

Các pp Runge – Kutta hiện
giải bài toán Cauchy cho
phương trình vi phân thường

Bài toán Cauchy

$$\begin{cases} y' = f(x, y), x \in I = [x_0, X], \\ y \in C^1(I, R^k) \\ y(x_0) = y_0 \end{cases}$$

Phương trình tích phân

$$y(x) = y(x_0) + \int_{x_0}^x f(t, y(t)) dt$$

$$y(x_{k+1}) = y(x_k) + \int_{x_k}^{x_{k+1}} f(t, y(t)) dt$$

- Euler forward (hiện)

$$y_{n+1} = y_n + hf(x_n, y_n)$$

- Euler backward (ẩn)

$$y_{n+1} = y_n + hf(x_{n+1}, y_{n+1})$$

- Công thức hình thang

$$y_{n+1} = y_n + \frac{h}{2} \left(f(x_n, y_n) + f(x_{n+1}, y_{n+1}) \right)$$

R-K làm gì?

- Tính tích phân trong phương trình tích phân qua s các trung gian
- Đảm bảo việc tính thông qua các các trung gian có hiệu quả giống như khai triển Taylor hàm $y(x)$ đến bậc cao

Công thức R-K tổng quát

$$y_{n+1} = y_n + r_1 k_1^{(n)} + r_2 k_2^{(n)} + \dots + r_s k_s^{(n)}$$

$$k_i^{(n)} = hf \left(x_n + \alpha_i h, y_n + \beta_{i1} k_1^{(n)} + \dots + \beta_{ii-1} k_{i-1}^{(n)} \right)$$

$$\alpha_1 = 0, \alpha_i \in [0, 1]$$

R-K 1 nấc

$$s = 1$$

$$y_{n+1} = y_n + r_1 k_1^{(n)}$$

$$k_1^{(n)} = hf(x_n, y_n)$$

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + O(h^2)$$

$$\Rightarrow r_1 = 1$$

R-K 2 nấc

$$s=2$$

$$y_{n+1} = y_n + r_1 k_1^{(n)} + r_2 k_2^{(n)}$$

$$k_1^{(n)} = hf(x_n, y_n) = hf_n$$

$$k_2^{(n)} = hf\left(x_n + \alpha_2 h, y_n + \beta_{11} k_1^{(n)}\right)$$

$$\Rightarrow k_2^{(n)} = h\left[f_n + \alpha_2 hf'_{x,n} + \beta_{11} k_1^{(n)} f'_{y,n} + O(h^2)\right]$$

$$y(x_{n+1}) = y(x_n) + hf_n + \frac{h^2}{2}\left[f'_{x,n} + f'_{y,n} \cdot f_n\right] + O(h^3)$$

R-K 2 nấc

$$r_1 + r_2 = 1; r_2 \alpha_2 = \frac{1}{2}; r_2 \beta_{11} = \frac{1}{2}$$

$$r_1 = 0; r_2 = 1; \alpha_2 = \frac{1}{2}; \beta_{11} = \frac{1}{2}$$

$$r_1 = r_2 = \frac{1}{2}; \alpha_2 = \beta_{11} = 1$$

$$r_1 = \frac{1}{3}; r_2 = \frac{2}{3}; \alpha_2 = \beta_{11} = \frac{3}{4}$$

.....

R-K 3 nấc

$$y_{n+1} = y_n + r_1 k_1^{(n)} + r_2 k_2^{(n)} + r_3 k_3^{(n)}$$

$$k_1^{(n)} = hf \left(x_n, y_n \right)$$

$$k_2^{(n)} = hf \left(x_n + \alpha_2 h, y_n + \beta_{11} k_1^{(n)} \right)$$

$$k_3^{(n)} = hf \left(x_n + \alpha_3 h, y_n + \beta_{21} k_1^{(n)} + \beta_{22} k_2^{(n)} \right)$$

R-K 3 nấc

$$k_2^{(n)} = h \left[\begin{aligned} &f_n + \alpha_2 h f'_{x,n} + \beta_{11} h f_n f'_{y,n} + \\ &+ \frac{h^2}{2} \alpha_2^2 f''_{x,n} + h^2 \alpha_2 \beta_{21} f_n f''_{x,y} + \frac{h^2}{2} \beta_{11}^2 f_n^2 f''_{y,n} + O(h^3) \end{aligned} \right]$$

$$k_3^{(n)} = h \left[\begin{aligned} &f_n + \alpha_3 h f'_{x,n} + \left(\beta_{21} k_1^{(n)} + \beta_{22} k_2^{(n)} \right) f'_{y,n} + \frac{h^2}{2} \alpha_3^2 f''_{xx} + \\ &+ \alpha_3 h \left(\beta_{21} k_1^{(n)} + \beta_{22} k_2^{(n)} \right) f''_{xy} + \left(\beta_{21} k_1^{(n)} + \beta_{22} k_2^{(n)} \right)^2 f''_{yy} + O(h^3) \end{aligned} \right]$$

$$y(x_{n+1}) = y(x_n) + h f_n + \frac{h^2}{2} \left[f'_{x,n} + f'_{y_n} \cdot f_n \right]$$

$$+ \frac{h^3}{6} \left[f''_{xx} + f''_{xy} f_n + f''_{yy} f_n^2 + f'_y f'_x + f'^2_y f_n \right] + O(h^4)$$

$$r_1 + r_2 + r_3 = 1$$

$$r_2\alpha_2 + r_3\alpha_3 = \frac{1}{2}$$

$$r_2\beta_{11} + r_3(\beta_{21} + \beta_{22}) = \frac{1}{2}$$

$$\frac{1}{2}r_2\alpha_2^2 + \frac{1}{2}r_3\alpha_3^2 = \frac{1}{6}$$

$$r_2\alpha_2\beta_{21} + r_3\alpha_3(\beta_{21} + \beta_{22}) = \frac{1}{6}$$

$$r_2\beta_{11}^2 + r_3(\beta_{21}^2 + \beta_{22}^2) = \frac{1}{6}$$

$$\beta_{22}\alpha_2 = \frac{1}{6}$$

$$\beta_{11}\beta_{22} = \frac{1}{6}$$

R-K3 thường dùng

$$r_1 = \frac{1}{6}; r_2 = \frac{2}{3}; r_3 = \frac{1}{6}; \alpha_2 = \frac{1}{2}; \alpha_3 = 1; \beta_{11} = \frac{1}{2}; \beta_{21} = -1; \beta_{22} = 2$$

$$y_{n+1} = y_n + \frac{1}{6} \left(k_1^{(n)} + 4k_2^{(n)} + k_3^{(n)} \right)$$

$$k_1^{(n)} = hf \left(x_n, y_n \right)$$

$$k_2^{(n)} = hf \left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1^{(n)} \right)$$

$$k_3^{(n)} = hf \left(x_n + h, y_n - k_1^{(n)} + 2k_2^{(n)} \right)$$

R-K3 thường dùng (Heun)

$$r_1 = \frac{1}{4}; r_2 = 0; r_3 = \frac{3}{4}; \alpha_2 = \beta_{11} = \frac{1}{3}; \alpha_3 = \beta_{22} = \frac{2}{3}; \beta_{21} = 0$$

$$y_{n+1} = y_n + \frac{1}{4} \left(k_1^{(n)} + 3k_3^{(n)} \right)$$

$$k_1^{(n)} = hf \left(x_n, y_n \right)$$

$$k_2^{(n)} = hf \left(x_n + \frac{1}{3}h, y_n + \frac{1}{3}k_1^{(n)} \right)$$

$$k_3^{(n)} = hf \left(x_n + \frac{2}{3}h, y_n + \frac{2}{3}k_2^{(n)} \right)$$

R-K 4 thường dùng

$$y_{n+1} = y_n + \frac{1}{6} \left(k_1^{(n)} + 2k_2^{(n)} + 2k_3^{(n)} + k_4^{(n)} \right)$$

$$k_1^{(n)} = hf \left(x_n, y_n \right)$$

$$k_2^{(n)} = hf \left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1^{(n)} \right)$$

$$k_3^{(n)} = hf \left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2^{(n)} \right)$$

$$k_4^{(n)} = hf \left(x_n + h, y_n + k_3^{(n)} \right)$$

Bậc cao nhất của các công thức R_K s nắc

s	1	2	3	4	5	6	7	8	9
p	1	2	3	4	4	5	6	6	7

Ví dụ mô hình hệ thú mồi

$$\begin{cases} x' = rn \left(1 - \frac{n}{K} \right) - ap \\ p' = -\mu p + anp \end{cases}$$