CMPE 16 Homework #4

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1. Bridge is a card game which uses a standard deck of 52 cards. All 52 cards are dealt to four players, so each player receives a 13-card hand. The order of the cards within a hand is not important.

   (a) How many 13-card hands have exactly 9-spades?

$$\binom{13}{9} \times \binom{39}{4} = \frac{13!}{9!4!} \times \frac{39!}{35!4!} = 58809465$$

   (b) How many 13-card hands have exactly 2 aces, 2 kings and 4 queens?

$$\binom{4}{2} \times \binom{4}{2} \times \binom{4}{4} = \frac{4!}{2!2!} \times \frac{4!}{2!2!} = 6^2 = 36$$

   (c) How many 13-card hands have exactly 7 face cards?

$$\binom{12}{7} \times \binom{40}{6} = \frac{12!}{5!7!} \times \frac{40!}{35!5!} = 792 \times \frac{40 * 39 * 38 * 37 * 36}{5!} = 521,142336$$

   (d) How many 13-card hands have at least 4 spades and exactly 5 hearts?

$$\binom{13}{4} \times \binom{13}{5} \times \binom{35}{4}^1 = \frac{13!}{9!4!} \times \frac{13!}{8!5!} \times \frac{35!}{31!4!} = 3.81819338 \times 10^{10}$$

   (e) How many 13-card hands have 4 aces or 4 kings?

$$A = \binom{4}{4} \times \binom{38}{9} = 163,011,640 \qquad B = \binom{4}{4} \times \binom{38}{9} = 163,011,640$$

$$|A \cap B| = \binom{4}{4} \times \binom{4}{4} \times \binom{34}{5} = \frac{34!}{29!}5! = 278,256$$

$$|A \cup B| = |A| + |B| - |A \cap B| = (163,011,640)^2 - 278,256 = 325,745,024$$

   (f) How many 13-card hands have cards from at most two suits?
   So first you have to choose which 2 suits of the four available you want to select 13 cards from. Then you can select any 13 cards from these two suits, including 13 cards that all belong to one suit.

$$\binom{4}{2} \times \binom{26}{13} = 6 \times \frac{26!}{13!^2} = 62,403,600$$

[1] 52 cards total, minus 13 hearts because we can't have any more of those, minus 4 spades because we can have more than 4 spades.

2. In this problem you will calculate the number of UCSC ID numbers that meet certain criteria, yet again. But this time, to make things simpler we will allow the first digit to be 0. So UCSC ID numbers have 7 digits and each digit can be 0 through 9. Be careful! There are still subtleties lurking here.

   (a) How many UCSC ID numbers have exactly three 4s ?

   $$\binom{7}{3} \times \binom{1}{1}^3 \times \binom{9}{1}^4 = 229,635$$

   (b) How many UCSC ID numbers have exactly two odd digits?

   $$\binom{7}{2} \times \binom{4}{1}^2 \times \binom{5}{1}^5 = \frac{7!}{5!2!} \times 16 \times 3125 = 1,050,000$$

   (c) How many UCSC ID numbers have at least 5 even digits? Choose which 5 digits must be even, select one of 5 for each of those 5 digits. For the last digits, select 1 of 10 numbers each.

   $$\binom{7}{5} \times \binom{5}{1}^5 \times \binom{10}{1}^2 = \frac{7!}{2!5!} \times 3125 \times 100 = 6,562,500$$

   (d) How many UCSC ID numbers have exactly two 5s or exactly three 8s?

   $$A = \text{ID's with two 2's} = \binom{7}{2} \times \binom{1}{1}^2 \times \binom{9}{1}^5 = 1,240,029$$

   $$B = \text{ID's with three 8's} = \binom{7}{3} \times \binom{1}{1}^3 \times \binom{9}{1}^4 = 229,635$$

   (e) How many UCSC ID numbers do not have 3 consecutive 6s?
   I will find the inverse and subtract that from 10^7 (# of possible ID's)
   There are 15 ways to place (at least) 3 consecutive items in 7 consecutive slots (no proof, I just counted by hand).

   $$15 * \binom{1}{1}^3 *$$

   (f) How many UCSC ID numbers with no repeated digits and have digits in alphabetical order (i.e. 8, 5, 4, 9,1, 7, 6, 3, 2, 0)?

3. In each case below give the coefficient of the specified term

   (a) What is the coefficient of x^4y^3 in $(x + y)^7$
   Answer : $\binom{7}{3} = \frac{7!}{3!4!} = 35$

   (b) What is the coefficient of x^7y^4 in $(x + y)^{11}$
   Answer : $\binom{11}{4} = \frac{11!}{7!4!} = 330$

   (c) What is the coefficient of x^4 in $(x + 2)^{10}$
   Answer : $\binom{10}{6} \times 2^6 = 13,440$

   (d) What is the coefficient of x^3 in $(2x + 1)^{13}$
   Answer : $\binom{13}{10} \times 2^3 = 2,288$

4. In this problem you will prove the following statement in two ways

$$\forall i \in \mathbb{N},\ \forall j \in \mathbb{N}, i\binom{i-1}{j-1} = j\binom{i}{j}$$

(a) By using the formula for $\binom{n}{k}$
Left Hand Side :

$$i\binom{i-1}{j-1} = i\frac{(i-1)!}{i-1-(j-1))!(j-1)!}$$

$$i\binom{i-1}{j-1} = \frac{i!}{(i-j)!(j-1)!}$$

Right Hand Side :

$$j\binom{i}{j} = j\frac{i!}{(i-j)!j!}$$

$$j\binom{i}{j} = \frac{i!}{(i-j)!(j-1)!}$$

Thus the two sides are equal.

(b) Using a combinatorial argument

5. Provide direct proofs for each of the following statements.

(a) if n is an odd integer, the n^3 is an odd integer.

$n = 2x+1, x \in \mathbb{Z}$	Definition of odd integer
$n^3 = (2x+1)^3 = 8x^3 + 12x^2 + 6x + 1$	Just expanding
$8x^3 + 12x^2 + 6x + 1 = 2(4x^3 + 6x^2 + 3x) + 1$	Factored out a 2
$x \in \mathbb{Z} \implies (4x^3 + 6x^2 + 3x) \in \mathbb{Z}$	Integers closed under add. and mult
$n^3 = 2(4x^3 + 6x^2 + 3x) + 1 = 2y + 1 : y \in \mathbb{Z}$	n^3 also satisfies def. of odd number.

Thus if n is an odd integer, n^3 is also an odd integer.

(b) If n is an odd integer, then 4 divides $n^2 - 1$
If $a|b$, then $\frac{b}{a} = k, k \in \mathbb{Z}$ thus $b = ka$

$$a = n^2 - 1, b = 4, \exists! q \in \mathbb{Z} : a = qb$$

$n = 2x+1, x \in \mathbb{Z}$	Definition of odd integer
$n^2 - 1 = (2x+1)^2 - 1 = 4x^2 + 4x$	Substitution and Simplification
$4x^2 + 4x = 4(x^2 + x)$	Simplifying
$x \in \mathbb{Z} \implies (x^2 + x) \in \mathbb{Z}$	Integers closed under add. and mult
$n^2 - 1 = (x^2 + x)4 = q4 \ ; \ q \in \mathbb{Z}$	Completing the proof

The above shows that we can always find $q, r \in \mathbb{Z}$ such that $4|n^2 - 1$ satisfies the division theorum.

(c)

(d) for any integers a, b, and c, if $a|b$ and $a|(b+c)$ then $a|c$

$$a|b \implies b = qa \quad : q \in \mathbb{Z} \qquad a|(b+c) \implies (b+c) = ta \quad : t \in \mathbb{Z}$$

To finish the proof, we need to show $c = ya, y \in \mathbb{Z}$

$b = qa$	Restating the facts
$b + c = ta : \ qa + c = ta$	Substituting
$c = ta - qa$	Balancing sides
$c = (t-q)a : t, q \in \mathbb{Z}$	Factoring
$c = ya, y \in \mathbb{Z}$	subtraction of integers produces an integer

QED