CMPE 16 Homework #8

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1. For each of the following relations :

   - Give 3 pairs of elements that are related, Determine whether the relation is reflective, Determine whether the relation is symmetric, determine whether the relation is transitive

   (a) $R_1 = \{(n, m) : n, m \in \mathbb{Z} \text{ and } n * m \geq 0\}$

   Part 1.) (1, 2), (3, 4), (5, 6)
   Part 2.) True. For any $a \in \mathbb{Z}$, $a * a \geq 0$. Thus (a, a) exists for all $a \in \mathbb{Z}$
   Part 3.) True. $\forall (x, y) \in \mathbb{Z} \times \mathbb{Z} : x * y \geq 0 \implies y * x \geq 0$
   Multiplication is communative, so $x * y \geq 0$ necessarily implies that $y * x \geq 0$, because the two statements are equivalent.
   Part 4.) True. To prove : $\forall x \forall y \forall z[([(x, y) \in R_1] \wedge [(y, z) \in R_1] \implies (x, z) \in R_1)]$
   This proof will use two cases, one where $x < 0$ and one where $x \geq 0$
   Case 1 : $x < 0$
   If $x < 0$, then $y \leq 0$ for the relation to hold.
   Since $y \leq 0$, then $z < 0$ for the relation to hold.
   Since $x < 0$ and $z < 0$, $x * z > 0$, which proves transitivity under this relation
   Case 2 : $x \geq 0$
   If $x \geq 0$, then $y \geq 0$ for the relation to hold.
   Since $y \geq 0$, then $z \geq 0$ for the relation to hold.
   Since $x \geq 0$ and $z \geq 0$, $x * z \geq 0$, which proves transitivity under this relation

   (b) $R_2 = \{(a, b) : a, b \in \mathbb{Z} \text{ and } a * b > 0\}$

   Part 1.) (1, 2), (3, 4), (5, 6)
   Part 2.) True. For any $a \in (\mathbb{Z} - \{0\})$, $a * a > 0$. Thus (a, a) exists for all $a \in \mathbb{Z}$
   Part 3.) True. $\forall (x, y) \in \mathbb{Z} \times \mathbb{Z} : x * y > 0 \implies y * x > 0$
   Multiplication is communative, so $x * y > 0$ necessarily implies that $y * x \geq 0$, because the two statements are equivalent.
   Part 4.) True. $\forall x \forall y \forall z[([(x, y) \in R_2] \wedge [(y, z) \in R_2] \implies (x, z) \in R_2)]$
   This proof will use two cases, one where $x < 0$ and one where $x > 0$
   Case 1 : $x < 0$
   If $x < 0$, then $y < 0$ for the relation to hold.
   Since $y < 0$, then $z < 0$ for the relation to hold.
   Since $x < 0$ and $z < 0$, $x * z > 0$, which proves transitivity under this relation
   Case 2 : $x > 0$
   If $x > 0$, then $y > 0$ for the relation to hold.
   Since $y > 0$, then $z > 0$ for the relation to hold.
   Since $x > 0$ and $z > 0$, $x * z > 0$, which proves transitivity under this relation

   (c) $R_3 = \{(i, j) : i, j \in \mathbb{N} \text{ and } i/j \geq 1\}$

   Part 1.) (10, 2), (20, 8), (1024, 2)
   Part 2.) True. $\forall i \in \mathbb{N}, i/i = 1$.
   Thus $\forall i \in \mathbb{N} : (i, i) \in R_3$
   Part 3.) False. A counter example would be $i = 4, j = 2$, $i/j > 1$ so $(i, j) \in R_3$ but, $j/i < 1$ so $(j, i) \notin R_3$
   Part 4.) True. I will attemp a direct proof
   To prove : $\forall x \forall y \forall z[([(x, y) \in R_3] \wedge [(y, z) \in R_3] \implies (x, z) \in R_3)]$
   $(x, y) \in R_3 \implies x/y \geq 1$, $(y, z) \in R_3 \implies y/z \geq 1$
   $x/y \geq 1$, $x \geq y$
   $y/z \geq 1$, $y \geq z$
   $x \geq y \geq z \geq 1$ ($z \geq 1$ by def. of natural numbers)
   $x/z \geq y/z \geq 1 \geq 1/z$
   $x/z \geq 1$, that which was to be shown

(d) $R_4 = \{(x, y) : x, y \in \mathbb{R} \text{ and } \lceil x \rceil = \lceil y \rceil\}$

Part 1.) (1.09, 1.85), (3.90, 3.95), (4.90, 4.95)
Part 2.)
Part 3.)
Part 4.)

2. In each case below explain why the relation between the set S and the set T is **not** a function.

(a) S is the set of all people at least 21 years old on October 25, 2014 and T is the set of all automobiles. A person is associated with their first car.
Answer : Not a function because there are people at the age of 21 or older who have never owned a car, and thus aren't associated with a first car.

(b) S is the set of all ordered pairs of integers ($\mathbb{Z} \times \mathbb{Z}$) and T is the set of all rational number ($\mathbb{Q}$). An ordered pair of integers (m, n) is associated with n/m
Answer : This is not a function because there is no defined mapping from $m \in \mathbb{Z}, n = 0$ to a rational number. (Rational numbers cannot have a zero in the denominator).

(c) S is the set of all bit strings and T is the set of integers. A bit string is associated with the integer n if it's nth bit is the rightmost bit which is a zero.
Answer : This is not a function because the relation is undefined if the bitstring contains no zeros, ex `0xFFFF`.

(d) S is the set of all integers and T is the set of all real numbers. An integer n is associated with a real number x if $\sqrt{n} = x$.
Answer : This is not a function because $\sqrt{n^2} = \pm n$, thus one input maps to more than one output.

3. In each case below, determine whether the function given is injective (one-to-one) and prove your answer.

(a) $f : \mathbb{Z}^+ \to \mathbb{R}$ where $f(x) = (3x - 4)/8$
Answer : : True, f is one-to-one. I will attempt to prove this using induction. To do this, I will show that the function f is always strictly increasing over it's domain, which means that $f(x) > f(x-1) > f(x-2) > \ldots > f(1)$, which implies that all of the inputs have a different output over f

$\forall x \in \mathbb{Z}^+ : f(x-1) < f(x)$	Inductive Hypothesis
$\forall x \in \mathbb{Z}^+ : f(x) < f(x+1)$	Inductive Conclusion
$f(1) = -1/8,\ f(2) = 1/4,\ f(1) < f(2)$	Base Case
$f(x) = (3x-4)/8 = \frac{3}{8}x - \frac{4}{8}$	Simplifying $f(x)$ for later use
$f(x+1) = (3(x+1)-4)/8 = (3x-1)/8 = \frac{3}{8}x - \frac{1}{8}$	Inputing $x+1$ into f
$f(x+1) = \frac{3}{8}x - \frac{4}{8} + \frac{3}{8} = f(x) + \frac{3}{8}$	Substituting in $f(x)$
$f(x+1) = f(x) + \frac{3}{8} \implies f(x+1) > f(x)$	The inductive conclusion has been shwon

(b) $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Q}^+$ where $f(a, b) = a/b$

(c) $f : \mathbb{N} \to \mathbb{R}$ where $f(n) = \frac{1}{n}$

(d) $f : \mathbb{R} \to \mathbb{Z}$ where $f(x) = \lfloor x \rfloor$