CMPE 16 Homework #6

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1. Prove or disprove that for any real number x if $\sqrt{x}$ is irrational then x is irrational.
   Proposition : For any real number x if $\sqrt{x}$ is irrational then x is irrational. This propsition is **False**, a single counter example will show this.

   Let $\sqrt{x} \in \mathbb{R} = \sqrt{2}$, which is irrational. [1]
   Then $x = \sqrt{x}^2 = \sqrt{2}^2 = 2 = \frac{2}{1}$. Thus $\exists a, b \in \mathbb{Z} : x = \frac{a}{b}$, which shows that x is rational, which is a counter example to the proposition above.

2. Prove or disprove that for any real number x if x is irrational the $\sqrt{x}$ is irrational.
   Proposition : For any real number x, is x is irrational then $\sqrt{x}$ is irrational.
   This proposition is **True**, proof given below

   Assume for the purposes of contradiction that x is irrational but $\sqrt{x}$ is rational.

$\forall a, b \in \mathbb{Z} : x \neq \frac{a}{b}$	x is not a rational number
$\exists c, d \in \mathbb{Z} : \sqrt{x} = \frac{c}{d}$	$\sqrt{x}$ is a rational number
$\sqrt{x}^2 = (\frac{c}{d})^2 = \frac{c^2}{d^2}$	Taking the square of $\sqrt{x}$
$\sqrt{x}^2 = x = \frac{c^2}{d^2} = \frac{j}{k} : j, k \in \mathbb{Z}$	Substituting
$\exists j, k \in \mathbb{Z} : x = \frac{j}{k}$	Contadicts our assumption that x is irrational

   Thus our assumption is false, and thus if x is irrational then $\sqrt{x}$ must also be irrational.

3. Prove or disprove that the product of an irrational number and a non-zero rational number is irrational.
   Proposition : Product of an irrational number and a non-zero rational number yields an irrational number.

   Assume for the purposes of contradiction that the product of an irrational number and a non-zero rational yields a rational number.

$\forall a, b \in \mathbb{Z} : x \neq \frac{a}{b}$	x is not a rational number
$\exists c, d \in \mathbb{Z} : y = \frac{c}{d} \land c \neq 0$	y is a rational number
$x * y = z : z = \frac{j}{k}$ where $j, k \in \mathbb{Z}$	$x * y$ yields a rational number
$x * \frac{c}{d} = \frac{j}{k}$	Substituting in for y and z
$x = \frac{jd}{kc}$	Cross multiplying
$jd = a : a \in \mathbb{Z}$	Product of to integers is an integer
$kc = b : b \in \mathbb{Z}$	Product of to integers is an integer
$\exists a, b \in \mathbb{Z} : x = \frac{a}{b}$	This contadicts our original assumption that x is irrational

   Thus the product of an irrational number and a non-zero rational number always yields another irrational number.

[1]See proof that $\sqrt{2}$ is irrational on bottom of page

4. In each of the following give a value for $x > -1$ and $x <$ mod (the number after mod inside the parentheses).

   (a) $x \equiv -75(\mod 11)$ **Answer :** $x = 2$

   (b) $x \equiv 895(\mod 7)$ **Answer :** $x = 6$

   (c) $x \equiv 2^{126}(\mod 5)$

   (d) $x^2 \equiv 9(\mod 11)$ **Answer :** $x^2 = 9, x = 3$

5. Prove the following theorum

6. Prove the following theorum
   For an integer n, n is an odd number iff $n^2 - 1$ is a multiple of 4.
   This proof will entail two sub-proofs, if both are true than the above iff statement will be true.

   Proof #1 - For an integer n, n being odd implies $n^2 - 1$ is a multiple of 4.

$\exists k \in \mathbb{Z} : n = 2k + 1$	Defintion of an off number
$n^2 - 1 = (2k+1)^2 - 1 = 4k^2 + 4k$	Substituting and expanding
$n^2 - 1 = 4k^2 + 4k = 4(k^2 + k)$	Factoring
$n^2 - 1 = 4(k^2 + k) = 4j : j \in \mathbb{Z}$	$k^2 + k$ is an integer because integers are closed under $*, +$
$n = (2k+1) \implies n^2 - 1 = 4j : j \in \mathbb{Z}$	If an integer n is odd, $n^2 - 1$ is a mult. of 4.

   Proof #2 - For an integer n, $n^2 - 1$ being a multiple of 4 implies n is an odd number.

   I will use a proof by contrapositive. This will entail assuming n is an even number, and showing this implies $n^2 - 1$ is not a multiple of 4.

$\exists j \in \mathbb{Z} : n = 2j$	n is an even number
$n^2 - 1 = (2j)^2 - 1 = 4(j^2) - 1$	Substituting and expanding
$n^2 - 1 = 4j^2 - 1 \neq 4m : \forall m \in \mathbb{Z}$	$4m - 1$ cannot be a multiple of 4

7. Consider the two sets S_1 and S_2

$$S_1 = \{k^2 : k \text{ is an odd integer }\} \text{ and } S_2 = \{4m + 1 : \ m \text{ is an integer}\}$$

   (a) Prove that S_1 is a subset of S_2

$S_1 = \{k^2 : k = (2n+1)\}$	S_1 contains the squares of all odd integers
$S_1 = \{(2n+1)^2 : n \in \mathbb{Z}\}$	Substituting
$S_1 = \{4n^2 + 4n + 1 : n \in \mathbb{Z}\}$	Expanding
$S_1 = \{4(n^2 + n) + 1 : n \in \mathbb{Z}\}$	Factoring
$S_1 = \{4j + 1 : j = n^2 + n, n \in \mathbb{Z}\}$	Substituting

   Since the set $X = \{n^2 + n : n \in \mathbb{Z}\} \subseteq \mathbb{Z}$, then the set $S_1 = \{4j + 1 : j = X\} \subseteq S_2$

   (b) Prove that S_2 is not a subset of S_1
   The number 3 is in the set S_2 (take $m = 1$ to show this), but the number 3 is not in the set S_1. This is because for 3 to be in S_1 would imply $3 = n^2$ for some odd number n. But 3 is not the square of an integer, none the less an odd integer, so 3 cannot be in the set S_1.

8. Prove that the two sets A and B below are equal

$$A = \{7m - 5 : m \in \mathbb{Z}\} \; B = \{14k + b : k \in \mathbb{Z}, b \in \{2, 9\}\}$$

The proof will need two very similar cases, one for $b = 2$, and one for $b = 9$.

If $b = 9$ then $B = \{14k + 9 : k \in \mathbb{Z}\}$