CMPE 16 Homework #7

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1. Prove by induction that
$$\forall n \in \mathbb{N} : \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

   Base Case : $n = 1$, $\sum_{i=1}^{1} i^3 = 1^3 = \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$

   Inductive Hypothesis : $n \in \mathbb{N}$, $\sum_{i=1}^{n} i^3 = \frac{(n)^2(n+1)^2}{4}$

   Inductive Conclusion : $n \in \mathbb{N}$, $\sum_{i=1}^{n+1} i^3 = \frac{(n+1)^2(n+2)^2}{4}$

$\sum_{i=1}^{n+1} i^3 = 1^3 + 2^3 + 3^3 \ldots + n^3 + (n+1)^3$	Defintion of a sum
$\sum_{i=1}^{n+1} i^3 = \sum_{i=1}^{n} i^3 + (n+1)^3$	Subsituting in the previous case
$\sum_{i=1}^{n+1} i^3 = \frac{(n)^2(n+1)^2}{4} + (n+1)^3$	We know what the sum of cubes from 1 to n is.
$\sum_{i=1}^{n+1} i^3 = (n+1)^2(\frac{n^2}{4} + (n+1))$	Factoring out a $(n+1)^2$
$\sum_{i=1}^{n+1} i^3 = (n+1)^2(\frac{n^2+4n+4}{4})$	Put it all over one denominator
$\sum_{i=1}^{n+1} i^3 = (n+1)^2\frac{(n+2)^2}{4}$	$n^2 + 4n + 4 = (n+2)^2$

   That which was to be shown has been thus shown.

2. Prove by induction that
$$\forall k \in \mathbb{N} : \sum_{k=1}^{n} \frac{k}{2^k} = 2 - \frac{n+2}{2^n}$$

   Base Case : $n = 1$, $\sum_{k=1}^{1} \frac{k}{2^k} = \frac{1}{2}$, $2 - \frac{1+2}{2^1} = 2 - \frac{3}{2} = \frac{1}{2}$

   Inductive Hypothesis : $n \in \mathbb{N}$, $\sum_{k=1}^{n} \frac{k}{2^k} = 2 - \frac{n+2}{2^n}$

   Inductive Conclusion : $n \in \mathbb{N}$, $\sum_{k=1}^{n+1} \frac{k}{2^k} = 2 - \frac{n+3}{2^{n+1}}$

$\sum_{k=1}^{n+1} \frac{k}{2^k} = \frac{1}{2} + \frac{2}{4} + \ldots + \frac{n}{2^n} + \frac{n+1}{2^{n+1}}$	Defintion of our sum
$\sum_{k=1}^{n+1} \frac{k}{2^k} = \sum_{k=1}^{n} \frac{k}{2^k} + \frac{n+1}{2^{n+1}}$	Subsituting summation expression for first n terms
$\sum_{k=1}^{n+1} \frac{k}{2^k} = 2 - \frac{n+2}{2^n} + \frac{n+1}{2^{n+1}}$	Summation expressioni s known
$\sum_{k=1}^{n+1} \frac{k}{2^k} = 2 - 2^{-n}(\frac{n+2}{1} - \frac{n+1}{2})$	Factor out 2^n
$\sum_{k=1}^{n+1} \frac{k}{2^k} = 2 - 2^{-n}(\frac{2n+4-n+1}{2})$	Common denominator
$\sum_{k=1}^{n+1} \frac{k}{2^k} = 2 - 2^{-n}(\frac{n+3}{2})$	Combine like terms
$\sum_{k=1}^{n+1} \frac{k}{2^k} = 2 - \frac{n+3}{2^{n+1}}$	Distribute leading term

   That which was to be shown has been thus shown.

3. Prove by induction that

$$\forall n \in \mathbb{N} : (n \geq 2) \implies = ((\sqrt{2}^n) \leq n!)$$

Base Case : $n = 2$, $\sqrt{2}^2 \leq 3! : 2 \leq 6$

Inductive Hypothesis : $n \geq 2 \in \mathbb{N}$, $\sqrt{2}^n \leq (n)!$

Inductive Conclusion : $n \geq 2 \in \mathbb{N}$, $\sqrt{2}^{n+1} \leq (n+1)!$

$\sqrt{2}^n \leq n!$	Our original assumption
$n \geq 2 : \sqrt{2} < (n+1)$	$\sqrt{2} < 2 < (n+1)$
$\sqrt{2^n}\sqrt{2} \leq (n+1)n!$	Combining the two above equations. [1]
$\sqrt{2^{n+1}} \leq (n+1)!$	Simplifying the above equation

Thus the inductive conclusion has been shown.

4. Prove by induction that

$$\forall n \in \mathbb{N} : 2n^3 + 4n \text{ is a multiple of } 3$$

Base Case : $n = 1$, $2 + 4 = 6 = 2 * 3$

Inductive Hypothesis : $n, k \in \mathbb{N}$, $2n^3 + 4n = k * 3$

Inductive Conclusion : $n, j \in \mathbb{N}$, $2(n+1)^3 + 4(n+1) = j * 3$

$2(n+1)^3 + 4(n+1) = 2(n^3 + 3n^2 + 3n + 1) + 4n + 4$	Distributing terms
$2(n+1)^3 + 4(n+1) = 2n^3 + 6n^2 + 6n + 2 + 4n + 4$	Distributing ..
$2(n+1)^3 + 4(n+1) = 2n^3 + 4n + (6n^2 + 6n + 6)$	Grouping terms
$2(n+1)^3 + 4(n+1) = k * 3 + (6n^2 + 6n + 6)$	See inductive hypthesis
$2(n+1)^3 + 4(n+1) = k * 3 + 3(2n^2 + 2n + 2)$	Factor out a 3
$2(n+1)^3 + 4(n+1) = 3k + 3j = 3r$	$k, j, r \in \mathbb{N}$

The Inductive conclusion has been shown.

5. Rancher Pat is planning to raise ducks. Based on extensive research Pat estimates that every year there will be at least 1 new baby duck for every 4 ducks in his ranch, and that he will lose no more than 10 ducks each year. To be exact there will be $\lceil D/4 \rceil$ new baby ducks next year if there are D ducks in the current year. If Pat starts with 60 ducks in year 0, prove by induction that in year n Pat will have at least

$$20(\frac{5}{4})^n + 40 \text{ Ducks } s$$

Base Case : $n = 0$, $\sum_{i=0}^{0} 60(\frac{5}{4})^0 = 60$, $20(\frac{5}{4})^0 + 40 = 60$

Inductive Hypothesis : $n \in \mathbb{N}$, $\sum_{i=0}^{n} 60(\frac{5}{4})^n = 20(\frac{5}{4})^n + 40$

Inductive Conclusion : $n \in \mathbb{N}$, $\sum_{i=0}^{n+1} 60(\frac{5}{4})^{n+1} = 20(\frac{5}{4})^{n+1} + 40$

6. In this problem you will prove Fermats Little Theorem in a different manner than we did in class

   (a) Begin by first showing that for any prime number p and integers a and b

$$(a+b)^p \equiv a^p + b^p \bmod p$$

(b) Prove by induction on a that for any natural number a

$$a^p \equiv a \bmod p$$

7. Prove by induction that for $n \in \mathbb{N}$ (where F_n is the n^{th} Fib. number)

$$F_1^2 + F_2^2 + \ldots + F_n^2 = F_n F_{n+1}$$

Base Case : $n = 0$, $F_0 = 1$; $F_1 = 1$; $F_0 F_1 = 1$

Inductive Hypothesis : $n \in \mathbb{N}$, $\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}$

Inductive Conclusion : $n \in \mathbb{N}$, $\sum_{i=0}^{n+1} F_i^2 = F_{n+1} F_{n+2}$

$\sum_{i=0}^{n+1} F_i^2 = F_1^2 + F_2^2 + \ldots + F_n^2 + F_{n+1}^2$	Defintion of our summation
$\sum_{i=0}^{n+1} F_i^2 = \sum_{i=0}^{n} F_i^2 + F_{n+1}^2$	Substitute summation for first n terms
$\sum_{i=0}^{n+1} F_i^2 = F_n F_{n+1} + F_{n+1}^2$	Substitute Inductive Hypothesis
$\sum_{i=0}^{n+1} F_i^2 = F_n F_{n+1} + F_{n+1} F_{n+1}$	Simplifying-ish
$\sum_{i=0}^{n+1} F_i^2 = F_{n+1}(F_n + F_{n+1})$	Factoring
$\sum_{i=0}^{n+1} F_i^2 = F_{n+1} F_{n+2}$	$F_{n+2} = F_n + F_{n+1}$

Thus the inductive conclusion has been shown.