Homework 7

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1 Redo of homework 6, Lagrange Multipliers

$$min \sum_i a_i^2 x_i \texttt{subject to} \quad \sum_i \frac{1}{x_i - B} = \frac{1}{C}$$

We first setup the Lagrangian equation and solve for the λ. We then plug this λ back into the constraint equations to find x.

$$L = \sum_i a_i^2 x_i - \lambda(\frac{1}{x_i - B} - \frac{1}{C})$$

Differentiate and set equal to zero:

$$\frac{dL}{dx} = \sum_i a_i^2 + \frac{\lambda}{(\sum_i x_i - B)^2} = 0$$

solve for x:

$$a_i = -\frac{\sqrt{\lambda}}{x_i - B}$$

$$-(x_i - B)a_i = \sqrt{\lambda}$$

$$x_i = B + \frac{\sqrt{\lambda}}{a_i}$$

Solve for lambda, plug x_i into constraint equations:

$$\sum \frac{1}{x_i - B} - \frac{1}{C} = 0$$

$$\sum_i x_i + \frac{\sqrt{\lambda}}{a_i} - B = C$$

$$\sqrt{\lambda} = C \sum_i a_i$$

Finally we plug in this $\sqrt{\lambda}$ back into the x_i equation.

$$x_i = B + \frac{C \sum_i a_i}{a_i}$$

We now multiply by a_i^2

$$\sum_i a_i^2 x_i = b \sum_i a_i^2 + c \sum_i a_i^2$$

2 KKT conditions

$$\min x_1^2 + x_2^2 + 2ax_1x_2$$
$$\texttt{SubjectTo} \quad |x_1| + |x_2| \leq 1|$$

We define the Lagrangian as:

$$L = x_1^2 + x_2^2 + 2ax_1x_2 + \lambda(|x_1| + |x_2| - 1)$$

We differentiate and set equal to zero, and set up the KKT conditions:

1. $\frac{d}{dx} = 2x_1 + 2ax_2 + \lambda = 0$

2. $\frac{d}{dy} = 2x_2 + 2ax1 + \lambda = 0$
3. $\lambda(|x_1| + |x_2| - 1) = 0$
4. $\lambda \geq 0$
5. $|x_1| + |x_2| \leq 1$

Consider the case when $\lambda = 0$ and solve the lagrangain equations for x:

$$2x_1 + 2ax_2 = 0$$

and

$$2x_2 + 2ax_1 = 0$$

This gives us: $x_1(a-1) = x_2(a-1)$ and $|x_1| + |x_1| = 1$ we can solve for:

$$|x_1| = \frac{1}{2}$$

$$|x_2| = \frac{1}{2}$$

Solving for λ in equation 1 or 2 we get $|1-a|$.We consider the values that follows our constraints:

$$\begin{bmatrix} x_1 & x_2 & \lambda \\ \frac{1}{2} & \frac{1}{2} & -1-a \\ -\frac{1}{2} & -\frac{1}{2} & 1+a \\ \frac{1}{2} & -\frac{1}{2} & -1+a \\ -\frac{1}{2} & \frac{1}{2} & 1-a \end{bmatrix}$$

So our kkt conditions, and hence our optimal solution, are dependent on our choice of a. Our choice of a must keep $\lambda \geq 0$.

3 Linear Programming

This problem closely follows class notes from Thurs Oct 6.

$$\min x^T Ax \quad \texttt{Subject to:} \quad x^T x = 1$$

We start by writing the lagrangian equations; set $x^t x = a^t x$ with a in the constraint equation, b is ommited in the equations as it is just a vector of ones and will not change the result (as far as I could see):

$$L = x^T Ax + \lambda^T(a^t x - 1)$$

$$\nabla L = Ax + a\lambda = 0$$
$$x = -A^{-1}a\lambda$$

from $a^T x = 1$ we get

$$a^T x = a^T A^{-1} a\lambda = 1$$
$$\lambda = (a^T A^{-1} a)^{-1}$$

We plug this λ into our equation for x to get:

$$x = A^{-1}a\lambda = A^{-1}a(a^T A^{-1}a)$$
$$a^t x - 1 = (a^T A^{-1}a)(a^T A^{-1}a)^{-1} = 1$$
$$L = x^t Ax = (a^t A^{-1}(a^T A^{-1}AA^{-1}a)(a^T A^{-1}a)^{-1} = (a^t A^{-1}a)^{-1}$$

4 Min and Duel

Note, I'm not sure I'm treating the a properly in this problem. I'm not sure how to deal with an inequality in the constraint, solving for the duel. I followed the example/theory of the book Example 12.10. They seem to just use equalities and I'm not sure why.

4.1 min

$$\min_{x \in R^3} \sum_{i=1}^{3} (x_i^2 + x_i)$$
$$\texttt{subject to} \quad x_1 - x_2 + 2x_3 = 1 \texttt{ and } 2x_1 + x_2 - 3x_3 \leq a$$
$$L = \sum_{i=1}^{3} (x_i^2 + x_i) - \lambda(x_1 - x_2 + 2x_3 - 1) - \mu(2x_1 + x_2 - 3x_3 - a)$$

Take derivatives of L, we can list our KKT conditions:

1. $\nabla_{x_1} = 2x_1 + 1 - \lambda_1 - 2\mu_2 = 0$
2. $\nabla_{x_2} = 2x_2 + 1 + \lambda_1 - \mu_2 = 0$
3. $\nabla_{x_3} = 2x_3 + 1 - 2\lambda_1 - 3\mu_2 = 0$
4. $\lambda(x_1 - x_2 + 2x_3 - 1) = 0$
5. $\mu(2x_1 + x_2 - 3x_3 - a) = 0$
6. $\mu \geq 0$
7. $\lambda \geq 0$

Then solving for x from the previous equations, we then list out our KKT conditions:

$$x1 = -\frac{1}{2} + \frac{\lambda}{2} + \mu$$
$$x2 = -\frac{1}{2} - \frac{\lambda}{2} + \frac{\mu}{2}$$
$$x3 = -\frac{1}{2} + \lambda + \frac{3\mu}{2}$$

We can now plug these values into the constraint equations $x_1 - x_2 + 2x_3 = 1$ `and` $2x_1 + x_2 - 3x_3 \leq a$ to solve for λ

$$(-\frac{1}{2} + \frac{\lambda}{2} + \mu) - (-\frac{1}{2} - \frac{\lambda}{2} + \frac{\mu}{2}) + 2(\frac{1}{2} + \lambda + \frac{3\mu}{2}) = 1$$

$$3\lambda - \frac{7}{2}\mu = 2$$

$$2(-\frac{1}{2} + \frac{\lambda}{2} + \mu) + (-\frac{1}{2} - \frac{\lambda}{2} + \frac{\mu}{2}) - 3(-\frac{1}{2} + \lambda + \frac{3\mu}{2}$$

$$-\frac{\lambda}{2} - 2\mu \leq a - 3$$

solving for λ from the two simplified equations above we get:

$$\lambda = 6 + \frac{21\mu}{2}$$

Plug in λ back into our equations for x we get:

$$x_1 = -\frac{5}{2} + \frac{29}{4}\mu$$

$$x_2 = -\frac{7}{2} + \frac{25}{4}\mu$$

$$x_3 = \frac{11}{2} + 10\mu$$

We then consider values of μ that would satisfy all of the KKT conditions above.(I didn't not finish this part).

4.2 dual

For a dual we transform the objective function to a max (if primal was a min) and optimize over λ.

$$L = \sum_{i=1}^{3}(x_i^2 + x_i) - \lambda(x_1 - x_2 + 2x_3 - 1) - \mu(2x_1 + x_2 - 3x_3 - a)$$

Take derivatives of L, we can list our partial derivatives:

1. $\nabla_{x_1} = 2x_1 + 1 - \lambda_1 - 2\mu_2 = 0$
2. $\nabla_{x_2} = 2x_2 + 1 + \lambda_1 - \mu_2 = 0$
3. $\nabla_{x_3} = 2x_3 + 1 - 2\lambda_1 - 3\mu_2 = 0$

Solving for x we get:

$$x1 = -\frac{1}{2} + \frac{\lambda}{2} + \mu$$

$$x2 = -\frac{1}{2} - \frac{\lambda}{2} + \frac{\mu}{2}$$

$$x3 = -\frac{1}{2} + \lambda + \frac{3\mu}{2}$$

To obtain dual objective we plug these values into the Lagrangian.

$$L = \sum_{i=1}^{3}(x_i^2 + x_i) - \lambda(x_1 - x_2 + 2x_3 - 1) - \mu(2x_1 + x_2 - 3x_3 - a)$$

Using wolframalpha to plug in the values and simplify the tedious algebra we get:

$$\max_{\lambda,\mu}\left(\frac{3\lambda^2}{2} + \frac{7\lambda\mu}{2} + \frac{7\mu^2}{2} - \frac{3}{4}\right)$$

$$\texttt{assuming:}\lambda \geq 0, \mu \geq 0$$