necessary to state this equation – it is implied by the budget constraints – it is useful as a consistency check in the numerical solution. Note further that aggregate consumption is defined by $C_t \equiv \sum_{j=1}^{J^E-1} N_{t,j} c_{t,j,S} + \sum_{j=J^E}^{J} p^E N_{t,j} c_{t,j,E} + \sum_{j=J^E}^{J} (1-p^E) N_{t,j} c_{t,j,S}$, where the subscripts $S$ and $E$ indicate consumption by savers and entrepreneurs, respectively.

unconditional mean, $Z_t^Y = 1$, with probability one and where bubbles are non-stochastic. The economy features long-run population growth and exogenous labor-augmenting technological growth given by $n$ and $g$, respectively. Variables have therefore to be normalized for a stationary system to exist. The normalized system is provided in appendix A.3. I first abstract from the inferior production technology by assuming $\gamma = 0$.

Equating the entrepreneurs' Euler equations for capital $k$ and debt $d$ as derived in appendix A.1 and normalized in appendix A.3 yields the following relation between the return on credit $R^d$ and the return on capital $R$

$$
\begin{aligned}
\beta \varrho_{j+1}\left(R-R^d\right)\left(\widetilde{c}'-\tilde{G}'\right)^{-\sigma} &= \widetilde{\omega}\left[R^d-(1-\delta)\right] \\
0 &= \widetilde{\omega}\left[(1-\delta)\widetilde{k}'+Q'\left(\widetilde{b}'+(\widetilde{b}^N)'\right)-R^{d\prime}\widetilde{d}'\right]
\end{aligned}
$$

where $\widetilde{x}_t \equiv \frac{x_t}{g^t}$ denotes a growth-adjusted variable and $\widetilde{\omega}$ is the growth-adjusted multiplier associated with the complementary slackness condition. The second line is the complementary slackness condition. First, it has to hold that $R^d > 1-\delta$, because otherwise the entrepreneurs' credit demand would be infinite and the capital stock would tend to infinity. Second, it has to hold that $R \geq R^d$.[15] If $R = R^d$, the borrowing constraint is not binding and if $R > R^d$, the borrowing constraint binds. In the first case, $R = R^d$, the multiplier $\widetilde{\omega}$ is zero as prescribed by the Euler equation. Intuitively, entrepreneurs are just indifferent between capital $k$ and bonds in the form of negative debt $-d$. In the second case, $R > R^d$, the Euler equation implies that $\widetilde{\omega} > 0$. Then, entrepreneurs borrow as much as possible because borrowing one unit at $R^d$, investing it into one unit of capital stock, obtaining $R > R^d$ units from producing and selling the capital stock in the next period, and clearing debts yields a profit of $R - R^d > 0$.

What is the relation between the return on bubbles and the other assets' returns? Equating the entrepreneurs' Euler equations for bubbles, $b$, and debt, $d$, from appendix A.1 in the same fashion yields the following relation between the return on credit, $R^d$, and the return on bubbles, $Q$:

$$
\left[\beta \varrho_{j+1}\left(\widetilde{c}'-\tilde{G}'\right)^{-\sigma}+\widetilde{\omega}\right]\left(Q-R^d\right)=0.
$$

When bubbles exist, then their return has to be equal to the return on debt, i.e. $Q = R^d$.[16] The intuition is that $Q > R^d$ would imply that the entrepreneurs' demand for bubbles is infinite, financed by an infinite demand for credit. Both is not possible in general equilibrium. Similarly, if $Q < R^d$, then each entrepreneur would demand negative amounts of the bubble, violating

[15] The return on capital can never be smaller than the return on credit, because this would imply that the multiplier $\widetilde{\omega}$ is negative. Intuitively, with $R^d > R$ entrepreneurs would hold zero units of capital, and save only through bonds in the form of negative debt. Since every entrepreneur would behave like that, the aggregate capital stock would tend to zero and the rate of return on capital to infinity, ruling $R^d > R$ out.

[16] The left hand side of the above expression can only be equal to zero if this holds, because marginal utility is strictly positive and $\widetilde{\omega}$ is non-negative such that the expression in square brackets is strictly positive.

| | Constraint binding | Constraint not binding |
|---|---|---|
| No bubble | $R^d < R$ | $R^d = R$ |
| Bubbly, no bubble creation | $Q = R^d = gn < R$ | $Q = R^d = gn = R$ |
| Bubbly, bubble creation | $Q = R^d < \min\{gn, R\}$ | $Q = R^d = R < gn$ |

**Table I:** Rates of return and growth rates in the steady state

the equilibrium condition that the aggregate bubble is non-negative.

Further restrictions on the return on bubbles can be derived from the law of motion of aggregate bubbles in normalized units[17] as given in appendix A.3

$$gnb_{t+1} = Q_{t+1}(b_t + b_t^N). \tag{12}$$

In the (normalized) steady state it holds that $Q_{t+1} = Q_t$, $b_{t+1} = b_t$, and $b_t^N = const.$. First, if no bubble creation exists, $b_t^N = 0$, then bubbles must grow at the growth rate of the economy, i.e. $Q = gn$. Since bubbles have also to yield the same return as debt, it holds therefore that also the return on debt is equal to the economy's growth rate, i.e. $R^d = Q = gn$. Second, if bubble creation exists, $b_t^N > 0$, then the return to bubbles has to be smaller than the growth rate of the economy, because new bubbles are added every period. This implies that also the return on debt is smaller then the economy's growth rate, i.e. $R^d = Q < gn$.

If bubble creation exists, $b^N > 0$, then the aggregate bubble in the steady state can be expressed as a function of $R^d$ by rearranging (12)

$$b = \frac{R^d}{R^d - gn} b^N.$$

This shows that the size of the bubble in equilibrium depends on the amount of newly created bubbles $b^N$ and on the difference between the economy's growth rate and the return on credit. The closer the return on credit is to $gn$, the larger is the equilibrium bubble.

The discussion of this section is summarized in table I. Note, in a steady state with bubbles the rate of return on capital can only be larger then the growth rate of the economy when the borrowing constraint is binding and hence financial frictions are necessary for $R > gn$.

How do the results change when $\gamma > 0$? The previous results would still hold except that the inequality $R^d \geq \gamma$ has also to hold. Otherwise, savers would not be willing to lend and entrepreneurs would demand credit such that $R^d$ would have to increase to clear the credit market. If $R^d > \gamma$, the inferior technology is not used and the results are the same as with $\gamma = 0$. If $R^d = \gamma$, it has to hold that $\gamma = gn$ for a bubbly steady state without bubble creation to exist and $\gamma < gn$ for a bubbly steady state with bubble creation to exist. Hence, if $R^d = \gamma > gn$,

[17] Normalized variables are expressed by their corresponding lower-case letters.

then bubbles cannot exist in equilibrium.

The previous results provide testable implications for the relation between $gn$, $R$ and $R^d$. What are the corresponding empirical values for the US? The growth factor of average annual real GDP in the US from 1950 to 2016 is 1.032. Data for the rate of returns are taken from Jorda *et al.* (2018, Table 11).[18] The (gross) rates of return for the US from 1950–2016 are $R^d = 1.017$ and $R = 1.067$. According to the data it hence holds that $R^d < gn < R$, implying that according to the model i) rational bubbles might have existed in the US during this period, ii) the borrowing constraint has to be binding and bubble creation has to exist in the steady state because otherwise $R^d$ would be equal to $gn$.

## 3. Parameterization of the model

The model is calibrated with respect to the postwar US economy (1950-2017) at an annual frequency. I calibrate some parameters to match cross-sectional moments of the US in 1995 because this year lies roughly in the middle of the postwar US period. Since the previous discussion implied that $R^d < gn < R$ holds on average for the postwar US, I will focus on equilibria with bubble creation and a binding borrowing constraint. I will discuss deviations from this equilibrium in section 4. In the following I will also abstract from the inferior production technology, i.e. $\gamma = 0$.

### 3.1. Specification of stochastic bubble process

This economy features multiple equilibria, depending on the existence and size of bubbles. I add structure to the model by assuming that the equilibrium is pinned down by an exogenous process that drives the rate of return on bubbles. This exogenous process is best thought of as market sentiments denoted by the exogenous and stochastic variable $Z_t^b \geq 0$. High realizations of $Z_t^b$ imply that entrepreneurs are optimistic, while low values correspond to pessimism. In order to be as parsimonious as possible, I assume that market sentiment, $Z_t^B$, follows a simple normally-distributed white noise process with variance $\sigma^B$. Although this process is not persistent it can generate very persistent *bubble-driven* business cycles as will become evident.

The ex-post return to holding bubbles is now conceptualized as

$$Q_t = Z_t^b \bar{Q}_t. \tag{13}$$

It consists of an exogenous component – market sentiments $Z_t^b$ – and an endogenous component – $\bar{Q}_t$. The return $\bar{Q}_t$ is predetermined and adjusts such that the market for bubbles clears in each period.

[18] Jorda *et al.* (2018) provide average returns for the period 1950–1980 and 1980–2015. I obtain the average return for the entire period 1950–2015 from $R = \left(R_{50-80}^{30} \times R_{80-15}^{35}\right)^{\frac{1}{65}}$.

Exogenous bubble creation is assumed to be given exogenously and to grow at the factor $g$

$$b_{t+1}^N = g^{t+1}\nu.$$

Bubble creation per entrepreneur is hence equal to $\nu \geq 0$ in the growth-adjusted system.

### 3.2. Parameters calibrated outside the model

**Demographics.** Individuals enter the economy at age 22, retire at age 65, and die before the age of 100.[19] Age 22 corresponds to $j = 1$ in the model and I hence set $J = 79$ and $J^w = 43$. According to a recent study by Azoulay *et al.* (2018) the average age of founding entrepreneurs is 42 years in the US. I therefore set the period when the entrepreneur shock realizes, $J^E$, to 21. US population grew from 159 to 320 million between 1950 and 2015 (UN, 2017). This implies an average annual growth rate of 1.1 percent such that $n = 1.011$. Survival probabilities $\{\zeta_j\}_{j=1}^J$ are taken from Anderson (1999) for the year 1997.

**Government.** I assume that the net-replacement ratio $\xi$ is constant over time such that

$$pen_t = \xi \frac{(1-\tau_t)w_t L_t}{\sum_{j=1}^{J^w} N_{t,j}} = \xi(1-\tau_t)w_t(1+\phi)\frac{L_t}{n^t}. \tag{14}$$

The last expression states that pensions are a constant share, $\xi$, of the current average after-tax labor income in the economy. Inserting (14) into (2) and rearranging yields the tax rate as a function of the replacement rate and the old-age dependency ratio according to

$$\tau = \frac{(1+\phi)\xi}{(1+\phi^{-1}) + (1+\phi)\xi}.$$

The tax rate is constant over time, which results from the assumption that the replacement rate is constant. This implies that fluctuations in wages or labor supply translate into fluctuations in pensions. Following İmrohoroglu *et al.* (1995) I set the net-replacement ratio $\xi$ to 0.5, implying a tax rate of 11.24 percent.

**Production.** In the U.S. economy the labor income share is about 2/3. The capital income share is therefore set to $\alpha = 1/3$, a standard value in the literature. The average growth rate of real GDP per capita between 1950 and 2017 was 1.96 percent annually in the US such that $g = 1.02$. I set the depreciation rate to match the average rate of return on capital of 1.067, as discussed in the previous section. From Table 1.1. of the Fixed Assets and Consumer Durable Goods of the Bureau of Economic Analysis and Table 1.1.5 from the National Income

[19] I subsume all individuals older than 100 years in the oldest cohort. This does not seem to be very critical since only 1.73 in 10000 US inhabitants were of age 100 or older in 2010 (Meyer, 2012).

and Product Accounts (NIPA) I obtain an average annual capital to output ratio of 2.8 (fixed assets) between 1950 and 2015. The value for $\beta$ will be calibrated internally to match this value. Given that this value is matched, the depreciation rate is obtained by rearranging the equation for the interest rate $R$ as follows

$$\delta = 1 + \alpha \left( \frac{K}{Y} \right)^{-1} - R.$$

This yields a value of $\delta = 0.052$, which is in the range of the values for $\delta$ commonly chosen in the literature. An alternative way to calibrate $\delta$ is to rearrange the law-of motion of capital in the steady state to

$$\delta = \frac{I/Y}{K/Y} + 1 - ng.$$

Given $n = 1.011$, $g = 1.02$, $\delta = 0.052$, and $K/Y = 2.8$ the implied investment output ratio is 0.23. This is close to the value observed in the data. According to NIPA table 1.10.1 line 7 the annual investment to output ratio is 17.4 percent on average between 1950 and 2015. The parameters of the AR(1) process governing the dynamics of TFP are taken from Prescott (1986) and are adjusted to an annual frequency. The volatility $\sigma^B$ of market sentiments is set to the same value as the volatility of TFP, but this choice is irrelevant for the results because i) I will consider impulse response functions to different sizes of this shocks, ii) I will use empirical observations to obtain sequences of $Z_t^B$ in section 5.4, and iii) I do not study second moments.

**Preferences and productivity profile.** According to the empirical meta-analysis by Havránek (2015) the intertemporal elasticity of substitution is far below unity, and the majority of evidence yields values in the range of 0.2 and 0.4. I therefore set the IES to 1/4 implying $\sigma = 4$. I set $\chi$ to a conservative value of 3, which implies a Frisch elasticity of labor supply equal to 1/3. This value is in the broad range of most micro-studies.[20] The model will therefore not be able to account for all of the observed volatility in total hours, but it is micro-consistent. The labor earnings profile $\{e_j\}_{j=1}^J$ is calibrated to match the empirical earnings profile as given by the Survey of Consumer Finances (SCF) for the year 1995.[21] Similarly to Glover *et al.* (2017) I recategorize capital income for the first cohort as labor income because in the model capital income is zero for $j = 1$. To be precise, I am matching labor earnings of a given age group as a share of total average labor earnings. This earnings' share is given in the model by

$$\mathcal{E}_j = \frac{(1-\tau)\widetilde{w}_t e_j l_j}{\sum_{j=1}^{J^w} (1-\tau)\widetilde{w}_t e_j l_j N_{0,j}} = \frac{e_j^{\frac{1+\chi}{\chi}}}{\sum_{j=1}^{J^w} e_j^{\frac{1+\chi}{\chi}} N_{0,j}}.$$

[20] See, e.g., Peterman (2016) for a recent discussion of the evidence on micro and macro estimations of the Frisch elasticity of labor supply.

[21] I use the SCF data as given in the online data appendix accompanying Kuhn and Rios-Rull (2016).

| Parameter | | Value | Explanation/Target |
|---|---|---|---|
| Life span | $J$ | 79 | life span of 85 years |
| Period of entrepr. shock | $J^E$ | 21 | average age of founding entrepr.: 42 (Azoulay *et al.*, 2018) |
| Retirement | $J^w$ | 43 | retirement at 65 |
| Population growth | $n$ | 1.011 | UN (2017) |
| Replacement ratio | $\xi$ | 0.5 | İmrohoroglu *et al.* (1995) |
| Inverse of IES | $\sigma$ | 4 | Havránek (2015) |
| Inverse of Frisch elasticity | $\chi$ | 3 | standard value |
| Productivity profile | $\{e_j\}_{j=1}^J$ | see text | Earnings profile in 1995 (SCF) |
| Survival probabilities | $\{\zeta_j\}_{j=1}^J$ | see text | Anderson (1999) |
| Capital income share | $\alpha$ | 1/3 | standard value |
| Technological growth | $g$ | 1.02 | per-capita GDP growth |
| Depreciation | $\delta$ | 0.052 | Return on capital 6.7% |
| TFP shock autocorr. | $\rho^Y$ | 0.814 | Prescott (1986) |
| TFP shock volatility | $\sigma^Y$ | 0.014 | Prescott (1986) |
| Bubble shock volatility | $\sigma^B$ | 0.014 | baseline |

**Table II:** Parameters calibrated outside the model

In the numerator are earnings for a household of age $j$ and in the denominator are total earnings per capita. The second equality makes use of the optimal labor supply as given by (16). Most importantly, the households' earnings share depends only on parameters. Note further that scaling each $e_j$ by the same constant has no effect on $\mathscr{E}_j$. I therefore normalize, without loss of generality, $\sum_{j=1}^{J^w} e_j^{\frac{1+\chi}{\chi}} N_{0,j}$ to unity. Then cohort $j$'s labor productivity is directly given by

$$e_j = \mathscr{E}_j^{\frac{\chi}{1+\chi}}.$$

All externally calibrated parameters are summarized in table II.

### 3.3. Parameters calibrated inside the model

The following four parameters are calibrated jointly by numerically solving the model. The parameter reflecting the weight of disutility derived from labor, $\theta$, is chosen in the standard fashion such that households spend on average one third of their time endowment working (Cooley and Prescott, 1995). The discount factor $\beta$ is calibrated in order to match the capital

| Parameter | | Value | Target | Value |
|---|---|---|---|---|
| Disutility from labor | $\theta$ | 31.105 | time spent working | $\frac{1}{3}$ |
| Discount factor | $\beta$ | 1.116 | capital output ratio | 2.8 |
| Share entrepreneurs | $\eta$ | 0.001 | return differential | 0.05 |
| Bubble creation | $\nu$ | 0.452 | bubble share in wealth | 0.01 |

**Table III:** Parameters calibrated (jointly) inside the model

to output ratio of 2.8, which is obtained from Table 1.1. of the Fixed Assets and Consumer Durable Goods of the Bureau of Economic Analysis and Table 1.1.5 from the NIPA for the period 1950–2015. One could also calibrate $\beta$ to match the wealth to earnings ratio of 5.5 as given by the SCF for the year 1995. Without aiming to match it the model produces a very close value of 5.0. Let $\eta$ denote the population share of entrepreneurs. It is constant over time and given by

$$\eta \equiv \frac{\sum_{j=J^E}^{J} p^E N_{t,j}}{N_t} = p^E \sum_{j=J^E}^{J} N_{0,j}. \tag{15}$$

The population share of entrepreneurs is chosen such that the model reproduces the return differential between bonds and stocks, $R{-}R^d$, of 5 percent as observed in Jorda *et al.* (2018) and discussed in the previous section. For a given $\eta$ the probability of becoming an entrepreneur $p^E$ is obtained from equation (15). Since I also match the rate of return on capital, $R$, of 1.067, this implies that the model matches the observed rate of return on bonds of 1.017. Lastly, the size of the bubble in the steady state is directly affected by the size of new bubble creation $\nu$. Since measuring the size of a rational bubble at the aggregate level is still empirically challenging, I take an agnostic stance and assume that in the steady state one percent of the entrepreneurs' wealth consists of bubbles. This implies that quantitatively almost no bubble exists in the steady state, but it allows me to study deviations from it where the bubble can grow considerably in size. All internally calibrated parameters are summarized in table III and relative differences between model and data moments are always less than $1 * 10^{-7}$.

### 3.4. Model fit

#### 3.4.1. Average earnings and wealth profiles

The model replicates the empirical age profile of net wealth quite well. Figure IX in the appendix shows the age profiles of earnings and wealth produced by the model and their empirical counterparts as given from the SCF for the year 1995. Within a cohort values correspond to averages across entrepreneurs and savers. The earnings profile is matched perfectly by construction, but the wealth profile is not targeted by the calibration. The calibrated model closely replicates the empirical wealth profile. The differences might be explained by the absence of voluntary bequests or idiosyncratic labor income shocks that force households to start accumulating a buffer stock of wealth early in their life.

### 3.4.2. Savers and entrepreneurs: wealth and consumption profiles

Entrepreneurs and savers live very different lives as can be seen from their respective wealth and consumption profiles in figure II. Households that are lucky and become entrepreneurs at the age of 42, indicated by the vertical line, are able to accumulate much more wealth as they can realize very large and leveraged returns. As a result, entrepreneurial wealth is on average larger by a factor of 232 than a saver's wealth and it is increasing almost all the time. The savers' wealth profile already starts declining before retirement because they reach the peak of their labor-productivity profile.

Entrepreneurs consume more than savers in every period of their life and find it optimal to accumulate wealth throughout almost all of their life. The savers' overall consumption profile is hump-shaped with a strong drop after retirement. This is in line with evidence from Fernández-Villaverde and Krueger (2011) who show that the empirical consumption profile is hump-shaped. The immediate drop in consumption is the result of a substitution between consumption and leisure as leisure increases immediately when households enter retirement. This immediate drop is a result of the assumption that all individuals retire at the same period. In reality this drop in consumption is smoothed out over several periods because individuals do not all retire at the same age. After the drop in consumption at retirement the profile is first increasing before survival probabilities become very large acting like a strong discount rate of future consumption. Very old savers, age 97 to 100, would even like to borrow against their expected future pension income, hitting their zero-borrowing constraint such that wealth is zero. Hence, very old savers are borrowing constrained hand-to-mouth consumers.

The benefit of being an entrepreneur is further highlighted by computing the gain in consumption associated with living the life of an entrepreneur instead of a saver's life. Accordingly, a savers' consumption would have to increase by 289 percent in every period in order to be indifferent between living the life of an entrepreneur or that of a saver. This reflects the fact that entrepreneurs face better investment opportunities than savers, but otherwise the same labor earnings and pension profiles.

### 3.4.3. Wealth inequality

The model can explain a considerable share of the empirically observed wealth inequality. From the internal calibration the share of entrepreneurs in the population is given by 0.1 percent. This very small group of *superstars* runs the production in the economy and is very rich. The top 0.01, 0.1, and 1 percent wealth shares are 4, 6, and 8 percent in the model. This is still

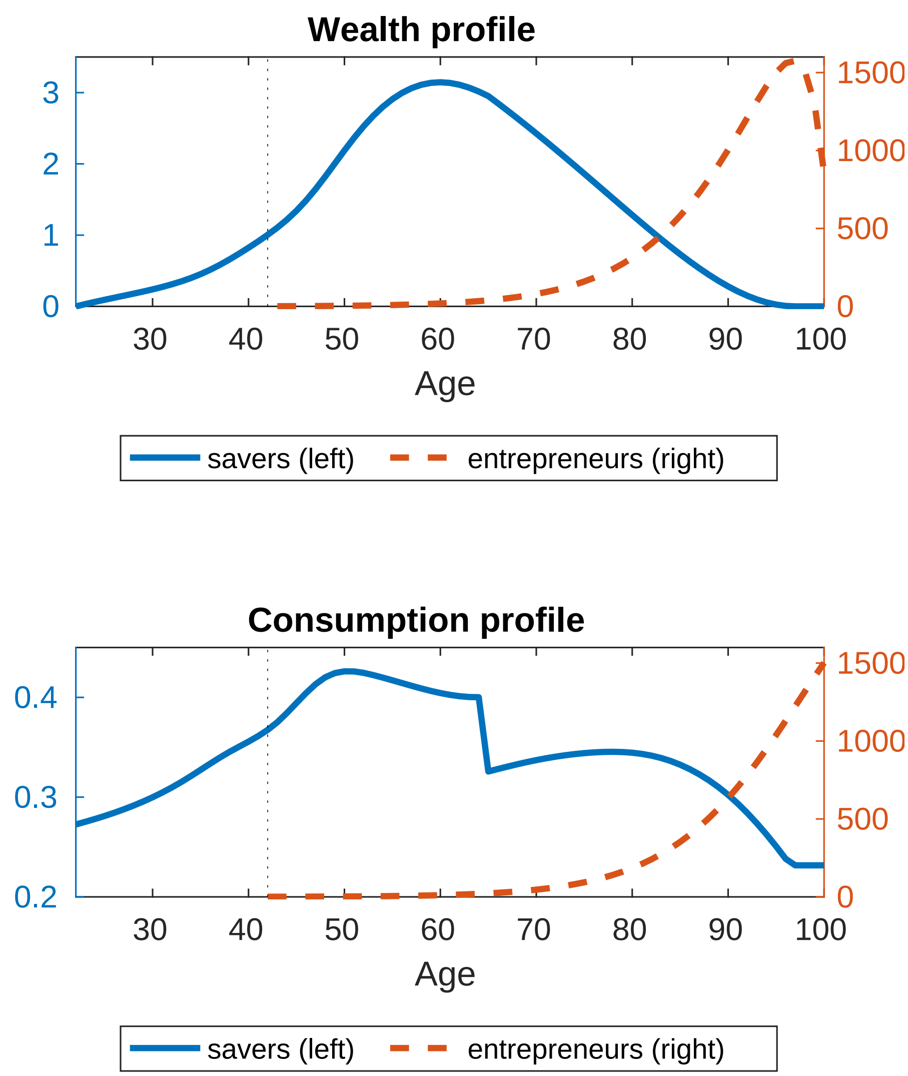



**Figure II:** Age profiles for savers and entrepreneurs

below the the empirical counterparts of 11, 22, and 41 percent (Saez and Zucman, 2016, table 1, 2012), but much larger than the values obtained in the friction- and bubble-less economy. By setting the population share of entrepreneurs sufficiently high, the borrowing constraint becomes slack, $R = R^d$, and financial frictions do effectively not exist. Hence, I set the population share of entrepreneurs to 10 percent and obtain top 0.01, 0.1, and 1 percent wealth shares are of 0.02, 0.2, and 2.2. These values are much smaller than the values obtained in the benchmark calibration. Hence, adding financial frictions and rational bubbles implies also that the life-cycle RBC model (Ríos-Rull, 1996) can generate more wealth inequality. If the goal of this paper were to explain the observed wealth distribution one would have to add idiosyncratic income shocks, idiosyncratic shocks within the groups of entrepreneurs and voluntary bequest that would imply that individuals are not identical at $t = 0$.[22] At this point, however, it is important that the model is micro-consistent by replicating several cross-sectional moments.

### 3.5. Computation

The general approach to the steady state solution is similar to the approach explained in detail in Heer and Maussner (2009, ch. 9). The solution consists of an outer and inner loop. The outer loop solves for $R, R^d$, and $beq$. If these three variables are given, then all other aggregate variables like the wage rate can be calculated from analytical expressions. The inner loop takes all aggregate prices and quantities as given and solves the household problem with the endogenous grid point method (EGM) (Carroll, 2006). The EGM is well suited for dealing with inequality constraints. Note, all idiosyncratic risk (entrepreneur shocks and death) are easily incorporated into the solution and no discretization is necessary. Since the entrepreneur shock occurs only once and has only two outcomes, one has just to calculate the expected value of margin utility in period $J^E$ at different grid points.

## 4. The effect of bubbles: liquidity vs. bubble-creation vs. crowding-out channel

When can bubbles exist in general equilibrium and what is their effect on the economy? The existence of bubbles implies that all general equilibrium conditions are satisfied, i.e. at the micro-level agents are willing to hold these assets as discussed in section 2.3, and at the macro-level bubbles are positive and not too large for the capital stock becoming negative. When bubbles exist they can exert different effects on capital and output. A bubble is called *expansionary* if output in the steady state with bubbles is larger than in the steady without. A *contractionary* bubble describes the opposite.

### 4.1. Decomposition of the effect of bubbles

Bubbles affect the economy through three different and in part opposing channels: the *bubble-creation, liquidty,* and *crowding-out channel*. In the following I will discuss these channels and their relative strength in greater detail by conducting a numerical decomposition into partial equilibrium effects. I will first consider the case when $\gamma = 0$ and the inferior production technology is not used.

The decomposition compares a fundamental steady state where bubbles do not exist with the calibrated bubbly steady state. Starting at the general equilibrium in the fundamental steady state I gradually add prices from the steady state with bubbles, solve the household

[22] See Cagetti and De Nardi (2006) for a richer OLG model with entrepreneurs that is able to replicate the empirically observed wealth inequality.

| | General equilibrium: fundamental | Partial equilibrium | | | | General equilibrium: bubbly |
|---|---|---|---|---|---|---|
| | (1) | (2) | (3) | (4) | (5) | (6) |
| | | $+b^N$ | $+w$ $+pen$ $+beq$ | $+R$ | + savers' $R^d$ | + entrepreneurs' $R^d$ |
| Output, $Y$ | 0.0 | 7.6 | 9.5 | 4.5 | 4.7 | 0.9 |
| Capital, $K$ | 0.0 | 24.6 | 25.3 | 11.6 | 12.3 | 2.2 |
| Labor, $L$ | 0.0 | 0.0 | 2.3 | 1.1 | 1.2 | 0.2 |
| Credit, $D$ | 0.0 | 24.6 | 25.3 | 11.6 | 12.3 | 2.2 |
| Savers' wealth, $A$ | 0.0 | -0.001 | 0.8 | 0.8 | 2.2 | 2.2 |
| Entrep. wealth, $A^E$ | 0.0 | 24.6 | 25.3 | 11.6 | 12.3 | 3.2 |
| Bubble, $B$ | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 |

*Notes: All variables are expressed relative to their fundamental steady state values, in percentage points, except B, which is expressed as a ratio of entrepreneurs wealth $A^E$. Prices change according to: $b^N\uparrow$, $w\uparrow$, $pen\uparrow$, $beq\uparrow$, $R\downarrow$, $R^d\uparrow$.*

**Table IV:** The effect of bubbles decomposed into different channels

problem, and aggregate household sequences in order to compute output and other aggregate quantities. The results, depicted in table IV, allow to understand how bubbles affect the economy through different partial equilibrium channels. Column (1) depicts the fundamental steady state and column (6) the bubbly steady state. Comparing these two columns shows that the bubble is expansionary in the calibrated model. A small bubble that corresponds to 1 percent of the entrepreneurs' wealth implies that output is by 0.9 percent larger in the bubbly than in the fundamental steady state. The specific partial equilibrium channels through which bubbles affect output can be inferred from columns (2) to (5).

The *bubble-creation channel* has been put forward by Martin and Ventura (2016, 2012), who also label it "the wealth effect of bubble creation" in Martin and Ventura (2018). When entrepreneurs create new bubbles, this raises the expected collateral value of their firm value as can be seen from the borrowing constraint (7). If entrepreneurs are borrowing constrained, which is the case in the calibrated model, then they will demand more credit in order to invest into additional capital. This is exactly what happens between column (1) and (2) in table IV. In column (2) new bubble creation is added to the household problem in the fundamental steady state. Entrepreneurial wealth and hence aggregate capital stock increase by 24.6 percent. Note, the market for bubbles does not clear, as nobody (within this economy) purchases these newly created bubbles. Column (2) therefore depicts the pure *bubble-creation channel* of bubbles. A higher aggregate capital stock implies higher wages, higher labor supply and subsequently a by 7.6 percent larger output.[23] Savers take the low-probability event of becoming an entrepreneur into account, anticipate larger windfall gains from bubble-creation and therefore save 0.001 percent less. When the probability of becoming an entrepreneur is larger, this number would increase.

In the next step, column (3), the values for wages, pensions and bequests from the bubbly steady state are added. This captures the general equilibrium feedback that a higher capital stock exerts on wages, pensions, and hence also on labor supply, as well as larger bequests as a result of higher overall wealth. Output and all other aggregate variables, except the aggregate bubble, increase further. The aggregate bubble is yet equal to zero because, by assumption, the market for bubbles is not cleared.

Column (4) adds the equilibrium feedback effect through the return on capital. Because the capital stock increases from the fundamental to the bubbly steady state the return to capital, $R$, declines. This feedback effect through $R$ is slightly contractionary, reducing output and other aggregates. However, the overall effect of bubble creation remains still positive as output is by 4.5 percent larger than in the fundamental steady state.

In column (5) the higher return on credit is added to the saver's optimization problem, keeping the return on credit unchanged for entrepreneurs. This implies that borrowers face a lower rate of return than lenders. When a bubble exists entrepreneurs borrow more in order to finance it, which raises the return on credit $R^d$. A higher return on credit $R^d$ in turn affects the savings decision by savers. If savers do not reduce the amount of savings too much, then a higher $R^d$ implies that they start with larger net wealth level when some of them become entrepreneurs in period $J^E$. Larger wealth translates directly into a larger capital stock, hence raising output. This is the *liquidity channel* put forward by Farhi and Tirole (2012). This channel can only exist if entrepreneurs are first savers and become entrepreneurs later, i.e. when $J^E > 1$. Quantitatively, the *liquidity channel* is very weak and much less pronounced than the *bubble-creation channel* as output increases only by modest 0.2 percentage points from column (4) to (5). It can be seen that this is the result of higher average wealth of savers, $A$, which increases from 0.8 to 2.2 percentage points.

Column (6) finally adds the last price to the household problem, the entrepreneurs' borrowing rate $R^d$. Now all prices that are inserted into the household problem correspond to the general equilibrium with bubbles. This last step shows how bubbles affect the economy through

[23] Note further that in equilibrium the savers' wealth, $A$, has to be equal to the entrepreneurs' demand for credit, $D$, which is not the case in this partial equilibrium analysis. The demand for credit, $D$, increases with the bubble, but the supply, $A$, is yet unchanged because the savers' problem remained the same. Hence, the credit market does not clear in this partial equilibrium.

the *crowding-out channel*. When the entrepreneurs' borrowing cost $R^d$ increase, their financial multiplier is reduced, leading to lower leverage and hence to a lower capital stock and output. This is the classical crowding-out effect of bubbles going back to Tirole (1985), which works through leverage and credit in this economy. The *crowding-out channel* is contractionary, while the *bubble-creation* and *liquidity channel* are expansionary. For expansionary bubbles to exist the *bubble-creation* and *liquidity channel* have to dominate the *crowding-out channel*, as is the case in the calibrated model. When comparing their relative strength, the *bubble-creation channel* is almost entirely responsible for bubbles being expansionary, while the *liquidity channel* is very weak. The obtained results are robust to the specific ordering with which prices series are studied in partial equilibrium, as shown in the appendix.

When the inferior production technology is relevant, $\gamma > 1.02$, then the *liquidity channel* does not operate because credit supply is inelastic (changes in $R^d$ do not affect credit supply as long as $\gamma = R^d$). Further, the *crowding-out channel* does also not exist. This is because bubbles increase the entrepreneurs' demand for credit, but with storage the credit supply curve is inelastic for $\gamma = R^d$, such that this has no feedback effect through $R^d$. Hence, with $\gamma = R^d$ only the *bubble-creation channel* is operating, as can be seen in table B.1 in the appendix.

### 4.2. The effects of bubbles under different parameter values

How does the effect of bubbles change when varying some of the most relevant parameters? Figure III shows how the parameter governing new bubble creation ,$\nu$, affects the economy. The steady state has been solved for 100 equispaced values of $\nu$ in the interval $[1, 10]$. The three lines depict a bubbly steady with bubble creation ($\nu > 0$), a bubbly steady without bubble creation ($\nu = 0$), and the fundamental steady at different values of $\nu$. Changes in $\nu$ only affect the bubbly economy with bubble creation and hence the other lines are flat. First, in the economy without bubble creation the bubble is very large and contractionary. This can be seen in the graph for output where the yellow dashed line lies above the dotted red line. Theoretically, that does not need to be the case. Although the *collateral channel* is absent without bubble creation, the *liquidity channel* could still dominate the contractionary *crowding-out channel*. Quantitatively, however, this is the case, indicating that the *liquidity channel* alone is not able generate expansionary bubbles. Second, the bubbly economy with bubble creation is always expansionary. For most values of $\nu$ output is increasing in $\nu$, but at some point the bubble becomes too large and output starts to decline. For values of $\nu$ close to 20 the bubble becomes too large and a bubbly steady state cannot exist anymore, explaining why the blue line is not plotted for all values of $\nu$ between 0 and 20. This can be best understood by looking at the equation for the steady state bubble

$$b = \frac{R^d}{gn - R^d} b^N = \frac{R^d}{gn - R^d} p^E N_{0,j} \nu.$$

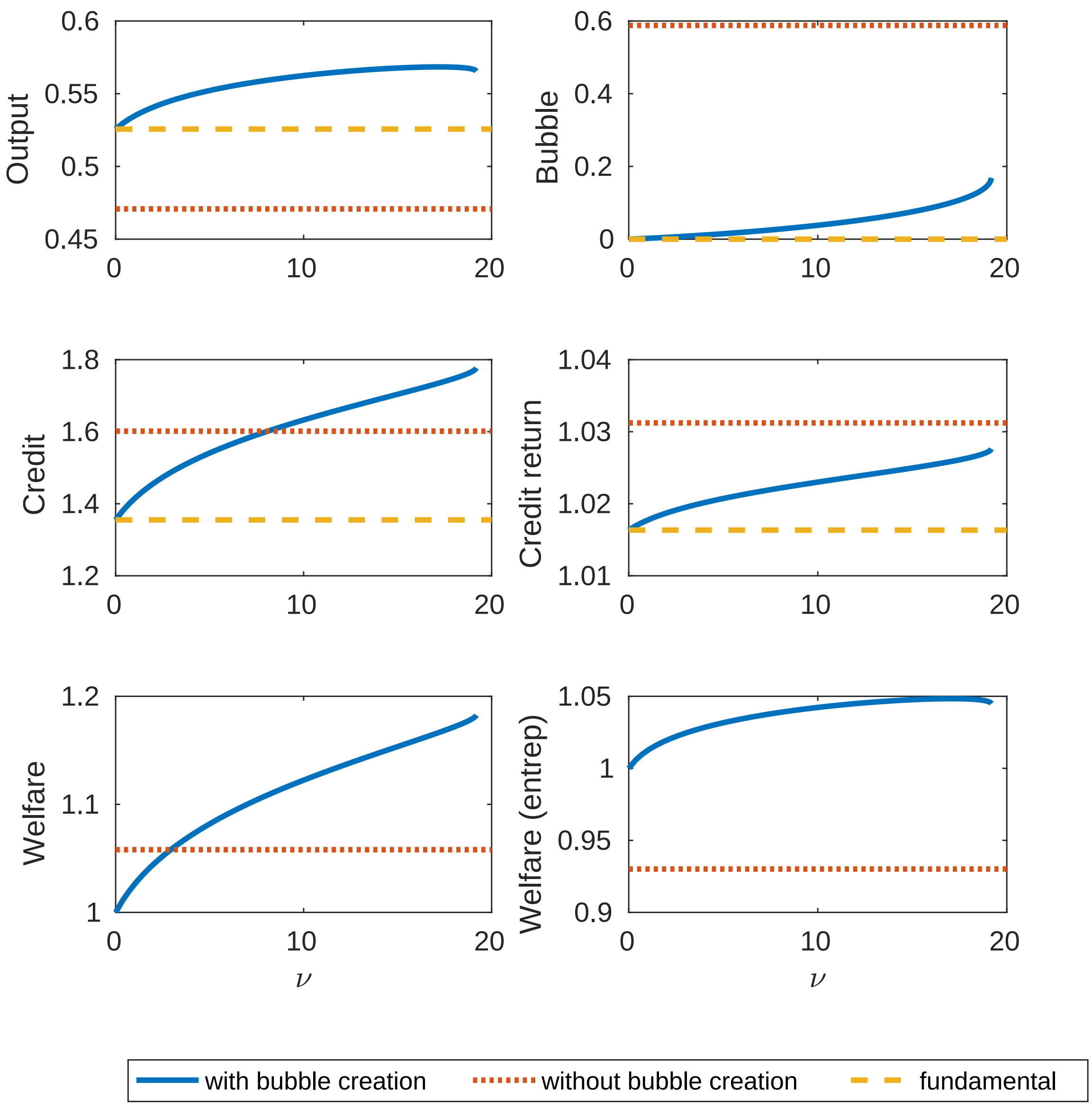



**Figure III:** Steady state under different degrees of bubble creation

The larger new bubble creation, the larger is the overall bubble, and the larger is the demand for credit. This increases the credit return $R^d$, as can be seen in figure III. The above equation, however, shows that the bubbles starts to explode when $R^d$ approaches the economy's growth rate $gn$, which is given by the red line in the figure for the credit return $R^d$. This can be observed in the figure III, where the bubble starts to explode when $R^d$ gets close to the economy's growth factor $gn$.

Figure III also plots the consumption equivalent variation between the bubbly and the

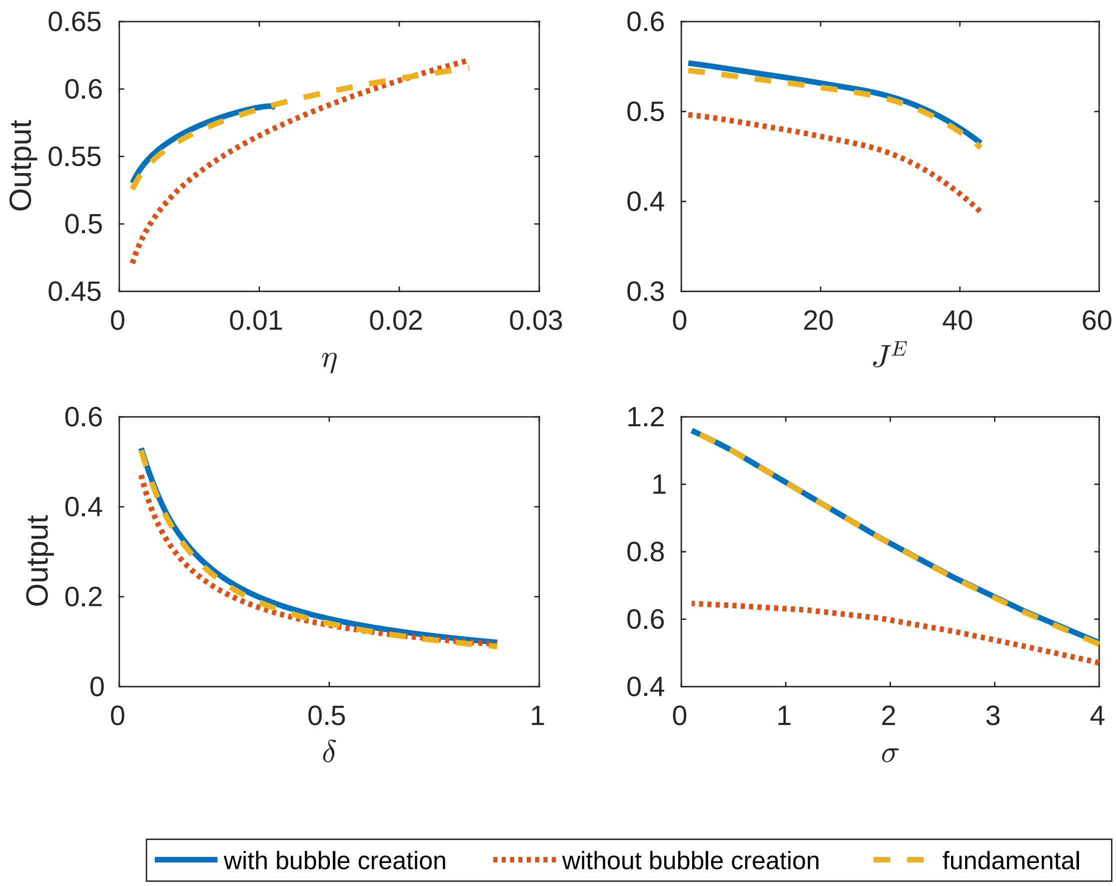



**Figure IV:** Properties of bubbles for different parameter values

fundamental economy. A value above one means that consumption in the fundamental steady state has to be increased in order for the agent being indifferent between living in the fundamental or the bubbly economy. Bubbles always raise overall welfare for the given parameter values as shown by the lower left graph. When looking only at the ex-post realized utility of entrepreneurs, then the picture slightly changes. In the bubbly economy with bubble creation entrepreneurs are always better off, but less then the average. In the bubbly economy without bubble creation entrepreneurs are actually worse off. This is explained by lower wages, lower pensions, and higher borrowing costs.

The effects of isolated changes in other parameters are summarized in figure IV.[24] The bubbly steady state without bubble creation is almost always contractionary, implying again that the *liquidity channel* alone is not strong enough to dominate the *crowding-out channel*. The bubbly steady state with bubble creation, on the other hand, is almost always expansionary.

The effect of increasing the population share of entrepreneurs, $\eta$, is that capital stock and

[24]See figure X to XIII in the appendix for more details.

output always increase while the severeness of the financial friction is reduced. In the upper left figure it seems that for large values of $\eta$ the bubbly steady state without bubble creation is expansionary, but as soon as the red dotted line crosses the yellow dashed line the bubble becomes negative, as can be seen in figure X in the appendix, implying that a bubbly equilibrium ceases to exist. Further, the share of entrepreneurs $\eta$ in the population has to be sufficiently small (below 10 percent with bubble creation and below 21 percent without bubble creation) for bubbles to exist. When the share of entrepreneurs rises, capital stock increases, the demand for credit rises, while the supply declines, increasing $R^d$ until $R^d \leq gn$ (condition for bubbles to exist) is not satisfied anymore.

Varying the period when the entrepreneur shock materializes, $J^E$, is another way to study the strength of the *liquidity channel*. When the *liquidity channel* is strong, one should observe that the difference of output between the fundamental steady and the bubbly steady state without bubble creation should decrease and become negative. This is, however, not the case as the gap remains fairly constant. Similarly, the output gap between the bubbly steady state with bubble creation and the fundamental steady state is decreasing in $J^E$.

The lower left figure shows one case when the *liquidity channel* can be strong enough for a bubbly economy without bubble creation to be expansionary. This is the case when the depreciation rate $\delta$ is above 0.63, which is an empirically highly implausible value at annual frequency.

Changing $\sigma$ and hence the intertemporal elasticity of substitution affects the credit supply and demand elasticities and might therefore change the strength of the *liqiudity channel*. This is, however, not the case. For values of $\sigma$ between 0.1 and 4 output in the bubbly steady state without bubble creation is always below output in the fundamental steady state.

## 5. *Bubble-driven* business cycles

### 5.1. Computation

The dynamic model is solved with the method proposed by Boppart *et al.* (2018). Accordingly, the solution to the stochastic dynamic system is obtained by solving transition paths of the non-stochastic system, starting from an economy that is being hit by a so-called MIT shock. An MIT shock is a small and unanticipated shock. Assuming that the dynamic system is linear in the aggregate shocks, the scaled transition path is then used to study the response of the economy to different combinations of shocks. The benefit of the method is that all nonlinearities that are not associated with aggregate uncertainty are captured, allowing to solve complex heterogeneous agent models with many aggregate shocks.

The solution of a transition for a given MIT shock is similar to the steady state solution, except that the household problem incorporates now time-dependent aggregate prices and that the outer loop with $R_t, R_t^d$ and $beq_t$ has to be solved for a transition of $T$ periods. I set $T = 3 \times J$, yielding a system of $9 \times J = 711$ equations. This system is solved with a modified Netwon-Raphson method. I calculate the Jacobi matrix once and update it with Broyden's method. Global convergence is further improved by using a line search algorithm and backtracking.[25] Since the Jacobi matrix is block-diagonal, I reduce the computational time by a factor of around 15 by calculating only some columns of the Jacobi matrix with a forward difference formula. Then, the remaining elements of the Jacobi are obtained by linearly inter- and extrapolating along all k-diagonals which makes use of the fact that derivatives are similar and converge to a steady state value along the diagonal dimension.[26]

The crucial assumption of this approach is that the model is linear in the aggregate shocks. As recommended by Boppart *et al.* (2018), I have tested the linearity assumption by computing impulse response functions (IRF) to innovations of different magnitudes and opposing signs. Figure XIV in the appendix shows IRFs of output to positive and negative innovations of 0.1 and 2 times the standard deviation of the TFP process $Z^Y$. Variables are expressed as log-differences that are scaled by their respective innovation. The lines are almost identical, supporting the linearity assumption for TFP shocks. Figure XV in the appendix repeats the same exercise with sentiment shocks $Z^B$. For small shocks the linearity assumption seems to be valid as the negative and positive shock of 0.1 standard deviations yield the same IRF. For large positive (negative) shocks, however, the IRF of output is becoming larger (smaller). It would be insightful to compare these results to a global (projection) method. The results partially support the linearity assumption in this model, showing that the method put forward by Boppart *et al.* (2018) can also be applied to large-scale overlapping generation models with aggregate uncertainty and financial frictions.

In the following I focus on the economy where the inferior production technology is relevant by setting $\gamma$ to a value of 1.02, which is slightly above the steady state value of $R^d$. I will show how the impulse response functions change when $\gamma = 0$. All other parameters remain untouched and the matched moments do almost not change.

### 5.2. IRF to TFP shock

Figure V shows the IRF to a one percent TFP shock. Output, wages and the return on capital all increase on impact. As labor supply depends positively on wages, it increases also immediately, amplifying the effect of TFP on output. Since a higher return on capital raises the income of entrepreneurs, their demand for credit rises and investment increases strongly because entrepreneurs are borrowing constrained – a result of the underlying financial multiplier mechanism (Kiyotaki and Moore, 2012). Wealth, in contrast to output, increases only

[25] See Press *et al.* (2007) for details on the numerical methods.

[26] I am grateful to Matthew Rognlie for making me aware of this faster calculation of the Jacobi matrix.

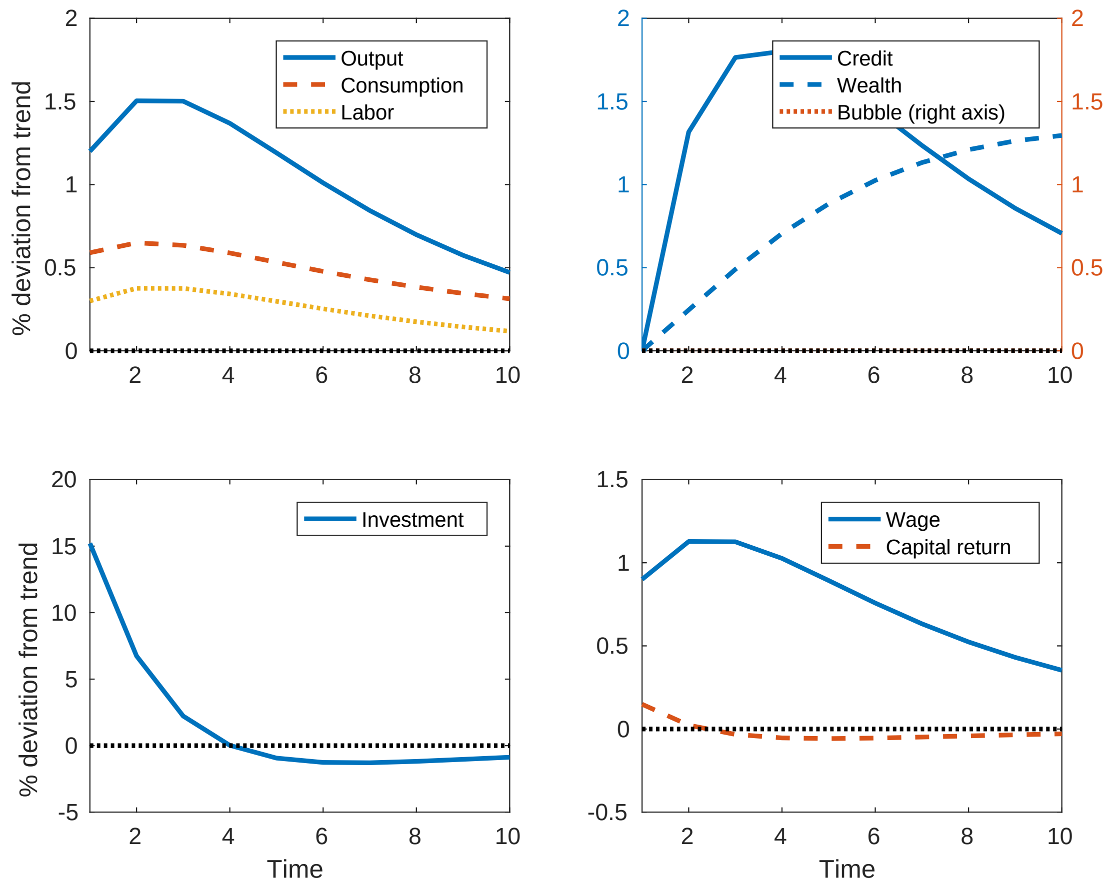



**Figure V:** TFP-driven business cycle

slowly and not more than by 1 percent because it is driven by capital accumulation only. The volatility of labor, which empirically is in the range of output, is too low, because I calibrated the Frisch elasticity of labor supply according to more conservative micro-studies, and not in order to match the macro-elasticity. TFP shocks are amplified by the financial friction, but cannot explain strong movements in aggregate wealth.

### 5.3. IRF to bubble shock

Figure V shows the IRF to a positive market sentiment shock. The economy enters into a boom as output, labor, consumption, and wages increase. Agents become very optimistic, implying that the bubble, which is very small in the steady state, grows in the first period by 600 percent. This explains the immediate increase in credit and aggregate wealth. Credit increasing because the bubble is financed by credit and therefore the increase in wealth appears small in the figure, but wealth increases on impact by more than 0.5 percent. The positive sentiment shock increases the value of new bubbles, raising entrepreneurial collateral, and implying higher credit demand and investment. This is the *bubble-creation channel*. After the initial boom the capital stock rises, pushing down the rate of return on capital, and leading to investment that is below the steady state level in the subsequent periods.

In this stochastic setting another channel through which bubbles affect output operates. When the existing bubble increases in its value after the positive sentiment shock, this raises wealth of the entrepreneurs that hold the existing bubble, implying that their firm value increases and that they hence invest part of this additional wealth in new capital.[27] When a bubble bursts, this *stochastic channel* of bubbles is reversed, implying that entrepreneurial wealth is destroyed and investment declines sharply.

When the inferior technology is not used, $\gamma = 0$, then sentiment shocks that raise the value of bubbles reduce output for the first 18 periods. The main difference is that with $\gamma = 1.02$ it holds that $R^d_t = \gamma = 1.02$, but with $\gamma = 0$ the rate of return on credit is not equal to $\gamma$ and fluctuates over time. Hence, under $\gamma = 0$ bubbles are contractionary for 18 periods before becoming expansionary, as can be seen in figure XVI in the appendix.[28] Since the model is better capable of generating *bubble-driven* business cycles under $\gamma = 1.02$, I follow this assumption in the remaining analysis.

### 5.4. The two recent US boom-bust episodes

I apply the model to study the recent US Dotcom and housing boom-bust episodes by computing the sequences of innovations to TFP and sentiment shocks, $\{\epsilon^Y_t, \epsilon^B_t\}^{2010}_{t=1990}$, that are necessary for the model to exactly replicate the observed time series of output and wealth,$\{Y_t, W_t\}^{2010}_{t=1990}$, during the 21 years from 1990 to 2010. This procedures is very convenient when the model is solved with the solution method by Boppart *et al.* (2018) because the IRFs are linear in the innovations. Linearity in innovations of IRF for output and wealth then implies that one can determine the sequence of innovations merely by solving a linear equation system.

Determining 42 innovations to match 42 data points is in itself not very interesting, but what does the model say about the aggregate bubble during these boom-bust episodes? This can be determined by the following decomposition of the IRF of wealth. Net worth $\mathscr{W}$ is given by the sum of the aggregate capital stock, bubbles, minus total debt, plus wealth of savers (part

[27]This channel is absent in a deterministic setting and also in the models by Farhi and Tirole (2012) and Martin and Ventura (2016, 2018). The reason is that if entrepreneurs exist only for two periods, then an unexpectedly strong growing bubble only increases their consumption, but not their investment in capital.

[28]The system displays oscillatory convergence towards the steady state. This is the result of the financial friction and is also present in the original paper by Kiyotaki and Moore (1997). When the model is solved for a population share of entrepreneurs of 10 percent the constraint is not binding, $R = R^d$, and the economy resembles the frictionless life-cycle RBC model (Ríos-Rull, 1996). Then the economy convergences without oscillations (not shown).

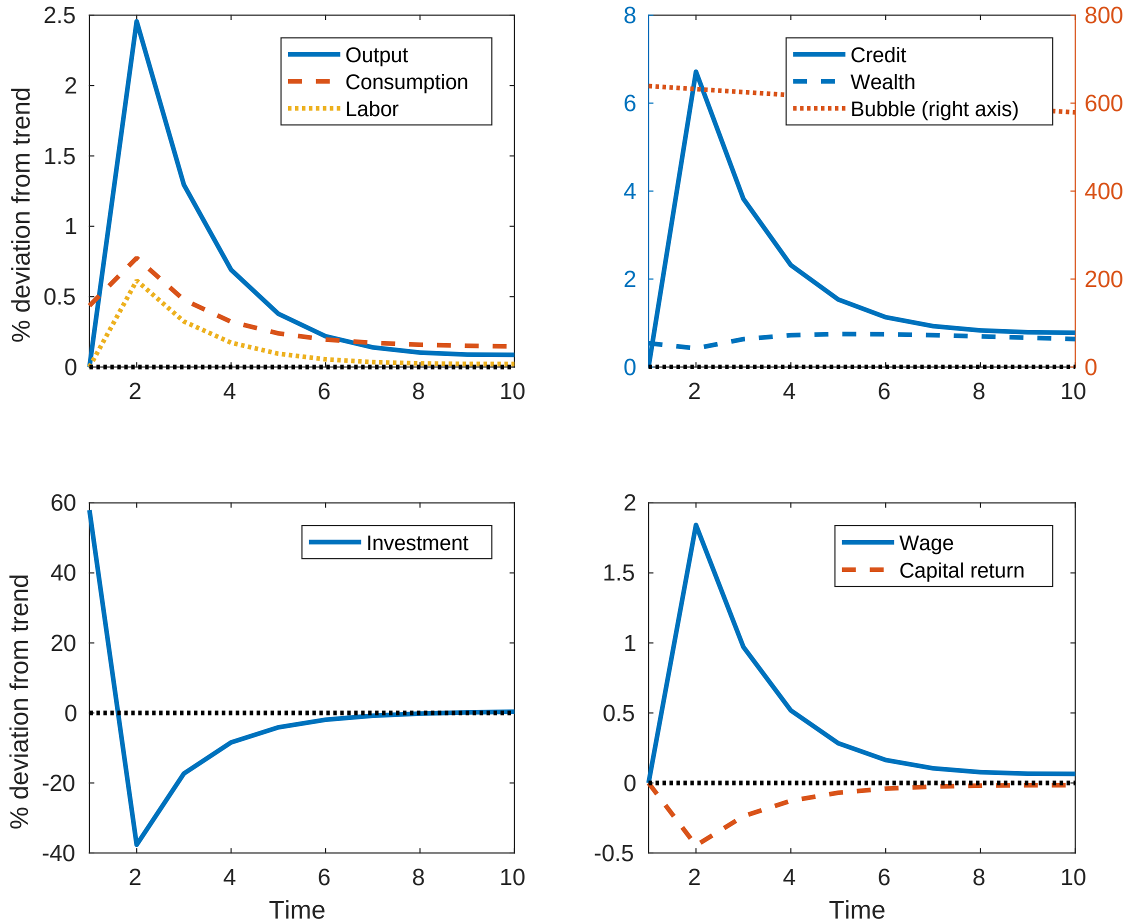



**Figure VI:** A bubble-driven business cycle

of which might be invested in the inferior production technology)

$$\mathscr{W}_t \equiv K_t + B_t - D_t + A_t.$$

The deviation of wealth from its trend can then be decomposed into a bubble component and a fundamental component

$$\underbrace{\frac{\mathscr{W}_t - \bar{\mathscr{W}}}{\bar{\mathscr{W}}}}_{\text{Total wealth}} = \underbrace{\frac{B_t - \bar{B}}{\bar{\mathscr{W}}}}_{\text{bubbly wealth}} + \underbrace{\frac{(K_t - \bar{K}) - (D_t - \bar{D}) + (A_t - \bar{A})}{\bar{\mathscr{W}}}}_{\text{fundamental wealth}},$$

where bars on top of variables denote steady state values. The bubble component shows how much of the percentage deviation of total wealth from its trend is due to the bubble. Figure VII plots both the total wealth change, $\frac{\mathscr{W}_t - \bar{\mathscr{W}}}{\bar{\mathscr{W}}}$, and the part driven by the bubble, $\frac{B_t - \bar{B}}{\bar{\mathscr{W}}}$. On average, 31 percent of the deviations of aggregate wealth from its trend are due to the rational bubble, as predicted by the model.

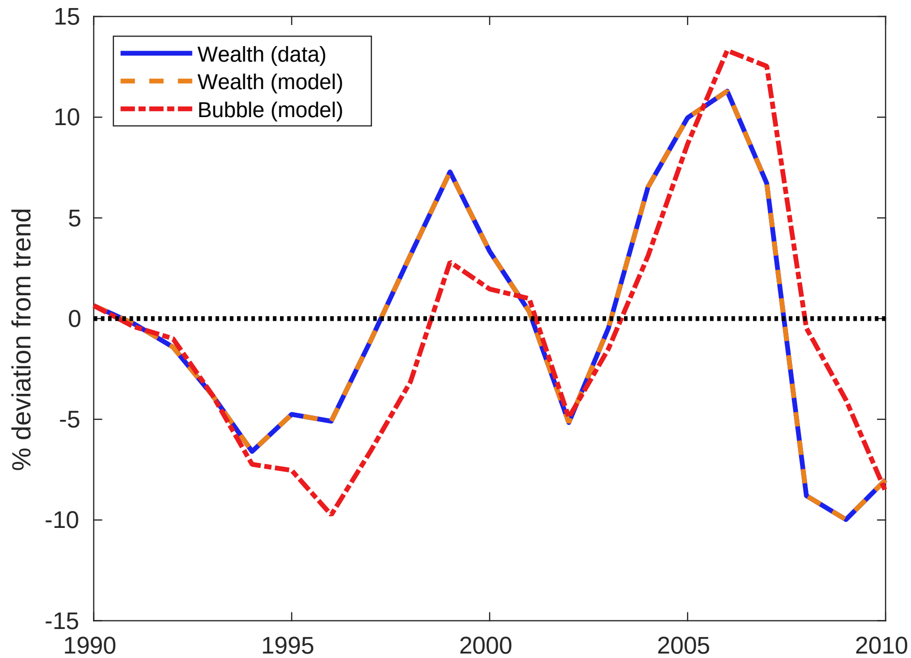



**Figure VII:** Model-implied bubble during the last two US boom-bust episodes

Repeating the same exercise only with TFP shocks and matching the empirical series for GDP shows that TFP shocks alone are not sufficient to explain both the boom-bust in wealth and the fluctuations in GDP during this episode. As can be seen in figure XVII in the appendix the model implies that the wealth series fluctuates far too little and often moves in the other direction than its empirical counterpart. This supports the conclusion that rational bubbles were the main driver of the two recent boom-bust episodes in the US.

## 6. Conclusion

Boom-bust episodes feature strong fluctuations in aggregate wealth, driven by asset price dynamics, together with co-moving fluctuations in real output. This paper explains these observed boom-bust episodes through the lens of a quantitative life-cycle RBC model with rational and stochastic bubbles. The calibrated model can generate *bubble-driven* business cycles, where growing bubbles go hand in hand with rising labor, capital, output, and consumption and where bursting bubbles lead to recessions.

It has been shown that bubbles can only exist in this economy when the rate of return to credit is smaller than the growth rate of the economy, which was on average the case for the postwar US economy. In order to endogenously obtain sufficiently low rates of return on credit, the financial friction has to be sufficiently pronounced, captured by a population share of entrepreneurs equal to 0.1 percent. The effect of bubbles on output works through three different channels: the *crowding-out, liquidity*, and *bubble-creation channels*. Through the *crowding-out channel* bubbles have a contractionary effect on the economy, while bubbles can lead to expansions through both the *liquidity* and *bubble-creation channels*. In the calibrated model bubbles increase real economic activity when they grow, and reduce economic activity when bursting, hence the *crowding-out channel* is dominated by the other two channels. More interestingly, I show that the *liquidity channel* is very weak under most of the plausible parameter constellations and that the *bubble-creation channel* is necessary for plausible bubbles to exist.

The model is also applied to study the two recent boom-bust episodes of the US history. By replicating both the fluctuations in real output and aggregate wealth I am able to use the model in order to study the relevance of rational bubbles during this episode. I show that *bubble-driven* business cycles can explain on average one third of the deviations of aggregate wealth from its trend during these two boom-bust episodes and that this also resulted in pronounced fluctuations in real GDP.

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# Appendix

## A. Model

### A.1. Household problem

**Savers** The vector $\Omega$ contains all aggregate state variables that are relevant for the household. The saver's problem in period $j = J^E - 1$ is different from the saver's problem in other periods because in $j = J^E - 1$ savers take the probability of becoming an entrepreneur in the next period into account. I first show the problem for all other periods, and then for the period before the entrepreneur shock realizes. The dynamic program for all savers of age $j \neq J^E - 1$ hence reads[29]

$$V^S(j, m; \Omega) = \max_{c,l,a'} u(c, l) + \beta \varrho_{j+1} \mathbb{E} V^S(j+1, m'; \Omega')$$
$$s.t. \;\; m' = (R^d)' a' \text{ and } (9).$$

The Lagrangian reads

$$\begin{aligned} \mathscr{L} =& u(c, l) + \beta \varrho_{j+1} \mathbb{E} V^S(j+1, (R^d)' a'; \Omega') \\ &+ \lambda \left[ (1-\tau) e_j l w + I_j pen + R^d a - c - a' \right] + \omega a'. \end{aligned}$$

The FOC are

$$\begin{aligned} l &= \left[ \frac{(1-\tau) e_j w}{g^t \theta} \right]^{\frac{1}{\chi}} \qquad (16) \\ u_c(c, l) &= \beta \varrho_{j+1} \mathbb{E} \frac{\partial V^S(j+1, R^d a'; \Omega')}{\partial a'} + \omega \\ \omega a' &= 0. \end{aligned}$$

The first equation shows that individual labor supply is a function of after-tax wages and labor productivity. Since for retired households $e_j = 0$ this also implies that $l_j = 0$ for retired households.

The Envelope condition yields

$$\frac{\partial V^S(j, R^d a; \Omega)}{\partial a} = u_c(c, l) R^d.$$

Hence, the Euler equation reads

$$u_c(c, l) = \beta \varrho_{j+1} \mathbb{E} u_c(c', l') (R^d)' + \omega.$$

[29] I omit indexes on variables and let variables with primes denote next periods values.

In period $j = J^E - 1$ the saver knows that she might become an entrepreneur in the next period and the problem therefore is given by

$$V^S(J^E-1,m;\Omega) = \max_{c,l,a'} u(c,l) + p^E \beta \varrho_{J^E} \mathbb{E} V^E(J^E,m';\Omega') + (1-p^E)\beta \varrho_{J^E} \mathbb{E} V^S(J^E,m';\Omega')$$
$$s.t. \quad m' = (R^d)'a' \text{ and (9).}$$

The Lagrangian reads

$$\begin{aligned}\mathscr{L} =& u(c,l) + p^E \beta \varrho_{J^E} \mathbb{E} V^E(J^E,(R^d)'a';\Omega') + (1-p^E)\beta \varrho_{J^E} \mathbb{E} V^S(J^E,(R^d)'a';\Omega') \\ &+ \lambda_1 \left[(1-\tau)e_j l w + R^d a - c - a'\right] + \omega a'.\end{aligned}$$

The FOC are

$$\begin{aligned} l &= \left[\frac{(1-\tau)e_j w}{\theta g^t}\right]^{\frac{1}{\chi}} \\ u_c(c,l) &= (1-p^E)\beta \varrho_{J^E} \mathbb{E} \frac{\partial V^S(J^E,(R^d)'a';\Omega')}{\partial a'} + p^E \beta \varrho_{J^E} \mathbb{E} \frac{\partial V^E(J^E,(R^d)'a';\Omega')}{\partial a'} + \omega \\ \omega a' &\geq 0. \end{aligned}$$

Applying the savers' and entrepreneurs' envelope theorems (see next paragraph) gives the Euler equation

$$u_c(c,l) = \beta \varrho_{J^E} \mathbb{E}\left[p^E u_c({c_E}',l') + (1-p^E) u_c({c_S}',l')\right](R^d)' + \omega,$$

where the subscripts $E,S$ denote next period consumption supply by entrepreneurs and savers. The expression in square brackets is expected marginal utility of consumption.

**Entrepreneurs** The entrepreneur's dynamic program reads

$$V^E(j,m;\Omega) = \max_{c,l,k^{\nu\prime},b',d'} u(c,l) + \beta \varrho_{j+1} \mathbb{E} V^E(j+1,m';\Omega')$$
$$s.t. \quad m' = R'k^{\nu\prime} - (R^d)'d' + Q'\left[b' + (b^N)'\right] \text{ and (11).}$$

The Lagrangian is

$$\begin{aligned}\mathscr{L} =& u(c,l) + \beta \varrho_{j+1} \mathbb{E} V^E(j+1,m';\Omega') + \lambda\left[(1-\tau)e_j l w + m - c - k^{\nu\prime} - b' + d'\right] \\ &+ \mu\left[R'k^{\nu\prime} - (R^d)'d' + Q'\left(b' + b^{N\prime}\right) - m'\right] \\ &+ \omega\left[\frac{\mathbb{E}p'}{p}(1-\delta)k^{\nu\prime} + \mathbb{E}Q'\left(b' + b^{N\prime}\right) - R^{d\prime} d'\right].\end{aligned}$$

The FOC yield the same labor supply (16) as for savers and

$$
\begin{aligned}
u_c(c,l) &= \mathbb{E}\left[\mu R' + \omega \frac{p'}{p}(1-\delta)\right] \\
u_c(c,l) &= \mathbb{E}\left[(\mu+\omega)Q'\right] \\
u_c(c,l) &= \mathbb{E}\left[(\mu+\omega)R^{d\prime}\right] \\
\mu &= \beta \varrho_{j+1} \mathbb{E} \frac{\partial V^E(j+1,m';\Omega')}{\partial m'} \\
0 &= \omega\left[\frac{\mathbb{E}p'}{p}(1-\delta)k' + \mathbb{E}Q'\left(b'+b^{N\prime}\right) - R^{d\prime}d'\right].
\end{aligned}
$$

The Envelope condition yields

$$
\frac{\partial V^E(j,m;\Omega)}{\partial m} = \lambda = u_c(c,l).
$$

The FOC consists of three Euler equations, one for each asset, and a complementary slackness condition associated with the borrowing constraint

$$
\begin{aligned}
u_c(c,l) &= \mathbb{E}\left[\beta \varrho_{j+1} u_c(c',l')R' + \omega \frac{p'}{p}(1-\delta)\right] \\
u_c(c,l) &= \mathbb{E}\left[\beta \varrho_{j+1} Q' u_c(c',l') + \omega Q'\right] \\
u_c(c,l) &= \mathbb{E}\left[\beta \varrho_{j+1} R^{d\prime} u_c(c',l') + \omega R^{d\prime}\right] \\
0 &= \omega\left[\frac{\mathbb{E}p'}{p}(1-\delta)k' + \mathbb{E}Q'\left(b'+b^{N\prime}\right) - R^{d\prime}d'\right].
\end{aligned}
$$

### A.2. Labor market

Aggregating labor supply across individuals yields

$$
L_t = \sum_{j=1}^{J^w} l_{t,j} e_j N_{t,j} = \left[\frac{(1-\tau)w_t/g^t}{\theta}\right]^{\frac{1}{\chi}} n^t \sum_{j=1}^{J^w} e_j^{\frac{1+\chi}{\chi}} N_{0,j}.
$$

The ratio of labor to total population, $L_t/n^t$, is constant if wages grow with the factor $g$, which is the case along the balanced growth path. Inserting the competitive wage given by (4) and rearranging yields $L_t$ as a function of only one endogenous variable, $K_t$:

$$
L_t = n^t \left(\frac{1-\alpha}{\theta}(1-\tau)Z_t^Y\right)^{\frac{1}{\alpha+\chi}} \left(\sum_{j=1}^{J^w} e_j^{\frac{1+\chi}{\chi}} N_{0,j}\right)^{\frac{\chi}{\alpha+\chi}} \left(\frac{K_t}{g^t n^t}\right)^{\frac{\alpha}{\alpha+\chi}}.
$$

### A.3. Normalization

The notation of normalized variables is as follows: A normalized aggregate variable $X_t$ is defined by $x_t \equiv \frac{X_t}{g^t n^t}$. A normalized individual-level variable $x_{t,j}$ is defined as $\widetilde{x}_{t,j} \equiv \frac{x_{t,j}}{g^t}$.

**Production**

$$l_t \equiv \frac{L_t}{n^t} = \left(\frac{1-\alpha}{\theta}(1-\tau)Z_t^Y\right)^{\frac{1}{\alpha+\chi}} \left(\sum_{j=1}^{J^w} e_j^{\frac{1+\chi}{\chi}} N_{0,j}\right)^{\frac{\chi}{\alpha+\chi}} k_t^{\frac{\alpha}{\alpha+\chi}}$$

$$y_t \equiv \frac{Y_t}{g^t n^t} = Z_t^Y k_t^\alpha l_t^{1-\alpha}$$

$$R_t = \frac{p_{t+1}(1-\delta) + \alpha Z_t^Y k_t^{\alpha-1} l_t^{1-\alpha}}{p_t}$$

$$\widetilde{w}_t \equiv \frac{w_t}{g^t} = (1-\alpha) Z_t^Y k_t^\alpha l_t^{-\alpha}$$

**Capital producer**

$$\left[1 - \frac{\psi}{2}(gn)^2\left(\frac{i_t}{i_{t-1}}-1\right)\left(3\frac{i_t}{i_{t-1}}-1\right)\right] q_t = 1 + \beta \mathbb{E}_t \phi\left(\frac{i_{t+1}}{i_t}-1\right)(gn)^3\left(\frac{i_{t+1}}{i_t}\right)^2 q_{t+1}.$$

$$gnk_{t+1} = (1-\delta)k_t + i_t\left[1 - \frac{\phi}{2}(gn)^2\left(\frac{i_t}{i_{t-1}}-1\right)^2\right]$$

**Equilibrium**

$$\sum_{j=1}^{J} e_j l_{j,t} N_{0,j} = \sum_{j=J^E+1}^{J} h_{j,t} \frac{p^E N_{0,j}}{\varrho_j} = l_t$$

$$k_t = \sum_{j=J^E+1}^{J} \frac{p^E N_{0,j}}{\varrho_j} \widetilde{k}_{t,j}$$

$$b_t = \sum_{j=J^E}^{J} \frac{p^E N_{0,j}}{\varrho_j} \widetilde{b}_{t,j}$$

$$\sum_{j=J^E+1}^{J} \frac{p^E N_{0,j}}{\varrho_j} \widetilde{d}_{t,j} = d_t = \sum_{j=1}^{J^E} \frac{N_{0,j}}{\varrho_j} \widetilde{a}_{t,j} + \sum_{s=J^E+1}^{J} \frac{(1-p^E)N_{0,j}}{\varrho_j} \widetilde{a}_{t,j} + \sum_{s=J^w+1}^{J} \frac{p^E N_{0,j}}{\varrho_j} \widetilde{a}^E_{t,j}$$

$$y_t = c_t + i_t - (1-\delta)k_t$$

$$b_t \geq 0, k_t \geq 0.$$

**Bubbles**

$$gnb_{t+1} \equiv gn\frac{B_t}{g^t n^t} = Q_t(b_t + b_t^N)$$
$$\widetilde{b}_{t+1}^N \equiv \frac{b_{t+1}^N}{g^t} = Z_t^B \nu$$
$$b_t^N \equiv \frac{B_t^N}{g^t n^t} = p^E N_{0,J^E} \widetilde{b}_{t+1}^N.$$

**Government**

$$\widetilde{pen}_t \equiv \frac{pen_t}{g^t} = \xi(1-\tau)\widetilde{w}_t(1+\phi)l_t.$$

**Accidental bequests**

$$\widetilde{m}_{t,1,S} = \frac{1-\varrho_j}{\varrho_j}\left\{\sum_{j=2}^{J^E}\frac{N_{0,j}}{N_{0,1}}\widetilde{m}_{t,j,S} + \sum_{j=J^E+1}^{J}\frac{N_{0,j}}{N_{0,1}}\left[\left(1-p^E\right)\widetilde{m}_{t,j,S} + p^E\widetilde{m}_{t,j,E}\right]\right\}.$$

**Savers** The savers' constraints are given by

$$\widetilde{c}_{t,j} + g\widetilde{a}_{t+1,j+1} = (1-\tau)\widetilde{w}_t e_j l_{j,t} + R_t^d \widetilde{a}_{t,j} + \mathbb{I}_j^r \widetilde{pen}_{t,j}.$$

Individual labor supply for both savers and entrepreneurs is given by

$$l_{t,j} = \left[\frac{(1-\tau)e_j\widetilde{w}_t}{\theta}\right]^{\frac{1}{\chi}}.$$

Marginal utility of consumption can then be expressed as

$$u_c(\bullet) = (g^t)^{-\sigma}\left[\widetilde{c}_{t,j} - \tilde{G}_{t,j}\right]^{-\sigma},$$

with

$$\tilde{G}_{t,j} \equiv \frac{\theta^{-\frac{1}{\chi}}}{1+\chi}\left[(1-\tau)e_j\widetilde{w}_t\right]^{\frac{1+\chi}{\chi}}.$$

The symbol " $\bullet$ " represents all arguments of the respective function.

The savers' Euler equations read

$$g^\sigma\left(\widetilde{c} - \tilde{G}\right)^{-\sigma} = \beta\varrho_{j+1}\mathbb{E}\left(\widetilde{c}' - \tilde{G}'\right)^{-\sigma}(R^d)' + \widetilde{\omega} \qquad j \neq J^E - 1$$
$$g^\sigma\left(\widetilde{c} - \tilde{G}\right)^{-\sigma} = \beta\varrho_{J^E}\mathbb{E}\left[p^E\left(\widetilde{c}_E' - \tilde{G}'\right)^{-\sigma} + (1-p^E)\left(\widetilde{c}_S' - \tilde{G}'\right)^{-\sigma}\right](R^d)' + \widetilde{\omega} \qquad j = J^E - 1,$$

where $\widetilde{\omega} \equiv g^{(1+t)\sigma}\omega$.

**Entrepreneurs** Cash-on-hand reads

$$\widetilde{m}_{t,j}=\begin{cases}R^d_t\widetilde{a}_{t,j} & \text{if } j=J^E\\ R_t\widetilde{k}^v_{t,j}-R^d_t\widetilde{d}_{t,j}+Q_t(\widetilde{b}_{t,j}+\widetilde{b}^N_{t,j}) & \text{if } j\in\{J^E+1,J^E+2,...,J^w+1\}\\ R^d_t\widetilde{a}^E_{t,j} & \text{if } j>J^w+1\end{cases}$$

and the entrepreneurs' constraints are given by

$$\begin{aligned}&\widetilde{c}_{t,j}+\mathbb{I}^r_j g\widetilde{a}^E_{t+1,j+1}+\left(1-\mathbb{I}^r_j\right)g\left[\widetilde{k}^v_{t+1,j+1}+\widetilde{b}_{t+1,j+1}-\widetilde{d}_{t+1,j+1}\right]=(1-\tau)\widetilde{w}_t e_j l_{j,t}\\ &+\widetilde{m}_{t,j}+\mathbb{I}^r_j\widetilde{pen}_{t,j}\\ &R^d_{t+1}\widetilde{d}_{t+1,j+1}\leq\frac{p_{t+1}}{p_t}(1-\delta)\widetilde{k}^v_{t+1,j+1}+Q_{t+1}(\widetilde{b}_{t+1,j+1}+\widetilde{b}^N_{t+1,j+1}).\end{aligned}$$

The FOC for entrepreneurs are given by

$$\begin{aligned}g^\sigma\left(\widetilde{c}-\tilde{G}\right)^{-\sigma}&=\mathbb{E}\left[\beta\varrho_{j+1}\left(\widetilde{c}'-\tilde{G}'\right)^{-\sigma}R'+\widetilde{\omega}\frac{p'}{p}(1-\delta)\right]\\ g^\sigma\left(\widetilde{c}-\tilde{G}\right)^{-\sigma}&=\mathbb{E}\left[\beta\varrho_{j+1}Q'\left(\widetilde{c}'-\tilde{G}'\right)^{-\sigma}+\widetilde{\omega}Q'\right]\\ g^\sigma\left(\widetilde{c}-\tilde{G}\right)^{-\sigma}&=\mathbb{E}\left[\beta\varrho_{j+1}R^{d'}\left(\widetilde{c}'-\tilde{G}'\right)^{-\sigma}+\widetilde{\omega}R^{d'}\right]\\ 0&=\widetilde{\omega}\left[\frac{\mathbb{E}p'}{p}(1-\delta)\widetilde{k}'+\mathbb{E}Q'\left(\widetilde{b}'+(\widetilde{b}^N)'\right)-R^{d'}\widetilde{d}'\right].\end{aligned}$$

### A.4. Computation of household problem

This section describes the computation of the household problems in the economy without uncertainty. In the computation an outer loop solves for market clearing prices and aggregate quantities while an inner loop takes these as given and solves the household problems.

**Savers** Given the series of wages $\{w_t\}_{t=0}^{\infty}$ calculate labor supply and disutility from working

$$\begin{aligned}l_j&=\left[\frac{(1-\tau)e_j\widetilde{w}_j}{\theta}\right]^{\frac{1}{\chi}}\\ \widetilde{G}_j&=\frac{\theta^{-\frac{1}{\chi}}}{1+\chi}\left[(1-\tau)e_j\widetilde{w}_j\right]^{\frac{1+\chi}{\chi}}\end{aligned}$$

where I index prices and aggregate quantities not by time $t$ but by household age $j$ (given the year of birth one can obtain time $t$).

Define labor and pension income as $y_j \equiv (1-\tau)\widetilde{w}_j e_j l_j + \mathbb{I}_j^r \widetilde{pen}_j$ and write the FOC

$$\begin{aligned}
\widetilde{c}_j\left(\widetilde{m}_j\right) + g\widetilde{a}_{j+1}\left(\widetilde{m}_j\right) &= y_j + \widetilde{m}_j \\
g^{\sigma}\left(\widetilde{c}_j\left(\widetilde{m}_j\right) - \widetilde{G}_j\right)^{-\sigma} &= \begin{cases} \beta\varrho_{j+1}\left(\widetilde{c}_{j+1}\left(\widetilde{m}_{j+1}\right) - \widetilde{G}_{j+1}\right)^{-\sigma} R_{j+1}^d + \widetilde{\omega}_{j+1} & \text{if } j \neq J^E - 1 \\ \beta\varrho_{j+1}\left[p^E\left(\widetilde{c}_{E,j+1}\left(\widetilde{m}_{j+1}\right) - \tilde{G}_{j+1}\right)^{-\sigma} + (1-p^E)\left(\widetilde{c}_{S,j+1}\left(\widetilde{m}_{j+1}\right) - \tilde{G}_{j+1}\right)^{-\sigma}\right] R_{j+1}^d + \widetilde{\omega}_{j+1} & \text{else} \end{cases} \\
\widetilde{m}_{j+1} &= R_{j+1}^d a_{j+1} \\
\widetilde{\omega}_{j+1}\widetilde{m}_{j+1} &= 0
\end{aligned}$$

where $\widetilde{c}_j\left(\widetilde{m}_j\right)$ and $\widetilde{a}_{j+1}\left(\widetilde{m}_j\right)$ are the policy functions.

**Entrepreneurs** Labor supply, disutility from work, and earning ($y_j$) are the same as for savers. The FOC read (note that $Q_{j+1} = R_{j+1}^d$)

$$\begin{aligned}
\widetilde{c}_j + g\left[\widetilde{k}_{j+1}^v + \widetilde{b}_{j+1} - \widetilde{d}_{j+1}\right] &= y_j + \widetilde{m}_j \\
g^{\sigma}\left(\widetilde{c}_j - \tilde{G}_j\right)^{-\sigma} &= \beta\varrho_{j+1} R_{j+1}\left(\widetilde{c}_{j+1} - \tilde{G}_{j+1}\right)^{-\sigma} + \widetilde{\omega}_{j+1}\frac{p_{j+1}}{p_j}(1-\delta) \\
g^{\sigma}\left(\widetilde{c}_j - \tilde{G}_j\right)^{-\sigma} &= \beta\varrho_{j+1} R_{j+1}^d\left(\widetilde{c}_{j+1} - \tilde{G}_{j+1}\right)^{-\sigma} + \widetilde{\omega}_{j+1} R_{j+1}^d \\
0 &= \widetilde{\omega}_{j+1}\left[\frac{p_{j+1}}{p_j}(1-\delta)\widetilde{k}_{j+1} + R_{j+1}^d\left(\widetilde{b}_{j+1} + \widetilde{b}_{j+1}^N - \widetilde{d}_{j+1}\right)\right] \\
\widetilde{m}_{j+1} &= R_{j+1}\widetilde{k}_{j+1}^v + R_{j+1}^d\left(\widetilde{b}_{j+1} + \widetilde{b}_{j+1}^N - \widetilde{d}_{j+1}\right)
\end{aligned}$$

Cash-on-hand in the first period, $\widetilde{m}_{J^E} = R_{J^E}^d a_{J^E}$, is given from the saver problem.

If $R_{j+1} = R_{j+1}^d$, then $\widetilde{\omega}_{j+1} = 0$ and the constraint is slack. If $R_{j+1} > R_{j+1}^d$, then $\widetilde{\omega}_{j+1} > 0$ and the constraint is binding. In the latter case it has to hold that $R^d \in \left((1-\delta)p_{j+1}/p_j, R\right)$ because otherwise entrepreneurs could run an infinite investment because the marginal collateral is larger than the marginal debt cost.

Define total end-of-period wealth of entrepreneurs as $\widetilde{a}_{E,j+1}$ ($m_j$ is beginning of period wealth). When the constraint is slack the problem simplifies to

$$\begin{aligned}
\widetilde{c}_j + g\widetilde{a}_{E,j+1} &= y_j + \widetilde{m}_j \\
g^{\sigma}\left(\widetilde{c}_j - \tilde{G}_j\right)^{-\sigma} &= \beta\varrho_{j+1} R_{j+1}^d\left(\widetilde{c}_{j+1} - \tilde{G}_{j+1}\right)^{-\sigma} \\
\widetilde{m}_{j+1} &= R_{j+1}^d\left(\widetilde{a}_{E,j+1} + \widetilde{b}_{j+1}^N\right)
\end{aligned}$$

This is exactly the same problem as for savers except that income is generated from $Q_j b_j^N$ (in the computation I feed in $\{b_j^N\}_{j=1}^J$ and set this sequence to zero for savers).

When the constraint is binding the problem simplifies to

$$\widetilde{c}_j + g\widetilde{a}_{E,j+1} = y_j + \widetilde{m}_j$$

$$
\begin{aligned}
g^{\sigma}\left(\widetilde{c}_j-\tilde{G}_j\right)^{-\sigma} &= \beta\varrho_{j+1}R^E_{j+1}\left(\widetilde{c}_{j+1}-\tilde{G}_{j+1}\right)^{-\sigma}\\
\widetilde{m}_{j+1} &= R^E_{j+1}\left(\widetilde{a}_{E,j+1}+\widetilde{b}^N_{j+1}\right)
\end{aligned}
$$

where the effective return to wealth faced by entrepreneurs reads

$$
R^E_{j+1}\equiv\frac{R_{j+1}-\frac{p_{j+1}}{p_j}(1-\delta)}{R^d_{j+1}-\frac{p_{j+1}}{p_j}(1-\delta)}R^d_{j+1}>R^d_{j+1}.
$$

From entrepreneurial wealth one obtains capital and debt as follows (indifferent wrt bubbles and debt)

$$
\begin{aligned}
\widetilde{k}^v_{j+1} &= \frac{R^d_{j+1}}{R^d_{j+1}-\frac{p_{j+1}}{p_j}(1-\delta)}\left(\widetilde{a}_{E,j+1}+\widetilde{b}^N_{j+1}\right)\\
\widetilde{d}_{j+1} &= \frac{\frac{p_{j+1}}{p_j}(1-\delta)}{R^d_{j+1}}\widetilde{k}_{j+1}+\left(\widetilde{b}_{j+1}+\widetilde{b}^N_{j+1}\right)
\end{aligned}
$$

If $R_{j+1}=R^d_{j+1}$, it follows that $R^E_{j+1}=R^d_{j+1}$ such that one has only to provide the sequence of $R^E_j$ and the constrained solution encompasses the unconstrained solution (no need to solve both). Backing out capital, debt, and bubbles from total wealth $a_E$ is, however, different. When the constraint is binding capital and debt are given by the above equations, otherwise they are indeterminate (the bubble is always indeterminate).

To summarize, for $j\neq J^E-1$ one can solve a generic problem

$$
\begin{aligned}
\widetilde{c}_j\left(\widetilde{m}_j\right)+g\widetilde{a}_{j+1}\left(\widetilde{m}_j\right) &= y_j+\widetilde{m}_j\\
g^{\sigma}\left(\widetilde{c}_j\left(\widetilde{m}_j\right)-\tilde{G}_j\right)^{-\sigma} &= \beta\varrho_{j+1}R_{j+1}\left(\widetilde{c}_{j+1}\left(\widetilde{m}_{j+1}\right)-\tilde{G}_{j+1}\right)^{-\sigma}+\widetilde{\mu}_{j+1}\\
\widetilde{m}_{j+1}\left(\widetilde{m}_j\right) &= R_{j+1}\left(\widetilde{a}_{j+1}\left(\widetilde{m}_j\right)+\widetilde{b}^N_{j+1}\right)\\
\widetilde{\mu}_{j+1}\widetilde{m}_{j+1}\left(\widetilde{m}_j\right) &= 0
\end{aligned}
$$

where $R_{j+1}=R^d_{j+1}$ for savers, $R_{j+1}=R^E_{j+1}$ for entrepreneurs and $\widetilde{b}^N_{j+1}=0$ for savers.

With the endogenous grid-point method $\widetilde{m}_{j+1}$ is exogenous and $\widetilde{m}_j$ endogenous

$$
\begin{aligned}
\widetilde{a}_{j+1}\left(\widetilde{m}_{j+1}\right) &= \frac{\widetilde{m}_{j+1}}{R_{j+1}}-\widetilde{b}^N_{j+1}\\
g^{\sigma}\left(\widetilde{c}_j\left(\widetilde{m}_{j+1}\right)-\tilde{G}_j\right)^{-\sigma} &= \beta\varrho_{j+1}R_{j+1}\left(\widetilde{c}_{j+1}\left(\widetilde{m}_{j+1}\right)-\tilde{G}_{j+1}\right)^{-\sigma}+\widetilde{\mu}_{j+1}\\
\widetilde{m}_j\left(\widetilde{m}_{j+1}\right) &= \widetilde{c}_j\left(\widetilde{m}_{j+1}\right)+g\widetilde{a}_{j+1}\left(\widetilde{m}_{j+1}\right)-y_j\\
\widetilde{\mu}_{j+1}\widetilde{m}_{j+1} &= 0
\end{aligned}
$$

When the constraint is not binding we get

$$\begin{aligned}
\widetilde{a}_{j+1}\left(\widetilde{m}_{j+1}\right) &= \frac{\widetilde{m}_{j+1}}{R_{j+1}} - \widetilde{b}^N_{j+1} \\
\widetilde{c}_j\left(\widetilde{m}_{j+1}\right) &= g\left(\beta \varrho_{j+1} R_{j+1}\right)^{-\frac{1}{\sigma}}\left(\widetilde{c}_{j+1}\left(\widetilde{m}_{j+1}\right) - \tilde{G}_{j+1}\right) + \tilde{G}_j \\
\widetilde{m}_j\left(\widetilde{m}_{j+1}\right) &= \widetilde{c}_j\left(\widetilde{m}_{j+1}\right) + g\widetilde{a}_{j+1}\left(\widetilde{m}_{j+1}\right) - y_j
\end{aligned}$$

Not binding ($\widetilde{\mu}_{j+1} = 0$) and $\widetilde{m}_{j+1} = 0$ (at this point the constraint just starts to become binding):

$$\begin{aligned}
\widetilde{a}_{j+1}\left(\widetilde{m}_{j+1} = 0\right) &= -\widetilde{b}^N_{j+1} \\
\widetilde{c}_j\left(\widetilde{m}_{j+1} = 0\right) &= g\left(\beta \varrho_{j+1} R_{j+1}\right)^{-\frac{1}{\sigma}}\left(\widetilde{c}_{j+1}\left(\widetilde{m}_{j+1}\right) - \tilde{G}_{j+1}\right) + \tilde{G}_j \\
\widetilde{m}_j\left(\widetilde{m}_{j+1} = 0\right) &= \widetilde{c}_j\left(\widetilde{m}_{j+1} = 0\right) - g\widetilde{b}^N_{j+1} - y_j
\end{aligned}$$

For values of $m_j$ that are smaller than the critical $\widetilde{m}_j\left(\widetilde{m}_{j+1} = 0\right)$, the constraint is binding, $\widetilde{m}_{j+1} = 0$, and consumption is simply $\widetilde{c}_j = m_j + g b^N_{j+1} + y_j$. Note, the lower bound on $m_j$ is $\underline{\mathrm{m}}_j \equiv -g b^N_{j+1} - y_j$ (otherwise consumption would be negative and therefore no household would achieve this low level of cash on hand in the first place). In the computation I add two grid points at $m_{j+1} = 0$, one where the constraint is not binding and one with a value of $m_j$ just slightly below the other one (because policy functions are linear in the constraint region and later one can just linearly interpolate and obtain very accurate solutions).

### A.5. Market clearing

I show that the n-th market is always cleared in this economy. First, I assume that profits from the capital producers are taxed lump sum and redistributed lump sum across households together with accidental bequests. These profits are denoted by

$$\pi^K_t \equiv p_t I^K_t - I_t.$$

Second, I aggregate the savers' budget constraints

$$\begin{aligned}
c_{t,j,S} + a_{t+1,j+1,S} &= (1-\tau) w_t e_j l_{j,t} + R^d_t a_{t,j,S} + \mathbb{I}^r_j pen_t + beq_t \\
N_{t,j} c_{t,j,S} + N_{t,j} a_{t+1,j+1,S} &= (1-\tau) w_t e_j N_{t,j} l_{j,t} + R^d_t N_{t,j} a_{t,j,S} + N_{t,j} \mathbb{I}^r_j pen_t + N_{t,j} beq_t \\
\sum_{j=1}^{J^E-1} N_{t,j} c_{t,j,S} + \sum_{j=1}^{J^E-1} N_{t,j} a_{t+1,j+1,S} &= (1-\tau) w_t \sum_{j=1}^{J^E-1} e_j N_{t,j} l_{j,t} + R^d_t \sum_{j=1}^{J^E-1} \varrho_j N_{t-1,j-1} a_{t,j,S} + beq_t \sum_{j=1}^{J^E-1} N_{t,j}
\end{aligned}$$

$$
\begin{aligned}
(1-p^E)\sum_{j=J^E}^{J} N_{t,j}c_{t,j,S} + (1-p^E)\sum_{j=J^E}^{J} N_{t,j}a_{t+1,j+1,S} = (1-\tau)w_t(1-p^E)\sum_{j=J^E}^{J} e_j N_{t,j} l_{j,t} + R_t^d(1-p^E)\sum_{j=J^E}^{J} \varrho_j N_{t-1,j-1}a_{t,j,S} \\
+ beq_t(1-p^E)\sum_{j=J^E}^{J} N_{t,j} + pen_t(1-p^E)\sum_{j=J^w+1}^{J} N_{t,j}
\end{aligned}
$$

Third, aggregate the entrepreneurs' budget constraints

$$
\begin{aligned}
c_{t,j} + p_t k_{t+1,j+1} + b_{t+1,j+1} - d_{t+1,j+1} &= (1-\tau)w_t e_j l_{j,t} + m_{t,j,E} + \mathbb{I}_j^r pen_t + beq_t \\
p^E N_{t,j}c_{t,j} + p_t p^E N_{t,j}k_{t+1,j+1} + p^E N_{t,j}b_{t+1,j+1} - p^E N_{t,j}d_{t+1,j+1} &= (1-\tau)w_t p^E N_{t,j}e_j l_{j,t} + p^E N_{t,j}m_{t,j,E} \\
&\quad + p^E N_{t,j}\mathbb{I}_j^r pen_t + p^E N_{t,j} beq_t \\
p^E \sum_{j=J^E}^{J} N_{t,j}c_{t,j} + p_t p^E \sum_{j=J^E}^{J} N_{t,j}k_{t+1,j+1} + p^E \sum_{j=J^E}^{J} N_{t,j}b_{t+1,j+1} - p^E \sum_{j=J^E}^{J} N_{t,j}d_{t+1,j+1} &= (1-\tau)w_t p^E \sum_{j=J^E}^{J} N_{t,j}e_j l_{j,t} + p^E \sum_{j=J^E}^{J} N_{t,j}m_{t,j,E} \\
&\quad + p^E \sum_{j=J^E}^{J} N_{t,j}\mathbb{I}_j^r pen_t + p^E \sum_{j=J^E}^{J} N_{t,j} beq_t \\
p^E \sum_{j=J^E}^{J} N_{t,j}c_{t,j} + p_t K_{t+1} + B_{t+1} - D_{t+1} &= (1-\tau)w_t p^E \sum_{j=J^E}^{J} N_{t,j}e_j l_{j,t} + p^E \sum_{j=J^E}^{J} \varrho_j N_{t-1,j-1}m_{t,j,E} \\
&\quad + p^E \sum_{j=J^w+1}^{J} N_{t,j} pen_t + p^E \sum_{j=J^E}^{J} N_{t,j} beq_t
\end{aligned}
$$

Now sum the lhs and rhs of the aggregated budget constraints from savers before $J^E$, after $J^E$, and from entrepreneurs. First, the lhs is

$$
\begin{aligned}
\sum_{j=1}^{J^E-1} N_{t,j}c_{t,j,S} + \sum_{j=1}^{J^E-1} N_{t,j}a_{t+1,j+1,S} \\
+(1-p^E)\sum_{j=J^E}^{J} N_{t,j}c_{t,j,S} + (1-p^E)\sum_{j=J^E}^{J} N_{t,j}a_{t+1,j+1,S} \\
+p^E \sum_{j=J^E}^{J} N_{t,j}c_{t,j} + p_t K_{t+1} + B_{t+1} - D_{t+1} \\
= C_t + A_{t+1}^S + p_t K_{t+1} + B_{t+1} - D_{t+1}
\end{aligned}
$$

Second, the rhs reads

$$
\begin{aligned}
(1-\tau)w_t \sum_{j=1}^{J^E-1} e_j N_{t,j} l_{j,t} + R_t^d \sum_{j=1}^{J^E-1} \varrho_j N_{t-1,j-1}a_{t,j,S} + beq_t \sum_{j=1}^{J^E-1} N_{t,j} \\
+(1-\tau)w_t(1-p^E)\sum_{j=J^E}^{J} e_j N_{t,j} l_{j,t} + R_t^d(1-p^E)\sum_{j=J^E}^{J} \varrho_j N_{t-1,j-1}a_{t,j,S} + beq_t(1-p^E)\sum_{j=J^E}^{J} N_{t,j} + pen_t(1-p^E)\sum_{j=J^W+1}^{J} N_{t,j} \\
+(1-\tau)w_t p^E \sum_{j=J^E}^{J} N_{t,j}e_j l_{j,t} + p^E \sum_{j=J^E+1}^{J} \varrho_j N_{t-1,j-1}m_{t,j,E} + R_t^d p^E \varrho_j N_{t-1,j-1}a_{t,J^E,S} + p^E \sum_{j=J^W+1}^{J} N_{t,j} pen_t + p^E \sum_{j=J^E}^{J} N_{t,j} beq_t \\
= (1-\tau)w_t L_t + R_t^d A_t^S + N_t beq_t + \underbrace{pen_t \sum_{j=J^W+1}^{J} N_{t,j}}_{=\tau w_t L_t} + p^E \sum_{j=J^E+1}^{J} N_{t,j}m_{t,j,E}
\end{aligned}
$$

$$
\begin{aligned}
&= w_t L_t + R_t^d \sum_{j=1}^{J^E} \varrho_j N_{t-1,j-1} a_{t,j,S} + (1-p^E) R_t^d \sum_{j=J^E+1}^{J} \varrho_j N_{t-1,j-1} a_{t,j,S} + N_t beq_t + p^E \sum_{j=J^E+1}^{J} \varrho_j N_{t-1,j-1} \left[ R_t p_{t-1} k_{t,j} - R_t^d d_{t,j} + Q_t (b_{t,j} + b_{t,j}^N) \right] \\
&= w_t L_t + R_t^d \sum_{j=1}^{J^E} N_{t-1,j-1} a_{t,j,S} + (1-p^E) R_t^d \sum_{j=J^E+1}^{J} N_{t-1,j-1} a_{t,j,S} + p^E \sum_{j=J^E+1}^{J} N_{t-1,j-1} \left[ R_t p_{t-1} k_{t,j} - R_t^d d_{t,j} + Q_t (b_{t,j} + b_{t,j}^N) \right] \\
&= w_t L_t + R_t^d A_t^S + N_t X_t + R_t p_{t-1} K_t - R_t^d D_t + Q_t (B_{t,j} + B_{t,j}^N)
\end{aligned}
$$

Note, $X_t$ are other potential lump-sump transfers (profits), that'll be declared later.

Equate lhs and rhs

$$
C_t + A_{t+1}^S + p_t K_{t+1} + B_{t+1} - D_{t+1} = w_t L_t + R_t^d A_t^S + N_t X_t + R_t p_{t-1} K_t - R_t^d D_t + Q_t (B_t + B_t^N)
$$

From the definition of $R_t$ we can rewrite

$$
\begin{aligned}
w_t L_t + R_t p_{t-1} K_t &= (1-\alpha) \frac{Y_t}{L_t} L_t + [\alpha \frac{Y_t}{K_t} + (1-\delta) p_t] K_t \\
&= (1-\alpha) Y_t + \alpha Y_t + (1-\delta) p_t K_t \\
&= Y_t + (1-\delta) p_t K_t
\end{aligned}
$$

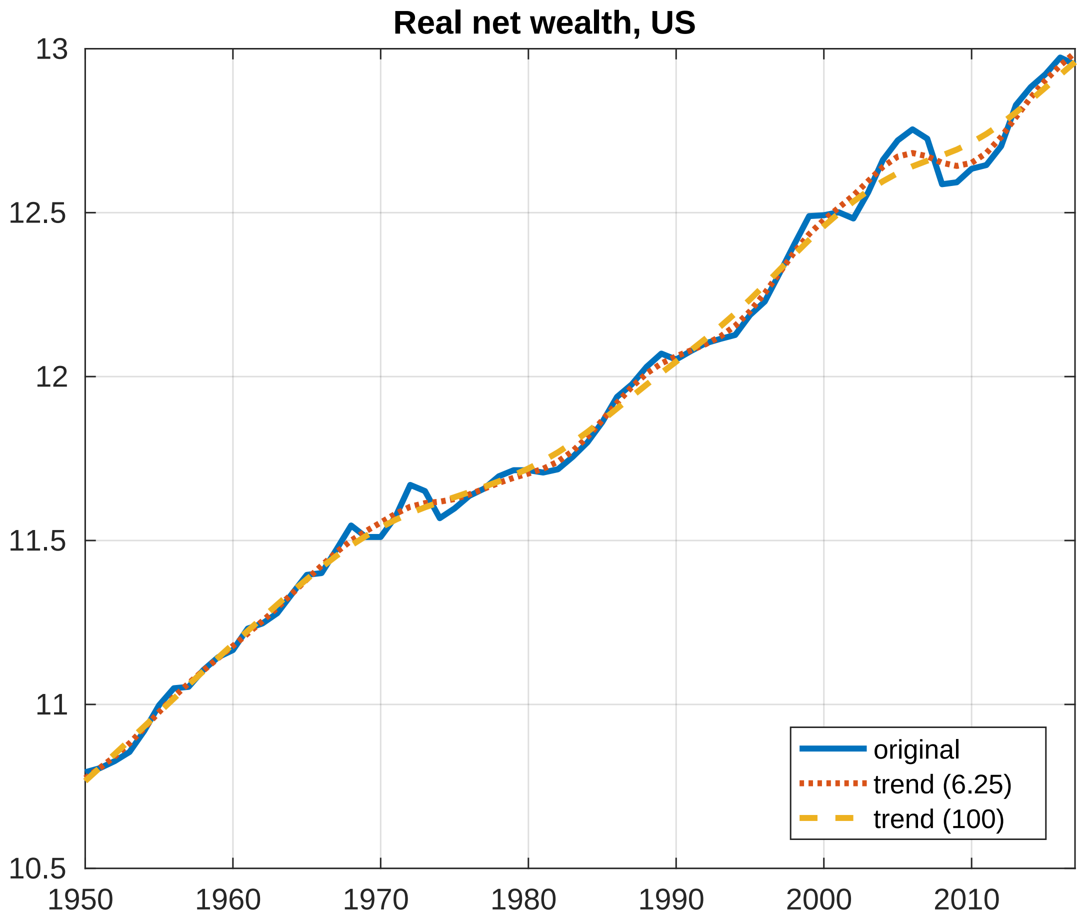
Noting this, the law of motion of aggregate bubbles yields, and defining storage as $S_t \equiv A_t^S - D_t$

$$
\begin{aligned}
C_t + S_{t+1} + p_t K_{t+1} &= Y_t + (1-\delta) p_t K_t + \gamma S_t + N_t X_t \\
\Leftrightarrow Y_t &= C_t + p_t [K_{t+1} - (1-\delta) K_t] + S_{t+1} - \gamma S_t - N_t X_t \\
Y_t &= C_t + p_t I_t^K + S_{t+1} - \gamma S_t - N_t X_t
\end{aligned}
$$

If profits from capital producers are redistributed lump sum, then $N_t X_t = \pi_t^K$ and the last equation can be rewritten as

$$
\begin{aligned}
Y_t &= C_t + p_t I_t^K + S_{t+1} - \gamma S_t - \pi_t^K \\
&= C_t + I_t + S_{t+1} - \gamma S_t .
\end{aligned}
$$

## B. Figures




*Notes: A conventional HP-filter value of 6.25 would imply a non-monotonous trend. In order to yield a monotonous trend the smoother filter of 100 is therefore chosen.*

**Figure VIII:** HP-filtered US wealth

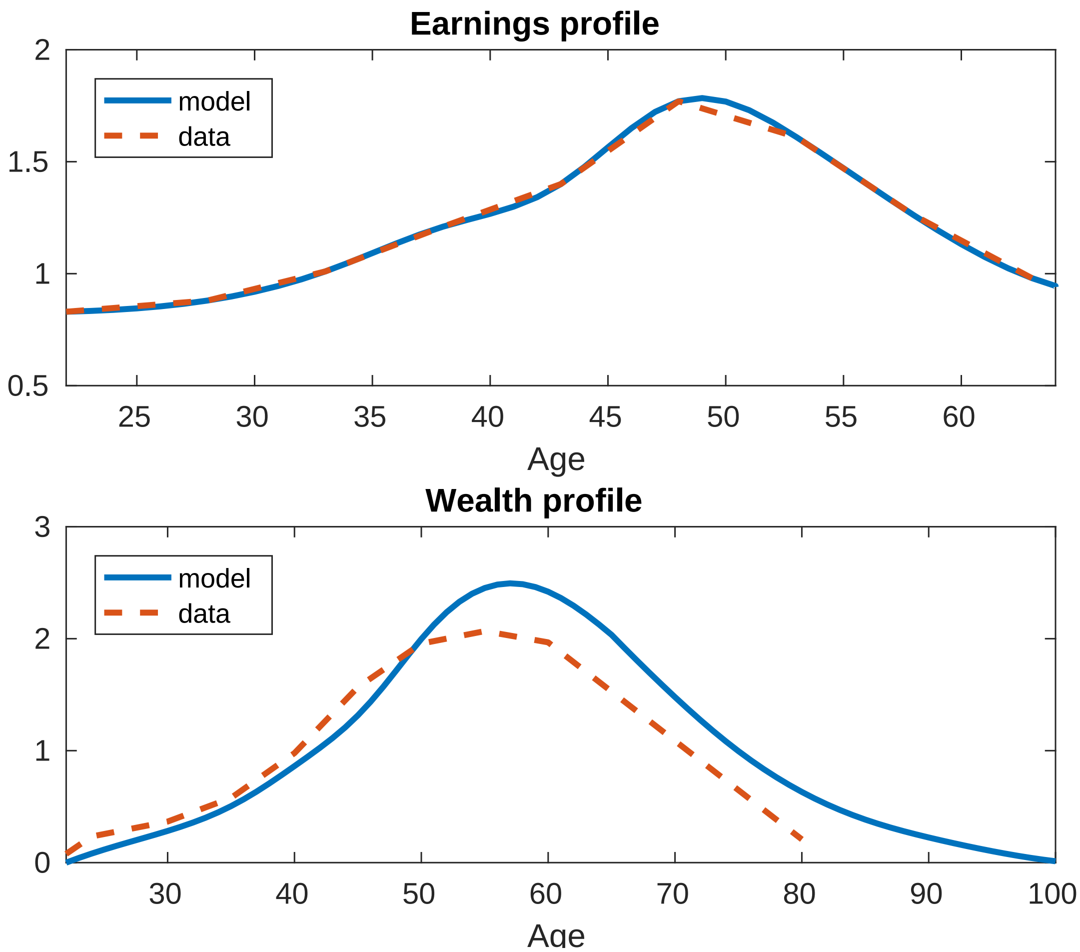



**Figure IX:** Age profiles, within cohort averages

| | General equilibrium: fundamental | Partial equilibrium | | | | General equilibrium: bubbly |
|---|---|---|---|---|---|---|
| | (1) | (2) | (3) | (4) | (5) | (6) |
| | | $+b^N$ | $+w$ $+pen$ $+beq$ | $+R$ | + savers' $R^d$ | + entrepreneurs' $R^d$ |
| Output $Y$ | 0.0 | 7.5 | 9.5 | 1.4 | 1.4 | 1.4 |
| Capital $K$ | 0.0 | 24.1 | 25.4 | 3.7 | 3.7 | 3.7 |
| Labor $L$ | 0.0 | 0.0 | 2.3 | 0.4 | 0.4 | 0.4 |
| Credit $D$ | 0.0 | 24.1 | 25.4 | 3.7 | 3.7 | 3.7 |
| Savers' wealth $A$ | 0.0 | -0.001 | 1.3 | 1.3 | 1.3 | 1.3 |
| Entrep' wealth $A^E$ | 0.0 | 24.1 | 25.4 | 3.7 | 3.7 | 3.7 |
| Bubble $B$ | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.3 |

*Notes: All variables are expressed relative to their fundamental steady state values, in percentage points, except B, which is expressed as a ratio of entrepreneurs wealth $A^E$. Prices change according to: $b^N \uparrow$, $w \uparrow$, $pen \uparrow$, $beq \uparrow$, $R \downarrow$, $R^d = const$.*

**Table B.1:** The effect of bubbles decomposed into different channels with $\gamma >> 0$

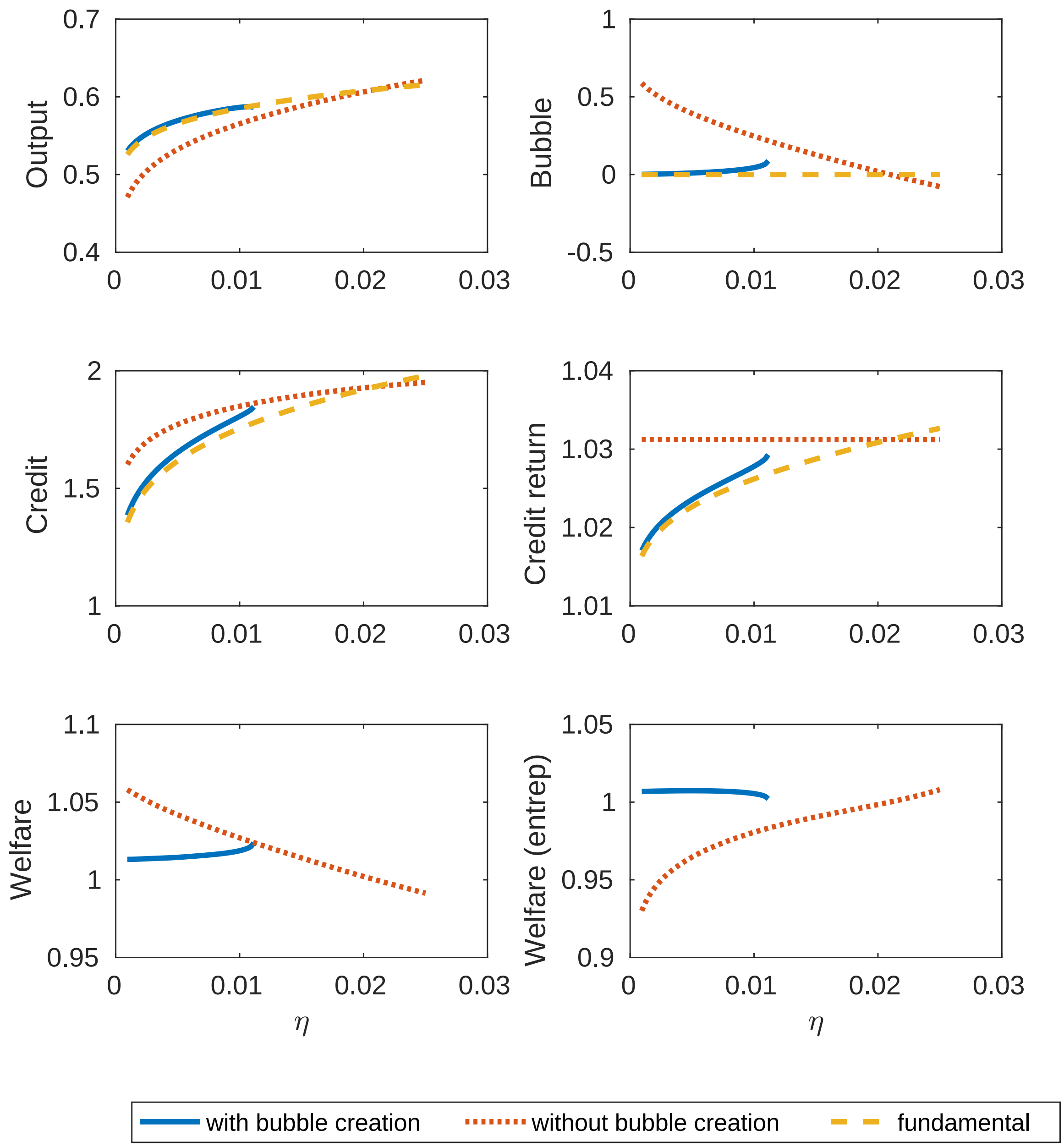



**Figure X:** Steady state with different population shares of entrepreneurs

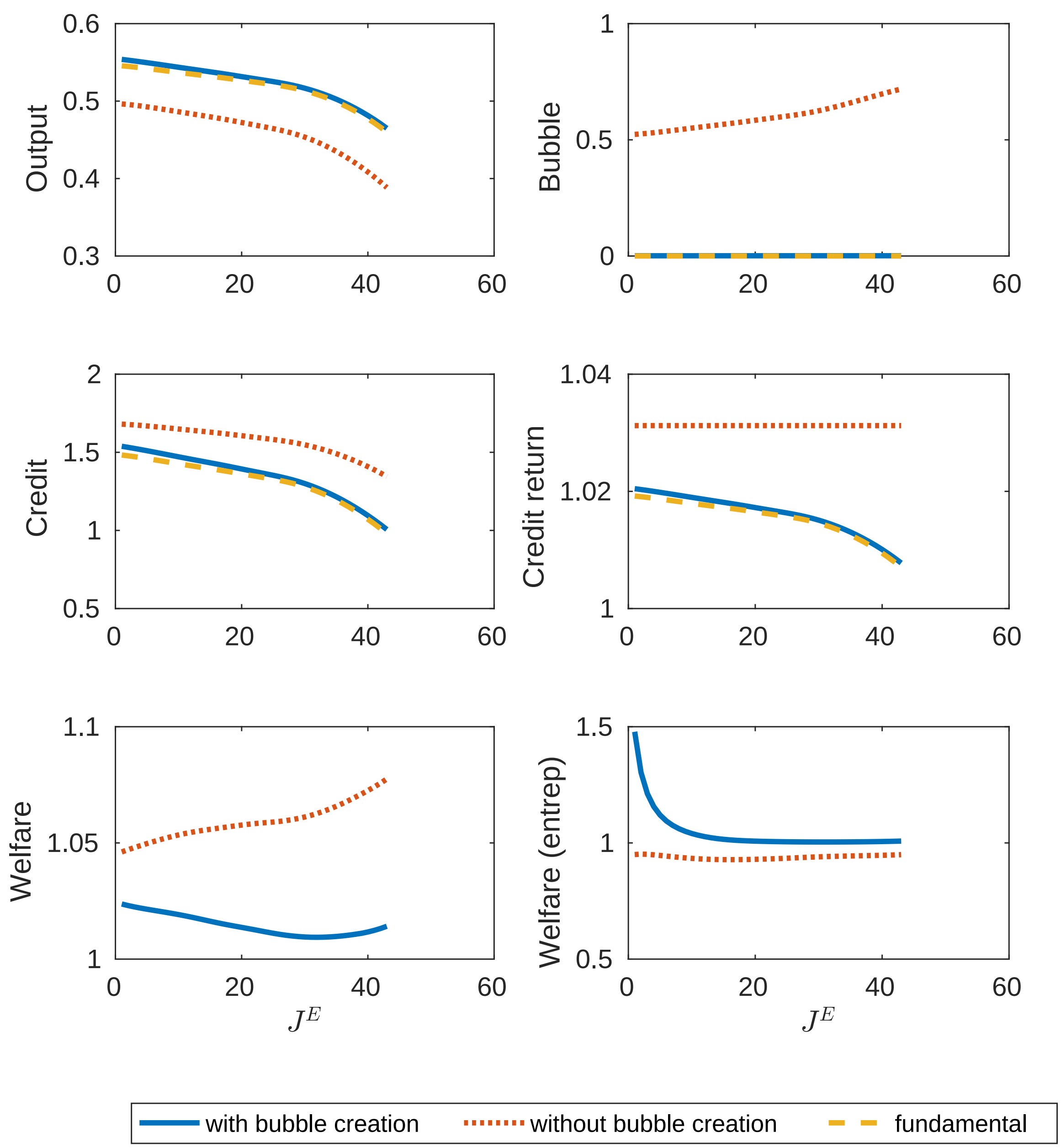



**Figure XI:** Steady state with different timings of entrepreneur shock

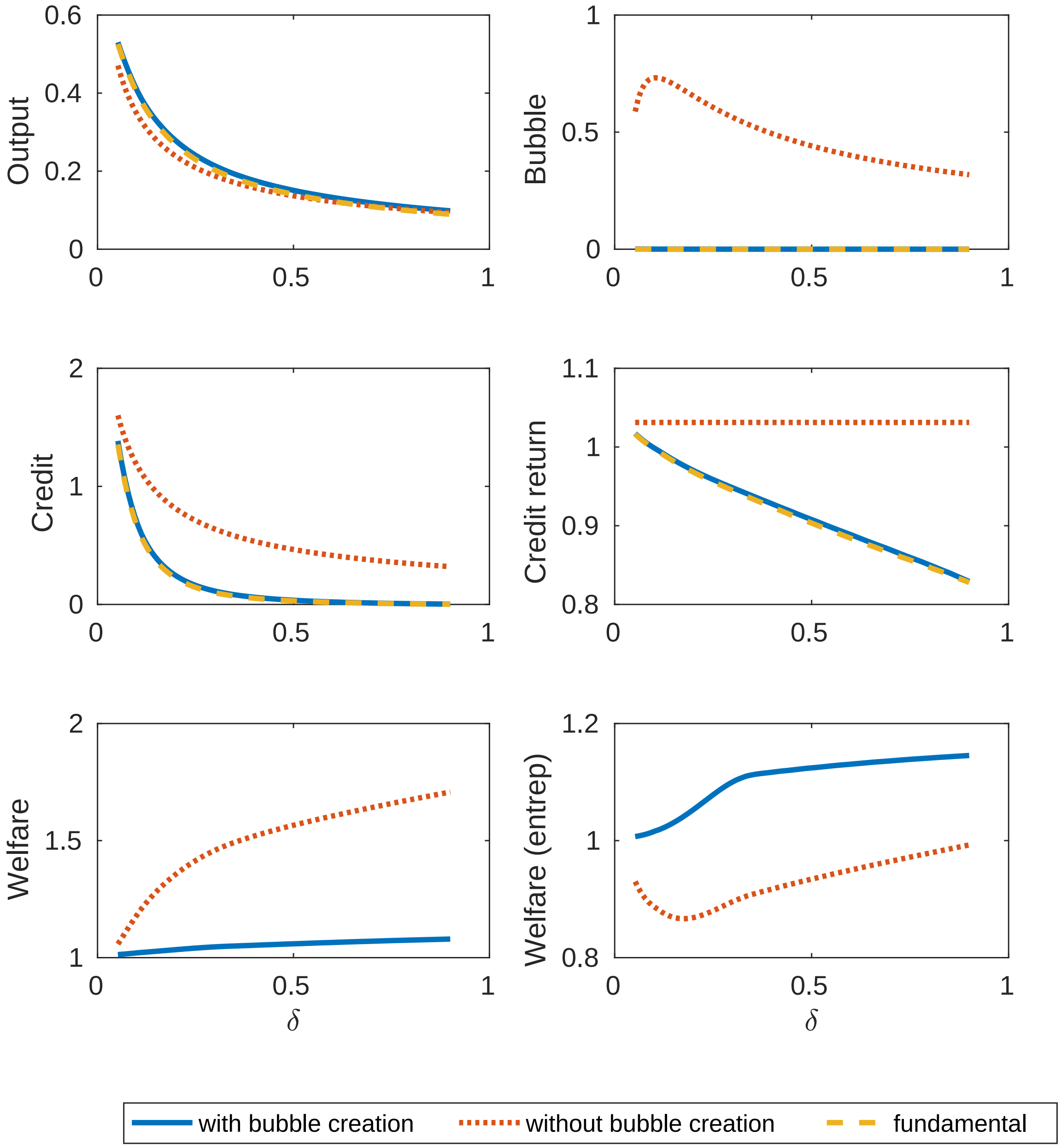



**Figure XII:** Steady state with different depreciation rates

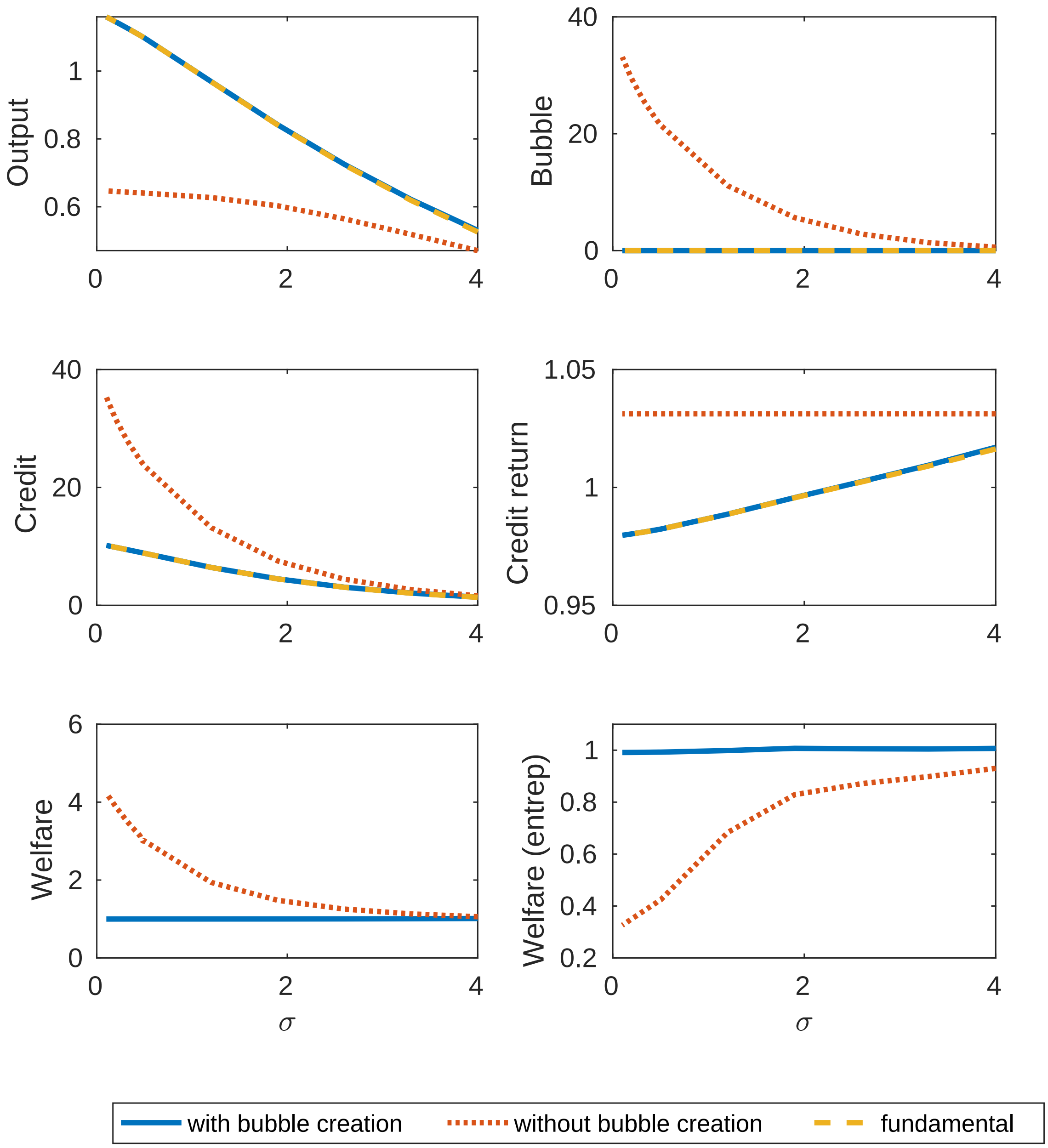



**Figure XIII:** Steady state with different elasticities of substitution

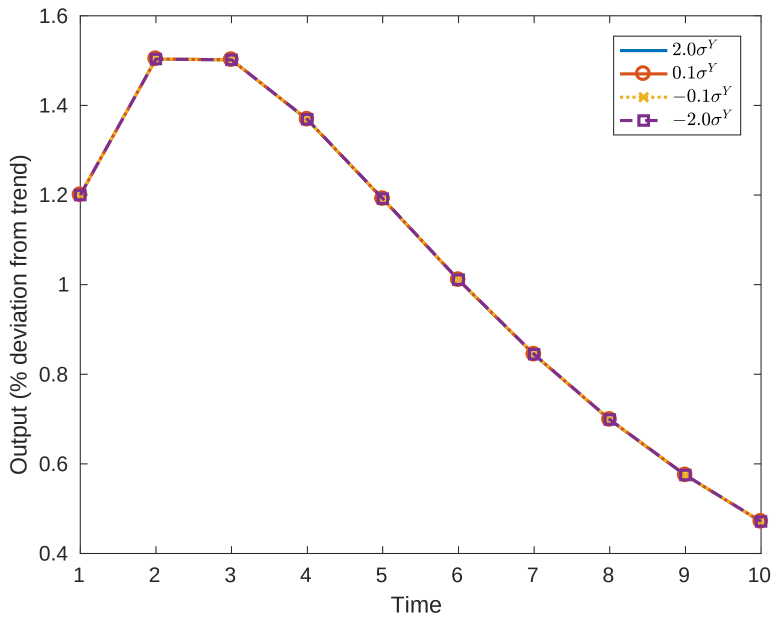



**Figure XIV:** IRF of Output to TFP shocks of different sizes and signs

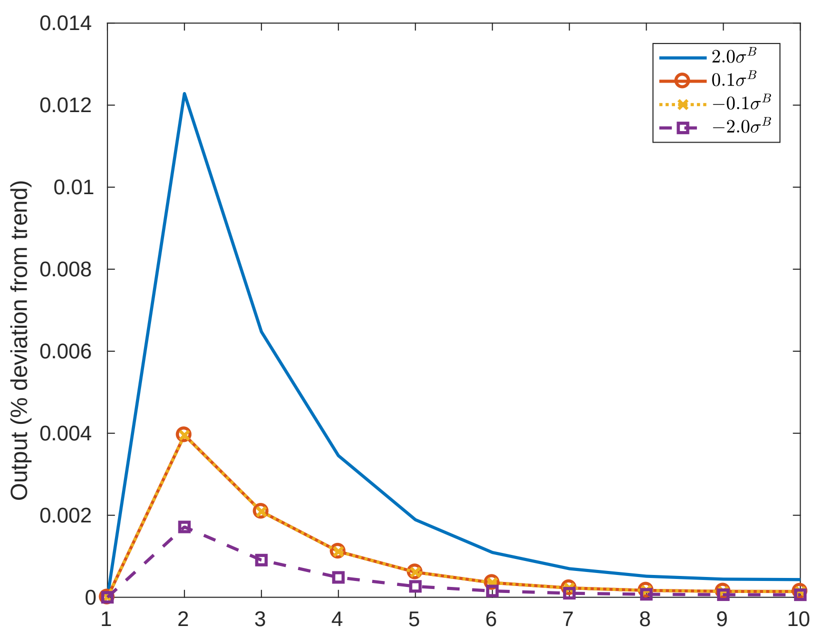



**Figure XV:** IRF of Output to bubble shocks of different sizes and signs

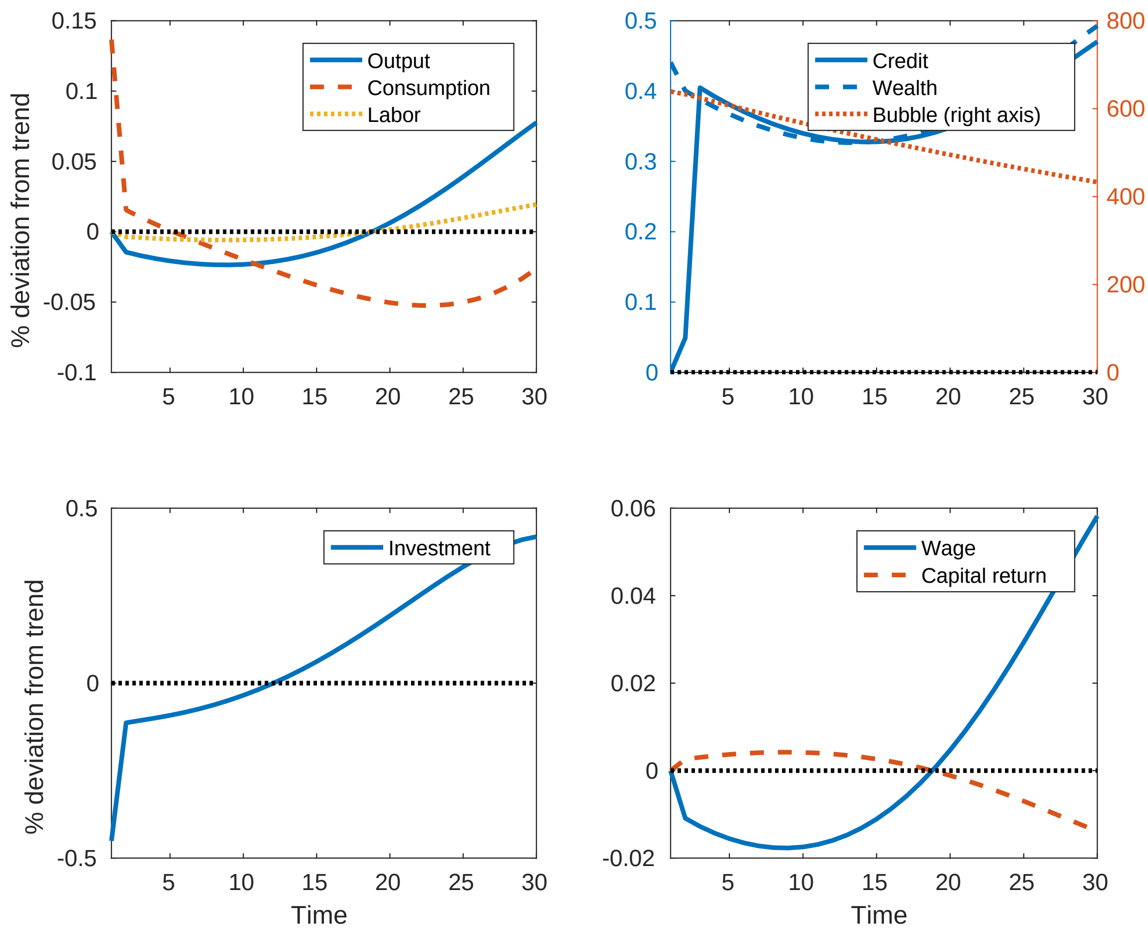



**Figure XVI:** A bubble-driven business cycle for $\gamma = 0$

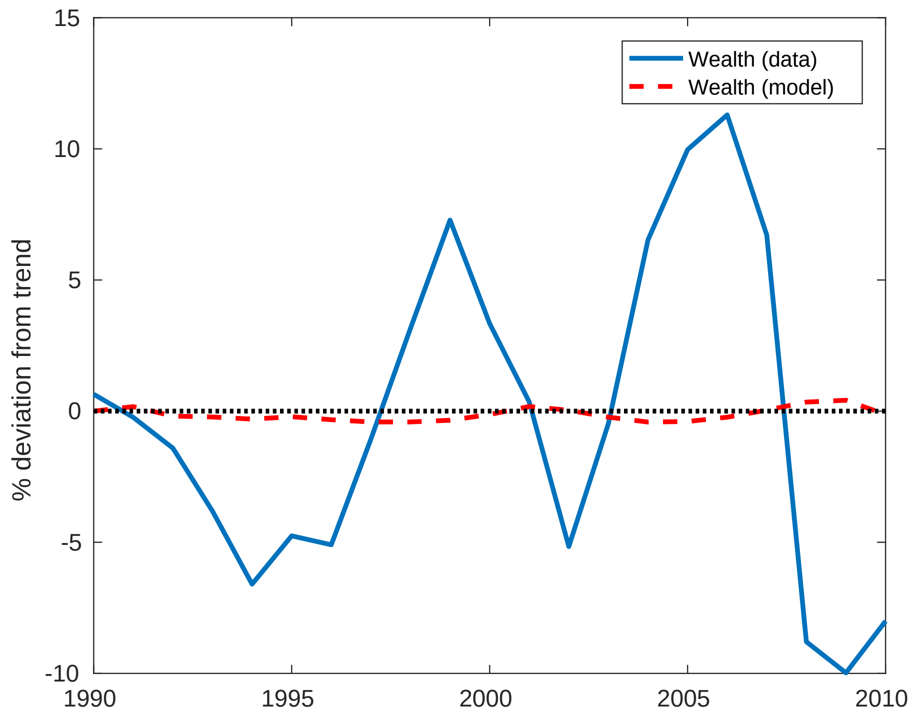



**Figure XVII:** Wealth series from TFP shocks that let the model match the empirical GDP series