MAE 6500 Final Project

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The Joukowsky transform relation can be used to model the flow around and geometry of a circular cylinder in the complex plane to an ellipse in the real plane. Using this relation, a sphere in the three-dimensional complex region can be mapped to an ellipsoid in the three-dimensional real region. The relation for this geometry transform is outlined. Examples are presented, demonstrating this transform. Further work is outlined for the study of three-dimensional potential flow using the Joukowsky transform.

I. Nomenclature

ε	=	quaternion - ellipsoid eccentricity
ε_m	=	ellipsoid eccentricity of the m coordinate
ε_x	=	ellipsoid eccentricity of the x coordinate
ε_y	=	ellipsoid eccentricity of the y coordinate
ε_z	=	ellipsoid eccentricity of the z coordinate
m	=	real coordinate orthogonal to the xyz region
μ	=	complex coordinate orthogonal to the $\chi\psi\omega$ region
μ_0	=	μ coordinate offset from the origin
q	=	quaternion - real region number
R	=	sphere radius
ω	=	complex coordinate normal to the freestream
ω_0	=	ω coordinate offset from the origin
x	=	real coordinate parallel to the freestream
χ	=	complex coordinate parallel to the freestream
χ_0	=	χ coordinate offset from the origin
y	=	real coordinate orthogonal to the xy plane
ψ	=	complex coordinate orthogonal to the $\chi\omega$ plane
ψ_0	=	ψ coordinate offset from the origin
z	=	real coordinate normal to the freestream
ζ	=	quaternion - complex region number
ζ_0	=	ζ quaternion offset from the origin

II. Introduction

The Joukowsky relation [1] used to map between the two-dimensional complex and real plane is defined as

$$z = \zeta + \frac{(R-\varepsilon)^2}{\zeta} \tag{1}$$

where z is the complex coordinate in the real plane, ζ is the complex coordinate in the complex plane, R is the elliptical-cylinder radius, and ε is the elliptical-cylinder eccentricity. This relation can be used to map the potential flow around a cylinder to an airfoil.

Other groups have examined the translation of a sphere to a spheroid [2]. However, to more fully approximate a three-dimensional wing, an ellipsoid must be studied. In order to determine the potential flow around a sphere, in a three-dimensional region, a similar complex number can be used. The quaternion is the three-dimensional equivalent of

*Graduate Research Assistant, Mechanical and Aerospace Engineering, 4130 Old Main Hill, AIAA Student Member the complex number. The quaternion is defined as

$$q = m + \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z \tag{2}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are the fundamental quaternion unit vectors in x, y, and z directions, respectively. A similar quaternion can be defined for the complex region as

$$\zeta = \mu + \boldsymbol{i}\chi + \boldsymbol{j}\psi + \boldsymbol{k}\omega \tag{3}$$

A variation of the Joukowsky transform can be used to map the between the three-dimensional complex and real regions [2–4]. This relation is

$$q = \zeta + \frac{(R-\varepsilon)^2}{\zeta} \tag{4}$$

and can be used to map a three-dimensional sphere in the complex region to an ellipsoid in the real region. It should be noted that the ellipsoid eccentricity ε is also a quaternion. A drawing of the coordinate system transformation can be seen in Fig. 1.

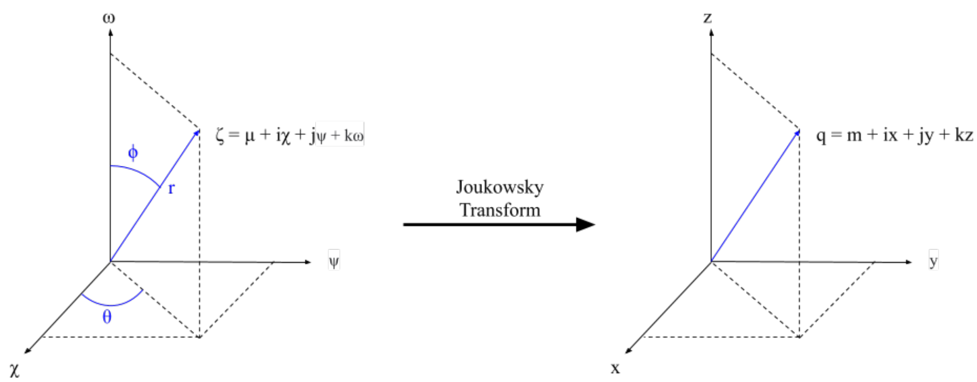

Fig. 1 Complex to real region transformation.

III. Relation Derivation

The Joukowsky transform can be used to map a complex coordinate to the real plan in quaternion form. First, the complex quaternion must be expanded, as

$$q = \mu + \boldsymbol{i}\chi + \boldsymbol{j}\psi + \boldsymbol{k}\omega + \frac{(R-\varepsilon)^2}{\mu + \boldsymbol{i}\chi + \boldsymbol{j}\psi + \boldsymbol{k}\omega} \tag{5}$$

after which the fraction on the right-hand side can be multiplied by the complex quaternion's complex conjugate. The remnant of the right-hand side can be similarly treated to create a single fraction as

$$q = \mu + \boldsymbol{i}\chi + \boldsymbol{j}\psi + \boldsymbol{k}\omega + \frac{(R-\varepsilon)^2}{\mu + \boldsymbol{i}\chi + \boldsymbol{j}\psi + \boldsymbol{k}\omega}\frac{\mu - \boldsymbol{i}\chi - \boldsymbol{j}\psi - \boldsymbol{k}\omega}{\mu - \boldsymbol{i}\chi - \boldsymbol{j}\psi - \boldsymbol{k}\omega} \tag{6}$$

$$= (\mu + \boldsymbol{i}\chi + \boldsymbol{j}\psi + \boldsymbol{k}\omega)\frac{\mu^2+\chi^2+\psi^2+\omega^2}{\mu^2+\chi^2+\psi^2+\omega^2} + \frac{(R-\varepsilon)^2(\mu - \boldsymbol{i}\chi - \boldsymbol{j}\psi - \boldsymbol{k}\omega)}{\mu^2+\chi^2+\psi^2+\omega^2} \tag{7}$$

$$= \frac{(\mu + \boldsymbol{i}\chi + \boldsymbol{j}\psi + \boldsymbol{k}\omega)(\mu^2+\chi^2+\psi^2+\omega^2) + (R-\varepsilon)^2(\mu - \boldsymbol{i}\chi - \boldsymbol{j}\psi - \boldsymbol{k}\omega)}{\mu^2+\chi^2+\psi^2+\omega^2} \tag{8}$$

$$= \frac{(\mu + \boldsymbol{i}\chi + \boldsymbol{j}\psi + \boldsymbol{k}\omega)(\mu^2+\chi^2+\psi^2+\omega^2) + (R-\varepsilon)^2(\mu - \boldsymbol{i}\chi - \boldsymbol{j}\psi - \boldsymbol{k}\omega)}{\mu^2+\chi^2+\psi^2+\omega^2} \tag{9}$$

The four distinct parts of the fraction can be split to resemble quaternion form as

$$q = \mu \frac{\mu^2 + \chi^2 + \psi^2 + \omega^2 + (R - \varepsilon_m)^2}{\mu^2 + \chi^2 + \psi^2 + \omega^2} + \boldsymbol{i}\chi \frac{\mu^2 + \chi^2 + \psi^2 + \omega^2 - (R - \varepsilon_y)^2}{\mu^2 + \chi^2 + \psi^2 + \omega^2} + \boldsymbol{j}\psi \frac{\mu^2 + \chi^2 + \psi^2 + \omega^2 - (R - \varepsilon_y)^2}{\mu^2 + \chi^2 + \psi^2 + \omega^2} + \boldsymbol{k}\omega \frac{\mu^2 + \chi^2 + \psi^2 + \omega^2 - (R - \varepsilon_z)^2}{\mu^2 + \chi^2 + \psi^2 + \omega^2} \quad (10)$$

which can be simplified as

$$q = \mu \left(1 + \frac{(R - \varepsilon_m)^2}{|\zeta|}\right) + \boldsymbol{i}\chi \left(1 - \frac{(R - \varepsilon_x)^2}{|\zeta|}\right) + \boldsymbol{j}\psi \left(1 - \frac{(R - \varepsilon_y)^2}{|\zeta|}\right) + \boldsymbol{k}\omega \left(1 - \frac{(R - \varepsilon_z)^2}{|\zeta|}\right) \quad (11)$$

It should be noted that a similar derivation process can be followed to allow for a complex region offset from the origin. This relation is given as

$$q = (\mu + \mu_0) \left(1 + \frac{(R - \varepsilon_m)^2}{|\zeta|}\right) + \boldsymbol{i}(\chi + \chi_0) \left(1 - \frac{(R - \varepsilon_x)^2}{|\zeta|}\right) + \boldsymbol{j}(\psi + \psi_0) \left(1 - \frac{(R - \varepsilon_y)^2}{|\zeta|}\right) + \boldsymbol{k}(\omega + \omega_0) \left(1 - \frac{(R - \varepsilon_z)^2}{|\zeta|}\right) \quad (12)$$

A simple way to examine the geometry of a sphere can be done using spherical coordinates, which can be related to Cartesian coordinates as

$$\chi = r \cos\theta \sin\phi \quad (13)$$

$$\psi = r \sin\theta \sin\phi \quad (14)$$

$$\omega = r \cos\phi \quad (15)$$

IV. Mapping Results

Equation (12) can be used to transform a sphere in the complex region to an ellipsoid in the real region, based on a set of inputs. These inputs consist in range numbers of the spherical coordinates n_θ and n_ϕ values, as well as a sphere radius R, an eccentricity quaternion ε and an offset quaternion ζ_0. The θ and ϕ values range from 0 to 2π and from 0 to π respectively.

A. Ellipsoid

The first example presented here is that of an ellipse which has no offset. The creation parameters for this design are given in Table 1

n_θ	n_ϕ	R	$\varepsilon/2R$	ζ_0
15	15	0.5	[0.00,1.60,1.25,0.20]	[0.00,0.00,0.00,0.00]

Table 1 Parameter inputs for ellipsoid example.

The ellipsoid created from these inputs is shown in Fig. 2

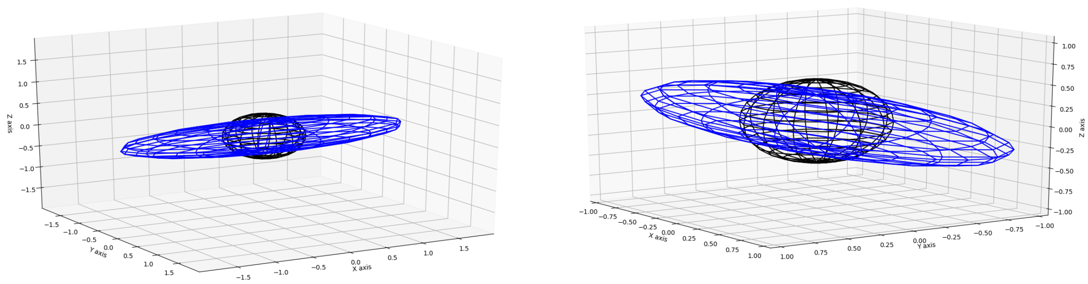

Fig. 2 Complex sphere mapped to the ellipsoid defined by Table 1.

Figure 2 demonstrates the Joukowsky transform to have a viable solution in the three-dimensional region. It should be noted this solution has no complex-region initial-offset.

B. Ellipsoid with Offset

The second example presented here is that of an ellipse with a complex offset. The creation parameters for this design are given in Table 2

n_θ	n_ϕ	R	$\varepsilon/2R$	ζ_0
15	15	0.5	[0.00,1.60,1.25,0.20]	[0.00,-0.10,0.00,0.10]

Table 2 Parameter inputs for ellipsoid with offset example.

The ellipsoid created from these inputs is shown in Fig. 3.

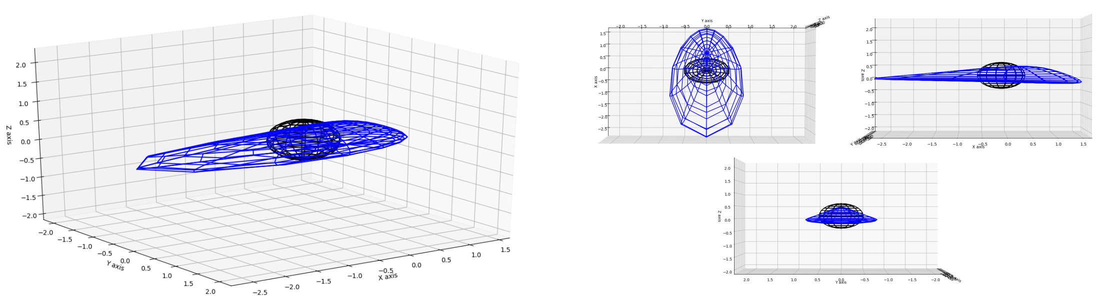

Fig. 3 Complex sphere mapped to the ellipsoid defined by Table 2.

This example demonstrates the effect of the offset on the ellipsoidal shape. The χ_0 and ω_0 offsets cause the ellipsoid shape to change with respect to the axes of the offsets.

C. Joukowsky Pseudo-Wing

The final example presented here is that of an ellipse with a complex offset which approximates a wing shape. The creation parameters for this design are given in Table 3.

n_θ	n_ϕ	R	$\varepsilon/2R$	ζ_0
15	15	0.5	[0.00,0.50,2.0,0.11]	[0.00,-0.10,0.00,0.10]

Table 3 Parameter inputs for pseudo-wing example.

The ellipsoid created from these inputs is shown in Fig. 4.

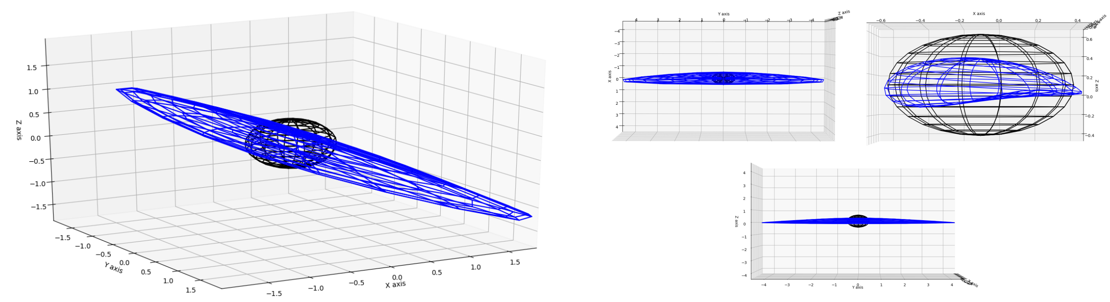

Fig. 4 Complex sphere mapped to the ellipsoid defined by Table 3.

While the values used to recreate this ellipsoid shape are simple guesses, they demonstrate the power of this 3D mapping technique in creating 3D wing-shapes mapped from a sphere in the complex region.

V. Further Work

Further study into the 3D mapping of a sphere to a wing is recommended. This further work should examine the development of a complex to real velocity equation for the potential flow over a sphere. Such a velocity flow equation should be of the form

$$\Phi = \Delta + \boldsymbol{i}\phi + \boldsymbol{j}\Psi_1 + \boldsymbol{k}\Psi_2 \tag{16}$$

where Φ is the complex quaternion potential, Δ is a negligible function, ϕ is the quaternion velocity potential, Ψ_1 is a quaternion stream surface, and Ψ_2 is a separate quaternion stream surface.

The velocity potential and stream surfaces for uniform, three-dimensional point doublet, and three-dimensional vortex line flow must be determined. Using these components, the complex quaternion potential can then be determined.

To determine the stream surfaces, the author recommends study into the integration of each potential flow velocity vector function. The author assumes the integration will yield two solutions which will be the two stream surface functions.

VI. Conclusion

The Joukowsky transform can be used to model the real version of a complex sphere in the real region. This solution results in an ellipsoid determined by the sphere radius, center offset, and complex eccentricity. Further study should examine the development of three-dimensional potential flow equations for a sphere. These equations can then be used to determine the transformed flow around an ellipsoid.

References

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[3] Malonek, H. R., and De Almeida, R., "A note on a generalized Joukowski transformation," *Applied Mathematics Letters*, Vol. 23, No. 10, 2010, pp. 1174–1178.

[4] Jeyabalan, S., "Quaternions in Joukowski Transformation," Master's thesis, Tampere University Of Technology, 2011.