

# Multivariate localization

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Last update: November 9, 2022

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# 1 Framework

## 1.1 State vector definition

The state vector  $\mathbf{x} \in \mathbb{R}^n$  is split into  $Q$  concatenated sub-vectors  $\mathbf{x}_q$  (called “groups”), each one gathering  $P_q$  sub-vectors  $\mathbf{x}_{p,q}$  of size  $n_q$  (called “variables”). Each sub-vector  $\mathbf{x}_{p,q}$  corresponds to the data for a given physical variable (e.g. zonal wind, temperature, humidity, etc.). Thus:

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_Q \end{pmatrix} \quad \text{and} \quad \mathbf{x}_q = \begin{pmatrix} \mathbf{x}_{1,q} \\ \vdots \\ \mathbf{x}_{P_q,q} \end{pmatrix} \quad (1)$$

As a consequence:

- The total number of variable sub-vectors is  $P = \sum_{q=1}^Q P_q$ .
- Each group sub-vector  $\mathbf{x}_q$  has a size  $P_q n_q$ , so the total state vector size is  $n = \sum_{q=1}^Q P_q n_q$ .

The main idea behind this distribution of variables into groups is to apply the same localization to all variables within a given group.

## 1.2 Illustration setup

An simple example will be used as illustration throughout this note. We define a vector  $\mathbf{x}$  with  $Q = 3$  groups, where  $P_1 = 3$ ,  $P_2 = 1$  and  $P_3 = 2$ :

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_{1,1} \\ \mathbf{x}_{2,1} \\ \mathbf{x}_{3,1} \\ \mathbf{x}_{1,2} \\ \mathbf{x}_{1,3} \\ \mathbf{x}_{2,3} \end{pmatrix} \left\{ \begin{array}{l} \mathbf{x}_1 \text{ (1<sup>st</sup> group of variables)} \\ \mathbf{x}_2 \text{ (2<sup>nd</sup> group of variables)} \\ \mathbf{x}_3 \text{ (3<sup>rd</sup> group of variables)} \end{array} \right. \quad (2)$$

## 1.3 Localization square-root

The localization matrix  $\mathbf{L} \in \mathbb{R}^{n \times n}$  is applied to  $\mathbf{x}$ . The main issue is to design a positive semi-definite  $\mathbf{L}$  matrix, and the simplest solution is to build it as:

$$\mathbf{L} = \mathbf{U}\mathbf{U}^T \quad (3)$$

where  $\mathbf{U} \in \mathbb{R}^{n \times m}$  is called a “square-root” of the localization matrix  $\mathbf{L}$ . The control vector  $\mathbf{v} \in \mathbb{R}^m$  is such that:

$$\mathbf{x} = \mathbf{U}\mathbf{v} \quad (4)$$

We can define a square-root component  $\mathbf{U}_q \in \mathbb{R}^{n_q \times m_q}$  for each group.

## 2 Univariate localization

For the case where there is no cross-localization between variables, the square-root  $\mathbf{U}$  can be defined as:

$$\mathbf{U} = \left( \begin{array}{ccc|c|cc} \mathbf{U}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{U}_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{U}_1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \mathbf{U}_2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \mathbf{U}_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{U}_3 \end{array} \right) \quad (5)$$

Thus:

$$\mathbf{L} = \left( \begin{array}{ccc|c|cc} \mathbf{U}_1 \mathbf{U}_1^T & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{U}_1 \mathbf{U}_1^T & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{U}_1 \mathbf{U}_1^T & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \mathbf{U}_2 \mathbf{U}_2^T & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \mathbf{U}_3 \mathbf{U}_3^T & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{U}_3 \mathbf{U}_3^T \end{array} \right) \quad (6)$$

In this case, the total control vector size is  $m = \sum_{q=1}^Q P_q m_q$

## 3 Block-univariate localization

### 3.1 With duplicates

It is possible to set the cross-localization between the variables of a same group as duplicates of the diagonal localization, by defining  $\mathbf{U}$  as:

$$\begin{aligned} \mathbf{U} &= \left( \begin{array}{c|c|c} \mathbf{U}_1 & 0 & 0 \\ \mathbf{U}_1 & 0 & 0 \\ \mathbf{U}_1 & 0 & 0 \\ \hline 0 & \mathbf{U}_2 & 0 \\ \hline 0 & 0 & \mathbf{U}_3 \\ 0 & 0 & \mathbf{U}_3 \end{array} \right) \\ &= \left( \begin{array}{c|c|c} \mathbf{I}_1 & 0 & 0 \\ \mathbf{I}_1 & 0 & 0 \\ \mathbf{I}_1 & 0 & 0 \\ \hline 0 & \mathbf{I}_2 & 0 \\ \hline 0 & 0 & \mathbf{I}_3 \\ 0 & 0 & \mathbf{I}_3 \end{array} \right) \left( \begin{array}{c|c|c} \mathbf{U}_1 & 0 & 0 \\ \hline 0 & \mathbf{U}_2 & 0 \\ \hline 0 & 0 & \mathbf{U}_3 \end{array} \right) \end{aligned} \quad (7)$$

where  $\mathbf{I}_q \in \mathbb{R}^{n_q, n_q}$  is the identity matrix of size  $n_q$ . Thus:

$$\mathbf{L} = \left( \begin{array}{ccc|c|cc} \mathbf{U}_1 \mathbf{U}_1^T & \mathbf{U}_1 \mathbf{U}_1^T & \mathbf{U}_1 \mathbf{U}_1^T & 0 & 0 & 0 \\ \mathbf{U}_1 \mathbf{U}_1^T & \mathbf{U}_1 \mathbf{U}_1^T & \mathbf{U}_1 \mathbf{U}_1^T & 0 & 0 & 0 \\ \mathbf{U}_1 \mathbf{U}_1^T & \mathbf{U}_1 \mathbf{U}_1^T & \mathbf{U}_1 \mathbf{U}_1^T & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \mathbf{U}_2 \mathbf{U}_2^T & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \mathbf{U}_3 \mathbf{U}_3^T & \mathbf{U}_3 \mathbf{U}_3^T \\ 0 & 0 & 0 & 0 & \mathbf{U}_3 \mathbf{U}_3^T & \mathbf{U}_3 \mathbf{U}_3^T \end{array} \right) \quad (8)$$

In this case, the total control vector size is  $m = \sum_{q=1}^Q m_q$

### 3.2 With weighted duplicates

A variant of the previous cases uses group-specific lower triangular matrices  $\mathbf{S}^q \in \mathbb{R}^{P_q \times P_q}$  to define  $\mathbf{U}$  as:

$$\begin{aligned} \mathbf{U} &= \left( \begin{array}{ccc|c|cc} S_{1,1}^1 \mathbf{U}_1 & 0 & 0 & 0 & 0 & 0 \\ S_{2,1}^1 \mathbf{U}_1 & S_{2,2}^1 \mathbf{U}_1 & 0 & 0 & 0 & 0 \\ S_{3,1}^1 \mathbf{U}_1 & S_{3,2}^1 \mathbf{U}_1 & S_{3,3}^1 \mathbf{U}_1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & S_{1,1}^2 \mathbf{U}_2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & S_{1,1}^3 \mathbf{U}_3 & 0 \\ 0 & 0 & 0 & 0 & S_{2,1}^3 \mathbf{U}_3 & S_{2,2}^3 \mathbf{U}_3 \end{array} \right) \\ &= \left( \begin{array}{ccc|c|cc} S_{1,1}^1 \mathbf{I}_1 & 0 & 0 & 0 & 0 & 0 \\ S_{2,1}^1 \mathbf{I}_1 & S_{2,2}^1 \mathbf{I}_1 & 0 & 0 & 0 & 0 \\ S_{3,1}^1 \mathbf{I}_1 & S_{3,2}^1 \mathbf{I}_1 & S_{3,3}^1 \mathbf{I}_1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & S_{1,1}^2 \mathbf{I}_2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & S_{1,1}^3 \mathbf{I}_3 & 0 \\ 0 & 0 & 0 & 0 & S_{2,1}^3 \mathbf{I}_3 & S_{2,2}^3 \mathbf{I}_3 \end{array} \right) \left( \begin{array}{ccc|c|cc} \mathbf{U}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{U}_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{U}_1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \mathbf{U}_2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \mathbf{U}_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{U}_3 \end{array} \right) \end{aligned} \quad (9)$$

Thus:

$$\mathbf{L} = \left( \begin{array}{ccc|c|cc} W_{1,1}^1 \mathbf{U}_1 \mathbf{U}_1^T & W_{1,2}^1 \mathbf{U}_1 \mathbf{U}_1^T & W_{1,3}^1 \mathbf{U}_1 \mathbf{U}_1^T & 0 & 0 & 0 \\ W_{2,1}^1 \mathbf{U}_1 \mathbf{U}_1^T & W_{2,2}^1 \mathbf{U}_1 \mathbf{U}_1^T & W_{2,3}^1 \mathbf{U}_1 \mathbf{U}_1^T & 0 & 0 & 0 \\ W_{3,1}^1 \mathbf{U}_1 \mathbf{U}_1^T & W_{3,2}^1 \mathbf{U}_1 \mathbf{U}_1^T & W_{3,3}^1 \mathbf{U}_1 \mathbf{U}_1^T & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & W_{1,1}^2 \mathbf{U}_2 \mathbf{U}_2^T & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & W_{1,1}^3 \mathbf{U}_3 \mathbf{U}_3^T & W_{1,2}^3 \mathbf{U}_3 \mathbf{U}_3^T \\ 0 & 0 & 0 & 0 & W_{2,1}^3 \mathbf{U}_3 \mathbf{U}_3^T & W_{2,2}^3 \mathbf{U}_3 \mathbf{U}_3^T \end{array} \right) \quad (10)$$

where  $\mathbf{W}^q = \mathbf{S}^q \mathbf{S}^{qT}$  provides the weights applied to each variable block with the group  $q$ . In practice, it is easier to set  $\mathbf{W}^q$  and to compute  $\mathbf{S}^q$  using a Cholesky decomposition. It also seems reasonable to set the diagonal elements of  $\mathbf{W}^q$  to one, even if it is not mandatory.

In this case, the total control vector size is  $m = \sum_{q=1}^Q P_q m_q$

## 4 Multivariate localization

### 4.1 Definition

For the case where the cross-localizations between variables are implicitly given as product of square-roots,  $\mathbf{U}$  can be reduced to a column:

$$\begin{aligned} \mathbf{U} &= \begin{pmatrix} \mathbf{U}_1 \\ \mathbf{U}_1 \\ \mathbf{U}_1 \\ \frac{\mathbf{U}_1}{\mathbf{U}_2} \\ \frac{\mathbf{U}_1}{\mathbf{U}_3} \\ \mathbf{U}_3 \end{pmatrix} \\ &= \left( \begin{array}{c|c|c} \mathbf{I}_1 & 0 & 0 \\ \mathbf{I}_1 & 0 & 0 \\ \mathbf{I}_1 & 0 & 0 \\ \hline 0 & \mathbf{I}_2 & 0 \\ \hline 0 & 0 & \mathbf{I}_3 \\ 0 & 0 & \mathbf{I}_3 \end{array} \right) \begin{pmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \mathbf{U}_3 \end{pmatrix} \end{aligned} \quad (11)$$

An additional assumption is required here: the size of the group sub-vectors must be the same for all groups, i.e.  $m_1 = \dots = m_Q = \bar{m}$ . Thus:

$$\mathbf{L} = \left( \begin{array}{ccc|ccc} \mathbf{U}_1 \mathbf{U}_1^T & \mathbf{U}_1 \mathbf{U}_1^T & \mathbf{U}_1 \mathbf{U}_1^T & \mathbf{U}_1 \mathbf{U}_2^T & \mathbf{U}_1 \mathbf{U}_3^T & \mathbf{U}_1 \mathbf{U}_3^T \\ \mathbf{U}_1 \mathbf{U}_1^T & \mathbf{U}_1 \mathbf{U}_1^T & \mathbf{U}_1 \mathbf{U}_1^T & \mathbf{U}_1 \mathbf{U}_2^T & \mathbf{U}_1 \mathbf{U}_3^T & \mathbf{U}_1 \mathbf{U}_3^T \\ \mathbf{U}_1 \mathbf{U}_1^T & \mathbf{U}_1 \mathbf{U}_1^T & \mathbf{U}_1 \mathbf{U}_1^T & \mathbf{U}_1 \mathbf{U}_2^T & \mathbf{U}_1 \mathbf{U}_3^T & \mathbf{U}_1 \mathbf{U}_3^T \\ \hline \mathbf{U}_2 \mathbf{U}_1^T & \mathbf{U}_2 \mathbf{U}_1^T & \mathbf{U}_2 \mathbf{U}_1^T & \mathbf{U}_2 \mathbf{U}_2^T & \mathbf{U}_2 \mathbf{U}_3^T & \mathbf{U}_2 \mathbf{U}_3^T \\ \hline \mathbf{U}_3 \mathbf{U}_1^T & \mathbf{U}_3 \mathbf{U}_1^T & \mathbf{U}_3 \mathbf{U}_1^T & \mathbf{U}_3 \mathbf{U}_2^T & \mathbf{U}_3 \mathbf{U}_3^T & \mathbf{U}_3 \mathbf{U}_3^T \\ \mathbf{U}_3 \mathbf{U}_1^T & \mathbf{U}_3 \mathbf{U}_1^T & \mathbf{U}_3 \mathbf{U}_1^T & \mathbf{U}_3 \mathbf{U}_2^T & \mathbf{U}_3 \mathbf{U}_3^T & \mathbf{U}_3 \mathbf{U}_3^T \end{array} \right) \quad (12)$$

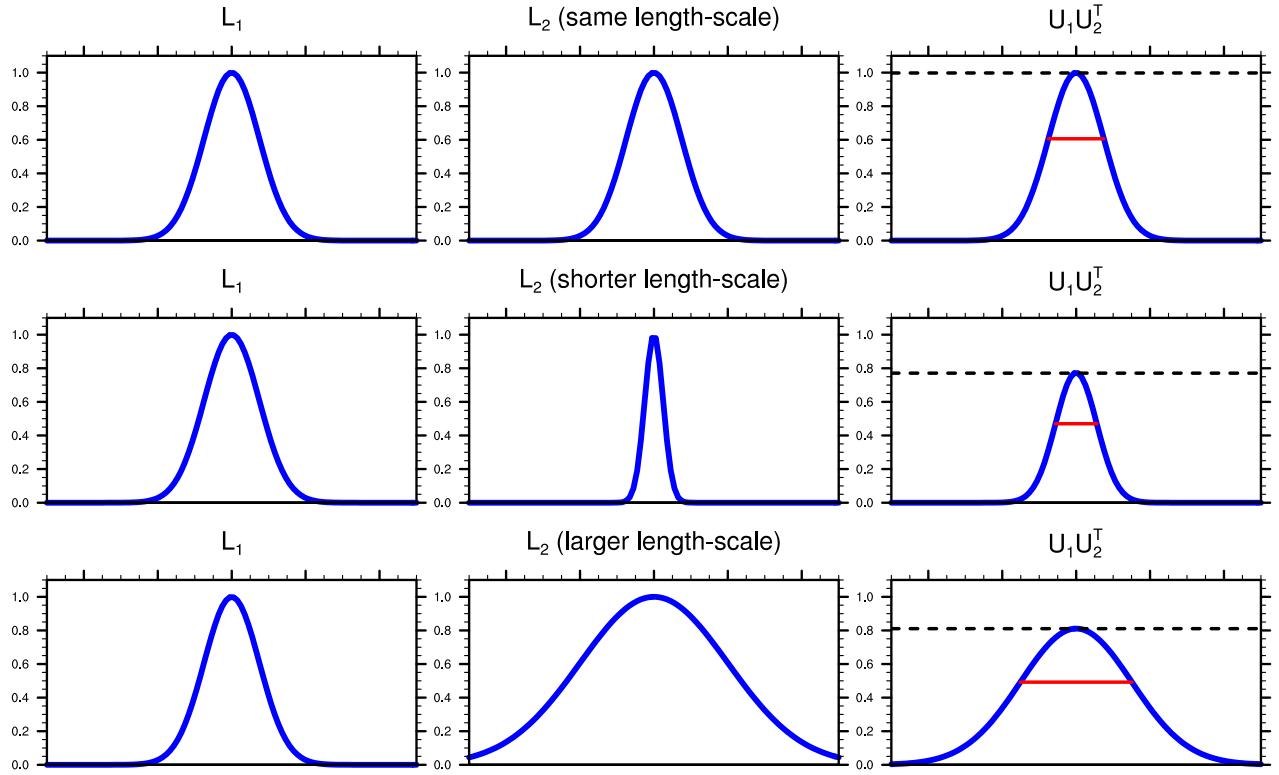
In this case, the total control vector size is  $m = \bar{m}$ .

### 4.2 Cross-localization amplitude

For  $1 \leq q, q' \leq Q$ , we denote  $\mathbf{u}$  the  $k^{\text{th}}$  column of  $\mathbf{U}_q$  and  $\mathbf{u}'$  the  $k^{\text{th}}$  column of  $\mathbf{U}_{q'}$ . The  $k^{\text{th}}$  diagonal coefficient of  $\mathbf{U}_q \mathbf{U}_{q'}^T$  is given by  $\langle \mathbf{u}, \mathbf{u}' \rangle$ , where  $\langle \cdot, \cdot \rangle$  denotes the canonical inner product. The Cauchy-Schwartz inequality requires that:

$$|\langle \mathbf{u}, \mathbf{u}' \rangle| \leq \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle \langle \mathbf{u}', \mathbf{u}' \rangle} \quad (13)$$

Thus, the cross-localization amplitude between groups  $q$  and  $q'$  is necessarily smaller than the geometric mean of the auto-localization amplitudes for groups  $q$  and  $q'$ . Besides, the more the vectors  $\mathbf{u}$  and  $\mathbf{u}'$  have different shapes, the smaller their inner product and the smaller the cross-localization amplitude.



**Figure 1:** 1D illustration of the product of two localization square-root, with Gaussian localization functions. The dashed black line shows the function amplitude, while the solid red line shows the function length-scale.