Diffusion and Matern functions

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1 Goal

The aim is to find a practical way to compute the Matern function:

$$c_\nu(\tilde{r}) = \alpha_\nu \tilde{r}^\nu K_\nu(\tilde{r}) \tag{1}$$

where K_ν is the modified Bessel function of second kind and the factor α_ν ensures that c_ν is normalized ($c_\nu(0) = 1$).

2 Explicit formula

For the particular case where $\nu - 1/2 \in \mathbb{Z}$, there is a special formula for the modified Bessel function of second kind (http://functions.wolfram.com/Bessel-TypeFunctions/BesselK/introductions/Bessels/05/):

$$K_\nu(\tilde{r}) = \sqrt{\pi} e^{-\tilde{r}} \sum_{j=0}^{\lfloor |\nu| - 1/2 \rfloor} \frac{(|\nu| - 1/2 + j)!}{j!\,(|\nu| - 1/2 - j)!} (2\tilde{r})^{-j-1/2} \tag{2}$$

Thus for $\nu > 0$:

$$c_\nu(\tilde{r}) = \alpha_\nu \tilde{r}^\nu K_\nu(\tilde{r}) \tag{3}$$

$$= \alpha_\nu \frac{\sqrt{\pi}}{2^\nu} e^{-\tilde{r}} \sum_{j=0}^{\nu-1/2} \frac{(\nu - 1/2 + j)!}{j!\,(\nu - 1/2 - j)!} (2\tilde{r})^{\nu-1/2-j} \tag{4}$$

$$\tag{5}$$

This formula seems consistent with the Wikipedia page (https://en.wikipedia.org/wiki/Mat%C3%A9rn_covariance_function#Simplification_for_.CE.BD_half_integer), via a simple change of variable:

$$\tilde{r} = \sqrt{2\nu}\, \frac{d}{\rho} \tag{6}$$

and $\nu = p + 1/2$ (and also with the appropriate normalization).

3 Inconsitency

However for Weaver and Mirouze (2013), $\nu = M - d/2$ so there equation (56) gives:

$$\begin{aligned}
\tilde{r} &= \sqrt{(\mathbf{x}-\mathbf{x}')^{\mathrm{T}}\boldsymbol{\kappa}^{-1}(\mathbf{x}-\mathbf{x}')} && (7)\\
&= \sqrt{2M-d-2}\,\sqrt{(\mathbf{x}-\mathbf{x}')^{\mathrm{T}}(\boldsymbol{D}_d^w)^{-1}(\mathbf{x}-\mathbf{x}')} && (8)\\
&= \sqrt{2\nu-2}\,\sqrt{(\mathbf{x}-\mathbf{x}')^{\mathrm{T}}(\boldsymbol{D}_d^w)^{-1}(\mathbf{x}-\mathbf{x}')} && (9)
\end{aligned}$$

which is not consistent with the previous equation. Maybe the "-2" of the third line of equation (56) is erroneous?

4 Practical computation

Anyway, for $d = 1$ or $d = 3$ we get:

$$c_\nu(\tilde{r}) = \alpha_\nu \frac{\sqrt{\pi}}{2^{M-d/2}} e^{-\tilde{r}} \sum_{j=0}^{M-(d+1)/2} \beta_{j,M-(d+1)/2} (2\tilde{r})^{M-(d+1)/2-j} \tag{10}$$

where

$$\beta_{j,M} = \frac{(M+j)!}{j!\,(M-j)!} \tag{11}$$

can be computed recursively:

$$\beta_{0,M} = 1 \tag{12}$$

and

$$\begin{aligned}
\beta_{j+1,M} &= \frac{(M+j+1)!}{(j+1)!\,(M-(j+1))!} && (13)\\
&= \frac{(M+j+1)(M-j)}{j+1}\beta_{j,M} && (14)
\end{aligned}$$

Thus, the normalization becomes:

$$c_\nu(0) = \alpha_\nu \frac{\sqrt{\pi}}{2^{M-d/2}} \beta_{M-(d+1)/2,M-(d+1)/2} = 1 \tag{15}$$

leading to

$$\alpha_\nu = \frac{2^{M-d/2}}{\sqrt{\pi}} \frac{1}{\beta_{M-(d+1)/2,M-(d+1)/2}} \tag{16}$$

And finally:

$$c_\nu(\tilde{r}) = \frac{1}{\beta_{M-(d+1)/2,M-(d+1)/2}} e^{-\tilde{r}} \sum_{j=0}^{M-(d+1)/2} \beta_{j,M-(d+1)/2} (2\tilde{r})^{M-(d+1)/2-j} \tag{17}$$

The case $d = 2$ would require another method.

References

Weaver AT, Mirouze I. 2013. On the diffusion equation and its application to isotropic and anisotropic correlation modelling in variational assimilation. *Quarterly Journal of the Royal Meteorological Society* **139**: 242–260.