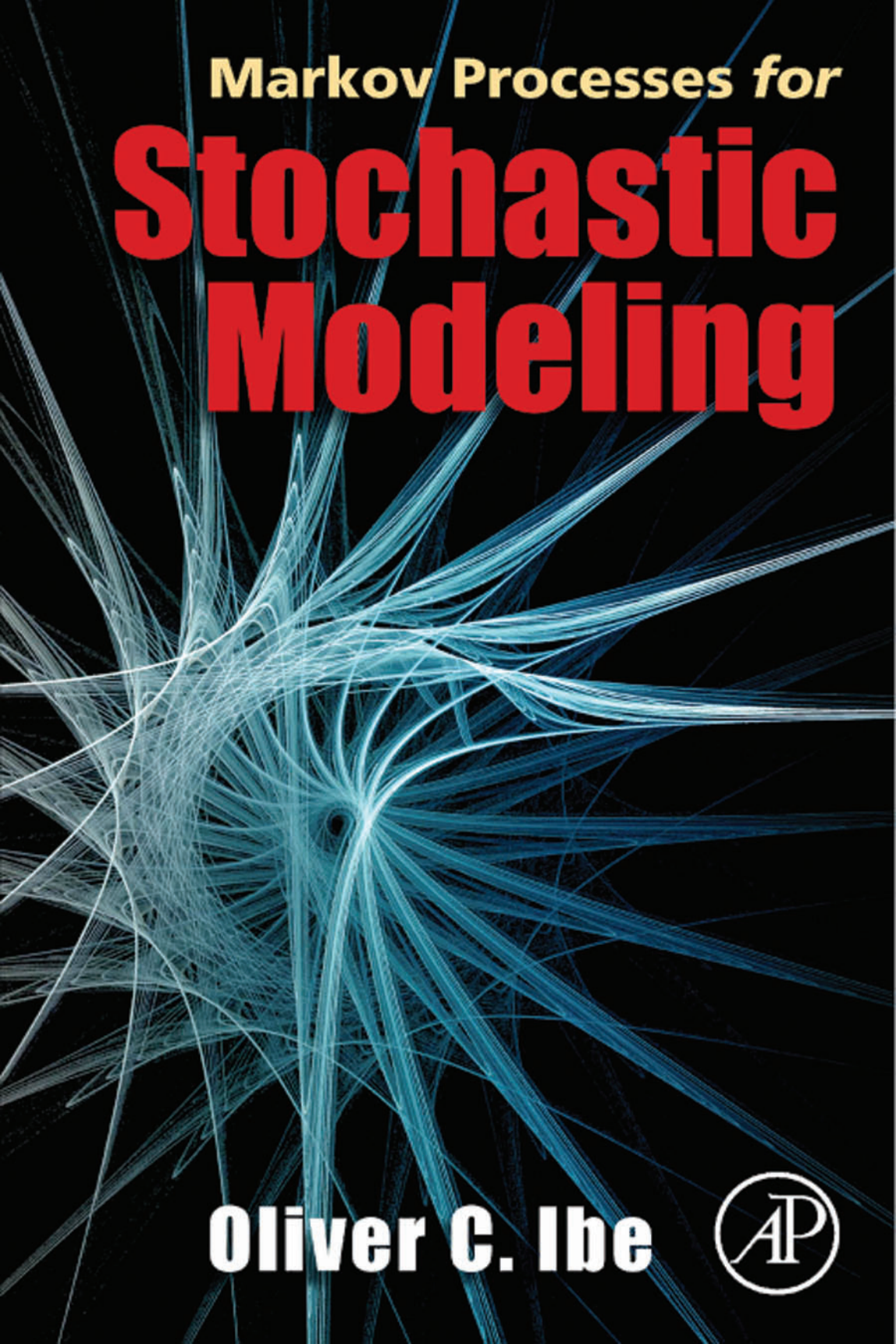
Markov Processes for
Stochastic Modeling
Oliver C. Ibe
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Preface

Markov processes are used to model systems with limited memory. They are used in many areas including communication systems, transportation networks, image segmentation and analysis, biological systems and DNA sequence analysis, random atomic motion and diffusion in physics, social mobility, population studies, epidemiology, animal and insect migration, queueing systems, resource management, dams, financial engineering, actuarial science, and decision systems.

Different books have been written specifically for different types of Markov processes. For example, books on bioinformatics discuss hidden Markov models, books on financial markets and economics discuss random walks and Brownian motion, and books on image analysis discuss Markov random fields and Markov point processes.

The purpose of this book is to bring into one volume the different Markovian models that are individually scattered across many books. The book is written for graduate students and researchers in the different fields where Markov processes are used. It is particularly designed for students in traffic engineering, image analysis, bioinformatics, biostatistics, financial engineering, and computational biology. It is a combination of theory and applications of Markov models and presents the essential details of these models. Therefore, any reader who is interested in more information than what is presented in any particular topic discussed in the book is advised to consult any of the specialized books listed in the references.

The book is organized as follows. Chapter 1 deals with basic concepts in probability and stochastic processes. Chapter 2 discusses general properties of Markov processes. Chapter 3 discusses discrete-time Markov chains while Chapter 4 discusses continuous-time Markov chains. Chapter 5 discusses Markovian queueing systems, which are essentially those queueing systems where either the interarrival times or service times or both are exponentially distributed. Chapter 6 deals with Markov renewal processes including semi-Markov processes.Chapter 7 discusses Markovian arrival processes while Chapter 8 deals with the random walk. Chapter 9 discusses Brownian motion and diffusion processes. Chapter 10 discusses controlled Markov processes, which include Markov decision processes, semi-Markov decision processes, and partially observable Markov decision processes. Chapter 11 discusses hidden Markov models, Chapter 12 discusses Markov random fields, and Chapter 13 discusses Markov point processes. Finally, Chapter 14 deals with Markov chain Monte Carlo.

The first nine chapters of the book are appropriate for a one-semester graduate course on applied stochastic processes. Such a course would usually include Markov chain Monte Carlo. The remainder of the book can be used according to the interests of the students.

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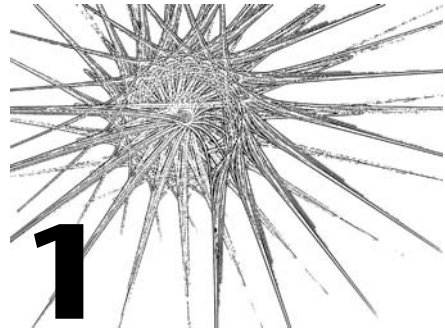
Finally, I would like to thank my wife, Christina, for enduring another intrusion into our family life by yet another book. I would also like to thank our children Chidinma, Ogechi, Amanze, and Ugonna, for the joy they bring to my life.

Basic Concepts



1.1 Review of Probability

The concepts of *experiments* and *events* are very important in the study of probability. In probability, an experiment is any process of trial and observation. An experiment whose outcome is uncertain before it is performed is called a *random* experiment. When we perform a random experiment, the collection of possible elementary outcomes is called the *sample space* of the experiment, which is usually denoted by Ω. We define these outcomes as elementary outcomes because exactly one of the outcomes occurs when the experiment is performed. The elementary outcomes of an experiment are called the *sample points* of the sample space and are denoted by $w_i, i = 1, 2, \ldots$. If there are n possible outcomes of an experiment, then the sample space is $\Omega = \{w_1, w_2, \ldots, w_n\}$.

An *event* is the occurrence of either a prescribed outcome or any one of a number of possible outcomes of an experiment. Thus, an event is a subset of the sample space. For example, if we toss a die, any number from 1 to 6 can appear. Therefore, in this experiment the sample space is defined by

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

The event "the outcome of the toss of a die is an even number" is a subset of Ω and is defined by

$$E = \{2, 4, 6\}$$

For a second example, consider a coin-tossing experiment in which each toss can result in either a head (H) or a tail (T). If we toss a coin three times and let

the triplet xyz denote the outcome "x on the first toss, y on the second toss, and z on the third toss," then the sample space of the experiment is

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

The event "one head and two tails" is a subset of Ω and is defined by

$$E = \{HTT, THT, TTH\}$$

For any two events A and B defined on a sample space Ω, we can define the following new events:

- $A \cup B$ is the event that consists of all sample points that are either in A or in B or in both A and B. The event $A \cup B$ is called the *union* of events A and B.
- $A \cap B$ is the event that consists of all sample points that are in both A and B. The event $A \cap B$ is called the *intersection* of events A and B. Two events are defined to be *mutually exclusive* if their intersection does not contain a sample point; that is, they have no outcomes in common. Events $A_1, A_2, A_3, \ldots$, are defined to be mutually exclusive if no two of them have any outcomes in common and the events collectively have no outcomes in common.
- $A - B$ is the event that consists of all sample points that are in A but not in B. The event $A - B$ is called the *difference* of events A and B. Note that $A - B$ is different from $B - A$. The difference $A - B$ is sometimes denoted by $A/B = A \cap \overline{B}$, where $\overline{B}$ is defined as the *complement* of event B.

Consider an abstract space Ω; that is, Ω is a space without any special structure. Let F be a family of subsets of Ω with the following properties:

1. $\emptyset \in F$ and $\Omega \in F$, where $\emptyset$ is the empty space called null event
2. If $A \in F$, then $\overline{A} \in F$, where $\overline{A}$ is the complement of A; that is, $\overline{A} = F - A$.
3. F is closed under countable unions and intersections; that is, if $A_1, A_2, \ldots$ are events in F, then $\bigcup_{k=1}^{\infty} A_k$ and $\bigcap_{k=1}^{\infty} A_k$ are both in F.

Under these conditions, F is defined to be a σ-algebra (or σ-field). A probability measure defined on a σ-algebra F of Ω is a function P that maps points in F onto the closed interval [0, 1]. Thus, for an event A in F, the function $P[A]$ is called the *probability* of event A. The probability measure P satisfies the following Kolmogorov axioms:

1. As stated earlier, for any event $A \in F$, $0 \leq P[A] \leq 1$
2. $P[\Omega] = 1$, which means that with probability 1, the outcome will be a sample point in the sample space.

3. For any set of n disjoint events $A_1, A_2, \ldots, A_n$ in F,

$$P[A_1 \cup A_2 \cup \ldots \cup A_n] = P[A_1] + P[A_2] + \cdots + P[A_n]$$

That is, for any set of mutually exclusive events defined on the same space, the probability of at least one of these events occurring is the sum of their respective probabilities.

The triple (Ω, F, P) is called a *probability space*. The following results are additional properties of a probability measure:

1. $P[\overline{A}] = 1 - P[A]$, which states that the probability of the complement of A is one minus the probability of A.
2. $P[\emptyset] = 0$, which states that the impossible (or null) event has probability zero.
3. If $A \subset B$, then $P[A] \leq P[B]$. That is, if A is a subset of B, the probability of A is at most the probability of B (or the probability of A cannot exceed the probability of B).
4. If $A = A_1 \cup A_2 \cup \ldots \cup A_n$, where $A_1, A_2, \ldots, A_n$ are mutually exclusive events, then

$$P[A] = P[A_1] + P[A_2] + \cdots + P[A_n]$$

5. For any two events A and B, $P[A] = P[A \cap B] + P[A \cap \overline{B}]$, which follows from the set identity: $A = (A \cap B) \cup (A \cap \overline{B})$. Since $A \cap B$ and $A \cap \overline{B}$ are mutually exclusive events, the result follows.
6. For any two events A and B, $P[A \cup B] = P[A] + P[B] - P[A \cap B]$.
7. We can extend Property 6 to the case of three events. If A_1, A_2, A_3 are three events in F, then

$$P[A_1 \cup A_2 \cup A_3] = P[A_1] + P[A_2] + P[A_3] - P[A_1 \cap A_2] - P[A_1 \cap A_3] - P[A_2 \cap A_3] + P[A_1 \cap A_2 \cap A_3]$$

This can be further generalized to the case of n arbitrary events in F as follows:

$$P[A_1 \cup A_2 \cup \ldots \cup A_n] = \sum_{i=1}^{n} P[A_i] - \sum_{1 \leq i \leq j \leq n} P[A_i \cap A_j] + \sum_{1 \leq i \leq j \leq k \leq n} P[A_i \cap A_j \cap A_k] - \cdots$$

That is, to find the probability that at least one of the n events A_i occurs, first add the probability of each event, then subtract the probabilities of all possible two-way intersections, then add the probabilities of all possible three-way intersections, and so on.

1.1.1 Conditional Probability

Let A denote an event and let B denote another event. The conditional probability of event A given event B, denoted by $P[A|B]$, is defined by

$$P[A|B] = \frac{P[A \cap B]}{P[B]} \quad P[B] > 0$$

For example, if A denotes the event that the sum of the outcomes of tossing two dice is 7 and B denotes the event that the outcome of the first die is 4, then the conditional probability of event A given event B is defined by

$$\begin{aligned} P[A|B] &= \frac{P[A \cap B]}{P[B]} \\ &= \frac{P[\{4,3\}]}{P[\{4,1\}] + P[\{4,2\}] + P[\{4,3\}] + P[\{4,4\}] + P[\{4,5\}] + P[\{4,6\}]} \\ &= \frac{(1/36)}{(1/6)} = \frac{1}{6} \end{aligned}$$

1.1.2 Independence

Two events A and B are defined to be independent if the knowledge that one has occurred does not change or affect the probability that the other will occur. In particular, if events A and B are independent, the conditional probability of event A, given event B, $P[A|B]$, is equal to the probability of event A. That is, events A and B are independent if

$$P[A|B] = P[A]$$

Because by definition $P[A \cap B] = P[A|B]P[B]$, an alternative definition of independence of events is that events A and B are independent if

$$P[A \cap B] = P[A]P[B]$$

The definition of independence can be extended to multiple events. The n events $A_1, A_2, \ldots, A_n$ are said to be independent if the following conditions are true:

$$P[A_i \cap A_j] = P[A_i]P[A_j]$$

$$P[A_i \cap A_j \cap A_k] = P[A_i]P[A_j]P[A_k]$$

$$\ldots$$

$$P[A_1 \cap A_2 \cap \ldots \cap A_n] = P[A_1]P[A_2]\ldots P[A_n]$$

This is true for all $1 \leq i < j < k < \ldots \leq n$. That is, these events are pairwise independent, independent in triplets, and so on.

1.1.3 Total Probability and the Bayes' Theorem

A partition of a set A is a set $\{A_1, A_2, \ldots, A_n\}$ with the following properties:

a. $A_i \subseteq A, i = 1, 2, \ldots, n$, which means that A is a set of subsets
b. $A_i \cap A_k = \emptyset, i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, n;\ i \neq k$, which means that the subsets are mutually (or pairwise) disjoint; that is, no two subsets have any element in common.
c. $A_1 \cup A_2 \cup \ldots \cup A_n = A$, which means that the subsets are collectively exhaustive. That is, the subsets together include all possible values of the set A.

Let $\{A_1, A_2, \ldots, A_n\}$ be a partition of the sample space Ω, and suppose each one of the events $A_1, A_2, \ldots A_n$, has nonzero probability of occurrence. Let A be any event. Then

$$\begin{aligned} P[A] &= P[A|A_1]P[A_1] + P[A|A_2]P[A_2] + \cdots + P[A|A_n]P[A_n] \\ &= \sum_{i=1}^{n} P[A|A_i]P[A_i] \end{aligned}$$

This result is defined as the *total probability* of event A.

Suppose event A has occurred, but we do not know which of the mutually exclusive and collectively exhaustive events $A_1, A_2, \ldots, A_n$ holds true. The conditional probability that event A_k occurred, given that A occurred, is given by

$$P[A_k|A] = \frac{P[A_k \cap A]}{P[A]} = \frac{P[A_k \cap A]}{\sum_{i=1}^{n} P[A|A_i]P[A_i]}$$

where the second equality follows from the total probability of event A. Because $P[A_k \cap A] = P[A|A_k]P[A_k]$, the preceding equation can be rewritten as follows:

$$P[A_k|A] = \frac{P[A_k \cap A]}{P[A]} = \frac{P[A|A_k]P[A_k]}{\sum_{i=1}^{n} P[A|A_i]P[A_i]}$$

This result is called the *Bayes' formula* (or *Bayes' rule*).

1.2 Random Variables

Consider a random experiment with sample space Ω. Let w be a sample point in Ω. We are interested in assigning a real number to each $w \in \Omega$. A random variable, $X(w)$, is a single-valued real function that assigns a real number, called the value of $X(w)$, to each sample point $w \in \Omega$. That is, it is a mapping of the sample space onto the real line.

Generally a random variable is represented by a single letter X instead of the function $X(w)$. Therefore, in the remainder of the book we use X to denote a random variable. The sample space Ω is called the *domain* of the random variable X. Also, the collection of all numbers that are values of X is called the *range* of the random variable X.

Let X be a random variable and x a fixed real value. Let the event A_x define the subset of Ω that consists of all real sample points to which the random variable X assigns the number x. That is,

$$A_x = \{w|X(w) = x\} = [X = x]$$

Because A_x is an event, it will have a probability, which we define as follows:

$$p = P[A_x]$$

1.2.1 Distribution Functions

Let X be a random variable and x be a number. As stated earlier, we can define the event $[X \leq x] = \{x|X(w) \leq x\}$. The distribution function (or the cumulative distribution function (CDF)) of X is defined by

$$F_X(x) = P[X \leq x] \quad -\infty < x < \infty$$

That is, $F_X(x)$ denotes the probability that the random variable X takes on a value that is less than or equal to x.

1.2.2 Discrete Random Variables

A discrete random variable is a random variable that can take on at most a countable number of possible values. For a discrete random variable X, the *probability mass function* (PMF), $p_X(x)$, is defined as follows:

$$p_X(x) = P[X = x]$$

The PMF is nonzero for at most a countable or countably infinite number of values of x. In particular, if we assume that X can only assume one of the values $x_1, x_2, \ldots, x_n$, then

$$p_X(x_i) \geq 0 \qquad i = 1, 2, \ldots, n$$

$$p_X(x) = 0 \qquad \text{otherwise}$$

The CDF of X can be expressed in terms of $p_X(x)$ as follows:

$$F_X(x) = \sum_{k \leq x} p_X(k)$$

The CDF of a discrete random variable is a staircase-shaped function. That is, if X takes on values $x_1, x_2, x_3, \ldots$, where $x_1 < x_2 < x_3 < \ldots$, then the value of $F_X(x)$ is constant in the interval between x_{i-1} and x_i and then takes a jump of size $p_X(x_i)$ at $x_i, i = 2, 3, \ldots$. Thus, in this case, $F_X(x)$ represents the sum of all the probability masses we have encountered as we move from $-\infty$ to slightly to the right of x.

1.2.3 Continuous Random Variables

Discrete random variables have a set of possible values that are either finite or countably infinite. However, there exists another group of random variables that can assume an uncountable set of possible values. Such random variables are called continuous random variables. Thus, we define a random variable X to be a continuous random variable if there exists a nonnegative function $f_X(x)$, defined for all real $x \in (-\infty, \infty)$, having the property that for any set A of real numbers,

$$P(X \in A) = \int_A f_X(x)dx$$

The function $f_X(x)$ is called the *probability density function* (PDF) of the random variable X and is defined by

$$f_X(x) = \frac{dF_X(x)}{dx}$$

1.2.4 Expectations

If X is a random variable, then the *expectation* (or *expected value* or *mean*) of X, denoted by $E[X]$ or $\overline{X}$, is defined by

$$E[X] = \begin{cases} \sum_i x_i p_X(x_i) & X \text{ discrete} \\ \int_{-\infty}^{\infty} x f_X(x) dx & X \text{ continuous} \end{cases}$$

Thus, the expected value of X is a weighted average of the possible values that X can take, where each value is weighted by the probability that X takes that value.

1.2.5 Expectation of Nonnegative Random Variables

Some random variables assume only nonnegative values. For example, the time X until a component fails cannot be negative. For a nonnegative random variable X with the PDF $f_X(x)$ and the CDF is $F_X(x)$, the expected value is given by

$$E[X] = \int_0^{\infty} P[X > x] dx = \int_0^{\infty} [1 - F_X(x)] dx$$

For a discrete random variable X that assumes only nonnegative values,

$$E[X] = \sum_{k=0}^{\infty} P[X > k]$$

1.2.6 Moments of Random Variables and the Variance

The nth moment of the random variable X, denoted by $E[X^n] = \overline{X^n}$, is defined by

$$E[X^n] = \overline{X^n} = \begin{cases} \sum_i x_i^n p_X(x_i) & X \text{ discrete} \\ \int_{-\infty}^{\infty} x^n f_X(x) dx & X \text{ continuous} \end{cases}$$

for $n = 1, 2, \ldots$. The first moment, $E[X]$, is the expected value of X.

We can also define the *central moments* (or *moments about the mean*) of a random variable. These are the moments of the difference between a random

variable and its expected value. The nth central moment is defined by

$$E[(X-\overline{X})^n] = \overline{(X-\overline{X})^n} = \begin{cases} \sum_i (x_i - \overline{X})^n p_X(x_i) & X \text{ discrete} \\ \int_{-\infty}^{\infty} (x-\overline{X})^n f_X(x)dx & X \text{ continuous} \end{cases}$$

The central moment for the case of $n = 2$ is very important and carries a special name, the *variance*, which is usually denoted by σ_X^2. Thus,

$$\sigma_X^2 = E[(X-\overline{X})^2] = \overline{(X-\overline{X})^2} = \begin{cases} \sum_i (x_i - \overline{X})^2 p_X(x_i) & X \text{ discrete} \\ \int_{-\infty}^{\infty} (x-\overline{X})^2 f_X(x)dx & X \text{ continuous} \end{cases}$$

1.3 Transform Methods

Different types of transforms are used in science and engineering. These include the z-transform, Laplace transform, and Fourier transform. We consider two types of transforms: the z-transform (or moment-generating function) of PMFs and the s-transform (or unilateral Laplace transform) of PDFs.

1.3.1 The s-Transform

Let $f_X(x)$ be the PDF of the continuous random variable X that takes only non-negative values; that is, $f_X(x) = 0$ for $x < 0$. The s-transform of $f_X(x)$, denoted by $M_X(s)$, is defined by

$$M_X(s) = E[e^{-sX}] = \int_0^{\infty} e^{-sx} f_X(x)dx$$

One important property of an s-transform is that when it is evaluated at the point $s = 0$, its value is equal to 1. That is,

$$M_X(s)|_{s=0} = \int_0^{\infty} f_X(x)dx = 1$$

For example, the value of K for which the function $A(s) = \dfrac{K}{s+5}$ is a valid s-transform of a PDF is obtained by setting $A(0) = 1$, which gives

$$K/5 = 1 \Rightarrow K = 5$$

One of the primary reasons for studying the transform methods is to use them to derive the moments of the different probability distributions. By definition

$$M_X(s) = \int_0^\infty e^{-sx} f_X(x)dx$$

Taking different derivatives of $M_X(s)$ and evaluating them at $s = 0$, we obtain the following results:

$$\begin{aligned}
\frac{d}{ds}M_X(s) &= \frac{d}{ds}\int_0^\infty e^{-sx} f_X(x)dx = \int_0^\infty \frac{d}{ds}e^{-sx} f_X(x)dx \\
&= -\int_0^\infty xe^{-sx} f_X(x)dx \\
\frac{d}{ds}M_X(s)|_{s=0} &= -\int_0^\infty xf_X(x)dx \\
&= -E[X] \\
\frac{d^2}{ds^2}M_X(s) &= \frac{d}{ds}(-1)\int_{-\infty}^\infty xe^{-sx} f_X(x)dx = \int_0^\infty x^2 e^{-sx} f_X(x)dx \\
\frac{d^2}{ds^2}M_X(s)|_{s=0} &= \int_0^\infty x^2 f_X(x)dx \\
&= E[X^2]
\end{aligned}$$

In general,

$$\frac{d^n}{ds^n}M_X(s)|_{s=0} = (-1)^n E[X^n]$$

1.3.2 The z-Transform

Let $p_X(x)$ be the PMF of the discrete nonnegative random variable X. The z-transform of $p_X(x)$, denoted by $G_X(z)$, is defined by

$$G_X(z) = E[z^X] = \sum_{x=0}^\infty z^x p_X(x)$$

The sum is guaranteed to converge and, therefore, the z-transform exists, when evaluated on or within the unit circle (where $|z| \leq 1$). Note that

$$G_X(1) = \sum_{x=0}^\infty p_X(x) = 1$$

This means that a valid z-transform of a PMF reduces to unity when evaluated at $z = 1$. However, this is a necessary but not sufficient condition for a function to

the z-transform of a PMF. By definition,

$$G_X(z) = \sum_{x=0}^{\infty} z^x p_X(x)$$
$$= p_X(0) + z p_X(1) + z^2 p_X(2) + z^3 p_X(3) + \cdots$$

This means that $P[X = k] = p_X(k)$ is the coefficient of z^k in the series expansion. Thus, given the z-transform of a PMF, we can uniquely recover the PMF. The implication of this statement is that not every polynomial that has a value 1 when evaluated at $z = 1$ is a valid z-transform of a PMF. For example, consider the function $A(z) = 2z - 1$. Although $A(1) = 1$, the function contains invalid coefficients in the sense that these coefficients either have negative values or positive values that are greater than one. Thus, for a function of z to be a valid z-transform of a PMF, it must have a value of 1 when evaluated at $z = 1$, and the coefficients of z must be nonnegative numbers that cannot be greater than 1.

The individual terms of the PMF can also be determined as follows:

$$p_X(x) = \frac{1}{x!}\left[\frac{d^x}{dz^x} G_X(z)\right]_{z=0} \qquad x = 0, 1, 2, \ldots$$

This feature of the z-transform is the reason it is sometimes called the *probability generating function*.

As stated earlier, one of the major motivations for studying transform methods is their usefulness in computing the moments of the different random variables. Unfortunately the moment-generating capability of the z-transform is not as computationally efficient as that of the s-transform. The moment-generating capability of the z-transform lies in the results obtained from evaluating the derivatives of the transform at $z = 1$. For a discrete random variable X with PMF $p_X(x)$, we have that

$$G_X(z) = \sum_{x=0}^{\infty} z^x p_X(x)$$
$$\frac{d}{dz} G_X(z) = \frac{d}{dz} \sum_{x=0}^{\infty} z^x p_X(x) = \sum_{x=0}^{\infty} \frac{d}{dz} z^x p_X(x)$$
$$= \sum_{x=0}^{\infty} x z^{x-1} p_X(x) = \sum_{x=1}^{\infty} x z^{x-1} p_X(x)$$
$$\frac{d}{dz} G_X(z)|_{z=1} = \sum_{x=1}^{\infty} x p_X(x) = \sum_{x=0}^{\infty} x p_X(x) = E[X]$$

Similarly,

$$\frac{d^2}{dz^2}G_X(z) = \frac{d}{dz}\sum_{x=1}^{\infty} xz^{x-1}p_X(x) = \sum_{x=1}^{\infty} x\frac{d}{dz}z^{x-1}p_X(x)$$

$$= \sum_{x=1}^{\infty} x(x-1)z^{x-2}p_X(x)$$

$$\frac{d^2}{dz^2}G_X(z)|_{z=1} = \sum_{x=1}^{\infty} x(x-1)p_X(x)$$

$$= \sum_{x=0}^{\infty} x(x-1)p_X(x) = \sum_{x=0}^{\infty} x^2 p_X(x) - \sum_{x=0}^{\infty} xp_X(x)$$

$$= E[X^2] - E[X]$$

$$E[x^2] = \frac{d^2}{dz^2}G_X(z)|_{z=1} + \frac{d}{dz}G_X(z)|_{z=1}$$

We can obtain higher moments in a similar manner.

1.4 Bivariate Random Variables

Consider two random variables X and Y defined on the same sample space. For example, X can denote the grade of a student and Y can denote the height of the same student. The joint cumulative distribution function (joint CDF) of X and Y is given by

$$F_{XY}(x, y) = P[X \leq x, Y \leq y]$$

The pair (X, Y) is referred to as a *bivariate* random variable. If we define $F_X(x) = P[X \leq x]$ as the *marginal* CDF of X and $F_Y(y) = P[Y \leq y]$ as the *marginal* CDF of Y, then we define the random variables X and Y to be independent if

$$F_{XY}(x, y) = F_X(x)F_Y(y)$$

for every value of x and y. The marginal CDFs are obtained as follows:

$$F_X(x) = F_{XY}(x, \infty)$$

$$F_Y(y) = F_{XY}(\infty, y)$$

From the above properties we can answer questions about X and Y.

1.4.1 Discrete Bivariate Random Variables

When both X and Y are discrete random variables, we define their joint PMF as follows:

$$p_{XY}(x, y) = P[X = x, Y = y]$$

The marginal PMFs are obtained as follows:

$$p_X(x) = \sum_y p_{XY}(x, y) = P[X = x]$$

$$p_Y(y) = \sum_x p_{XY}(x, y) = P[Y = y]$$

If X and Y are independent random variables,

$$p_{XY}(x, y) = p_X(x)\, p_Y(y)$$

1.4.2 Continuous Bivariate Random Variables

If both X and Y are continuous random variables, their joint PDF is given by

$$f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y)$$

The joint PDF satisfies the following condition:

$$F_{XY}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(u, v) dv du$$

The marginal PDFs are given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y) dx$$

If X and Y are independent random variables, then

$$f_{XY}(x, y) = f_X(x)\, f_Y(y)$$

1.4.3 Covariance and Correlation Coefficient

Consider two random variables X and Y with expected values $E[X] = \mu_x$ and $E[Y] = \mu_Y$, respectively, and variances σ_X^2 and σ_Y^2, respectively. The *covariance* of X and Y, which is denoted by $Cov(X, Y)$ or σ_{XY}, is defined by

$$\begin{aligned} Cov(X, Y) = \sigma_{XY} &= E[(X - \mu_X)(Y - \mu_Y)] \\ &= E[XY - \mu_Y X - \mu_X Y + \mu_X \mu_Y] \\ &= E[XY] - \mu_X \mu_Y - \mu_X \mu_Y + \mu_X \mu_Y \\ &= E[XY] - \mu_X \mu_Y \end{aligned}$$

If X and Y are independent, then $E[XY] = \mu_X \mu_Y$ and $Cov(X, Y) = 0$. However, the converse is not true; that is, if the covariance of X and Y is zero, it does not mean that X and Y are independent random variables. If the covariance of two random variables is zero, we define the two random variables to be *uncorrelated*.

We define the *correlation coefficient* of X and Y, denoted by $\rho(X, Y)$ or ρ_{XY}, as follows:

$$\rho_{XY} = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

The correlation coefficient has the property that

$$-1 \leq \rho_{XY} \leq 1$$

1.5 Many Random Variables

In the previous sections we considered a system of two random variables. In this section we extend the concepts developed for two random variables to systems of more than two random variables.

Let $X_1, X_2, \ldots, X_n$ be a set of random variables that are defined on the same sample space. Their joint CDF is defined as

$$F_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n) = P[X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n]$$

If all the random variables are discrete random variables, their joint PMF is defined by

$$p_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n) = P[X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n]$$

The marginal PMFs are obtained by summing the joint PMF over the appropriate ranges. For example, the marginal PMF of X_n is given by

$$p_{X_n}(x_n) = \sum_{x_1} \sum_{x_2} \cdots \sum_{x_{n-1}} p_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n)$$

The conditional PMFs are similarly obtained. For example,

$$\begin{aligned} p_{X_n|X_1 \ldots X_{n-1}}(x_n|x_1, x_2, \ldots, x_{n-1}) &= P[X_n = x_n \\ &\quad |X_1 = x_1, X_2 = x_2, \ldots, X_{n-1} = x_{n-1}] \\ &= \frac{p_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n)}{p_{X_1 X_2 \ldots X_{n-1}}(x_1, x_2, \ldots, x_{n-1})} \end{aligned}$$

The random variables are defined to be mutually independent if

$$p_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} p_{X_i}(x_i)$$

If all the random variables are continuous random variables, their joint PDF can be obtained from the joint CDF as follows:

$$f_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n) = \frac{\partial^n}{\partial x_1 \partial x_2 \ldots \partial x_n} F_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n)$$

The conditional PDFs can also be defined. For example,

$$f_{X_n|X_1 \ldots X_{n-1}}(x_n|x_1, x_2, \ldots, x_{n-1}) = \frac{f_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n)}{f_{X_1 X_2 \ldots X_{n-1}}(x_1, x_2, \ldots, x_{n-1})}$$

If the random variables are mutually independent, then

$$f_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i)$$

1.6 Fubini's Theorem

The expectation of a function of random vectors (i.e., d-dimensional random variables, where $d > 1$) is obtained as a multidimensional integral. For example, if

g is a function defined in the probability space (Ω, F, P) and (X, Y) is a random vector, then

$$E[g(X,Y)] = \int_{\Omega} g(X,Y)dP = \int_{R^2} g(x,y)dF_{XY}(x,y)$$
$$= \begin{cases} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} g(x_i,y_j)p_{XY}(x_i,y_i) & \text{discrete case} \\ \int_{R^2} g(x,y)f_{XY}(x,y)dxdy & \text{continuous case} \end{cases}$$

Fubini's theorem allows us to compute expectations of functions of random variables in a rather simpler manner when the probability spaces are product spaces. Specifically, in the case of a function of two random variables, the theorem allows us to evaluate the expectation, which involves double integrals, as iterated single integrals. We state the theorem without proof.

Theorem 1.1 (Fubini's Theorem) Let (X, Y) be a two-dimensional random variable where $(X, Y) \in \{(x, y) \in R^2\}$ and assume that g is $R^2 = R \times R$-measurable and is nonnegative and integrable. Then

$$E[g(X,Y)] = \int\int_{R^2} g(x,y)dF_X(x)dF_Y(y)$$
$$= \int_R \left\{ \int_R g(x,y)dF_X(x) \right\} dF_Y(y)$$
$$= \int_R \left\{ \int_R g(x,y)dF_Y(y) \right\} dF_X(x)$$

This means that to compute the double integral with respect to the product measure, we integrate first with respect to one variable and then with respect to the other variable. ■

1.7 Sums of Independent Random Variables

Consider two independent continuous random variables X and Y. We are interested in computing the CDF and PDF of their sum $g(X, Y) = S = X + Y$. The random variable S can be used to model the reliability of systems with stand-by connections. In such systems, the component A whose time-to-failure is represented by the random variable X is the primary component, and the component B whose time-to-failure is represented by the random variable Y is the

backup component that is brought into operation when the primary component fails. Thus, S represents the time until the system fails, which is the sum of the lifetimes of both components.

The CDF of S can be obtained as follows:

$$F_S(s) = P[S \leq s] = P[X + Y \leq s] = \int_D \int f_{XY}(x, y)dxdy$$

where D is the set $D = \{(x, y)|x + y \leq s\}$. Thus,

$$\begin{aligned} F_S(s) &= \int_{-\infty}^{\infty} \int_{-\infty}^{s-y} f_{XY}(x, y)dxdy = \int_{-\infty}^{\infty} \int_{-\infty}^{s-y} f_X(x) f_Y(y)dxdy \\ &= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{s-y} f_X(x)dx \right\} f_Y(y)dy \\ &= \int_{-\infty}^{\infty} F_X(s - y) f_Y(y)dy \end{aligned}$$

The PDF of S is obtained by differentiating the CDF, as follows:

$$\begin{aligned} f_S(s) &= \frac{d}{ds} F_S(s) = \frac{d}{ds} \int_{-\infty}^{\infty} F_X(s - y) f_Y(y)dy \\ &= \int_{-\infty}^{\infty} \frac{d}{ds} F_X(s - y) f_Y(y)dy \\ &= \int_{-\infty}^{\infty} f_X(s - y) f_Y(y)dy \end{aligned}$$

where we have assumed that we can interchange differentiation and integration. The expression on the right-hand side is a well-known result in signal analysis called the *convolution integral*. Thus, we find that the PDF of the sum S of two independent random variables X and Y is the convolution of the PDFs of the two random variables; that is,

$$f_S(s) = f_X(s) * f_Y(s)$$

In general, if S is the sum on n mutually independent random variables $X_1, X_2, \ldots, X_n$ whose PDFs are $f_{X_i}(x), i = 1, 2, \ldots, n$, then we have that

$$\begin{aligned} S &= X_1 + X_2 + \cdots + X_n \\ f_S(s) &= f_{X_1}(s) * f_{X_2}(s) \cdots * f_{X_n}(s) \end{aligned}$$

Thus, the s-transform of the PDF of S is given by

$$M_S(s) = \prod_{i=1}^{n} M_{X_i}(s)$$

1.8 Some Probability Distributions

Random variables with special probability distributions are encountered in different fields of science and engineering. In this section we describe some of these distributions, including their expected values, variances, and s-transforms (or z-transforms, as the case may be). These include the Bernoulli distribution, binomial distribution, geometric distribution, Pascal distribution, Poisson distribution, exponential distribution, Erlang distribution, and uniform distribution.

1.8.1 The Bernoulli Distribution

A Bernoulli trial is an experiment that results in two outcomes: *success* and *failure*. One example of a Bernoulli trial is the coin tossing experiment, which results in heads or tails. In a Bernoulli trial we define the probability of success and probability of failure as follows:

$$P[\text{success}] = p \qquad 0 \le p \le 1$$
$$P[\text{failure}] = 1 - p$$

Let us associate the events of the Bernoulli trial with a random variable X such that when the outcome of the trial is a success, we define $X = 1$, and when the outcome is a failure, we define $X = 0$. The random variable X is called a Bernoulli random variable, and its PMF is given by

$$p_X(x) = \begin{cases} 1 - p & x = 0 \\ p & x = 1 \end{cases}$$

An alternative way to define the PMF of X is as follows:

$$p_X(x) = p^x(1-p)^{1-x} \qquad x = 0, 1$$

The CDF is given by

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - p & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

The expected value of X is given by

$$E[X] = 0(1-p) + 1(p) = p$$

Similarly, the second moment of X is given by

$$E[X^2] = 0^2(1-p) + 1^2(p) = p$$

Thus, the variance of X is given by

$$\sigma_X^2 = E[X^2] - \{E[X]\}^2 = p - p^2 = p(1-p)$$

The z-transform of the PMF is given by

$$\begin{aligned} G_X(z) &= \sum_{x=0}^{\infty} z^x p_X(x) = \sum_{x=0}^{1} z^x p_X(x) = z^0(1-p) + z^1 p \\ &= 1 - p + zp \end{aligned}$$

1.8.2 The Binomial Distribution

Suppose we conduct n independent Bernoulli trials, and we represent the number of successes in those n trials by the random variable $X(n)$. Then $X(n)$ is defined as a binomial random variable with parameters (n, p). The PMF of a random variable $X(n)$ with parameters (n, p) is given by

$$p_{X(n)}(x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, 2, \ldots, n$$

The binomial coefficient, $\binom{n}{x}$, represents the number of ways of arranging x successes and $n - x$ failures.

Because $X(n)$ is essentially the sum of n independent Bernoulli random variables, its CDF, mean, variance, and the z-transform of its PMF are given by

$$F_{X(n)}(x) = P[X(n) \le x] = \sum_{k=0}^{x} \binom{n}{k} p^k (1-p)^{n-k}$$

$$E[X(n)] = np$$

$$E[X^2(n)] = n(n-1)p^2 + np$$

$$\sigma_{X(n)}^2 = E[X^2(n)] - \{E[X(n)]\}^2 = np(1-p)$$

$$G_{X(n)}(z) = (zp + 1 - p)^n$$

1.8.3 The Geometric Distribution

The geometric random variable is used to describe the number of Bernoulli trials until the first success occurs. Let X be a random variable that denotes the number of Bernoulli trials until the first success. If the first success occurs on the xth trial, then we know that the first $x-1$ trials resulted in failures. Thus, the PMF of a geometric random variable, X, is given by

$$p_X(x) = p(1-p)^{x-1} \qquad x = 1, 2, 3, \ldots$$

The CDF, mean, and variance of X and the z-transform of its PMF are given by

$$\begin{aligned} F_X(x) &= P[X \le x] = 1 - (1-p)^x \\ E[X] &= 1/p \\ E[X^2] &= \frac{2-p}{p^2} \\ \sigma_X^2 &= E[X^2] - \{E[X]\}^2 = \frac{1-p}{p^2} \\ G_X(z) &= \frac{zp}{1-z(1-p)} \end{aligned}$$

1.8.4 The Pascal Distribution

The Pascal random variable is an extension of the geometric random variable. It describes the number of trials until the kth success, which is why it is sometimes called the "kth-order interarrival time for a Bernoulli process." The Pascal distribution is also called the *negative binomial distribution*.

Let X_k be a kth-order Pascal random variable. Then its PMF is given by

$$p_{X_k}(n) = \binom{n-1}{k-1} p^k (1-p)^{n-k} \qquad k = 1, 2, \ldots; n = k, k+1, \ldots$$

X_k is essentially the sum of k independent geometric random variables. Therefore, its CDF, mean, variance and the z-transform of its PMF are given by

$$F_{X_k}(x) = P[X_k \le x] = \sum_{n=k}^{x} \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

$$E[X_k] = k/p$$

$$E[X_k^2] = \frac{k^2 + k(1-p)}{p^2}$$

$$\sigma_{X_k}^2 = E[X_k^2] - \{E[X_k]\}^2 = \frac{k(1-p)}{p^2}$$

$$G_{X_k}(z) = \left[\frac{zp}{1 - z(1-p)}\right]^k$$

Note that the geometric random variable is the *first-order* Pascal random variable.

1.8.5 The Poisson Distribution

A discrete random variable K is called a Poisson random variable with parameter λ, where $\lambda > 0$, if its PMF is given by

$$p_K(k) = \frac{\lambda^k}{k!} e^{-\lambda} \qquad k = 0, 1, 2, \ldots$$

The CDF, mean, and variance of K and the z-transform of its PMF are given by

$$F_K(k) = P[K \le k] = \sum_{r=0}^{k} \frac{\lambda^r}{r!} e^{-\lambda}$$

$$E[K] = \lambda$$

$$E[K^2] = \lambda^2 + \lambda$$

$$\sigma_K^2 = E[K^2] - \{E[K]\}^2 = \lambda$$

$$G_K(z) = e^{\lambda(z-1)}$$

1.8.6 The Exponential Distribution

A continuous random variable X is defined to be an exponential random variable (or X has an exponential distribution) if for some parameter $\lambda > 0$ its PDF is given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & x < 0 \end{cases}$$

The CDF, mean, and variance of X and the s-transform of its PDF are given by

$$\begin{aligned} F_X(x) &= P[X \le x] = 1 - e^{-\lambda x} \\ E[X] &= 1/\lambda \\ E[X^2] &= 2/\lambda^2 \\ \sigma_X^2 &= E[X^2] - \{E[X]\}^2 = 1/\lambda^2 \\ M_X(s) &= \frac{\lambda}{s+\lambda} \end{aligned}$$

1.8.7 The Erlang Distribution

The Erlang distribution is a generalization of the exponential distribution. While the exponential random variable describes the time between adjacent events, the Erlang random variable describes the time interval between any event and the kth following event. A random variable X_k is referred to as a kth-order Erlang (or Erlang-k) random variable with parameter λ if its PDF is given by

$$f_{X_k}(x) = \begin{cases} \dfrac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} & k = 1, 2, 3, \ldots; x \ge 0 \\ 0 & x < 0 \end{cases}$$

X_k is the sum of k independent exponential random variables. Thus, its CDF, mean, variance, and the s-transform of its PDF are given by

$$\begin{aligned} F_{X_k}(x) &= P[X_k \le x] = 1 - \sum_{j=0}^{k-1} \frac{(\lambda x)^j e^{-\lambda x}}{j!} \\ E[X_k] &= k/\lambda \\ E[X_k^2] &= \frac{k(k+1)}{\lambda^2} \\ \sigma_{X_k}^2 &= E[X_k^2] - \{E[X_k]\}^2 = \frac{k}{\lambda^2} \\ M_{X_k}(s) &= \left[\frac{\lambda}{s+\lambda}\right]^k \end{aligned}$$

The exponential random variable is the first-order Erlang random variable.

1.8.8 Normal Distribution

A continuous random variable X is defined to be a normal random variable with parameters μ_X and σ_X^2 if its PDF is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-(x-\mu_X)^2/2\sigma_X^2} \qquad -\infty < x < \infty$$

The PDF is a bell-shaped curve that is symmetric about μ_X, which is the mean of X. The parameter σ_X^2 is the variance. Figure 1.1 illustrates the shape of the PDF. Because the variance (or more precisely, the standard deviation) is a measure of the spread around the mean, the larger the variance, the lower the peak of the curve and the more spread out it will be.

The CDF of X is given by

$$F_X(x) = P[X \le x] = \frac{1}{\sigma_X\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(u-\mu_X)^2/2\sigma_X^2} du$$

The normal random variable X with parameters μ_X and σ_X^2 is usually designated $X \sim N(\mu_X, \sigma_X^2)$. The special case of zero mean and unit variance (i.e., $\mu_X = 0$ and $\sigma_X^2 = 1$) is designated $X \sim N(0, 1)$ and is called the *standard normal random variable*. Let $y = (u - \mu_X)/\sigma_X$. Then $du = \sigma_X dy$, and the CDF of X becomes

$$F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(x-\mu_X)/\sigma_X} e^{-y^2/2} dy$$

Thus, with the above transformation, X becomes a standard normal random variable. The above integral cannot be evaluated in closed form. It is usually evaluated numerically through the function $\Phi(x)$, which is defined as

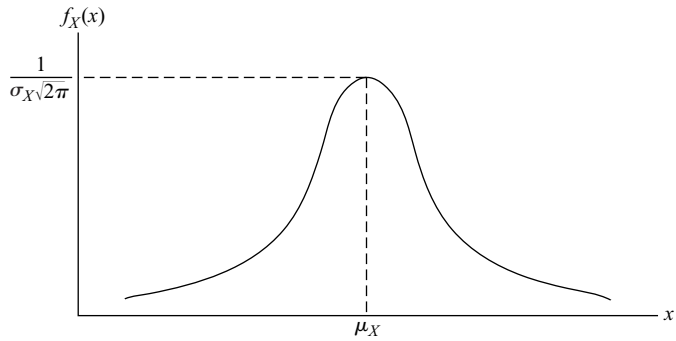


Figure 1.1. PDF of the normal random variable.

follows:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$$

Thus, the CDF of X is given by

$$F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(x-\mu_X)/\sigma_X} e^{-y^2/2} dy = \Phi\left(\frac{x-\mu_X}{\sigma_X}\right)$$

The values of $\Phi(x)$ are usually given for nonnegative values of x. For negative values of x, $\Phi(x)$ can be obtained from the following relationship:

$$\Phi(-x) = 1 - \Phi(x)$$

Values of $\Phi(x)$ are usually available in standard probability and statistics textbooks.

1.9 Introduction to Stochastic Processes

Stochastic processes deal with the dynamics of probability theory. The concept of stochastic processes enlarges the random variable concept to include time. Thus, instead of thinking of a random variable X that maps an event $s \in \Omega$, where Ω is the sample space, to some number $X(s)$, we think of how the random variable maps the event to different numbers at different times. This implies that instead of the number $X(s)$ we deal with $X(t, s)$, where $t \in T$ and T is called the *parameter set* of the process and is usually a set of times.

If we fix the sample point s, $X(t)$ is some real function of time. For each s, we have a function $X(t)$. Thus, $X(t, s)$ can be viewed as a collection of time functions, one for each sample point s. On the other hand, if we fix t, we have a function $X(s)$ that depends only on s and thus is a random variable. Therefore, a stochastic process becomes a random variable when time is fixed at some particular value. With many values of t we obtain a collection of random variables. Thus, we can define a stochastic process as a family of random variables $\{X(t, s) | t \in T, s \in \Omega\}$ defined over a given probability space and indexed by the time parameter t. A stochastic process is also called a *random process*. Thus, we will use the terms "random process" and "stochastic process" interchangeably.

1.10 Classification of Stochastic Processes

A stochastic process can be classified according to the nature of the time parameter and the values that $X(t, s)$ can take on. As discussed earlier, T is called the parameter set of the stochastic process and is usually a set of times. If T is an interval of real numbers and hence is continuous, the process is called a *continuous-time* stochastic process. Similarly, if T is a countable set and hence is discrete, the process is called a *discrete-time* stochastic process. A discrete-time stochastic process is also called a *random sequence*, which is denoted by $\{X[n]|n = 1, 2, \ldots\}$.

The values that $X(t, s)$ assumes are called the *states* of the stochastic process. The set of all possible values of $X(t, s)$ forms the *state space*, E, of the stochastic process. If E is continuous, the process is called a *continuous-state* stochastic process. Similarly, if E is discrete, the process is called a *discrete-state* stochastic process.

1.11 Characterizing a Stochastic Process

In the remainder of the discussion we will represent the stochastic process $X(t, s)$ by $X(t)$; that is, we will suppress s, the sample space parameter. A stochastic process is completely described or characterized by the joint CDF. Because the value of a stochastic process $X(t)$ at time t_i, $X(t_i)$ is a random variable, let

$$\begin{aligned} F_X(x_1, t_1) &= F_X(x_1) = P[X(t_1) \leq x_1] \\ F_X(x_2, t_2) &= F_X(x_2) = P[X(t_2) \leq x_2] \\ &\cdots \\ F_X(x_n, t_n) &= F_X(x_n) = P[X(t_n) \leq x_n] \end{aligned}$$

where $0 < t_1 < t_2 < \cdots < t_n$. Then the joint CDF, which is defined by

$$\begin{aligned} F_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = P[X(t_1) \leq x_1, X(t_2) \leq x_2, \ldots, \\ X(t_n) \leq x_n] \qquad \text{for all } n \end{aligned}$$

completely characterizes the stochastic process. If $X(t)$ is a continuous-time stochastic process, then it is specified by a collection of PDFs:

$$f_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = \frac{\partial^n}{\partial x_1 \partial x_2 \cdots \partial x_n} F_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n)$$

Similarly, if $X(t)$ is a discrete-time stochastic process, then it is specified by a collection of PMFs:

$$p_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = P[X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n]$$

1.11.1 Mean and Autocorrelation Function of a Stochastic Process

The mean of $X(t)$ is a function of time called the *ensemble average* and is denoted by

$$\mu_X(t) = E[X(t)]$$

The autocorrelation function provides a measure of similarity between two observations of the stochastic process $X(t)$ at different points in time t and s. The autocorrelation function of $X(t)$ and $X(s)$ is denoted by $R_{XX}(t, s)$ and is defined as follows:

$$R_{XX}(t, s) = E[X(t)X(s)] = E[X(s)X(t)] = R_{XX}(s, t)$$
$$R_{XX}(t, t) = E[X^2(t)]$$

It is common to define $s = t + \tau$, which gives the autocorrelation function as

$$R_{XX}(t, t + \tau) = E[X(t)X(t + \tau)]$$

The parameter τ is sometimes called the *delay time* (or *lag time*). The autocorrelation function of a deterministic periodic function of period T is given by

$$R_{XX}(t, t + \tau) = \frac{1}{2T} \int_{-T}^{T} f_X(t) f_X(t + \tau) dt$$

Similarly, for an aperiodic function the autocorrelation function is given by

$$R_{XX}(t, t + \tau) = \int_{-\infty}^{\infty} f_X(t) f_X(t + \tau) dt$$

Basically the autocorrelation function defines how much a signal is similar to a time-shifted version of itself. A random process $X(t)$ is called a *second order process* if $E[X^2(t)] < \infty$ for each $t \in T$.

1.12 Stationary Stochastic Processes

There are several ways to define a stationary stochastic process. At a high level, it is a process whose statistical properties do not vary with time. In this book we consider only two types of stationary processes. These are the *strict-sense stationary* processes and the *wide-sense stationary* processes.

1.12.1 Strict-Sense Stationary Processes

A random process is defined to be a strict-sense stationary process if its CDF is invariant to a shift in the time origin. This means that the process $X(t)$ with the CDF $F_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n)$ is a strict-sense stationary process if its CDF is identical to that of $X(t + \varepsilon)$ for any arbitrary ε. Thus, being a strict-sense stationary process implies that for any arbitrary ε,

$$F_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = F_X(x_1, x_2, \ldots, x_n; t_1 + \varepsilon, t_2 + \varepsilon, \ldots, t_n + \varepsilon) \quad \text{for all } n$$

When the CDF is differentiable, the equivalent condition for strict-sense stationarity is that the PDF is invariant to a shift in the time origin; that is,

$$f_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = f_X(x_1, x_2, \ldots, x_n; t_1 + \varepsilon, t_2 + \varepsilon, \ldots, t_n + \varepsilon) \quad \text{for all } n$$

If $X(t)$ is a strict-sense stationary process, then the CDF $F_{X_1X_2}(x_1, x_2; t_1, t_1 + \tau)$ does not depend on t, but it might depend on τ. Thus, if $t_2 = t_1 + \tau$, then $F_{X_1X_2}(x_1, x_2; t_1, t_2)$ might depend on $t_2 - t_1$, but not on t_1 and t_2 individually. This means that if $X(t)$ is a strict-sense stationary process, then the autocorrelation and autocovariance functions do not depend on t. Thus, we have that for all $\tau \in T$:

$$\mu_X(t) = \mu_X(0)$$

$$R_{XX}(t, t + \tau) = R_{XX}(0, \tau)$$

$$C_{XX}(t, t + \tau) = C_{XX}(0, \tau)$$

where $C_{XX}(t_1, t_2) = E[\{X(t_1) - \mu_X(t_1)\}\{X(t_2) - \mu_X(t_2)\}]$ is the autocovariance function. If the condition $\mu_X(t) = \mu_X(0)$ holds for all t, the mean is constant and denoted by μ_X. Similarly, if the function $R_{XX}(t, t + \tau)$ does not depend on t but is a function of τ, we write $R_{XX}(0, \tau) = R_{XX}(\tau)$. Finally, whenever the condition $C_{XX}(t, t + \tau) = C_{XX}(0, \tau)$ holds for all t, we write $C_{XX}(0, \tau) = C_{XX}(\tau)$.

1.12.2 Wide-Sense Stationary Processes

Many practical problems that we encounter require that we deal with only the mean and autocorrelation function of a random process. Solutions to these problems are simplified if these quantities do not depend on absolute time. Random processes in which the mean and autocorrelation functions do not depend on absolute time are called wide-sense stationary (WSS) processes. Thus, for a wide-sense stationary process $X(t)$,

$$E[X(t)] = \mu_X \qquad \text{(constant)}$$
$$R_{XX}(t, t+\tau) = R_{XX}(\tau)$$

Note that a strict-sense stationary process is also a wide-sense stationary process. However, in general the converse is not true; that is, a WSS process is not necessarily stationary in the strict sense.

1.13 Ergodic Stochastic Processes

One desirable property of a stochastic process is the ability to estimate its parameters from measurement data. Consider a random process $X(t)$ whose observed samples are $x(t)$. The time average of a function of $x(t)$ is defined by

$$\overline{x} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)dt$$

The statistical average of the random process $X(t)$ is the expected value $E[X(t)]$ of the process. The expected value is also called the *ensemble average*. An ergodic stochastic process is a stationary process in which every member of the ensemble exhibits the same statistical behavior as the ensemble. This implies that it is possible to determine the statistical behavior of the ensemble by examining only one typical sample function. Thus, for an ergodic stochastic process, the mean values and moments can be determined by time averages as well as by ensemble averages (or expected values), which are equal. That is,

$$E[X^n] = \overline{X^n} = \int_{-\infty}^{\infty} x^n f_X(x)dx = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^n(t)dt$$

A stochastic process $X(t)$ is defined to be *mean-ergodic* (or *ergodic in the mean*) if $E[X(t)] = \overline{x}$.

1.14 Some Models of Stochastic Processes

In this section we consider some examples of random processes that we will encounter in the remainder of the book.

1.14.1 Martingales

A stochastic process $\{X_n, n = 1, 2, \ldots\}$ is defined to be a martingale process (or a martingale) if it has the following properties:

- $E[|X_n|] < \infty$ for all n; that is, it has finite means, and
- $E[X_{n+1}|X_1, X_2, \ldots, X_n] = X_n$; that is, the best prediction of its future value is its present value.

If $E[X_{n+1}|X_1, X_2, \ldots, X_n] \leq X_n$, then $\{X_n, n = 1, 2, \ldots\}$ is called a *supermartingale*. Similarly, if $E[X_{n+1}|X_1, X_2, \ldots, X_n] \geq X_n$, then $\{X_n, n = 1, 2, \ldots\}$ is called a *submartingale*. Thus, a martingale satisfies the conditions for both a supermartingale and a submartingale.

Sometimes the martingale property is defined with respect to another stochastic process. Specifically, let $\{X_n, n = 1, 2, \ldots\}$ and $\{Y_n, n = 1, 2, \ldots\}$ be stochastic processes. $\{X_n\}$ is defined to be a martingale with respect to $\{Y_n\}$ if, for $n = 1, 2, \ldots$, the following conditions hold:

- $E[|X_n|] < \infty$
- $E[X_{n+1}|Y_1, Y_2, \ldots, Y_n] = X_n$

A martingale captures the essence of a fair game in the sense that regardless of a player's current and past fortunes, his expected fortune at any time in the future is the same as his current fortune. Thus, on the average he neither wins nor loses any money. Also, martingales fundamentally deal with conditional expectation. If we define $\Im_n = \{Y_1, Y_2, \ldots, Y_n\}$, then $\Im_n$ can be considered the potential information that is being revealed as time progresses. Therefore, we can consider a martingale as a process whose expected value, conditional on some potential information, is equal to the value revealed by the last available information. Similarly, a submartingale represents a favorable game because the expected fortune increases in the future, while a supermartingale represents an unfavorable game because the expected fortune decreases in the future.

Martingales occur in many stochastic processes. They have also become an important tool in modern financial mathematics because martingales provide one idea of fair value in financial markets.

Theorem 1.2 If $\{X_n, n \geq 0\}$ is a martingale, then $E[X_n] = E[X_0]$ for all $n \geq 0$.

Proof Let $\Im_n$ be as defined earlier. We know that $E[E[X|Y]] = E[X]$. Also, because $\{X_n, n \geq 0\}$ is a martingale, $E[X_n|\Im_0] = X_0$. Thus, we have that

$$E[X_n] = E[E[X_n|\Im_0]] = E[X_0]$$ ■

Example 1.1 Let $X_1, X_2, \ldots$ be independent random variables with mean 0, and let $Y_n = \sum_{k=1}^{n} X_k$. We show that the process $\{Y_n, n \geq 1\}$ is a martingale as follows:

$$\begin{aligned} E[Y_{n+1}|Y_1, \ldots Y_n] &= E[Y_n + X_{n+1}|Y_1, \ldots Y_n] \\ &= E[Y_n|Y_1, \ldots Y_n] + E[X_{n+1}|Y_1, \ldots Y_n] \\ &= Y_n + E[X_{n+1}] \\ &= Y_n \end{aligned}$$ ■

Example 1.2 Consider the variance of a sum of random variables. Specifically, let $X_0 = 0$ and $X_1, X_2, \ldots$ be independent and identically distributed random variables with mean $E[X_k] = 0$ and finite variance $E[X_k^2] = \sigma^2, k \geq 1$. If we let $Y_0 = 0$ and define

$$Y_n = \left(\sum_{k=1}^{n} X_k\right)^2 - n\sigma^2$$

we show that $\{Y_n\}$ is a martingale with respect to $\{Y_n\}$ as follows:

$$\begin{aligned} &E[Y_{n+1}|X_0, X_1, \ldots n_n] \\ &= E\left[\left(X_{n+1} + \sum_{k=1}^{n} X_k\right)^2 - (n+1)\sigma^2|X_0, X_1, \ldots, X_n\right] \\ &= E\left[X_{n+1}^2 + 2X_{n+1}\sum_{k=1}^{n} X_k + \left(\sum_{k=1}^{n} X_k\right)^2 - (n+1)\sigma^2|X_0, X_1, \ldots X_n\right] \\ &= E\left[\left\{\left(\sum_{k=1}^{n} X_k\right)^2 - n\sigma^2\right\} + X_{n+1}^2 + 2X_{n+1}\sum_{k=1}^{n} X_k - \sigma^2|X_0, X_1, \ldots X_n\right] \end{aligned}$$

$$= Y_n + E[X_{n+1}^2 | X_0, X_1, \ldots X_n] + 2\left(\sum_{k=1}^{n} X_k\right) E[X_{n+1} | X_0, X_1, \ldots X_n] - \sigma^2$$

$$= Y_n + \sigma^2 + 0 - \sigma^2 = Y_n \qquad \blacksquare$$

Example 1.3 Let $X_1, X_2, \ldots$ be independent random variables with mean $E[X_k] = 1, k \geq 1$, and let $Y_n = \prod_{k=1}^{n} X_k$. We show that the process $\{Y_n, n \geq 1\}$ is a martingale as follows:

$$\begin{aligned} E[Y_{n+1} | Y_1, \ldots, Y_n] &= E[Y_n X_{n+1} | Y_1, \ldots Y_n] = Y_n E[X_{n+1} | Y_1, \ldots Y_n] \\ &= Y_n E[X_{n+1}] \\ &= Y_n \end{aligned} \qquad \blacksquare$$

Stopping Times

Consider a stochastic process $\{X_n, n \geq 0\}$. The nonnegative integer-valued random variable T is called a stopping time for X if the event $\{T = n\}$ depends only on $\{X_0, X_1, \ldots, X_n\}$ and does not depend on $\{X_{n+k}, k \geq 1\}$. If T_k is a stopping time, then we have that

$$\{T_k = n\} = \{X_1 \neq k, \ldots, X_{n-1} \neq k, X_n = k\}$$

The use of stopping times in martingales is given by the following proposition, which is stated without proof.

Proposition 1.1 Let T be a stopping time for a stochastic process $\{X_n\}$, and let $a \wedge b = \min(a, b)$.

1. If $\{X_n\}$ is a martingale, then so is $\{X_{T \wedge n}\}$.
2. If $\{X_n\}$ is a supermartingale, then so is $\{X_{T \wedge n}\}$.
3. If $\{X_n\}$ is a submartingale, then so is $\{X_{T \wedge n}\}$.

Stopping times can be thought of as the time when a given event occurs. If it has the value $T = \infty$, then the event never occurs. For example, we might be interested in the first time the value of a random sequence that is known to be a martingale is 6. Then we consider the martingale $\{X_n, n \geq 0\}$ and a random variable T that is defined by

$$T = \begin{cases} \inf_{n \geq 0}\{n | X_n = 6\} & \text{if } X_n = 6 \text{ for some } n \in \aleph \\ \infty & \text{otherwise} \end{cases}$$

where $\aleph$ is the set of positive integers.

Theorem 1.3 (Optional Stopping Theorem) Let T be a stopping time for the martingale $\{X_n\}$. Then $E[X_T] = E[X_0]$ if at least one of the following conditions holds:

a. T is finite (that is, $P[T < \infty] = 1$) and there exists a finite constant C_1 such that $|X_n| \leq C_1$ for all $n \leq T$.
b. T is bounded; that is, there exists a finite constant C_2 so that with probability 1, $T \leq C_2$.
c. $E[T]$ is finite and there exists a finite constant C_3 such that $E[|X_{n+1} - X_n||X_1, \ldots X_n] < C_3$ for $n = 0, 1, \ldots$.

This theorem is also called the *stopping time theorem.*

Proof The proof of this theorem can be found in any standard stochastic processes book, such as Grimmett and Stirzaker (2001). ■

1.14.2 Counting Processes

A stochastic process $\{X(t)|t \geq 0\}$ is called a counting process if $X(t)$ represents the total number of "events" that have occurred in the interval $[0, t)$. An example of a counting process is the number of customers that arrive at a bank from the time the bank opens its doors for business until some time t. A counting process satisfies the following conditions:

1. $X(t) \geq 0$, which means that it has nonnegative values.
2. $X(0) = 0$, which means that the counting of events begins at time 0.
3. $X(t)$ is integer-valued.
4. If $s < t$, then $X(s) \leq X(t)$, which means that it is a nondecreasing function of time.
5. $X(t) - X(s)$ represents the number of events that have occurred in the interval $[s, t]$.

Figure 1.2 represents a sample path of a counting process. The first event occurs at time t_1, and subsequent events occur at times t_2, t_3, and t_4. Thus, the number of events that occur in the interval $[0, t_4]$ is 4.

1.14.3 Independent Increment Processes

A counting process is defined to be an independent increment process if the number of events that occur in disjoint time intervals is an independent random variable. For example, in Figure 1.2, consider the two nonoverlapping (i.e., disjoint) time intervals $[0, t_1]$ and $[t_2, t_4]$. If the number of events occurring in one interval is independent of the number of events that occur in the other, then the process is an independent increment process. Thus, $X(t)$ is an independent increment

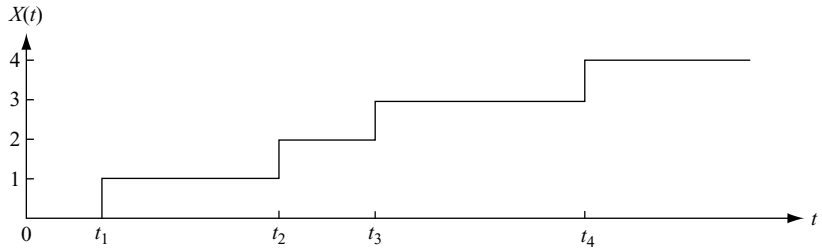


Figure 1.2. Sample function of a counting process.

process if for every set of time instants $t_0 = 0 < t_1 < t_2 < \ldots < t_n$ the increments $X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are mutually independent random variables.

1.14.4 Stationary Increment Process

A counting process $X(t)$ is defined to possess stationary increments if for every set of time instants $t_0 = 0 < t_1 < t_2 < \ldots < t_n$ the increments $X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are identically distributed. In general, the mean of an independent increment process $X(t)$ with stationary increments has the form

$$E[X(t)] = mt$$

where the constant m is the value of the mean at time $t = 1$. That is, $m = E[X(1)]$. Similarly, the variance of an independent increment process $X(t)$ with stationary increments has the form

$$\text{Var}[X(t)] = \sigma^2 t$$

where the constant σ^2 is the value of the variance at time $t = 1$; that is, $\sigma^2 = \text{Var}[X(1)]$.

1.14.5 Poisson Processes

Poisson processes are widely used to model arrivals (or occurrence of events) in a system. For example, they are used to model the arrival of telephone calls at a switchboard, the arrival of customers' orders at a service facility, and the random failures of equipment.

There are two ways to define a Poisson process. The first definition of the process is that it is a counting process $X(t)$ in which the number of events in any interval

of length t has a Poisson distribution with mean λt. Thus, for all $s, t > 0$,

$$P[X(s+t) - X(s) = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \qquad n = 0, 1, 2, \ldots$$

The second way to define the Poisson process $X(t)$ is that it is a counting process with stationary and independent increments such that for a rate $\lambda > 0$ the following conditions hold:

1. $P[X(t + \Delta t) - X(t) = 1] = \lambda \Delta t + o(\Delta t)$, which means that the probability of one event within a small time interval Δt is approximately $\lambda \Delta t$, where $o(\Delta t)$ is a function of Δt that goes to zero faster than Δt does. That is,

$$\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$$

2. $P[X(t + \Delta t) - (X(t) \geq 2)] = o(\Delta t)$, which means that the probability of two or more events within a small time interval Δt is $o(\Delta t)$.
3. $P[X(t + \Delta t) - X(t) = 0] = 1 - \lambda \Delta t + o(\Delta t)$.

These three properties enable us to derive the PMF of the number of events in a time interval of length t as follows:

$$\begin{aligned}
P[X(t+\Delta t) = n] &= P[X(t) = n] P[X(\Delta t) = 0] \\
&\quad + P[X(t) = n-1] P[X(\Delta t) = 1] \\
&= P[X(t) = n](1 - \lambda \Delta t) \\
&\quad + P[X(t) = n-1] \lambda \Delta t \\
P[X(t+\Delta t) = n] - P[X(t) = n] &= -\lambda P[X(t) = n] \Delta t \\
&\quad + \lambda P[X(t) = n-1] \Delta t \\
\frac{P[X(t+\Delta t) = n] - P[X(t) = n]}{\Delta t} &= -\lambda P[X(t) = n] + \lambda P[X(t) = n-1] \\
\lim_{\Delta t \to 0} \left\{ \frac{P[X(t+\Delta t) = n] - P[X(t) = n]}{\Delta t} \right\} &= \frac{d}{dt} P[X(t) = n] = -\lambda P[X(t) = n] \\
&\quad + \lambda P[X(t) = n-1] \\
\frac{d}{dt} P[X(t) = n] + \lambda P[X(t) = n] &= \lambda P[X(t) = n-1]
\end{aligned}$$

The last equation can be solved iteratively for $n = 0, 1, 2, \ldots$, subject to the initial conditions

$$P[X(0) = n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

This gives the PMF of the number of events (or "arrivals") in an interval of length t as

$$p_X(n, t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \qquad t \geq 0, n = 0, 1, 2, \ldots$$

From the results obtained for Poisson random variables earlier in the chapter, we have that

$$G_{X(t)}(z) = e^{\lambda t(z-1)}$$
$$E[X(t)] = \lambda t$$
$$\sigma^2_{X(t)} = \lambda t$$

The fact that the mean $E[X(t)] = \lambda t$ indicates that λ is the expected number of arrivals per unit time in the Poisson process. Thus, the parameter λ is called the *arrival rate* for the process. If λ is independent of time, the Poisson process is called a *homogeneous Poisson process*. Sometimes the arrival rate is a function of time, and we represent it as $\lambda(t)$. Such processes are called *nonhomogeneous Poisson processes*. In this book we are concerned mainly with homogeneous Poisson processes.

Interarrival Times for the Poisson Process

Let L_r be a continuous random variable that is defined to be the interval between any event in a Poisson process and the rth event after it. Then L_r is called the rth-order interarrival time. Let $f_{L_r}(l)$ be the PDF of L_r. To derive the expression for $f_{L_r}(l)$, we consider time of length l over which we know that $r - 1$ events have occurred. Assume that the next event (that is, the rth event) occurs during the next time of length Δl, as shown in Figure 1.3.

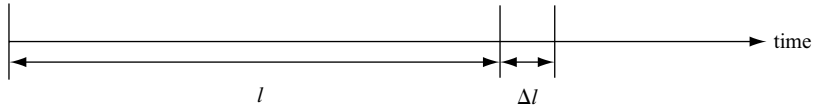


Figure 1.3. Definition of event intervals.

Because the intervals l and Δl are nonoverlapping, the number of events that occur within one interval is independent of the number of events that occur within the other interval. Thus, the PDF of L_r can be obtained as follows:

$$\begin{aligned} f_{L_r}(l)\Delta l &= P[l < L_r \le l + \Delta l] \\ &= P[\{X(l) = r-1\} \cap \{X(\Delta l) = 1\}] \\ &= P[X(l) = r-1]P[X(\Delta l) = 1] \\ &= \frac{(\lambda l)^{r-1}}{(r-1)!} e^{-\lambda l} \lambda \Delta l \end{aligned}$$

Thus, $f_{L_r}(l)$ is given by

$$f_{L_r}(l) = \frac{(\lambda l)^{r-1}}{(r-1)!} e^{-\lambda l} \lambda = \frac{\lambda^r l^{r-1}}{(r-1)!} e^{-\lambda l} \qquad l \ge 0; r = 1, 2, \ldots$$

which is the Erlang-r (or rth-order Erlang) distribution. The special case of $r = 1$ is the exponential distribution. That is,

$$f_{L_1}(l) = \lambda e^{-\lambda l} \qquad l \ge 0$$

This result provides another definition of a Poisson process: It is a counting process with stationary and independent increments in which the intervals between consecutive events are exponentially distributed.

Compound Poisson Process

Let $\{N(t)|t \ge 0\}$ be a Poisson process with arrival rate λ. Let $\{Y_i, i = 1, 2, \ldots\}$ be a family of independent and identically distributed random variables. Assume that the Poisson process $\{N(t)|t \ge 0\}$ and the sequence $\{Y_i, i = 1, 2, \ldots\}$ are independent. We define a stochastic process $\{X(t)|t \ge 0\}$ to be a *compound Poisson process* if, for $t \ge 0$, it can be represented by

$$X(t) = \sum_{i=1}^{N(t)} Y_i$$

Thus, $X(t)$ is a Poisson sum of random variables. One example of the concept of compound Poisson process is the following. Assume students arrive at the university bookstore to buy books in a Poisson manner. If the number of books that each of these students buys is an independent and identically distributed random variable, then the number of books bought by time t is a compound Poisson process.

Because the compound Poisson process has a rate that takes on a stochastic nature, it is also called a *doubly stochastic Poisson process.* This term is used to emphasize the fact that the process involves two kinds of randomness: There is a randomness that is associated with the main process that is sometimes called the *Poisson point process*, and there is another independent randomness that is associated with its rate.

Assume that the Y_i are discrete random variables with the PMF $p_Y(y)$. The value of $X(t)$, given that $N(t) = n$, is $X(t) = Y_1 + Y_2 + \cdots + Y_n$. Thus, the conditional z-transform of the PMF of $X(t)$, given that $N(t) = n$, is given by

$$G_{X(t)|N(t)}(z|n) = E[z^{Y_1+Y_2+\cdots+Y_n}] = (E[z^Y])^n = [G_Y(z)]^n$$

where the last two equalities follow from the fact that the Y_i are independent. Thus, the unconditional z-transform of the PMF of $X(t)$ is given by

$$\begin{aligned} G_{X(t)}(z) &= \sum_{n=0}^{\infty} G_{X(t)|N(t)}(z|n)\, p_{N(t)}(n) = \sum_{n=0}^{\infty} [G_Y(z)]^n\, p_{N(t)}(n) \\ &= \sum_{n=0}^{\infty} [G_Y(z)]^n \frac{(\lambda t)^n}{n!} e^{-\lambda t} = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{[G_Y(z)\lambda t]^n}{n!} = e^{-\lambda t} e^{G_Y(z)\lambda t} \\ &= e^{-\lambda t[1-G_Y(z)]} \end{aligned}$$

The mean and variance of $X(t)$ can be obtained through differentiating the above function. These are given by

$$\begin{aligned} E[X(t)] &= \frac{d}{dz} G_{X(t)}(z)|_{z=1} = \lambda t E[Y] \\ E[X^2(t)] &= \frac{d^2}{dz^2} G_{X(t)}(z)|_{z=1} + \frac{d}{dz} G_{X(t)}(z)|_{z=1} = \lambda t E[Y^2] + \{\lambda t E[Y]\}^2 \\ \sigma^2_{X(t)} &= E[X^2(t)] - (E[X(t)])^2 = \lambda t E[Y^2] \end{aligned}$$

Thus, we have that

$$\begin{aligned} E[X(t)] &= E[N(t)]E[Y] = \lambda t E[Y] \\ \sigma^2_{X(t)} &= E[N(t)]\sigma^2_Y + (E[Y])^2 \sigma^2_{N(t)} = \lambda t \sigma^2_Y + \lambda t (E[Y])^2 \\ &= \lambda t\{\sigma^2_Y + (E[Y])^2\} = \lambda t E[Y^2] \end{aligned}$$

Note also that in the case when the Y_i are continuous random variables, the result would be $M_{X(t)}(s) = e^{\lambda t[M_Y(s)-1]}$ and the above results still hold.

1.15 Problems

1.1 A sequence of Bernoulli trials consists of choosing seven components at random from a batch of components. A selected component is classified as either defective or nondefective. A nondefective component is considered to be a success, while a defective component is considered to be a failure. If the probability that a selected component is nondefective is 0.8, what is the probability of three successes?

1.2 The probability that a patient recovers from a rare blood disease is 0.3. If 15 people are known to have contracted this disease, find the following probabilities:

- **a.** At least 10 survive.
- **b.** From three to eight survive.
- **c.** Exactly six survive.

1.3 A sequence of Bernoulli trials consists of choosing components at random from a batch of components. A selected component is classified as either defective or nondefective. A nondefective component is considered to be a success, while a defective component is considered to be a failure. If the probability that a selected component is nondefective is 0.8, determine the probabilities of the following events:

- **a.** The first success occurs on the fifth trial.
- **b.** The third success occurs on the eighth trial.
- **c.** There are 2 successes by the fourth trial, there are four successes by the 10th trial, and there are 10 successes by the 18th trial.

1.4 A lady invites 12 people for dinner at her house. Unfortunately the dining table can only seat six people. Her plan is that if six or fewer guests come, then they will be seated at the table (i.e., they will have a sit-down dinner); otherwise she will set up a buffet-style meal. The probability that each invited guest will come to dinner is 0.4, and each guest's decision is independent of other guests' decisions. Determine the following:

- **a.** The probability that she has a sit-down dinner.
- **b.** The probability that she has a buffet-style dinner.
- **c.** The probability that there are at most three guests.

1.5 A Girl Scout troop sells cookies from house to house. One of the parents of the girls figured out that the probability that they sell a set of packs of cookies at any house they visit is 0.4, where it is assumed that they sell exactly one set to each house that buys their cookies.

a. What is the probability that the first house where they make their first sale is the fifth house they visit?
b. Given that they visited 10 houses on a particular day, what is the probability that they sold exactly six sets of cookie packs?
c. What is the probability that on a particular day the third set of cookie packs is sold at the seventh house that the girls visit?

1.6 Students arrive for a lab experiment according to a Poisson process with a rate of 12 students per hour. However, the lab attendant opens the door to the lab when at least four students are waiting at the door. What is the probability that the waiting time of the first student to arrive exceeds 20 minutes? (By waiting time we mean the time that elapses from when a student arrives until the door is opened by the lab attendant.)

1.7 Cars arrive at a gas station according to a Poisson process at an average rate of 12 cars per hour. The station has only one attendant. If the attendant decides to take a two-minute coffee break when there were no cars at the station, what is the probability that one or more cars will be waiting when he comes back from the break, given that any car that arrives when he is on coffee break waits for him to get back?

1.8 Suppose $X(t)$ is a Gaussian random process with a mean $E[X(t)] = 0$ and autocorrelation function $R_{XX}(\tau) = e^{-|\tau|}$. Assume that the random variable A is defined as follows:

$$A = \int_0^1 X(t)dt$$

Determine the following:

a. $E[A]$
b. σ_A^2

1.9 Suppose $X(t)$ is a Gaussian random process with a mean $E[X(t)] = 0$ and autocorrelation function $R_{XX}(\tau) = e^{-|\tau|}$. Assume that the random variable A is defined as follows:

$$A = \int_0^B X(t)dt$$

where B is a uniformly distributed random variable with values between 1 and 5 and is independent of the random process $X(t)$. Determine the following:

a. $E[A]$
b. σ_A^2

1.10 An insurance company pays out claims on its life insurance policies in accordance with a Poisson process with an average rate of five claims per week. If the amount of money paid on each policy is uniformly distributed between \$2000 and \$10,000, what is the mean of the total amount of money that the company pays out in a four-week period?

1.11 Customers arrive at the neighborhood bookstore according to a Poisson process with an average rate of 10 customers per hour. Independent of other customers, each arriving customer buys a book with probability 1/8.

a. What is the probability that the bookstore sells no book during a particular hour?

b. What is the PDF of the time until the first book is sold?

1.12 Joe is a student who is conducting experiments with a series of lightbulbs. He started with 10 identical lightbulbs, each of which has an exponentially distributed lifetime with a mean of 200 hours. Joe wants to know how long it will take until the last bulb burns out (or fails). At noontime Joe stepped out to get some lunch with six bulbs still on. Assume that Joe came back and found that none of the six bulbs has failed.

a. After Joe came back, what is the expected time until the next bulb failure?

b. What is the expected length of time between the fourth bulb failure and the fifth bulb failure?

1.13 Three customers A, B, and C simultaneously arrive at a bank with two tellers on duty. The two tellers were idle when the three customers arrived, and A goes directly to one teller, B goes to the other teller, and C waits until either A or B leaves before she can begin receiving service. If the service times provided by the tellers are exponentially distributed with a mean of four minutes, what is the probability that customer A is still in the bank after the other two customers leave?

1.14 Students arrive at the professor's office for extra help according to a Poisson process with an average rate of four students per hour. The professor does not start the tutorial until at least three students are available. Students who arrive while the tutorial is going on will have to wait for the next session.

a. Given that a tutorial has just ended and there are no students currently waiting for the professor, what is the mean time until another tutorial can start?

b. Given that one student was waiting when the tutorial ended, what is the probability that the next tutorial does not start within the first two hours?

1.15 Customers arrive at a bank according to a Poisson process with an average rate of six customers per hour. Each arriving customer is either a man with probability p or a woman with probability $1 - p$. It was found that in the first

two hours the average number of men who arrived at the bank was eight. What is the average number of women who arrived over the same period?

1.16 Chris is conducting an experiment to test the mean lifetimes of two sets of electric bulbs labeled A and B. The manufacturer claims that the mean lifetime of bulbs in set A is 200 hours, while the mean lifetime of the bulbs in set B is 400 hours. The lifetimes for both sets are exponentially distributed. Chris' experimental procedure is as follows. He started with one bulb from each set. As soon as a bulb from a given set fails (or burns out), he immediately replaces it with a new bulb from the same set and writes down the lifetime of the burnt-out bulb. Thus, at any point in time he has two bulbs on, one from each set. If at the end of the week Chris tells you that 8 bulbs have failed, determine the following:

- **a.** The probability that exactly five of those eight bulbs are from set B.
- **b.** The probability that no bulb will fail in the first 100 hours.
- **c.** The mean time between two consecutive bulb failures.

1.17 Bob has a pet that requires the light in his apartment to always be on. To achieve this, Bob keeps three lightbulbs on with the hope that at least one bulb will be operational when he is not at the apartment. The lightbulbs have independent and identically distributed lifetimes T with PDF $f_T(t) = \lambda e^{-\lambda t}, t \geq 0$.

- **a.** Probabilistically speaking, given that Bob is about to leave the apartment and all three bulbs are working fine, what does he gain by replacing all three bulbs with new ones before he leaves?
- **b.** Suppose X is the random variable that denotes the time until the first bulb fails. What is the PDF of X?
- **c.** Given that Bob is going away for an indefinite period of time and all three bulbs are working fine before he leaves, what is the PDF of Y, the time until the third bulb failure after he leaves?
- **d.** What is the expected value of Y?

1.18 Joe replaced two lightbulbs, one of which is rated 60 watts with an exponentially distributed lifetime whose mean is 200 hours, and the other is rated 100 watts with an exponentially distributed lifetime whose mean is 100 hours.

- **a.** What is the probability that the 60-watt bulb fails before the 100-watt bulb?
- **b.** What is the mean time until the first of the two bulbs fails?
- **c.** Given that the 60-watt bulb has not failed after 300 hours, what is the probability that it will last at least another 100 hours?

1.19 A five-motor machine can operate properly if at least three of the five motors are functioning. If the lifetime X of each motor has the PDF

$f_X(x) = \lambda e^{-\lambda x}$, $x \geq 0$, $\lambda > 0$, and if the lifetimes of the motors are independent, what is the mean of the random variable Y, the time until the machine fails?

1.20 Suzie has two identical personal computers, which she never uses at the same time. She uses one PC at a time, and the other is a backup. If the one she is currently using fails, she turns it off, calls the PC repairman, and turns on the backup PC. The time until either PC fails when it is in use is exponentially distributed with a mean of 50 hours. The time between the moment a PC fails until the repairman comes and finishes repairing it is also exponentially distributed with a mean of three hours. What is the probability that Suzie is idle because neither PC is operational?

1.21 Cars arrive from the northbound section of an intersection in a Poisson manner at the rate of λ_N cars per minute and from the eastbound section in a Poisson manner at the rate of λ_E cars per minute.

- **a.** Given that there is currently no car at the intersection, what is the probability that a northbound car arrives before an eastbound car?
- **b.** Given that there is currently no car at the intersection, what is the probability that the fourth northbound car arrives before the second eastbound car?

1.22 A one-way street has a fork in it, and cars arriving at the fork can either bear right or left. A car arriving at the fork will bear right with probability 0.6 and will bear left with probability 0.4. Cars arrive at the fork in a Poisson manner with a rate of eight cars per minute.

- **a.** What is the probability that at least four cars bear right at the fork in three minutes?
- **b.** Given that three cars bear right at the fork in three minutes, what is the probability that 2 cars bear left at the fork in three minutes?
- **c.** Given that 10 cars arrive at the fork in three minutes, what is the probability that four of the cars bear right at the fork?

1.23 Let the random variable S_n be defined as follows:

$$S_n = \begin{cases} 0 & n = 0 \\ \sum_{k=1}^{n} X_k & n \geq 1 \end{cases}$$

where X_k is the kth outcome of a Bernoulli trial such that $P[X_k = 1] = p$ and $P[X_k = -1] = q = 1 - p$, and the X_k are independent and identically distributed. Consider the process $\{S_n : n = 1, 2, \ldots\}$.

- **a.** For what values of p (relative to q) is $\{S_n\}$ a martingale?
- **b.** For what values of p is $\{S_n\}$ a submartingale?
- **c.** For what values of p is $\{S_n\}$ a supermartingale?

1.24 Let $X_1, X_2, \ldots$ be independent and identically distributed Bernoulli random variables with values ± 1 that have equal probability of 1/2. Show that the partial sums

$$S_n = \sum_{k=1}^{n} \frac{X_k}{k} \quad n = 1, 2, \ldots$$

form a martingale with respect to $\{X_n\}$.

1.25 Let $X_1, X_2, \ldots$ be independent and identically distributed Bernoulli random variables with values ± 1 that have equal probability of 1/2. Let K_1 and K_2 be positive integers, and define N as follows:

$$N = \min\{n : S_n = K_1 \text{ or } -K_2\}$$

where

$$S_n = \sum_{k=1}^{n} X_k \quad n = 1, 2, \ldots$$

is called a symmetric random walk.

a. Show that $E[N] < \infty$.
b. Show that $P[S_n = K_1] = \frac{K_2}{K_1 + K_2}$.

1.26 A symmetric random walk $\{S_n : n = 0, 1, 2, \ldots\}$ starts at the position $S_0 = k$ and ends when the walk first reaches either the origin or the position m, where $0 < k < m$. Let T be defined by

$$T = \min\{n : S_n = 0 \text{ or } m\}$$

That is, T is the stopping time.

a. Show that $E[S_T] = k$.
b. Define $Y_n = S_n^2 - n$ and show that $\{Y_n\}$ is a martingale with respect to $\{S_n\}$.

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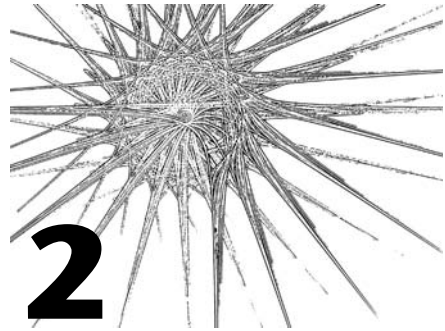
Introduction to Markov Processes



2.1 Introduction

A stochastic process $\{X(t)|t \in T\}$ is called a first-order Markov process if for any $t_0 < t_1 < \ldots < t_n$ the conditional CDF of $X(t_n)$ for given values of $X(t_0)$, $X(t_1), \ldots, X(t_{n-1})$ depends only on $X(t_{n-1})$. That is,

$$P[X(t_n) \leq x_n | X(t_{n-1}) = x_{n-1}, X(t_{n-2}) = x_{n-2}, \ldots, X(t_0) = x_0]$$
$$= P[X(t_n) \leq x_n | X(t_{n-1}) = x_{n-1}]$$

This means that, given the present state of the process, the future state is independent of the past. This property is usually referred to as the *Markov property*. In second-order Markov processes the future state depends on both the current state and the last immediate state, and so on for higher-order Markov processes. In this chapter we consider only first-order Markov processes.

Markov processes are classified according to the nature of the time parameter and the nature of the state space. With respect to state space, a Markov process can be either a discrete-state Markov process or continuous-state Markov process. A discrete-state Markov process is called a *Markov chain*. Similarly, with respect to time, a Markov process can be either a discrete-time Markov process or a continuous-time Markov process. Thus, there are four basic types of Markov processes:

1. Discrete-time Markov chain (or discrete-time discrete-state Markov process)
2. Continuous-time Markov chain (or continuous-time discrete-state Markov process)

		State Space	
		Discrete	Continuous
Time	Discrete	Discrete-Time Markov Chain	Discrete-Time Markov Process
	Continuous	Continuous-Time Markov Chain	Continuous-Time Markov Process

Figure 2.1. Classification of Markov processes.

3. Discrete-time Markov process (or discrete-time continuous-state Markov process)
4. Continuous-time Markov process (or continuous-time continuous-state Markov process)

This classification of Markov processes is illustrated in Figure 2.1.

In this chapter we present a brief discussion of the different types of Markov processes. More detailed discussion of each type is presented later in the book. Specifically, discrete-time Markov chains are discussed in Chapter 3, continuous-time Markov chains are discussed in Chapter 4, and continuous-state Markov processes are discussed in Chapter 9. Continuous-state processes include the Brownian motion and diffusion processes.

2.2 Structure of Markov Processes

A jump process is a stochastic process that makes transitions between discrete states at times that can be fixed or random. In such a process the system enters a state, spends an amount of time called the *holding time* (or *sojourn time*) and then jumps to another state where it spends another holding time, and so on. If the jump times are $T_0 = 0 < T_1 < T_2 < \ldots$, then the sample path of the process is constant between T_k and T_{k+1}. If the jump times are discrete, the jump process is called a *jump chain*.

There are two types of jump processes: *pure* (or *nonexplosive*) and *explosive*. In an explosive jump process, the process makes an infinitely many jumps within a finite time interval. In a pure jump process, there are a finite number of jumps in a finite interval. Figure 2.2 illustrates a realization of a pure jump process.

If the holding times of a continuous-time jump process are exponentially distributed, the process is called a Markov jump process. As will be discussed in

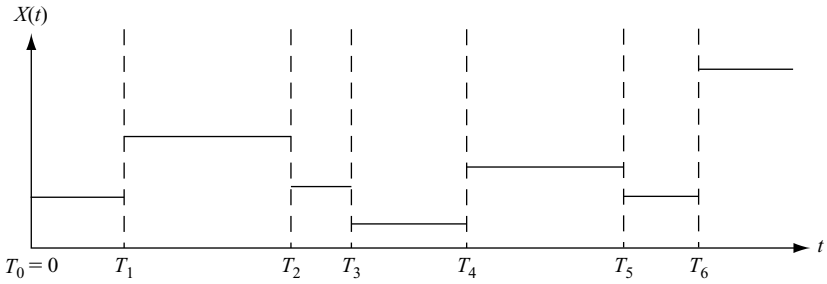


Figure 2.2. Realization of a pure jump process.

Chapter 4, a Markov jump process is a continuous-time Markov chain if the holding time depends only on the current state. If the holding times of a discrete-time jump process are geometrically distributed, the process is called a *Markov jump chain*. However, not all discrete-time Markov chains are Markov jump chains. For many discrete-time Markov chains, transitions occur in equally spaced intervals, such as every day, every week, every year, and so on. Such Markov chains are not Markov jump chains.

Unfortunately, not every physical system can be modeled by a jump process. Such systems can be modeled by processes that move continuously between all possible states that lie in some interval of the real line. Thus, such processes have continuous space and continuous time. One example of a continuous-time continuous-space process is the Brownian motion, which was first described in 1828 by the botanist Robert Brown, who observed that pollen particles suspended in a fluid moved in an irregular random manner. In his mathematical theory of speculation, Bachelier (1900) used the Brownian motion to model the movement of stock prices. Arguing that the Brownian motion is caused by the bombardment of the particles by the molecules of the fluid, Einstein (1905) obtained the equation for Brownian motion. Finally, Wiener (1923) established the mathematical foundation of the Brownian motion as a stochastic process. Brownian motion is also called the *Wiener process* and is discussed in greater detail in Chapter 9.

The diffusion process is a probabilistic model of the physical model of diffusion. The Brownian motion is usually considered to be a probabilistic model of diffusion in a homogeneous medium. To motivate the discussion, consider a physical system with state $x(t), t \geq 0$. The behavior of the system when an input $w(t), t \geq 0$ is presented to it is governed by a differential equation of the following form that gives the rate of change of the state:

$$\frac{dx(t)}{dt} = a\,(x(t), t) + b\,(x(t), t)\,w(t) \qquad t \geq 0$$

where the functions a and b depend on the system properties. The previous equation assumes that the system properties and the input are perfectly known and deterministic. However, when the input is a random function, the state function will be a stochastic process. Under this condition, it is common practice to assume that the input is a white noise process. Also, instead of dealing with a differential equation, we deal with increments in the system state. Thus, the evolution of the state $X(t)$ is given by the following *stochastic differential equation*:

$$dX(t) = a\,(X(t), t)\,dt + b\,(X(t), t)\,dW(t) \quad t \geq 0$$

For a diffusion process, the function a is called the *drift coefficient*, the function b is called the *diffusion coefficient*, and $W(t)$ is the Brownian motion. Thus, a stochastic differential equation can be regarded as a mathematical description of the motion of a particle in a moving fluid. The Markov property of the diffusion process is discussed in Chapter 9. The solution to the stochastic differentiation is obtained via the following stochastic integral equation:

$$X(t) = X(0) + \int_0^t a\,(X(u), u)\,du + \int_0^t b\,(X(u), u)\,dW(u) \quad t \geq 0$$

Different types of diffusion processes are discussed in Chapter 9, and they differ in the way the drift and diffusion coefficients are defined.

2.3 Strong Markov Property

The Markov property implies that for all t, the process $\{X(t+s) - X(t) | s \geq 0\}$ has the same distribution as the process $\{X(s) | s \geq 0\}$ and is independent of $\{X(s) | 0 \leq s \leq t\}$. Thus, when the state of the process is known at time t, the probability law of the future change of state of the process will be determined as if the process started at time t, independently of the history of the process up to time t. While the time t is arbitrary, it is constant. The strong Markov property allows us to replace the fixed time t with a nonconstant random time. Before we state the strong Markov property we first revisit the concept of stopping time that was discussed in Chapter 1.

Consider a stochastic process $\{X_k, k \geq 0\}$. The nonnegative integer-valued random variable T is called a stopping time for $\{X_k\}$ if, for all n, the event $\{T = n\}$ depends only on $\{X_0, X_1, \ldots, X_n\}$ and does not depend on $\{X_{n+m}, m \geq 1\}$. Thus, the event $\{T = n\}$ is nonanticipating in the sense that it is required to be independent of the future; it does not depend on $\{X_{n+1}, X_{n+2}, \ldots\}$. If T_r is a stopping time, then we have that

$$\{T_r = n\} = \{X_1 \neq r, \ldots, X_{n-1} \neq r, X_n = r\}$$

For example, if we define the *recurrence time* of state i, $f_{ii}(n)$, as the conditional probability that given that the process is presently in state i, the first time it will return to state i occurs in exactly n transitions, then we have that

$$\{T_i = n\} = \{X_0 = i, X_1 \neq i, \ldots, X_{n-1} \neq i, X_n = i\}$$

Similarly, if we define the *first passage time* between state i and state j, $f_{ij}(n)$, as the conditional probability that given that the process is presently in state i, the first time it will enter state j occurs in exactly n transitions, then we have that

$$\{T_{ij} = n\} = \{X_0 = i, X_1 \neq j, \ldots, X_{n-1} \neq j, X_n = j\}$$

Thus, both the recurrence time and the first passage time are stopping times.

With respect to the Markov process, the strong Markov property states as follows. Let $T < \infty$ be a stopping time with respect to the Markov chain $X = \{X_k, k \geq 0\}$. Given that $X_T = m$, the sequence $X_{T+1}, X_{T+2}, \ldots$ is a Markov chain that behaves as if X started at m, independently of $X_0, X_1, \ldots, X_T$. More specifically, we state the following theorem without proof. The proof can be found in Iosifescu (1980), Norris (1997), and Stirzaker (2005), among other books.

Theorem 2.1 Let T be a stopping time for a Markov chain $\{X_k, k \geq 0\}$. If $T < \infty$ and E is a random event prior to T, then

$$P[X_{T+1} = j | X_T = i, E] = P[X_{T+1} = j | X_T = 1] \quad \blacksquare$$

2.4 Applications of Discrete-Time Markov Processes

Discrete-time Markov processes have applications in many systems. Some of these applications have already been identified in the preceding discussion. In this section we discuss other areas.

2.4.1 Branching Processes

Consider a system that initally consists of a finite set of elements. As time passes, each element can independently disappear with probability p_0 or produce k other elements with probability p_k, where $k = 1, 2, \ldots$. The behavior of each of these k elements is similar to that of their parents. Let X_n denote the size of the population after n such events. The process $\{X_n | (n \geq 0)\}$ is a Markov chain called a branching process.

Branching processes are used to model many problems in science and engineering. These problems include population growth, the spread of epidemics, and nuclear fission. A good discussion on the application of Markov chains in biology can be found in Norris (1997).

2.4.2 Social Mobility

Prais (1955) reports on how sociologists have used Markov chains to determine how the social class of the father, grandfather, etc., affects the social class of a son. Such a determination is based on the fact that people can be classified into three social classes: upper class, middle class, and lower class. Thus, when the conditional probabilities are known, they can be used to represent the transition probabilities between social classes of the successive generations in a family thereby modeling the social mobility between classes by a Markov chain.

2.4.3 Markov Decision Processes

A Markov decision process is used to model an uncertain dynamic system whose states change with time. In such a system, a decision maker is required to make a sequence of decisions over time with uncertain outcomes. Each action taken by the decision maker can either yield a reward or incur a cost. Thus, the goal is to find an optimal sequence of actions that the decision maker must take to maximize the expected reward over a given time interval, which can be finite or infinite.

2.5 Applications of Continuous-Time Markov Processes

Like their discrete-time counterpart, continuous-time Markov processes have applications in many systems. Some of these applications have already been identified in the preceding discussion. In this section we discuss other areas.

2.5.1 Queueing Systems

A queue is a waiting line, and queueing systems are encountered almost everywhere, including checkout counters in grocery stores and people waiting for service at banks, post offices, movie theaters, and cafeterias. A queueing system consists of one or more servers who attend to customers that arrive according to a well-defined stochastic process. Any customer who arrives when at least one server is idle goes and receives service from a server without having to wait. Customers that arrive when all the servers are busy wait to be served according to a specified service policy, such as first-come first-served. Let n denote the number of customers in the system. If the arrival process is a Poisson process and the service time is exponentially distributed, then the process $\{n|n = 0, 1, \ldots\}$ is a Markov chain.

Sometimes the service center is organized in stages such that after a customer has finished receiving service at one stage he can proceed to receive additional

service at other stages or exit the system. In this case we have a network of queues. Basic Markovian queueing systems are discussed in Chapter 5.

2.5.2 Continuous-Time Markov Decision Processes

As discussed in the previous section, the Markov decision process is used to model an uncertain dynamic system whose states change with time. A decision maker is required to make a sequence of decisions over time with uncertain outcomes, and an action can either yield a reward or incur a cost. Thus, the goal is to find an optimal sequence of actions that the decision maker must take to maximize the expected reward over a given time interval, which can be finite or infinite. In discrete-time Markov decison processes, actions are taken at discrete intervals. However, in continuous-time Markov decision processes, actions are taken over exponentially distributed time intervals.

2.5.3 Stochastic Storage Systems

Inventory systems, dams, and insurance claims involve activities where some resource is kept in storage until it is used. Because the demand for these resources is generally random, they are generically referred to as stochastic storage systems.

In an inventory system, the goal is to coordinate the production rate and the inventory to cope with the random fluctuations in demand. There are different types of stochastic inventory control models, including the continuous review model, the single-period model, and the multi-period model. In the continuous review model, the stock is continuously reviewed, and an order of a fixed size is placed whenever the stock level reaches a certain reorder point. In the single-period model, an item is ordered once only to satisfy the demand of a specific period. In the multi-period model, an order is placed whenever the quantity in stock cannot satisfy the current demand.

We consider the single-period model. Let c denote the capacity of the warehouse. Assume that in each period n, there is a demand for D_n units. Let Y_n denote the residual stock at the end of period n. Consider the policy that requires the warehouse manager to restock to capacity for the beginning of period $n+1$ whenever $Y_n \leq m$, where m is a fixed threshold value. Then we have that

$$Y_{n+1} = \begin{cases} \max\{0, c - D_{n+1}\} & \text{if } Y_n \leq m \\ \max\{0, Y_n - D_{n+1}\} & \text{if } Y_n > m \end{cases}$$

If the $D_1, D_2, \ldots$ are independent and identically distributed, then $\{Y_n | (n \geq 0)\}$ is a Markov chain.

We can extend the same discussion to dams and insurance risks, which Tijms (1986) and Norris (1997) have shown to belong to a class of queueing systems called M/G/1 queue, which is discussed in Chapter 5.

2.6 Applications of Continuous-State Markov Processes

Like their discrete-state counterparts, continuous-state Markov processes are used to model many physical systems.

2.6.1 Application of Diffusion Processes to Financial Options

A financial option is a contract that gives a person the right, but not the obligation, to buy (or what is known as a *call option*) or sell (or what is known as a *put option*) an *underlying asset*, such as a stock or commodities, at a given price (known as the *strike price*) at a future date, in the case of the so-called *European option*, or before a future date, in the case of the so-called *American option*.

For a financial market that consists of stocks, which are risky investments, and bonds, which are risk-free investments, the celebrated Black-Scholes model of the price process of stocks assumes that the a and b functions of the stochastic differential equation

$$dX(t) = a(X(t), t)\,dt + b(X(t), t)\,dW(t) \quad t \geq 0$$

are given by $a(X(t), t) = \alpha X(t)$ and $b(X(t), t) = \beta X(t)$, where α and β are constants. Thus, the price dynamics of stocks for the European option are given by

$$dX(t) = \alpha X(t)dt + \beta X(t)dW(t) \quad t \geq 0$$

While different refinements of the model have been suggested, a basic tool used in financial mathematics is the diffusion process. The Black-Scholes model is discussed in Chapter 9.

2.6.2 Applications of Brownian Motion

Apart from its being an integral part of the diffusion process, the Brownian motion is used to model many physical systems. In this section we review some of these applications.

Fractal geometry enables fractal image models to be used in medical image processing. Medical images have a degree of randomness associated with both the natural random nature of the underlying structure and the random noise superimposed on the image. The fractional Brownian motion model, which was developed by Mandelbroth (1968), regards naturally occurring surfaces as the result of random walks. This has permitted the intensity of a medical image to be treated fractionally by the Brownian motion model.

An application of the Brownian motion in robotics is reported in Arakawa and Krotkov (1994). In Wein (1990), the Brownian motion is used to model a flexible manufacturing system (FMS) as a network of queues. This enables the FMS scheduling problem to be modeled as a problem of controlling the flow in a queueing network. Another application is in decision making, which is reported in Brekke and Oksendal (1991) and Romanow (1984).

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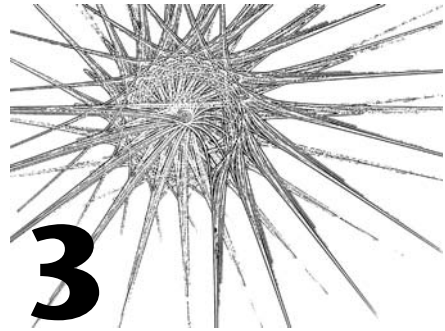
Discrete-Time Markov Chains



3

3.1 Introduction

The discrete-time process $\{X_k, k = 0, 1, 2, \ldots\}$ is called a Markov chain if for all $i, j, k, \ldots, m$, the following is true:

$$P[X_k = j | X_{k-1} = i, X_{k-2} = n, \ldots, X_0 = m] = P[X_k = j | X_{k-1} = i] = p_{ijk}$$

The quantity p_{ijk} is called the *state transition probability*, which is the conditional probability that the process will be in state j at time k immediately after the next transition, given that it is in state i at time $k - 1$. A Markov chain that obeys the preceding rule is called a *nonhomogeneous Markov chain*. In this book we will consider only *homogeneous Markov chains*, which are Markov chains in which $p_{ijk} = p_{ij}$. This means that homogeneous Markov chains do not depend on the time unit, which implies that

$$P[X_k = j | X_{k-1} = i, X_{k-2} = n, \ldots, X_0 = m] = P[X_k = j | X_{k-1} = i] = p_{ij}$$

This is the so-called Markov property mentioned in Chapter 2. The *homogeneous state transition probability* p_{ij} satisfies the following conditions:

1. $0 \leq p_{ij} \leq 1$
2. $\sum_j p_{ij} = 1$, $i = 1, 2, \ldots, n$, which follows from the fact that the states are mutually exclusive and collectively exhaustive

From the preceding definition we obtain the following *Markov chain rule*:

$$
\begin{aligned}
&P[X_k = j, X_{k-1} = i_1, X_{k-2}, \ldots, X_0] \\
&= P[X_k = j | X_{k-1} = i_1, X_{k-2}, \ldots, X_0] P[X_{k-1} = i_1, X_{k-2} = i_2, \ldots, X_0 = i_k] \\
&= P[X_k = j | X_{k-1} = i_1] P[X_{k-1} = i_1, X_{k-2}, \ldots, X_0 = i_k] \\
&= P[X_k = j | X_{k-1} = i_1] P[X_{k-1} = i_1 | X_{k-2} = i_2, \ldots, X_0] P[X_{k-2} = i_2, \ldots, X_0] \\
&= P[X_k = j | X_{k-1} = i_1] P[X_{k-1} = i_1 | X_{k-2} = i_2], \ldots, P[X_1 = i_{k-1} | X_0] \\
&\quad P[X_0 = i_k] \\
&= p_{i_1 j} p_{i_2 i_1} p_{i_3 i_2} \cdots p_{i_k i_{k-1}} P[X_0 = i_k]
\end{aligned}
$$

Thus, when we know the initial state X_0 and the state transition probabilities, we can evaluate the joint probability $P[X_k, X_{k-1}, \ldots, X_0]$.

3.2 State Transition Probability Matrix

It is customary to display the state transition probabilities as the entries of an $n \times n$ matrix P, where p_{ij} is the entry in the ith row and jth column:

$$
P = \begin{bmatrix}
p_{11} & p_{12} & \cdots & p_{1n} \\
p_{21} & p_{22} & \cdots & p_{2n} \\
\cdots & \cdots & \cdots & \cdots \\
p_{n1} & p_{n2} & \cdots & p_{nn}
\end{bmatrix}
$$

P is called the *transition probability matrix*. It is a *stochastic matrix* because for any row i, $\sum_j p_{ij} = 1$.

3.2.1 The *n*-Step State Transition Probability

Let $p_{ij}(n)$ denote the conditional probability that the process will be in state j after exactly n transitions, given that it is presently in state i. That is,

$$
\begin{aligned}
p_{ij}(n) &= P[X_{m+n} = j | X_m = i] \\
p_{ij}(0) &= \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \\
p_{ij}(1) &= p_{ij}
\end{aligned}
$$

Consider the two-step transition probability $p_{ij}(2)$, which is defined by

$$p_{ij}(2) = P[X_{m+2} = j | X_m = i]$$

Assume that $m = 0$, then

$$\begin{aligned} p_{ij}(2) &= P[X_2 = j | X_0 = i] \\ &= \sum_k P[X_2 = j, X_1 = k | X_0 = i] \\ &= \sum_k P[X_2 = j | X_1 = k, X_0 = i] P[X_1 = k | X_0 = i] \\ &= \sum_k P[X_2 = j | X_1 = k] P[X_1 = k | X_0 = i] \\ &= \sum_k p_{kj} p_{ik} = \sum_k p_{ik} p_{kj} \end{aligned}$$

where the second to the last equality is due to the Markov property. The final equation states that the probability of starting in state i and being in state j at the end of the second transition is the probability that we first go immediately from state i to an intermediate state k and then immediately from state k to state j; the summation is taken over all possible intermediate states k.

The following proposition deals with a class of equations called the *Chapman-Kolmogorov equations*, which provide a generalization of the preceding results obtained for the two-step transition probability.

Proposition 3.1 For all $0 < r < n$,

$$p_{ij}(n) = \sum_k p_{ik}(r) p_{kj}(n - r)$$

This proposition states that the probability that the process starts in state i and finds itself in state j at the end of the nth transition is the product of the probability that the process starts in state i and finds itself in an intermediate state k after r transitions and the probability that it goes from state k to state j after additional $n - r$ transitions.

Proof The proof is a generalization of the proof for the case of $n = 2$ and is as follows:

$$\begin{aligned} p_{ij}(n) &= P[X_n = j | X_0 = i] \\ &= \sum_k P[X_n = j, X_r = k | X_0 = i] \end{aligned}$$

$$= \sum_k P[X_n = j | X_r = k, X_0 = i] P[X_r = k | X_0 = i]$$

$$= \sum_k P[X_n = j | X_r = k] P[X_r = k | X_0 = i] = \sum_k p_{kj}(n-r)\, p_{ik}(r)$$

$$= \sum_k p_{ik}(r)\, p_{kj}(n-r)$$

This completes the proof. ■

It can be shown that $p_{ij}(n)$ is the ijth entry (ith row, jth column) in the matrix P^n. That is, for an N-state Markov chain, P^n is the matrix

$$P^n = \begin{bmatrix} p_{11}(n) & p_{12}(n) & p_{13}(n) & \dots & p_{1N}(n) \\ p_{21}(n) & p_{22}(n) & p_{23}(n) & \dots & p_{2N}(n) \\ p_{31}(n) & p_{32}(n) & p_{33}(n) & \dots & p_{3N}(n) \\ \dots & \dots & \dots & \dots & \dots \\ p_{N1}(n) & p_{N2}(n) & p_{N3}(n) & \dots & p_{NN}(n) \end{bmatrix}$$

3.3 State Transition Diagrams

Consider the following problem. It has been observed via a series of tosses of a particular biased coin that the outcome of the next toss depends on the outcome of the current toss. In particular, given that the current toss comes up heads, the next toss will come up heads with probability 0.6 and tails with probability 0.4. Similarly, given that the current toss comes up tails, the next toss will come up heads with probability 0.35 and tails with probability 0.65.

If we define state 1 to represent heads and state 2 to represent tails, then the transition probability matrix for this problem is the following:

$$P = \begin{bmatrix} 0.6 & 0.4 \\ 0.35 & 0.65 \end{bmatrix}$$

All the properties of the Markov chain can be determined from this matrix. However, the analysis of the problem can be simplified by the use of the *state-transition diagram* in which the states are represented by circles, and directed arcs represent transitions between states. The state transition probabilities are labeled on the appropriate arcs. Thus, with respect to the preceding problem, we obtain the state-transition diagram shown in Figure 3.1.

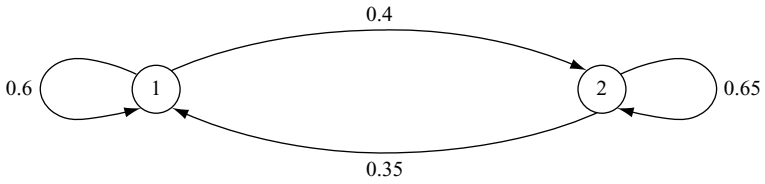


Figure 3.1. Example of state-transition diagram.

3.4 Classification of States

A state j is said to be *accessible* (or *can be reached*) from state i if, starting from state i, it is possible that the process will ever enter state j. This implies that $p_{ij}(n) > 0$ for some $n > 0$. Thus, the n-step probability enables us to obtain reachability information between any two states of the process.

Two states that are accessible from each other are said to *communicate* with each other. The concept of communication divides the state space into different classes. Two states that communicate are said to be in the same *class*. All members of one class communicate with one another. If a class is not accessible from any state outside the class, we define the class to be a *closed communicating class*. A Markov chain in which all states communicate, which means that there is only one class, is called an *irreducible* Markov chain. For example, the Markov chain shown in Figure 3.1 is an irreducible chain.

The states of a Markov chain can be classified into two broad groups: those that the process enters infinitely often and those that it enters finitely often. In the long run, the process will be found to be in only those states that it enters infinitely often. Let $f_{ij}(n)$ denote the conditional probability that given that the process is presently in state i, the first time it will enter state j occurs in exactly n transitions (or steps). We call $f_{ij}(n)$ the probability of *first passage* from state i to state j in n transitions. The parameter f_{ij}, which is given by

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}(n)$$

is the probability of first passage from state i to state j. It is the conditional probability that the process will ever enter state j, given that it was initially in state i. Obviously $f_{ij}(1) = p_{ij}$, and a recursive method of computing $f_{ij}(n)$ is

$$f_{ij}(n) = \sum_{l \neq j} p_{il} f_{lj}(n-1)$$

The quantity f_{ii} denotes the probability that a process that starts at state i will ever return to state i. Any state i for which $f_{ii} = 1$ is called a *recurrent state*, and

any state i for which $f_{ii} < 1$ is called a *transient state*. More formally, we define these states as follows:

a. A state j is called a *transient* (or *nonrecurrent*) state if there is a positive probability that the process will never return to j again after it leaves j.
b. A state j is called a *recurrent* (or *persistent*) state if, with probability 1, the process will eventually return to j after it leaves j. A set of recurrent states forms a *single chain* if every member of the set communicates with all other members of the set.
c. A recurrent state j is called a *periodic* state if there exists an integer d, $d > 1$, such that $p_{jj}(n)$ is zero for all values of n other than $d, 2d, 3d, \ldots$; d is called the period. If $d = 1$, the recurrent state j is said to be *aperiodic*.
d. A recurrent state j is called a *positive recurrent* state if, starting at state j, the expected time until the process returns to state j is finite. Otherwise, the recurrent state is called a *null recurrent* state.
e. Positive recurrent, aperiodic states are called *ergodic* states.
f. A chain consisting of ergodic states is called an *ergodic chain*.
g. A state j is called an *absorbing* (or *trapping*) state if $p_{jj} = 1$. Thus, when the process enters a trapping or absorbing state, it never leaves the state, which means that it is "trapped."

Example 3.1 Consider the Markov chain with the state-transition diagram shown in Figure 3.2. State 4 is the only transient state; states 1, 2, and 3 are recurrent states; there is no periodic state. There is a single chain, which is $\{1, 2, 3\}$. ■

Example 3.2 Consider the state-transition diagram of Figure 3.3. The transition is now from state 2 to state 4 instead of from state 4 to state 2. For this case, states 1, 2, and 3 are now transient states because when the process enters state 2 and makes a transition to state 4, it does not return to these states again. Also, state 4 is a trapping (or absorbing) state because when the process enters the state, the process never leaves the state. As stated in the definition, we identify a trapping state from the fact that, as in this example, $p_{44} = 1$ and $p_{4k} = 0$ for k not equal to 4. ■

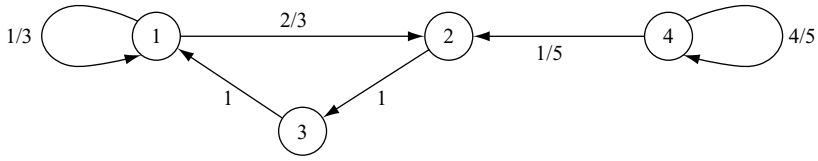


Figure 3.2. State-transition diagram for Example 3.1.

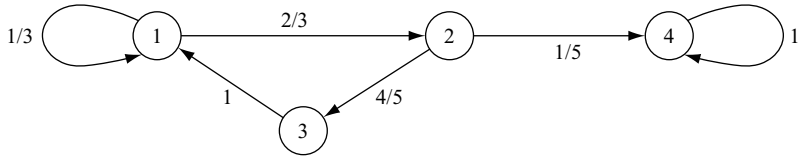


Figure 3.3. State-transition diagram for Example 3.2.

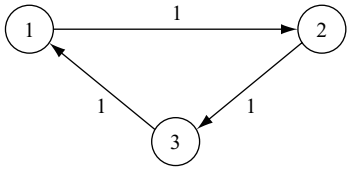


Figure 3.4. State-transition diagram for Example 3.3.

Example 3.3 The Markov chain whose state-transition diagram is shown in Figure 3.4 has a single chain {1, 2, 3}, and the three states are periodic with a period of 3. ■

3.5 Limiting-State Probabilities

Recall that the n-step state transition probability $p_{ij}(n)$ is the conditional probability that the process will be in state j after exactly n transitions, given that it is presently in state i. The n-step transition probabilities can be obtained by multiplying the transition probability matrix by itself n times. For example, consider the following transition probability matrix:

$$P = \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$$

$$P^2 = P \times P = \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{bmatrix} \times \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.34 & 0.37 & 0.29 \\ 0.33 & 0.32 & 0.35 \\ 0.33 & 0.31 & 0.36 \end{bmatrix}$$

$$P^3 = P^2 \times P = \begin{bmatrix} 0.34 & 0.37 & 0.29 \\ 0.33 & 0.32 & 0.35 \\ 0.33 & 0.31 & 0.36 \end{bmatrix} \times \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.334 & 0.339 & 0.327 \\ 0.333 & 0.331 & 0.336 \\ 0.333 & 0.330 & 0.337 \end{bmatrix}$$

From the matrix P^2 we obtain the $p_{ij}(2)$. For example, $p_{23}(2) = 0.35$, which is the entry in the second row and third column of the matrix P^2. Similarly, the entries of the matrix P^3 are the $p_{ij}(3)$.

For this particular matrix and matrices for a large number of Markov chains, we find that as we multiply the transition probability matrix by itself many times the entries remain constant. More importantly, all the members of one column will tend to converge to the same value.

If we define $P[X(0) = i]$ as the probability that the process is in state i before it makes the first transition, then the set $\{P[X(0) = i]\}$ defines the initial condition for the process, and for an N-state process,

$$\sum_{i=1}^{N} P[X(0) = i] = 1$$

Let $P[X(n) = j]$ denote the probability that it is in state j at the end of the first n transitions, then for the N-state process,

$$P[X(n) = j] = \sum_{i=1}^{N} P[X(0) = i] p_{ij}(n)$$

For the class of Markov chains previously referenced, it can be shown that as $n \to \infty$ the n-step transition probability $p_{ij}(n)$ does not depend on i, which means that $P[X(n) = j]$ approaches a constant as $n \to \infty$ for this class of Markov chains. That is, the constant is independent of the initial conditions. Thus, for the class of Markov chains in which the limit exists, we define the *limiting-state probabilities* as follows:

$$\lim_{n \to \infty} P[X(n) = j] = \pi_j \qquad j = 1, 2, \ldots, N$$

Because the n-step transition probability can be written in the form

$$p_{ij}(n) = \sum_{k} p_{ik}(n-1) p_{kj}$$

then if the limiting-state probabilities exist and do not depend on the initial state, we have that

$$\lim_{n\to\infty} p_{ij}(n) = \pi_j = \lim_{n\to\infty} \sum_k p_{ik}(n-1)\, p_{kj}$$

$$= \sum_k \pi_k p_{kj}$$

If we define the limiting-state probability vector $\pi = [\pi_1, \pi_2, \ldots, \pi_N]$, then we have that

$$\pi_j = \sum_k \pi_k p_{kj}$$

$$\pi = \pi P$$

$$1 = \sum_j \pi_j$$

where the last equation is due to the law of total probability. Each of the first two equations, together with the last equation, gives a system of linear equations that the π_j must satisfy. The following propositions specify the conditions for the existence of the limiting-state probabilities:

a. In any irreducible, aperiodic Markov chain the limits $\pi_j = \lim_{n\to\infty} p_{ij}(n)$ exist and are independent of the initial distribution.
b. In any irreducible, periodic Markov chain the limits $\pi_j = \lim_{n\to\infty} p_{ij}(n)$ exist and are independent of the initial distribution. However, they must be interpreted as the long-run probability that the process is in state j.

Example 3.4 Recall the biased coin problem whose state transition diagram is given in Figure 3.1 and reproduced in Figure 3.5. Find the limiting-state probabilities. ■

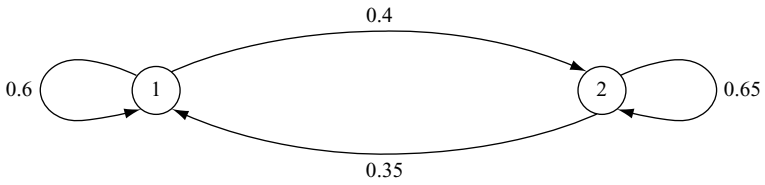


Figure 3.5. State-transition diagram for Example 3.4.

Solution: There are three equations associated with the preceding Markov chain, and they are

$$\pi_1 = 0.6\pi_1 + 0.35\pi_2$$
$$\pi_2 = 0.4\pi_1 + 0.65\pi_2$$
$$1 = \pi_1 + \pi_2$$

Because there are three equations and two unknowns, one of the equations is redundant. Thus, the rule of thumb is that for an N-state Markov chain, we use the first $N-1$ linear equations from the relation $\pi_j = \sum_k \pi_k p_{kj}$ and the total probability, $1 = \sum_j \pi_j$. For the given problem we have

$$\pi_1 = 0.6\pi_1 + 0.35\pi_2$$
$$1 = \pi_1 + \pi_2$$

From the first equation we obtain $\pi_1 = (0.35/0.4)\pi_2 = (7/8)\pi_2$. Substituting for π_1 and solving for π_2 in the second equation, we obtain the result $\pi = \{\pi_1, \pi_2\} = \{7/15, 8/15\}$.

Suppose we are also required to compute $p_{12}(3)$, which is the probability that the process will be in state 2 at the end of the third transition, given that it is presently in state 1. We can proceed in two ways: the direct method and the matrix method. We consider both methods.

(a) **Direct Method:** Under this method we exhaustively enumerate all the possible ways of a state 1 to state 2 transition in three steps. If we use the notation $a \to b \to c$ to denote a transition from state a to state b and then from state b to state c, the desired result is the following:

$$p_{12}(3) = P[\{1 \to 1 \to 1 \to 2\} \cup \{1 \to 1 \to 2 \to 2\} \cup \{1 \to 2 \to 1 \to 2\} \cup \{1 \to 2 \to 2 \to 2\}]$$

Because the different events are mutually exclusive, we obtain

$$\begin{aligned} p_{12}(3) &= P[1 \to 1 \to 1 \to 2] + P[1 \to 1 \to 2 \to 2] + P[1 \to 2 \to 1 \to 2] \\ &\quad + P[1 \to 2 \to 2 \to 2] \\ &= (0.6)(0.6)(0.4) + (0.6)(0.4)(0.65) + (0.4)(0.35)(0.4) \\ &\quad + (0.4)(0.65)(0.65) \\ &= 0.525 \end{aligned}$$

(b) **Matrix Method:** One of the limitations of the direct method is that it is difficult to exhaustively enumerate the different ways of going from state 1 to

state 2 in n steps, especially when n is large. This is where the matrix method becomes very useful. As discussed earlier, $p_{ij}(n)$ is the ijth entry in the matrix P^n. Thus, for the current problem, we are looking for the entry in the first row and second column of the matrix P^3. Therefore, we have

$$P = \begin{bmatrix} 0.6 & 0.4 \\ 0.35 & 0.65 \end{bmatrix}$$

$$P^2 = P \times P = \begin{bmatrix} 0.6 & 0.4 \\ 0.35 & 0.65 \end{bmatrix} \times \begin{bmatrix} 0.6 & 0.4 \\ 0.35 & 0.65 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.4375 & 0.5625 \end{bmatrix}$$

$$P^3 = P \times P^2 = \begin{bmatrix} 0.6 & 0.4 \\ 0.35 & 0.65 \end{bmatrix} \times \begin{bmatrix} 0.5 & 0.5 \\ 0.4375 & 0.5625 \end{bmatrix}$$

$$= \begin{bmatrix} 0.475 & 0.525 \\ 0.459375 & 0.540625 \end{bmatrix}$$

The required result (first row, second column) is 0.525, which is the result obtained via the direct method.

3.5.1 Doubly Stochastic Matrix

A transition probability matrix P is defined to be a doubly stochastic matrix if each of its columns sums to 1. That is, not only does each row sum to 1, each column also sums to 1. Thus, for every column j of a doubly stochastic matrix, we have that $\sum_i p_{ij} = 1$.

Doubly stochastic matrices have interesting limiting-state probabilities, as the following theorem shows.

Theorem 3.1 If P is a doubly stochastic matrix associated with the transition probabilities of a Markov chain with N states, then the limiting-state probabilities are given by $\pi_i = 1/N$, $i = 1, 2, \ldots, N$.

Proof We know that the limiting-state probabilities satisfy the condition

$$\pi_j = \sum_k \pi_k p_{kj}$$

To check the validity of the theorem, we observe that when we substitute $\pi_i = 1/N$, $i = 1, 2, \ldots, N$, in the preceding equation we obtain

$$\frac{1}{N} = \frac{1}{N} \sum_k p_{kj} \Rightarrow 1 = \sum_k p_{kj}$$

This shows that $\pi_i = 1/N$ satisfies the condition $\pi = \pi P$, which the limiting-state probabilities are required to satisfy. Conversely, from the preceding equation,

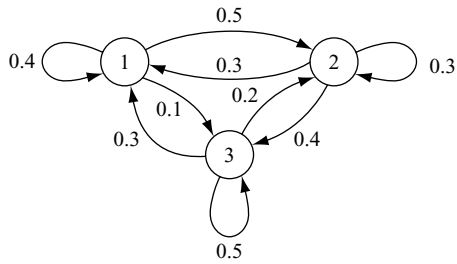


Figure 3.6. State-transition diagram for Example 3.5.

we see that if the limiting-state probabilities are given by $1/N$, then each column j of P sums to 1; that is, P is doubly stochastic. This completes the proof. ■

Example 3.5 Find the transition probability matrix and the limiting-state probabilities of the process represented by the state-transition diagram shown in Figure 3.6. ■

Solution: The transition probability matrix is given by

$$P = \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$$

It can be seen that each row of the matrix sums to 1, and each column also sums to 1; that is, it is a doubly stochastic matrix. Because the process is an irreducible, aperiodic Markov chain, the limiting-state probabilities exist and are given by $\pi_1 = \pi_2 = \pi_3 = 1/3$.

3.6 Sojourn Time

Consider a state i for which $p_{ii} > 0$. We are interested in the probability that the process remains in the state for exactly d time units. Thus, if we let the random variable D_i denote the number of time units that the process remains in state i before leaving the state, given that it enters the state, then the PMF of D_i is given by

$$\begin{aligned} p_{D_i}(d) &= P[D_i = d] = P[X_0 = i, X_1 = i, X_2 = i, \ldots, X_{d-1} = i, X_d \neq i] \\ &= p_{ii}^{d-1}(1 - p_{ii}) \end{aligned}$$

where we have used the Markov chain rule. If the state of the process denotes the members of an observation sequence, then $p_{D_i}(d)$ represents the probability that

the sequence remains unchanged exactly d times. Because D_i is a geometrically distributed random variable, the mean sojourn time in state i is given by

$$E[D_i] = \frac{1}{1 - p_{ii}}$$

Note that if i is a trapping state, then $p_{ii} = 1$ and $E[D_i] = \infty$, which is true because the process remains in the state indefinitely. For $p_{ii} \neq 1$, $E[D_i]$ is finite.

3.7 Transient Analysis of Discrete-Time Markov Chains

Recall that the n-step transition probability $p_{ij}(n)$, which is the conditional probability that the system will be in state j after exactly n transitions given that it is presently in state i, is given by

$$p_{ij}(n) = \sum_k p_{ik}(r)\, p_{kj}(n - r)$$

In particular, for an N-state Markov chain we have that

$$p_{ij}(n + 1) = \sum_{k=1}^{N} p_{ik}(n)\, p_{kj} \qquad n = 0, 1, \ldots$$

Let $g_{ij}(z)$ denote the z-transform of $p_{ij}(n), n = 0, 1, \ldots$. That is,

$$g_{ij}(z) = \sum_{n=0}^{\infty} p_{ij}(n) z^n$$

Then, taking the z-transform on both sides of the previous equation we obtain

$$z^{-1}[g_{ij}(z) - p_{ij}(0)] = \sum_{k=1}^{N} g_{ik}(z)\, p_{kj}$$

Let $G(z)$ denote the matrix of the $g_{ij}(z)$. Recall that $p_{ij}(0) = 1$ if $i = j$, and $p_{ij}(0) = 0$ otherwise. Thus, if $P(0)$ is the matrix of the $p_{ij}(0)$, then $P(0) = I$, where I is the identity matrix, and we have that

$$z^{-1}[G(z) - I] = G(z)P$$

$$G(z) - I = G(z)Pz$$

$$G(z)\{I - Pz\} = I$$

$$G(z) = [I - Pz]^{-1}$$

We obtain P^n as the inverse of $G(z)$. In general P^n obtained via this operation consists of two sets of components: a constant term C and a transient term $T(n)$ that is a function of n. That is,

$$P^n = C + T(n)$$

The constant term C has the characteristic that all the n rows are identical, and the elements of the rows are the limiting-state probabilities. This means that $G(z)$ can be written as follows:

$$G(z) = [I - Pz]^{-1} = \frac{1}{1-z}C + T(z)$$

where $T(z)$ is the z-transform of $T(n)$.

Example 3.6 Consider the transition probability matrix given by

$$P = \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$$

We would like to obtain P^n. ■

Solution: We proceed as follows:

$$I - Pz = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - z \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - 0.4z & -0.5z & -0.1z \\ -0.3z & 1 - 0.3z & -0.4z \\ -0.3z & -0.2z & 1 - 0.5z \end{bmatrix}$$

The determinant of $1 - Pz$ is

$$|I - Pz| = 1 - 1.2z + 0.21z^2 - 0.01z^3 = (1 - z)(1 - 0.2z + 0.01z^2)$$

$$= (1 - z)(1 - 0.1z)^2$$

$$[I - Pz]^{-1} = \frac{1}{(1-z)(1-0.1z)^2} \begin{bmatrix} 1 - 0.8z + 0.07z^2 & 0.5z - 0.23z^2 & 0.1z + 0.17z^2 \\ 0.3z - 0.03z^2 & 1 - 0.9z + 0.17z^2 & 0.4z - 0.13z^2 \\ 0.3z - 0.03z^2 & 0.2z + 0.07z^2 & 1 - 0.7z - 0.03z^2 \end{bmatrix}$$

$$= \frac{1}{1-z}\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} + \frac{1}{1-0.1z}\begin{bmatrix} 2/3 & -7/3 & 5/3 \\ -1/3 & 5/3 & -4/3 \\ -1/3 & 2/3 & -1/3 \end{bmatrix}$$

$$+ \frac{1}{(1-0.1z)^2}\begin{bmatrix} 0 & 2 & -2 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

Thus, we obtain

$$P = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} + (0.1)^n \begin{bmatrix} 2/3 & -7/3 & 5/3 \\ -1/3 & 5/3 & -4/3 \\ -1/3 & 2/3 & -1/3 \end{bmatrix}$$

$$+ (n+1)(0.1)^n \begin{bmatrix} 0 & 2 & -2 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \qquad n = 0, 1, \ldots$$

Note that the matrix associated with the root $1 - z$ gives the limiting-state probabilities, which can be seen to be $\pi = \{1/3, 1/3, 1/3\}$. The reason why the limiting-state probabilities are equal is because P is a doubly stochastic matrix. Also, each row in the two other matrices sums to zero. Finally, it must be observed that when $n = 0$ we obtain the identity matrix, and when $n = 1$ we obtain P.

3.8 First Passage and Recurrence Times

We have earlier defined the first passage time from state i to state j, T_{ij}, as

$$T_{ij} = \min\{n \geq 1 : X_n = j | X_0 = i\}$$

Thus, the probability of first passage time from state i to state j in n transitions, $f_{ij}(n)$, is the conditional probability that given that the process is presently in state i, the first time it will enter state j occurs in exactly n transitions (or steps). The probability of first passage from state i to state j, f_{ij}, is also defined as follows:

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}(n)$$

Thus, f_{ij} is the conditional probability that the process will ever enter state j, given that it was initially in state i. Obviously $f_{ij}(1) = p_{ij}$ and a recursive method of computing $f_{ij}(n)$ is

$$f_{ij}(n) = \sum_{l \neq j} p_{il} f_{lj}(n-1)$$

When $i = j$, the first passage time becomes the *recurrence time* for state i. That is, $f_{ii}(n)$ is the conditional probability that given that the process is presently in

state i, the first time it will return to state i occurs in exactly n transitions. Thus, f_{ii} is the conditional probability that a process that starts in state i will ever return to state i.

The relationship between the n-step transition probability $p_{ij}(n)$ and the first passage time probability $f_{ij}(n)$ can be obtained as follows. A process that starts in state i can be in state j in n transitions if it entered state j for the first time after $m \leq n$ transitions and reached state j again after another $n - m$ transitions. Thus, we have that

$$p_{ij}(n) = \sum_{m=1}^{n} f_{ij}(m) p_{jj}(n-m) \qquad n = 1, 2, \ldots$$

This expression can also be written in the following form:

$$p_{ij}(n) = \sum_{m=1}^{n-1} f_{ij}(m) p_{jj}(n-m) + f_{ij}(n) \qquad n = 2, 3, \ldots$$

where the last equation follows from the fact that $p_{jj}(0) = 1$. From this we have that

$$f_{ij}(n) = \begin{cases} 0 & n = 0 \\ p_{ij}(n) & n = 1 \\ p_{ij}(n) - \sum_{m=1}^{n-1} f_{ij}(m) p_{jj}(n-m) & n = 2, 3, \ldots \end{cases}$$

If we define m_{ij} as the mean first passage time from state i to state j, then it can be shown that

$$m_{ij} = \sum_{n=1}^{\infty} n f_{ij}(n) = 1 + \sum_{k \neq j} p_{ik} m_{kj} \qquad i \neq j$$

The meaning of the second equation is as follows. Because the time the process spends in each state (called the *holding time*) is 1, the equation says that the mean first passage time from state i to j is the holding time in state i plus the mean first passage time from state k to state j, $k \neq j$, given that the next state the process visits when it leaves state i is state k. The probability of this transition is, of course, p_{ik}, and we sum over all $k \neq j$. Similarly, the mean reccurence time is given by

$$m_{ii} = \sum_{n=1}^{\infty} n f_{ii}(n) = 1 + \sum_{k \neq i} p_{ik} m_{ki}$$

Example 3.7 Consider the transition probability matrix associated with Example 3.5. We would like to obtain the mean first passage time m_{13}.

$$P = \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$$

■

Solution: We have that $m_{13} = 1 + p_{11}m_{13} + p_{12}m_{23}$. Thus, to compute m_{13} we must obtain m_{23}. Therefore, we must solve the following system of equations:

$$\begin{aligned} m_{13} &= 1 + p_{11}m_{13} + p_{12}m_{23} \\ &= 1 + 0.4m_{13} + 0.5m_{23} \Rightarrow 0.6m_{13} = 1 + 0.5m_{23} \\ m_{23} &= 1 + p_{21}m_{13} + p_{22}m_{23} \\ &= 1 + 0.3m_{13} + 0.2m_{23} \Rightarrow 0.8m_{23} = 1 + 0.3m_{13} \end{aligned}$$

From this we obtain

$$m_{13} = 3.939$$

$$m_{23} = 2.737$$

We can also obtain the mean recurrence time m_{11} as follows:

$$\begin{aligned} m_{11} &= 1 + p_{12}m_{21} + p_{13}m_{31} = 1 + 0.5m_{21} + 0.1m_{31} \\ m_{21} &= 1 + p_{22}m_{21} + p_{23}m_{31} \\ &= 1 + 0.3m_{21} + 0.4m_{31} \Rightarrow 0.7m_{21} = 1 + 0.4m_{31} \\ m_{31} &= 1 + p_{32}m_{21} + p_{33}m_{31} \\ &= 1 + 0.2m_{21} + 0.5m_{31} \Rightarrow 0.5m_{31} = 1 + 0.2m_{21} \end{aligned}$$

The solution to the system of equations is

$$m_{11} = 1.7187$$

$$m_{21} = 2.8125$$

$$m_{31} = 3.1250$$

3.9 Occupancy Times

Consider a discrete-time Markov chain $\{X_n, n \geq 0\}$. Let $N_i(n)$ denote the number of times that the process visits state i in the first n transitions, $n = 1, 2, \ldots$. Let $\phi_{ik}(n)$ be defined by

$$\phi_{ik}(n) = E[N_k(n)|X_0 = i]$$

That is, $\phi_{ik}(n)$ is the expected number of times that the process visits state k in the first n transitions, given that it starts in state i, and is called the *mean occupancy time* of state k up to n transitions given that the process started from state i. It can be shown that $\phi_{ik}(n)$ is given by

$$\phi_{ik}(n) = \sum_{r=0}^{n} p_{ik}(r)$$

where $p_{ik}(r)$ is the r-step transition probability from state i to state k. Because $p_{ik}(r)$ is the ikth entry in the matrix P^r, we can define the matrix for an N-state Markov chain

$$\Phi(n) = \begin{bmatrix} \phi_{11}(n) & \phi_{12}(n) & \phi_{13}(n) & \ldots & \phi_{1N}(n) \\ \phi_{21}(n) & \phi_{22}(n) & \phi_{23}(n) & \ldots & \phi_{2N}(n) \\ \phi_{31}(n) & \phi_{32}(n) & \phi_{33}(n) & \ldots & \phi_{3N}(n) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \phi_{N1}(n) & \phi_{N2}(n) & \phi_{N3}(n) & \ldots & \phi_{NN}(n) \end{bmatrix}$$

Then we have that

$$\Phi(n) = \sum_{r=0}^{n} P^r$$

Example 3.8 Consider the transition probability matrix associated with Example 3.6. We would like to obtain the mean first passage time $\phi_{13}(5)$.

$$P = \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$$

■

Solution: The matrix $\Phi(5)$ is given by

$$\Phi(5) = \sum_{r=0}^{5} P^r = \sum_{r=0}^{5} \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}^r$$

From Example 3.6, we have that

$$P^r = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + (0.1)^r \begin{bmatrix} 2/3 & -7/3 & 5/3 \\ -1/3 & 5/3 & -4/3 \\ -1/3 & 2/3 & -1/3 \end{bmatrix}$$

$$+ (r+1)(0.1)^r \begin{bmatrix} 0 & 2 & -2 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

Thus,

$$\phi_{13}(5) = \sum_{r=0}^{5} p_{13}(r) = \sum_{r=0}^{5} \left\{ \frac{1}{3} + \frac{5}{3}(0.1)^r - 2(r+1)(0.1)^r \right\}$$

$$= 2 - \frac{1}{3}\left\{ \frac{1-(0.1)^6}{1-0.1} \right\} - 2\sum_{r=0}^{5} r(0.1)^r = 2 - 0.37037 - 0.24690$$

$$= 1.38273$$

3.10 Absorbing Markov Chains and the Fundamental Matrix

An absorbing Markov chain is a Markov chain with at least one absorbing state. A problem of interest in the study of absorbing Markov chains is the probability that the process eventually reaches an absorbing state. We might also be interested in how long it will take for the process to be absorbed. To study this class of Markov chains we use the canonical form used in Kemeny and Snell (1976) and Doyle and Snell (1984).

Let P be an $N \times N$ transition probability matrix of an absorbing Markov chain, and assume that there are k absorbing states and $m = N - k$ nonabsorbing (or transient) states. If we reorder the states so that the absorbing states (A) come first and the transient states (T) come last, we obtain the following canonical form:

$$P = \begin{array}{c} \\ A \\ T \end{array} \begin{array}{c} \begin{array}{cc} A & T \end{array} \\ \left[\begin{array}{c|c} I & 0 \\ \hline R & Q \end{array}\right] \end{array}$$

Here I is a $k \times k$ identity matrix, 0 is a $k \times m$ matrix whose entries are all 0, R is an $m \times k$ matrix, and Q is an $m \times m$ matrix. We observe that

$$P^2 = \begin{bmatrix} I & 0 \\ R + QR & Q^2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ (I + Q)R & Q^2 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} I & 0 \\ R + QR + Q^2R & Q^3 \end{bmatrix} = \begin{bmatrix} I & 0 \\ (I + Q + Q^2)R & Q^3 \end{bmatrix}$$

$$\cdots$$

$$P^n = \begin{bmatrix} I & 0 \\ (I + Q + \cdots + Q^{n-1})R & Q^n \end{bmatrix}$$

A primary parameter of interest is the quantity N_{ij}, which is the mean number of times the process is in transient state j before hitting an absorbing state, given that it starts in transient state i. Note that the emphasis is on both state i and j being transient states. If, for example, state i is an absorbing state and $i \neq j$, then the quantity is zero. Similarly, if state j is an absorbing state and $i \neq j$, then the quantity is infinity if state j is accessible from state i. The following theorem, which is proved in Grinstead and Snell (1997), establishes the relationship between $N = [N_{ij}]$ (i.e., N is the matrix of the N_{ij}) and Q.

Theorem 3.2

$$N = \sum_{k=0}^{\infty} Q^k = [I - Q]^{-1}$$

The matrix $N = [I - Q]^{-1}$ is called the *fundamental matrix* for P, the transition probability matrix for the absorbing Markov chain.

3.10.1 Time to Absorption

The mean time to absorption is defined as the expected number of transitions until the process reaches a nontransient state, given that it starts from a particular transient state. The following recursive equation gives the expected absorption time μ_i for a process that starts in state $i \in T$:

$$\mu_i = \begin{cases} 0 & i \in R \\ 1 + \sum_{j \in T} p_{ij}\mu_j & i \in T \end{cases}$$

This equation can be explained as follows. Given that the process is currently in state i, the mean number of transitions until absorption is 1 plus the mean number

of transitions until absorption from state j given that the next transition from state i is state j. This is true for all j and so we sum over all $j \in T$.

The mean time to absorption can also be computed directly from the fundamental matrix. The following theorem, which is proved in Grinstead and Snell (1997), defines how to compute the mean times to absorption for the different transient states.

Theorem 3.3 Let μ_i denote the number of transitions before the process hits an absorption state, given that the chain starts in state i, and let M be the column vector whose ith entry is μ_i. Then

$$M = N1 = [I - Q]^{-1}1$$

where 1 is a column vector whose entries are all 1.

Example 3.9 Consider the Markov chain whose state-transition diagram is shown in Figure 3.7. Find μ_3. ■

Solution: The sets of nontransient and transient states are as follows:

$$A = \{4\}$$
$$T = \{1, 2, 3\}$$

Thus, using the direct method we obtain

$$\mu_3 = 1 + p_{31}\mu_1 + p_{32}\mu_2 = 1 + \mu_1$$
$$\mu_2 = 1 + p_{21}\mu_1 + p_{23}\mu_3 = 1 + \frac{4}{5}\mu_3$$
$$\mu_1 = 1 + p_{11}\mu_1 + p_{12}\mu_2 = 1 + \frac{1}{3}\mu_1 + \frac{2}{3}\mu_2$$

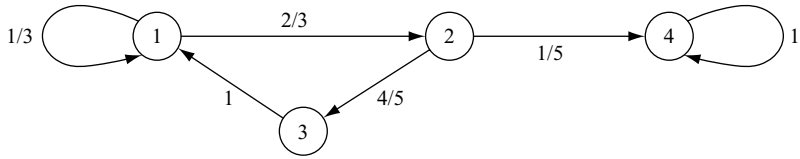


Figure 3.7. State-transition diagram for Example 3.9.

From this system of equations we obtain

$$\mu_3 = 17.5$$
$$\mu_2 = 15.0$$
$$\mu_1 = 16.5$$

Alternatively, the matrix Q associated with the transient states and the fundamental matrix are given by

$$Q = \begin{bmatrix} 1/3 & 2/3 & 0 \\ 0 & 0 & 4/5 \\ 1 & 0 & 0 \end{bmatrix}$$

$$I - Q = \begin{bmatrix} 2/3 & -2/3 & 0 \\ 0 & 1 & -4/5 \\ -1 & 0 & 0 \end{bmatrix}, \qquad |I - Q| = \frac{2}{15}$$

$$[I - Q]^{-1} = \frac{15}{2}\begin{bmatrix} 1 & 2/3 & 8/15 \\ 4/5 & 2/3 & 8/15 \\ 1 & 2/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 15/2 & 5 & 4 \\ 6 & 5 & 4 \\ 15/2 & 5 & 5 \end{bmatrix} = N$$

Thus,

$$M = N1 = \begin{bmatrix} 15/2 & 5 & 4 \\ 6 & 5 & 4 \\ 15/2 & 5 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 33/2 \\ 15 \\ 35/2 \end{bmatrix} = \begin{bmatrix} 16.5 \\ 15 \\ 17.5 \end{bmatrix}$$

Example 3.10 Consider the Markov chain whose state-transition diagram is shown in Figure 3.8. Find the μ_i for $i = 2, 3, 4$. ■

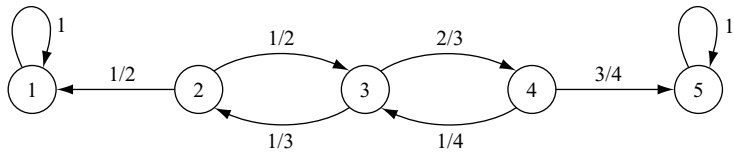


Figure 3.8. State-transition diagram for Example 3.10.

Solution: Because the transient states are $T = \{2, 3, 4\}$ and the absorbing states are $A = \{1, 5\}$, the P, Q, and R matrices are as follows:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/3 & 0 & 2/3 \\ 0 & 3/4 & 0 & 1/4 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 1/2 & 0 \\ 1/3 & 0 & 2/3 \\ 0 & 1/4 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \\ 0 & 3/4 \end{bmatrix}$$

Thus,

$$I - Q = \begin{bmatrix} 1 & -1/2 & 0 \\ -1/3 & 1 & -2/3 \\ 0 & -1/4 & 1 \end{bmatrix}$$

$$|I - Q| = \frac{2}{3}$$

$$N = [I - Q]^{-1} = \frac{3}{2} \begin{bmatrix} 5/6 & 1/2 & 1/3 \\ 1/3 & 1 & 2/3 \\ 1/12 & 1/4 & 5/6 \end{bmatrix} = \begin{bmatrix} 5/4 & 3 & 1/2 \\ 1/2 & 3/2 & 1 \\ 1/8 & 3/8 & 5/4 \end{bmatrix}$$

From this we obtain

$$M = N1 = \begin{bmatrix} 5/4 & 3 & 1/2 \\ 1/2 & 3/2 & 1 \\ 1/8 & 3/8 & 5/4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 19/4 \\ 3 \\ 7/4 \end{bmatrix} = \begin{bmatrix} 4.75 \\ 3 \\ 1.75 \end{bmatrix}$$

That is, $\mu_2 = 4.75$, $\mu_3 = 3$, $\mu_4 = 1.75$.

3.10.2 Absorption Probabilities

For an absorbing Markov chain, the probability that the chain that starts in a transient state i will be absorbed in state j is denoted by b_{ij}. Let B be the $m \times k$ matrix whose entries are b_{ij}. Then B is given by

$$B = [I - Q]^{-1} R = NR$$

where N is the fundamental matrix and R is the $m \times k$ matrix whose entries are transition probabilities from the transient states to the absorbing states.

Example 3.11 For the Markov chain whose state-transition diagram is shown in Figure 3.8, find the absorption probabilities b_{ij} for $i = 2, 3, 4$ and $j = 1, 5$. ■

Solution: Using $B = [I - Q]^{-1}R = NR$, where N and R are as defined in Example 3.10, we obtain

$$B = \begin{bmatrix} 5/4 & 3 & 1/2 \\ 1/2 & 3/2 & 1 \\ 1/8 & 3/8 & 5/4 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \\ 0 & 3/4 \end{bmatrix} = \begin{bmatrix} 5/8 & 3/8 \\ 1/4 & 3/4 \\ 1/16 & 15/16 \end{bmatrix}$$

That is, $b_{21} = 5/8, b_{25} = 3/8, b_{31} = 1/4, b_{35} = 3/4, b_{41} = 1/16, b_{45} = 15/16$.

3.11 Reversible Markov Chains

A Markov chain $\{X_n\}$ is defined to be a reversible Markov chain if the sequence of states $\ldots, X_{n+1}, X_n, X_{n-1}, \ldots$ has the same probabilistic structure as the sequence $\ldots, X_{n-1}, X_n, X_{n+1}, \ldots$. That is, the sequence of states looked at backward in time has the same structure as the sequence running forward in time. Consider a Markov chain $\{X_n\}$ with limiting state probabilities $\{\pi_1, \pi_2, \pi_3, \ldots\}$ and transition probabilities p_{ij}. Suppose that starting at time n we consider the sequence $X_n, X_{n-1}, \ldots$, and let $\hat{p}_{ij}$ be the transition probabilities of the reversed process. That is,

$$\begin{aligned}
\hat{p}_{ij} &= P[X_m = j | X_{m+1} = i, X_{m+2} = i_2, \ldots, X_{m+k} = i_k] \\
&= \frac{P[X_m = j, X_{m+1} = i, X_{m+2} = i_2, \ldots, X_{m+k} = i_k]}{P[X_{m+1} = i, X_{m+2} = i_2, \ldots, X_{m+k} = i_k]} \\
&= \frac{P[X_m = j]P[X_{m+1} = i | X_m = j]P[X_{m+2} = i_2, \ldots, X_{m+k} = i_k | X_m = j, X_{m+1} = i]}{P[X_{m+1} = i]P[X_{m+2} = i_2, \ldots, X_{m+k} = i_k | X_{m+1} = i]} \\
&= \frac{\pi_j p_{ji} P[X_{m+2} = i_2, \ldots, X_{m+k} = i_k | X_{m+1} = i]}{\pi_i P[X_{m+2} = i_2, \ldots, X_{m+k} = i_k | X_{m+1} = i]} \\
&= \frac{\pi_j p_{ji}}{\pi_i}
\end{aligned}$$

Thus, the backward chain is homogeneous if the forward process is in steady state, and the backward transition probabilities $\hat{p}_{ij}$ are given by

$$\hat{p_{ij}} = \frac{\pi_j p_{ji}}{\pi_i}$$

A Markov chain is said to be *reversible* if $\hat{p}_{ij} = p_{ij}$ for all i and j. Thus, for a reversible Markov chain,

$$\pi_i p_{ij} = \pi_j p_{ji} \qquad \forall i, j$$

This condition simply states that for states i and j, the rate $\pi_i p_{ij}$ at which the process goes from state i to state j is equal to the rate $\pi_j p_{ji}$ at which the process goes from state j to state i. This "local balance" is particularly used in the steady-state analysis of birth-and-death processes that are discussed in Chapter 4.

3.12 Problems

3.1 Consider the following transition probability matrix:

$$P = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.0 & 0.3 & 0.7 \end{bmatrix}$$

a. Give the state-transition diagram.
b. Given that the process is currently in state 1, what is the probability that it will be in state 2 at the end of the third transition?
c. Given that the process is currently in state 1, what is the probability that the first time it enters state 3 is the fourth transition?

3.2 Consider the following social mobility problem. Studies indicate that people in a society can be classified as belonging to the upper class (state 1), middle class (state 2), and lower class (state 3). Membership in any class is inherited in the following probabilistic manner. Given that a person is raised in an upper-class family, he will have an upper-class family with probability 0.45, a middle-class family with probability 0.48, and a lower-class family with probability 0.07. Given that a person is raised in a middle-class family, he will have an upper-class family with probability 0.05, a middle-class family with probability 0.70, and a lower-class family with probability 0.25. Finally, given that a person is raised in a lower-class family, he will have an upper-class family with probability 0.01, a middle-class family with probability 0.50, and a lower-class family with probability 0.49. Determine the following:

a. The state-transition diagram of the process.
b. The transition probability matrix of the process.
c. The limiting-state probabilities. Interpret what they mean to the layperson.

3.3 A taxi driver conducts his business in three different towns 1, 2, and 3. On any given day, when he is in town 1, the probability that the next passenger he picks up is going to a place in town 1 is 0.3, the probability that the next passenger he picks up is going to town 2 is 0.2, and the probability that the next passenger he picks up is going to town 3 is 0.5. When he is in town 2,

the probability that the next passenger he picks up is going to town 1 is 0.1, the probability that the next passenger he picks up is going to town 2 is 0.8, and the probability that the next passenger he picks up is going to town 3 is 0.1. When he is in town 3, the probability that the next passenger he picks up is going to town 1 is 0.4, the probability that the next passenger he picks up is going to town 2 is 0.4, and the probability that the next passenger he picks up is going to town 3 is 0.2.

a. Determine the state-transition diagram for the process.
b. Give the transition probability matrix for the process.
c. What are the limiting-state probabilities?
d. Given that the taxi driver is currently in town 2 and is waiting to pick up his first customer for the day, what is the probability that the first time he picks up a passenger to town 2 is when he picks up his third passenger for the day?
e. Given that he is currently in town 2, what is the probability that his third passenger from now will be going to town 1?

3.4 The New England fall weather can be classified as sunny, cloudy, or rainy. A student conducted a detailed study of the weather conditions and came up with the following conclusion: Given that it is sunny on any given day, then on the following day it will be sunny again with probability 0.5, cloudy with probability 0.3, and rainy with probability 0.2. Given that it is cloudy on any given day, then on the following day it will be sunny with probability 0.4, cloudy again with probability 0.3, and rainy with probability 0.3. Finally, given that it is rainy on any given day, then on the following day it will be sunny with probability 0.2, cloudy with probability 0.5, and rainy again with probability 0.3.

a. Give the state-transition diagram of New England fall weather with the state "sunny" as state 1, the state "cloudy" as state 2, and the state "rainy" as state 3.
b. Using the same convention as in part (a), give the transition probability matrix of New England fall weather.
c. Given that it is sunny today, what is the probability that it will be sunny four days from now?
d. Determine the limiting-state probabilities of the weather.

3.5 Consider the following transition probability matrix:

$$P = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.1 & 0.8 & 0.1 \\ 0.4 & 0.4 & 0.2 \end{bmatrix}$$

a. What is P^n?
b. Obtain $\phi_{13}(5)$, the mean occupancy time of state 3 up to five transitions given that the process started from state 1.

3.6 Consider the following transition probability matrix:

$$P = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.3 & 0.3 & 0.4 \\ 0.25 & 0.5 & 0.25 \end{bmatrix}$$

a. Calculate $p_{13}(3)$, $p_{22}(2)$, and $p_{32}(4)$.
b. Calculate $p_{32}(\infty)$.

3.7 Consider the following transition probability matrix:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.75 & 0 & 0.25 & 0 \\ 0 & 0.25 & 0 & 0.75 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

a. Put the matrix in the canonical form $P \begin{bmatrix} I & 0 \\ R & Q \end{bmatrix}$.
b. Calculate the expected absorption times μ_2 and μ_3.

3.8 Consider the following transition probability matrix:

$$P = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.3 & 0.3 & 0.4 \\ 0.25 & 0.5 & 0.25 \end{bmatrix}$$

a. Calculate $f_{13}(4)$, the probability of first passage from state 1 to state 3 in four transitions.
b. Calculate the mean sojourn time in state 2.

3.9 Let $\{X_n\}$ be a Markov chain with the state space $\{1, 2, 3\}$ and transition probability matrix

$$P = \begin{bmatrix} 0 & 0.4 & 0.6 \\ 0.25 & 0.75 & 0 \\ 0.4 & 0 & 0.6 \end{bmatrix}$$

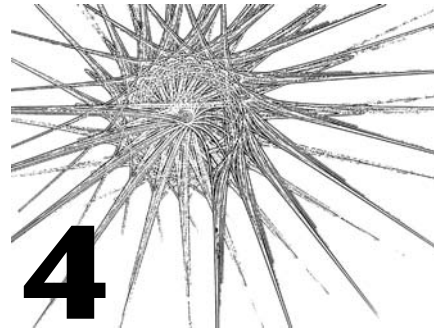
Let the initial distribution be $p(0) = [p_1(0), p_2(0), p_3(0)] = [0.4, 0.2, 0.4]$. Calculate the following probabilities:

a. $P[X_1 = 2, X_2 = 2, X_3 = 1 | X_0 = 1]$
b. $P[X_1 = 2, X_2 = 2, X_3 = 1]$
c. $P[X_1 = 2, X_4 = 2, X_6 = 2]$

3.10 On a given day Mark is either cheerful, so-so, or glum. Given that he is cheerful on a given day, then he will be cheerful the next day with probability 0.6, so-so with probability 0.2, and glum with probability 0.2. Given that he is so-so on a given day, then he will be cheerful the next day with probability 0.3, so-so with probability 0.5, and glum with probability 0.2. Given that he is glum on a given day, then he will be so-so the next day with probability 0.5, and glum with probability 0.5. Let state 1 denote the cheerful state, state 2 denote the so-so state, and state 3 denote the glum state. Let X_n denote Mark's mood on the nth day, then $\{X_n{:}n = 0, 1, 2, \ldots\}$ is a three-state Markov chain.

a. Draw the state-transition diagram of the process.
b. Give the state-transition probability matrix.
c. Given that Mark was so-so on Monday, what is the probability that he will be cheerful on Wednesday and Friday and glum on Sunday?
d. On the long run, what proportion of time is Mark in each of his three moods?

Continuous-Time Markov Chains



4

4.1 Introduction

A stochastic process $\{X(t)|t \geq 0\}$ is a continuous-time Markov chain if, for all $s, t \geq 0$ and nonnegative integers i, j, k,

$$P[X(t+s) = j|X(s) = i, X(u) = k, 0 \leq u \leq s] = P[X(t+s) = j|X(s) = i]$$

This means that in a continuous-time Markov chain the conditional probability of the future state at time $t + s$ given the present state at s and all past states depends only on the present state and is independent of the past. If in addition $P[X(t+s) = j|X(s) = i]$ is independent of s, then the process $\{X(t)|t \geq 0\}$ is said to be *time homogeneous* or have the *time homogeneity property*. Time homogeneous Markov chains have stationary (or homogeneous) transition probabilities. Let

$$p_{ij}(t) = P[X(t+s) = j|X(s) = i]$$

$$p_j(t) = P[X(t) = j]$$

That is, $p_{ij}(t)$ is the probability that a Markov chain that is presently in state i will be in state j after an additional time t, and $p_j(t)$ is the probability that a Markov chain is in state j at time t. Thus, the $p_{ij}(t)$ are the *transition probability*

functions that satisfy the condition $0 \leq p_{ij}(t) \leq 1$. Also,

$$\sum_j p_{ij}(t) = 1$$

$$\sum_j p_j(t) = 1$$

The last equation follows from the fact that at any given time the process must be in some state. Furthermore,

$$\begin{aligned}
p_{ij}(t+s) &= \sum_k P[X(t+s) = j, X(t) = k | X(0) = i] \\
&= \sum_k \frac{P[X(0) = i, X(t) = k, X(t+s) = j]}{P[X(0) = i]} \\
&= \sum_k \left\{ \frac{P[X(0) = i, X(t) = k]}{P[X(0) = i]} \right\} \left\{ \frac{P[X(0) = i, X(t) = k, X(t+s) = j}{P[X(0) = i, X(t) = k]} \right\} \\
&= \sum_k P[X(t) = k | X(0) = i] P[X(t+s) = j | X(0) = i, X(t) = k] \\
&= \sum_k P[X(t) = k | X(0) = i] P[X(t+s) = j | X(t) = k] \\
&= \sum_k p_{ik}(t) p_{kj}(s)
\end{aligned}$$

This equation is called the Chapman-Kolmogorov equation for the continuous-time Markov chain. Note that the second to last equation is due to the Markov property. If we define $P(t)$ as the matrix of the $p_{ij}(t)$, that is,

$$P(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) & p_{13}(t) & \cdots \\ p_{21}(t) & p_{22}(t) & p_{23}(t) & \cdots \\ p_{31}(t) & p_{32}(t) & p_{33}(t) & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

then the Chapman-Kolmogorov equation becomes

$$P(t+s) = P(t)P(s)$$

Whenever a continuous-time Markov chain enters a state i, it spends an amount of time called the *dwell time* (or *holding time*) in that state. The holding time

in state i is exponentially distributed with mean $1/\nu_i$. At the expiration of the holding time the process makes a transition to another state j with probability p_{ij}, where

$$\sum_j p_{ij} = 1$$

Because the mean holding time in state i is $1/\nu_i$, ν_i represents the rate at which the process leaves state i, and $\nu_i p_{ij}$ represents the rate when in state i that the process makes a transition to state j. Also, because the holding times are exponentially distributed, the probability that when the process is in state i a transition to state $j \neq i$ will take place in the next small time Δt is $p_{ij}\nu_i \Delta t$. The probability that no transition out of state i will take place in Δt given that the process is presently in state i is $1 - \sum_{j\neq i} p_{ij}\nu_i \Delta t$, and $\sum_{i\neq i} p_{ij}\nu_i \Delta t$ is the probability that it leaves state i in Δt.

With these definitions we consider the state-transition diagram for the process, which is shown in Figure 4.1 for state i. We consider the transition equations for state i for the small time interval Δt.

From Figure 4.1, we obtain the following equation:

$$p_i(t + \Delta t) = p_i(t) \left\{ 1 - \sum_{j\neq i} p_{ij}\nu_i \Delta t \right\} + \sum_{j\neq i} p_j(t)\, p_{ji}\nu_j \Delta t$$

$$p_i(t + \Delta t) - p_i(t) = -p_i(t) \sum_{j\neq i} p_{ij}\nu_i \Delta t + \sum_{j\neq i} p_j(t)\, p_{ji}\nu_j \Delta t$$

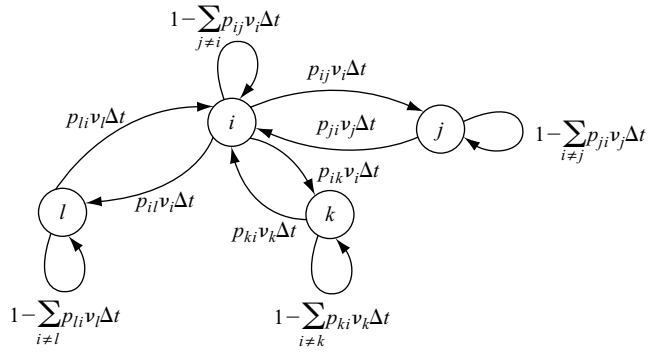


Figure 4.1. State-transition diagram for state i over small time Δt.

$$\frac{p_i(t+\Delta t) - p_i(t)}{\Delta t} = -\nu_i p_i(t) \sum_{j \neq i} p_{ij} + \sum_{j \neq i} p_j(t) p_{ji} \nu_j$$

$$\lim_{\Delta t \to 0} \left\{ \frac{p_i(t+\Delta t) - p_i(t)}{\Delta t} \right\} = \frac{dp_i(t)}{dt} = -\nu_i p_i(t) \sum_{j \neq i} p_{ij} + \sum_{j \neq i} p_j(t) p_{ji} \nu_j$$

In the steady state, $p_j(t) \to p_j$ and

$$\lim_{t \to \infty} \left\{ \frac{dp_i(t)}{dt} \right\} = 0$$

Thus, we obtain

$$0 = -\nu_i p_i \sum_{j \neq i} p_{ij} + \sum_{j \neq i} p_j p_{ji} \nu_j$$

$$1 = \sum_i p_i$$

Alternatively, we can write

$$\nu_i p_i \sum_{j \neq i} p_{ij} = \sum_{j \neq i} p_j p_{ji} \nu_j$$

$$1 = \sum_i p_i$$

The left side of the first equation is the rate of transition out of state i, while the right side is the rate of transition into state i. This "balance" equation states that in the steady state the two rates are equal for any state in the Markov chain.

4.2 Transient Analysis

Recall the following transition equation for state i that we developed earlier:

$$\frac{dp_i(t)}{dt} = -\nu_i p_i(t) \sum_{j \neq i} p_{ij} + \sum_{j \neq i} p_j(t) p_{ji} \nu_j$$

We define the following parameters:

$$q_{ji} = p_{ji}\nu_j$$

$$q_i = \nu_i \sum_{j \neq i} p_{ij} = \sum_{j \neq i} q_{ij}$$

Then the above equation becomes

$$\frac{dp_i(t)}{dt} = -q_i p_i(t) + \sum_{j \neq i} p_j(t) q_{ji}$$

We further define the following vectors and matrix:

$$p(t) = [p_1(t), p_2(t), p_3(t), \ldots]$$

$$\frac{dp(t)}{dt} = \left[\frac{dp_1(t)}{dt}, \frac{dp_2(t)}{dt}, \frac{dp_3(t)}{dt}, \ldots\right]$$

$$Q = \begin{bmatrix} -q_1 & q_{12} & q_{13} & q_{14} & \cdots \\ q_{21} & -q_2 & q_{23} & q_{24} & \cdots \\ q_{31} & q_{32} & -q_3 & q_{34} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Then the equation becomes

$$\frac{dp(t)}{dt} = p(t)Q$$

Q is usually called the *infinitesimal generator matrix* (or the *intensity matrix*). Under the initial condition that $p(0) = I$, where I is the identity matrix, the solution to this matrix equation is

$$p(t) = e^{Qt} = I + \sum_{k=1}^{\infty} \frac{Q^k t^k}{k!}$$

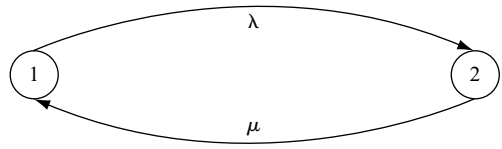


Figure 4.2. State-transition-rate diagram for Example 4.1.

Example 4.1 Find the transition probability functions for the 2-state Markov chain shown in Figure 4.2. ■

Solution: In this example, $\nu_1 = \lambda$ and $\nu_2 = \mu$. Also, the Q-matrix is given by

$$Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

Thus,

$$Q^2 = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix} \times \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix} = \begin{bmatrix} \lambda^2 + \lambda\mu & -\lambda^2 - \lambda\mu \\ -\mu^2 - \lambda\mu & \mu^2 + \lambda\mu \end{bmatrix}$$

$$= -(\lambda + \mu) \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix} = -(\lambda + \mu) Q$$

$$Q^3 = Q \times Q^2 = -(\lambda + \mu) \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix} \times \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix} = (\lambda + \mu)^2 \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

$$= (\lambda + \mu)^2 Q$$

$$\cdots$$

$$Q^k = [-(\lambda + \mu)]^{k-1} Q$$

With this we obtain

$$p(t) = I + \sum_{k=1}^{\infty} \frac{[-(\lambda + \mu)]^{k-1} Q t^k}{k!} = I - \frac{1}{\lambda + \mu} \sum_{k=1}^{\infty} \frac{[-(\lambda + \mu)t]^k}{k!} Q$$

$$= I - \frac{1}{\lambda + \mu} \{e^{-(\lambda+\mu)t} - 1\} Q = I + \frac{1}{\lambda + \mu} Q - \frac{1}{\lambda + \mu} P e^{-(\lambda+\mu)t}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{-\lambda}{\lambda + \mu} & \frac{\lambda}{\lambda + \mu} \\ \frac{\mu}{\lambda + \mu} & \frac{-\mu}{\lambda + \mu} \end{bmatrix} - \begin{bmatrix} \frac{-\lambda}{\lambda + \mu} & \frac{\lambda}{\lambda + \mu} \\ \frac{\mu}{\lambda + \mu} & \frac{-\mu}{\lambda + \mu} \end{bmatrix} e^{-(\lambda+\mu)t}$$

$$= \begin{bmatrix} \dfrac{\mu}{\lambda+\mu} & \dfrac{\lambda}{\lambda+\mu} \\ \dfrac{\mu}{\lambda+\mu} & \dfrac{\lambda}{\lambda+\mu} \end{bmatrix} + \begin{bmatrix} \dfrac{\lambda}{\lambda+\mu} & \dfrac{-\lambda}{\lambda+\mu} \\ \dfrac{-\mu}{\lambda+\mu} & \dfrac{\mu}{\lambda+\mu} \end{bmatrix} e^{-(\lambda+\mu)t}$$

Continuing the discussion, in the limit as t becomes very large, we have that

$$\lim_{t\to\infty} p(t) = p$$

$$\lim_{t\to\infty} \frac{dp(t)}{dt} = 0$$

$$pQ = 0$$

$$pe^T = 1$$

where $e^T = [1, 1, 1, \ldots]^T$ and p is the vector of the limiting-state probabilities.

Another method of analysis is via the s-transform. Consider again the equation

$$\frac{dp(t)}{dt} = p(t)Q$$

Let $M_{P(t)}(s)$ denote the s-transform of $p(t)$. Then taking the s-transform of both sides of the equation we obtain

$$sM_{P(t)}(s) - p(0) = M_{P(t)}(s)Q \Rightarrow M_{P(t)}(s)[sI - Q] = p(0)$$

From this we obtain

$$M_{P(t)}(s) = p(0)[sI - Q]^{-1}$$

The matrix $[sI - Q]^{-1}$ is generally of the form

$$[sI - Q]^{-1} = \frac{1}{s}p + T(s)$$

where p is the matrix of the limiting state probabilities and $T(s)$ represents transient components of the form e^{-qt}, te^{-qt}, t^2e^{-qt}, and so on. These transient components vanish as t goes to infinity.

For example, recalling Example 4.1 we have that

$$Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

$$sI - Q = \begin{bmatrix} s+\lambda & -\lambda \\ -\mu & s+\mu \end{bmatrix}$$

$$|sI - Q| = s(s+\lambda+\mu)$$

$$[sI - Q]^{-1} = \frac{1}{s(s+\lambda+\mu)} \begin{bmatrix} s+\mu & \lambda \\ \mu & s+\lambda \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\mu/(\lambda+\mu)}{s} + \dfrac{\lambda/(\lambda+\mu)}{s+\lambda+\mu} & \dfrac{\lambda/(\lambda+\mu)}{s} - \dfrac{\lambda/(\lambda+\mu)}{s+\lambda+\mu} \\ \dfrac{\mu/(\lambda+\mu)}{s} - \dfrac{\mu/(\lambda+\mu)}{s+\lambda+\mu} & \dfrac{\lambda/(\lambda+\mu)}{s} + \dfrac{\mu/(\lambda+\mu)}{s+\lambda+\mu} \end{bmatrix}$$

$$= \frac{1}{s} \begin{bmatrix} \dfrac{\mu}{\lambda+\mu} & \dfrac{\lambda}{\lambda+\mu} \\ \dfrac{\mu}{\lambda+\mu} & \dfrac{\lambda}{\lambda+\mu} \end{bmatrix} + \frac{1}{s+\lambda+\mu} \begin{bmatrix} \dfrac{\lambda}{\lambda+\mu} & -\dfrac{\lambda}{\lambda+\mu} \\ -\dfrac{\mu}{\lambda+\mu} & \dfrac{\mu}{\lambda+\mu} \end{bmatrix}$$

If, as before, we assume that $p(0) = I$, we obtain

$$p(t) = \begin{bmatrix} \dfrac{\mu}{\lambda+\mu} & \dfrac{\lambda}{\lambda+\mu} \\ \dfrac{\mu}{\lambda+\mu} & \dfrac{\lambda}{\lambda+\mu} \end{bmatrix} + \begin{bmatrix} \dfrac{\lambda}{\lambda+\mu} & -\dfrac{\lambda}{\lambda+\mu} \\ -\dfrac{\mu}{\lambda+\mu} & \dfrac{\mu}{\lambda+\mu} \end{bmatrix} e^{-(\lambda+\mu)t}$$

which is the same result we obtained earlier.

4.3 Birth and Death Processes

Birth and death processes are a special type of continuous-time Markov chain. Consider a continuous-time Markov chain with states 0, 1, 2, ... If $p_{ij} = 0$ whenever $j \neq i - 1$ or $j \neq i + 1$, then the Markov chain is called a birth and

death process. Thus, a birth and death process is a continuous-time Markov chain with states 0, 1, 2, ..., in which transitions from state i can only go to either state $i+1$ or state $i-1$. That is, a transition either causes an increase in state by one or a decrease in state by one. A birth is said to occur when the state increases by one, and a death is said to occur when the state decreases by one. For a birth and death process, we define the following *transition rates* from state i:

$$\lambda_i = \nu_i p_{i(i+1)}$$

$$\mu_i = \nu_i p_{i(i-1)}$$

Thus, λ_i is the rate at which a birth occurs when the process is in state i, and μ_i is the rate at which a death occurs when the process is in state i. The sum of these two rates is $\lambda_i + \mu_i = \nu_i$, which is the rate of transition out of state i. The *state-transition-rate diagram* of a birth and death process is shown in Figure 4.3. It is called a state-transition-rate diagram as opposed to a state-transition diagram because it shows the rate at which the process moves from state to state and not the probability of moving from one state to another. Note that $\mu_0 = 0$, because there can be no death when the process is in empty state.

The actual state-transition probabilities when the process is in state i are $p_{i(i+1)}$ and $p_{i(i-1)}$. By definition, $p_{i(i+1)} = \lambda_i/(\lambda_i + \mu_i)$ is the probability that a birth occurs before a death when the process is in state i. Similarly, $p_{i(i-1)} = \mu_i/(\lambda_i + \mu_i)$ is the probability that a death occurs before a birth when the process is in state i.

Recall that the rate at which the probability of the process being in state i changes with time is given by

$$\frac{dp_i(t)}{dt} = -\nu_i p_i(t) \sum_{j \neq i} p_{ij} + \sum_{j \neq i} p_j(t) p_{ji} \nu_j$$

$$= -(\lambda_i + \mu_i) p_i(t) + \mu_{i+1} p_{i+1}(t) + \lambda_{i-1} p_{i-1}(t)$$

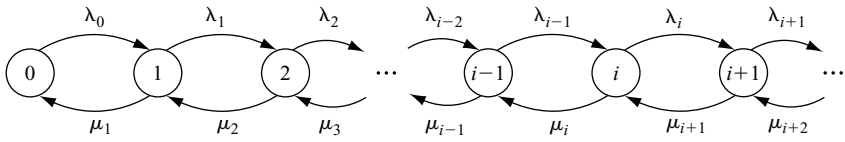


Figure 4.3. State-transition-rate diagram for birth and death process.

Thus, for the birth and death process we have that

$$\frac{dp_0(t)}{dt} = -\lambda_0 p_0(t) + u_1 p_1(t)$$

$$\frac{dp_i(t)}{dt} = -(\lambda_i + \mu_i) p_i(t) + \mu_{i+1} p_{i+1}(t) + \lambda_{i-1} p_{i-1}(t), \qquad i > 0$$

In the steady state,

$$\lim_{t\to\infty} \left\{ \frac{dp_i(t)}{dt} \right\} = 0$$

If we assume that the limiting probabilities $\lim_{t\to\infty} p_{ij}(t) = p_j$ exist, then from the preceding equation we obtain the following:

$$\lambda_0 p_0 = \mu_1 p_1$$

$$(\lambda_i + \mu_i) p_i = \mu_{i+1} p_{i+1} + \lambda_{i-1} p_{i-1}, i = 1, 2, \ldots$$

$$\sum_i p_i = 1$$

The equation states that the rate at which the process leaves state i either through a birth or a death is equal to the rate at which it enters the state through a birth when the process is in state $i-1$ or through a death when the process is in state $i+1$. This is called the *balance equation* because it balances (or equates) the rate at which the process enters state i with the rate at which it leaves state i.

Example 4.2 A machine is operational for an exponentially distributed time with mean $1/\lambda$ before breaking down. When it breaks down, it takes a time that is exponentially distributed with mean $1/\mu$ to repair it. What is the fraction of time that the machine is operational (or available)? ■

Solution: This is a two-state birth and death process. Let U denote the up state and D the down state. Then, the state-transition-rate diagram is shown in Figure 4.4.

Let p_U denote the steady-state probability that the process is in the operational state, and let p_D denote the steady-state probability that the process is in the down state. Then the balance equations become

$$\lambda p_U = \mu p_D$$

$$p_U + p_D = 1 \Rightarrow p_D = 1 - p_U$$

Substituting $p_D = 1 - p_U$ in the first equation gives $p_U = \mu/(\lambda + \mu)$.

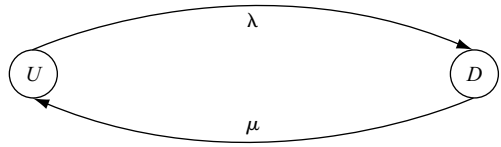


Figure 4.4. State-transition-rate diagram for Example 4.2.

Example 4.3 Customers arrive at a bank according to a Poisson process with rate λ. The time to serve each customer is exponentially distributed with mean $1/\mu$. There is only one teller at the bank, and an arriving customer who finds the teller busy when she arrives will join a single queue that operates on a first-come first-served basis. Determine the limiting-state probabilities given that $\mu > \lambda$. ■

Solution: This is a continuous-time Markov chain in which arrivals constitute births and service completions constitute deaths. Also, for all i, $\mu_i = \mu$ and $\lambda_i = \lambda$. Thus, if p_k denotes the steady-state probability that there are k customers in the system, the balance equations are as follows:

$$\lambda p_0 = \mu p_1 \Rightarrow p_1 = \left(\frac{\lambda}{\mu}\right) p_0$$

$$(\lambda + \mu) p_1 = \lambda p_0 + \mu p_2 \Rightarrow p_2 = \left(\frac{\lambda}{\mu}\right) p_1 = \left(\frac{\lambda}{\mu}\right)^2 p_0$$

$$(\lambda + \mu) p_2 = \lambda p_1 + \mu p_3 \Rightarrow p_3 = \left(\frac{\lambda}{\mu}\right) p_2 = \left(\frac{\lambda}{\mu}\right)^3 p_0$$

Similarly, it can be shown that

$$p_k = \left(\frac{\lambda}{\mu}\right)^k p_0 \qquad k = 0, 1, 2, \ldots$$

Now,

$$\sum_{k=0}^{\infty} p_k = 1 = p_0 \sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k = \frac{p_0}{1 - \frac{\lambda}{\mu}}$$

Thus,

$$p_0 = 1 - \frac{\lambda}{\mu}$$

$$p_k = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^k \qquad k = 0, 1, 2, \ldots$$

4.3.1 Local Balance Equations

Recall that the steady-state solution of the birth and death process is given by

$$\lambda_0 p_0 = \mu_1 p_1$$

$$(\lambda_i + \mu_i) p_i = \mu_{i+1} p_{i+1} + \lambda_{i-1} p_{i-1}, \quad i = 1, 2, \ldots$$

$$\sum_i p_i = 1$$

For $i = 1$, we obtain $(\lambda_1 + \mu_1) p_1 = \mu_2 p_2 + \lambda_0 p_0$. Because we know from the first equation that $\lambda_0 p_0 = \mu_1 p_1$, this equation becomes

$$\lambda_1 p_1 = \mu_2 p_2$$

Similarly, for $i = 2$, we have that $(\lambda_2 + \mu_2) p_2 = \mu_3 p_3 + \lambda_1 p_2$. Applying the last result we obtain

$$\lambda_2 p_2 = \mu_3 p_3$$

Repeated application of this method yields the general result

$$\lambda_i p_i = \mu_{i+1} p_{i+1} \qquad i = 0, 1, \ldots$$

This result states that when the process is in the steady state, the rate at which it makes a transition from state i to state $i + 1$, which we refer to the rate of flow from state i to state $i + 1$, is equal to the rate of flow from state $i + 1$ to state i. This property is referred to as *local balance* condition. Recall that it is an application of the reversibility property discussed in Chapter 3. Direct application of the property allows us to solve for the steady-state probabilities of the birth and death process recursively as follows:

$$p_{i+1} = \frac{\lambda_i}{\mu_{i+1}} p_i$$

$$= \frac{\lambda_i \lambda_{i-1}}{\mu_{i+1} \mu_i} p_{i-1}$$

$$\ldots$$

$$= \frac{\lambda_i \lambda_{i-1} \ldots \lambda_0}{\mu_{i+1} \mu_i \ldots \mu_1} p_0$$

$$1 = p_0 \left[1 + \sum_{i=1}^{\infty} \frac{\lambda_i \lambda_{i-1} \dots \lambda_0}{\mu_{i+1} \mu_i \dots \mu_1} \right]$$

$$p_0 = \left[1 + \sum_{i=1}^{\infty} \frac{\lambda_i \lambda_{i-1} \dots \lambda_0}{\mu_{i+1} \mu_i \dots \mu_1} \right]^{-1}$$

$$p_i = \frac{\lambda_i \lambda_{i-1} \dots \lambda_0}{\mu_{i+1} \mu_i \dots \mu_1} \left[1 + \sum_{i=1}^{\infty} \frac{\lambda_i \lambda_{i-1} \dots \lambda_0}{\mu_{i+1} \mu_i \dots \mu_1} \right]^{-1}, \quad i \geq 1$$

When $\lambda_i = \lambda$ for all i and $\mu_i = \mu$ for all i, we obtain the result

$$p_0 = \left[1 + \sum_{i=1}^{\infty} \left(\frac{\lambda}{\mu} \right)^i \right]^{-1}$$

The sum converges if and only if $\lambda/\mu < 1$, which is equivalent to the condition that $\lambda < \mu$. Under this condition we obtain the solutions

$$p_0 = 1 - \frac{\lambda}{\mu}$$

$$p_i = \left(1 - \frac{\lambda}{\mu} \right) \left(\frac{\lambda}{\mu} \right)^i, \quad i \geq 1$$

In Chapter 5, we will refer to this special case of the birth and death process as an M/M/1 queueing system.

4.3.2 Transient Analysis of Birth and Death Processes

Recall that

$$\frac{dp(t)}{dt} = p(t) Q$$

For the birth and death process we have that

$$Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \dots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & 0 & \dots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & 0 & \dots \\ 0 & 0 & \mu_3 & -(\lambda_3 + \mu_3) & \lambda_3 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

From this we obtain the following system of differential equations:

$$\frac{dp_0(t)}{dt} = -\lambda_0 p_0(t) + \mu_1 p_1(t)$$

$$\frac{dp_i(t)}{dt} = -(\lambda_i + \mu_i) p_i(t) + \mu_{i+1} p_{i+1}(t) + \lambda_{i-1} p_{i-1}(t), \quad i > 0$$

4.4 First Passage Time

Consider the continuous-time Markov chain $\{X(t)|t \geq 0\}$ with state space $1, 2, \ldots$. The first passage time T_k into state k is defined as follows:

$$T_k = \min\{t \geq 0 | X(t) = k\}$$

Let m_{ik} be defined as follows:

$$m_{ik} = E[T_k | X(0) = i]$$

That is, m_{ik} is the mean first passage time to state k given that the process started in state i. It can be shown that if ν_i is the total rate of transition out of state i, and ν_{ij} is the rate of transition from state i to state j, then in a manner similar to the discrete-time case,

$$\nu_i m_{ik} = 1 + \sum_{j \neq k} \nu_{ij} m_{jk}$$

The intuitive meaning of this equation can be understood from recognizing the fact that $1/\nu_i$ is the mean holding time (or mean waiting time) in state i. Thus, given that the process started in state i, it would spend a mean time of $1/\nu_i$ in that state and then move into state j with probability $p_{ij} = \nu_{ij}/\nu_i$. Then from state j it takes a mean time of m_{jk} to reach state k. Thus, the equation can be rearranged as follows:

$$m_{ik} = \frac{1}{\nu_i} + \sum_{j \neq k} \frac{\nu_{ij}}{\nu_i} m_{jk} = \frac{1}{\nu_i} + \sum_{j \neq k} p_{ij} m_{jk}$$

This form of the equation is similar to that for the mean first passage time for the discrete-time Markov chain that is discussed in Chapter 3.

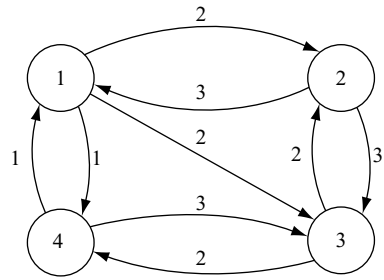


Figure 4.5. State-transition-rate diagram for Example 4.4.

Example 4.4 Consider the Markov chain whose state-transition-rate diagram is given in Figure 4.5. Find m_{14}. ■

Solution: Because the transition rates v_{ik} are specified in the figure, we first obtain the rates v_i, which are as follows:

$$v_1 = 1 + 2 + 2 = 5$$

$$v_2 = 3 + 3 = 6$$

$$v_3 = 2 + 2 = 4$$

$$v_4 = 1 + 3 = 4$$

Thus,

$$5m_{14} = 1 + m_{14} + 2m_{24} + 2m_{34}$$

$$6m_{24} = 1 + 3m_{14} + 3m_{34}$$

$$4m_{34} = 1 + 2m_{24}$$

The solution to the system of equations is

$$m_{14} = 1.3333$$

$$m_{24} = 1.2778$$

$$m_{34} = 0.8889$$

Thus, it takes 1.3333 units of time to go from state 1 to state 4.

4.5 The Uniformization Method

Uniformization is fundamentally used to transform a continuous-time Markov chain into a discrete-time analog that is more amenable to numerical calculation with respect to transient solutions. Uniformization is sometimes called *randomization*, and the method has probabilistic interpretations that can be used to derive various results.

In our discussion on the continuous-time Markov chain, we have assumed that a transition from any state to itself is not allowed; that is, we assumed that $p_{ii} = 0$ for all i. The uniformization method allows us to remove this restriction. The method works by making the total transition rates ν_i the same for all i (that is, it *uniformizes* the transition rates). It chooses a rate ν such that

$$\nu_i \leq \nu \qquad \forall i$$

For example, a good choice of ν is

$$\nu = \max_i\{\nu_i\}$$

Recall that when the process is in state i, it leaves the state at the rate ν_i. However, because we have augmented the rate to ν, we now suppose that the residual rate $\nu - \nu_i$ is the fictitious rate at which it makes a transition from state i back to state i. Thus, we have a Markov chain $\{\hat{X}(t)\}$ that spends an exponentially distributed amount of time with mean $1/\nu$ in state i and then makes transitions with probabilities that are governed by an imbedded discrete-time Markov chain $\{\hat{X}_n\}$ whose transition probabilities $\hat{p}_{ij}$ are defined as follows:

$$\hat{p}_{ij} = \begin{cases} 1 - \dfrac{\nu_i}{\nu} & j = i \\ \dfrac{\nu_i}{\nu} p_{ij} & j \neq i \end{cases}$$

One of the applications of the uniformization method is in computation of the transition probability functions $p_{ij}(t)$. Let $\hat{P}$ denote the transition probability matrix of $\{\hat{X}_n\}$; that is,

$$\hat{P} = \begin{bmatrix} \hat{p}_{11} & \hat{p}_{12} & \hat{p}_{13} & \cdots \\ \hat{p}_{21} & \hat{p}_{22} & \hat{p}_{23} & \cdots \\ \hat{p}_{31} & \hat{p}_{32} & \hat{p}_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Then the transition probability functions are given by

$$p_{ij}(t) = \sum_{n=0}^{\infty} e^{-\nu t} \left\{ \frac{(\nu t)^n}{n!} \right\} \hat{p}_{ij}^n$$

where $\hat{p}_{ij}^n$ is the ijth entry of the n-step transition probability of $\{\hat{X}_n\}$. This provides another way to obtain the transition probability functions $p_{ij}(t)$. In the matrix form this becomes

$$p(t) = \sum_{n=0}^{\infty} e^{-\nu t} \left\{ \frac{(\nu t)^n}{n!} \right\} \hat{P}^n$$

4.6 Reversible Continuous-Time Markov Chains

A continuous-time Markov chain $\{X(t), -\infty < t < \infty\}$ is said to be a reversible Markov chain if for any fixed τ and integer $n \geq 0$ the sequence of states $X(t_1), X(t_2), \ldots, X(t_n)$ has the same probabilistic structure as the sequence $X(\tau - t_1), X(\tau - t_2), \ldots, X(\tau - t_n)$. As discussed in Chapter 3, this means that a sequence of states when looked at backward in time has the same structure as the sequence running forward in time. As in the discrete-time analog discussed in Chapter 3, a continuous-time Markov chain is reversible if

$$\nu_{ij} p_i = \nu_{ji} p_j$$

As we discussed earlier, in the steady state the local balance condition for a birth and death process $\{X(t), t \geq 0\}$ states that

$$\lambda_i p_i = \mu_{i+1} p_{i+1} \qquad i = 0, 1, \ldots$$

Because for a birth-and-death process $\nu_{ij} = \lambda_i$ and $\nu_{ji} = \mu_{i+1}$, we observe that the reversibility condition has been met. Thus, a birth-and-death process is a reversible continuous-time Markov chain.

4.7 Problems

4.1 A small company has two identical PCs that are running at the same time. The time until either PC fails is exponentially distributed with a mean of $1/\lambda$. When a PC fails, a technician starts repairing it immediately. The two PCs fail independently of each other. The time to repair a failed PC is exponentially distributed with a mean of $1/\mu$. As soon as the repair is completed, the PC is brought back online and is assumed to be as good as new.

a. Give the state-transition-rate diagram of the process.
b. What is the fraction of time that both machines are down?

4.2 Customers arrive at Mike's barber shop according to a Poisson process with rate λ customers per hour. Unfortunately Mike, the barber, has only five chairs in his shop for customers to wait when there is already a customer receiving a haircut. Customers who arrive when Mike is busy and all the chairs are occupied leave without waiting for a haircut. Mike is the only barber in the shop, and the time to complete a haircut is exponentially distributed with a mean of $1/\mu$ hours.

a. Give the state-transition-rate diagram of the process.
b. What is the probability that there are three customers waiting in the shop?
c. What is the probability that an arriving customer leaves without receiving a haircut?
d. What is the probability that an arriving customer does not have to wait?

4.3 A small company has two PCs A and B. The time to failure for PC A is exponentially distributed with a mean of $1/\lambda_A$ hours, and the time to failure for PC B is exponentially distributed with a mean of $1/\lambda_B$ hours. The PCs also have different repair times. The time to repair PC A when it fails is exponentially distributed with a mean of $1/\mu_A$ hours, and the time to repair PC B when it fails is exponentially distributed with a mean of $1/\mu_B$ hours. There is only one repair person available to work on both machines when failure occurs, and each machine is considered to be as good as new after it has been repaired.

a. Give the state-transition-rate diagram of the process.
b. What is the probability that both PCs are down?
c. What is the probability that PC A is the first to fail given that both PCs have failed?
d. What is the probability that both PCs are up?

4.4 Lazy Chris has three identical lightbulbs in his living room that he keeps on all the time. Because of his laziness, Chris does not replace a lightbulb when it fails. (Maybe Chris does not even notice that the bulb has failed!) However, when all three bulbs have failed, Chris replaces them at the same time. The lifetime of each bulb is exponentially distributed with a mean of $1/\lambda$, and the time to replace all three bulbs is exponentially distributed with a mean of $1/\mu$.

a. Give the state-transition-rate diagram of the process.
b. What is the probability that only one lightbulb is working?
c. What is the probability that all three lightbulbs are working?

4.5 A switchboard has two outgoing lines serving four customers who never call each other. When a customer is not talking on the phone, he or she generates calls according to a Poisson process with rate λ calls per minute. Call lengths are exponentially distributed with a mean of $1/\mu$ minutes. If a customer finds the switchboard blocked (i.e., both lines are busy) when attempting to make a call, he or she never tries to make that particular call again; that is, the call is lost.

a. Give the state-transition-rate diagram of the process.
b. What is the fraction of time that the switchboard is blocked?

4.6 A service facility can hold up to six customers who arrive according to a Poisson process with a rate of λ customers per hour. Customers who arrive when the facility is full are lost and never make an attempt to return to the facility. Whenever there are two or fewer customers in the facility, there is only one attendant serving them. The time to service each customer is exponentially distributed with a mean of $1/\mu$ hours. Whenever there are three or more customers, the attendant is joined by a colleague, and the service time is still the same for each customer. When the number of customers goes down to two, the last attendant to complete service will stop serving. Thus, whenever there are two or less customers in the facility, only one attendant can serve.

a. Give the state-transition-rate diagram of the process.
b. What is the probability that both attendants are busy attending to customers?
c. What is the probability that neither attendant is busy?

4.7 A taxicab company has a small fleet of three taxis that operate from the company's station. The time it takes a taxi to take a customer to his or her location and return to the station is exponentially distributed with a mean of $1/\mu$ hours. Customers arrive according to a Poisson process with average rate of λ customers per hour. If a potential customer arrives at the station and finds that no taxi is available, he or she goes to another taxicab company. The taxis always return to the station after dropping off a customer without picking up any new customers on their way back.

a. Give the state-transition-rate diagram of the process.
b. What is the probability that an arriving customer sees exactly one taxi at the station?
c. What is the probability that an arriving customer goes to another taxicab company?

4.8 Consider a collection of particles that act independently in giving rise to succeeding generations of particles. Suppose that each particle, from the time it appears, waits a length of time that is exponentially distributed

with a mean of $1/\lambda$ and then either splits into two identical particles with probability p or disappears with probability $1 - p$. Let $X(t)$, $0 \leq t < \infty$, denote the number of particles that are present at time t.

a. Find the birth and death rates of the process.
b. Give the state-transition-rate diagram of the process.

4.9 An assembly line consists of two stations in tandem. Each station can hold only one item at a time. When an item is completed in station 1, it moves into station 2 if the latter is empty; otherwise it remains in station 1 until station 2 is free. Items arrive at station 1 according to a Poisson process with rate λ. However, an arriving item is accepted only if there is no other item in the station; otherwise it is lost from the system. The time that an item spends at station 1 is exponentially distributed with mean $1/\mu_1$, and the time that it spends at station 2 is exponentially distributed with mean $1/\mu_2$. Let the state of the system be defined by (m, n), where m is the number of items in station 1 and n is the number of items in station 2.

a. Give the state-transition-rate diagram of the process.
b. Calculate the limiting state probabilities p_{mn}.

4.10 Trucks bring crates of goods to a warehouse that has a single attendant. It is the responsibility of each truck driver to offload his truck, and the time that it takes to offload a truck is exponentially distributed with mean $1/\mu_1$. When a truck is offloaded, it leaves the warehouse and takes a time that is exponentially distributed with mean $1/\lambda$ to return to the warehouse with another set of crates. When a truck driver is done with offloading his truck, the warehouse attendant takes an additional time that is exponentially distributed with mean $1/\mu_2$ to arrange the crates before the next truck in line can start offloading. Assume that there are N trucks that bring crates to the warehouse. Denote the state of the system by (m, n), where m is the number of trucks in the system and n is the state of the attendant: $n = 1$ if the attendant is busy arranging the crates and $n = 0$ otherwise.

a. Formulate the problem as a continuous-time Markov chain, specifying the state variables and giving the state-transition-rate diagram of the process.
b. From the diagram identify the steady-state probability that the attendant is idle with no truck in the warehouse.

4.11 Consider a system consisting of two birth and death processes labeled system 1 and system 2. Customers arrive at system 1 according to a Poisson process with rate λ_1, and customers arrive at system 2 according to a Poisson process with rate λ_2. Each system has two identical attendants. The time it takes an attendant in system 1 to serve a customer is exponentially distributed with mean $1/\mu_1$, and the time it takes an attendant in system

2 to serve a customer is exponentially distributed with mean $1/\mu_2$. Any customer that arrives when the two attendants in its group are busy can receive service from the other group, provided that there is at least one free attendant in that group; otherwise it is lost. Let the state of the system be denoted by (m, n), where m is the number of customers in system 1 and n is the number of customers in system 2. Give the state-transition-rate diagram of the process and specify the probability that a customer receives service from a group that is different from its own group.

4.12 Cars arrive at a parking lot according to a Poisson process with rate λ. There are only four parking spaces, and any car that arrives when all the spaces are occupied is lost. The parking duration of a car is exponentially distributed with mean $1/\mu$. Let $p_k(t)$ denote the probability that k cars are parked in the lot at time t, $k = 0, 1, \ldots, 4$.

a. Give the differential equation governing $p_k(t)$.

b. What are the steady-state values of these probabilities?

c. What is m_{14}, the mean first passage time to state 4 given that the process started in state 1?

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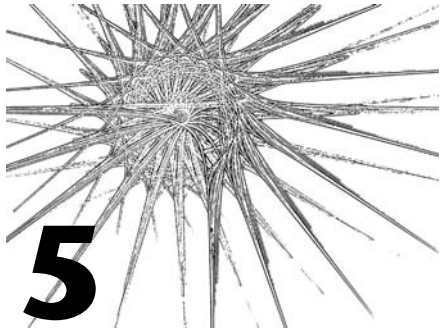
Markovian Queueing Systems



5

5.1 Introduction

A queue is a waiting line. Queues arise in many of our daily activities. For example, we join a queue to buy stamps at the post office, to cash checks or deposit money at the bank, to pay for groceries at the grocery store, to purchase tickets for movies or games, or to get a table at the restaurant. This chapter discusses a class of queueing systems called Markovian queueing systems. They are characterized by the fact that either the service times are exponentially distributed or customers arrive at the system according to a Poisson process or both. The emphasis in this chapter is on the steady-state analysis.

5.2 Description of a Queueing System

In a queueing system, *customers* from a specified *population* arrive at a *service facility* to receive service. The service facility has one or more *servers* who attend to arriving customers. If a customer arrives at the facility when all the servers are busy attending to earlier customers, the arriving customer joins the queue until a server is free. After a customer has been served, he leaves the system and will not join the queue again. That is, service with feedback is not allowed. We consider systems that obey the *work conservation rule*: A server cannot be idle when there are customers to be served. Figure 5.1 illustrates the different components of a queueing system.

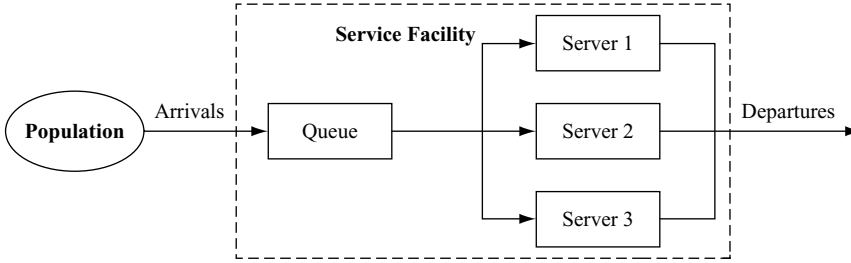


Figure 5.1. Components of a queueing system.

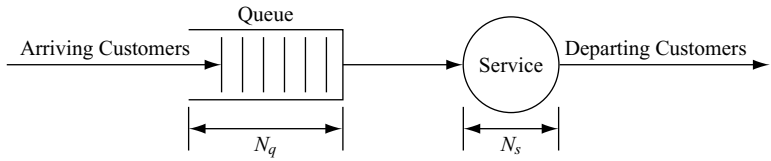


Figure 5.2. The queueing process.

When a customer arrives at a service facility, a server commences service on the customer if the server is currently idle. Otherwise, the customer joins a queue that is attended to in accordance with a specified service policy such as first-come first-served, last-come first-served, priority, etc. Thus the time a customer spends waiting for service to begin is dependent on the service policy. Also, because there can be one or more servers in the facility, more than one customer can be receiving service at the same time. The following notation is used to represent the random variables associated with a queueing system:

a. N_q is the number of customers in a queue.
b. N_s is the number of customers receiving service.
c. N is the total number of customers in the system: $N = N_q + N_s$.
d. W is the time a customer spends in a queue before going to service; W is the waiting time.
e. X is the time a customer spends in actual service.
f. T is the total time a customer spends in the system (also called the sojourn time): $T = W + X$.

Figure 5.2 is a summary of the queueing process at the service facility.
A queueing system is characterized as follows:

a. ***Population***, which is the source of the customers arriving at the service facility. The population can be finite or infinite.

b. ***Arriving pattern***, which defines the customer interarrival process.

c. ***Service time distribution***, which defines the time taken to serve each customer.

d. ***Capacity of the queueing facility***, which can be finite or infinite. If the capacity is finite, customers that arrive when the system is full are lost (or blocked). Thus, a finite-capacity system is a blocking system.

e. ***Number of servers***, which can be one or more than one. A queueing system with one server is called a single-server system; otherwise it is called a multi-server system. A single-server system can serve only one customer at a time while multi-server systems can serve multiple customers simultaneously. In a multi-server system, the servers can be identical, which means that their service rates are identical and it does not matter which server a particular customer receives service from. On the other hand, the servers can be heterogeneous in the sense that some provide faster service than others. In this case the time a customer spends in service depends on which server provides the service. A special case of a multi-server system is the infinite-server system where each arriving customer is served immediately; that is, there is no waiting in queue.

f. ***Queueing discipline***, which is also called the *service discipline*. It defines the rule that governs how the next customer to receive service is selected after a customer who is currently receiving service leaves the system. Specific disciplines that can be used include the following:

   1. First-come first-served (FCFS), which means that customers are served in the order they arrived. The discipline is also called first-in first-out (FIFO).
   2. Last-come first-served (LCFS), which means that the last customer to arrive receives service before those that arrived earlier. The discipline is also called last-in first-out (LIFO).
   3. Service in random order (SIRO), which means that the next customer to receive service after the current customer has finished receiving service will be selected in a probabilistic manner, such as tossing a coin, rolling a die, etc.
   4. Priority, which means that customers are divided into ordered classes such that a customer in a higher class will receive service before a customer in a lower class, even if the higher-class customer arrives later than the lower-class customer. There are two types of priority: *preemptive* and *nonpreemptive*. In preemptive priority, the service of a customer currently receiving service is suspended upon the arrival of a higher-priority customer; the latter goes straight to receive service. The preempted customer goes in to receive service upon the completion of service of the higher-priority customer, if no higher-priority customer arrived while the high-priority customer was being served. How the service of a preempted

customer is continued when the customer goes to complete his service depends on whether we have a *preemptive repeat* or *preemptive resume* policy. In preemptive repeat, the customer's service is started from the beginning when the customer enters to receive service again, regardless of how many times the customer is preempted. In preemptive resume, the customer's service continues from where it stopped before being preempted. Under nonpreemptive priority, an arriving high-priority customer goes to the head of the queue and waits for the current customer's service to be completed before he enters to receive service ahead of other waiting lower-priority customers.

Thus, the time a customer spends in the system is a function of the preceding parameters and service policies.

5.3 The Kendall Notation

The Kendall notation is a shorthand notation that is used to describe queueing systems. It is written in the form:

A/B/c/D/E/F

- "A" describes the arrival process (or the interarrival time distribution), which can be an exponential or nonexponential (i.e., general) distribution.
- "B" describes the service time distribution.
- "c" describes the number of servers.
- "D" describes the system capacity, which is the maximum number of customers allowed in the system including those currently receiving service; the default value is infinity.
- "E" describes the size of the population from where arrivals are drawn; the default value is infinity.
- "F" describes the queueing (or service) discipline. The default is FCFS.
- When default values of D, E, and F are used, we use the notation A/B/c, which means a queueing system with infinite capacity, customers arrive from an infinite population and are served in an FCFS manner. Symbols traditionally used for A and B are:
  - GI, which stands for general independent interarrival time.
  - G, which stands for general service time distribution.
  - M, which stands for memoryless (or exponential) interarrival time or service time distribution. Note that an exponentially distributed interarrival time means that customers arrive according to a Poisson process.

- D, which stands for deterministic (or constant) interarrival time or service time distribution.

For example, we can have queueing systems of the following form:

- M/M/1 queue, which is a queueing system with exponentially distributed interarrival time, exponentially distributed service time, a single server, infinite capacity, customers are drawn from an infinite population, and service is on a FCFS basis.
- M/D/1 queue, which is a queueing system with exponentially distributed interarrival time, constant service time, a single server, infinite capacity, customers are drawn from an infinite population, and service is on an FCFS basis.
- M/G/3/20 queue, which is a queueing system with exponentially distributed interarrival time, general (or nonexponentially) distributed service time, three servers, a finite capacity of 20 (i.e., a maximum of 20 customers can be in the system, including the three that can be in service at the same time), customers are drawn from an infinite population, and service is on an FCFS basis.

5.4 The Little's Formula

The Little's formula is a statement on the relationship between the mean number of customers in the system, the mean time spent in the system, and the average rate at which customers arrive at the system. Let λ denote the mean arrival rate, $E[N]$ the mean number of customers in the system, $E[T]$ the mean total time spent in the system, $E[N_q]$ the mean number of customers in queue, and $E[W]$ the mean waiting time. Then Little's formula states that

$$E[N] = \lambda E[T]$$

which says that the mean number of customers in the system (including those currently being served) is the product of the average arrival rate and the mean time a customer spends in the system. The formula can also be stated in terms of the number of customers in queue, as follows:

$$E[N_q] = \lambda E[W]$$

which says that the mean number of customers in queue (or waiting to be served) is equal to the product of the average arrival rate and the mean waiting time.

5.5 The PASTA Property

Markovian queueing systems with Poisson arrivals possess the PASTA (Poisson Arrivals See Time Averages) property. This property, which was proposed in Wolff (1982), asserts that customers with Poisson arrivals see the system as if they arrived at an arbitrary point in time despite the fact that they induce transitions in the system. This phenomenon arises from the lack of memory of exponential interarrival times with the result that the arrival history just before a tagged arrival instant is stochastically identical to that of a random instant as well as that of the arrival instant. PASTA is a powerful tool used in the analysis of many queueing systems.

5.6 The M/M/1 Queueing System

This is the simplest queueing system in which customers arrive according to a Poisson process to a single-server service facility, and the time to serve each customer is exponentially distributed. The model also assumes the various default values: infinite capacity at the facility, customers are drawn from an infinite population, and service is on an FCFS basis.

Because we are dealing with a system that can increase or decrease by at most one customer at a time, it is a birth-and-death process with homogeneous birth rate λ and homogeneous death rate μ. This means that the mean service time is $1/\mu$. Thus, the state-transition-rate diagram is shown in Figure 5.3.

Let p_n be the limiting state probability that the process is in state n, $n = 0, 1, 2, \ldots$. Then applying the balance equations we obtain:

$$\lambda p_0 = \mu p_1 \Rightarrow p_1 = \left(\frac{\lambda}{\mu}\right) p_0 = \rho p_0$$

$$(\lambda + \mu) p_1 = \lambda p_0 + \mu p_2 \Rightarrow p_2 = \rho p_1 = \rho^2 p_0$$

$$(\lambda + \mu) p_2 = \lambda p_1 + \mu p_3 \Rightarrow p_3 = \rho p_2 = \rho^3 p_0$$

where $\rho = \lambda/\mu$. Similarly, it can be shown that

$$p_n = \rho^n p_0 \quad n = 0, 1, 2, \ldots$$

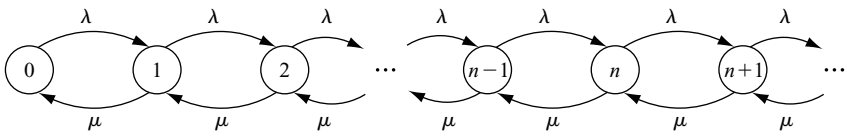


Figure 5.3. State-transition-rate diagram for M/M/1 queue.

Because

$$\sum_{n=0}^{\infty} p_n = 1 = p_0 \sum_{n=0}^{\infty} \rho^n = \frac{p_0}{1-\rho}$$

we obtain

$$p_0 = 1 - \rho$$
$$p_n = (1-\rho)\rho^n \quad n = 0, 1, 2, \ldots;\ \rho < 1$$

Because $\rho = 1 - p_0$, which is the probability that the system is not empty and hence the server is not idle, we call ρ the *server utilization* (or *utilization factor*).

The expected number of customers in the system is given by

$$E[N] = \sum_{n=0}^{\infty} n p_n = \sum_{n=0}^{\infty} n(1-\rho)\rho^n = (1-\rho)\sum_{n=0}^{\infty} n\rho^n$$

But

$$\frac{d}{d\rho}\sum_{n=0}^{\infty} \rho^n = \sum_{n=0}^{\infty} \frac{d}{d\rho}\rho^n = \sum_{n=1}^{\infty} n\rho^{n-1} = \frac{1}{\rho}\sum_{n=0}^{\infty} n\rho^n$$

Thus,

$$\sum_{n=0}^{\infty} n\rho^n = \rho\frac{d}{d\rho}\sum_{n=0}^{\infty} \rho^n = \rho\frac{d}{d\rho}\left(\frac{1}{1-\rho}\right) = \frac{\rho}{(1-\rho)^2}$$

Therefore,

$$E[N] = (1-\rho)\sum_{n=0}^{\infty} n\rho^n = (1-\rho)\frac{\rho}{(1-\rho)^2} = \frac{\rho}{1-\rho}$$

We can obtain the mean time in the system from Little's formula as follows:

$$E[T] = E[N]/\lambda = \frac{\lambda/\mu}{\lambda(1-\rho)}$$
$$= \frac{1}{\mu(1-\rho)} = \frac{E[X]}{1-\rho}$$

where the last result follows from the fact that the mean service time is $E[X] = 1/\mu$. Similarly, the mean waiting time and mean number of customers in queue are given by

$$\begin{aligned} E[W] &= E[T] - E[X] = \frac{E[X]}{1-\rho} - E[X] \\ &= \frac{\rho E[X]}{1-\rho} = \frac{\rho}{\mu(1-\rho)} \\ E[N_q] &= \lambda E[W] = \frac{\lambda\rho}{\mu(1-\rho)} \\ &= \frac{\rho^2}{1-\rho} \end{aligned}$$

Recall that the mean number of customers in service is $E[N_s]$. Using the above results we obtain

$$E[N_s] = E[N] - E[N_q] = \frac{\rho}{1-\rho} - \frac{\rho^2}{1-\rho} = \rho$$

Thus, the mean number of customers in service is ρ, the probability that the server is busy. Note that the mean waiting time, the mean time in the system, the mean number of customers in the system, and the mean number of customers in queue become extremely large as the server utilization ρ approaches one. Figure 5.4 illustrates this for the case of the expected waiting time.

Example 5.1 Students arrive at the campus post office according to a Poisson process with an average rate of one student every 4 minutes. The time required to

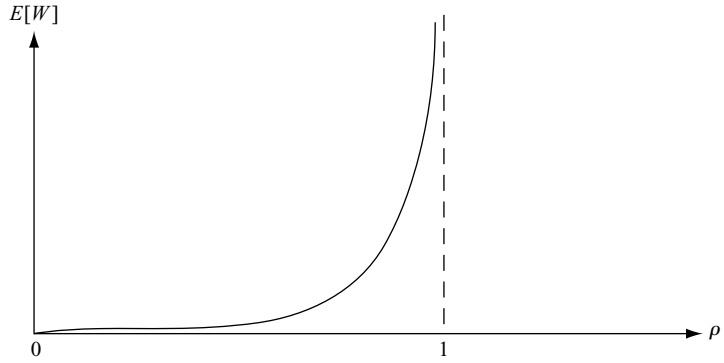


Figure 5.4. Mean waiting time versus server utilization.

serve each student is exponentially distributed with a mean of 3 minutes. There is only one postal worker at the counter, and any arriving student that finds the worker busy joins a queue that is served in an FCFS manner.

a. What is the probability that an arriving student has to wait?
b. What is the mean waiting time of an arbitrary student?
c. What is the mean number of waiting students at the post office? ■

Solution: Example 5.1 is an M/M/1 queue with the following parameters:

$$\lambda = 1/4$$
$$\mu = 1/3$$
$$\rho = \lambda/\mu = 3/4 = 0.75$$

a. P[arriving student waits] = P[server is busy] = $\rho = 0.75$
b. $E[W] = \rho E[X]/(1-\rho) = (0.75)(3)/(1-0.75) = 9$ minutes
c. $E[N_q] = \lambda E[W] = (1/4)(9) = 2.25$ students

Example 5.2 Customers arrive at a checkout counter in a grocery store according to a Poisson process with an average rate of 10 customers per hour. There is only one clerk at the counter, and the time to serve each customer is exponentially distributed with a mean of 4 minutes.

a. What is the probability that a queue forms at the counter?
b. What is the average time a customer spends at the counter?
c. What is the average queue length at the counter? ■

Solution: This is an M/M/1 queueing system. We must first convert the arrival and service rates to the same unit of customers per minute because service time is in minutes. Thus, the parameters of the system are as follows:

$$\lambda = 10/60 = 1/6$$
$$\mu = 1/4 = 1/E[X]$$
$$\rho = \lambda/\mu = 2/3$$

a. P[queue forms] = P[server is busy] = $\rho = 2/3$
b. Average time at the counter = $E[T] = E[X]/(1-\rho) = 4/(1/3) =$ 12 minutes
c. Average queue length = $E[N_q] = \rho^2/(1-\rho) = (4/9)/(1/3) = 4/3$

5.6.1 Stochastic Balance

A shortcut method of obtaining the steady-state equations in an M/M/1 queueing system is by means of a flow balance procedure called *stochastic balance*. The idea behind stochastic balance is that in any steady-state condition, the rate at which the process moves from left to right across a "probabilistic wall" is equal to the rate at which it moves from right to left across that wall. For example, Figure 5.5 shows three states $n-1$, n, and $n+1$ in the state-transition-rate diagram of an M/M/1 queue.

The rate at which the process crosses wall A from left to right is λp_{n-1}, which is true because this can only happen when the process is in state $n-1$. Similarly, the rate at which the process crosses wall A from right to left is μp_n. By stochastic balance we mean that $\lambda p_{n-1} = \mu p_n$ or $p_n = (\lambda/\mu) p_{n-1} = \rho p_{n-1}$, which is the result we obtained earlier in the analysis of the system.

5.6.2 Total Time and Waiting Time Distributions of the M/M/1 Queueing System

Consider a tagged customer, say customer k, that arrives at the system and finds it in state n. If $n > 0$, then because of the fact that the service time X is exponentially distributed, the service time of the customer receiving service when the tagged customer arrives "starts from scratch." Thus, the total time that the tagged customer spends in the system is given by

$$T_k = \begin{cases} X_k + X_1 + X_2 + \cdots + X_n & n > 0 \\ X_k & n = 0 \end{cases}$$

Because the X_k are identically distributed, the s-transform of the PDF of T_k is given by

$$M_{T|n}(s|n) = \{M_X(s)\}^{n+1} \quad n = 0, 1, \ldots$$

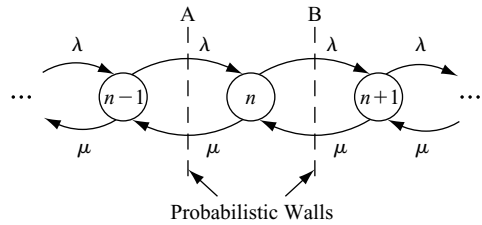


Figure 5.5. Stochastic balance concept.

Thus, the unconditionals-transform of the PDF of the total time in the system is given by

$$M_T(s) = \sum_{n=0}^{\infty} M_{T|n}(s|n)\, p_n = \sum_{n=0}^{\infty} \{M_X(s)\}^{n+1}\, p_n = M_X(s) \sum_{n=0}^{\infty} \{M_X(s)\}^n\, (1-\rho)\rho^n$$

$$= (1-\rho) M_X(s) \sum_{n=0}^{\infty} \{\rho M_X(s)\}^n = \frac{(1-\rho) M_X(s)}{1 - \rho M_X(s)}$$

Because $M_X(s) = \mu/(s+\mu)$, we obtain the following result:

$$M_T(s) = \frac{\mu(1-\rho)}{s + \mu(1-\rho)}$$

This shows that T is an exponentially distributed random variable with PDF and CDF given respectively by

$$f_T(t) = \mu(1-\rho) e^{-\mu(1-\rho)t}$$

$$F_T(t) = 1 - e^{-\mu(1-\rho)t}$$

and mean $1/\mu(1-\rho)$, as shown earlier. Similarly, the waiting time distribution can be obtained by considering the experience of the tagged customer in the system, as follows:

$$W_k = \begin{cases} X_1 + X_2 + \cdots + X_n & n > 0 \\ 0 & n = 0 \end{cases}$$

Thus, the s-transform of the PDF of W_k, given that there are n customers when the tagged customer arrives, is given by

$$M_{W|n}(s|n) = \begin{cases} \{M_X(s)\}^n & n = 1, 2, \ldots \\ 1 & n = 0 \end{cases}$$

$$= \{M_X(s)\}^n \qquad n = 0, 1, 2, \ldots$$

Therefore,

$$M_W(s) = \sum_{n=0}^{\infty} M_{W|n}(s|n)\, p_n = \sum_{n=0}^{\infty} \{M_X(s)\}^n p_n = \sum_{n=0}^{\infty} \{M_X(s)\}^n (1-\rho)\rho^n$$

$$= (1-\rho) \sum_{n=0}^{\infty} \{\rho M_X(s)\}^n = (1-\rho) \left\{ \frac{1}{1 - \rho M_X(s)} \right\} = \frac{(1-\rho)(s+\mu)}{s + \mu(1-\rho)}$$

$$= \frac{(1-\rho)\{s + \mu(1-\rho)\} + \rho\mu(1-\rho)}{s + \mu(1-\rho)}$$

$$= (1-\rho) + \frac{\rho\mu(1-\rho)}{s + \mu(1-\rho)}$$

Thus, the PDF and CDF of W are given by

$$\begin{aligned} f_W(w) &= (1-\rho)\delta(w) + \rho\mu(1-\rho)e^{-\mu(1-\rho)w} \\ &= (1-\rho)\delta(w) + \lambda(1-\rho)e^{-\mu(1-\rho)w} \\ F_W(w) &= P[W \le w] = (1-\rho) + \rho\left\{1 - e^{-\mu(1-\rho)w}\right\} \\ &= 1 - \rho e^{-\mu(1-\rho)w} \end{aligned}$$

where $\delta(w)$ is the impulse function. Observe that the mean waiting time is given by

$$E[W] = \int_0^\infty \{1 - F_W(w)\}\, dw = \int_0^\infty \rho e^{-\mu(1-\rho)w} dw = \frac{\rho}{\mu(1-\rho)}$$

which is the result we obtained earlier. An alternative method of obtaining the preceding results is as follows. Let $r_{n+1}(t)$ denote the probability that $n+1$ service completions are made in a time of less than or equal to t, given that a tagged customer found n customers in the system when it arrived. Then

$$\begin{aligned} F_T(t) = P[T \le t] &= \sum_{n=0}^{\infty} p_n r_{n+1}(t) = \sum_{n=0}^{\infty} p_n \int_0^t \frac{\mu^{n+1}x^n}{n!} e^{-\mu x} dx \\ &= \sum_{n=0}^{\infty} (1-\rho)\rho^n \int_0^t \frac{\mu^{n+1}x^n}{n!} e^{-\mu x} dx \\ &= (1-\rho)\int_0^t \left\{\sum_{n=0}^{\infty} \rho^n \frac{\mu^{n+1}x^n}{n!}\right\} e^{-\mu x} dx \\ &= \mu(1-\rho)\int_0^t \left\{\sum_{n=0}^{\infty} \frac{(\rho\mu x)^n}{n!}\right\} e^{-\mu x} dx \\ &= \mu(1-\rho)\int_0^t e^{\rho\mu x} e^{-\mu x} dx = \mu(1-\rho)\int_0^t e^{-\mu(1-\rho)x} dx \\ &= 1 - e^{-\mu(1-\rho)t} \end{aligned}$$

Similarly,

$$F_W(w) = P[W \le w] = \sum_{n=1}^{\infty} p_n r_n(t) + p_0 = \sum_{n=1}^{\infty} p_n \int_0^w \frac{\mu^n x^{n-1}}{(n-1)!} e^{-\mu x} dx + p_0$$

$$= \sum_{n=1}^{\infty} (1-\rho)\rho^n \int_0^w \frac{\mu^n x^{n-1}}{(n-1)!} e^{-\mu x} dx + (1-\rho)$$

$$= \mu(1-\rho)\rho \int_0^w \left\{ \sum_{n=1}^{\infty} \frac{(\rho\mu x)^n}{n!} \right\} e^{-\mu x} dx$$

$$= \mu(1-\rho)\rho \int_0^w e^{\rho\mu x} e^{-\mu x} e^{-\mu x} dx = \mu(1-\rho)\rho \int_0^w e^{-\mu(1-\rho)x} dx$$

$$= 1 - \rho e^{-\mu(1-\rho)w}$$

5.7 Examples of Other M/M Queueing Systems

The goal of this section is to describe some relatives of the M/M/1 queueing systems without rigorously analyzing them as we did for the M/M/1 queueing system. These systems can be used to model different human behaviors and they include:

a. *Blocking* from entering a queue, which is caused by the fact that the system has a finite capacity and a customer that arrives when the system is full is rejected.
b. *Defections* from a queue, which can be caused by the fact that a customer has spent too much time in queue and leaves out of frustration without receiving service. Defection is also called *reneging* in queueing theory.
c. *Jockeying* for position among many queues, which can arise when in a multi-server system each server has its own queue and some customer in one queue notices that another queue is being served faster than his own, thus he leaves his queue and moves into the supposedly faster queue.
d. *Balking* before entering a queue, which can arise if the customer perceives the queue to be too long and chooses not to join it at all.
e. *Bribing* for queue position, which is a form of dynamic priority because a customer pays some "bribe" to improve his position in the queue. Usually the more bribe he pays, the better position he gets.
f. *Cheating* for queue position, which is different from bribing because in cheating, the customer uses trickery rather than his personal resources to improve his position in the queue.

g. *Bulk service*, which can be used to model table assignment in restaurants. For example, a queue at a restaurant might appear to be too long, but in actual fact when a table is available (i.e., a server is ready for the next customer), it can be assigned to a family of four, which is identical to serving the four people in queue together.
h. *Batch arrival*, which can be used to model how friends arrive in groups at a movie theatre, concert show, or a ball game; or how families arrive at a restaurant. Thus, the number of customers in each arriving batch can be modeled by some probabilistic law.

From this list, it can be seen that queueing theory is a very powerful modeling tool that can be applied to all human activities and hence all walks of life. In the following sections we describe some of the different queueing models that are based on Poisson arrivals and/or exponentially distributed service times.

5.7.1 The M/M/c Queue: The c-Server System

In this scheme there are c identical servers. When a customer arrives he is randomly assigned to one of the idle servers until all servers are busy when a single queue is formed. Note that if a queue is allowed to form in front of each server, then we have an M/M/1 queue with modified arrival because customers join a server's queue in some probabilistic manner. In the single queue case, we assume that there is an infinite capacity and service is based on FCFS policy.

The service rate in the system is dependent on the number of busy servers. If only one server is busy, the service rate is μ; if two servers are busy, the service rate is 2μ; and so on until all servers are busy when the service rate becomes $c\mu$. Thus, until all servers are busy the system behaves like a heterogeneous queueing system in which the service rate in each state is different. When all servers are busy, it behaves like a homogeneous queueing system in which the service rate is the same in each state. The state-transition-rate diagram of the system is shown in Figure 5.6.

Thus, the service rate is

$$\mu_n = \begin{cases} n\mu & 0 \le n < c \\ c\mu & n \ge c \end{cases}$$

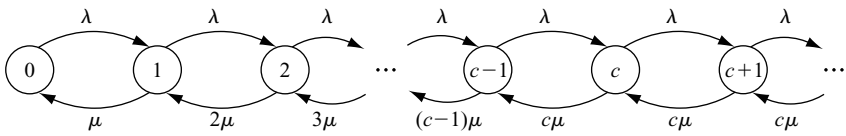


Figure 5.6. State-transition-rate diagram for M/M/c queue.

Using stochastic balance equations we obtain

$$\lambda p_{n-1} = \min(n, c)\mu p_n \quad n = 1, 2, \ldots$$

Iterating on this equation we obtain the limiting-state probability of being in state n as

$$p_n = \begin{cases} \left(\frac{\lambda}{\mu}\right)^n \left(\frac{1}{n!}\right) p_0 & n = 0, 1, \ldots, c \\ \frac{(\lambda/\mu)^n}{c! c^{n-c}} p_0 & n \geq c \end{cases}$$

$$\sum_{n=0}^{\infty} p_n = 1 \Rightarrow p_0 = \frac{1}{\sum_{n=0}^{c-1} \left(\frac{\lambda}{\mu}\right)^n \left(\frac{1}{n!}\right) + \left(\frac{\lambda}{\mu}\right)^c \left(\frac{1}{c!}\right) \left(\frac{c\mu}{c\mu - \lambda}\right)} \quad \lambda < c\mu$$

Note that queues can only form when the process is in state c or any state higher than c. Thus, arriving customers who see the system in any state less than c do not have to wait. The probability that an arriving customer has to wait, which is usually referred to as the *delay probability*, is obtained using PASTA as

$$P_W = p_c + p_{c+1} + p_{c+2} + \cdots = \frac{p_c}{1 - \frac{\lambda}{c\mu}} = \frac{p_c}{1 - \rho}$$

$$= \frac{(c\rho)^c}{c!} \left\{ (1 - \rho) \sum_{n=0}^{c-1} \frac{(c\rho)^n}{n!} + \frac{(c\rho)^c}{c!} \right\}^{-1}$$

where $\rho = \lambda / c\mu$. The mean queue length is given by

$$E\left[N_q\right] = \sum_{n=c}^{\infty} n p_n = p_c \sum_{n=0}^{\infty} n \rho^n = \frac{p_c}{1 - \rho} \sum_{n=0}^{\infty} n (1 - \rho) \rho^n$$

$$= \frac{\rho p_c}{(1 - \rho)^2} = \frac{\rho P_W}{1 - \rho}$$

From Little's formula we obtain the mean waiting time as

$$E[W] = \frac{p_c}{c\mu (1 - \rho)^2} = \frac{P_W}{c\mu (1 - \rho)}$$

We can also obtain the distribution of the waiting time as follows. Let $F_W(t) = P[W \leq t]$ denote the CDF of the waiting time. When there are at least c customers

in the system, the composite service rate is $c\mu$; thus the interdeparture times are exponentially distributed with mean $1/c\mu$. Therefore, when there are $n \geq c$ customers in the system, the total service time of the n customers is Erlang of order $n - c + 1$, and the PDF of the total service time, s_n, is given by

$$f_{S_n}(t) = \frac{(c\mu)^{n-c+1} t^{n-c}}{(n-c)!} e^{-c\mu t} \quad t \geq 0, n \geq c$$

Now,

$$F_W(t) = F_W(0) + P[0 < W \leq t]$$

where

$$F_W(0) = P\left[N_q = 0\right] = P[N < c] = p_0 \sum_{n=0}^{c-1} \left(\frac{\lambda}{\mu}\right)^n \left(\frac{1}{n!}\right)$$

Because

$$p_0 \left\{ \sum_{n=0}^{c-1} \left(\frac{\lambda}{\mu}\right)^n \left(\frac{1}{n!}\right) + \left(\frac{\lambda}{\mu}\right)^c \left(\frac{1}{c!}\right) \left(\frac{c\mu}{c\mu - \lambda}\right) \right\} = 1$$

we have that

$$F_W(0) = p_0 \sum_{n=0}^{c-1} \left(\frac{\lambda}{\mu}\right)^n \left(\frac{1}{n!}\right) = 1 - p_0 \left(\frac{\lambda}{\mu}\right)^c \left(\frac{1}{c!}\right) \left(\frac{c\mu}{c\mu - \lambda}\right) = 1 - \frac{(c\rho)^c p_0}{(1-\rho)c!}$$

Similarly,

$$\begin{aligned}
P[0 < W \leq t] &= \sum_{n=c}^{\infty} p_n \int_0^t f_{S_n}(u)du = \sum_{n=c}^{\infty} p_n \int_0^t \frac{(c\mu)^{n-c+1} u^{n-c}}{(n-c)!} e^{-c\mu u} du \\
&= \int_0^t e^{-c\mu u} \left\{ \sum_{n=c}^{\infty} \frac{(\lambda/\mu)^n}{c! c^{n-c}} \frac{(c\mu)^{n-c+1} u^{n-c}}{(n-c)!} \right\} du \\
&= \frac{p_0}{c!} (\lambda/\mu)^c \int_0^t c\mu e^{-c\mu u} \left\{ \sum_{n=c}^{\infty} \frac{(\lambda u)^{n-c}}{(n-c)!} \right\} du \\
&= \frac{p_0}{c!} (\lambda/\mu)^c \int_0^t c\mu e^{-c\mu u} \left\{ e^{\lambda u} \right\} du = \frac{(c\rho)^c p_0}{c!} \int_0^t c\mu e^{-c\mu(1-\rho)u} du \\
&= \frac{(c\rho)^c p_0}{(1-\rho)c!} \left\{ 1 - e^{-c\mu(1-\rho)t} \right\}
\end{aligned}$$

Thus,

$$F_W(t) = 1 - \frac{(c\rho)^c p_0}{(1-\rho)c!} + \frac{(c\rho)^c p_0}{(1-\rho)c!}\left\{1 - e^{-c\mu(1-\rho)t}\right\} = 1 - \frac{(c\rho)^c p_0}{(1-\rho)c!}e^{-c\mu(1-\rho)t}$$
$$= 1 - \frac{(\lambda/\mu)^c p_0}{(1-\rho)c!}e^{-c\mu(1-\rho)t} = 1 - \frac{p_c}{(1-\rho)}e^{-c\mu(1-\rho)t}$$

Note that we can obtain the expected waiting time from the preceding equation as follows:

$$E[W] = \int_0^\infty \{1 - F_W(t)\}dt = \frac{p_c}{(1-\rho)}\int_0^\infty e^{-c\mu(1-\rho)t}dt = \frac{p_c}{c\mu(1-\rho)^2}$$

which is the result we obtained earlier.

Example 5.3 Students arrive at a checkout counter in the college cafeteria according to a Poisson process with an average rate of 15 students per hour. There are two cashiers at the counter, and they provide identical service to students. The time to serve a student by either cashier is exponentially distributed with a mean of 3 minutes. Students that find both cashiers busy on their arrival join a single queue. What is the probability that an arriving student does not have to wait? ■

Solution: This is an M/M/2 queueing problem with the following parameters with λ and μ in students per minute:

$$\lambda = 15/60 = 1/4$$
$$\mu = 1/3$$
$$p_0 = \frac{1}{1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2\left[\frac{2\mu}{2(2\mu-\lambda)}\right]} = \frac{5}{11}$$
$$p_1 = \left(\frac{\lambda}{\mu}\right)p_0 = \left(\frac{3}{4}\right)\left(\frac{5}{11}\right) = \frac{15}{44}$$

An arriving student does not have to wait if he finds the system either empty or with only one server busy. The probability of this event is $p_0 + p_1 = 35/44$.

5.7.2 The M/M/1/K Queue: The Single-Server Finite-Capacity System

In this system, arriving customers that see the system in state K are lost. The state-transition-rate diagram is shown in Figure 5.7.

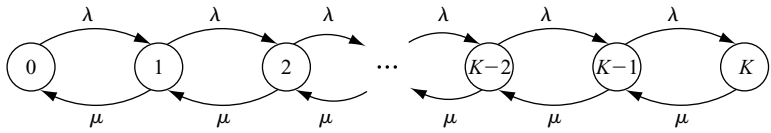


Figure 5.7. State-transition-rate diagram for M/M/1/K queue.

Using stochastic balance equations we obtain the steady-state probability that the process is in state n as

$$p_n = \left(\frac{\lambda}{\mu}\right)^n p_0 \qquad n = 0, 1, 2, \ldots, K$$

$$\sum_{n=0}^{K} p_n = 1 \Rightarrow p_0 = \frac{1 - (\lambda/\mu)}{1 - (\lambda/\mu)^{K+1}}$$

where $\lambda < \mu$. If we define $\rho = \lambda/\mu$, the mean number of customers in the system and the mean number in queue are given respectively by

$$\begin{aligned} E[N] &= \sum_{n=0}^{K} n p_n = \sum_{n=1}^{K} n\rho^n p_0 = \rho p_0 \frac{d}{d\rho}\left\{\frac{1-\rho^{K+1}}{1-\rho}\right\} \\ &= \rho p_0 \left\{\frac{1 - \rho^{K+1} - (1-\rho)(K+1)\rho^K}{(1-\rho)^2}\right\} \\ &= \rho p_0 \left\{\frac{1 - (K+1)\rho^K + K\rho^{K+1}}{(1-\rho)^2}\right\} \\ &= \frac{\rho}{1-\rho^{K+1}}\left\{\frac{1-(K+1)\rho^K + K\rho^{K+1}}{1-\rho}\right\} = \frac{\rho}{1-\rho} - \frac{(K+1)\rho^{K+1}}{1-\rho^{K+1}} \end{aligned}$$

$$E\left[N_q\right] = E[N] - (1 - p_0) = E[N] = \frac{\rho(1-\rho^K)}{1-\rho^{K+1}} = \frac{\rho}{1-\rho} - \frac{\rho(1+K\rho^K)}{1-\rho^{K+1}}$$

Using L'Hôpital's rule we obtain the values when $\rho = 1$ as follows:

$$\lim_{\rho \to 1} p_k = \frac{1}{K+1} \qquad k = 0, 1, \ldots, K$$

$$\lim_{\rho \to 1} E[N] = \frac{K}{2}$$

$$\lim_{\rho \to 1} E[N_q] = \frac{K(K-1)}{2(K+1)}$$

Note that not all the traffic arriving at the system enters the system because customers are not allowed into the system when there are already K customers in the system. That is, customers are turned away with probability

$$p_K = \frac{(1-\rho)\rho^K}{1-\rho^{K+1}}$$

Thus, we define the *actual rate* at which customers arrive into the system, λ_a, as

$$\lambda_a = \lambda(1-p_K) = \frac{\lambda\left(1-\rho^K\right)}{1-\rho^{K+1}}$$

We can then apply Little's formula to obtain

$$E[T] = E[N]/\lambda_a = \frac{1}{\mu}\left\{\frac{1}{1-\rho} - \frac{K\rho^K}{1-\rho^K}\right\}$$

$$E[W] = E[N_q]/\lambda_a = \frac{1}{\mu}\left\{\frac{\rho}{1-\rho} - \frac{K\rho^K}{1-\rho^K}\right\}$$

The CDF of the waiting time can be obtained by noting that when the system is in equilibrium, the probability that an arriving customer joins the queue is $1-p_K$. Thus, the probability that an arriving customer who finds n customers in the system, where $n < K$, joins the queue is $p_n/(1-p_K)$. Thus, the CDF of W is given by

$$F_W(t) = P[W \le t] = F_W(0) + P[0 < W \le t]$$

where

$$F_W(0) = \frac{p_0}{1-p_K} = \frac{1-\rho^{K+1}}{1-\rho^K}\left\{\frac{1-\rho}{1-\rho^{K+1}}\right\} = \frac{1-\rho}{1-\rho^K}$$

When there are n customers in the system, the time to complete their service is an Erlang random variable of order n. Thus,

$$\begin{aligned} P[0 < W \le t] &= \sum_{n=1}^{K-1} \frac{p_n}{1-p_K}\int_0^t \frac{\mu^n u^{n-1}}{(n-1)!}e^{-\mu u}du \\ &= \sum_{n=1}^{K-1} \frac{p_n}{1-p_K}\left\{1 - \int_t^\infty \frac{\mu^n u^{n-1}}{(n-1)!}e^{-\mu u}du\right\} \end{aligned}$$

Now,

$$\int_t^\infty \frac{\mu^n u^{n-1}}{(n-1)!} e^{-\mu u} du = \sum_{j=0}^{n-1} \frac{(\mu t)^j}{j!} e^{-\mu t}$$

Thus,

$$P[0 < W \le t] = \sum_{n=1}^{K-1} \frac{p_n}{1-p_K} \left\{ 1 - \sum_{j=0}^{n-1} \frac{(\mu t)^j}{j!} e^{-\mu t} \right\}$$

which gives

$$\begin{aligned} F_W(t) &= \frac{1-\rho}{1-\rho^K} + \sum_{n=1}^{K-1} \frac{p_n}{1-p_K} - \frac{1}{1-p_K} \sum_{n=1}^{K-1} p_n \sum_{j=0}^{n-1} \frac{(\mu t)^j}{j!} e^{-\mu t} \\ &= 1 - \frac{1-\rho}{1-\rho^K} \sum_{n=1}^{K-1} \rho^n \sum_{j=0}^{n-1} \frac{(\mu t)^j}{j!} e^{-\mu t} \end{aligned}$$

From this we obtain the mean waiting time as

$$\begin{aligned} E[W] &= \int_0^\infty \{1 - F_W(t)\} dt = \int_0^\infty \frac{1-\rho}{1-\rho^K} \left(\sum_{n=1}^{K-1} \rho^n \sum_{j=0}^{n-1} \frac{(\mu t)^j}{j!} e^{-\mu t} \right) dt \\ &= \frac{1-\rho}{1-\rho^K} \sum_{n=1}^{K-1} \frac{\rho^n}{\mu} \sum_{j=0}^{n-1} \frac{1}{j!} \int_0^\infty (\mu t)^j e^{-\mu t} d\mu t = \frac{1-\rho}{\mu\left(1-\rho^K\right)} \sum_{n=1}^{K-1} n\rho^n \\ &= \begin{cases} \dfrac{1}{\mu} \left\{ \dfrac{\rho}{1-\rho} - \dfrac{K\rho^K}{1-\rho^K} \right\} & \rho \neq 1 \\ \dfrac{K-1}{2\mu} & \rho = 1 \end{cases} \end{aligned}$$

Similarly, the CDF of the total time in the system can be obtained as follows:

$$
\begin{aligned}
F_T(t) = P[T \le t] &= \sum_{n=0}^{K-1} \frac{p_n}{1-p_K} \int_0^t \frac{\mu^{n+1}u^n}{n!} e^{-\mu u} du \\
&= \sum_{n=0}^{K-1} \frac{p_n}{1-p_K} \left\{ 1 - \int_t^\infty \frac{\mu^{n+1}u^n}{n!} e^{-\mu u} du \right\} \\
&= \sum_{n=0}^{K-1} \frac{p_n}{1-p_K} \left\{ 1 - \sum_{j=0}^{n} \frac{(\mu t)^j}{j!} e^{-\mu t} \right\} \\
&= \sum_{n=0}^{K-1} \frac{p_n}{1-p_K} - \sum_{n=0}^{K-1} \frac{p_n}{1-p_K} \left(\sum_{j=0}^{n} \frac{(\mu t)^j}{j!} e^{-\mu t} \right) \\
&= 1 - \sum_{n=0}^{K-1} \frac{p_n}{1-p_K} \left(\sum_{j=0}^{n} \frac{(\mu t)^j}{j!} e^{-\mu t} \right) = 1 - \frac{1-\rho}{1-\rho^K} \sum_{n=0}^{K-1} \rho^n \sum_{j=0}^{n} \frac{(\mu t)^j}{j!} e^{-\mu t}
\end{aligned}
$$

The mean total time in the system is given by

$$
\begin{aligned}
E[T] &= \int_0^\infty \{1 - F_T(t)\} dt = \int_0^\infty \frac{1-\rho}{1-\rho^K} \left(\sum_{n=0}^{K-1} \rho^n \sum_{j=0}^{n} \frac{(\mu t)^j}{j!} e^{-\mu t} \right) dt \\
&= \int_0^\infty \frac{1-\rho}{1-\rho^K} \left(\sum_{n=0}^{K-1} \rho^n \sum_{j=0}^{n} \frac{(\mu t)^j}{j!} e^{-\mu t} \right) dt \\
&= \frac{1-\rho}{1-\rho^K} \sum_{n=0}^{K-1} \frac{\rho^n}{\mu} \sum_{j=0}^{n} \frac{1}{j!} \int_0^\infty (\mu t)^j e^{-\mu t} d\mu t \\
&= \frac{1-\rho}{\mu\left(1-\rho^K\right)} \sum_{n=0}^{K-1} (n+1)\rho^n = \frac{1}{\mu} \left\{ 1 + \frac{\rho}{1-\rho} - \frac{K\rho^K}{1-\rho^K} \right\} \\
&= \begin{cases} \dfrac{1}{\mu} \left\{ \dfrac{1}{1-\rho} - \dfrac{K\rho^K}{1-\rho^K} \right\} & \rho \neq 1 \\ \dfrac{K+1}{2\mu} & \rho = 1 \end{cases}
\end{aligned}
$$

Example 5.4 Each morning people arrive at Ed's garage to have their cars fixed. Ed's garage can only accommodate four cars. Anyone arriving when there

are already four cars in the garage has to go away without leaving his car for Ed to fix. Ed's customers arrive according to a Poisson process with a rate of one customer per hour, and the time it takes Ed to service a car is exponentially distributed with a mean of 45 minutes.

1. What is the probability that an arriving customer finds Ed idle?
2. What is the probability that an arriving customer leaves without getting his car fixed?
3. What is the expected waiting time at Ed's garage? ■

Solution: This is an M/M/1/4 queue with the following parameters:

$$\lambda = 1$$

$$\mu = 60/45 = 4/3$$

$$\rho = \lambda/\mu = 3/4$$

$$p_0 = \frac{1-\rho}{1-\rho^5} = \frac{1-(3/4)}{1-(3-4)^5} = \frac{0.25}{0.7627} = 0.3278$$

1. The probability that an arriving customer finds Ed idle is $p_0 = 0.3278$.
2. The probability that a customer leaves without getting his car fixed is the probability that he finds the garage full when he arrived, which is $p_4 = \rho^4 + p_0 = 0.1037$.
3. The expected waiting time at Ed's garage is

$$E[W] = \frac{3}{4}\left\{\frac{3/4}{1-3/4} - \frac{4(3/4)^4}{1-(3/4)^4}\right\} = 0.8614$$

5.7.3 The M/M/c/c Queue: The c-Server Loss System

This is a very useful model in telephony. It is used to model calls arriving at a telephone switchboard, which usually has a finite capacity. It is assumed that the switchboard can support a maximum of c simultaneous calls (i.e., it has a total of c channels available). Any call that arrives when all c channels are busy will be lost. This is usually referred to as the *blocked-calls-lost* model. The state-transition-rate diagram is shown in Figure 5.8.

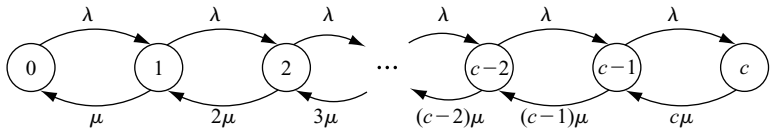


Figure 5.8. State-transition-rate diagram for M/M/c/c queue.

The steady-state probability that the process is in state n is given by

$$p_n = \frac{1}{n!}(\lambda/\mu)^n p_0$$

$$1 = \sum_{n=0}^{c} p_n = p_0 \sum_{n=0}^{c} \frac{(\lambda/\mu)^n}{n!}$$

Thus,

$$p_0 = \left\{ \sum_{n=0}^{c} \frac{(\lambda/\mu)^n}{n!} \right\}^{-1}$$

and we obtain

$$p_n = \frac{(\lambda/\mu)^n/n!}{\sum_{n=0}^{c} (\lambda/\mu)^n/n!} \quad 0 \le n \le c$$

The probability that the process is in state c, p_c, is called the *Erlang's loss formula*, which is given by

$$p_c = \frac{(\lambda/\mu)^c/c!}{\sum_{n=0}^{c} (\lambda/\mu)^n/n!} = \frac{(c\rho)^c/c!}{\sum_{n=0}^{c} (c\rho)^n/n!}$$

where $\rho = \lambda/c\mu$ is the utilization factor of the system.

As in the M/M/1/K queueing system, not all traffic enters the system. The actual average arrival rate into the system is

$$\lambda_a = \lambda(1 - p_c)$$

Because no customer is allowed to wait, $E[W]$ and $E[N_q]$ are both zero. However, the mean number of customers in the system is

$$E[N] = \sum_{n=0}^{c} n p_n = p_0 \sum_{n=1}^{c} n \frac{(\lambda/\mu)^n}{n!}$$

$$= (\lambda/\mu) p_0 \sum_{n=1}^{c} \frac{(\lambda/\mu)^{n-1}}{(n-1)!} = (\lambda/\mu) p_0 \sum_{n=0}^{c-1} \frac{(\lambda/\mu)^n}{n!} = (\lambda/\mu)\,[1 - p_c]$$

By Little's formula,

$$E[T] = E[N]/\lambda_a = 1/\mu$$

This confirms that the mean time a customer admitted into the system spends in the system is the mean service time.

Example 5.5 Bob established a dial-up service for Internet access in his cyber cafe. As a small businessman, Bob can only support four lines for his customers. Any of Bob's customers that arrive at the cafe when all four lines are busy are blocked. Bob's studies indicate that customers arrive at the cafe according to a Poisson process with an average rate of eight customers per hour, and the duration of each customer's Internet use is exponentially distributed with a mean of 10 minutes. If Jay is one of Bob's customers, what is the probability that on one particular trip to the cafe he could not use the Internet service? ■

Solution: This is an example of an M/M/4/4 queueing system. The parameters of the model are as follows:

$$\lambda = 8/60 = 2/15$$
$$\mu = 1/10$$
$$\rho = \lambda/4\mu = 1/3$$
$$c\rho = 4/3$$

The probability that Jay was blocked is the probability that he arrived when the process was in state 4. This is given by

$$p_4 = \frac{(c\rho)^4/4!}{\sum_{k=0}^{4}(c\rho)^k/k!} = \frac{(4/3)^4/24}{1 + (4/3) + \frac{(4/3)^2}{2} + \frac{(4/3)^3}{6} + \frac{(4/3)^4}{24}}$$
$$= \frac{0.1317}{1 + 1.3333 + 0.8889 + 0.3951 + 0.1317}$$
$$= 0.0351$$

5.7.4 The M/M/1//K Queue: The Single-Server Finite-Customer Population System

In the previous examples we assumed that the customers are drawn from an infinite population because the arrival process has a Poisson distribution. Assume that there are K potential customers in the population. An example is where we have a total of K machines that can be either operational or down needing a serviceman to fix them. If we assume that the customers act independently of each other and that given that a customer has not yet come to the service facility the time until he comes to the facility is exponentially distributed with mean $1/\lambda$, then the number of arrivals when n customers are already in the service facility is Poisson with

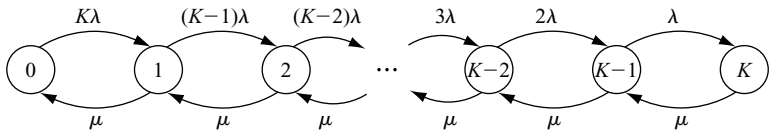


Figure 5.9. State-transition-rate diagram for M/M/1//K queue.

parameter $\lambda(K-n)$. When $n = K$, there are no more customers left to draw from, which means that the arrival rate becomes zero. Thus, the state-transition-rate diagram is as shown in Figure 5.9.

The arrival rate when the process is in state n is

$$\lambda_n = \begin{cases} (K-n)\lambda & 0 \le n < K \\ 0 & n > K \end{cases}$$

It can be shown that the steady-state probabilities are given by

$$p_n = n!\left(\frac{\lambda}{\mu}\right)^n p_0 \qquad n = 0, 1, 2, \ldots, K$$

$$\sum_{n=0}^{K} p_n = 1 \Rightarrow p_0 \frac{1}{\sum_{n=0}^{K} n!\left(\frac{\lambda}{\mu}\right)^n}$$

Other schemes can easily be derived from the preceding models. For example, we can obtain the state-transition-rate diagram for the c-server finite population system with population $K > c$ by combining the arriving process on the M/M/1//K queueing system with the service process of the M/M/c queueing system.

Example 5.6 A small organization has three old PCs, each of which can be working (or operational) or down. When any PC is working, the time until it fails is exponentially distributed with a mean of 10 hours. When a PC fails, the repairman immediately commences servicing it to bring it back to the operational state. The time to service each failed PC is exponentially distributed with a mean of 2 hours. If there is only one repairman in the facility and the PCs fail independently, what is the probability that the organization has only two PCs working? ■

Solution: This is an M/M/1//3 queueing problem in which the arrivals are PCs that have failed and the single server is the repairman. Thus, when the process is in state 0, all PCs are working; when it is in state 1, two PCs are working; when it is in state 2, only one PC is working; and when it is in state 3,

all PCs are down awaiting repair. The parameters of the problem are:

$$\lambda = 1/10$$
$$\mu = 1/2$$
$$\lambda/\mu = 0.2$$
$$p_0 = \frac{1}{\sum_{n=0}^{3} n! \left(\frac{\lambda}{\mu}\right)^n} = \frac{1}{1 + 0.2 + 2(0.2)^2 + 6(0.2)^3} = 0.7530$$

As stated earlier, the probability that two computers are working is the probability that the process is in state 1, which is given by

$$p_1 = \left(\frac{\lambda}{\mu}\right) p_0 = (0.2)(0.7530) = 0.1506$$

5.8 M/G/1 Queue

In this system, customers arrive according to a Poisson process with rate λ and are served by a single server with a general service time X whose PDF is $f_X(x)$, $x \geq 0$, mean is $E[X]$, second moment is $E[X^2]$, and variance is σ_X^2. The capacity of the system is infinite, and customers are served on a first-come first-served basis. Thus, the service time distribution does not have the memoryless property of the exponential distribution, and the number of customers in the system time t, $N(t)$, is not a Poisson process. Therefore, a more appropriate description of the state at time t includes both $N(t)$ and the residual life of the service time of the current customer. That is, if R denotes the residual life of the current service, then the set of pairs $\{(N, R)\}$ provides the description of the state space. Thus, we have a two-dimensional state space, which is a somewhat complex way to proceed with the analysis. However, the analysis is simplified if we can identify those points in time where the state is easier to describe. Such points are usually chosen to be those time instants at which customers leave the system, which means that $R = 0$.

To obtain the steady-state analysis of the system, we proceed as follows. Consider the instant the kth customer arrives at the system. Assume that the ith customer was receiving service when the kth customer arrived. Let R_i denote the residual life of the service time of the ith customer at the instant the kth customer arrived, as shown in Figure 5.10.

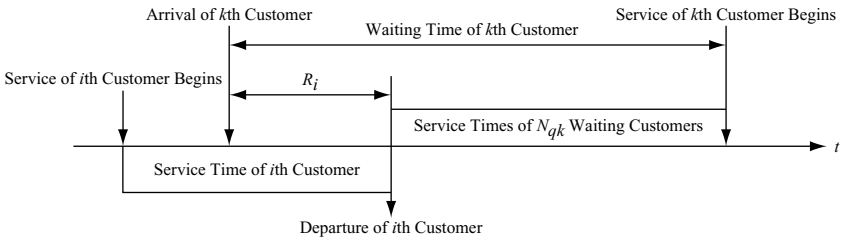


Figure 5.10. Service experience of the ith customer in M/G/1 queue.

Assume that N_{qk} customers were waiting when the kth customer arrived. Because the service times are identically distributed, the waiting time W_K of the kth customer is given by

$$W_k = R_i u(k) + N_{qk} X$$

where $u(k)$ is an indicator function that has a value of 1 if the server was busy when the kth customer arrived and zero otherwise. That is, if N_k defines the total number of customers in the system when the kth customer arrived, then

$$u(k) = \begin{cases} 1 & N_k > 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus, taking expectations on both sides and noting that N_{qk} and X are independent random variables, and also that $u(k)$ and R_i are independent random variables, we obtain

$$E[W_k] = E[R_i]E[u(k)] + E[N_{qk}]E[X]$$

In Chapter 6 we show from the principle of random incidence that

$$E[R_i] = \frac{E[X^2]}{2E[X]}$$

Also,

$$E[u(k)] = 0P[N_k = 0] + 1P[N_k > 0] = P[N_k > 0] = 1 - p_0 = \rho$$

Finally, from Little's formula, $E[N_{qk}] = \lambda E[W_k]$. Thus, the mean waiting time of the kth customer is given by

$$E[W_k] = \frac{\rho E[X^2]}{2E[X]} + \lambda E[W_k]E[X] = \frac{\rho E[X^2]}{2E[X]} + \rho E[W_k]$$

$$= \frac{\rho E[X^2]}{2(1-\rho)E[X]} = \frac{\lambda E[X^2]}{2(1-\rho)}$$

Because the experience of the kth customer is a typical experience, we conclude that the mean waiting time in an M/G/1 queue is given by

$$E[W] = \frac{\rho E[X^2]}{2(1-\rho)E[X]} = \frac{\lambda E[X^2]}{2(1-\rho)}$$

Thus, the expected number of customers in the system is given by

$$E[N] = \rho + E[N_q] = \rho + \frac{\lambda^2 E[X^2]}{2(1-\rho)}$$

This expression is called the *Pollaczek-Khinchin formula*. It is sometimes written in terms of the coefficient of variation C_X of the service time. The square of C_X is defined as follows:

$$C_X^2 = \frac{\sigma_X^2}{(E[X])^2} = \frac{E[X^2] - (E[X])^2}{(E[X])^2} = \frac{E[X^2]}{(E[X])^2} - 1$$

Thus, the second moment of the service time becomes $E[X^2] = \left(1 + C_X^2\right)(E[X])^2$, and the Pollaczek-Khinchin formula becomes

$$E[N] = \rho + \frac{\lambda^2 (E[X])^2 (1 + C_X^2)}{2(1-\rho)} = \rho + \frac{\rho^2(1 + C_X^2)}{2(1-\rho)}$$

Similarly, the mean waiting time becomes

$$E[W] = \frac{\lambda E[X^2]}{2(1-\rho)} = \frac{\lambda(1 + C_X^2)(E[X])^2}{2(1-\rho)} = \frac{\rho(1 + C_X^2)E[X]}{2(1-\rho)}$$

5.8.1 Waiting Time Distribution of the M/G/1 Queue

We can obtain the distribution of the waiting time as follows. Let N_k denote the number of customers left behind by the kth departing customer, and let A_k denote the number of customers that arrive during the service time of the kth customer. Then we obtain the following relationship:

$$N_{k+1} = \begin{cases} N_k - 1 + A_{k+1} & N_k > 0 \\ A_{k+1} & N_k = 0 \end{cases}$$

Thus, we see that $\{N_k\}_{k=0}^{\infty}$ forms a Markov chain called the *imbedded M/G/1 Markov chain*. Let the transition probabilities of the imbedded Markov chain be defined as follows:

$$P_{ij} = P\left[N_{k+1} = j | N_k = i\right]$$

Because N_k cannot be greater than $N_{k+1} + 1$, we have that $p_{ij} = 0$ for all $j < i - 1$. For $j \geq i - 1$, p_{ij} is the probability that exactly $j - i + 1$ customers arrived during the service time of the $(k+1)$th customer, $i > 0$. Also, because the kth customer left the system empty in state 0, p_{0j} represents the probability that exactly j customers arrived while the $(k+1)$th customer was being served. Similarly, because the kth customer left one customer behind, which is the $(k+1)$th customer, in state 1, p_{1j} is the probability that exactly j customers arrived while the $(k+1)$th customer was being served. Thus, $p_{0j} = p_{1j}$ for all j. Let the random variable A_S denote the number of customers that arrive during a service time. Then the PMF of A_S is given by

$$p_{A_S}(n) = P[A_S = n] = \int_{x=0}^{\infty} \frac{(\lambda x)^n}{n!} e^{-\lambda x} f_X(x) dx \quad n = 0, 1, \ldots$$

If we define $\alpha_n = P[A_S = n]$, then the state-transition matrix of the imbedded Markov chain is given as follows:

$$P = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \cdots \\ \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \cdots \\ 0 & \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \cdots \\ 0 & 0 & \alpha_0 & \alpha_1 & \cdots & \cdots \\ 0 & 0 & 0 & \alpha_0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

The state-transition diagram is shown in Figure 5.11.

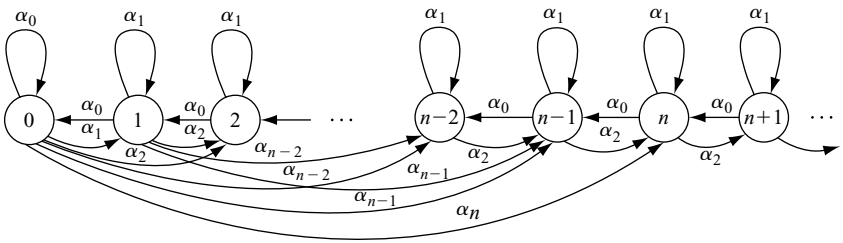


Figure 5.11. *Partial state-transition* diagram for M/G/1 imbedded Markov chain.

Observe that the z-transform of the PMF of A_S is given by

$$\begin{aligned} G_{A_S}(z) &= \sum_{n=0}^{\infty} z^n p_{A_S}(n) = \sum_{n=0}^{\infty} z^n \int_{x=0}^{\infty} \frac{(\lambda x)^n}{n!} e^{-\lambda x} f_X(x) dx \\ &= \int_{x=0}^{\infty} \left\{ \sum_{n=0}^{\infty} \frac{(\lambda x z)^n}{n!} \right\} e^{-\lambda x} f_X(x) dx \\ &= \int_{x=0}^{\infty} e^{\lambda x z} e^{-\lambda x} f_X(x) dx = \int_{x=0}^{\infty} e^{-(\lambda - \lambda z)x} f_X(x) dx \\ &= M_X(\lambda - \lambda z) \end{aligned}$$

where $M_X(s)$ is the s-transform of the PDF of X, the service time. That is, the z-transform of the PMF of A_S is equal to the s-transform of the PDF of X evaluated at the point $s = \lambda - \lambda z$. Let $f_T(t)$ denote the PDF of T, the total time in the system. Let K be a random variable that denotes the number of customers that a tagged customer leaves behind. This is the number of customers that arrived during the total time that the tagged customer was in the system. Thus, the PMF of K is given by

$$p_K(n) = P[K = n] = \int_{x=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} f_T(t) dt \qquad n = 0, 1, \ldots$$

As in the case of A_S, it is easy to show that the z-transform of the PMF of K is given by

$$G_K(z) = M_T(\lambda - \lambda z)$$

Recall that N_k, the number of customers left behind by the kth departing customer, satisfies the relationship

$$N_{k+1} = \begin{cases} N_k - 1 + A_{k+1} & N_k > 0 \\ A_{k+1} & N_k = 0 \end{cases}$$

where A_k denotes the customers that arrive during the service time of the kth customer. Thus, K is essentially the value of N_k for our tagged customer.

In Kleinrock (1975) it is shown that the z-transform of the PMF of K is given by

$$G_K(z) = \frac{(1-\rho)M_X(\lambda - \lambda_z)(1-z)}{M_X(\lambda - \lambda z) - z}$$

Thus, we have that

$$M_T(\lambda - \lambda z) = \frac{(1-\rho)M_X(\lambda - \lambda_z)(1-z)}{M_X(\lambda - \lambda z) - z}$$

If we set $s = \lambda - \lambda z$, we obtain the following:

$$M_T(s) = \frac{s(1-\rho)M_X(s)}{s - \lambda + \lambda M_X(s)}$$

This is one of the equations that is usually called the *Pollaczek-Khinchin formula*. Finally, because $T = W + X$, which is the sum of two independent random variables, we have that the s-transform of T is given by

$$M_T(s) = M_W(s)M_X(s)$$

From this we obtain the s-transform of the PDF of W as

$$M_W(s) = \frac{M_T(s)}{M_X(s)} = \frac{s(1-\rho)}{s - \lambda + \lambda M_X(s)}$$

This is also called the Pollaczek-Khinchin formula.

5.8.2 The M/E_k/1 Queue

The M/E_k/1 queue is an M/G/1 queue in which the service time has the Erlang-k distribution. It is usually modeled by a process in which service consists of a customer passing, stage by stage, through a series of k independent and identically distributed subservice centers, each of which has an exponentially distributed service time with mean $1/k\mu$. Thus, the total mean service time is $k \times (1/k\mu) = 1/\mu$. The state-transition-rate diagram for the system is shown in Figure 5.12. Note that the states represent service stages. Thus, when the system is in state 0, an arrival causes it to go to state k; when the system is in state 1, an arrival causes it to enter state $k+1$, and so on. A completion of service at state j leads to a transition to state $j-1$, $j \geq 1$.

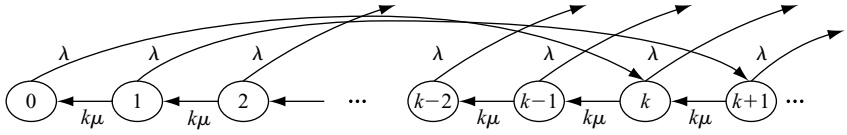


Figure 5.12. State-transition-rate diagram for the M/E_k/1 queue.

While we can analyze the system from scratch, we can also apply the results obtained for the M/G/1 queue. We know for an Erlang-k random variable X, the following results can be obtained:

$$f_X(x) = \frac{(k\mu)^k x^{k-1} e^{-k\mu x}}{(k-1)!}$$

$$M_X(s) = \left(\frac{k\mu}{s+k\mu}\right)^k$$

$$E[X] = \frac{1}{\mu}$$

$$\sigma_X^2 = \frac{k}{(k\mu)^2} = \frac{1}{k\mu^2}$$

$$C_X^2 = \frac{\sigma_X^2}{(E[X])^2} = \frac{1}{k}$$

$$\rho = \lambda E[X] = \lambda k/\mu < 1$$

Thus, we obtain the following results:

a. The mean waiting time is

$$E[W] = \frac{\rho(1+C_X^2)E[X]}{2(1-\rho)} = \frac{\rho(k+1)}{2k\mu(1-\rho)}$$

b. The s-transform of the PDF of the waiting time is

$$M_W(s) = \frac{s(1-\rho)}{s-\lambda+\lambda M_X(s)} = \frac{s(1-\rho)}{s-\lambda+\lambda\left(\frac{k\mu}{s+k\mu}\right)^k}$$

c. The mean total number of customers in the system is

$$E[N] = \rho + \frac{\lambda^2(E[X])^2(1+C_X^2)}{2(1-\rho)} = \rho + \frac{\rho^2(k+1)}{2k(1-\rho)}$$

d. The s-transform of the total time in the system is

$$M_T(s) = \left.\frac{s(1-\rho)M_X(s)}{s-\lambda+\lambda M_X(s)}\right|_{M_X(s)=\left(\frac{k\mu}{s+k\mu}\right)^k}$$

5.8.3 The M/D/1 Queue

The M/D/1 queue is an M/G/1 queue deterministic (or fixed) service time. We can analyze the queueing system by applying the results for the M/G/1 queueing system as follows:

$$f_X(x) = \delta(x - 1/\mu)$$
$$M_X(s) = e^{-s/\mu}$$
$$E[X] = \frac{1}{\mu}$$
$$\sigma_X^2 = 0$$
$$C_X^2 = \frac{\sigma_X^2}{(E[X])^2} = 0$$
$$\rho = \lambda E[X] < 1$$

Thus, we obtain the following results:

a. The mean waiting time is

$$E[W] = \frac{\rho\left(1+C_X^2\right)E[X]}{2(1-\rho)} = \frac{\rho}{2\mu(1-\rho)}$$

b. The s-transform of the PDF of the waiting time

$$M_W(s) = \frac{s(1-\rho)}{s-\lambda+\lambda M_X(s)} = \frac{s(1-\rho)}{s-\lambda+\lambda e^{-s/\mu}}$$

c. The mean total number of customers in the system is

$$E[N] = \rho + \frac{\lambda^2(E[X])^2 + \left(1+C_X^2\right)}{2(1-\rho)} = \rho + \frac{\rho^2}{2(1-\rho)}$$

d. The s-transform of the PDF of the total time in the system is

$$M_T(s) = \frac{s(1-\rho)e^{-s/\mu}}{s-\lambda+\lambda e^{-s/\mu}}$$

5.8.4 The M/M/1 Queue Revisited

The M/M/1 queue is also an example of the M/G/1 queue with the following parameters:

$$f_X(x) = \mu e^{-\mu x}$$

$$M_X(s) = \frac{\mu}{s+\mu}$$

$$E[X] = \frac{1}{\mu}$$

$$\sigma_X^2 = \frac{1}{\mu^2}$$

$$C_X^2 = \frac{\sigma_X^2}{(E[X])^2} = 1$$

$$\rho = \lambda E[X] < 1$$

When we substitute for these parameters in the equations for M/G/1, we obtain the results previously obtained for the M/M/1 queueing system.

5.8.5 The M/H_K/1 Queue

This is a single-server, infinite-capacity queueing system with Poisson arrivals with hyperexponentially distributed service time of order k. That is, with probability θ_j an arriving customer will choose to receive service from server j whose service time is exponentially distributed with a mean of $1/\mu_j$, $1 \leq j \leq k$, where

$$\sum_{j=1}^{k} \theta_j = 1$$

The system is illustrated in Figure 5.13.
For this system we have that

$$f_X(x) = \sum_{j=1}^{k} \theta_j \mu_j e^{-\mu_j x}$$

$$M_X(s) = \sum_{j=1}^{k} \theta_j \left\{ \frac{\mu_j}{s+\mu_j} \right\}$$

$$E[X] = \sum_{j=1}^{k} \left\{ \frac{\theta_j}{\mu_j} \right\}$$

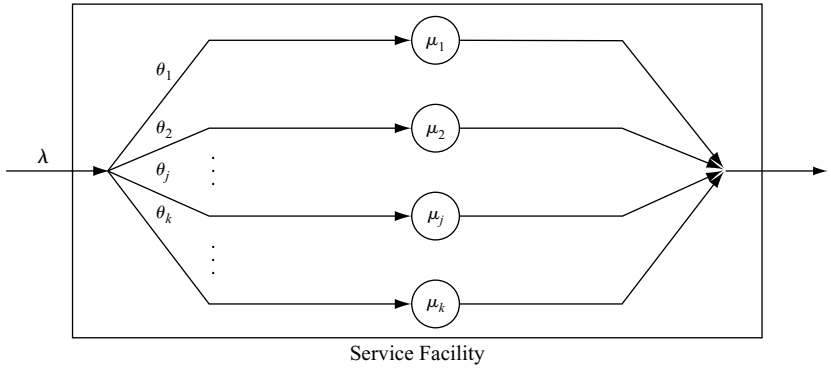


Figure 5.13. A k-stage hyperexponential server.

$$E\left[X^2\right] = 2\sum_{j=1}^{k} \frac{\theta_j}{\mu_j^2}$$

$$C_X^2 = \frac{E\left[X^2\right] - (E[X])^2}{(E[X])^2} = \frac{E[X^2]}{(E[X])^2} - 1$$

$$= \frac{2\sum_{j=1}^{k} \frac{\theta_j}{\mu_j^2}}{\left(\sum_{j=1}^{k} \frac{\theta_j}{\mu_j}\right)^2} - 1$$

For the special case of $k = 2$, that is, for the M/H_2/1 queue, we have the following results:

$$f_X(x) = \mu_1 e^{-\mu_1 x} + \mu_2 e^{-\mu_2 x}$$

$$M_X(s) = \frac{\theta_1 \mu_1}{s + \mu_1} + \frac{\theta_2 \mu_2}{s + \mu_2}$$

$$E[X] = \frac{\theta_1}{\mu_1} + \frac{\theta_2}{\mu_2}$$

$$\sigma_X^2 = 2\left(\frac{\theta_1}{\mu_1^2} + \frac{\theta_2}{\mu_2^2}\right)$$

$$C_X^2 = \left(\frac{\theta_1 \mu_2 - \theta_2 \mu_1}{\theta_1 \mu_2 + \theta_2 \mu_1}\right)^2$$

$$\rho = \lambda E[X] < 1$$

From this we obtain the following performance parameters:

a. The mean waiting time is

$$E[W] = \frac{\rho\left(1 + C_X^2\right) E[X]}{2(1-\rho)} = \frac{\rho\left(\frac{\theta_1}{\mu_1} + \frac{\theta_2}{\mu_2}\right)\left\{1 + \left(\frac{\theta_1\mu_2 - \theta_2\mu_1}{\theta_1\mu_2 + \theta_2\mu_1}\right)^2\right\}}{2(1-\rho)}$$

b. The s-transform of the PDF of the waiting time is

$$M_W(s) = \frac{s(1-\rho)}{s - \lambda + \lambda M_X(s)} = \frac{s(1-\rho)}{s - \lambda + \lambda\left\{\frac{\theta_1\mu_1}{s+\mu_1} + \frac{\theta_2\mu_2}{s+\mu_2}\right\}}$$

c. The mean total number of customers in the system is

$$E[N] = \rho + \frac{\lambda^2 (E[X])^2 \left(1 + C_X^2\right)}{2(1-\rho)} = \rho + \frac{\rho^2\left\{1 + \left(\frac{\theta_1\mu_2 - \theta_2\mu_2}{\theta_1\mu_2 - \theta_2\mu_1}\right)^2\right\}}{2(1-\rho)}$$

d. The s-transform of the PDF of the total time in the system is

$$M_T(s) = \frac{s(1-\rho)\left(\frac{\theta_1\mu_1}{s+\mu_1} + \frac{\theta_2\mu_2}{s+\mu_2}\right)}{s - \lambda + \lambda\left(\frac{\theta_1\mu_1}{s+\mu_1} + \frac{\theta_2\mu_2}{s+\mu_2}\right)}$$

5.9 G/M/1 Queue

The G/ M/1 queue is the dual of the M/G/1 queue. In this system, customers arrive according to a general arrival process with independent and identically distributed interarrival times A with PDF $f_A(t)$ and mean $1/\lambda$. The facility has a single server, and the time X to serve arriving customers is exponentially distributed with mean $1/\mu$. As in the case of the M/G/1 queue, the number $N(t)$ of customers in the system is not Markovian. In this case, the reason is that to completely define a state we need both $N(t)$ and the time that has elapsed since the last arrival. Thus, if Y is the time that has elapsed since the last arrival, then the state of the G/M/1 queue can be defined by the set of pairs $\{(N, Y)\}$, which means that we need a complicated two-dimensional state description. As in the M/G/1 queue, we look for those special points in time where a much easier state description can be formulated.

Because of the memoryless nature of the service process, such points that provide an easier state description are those time instants at which customers arrive. At these points, $Y = 0$, which means the state description is captured by N only. Let N_K denote the number of customers that the ith arriving customer sees upon joining the queue, where $N_k = 0, 1, 2, \ldots$. Let T_k denote the time between the kth arrival

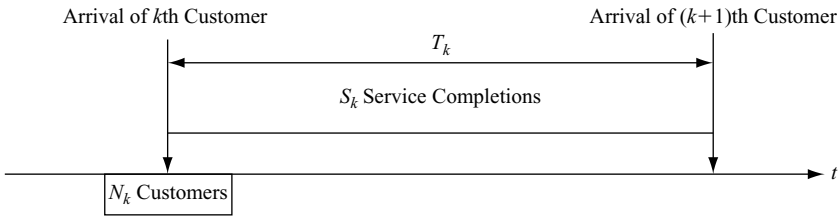


Figure 5.14. Service experience of the kth customer in G/M/1 queue.

and the $(k+1)$th arrival. Let S_k denote the number of service completions during T_k, as illustrated in Figure 5.14.

Thus, we obtain the following equation:

$$N_{k+1} = \max(N_k + 1 - S_k, 0) \quad k = 1, 2, \ldots$$

The initial condition is $N_1 = 0$, and we see that the sequence $\{N_k\}_{k=1}^{\infty}$ forms a Markov chain called the G/M/1 *imbedded Markov chain*. Let the transition probabilities be defined by

$$p_{ij} = p[N_{k+1} = j | N_k = i]$$

It is clear that $p_{ij} = 0$ for all $j > i+1$. For $j \leq i+1$, p_{ij} represents the probability that exactly $i + 1 - j$ customers are served during the interval between the kth arrival and the $(k+1)$th arrival, given that the server is busy during this interval. Let r_n denote the probability that n customers are served during an interarrival time. Then r_n is given by

$$r_n = \int_{t=0}^{\infty} \frac{(\mu t)^n}{n!} e^{-\mu t} f_A(t) dt$$

Thus, the transition probability matrix is given by

$$P = [p_{ij}] = \begin{bmatrix} 1 - r_0 & r_0 & 0 & 0 & 0 & 0 & \cdots \\ 1 - \sum_{k=0}^{1} r_k & r_1 & r_0 & 0 & 0 & 0 & \cdots \\ 1 - \sum_{k=0}^{2} r_k & r_2 & r_1 & r_0 & 0 & 0 & \cdots \\ 1 - \sum_{k=0}^{3} r_k & r_3 & r_2 & r_1 & r_0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

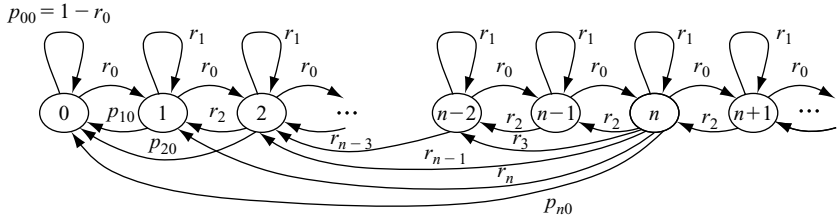


Figure 5.15. Partial state-transition diagram for G/M/1 imbedded Markov chain.

The partial state-transition diagram is illustrated in Figure 5.15. In the figure,

$$p_{m0} = 1 - \sum_{k=0}^{m} r_k$$

$$p_{00} = 1 - r_0$$

Let π_n denote the limiting state probability that the queue is in state n. Then these probabilities must satisfy the following balance equation:

$$\pi_0 = \pi_0 p_{00} + \pi_1 p_{10} + \pi_2 p_{20} + \cdots = \sum_{i=0}^{\infty} \pi_i p_{i0}$$

$$\pi_n = \pi_{n-1} r_0 + \pi_n r_1 + \pi_{n+1} r_2 + \cdots = \sum_{i=0}^{\infty} \pi_{n-1+i}\, r_i,\ n = 1, 2, \ldots$$

The solution to this system of equations is of the form

$$\pi_n = c\beta^n \quad n = 0, 1, \ldots$$

where c is some constant. Thus, substituting this in the previous equation we obtain

$$\beta^n = \sum_{i=0}^{\infty} \beta^{n-1+i} r_i \Rightarrow \beta = \sum_{i=0}^{\infty} \beta^i r_i$$

Because $r_n = \int_{t=0}^{\infty} \frac{(\mu t)^n}{n!} e^{-\mu t} f_A(t)dt$, we obtain

$$\beta = \sum_{i=0}^{\infty} \beta^i \int_{t=0}^{\infty} \frac{(\mu t)^i}{i!} e^{-\mu t} f_A(t)dt = \int_{t=0}^{\infty} \left\{ \sum_{i=0}^{\infty} \frac{(\beta\mu t)^i}{i!} \right\} e^{-\mu t} f_A(t)dt$$

$$= \int_{t=0}^{\infty} e^{\mu\beta t} e^{-\mu t} f_A(t)dt$$

$$= \int_{t=0}^{\infty} e^{-(\mu - \mu\beta)t} f_A(t)dt$$

$$= M_A(\mu - \mu\beta)$$

where $M_A(s)$ is the s-transform of the PDF $f_A(t)$. Because we know that $M_A(0) = 1, \beta = 1$ is a solution to the functional equation $\beta = M_A(\mu - \mu\beta)$. It can be shown that as long as $\lambda/\mu = \rho < 1$, there is a unique real solution for β in the range $0 < \beta < 1$, which is the solution we are interested in. Now, we know that

$$\sum_{n=0}^{\infty} \pi_n = c \sum_{n=0}^{\infty} \beta^n = 1 = \frac{c}{1-\beta}$$

Thus, $c = 1 - \beta$, and we obtain

$$\pi_n = (1-\beta)\beta^n \quad n \geq 0$$

To find the mean total time in the system (or the sojourn time), let n denote the number of customers in the system when some tagged customer arrived. Because the service time is exponentially distributed, the total time that the tagged customer spends in the system is the time to serve $n + 1$ customers, including the tagged customer, which is given by the following random sum of random variables:

$$T = \sum_{k=1}^{n+1} X_k$$

where the X_k are independent and identically distributed with the PDF $f_X(x)\mu e^{-\mu x}, x \geq 0$. Thus, the s-transform of the PDF of T, $M_T(s)$, can be obtained as follows:

$$\begin{aligned}
T|_{N=n} &= X_1 + X_2 + \cdots + X_{n+1} \\
M_{T|N=n}(s|n) &= \{M_X(s)\}^{n+1} \\
M_T(s) &= \sum_{n=0}^{\infty} M_{T|N=n}(s|n)\pi_n = \sum_{n=0}^{\infty} \{M_X(s)\}^{n+1} \pi_n \\
&= \sum_{n=0}^{\infty} \left\{\frac{\mu}{s+\mu}\right\}^{n+1} (1-\beta)\beta^n = \frac{\mu(1-\beta)}{s+\mu} \sum_{n=0}^{\infty} \left\{\frac{\mu\beta}{s+\mu}\right\}^n \\
&= \frac{\mu(1-\beta)}{s+\mu(1-\beta)}
\end{aligned}$$

This means that the sojourn time is exponentially distributed with mean $1/\mu(1-\beta)$. From this we obtain the following results:

$$E[W] = \frac{1}{\mu(1-\beta)} = \frac{1}{\mu} = \frac{\beta}{\mu(1-\beta)}$$

$$E[N_q] = \lambda E[W] = \frac{\beta}{1-\beta}$$

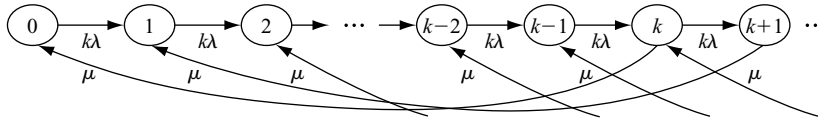


Figure 5.16. State-transition-rate diagram for the E_k/M/1 queue.

5.9.1 The E_k/M/1 Queue

The E_k/M/1 queue is a G/ M/1 queue in which the interarrival time has the Erlang-k distribution. It is usually modeled by a process in which an arrival consists of a customer passing, stage by stage, through a series of k independent and identically distributed substations, each of which has an exponentially distributed service time with mean $1/k\lambda$. Thus, the total mean interarrival time is $k \times (1/k\lambda) = 1/\lambda$. The state-transition-rate diagram for the system is shown in Figure 5.16. Note that the states represent arrival stages. Thus, when the system is in state 0, an arrival is complete only when the system enters state k and then service can commence. A completion of service causes the system to jump k states to the left.

As an example of a G/M/1 queue, we analyze the system by applying the results of the G/M/1 analysis as follows:

$$f_A(t) = \frac{(k\lambda)^k t^{k-1} e^{-k\lambda x}}{(k-1)!}$$

$$M_A(s) = \left(\frac{k\lambda}{s+k\lambda}\right)^k$$

$$\beta = M_A(\mu - \mu\beta) = \left(\frac{k\lambda}{\mu - \mu\beta + k\lambda}\right)^k = \left(\frac{k\rho}{1-\beta+k\rho}\right)^k$$

For the special case of $k = 1$, the equation $\beta = M_A(\mu - \mu\beta)$ has the solutions $\beta = 1$ and $\beta = \rho$. Because we seek a solution $0 < \beta < 1$, we accept the solution $\beta = \rho$. Similarly, when $k = 2$, the equation becomes

$$(\beta - 1)\left\{\beta^2 - \beta(1+4\rho) + 4\rho^2\right\} = 0$$

whose solutions are

$$\beta_1 = 1$$

$$\beta_2 = \frac{1+4\rho+\sqrt{1+8\rho}}{2} = 2\rho + \frac{1}{2} + \sqrt{2\rho + \frac{1}{4}} \in (1,4)$$

$$\beta_3 = \frac{1 + 4\rho - \sqrt{1 + 8\rho}}{2} = 2\rho + \frac{1}{2} - \sqrt{2\rho + \frac{1}{4}} \in (0, 1)$$

Because there exist values of $0 \leq \rho < 1$ for which β_2 has values greater than 1 and we seek a solution $0 < \beta < 1$, we accept β_3 as the only valid solution. Thus, with this value of β we can obtain all the relevant performance parameters.

5.9.2 The D/M/1 Queue

In this case, the interarrival times are deterministic with a constant value of $1/\lambda$, and service times are exponentially distributed with mean $1/\mu$. There is one server and infinite capacity. This can be used to model a system with a periodic arrival stream of customers, as is the case with time-division multiplexing (TDM) voice communication systems. Thus, we have that

$$f_A(t) = \delta(t - 1/\lambda)$$
$$M_A(s) = e^{-s/\lambda}$$
$$\beta = M_A(\mu - \mu\beta) = e^{-(\mu - \mu\beta)/\lambda} = e^{-(1-\beta)/\rho} \Rightarrow \rho \ln \beta = \beta - 1$$

The last equation can be solved iteratively for a fixed value of ρ to obtain a solution for β in the range $0 < \beta < 1$ that can be used to obtain the performance parameters of the system. Table 5.1 shows the solutions to the preceding equation for different values of ρ.

Table 5.1 Values of β for different values of ρ

ρ	β
0.1	0.00004
0.2	0.00698
0.3	0.04088
0.4	0.10735
0.5	0.20319
0.6	0.32424
0.7	0.46700
0.8	0.62863
0.9	0.80690

That is, for a given value of ρ we can obtain the mean total time in the system, the mean waiting time, and the mean number of customers in queue, respectively, as

$$E[T] = \frac{1}{\mu(1-\beta)}$$
$$E[W] = \frac{1}{\mu(1-\beta)} - \frac{1}{\mu} = \frac{\beta}{\mu(1-\beta)}$$
$$E[N_q] = \lambda E[W] = \frac{\beta}{1-\beta}$$

5.9.3 The H_2/M/1 Queue

This is a G/M/1 queueing system with the second-order hyperexponential arrival process with rates λ_1 and λ_2. Thus, we have the following parameters:

$$f_A(t) = \theta_1\lambda_1 e^{-\lambda_1 t} + \theta_2\lambda_2 e^{-\lambda_2 t}$$
$$E[A] = \frac{1}{\lambda} = \frac{\theta_1}{\lambda_1} + \frac{\theta_2}{\lambda_2} \Rightarrow \lambda = \frac{\lambda_1\lambda_2}{\theta_1\lambda_2 + \theta_2\lambda_1}$$
$$\rho = \lambda/\mu = \frac{\lambda_1\lambda_2}{\mu(\theta_1\lambda_2 + \theta_2\lambda_1)}$$
$$M_A(s) = \frac{\theta_1\lambda_1}{s+\lambda_1} + \frac{\theta_2\lambda_2}{s+\lambda_2}$$
$$\beta = M_A(\mu - \mu\beta) = \frac{\theta_1\lambda_1}{\mu - \mu\beta + \lambda_1} + \frac{\theta_2\lambda_2}{\mu - \mu\beta + \lambda_2}$$

where $\theta_1 + \theta_2 = 1$. The last equation implies that

$$\mu^2\beta^3 - \mu(2\mu + \lambda_1 + \lambda_2)\beta^2 + \{(\mu+\lambda_1)(\mu+\lambda_2) + \mu(\theta_1\lambda_1 + \theta_2\lambda_2)\}\beta - \{\theta_1\lambda_1(\mu+\lambda_2) + \theta_2\lambda_2(\mu+\lambda_1)\} = 0$$

Observe that as discussed earlier, $\beta = 1$ is a solution to the preceding equation. Thus, we can rewrite the equation as follows:

$$(\beta - 1)\{\mu^2\beta^2 - \mu(\mu + \lambda_1 + \lambda_2)\beta + \lambda_1\lambda_2 + \mu(\theta_1\lambda_1 + \theta_2\lambda_2)\} = 0$$

Because we exclude the solution $\beta = 1$, we seek the solution to the equation

$$\mu^2\beta^2 - \mu(\mu + \lambda_1 + \lambda_2)\beta + \lambda_1\lambda_2 + \mu(\theta_1\lambda_1 + \theta_2\lambda_2) = 0$$

that satisfies the condition $0 < \beta < 1$. Now, the solutions to the preceding equation are

$$\beta = \frac{\mu(\mu + \lambda_1 + \lambda_2) \pm \sqrt{\{\mu(\mu + \lambda_1 + \lambda_2)\}^2 - 4\mu^3(\theta_1\lambda_1 + \theta_2\lambda_2)}}{2\mu^2}$$

$$= \frac{(\mu + \lambda_1 + \lambda_2) \pm \sqrt{(\mu + \lambda_1 + \lambda_2)^2 - 4\mu(\theta_1\lambda_1 + \theta_2\lambda_2)}}{2\mu}$$

$$\beta_1 = \frac{\mu + \lambda_1 + \lambda_2}{2\mu} + \sqrt{\left\{\frac{\mu + \lambda_1 + \lambda_2}{2\mu}\right\}^2 - \frac{\theta_1\lambda_1 + \theta_2\lambda_2}{\mu}}$$

$$\beta_2 = \frac{\mu + \lambda_1 + \lambda_2}{2\mu} - \sqrt{\left\{\frac{\mu + \lambda_1 + \lambda_2}{2\mu}\right\}^2 - \frac{\theta_1\lambda_1 + \theta_2\lambda_2}{\mu}}$$

Thus, we choose β_2 as the right solution. Consider the special case where $\theta_1 = \theta_2 = 0.5$ and $\lambda_1 = 2\lambda_2$, which gives $\rho = 2\lambda_1/3\mu$. The solution becomes

$$\beta = \frac{1}{2} + \frac{9\rho}{8} - \sqrt{\frac{1}{4} + \frac{81\rho^2}{64}}$$

With this value of β we can obtain the performance parameters.

5.10 Applications of Markovian Queues

The application areas of queueing theory have changed over the years. In the early days of the development of the theory, many of the application areas were service networks in operations research. These included inventory systems, dams, and reliability studies. Today, the theory is widely used in computer systems, telecommunication networks, and manufacturing systems. Much of the current research on queueing theory is motivated by the need to understand and control the behavior of these systems in order to improve their design and performance. Many queueing problems that arise in practice involve making one or more of three basic decisions:

- Number of servers c required in a service facility to provide an acceptable quality of service.
- Efficiency of the servers, which is reflected by ρ, the system utilization.
- How to minimize the mean waiting time of customers for a fixed number of servers.

M/M/1 queue and its variants are used in many telecommunication systems to model the arrival of messages at a transmission channel with input buffers. Here, the service time is the message length divided by the channel capacity. Multi-server queues are used for capacity planning of telephone networks, checkout counters, and banking systems. The M/M/c queue is used to model the so-called *blocked-calls-delayed* systems, and the M/M/c/c queue is used to model the so-called *blocked-calls-lost* systems.

5.11 Problems

5.1 People arrive to buy tickets at a movie theater according to a Poisson process with an average rate of 12 customers per hour. The time it takes to complete the sale of a ticket to each person is exponentially distributed with a mean of 3 minutes. There is only one cashier at the ticket window, and any arriving customer that finds the cashier busy joins a queue that is served in an FCFS manner.

- **a.** What is the probability that an arriving customer does not have to wait?
- **b.** What is the mean number of waiting customers at the window?
- **c.** What is the mean waiting time of an arbitrary customer?

5.2 Cars arrive at a car wash according to a Poisson process with a mean rate of 8 cars per hour. The policy at the car wash is that the next car cannot pass through the wash procedure until the car in front of it is completely finished. The car wash has a capacity to hold 10 cars, including the car being washed, and the time it takes a car to go through the wash process is exponentially distributed with a mean of 6 minutes. What is the average number of cars lost to the car wash company every 10-hour day as a result of the capacity limitation?

5.3 A shop has five identical machines that break down independently of each other. The time until a machine breaks down is exponentially distributed with a mean of 10 hours. There are two repairmen who fix the machines when they fail. The time to fix a machine when it fails is exponentially distributed with a mean of 3 hours, and a failed machine can be repaired by either of the two repairmen. What is the probability that exactly one machine is operational at any one time?

5.4 People arrive at a phone booth according to a Poisson process with a mean rate of five people per hour. The duration of calls made at the phone booth is exponentially distributed with a mean of 4 minutes.

- **a.** What is the probability that a person arriving at the phone booth will have to wait?

b. The phone company plans to install a second phone at the booth when it is convinced that an arriving customer would expect to wait at least 3 minutes before using the phone. At what arrival rate will this occur?

5.5 People arrive at a library to borrow books according to a Poisson process with a mean rate of 15 people per hour. There are two attendants at the library, and the time to serve each person by either attendant is exponentially distributed with a mean of 3 minutes.

a. What is the probability that an arriving person will have to wait?
b. What is the probability that one or both attendants are idle?

5.6 A clerk provides exponentially distributed service to customers who arrive according to a Poisson process with an average rate of 15 per hour. If the service facility has an infinite capacity, what is the mean service time that the clerk must provide in order that the mean waiting time shall be no more than 10 minutes?

5.7 A company is considering how much capacity K to provide in its new service facility. When the facility is completed, customers are expected to arrive at the facility according to a Poisson process with a mean rate of 10 customers per hour, and customers that arrive when the facility is full are lost. The company hopes to hire an attendant to serve at the facility. Because the attendant is to be paid by the hour, the company hopes to get its money's worth by making sure that the attendant is not idle for more than 20% of the time she should be working. The service time is expected to be exponentially distributed with a mean of 5.5 minutes.

a. How much capacity should the facility have to achieve this goal?
b. With the capacity obtained in part (a), what is the probability that an arriving customer is lost?

5.8 A small PBX serving a startup company can only support five lines for communication with the outside world. Thus, any employee who wants to place an outside call when all five lines are busy is blocked and will have to hang up. A blocked call is considered to be lost because the employee will not make that call again. Calls to the outside world arrive at the PBX according to a Poisson process with an average rate of six calls per hour, and the duration of each call is exponentially distributed with a mean of 4 minutes.

a. What is the probability that an arriving call is blocked?
b. What is the actual arrival rate of calls to the PBX?

5.9 A machine has four identical components that fail independently. When a component is operational, the time until it fails is exponentially distributed with a mean of 10 hours. There is one resident repairman at the site so that when a component fails he immediately swaps it out and commences

servicing it to bring it back to the operational state. When he has finished repairing a failed component, the repairman immediately swaps it back into the machine. The time to service each failed component is exponentially distributed with a mean of 2 hours. Assume the machine needs at least two operational components to work.

a. What is the probability that only one component is working?
b. What is the probability that the machine is down?

5.10 A cyber cafe has six PCs that customers can use for Internet access. These customers arrive according to a Poisson process with an average rate of six per hour. Customers who arrive when all six PCs are being used are blocked and have to go elsewhere for their Internet access. The time that a customer spends using a PC is exponentially distributed with a mean of 8 minutes.

a. What fraction of arriving customers are blocked?
b. What is the actual arrival rate of customers at the cafe?
c. What fraction of arriving customers would be blocked if one of the PCs is out of service for a very long time?

5.11 Consider a birth-and-death process representing a multi-server finite population system with the following birth-and-death rates:

$$\lambda_k = (4-k)\lambda \quad k = 0, 1, 2, 3, 4$$
$$\mu_k = k\mu \qquad\quad k = 1, 2, 3, 4$$

a. Find the $p_k, k = 0, 1, 2, 3, 4$ in terms of λ and μ.
b. Find the average number of customers in the system.

5.12 Students arrive at a checkout counter in the college cafeteria according to a Poisson process with an average rate of 15 students per hour. There are three cashiers at the counter, and they provide identical service to students. The time to serve a student by any cashier is exponentially distributed with a mean of 3 minutes. Students who find all cashiers busy on their arrival join a single queue. What is the probability that at least one cashier is idle?

5.13 Customers arrive at a checkout counter in a grocery store according to a Poisson process with an average rate of 10 customers per hour. There are two clerks at the counter, and the time either clerk takes to serve each customer is exponentially distributed with an unspecified mean. If it is desired that the probability that both cashiers are idle is to be no more than 0.4, what will be the mean service time?

5.14 Consider an M/M/1/5 queueing system with mean arrival rate λ and mean service time $1/\mu$ that operates in the following manner. When any customer is in queue, the time until he defects (i.e., leaves the queue without receiving service) is exponentially distributed with a mean of $1/\beta$. It is assumed that when a customer begins receiving service he does not defect.

a. Give the state-transition-rate diagram of the process.
b. What is the probability that the server is idle?
c. Find the average number of customers in the system.

5.15 Consider an M/M/1 queueing system with mean arrival rate λ and mean service time $1/\mu$ that operates in the following manner. When the number of customers in the system is greater than three, a newly arriving customer joins the queue with probability p and balks (i.e., leaves without joining the queue) with probability $1 - p$.

a. Give the state-transition-rate diagram of the process.
b. What is the probability that the server is idle?
c. Find the actual arrival rate of customers in the system.

5.16 Consider an M/M/1 queueing system with mean arrival rate λ and mean service time $1/\mu$. The system provides bulk service in the following manner. When the server completes any service, the system returns to the empty state if there are no waiting customers. Customers who arrive while the server is busy join a single queue. At the end of a service completion, all waiting customers enter the service area to begin receiving a bulk service.

a. Give the state-transition-rate diagram of the process.
b. What is the probability that the server is idle?

5.17 Consider an M/M/2 queueing system with hysteresis. Specifically, the system operates as follows. Customers arrive according to a Poisson process with rate λ customers per second. There are two identical servers, each of which serves at the rate of μ customers per second, but as long as the number of customers in the system is less than eight, only one server is busy serving them. When the number of customers exceeds eight; the second server is brought in and the two will continue to serve until the number of customers in the system decreases to four when the server that has just completed a service is retired and only one server is allowed in the system.

a. Give the state-transition-rate diagram of the process.
b. What is the probability that both servers are idle?
c. What is the probability that exactly one server is busy?
d. What is the expected waiting time in the system?

5.18 Consider an M/G/1 queueing system where service is rendered in the following manner. Before a customer is served, a biased coin whose probability of heads is p is flipped. If it comes up heads, the service time is exponentially distributed with mean $1/\mu_1$. If it comes up tails, the service time is constant at d. Calculate the following:

a. The mean service time, $E[X]$
b. The coefficient of variation, C_X, of the service time

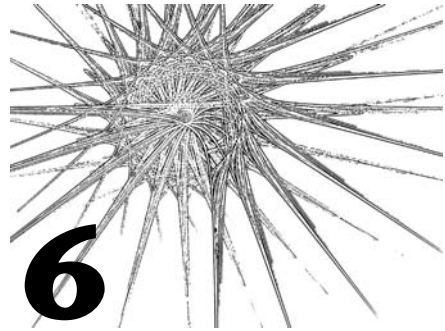
c. The expected waiting time, $E[W]$
d. The s-transform of the PDF of W

5.19 Consider a finite-capacity G/M/1 queueing that allows at most 3 customers in the system including the customer receiving service. The time to serve a customer is exponentially distributed with mean $1/\mu$. As usual, let r_n denote the probability that n customers are served during an interarrival time, $n = 0, 1, 2, 3$, and let $f_A(t)$ be the PDF of the interarrival times. Find r_n in terms of μ and $M_A(s)$, where $M_A(s)$ is the s-transform of $f_A(t)$.

5.20 Consider a queueing system in which the interarrival times of customers are the third-order Erlang random variable with parameter λ. The time to serve a customer is exponentially distributed with mean $1/\mu$. What is the expected waiting time of a customer?

5.21 Consider a queueing system in which customers arrive according to a Poisson process with rate λ. The time to serve a customer is a third-order Erlang random variable with parameter μ. What is the expected waiting time of a customer?

Markov Renewal Processes



6.1 Introduction

Consider an experiment that involves a set of identical lightbulbs whose lifetimes are independent. The experiment consists of using one lightbulb at a time, and when it fails it is immediately replaced by another lightbulb from the set. Each time a failed lightbulb is replaced constitutes a *renewal event*. Let X_i denote the lifetime of the ith lightbulb, $i = 1, 2, \ldots$, where $X_0 = 0$. Because the lightbulbs are assumed to be identical, the X_i are independent and identically distributed with PDF $f_X(x)$, $x \geq 0$, and mean $E[X]$.

Let $N(t)$ denote the number of renewal events up to and including the time t, where it is assumed that the first lightbulb was turned on at time $t = 0$. The time to failure T_n of the first n lightbulbs is given by

$$
\begin{aligned}
T_0 &= 0 \\
T_1 &= X_1 \\
T_2 &= X_1 + X_2 \\
&\cdots \\
T_n &= X_1 + \cdots + X_n
\end{aligned}
$$

The relationship between the interevent times X_n and the T_n is illustrated in Figure 6.1, where E_k denotes the kth event.

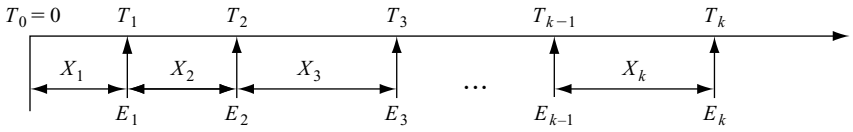


Figure 6.1. Interarrival times of a renewal process.

T_n is called the time of the nth renewal, and we have that

$$N(t) = \max\{n | T_n \leq t\}$$

Thus, the process $\{N(t) | t \geq 0\}$ is a counting process known as a *renewal process*, and $N(t)$ denotes the number of renewals up to time t. Observe that the event that the number of renewals up to and including the time t is less than n is equivalent to the event that the nth renewal occurs at a time that is later than t. Thus, we have that

$$\{N(t) < n\} = \{T_n > t\}$$

Therefore, $P[N(t) < n] = P[T_n > t]$. Let $f_{T_n}(t)$ and $F_{T_n}(t)$ denote the PDF and CDF, respectively, of T_n. Thus, we have that

$$P[N(t) < n] = P[T_n > t] = 1 - F_{T_n}(t)$$

Because $P[N(t) = n] = P[N(t) < n+1] - P[N(t) < n]$, we obtain the following result for the PMF of $N(t)$:

$$\begin{aligned} p_{N(t)}(n) = P[N(t) = n] &= P[N(t) < n+1] - P[N(t) < n] \\ &= 1 - F_{T_{n+1}}(t) - \{1 - F_{T_n}(t)\} = F_{T_n}(t) - F_{T_{n+1}}(t) \end{aligned}$$

6.2 The Renewal Equation

The expected number of renewals by time t is called the *renewal function*. It is denoted by $H(t)$ and given by

$$\begin{aligned} H(t) = E[N(t)] &= \sum_{n=0}^{\infty} n P[N(t) = n] = \sum_{n=0}^{\infty} n\{F_{T_n}(t) - F_{T_{n+1}}(t)\} \\ &= \{F_{T_1}(t) + 2F_{T_2}(t) + 3F_{T_3}(t) + \cdots\} - \{F_{T_2}(t) + 2F_{T_3}(t) + \cdots\} \\ &= F_{T_1}(t) + F_{T_2}(t) + F_{T_3}(t) + \cdots = \sum_{n=1}^{\infty} F_{T_n}(t) \end{aligned}$$

If we take the derivative of each side we obtain

$$h(t) = \frac{dH(t)}{dt} = \frac{d}{dt}\sum_{n=1}^{\infty} F_{T_n}(t) = \sum_{n=1}^{\infty}\frac{d}{dt}F_{T_n}(t) = \sum_{n=1}^{\infty} f_{T_n}(t)$$

where $h(t)$ is called the *renewal density*. Let $M_h(s)$ denote the Laplace transform of $h(t)$ and $M_{T_n}(s)$ the s-transform of $f_{T_n}(t)$. Because T_n is the sum of n independent and identically distributed random variables, the PDF $f_{T_n}(t)$ is the n-fold convolution of the PDF of X. Thus, we have that $M_{T_n}(s) = \{M_X(s)\}^n$. From this we obtain $M_h(s)$ as follows:

$$\begin{aligned} M_h(s) &= \sum_{n=1}^{\infty} M_{T_n}(s) = \sum_{n=1}^{\infty}\{M_X(s)\}^n \\ &= \frac{1}{1 - M_X(s)} - 1 = \frac{M_X(s)}{1 - M_X(s)} \end{aligned}$$

This gives

$$M_h(s) = M_X(s) + M_h(s)M_X(s)$$

Taking the inverse transform we obtain

$$h(t) = f_X(t) + \int_{u=0}^{t} h(t-u)\, f_X(u)du$$

Finally, integrating both sides of the equation, we obtain

$$H(t) = F_X(t) + \int_{u=0}^{t} H(t-u)\, f_X(u)du$$

This equation is called the *fundamental equation of renewal theory*.

Example 6.1 Assume that X is exponentially distributed with mean $1/\lambda$. Then we obtain

$$\begin{aligned} f_X(t) &= \lambda e^{-\lambda t} \\ M_X(s) &= \frac{\lambda}{s+\lambda} \\ M_h(s) &= \frac{M_X(s)}{1 - M_X(s)} = \frac{\lambda}{s} \\ h(t) &= L^{-1}\{M_h(s)\} = \lambda \\ H(t) &= \int_{u=0}^{t} h(u)du = \lambda t \end{aligned}$$

where $L^{-1}\{M_h(s)\}$ is the inverse Laplace transform of $M_h(s)$. ■

6.2.1 Alternative Approach

An alternative method of studying the renewal process is as follows. With respect to the lightbulbs, the renewal density $h(t)$ represents the conditional failure rate; that is, $h(t)dt$ is the probability that a lightbulb will fail in the interval $(t, t+dt)$, given that it has survived up to time t. Thus, we can write

$$h(t)dt = P[X \le t + dt | X > t] = \frac{P[t < X \le t + dt]}{P[X > t]} = \frac{F_X(t+dt) - F_X(t)}{1 - F_X(t)}$$

$h(t)$ is sometimes called the *hazard function* of X and represents the instantaneous rate at which an item will fail, given that it survived up to time t. If we define $\overline{F}_X(t) = 1 - F_X(t)$ then we obtain

$$\begin{aligned}\{1 - F_X(t)\}h(t)dt &= F_X(t+dt) - F_X(t) = F_X(t+dt) - F_X(t) - 1 + 1 \\ &= \{1 - F_X(t)\} - \{1 - F_X(t+dt)\}\end{aligned}$$

That is,

$$\overline{F}_X(t) - \overline{F}_X(t+dt) = \overline{F}_X(t)h(t)dt$$

Thus,

$$\frac{\frac{d}{dt}\overline{F}_X(t)}{\overline{F}_X(t)} = \frac{d}{dt}\ln\{\overline{F}_X(t)\} = -h(t)$$

Integrating both sides yields

$$[\ln\{\overline{F}_X(u)\}]_{u=0}^{t} = -\int_0^t h(u)du$$

Because $\overline{F}_X(0) = 1$, we obtain $\ln\{\overline{F}_X(t)\} = -\int_0^t h(u)du$ and

$$\overline{F}_X(t) = \exp\left\{-\int_0^t h(u)du\right\}$$

The probability density function (PDF) of the age $X(t)$ is given by

$$f_X(t) = -\frac{d\overline{F}_X(t)}{dt} = h(t)\exp\left\{-\int_0^t h(u)du\right\} = h(t)e^{-H(t)}$$

This implies that $h(t)$, $F_X(t)$, or $f_X(t)$ can be used to characterize the distribution of the lifetimes. Because the renewal function $H(t) = E[N(t)]$, the previous results can be expressed as follows:

$$\overline{F}_X(t) = \exp\left\{-\int_0^t h(u)du\right\} = e^{-H(t)} = e^{-E[N(t)]}$$

$$f_X(t) = -\frac{\overline{F}_X(t)}{dt} = h(t)\exp\left\{-\int_0^t h_X(u)du\right\} = h_X(t)e^{-E[N(t)]}$$

Now we have that

$$F_{T_1}(t) = P[T_1 \le t] = F_X(t)$$
$$F_{T_2}(t) = P[T_2 \le t] = P[X_1 + X_2 \le t] = F_X^{(2)}(t)$$
$$F_{T_n}(t) = P[T_n \le t] = P[X_1 + \cdots + X_n \le t] = F_X^{(n)}(t)$$

where $F_X^{(k)}(t)$ is the k-fold time convolution of $F_X(t)$. From these equations we obtain

$$P[N(t) \ge k] = P[T_k \le t] = F_X^{(k)}(t)$$
$$F_{N(t)}(k) = P[N(t) \le k] = 1 - P[N(t) \ge k+1] = 1 - P[T_{k+1} \le t]$$
$$= 1 - F_X^{(k+1)}(t)$$

Thus,

$$p_{N(t)}(k) = P[N(t) = k] = P[N(t) \le k] - P[N(t) \le k-1]$$
$$= \{1 - F_X^{(k+1)}(t)\} - \{1 - F_X^{(k)}(t)\}$$
$$= F_X^{(k)}(t) - F_X^{(k+1)}(t)$$
$$p_{N(t)}(0) = 1 - F_X(t)$$

The renewal function is then given by

$$H(t) = E[N(t)] = \sum_{k=0}^{\infty} k p_{N(t)}(k) = \sum_{k=0}^{\infty} k p_{N(t)}(k)$$
$$= \{F_X(t) - F_X^{(2)}(t)\} + \{2F_X^{(2)}(t) - 2F_X^{(3)}(t)\} + \{3F_X^{(3)}(t) - 3F_X^{(4)}(t)\} + \cdots$$
$$= \sum_{k=1}^{\infty} F_X^{(k)}(t) = \sum_{k=1}^{\infty} F_{T_k}(t)$$

which is the result we obtained earlier. If $F_X(t)$ is absolutely continuous, then $F_X^{(k)}(t)$ is absolutely continuous and we obtain

$$H(t) = \sum_{k=1}^{\infty} F_X^{(k)}(t) = \sum_{k=1}^{\infty} \int_0^t f_X^{(k)}(u)du$$

Using Fubini's theorem, we obtain

$$H(t) = \int_0^t \left\{ \sum_{k=1}^{\infty} f_X^{(k)}(u) \right\} du \equiv \int_0^t h(u)du$$

Thus, we have that

$$h(u) = \sum_{k=1}^{\infty} f_X^{(k)}(u)$$

Example 6.2 X is a second-order Erlang random variable with parameter λ. Thus, $f_X(t) = \lambda^2 te^{-\lambda t}$ and $M_X(s) = [\lambda/(s+\lambda)]^2$. From this we obtain

$$M_h(s) = \frac{[\lambda/(s+\lambda)]^2}{1-[\lambda/(s+\lambda)]^2} = \frac{\lambda^2}{s(s+2\lambda)} = \frac{\lambda}{2}\left\{\frac{1}{s} - \frac{1}{s+2\lambda}\right\}$$

Thus,

$$h(t) = \frac{\lambda}{2}\{1 - e^{-2\lambda t}\} \qquad t > 0$$

Note that $\lim_{t\to\infty} h(t) = \lambda/2$. ■

6.3 The Elementary Renewal Theorem

We state the following theorem called the elementary renewal theorem without proof:

$$\lim_{t\to\infty} \frac{H(t)}{t} = \frac{1}{E[X]}$$

6.4 Random Incidence and Residual Time

Consider a renewal process $N(t)$ in which events (or arrivals) occur at times $0 = T_0, T_1, T_2, \ldots$. As discussed earlier, the interevent times X_k can be defined in terms of the T_k as follows:

$$X_1 = T_1 - T_0 = T_1$$
$$X_2 = T_2 - T_1$$
$$\cdots$$
$$X_k = T_k - T_{k-1}$$

Note that the X_k are mutually independent and identically distributed.

Consider the following problem in connection with the X_k. Assume the T_k are the points in time that buses arrive at a bus stop. A passenger arrives at the bus stop at a *random time* and wants to know how long he or she will wait until the next bus arrival. This problem is usually referred to as the *random incidence problem*, because the subject (or passenger in this example) is incident to the process at a random time. Let R be the random variable that denotes the time from the moment the passenger arrived until the next bus arrival. R is referred to as the *residual life* of the renewal process. Also, let W denote the length of the interarrival gap that the passenger entered by random incidence. Figure 6.2 illustrates the random incidence problem.

Let $f_X(x)$ denote the PDF of the interarrival times; let $f_W(w)$ denote the PDF of W, the gap entered by random incidence; and let $f_R(r)$ denote the PDF of the residual life, R. The probability that the random arrival occurs in a gap of length between w and $w + dw$ can be assumed to be directly proportional to the length w of the gap and relative occurrence $f_X(w)dw$ of such gaps. That is,

$$f_W(w)dw = \beta w f_X(w)dw$$

where β is a constant of proportionality. Thus, $f_W(w) = \beta w f_X(w)$. Because $f_W(w)$ is a PDF, we have that

$$\int_{-\infty}^{\infty} f_W(w)dw = 1 = \beta \int_{-\infty}^{\infty} w f_X(w)dw = \beta E[X]$$

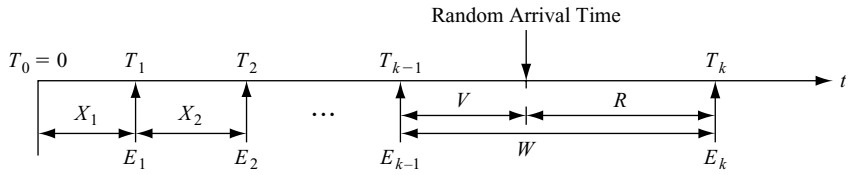


Figure 6.2. Random incidence.

Thus, $\beta = 1/E[X]$, and we obtain

$$f_W(w) = \frac{wf_X(w)}{E[X]}$$

The expected value of W is given by $E[W] = E[X^2]/E[X]$. This result applies to all renewal processes.

A Poisson process is an example of a renewal process in which X is exponentially distributed with $E[X] = 1/\lambda$ and $E[X^2] = 2/\lambda^2$. Thus, for a Poisson process we obtain

$$f_W(w) = \lambda w f_X(w) = \lambda^2 w e^{-\lambda w} \qquad w \geq 0$$
$$E[W] = 2/\lambda$$

This means that for a Poisson process the gap entered by random incidence has the second-order Erlang distribution; thus, the expected length of the gap is twice the expected length of an interarrival time. This is often referred to as the *random incidence paradox*. The reason for this fact is that the passenger is more likely to enter a large gap than a small gap; that is, the gap entered by random incidence is not a typical interval.

Next, we consider the PDF of the residual life R of the process. Given that the passenger enters a gap of length w, he or she is equally likely to be anywhere within the gap. Thus, the conditional PDF of R, given that $W = w$, is given by

$$f_{R|W}(r|w) = \frac{1}{w} \qquad 0 \leq r \leq w$$

When we combine this result with the previous one, we get the joint PDF of R and W as follows:

$$f_{RW}(r, w) = f_{R|W}(r|w) f_W(w) = \frac{1}{w}\left\{\frac{wf_X(w)}{E[X]}\right\}$$
$$= \frac{f_X(w)}{E[X]} \qquad 0 \leq r \leq w \leq \infty$$

The marginal PDF of R and its expected value become

$$f_R(r) = \int_{-\infty}^{\infty} f_{RW}(r, w)dw = \int_r^{\infty} \frac{f_X(w)}{E[X]}dw$$
$$= \frac{1 - F_X(r)}{E[X]} \qquad r \geq 0$$
$$E[R] = \int_0^{\infty} rf_R(r)dr = \frac{1}{E[X]}\int_{r=0}^{\infty} r\int_{w=r}^{\infty} f_X(w)dwdr$$
$$= \frac{1}{E[X]}\int_{w=0}^{\infty}\int_{r=0}^{w} rf_X(w)drdw$$

$$= \frac{1}{E[X]} \int_{w=0}^{\infty} f_X(w) \left[\frac{r^2}{2} \right]_0^w dw$$

$$= \frac{1}{2E[X]} \int_{w=0}^{\infty} w^2 f_X(w) dw = \frac{E[X^2]}{2E[X]}$$

For the Poisson process, X is exponentially distributed and $1 - F_X(r) = e^{-\lambda r}$, which means that

$$f_R(r) = \lambda e^{-\lambda r} \qquad r \geq 0$$

Thus, for a Poisson process, the residual life of the process has the same distribution as the interarrival time, which can be expected from the "forgetfulness" property of the exponential distribution. In Figure 6.2 the random variable V denotes the time between the last bus arrival and the passenger's random arrival. Because $W = V + R$, the expected value of V is $E[V] = E[W] - E[R]$. For a Poisson process, $E[V] = 2/\lambda - 1/\lambda = 1/\lambda$.

6.5 Markov Renewal Process

We have earlier defined the renewal process $\{N(t)|t \geq 0\}$ as a counting process that denotes the number of renewals up to time t. The Markov renewal process is a generalization of the renewal process in which the times between renewals are chosen according to a Markov chain. It is the framework that results when one analyzes non-Markovian models such as the PH/G/1 or G/PH/1 queues through the "embedded chain" technique.

Consider a random variable X_n that takes values in a countable set Ω, and a random variable T_n that takes values in the interval $[0, \infty)$ such that $0 = T_0 \leq T_1 \leq T_2 \leq \ldots$. The stochastic process $\{(X_n, T_n)|n \in \Omega\}$ is defined to be a Markov renewal process with state space Ω if

$$P[X_{n+1} = j, T_{n+1} - T_n \leq t | X_0, \ldots, X_n; T_0, \ldots, T_n]$$
$$= P[X_{n+1} = j, T_{n+1} - T_n \leq t | X_n]$$

for $n = 0, 1, \ldots$; $j \in \Omega$; and $t \in [0, \infty)$. An alternative definition for the Markov renewal process is as follows. Let $N_k(t)$ denote the number of times the process $\{(X_n, T_n)\}$ visits state $X_n = k$ in the interval $(0, t]$, then $\{N_k(t), k \in \Omega, t \geq 0\}$ is a Markov renewal process. In particular, if we assume that the initial state is k, then transitions into state k constitute renewals, which means that successive times between transitions into state k are independent and identically distributed.

The interval $H_n = T_{n+1} - T_n, n \geq 0$, is called the holding time or waiting time in state X_n.

We define the following function:

$$V_k(n,t) = \begin{cases} 1 & \text{if } X_n = k,\ H_0 + \cdots + H_{n-1} \leq t, n \in I_+ \\ 0 & \text{otherwise} \end{cases}$$

where $I_+ = \{1, 2, \ldots\}$, $k \in \Omega$, and $t \geq 0$. Then

$$N_k(t) = \sum_{n=0}^{\infty} V_k(n,t) \qquad k \in \Omega,\ t \geq 0$$

6.5.1 The Markov Renewal Function

The function

$$M_{ik}(t) = E[N_k(t)|X_0 = i] \qquad i, k \in \Omega,\ t \geq 0$$

is called the *Markov renewal function*. Substituting the value of $N_k(t)$ we obtain

$$M_{ik}(t) = E\left[\sum_{n=0}^{\infty} V_k(n,t)|X_0 = i\right] = \sum_{n=0}^{\infty} E[V_k(n,t)|X_0 = i]$$

$$= \sum_{n=1}^{\infty} P[X_n = k,\ J_n \leq t|X_0 = i]$$

where $J_n = H_0 + \cdots + H_{n-1}$ is the time from the beginning until the process enters state X_n; alternatively, it is the epoch of the nth transition of the process $\{(X_n, T_n)\}$. We define the *one-step transition probability* $Q_{ij}(t)$ of the semi-Markov process by

$$Q_{ij}(t) = P[X_{n+1} = j,\ H_n \leq t|X_n = i] \qquad t \geq 0$$

independent of n. Thus, $Q_{ij}(t)$ is the conditional probability that the process will be in state j next, given that it is currently in state i and the waiting time in the current state i is no more than t. The family of probabilities $Q = \{Q_{ij}(t), i, j \in \Omega; t \geq 0\}$ is called the *semi-Markov kernel* over Ω. In particular,

$$Q_{ik}(t) = P[X_1 = k,\ H_0 \leq t|X_0 = i] \qquad t \geq 0$$

Thus,

$$\begin{aligned} M_{ik}(t) &= \sum_{n=1}^{\infty} P[X_n = k, J_n \le t | X_0 = i] \\ &= P[X_1 = k, J_1 \le t | X_0 = i] + \sum_{n=2}^{\infty} P[X_n = k, J_n \le t | X_0 = i] \\ &= P[X_1 = k, H_0 \le t | X_0 = i] + \sum_{n=2}^{\infty} P[X_n = k, J_n \le t | X_0 = i] \\ &= Q_{ik}(t) + \sum_{n=2}^{\infty} P[X_n = k, J_n \le t | X_0 = i] \end{aligned}$$

If we define

$$Q_{ik}^{(n)}(t) = P[X_n = k, J_n \le t | X_0 = i]$$

$$Q_{ik}^{(0)}(t) = \begin{cases} 0 & \text{if } i \ne k \\ 1 & \text{if } i = k \end{cases}$$

then $Q_{ik}^{(1)}(t) = Q_{ik}(t)$ and

$$\begin{aligned} Q_{ik}^{(2)}(t) &= P[X_2 = k, J_2 \le t | X_0 = i] \\ &= \sum_{j \in S} \int_0^t P[X_2 = k, H_1 \le t - u | X_1 = j] dQ_{ij}(u) \\ &= \sum_{j \in S} \int_0^t Q_{ij}(t-u) dQ_{jk}(u) \end{aligned}$$

From this we can easily obtain the following recursive relationship:

$$Q_{ik}^{(n+1)}(t) = \sum_{j \in S} \int_0^t Q_{ij}^{(n)}(t-u) dQ_{jk}(u)$$

If we define the matrix $Q = [Q_{ik}]$, then the above expression is the convolution of $Q^{(n)}$ and Q. That is,

$$Q^{(n+1)}(t) = Q^{(n)}(t) * Q(t)$$

Thus, if we define the matrix $M(t) = [M_{ik}(t)]$, we obtain

$$M(t) = \sum_{n=1}^{\infty} Q^{(n)}(t) \qquad t \geq 0$$

We can rewrite this equation as follows:

$$\begin{aligned} M(t) &= Q(t) + \sum_{n=2}^{\infty} Q^{(n)}(t) = Q(t) + \sum_{n=1}^{\infty} Q^{(n)}(t) * Q(t) \\ &= Q(t) + \left(\sum_{n=1}^{\infty} Q^{(n)}(t) \right) * Q(t) \\ &= Q(t) + M(t) * Q(t) = Q(t) + Q(t) * M(t) \end{aligned}$$

If we take the Laplace transform of both sides we obtain

$$\begin{aligned} M^*(s) &= Q^*(s) + Q^*(s)M^*(s) \\ [I - Q^*(s)]M^*(s) &= Q^*(s) \end{aligned}$$

where $M^*(s)$ is the Laplace transform of $M(t)$ and $Q^*(s)$ is the Laplace transform of $Q(t)$. Thus,

$$M^*(s) = [I - Q^*(s)]^{-1} Q^*(s) = \sum_{n=1}^{\infty} [Q^*(s)]^n = [I - Q^*(s)]^{-1} - I$$

Note that from the equation $M^*(s) = Q^*(s) + Q^*(s)M^*(s) = Q^*(s)[I + M^*(s)]$ we also obtain

$$Q^*(s) = M^*(s)[I + M^*(s)]^{-1}$$

6.6 Semi-Markov Processes

A semi-Markov process is both a Markov renewal process as well as a generalization of the Markov process. While a Markov renewal process is concerned with the generalized renewal random variables (i.e., it records the number of visits to each state of the process), a semi-Markov process is concerned with the random variables that describe the state of the process at some time.

In a discrete-time Markov process, we assume that the amount of time spent in each state before a transition to the next state occurs is a unit time. Similarly, in a continuous-time Markov process, we assume that the amount of time spent in a state before a transition to the next state occurs is exponentially distributed. A semi-Markov process is a process that makes transitions from state to state like a Markov process. However, the amount of time spent in each state before a transition to the next state occurs is an arbitrary random variable that depends on the next state the process will enter. Thus, at transition instants a semi-Markov process behaves like a Markov process.

6.6.1 Discrete-Time Semi-Markov Processes

Consider a finite-state discrete-time random process $\{X_n | n = 0, 1, 2, \ldots, N\}$. That is, the state space is $\Omega = \{0, 1, \ldots, N\}$. Assume that when the process enters state i it chooses its next state as state j with probability p_{ij}, where

$$p_{ij} \geq 0 \qquad i, j \in \Omega$$

$$\sum_{j=0}^{N} p_{ij} = 1$$

Let $T_0, T_1, \ldots, T_N$ denote the transition epochs on the nonnegative real line such that $0 = T_0 \leq T_1 \leq T_2 \leq \cdots \leq T_N$. Define the interval $T_{i+1} - T_i = H_i$ to be the *holding time* (or *waiting time*) in state $i \in S$. H_i can be explained as follows. After choosing j, the process spends a holding time H_{ij} before making the transition, where H_{ij} is a positive, integer-valued random variable with the PMF $p_{H_{ij}}(m) = P[H_{ij} = m]$, $m = 1, 2, \ldots$. It is assumed that $E[H_{ij}] < \infty$ and $p_{H_{ij}}(0) = 0$ for all i and j, which means that the system spends at least one unit of time before making a transition. Note that if we focus only on the transitions and ignore the times between transitions, we will have a Markov process. However, when we include the holding times, the process will no longer satisfy the Chapman-Kolmogorov equation. We call the process that governs transitions between states the *embedded Markov process*. Thus, the PMF of the holding time H_i in state i is given by

$$p_{H_i}(m) = \sum_{j=0}^{N} p_{ij} p_{H_{ij}}(m) \qquad m = 1, 2, \ldots$$

The mean holding time in state i is given by

$$E[H_i] = \sum_{j=0}^{N} p_{ij} E[H_{ij}] \qquad i = 1, 2, \ldots, N$$

The two-dimensional stochastic process $\{(X_n, T_n) | n = 0, 1, \ldots, N\}$ is called a discrete-time semi-Markov process if the following conditions are satisfied:

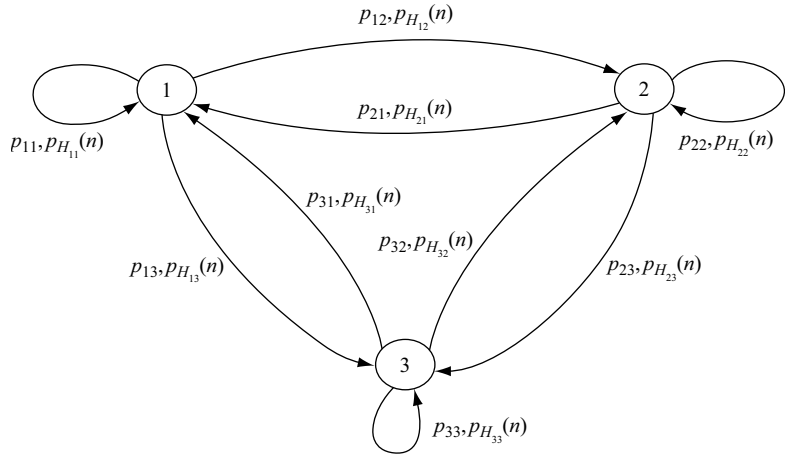


Figure 6.3. State transition diagram of a discreate-time semi-Markov process.

a. $\{X_n | n = 0, 1, \ldots, N\}$ is a Markov chain, which is called the embedded Markov chain
b. $P[X_{n+1} = j, T_{n+1} - T_n \leq m | X_0, X_1, \ldots, X_n = i; T_0, \ldots, T_n] = P[X_{n+1} = j, H_n \leq m | X_n = i], m = 0, 1, \ldots$, where $H_n = T_{n+1} - T_n$

We can represent a discrete-time semi-Markov process as shown in Figure 6.3, which is the state transition diagram of a discrete-time semi-Markov process whose embedded Markov process is a Markov chain with three states. A transition arc contains both the transition probability and the PMF of the conditional holding time.

State Probabilities

Let $\phi_{ij}(n)$ denote the probability that the process is in state j at time n given that it entered state i at time zero. In Howard (1960) $\phi_{ij}(n)$ is referred to as the "interval-transition probabilities of the process." However, a more general name is the *transition probability function* from state i to state j. To get an expression for $\phi_{ij}(n)$ we consider two cases:

a. For $i \neq j$, the process makes a transition to some state k at time m, and then in the remaining time $n - m$ it travels to state j. The probability of this event is

$$\phi_{ij}(n) = \sum_{k=1}^{N} p_{ik} \sum_{m=0}^{n} p_{H_{ik}}(m) \phi_{kj}(n - m) \quad i \neq j$$

b. For $i = j$ we have an additional probability that the process never left state i during the interval of interest. The probability of this additional event is

$$P[H_i > n] = 1 - P[H_i \leq n] = 1 - F_{H_i}(n)$$

Because the event that it never left state i during the interval and the event that it left i for some state k are mutually exclusive, we have that

$$\phi_{ij}(n) = 1 - F_{H_i}(n) + \sum_{k=1}^{N} p_{ik} \sum_{m=0}^{n} p_{H_{ik}}(m)\phi_{ki}(n-m)$$

If we define

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

then for $i, j \in S$, and $n \geq 0$ we obtain

$$\begin{aligned} \phi_{ij}(n) &= \delta_{ij}\{1 - F_{H_i}(n)\} + \sum_{k=1}^{N} p_{ik} \sum_{m=0}^{n} p_{H_{ik}}(m)\phi_{kj}(n-m) \\ &= \delta_{ij}\{1 - F_{H_i}(n)\} + \sum_{k=1}^{N} p_{ik} p_{H_{ik}}(n) * \phi_{kj}(n) \end{aligned}$$

where $\phi_{ij}(0) = \delta_{ij}$ and $*$ is the convolution operator. The preceding equation is referred to as the *backward Chapman-Kolmogorov equation* for the semi-Markov process. Let $P_H(n)$ denote the matrix of the $p_{H_{ik}}(n)$, and let P denote the state-transition matrix of the embedded Markov chain. Using the terminology of Howard (1971b), we define the *discrete-time core matrix* $C(n)$ by

$$C(n) = P \square P_H(n)$$

That is, the elements of $C(n)$ are $c_{ij}(n) = p_{ij} p_{H_{ij}}(n)$. Thus, the transition probability function becomes

$$\phi_{ij}(n) = \delta_{ij}\{1 - F_{H_i}(n)\} + \sum_{k=1}^{N} c_{ik}(n) * \phi_{kj}(n)$$

Taking the z-transform of both sides we obtain

$$\phi_{ij}(z) = \frac{\delta_{ij}\{1 - G_{H_i}(z)\}}{1 - z} + \sum_{k=1}^{N} c_{ik}(z)\phi_{kj}(z)$$

where $G_{H_i}(z)$ is the z-transform of $p_{H_i}(n)$. Let $D(z)$ be the $N \times N$ diagonal matrix whose ith element is $\{1 - G_{H_i}(z)\}/(1 - z)$. Then in the matrix form the equation becomes

$$\Phi(z) = D(z) + C(z)\Phi(z)$$

which gives

$$\Phi(z) = [I - C(z)]^{-1} D(z)$$

If we define

$$\phi_{ij} = \lim_{n \to \infty} \phi_{ij}(n)$$

then from Howard (1971b) it can be shown that

$$\phi_{ij} = \frac{\pi_j E[H_j]}{\sum_{k=1}^{N} \pi_k E[H_k]}$$

where π_j is limiting state probability of the embedded Markov chain. Note that the right-hand side of the equation is independent of i. Thus, we define the limiting probability of the semi-Markov process by

$$\phi_j = \frac{\pi_j E[H_j]}{\sum_{k=1}^{N} \pi_k E[H_k]}$$

The limiting state probability is also called the *occupancy distribution* because ϕ_j gives the long-run fraction of time that the process spends in state j.

First Passage Times

Let T_{ij} denote the first passage time from state i to state j. That is,

$$T_{ij} = \min\{n > 0 | X_n = j, X_0 = i\}$$

Let $m_{ij} = E[T_{ij}]$; that is, m_{ij} is the mean first passage time from state i to state j. We can use the same arguments used in Chapters 3 and 4 to obtain m_{ij} recursively as follows. Because the mean waiting time in state i is $E[H_i]$, then given that the process starts in state i, it will spend a mean time $E[H_i]$ before making a transition

to some state k with probability p_{ik}. Then from state k it takes a mean time m_{kj} to reach state j. Thus, we obtain

$$m_{ij} = E[H_i] + \sum_{k \neq j} p_{ik} m_{kj}$$

The mean recurrence time is obtained when $j = i$. Specifically, we note that because the limiting state probabilities of the embedded Markov chain exist, they satisfy the balance equations

$$\pi_i = \sum_{k=0}^{N} \pi_k p_{ki}$$

Thus, multiplying both sides of the mean first passage time equation by π_i and summing over i we obtain

$$\sum_{i=0}^{N} \pi_i m_{ij} = \sum_{i=0}^{N} \pi_i E[H_i] + \sum_{i=0}^{N} \pi_i \sum_{\substack{k=0 \\ k \neq j}}^{N} p_{ik} m_{kj} = \sum_{i=0}^{N} \pi_i E[H_i] + \sum_{\substack{k=0 \\ k \neq j}}^{N} m_{kj} \sum_{i=0}^{N} \pi_i p_{ik}$$

$$= \sum_{i=0}^{N} \pi_i E[H_i] + \sum_{\substack{k=0 \\ k \neq j}}^{N} m_{kj} \pi_k = \sum_{i=0}^{N} \pi_i E[H_i] + \sum_{\substack{k=0 \\ k \neq j}}^{N} \pi_k m_{kj}$$

$$= \sum_{i=0}^{N} \pi_i E[H_i] + \sum_{i=0}^{N} \pi_i m_{ij} - \pi_j m_{jj}$$

where in the third equation we have made use of the balance equation. Canceling the like terms on both sides of the equation we obtain

$$m_{jj} = \frac{\sum_{i=0}^{N} \pi_i E[H_i]}{\pi_j} \qquad j = 0, 1, \ldots, N$$

From these results we obtain the following definitions:

a. Two states i and j in a semi-Markov process are said to communicate if they communicate in the embedded Markov chain. For two such states, $P[T_{ij} < \infty] \times P[T_{ji} < \infty] > 0$.
b. A state i in a semi-Markov process is said to be a recurrent state if it is a recurrent state in the embedded Markov chain. For such a state, $P[T_{ii} < \infty] > 0$.
c. A state i in a semi-Markov process is said to be a transient state if it is a transient state in the embedded Markov chain. For such a state, $P[T_{ii} < \infty] = 0$.

6.6.2 Continuous-Time Semi-Markov Processes

Consider a finite-state continuous-time stochastic process $\{X(t), t \geq 0\}$ with state space $\Omega = \{0, 1, \ldots, N\}$. Assume that the process just entered state i at time $t = 0$, then it chooses the next state j with probability p_{ij}, where

$$p_{ij} \geq 0 \quad i \in \Omega, j \in \Omega$$

$$\sum_{j=0}^{N} p_{ij} = 1$$

Given that the next transition out of state i will be to state j, the time H_{ij} that the process spends in state i until the next transition has the PDF $f_{H_{ij}}(t)$, $t \geq 0$. The random variable H_{ij} is called the *holding time* for a transition from i to j, and is it assumed that $E[H_{ij}] < \infty$. As discussed in the discrete-time case, if we focus only on the transitions and ignore the times between transitions, we will have a Markov process. However, when we include the holding times, the process will no longer satisfy the Chapman-Kolmogorov equation unless the holding times are exponentially distributed. We call the process that governs transitions between states the *embedded Markov process*. The time H_i that the process spends in state i before making a transition is called the *waiting time* in state i, and its PDF is given by

$$f_{H_i}(t) \sum_{j=0}^{N} p_{ij} f_{H_{ij}}(t) \qquad t \geq 0$$

Thus, the mean waiting time in state i is

$$E[H_i] = \sum_{j=0}^{N} p_{ij} E[H_{ij}] \qquad i = 1, 2, \ldots, N$$

The two-dimensional stochastic process $\{(X_n, T_n) | n = 0, 1, \ldots, N\}$ is called a continuous-time semi-Markov process if the following conditions are satisfied:

a. $\{X_n | n = 0, 1, \ldots, N\}$ is a Markov chain, the embedded Markov chain
b. $P[X_{n+1} = j, T_{n+1} - T_n \leq t | X_0, \ldots, X_n = i; T_0, \ldots, T_n] = P[X_{n+1} = j, H_n \leq t | X_n = i], t \geq 0$, where $H_n = T_{n+1} - T_n$

As in the discrete-time case, we can represent a continuous-time semi-Markov process as shown in Figure 6.4, which is the state transition diagram of a semi-Markov process whose embedded Markov process is a Markov chain with three

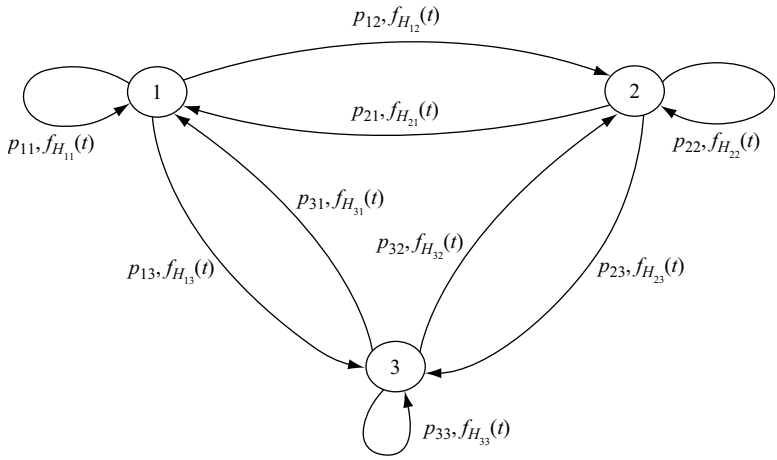


Figure 6.4. State transition diagram of a continuous-time semi-Markov process.

states. In this case, a transition arc contains both the transition probability and the PDF of the conditional holding time.

We can now see the difference between a continuous-time Markov process and a continuous-time semi-Markov process. In a continuous-time Markov process, the holding times (or sojourn times) are exponentially distributed and depend only on the current state. In a continuous-time semi-Markov process, the holding times can have an arbitrary distribution and can depend on both the current state and the state to be visited next.

A transition from state i to state j is called a *real transition* if $i \neq j$, and it is called a *virtual transition* if $i = j$. As discussed earlier, the one-step transition probability $Q_{ij}(t)$ of the semi-Markov process is defined by

$$Q_{ij}(t) = P[X_{n+1} = j, H_n \leq t | X_n = i] \qquad t \geq 0$$

independent of n. Thus, $Q_{ij}(t)$ is the conditional probability that the process will be in state j next, given that it is currently in state i and the waiting time in the current state i is no more than t. From this we obtain the transition probability p_{ij} of the embedded Markov chain and the cumulative distribution function of the waiting time at state i, $F_{H_i}(t)$, as follows:

$$\begin{aligned} p_{ij} &= P[X_{n+1} = j | X_n = i] = \lim_{t \to \infty} P[X_{n+1} = j, H_n \leq t | X_n = i] \\ &= Q_{ij}(\infty) \end{aligned}$$

$$F_{H_i}(t) = P[H_i \le t] = P[H_n \le t | X_n = i] = \sum_{j \in \Omega} P[X_{n+1} = j, H_n \le t | X_n = i]$$

$$= \sum_{j \in \Omega} Q_{ij}(t)$$

Note that it is normal practice to define the matrix of the $Q_{ij}(t)$ as $Q = [Q_{ij}(t)]$. However, this Q matrix is not the same as the intensity matrix or the infinitesimal generator of the continuous-time Markov chain. The convention is historical, and the two have historically been defined differently. Note also that

$$P[X_{n+1} = j, H_n \le t | X_n = i] = Q_{ij}(t)$$
$$= P[H_n \le t | X_n = i, X_{n+1} = j] P[X_{n+1} = j | X_n = i]$$

Because $P[X_{n+1} = j | X_n = i] = p_{ij}$, if we denote $P[H_n \le t | X_n = i, X_{n+1} = j] = G_{ij}(t)$ then we have

$$Q_{ij}(t) = p_{ij} G_{ij}(t)$$

There are several systems that can be modeled by a continuous-time semi-Markov process. These include any system that can be modeled by a continuous-time Markov process because from the foregoing discussion we observe that a continuous-time Markov process is a semi-Markov process in which

$$f_{H_{ij}}(t) = \lambda_{ij} e^{-\lambda_{ij} t} \qquad t \ge 0,\ 1 \le i, j \le N$$

A major area of application of continuous-time semi-Markov processes is in reliability and availability studies, see Limnios and Oprisan (2001) and Osaki (1985). For example, consider a system that is subject to failures and repairs. Assume that when the system is up (or functioning) the time until it fails, which is called the time to failure, has an arbitrary distribution. After it has failed, the time until it is repaired, which is called the time to repair, has another arbitrary distribution. If we denote the up state by state 1 and the down state by state 2, then the behavior of the system can be modeled by a semi-Markov process with the transition probability matrix of the embedded Markov chain given by

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

More sophisticated failure modes can similarly be modeled by the semi-Markov process.

In an M/G/1 queue, if we define $X(t)$ as the number of customers in the system at time t, the process $\{X(t), t \ge 0\}$ is not a semi-Markov process because the state

of the system after a transition depends not only on $X(t)$ but also on the length of time the customer currently being served has been receiving service. If we define transitions as occuring only when a customer has completed service, then the process becomes a semi-Markov process. In our study of the M/G/1 queue, we basically considered the process $\{N(t), t \geq 0\}$, where $N(t) = n$ if n customers were left in the system the last time prior to t that a customer completed service. Such a process is a semi-Markov process.

State Probabilities

Let $\phi_{ij}(t)$ denote the probability that the process is in state j at time t given that it entered state i at time zero. As mentioned earlier, $\phi_{ij}(t)$ is referred to as the *transition probability function* from state i to state j. Following the technique used for the discrete-time case, we consider two cases:

a. For $i \neq j$, the process makes a transition to some state k at time u, and then in the remaining time $t - u$ it travels to state j. The probability of this event is

$$\phi_{ij}(t) = \sum_{k=0}^{N} p_{ik} \int_{u=0}^{t} f_{H_{ik}}(u)\phi_{kj}(t-u)du \qquad i \neq j$$

b. For $i = j$ we have an additional probability that the process never left state i during the interval of interest. The probability of this additional event is

$$P[H_i > t] = 1 - P[H_i \leq t] = 1 - F_{H_i}(t)$$

Because the event that it never left state i during the interval and the event that it left i for some state k are mutually exclusive, we have that

$$\phi_{ii}(t) = 1 - F_{H_i}(t) + \sum_{k=0}^{N} p_{ik} \int_{u=0}^{t} f_{H_{ik}}(u)\phi_{ki}(t-u)du$$

Thus, for $i, j \in S$, and $t \geq 0$ we obtain

$$\phi_{ij}(t) = \delta_{ij}\{1 - F_{H_i}(t)\} + \sum_{k=0}^{N} p_{ik} \int_{u=0}^{t} f_{H_{ik}}(u)\phi_{kj}(t-u)du$$

where $\phi_{ij}(0) = \delta_{ij}$. Because the integral term is essentially a convolution integral, we can write

$$\phi_{ij}(t) = \delta_{ij}\{1 - F_{H_i}(t)\} + \sum_{k=0}^{N} p_{ik} f_{H_{ik}}(t) * \phi_{kj}(t)$$

$$= \delta_{ij}\{1 - F_{H_i}(t)\} + \sum_{k=0}^{N} c_{ik}(t) * \phi_{kj}(t)$$

where, as in the discrete-time case, we define the *continuous-time core matrix*

$$C(t) = [c_{ij}(t)] = P \square f_H(t) = [p_{ij} f_{H_{ij}}(t)]$$

We define $\phi_{ij}(s)$ as the s-transform of $\phi_{ij}(t)$, $c_{ij}(s)$ as the s-transform of $c_{ij}(t)$, and note that the s-transform of $F_{H_i}(t)$ is $M_{H_i}(s)/s$, where $M_{H_i}(s)$ is the s-transform of $f_{H_i}(t)$. Thus, taking the s-transform of both sides of the above equation we obtain the following:

$$\phi_{ij}(s) = \frac{\delta_{ij}}{s}\{1 - M_{H_i}(s)\} + \sum_{k=0}^{N} c_{ik}(s)\phi_{kj}(s)$$

$$= \frac{\delta_{ij}}{s}\left\{1 - \sum_{k=0}^{N} p_{ik} M_{H_{ik}}(s)\right\} + \sum_{k=0}^{N} c_{ik}(s)\phi_{kj}(s)$$

Finally, if $D(s)$ is the $N \times N$ diagonal matrix with entries $\{1 - M_{H_i}(s)\}/s$, the matrix form of the equation becomes

$$\Phi(s) = D(s) + C(s)\Phi(s)$$

which gives

$$\Phi(s) = [I - C(s)]^{-1} D(s)$$

If we define

$$\phi_{ij} = \lim_{t \to \infty} \phi_{ij}(t)$$

then from Heyman and Sobel (1982) and Howard (1960, 1971b) it can be shown that

$$\phi_{ij} = \frac{\pi_j E[H_j]}{\sum_{k=0}^{N} \pi_k E[H_k]}$$

where π_j is limiting state probability of the embedded Markov chain. Because the right-hand side of the equation is independent of i, we obtain the limiting probability of the semi-Markov process as

$$\phi_j = \frac{\pi_j E[H_j]}{\sum_{k=0}^{N} \pi_k E[H_k]}$$

As stated earlier, the limiting state probability is also called the *occupancy distribution* because ϕ_j gives the long-run fraction of time that the process spends in state j.

First Passage Times

Let T_{ij} denote the first passage time from state i to state j. That is,

$$T_{ij} = \min\{t > 0 | X(t) = j, X(0) = i\}$$

Let $m_{ij} = E[T_{ij}]$; that is, m_{ij} is the mean first passage time from state i to state j. From earlier discussion in the discrete-time case, we have that

$$m_{ij} = E[H_i] + \sum_{k \neq j} p_{ik} m_{kj}$$

Finally, using the same method used in the discrete-time case we obtain the mean recurrence time as

$$m_{jj} = \frac{\sum_{i=0}^{N} \pi_i E[H_i]}{\pi_j} \qquad j = 1, 2, \ldots, N$$

6.7 Markov Jump Processes

Recall from Chapter 2 that a jump process is a stochastic process that makes transitions between discrete states in the following manner: It spends an amount of time called the holding time (or sojourn time) in the current state and then jumps to another state where it spends another holding time, and so on. A Markov jump process is a jump process in which the holding times are exponentially distributed. Thus, if the holding times depend on both the current state and the state to be visited next, then a Markov jump process is a semi-Markov process. On the other hand, if the holding times depend only on the current state, then a Markov jump process is a continuous-time Markov process.

Consider a process that starts at state $X(0) = x$ at time $t = 0$ and waits until time $t = T_1$ when it makes a jump of size θ_1, which is not necessarily positive. The process then waits until time $t = T_2$, when it makes another jump of size θ_2, and so on. The jump sizes θ_i are also assumed to be independent and identically distributed. The times $T_1, T_2, \ldots$ are the instants when the process makes jumps, and the intervals $\tau_i = T_i - T_{i-1}, i = 1, 2, \ldots$, called the *waiting times* (or *pausing times*), are assumed to be independent and exponentially distributed with rates $\lambda_i, i = 1, 2, \ldots$. The time at which the nth jump occurs, T_n, is given by

$$T_n = \sum_{i=1}^{n} \tau_i \qquad n = 1, 2, \ldots; t_0 = 0$$

The time the process spends in any state x (i.e., the waiting time in state x) and the choice of the next state y are independent. Let $T(t)$ denote the cumulative waiting time from time $t = 0$ to the instant the process changes the current state $X(t)$, and let the function $p_{ij}(s, t)$ denote the probability that the process is in state j at time t, given that it was in state i at time s; that is, for $t \geq s$,

$$p_{ij}(s, t) = P[X(t) = j | X(s) = i]$$

Then we have that for $\tau > t$,

$$\begin{aligned} P[X(t) = y, T(t) \leq \tau | X(0) = x] &= P[X(t) = y | X(0) = x] P[T(t) \leq \tau] \\ &= P[X(t) = y | X(0) = x] P[t < w(y) \leq \tau] \\ &= p_{xy}(0, t) \int_t^\tau \lambda_y \exp(-\lambda_y u) du \\ &= p_{xy}(0, t)\{\exp(-\lambda_y t) - \exp(-\lambda_y \tau)\} \end{aligned}$$

where $w(y)$ is the waiting time in state y. A realization of the process is illustrated in Figure 6.5.

The state of the process at time t relative to its initial value is given by

$$\triangle X(t) = \sum_{i=1}^{N(t)} \Theta_i$$

where the upper limit $N(t)$ is a random function of time that denotes the number of jumps up to time t and is given by

$$N(t) = \max\{n : T_n \leq t\}$$

We have used the notation $\triangle X(t)$ to denote the change in state by time t relative to the initial value $X(0)$ because $X(0)$ might not be zero; but if it is zero, then $X(t) = X(0) + \triangle X(t) = \triangle X(t)$.

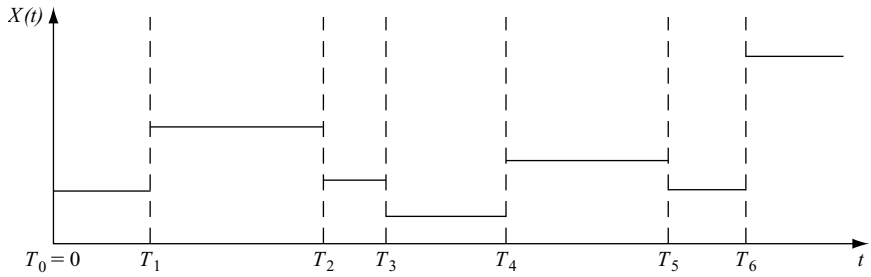


Figure 6.5. Interarrival times of a renewal process.

Because $X(t)$ is a Markov process, the Markov property holds. That is, given the times $0 < t_1 < t_2 < \cdots < t_n < s < t$, we have that

$$P[X(t) = y | X(s) = x, X(t_n) = x_n, \ldots, X(t_1) = x_1] = P[X(t) = y | X(s) = x]$$

Let $\pi_k(t)$ denote that probability that the process is in state k at time t. Then for $s, t > 0$,

$$\begin{aligned}
&P[X(t+s) = j, X(0) = i] \\
&\quad = \sum_k P[X(t+s) = j, X(s) = k, X(0) = i] \\
&\quad = \sum_k P[X(t+s) = j | X(s) = k, X(0) = i] P[X(s) = k, X(0) = i] \\
&\quad = \sum_k P[X(t+s) = j | X(s) = k] P[X(s) = k, X(0) = i] \\
&\quad = \sum_k P[X(t+s) = j | X(s) = k] P[X(s) = k | X(0) = i] P[X(0) = i] \\
&\quad = \sum_k p_{ik}(0, s)\, p_{kj}(s, t) \pi_i(0)
\end{aligned}$$

where the third equality is due to the Markov property of $X(t)$. Now, we know that

$$\begin{aligned}
P[X(t+s) = j, X(0) = i] &= P[X(t+s) = j | X(0) = i] P[X(0) = i] \\
&= p_{ij}(0, t+s) \pi_i(0)
\end{aligned}$$

Combining the two results we obtain

$$p_{ij}(0, t+s) = \sum_k p_{ik}(0, s)\, p_{kj}(s, t)$$

which is the Chapman-Kolmogorov equation for Markov jump processes.

6.7.1 The Homogeneous Markov Jump Process

We now consider the special case where $X(0) = 0$, and the waiting times are identically distributed with parameter λ. Let T denote the waiting time and Θ the jump size that we assume to be nonnegative. Similarly, let $p_{X(t)}(x, t)$ denote the PMF of $X(t)$, $p_\Theta(\theta)$ the PMF of Θ, $f_T(\tau)$ the PDF of T, and $p_{N(t)}(n, t)$ the PMF of $N(t)$. Also, let $G_{X(t)}(z)$ denote the z-transform of $\triangle X(t)$, let $G_\Theta(z)$ denote the z-transform of Θ, and let $G_{N(t)}(z)$ denote the z-transform of $N(t)$, where

$$G_\Theta(z) = E[z^\Theta] = \sum_{\theta=0}^{\infty} z^\theta p_\Theta(\theta)$$

$$G_{N(t)}(z) = E[z^{N(t)}] = \sum_{n=0}^{\infty} z^n p_{N(t)}(n, t)$$

Because $X(t) = \sum_{i=1}^{N(t)} \Theta_i$, we know from probability theory (see Ibe (2005), for example) that

$$G_{X(t)}(z) = G_{N(t)}(G_\Theta(z))$$

Thus, if $p_{N(t)}(n, t)$ and $p_\Theta(\theta)$ are known, we can determine $G_{X(t)}(z)$ and consequently $p_{X(t)}(x, t)$.

The expected value of $X(t)$ is given by

$$E[X(t)] = E[N(t)]E[\Theta]$$

Because the intervals τ_i are independent and exponentially distributed, we note that the process $\{N(t)|t \geq 0\}$ is a renewal process and T_n is the time of the nth renewal. Thus, we have that

$$\begin{aligned} p_{N(t)}(n, t) &= P[N(t) = n] = P[N(t) < n + 1] - P[N(t) < n] \\ &= 1 - F_{T_{n+1}}(t) - \{1 - F_{T_n}(t)\} = F_{T_n}(t) - F_{T_{n+1}}(t) \end{aligned}$$

and the expected value of $N(t)$ is given by

$$\begin{aligned} E[N(t)] &= \sum_{n=0}^{\infty} n p_{N(t)}(n, t) = \sum_{n=0}^{\infty} n\{F_{T_n}(t) - F_{T_{n+1}}(t)\} \\ &= \{F_{T_1}(t) + 2F_{T_2}(t) + 3F_{T_3}(t) + \cdots\} - \{F_{T_2}(t) + 2F_{T_3}(t) + \cdots\} \\ &= F_{T_1}(t) + F_{T_2}(t) + F_{T_3}(t) + \cdots = \sum_{n=1}^{\infty} F_{T_n}(t) \end{aligned}$$

$$= \sum_{n=1}^{\infty} \left\{ 1 - \sum_{n=0}^{n-1} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \right\} = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

Recall that $p_{0x}(0, t)$ is the probability that the state of the process at time t is x, given that it was initially in state 0; that is,

$$p_{0x}(0, t) = P[X(t) = x | X(0) = 0]$$

Because T and Θ are assumed to be independent, we can obtain $p_{0x}(0, t)$ as follows:

$$p_{0x}(0, t) = \delta(x) R(t) + \int_0^t \sum_{k=0}^{\infty} p_{0k}(0, \tau) f_T(t - \tau) p_{\Theta}(x - k) d\tau$$

This equation is usually referred to as the *master equation* of the Markov jump process. In the above equation, $\delta(x)$ is the Dirac delta function, and $R(t) = P[T_0 > t] = 1 - F_{T_0}(t)$ is called the *survival probability* for state $X(t) = 0$, which is the probability that the waiting time when the process is in state 0 is greater than t. The equation states that the probability that $X(t) = x$ is equal to the probability that the process was in state $X(t) = 0$ up to time t plus the probability that the process was at some state k at time τ, where $0 < \tau \leq t$, and within the waiting time $t - \tau$ a jump of size $x - k$ took place. Note that

$$R(t) = \int_t^{\infty} f_T(v) dv = 1 - \int_0^t f_T(v) dv = e^{-\lambda t}$$

We define joint z-s transform of $p_{0x}(0, t)$ as follows:

$$\tilde{P}(z, s) = \int_0^{\infty} e^{-st} \left\{ \sum_{x=0}^{\infty} z^x p_{0x}(0, t) \right\} dt$$

Thus, the master equation becomes transformed into the following:

$$\begin{aligned} \tilde{P}(z, s) &= \int_0^{\infty} e^{-st} \sum_{x=0}^{\infty} z^x \left\{ \delta(x) R(t) + \int_0^t \sum_{k=0}^{\infty} p_{0k}(0, \tau) f_T(t - \tau) p_{\Theta}(x - k) d\tau \right\} dt \\ &= R(s) + \tilde{P}(z, s) G_{\Theta}(z) \Phi_T(s) \\ &= \frac{R(s)}{1 - G_{\Theta}(z) \Phi_T(s)} \end{aligned}$$

Because

$$R(s) = \frac{1 - \Phi_T(s)}{s}$$

we have that

$$\tilde{P}(z, s) = \frac{1 - \Phi_T(s)}{s[1 - G_\Theta(z)\Phi_T(s)]}$$

Similarly, because T is exponentially distributed with a mean of $1/\lambda$, we have that $\Phi_T(s) = \lambda/(s + \lambda)$ and

$$\tilde{P}(z, s) = \frac{1 - \lambda/(s + \lambda)}{s[1 - \lambda G_\Theta(z)/(s + \lambda)]} = \frac{1}{s + \lambda - \lambda G_\Theta(z)}$$

From this we obtain the inverse transform as follows:

$$P(z, t) = e^{-\lambda t(1 - G_\Theta(z))}$$

$$p_{0x}(0, t) = \frac{1}{x!}\left[\frac{d^x}{dz^x} P(z, t)\right]_{z=0}$$

This solution assumes that $X(t)$ takes only nonnegative values. When negative values are allowed, the z-transform must be replaced by the characteristic function. For the special case when Θ is a Poisson random variable, $G_\Theta(z) = e^{-\lambda(1-z)}$, and we obtain

$$\tilde{P}(z, s) = \frac{1}{s + \lambda - \lambda G_\Theta(z)} = \frac{1}{s + \lambda - \lambda e^{-\lambda(1-z)}}$$

An alternative method of deriving $p_{0x}(0, t)$ is from our earlier observation that $G_{X(t)}(z) = G_{N(t)}(G_\Theta(z))$. We know from earlier discussion that the PMF of $N(t)$ is given by

$$\begin{aligned} p_{N(t)}(n) &= F_{T_n}(t) - F_{T_{n+1}}(t) \\ &= \left\{1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k e^{-\lambda t}}{k!}\right\} - \left\{1 - \sum_{k=0}^{n} \frac{(\lambda t)^k e^{-\lambda t}}{k!}\right\} = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \end{aligned}$$

Thus, the z-transform of $p_{N(t)}(n)$ is given by

$$G_{N(t)}(z) = e^{-\lambda t(1-z)}$$

This gives

$$G_{X(t)}(z) \equiv P(z,t) = G_{N(t)}(G_\Theta(z)) = e^{-\lambda t(1-G_\Theta(z))}$$

which is the same result that we obtained earlier, and from this we can obtain $p_{0x}(0,t)$. The conditional expected value $E[X(t)|X(0)=0]$ is given by

$$E[X(t)|X(0)=0] = \left[\frac{dG_{X(t)}(z)}{dz}\right]_{z=1}$$

Similarly, the conditional variance $\sigma^2_{X(t)|X(0)}$ is given by

$$\sigma^2_{X(t)|X(0)} = \left[\frac{d^2G_{X(t)}(z)}{dz^2} + \frac{dG_{X(t)}(z)}{dz} - \left\{\frac{dG_{X(t)}(z)}{dz}\right\}^2\right]_{z=1}$$

We can generalize the result for the case when $X(0)=x$ by defining the probability transition function $p_{xy}(0,t)$ as follows:

$$\begin{aligned} p_{xy}(0,t) &= P[X(t)=y|X(0)=x] \\ &= \delta(y-x)R(t) + \int_0^t \sum_{k=0}^{\infty} p_{xk}(0,\tau) f_T(t-\tau) p_\Theta(y-x-k)d\tau \end{aligned}$$

Using the same technique we used for the case when $X(0)=0$ we can obtain the z-s transform of the probability transition function.

6.8 Problems

6.1 Consider a machine that is subject to failure and repair. The time to repair the machine when it breaks down is exponentially distributed with mean $1/\mu$. The time the machine runs before breaking down is also exponentially distributed with mean $1/\lambda$. When repaired the machine is considered to be as good as new. The repair time and the running time are assumed to be independent. If the machine is in good condition at time 0, what is the expected number of failures up to time t?

6.2 The Merrimack Airlines company runs a commuter air service between Manchester, New Hampshire, and Cape Cod, Massachusetts. Because the company is a small one, there is no set schedule for their flights, and no reservation is needed for the flights. However, it has been determined that their planes arrive at the Manchester airport according to a Poisson process with an average rate of two planes per hour. Gail arrived at the Manchester airport and had to wait to catch the next flight.

- **a.** What is the mean time between the instant Gail arrived at the airport until the time the next plane arrived?
- **b.** What is the mean time between the arrival time of the last plane that took off from the Manchester airport before Gail arrived and the arrival time of the plane that she boarded?

6.3 Victor is a student who is conducting experiments with a series of lightbulbs. He started with 10 identical lightbulbs, each of which has an exponentially distributed lifetime with a mean of 200 hours. Victor wants to know how long it will take until the last bulb burns out (or fails). At noontime he stepped out to get some lunch with 6 bulbs still on. Assume that he came back and found that none of the 6 bulbs has failed.

- **a.** After Victor came back, what is the expected time until the next bulb failure?
- **b.** What is the expected length of time between the fourth bulb failure and the fifth bulb failure?

6.4 A machine has three components labeled 1, 2, and 3, whose times between failure are exponentially distributed with mean $1/\lambda_1, 1/\lambda_2$, and $1/\lambda_3$, respectively. The machine needs all three components to work, thus when a component fails the machine is shut down until the component is repaired and the machine is brought up again. When repaired, a component is considered to be as good as new. The time to repair component 1 when it fails is exponentially distributed with mean $1/\mu_1$. The time to repair component 2 when it fails is constant at $1/\mu_2$, and the time to repair component 3 when it fails is a third-order Erlang random variable with parameter μ_3.

- **a.** What fraction of time is the machine working?
- **b.** What fraction of time is component 2 being repaired?
- **c.** What fraction of time is component 3 idle but has not failed?
- **d.** Given that Bob arrived when component 1 was being repaired, what is the expected time until the machine is operational again?

6.5 A high school student has two favorite brands of bag pack labeled X and Y. She continuously chooses between these brands in the following manner. Given that she currently has brand X, the probability that she will buy brand X again is 0.8, and the probability that she will buy brand Y next is 0.2.

Similarly, given that she currently has brand Y, the probability that she will buy brand X next is 0.3, and the probability that she will buy brand Y again is 0.7. With respect to the time between purchases, if she currently has a brand X bag pack, the time until the next purchase is exponentially distributed with a mean of 6 months. Similarly, given that she currently has a brand Y bag pack, the time until the next purchase is exponentially distributed with a mean of 8 months.

a. What is the long-run probability that she has a brand Y bag pack?
b. Given that she has just purchased brand X, what is the probability that t months later her last purchase was brand Y?

6.6 Larry is a student who cannot seem to make up his mind whether to live in the city or in the suburb. Every time he lives in the city, he moves to the suburb after one semester. Half of the time he lives in the suburb, he moves back to the city after one semester. The other half of Larry's suburban living results in his moving to a new apartment in the suburb where he lives for a time that is geometrically distributed with a mean of two semesters.

a. Model Larry's living style by a two-state discrete-time semi-Markov process, giving the state-transition diagram.
b. Assume that Larry lives in the city at the beginning of the current semester. What is the probability that he will be living in the suburb k semesters from now?
c. What is the probability that the total duration of any uninterrupted stay in the suburb is k semesters? What are the mean and variance of the duration of one such stay in the suburb?

6.7 Consider a Markov renewal process with the semi-Markov kernel Q given by

$$Q = \begin{bmatrix} 0.6(1 - e^{-5t}) & 0.4 - 0.4e^{-2t} \\ 0.5 - 0.2e^{-3t} - 0.3e^{-5t} & 0.5 - 0.5e^{-2t} - te^{-2t} \end{bmatrix}$$

a. Determine the state-transition probability matrix P for the Markov chain $\{X_n\}$.
b. Determine the conditional distributions $G_{ij}(t)$ of the waiting time in state i given that the next state is j for all i, j.

6.8 A machine can be in one of three states: good, fair, and broken. When it is in a good condition, it will remain in this state for a time that is exponentially distributed with mean $1/\mu_1$ before going to the fair state with probability $4/5$ and to the broken state with probability $1/5$. When it is in a fair condition, it will remain in this state for a time that is exponentially distributed with mean $1/\mu_2$ before going to the broken state. When it is in the broken state, it will take a time that is exponentially distributed with mean $1/\mu_3$ to be

repaired before going to the good state with probability 3/4 and to the fair state with probability 1/4. What is the fraction of time that the machine is in each state?

6.9 Customers arrive at a taxi depot according to a Poisson process with rate λ. The dispatcher sends for a taxi when there are N customers waiting at the station. It takes M units of time for a taxi to arrive at the depot. When it arrives, the taxi picks up all waiting customers. The taxi company incurs a cost at a rate of nk per unit time whenever n customers are waiting. What is the steady-state average cost that the company incurs?

6.10 A component is replaced every T time units and upon its failure. The lifetimes of successive components are independent and identically distributed random variables with PDF $f_X(x)$. A cost $c_1 > 0$ is incurred for each planned replacement, and a fixed cost $c_2 > c_1$ is incurred for each failure replacement. What is the long-run average cost per unit time?

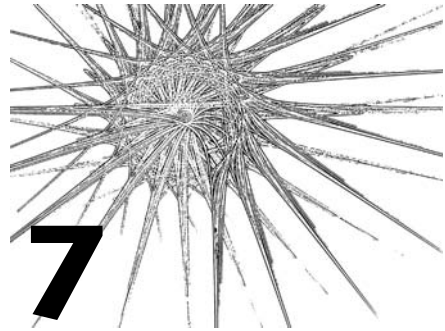
6.11 In her retirement days, a mother of three grown-up children splits her time living with her three children who live in three different states. It has been found that her choice of where to spend her time next can be modeled by a semi-Markov chain. Thus, if the children are labeled by ages as child 1, child 2, and child 3, the transition probabilities are as follows. Given that she is currently staying with child 1, the probability that she will stay with child 1 next is 0.3, the probability that she will stay with child 2 next is 0.2, and the probability that she will stay with child 3 next is 0.5. Similarly, given that she is currently staying with child 2, the probability that she will stay with child 1 next is 0.1, the probability that she will stay with child 2 next is 0.8, and the probability that she will stay with child 3 next is 0.1. Finally, given that she is currently staying with child 3, the probability that she will stay with child 1 next is 0.4, the probability that she will stay with child 2 next is 0.4, and the probability that she will stay with child 3 next is 0.2. The length of time that she spends with child 1 is geometrically distributed with mean 2 months, the length of time she spends with child 2 is geometrically distributed with mean 3 months, and the time she spends with child 3 is geometrically distributed with mean 1 month.

a. Obtain the transition probability functions of the process; that is, obtain the set of probabilities $\{\phi_{ij}(n)\}$, where $\phi_{ij}(n)$ is the probability that the process is in state j at time n given that it entered state i at time zero.

b. What is the occupancy distribution of the process?

6.12 Consider Problem 6.11. Assume that the time she spends with each of her children is exponentially distributed with the same means as specified. Obtain the transition probability functions $\{\phi_{ij}(t)\}$ of the process.

Markovian Arrival Processes



7.1 Introduction

Modern telecommunication networks are designed to support multimedia traffic including voice, data, and video. One major feature of these networks is that the input traffic is usually highly bursty. Also, because these networks operate in a packet switching mode, there is usually a strong correlation between packet arrivals. Thus, for these networks the traditional Poisson traffic model cannot be applied because the presence of correlation between traffic arrivals violates the independence assumption associated with the Poisson process. For example, in an Asynchronous Transfer Mode (ATM) network, different types of traffic from different sources arrive at an ATM switch that statistically multiplexes the traffic and transmits them as fixed length packets called cells, as discussed in several books such as Ibe (1997). Thus, there is a high degree of correlation between the individual user traffic and the aggregate arrival process that cannot be captured by traditional Poisson models.

In fact, traffic measurement studies reported in Leland (1994) and Crovella and Bestavros (1997) indicate that Ethernet and Internet traffic displays behavior that is associated with *long-range dependence* (LRD) and *self-similarity*. Self-similarity is a feature whereby parts of an object show the same statistical properties as the object at many scales. For example, in the case of Internet Protocol (IP) traffic, self-similarity means that similar looking traffic bursts can be seen at every time scale ranging from a few milliseconds to minutes and even hours. Thus, self-similar processes tend to exhibit long-range dependence, which means that values at any instant tend to be positively correlated with values at several future instants.

The Internet has become a multiservice network, and one of the features of such networks is the burstiness exhibited by the different services, such as voice, compressed video, and file transfer, that use the network. A traffic process is defined to be bursty if the traffic arrival points $\{t_n\}$ tend to form visual clusters. This means the $\{t_n\}$ tends to consist of a bunch of several relatively short interarrival times followed by a relatively long one.

To deal with the new multiservice network traffic pattern, teletraffic systems analysts have developed a set of traffic models that have been shown to be analytically tractable while capturing the true nature of the traffic better than the traditional Poisson model does. These models are characterized by the fact that they are doubly stochastic Poisson processes that are obtained as a natural extension of the homogeneous Poisson process by allowing the arrival rate to be a stochastic process.

This chapter deals with some of these traffic models, which include the Markovian arrival process (MAP), the batch Markovian arrival process (BMAP), the Markov-modulated Poisson process (MMPP), and the Markov-modulated Bernoulli process (MMBP).

Most of the models discussed in this chapter are usually analyzed via the matrix-analytic method proposed by Neuts (1981) and discussed in greater detail by Latouche and Ramaswami (1999). Thus, we begin by providing a brief discussion on quasi-birth-and-death processes and the matrix-analytic method.

7.2 Overview of Matrix-Analytic Methods

Recall from Chapter 4 that the infinitesimal generator (or intensity) matrix Q for a continuous-time Markov chain (CTMC) is given by

$$Q = \begin{bmatrix} -q_1 & q_{12} & q_{13} & q_{14} & \cdots \\ q_{21} & -q_2 & q_{23} & q_{24} & \cdots \\ q_{31} & q_{32} & -q_3 & q_{34} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

where

$$q_i = \sum_{j \neq i} q_{ij} \qquad i = 1, 2, \ldots$$

For the special case of a birth-and-death process, it has been shown in Chapter 4 that the intensity matrix is given by

$$Q_{\mathrm{BD}} = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \dots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & 0 & \dots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & 0 & \dots \\ 0 & 0 & \mu_3 & -(\lambda_3+\mu_3) & \lambda_3 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

Thus, from this we obtain the intensity matrix for the M/M/1 queue as follows:

$$Q_{\mathrm{M/M/1}} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & \dots \\ \mu & -(\lambda+\mu) & \lambda & 0 & 0 & 0 & \dots \\ 0 & \mu & -(\lambda+\mu) & \lambda & 0 & 0 & \dots \\ 0 & 0 & \mu & -(\lambda+\mu) & \lambda & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

We observe that Q for the M/M/1 queue is a tridiagonal matrix in which all elements above the main diagonal are equal and all elements below the main diagonal are equal. With the exception of the topmost element, all elements of the main diagonal are also equal and are negative sums. The topmost element of the main diagonal is different because that state is a boundary state that has no transition to a lower state. The importance of Q can be explained as follows. Let $P(t)$ be an $n \times n$ matrix such that $p_{ij}(t) \in P(t)$ is defined by

$$p_{ij}(t) = P[X(t) = j | X(0) = i]$$

Then from the forward Kolmogorov equation we obtain

$$\frac{dP(t)}{dt} = P(t)Q$$

The solution to this matrix equation is

$$P(t) = e^{Qt} = \sum_{k=0}^{\infty} \frac{t^n}{n!} Q^n \quad t \geq 0$$

Let $p_i(t) = P[X(t) = i]$ and $p(t) = [p_1(t), p_2(t), \dots, p_n(t)]$. Then,

$$p(t) = p(0)P(t)$$

Thus, while the solution to the matrix equation might not be easy to obtain in practice, knowledge of Q enables us to obtain the $p_i(t)$.

A quasi-birth-and-death (QBD) process is a special case of an infinite-state CTMC that provides a two-dimensional state space version of the birth-and-death

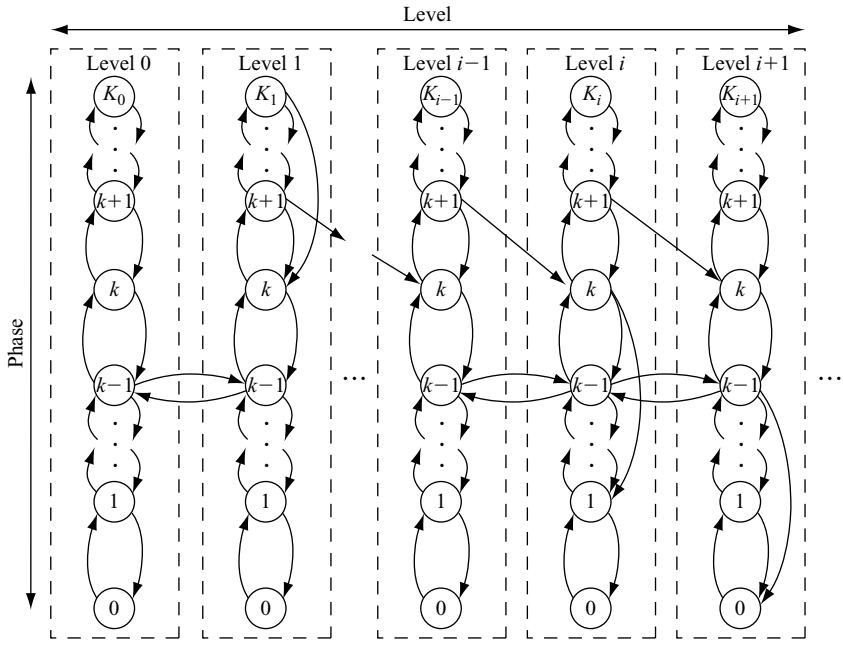


Figure 7.1. Example of the QBD process.

process. In this process, states are grouped into *levels*, and transitions are allowed only between levels and within a level. A level l_i consists of m_i *phases*, where m_i can be finite or infinite. An example of the *QBD* is shown in Figure 7.1.

Like the CTMC, all levels are alike except for the first level, which can be different because it is the *boundary level* while others are *repeating levels* that usually have the same transition structure. Sometimes the second level is referred to as the *border level* that can have some slightly different structure from the other repeating levels. In Figure 7.1 level 0 is the boundary level and level 1 is the border level. Also, for a homogeneous process, $m_i = m$ for $i \geq 1$, if we assume that l_0 is the boundary level. Thus, the Q-matrix for a homogeneous QBD process is of the following tridiagonal form:

$$Q_{QBD} = \begin{bmatrix} D_0 & D_1 & 0 & 0 & 0 & 0 & \dots \\ D_2 & A_0 & A_1 & 0 & 0 & 0 & \dots \\ 0 & A_2 & A_0 & A_1 & 0 & 0 & \dots \\ 0 & 0 & A_2 & A_0 & A_1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

where A_0, A_1, A_2 are $m \times m$ matrices, where m is the number of *phases* in a level that is not the boundary level; D_0 is an $n \times n$ submatrix, where n is the number of phases in the boundary level; D_1 is an $n \times m$ submatrix; and D_2 is an $m \times n$ submatrix. The states in a given level are called the phases of the level. In general A_0 and D_0 have nonnegative off-diagonal elements and strictly negative diagonal elements while A_1, A_2, D_1, and D_2 are nonnegative matrices.

One way to visualize these matrices is from Figure 7.2 where the submatrix D_1 deals with the transition rates of the transitions from the boundary level to the border level, D_2 deals with the transition rates of transitions from the border level to the boundary level, A_1 deals with the transition rates of transitions from a repeating level to the next higher repeating level, and A_2 deals with the transition rates of transitions from a repeating level to a preceding repeating level. D_0 and A_0 can be likened to self-loops that deal with intralevel transitions: D_0 for the boundary level and A_0 for the repeating levels.

Example 7.1 Consider an $M/H_2/1$ queue, which is a single-server queue to which customers arrive according to a Poisson process with rate λ, and each customer requires two exponential stages of service with service rates μ_1 and μ_2. Each state is usually represented by (k, s), where k is the number of customers in the system and s is the stage of service of the customer who is currently receiving service. The state-transtion-rate diagram for the queue is given in Figure 7.3.

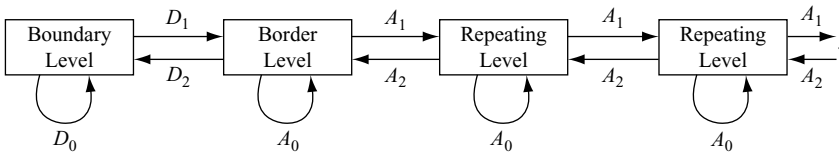


Figure 7.2. Visualization of the roles of the QBD submatrices.

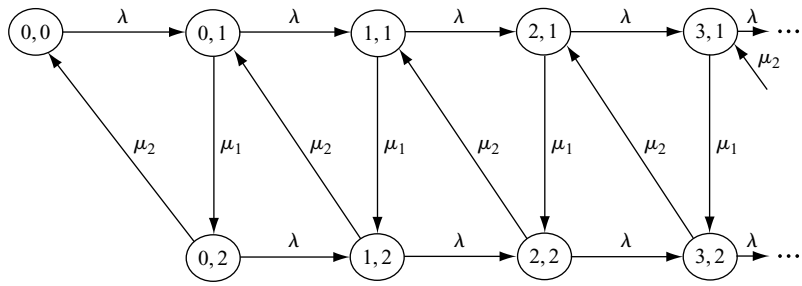


Figure 7.3. State-transition-rate diagram of the $M/H_2/1$ queue.

If we lexicographically order the states such that (k_1, s_1) precedes (k_2, s_2) if and only if $k_1 < k_2$ or $\{k_1 = k_2$ and $s_1 < s_2\}$, we obtain the state $S = \{(0,0), (0,1), (0,2), (1,1), (1,2), (2,1), (2,2), \ldots\}$, and the Q-matrix for the process becomes

$$Q_{M|H_2|1} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & -(\lambda+\mu_1) & \mu_1 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \mu_2 & 0 & -(\lambda+\mu_2) & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & -(\lambda+\mu_1) & \mu_1 & \lambda & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & \mu_2 & 0 & 0 & -(\lambda+\mu_2) & 0 & \lambda & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_1) & \mu_1 & \lambda & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & \mu_2 & 0 & 0 & -(\lambda+\mu_2) & 0 & \lambda & 0 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}$$

$$= \begin{bmatrix} D_0 & D_1 & 0 & 0 & 0 & 0 & \ldots \\ D_2 & A_0 & A_1 & 0 & 0 & 0 & \ldots \\ 0 & A_2 & A_0 & A_1 & 0 & 0 & \ldots \\ 0 & 0 & A_2 & A_0 & A_1 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}$$

where

$$D_0 = \begin{bmatrix} -\lambda & \lambda & 0 \\ 0 & -(\lambda+\mu_1) & \mu_1 \\ \mu_2 & 0 & -(\lambda+\mu_2) \end{bmatrix} \qquad D_1 = \begin{bmatrix} 0 & 0 \\ \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$D_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mu_2 & 0 \end{bmatrix} \qquad A_0 = \begin{bmatrix} -(\lambda+\mu_1) & \mu_1 \\ 0 & -(\lambda+\mu_2) \end{bmatrix}$$

$$A_1 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 0 \\ \mu_2 & 0 \end{bmatrix}$$

Thus, for this example, $m = 2$ and $n = 3$. ■

QBDs are usually analyzed via matrix analytic methods whose fundamental premise is that the intensity matrix Q of many complex Markov processes has an internal structure that can be exploited to simplify their analysis. In particular, the matrix Q for QBDs can be written as a block-tridiagonal matrix that is similar to the scalar-tridiagonal Q-matrix of the CTMC.

Recall that for an M/M/1 queue the limiting state probabilities are the geometric distribution obtained as follows:

$$\pi_i = \rho\pi_{i-1} = \rho^i\pi_0 \quad i = 1, 2, \ldots$$

where $\rho = \lambda/\mu$. From the law of total probability we obtain

$$1 = \sum_{i=0}^{\infty} \pi_i = \pi_0 \sum_{i=0}^{\infty} \rho^i = \frac{\pi_0}{1-\rho}$$

$$\pi_0 = 1 - \rho$$
$$\pi_i = \pi_0 \rho^i = (1 - \rho)\rho^i$$

where $i = 0, 1, 2, \ldots$. Observe that the Q-matrix for the homogeneous QBD is similar to that of the M/M/1 queue. Let the stationary probability vector be π that is partitioned into subvectors π_k, where $\pi_0 = \{\pi(0,0), \pi(0,1), \pi(0,2)\}$, and for $k \geq 1$, $\pi_k = \{\pi(k,1), \pi(k,2)\}$; $\pi(i,j)$ is the stationary probability of being in state (i,j). The key to obtaining π is the fact that a geometric relationship exists among the π_k, which is that

$$\pi_k = \pi_{k-1} R \qquad k \geq 2$$

This solution is called a *matrix geometric solution*, and the matrix R is called the *geometric coefficient*. Applying successive substitution we obtain

$$\pi_k = \pi_1 R^{k-1} \qquad k \geq 1$$

The balance equations of the QBD process are given by $\pi Q = 0$, which means that

$$\pi_0 D_0 + \pi_1 D_2 = 0$$
$$\pi_0 D_1 + \pi_1 A_0 + \pi_2 A_2 = 0$$
$$\pi_{k-1} A_1 + \pi_k A_0 + \pi_{k+1} A_2 = 0 \quad k \geq 2$$

Substituting $\pi_k = \pi_1 R^{k-1}$ we obtain

$$\pi_0 D_0 + \pi_1 D_2 = 0$$
$$\pi_0 D_1 + \pi_1 A_0 + \pi_1 R A_2 = 0$$
$$A_1 + R A_0 + R^2 A_2 = 0$$

where the last equation follows from the fact that π_1 cannot be identically zero. If we can find a matrix R that solves the above equations, then the proposition that $\pi_k = \pi_1 R^{k-1}$ is correct. A number of iterative methods have been proposed for solving the above quadratic matrix equations. These are given in Latouche and Ramaswami (1999) and will not be repeated here. Another method of solution is proposed in Servi (2002).

7.3 Markovian Arrival Process

The Markovian arrival process was introduced by Lucantoni (1990) as a simpler version of an earlier model proposed by Neuts (1989). It is a generalization of the

Markov process where arrivals are governed by an underlying m-state Markov chain. MAP includes phase-type renewal processes and the Markov-modulated Poisson process. The discrete-time version of the process is called DMAP, and a version that includes batch arrivals is called BMAP, which is discussed later in this chapter. One important property of both MAP and BMAP is that the superpositions of independent processes of these types are also processes of the same type.

MAP generalizes the Poisson process by permitting interarrival times that are not exponential while maintaining its Markovian structure. Consider a Poisson process $\{N(t)\}$ with rate λ, where $N(t)$ is the number of arrivals in $(0, t]$ and thus takes nonnegative integer values. The state space of $N(t)$ is $\{0, 1, 2, \ldots\}$, and the state-transition-rate diagram of the process is given in Figure 7.4.

Thus, the Q-matrix for a Poisson process is given by

$$Q_{\text{Poisson}} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & \ldots \\ 0 & -\lambda & \lambda & 0 & 0 & 0 & \ldots \\ 0 & 0 & -\lambda & \lambda & 0 & 0 & \ldots \\ 0 & 0 & 0 & -\lambda & \lambda & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix} = \begin{bmatrix} d_0 & d_1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & d_0 & d_1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & d_0 & d_1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & d_0 & d_1 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}$$

where $d_0 = -\lambda$ and $d_1 = \lambda$. Let $\{J(t)\}$ be an additional process, called the phase process, that takes values in $\{1, 2, \ldots, m\}$ such that when the process is in state $j \in J(t)$ the Poisson arrival rate is λ_j. Additionally, the state transition rate from state j to k is α_{jk}, where $j, k \in J(t)$. Thus, $\{J(t)\}$ is an irreducible continuous-time Markov chain. The two-dimensional process $\{N(t), J(t)\}$ is a MAP, which represents a Markov process on the state space $\{(i, j)|i = 0, 1, \ldots; 1 \leq j \leq m\}$. $N(t)$ counts the number of arrivals during $[0, t)$, and $J(t)$ represents the phase of the arrival process. The value of m defines the order of a MAP. For example, a MAP that has $m = 2$ is called a MAP of order 2 and is denoted by MAP(2). The state-transition-rate diagram for MAP(4) is shown in Figure 7.5.

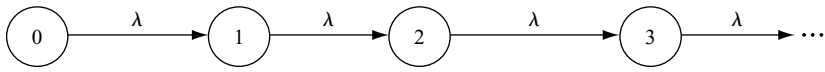


Figure 7.4. State-transition-rate diagram of a poisson process.

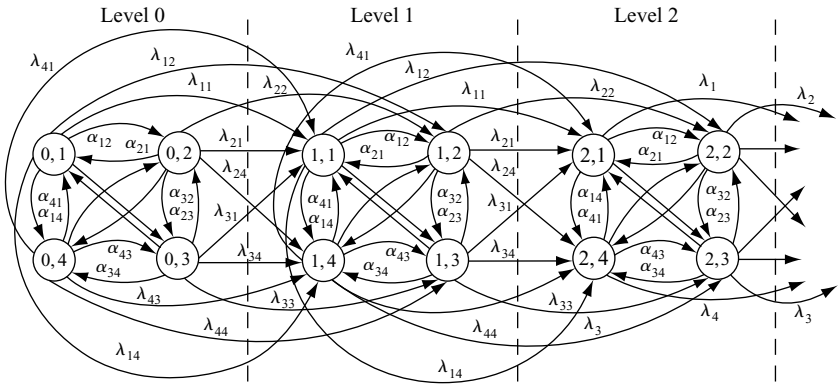


Figure 7.5. State-transition-rate diagram of MAP(4).

If we arrange the states in a lexicographical order, then the infinitesimal generator matrix Q is given by

$$Q_{\mathrm{MAP}} = \begin{bmatrix} -(\lambda_1+\alpha_1) & \alpha_{12} & \alpha_{13} & \dots & \alpha_{1m} & \lambda_{11} & \lambda_{12} & \lambda_{13} & \dots & \lambda_{1m} & 0 & \dots \\ \alpha_{21} & -(\lambda_2+\alpha_2) & \alpha_{23} & \dots & \alpha_{2m} & \lambda_{21} & \lambda_{22} & \lambda_{23} & \dots & \lambda_{2m} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_{m1} & \alpha_{m2} & \alpha_{m3} & \dots & -(\lambda_m+\alpha_m) & \lambda_{m1} & \lambda_{m2} & \lambda_{m3} & \dots & \lambda_{mm} & 0 & \dots \\ 0 & 0 & 0 & \dots & 0 & -(\lambda_1+\alpha_1) & \alpha_{12} & \alpha_{13} & \dots & \alpha_{1m} & \lambda_{11} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

where, for $i = 1, 2, \dots, m$,

$$\lambda_i = \sum_{k=1}^{m} \lambda_{ik}$$

$$\alpha_i = \sum_{k=1}^{m} \alpha_{ik}$$

If we define D_0 and D_1 by the following $m \times m$ matrices

$$D_0 = \begin{bmatrix} -(\lambda_1+\alpha_1) & \alpha_{12} & \alpha_{13} & \dots & \alpha_{1m} \\ \alpha_{21} & -(\lambda_2+\alpha_2) & \alpha_{23} & \dots & \alpha_{2m} \\ \alpha_{31} & \alpha_{32} & -(\lambda_3+\alpha_3) & \dots & \alpha_{3m} \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{m1} & \alpha_{m2} & \alpha_{m3} & \dots & -(\lambda_m+\alpha_m) \end{bmatrix}$$

$$D_1 = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \dots & \lambda_{1m} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} & \dots & \lambda_{2m} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} & \dots & \lambda_{3m} \\ \dots & \dots & \dots & \dots & \dots \\ \lambda_{m1} & \lambda_{m2} & \lambda_{m3} & \dots & \lambda_{mm} \end{bmatrix}$$

then Q_{MAP} can be represented in the following block form

$$Q_{\text{MAP}} = \begin{bmatrix} D_0 & D_1 & 0 & 0 & 0 & \dots \\ 0 & D_0 & D_1 & 0 & 0 & \dots \\ 0 & 0 & D_0 & D_1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

that has a structure similar to Q_{Poisson}. As can be observed, D_0 has negative diagonal elements and nonnegative off-diagonal elements, and its elements correspond to state transitions without an arrival; that is, they are phase transitions. Similarly, D_1 is a nonnegative matrix whose elements represent state transitions with one arrival. Because the Poisson process is a pure birth process, the structure of Q_{MAP} suggests that MAP behaves like a quasi-birth process. Sometimes MAP is denoted by MAP(D_0, D_1) to stress the fact that it is completely characterized by these two matrices.

7.3.1 Properties of MAP

In this section we discuss some of the properties of MAP. The first property of MAP is that the process is so broad that any stationary point process can be approximated arbitrarily closely by a MAP. The second property of MAP is that the superposition of two independent MAPs, say MAP(C_0, C_1) and MAP(D_0, D_1), is another MAP, MAP(E_0, E_1), where

$$E_0 = C_0 \oplus D_0$$
$$E_1 = C_1 \oplus D_1$$

where $\oplus$ represents the *Kronecker sum*, which is defined as follows. Let A be a $k \times k$ matrix and B be an $n \times n$ matrix. Let I_k and I_n be identity matrices of order k and n, respectively. Then,

$$A \oplus B = (A \otimes \text{I}_n) + (\text{I}_k \otimes B)$$

where $\otimes$ is the *Kronecker product*, which is given by

$$G \otimes F = \begin{bmatrix} g_{11}F & g_{12}F & \cdots & g_{1m}F \\ g_{21}F & g_{22}F & \cdots & g_{2m}F \\ \cdots & \cdots & \cdots & \cdots \\ g_{n1}F & g_{n2}F & \cdots & g_{nm}F \end{bmatrix}$$

where G is an $n \times m$ and F is a $p \times q$ matrix. Thus, $G \otimes F$ is an $np \times mq$ matrix. This construction can be extended to the superpositions of $n > 2$ MAPs.

Let the matrix D be defined as follows:

$$D = D_0 + D_1$$

Then D is the irreducible infinitesimal generator of the underlying Markov chain $\{J(t)\}$. Let π be the stationary probability vector in the Markov chain with infinitesimal generator D. We know that if we define $\mathbf{e}$ as the column vector of 1's (i.e., $\mathrm{e} = [1, 1, \ldots, 1]^T$ of length m), then

$$\pi D = 0$$
$$\pi \mathrm{e} = 1$$

The average rate of events in a MAP, which is called the *fundamental rate* of the MAP, is given by

$$\lambda = \pi D_1 \mathrm{e}$$

Let X_n denote the time between the nth arrival and the $(n+1)$th arrival, and let $J_n, n \geq 1$, denote the state of the Markov chain with infinitesimal generator D. Then $\{(J_n, X_n) | n \geq 1\}$ is a Markov renewal sequence with the transition probability matrix $F(x)$ whose (i, j)th element is

$$F_{ij}(x) = P[X_n \leq x, J_n = j | J_{n-1} = i]$$

In Neuts (1992) it is shown that, for $x \geq 0$,

$$\begin{aligned} F(x) &= \int_0^x \exp(D_0 u) du D_1 \\ &= \{\mathrm{I} - \exp(D_0 x)\}(-D_0)^{-1} D_1 \end{aligned}$$

The sequence $\{J_n | n \geq 1\}$ forms a Markov chain with state-transition probability matrix P given by

$$P = F(\infty) = (-D_0)^{-1} D_1$$

Let p denote the stationary distribution vector of P. The relationship between p and π is

$$p = \frac{1}{\lambda} \pi D_1$$

Also, as defined earlier, N_t is the number of arrivals in $(0, t]$. Let J_t denote the phase of the MAP at time t. Let the (i, j)th element of the $m \times m$ matrix $P(n, t)$ be defined as follows:

$$P_{ij}(n, t) = P[N_t = n, J_t = j | N_0 = 0, J_0 = i\}]$$

In Lucantoni (1993) it is shown that the matrix-generating function (or z-transform) of $P(n, t)$ is given by

$$G_P(z, t) = \sum_{n=0}^{\infty} z^n P(n, t) = \exp\{(D_0 + zD_1)t\}, \quad t \geq 0$$

Finally, the z-transform of the PMF of N_t is given by

$$G_{N_t}(z, t) = \pi G_P(z, t) \mathrm{e}$$

7.4 Batch Markovian Arrival Process

Batch Markovian arrival process (BMAP) was proposed by Lucantoni (1991) as an extension of MAP that provides a far more accurate view of the Internet Protocol (IP) traffic because it captures two important statistical properties of the IP traffic, namely self-similarity and burstiness. As stated earlier in the chapter, Crovella and Bestavros (1997) show that the World Wide Web traffic exhibits self-similarity, which means that the Poisson process cannot be used to model such traffic because it cannot effectively capture the dependence and correlation of the traffic arrival process in the Internet.

BMAP is particularly useful in modeling interactive data transfer in ATM networks. A bulk data transmission in an ATM network results in a large number of ATM cells. That is, for example, a single file transfer request results in a batch arrival of several cells. Thus, BMAP jointly characterizes the traffic arrival process and batch size distribution.

We motivate our discussion on BMAP by considering a batch Poisson process (BPP) in which batches arrive with rate λ. Assume that the batch size B is a discrete random variable with the PMF

$$p_B(k) = \sum_{k=1}^{m} p_k = 1 \quad P[B = k] \equiv p_k \quad 1 \le k \le m$$

where m can be infinite. Furthermore, let $N(t)$ denote the number of arrivals in the interval $(0, t]$. Then $\{N(t)|t \ge 0\}$ is a continuous-time Markov chain with state space $\{0, 1, 2, \ldots\}$ whose state-transition-rate diagram is shown in Figure 7.6.

The infinitesimal generator is given by

$$Q_{BPP} = \begin{bmatrix} -\lambda & \lambda p_1 & \lambda p_2 & \ldots & \lambda p_m & 0 & 0 & 0 & 0 & \ldots \\ 0 & -\lambda & \lambda p_1 & \lambda p_2 & \ldots & \lambda p_m & 0 & 0 & 0 & \ldots \\ 0 & 0 & -\lambda & \lambda p_1 & \lambda p_2 & \ldots & \lambda p_m & 0 & 0 & \ldots \\ 0 & 0 & 0 & -\lambda & \lambda p_1 & \lambda p_2 & \ldots & \lambda p_m & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}$$

$$= \begin{bmatrix} d_0 & d_1 & d_2 & \ldots & d_m & 0 & 0 & 0 & 0 & \ldots \\ 0 & d_0 & d_1 & d_2 & \ldots & d_m & 0 & 0 & 0 & \ldots \\ 0 & 0 & d_0 & d_1 & d_2 & \ldots & d_m & 0 & 0 & \ldots \\ 0 & 0 & 0 & d_0 & d_1 & d_2 & \ldots & d_m & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix},$$

where $d_0 = -\lambda$ and $d_k = \lambda p_k, k = 1, 2, \ldots, m$.

As defined by Lucantoni (1991), a BMAP is a doubly stochastic process that operates as follows. There is an ergodic continuous-time Markov chain with a finite state space $\{1, 2, \ldots, m\}$. When the process is in state i, the sojourn time of the process is exponentially distributed with mean $1/\lambda_i$. At the end of the sojourn time, a batch of size $l \ge 1$ can arrive with probability $p_{ij}(l)$, and the Markov chain moves to state $j \ne i$. Thus, the BMAP is a two-dimensional Markov process $X(t) = \{N(t), J(t)\}$ on the state space $\{(i, j)|i \ge 0, 1 \le j \le m\}$, where $N(t)$ defines the continuous-time Markov chain and $J(t)$ is the phase process. Figure 7.7

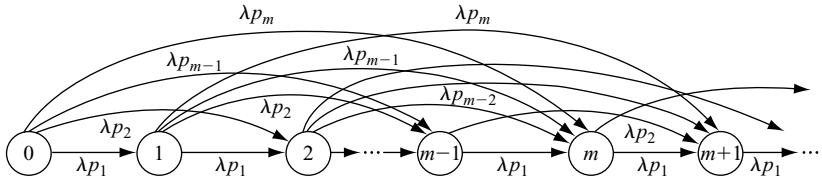


Figure 7.6. State-transition-rate diagram of batch poisson process.

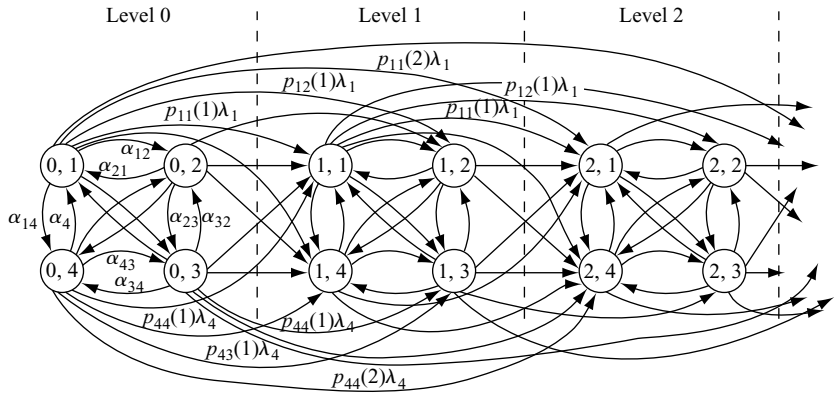


Figure 7.7. State-transition-rate diagram of BMAP(4).

illustrates the state-transition-rate diagram of BMAP for the case of $m = 4$, α_{jk} has the same notation that we used for MAP.

Arranging the states in a lexicographic order we obtain the infinitesimal generator of the process as follows:

$Q_{\text{BMAP}} =$

$$\begin{bmatrix}
-(\lambda_1+\alpha_1) & \alpha_{12} & \alpha_{13} & \dots & \alpha_{1m} & p_{11}(1)\lambda_1 & p_{12}(1)\lambda_1 & \dots & p_{1m}(1)\lambda_1 & p_{11}(2)\lambda_1 & \dots\ \dots \\
\alpha_{21} & -(\lambda_2+\alpha_2) & \alpha_{23} & \dots & \alpha_{2m} & p_{21}(1)\lambda_2 & p_{22}(1)\lambda_2 & \dots & p_{2m}(1)\lambda_2 & p_{21}(2)\lambda_2 & \dots\ \dots \\
\alpha_{31} & \alpha_{32} & -(\lambda_3+\alpha_3) & \dots & \alpha_{3m} & p_{31}(1)\lambda_3 & p_{32}(1)\lambda_3 & \dots & p_{3m}(1)\lambda_3 & p_{31}(2)\lambda_3 & \dots\ \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\ \dots \\
\alpha_{m1} & \alpha_{m2} & \alpha_{m3} & \dots & -(\lambda_m+\alpha_m) & p_{m1}(1)\lambda_m & p_{m2}(1)\lambda_m & \dots & p_{mm}(1)\lambda_m & p_{m1}(2)\lambda_3 & \dots\ \dots \\
0 & 0 & 0 & \dots & 0 & -(\lambda_1+\alpha_1) & \alpha_{12} & \dots & \alpha_{1m} & p_{11}(1)\lambda_1 & \dots\ \dots \\
0 & 0 & 0 & \dots & 0 & \alpha_{21} & -(\lambda_2+\alpha_2) & \dots & \alpha_{2m} & p_{21}(1)\lambda_2 & \dots\ \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\ \dots \\
0 & 0 & 0 & \dots & 0 & \alpha_{m1} & \alpha_{m2} & \dots & -(\lambda_m+\alpha_m) & p_{m1}(1)\lambda_m & \dots\ \dots \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & -(\lambda_1+\alpha_1) & \dots\ \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\ \dots
\end{bmatrix}$$

If we define D_0 and D_i, $i = 1, 2, \dots$, by the following $m \times m$ matrices

$$D_0 = \begin{bmatrix}
-(\lambda_1+\alpha_1) & \alpha_{12} & \alpha_{13} & \dots & \alpha_{1m} \\
\alpha_{21} & -(\lambda_2+\alpha_2) & \alpha_{23} & \dots & \alpha_{2m} \\
\alpha_{31} & \alpha_{32} & -(\lambda_3+\alpha_3) & \dots & \alpha_{3m} \\
\dots & \dots & \dots & \dots & \dots \\
\alpha_{m1} & \alpha_{m2} & \alpha_{m3} & \dots & -(\lambda_m+\alpha_m)
\end{bmatrix}$$

$$D_i = \begin{bmatrix}
p_{11}(i)\lambda_1 & p_{12}(i)\lambda_1 & p_{13}(i)\lambda_1 & \dots & p_{1m}(i)\lambda_1 \\
p_{21}(i)\lambda_2 & p_{22}(i)\lambda_2 & p_{23}(i)\lambda_2 & \dots & p_{2m}(i)\lambda_2 \\
p_{31}(i)\lambda_3 & p_{32}(i)\lambda_3 & p_{33}(i)\lambda_3 & \dots & p_{3m}(i)\lambda_3 \\
\dots & \dots & \dots & \dots & \dots \\
p_{m1}(i)\lambda_m & p_{m2}(i)\lambda_m & p_{m3}(i)\lambda_m & \dots & p_{mm}(i)\lambda_m
\end{bmatrix} \qquad i = 1, 2, \dots, m$$

then Q_{MAP} can be represented in the following block form

$$Q_{\text{BMAP}} = \begin{bmatrix} D_0 & D_1 & D_2 & D_3 & D_4 & \dots & D_m & 0 & 0 & \dots \\ 0 & D_0 & D_1 & D_2 & D_3 & \dots & D_{m-1} & D_m & 0 & \dots \\ 0 & 0 & D_0 & D_1 & D_2 & \dots & D_{m-2} & D_{m-1} & D_m & \dots \\ 0 & 0 & 0 & D_0 & D_1 & \dots & D_{m-3} & D_{m-2} & D_{m-1} & \dots \\ 0 & 0 & 0 & 0 & D_0 & \dots & D_{m-4} & D_{m-3} & D_{m-2} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

Thus, Q_{BMAP} has a structure that is similar to that of Q_{BPP}. Observe that D_0 has negative entries in the main diagonal, and all other entries are nonnegative. Each D_i has only nonnegative entries. To ensure that D_0 is a nondegenerate and stable matrix and thus invertible, we require that Q_{BMAP} be irreducible and $Q_{\text{BMAP}} \neq D_0$, which ensures that arrivals will occur.

BMAP has many applications and variants. Special cases of the process include Poisson process, MMPP, phase-type renewal processes, and MAP, which is a BMAP with a batch size of 1. Thus, BMAP can be considered to be a generalization of the Poisson process.

7.4.1 Properties of BMAP

Let D be the sum of the D_k; that is,

$$D = \sum_{k=0}^{\infty} D_k$$

We observe that D is the infinitesimal generator for the phase process. Let π_{BMAP} be the stationary probability vector in the Markov chain with infinitesimal generator D.

$$\pi_{\text{BMAP}} D = 0$$

$$\pi_{\text{BMAP}} \mathrm{e} = 1$$

The fundamental arrival rate of the BMAP is given by

$$\lambda_{\text{BMAP}} = \pi_{\text{BMAP}} \sum_{k=1}^{\infty} k D_k \mathrm{e}$$

Performance measures that are related to the interarrival times of batches are usually obtained from a MAP that is derived from BMAP by setting all nonzero batches to size one. Thus, the batch arrival rate is given by

$$\lambda_{\text{B}} = \pi_{\text{BMAP}} (-D_0) \mathrm{e}$$

Similarly, the squared coefficient of variation of the interbatch arrival time X is given by

$$c^2_{\text{BMAP}} = \frac{E[X^2]}{(E[X]^2)} - 1 = 2\lambda_{\text{B}}\pi_{\text{BMAP}}(-D_0)^{-1}\text{e} - 1$$

Let X_0 and X_k be two interbatch times that are k lag times apart, where $k > 0$. In Neuts (1995), the lag-k coefficients of correlation are obtained as

$$\begin{aligned}\text{corr}[X_0, X_k] &= \frac{E[X_0 - E[X]]E[X_k - E[X]]}{\text{Var}[X]} \\ &= \frac{\lambda_{\text{B}}\pi_{\text{BMAP}}\{(-D_0)^{-1}(D - D_0)\}^k(-D_0)^{-1}\text{e} - 1}{2\lambda_{\text{B}}\pi_{\text{BMAP}}(-D_0)^{-1}\text{e} - 1}\end{aligned}$$

Also, let $D(z)$ be the z-transform of the matrices $\{D_k : k = 0, 1, \ldots\}$. That is,

$$D(z) = \sum_{k=0}^{\infty} z^k D_k$$

Then the z-transform of $P(n, t)$ is given by

$$G_P(z, t) = \sum_{n=0}^{\infty} z^n P(n, t) = e^{D(z)t} \qquad t \geq 0$$

Finally, assume that the BMAP starts with the initial phase distribution π. That is, $\pi = \{\pi_j,\ j \in J(0)\}$. Let $\mathbf{1}$ be the column vector whose entries are all 1. Then the expected number of arrivals up to time t is given by

$$E_\pi[N_t] = t\pi \sum_{k=1}^{\infty} kD_k 1$$

7.5 Markov-Modulated Poisson Process

The Markov-modulated Poisson process (MMPP) is a doubly stochastic Poisson process whose rate varies according to a continuous-time Markov process. As with MAP and BMAP, the use of MMPP permits modeling of time-varying systems while keeping the analytical solution tractable. A review of MMPP is given in Fischer and Meier-Hellstern (1992).

MMPP is a variation of MAP and hence of BMAP, and it is a MAP(D_0, D_1) whose D_1 is diagonal. That is, in the Markov chain that defines an MMPP, all the

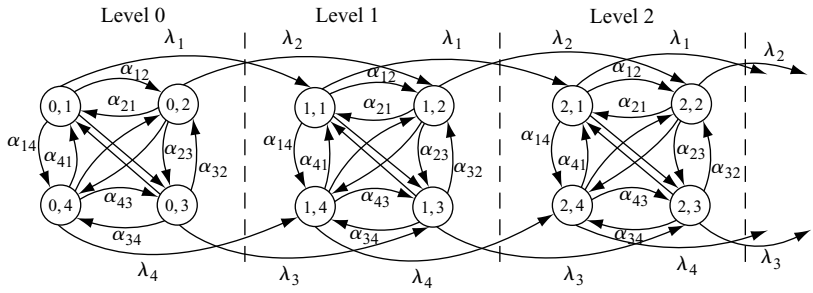


Figure 7.8. State-transition-rate diagram of MMPP(4).

transitions that are associated with events do not change the phase of a state. The state-transition-rate diagram for MMPP(4) is shown in Figure 7.8.

Thus, we obtain

$$D_o = \begin{bmatrix} -(\lambda_1+\alpha_1) & \alpha_{12} & \alpha_{13} & \dots & \alpha_{1m} \\ \alpha_{21} & -(\lambda_2+\alpha_2) & \alpha_{23} & \dots & \alpha_{2m} \\ \alpha_{31} & \alpha_{32} & -(\lambda_3+\alpha_3) & \dots & \alpha_{3m} \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{m1} & \alpha_{m2} & \alpha_{m3} & \dots & -(\lambda_m+\alpha_m) \end{bmatrix}$$

$$D_1 = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda_m \end{bmatrix} = \Lambda$$

$$Q_{\text{MMPP}} = \begin{bmatrix} D_0 & D_1 & 0 & 0 & 0 & \dots \\ 0 & D_0 & D_1 & 0 & 0 & \dots \\ 0 & 0 & D_0 & D_1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

where Λ is the $m \times m$ diagonal matrix whose elements are the arrival rates λ_i, $i = 1, 2, \dots, m$; that is, $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_m)$.

7.5.1 The Interrupted Poisson Process

The most basic type of MMPP is a Poisson process that is controlled by a two-state Markov chain, which is typically associated with a voice source that alternates between a talkspurt mode and a silence mode. The generic names of the states are the ON state and the OFF state. When the chain is in the ON state, it is said to

be in the talkspurt mode that generates voice traffic. Similarly, when it is in the OFF state, it is said to be in the silence mode and does not generate any traffic. The time spent in the ON state is exponentially distributed with mean $1/\beta$, and the time spent in the OFF state is independent of the time in the ON state and is also exponentially distributed with mean $1/\alpha$. Such a process in which arrivals are blocked in the OFF state is called an *interrupted Poisson process* (IPP). If we denote the ON state by state 0 and the OFF state by state 1, then we can represent IPP by the state-transition-rate diagram of IPP shown in Figure 7.9.

The infinitesimal generator for IPP is given by

$$Q_{\text{IPP}} = \begin{bmatrix} D_0 & D_1 & 0 & 0 & 0 & \cdots \\ 0 & D_0 & D_1 & 0 & 0 & \cdots \\ 0 & 0 & D_0 & D_1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

where

$$D_0 = \begin{bmatrix} -\alpha & \alpha \\ \beta & -(\lambda+\beta) \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix} = \Lambda$$

When N IPP sources are multiplexed, the number of sources in the ON state, n, is represented as a birth-and-death process with birth rate $\lambda(n)$ and death rate $\mu(n)$ given by

$$\begin{aligned} \lambda(n) &= (N-n)\alpha \\ \mu(n) &= n\beta \end{aligned} \qquad n = 0, 1, 2, \ldots, N$$

The state-transition-rate diagram for the number of ON sources is shown in Figure 7.10. The probability P_n that n of the sources are in the ON state can be obtained using the techniques developed in Chapter 3.

IPP has been used to model overflow systems in Kuczura (1973) and Meier-Hellstern (1989). Such systems operate as follows. Assume we have a queueing system with two facilities labeled primary facility and overflow facility such that arriving customers first try to receive service at the primary facility. If the primary facility is busy (or full), the arrivals are directed to the overflow system. Thus, during the busy period, customers arrive at the overflow facility according to a Poisson process; during nonbusy periods, no customers arrive. This is illustrated in Figure 7.11.

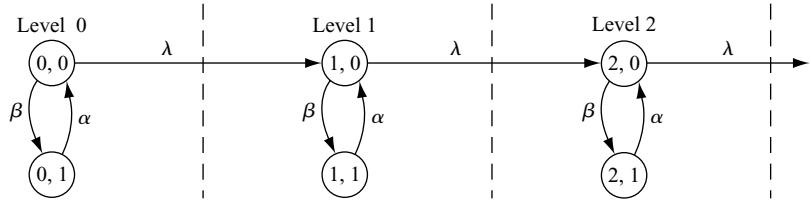


Figure 7.9. State-transition-rate diagram of IPP.

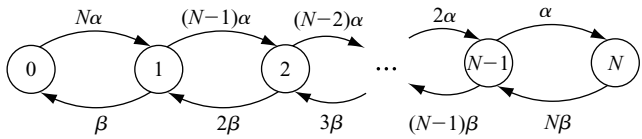


Figure 7.10. State-transition-rate diagram for number of ON sources.

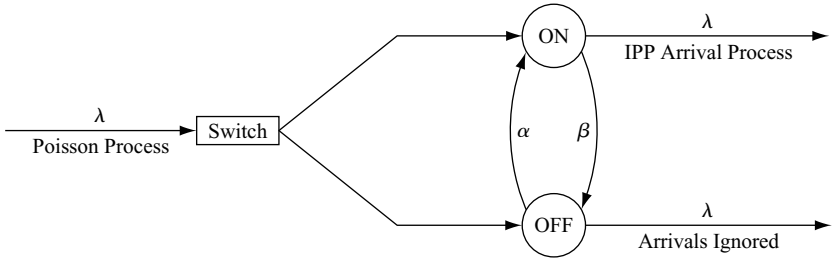


Figure 7.11. Illustration of an interrupted Poisson process.

7.5.2 The Switched Poisson Process

The switched Poisson process (SPP) is very closely related to IPP. Like the latter, it has two states, but unlike IPP, SPP permits traffic to be generated in both states, but with two different rates, λ_1 and λ_2. Thus, it is essentially an MMPP(2), and its state-transition-rate diagram is shown in Figure 7.12.

The infinitesimal generator for SPP is given by

$$Q_{\text{SPP}} = \begin{bmatrix} D_0 & D_1 & 0 & 0 & 0 & \cdots \\ 0 & D_0 & D_1 & 0 & 0 & \cdots \\ 0 & 0 & D_0 & D_1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

where

$$D_0 = \begin{bmatrix} -(\lambda_1 + \alpha) & \alpha \\ \beta & -(\lambda_2 + \beta) \end{bmatrix} \qquad D_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \Lambda$$

7.5.3 Properties of MMPP

The properties of MMPP are similar to those of MAP and BMAP. First, the superposition of n MMPPs with individual infinitesimal generators D_{0i} and rate matrices

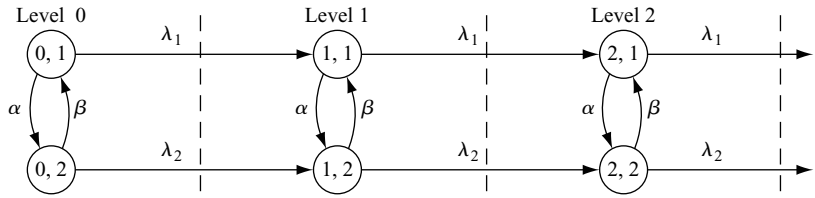


Figure 7.12. State-transition-rate diagram for SPP.

Λ_i is a map with infinitesimal generator and rate matrix given by

$$D_0 = D_{01} \oplus D_{02} \oplus \ldots \oplus D_{0n}$$
$$\Lambda = \Lambda_1 \oplus \Lambda_2 \oplus \ldots \oplus \Lambda_n$$

Example 7.2 Consider two *superposed* MMPP(2) systems, MMPP(D_{01}, Λ_1) and MMPP(D_{02}, Λ_2), where

$$D_{01} = \begin{bmatrix} -(\lambda_1 + \alpha_{12}) & \alpha_{12} \\ \alpha_{21} & -(\lambda_2 + \alpha_{21}) \end{bmatrix} \quad \Lambda_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$
$$D_{02} = \begin{bmatrix} -(\lambda_3 + \beta_{12}) & \beta_{12} \\ \beta_{21} & -(\lambda_4 + \beta_{21}) \end{bmatrix} \quad \Lambda_2 = \begin{bmatrix} \lambda_3 & 0 \\ 0 & \lambda_4 \end{bmatrix}$$

The resulting process is MMPP(D_0, Λ), where

$$D_0 = D_{01} \oplus D_{02} = (D_{01} \otimes \mathrm{I}_2) + (\mathrm{I}_2 \otimes D_{02})$$
$$= \begin{bmatrix} -(\lambda_1 + \alpha_{12}) & 0 & \alpha_{12} & 0 \\ 0 & -(\lambda_1 + \alpha_{12}) & 0 & \alpha_{12} \\ \alpha_{21} & 0 & -(\lambda_2 + \alpha_{21}) & 0 \\ 0 & \alpha_{21} & 0 & -(\lambda_2 + \alpha_{21}) \end{bmatrix}$$
$$+ \begin{bmatrix} -(\lambda_3 + \beta_{12}) & \beta_{12} & 0 & 0 \\ \beta_{21} & -(\lambda_4 + \beta_{21}) & 0 & 0 \\ 0 & 0 & (\lambda_3 + \beta_{12}) & \beta_{12} \\ 0 & 0 & \beta_{21} & -(\lambda_4 + \beta_{21}) \end{bmatrix}$$

$$= \begin{bmatrix} -(\lambda_1+\lambda_3+\alpha_{12}+\beta_{12}) & \beta_{12} & \alpha_{12} & 0 \\ \beta_{21} & -(\lambda_1+\lambda_4+\alpha_{12}+\beta_{21}) & 0 & \alpha_{12} \\ \alpha_{21} & 0 & -(\lambda_2+\lambda_3+\alpha_{21}+\beta_{12}) & \beta_{12} \\ 0 & \alpha_{21} & \beta_{21} & -(\lambda_2+\lambda_4+\alpha_{21}+\beta_{21}) \end{bmatrix}$$

$$\Lambda = \Lambda_1 \oplus \Lambda_2 = (\Lambda_1 \otimes I_2) + (I_2 \otimes \Lambda_2)$$

$$= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \lambda_3 & 0 \\ 0 & \lambda_4 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix} + \begin{bmatrix} \lambda_3 & 0 & 0 & 0 \\ 0 & \lambda_4 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1+\lambda_3 & 0 & 0 & 0 \\ 0 & \lambda_1+\lambda_4 & 0 & 0 \\ 0 & 0 & \lambda_2+\lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_2+\lambda_4 \end{bmatrix}$$

■

7.5.4 The MMPP(2)/M/1 Queue

We illustrate the application of MMPP in performance analysis by considering the MMPP(2)/M/1 queue. MMPP(2) is a process that behaves like a Poisson process with parameter λ_1 for a time that is exponentially distributed with a mean of $1/\alpha$. Then it switches to a Poisson process with parameter λ_2 for a time period that is exponentially distributed with a mean of $1/\beta$. It then switches back to a Poisson process with parameter λ_1 for a time that is exponentially distributed with a mean of $1/\alpha$, and so on. To motivate the discussion on the MMPP(2)/M/1 queue we consider the following situation.

The weather condition in a certain city is very unpredictable. On any given day it constantly switches between being sunny and being showery. The rate at which people in the city arrive at a local video rental store to rent videos depends on the weather condition. The duration of a sunny spell is exponentially distributed with a mean of $1/\alpha$, and the duration of a showery spell is exponentially distributed with a mean of $1/\beta$. During a sunny spell, people arrive at the video store according a Poisson process with rate λ_1. Similarly, during a showery spell, people arrive at the video store according to a Poisson process with rate λ_2. Regardless of the prevailing weather condition, the time to serve each customer at the store is exponentially distributed with a mean of $1/\mu$. Thus, we can model the video store by an MMPP(2)/M/1 queue whose state-transition-rate diagram is shown in Figure 7.13, where the "sunny" condition is state 1 and the "showery" condition is state 2.

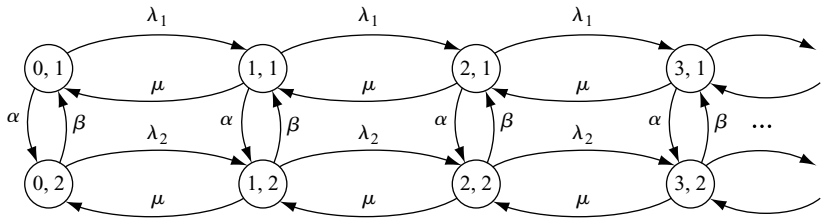


Figure 7.13. State-transition-rate diagram for MMPP(2)/M/1 queue.

From the figure we observe that if lexicographic ordering of the states is used, then the infinitesimal generator is given by

$$Q = \begin{bmatrix} -(\lambda_1+\alpha) & \alpha & \lambda_1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \beta & -(\lambda_2+\beta) & 0 & \lambda_2 & 0 & 0 & 0 & 0 & \dots \\ \mu & 0 & -(\lambda_1+\alpha+\mu) & \alpha & \lambda_1 & 0 & 0 & 0 & \dots \\ 0 & \mu & \beta & -(\lambda_2+\beta+\mu) & 0 & \lambda_2 & 0 & 0 & \dots \\ 0 & 0 & \mu & 0 & -(\lambda_1+\alpha+\mu) & \alpha & \lambda_1 & 0 & \dots \\ 0 & 0 & 0 & \mu & \beta & -(\lambda_2+\beta+\mu) & 0 & \lambda_2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

$$= \begin{bmatrix} D_0 & D_1 & 0 & 0 & 0 & 0 & \dots \\ A_2 & A_0 & A_1 & 0 & 0 & 0 & \dots \\ 0 & A_2 & A_0 & A_1 & 0 & 0 & \dots \\ 0 & 0 & A_2 & A_0 & A_1 & 0 & \dots \\ 0 & 0 & 0 & A_2 & A_0 & A_1 & \dots \\ 0 & 0 & 0 & 0 & A_2 & A_0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

where

$$D_0 = \begin{bmatrix} -(\lambda_1+\alpha) & \alpha \\ \beta & -(\lambda_2+\beta) \end{bmatrix} \qquad D_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$A_0 = \begin{bmatrix} -(\lambda_1+\alpha+\mu) & \alpha \\ \beta & -(\lambda_2+\beta+\mu) \end{bmatrix} \quad A_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad A_2 = \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix}$$

Let π_{ij} denote the steady-state probability of being in state (i, j) and p_m the probability that the process is in phase m, where $m = 1, 2$. Then we have that

$$\alpha p_1 = \beta p_2$$
$$1 = p_1 + p_2$$

Thus,

$$p_1 = \frac{\beta}{\alpha+\beta} \qquad p_2 = \frac{\alpha}{\alpha+\beta}$$

Note that

$$p_j = \sum_{i=0}^{\infty} \pi_{ij} \qquad j = 1, 2$$

Also, the average arrival rate is given by

$$\lambda = \lambda_1 p_1 + \lambda_2 p_2 = \frac{\lambda_1 \beta}{\alpha+\beta} + \frac{\lambda_2 \alpha}{\alpha+\beta}$$

Let $\pi_i = [\pi_{i1}, \pi_{i2}]$ denote the vector of the probabilities that the process is in level i, and let the vector π be defined by

$$\pi = [\pi_0, \pi_1, \pi_2, \ldots]$$

Because $\pi Q = 0$, we have that

$$\begin{aligned} \pi_0 D_0 + \pi_1 A_2 &= 0 \\ \pi_0 D_1 + \pi_1 A_0 + \pi_2 A_2 &= 0 \\ \pi_{k-1} A_1 + \pi_k A_0 + \pi_{k+1} A_2 &= 0 \qquad k \geq 1 \end{aligned}$$

As discussed earlier, the analysis of the QBD process is based on the fact that there exists a matrix R such that

$$\pi_k = \pi_{k-1} R \qquad k \geq 1$$

Then by successive substitution we have that

$$\pi_k = \pi_0 R^k \qquad k \geq 0$$

Thus, we have that

$$\begin{aligned} D_0 + R A_2 &= 0 \\ A_1 + R A_0 + R^2 A_2 &= 0 \end{aligned}$$

If we can find a matrix R that satisfies the above equations, then the proposition that $\pi_{k+1} = \pi_k R$ is correct.

We can also rewrite the preceding equation in the following matrix form:

$$\pi_0 D_0 + \pi_1 A_2 = 0$$
$$\pi_0 D_1 + \pi_1 A_0 + \pi_2 A_2 = \pi_0 A_1 + \pi_1 (A_0 + RA_2) = 0$$

This means that

$$\begin{bmatrix} \pi_0 \ \pi_1 \end{bmatrix} \begin{bmatrix} D_0 & D_1 \\ A_2 & A_0 + RA_2 \end{bmatrix} = 0$$

This equation can be uniquely solved together with the normalization equation

$$\sum_{k=0}^{\infty} \pi_k e = \pi_0 \sum_{k=0}^{\infty} R^k e = \pi_0 [I - R]^{-1} e = 1$$

where $\boldsymbol{e}$ is the column vector $e = [1\ 1]^T$. Note that $[I - R]^{-1}$ is not guaranteed to exist. Thus, when the matrix does not exist, it is common practice to use the iterative procedure that is derived from the equation

$$\pi_0 A_1 + \pi_1 A_0 + \pi_2 A_2 = \pi_0 \{A_1 + RA_0 + R^2 A_2\} = 0 \Rightarrow R = -\{A_1 + R^2 A_2\} A_0^{-1}$$

The recursive solution is given by

$$R(0) = 0$$
$$R(k+1) = -\{A_1 + R^2(k) A_2\} A_0^{-1}$$

where $R(k)$ is the value of R in the kth iteration. The iteration is repeated until the result of two successive iterations differs by less than a predefined parameter ε; that is,

$$\| R(k+1) - R(k) \| < \varepsilon$$

where $\| \cdot \|$ is a matrix norm. The mean total number of customers in the system is given by

$$E[N] = \sum_{j=1}^{2} \sum_{i=0}^{\infty} i \pi_{ij}$$
$$= \sum_{j=1}^{2} \sum_{i=0}^{\infty} i \pi_0 R_j^i$$

where R_j^n is the jth column of the matrix R^n. As stated earlier, several techniques have been proposed for computing the matrix R, and these can be found in Latouche and Ramaswami (1999).

7.6 Markov-Modulated Bernoulli Process

The Markov-modulated Bernoulli process (MMBP) is the discrete-time analog of the MMPP. It is particularly used to model traffic in ATM networks. In Ozekici (1997) and Ozekici and Soyer (2003), MMBP is used in reliability modeling where systems and components function in a randomly changing environment. For example, it is used in reliability assessment of power systems that are subject to fluctuating weather conditions over time. Without loss of generality we consider the use of MMBP in teletraffic applications. However, our formulation of the problem is modeled along the method used in Ozekici (1997) for a more generic system.

Assume that K is an m-state discrete-time Markov chain such that given that the process has just entered state $k \in K$, $1 \leq k \leq m$, the time it spends in the state is geometrically distributed with a mean of $1/p_k$. This implies that each time the process enters state k, the probability that it makes a transition to another state that is different from k in the next time slot is p_k. Thus, each transition is a Bernoulli trial with success probability of p_k in state k. For the simple case of $m = 3$ the state-transition matrix is as follows:

$$P = \begin{bmatrix} 1 - p_1 & p_{12} & p_{13} \\ p_{21} & 1 - p_2 & p_{23} \\ p_{31} & p_{32} & 1 - p_3 \end{bmatrix}$$

The state-transition diagram for K when $m = 3$ is shown in Figure 7.14. In the figure, $p_{12} + p_{13} = p_1$, $p_{21} + p_{23} = p_2$, and $p_{31} + p_{32} = p_3$.

Furthermore, given that the process is in state k in the current time slot, the probability that it will generate a packet in the next time slot is α_k, $1 \leq k \leq m$. Let N_n denote the number of packets that have arrived in the time interval $(0, n]$ and let K_n denote the state of the Markov chain in time slot n. The two-dimensional process $Y = \{N_n, K_n\}$ represents a Markov process on the state space $\{(i, k) | i = 0, 1, \ldots; k = 1, \ldots, m\}$ and is called a Markov-modulated Bernoulli

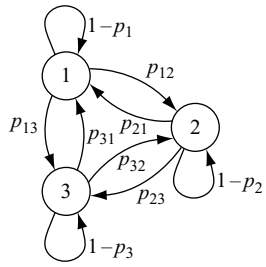


Figure 7.14. State-transition diagram of K for $m = 3$.

process (MMBP). Thus, K_n represents the phase of the process. The state-transition-rate diagram for Y for the case of $m = 3$, which is referred to as the three-state MMBP or MMBP(3), is shown in Figure 7.15.

Let S_k denote the sojourn time in phase k, which is the number of time slots the Markov chain spends in phase k before making a transition to another phase $l \neq k$. The probability mass function of S_k is the geometric distribution given by

$$p_{S_k}(x) = P[S_k = x] = p_k(1 - p_k)^{x-1} \qquad x = 1, 2, \ldots$$

and the number of slots until a packet is generated in phase k is geometrically distributed with mean $1/\alpha_k$.

7.6.1 The MMBP(2)

The special case of $m = 2$, which is the MMBP(2), is used to model cell arrivals in ATM networks. For this case we assume that $p_1 = p$ and $p_2 = q$. The state-transition diagram for the two phases is shown in Figure 7.16.

Thus, the state-transition matrix is given by

$$P = \begin{bmatrix} 1 - p & p \\ q & 1 - q \end{bmatrix}$$

Also, given that the system is in phase 1 in the current time slot, the probability that it will generate a packet in the next time slot is α. Similarly, given that it is in

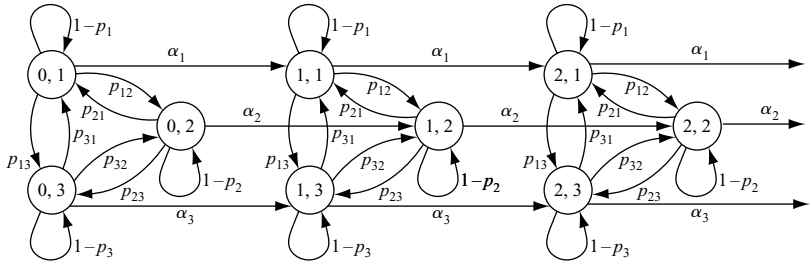


Figure 7.15. State-transition-rate diagram of MMBP(3).

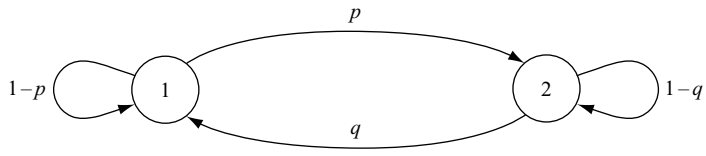


Figure 7.16. State-transition diagram of K for $m = 2$.

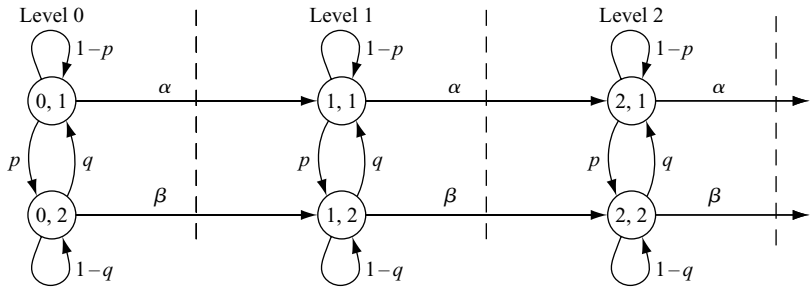


Figure 7.17. State-transition diagram of Y.

phase 2 in the current time slot, the probability that it will generate a packet in the next time slot is β. Thus, the state-transition diagram for Y is shown in Figure 7.17.

Let π_1 and π_2 denote the steady-state probabilities of being in state 1 and state 2, respectively. Solving the equation $\pi P = \pi$, where $\pi = (\pi_1, \pi_2)$, together with $\pi_1 + \pi_2 = 1$ gives the result

$$\pi_1 = \frac{q}{p+q}$$

$$\pi_2 = \frac{p}{p+q}$$

For ATM traffic it is often assumed that $\alpha = 0$; that is, traffic is only generated when the underlying Markov chain is in phase 2. In Viterbi (1986), the "burstiness" parameter of traffic, γ, is defined as $\gamma = p_{22} - p_{12}$, where p_{12} is the probability that a packet arrives in the current slot given that there was no arrival in the previous slot, and p_{22} is the probability that a packet arrives in the current slot given that there was an arrival in the previous slot. Thus, $p_{12} = p$ and $p_{22} = 1 - q$, which gives $\gamma = 1 - p - q$. From this we can express π_1 and π_2 as

$$\pi_1 = \frac{q}{p+q} = \frac{q}{1-\gamma}$$

$$\pi_2 = \frac{p}{p+q} = \frac{p}{1-\gamma}$$

Alternatively, we have that

$$p = \pi_2(1-\gamma)$$

$$q = \pi_1(1-\gamma)$$

The arrival rate of packets, λ, is the probability that the system is in phase 2. That is,

$$\lambda = \frac{p}{1-\gamma}$$

Another parameter of interest is the source utilization, ρ, which is the probability that a slot contains a packet (or an arrival) and is given by

$$\rho = \pi_1\alpha + \pi_2\beta = \pi_2\beta = \frac{p\beta}{p+q}$$

because we assume that $\alpha = 0$ in ATM traffic modeling.

7.7 Sample Applications of MAP and Its Derivatives

MAP has been widely used for modeling many types of queueing systems. Thus, different types of MAP-based queueing models have been developed, including MAP/M/1 queue, MAP/G/1 queue, and MAP/PH/1 queue.

One major advantage of BMAP over MAP is that batches associated with BMAP add to the modeling power and flexibility of MAP. This fact has been exploited by Klemm (2003) to model IP traffic. BMAP-based queueing systems have been extensively analyzed by many authors with respect to ATM traffic. Lucantoni (1993) provides a survey of the analysis of the BMAP/G/1 queue. As pointed out in Masuyama (2003), queues with batch Markovian arrival are so flexible that they can represent most of the queues studied in the past as special cases.

Many authors, including Heffes and Lucantoni (1986), Baiocchi (1991), Yamada and Sumita (1991), and Li and Hwang (1993), have used MMPP(2) to model the superposed ATM traffic. Their analysis deals with MMPP/M/1 or MMPP/G/1 queueing systems. Fischer and Meier-Hellstern (1992) discuss other applications of MMPP.

Zhou and Gans (1999) consider an M/MMPP/1 queue, which can be used to model a system that processes jobs from different sources. The time to process jobs from each source (or job type) is exponentially distributed, but each job type has a different mean service time. Moreover, after a job completion, the choice of the next job to be processed is governed by a Markov chain. Thus, while the aggregate arrival process is Poisson, the source from which a particular job comes is determined by an underlying Markov chain.

Muscariello (2005) uses a hierarchical MMPP traffic model that very closely approximates the long-range dependence (LRD) characteristics of Internet traffic

traces over relevant time scales. As stated earlier, the LRD property of Internet traffic means that values at any instant tend to be positively correlated with values at all future instants. This means that it has some sort of memory. However, long-term correlation properties, heavy tail distributions, and other characteristics are meaningful only over a limited range of time scale.

As discussed earlier, overflow traffic has been modeled using an interrupted Poisson process (IPP) by Kuczura (1973) and Meier-Hellstern (1989). Min (2001) analyzed adaptive worm-hole-routed torus networks with IPP traffic input.

A survey of these traffic models is given in Bae and Suda (1991), Frost and Melamed (1994), Michiel and Laevens (1997), and Adas (1997).

7.8 Problems

7.1 Give the state-transition-rate diagram for the BMAP(2)/M/1 queue with internal rates α_{12} and α_{21}, external arrival rates λ_1 and λ_2, and service rate μ. Specify the infinitesimal generator, Q, if the batch size is equally likely to be 1, 2, or 3.

7.2 Consider the superposition of two identical interrupted Poisson processes with internal rates α and β and external arrival rate λ. Obtain the infinitesimal generator and arrival rate matrix for the superposed system.

7.3 Consider an MMBP(2)/Geo/1 queueing system, which is a single-server queueing system with a second-order Markov-modulated Bernoulli arrival process with external arrival parameters α and β and internal switching probabilities p and q, where $q = 1 - p$, and geometric service times with parameter γ. Give the state-transition diagram.

7.4 Consider a queueing system in which the server is subject to breakdown and repair. When it is operational, the time until it fails is exponentially distributed with mean $1/\eta$. When it breaks down, the time until it is repaired and brought back to service is also exponentially distributed with mean $1/\gamma$. Customers arrive according to a Poisson process with rate λ. However, it has been found that the behavior of arriving customers depends on the state of the server. Specifically, when it is operational, all arriving customers stay in the system until they are served. But a customer that arrives when the server is down will balk (i.e., leave without receiving service) with probability p. Finally, the time to serve a customer when the system is operational is exponentially distributed with mean $1/\mu$. Give the state-transition-rate diagram of the process and determine the Q matrix, identifying the A and D submatrices.

7.5 Consider an m-server queueing system that operates in the following manner. There are two types of customers: type 1 and type 2. Type 1 customers

arrive according to a Poisson process with rate λ_1, and type 2 customers arrive according to a Poisson process with rate λ_2. All the m servers are identical, and the time each takes to serve a customer, regardless of its type, is exponentially distributed with mean $1/\mu$. As long as there is at least one idle server, all arriving customers are served without regard to their type. However, when all m servers are busy, type 2 customers are blocked; only type 1 customers may form a queue. When the number of customers in the system decreases to $k < m$ following an incidence of type 2 customer blocking, type 2 customers will once again be allowed to enter the system. Define the state of the system by (a, b), where a is the number of customers in the system and b is the phase of the system that takes the value 0 when both types of customers are allowed to enter the system and the value 1 when only type 1 customers are allowed. Give the state-transition-rate diagram of the process and determine the Q-matrix, including the D-submatrices.

7.6 Consider a system whose environment changes according to a Markov chain. Specifically, Y_n is the state of the environment at the beginning of the nth period, where $Y = \{Y_n : n \geq 1\}$ is a Markov chain with a state-transition probability matrix P. At the beginning of every period, a Bernoulli experiment is performed whose outcome depends on the state Y_n. Specifically, given that the process is in state Y_n, the probability of success is p_n and the probability of failure is $q_n = 1 - p_n$. Thus, the outcome of the experiment, X_n, depends on the state of the environment. Assume that the conditional PMF of X_n is given by

$$p_{X_n}(x|Y) = \begin{cases} p_n & \text{if } x = 1 \\ q_n & \text{if } x = -1 \end{cases}$$

Define the random variable K_n as follows:

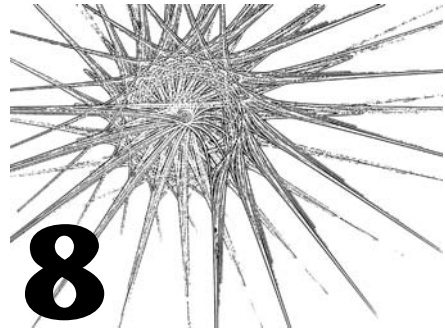
$$K_n = \begin{cases} 0 & n = 0 \\ X_1 + X_2 + \cdots + X_n & n \geq 1 \end{cases}$$

If we assume that a unit positive reward is associated with a success in the Bernoulli experiment, and a unit negative reward is associated with a failure, then K_n is the total reward at the end of the nth period. The bivariate process $\{(K_n, Y_n) : n \geq 0\}$ is a Bernoulli-modulated Markov process. Consider the case where P is the matrix

$$P = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} = \begin{bmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{bmatrix}$$

Give the state-transition diagram of the process, assuming that $p_n = p$.

Random Walk



8

8.1 Introduction

A random walk, which is also called the *drunkard's walk*, is derived from a sequence of Bernoulli trials. It is used in many fields including thermodynamics, biology, astronomy, and economics where it is used to model fluctuations in the stock market.

Consider a Bernoulli trial in which the probability of success is p and the probability of failure is $1 - p$. Assume that the experiment is performed every T time units, and let the random variable X_k denote the outcome of the kth trial. Furthermore, assume that the PMF of X_k is as follows:

$$p_{X_k}(x) = \begin{cases} p & x = 1 \\ 1 - p & x = -1 \end{cases}$$

That is, the X_k are independent and identically distributed random variables. Finally, let the random variable Y_n be defined as follows:

$$Y_0 = 0$$

$$Y_n = \sum_{k=1}^{n} X_k = Y_{n-1} + X_n \qquad n = 1, 2, \ldots$$

If X_k models a process in which we take a step to the right when the outcome of the kth trial is a success and a step to the left when the outcome is a failure, then the random variable Y_n represents the location of the process relative to the

starting point (or origin) at the end of the nth trial. The resulting trajectory of the process $\{Y_n\}$ as it moves through the xy plane, where the x coordinate represents the time and the y coordinate represents the location at a given time, is called a one-dimensional *random walk* generated by $\{X_n\}$. If we define the random process $Y(t) = \{Y_n | n \le t < n+1\}$, then Figure 8.1 shows an example of the sample path of $Y(t)$, where the length of each step is s. It is a staircase with discontinuities at $t = kT, k = 1, 2, \ldots$.

Suppose that at the end of the nth trial there are exactly k successes. Then there are k steps to the right and $n-k$ steps to the left. Thus,

$$Y(nT) = ks - (n-k)s = (2k-n)s = rs$$

where $r = 2k-n$. This implies that $Y(nT)$ is a random variable that assumes values rs, where $r = n, n-2, n-4, \ldots, -n$. Because the event $\{Y(nT) = rs\}$ is the event $\{k$ successes in n trials$\}$, where $k = (n+r)/2$, we have that

$$P[Y(nT) = rs] = P\left[\frac{n+r}{2} \text{ successes}\right] = \binom{n}{\frac{n+r}{2}} p^{\frac{n+r}{2}} (1-p)^{\frac{n-r}{2}}$$

Note that $(n+r)$ must be an even number. Also, because $Y(nT)$ is the sum of n independent Bernoulli random variables, its mean and variance are given as follows:

$$E[Y(nT)] = nE[X_k] = n[ps - (1-p)s] = (2p-1)ns$$

$$E[X_k^2] = ps^2 + (1-p)s^2 = s^2$$

$$\text{Var}[Y(nT)] = n\text{Var}[X_k] = n[s^2 - s^2(2p-1)^2] = 4p(1-p)ns^2$$

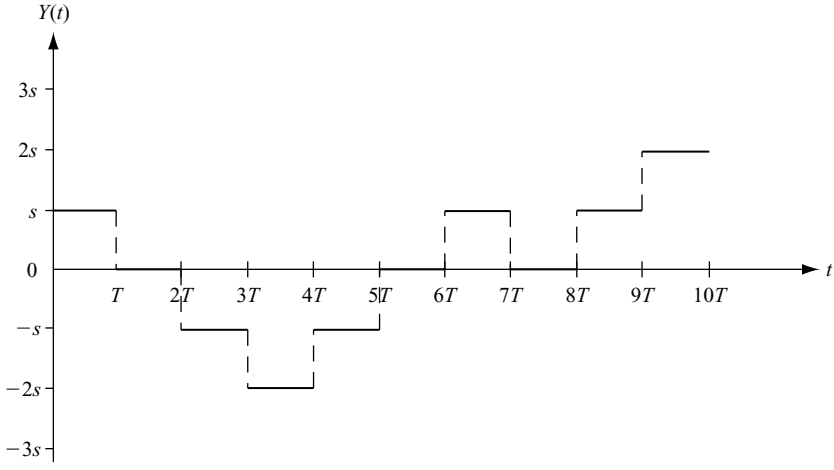


Figure 8.1. A sample path of the random walk.

In the special case where $p = 1/2$, $E[Y(nT)] = 0$, and $\text{Var}[Y(nT)] = ns^2$. In this case the random walk is called a *symmetric random walk*, which is also known as a *Bernoulli random walk*.

Example 8.1 Let the random variable Y_n be as defined above; that is, $Y = X_1 + X_2 + \cdots + X_n$, where the X_k are independent and identically distributed Bernoulli random variables. Find the value of $P[Y_n = c]$. ■

Solution: We assume that $X_k = 1$ corresponds to a step to the right in the kth trial and $X_k = -1$ corresponds to a step to the left. Suppose that out of the n steps the process takes r steps to the right and, therefore, $n - r$ steps to the left. Thus, we have that $Y_n = c = r - (n - r) = 2r - n$, which gives $r = (n + c)/2$. Using the binomial distribution we obtain

$$P[Y_n = c] = \begin{cases} \binom{n}{\frac{n+c}{2}} p^{(n+c)/2} q^{(n-c)/2} & n + c \text{ even} \\ 0 & n + c \text{ odd} \end{cases}$$

For the special case of $c = 0$ we have that

$$P[Y_n = 0] = \begin{cases} \binom{n}{\frac{n}{2}} (pq)^{n/2} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

which makes sense because this means that the process took as many steps to the right as it did to the left for it to return to the origin.

8.2 The Two-Dimensional Random Walk

A two-dimensional random walk is a stochastic process $\{(X_n, Y_n), n = 0, 1, \ldots\}$ whose state space is the set $\Omega = \{(i, j) | i, j \in 0, \pm 1, \pm 2, \ldots\}$ and such that

$$P[(X_{n+1}, Y_{n+1}) = (i_{n+1}, j_{n+1}) | (X_n, Y_n) = (i_n, j_n)]$$

$$= \begin{cases} p_1 & i_{n+1} = i_n + 1,\ j_{n+1} = j_n \\ p_2 & i_{n+1} = i_n,\ j_{n+1} = j_n + 1 \\ q_1 & i_{n+1} = i_n - 1,\ j_{n+1} = j_n \\ q_2 & i_{n+1} = i_n,\ j_{n+1} = j_n - 1 \\ 0 & \text{otherwise} \end{cases}$$

where $p_1 \geq 0,\ p_2 \geq 0,\ q_1 \geq 0,\ q_2 \geq 0,\ p_1 + p_2 + q_1 + q_2 = 1$. Thus, the walker only moves to one of its four neighbors: up, down, to the right, and to the left.

8.3 Random Walk as a Markov Chain

Let Y_n be as defined earlier. Then

$$
\begin{aligned}
&P[Y_{n+1} = k | Y_1, Y_2, \ldots, Y_{n-1}, Y_n = m] \\
&\quad = P[X_{n+1} = k - m | Y_1, Y_2, \ldots, Y_{n-1}, Y_n = m] \\
&\quad = P\left[X_{n+1} = k - m | X_1, X_1 + X_2, \ldots, \sum_{i=1}^{n-1} X_i, \sum_{i=1}^{n} X_i = m \right] \\
&\quad = P[X_{n+1} = k - m] \qquad \text{by the independence of } X_{n+1} \\
&\quad = P[X_{n+1} = k - m | X_1 + \cdots + X_n = m] \\
&\quad = P[X_{n+1} = k - m | Y_n = m] \\
&\quad = P[Y_{n+1} = k | Y_n = m]
\end{aligned}
$$

where the fourth equality is due to the independence of X_{n+1} and the other n random variables. Thus, the future of a random walk depends only on the most recent past outcome. Figure 8.2 shows the state-transition diagram of the Markov chain for a one-dimensional random walk.

If with probability 1 a random walker revisits its starting point, the walk is defined to be a *recurrent random walk*; otherwise, it is defined to be nonrecurrent. In Grimmett and Welsh (1986) and Grimmett and Stirzaker (2001), it is shown that the probability that a random walker ever revisits its starting point is

$$P[Y_n = 0 | Y_0 = 0, n = 1, 2, \ldots] = 1 - |p - q|$$

Thus, a random walk is recurrent only if it is a symmetric random walk; that is, only if $p = q = 1/2$.

One important feature of a recurrent random walk is the *mixing time*, which is the number of steps it takes for the random walk to reach the stationary distribution.

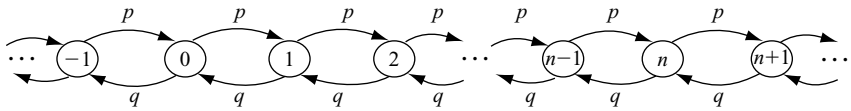


Figure 8.2. State-transition diagram of a random walk.

Associated with the mixing time is the *mixing rate*, which measures how fast the random walk converges to the stationary distribution. The computation of the mixing time and the mixing rate is not discussed in this book.

8.4 Symmetric Random Walk as a Martingale

Consider the random walk $\{Y_n | n = 0, 1, \ldots\}$ where $Y_0 = 0$ and

$$Y_n = \sum_{k=1}^{n} X_k = Y_{n-1} + X_n \qquad n = 1, 2, \ldots$$

Assume that $P[X_k = 1] = p$ and $P[X_k = -1] = 1 - p$. Then

$$\begin{aligned} E[Y_n | Y_0, \ldots, Y_k] &= E[Y_n - Y_k + Y_k | Y_0, \ldots, Y_k] \\ &= E[Y_n - Y_k | Y_0, \ldots, Y_k] + Y_k \\ &= \sum_{j=k+1}^{n} E[X_j | X_1, \ldots, X_k] + Y_k = \sum_{j=k+1}^{n} E[X_j] + Y_k \\ &= (n-k)(2p-1) + Y_k \end{aligned}$$

Now, for a symmetric random walk, $p = 1/2$ and the result becomes

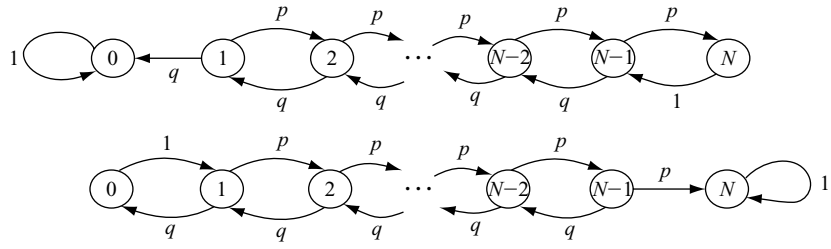
$$E[Y_n | Y_0, \ldots, Y_k] = Y_k$$

Thus, a symmetric random walk is a martingale, but a nonsymmetric random walk is not a martingale.

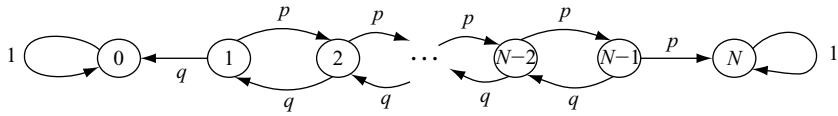
8.5 Random Walk with Barriers

The random walk previously described assumes that the process can continue forever; in other words, it is unbounded. If the walk is bounded, then the ends of the walk are called *barriers*. These barriers can impose different characteristics on the process. For example, they can be *reflecting barriers*, which means that on hitting them the walk turns around and continues. They can also be *absorbing barriers*, which means that the walk ends when a barrier is hit. Figure 8.3 shows the different types of barriers where it is assumed that the number of states is finite.

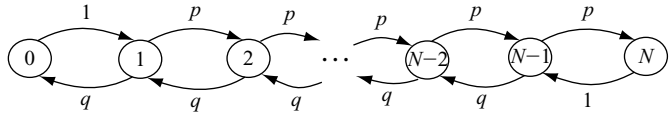
When absorbing barriers are present we obtain an absorbing Markov chain. We can then use the fundamental matrix method discussed in Chapter 3 to obtain the necessary performance measures, such as the hitting times and first passage times. One example of the random walk with barriers is the gambler's ruin problem that is discussed in the next section.



(a) Random Walk with One Absorbing Barrier and One Reflecting Barrier



(b) Random Walk with Two Absorbing Barriers



(c) Random Walk with Two Reflecting Barriers

Figure 8.3. Random walks with barriers.

8.6 Gambler's Ruin

Consider the following random walk with two absorbing barriers, which is generally referred to as the gambler's ruin. Suppose a gambler plays a sequence of independent games against an opponent. He starts out with $\$k$, and in each game he wins \$1 with probability p and loses \$1 with probability $q = 1 - p$. When $p > q$, the game is advantageous to the gambler either because he is more skilled than his opponent or the rules of the game favor him. If $p = q$, the game is fair; if $p < q$, the game is disadvantageous to the gambler.

8.6.1 Ruin Probability

Assume that the gambler stops when he has a total of \$0 or $\$N$. In the latter case he has additional $\$(N - k)$ over his initial $\$k$. (Another way to express this is that he plays against an opponent who starts out with $\$(N - k)$, and the game stops when either player has lost all his money.) We are interested in computing the probability r_k that the player will be ruined (or he has lost all his money) after starting

with $\$k$. The state-transition diagram of the process is illustrated in Figure 8.3(b) where the states represent the total amount the gambler currently has.

To solve the problem, we note that at the end of the first game, the player will have the sum of $\$(k+1)$ if he wins the game (with probability p) and the sum of $\$(k-1)$ if he loses the game (with probability q). Thus, if he wins the first game, the probability that he will eventually be ruined is r_{k+1}; if he loses his first game, the probability that he will be ruined is r_{k-1}. There are two boundary conditions in this problem. First, $r_0 = 1$ because he cannot gamble when he has no money. Second, $r_N = 0$ because he cannot be ruined. Thus, we obtain the following difference equation:

$$r_k = qr_{k-1} + pr_{k+1} \qquad 0 < k < N$$

Because $p + q = 1$, we obtain

$$(p+q)r_k = qr_{k-1} + pr_{k+1} \qquad 0 < k < N$$

which we can write as

$$p(r_{k+1} - r_k) = q(r_k - r_{k-1})$$

From this we obtain the following:

$$r_{k+1} - r_k = (q/p)(r_k - r_{k-1}) \qquad 0 < k < N$$

Noting that $r_2 - r_1 = (q/p)(r_1 - r_0) = (q/p)(r_1 - 1), r_3 - r_2 = (q/p)\ (r_2 - r_1) = (q/p)^2(r_1 - 1)$, and so on, we obtain the following:

$$r_{k+1} - r_k = (q/p)^k(r_1 - 1) \qquad 0 < k < N$$

Now,

$$\begin{aligned} r_k - 1 = r_k - r_0 &= (r_k - r_{k-1}) + (r_{k-1} - r_{k-2}) + \cdots + (r_1 - 1) \\ &= [(q/p)^{k-1} + (q/p)^{k-2} + \cdots + 1](r_1 - 1) \\ &= \begin{cases} \dfrac{1 - (q/p)^k}{1 - q/p}(r_1 - 1) & p \neq q \\ k(r_1 - 1) & p = q \end{cases} \end{aligned}$$

The boundary condition that $r_N = 0$ implies that

$$r_1 = \begin{cases} 1 - \dfrac{1-(q/p)}{1-(q/p)^N} & p \neq q \\ 1 - \dfrac{1}{N} & p = q \end{cases}$$

Thus,

$$r_k = \begin{cases} 1 - \dfrac{1-(q/p)^k}{1-(q/p)^N} = \dfrac{(q/p)^k - (q/p)^N}{1-(q/p)^N} & p \neq q \\ 1 - \dfrac{k}{N} & p = q \end{cases}$$

Example 8.2 Jack needs a total of \$100 to go on a trip. He currently has \$65 and believes that he can make up the difference by gambling in a local casino joint where he pays \$1 for each play and gets \$2 back if he wins and forfeits the \$1 if he loses. Jack's goal is to stop playing when he has exactly \$100 for the trip. Knowing that Jack can lose all the \$65 he initially had and thus will never go on the trip, what is the probability that he is able to go on the trip if the probability that he wins in each play is 0.6? ■

Solution: Using the results from the previous discussion, we have that $N = 100, k = 65$, and $p = 0.6$. Therefore, the probability that he goes on the trip is the complement of the ruin probability, which is

$$1 - r_{65} = \frac{1-(0.4/0.6)^{65}}{1-(0.4/0.6)^{100}} = 1$$

If we had $p = 0.5$, then the result would be $1 - r_{65} = 65/100 = 0.65$.

8.6.2 Duration of a Game

Let d_k denote the expected time until a gambler that starts with $\$k$ is ruined. Clearly, $d_0 = d_N = 0$. If the gambler wins the first game, the expected duration of the game is $d_{k+1} + 1$; if he loses the first game, the expected duration of the game is $d_{k-1} + 1$. Thus, d_k satisfies the following difference equation:

$$d_k = q(d_{k-1} + 1) + p(d_{k+1} + 1) = 1 + qd_{k-1} + pd_{k+1} \qquad 0 < k < N$$

Because $p+q=1$, we can rewrite the above difference equation as follows:

$$pd_{k+1} - pd_k - qd_k + qd_{k-1} + 1 = p(d_{k+1} - d_k) - q(d_k - d_{k-1}) + 1 = 0 \qquad 0 < k < N$$

That is,

$$p(d_{k+1} - d_k) = q(d_k - d_{k-1}) - 1 \qquad 0 < k < N$$

Let $m_k = d_k - d_{k-1}$. Then we have that

$$pm_{k+1} = qm_k - 1$$

Solving the preceding equation iteratively we obtain

$$pm_2 = qm_1 - 1 \Rightarrow m_2 = \frac{q}{p}m_1 - \frac{1}{p}$$

$$pm_3 = qm_2 - 1 \Rightarrow m_3 = \frac{q}{p}m_2 - \frac{1}{p} = \left(\frac{q}{p}\right)^2 m_1 - \frac{1}{p}\left\{1 + \frac{q}{p}\right\}$$

$$pm_4 = qm_3 - 1 \Rightarrow m_4 = \frac{q}{p}m_3 - \frac{1}{p} = \left(\frac{q}{p}\right)^3 m_1 - \frac{1}{p}\left\{1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2\right\}$$

$$pm_5 = qm_4 - 1 \Rightarrow m_5 = \frac{q}{p}m_4 - \frac{1}{p} = \left(\frac{q}{p}\right)^4 m_1 - \frac{1}{p}\left\{1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \left(\frac{q}{p}\right)^3\right\}$$

Thus, in general we have that

$$m_k = \left(\frac{q}{p}\right)^{k-1} m_1 - \frac{1}{p}\left\{1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \left(\frac{q}{p}\right)^3 + \cdots + \left(\frac{q}{p}\right)^{k-2}\right\}$$

$$= \left(\frac{q}{p}\right)^{k-1} m_1 - \frac{1}{p}\sum_{j=0}^{k-2}\left(\frac{q}{p}\right)^j$$

Now, $m_1 = d_1 - d_0 = d_1$. Thus,

$$d_k = \sum_{j=1}^{k} m_j = \sum_{j=1}^{k}\left\{\left(\frac{q}{p}\right)^{j-1} d_1 - \frac{1}{p}\sum_{i=0}^{j-2}\left(\frac{q}{p}\right)^i\right\}$$

$$= \begin{cases} \dfrac{1-(q/p)^k}{1-(q/p)}\left\{d_1 + \dfrac{1}{p-q}\right\} - \dfrac{k}{p-q} & p \neq q \\ k\{d_1 - (k-1)\} & p = q \end{cases}$$

Because $d_N = 0$, we have that

$$d_1 = \begin{cases} \dfrac{1-(q/p)}{1-(q/p)^N}\left\{\dfrac{N}{p-q}\right\} - \dfrac{1}{p-q} & p \neq q \\ N-1 & p = q \end{cases}$$

Thus, we finally obtain

$$d_k = \begin{cases} \dfrac{1-(q/p)^k}{1-(q/p)}\left\{d_1 + \dfrac{1}{p-q}\right\} - \dfrac{k}{p-q} & p \neq q \\ k(N-k) & p = q \end{cases}$$

8.7 First Return Times

Let T_0 denote the time until a random walk returns to the origin for the first time. That is, given that $Y_n = Y_{n-1} + X_n$, $Y_0 = 0$, $n = 1, 2, \ldots$,

$$T_0 = \min\{n \geq 1 : Y_n = 0\}$$

T_0 is the first return to zero and its PMF

$$p_{T_0}(n) = P[T_0 = n] = P[Y_1 \neq 0, Y_2 \neq 0, \ldots, Y_{n-1} \neq 0, Y_n = 0]$$

is given by the following theorem:

Theorem 8.1 The z-transform of the probability mass function of the first return to zero is given by

$$G_{T_0}(z) = 1 - (1 - 4pqz^2)^{1/2}$$

Proof Let $p_0(n) = P[Y_n = 0]$ be the probability that the process is at the origin after n steps. Let A be the event that $Y_n = 0$, and let B_k be the event that the first return to the origin occurs at the kth step. Because the B_k are mutually exclusive events, we have that

$$P[A] = \sum_{k=1}^{n} P[A|B_k]P[B_k] \qquad n \geq 1$$

Now, $P[B_k] = p_{T_0}(k)$ and $P[A|B_k] = p_0(n-k)$, which means that

$$p_0(n) = \sum_{k=1}^{n} p_0(n-k)\, p_{T_0}(k) \qquad n \geq 1$$

Let the z-transform of $p_0(n)$ be $G_{Y_n}(z)$. As the z-transform of $p_{T_0}(k)$ is given by $G_{T_0}(z)$, the preceding equation becomes

$$\begin{aligned} G_{Y_n}(z) &= \sum_{n=0}^{\infty} p_0(n) z^n = p_0(0) + \sum_{n=1}^{\infty} \sum_{k=1}^{n} p_0(n-k)\, p_{T_0}(k) z^n \\ &= 1 + G_{Y_n}(z) G_{T_0}(z) \end{aligned}$$

where the last equality follows from the fact that $p_0(0) = 1$. Thus, we obtain

$$G_{T_0}(z) = 1 - \frac{1}{G_{Y_n}(z)}$$

To obtain $Y_n = 0$ we must move an equal number of steps to the right and to the left. That is,

$$p_0(n) = \begin{cases} \binom{n}{n/2} (pq)^{n/2} & n \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

Thus, we have that

$$G_{Y_n}(z) = \sum_{n=0}^{\infty} \binom{2n}{n} (pq)^n z^{2n} = \sum_{n=0}^{\infty} \binom{2n}{n} (pqz^2)^n = (1 - 4pqz^2)^{-1/2}$$

where the last equality follows from the identity

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}$$

From this we obtain

$$G_{T_0}(z) = 1 - (1 - 4pqz^2)^{1/2}$$

which completes the proof. ■

By expanding $G_{T_0}(z)$ as a power series we obtain the distribution of T_0 as follows. We know that

$$(1-a)^N = \sum_{k\geq 0} \binom{N}{k} (-a)^k$$

Thus,

$$G_{T_0}(z) = 1 - (1-4pqz^2)^{1/2} = 1 - \sum_{k\geq 0} \binom{1/2}{k} (-4pqz^2)^k$$

$$= \sum_{k\geq 1} \binom{1/2}{k} (-1)^{k-1} (4pq)^k z^{2k}$$

which implies that

$$p_{T_0}(2k) = (-1)^{k-1} \binom{1/2}{k} (4pq)^k \qquad k = 1, 2, \ldots$$

If we define ξ_0 as the probability that the process ever returns to the origin, then we have that

$$\xi_0 = \sum_{n=1}^{\infty} p_{T_0}(n) = G_{T_0}(1) = P[T_0 < \infty] = 1 - (1-4pq)^{1/2}$$

Because $p+q=1$, we have that

$$(p+q)^2 = 1 = p^2 + q^2 + 2pq \Rightarrow 1 - 4pq = p^2 + q^2 - 2pq$$

$$= (p-q)^2 = (2p-1)^2$$

which gives $(1-4pq)^{1/2} = |p-q| = |2p-1|$, and hence

$$\xi_0 = 1 - |p-q| = 1 - |2p-1|$$

Also, for the symmetric random walk (i.e., $p = q = 1/2$), we have that

$$G_{T_0}(z) = 1 - (1-z^2)^{1/2}$$

In this case we have that $\xi_0 = 1$. However, because $dG_{T_0}(z)/dz|_{z=1} = \infty$, we have that $E[T_0] = \infty$ for a symmetric random walk.

8.8 First Passage Times

If $Y_n = r$ for some $n \geq 1$, we say that the random walk *visits* or *hits* level r at step n. Let T_r denote the first time the process enters the state r, given that it started in state 0. That is,

$$T_r = \min\{n \geq 1 : Y_n = r\}$$

T_r is referred to as the first passage time of the random walk. Let the PMF of T_r be $p_{T_r}(k) = P[T_r = k]$. We obtain $p_{T_r}(k)$ via the *reflection principle*.

A simple case of the reflection principle is illustrated in Figure 8.4. According to the principle, the number of paths from A to B that touch or cross the horizontal axis is equal to the number of all paths from $\bar{A}$ to B, where $\bar{A}$ is the reflection of the point A on the x-axis.

Let $N_{a,b}(p, q)$ denote the number of possible paths between the points (a, p) and (b, q). Then $N_{a,b}(p, q)$ can be computed as follows. Let a path consist of m steps to the right and l steps to the left. Thus, the total number of steps is $m + l = b - a$, and the difference between the number of rightward steps and the leftward steps is $m - l = q - p$. From this we obtain

$$m = \frac{1}{2}(b - a + p - q)$$

Because $N_{a,b}(p, q)$ can be defined as the number of "successes," m, in $m + l$ binomial trials, we have that

$$N_{a,b}(p, q) = \binom{b-a}{m}$$

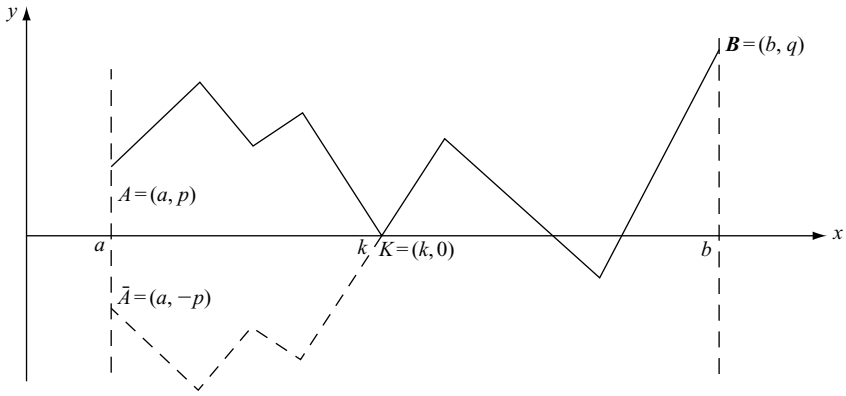


Figure 8.4. Illustration of the reflection principle.

where m is as defined above. Next, we assume that p and q are positive numbers and determine the number $N^0_{a,b}(p, q)$ of these paths that do not touch or intersect the x-axis. This number is the complement of the number $N^1_{a,b}(p, q)$ of paths that touch or intersect the x-axis, which, from the reflection principle, is given by

$$N^1_{a,b}(p, q) = N_{a,b}(-p, q)$$

Thus,

$$N^0_{a,b}(p, q) = N_{a,b}(p, q) - N^1_{a,b}(p, q) = N_{a,b}(p, q) - N_{a,b}(-p, q)$$

In a more general case, we consider a process that reaches the level m at time τ but above level m at time b. The reflection occurs on the line $y = m$, and the reflected path is constructed from the point τ where the process first reaches the level m onward, as illustrated in Figure 8.5. The reflection principle states that for every path with $y \geq m$ there exists another path obtained by reflection through the line $y = m$ from the time τ onward up to b such that the value of the terminal point on the path is at least m. Thus, there is a reflected path that is at level $q - 2(q - m) = 2m - q$ at time b, and

$$N^1_{a,b}(p, q) = N_{a,b}(p, 2m - q)$$

For our case where $(a, p) = (0, 0)$ and $(b, q) = (k, r)$, we assume that $r > 0$. Then for the process to be at state r by time k, either $Y_{k-1} = r + 1$ or $Y_{k-1} = r - 1$. Because the valid option for our problem is $Y_{k-1} = r - 1$, we are essentially concerned with the number of paths from $(0, 0)$ to $(k - 1, r - 1)$. Thus, the number of these paths that do not cross the state r is given by

$$N^0_{0,k-1}(0, r - 1) = N_{0,k-1}(0, r - 1) - N^1_{0,k-1}(0, r - 1)$$

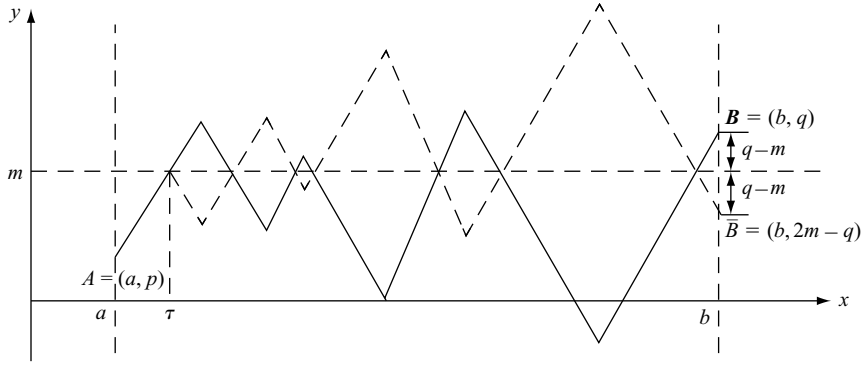


Figure 8.5. Second illustration of the reflection principle.

According to the reflection principle, $N^1_{0,k-1}(0, r-1) = N_{0,k-1}(0, r+1)$ (because $m = r$ and $q = r-1$ from our previous results), which means that

$$\begin{aligned} N^0_{0,k-1}(0, r-1) &= N_{0,k-1}(0, r-1) - N_{0,k-1}(0, r+1) \\ &= \binom{k-1}{\frac{k-r}{2}} - \binom{k-1}{\frac{k-r-2}{2}} \\ &= \frac{(k-1)!}{\left(\frac{k-r}{2}\right)!\left(\frac{k+r-2}{2}\right)!} - \frac{(k-1)!}{\left(\frac{k-r-2}{2}\right)!\left(\frac{k+r}{2}\right)!} \\ &= \frac{k!}{k\left(\frac{k-r}{2}\right)!\left(\frac{k+r}{2}\right)!}\left\{\frac{k+r}{2} - \frac{k-r}{2}\right\} = \frac{(r/k)k!}{\left(\frac{k-r}{2}\right)!\left(\frac{k+r}{2}\right)!} \\ &= \frac{rN_{0,k}(0, r)}{k} \end{aligned}$$

Because a similar result can also be obtained when $r < 0$, we have that the PMF of T_r is given by

$$p_{T_r}(k) = P[T_r = k] = \frac{|r|}{k}\binom{k}{\frac{k+r}{2}} p^{(k+r)/2} q^{(k-r)/2}$$

Observe that for $r > 0$, $N_{0,k}(0, r)\, p^{(k+r)/2} q^{(k-r)/2} = P[Y_k = r]$. Thus, when $r > 0$ we have that

$$p_{T_r}(k) = P[T_r = k] = \frac{r}{k} P[Y_k = r]$$

8.9 Maximum of a Random Walk

The maximum of a random walk Y_n up to time k is defined by

$$M_k = \max\{Y_n : n = 1, 2, \ldots, k\}$$

Theorem 8.2 For $r \geq 1$,

$$P[M_k \geq r, Y_k = b] = \begin{cases} P[Y_k = b] & b \geq r \\ \{q/p\}^{r-b} P[Y_k = 2r - b] & b < r \end{cases}$$

Proof The proof of this theorem is given in Grimmett and Stirzaker (2001) and is based on the reflection principle. ■

Thus,

$$\begin{aligned}
P[M_k \geq r] &= \sum_{b=-\infty}^{\infty} P[M_k \geq r, Y_k = b] = \sum_{b=r}^{\infty} P[M_k \geq r, Y_k = b] \\
&\quad + \sum_{b=-\infty}^{r-1} P[M_k \geq r, Y_k = b] \\
&= \sum_{b=r}^{\infty} P[Y_k = b] + \sum_{b=-\infty}^{r-1} (q/p)^{r-b} P[Y_k = 2r - b] \\
&= P[Y_k \geq r] + \sum_{m=r+1}^{\infty} (q/p)^{m-r} P[Y_k = m] \\
&= P[Y_k = r] + \sum_{b=r+1}^{\infty} P[Y_k = b] + \sum_{m=r+1}^{\infty} (q/p)^{m-r} P[Y_k = m] \\
&= P[Y_k = r] + \sum_{m=r+1}^{\infty} \{1 + (q/p)^{m-r}\} P[Y_k = m]
\end{aligned}$$

For the symmetric random walk in which $p = q = 1/2$, we obtain

$$P[M_k \geq r] = P[Y_k = r] + 2P[Y_k > r]$$

Let T_M denote the time at which the random walk $\{Y_n\}$ first attains the maximum value. That is,

$$T_M = \min\{n \leq k : Y_n = M_k\}$$

It can be shown (see Stirzaker (2005), for example) that the PDF of T_M is given by

$$f_{T_M}(x) = \frac{1}{\pi\sqrt{x(m-x)}} \quad 0 < x < m$$

which is the so-called arcsine law that states that the time T_M at which $\{Y_n\}$ attains its maximum value in the interval $(0, m)$ has the arcsine distribution, which has singularity near the extreme positions $x \to 0$ and $x \to m$, as shown in Figure 8.6.

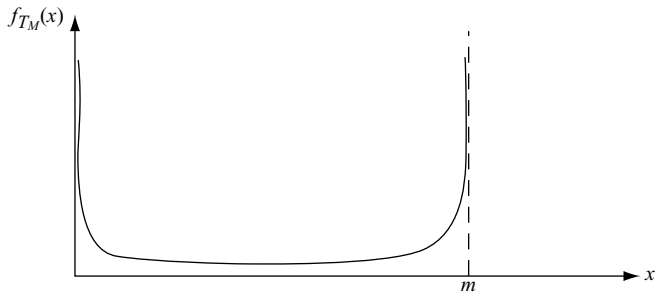


Figure 8.6. The arcsine distribution.

8.10 Random Walk on a Graph

A graph $G = (V, E)$ is a pair of sets V (or $V(G)$) and E (or $E(G)$) called vertices (or nodes) and edges (or arcs), respectively, where the edges join different pairs of vertices. The vertices are represented by points, and the edges are represented by lines joining the vertices. A graph is a mathematical concept that captures the notion of connection. A graph is defined to be a *simple graph* if there is at most one edge connecting any pair of vertices and an edge does not loop to connect a vertex to itself. When multiple edges are allowed between any pair of vertices, the graph is called a *multigraph*. Examples of a simple graph, a multigraph, and a graph with loop are shown in Figure 8.7.

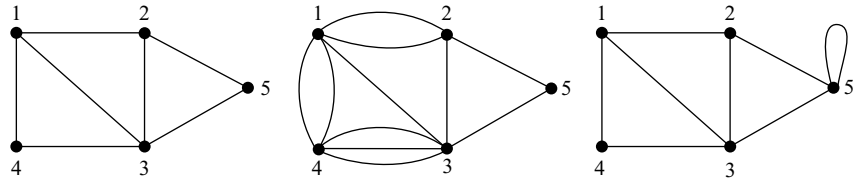
Two vertices are said to be *adjacent* if they are joined by an edge. For example, in Figure 8.7, vertices 1 and 2 are adjacent. An edge e that connects vertices a and b is denoted by (a, b). Such an edge is said to be *incident* with vertices a and b; the vertices a and b are called the *ends* or *endpoints* of e. If the edge $e = (a, b)$ exists, we sometimes call vertex b a *neighbor* of vertex a. The set of neighbors of vertex a is usually denoted by $\Gamma(a)$.

The *degree* (or *valency*) of a vertex x, which is denoted by $d(x)$, is the number of edges that are incident with x. For example, in Figure 8.7(a), $d(3) = 4$ and $d(4) = 2$. If a node x has $d(x) = 0$, then x is said to be *isolated*. It can be shown that

$$\sum_{x \in V(G)} d(x) = 2|E(G)|$$

where $|E(G)|$ is the number of edges in the graph.

A *subgraph* of G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, and the endpoints of an edge $e \in E(H)$ are the same as its endpoints in G. A complete graph K_n on n vertices is the simple graph that has all $\binom{n}{2}$ possible edges.



(a) Simple Graph (b) Multigraph (c) Graph with Loop

Figure 8.7. Examples of simple graph, multigraph, and graph with loop.

A *walk* in a graph is an alternating sequence $x_0, e_1, x_1, e_2, \ldots, x_{k-1}, e_k, x_k$ of vertices x_i, which are not necessarily distinct, and edges e_i such that the endpoints of e_i are x_{i-1} and x_i, $i = 1, \ldots, k$. A *path* is a walk in which the vertices are distinct. For example, in Figure 8.7(a), the path $\{1, 3, 5\}$ connects vertices 1 and 5. When a path can be found between every pair of distinct vertices, we say that the graph is a *connected graph*. A graph that is not connected can be decomposed into two or more connected subgraphs, each pair of which has no node in common. That is, a disconnected graph is the union of two or more disjoint subgraphs.

Random walk on a graph is used as a search technique in which a search proceeds from a start node by randomly selecting one of its neighbors, say k. At k the search randomly selects one of its neighbors, making an effort not to reselect the node from where it reached k, and so on. If the goal is to reach a particular destination node, the search terminates when this destination is reached.

Another way to describe a graph is in terms of the *adjacency matrix* $A(x, y)$, which has a value 1 in its (x, y) cell if nodes x and y are neighbors and zero otherwise, for all $x, y \in V$. Then the degree of vertex x is given by

$$d(x) = \sum_y A(x, y)$$

The *hitting time* from node v_i to node v_j is denoted by $H(v_i, v_j)$ and defined as the expected number of steps required to reach v_j for the first time from v_i. Similarly, the *cover time* $C(v_i)$ from node v_i is the expected number of steps required to visit all the nodes starting from v_i. The cover time for a graph is the maximum $C(v_i)$ over all nodes v_i and denoted by $C(G)$. The *commute time* $C(v_i, v_j)$ between node v_i and node v_j is the expected number of steps that it takes to go from v_i to v_j and back to v_i. That is,

$$C(v_i, v_j) = H(v_i, v_j) + H(v_j, v_i)$$

A bound for $C(G)$ was obtained by Kahn (1989) as $C(G) \leq 4n^2 d_{\text{ave}}/d_{\min}$, where n is the number of nodes in the graph, d_{ave} is the average degree of the graph, and

$d_{\min}$ is the minimum degree of the graph. In Bollobas (1998) it is shown that in a connected graph with m edges the mean *return time* to a vertex v, which is denoted by $H(v, v)$, is given by

$$H(v, v) = \frac{2m}{d(v)}$$

Graphs are often used to model relationships. When there is a special association in these relationships, the *undirected graphs* we have described so far do not convey this information; a *directed graph* is required. A directed graph (or *digraph*) is a graph in which an edge consists of an ordered vertex pair, giving it a direction from one vertex to the other. Generally in a digraph the edge (a, b) has a direction from vertex a to vertex b, which is indicated by an arrow in the direction from a to b. Figure 8.8 illustrates a simple digraph. When the directions are ignored, we obtain the underlying undirected graph shown in Figure 8.7(a).

Let $G = (V, E)$ be a connected undirected graph with n vertices and m edges. A random walk on G can be described as follows. We start at vertex v_0 and arrive at vertex v_i in the kth step. We move to vertex v_l, which is one of the neighbors of vertex v_i, with probability $1/d(v_i)$. The sequence of random vertices $\{v_t, t = 0, 1, \ldots\}$ is a Markov chain with transition probabilities p_{ij} given by

$$p_{ij} = \begin{cases} 1/d(i) & i, j \in V \\ 0 & \text{otherwise} \end{cases}$$

Let $P = [p_{ij}]_{i,j \in V}$ be the state transition probability matrix. Then for the simple graph in Figure 8.7(a), we have that

$$P = \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \end{bmatrix}$$

which corresponds to the state-transition diagram shown in Figure 8.9.

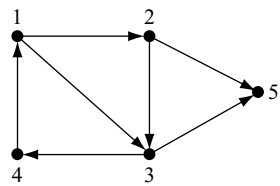


Figure 8.8. Example of a digraph.

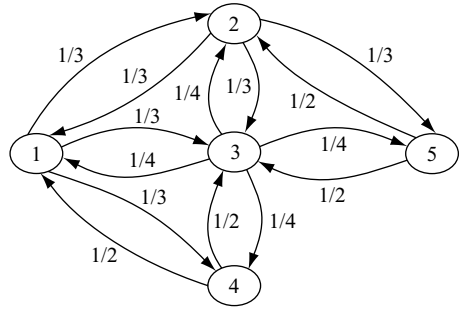


Figure 8.9. State-transition diagram of graph in Figure 8.7(a).

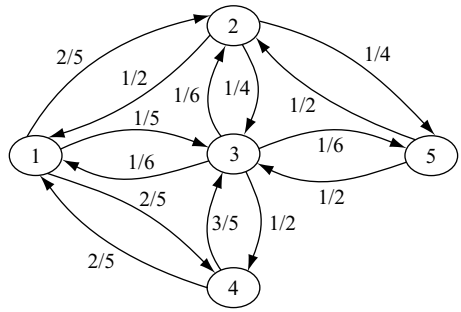


Figure 8.10. State-transition diagram of multigraph in Figure 8.7(b).

We can construct the Markov chain of the multigraph in a similar manner. In this case,

$$p_{ij} = \begin{cases} n_{ij}/d(i) & i, j \in V \\ 0 & \text{otherwise} \end{cases}$$

where n_{ij} is the number of edges between nodes i and j. For example, in the multigraph of Figure 8.7(b), we have that

$$P = \begin{bmatrix} 0 & 2/5 & 1/5 & 2/5 & 0 \\ 1/2 & 0 & 1/4 & 0 & 1/4 \\ 1/6 & 1/6 & 0 & 1/2 & 1/6 \\ 2/5 & 0 & 3/5 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \end{bmatrix}$$

The Markov chain of the multigraph is shown in Figure 8.10.

The Markov chain associated with a random walk on a graph is irreducible if and only if the graph is connected. Let m denote the number of edges in an undirected

connected graph $G = (V, E)$, and let $\{\pi_k\}$, $k \in V$, be the stationary distribution of the Markov chain associated with the graph. The stationary distribution of the Markov chain associated with $G = (V, E)$ is given by the following theorem:

Theorem 8.3 The stationary distribution of the Markov chain associated with the connected graph $G = (V, E)$ is given by $\pi_i = d(i)/2m$, $i = 1, \ldots, n$.

Proof The proof consists in our showing that the distribution $\pi = (\pi_1, \ldots, \pi_n)$ satisfies the equation $\pi P = \pi$. We prove the theorem with a multigraph, which is more general than the simple graph. Thus, we have that with respect to node j,

$$(\pi P)_j = \sum_i \pi_i p_{ij} = \sum_i \left\{ \frac{d(i)}{2m} \times \frac{n_{ij}}{d(i)} \right\} = \frac{1}{2m} \sum_i n_{ij} = \frac{d(j)}{2m} = \pi_j$$

This implies that by definition π is the stationary distribution of the unique Markov chain defined by P. ■

Note that for the simple graph we have that $n_{ij} = 1$, and the same result holds.

Example 8.3 Consider the simple graph of Figure 8.7(a). We have that $m = 7$, which means that the stationary distribution is given by

$$\pi = \left(\frac{3}{14}, \frac{3}{14}, \frac{4}{14}, \frac{2}{14}, \frac{2}{14} \right)$$

Similarly, for the multigraph of Figure 8.7(b), the number of edges is $m = 11$. Thus, the stationary distribution of the Markov chain in Figure 8.7(b) is given by

$$\pi = \left(\frac{5}{22}, \frac{4}{22}, \frac{6}{22}, \frac{5}{22}, \frac{2}{22} \right)$$

Note that a loop is considered to contribute twice to the degree of a node. Thus, in the case of the graph with loop shown in Figure 8.7(c), $m = 8$, and because $d(5) = 4$ we obtain the stationary distribution as follows:

$$\pi = \left(\frac{3}{16}, \frac{3}{16}, \frac{4}{16}, \frac{2}{16}, \frac{4}{16} \right)$$

Recall that the mean return time to a node v in a connected graph with m edges is given $H(v, v) = 2m/d(v)$. From the results on the stationary distributions we may then write

$$H(v, v) = 1/\pi_v$$ ■

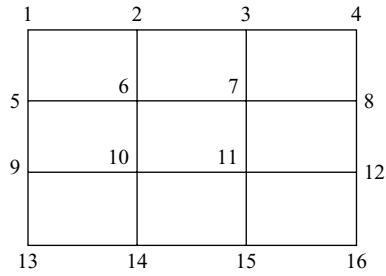


Figure 8.11. 4×4 checkerboard.

Example 8.4 Consider a random walk on a two-dimensional lattice consisting of the 4×4 checkerboard shown in Figure 8.11.

The number of edges is $m = 24$, and the degrees of the nodes are as follows:

$$d(1) = d(4) = d(13) = d(16) = 2$$

$$d(2) = d(3) = d(5) = d(8) = d(9) = d(12) = d(14) = d(15) = 3$$

$$d(6) = d(7) = d(10) = d(11) = 4$$

Thus, the stationary probabilities are

$$\pi_1 = \pi_4 = \pi_{13} = \pi_{16} = 1/24$$

$$\pi_2 = \pi_3 = \pi_5 = \pi_8 = \pi_9 = \pi_{12} = \pi_{14} = \pi_{15} = 1/16$$

$$\pi_6 = \pi_7 = \pi_{10} = \pi_{11} = 1/12$$ ■

8.10.1 Random Walk on a Weighted Graph

A more general random walk is that performed on a weighted graph. Specifically, we consider a connected graph with n nodes labeled $\{1, 2, \ldots, n\}$ with weight $w_{ij} \geq 0$ on the edge (i, j). If edge (i, j) does not exist, we set $w_{ij} = 0$. We assume that the graph is undirected so that $w_{ij} = w_{ji}$.

Assume that a particle walks from node to node in the graph in the following manner. Given that the particle is currently at node i, the next node j is chosen from among the neighbors of i with probability

$$p_{ij} = \frac{w_{ij}}{\sum_k w_{ik}}$$

Thus, we see that p_{ij} is proportional to the weight of the edge (i, j). Let the total weight of the edges emanating from node i be w_i, which is given by

$$w_i = \sum_j w_{ij}$$

Then the sum of the weights of all edges is

$$w = \sum_{i,j:j>1} w_{ij}$$

where the inequality in the summation is used to avoid double counting. It is easy to show that the stationary distribution is given by

$$\pi_i = \frac{w_i}{2w}$$

The proof consists as usual in verifying that the preceding distribution satisfies the relationship $\pi P = \pi$, which can be seen as follows:

$$\begin{aligned}\sum_i \pi_i P_{ij} &= \sum_i \left\{ \frac{w_i}{2w} \times \frac{w_{ij}}{w_i} \right\} = \sum_i \left\{ \frac{1}{2w} \times w_{ij} \right\} = \frac{1}{2w} \sum_i w_{ij} \\ &= \frac{w_j}{2w} = \pi_j\end{aligned}$$

Thus, the stationary probability of state i is proportional to the weight of the edges emanating from node i.

8.11 Markov Random Walk

A Markov random walk can be viewed as an extension of the random walk on a weighted graph such that the transition probability between two nodes is a measure of the similarity between the nodes. Let $G = (V, E, W)$ be a weighted graph such that $V = \{v_1, v_2, \ldots, v_n\}$ is a set of nodes, $E = \{(i, j)|v_i, v_j \in V\}$ is the set of edges, and $W = \{w_{ij}|v_i, v_j \in V\}$ is the set of weights, where w_{ij} is the weight between nodes v_i and v_j. w_{ij} defines the strength of the immediate connection between nodes v_i and v_j, which is sometimes called the *immediate affinity* between nodes v_i and v_j. A common choice of w_{ij} is the following:

$$w_{ij} = \exp\{-\beta d(i, j)^2\}$$

where β is a positive constant and $d(i, j)$ is the Euclidean distance between nodes v_i and v_j and is given by

$$d(i, j) = ||v_i - v_j||$$

The weights are symmetric in the sense that $w_{ij} = w_{ji}$. A Markov random walk on $G = (V, E, W)$ is characterized by a transition probability p_{ij} from node v_i to node v_j that is given by

$$p_{ij} = \frac{w_{ij}}{\sum_{j \neq i} w_{ij}}$$

where we set $p_{ij} = 0$ whenever $(i, j) \notin E$. While the weights are symmetric, the transition probabilities are not necessarily symmetric because the normalization factor $\sum_{j \neq i} w_{ij}$ varies from node to node.

Let P denote the state-transition matrix; that is, $P = [p_{ij}]$, where p_{ij} is the element in the ith row and jth column. Let $A = [a_{ij}] = P^n$; then a_{ij} is the probability of transitioning from node v_i to node v_j in n steps. If $a_{ij} > 0$ then node v_i is defined to be an n-step neighbor of node v_j. All the n-step neighbors of node v_i define the n-step neighborhood of v_i, which is denoted by $\Gamma_n(v_i)$ and usually expressed as follows:

$$\Gamma_n(v_i) = \{(v_j, a_{ij}) | a_{ij} > 0, v_j \in V\}$$

By increasing n we propagate the local influence of each node on its neighbors. Note that from our Markov chain notation, $a_{ij} = p_{ij}(n)$. If the graph is connected (that is, there is a path from any point to any other point in the graph), then

$$\lim_{n \to \infty} p_{ij}(n) = \phi_0(j)$$

where $\phi_0(k)$ is the unique stationary distribution

$$\phi_0(k) = \frac{d(v_k)}{\sum_{v_j \in V} d(v_j)}$$

As defined earlier, $d(v_k)$ is the degree of node v_k. Thus, $\phi_0(k)$ is proportional to the degree of node v_k. Also, the Markov chain is reversible in the sense that

$$\phi_0(i)\, p_{ij} = \phi_0(j)\, p_{ji}$$

8.11.1 Markov Random Walk in Semisupervised Machine Learning

A Markov random walk is popularly used in data classification and clustering in machine learning, which deals with the design and development of algorithms and techniques that allow computers to "learn" in order to be able to extract information automatically from a given data set. Many applications in multimedia retrieval, such as visual information retrieval, object recognition, and human activity modeling, require the labeling or annotating of data. Unfortunately, manually labeling data is not only a labor intensive and time-consuming task but also subject to human errors. Different machine learning schemes have been developed for data labeling. These include *unsupervised* learning, *supervised* learning, and *semisupervised* learning.

In supervised learning, the learner is trained under complete supervision and is initially provided with only labeled data, and the goal is to learn a classification rule that can be used to classify unlabeled data. In unsupervised learning, the learner's training is not supervised, and it is given only unlabeled data from which it is required to discover the structures, regularities, and categories in the data. Semisupervised learning combines supervised and unsupervised learning. It makes use of both labeled and unlabeled data for training, where the amount of labeled data is usually small while the amount of unlabeled data is usually large. A popular belief in the machine-learning research community is that unlabeled data, when used in conjunction with a small amount of labeled data, can produce considerable improvement in learning accuracy.

The Markov random walk model of semisupervised learning can be defined as follows. Given a partially labeled data set $\{(x_1, y_1), (x_2, y_2), \ldots, (x_l, y_l), x_{l+1}, x_{l+2}, \ldots, x_{l+n}\}$, where the labeled data subset is $X_L = \{x_1, x_2, \ldots, x_l\}$ with the corresponding set of labels $Y_L = \{y_1, y_2, \ldots, y_l\}$, and the unlabeled data subset is $X_U = \{x_{l+1}, x_{l+2}, \ldots, x_{l+n}\}$, find the label set $Y_U = \{y_{l+1}, y_{l+2}, \ldots, y_{l+n}\}$ for the unlabeled data subset. The Markov random walk uses the data points x_k as the nodes of the graph. Then using the labels $y_k, 1 \le k \le l$, it attempts to estimate the missing labels $y_k, l + 1 \le k \le l + n$. Note that the one-step transition probabilities must first be determined from the weights associated with the nodes by

$$p_{ik} = \frac{w_{ik}}{\sum_j w_{ij}}$$

where $p_{ik} = 0$ for any nonneighbor k. The steps used to estimate the missing labels are developed in Szummer and Jaakkola (2002). It is assumed that each node (or a data point) x has a label or a conditional probability distribution $P_{Y|X}(y|x)$ over

the class labels. These distributions are unknown and need to be estimated as follows:

1. Select an unlabeled node uniformly at random from the graph.
2. Sample a label y according to $P_{Y|X}(y|x)$.
3. Perform n transitions in the Markov chain, following edges according to the transition probabilities. Assume that we end up at node k at the end of the n transitions.
4. Emit the sampled y as if it were generated by node k.

Using this algorithm we can estimate $P_{Y|X}(y|x)$ at every node such that the labels emitted by the label points according to the preceding procedure match their observed labels as closely as possible.

8.12 Random Walk with Correlation

In the classical random walk described earlier, there is a constant probability p of incrementing the current state by 1 and, therefore, a constant probability $q = 1 - p$ of decrementing the current state by 1. This is due to the fact that the outcomes of the experiments are independent. A more general case is when there is a correlation between results of successive trials; that is, the outcome of the current trial depends on the outcome of the previous trial. Such random walks are called *correlated random walks* (CRWs) or *persistent random walks* and have been studied by many authors including Goldstein (1951), Gillis (1955), Mohan (1955), Seth (1963), Renshaw and Henderson (1981), Kehr and Argyrakis (1986), Lal and Bhat (1989), Argyrakis and Kehr (1992), Hanneken and Franceschetti (1998), and Bohm (2000).

To illustrate this process, we consider an experiment with two possible outcomes: a win or a loss. The probability of the first outcome is assumed to be fixed, and thereafter each subsequent outcome is governed by the following rule. Given that the current trial results in a win, the probability that the next trial will result in a win is p_1, and the probability that it will result in a loss is $1 - p_1 = q_1$. Similarly, given the current trial results in a loss, the probability that the next trial will result in a loss is p_0, and the probability that it will result in a win is $1 - p_0 = q_0$. Thus, the conditional probabilities are as follows:

$$P[\text{win}|\text{win}] = p_1$$

$$P[\text{loss}|\text{win}] = 1 - p_1 = q_1$$

$$P[\text{loss}|\text{loss}] = p_0$$

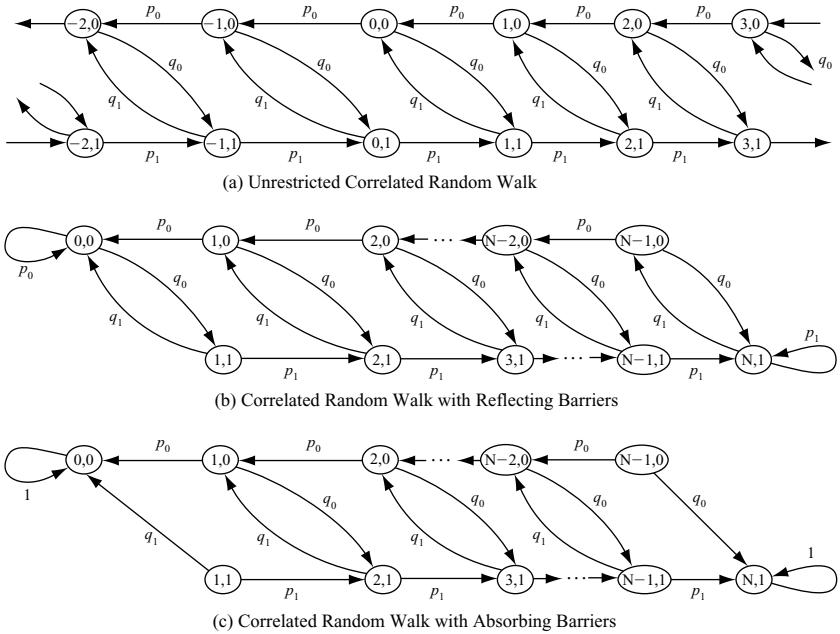
$$P[\text{win}|\text{loss}] = 1 - p_0 = q_0$$

The correlated random walk can be modeled as a bivariate Markov chain with the location of the walker and the result of the previous walk as the two variables. Specifically, we represent a state by (k, l), where k is the location of the walker, where a move to the right indicates a success and a move to the left indicates a failure; and l is the result index, which is defined by

$$l = \begin{cases} 1 & \text{if the previous trial resulted in a success} \\ 0 & \text{if the previous trial resulted in a failure} \end{cases}$$

The range of k depends on the type of correlated random walk. When the walk is unrestricted, then $k = \ldots, -2, -1, 0, 1, 2, \ldots$. Similarly, when barriers exist, we have that $k = 0, 1, \ldots, N$, where N can be infinite. Thus, the state-transition diagrams in Figure 8.12 show the cases of unrestricted correlated random walk, correlated random walk with reflecting barriers, and correlated random walk with absorbing barriers.

As can be seen from Figure 8.12, the correlated random walk is a quasi-birth-and-death process. In fact, if we order the states lexicographically, then for the case



(a) Unrestricted Correlated Random Walk

(b) Correlated Random Walk with Reflecting Barriers

(c) Correlated Random Walk with Absorbing Barriers

Figure 8.12. State-transition diagrams for correlated random walks.

of the correlated random walk with reflecting barriers the state-transition matrix is given by

$$P = \begin{bmatrix}
p_0 & 0 & q_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_0 & 0 & 0 & 0 & q_0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q_1 & 0 & 0 & 0 & p_1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & p_0 & 0 & 0 & 0 & 0 & q_0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & q_1 & 0 & 0 & 0 & 0 & p_1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & p_0 & 0 & 0 & 0 & 0 & q_0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q_1 & 0 & 0 & 0 & 0 & p_1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & p_0 & 0 & 0 & 0 & 0 & q_0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & q_1 & 0 & 0 & 0 & 0 & p_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & p_0 & 0 & 0 & 0 & q_0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & p_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & q_1 & 0 & p_1
\end{bmatrix}$$

$$= \begin{bmatrix}
D_0 & D_1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\
D_2 & 0 & A_0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A_1 & 0 & A_0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A_1 & 0 & A_0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & A_1 & 0 & A_0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & A_1 & 0 & B_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & B_1 & B_0
\end{bmatrix}$$

where

$$D_0 = \begin{bmatrix} p_0 & 0 & q_0 \\ p_0 & 0 & 0 \\ q_1 & 0 & 0 \end{bmatrix} \qquad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & q_0 & 0 \\ 0 & p_1 & 0 \end{bmatrix} \qquad D_2 = \begin{bmatrix} 0 & p_0 & 0 \\ 0 & q_1 & 0 \end{bmatrix}$$

$$A_0 = \begin{bmatrix} 0 & q_0 \\ 0 & p_1 \end{bmatrix} \qquad A_1 = \begin{bmatrix} p_0 & 0 \\ q_1 & 0 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 0 & 0 & q_0 \\ 0 & 0 & p_1 \\ q_1 & 0 & p_1 \end{bmatrix} \qquad B_1 = \begin{bmatrix} p_0 & 0 \\ q_1 & 0 \\ 0 & 0 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 0 & p_0 & 0 \\ 0 & q_1 & 0 \end{bmatrix} = D_2$$

Let $\pi_{k,0}(n)$ denote the probability of being in state $(k, 0)$ after n trials, and let $\pi_{k,1}(n)$ denote the probability of being in state $(k, 1)$ after n trials. Then we have

the following difference equations:

$$\pi_{k,0}(n+1) = p_0\pi_{k+1,0}(n) + q_1\pi_{k+1,1}(n)$$
$$\pi_{k,1}(n+1) = q_0\pi_{k-1,0}(n) + p_1\pi_{k-1,1}(n)$$

Let the z-transform of $\pi_{k,l}(n)$ be defined by

$$\Pi_l(z,n) = \sum_{k=-\infty}^{\infty} z^k \pi_{k,l}(n) \quad l = 0, 1; n = 1, 2, \ldots$$

Then applying the transform operation to the difference equations we obtain

$$\Pi_0(z, n+1) = \frac{p_0}{z}\Pi_0(z,\ n) + \frac{q_1}{z}\Pi_1(z,\ n)$$

$$\Pi_1(z, n+1) = zq_0\Pi_0(z,\ n) + zp_1\Pi_1(z,\ n)$$

which can be arranged in the following form:

$$\begin{bmatrix} \Pi_0(z, n+1) \\ \Pi_1(z, n+1) \end{bmatrix} = \begin{bmatrix} z^{-1}p_0 & z^{-1}q_1 \\ zq_0 & zp_1 \end{bmatrix} \begin{bmatrix} \Pi_0(z, n) \\ \Pi_1(z, n) \end{bmatrix}$$

The initial states $\pi_{k,0}(0)$ and $\pi_{k,l}(0)$ can be obtained by noting that the sequence of result indices $\{l\}$ of the state (k, l) constitutes a two-state Markov chain as shown in Figure 8.13.

Thus, if we let $\pi_0(0)$ denote the probability that the process initially started with the first trial being a failure and $\pi_1(0)$ the probability that the first trial was a success, then we have that

$$\pi_0(0) = \frac{q_1}{q_0 + q_1}$$

$$\pi_1(0) = \frac{q_0}{q_0 + q_1}$$

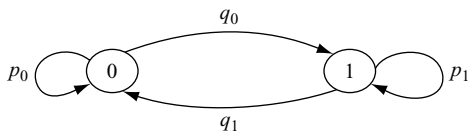


Figure 8.13. State-transition diagrams for result sequence.

The exact solution of the transform equation of the correlated walk after n trials is of the form

$$\Pi(z,n)=\begin{bmatrix}\Pi_0(z,n)\\ \Pi_1(z,n)\end{bmatrix}=\begin{bmatrix}z^{-1}p_0 & z^{-1}q_1\\ zq_0 & zp_1\end{bmatrix}^n\begin{bmatrix}\pi_0(0)\\ \pi_1(0)\end{bmatrix}$$

Let A be the coefficient matrix; that is,

$$A=\begin{bmatrix}z^{-1}p_0 & z^{-1}q_1\\ zq_0 & zp_1\end{bmatrix}$$

Then we have that

$$\Pi(z,n)=A^n\pi(0)$$

We solve the equation via the diagonalization method, which requires our obtaining the eigenvalues of A. The eigenvalues λ of A satisfy the equation

$$|\lambda I-A|=0$$

which gives

$$\lambda^2-(zp_1+z^{-1}p_0)\lambda+p_0p_1-q_0q_1=0$$

from which we obtain

$$\lambda=\frac{(zp_1+z^{-1}p_0)\pm\sqrt{\{zp_1-z^{-1}p_0\}^2+4q_0q_1}}{2}$$

Thus, the eigenvalues are

$$\lambda_1=\frac{(zp_1+z^{-1}p_0)+\sqrt{\{zp_1-z^{-1}p_0\}^2+4q_0q_1}}{2}$$

$$\lambda_2=\frac{(zp_1+z^{-1}p_0)-\sqrt{\{zp_1-z^{-1}p_0\}^2+4q_0q_1}}{2}$$

Let the matrix X be defined by $X=(X_1,X_2)$, where $X_i=(x_{1i},x_{2i})^T$, $i=1,2$. The eigenvectors belonging to λ_1 are obtained from

$$(\lambda_1 I-A)X_1=0$$

That is,

$$(\lambda_1-z^{-1}p_0)x_{11}-z^{-1}q_1x_{21}=0$$

$$-zq_0x_{11}+(\lambda_1-zp_1)x_{21}=0$$

If we set $x_{11}=1$ in the first equation, we obtain $x_{21}=(\lambda_1-z^{-1}p_0)/z^{-1}q_1=(\lambda_1 z-p_0)/q_1$. Note that $(\lambda_1-z^{-1}p_0)/z^{-1}q_1=q_0/(\lambda_1-zp_1)$ because this

relationship yields the equation

$$\lambda_1^2 - (zp_1 + z^{-1}p_0)\lambda_1 + p_0p_1 - q_0q_1 = 0$$

that we know to be true. Thus,

$$X_1 = (x_{11}, x_{21})^T = [1\ (\lambda_1 z - p_0)/q_1]^T$$

is an eigenvector. Similarly, the eigenvectors belonging to λ_2 are obtained from

$$(\lambda_2 I - A)X_2 = 0$$

which is similar to the equation for the eigenvectors belonging to λ_1. Thus, another eigenvector is

$$X_2 = (x_{12}, x_{22})^T = [1\ (\lambda_2 z - p_0)/q_1]^T$$

Then using the diagonalization method we obtain

$$\Pi(z, n) = A^n\pi(0) = S\Lambda^n S^{-1}\pi(0)$$

where $A = S\Lambda S^{-1}$, S is the matrix whose columns are the eigenvectors, and Λ is the diagonal matrix of the corresponding eigenvalues. That is,

$$S = \begin{bmatrix} 1 & 1 \\ (\lambda_1 z - p_0)/q_1 & (\lambda_2 z - p_0)/q_1 \end{bmatrix} \Rightarrow$$

$$S^{-1} = \frac{q_1}{z(\lambda_2 - \lambda_1)} \begin{bmatrix} (\lambda_2 z - p_0)/q_1 & -1 \\ -(\lambda_1 z - p_0)/q_1 & 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \Rightarrow \Lambda^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}$$

From this we can obtain the $\pi_{k,l}(n)$, which are the inverse z-transforms of the components of $\Pi(z, n)$.

Correlated random walk is popularly used in ecology to model animal and insect movement. For example, it is used in Kareiva and Shigesada (1983) to analyze insect movement. Similarly, it has been used in Byers (2001) to model animal dispersal and in Jonsen (2005) to model animal movement. CRW has also been used to model mobility in mobile ad hoc networks in Bandyopadhyay (2006).

8.13 Continuous-Time Random Walk

In the random walk models described earlier, a walker takes steps in a periodic manner, such as every second or minute or hour, or any other equal time interval. More importantly, a classical random walk is a Bernoulli process that allows only two possible events that have values of ± 1. A more general case is when the time between steps is a random variable and the step size is a random variable. In this case we obtain a continuous-time random walk (CTRW), which was introduced by Montroll and Weiss (1965). Continuous-time random walk has been applied in many physical phenomena. It is used in physics to model diffusion with instantaneous jumps from one position to the next. For example, it is used in Scher and Montroll (1975) to model the transport of amorphous materials, in Montroll and Shlesinger (1984) to model the transport in disordered media, and in Weiss (1998) to model the transport in turbid media. In Helmstetter and Sornette (2002) it is used to model earthquakes. Recently it has been used in econophysics to model the financial market, particularly to describe the movement of log-prices. Examples of the application of CTRW in econophysics are discussed in Masoliver and Montero (2003), Masoliver (2006), and Scalas (2006a, 2006b).

CTRW is essentially a point process with reward that permits intervals between successive walks to be independent and identically distributed. Specifically, assume that the walker starts at the point zero at time $T_0 = 0$ and waits until time T_1 when it makes a jump of size θ_1, which is not necessarily positive. The walker then waits until time T_2 when it makes another jump of size θ_2, and so on. The jump sizes θ_i are also assumed to be independent and identically distributed. Thus, we assume that the times $T_1, T_2, \ldots$ are the instants when the walker makes jumps. The intervals $\tau_i = T_i - T_{i-1}$, $i = 1, 2, \ldots$, are called the *waiting times* (or *pausing times*) and are assumed to be independent and identically distributed. Thus, one major difference between the classical or discrete-time random walk (DTRW) and the CTRW is that in CTRW the waiting time between jumps is not constant as in the case of DTRW, and the step size is a random variable unlike the DTRW where the step size is ± 1. The time at which the nth walk occurs, T_n, is given by

$$T_n = T_0 + \sum_{i=1}^{n} \tau_i \qquad n = 1, 2, \ldots; t_0 = 0$$

If we define the jump sizes as the rewards, then the cumulative reward at time t (alternatively, the position of the walker at time t) is given by

$$X(t) = \sum_{i=1}^{N(t)} \theta_i$$

where the upper limit $N(t)$ is a random function of time that denotes the number of jumps up to time t and is given by

$$N(t) = \max\{n : T_n \leq T\}$$

The CTRW is not Markovian except in the special case where the waiting time is exponentially distributed. A sample realization of $X(t)$ is illustrated in Figure 8.14.

Let T denote the waiting time and let Θ denote the jump size. Similarly, let $f_{X(t)}(x,t)$ denote the PDF of $X(t)$, $f_\Theta(\theta)$ the PDF of Θ, $f_T(\tau)$ the PDF of T, and $p_{N(t)}(n,t)$ the PMF of $N(t)$. Also, let $\Phi_{X(t)}(w)$ denote the characteristic function of $X(t)$, let $\Phi_\Theta(w)$ denote the characteristic function of Θ, and let $G_{N(t)}(z)$ denote the z-transform of $N(t)$, where

$$\Phi_{X(t)}(w) = E[e^{jwX(t)}] = \int_{-\infty}^{\infty} e^{jwx} f_{X(t)}(x,t)dx$$

$$\Phi_\Theta(w) = \int_{-\infty}^{\infty} e^{jw\theta} f_\Theta(\theta)d\theta$$

$$G_{N(t)}(z) = E[z^{N(t)}] = \sum_{n=0}^{\infty} z^n p_{N(t)}(n,t)$$

Then we know from probability theory (see Ibe (2005), for example) that

$$\Phi_{X(t)}(w) = G_{N(t)}(\Phi_\Theta(w))$$

Thus, if $p_{N(t)}(n,t)$ and $f_\Theta(\theta)$ are known, we can determine $\Phi_{X(t)}(w)$ and consequently $f_{X(t)}(x,t)$. Let $f_{\Theta T}(\theta,\tau)$ be the joint PDF of the jump size and waiting time. For the so-called *uncoupled continuous-time random walk*, it is assumed that Θ and T are independent, while in the *coupled CTRW* they are not independent.

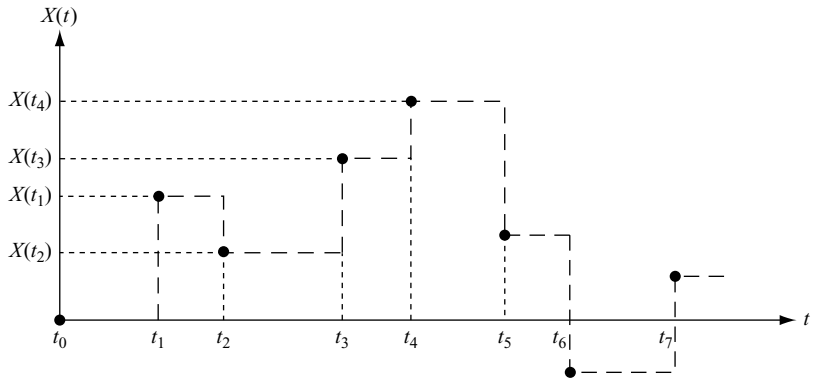


Figure 8.14. Sample realization of $X(t)$.

The expected value of $X(t)$ is given by

$$E[X(t)] = E[N(t)]E[\Theta]$$

The expected value of $N(t)$ can be obtained as follows. Because the intervals τ_i are independent and identically distributed, we note that the process $\{N(t)|t \geq 0\}$ is a renewal process and T_n is the time of the nth renewal. Using the results developed in Chapter 6, we have that

$$\begin{aligned} p_{N(t)}(n,t) &= P[N(t) = n] = P[N(t) < n+1] - P[N(t) < n] \\ &= 1 - F_{T_{n+1}}(t) - \{1 - F_{T_n}(t)\} \\ &= F_{T_n}(t) - F_{T_{n+1}}(t) \end{aligned}$$

and the expected value of $N(t)$ is given by

$$\begin{aligned} E[N(t)] &= \sum_{n=0}^{\infty} n p_{N(t)}(n,t) = \sum_{n=0}^{\infty} n\{F_{T_n}(t) - F_{T_{n+1}}(t)\} \\ &= \{F_{T_1}(t) + 2F_{T_2}(t) + 3F_{T_3}(t) + \cdots\} - \{F_{T_2}(t) + 2F_{T_3}(t) + \cdots\} \\ &= F_{T_1}(t) + F_{T_2}(t) + F_{T_3}(t) + \cdots \\ &= \sum_{n=1}^{\infty} F_{T_n}(t) \end{aligned}$$

Note that the dynamics of the continuous-time random walk are similar to those of the Markov jump process discussed in Chapter 6. One major difference between the two processes is that there is an imbedded Markov process in the Markov jump process. Also, the waiting times τ_k are identically distributed with a general distribution in the CTRW, but in the Markov jump process they are exponentially distributed with different means that depend on the state of the process. Another difference is that because the imbedded Markov process in the Markov jump process is a discrete-state process (i.e., it is a Markov chain), the jump size Θ is required to be a discrete random variable in the Markov jump process while it can be a continuous random variable in the case of the CTRW, as we have assumed here.

8.13.1 The Master Equation

Another method of analyzing the CTRW is as follows. Let $P(x,t)$ denote the probability that the position of the walker at time t is x, given that it was in position 0 at time $t = 0$; that is,

$$P(x,t) = P[X(t) = x|X(0) = 0]$$

For an uncoupled CTRW, $f_{\Theta T}(\theta, \tau) = f_{\Theta}(\theta) f_T(\tau)$; otherwise, we have that

$$f_{\Theta}(\theta) = \int_0^{\infty} f_{\Theta T}(\theta, \tau) d\tau$$

$$f_T(\tau) = \int_{-\infty}^{\infty} f_{\Theta T}(\theta, \tau) d\theta$$

The relationship between $P(x, t)$ and $f_{\Theta T}(\theta, \tau)$ is given by the following equation that is generally called the *master equation* of the CTRW:

$$P(x, t) = \delta(x) R(t) + \int_0^t \int_{-\infty}^{\infty} P(u, \tau) f_{\Theta T}(x - u, t - \tau) du d\tau$$

where $\delta(x)$ is the Dirac delta function and $R(t) = P[T > t] = 1 - F_T(t)$ is called the *survival probability*, which is the probability that the waiting time when the process is in a given state is greater than t. The equation states that the probability that $X(t) = x$ is equal to the probability that the process was in state 0 up to time t, plus the probability that the process was at some state u at time τ, where $0 < \tau \leq t$, and within the waiting time $t - \tau$, a jump of size $x - u$ took place. Note that

$$R(t) = \int_t^{\infty} f_T(v) dv = 1 - \int_0^t f_T(v) dv$$

$$f_T(t) = \frac{dR(t)}{dt}$$

For the uncoupled CTRW, the master equation becomes

$$P(x, t) = \delta(x) R(t) + \int_0^t \int_{-\infty}^{\infty} P(u, \tau) f_{\Theta}(x - u) f_T(t - \tau) du d\tau$$

Let the joint Laplace-Fourier transform of $P(x, t)$ be defined as follows:

$$P^*(w, s) = \int_0^{\infty} e^{-st} dt \int_{-\infty}^{\infty} e^{jwx} P(x, t) dx$$

Then the master equation is transformed into

$$P^*(w, s) = \int_0^{\infty} e^{-st} \int_{-\infty}^{\infty} e^{jwx} \left\{ \delta(x) R(t) + \int_0^t \int_{-\infty}^{\infty} P(u, \tau) f_{\Theta T}(x - u, t - \tau) du d\tau \right\} dx dt$$

$$= R(s) + P^*(w, s)\Phi_{\Theta T}(w, s)$$
$$= \frac{R(s)}{1 - \Phi_{\Theta T}(w, s)}$$

where $\Phi_{\Theta T}(w, s)$ is the joint characteristic function-Laplace transform of $f_{\Theta T}(\theta, \tau)$. For an uncoupled CTRW, $\Phi_{\Theta T}(w, s) = \Phi_\Theta(w)\Phi_T(s)$. Also,

$$R(s) = \frac{1 - \Phi_T(s)}{s}$$

Thus, for the uncoupled CTRW we have that

$$P^*(w, s) = \frac{1 - \Phi_T(s)}{s[1 - \Phi_\Theta(w)\Phi_T(s)]}$$

and for the coupled CTRW we have that

$$P^*(w, s) = \frac{1 - \Phi_T(s)}{s[1 - \Phi_{\Theta T}(w, s)]}$$

For the special case where T is exponentially distributed with a mean of $1/\lambda$, we have that $\Phi_T(s) = \lambda/(s + \lambda)$. Thus, for the uncoupled CTRW we obtain

$$P^*(w, s) = \frac{1 - \lambda/(s + \lambda)}{s[1 - \lambda\Phi_\Theta(w)/(s + \lambda)]} = \frac{1}{s + \lambda - \lambda\Phi_\Theta(w)}$$

Taking the inverse Laplace transform for this special case we obtain

$$P(w, t) = \exp\{-\lambda t[1 - \Phi_\Theta(w)]\}$$

Note that for the special case when $N(t)$ is a Poisson process, the CTRW becomes a compound Poisson process and, as shown in Chapter 1, the analysis is simplified as the characteristic function of $X(t)$ becomes

$$\Phi_{X(t)}(w) = e^{-\lambda t\{1 - \Phi_\Theta(w)\}} = P(w, t)$$

which is not surprising because the fact that $N(t)$ is a Poisson process implies that T is exponentially distributed. More detailed discussion on CTRW can be found in Weiss (1994). Note that in general the inverse Laplace transform of $P^*(w, s)$ for the uncoupled system is given by

$$P(w, t) = \Phi_{X(t)}(w) = G_{N(t)}(\Phi_\Theta(w))$$

8.14 Sample Applications of Random Walk

We consider some applications of random walk, which include the ballot problem, Web search, mobility in mobile networks, insurance risk, dam content, and cash management.

8.14.1 The Ballot Problem

The ballot problem can be defined as follows. Consider an election involving only two candidates: A and B. Assume that candidate A receives m votes and candidate B receives n votes, where $m > n$. What is the probability that candidate A is always ahead of candidate B in vote counts?

Let the probability that A always leads B in vote counts be $P_{m,n}$. Then we show that

$$P_{m,n} = \frac{m-n}{m+n}$$

We consider two methods of solving this problem, which are the conditional probability method and the random walk method.

The Conditional Probability Method

Let the probability that A is always ahead, given that A received the last vote, be denoted by $P[A|AL]$; let the probability that A is always ahead, given that B received the last vote, be $P[A|BL]$. If $P[AL]$ denotes the probability that A received the last vote, which is $m/(m+n)$, and $P[BL]$ denotes the probability that B received the last vote, which is $n/(m+n)$, then we have that

$$\begin{aligned} P_{m,n} &= P[A|AL]P[AL] + P[A|BL]P[BL] \\ &= P[A|AL]\left\{\frac{m}{m+n}\right\} + P[A|BL]\left\{\frac{n}{m+n}\right\} \\ &= P_{m-1,n}\left\{\frac{m}{m+n}\right\} + P_{m,n-1}\left\{\frac{n}{m+n}\right\} \end{aligned}$$

With the initial condition as $P_{1,0} = 1$, we derive the answer by induction. Define $m+n = K$. Then the equation is true for $K = 1$ because $P_{1,0} = (1-0)/(1+0) = 1$. Similarly, it is true for $K = 2$ because $P_{2,0} = P_{1,0}(2)/(2+0) = 1 = (2-0)/(2+0)$. Also, when $K = 3$ the valid configurations are $(m,n) = \{(2,1),(3,0)\}$. Now, because we want $P_{1,1} = 0$, we obtain

$$P_{2,1} = P_{2,0}\left(\frac{1}{3}\right) = \frac{1}{3} = \frac{2-1}{2+1}$$

Similarly,

$$P_{3,0} = P_{2,0}\left(\frac{3}{3}\right) = 1 = \frac{3-0}{3+0}$$

Assume that the solution holds for K. Let m and n be such that $m \geq n$ and $m + n = K + 1$. Then by the induction we have that

$$P_{m,n} = \left\{\frac{m-1-n}{m-1+n}\right\}\left\{\frac{m}{m+n}\right\} + \left\{\frac{m-n+1}{m+n-1}\right\}\left\{\frac{n}{m+n}\right\} = \frac{m-n}{m+n}$$

Thus, the proposition is also true for $m + n = K + 1$, which implies that it is valid for all m and n.

The Random Walk Method

The ballot problem can be viewed as random walk. Let the random variable X_i be equal to 1 if the ith vote was cast for A and -1 if the vote was cast for B, where $P[X_i = 1] = P[X_i = -1] = 1/2$; that is, each person casting a vote is equally likely to vote for either candidate. Then

$$Y_k = X_1 + X_2 + \cdots + X_k$$

is a random walk that represents the difference between A's vote count and B's vote count after k votes have been cast.

The proof of the proposition is based on the reflection principle, which, as stated earlier in the chapter, states as follows. With respect to a random walk of the type previously described that starts from the point (a, p) in the x-y plane and ends at the point (b, q), where $q > p$, as shown in Figure 8.15, the number of possible paths between these two points that do touch or cross the x-axis is equal to the number of all paths from the point $\bar{A} = (a, -p)$ to $B = (b, q)$.

The number of possible paths between these two points is defined by $N_{a,b}(p, q)$ and can be computed by assuming that a path consists of r steps to the right and l steps to the left. From earlier results we know that

$$r = \frac{1}{2}(b - a + p - q)$$

Because $N_{a,b}(p, q)$ can be defined as the number of "successes," r, in $r + l = b - a$ binomial trials, we have that

$$N_{a,b}(p, q) = \binom{b-a}{r}$$

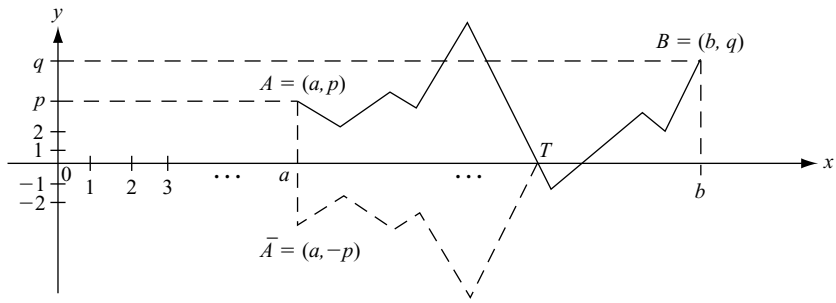


Figure 8.15. Sample path between (a, p) and (b, q).

where r is as previously defined. Let $N^0_{a,b}(p, q)$ be the number of these paths that do not touch or intersect the x-axis and let $N^1_{a,b}(p, q)$ be the number of paths that touch or intersect the x-axis. With respect to Figure 8.15, let the path from $A = (a, p)$ to $B = (b, q)$ touch the x-axis for the first time at the point T, and let $\bar{A}T$ be the reflection of the path AT on the x-axis. Then according to the reflection principle,

$$N^1_{a,b}(p, q) = N_{a,b}(-p, q)$$

Now, for the ballot problem, for m to be always greater than n it means that $X_1 = 1$. Thus, the number of paths that go from $(a, p) = (0, 0)$ to $(b, q) = (m + n, m - n)$, because $m + n$ is the total number of votes cast (or "trials"), is the number of paths that pass through $(1, 1)$ minus the number of paths that touch or cross the x-axis after passing through $(1, 1)$. Thus, using our earlier results we obtain

$$\begin{aligned}
N^0_{0,m+n}(0, m-n) &= N_{1,m+n}(1, m-n) - N^1_{1,m+n}(1, m-n) \\
&= N_{1,m+n}(1, m-n) - N_{1,m+n}(-1, m-n) \\
&= \binom{m+n-1}{m-1} - \binom{m+n-1}{m} \\
&= \frac{(m+n-1)!}{(m-1)!n!} - \frac{(m+n-1)!}{m!(n-1)!} \\
&= \frac{m(m+n-1)!}{m!n!} - \frac{n(m+n-1)!}{m!n!} \\
&= \frac{(m-n)(m+n-1)!}{m!n!}
\end{aligned}$$

where the second equality follows from the reflection principle. If we assume that all orderings of the vote count are equally likely, then $X_1, X_2, \ldots, X_{m+n}$ is equally likely to be any one of the

$$\binom{m+n}{m}$$

sequences of m plus ones and n minus ones. Thus, the probability that A always leads is

$$P_{m,n} = N^0_{0,m+n}(0, m-n) = \frac{(m-n)(m+n-1)!}{m!n!} \times \frac{m!n!}{(m+n)!} = \frac{m-n}{m+n}$$

8.14.2 Web Search

Information retrieval seeks to find all documents that are relevant to a user's query. Prior to the advent of the World Wide Web, information retrieval used only word-based techniques, but with the advent of the Web, new techniques have been developed. One example is the *hyperlink*, which is a reference of a Web page B that is contained in a Web page A. When the hyperlink for page B is clicked on page A, page B is displayed. Because almost every Web user can publish his own Web page, search engines face many difficult tasks. One such task is how to find and index the documents in the Web. Also, because the pages contain heterogeneous information with different formats and styles, a great deal of effort must be expended to create reliable and efficient indices. Another task is to provide high-quality and relevant results to users who merely use simple keyword-based interfaces. This is an important task because when several documents match a user's query, he can be flooded with so much information, some of which might not be very useful to him.

The analysis of hyperlink provides a way to derive the quality of the information on the Web. A great deal of effort has been made by several authors to provide structure to the hyperlink on the Web. Algorithms have been developed to compute reliable measures of authority from the topological structure of interconnections among the Web pages. One such algorithm is the PageRank, which is used by the Google search engine. PageRank uses a topologically-based ranking criterion that computes the authority of a page recursively as a function of the authorities of the pages that link to a target page.

We can view the Web as a graph whose nodes correspond to the Web pages and whose arcs are defined by hyperlinks between the Web pages. Consider a Web surfer who jumps from Web page to Web page in a random manner. The behavior of the surfer on each page depends on the contents of that page. Thus, from our

earlier discussion, such an action is essentially a random walk on a graph. Different models of the surfer's behavior have been analyzed in the literature. Examples of these models can be found in Page (1998), Henzinger (2001), Greco (2001), Diligenti (2004), and Bianchini (2005). Most of these models assign a weight to each hyperlink according to the relevance of the link to the user's query, thereby converting the problem to that of a random walk on a weighted graph.

8.14.3 Mobility Models in Mobile Networks

Several random walk models have been proposed for mobility in mobile networks. In this section we consider the correlated random walk model of mobility in ad hoc networks. As discussed earlier, this model has been used in Bandyopadhyay (2006). A mobile ad hoc network (MANET) is a collection of mobile users that communicate over wireless links that are not part of a preexisting network infrastructure. Because the nodes are mobile, the network topology usually changes unpredictably over time. An example of a node in a MANET is a wireless-equipped vehicle plying the streets of a city and communicating with similar vehicles.

Consider a MANET whose nodes move in a two-dimensional grid. We assume that a node's motion is subject to the following rules:

- There is a reflecting barrier at 0 such that on reaching the barrier a node is reflected with probability 1.
- A node takes a step in the same direction as its previous step with probability p_0, if the direction is away from the barrier, and takes a step in the opposite direction with probability q_0. Similarly, if the direction is toward the reflecting barrier, it takes a step in the same direction as its previous step with probability p_1, and it takes a step in the opposite direction with probability q_1.
- It takes a step in each of the two orthogonal directions with probability r, where $p_i + q_i + 2r = 1$, $i = 0, 1$.
- The time between epochs at which steps are taken are exponentially distributed with mean $1/\lambda$. This condition amounts to the fact that the node moves from one intersection to a neighboring intersection with a random velocity.

Thus, we can model the system by a continuous-time correlated random walk. We first consider the one-dimensional walk in which $r = 0$. As discussed earlier, we represent a state by (k, l), where k is the location of the walker, where a move to the previous direction indicates a success and a move to the opposite direction indicates a failure, and l is the result index, which is defined by

$$l = \begin{cases} 1 & \text{if the previous trial resulted in a success} \\ 0 & \text{if the previous trial resulted in a failure} \end{cases}$$

Figure 8.16 is the state-transition-rate diagram for the process.

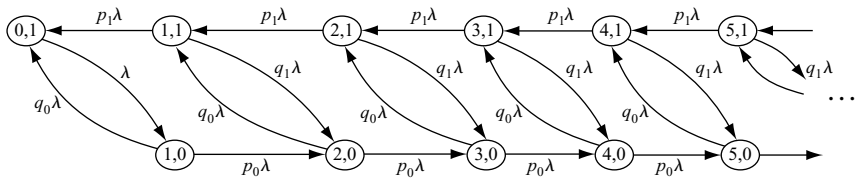


Figure 8.16. State-transition-rate diagram for the CRW.

We observe that the process is a quasi-birth-and-death process with the Q matrix given by

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ q_0\lambda & -\lambda & 0 & p_0\lambda & 0 & 0 & 0 & 0 & 0 & \dots \\ p_1\lambda & 0 & -\lambda & q_1\lambda & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & q_0\lambda & -\lambda & 0 & p_0\lambda & 0 & 0 & 0 & \dots \\ 0 & 0 & p_1\lambda & 0 & -\lambda & q_1\lambda & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & q_0\lambda & -\lambda & p_0\lambda & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & p_1\lambda & 0 & -\lambda & q_1\lambda & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

$$= \begin{bmatrix} D_0 & D_1 & 0 & 0 & 0 & \dots \\ A_2 & A_0 & A_1 & 0 & 0 & \dots \\ 0 & A_2 & A_0 & A_1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

where

$$D_0 = \begin{bmatrix} -\lambda & \lambda \\ q_0\lambda & -\lambda \end{bmatrix} \qquad D_1 = \begin{bmatrix} 0 & 0 \\ 0 & p_0\lambda \end{bmatrix}$$

$$A_0 = \begin{bmatrix} -\lambda & q_1\lambda \\ q_0\lambda & -\lambda \end{bmatrix} \qquad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & p_0\lambda \end{bmatrix} \qquad A_2 = \begin{bmatrix} p_1\lambda & 0 \\ 0 & 0\lambda \end{bmatrix}$$

Thus, using the matrix-analytic method discussed in Chapter 7, we can analyze the process. The case where r is not zero can be analyzed by decomposing it into two independent one-dimensional CRWs: one for the east-west direction and the other for the north-south direction. The probability of being in location (x, y) is the product of the probability of being in location x and the probability of being in location y in the east-west and north-south directions, respectively.

8.14.4 Insurance Risk

Consider an insurance company that starts at time 0 with a capital X_0. Assume that the company receives insurance premiums $Y_1, Y_2, \ldots$ from its customers and pays out compensations $V_1, V_2, \ldots$ at times $1, 2, \ldots$. Thus, at time n the actual capital available to the company is

$$X_n = X_0 + (Y_1 - V_1) + \cdots + (Y_n - V_n)$$

The company is bankrupt if $X_n < 0$. Let the random variable W_k be defined by $W_k = Y_k - V_k$. If we assume that the sequences $\{Y_n\}$ and $\{V_n\}$ are independent, then the company's capital behaves like a process starting at X_0 and having jumps W_k. Thus, the dynamics of the process is given by

$$X_n = \begin{cases} X_{n-1} + W_n & X_{n-1} > 0, X_{n-1} + W_n > 0 \\ 0 & \text{otherwise} \end{cases}$$

If we know the probability distribution of W_k we will be able to solve the problem as a random walk. One question that one might want to ask is the probability that the company becomes bankrupt.

8.14.5 Content of a Dam

Consider a basin behind a dam that has X_n amount of water at the end of the nth time period, which we assume to be a day. During day k a total of Y_k units of water flow into the basin, where Y_k has a well-defined probability distribution. The basin has a finite capacity b so that overflow occurs when Y_k exceeds b. Thus, the volume of water in the basin at the end of day n is

$$X_n = \min\{X_{n-1} + Y_n, b\}$$

Assume that the demand for water on day k is V_k. Then the effective additional volume of water in the basin is $U_k = Y_k - V_k$, where we assume that the sequences $\{Y_n\}$ and $\{V_n\}$ are independent. Because the content cannot be negative, we have that

$$X_n = \min\{\max\{X_{n-1} + U_n, 0\}, b\}$$

Thus, we have that

$$X_n = \begin{cases} X_{n-1} + U_n & 0 < X_{n-1} + U_n < b \\ 0 & X_{n-1} + U_n \leq 0 \\ b & X_{n-1} + U_n \geq b \end{cases}$$

This means that $\{X_n\}$ is a random walk with absorbing barriers at 0 and b. An overflow might occur when the process reaches level b. The probability distribution of the process can be determined if the probability distribution of U_n is known.

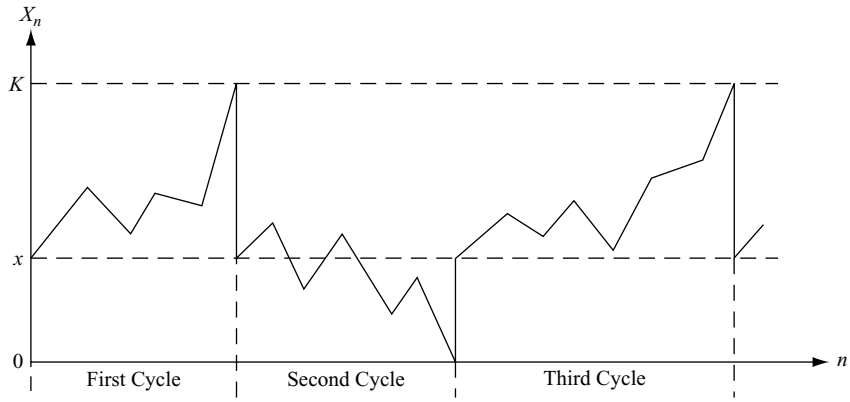


Figure 8.17. Cash management model.

8.14.6 Cash Management

Consider a company that attempts to maintain just enough cash balance to meet its operational needs. It periodically intervenes to ensure that it neither has too much cash on hand nor insufficient cash. To do this, the company has set a goal to periodically manage its cash as follows. If the cash available in a period exceeds the value K, the company buys treasury bills to reduce its cash level to x, where $0 < x < k$. Similarly, if the available cash is 0, the company sells sufficient treasury bills to boost the cash level to x. Thus, the cash level fluctuates as the process goes through a series of cycles, where the duration of each cycle is a random variable, as illustrated in Figure 8.17.

We might want to determine the mean length of a cycle. Also, given that an intervention is made, we might want to know that it is from a given level, say, level K. This is the probability that the process hits level K before level 0. Thus, if we assume that the cash demand is an asymmetric random walk, we can use the same techniques used to obtain the first passage time and the probability of a maximum to solve the problems associated with this model.

8.15 Problems

8.1 A bag contains four red balls, three blue balls, and three green balls. Jim plays a game in which he bets \$1 to draw a ball from the bag. If he draws a red ball, he wins \$1; otherwise he loses \$1. Assume that the balls are drawn with replacement and that Jim starts with \$50 with the hope of reaching \$100 at which point he stops playing. However, if he loses all his money before this, the game also ends and he becomes bankrupt. What is the probability that the game ends with Jim being \$50 richer than he was at the beginning?

8.2 Mark and Kevin play a series of games of cards. During each game each player bets \$1, and whoever wins the game gets the \$2. Sometimes a game ends in a tie in which case neither player loses his money. Mark is a better player than Kevin and has a probability 0.5 of wining each game, a probability 0.3 of losing each game, and a probability 0.2 of tying with Kevin. Initially Mark had \$9 and Kevin had \$6, and the game is over when either player is bankrupt.

a. Give the state-transition diagram of the process.
b. If r_k denotes the probability that Mark is ruined, given that the game is currently in state k, obtain an expression for r_k in the first game when the process is in state k.

8.3 Chris has \$20 and Dana has \$30. They decide to play a game in which each pledges \$1 and flips a fair coin. If both coins come up on the same side, Chris wins the \$2, and if they come up on different sides, Dana wins the \$2. The game ends when either of them has all the money. What is the probability that Chris wins the game?

8.4 Consider a single-server discrete-time queueing system that operates in the following manner. Let X_n denote the number of customers in the system at time $n \in \{0, 1, 2, \ldots\}$. If a customer is receiving service in time n, then the probability that he finishes receiving service before time $n + 1$ is q, where $0 \leq q \leq 1$. Let the random variable Y_n denote the number of customers that arrive between time n and $n + 1$, where the PMF of Y_n is given by

$$P_{Y_n}(k) = P[Y_n = k] = e^{-\lambda}\frac{\lambda^k}{k!} \qquad k = 0, 1, \ldots$$

a. Give an expression for the relationship between X_{n+1}, X_n, and Y_n.
b. Find the expression for the transition probabilities $P[X_{n+1} = j | X_n = i]$.

8.5 Consider the random walk $S_n = X_1 + X_2 + \cdots + X_n$, where the X_i are independent and identically distributed Bernoulli random variables that take on the value 1 with probability $p = 0.6$ and the value -1 with probability $q = 1 - p = 0.4$.

a. Find the probability $P[S_8 = 0]$.
b. What value of p maximizes $P[S_8 = 0]$?

8.6 Let N denote the number of times that an asymmetric random walk that takes a step to the right with probability p and a step to the left with probability $q = 1 - p$ revisits its starting point. Show that the PMF of N is given by

$$p_N(n) = P[N = n] = \beta(1 - \beta)^n \qquad n = 0, 1, 2, \ldots$$

where $\beta = |p - q|$.

8.7 Consider an asymmetric random walk that takes a step to the right with probability p and a step to the left with probability $q = 1 - p$. Assume that there are two absorbing barriers, a and b, and that the walk starts at the point k, where $b < k < a$.

a. What is the probability that the walk stops at b?

b. What is the mean number of steps until the walk stops?

8.8 Consider a cash management scheme in which a company needs to maintain the available cash to be no more than $\$K$. Whenever the cash level reaches K, the company buys treasury bills and reduces the cash level to x. Whenever the cash level reaches 0, the company sells enough treasury bills to bring the cash level up to x, where $0 < x < k$. Assume that in any given period the probability that the cash level increases by \$1 is p, and the probability that it decreases by \$1 is $q = 1 - p$. We define an intervention cycle as the period from the point when the cash level is x until the point when it is either 0 or K. Let T denote the time at which the available cash first reaches 0 or K, given that it starts at level x.

a. What is the expected value of T?

b. What is the mean number of visits to level m up to time T, where $0 < m < K$?

8.9 Consider a coordinated random walk with stay. That is, a walker can move to the right, to the left, or not move at all. Given that the move in the current step is to the right, then in the next step it will move to the right again with probability a, to the left with probability b, and remain in the current position with probability $1 - a - b$. Given that the walker did not move in the current step, then in the next step it will move to the right with probability c, to the left with probability d, and not move again with probability $1 - c - d$. Finally, given that the move in the current step is to the left, then in the next step it will move to the right with probability g, to the left again with probability h, and remain in the current position with probability $1 - g - h$. Let the process be represented by the bivariate process $\{(X_n, Y_n), n = 0, 1, 2, \ldots\}$, where X_n is the location of the walker after n steps and Y_n is the nature of the nth step (i.e., right, left, or no move). Let π_1 be the limiting probability that the process is in the "right" state, π_0 the limiting probability that it is in the "no move" state, and π_{-1} the limiting probability that it is in the "left" state. Let $\Pi = [\pi_1, \pi_0, \pi_{-1}]$, where $\pi_1 + \pi_0 + \pi_{-1} = 1$.

a. Find the values of π_1, π_0, and π_{-1}.

b. Obtain the transition probability matrix of the process and show that the process is a quasi-birth-and-death process.

8.10 Consider a continuous-time random walk $\{X(t) : t \geq 0\}$ in which the jump size, Θ, is normally distributed with mean μ and variance σ^2, and the waiting time, T, is exponentially distributed with mean $1/\lambda$. Obtain the master equation, $P(x, t)$, which is the probability that the position of the walker at time t is $X(t) = x$, given that it was in position 0 at time $t = 0$.

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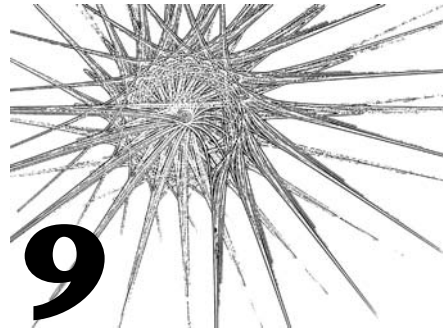
Brownian Motion and Diffusion Processes



9

9.1 Introduction

In this chapter we consider continuous-time continuous-state Markov processes, including the Brownian motion and the diffusion process. The dynamics of these processes are usually captured via stochastic differential equations. Thus, we discuss differential calculus in this chapter.

9.2 Brownian Motion

The Scottish botanist Robert Brown noticed that small particles that are suspended in a liquid move in an irregular manner. Since his observation of this phenomenon in the 1800s, biologists, chemists, and physicists have observed that the movements of charged particles in any fluid follow the same pattern. Over the years this phenomenon has been abstracted by a stochastic model known as the Brownian motion, which is now used to analyze environments with irregular movement, such as prices in the stock market.

More formally, the Brownian motion $\{W(t), t \geq 0\}$ is a stochastic process that models random continuous motion. It is considered to be the continuous-time analog of the random walk and can also be considered as a continuous-time Gaussian

process with independent increments. In particular, the Brownian motion has the following properties:

1. $W(0) = 0$; that is, it starts at zero.
2. $W(t)$ is continuous in $t \geq 0$; that is, it has continuous sample paths with no jumps.
3. It has both stationary and independent increments.
4. For $0 \leq s < t$, the random variable $W = W(t) - W(s)$ has a normal distribution with mean 0 and variance $\sigma_W^2 = \sigma^2(t-s)$. That is, $W \sim N(0, \sigma^2(t-s))$.

Brownian motion is an important building block for modeling continuous-time stochastic processes. In particular, it has become an important framework for modeling financial markets. The path of a Brownian motion is always continuous, but it is nowhere smooth; consequently, it is nowhere differentiable. The fact that the path is continuous means that a particle in Brownian motion cannot jump instantaneously from one point to another.

Because $W(0) = 0$, then according to property 3, $W(t) = W(t) - W(0) \sim N(0, \sigma^2(t-0)) = N(0, \sigma^2 t)$. Thus, $W(t-s) \sim N(0, \sigma^2(t-s))$; that is, $W(t) - W(s)$ has the same distribution as $W(t-s)$. This also means that another way to define a Brownian motion $\{W(t), t \geq 0\}$ is that it is a process that satisfies conditions 1, 2, and 3 along with the condition $W(t) = N(0, \sigma^2 t)$.

As discussed in Rogers and Williams (2000a), there are many reasons for studying the Brownian motion. As stated earlier, it is an important building block for modeling continuous-time stochastic processes because many classes of stochastic processes contain Brownian motion. It is a Markov process, a Gaussian process, a martingale, a diffusion process, as well as a Levy process. Over the years it has become a rich mathematical object. For example, it is the central theme of stochastic calculus.

A Brownian motion is sometimes called a *Wiener process*. A sample function of the Weiner process is shown in Figure 9.1.

Let $B(t) = W(t)/\sigma$. Then $E[B(t)] = 0$ and $\sigma_{B(t)}^2 = 1$. The stochastic process $\{B(t), t \geq 0\}$ is called the *standard Brownian motion*, which has the property

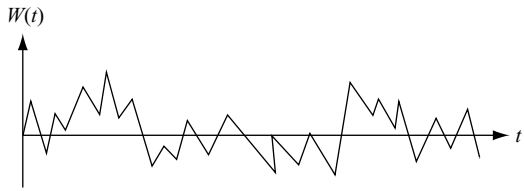


Figure 9.1. Sample function of a Wiener process.

that when sampled at regular intervals produces a symmetric random walk. Note that $B(t) = N(0, t)$. In the remainder of this chapter we refer to the Weiner process $\{W(t), t \geq 0\}$ as the *classical Brownian motion* and use the two terms interchangeably.

9.2.1 Brownian Motion with Drift

Brownian motion is used to model stock prices. Because stock prices do not generally have a zero mean, it is customary to include a *drift* measure that makes the following model with a *drift rate* $\mu > 0$ a better model than the classical Brownian motion:

$$Y(t) = \mu t + W(t) \qquad t \geq 0$$

where $W(t)$ is the classical Brownian motion. Note that $E[Y(t)] = \mu t$ and $\sigma^2_{Y(t)} = \sigma^2 t$, which means that $Y(t) \sim N(\mu t, \sigma^2 t)$. Note also that we can express $Y(t)$ in terms of the standard Brownian motion as follows:

$$Y(t) = \mu t + \sigma B(t) \qquad t \geq 0$$

9.2.2 Brownian Motion as a Markov Process

Let $W(t)$ be a classical Brownian motion, and let $\Im_s$ denote the information that is being revealed by watching the process up through the time $s < t$. Then the conditional expected value of $W(t)$ given $\Im_s$ can be obtained as follows:

$$E[W(t)|\Im_s] = E[W(s)|\Im_s] + E[W(t) - W(s)|\Im_s]$$

The first term on the right-hand side is equal to $W(s)$ because it is already revealed through $\Im_s$. Also, the increment $W(t) - W(s)$ is independent of $\Im_s$; thus $E[W(t) - W(s)|\Im_s] = E[W(t) - W(s)]$, which is zero. Therefore, we obtain

$$E[W(t)|\Im_s] = W(s) = E[W(t)|W(s)]$$

This means that to predict $W(t)$ given all the information up through time s, we only need to consider the value of the process at time s. Thus, a Brownian motion is a Markov process. This is not surprising because it is an independent increment process. As stated earlier, all independent increment processes have the Markov property.

Because for the classical Brownian motion the increment over an interval of length X has the Gaussian distribution with the PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$$

the classical Brownian motion is a Markov process with transition probability density function given by

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y-x)^2/2\sigma^2}$$

9.2.3 Brownian Motion as a Martingale

Let $0 \le s \le t$ and let $v \ge 0$. We show that $E[W(t+v)|W(s)] = W(t)$. We recall the Markov property that for $0 \le s \le t$, $E[W(t+v)|W(s)] = E[W(t+v)|W(t)]$. Therefore,

$$\begin{aligned} E[W(t+v)|W(s)] &= E[W(t+v)|W(t)] \\ &= E[W(t) + \{W(t+v) - W(t)\}|W(t)] \\ &= W(t) + E[W(t+v) - W(t)|W(t)] \\ &= W(t) + E[W(t+v) - W(t)] = W(t) + E[W(v)] = W(t) + 0 \\ &= W(t) \end{aligned}$$

where the fourth equality follows from the independent increments property.

9.2.4 First Passage Time of a Brownian Motion

Let T_k denote the time it takes a classical Brownian motion to go from $W(0) = 0$ to $W(t) = k \neq 0$; that is,

$$T_k = \min\{t > 0 : W(t) = k\}$$

Suppose $k > 0$. Then we obtain the PDF of T_k as follows:

$$P[W(t) \ge k] = P[W(t) \ge k|T_k \le t]P[T_k \le t] + P[W(t) \ge k|T_k > t]P[T_k > t]$$

By the definition of the first passage time, $P[W(t) \ge k|T_k > t] = 0$ for all t. Also, if $T_k \le t$, then we assume that there exists a $t_0 \in (0, t)$ with the property that

$W(t_0) = k$. We know that the process $\{W(t)|W(t_0) = k\}$ has a normal distribution with mean k and variance $\sigma^2(t - t_0)$. That is, the random variable

$$W(t)|\{W(t_0) = k\} \sim N(k, \sigma^2(t - t_0)) \qquad \text{for all } (t \geq t_0)$$

Thus, from the symmetry about k, we have that

$$P[W(t) \geq k|T_k \leq t] = \frac{1}{2}$$

which means that the CDF of T_k is given by

$$F_{T_k}(t) = P[T_k \leq t] = 2P[W(t) \geq k] = \frac{2}{\sigma\sqrt{2\pi t}} \int_k^\infty \exp\{-v^2/2\sigma^2 t\} dv$$

By symmetry, T_k and T_{-k} are identically distributed random variables, which means that

$$F_{T_k}(t) = \frac{2}{\sigma\sqrt{2\pi t}} \int_{|k|}^\infty \exp\{-v^2/2\sigma^2 t\} dv = 2\left\{1 - \Phi\left(\frac{|k|}{\sigma\sqrt{t}}\right)\right\} \qquad t > 0$$

where $\Phi(\cdot)$ is the CDF of the standard normal random variable. Let $y^2 = v^2/\sigma^2 t \Rightarrow dv = \sigma\sqrt{t}dy$. Then the CDF becomes

$$F_{T_k}(t) = \frac{2}{\sqrt{2\pi}} \int_{|k|/\sigma\sqrt{t}}^\infty \exp\{-y^2/2\} dy \qquad t > 0$$

Differentiating with respect to t we obtain the PDF of T_k as follows:

$$f_{T_k}(t) = \frac{|k|}{\sqrt{2\pi\sigma^2 t^3}} \exp\left\{-\frac{k^2}{2\sigma^2 t}\right\} \qquad t > 0$$

Because for the standard Brownian motion $\sigma = 1$, the PDF of T_k for the standard Brownian motion is given by

$$f_{T_k}(t) = \frac{|k|}{\sqrt{2\pi t^3}} \exp\left\{-\frac{k^2}{2t}\right\} \qquad t > 0$$

Example 9.1 The accuracy of the measurements from a certain sensor has been found to be deteriorating. Studies indicate that the deterioration can be modeled by a standard Brownian motion. What is the probability that the sensor reading will deviate from the true value by -8 units in

a. 6 months (or 180 days)
b. 1 year (or 365 days) ■

Solution: We are required to find $F_{T_{-8}}(x)$, where $x = 180$ and $x = 365$ for a standard Brownian process.

a. In 6 months we have

$$F_{T_{-8}}(180) = 2\left\{1 - \Phi\left(\frac{8}{\sqrt{180}}\right)\right\}$$
$$= 2\{1 - \Phi(0.60)\} = 2\{1 - 0.7257\} = 0.5486$$

where the $\Phi(\cdot)$ is obtained from any standard text in probability, such as Ibe (2005).

b. In 1 year we have

$$F_{T_{-8}}(365) = 2\left\{1 - \Phi\left(\frac{8}{\sqrt{365}}\right)\right\}$$
$$= 2\{1 - \Phi(0.42)\} = 2\{1 - 0.6628\} = 0.6744$$

9.2.5 Maximum of a Brownian Motion

Let $M(t)$ denote the maximum value of the classical Brownian motion $\{W(t)\}$ in the interval $[0, t]$; that is,

$$M(t) = \max\{W(u), 0 \le u \le t\}$$

The PDF of $M(t)$ is obtained by noting that

$$P[M(t) \ge x] = P[T_x \le t]$$

Thus, we have that

$$F_{M(t)}(x) = P[M(t) \le x] = 1 - P[T_x \le t] = 1 - 2\left\{1 - \Phi\left(\frac{|x|}{\sigma\sqrt{t}}\right)\right\}$$
$$= 1 - \frac{2}{\sqrt{2\pi}} \int_{|x|/\sigma\sqrt{t}}^{\infty} \exp\{-y^2/2\} dy$$

Upon differentiation we obtain the PDF as

$$f_{M(t)}(x) = \sqrt{\frac{2}{\pi\sigma^2 t}} \exp\left\{-\frac{x^2}{2\sigma^2 t}\right\} \qquad x \ge 0$$

Example 9.2 With respect to Example 9.1, what is the probability that the maximum deviation of the sensor reading will be at most eight units in 6 months? ■

Solution: We are required to find $F_{M(180)}(8)$, which is given by

$$F_{M(180)}(8) = 1 - 2\left\{1 - \Phi\left(\frac{8}{\sqrt{180}}\right)\right\} = 1 - 2\{1 - \Phi(0.60)\}$$
$$= 1 - 2\{1 - 0.7257\} = 0.4514$$

9.2.6 First Passage Time in an Interval

Let T_{ab} denote the time at which the classical Brownian motion $\{W(t), t \geq 0\}$ for the first time hits either the value a or the value b, where $b < 0 < a$, as shown in Figure 9.2.

Thus, we can write

$$T_{ab} = \min\{t: W(t) = a \text{ or } W(t) = b\} \qquad b < 0 < a < \infty$$

Let p_{ab} be the probability that $\{W(t)\}$ assumes the value a first; that is, $p_{ab} = P[W(T_{ab}) = a]$. Now, T_{ab} is a stopping time whose mean $E[T_{ab}]$ is finite. Thus, according to the stopping time theorem, $E[W(T_{ab})] = E[W(0)] = 0$. This means that

$$E[W(T_{ab})] = ap_{ab} + b(1 - p_{ab}) = 0$$

From this we obtain the probability that the process hits a before b as

$$p_{ab} = \frac{|b|}{a + |b|}$$

An alternative method of solving the problem is by seeing the Brownian motion as a limit of the symmetric random walk. The probability that the process hits a

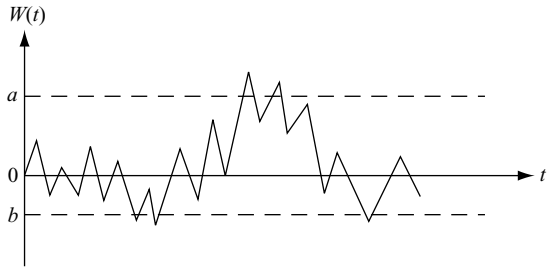


Figure 9.2. First passage time in an interval.

before b can then be likened to the gambler's ruin problem in which the process is equally likely to go up or down by a distance of Δh. Thus, with respect to the gambler's ruin problem of Chapter 8, $N = (a + |b|)/\Delta h$ and $i = |b|/\Delta h$. Because p_{ab} is the ruin probability, and for the symmetric walk the ruin probability is i/N, we have that

$$p_{ab} = \frac{i}{N} = \frac{|b|/\Delta h}{(a + |b|)/\Delta h} = \frac{|b|}{a + |b|}$$

9.2.7 The Brownian Bridge

The Brownian bridge is a classical Brownian motion defined on the interval $[0, 1]$ and conditioned on the event $W(1) = 0$. Thus, the Brownian bridge is the process $\{W(t), t \in [0, 1] | W(1) = 0\}$. One way to realize the process is by defining $X(t)$, the Brownian bridge, as follows:

$$X(t) = W(t) - tW(1) \qquad 0 \le t \le 1$$

The Brownian bridge is sometimes called the *tied-down Brownian motion* (or *tied-down Wiener process*). It is useful for modeling a system that starts at some given level and is expected to return to that level at some specified future time. We note that

$$X(0) = W(0) - 0W(1) = 0$$
$$X(1) = W(1) - 1W(1) = 0$$

Thus, $E[X(t)] = 0$. For $0 \le s < t \le 1$, the covariance of $X(t)$ and $X(s)$ is given by

$$\begin{aligned}
\text{Cov}\{X(s)X(t)\} &= E[\{X(s) - E[X(s)]\}\{X(t) - E[X(t)]\}] = E[X(s)X(t)] \\
&= E[\{W(s) - sW(1)\}\{W(t) - tW(1)\}] \\
&= E[W(s)W(t) - tW(s)W(1) - sW(1)W(t) + stW^2(1)] \\
&= \sigma^2(s \wedge t) - \sigma^2 t(s \wedge 1) - \sigma^2 s(1 \wedge t) + \sigma^2 st \\
&= \sigma^2(s - st - st + st) = \sigma^2(s - st) \\
&= \sigma^2 s(1 - t)
\end{aligned}$$

where we have used the fact that $E[W(s)W(t)] = \sigma^2 \min(s, t) = \sigma^2(s \wedge t)$. Thus, the Brownian bridge is not a wide-sense stationary process because the covariance $\text{Cov}\{X(s)X(t)\}$ is not a function of only the difference between s and t.

9.3 Introduction to Stochastic Calculus

One of the basic tools for analyzing the Brownian motion is stochastic calculus, which deals with infinitesimal calculus on nondifferentiable functions. It arises from the need to incorporate unpredictable random factors in system modeling. The primary focus of stochastic calculus is the Brownian motion because it is a model that is not only useful and permits explicit calculations to be performed, but also it is applicable to many naturally occurring phenomena.

The driving force behind stochastic calculus was the attempt to understand the motion driven by a series of small random impulses. In this so-called Brownian motion, the net distance traveled in time Δt by a particle is proportional to $\sqrt{\Delta t}$. Thus, in the stochastic differential equation that represents the dynamic behavior of the process, future changes are expressed as differentials, not as derivatives. For a process $\{X(t), t \geq 0\}$, the differential $dX(t)$ is defined by

$$dX(t) = X(t + dt) - X(t)$$

The integral form is the forward sum of uncountable and random increments over time and is given by

$$X(t) = \int_0^t dX(u)$$

9.3.1 The Ito Integral

The Ito integral deals with the integration of the expression

$$\int_0^t X(u)dW(u)$$

where $X(t)$ is a stochastic process and $W(t)$ is the classical Brownian motion. Thus, the above integral is referred to as the Ito integral of $X(t)$ with respect to the Brownian motion. If $W(t)$ were the function $w(t)$ that is differentiable, we would write $dw(u) = w_1(u)du$, where $w_1(u) = dw(u)/du$, and thus obtain the expression

$$\int_0^t X(u)dw(u) = \int_0^t X(u)w_1(u)du$$

that is known to be a standard integral. When the process $W(t)$ is not differentiable, as is the case when it is the Brownian motion, the integral becomes an unfamiliar integral. To evaluate the integral when $W(t)$ is not differentiable we first

divide the interval $[0, t]$ into n disjoint intervals at points $0 = t_0 < t_1 < \cdots < t_n = t$ and obtain

$$\int_0^1 X(u)dW(u) = \lim_{n\to\infty} \sum_{k=0}^{n=1} X(u_k)\{W(u_{k+1}) - W(u_k)\}$$

which is the *Ito integral*. However, for this to work $X(t)$ and $W(t)$ are required to satisfy certain conditions, which include the fact that $X(t)$ must be smooth enough for $X(t_k)$ to represent $X(t)$ in the interval (t_k, t_{k+1}). Also, the $X(t_k)$ are required to be independent of the increments $W(t_{k+1}) - W(t_k)$. In addition, $X(t)$ is required to be an *adapted process*, which is sometimes called a *nonanticipating process* because it cannot "see into the future."

To fully understand an adapted process we first define the concept of *filtration*. A filtration is a family of σ-algebras $\{F_t, 0 \le t < \infty\}$ that is increasing; that is, if $s \le t$, then $F_s \subset F_t$. The property that a filtration is increasing implies that information is not forgotten. The space $(\Omega, F, \{F_t\}, P)$ is called a *filtered probability space*. A stochastic process $\{X(t), 0 \le t \le T\}$ on a filtered probability space is called an adapted process if for any $t \in [0, T]$, F_t contains all the information about the random variable $X(t)$; alternatively, it is an adapted process if the random variable $X(t)$ is F_t-adapted.

Thus, a stochastic process $\{X(t)\}$ is defined to be *Ito integrable* on the interval $[0, T]$ if the following conditions are satisfied:

1. The random variable $X(t)$ is adapted for $t \in [0, T]$.
2. $\displaystyle\int_0^t E[X^2(u)]du < \infty.$

We state the following proposition without proof; the proof can be found in many books on stochastic processes such as Capasso and Bakstein (2005), Steele (2001), Oksendal (2005), Benth (2004), and Klebaner (2005).

Proposition 9.1 Let $f(t)$ and $g(t)$ be continuous functions in the interval $[a, b]$, and let $B(t)$ be a standard Brownian motion. Then

1. $\displaystyle E\left[\int_a^b f(t)dB(t)\right] = 0$
2. $\displaystyle E\left[\int_a^b f(t)dB(t)\int_a^b g(t)dB(t)\right] = \int_a^b E[f(t)g(t)]dt$
3. $\displaystyle E\left[\left\{\int_a^b f(t)dB(t)\right\}^2\right] = \mathrm{Var}\left(\int_a^b f(t)dB(t)\right) = \int_a^b E[\{f(t)\}^2]dt$; this is called the *Ito isometry* property.

Another result that relates to $B(t)$ is the following proposition that is also stated without proof:

Proposition 9.2 $\int_a^b B(t)dB(t) = \frac{1}{2}\{B^2(b) - B^2(a)\} - \frac{b-a}{2}.$

The proof can also be found in the books referenced earlier.

9.3.2 The Stochastic Differential

Let $\{X((t), t \in [0, T])\}$ be a stochastic process such that for $0 < t \leq T$,

$$X(t) = X(0) + \int_0^t a(u)du + \int_0^t b(u)dB(u)$$

where $a(t)$ and $b(t)$ are continuous functions in the interval $[0, T]$. Then we say that $X(t)$ has the stochastic differential

$$dX(t) = a(t)dt + b(t)dB(t)$$

in the interval $[0, T]$.

9.3.3 The Ito's Formula

The Ito's formula serves as a bridge between classical theory and stochastic theory. It is the stochastic equivalent of Taylor's theorem about the expansion of functions. Consider the stochastic process $X(t)$ with the stochastic differential

$$dX(t) = a(t)dt + b(t)dB(t)$$

in the interval $[0, T]$. Let $f(t, y)$ be a continuous function in the interval $[0, T]$ that is twice differentiable in y. Then the Ito's formula gives the stochastic differential of the function $y(t) = f(t, X(t))$ as follows:

$$dy(t) = \left\{ \frac{\partial f(t, X(t))}{\partial t} + a(t)\frac{\partial f(t, X(t))}{\partial X} + \frac{1}{2}b^2(t)\frac{\partial^2 f(t, X(t))}{\partial X^2} \right\} dt$$

$$+ b(t)\frac{\partial f(t, X(t))}{\partial X}dB(t)$$

9.3.4 Stochastic Differential Equations

A differential equation is a rule that allows us to calculate the value of some quantity at a later time given the value at some earlier time. Thus, a stochastic differential equation (SDE) can be viewed as the representation of the dynamic behavior of a stochastic process. The dynamic behavior of almost all important continuous stochastic processes can be expressed by an equation of the form

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dB(t), \qquad X(0) = x_0$$

where the functions $\mu(X(t), t)$ and $\sigma(X(t), t)$ are given, $X(t)$ is an unknown process, and $B(t)$ is the standard Brownian motion. Such a stochastic differential equation is said to be driven by Brownian motion. As stated earlier, Brownian motion-driven systems have become an important framework for modeling financial markets. The coefficients $\mu(X(t), t)$ and $\sigma(X(t), t)$ can be interpreted as measures of short-term growth and short-term variability respectively. Thus, adjusting them permits a modeler to construct stochastic processes that reflect real-life behavior. A solution to the preceding SDE is

$$X(t) = X(0) + \int_0^t \mu(X(u), u)du + \int_0^t \sigma(X(u), u)dB(u)$$

If the two integrals exist for all $t > 0$, $X(t)$ is called a strong solution of the SDE. In some processes, the coefficients $\mu(X(t), t)$ and $\sigma(X(t), t)$ are defined as follows: $\mu(X(t), t) = \mu X(t)$ where $-\infty < \mu < \infty$ and $\sigma(X(t), t) = \sigma X(t)$ where $\sigma > 0$. Thus, the stochastic differential equation for such processes is given by

$$dX(t) = \mu X(t)dt + \sigma X(t)dB(t), \qquad X(0) = x_0 > 0$$

The Ito's formula is the key to the solution of many SDEs. We illustrate the solution with the geometric Brownian motion.

9.4 Geometric Brownian Motion

Let $\{X(t)|t \geq 0\}$ be a Brownian motion with drift. The process $\{Y(t)|t \geq 0\}$, which is defined by

$$Y(t) = e^{X(t)}$$

is called the geometric Brownian motion. The stochastic differential equation of the process is obtained as follows. Because $\ln\{Y(t)\} = X(t)$, we have that

$$\frac{dY(t)}{Y(t)} = dX(t) = \mu dt + \sigma dB(t)$$

The equation can also be written as follows:

$$dY(t) = \mu Y(t)dt + \sigma Y(t)dB(t)$$

To obtain the dynamics of the process, let $Z(t) = \ln\{Y(t)\}$. Then using Ito's formula we have that

$$\begin{aligned} dZ(t) = d[\ln\{Y(t)\}] &= \frac{1}{Y(t)}dY(t) + \frac{1}{2}\left\{-\frac{1}{Y^2(t)}\right\}\sigma^2 Y^2(t)dt \\ &= \frac{1}{Y(t)}\{\mu Y(t)dt + \sigma Y(t)dB(t)\} - \frac{1}{2}\sigma^2 dt \\ &= \left\{\mu - \frac{1}{2}\sigma^2\right\}dt + \sigma dB(t) \end{aligned}$$

From this we obtain

$$Z(t) = Z(0) + \left\{\mu - \frac{1}{2}\sigma^2\right\}t + \sigma B(t)$$

Finally,

$$Y(t) = Y(0)\exp\{[\mu - (\sigma^2/2)]t + \sigma B(t)\} = Y(0)\exp\{[\mu - (\sigma^2/2)]t + W(t)\}$$

where $W(t)$ is the classical Brownian motion. Also, let $\tau > 0$ denote the interval between observations of the process. The τ-period *logarithmic return* is defined by

$$\ln\left(\frac{Y(t+\tau)}{Y(t)}\right) \equiv V(\tau) = [\mu - (\sigma^2/2)]\tau + \sigma\{B(t+\tau) - B(t)\}$$

Because the difference $B(t+\tau) - B(t)$ is normally distributed with mean zero and variance τ, $V(\tau)$ is normally distributed with mean and variance that are proportional to the observation interval. That is, $V(\tau) \sim N([\mu - (\sigma^2/2)]\tau, \sigma^2\tau)$.

9.5 Fractional Brownian Motion

The fractional Brownian motion (fBM) $\{B_H(t), t \geq 0\}$ is a generalization of the Brownian motion. Although the main principles of fBM were introduced earlier by Kolmogorov, the name was introduced in Mandelbroth (1968).

Fractional Brownian motion is a centered Gaussian process with stationary but not independent increments; it has independent increments only when it is a standard Brownian motion. The dependence property of the increments is modeled by the so-called *Hurst index* (or *Hurst parameter*), $0 < H < 1$. It satisfies the following condition:

$$B_H(0) = E[B_H(t)] = 0 \qquad t \geq 0$$

The covariance function of fBM, $R(s, t)$, is given by

$$R(s,t) = E[B_H(s)B_H(t)] = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}) \qquad s, t \geq 0$$

The value of H determines the kind of process fBM is. Generally three types of processes can be captured by the model:

- If $H = 1/2$, the process is a regular Brownian motion.
- If $H > 1/2$, the increments of the process are positively correlated, and the process is said to have a *long memory* (or *strong after-effects*).
- If $0 < H < 1/2$, the increments are negatively correlated. In this case the process is said to be *antipersistent*, which means that positive increments are followed by negative increments, and vice versa.

Another property of fBM is self-similarity. In general, a self-similar process is a stochastic process that is invariant in distribution under suitable scaling of time and space. Specifically, the stochastic process $\{X(t), t \geq 0\}$ is said to be self-similar if for any $a > 0$ there exists some $b > 0$ such that $\{X(at), t \geq 0\}$ has the same distribution as $\{bX(t), t \geq 0\}$. In the case of the fBM with a Hurst index H, a and b are related by $b = a^H$. Thus, fBM is a self-similar stochastic process in the sense that $\{B_H(at), t \geq 0\}$ has the same distribution as $\{a^H B_H(t), t \geq 0\}$ for all $a > 0$.

Also, fBM exhibits long-range dependence when $H > 1/2$, which means that if we define the autocorrelation function $\rho(n) = E[B_H(t_0)B_H(t_n)]$, then

$$\sum_{n=1}^{\infty} \rho(n) = \infty$$

When $H < 1/2$, $\rho(n)$ tends to decay exponentially; that is, $\rho(n) = \beta^n$ as n tends to infinity, where $\beta < 1$, which means that the process exhibits short-range dependence.

Fractional Brownian motion has been used in Norris (1995) and Mikosch (2002) to model aggregated connectionless traffic. It has also been used in Rogers (1997), Dasgupta (1998), and Sottinen (2001) to model financial markets. The motivation for using fBM to model financial markets is that empirical studies indicate that the so-called logarithmic returns $r_n = \log\{S(t_n)/S(t_{n-1})\}$, where $S(t)$ is the observed price of a given stock at time t, is normal. However, empirical studies indicate that r_n tends to have a strong after-effect. Thus, advocates of fBM claim that it is a better model than traditional Brownian motion by using the Hurst index to capture dependency.

9.6 Application of Brownian Motion to Option Pricing

The Brownian motion and its transformations are popularly used financial models. Specifically, they are used to model the evolution in time of prices of risky securities. An option is a financial contract that gives the holder the right, but not the obligation, to buy or sell an underlying asset at a fixed, predetermined price called the *strike price* or *exercise price* on or before a certain date called the *expiration date* or *maturity date*. Options can be characterized by the nature of the period during which the option can be exercised. An option that can be exercised at any time between the initiation of the contract and the expiration date is called an *American option.* An option that can be exercised only on the maturity date is called a *European option*.

Options can also be classified as *call options* and *put options*. A call option gives its holder the right to buy the underlying financial asset at the strike price before or at the maturity date. This right is obtained by paying to the seller of the option a certain amount of money when the contract is signed. If the option holder chooses not to exercise the option on the expiration date, he forfeits this money. A put option gives its holder the right to sell the underlying financial asset at the strike price whenever the buyer exercises the option. As in the call option, the put option holder pays a certain amount of money for this right at the initiation of the contract; the money is forefeited if the holder chooses not to exercise the option on the maturity date.

There are two fundamental reasons why investors use options. One reason is *speculation* in which the buyer believes that the value of the asset will go up or down by the time the option is exercised. The other reason is *hedging* that is used to limit the risk associated with the fluctuations in the price of the asset over time.

We limit our discussion to the European option. Consider a stock whose value at time $t \geq 0$ is $S(t)$. One major task in option pricing is to find the rational price $C(S_t, t)$, which is called the *contract function*, that the option holder should pay

at t when the stock price is $S(t)$. The Black-Scholes model, which was proposed independently by Black and Scholes (1973) and Merton (1973), was the first reliable solution to the problem. The solution made several assumptions, which include the following: (a) there is continuous security trading, (b) there are no transaction costs, (c) selling of securities is possible at any time, (d) the market interest rate r is constant, and (e) there are no dividends in the interval $[0, T]$. The model assumes that the stock price $S(t) \equiv S_t$ can be modeled by a geometric Brownian motion

$$S(t) = S(0) \exp(W_1(t))$$

where $W_1(t) = \mu t + \sigma B(t)$ is a Brownian motion with drift. Thus, applying the Ito formula we obtain the following stochastic differential equation:

$$dS(t) = \left(\mu + \frac{\sigma^2}{2}\right) S(t)dt + \sigma S(t)dB(t)$$

The Black-Scholes model states that the price the holder has to pay for the contract must solve the *Black and Scholes partial differential equation*, which is

$$rC(S_t, t) = \frac{\partial C(S_t, t)}{\partial t} + rS_t \frac{\partial C(S_t, t)}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C(S_t, t)}{\partial S_t^2}, \qquad S_t \geq 0$$

with the terminal condition on the contract function as the value of the call or put at the expiration time T, which is

$$C(S_T, T) = \begin{cases} \max\{S_T - K, 0\} & \text{for a call option} \\ \max\{K - S_T, 0\} & \text{for a put option} \end{cases}$$

where K is the strike price.

The final solution of the Black and Scholes partial differential equation for a European call option is given by

$$C(S_t, T) = S_t \Phi(d_1) - Ke^{-r(T-t)} \Phi(d_2)$$

where $\Phi(d)$ is the CDF of the standard normal random variable; that is,

$$\Phi(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-x^2/2} dx$$

and the two arguments are given by

$$d_1 = \frac{\ln(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\ln(S_t/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

Similarly, for a European put option we obtain

$$C(S_t, t) = -S_t[1 - \Phi(d_1)] + Ke^{-r(T-t)}[1 - \Phi(d_2)]$$

The results essentially indicate to the writer what price he should charge for an option at time t. The price itself depends on the parameters K and T of the contract and on the market characteristics r and σ, where σ is the volatility of the underlying asset and is one parameter of the model that cannot be observed directly. The preceding equations provide the necessary information to eliminate risk, which can be accomplished if the writer resorts to Δ-hedging; that is, he continuously adjusts the amount $\Delta(t)$ that is defined by

$$\Delta(t) = \frac{\partial C(S_t, t)}{\partial S_t} = \begin{cases} \Phi(d_1) & \text{for a call option} \\ -\{1 - \Phi(d_1)\} & \text{for a put option} \end{cases}$$

The different terms in $C(S_t, t)$ have interpretations. For example, in the case of the call option, we have that:

a. $\Phi(d_2)$ represents the probability of exercising the option in a risk-neutral world; that is, where the actual drift of a financial time series can be replaced by the risk-free rate r.
b. $K\Phi(d_2)$ is the expected amount of money to be paid under the contract, because $\Phi(d_2)$ is the probability that the strike price, K, will be paid.
c. $e^{-r(T-t)}$ is a discount factor that discounts the expected profit to the present-day value.

The Black-Scholes model represents a milestone in modern financial markets. However, it made some assumptions that do not fully reflect the stochastic behavior observed in real markets. For example, it assumes that the path of the underlying asset price is continuous, which is not always the case because the arrival of some relevant economic information can cause a jump, and hence a discontinuity, in the price. Also, the volatility of a given stock and the interest rate that are assumed to be constant are not actually constant. It must be emphasized, however, that the model remains the most widely used financial model in option trading despite all the simplifications used in its derivation.

9.7 Random Walk Approximation of Brownian Motion

While it is easy to construct a random walk, it is not easy to construct Brownian motion. Some intuition to the behavior of Brownian motion can be gained by comparing it with a simple symmetric random walk. Specifically, assume that the process $X(t)$ increases by Δx over each time increment Δt with probability p and decreases by Δx with probability $1 - p$. Then we have that

$$E[X(t + \Delta t) - X(t)] = p\Delta x - (1 - p)\Delta x = (2p - 1)\Delta x$$

$$E[\{X(t + \Delta t) - X(t)\}^2] = p(\Delta x)^2 + (1 - p)(-\Delta x)^2 = (\Delta x)^2$$

If this process is to approximate the Brownian motion $W(t) \sim N(\mu t, \sigma^2 t)$, we must have that

$$(2p - 1)\Delta x = \mu \Delta t$$

$$(\Delta x)^2 = \sigma^2 \Delta t$$

From these two equations we obtain

$$\Delta x = \sigma\sqrt{\Delta t}$$

$$p = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right)$$

Let $T > 0$ be some fixed interval and let $\Delta t = T/n$, where $n = 1, 2, \ldots$. Assume that the random variables Y_k, $k = 1, \ldots, n$, independently take values $\sigma\sqrt{\Delta t}$ with probability $p = \{1 + (\mu\sqrt{\Delta t})/\sigma\}/2$ and $-\sigma\sqrt{\Delta t}$ with probability $1 - \{1 + (\mu\sqrt{\Delta t})/\sigma\}/2$. Define S_n by

$$S_n = \sum_{k=1}^{n} Y_k$$

Then from the central limit theorem, $S_n \to W(t)$ as $n \to \infty$. Thus, the Brownian motion can be regarded as a symmetric random walk defined over an infinitesimally small step size and infinitesimally small time intervals between walks.

9.8 The Ornstein-Uhlenbeck Process

The Brownian motion is used to construct the Ornstein-Uhlenbeck process, which has become a popular tool for modeling interest rates. Recall that the derivative of the Brownian motion $X(t)$ does not exist at any point in time. Thus, if $X(t)$ represents the position of a particle, we might be interested in obtaining its velocity, which is the derivative of the motion. The Ornstein-Uhlenbeck process is an alternative model of the Brownian motion that overcomes the preceding problem. It does this by considering the velocity $V(t)$ of a Brownian motion at time t. Over a small time interval, two factors affect the change in velocity: the frictional resistance of the surrounding medium whose effect is proportional to $V(t)$ and the random impact of neighboring particles whose effect can be represented by a standard Wiener process. Thus, because mass times velocity equals force, we have that

$$mdV(t) = -\gamma V(t)dt + dB(t)$$

where $\gamma > 0$ is called the *friction coefficient* and $m > 0$ is the mass. If we define $\alpha = \gamma/m$ and $\beta = 1/m$ we obtain the Ornstein-Uhlenbeck process with the following differential equation:

$$dV(t) = -\alpha V(t)dt + \beta dB(t)$$

The Ornstein-Uhlenbeck process is used to describe the velocity of a particle in a fluid and is encountered in statistical mechanics. It is the model of choice for random movement toward a concentration point. It is sometimes called a *continuous-time Gauss-Markov process*, where a Gauss-Markov process is a stochastic process that satisfies the requirements for both a Gaussian process and a Markov process. Because a Wiener process is both a Gaussian process and a Markov process, in addition to being a stationary independent increment process, it can be considered a Gauss-Markov process with independent increments.

The Ornstein-Uhlenbeck process can be obtained from the standard Brownian process $B(t)$ by scaling and time change, as follows:

$$V(t) = e^{-\alpha t} B\left(\frac{\beta^2}{2\alpha} e^{2\alpha t}\right)$$

Thus, $E[V(t)] = 0$, and because $B(t)$ is Gaussian, its covariance is

$$\begin{aligned}\text{Cov}\{V(t+\tau)V(t)\} &= E\left[e^{-\alpha(t+\tau)}B\left(\frac{\beta^2}{2\alpha}e^{2\alpha(t+\tau)}\right)e^{-t\alpha}B\left(\frac{\beta^2}{2\alpha}e^{2\alpha t}\right)\right] \\ &= e^{(-2\alpha t+\tau)}E\left[B\left(\frac{\beta^2}{2\alpha}e^{2\alpha(t+\tau)}\right)B\left(\frac{\beta^2}{2\alpha}e^{2\alpha t}\right)\right] \\ &= e^{-(2\alpha t+\tau)}\min\left\{\frac{\beta^2}{2\alpha}e^{2\alpha(t+\tau)}, \frac{\beta^2}{2\alpha}e^{2\alpha t}\right\} \\ &= \frac{\beta^2}{2\alpha}e^{-(2\alpha t+\tau)}e^{2\alpha t} = \frac{\beta^2}{2\alpha}e^{-\tau}\end{aligned}$$

Thus, $V(t)$ is a stationary process. However, unlike the Wiener process, it does not have independent increments. For a *standard Ornstein-Uhlenbeck process*, $\alpha = \beta = 1$ so that

$$V(t) = e^{-t}B\left(\frac{1}{2}e^{2t}\right)$$

To solve this equation

$$dV(t) = -\alpha V(t)dt + \beta db(t)$$

we consider the function $e^{\alpha t}V(t)$, and take the differential

$$d(e^{\alpha t}V(t)) = \alpha e^{\alpha t}V(t)dt + e^{\alpha t}dV(t)$$

Thus,

$$e^{\alpha t}dV(t) = d(e^{\alpha t}V(t)) - \alpha e^{\alpha t}V(t)dt = e^{\alpha t}[-\alpha V(t)dt + \beta dB(t)]$$

This means that

$$d(e^{\alpha t}V(t)) = e^{\alpha t}\beta dB(t)$$

Given that $V(0) = v_0$ we obtain the solution as

$$e^{\alpha t}V(t) = v_0 + \int_0^t e^{\alpha u}\beta dB(u)$$

which gives

$$V(t) = v_0e^{-\alpha t} + \int_0^t \beta e^{-\alpha(t-u)}dB(u) = v_0e^{-\alpha t} + e^{-\alpha t}\int_0^t \beta e^{\alpha u}dB(u)$$

Because $B(t)$ is a Brownian process, we have that the mean of $V(t)$ is

$$E[V(t)] = e^{-\alpha t} v_0 + E\left[\int_0^t e^{-\alpha(t-u)} \beta dB(u)\right] = e^{-\alpha t} v_0$$

Using the Ito isometry we obtain the variance of $V(t)$ as

$$\sigma^2_{V(t)} = E\left[\left\{\int_0^t e^{-\alpha(t-u)} \beta dB(u)\right\}^2\right] = \int_0^t \{\beta e^{-\alpha(t-u)}\}^2 du$$
$$= \beta^2 \int_0^2 e^{-2\alpha(t-u)} du = \frac{\beta^2}{2\alpha}(1 - e^{-2\alpha t})$$

Thus, we have that

$$V(t) \sim N\left(v_0 e^{-\alpha t}, \frac{\beta^2}{2\alpha}\{1 - e^{-2\alpha t}\}\right)$$

From this result we observe that as $t \to \infty$, the influence of the initial value decays exponentially and $\sigma^2_{V(t)} \to \beta^2/2\alpha$. Thus, as $t \to \infty$, $V(t)$ converges exponentially to a Gaussian distribution with mean zero and variance $\beta^2/2\alpha$; that is,

$$\lim_{t\to\infty} V(t) \sim N(0, \beta^2/2\alpha)$$

Because the Ornstein-Uhlenbeck process is a Gaussian process that is completely characterized by the mean and the variance, an alternative method of analyzing the process is that due to Gillespie (1996), which is obtained by rewriting the stochastic differential equation for the process as follows:

$$dV(t) = V(t + dt) - V(t) = -\alpha V(t)dt + \beta dB(t)$$

Thus, taking expectations and remembering that $B(t)$ is a zero-mean process we obtain

$$E[V(t + dt)] - E[V(t)] = -\alpha E[V(t)]dt \Rightarrow \lim_{dt\to 0} \frac{E[V(t + dt)] - E[V(t)]}{dt}$$
$$= -\alpha E[V(t)]$$

The solution to the second equation is

$$E[V(t)] = v_0 e^{-\alpha t} \quad t \geq 0$$

Also, because $V(t+dt) = V(t) - \alpha V(t)dt + \beta dB(t)$, taking the square of both sides gives

$$V^2(t+dt) = V^2(t) + \alpha^2 V^2(t)(dt)^2 + \beta^2\{dB(t)\}^2 - 2\alpha V^2(t)dt + 2\beta V(t)dB(t) - 2\alpha\beta V(t)dB(t)$$

Taking expectations of both sides and recalling that $E[\{dB(t)\}^2] = \sigma^2_{dB(t)} = dt$ we obtain

$$E[V^2(t+dt)] = E[V^2(t)] + \alpha^2 E[V^2(t)](dt)^2 + \beta^2 dt - 2\alpha E[V^2(t)]dt$$

where we have assumed that $V(t)$ is statistically independent of $B(u)t \leq u$ so that $E[V(t)dB(t)] = E[V(t)]E[dB(t)] = 0$. Thus, taking the limits as $dt \to 0$ we obtain

$$E[V^2(t+dt)] - E[V^2(t)] = \alpha^2 E[V^2(t)](dt)^2 + \beta^2 dt - 2\alpha E[V^2(t)]dt$$

$$\lim_{dt\to 0} \frac{E[V^2(t+dt)] - E[V^2(t)]}{dt} = \beta^2 - 2\alpha E[V^2(t)]$$

If we assume that $V^2(0) = v_0^2$, then the solution to the second equation is given by

$$E[V^2(t)] = v_0^2 e^{-2\alpha t} + \frac{\beta^2}{2\alpha}\{1 - e^{-2\alpha t}\}$$

Thus, using the previous result for $E[V(t)]$, the variance of $V(t)$ is given by

$$\text{Var}\{V(t)\} = E[V^2(t)] - \{E[V(t)]\}^2 = \frac{\beta^2}{2\alpha}\{1 - e^{-2\alpha t}\}$$

This means that

$$V(t) \sim N\left(v_0 e^{-\alpha t}, \frac{\beta^2}{2\alpha}\{1 - e^{-2\alpha t}\}\right)$$

as we obtained earlier.

Note that the above Ornstein-Uhlenbeck process is referred to as the Brownian motion-driven Ornstein-Uhlenbeck process. Other types of the process are the Poisson-driven and gamma-driven Ornstein-Uhlenbeck processes.

9.8.1 Mean Reverting Ornstein-Uhlenbeck Process

A mean reverting Ornstein-Uhlenbeck process $X(t)$ is the solution to the following stochastic differential equation

$$dX(t) = \alpha\{\mu - X(t)\}dt + \beta\, dB(t)$$

where $B(t)$ is the standard Brownian process, m is the long-run mean of $X(t)$, and α is usually called the *rate of mean reversion*. Mean reversion refers to the fact that the process does not wander off to infinity. Instead it always tries to come back to a well-defined asymptotic mean value. For this reason, it is sometimes used to model processes such as prices, interest rates, and volatilities that tend to return to a mean or average value after reaching extremes.

As in previous sections, we solve this equation by considering $e^{\alpha t}X(t)$ and taking the differential

$$\begin{aligned} d(e^{\alpha t}X(t)) &= \alpha e^{\alpha t}X(t)dt + e^{\alpha t}dX(t) \\ &= \alpha e^{\alpha t}X(t)dt + e^{\alpha t}[\alpha\{\mu - X(t)\}dt + \beta dB(t)] \end{aligned}$$

This means that

$$d(e^{\alpha t}X(t)) = \alpha\mu e^{\alpha t}dt + e^{\alpha t}\beta dB(t)$$

Given that $X(0) = x_0$ we obtain the solution as

$$e^{\alpha t}X(t) = x_0 + \int_0^t \alpha\mu e^{\alpha u}du + \int_0^t e^{\alpha u}\beta dB(u)$$

which gives

$$\begin{aligned} X(t) &= x_0 e^{-\alpha t} + \int_0^t \alpha\mu e^{-\alpha(t-u)}du + \int_0^t e^{-\alpha(t-u)}\beta dB(u) \\ &= x_0 e^{-\alpha t} + \mu(1 - e^{-\alpha t}) + \int_0^t e^{-\alpha(t-u)}\beta dB(u) \\ &= \mu + e^{-\alpha t}(x_0 - \mu) + \int_0^t e^{-\alpha(t-u)}\beta dB(u) \end{aligned}$$

Because $B(t)$ is a Brownian motion, we have that the mean of $X(t)$ is

$$E[X(t)] = \mu + e^{-\alpha t}(x_0 - \mu) + E\left[\int_0^t e^{-\alpha(t-u)}\beta dB(u)\right] = \mu + e^{-\alpha t}(x_0 - \mu)$$

Using the Ito isometry we obtain the variance of $X(t)$ as

$$\sigma^2_{X(t)} = E\left[\left\{\int_0^t e^{-\alpha(t-u)}\beta dB(u)\right\}^2\right] = \int_0^t \{\beta e^{-\alpha(t-u)}\}^2 du$$

$$= \beta^2 \int_0^t e^{-2\alpha(t-u)} du = \frac{\beta^2}{2\alpha}(1 - e^{-2\alpha t})$$

Thus, we have that

$$X(t) \sim N\left(\mu + e^{-\alpha t}(x_0 - \mu), \frac{\beta^2}{2\alpha}(1 - e^{-2\alpha t})\right)$$

and

$$\lim_{t\to\infty} X(t) \sim N(\mu, \beta^2/2\alpha)$$

The difference between this process and the standard Ornstein-Uhlenbeck process is that as $t \to \infty$ the mean of the mean-reverting scheme is nonzero while that of the traditional scheme is zero.

9.8.2 Applications of the Ornstein-Uhlenbeck Process

The Ornstein-Uhlenbeck process has many applications in statistics, operations research, engineering, and physics. Daniels (1969) used the process to study the statistics of the minimum of stationary Gaussian processes superimposed on "U-shaped trends," as a model of, for example, winter temperatures. McNeil and Schach (1973) have shown that several birth-and-death type models can be approximated by Ornstein-Uhlenbeck processes as certain parameters become large.

In neurophysiology, one is interested in the excursions of the transmembrane potential subject to random additive inputs and exponential decay. The most readily observed variable is the time between firings. Johannesma (1968) and Feinberg (1970) used the first passage time of an Ornstein-Uhlenbeck process as a simple diffusion approximation to the neuron model. Di Nardo (2003) also discusses using the Gauss-Markov process to model neuronal firing.

The Ornstein-Uhlenbeck process is widely used in modeling short-term interest rates. In Brody (2002), Benth (2003), and Elliott (2003), the Ornstein-Uhlenbeck process is used to model weather derivatives.

The Ornstein-Uhlenbeck process is also used in communication and signal processing system modeling. In Tang and Tracy (1998), the process is used to model data compression and information retrieval systems. In Medard (2000) and Zhang (2006), the discrete-time Gauss-Markov model is used to model the correlation in communication over fading channels. Liu and Ulukus (2006) used the process to study optimal distortion and power trade-offs in sensor networks. Rasmussen and Williams (2006) describes application of the process in machine learning.

9.9 Diffusion Processes

Many random processes seem to move continuously between their possible states, which generally lie in some interval subset of the real line. One example is the Brownian motion. Another example is the diffusion process that is the subject of this section.

Consider a continuous-time continuous-state Markov process $\{X(t), t \geq 0\}$ whose transition probability distribution is given by

$$F(t, y|s, x) = P[X(t) \leq y|X(s) = x] \quad s < t$$

If the derivative

$$f(t, y|s, x) = \frac{\partial}{\partial y} F(t, y|s, x)$$

exists, then it is called the *transition density function* of the diffusion process. Since $\{X(t)\}$ is a Markov process, $f(t, y|s, x)$ satisfies the Chapman-Kolmogorov equation:

$$f(t, y|s, x) = \int_{-\infty}^{\infty} f(t, y|u, z) f(u, z|s, x) dz$$

The process also satisfies the following two Kolmogorov equations:

a. Forward diffusion equation: For a fixed initial state (s, x),

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial y}\{a(t, y) f\} + \frac{1}{2}\frac{\partial^2}{\partial y^2}\{b(t, y) f\}$$

where $f \equiv f(t, y|s, x)$. The equation gives the forward evolution of the process with respect to the final state (t, y) and is commonly called the *Fokker-Planck equation*.

b. Backward diffusion equation: For a fixed final state (t, y).

$$\frac{\partial f}{\partial s} = -a(s,x)\frac{\partial f}{\partial x} - \frac{1}{2}b(s,x)\frac{\partial^2 f}{\partial x^2}$$

The equation gives the backward evolution of the process with respect to the initial state (s, x).

We assume that the process satisfies the following conditions:

1. $P(|X(t+\Delta t) - X(t)| > \varepsilon | X(t)) = o(\Delta t)$, for $\varepsilon > 0$, which states that the sample path is continuous; alternatively we say that the process is continuous in probability.
2. $E[X(t+\Delta t) - X(t)|X(t) = x] = a(t,x)\Delta t + o(\Delta t)$ so that

$$\lim_{\Delta t \to 0} \frac{E[X(t+\Delta t) - X(t)|X(t) = x]}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{-\infty}^{\infty} (y - x) f(t + \Delta t, y | t, x) dy = a(t,x)$$

3. $E[\{X(t+\Delta t) - X(t)\}^2 | X(t) = x] = b(t,x)\Delta t + o(\Delta t)$ is finite so that

$$\lim_{\Delta t \to 0} \frac{E[\{X(t+\Delta t) - X(t)\}^2 | X(t) = x]}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{-\infty}^{\infty} (y - x)^2 f(t + \Delta t, y | t, x) dy = b(t,x)$$

A Markov process that satisfies these three conditions is called a diffusion process. The function $a(t, x)$ is called the *instantaneous* (or infinitesimal) *mean* (or drift) of $X(t)$, and the function $b(t, x)$ is called the *instantaneous* (or infinitesimal) *variance* of $X(t)$. Let the small increment in $X(t)$ over any small interval dt be denoted by $dX(t)$. Then it can be shown that if $B(t)$ is a standard Brownian motion, we can incorporate the above properties into the following stochastic differential equation

$$dX(t) = a(t,x)dt + [b(t,x)]^{1/2} dB(t)$$

where $dB(t)$ is the increment of $B(t)$ over the small interval $(t, t + dt)$.

Diffusion processes are used to model the price movements of financial instruments. The Black-Scholes model for pricing options assumes that the underlying instrument follows a traditional diffusion process with small, continuous, random movements.

9.10 Examples of Diffusion Processes

The following are some examples of the diffusion process. These processes differ only in their values of the infinitesimal mean and infinitesimal variance.

9.10.1 Brownian Motion

The Brownian motion is a diffusion process on the interval $(-\infty, \infty)$ with zero mean and constant variance. That is, for the standard Brownian motion, $a(t, x) = 0$ and $b(t, x) = \sigma^2$, for some $\sigma^2 > 0$. The forward diffusion equation becomes

$$\frac{\partial f}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial y^2}$$

This equation is an example of a heat equation, which describes the variation in temperature as heat diffuses through an isotropic and homogeneous physical medium in one-dimensional space. Similarly, the backward diffusion equation becomes

$$\frac{\partial f}{\partial s} = -\frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2}$$

A Solution of the Forward Diffusion Equation

If we define $\lambda^2 = \sigma^2/2$, the forward diffusion equation becomes

$$\frac{\partial}{\partial t} f(t, y) = \lambda^2 \frac{\partial^2}{\partial y^2} f(t, y) \quad 0 \le y \le L, \quad 0 \le t \le T$$

where

$$f(t, y) = \frac{\partial}{\partial y} P[X(t) \le y]$$

Assume that the initial condition is $f(0, y) = \phi(y), 0 \le y \le L$, and the boundary conditions are $f(t, 0) = f(t, L) = 0,\ 0 \le t \le T$. If we assume that $f(t, y)$ is a separable function of t and y, then we can write

$$f(t, y) = g(t)h(y)$$

with the boundary conditions $h(0) = h(L) = 0$. Thus, the differential equation becomes

$$\frac{dg(t)}{dt} h(y) = \lambda^2 g(t) \frac{d^2 h(y)}{dy^2}$$

which gives

$$\frac{1}{\lambda^2} \frac{dg(t)}{dt} \frac{1}{g(t)} = \frac{d^2 h(y)}{dy^2} \frac{1}{h(y)}$$

Because the left side of the equation is a function of t alone and the right side is a function of y alone, the equality can be satisfied only if the two sides are equal to some constant m; that is,

$$\frac{dg(t)}{dt} = m\lambda^2 g(t) \qquad 0 \le t \le T$$
$$\frac{d^2h(y)}{dy^2} = mh(y) \quad 0 \le y \le L$$

Using the method of characteristic equation, the solutions to these equations are given by

$$g(t) = C_0 e^{m\lambda^2 t}$$
$$h(y) = C_1 e^{y\sqrt{m}} + C_2 e^{-y\sqrt{m}}$$

where C_0, C_1, and C_2 are constants. To avoid a trivial solution obtained when $m = 0$, and to obtain a solution that does not increase exponentially with t, we require that $m < 0$. Thus, if we define $m = -p^2$, the solutions become

$$g(t) = C_0 e^{-\lambda^2 p^2 t}$$
$$h(y) = C_1 e^{jpy} + C_2 e^{-jpy} = B_1 \sin(py) + B_2 \cos(py)$$
$$f(t, y) = C_0 e^{-\lambda^2 p^2 t}\{B_1 \sin(py) + B_2 \cos(py)\}$$

From the boundary condition $f(t, 0) = 0$, we obtain $B_2 = 0$. Similarly, from the boundary condition $f(t, L) = 0$, we obtain $B_1 \sin(pL) = 0$, which gives $pL = k\pi$, $k = 1, 2, \ldots$. Thus,

$$p = \frac{k\pi}{L}, k = 1, 2, \ldots$$

Thus, we can define the following functions:

$$g_k(t) = C_{0k} \exp\left\{-\frac{\lambda^2 k^2 \pi^2 t}{L^2}\right\}$$
$$h_k(y) = B_{1k} \sin\left(\frac{k\pi y}{L}\right)$$
$$f_k(t, y) = C_k \exp\left\{-\frac{\lambda^2 k^2 \pi^2 t}{L^2}\right\} \sin\left(\frac{k\pi y}{L}\right)$$
$$f(t, y) = \sum_{k=1}^{\infty} f_k(t, y) = \sum_{k=1}^{\infty} C_k \exp\left\{-\frac{\lambda^2 k^2 \pi^2 t}{L^2}\right\} \sin\left(\frac{k\pi y}{L}\right)$$

where $C_k = C_{0k} \times B_{1k}$. From the initial condition, we obtain

$$f(0, y) = \phi(y) = \sum_{k=1}^{\infty} C_k \sin\left(\frac{k\pi y}{L}\right)$$

Because

$$\int_0^L \sin\left(\frac{m\pi v}{L}\right) \sin\left(\frac{k\pi v}{L}\right) dv = \begin{cases} 0 & \text{if } k \neq m \\ \dfrac{L}{2} & \text{if } k = m \end{cases}$$

that is, the functions $\{\sin(k\pi v/L)\}$ are orthogonal on the interval $0 \leq v \leq L$, we obtain

$$C_k = \frac{2}{L} \int_0^L \phi(v) \sin\left(\frac{k\pi v}{L}\right) dv$$

This means that

$$f(t, y) = \frac{2}{L} \sum_{k=1}^{\infty} \left\{ \int_0^L \phi(v) \sin\left(\frac{k\pi v}{L}\right) dv \right\} \exp\left\{ -\frac{\lambda^2 k^2 \pi^2 t}{L^2} \right\} \sin\left(\frac{k\pi y}{L}\right)$$

9.10.2 Brownian Motion with Drift

In this process, $a(t, x) = \mu$ and $b(t, x) = \sigma^2$, where μ is the drift rate. The forward diffusion equation becomes

$$\frac{\partial f}{\partial t} = -\mu \frac{\partial f}{\partial y} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial y^2}$$

Similarly, the backward diffusion equation becomes

$$\frac{\partial f}{\partial s} = -\mu \frac{\partial f}{\partial x} - \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2}$$

9.10.3 Levy Processes

Both the Poisson process and the Brownian motion have stationary and independent increments. However, they have different sample paths: the Brownian motion has continuous sample paths while the Poisson process has discontinuities (or jumps) of size 1. At a high level, Levy processes are stochastic processes with both stationary and independent increments. They constitute a wide class of stochastic processes whose sample paths can be continuous, continuous with occasional discontinuities, and purely discontinuous. The Brownian motion with drift, the Poisson process, and the compound Poisson process are examples of Levy processes. The Brownian motion with drift is the only continuous Levy process. Levy processes are widely used in the field of quantitative finance to model asset prices, and in physics, as reported in Barndorff-Nielsen (2001).

More formally, a stochastic process $\{X(t), t \geq 0\}$ is a Levy process if the following conditions are satisfied:

a. $X(0) = 0$.
b. $X(t)$ has independent increments.
c. $X(t)$ has stationary increments.
d. $X(t)$ is continuous in probability; that is, for all $t \geq 0$ and $\varepsilon > 0$,

$$\lim_{u \to t} P[|X(t) - X(u)| > \varepsilon] = 0$$

Infinite Divisibility

An interesting property of the Levy process is the *infinite divisibility* property. A random variable Y is said to be infinitely divisible if for every integer $n \geq 2$ there are n independent random variables $Y_{1,n}, \ldots, Y_{n,n}$ such that the sum $Y_n = Y_{1,n} + \cdots + Y_{n,n}$ has the same distribution as Y. Because the cumulative distribution function of the sum of n independent random variables is the n-way convolution of their CDFs, we have that

$$F_Y(y) = F_{Y_n}(y) = F_{Y_{1,n}}(y) * \ldots * F_{Y_{n,n}}(y)$$

In terms of the characteristic function, because the $Y_{k,n}$ are also identically distributed, we have that

$$\Phi_Y(w) = [\Phi_{Y_{1,n}}(w)]^n$$

$$\Phi_{Y_{1,n}}(w) = [\Phi_Y(w)]^{1/n}$$

Because $X(0) = 0$, we can write

$$\begin{aligned} X(t) &= \left\{ X\left(\frac{t}{n}\right) - X\left(\frac{0t}{n}\right) \right\} + \left\{ X\left(\frac{2t}{n}\right) - X\left(\frac{t}{n}\right) \right\} + \cdots \\ &\quad + \left\{ X\left(\frac{(n-1)t}{n}\right) - X\left(\frac{(n-2)t}{n}\right) \right\} + \left\{ X\left(\frac{nt}{n}\right) - X\left(\frac{(n-1)t}{n}\right) \right\} \\ &= \sum_{k=1}^{n} \left\{ X\left(\frac{kt}{n}\right) - X\left(\frac{(k-1)t}{n}\right) \right\} \end{aligned}$$

Thus, $X(t)$ is the sum of n independent random variables, all of which have the same distribution as $X(t/n)$. Because the condition is true for all $n \geq 1$, we conclude that $X(t)$ has an infinitely divisible distribution.

Infinite Divisibility of the Poisson Process

As stated earlier, the Poisson process is a Levy process. The characteristic function of the Poisson random variable $X(t)$ with mean λt is given by

$$\begin{aligned} \Phi_{X(t)}(w) &= E[e^{jwX(t)}] = \sum_{k=-\infty}^{\infty} e^{jwk} p_{X(t)}(k) = \sum_{k=0}^{\infty} e^{jwk} \left\{ \frac{e^{-\lambda t}(\lambda t)^k}{k!} \right\} \\ &= e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t e^{jw})^k}{k!} = e^{-\lambda t(1-e^{jw})} = e^{-\frac{\lambda}{n} t(1-e^{jw})n} = \left[e^{-\frac{\lambda}{n} t(1-e^{jw})} \right]^n \\ &= \left[\Phi_{Y(t)}(w) \right]^n \end{aligned}$$

where $Y(t)$ is a Poisson random variable with rate λ/n. Thus, the Poisson random variable has an infinitely divisible distribution.

Infinite Divisibility of the Compound Poisson Process

The compound Poisson process $X(t)$ is another example of a Levy process. Let $\Phi_Y(w)$ denote the characteristic function of the jump size density. It can be shown, using the random sum of the random variable method used in Ibe (2005), that the characteristic function of the compound Poisson process is given by

$$\begin{aligned} \Phi_{X(t)}(w) &= e^{-\lambda t(1-\Phi_Y(w))} = e^{-\frac{\lambda}{n} t(1-\Phi_Y(w))n} = \left[e^{-\frac{\lambda}{n} t(1-\Phi_Y(w))} \right]^n \\ &= [\Phi_{V(t)}(w)]^n \end{aligned}$$

where $V(t)$ is a Poisson random variable with rate λ/n. Thus, the compound Poisson random process has an infinite divisibility property.

Infinite Divisibility of the Brownian Motion with Drift

The Brownian motion with drift $X(t)$ is a Levy process. We know that

$$X(t) = \mu t + \sigma B(t) \sim N(\mu t, \sigma^2 t)$$

Thus, its characteristic function is given by

$$\begin{aligned}\Phi_{X(t)}(w) = E[e^{jwX(t)}] &= \int_{-\infty}^{\infty} e^{jwx} f_x(x)dx = \int_{-\infty}^{\infty} e^{jwx} \left\{ \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-\mu)^2}{2\sigma^2 t}} \right\} dx \\ &= e^{\left(j\mu tw - \frac{1}{2}\sigma^2 tw^2\right)} = e^{\left(\frac{j\mu tw}{n} - \frac{1}{2}\frac{\sigma^2 tw^2}{n}\right)n} = \left[e^{\left(\frac{j\mu tw}{n} - \frac{1}{2}\frac{\sigma^2 tw^2}{n}\right)} \right]^n \\ &= [\Phi_{U(t)}(w)]^n\end{aligned}$$

where $U(t) = \dfrac{\mu t}{n} + \dfrac{\sigma}{\sqrt{n}} W(t) \sim N\left(\dfrac{\mu t}{n}, \dfrac{\sigma^2 t}{n}\right)$.

9.11 Relationship Between the Diffusion Process and Random Walk

Let $P[x, t]$ denote the probability that at time t a particle undergoing a random walk is located at point x. Assume that the time step is Δt and the step size of the walk is Δx. Then

$$P[x, t + \Delta t] = pP[x - \Delta x, t] + qP[x + \Delta x, t]$$

Using Taylor series expansions we obtain

$$P[x, t + \Delta t] = P[x, t] + \Delta t \frac{\partial}{\partial t} P[x, t] + O([\Delta t]^2)$$

$$P[x \pm \Delta x, t] = P[x, t] \pm \Delta x \frac{\partial}{\partial x} P[x, t] + \frac{[\Delta x]^2}{2} \frac{\partial^2}{\partial x^2} P[x, t] + O([\Delta x]^3)$$

Substituting in the previous equation and omitting the negligible terms we obtain

$$\begin{aligned}P[x, t] + \Delta t \frac{\partial}{\partial t} P[x, t] = p &\left\{ P[x, t] - \Delta x \frac{\partial}{\partial x} P[x, t] + \frac{[\Delta x]^2}{2} \frac{\partial}{\partial x^2} P[x, t] \right\} \\ &+ q \left\{ P[x, t] + \Delta x \frac{\partial}{\partial x} P[x, t] + \frac{[\Delta x]^2}{2} \frac{\partial^2}{\partial x^2} P[x, t] \right\}\end{aligned}$$

From this we have that

$$\frac{\partial}{\partial t}P[x,t] = -(p-q)\frac{\Delta x}{\Delta t}\frac{\partial}{\partial x}P[x,t] + \frac{[\Delta x]^2}{2\Delta t}\frac{\partial^2}{\partial x^2}P[x,t]$$

Let Δx and Δx approach zero in such a way that

$$\frac{\Delta x}{\Delta t} = k$$
$$\frac{[\Delta x]^2}{2\Delta t} = D$$

Then we have that

$$\frac{\partial}{\partial t}P[x,t] = -(p-q)k\frac{\partial}{\partial x}P[x,t) + D\frac{\partial^2}{\partial x^2}P[x,t]$$

In the case of a symmetric random walk in which $p = q = 1/2$, we obtain

$$\frac{\partial}{\partial t}P[x,t] = D\frac{\partial^2}{\partial x^2}P[x,t]$$

which is the one-dimensional diffusion equation, and D is called the *diffusion constant*. For a nonsymmetric walk we obtain

$$\frac{\partial}{\partial t}P[x,t] = -K\frac{\partial}{\partial x}P[x,t] + D\frac{\partial^2}{\partial x^2}P[x,t]$$

which is a Fokker-Planck equation that describes the drift and diffusion of the density function. Here $K = k(p-q)$.

9.12 Problems

9.1 Assume that X and Y are independent random variables such that $X \sim N(0,\sigma^2)$ and $Y \sim N(0,\sigma^2)$. Consider the random variables $U = (X+Y)/2$ and $V = (X-Y)/2$. Show that U and V are independent with $U \sim N(0,\sigma^2/2)$ and $V \sim N(0,\sigma^2/2)$.

9.2 Suppose $X(t)$ is a standard Brownian motion and $Y(t) = tX(1/t)$. Show that $Y(t)$ is a standard Brownian motion.

9.3 Let $\{X(t), t \geq 0\}$ be a Brownian motion with drift rate μ and variance parameter σ^2. What is the conditional distribution of $X(t)$ given that $X(u) = b, u < t$?

9.4 Let $T = \min\{t{:}B(t) = 5 - 3t\}$. Use the martingale stopping theorem to find $E[T]$.

9.5 Let $Y(t) = \int_0^t B(u)du$; where $\{B(t), t \geq 0\}$ is the standard Brownian motion. Find

a. $E[Y(t)]$
b. $E[Y^2(t)]$
c. The conditional distribution of $Y(t)$, given that $B(t) = x$

9.6 Consider the Brownian motion with drift

$$Y(t) = \mu t + \sigma B(t) + x$$

where $Y(0) = x$ and $b < x < a$. Let $P_a(x)$ denote the probability that $Y(t)$ hits a before b. Show that

$$\frac{1}{2}\frac{d^2 p_a(x)}{dx^2} + \mu \frac{dp_a(x)}{dx} = 0$$

and deduce that $p_a(x) = \dfrac{e^{-2\mu b} - e^{-2\mu x}}{e^{-2\mu b} - e^{-2\mu a}}$. What is $p_a(x)$ when $\mu = 0$?

9.7 What is the mean value of the first passage time of the reflected Brownian motion $\{|B(t)|, t \geq 0\}$ with respect to a positive level x where $B(t)$ is the standard Brownian motion? Determine the CDF of $|B(t)|$.

9.8 Let the process $\{X(t), t \geq 0\}$ be defined by $X(t) = B^2(t) - t$, where $\{B(t), t \geq 0\}$ is a standard Brownian motion.

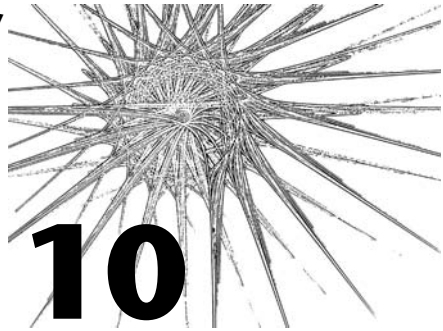
a. What is $E[X(t)]$?
b. Show that $\{X(t), t \geq 0\}$ is a martingale.

Hint: Start by computing $E[X(t)|B(v),\ 0 \leq v \leq s]$.

9.9 Let $\{B(t), t \geq 0\}$ be a standard Brownian motion, and define the process $Y(t) = e^{-t}B(e^{2t}), t \geq 0$; that is, $\{Y(t), t \geq 0\}$ is the Ornstein-Uhlenbeck process.

a. Show that $Y(t)$ is a Gaussian process.
b. Find the covariance function of the process.

Controlled Markov Processes



10

10.1 Introduction

Controlled Markov processes are a class of processes that deal with decision making under uncertainty. These processes, which include the Markov decision process (MDP), the semi-Markov decision process (SMDP), and the partially observable Markov decision process (POMDP), can be viewed as mathematical models that are concerned with optimal strategies of a decision maker who must make a sequence of decisions over time with uncertain outcomes. These three decision processes are the subject of this chapter.

10.2 Markov Decision Processes

In MDP a decision maker or *agent* can influence the state of the system by taking a sequence of *actions* that causes the system to optimize a predefined performance criterion. To do this the agent observes the state of the system at specified points in time called *decision epochs* and gathers information necessary to choose actions that the agent expects will enable the desired performance criterion to be met. Each action that the agent takes incurs a cost or a reward, and the action affects the system state thereby affecting future actions. Thus, by applying a chosen action to the system the agent incurs an immediate cost and the system changes to a new state according to a transition probability distribution. In general, the immediate cost and transition probability distribution depend on the state and the chosen action.

If we denote the set of decision epochs by T, then the decision process can be classified as a discrete-time decision process or a continuous-time decision process, depending on whether T is discrete or continuous. In a discrete-time decision process, decisions are only made at the decision epochs. Similarly, in a continuous-time decision process, decision can be made continuously or at random points when certain predefined events occur. In discrete-time decision processes, the set of decision epochs, T, can be finite or infinite. When T is finite, we have that $T = \{1, 2, \ldots, N\}$, where $N < \infty$ and the elements of T are the decision epochs that are denoted by $t \in T$. When T is infinite, we have that $T = \{1, 2, \ldots\}$, which means that decisions will be made indefinitely. When N is finite, the decision process is called a *finite-horizon* (or finite-stage) decision process; otherwise it is called an *infinite-horizon* (or infinite-stage) decision process.

The outcome of each decision is not fully predictable but can be anticipated to some extent before the next decision is made through the transition probability distribution. Also, as discussed earlier, the actions applied to the system have a long-term consequence because decisions made at the current decision epoch have an impact on decisions at the next decision epoch, and so on. Therefore, decisions cannot be viewed in isolation. Consequently, it is necessary to balance the desire for a low present cost against the undesirability of high future costs. Thus, good decision rules are needed to specify the actions that should be taken at any given decision epoch and state. A rule for making decisions at each decision epoch is called a *policy*. A policy used at decision epoch t could use the history of the system up to t (that is, the system's sequence of observed states and sequence of actions). However, in practice policies depend only on the observed state of the system at the decision epoch t. Thus, we can view a policy as a sequence of decision rules that prescribes the action to be taken at all decision epochs.

We denote a policy by $D = (d_1, d_2, \ldots, d_{N-1})$, where d_t is the action to be taken at the decision epoch $t \in T$. Policies can be classified as *stationary* or *nonstationary*. A stationary policy is one in which the same action a_i is taken whenever the system is in a given state i. For example, consider a decision process where the states of the process are the outcomes of a flip of a coin. If the policy requires the agent to bet \$2 whenever the outcome is a head and \$1 whenever the outcome is a tail, then it is a stationary policy. A nonstationary policy is one in which different actions can be taken when the system is in a given state. The action taken might depend on the decision epoch. For example, for a finite-horizon process, we can take one action at the beginning of the horizon when the process is in state k and a different action toward the end of the horizon when the system is in state k again.

Markov decision processes have been applied to a wide range of stochastic control problems, such as inspection-maintenance-replacement systems, inventory management, and economic planning. The topic is covered in several books including Bertsekas (1976, 1995a, 1995b), Borovkov (2003), Heyman and Sobel (1984), Howard (1960, 1971b), Kumar and Varaiya (1986), Puterman (1994),

Ross (1970, 1983), and Tijms (1995). Markov decision processes have been applied in modeling communication networks in Towsley (2000), finance and dynamic options in Schal (2001), water reservoir in Lamond (2001), and medical treatment in Schaefer (2004). We begin by presenting an overview of dynamic programming.

10.2.1 Overview of Dynamic Programming

Dynamic programming (DP) is a mathematical technique that is used for optimizing multistage decision problems. A multistage decision problem is a problem that can be separated into a number of stages (or steps), where each stage involves the optimization of exactly one variable. The computations at different stages are linked via a recursive algorithm that ensures that a feasible optimal solution to the entire problem is obtained when the last stage is reached. The optimization of each stage is based on a *decision* (which we defined earlier as an *action* taken), and a sequence of decisions is called a *policy*, as discussed earlier. Each stage has a number of states associated with it, where a state is any possible condition in which the system associated with the multistage problem can be in that stage. The number of states can be finite or infinite, and the effect of a decision at each stage is to transform the current state into a new state associated with the next stage.

A multistage decision problem usually has certain returns associated with each decision made; these returns can be costs or benefits. The objective of the solution to the problem is to determine the *optimal policy*, which is the policy that provides the best return. Dynamic programming is based on the Bellman's *principle of optimality* in Bellman (1957), which states as follows:

Principle of Optimality An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

This principle implies that given the current state, an optimal policy for the remaining stages is independent of the policy used in the previous stages. In other words, knowledge of the current state of the system embodies all the information about the past behavior that is necessary for determining the optimal policy from here on. This is essentially a Markovian property, as discussed in earlier chapters.

The implementation of this principle starts with finding the optimal policy for each state of the last stage. Then it moves backward stage by stage such that at each stage it determines the best policy for leaving each state of the stage using the results obtained in the previous stages.

Example of Dynamic Programming Problem

Consider a situation where we are given X units of a resource to be allocated to N activities. For example, we might want $\$X$ to be used to fund N projects. Suppose

we are also given a table that lists the return $r_i(x)$ to be realized from allocating x units of the resource to activity i, where $i = 1, \ldots, N$ and $x = 0, 1, \ldots, X$. Then the problem we are confronted with becomes the following:

$$\text{Maximize} \quad \sum_{i=1}^{N} r_i(x_i)$$

$$\text{Subject to} \quad \sum_{i=1}^{N} x_i = X$$

$$0 \le x_i \le X \quad i = 1, \ldots, N$$

To see this problem from the point of view of dynamic programming, we consider N stages labeled $1, 2, \ldots, N$ where each stage represents an activity, and we first allocate x_1 units of the total resource to activity 1 at stage 1, then x_2 units of the remaining $X - x_1$ resource to activity 2 at stage 2, and so on. To be able to optimize the remaining process, we must know the sum of units that have been allocated so far at each stage and the quantity left. We define the optimal value function $v_k(x)$ as the maximum return obtained from activities k through N, given that x units of the resource remain to be allocated. From the principle of optimality we obtain the following recurrence relation

$$v_k(x) = \max[r_k(x_k) + v_{k+1}(x - x_k)] \qquad x_k = 0, 1, \ldots, x$$

where x_k is the allocation to activity k and $x = 0, 1, \ldots, X$. The boundary condition is

$$v_N(x) = r_N(x)$$

and the solution to the problem is $v_1(x)$.

Example 10.1 For a numerical example, we consider $X = 6$ and $N = 3$. Table 10.1 shows the values of the returns $r_i(x)$. ■

Solution: Because there are $N = 3$ activities, we define 3 stages with the following boundary conditions: $v_3(0) = 0$, $v_3(1) = 1$, $v_3(2) = 3$, $v_3(3) = 5$, $v_3(4) = 8$, $v_3(5) = 12$, and $v_3(6) = 13$. These are essentially the values of $r_3(x_3)$. Let $m_k(x)$ denote the value of x_k that maximizes the right-hand side of the recurrence relation. Then using the recurrence relation we obtain the following results:

$$v_2(0) = 0 \quad m_2(0) = 0$$

$$v_2(1) = \max[r_2(x_k) + v_3(1 - x_k)] = \max[r_2(0) + v_3(1), r_2(1) + v_3(0)]$$

$$= \max[0 + 1, 2 + 0] = 2 \quad m_2(1) = 1$$

Table 10.1 Data for the numerical example

x_1	$r_1(x_1)$	x_2	$r_2(x_2)$	x_3	$r_3(x_3)$
0	0	0	0	0	0
1	3	1	2	1	1
2	6	2	4	2	3
3	9	3	6	3	5
4	12	4	9	4	8
5	16	5	11	5	12
6	16	6	13	6	13

$$
\begin{aligned}
v_2(2) &= \max[r_2(0) + v_3(2), r_2(1) + v_3(1), r_2(2) + v_3(0)] \\
&= \max[0 + 3, 2 + 1, 4 + 0] = 4 \quad m_2(2) = 2 \\
v_2(3) &= \max[r_2(0) + v_3(3), r_2(1) + v_3(2), r_2(2) + v_3(1), r_2(3) + v_3(0)] \\
&= \max[0 + 5, 2 + 3, 4 + 1, 6 + 0] = 6 \quad m_2(3) = 3 \\
v_2(4) &= \max[r_2(0) + v_3(4), r_2(1) + v_3(3), r_2(2) + v_3(2), r_2(3) \\
&\quad + v_3(1), r_2(4) + v_3(0)] \\
&= \max[0 + 8, 2 + 5, 4 + 3, 6 + 1, 9 + 0] = 9 \quad m_2(4) = 4 \\
v_2(5) &= \max[r_2(0) + v_3(5), r_2(1) + v_3(4), r_2(2) + v_3(3), r_2(3) + v_3(2), r_2(4) \\
&\quad + v_3(1), r_2(5) + v_3(0)] \\
&= \max[0 + 12, 2 + 8, 4 + 5, 6 + 3, 9 + 1, 11 + 0] = 12 \quad m_2(5) = 0 \\
v_2(6) &= \max[r_2(0) + v_3(6), r_2(1) + v_3(5), r_2(2) + v_3(4), r_2(3) + v_3(3), r_2(4) \\
&\quad + v_3(2), r_2(5) + v_3(1), r_2(6) + v_3(0)] \\
&= \max[0 + 13, 2 + 12, 4 + 8, 6 + 5, 9 + 3, 11 + 1, 13 + 0] \\
&= 14 \quad m_2(6) = 1
\end{aligned}
$$

Because we are starting with 6 units, we need only to compute $v_1(6)$, which is given by

$$
\begin{aligned}
v_1(6) &= \max[r_1(0) + v_2(6), r_1(1) + v_2(5), r_1(2) + v_2(4), r_1(3) + v_2(3), r_1(4) \\
&\quad + v_2(2), r_1(5) + v_2(1), r_1(6) + v_2(0)] \\
&= \max[0 + 14, 2 + 12, 6 + 9, 9 + 6, 12 + 4, 16 + 2, 16 + 0] \\
&= 18 \quad m_1(6) = 5
\end{aligned}
$$

Thus, the optimal return from the 6-unit total allocation is 18. The number of allocations are as follows: Because $m_1(6) = 5$, we allocate 5 units to activity 1 leaving 1 unit. Because $m_2(1) = 1$, we allocate 1 unit to activity 2, leaving a balance of 0; thus, we allocate no unit to activity 3. We can check to see that $r_1(5) + r_2(1) = 18$.

10.2.2 Markov Reward Processes

The Markov reward process (MRP) is an extension of the basic Markov process that associates each state of a Markov process with a reward. Specifically, let $\{X_k, k = 1, 2, \ldots, N\}$ be a discrete-time Markov chain with a finite-state space $\{1, 2, \ldots, N\}$ and transition probability matrix P. Assume that when the process enters state i it receives a reward r_{ij} when it makes a transition to state j, where r_{ij} can be positive or negative. Let the reward matrix R be defined as follows:

$$R = \begin{bmatrix} r_{11} & r_{11} & \cdots & r_{1N} \\ r_{21} & r_{22} & \cdots & r_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ r_{N1} & r_{N2} & \cdots & r_{NN} \end{bmatrix}$$

That is, R is the matrix of the rewards. We define the process $\{X_n, R\}$ to be a discrete-time Markov reward process.

Let $v_n(i)$ denote the expected total earnings in the next n transitions, given that the process is currently in state i. Assume that the process makes a transition to state j with probability p_{ij}. It receives an immediate reward of r_{ij}, where $j = 1, \ldots, N$. To compute $v_n(i)$, let the reward when there are n transitions to be made be represented by $\Re_n$, and let s_n denote the current state. Then we have that

$$\begin{aligned} v_n(i) &= E[\Re_n + \Re_{n-1} + \cdots + \Re_1 + \Re_0 | s_n = i] \\ &= E[\{\Re_n | s_n = i\} + \{\Re_{n-1} + \cdots + \Re_1 + \Re_0\} | s_n = i] \\ &= E[\Re_n | s_n = i] + E[\Re_{n-1} + \cdots + \Re_1 + \Re_0 | s_n = i] \\ &= \sum_{j=1}^{N} p_{ij} r_{ij} + \sum_{j=1}^{N} p_{ij} v_{n-1}(j) \end{aligned}$$

The interpretation of the above equation is as follows. The first sum denotes the expected immediate reward that accrues from making a transition from state i to any state. When this transition takes place, the number of remaining transitions out of the n transitions is $n - 1$. Thus, the second sum represents the expected total reward in these $n - 1$ transitions given that the process is now in state j, $v_{n-1}(j)$, over all possible j that a transition from state i can be made.

If we define the parameter q_i by

$$q_i = \sum_{j=1}^{N} p_{ij} r_{ij}$$

then q_i is basically the expected reward in the next transition out of state i. Thus, we obtain

$$v_n(i) = q_i + \sum_{j=1}^{N} p_{ij} v_{n-1}(j)$$

If we define the column vector $v_n = [v_n(1), v_n(2), \ldots, v_n(N)]^T$ and the column vector $q = [q_1, q_2, \ldots, q_N]^T$, then we can rewrite that equation in following matrix form:

$$v_n = q + P v_{n-1}$$

which is equivalent to the following:

$$v_{n+1} = q + P v_n$$

Finally, if we denote the z-transform of v_n by $G_{v_n}(z)$, then taking the z-transform of both sides of the above equation we obtain

$$\begin{aligned} z^{-1}[G_{v_n}(z) - v_0] &= \frac{1}{1-z} q + P G_{v_n}(z) \\ (I - zP) G_{v_n}(z) &= \frac{z}{1-z} q + v_0 \\ G_{v_n}(z) &= \frac{z}{1-z} (I - zP)^{-1} q + (I - zP)^{-1} v_0 \end{aligned}$$

where $v_0 = [v_0(1), v_0(2), \ldots, v_0(N)]^T$. From the nature of the problem we can determine v_0 and thus obtain the solution. Note that $v_0(i)$ is the terminal cost incurred when the process ends up at state i.

Recall that in Chapter 3 it was stated that the inverse transform $[I - Pz]^{-1}$ can be expressed in the form

$$[I - Pz]^{-1} = \frac{1}{1-z} C + B(z)$$

where the constant term C has the characteristic that all the n rows are identical, and the elements of the rows are the limiting-state probabilities of the system whose

transition probability matrix is P. Thus, if $b(n)$ is the sequence whose z-transform is $B(z)$ we have that

$$\begin{aligned} G_{v_n}(z) &= \frac{z}{(1-z)^2}Cq + \frac{z}{1-z}B(z)q + \frac{1}{1-z}Cv_0 + B(z)v_0 \\ &= \frac{z}{(1-z)^2}Cq + \left\{\frac{1}{1-z} - 1\right\}B(z)q + \frac{1}{1-z}C_{v_0} + B(z)v_0 \\ &= \frac{z}{(1-z)^2}Cq + \frac{1}{1-z}Cv_0 + B(z)[v_0 - q] + \frac{1}{1-z}B(z)q \end{aligned}$$

From this we obtain the solution

$$v_n = nCq + Cv_0 + b(n)[v_0 - q] + q\sum_{k=0}^{n} b(k)$$

If we define $g = Cq$, we obtain the solution

$$v_n = ng + Cv_0 + b(n)[v_0 - q] + q\sum_{k=0}^{n} b(k)$$

10.2.3 MDP Basics

MDP is an extension of both MRP and DP in which an agent takes a set of actions that can be used to control the system at each state with a view to maximizing the expected reward. MDP is a discrete-time probabilistic system that can be represented by the tuple (S, A, R, P), where

- S is a finite set of N states; that is $S = \{1, 2, \ldots, N\}$. In practice the state of a system is a set of parameters that can be used to describe the system. For example, the state of a robot can be the coordinates of the robot.
- A is a finite set of K actions that can be taken at any state; that is, $A = \{a_1, a_2, \ldots, a_K\}$. In the case of a robot that can move in discrete steps, for example, an action can be a statement like "go east" or "go west."
- R is the reward matrix, which can vary with the action taken. Thus, for action $a \in A$ we denote the reward associated with a transition from state i to state j when action a is taken by $r_{ij}(a)$.
- P is the transition probability matrix, which can be different for each action. Thus, for action $a \in A$ we denote the probability that the process moves from state i to state j when action a is taken by $p_{ij}(a)$.

For such a system we can see that

$$P[S_{n+1} = j | S_0, a_0, S_1, a_1, \ldots, S_n = i, a_n = a]$$
$$= P[S_{n+1} = j | S_n = i, a_n = a] = p_{ij}(a)$$

Thus, the transition probabilities and reward functions are functions only of the last state and the subsequent action. Any homogeneous Markov chain $\{S_n\}$ whose transition probabilities are $p_{ij}(a)$ is called a Markov decision process, where $\sum_j p_{ij}(a) = 1$ for all $i \in S$ and all $a \in A$.

As stated earlier, the actions taken at each state are usually chosen according to a well-defined policy. Thus, a policy D is a mapping from S to A; that is, we can formally define a policy as a rule for taking actions at each state during a decision epoch. Because the objective in the decision process is to maximize the expected value of the sum of the returns (called expected total return) over a given time span, we define the *optimal policy* as the policy that maximizes the total expected return for each starting state i and number of transitions n. We are interested in stationary policies where, as we defined earlier, a stationary policy is a policy that assigns to each state i a fixed action $a = R_i$ that is always used whenever the process is in that state. Note that many texts use π to represent policy. However, in this book we use D because in earlier chapters π has been used to represent limiting-state probabilities of discrete-time Markov chains.

To solve the problem we consider the decision epochs to be stages in the decision-making process. Let $v_n(i, d)$ denote the expected total return in the next n stages, given that the process is in state i and policy d is used. $v_n(i, d)$ is sometimes called the *value function*. To derive the recursive equation for $v_n(i, d)$ we consider Figure 10.1.

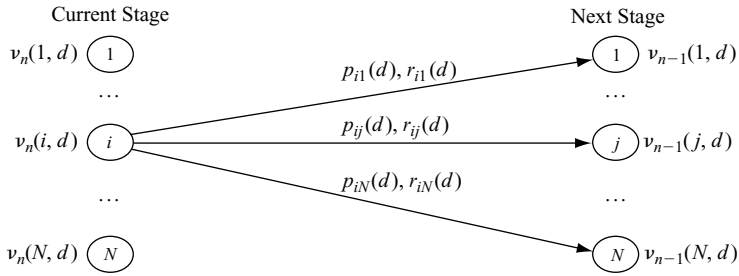


Figure 10.1. Stage-based expected return.

From the figure we observe that for a finite-stage system, the recursive equation relating $v_n(i, d)$ and $v_{n-1}(i, d)$ is given by

$$v_n(i,d) = \sum_{j=1}^{N} p_{ij}(d) r_{ij}(d) + \sum_{j=1}^{N} p_{ij}(d) v_{n-1}(j,d)$$

$$= q_i(d) + \sum_{j=1}^{N} p_{ij}(d) v_{n-1}(j,d)$$

where

$$q_i(d) = \sum_{j=1}^{N} p_{ij}(d) r_{ij}(d)$$

is the expected return in the next transition out of state i using policy d. Thus, applying the Bellman's principle of optimality, the optimal return with respect to state i is given by

$$v_n(i) = \max_d \left\{ q_i(d) + \sum_{j=1}^{N} p_{ij}(d) v_{n-1}(j) \right\} \qquad i = 1, 2, \ldots, N$$

10.2.4 MDPs with Discounting

In many economic systems, it is important to take the cost of money into consideration by introducing discounting. This is due to the fact that the value of \$1 now is not the same as its value in three years' time. This difference can be accounted for by introducing the so-called *discounted return*. Let the *discount factor* β denote the value at the beginning of a transition interval of a unit return received at the end of the transition, where $0 \leq \beta < 1$. Let the random variable $\Re_n$ be as previously defined. Then for the finite-stage system, $v_n(i, d, \beta)$ is given by

$$\begin{aligned} v_n(i,d,\beta) &= E[\Re_n + \beta\Re_{n-1} + \beta^2\Re_{n-2} + \cdots + \beta^{n-1}\Re_1 + \beta^n\Re_0 | s_n = i] \\ &= E[\{\Re_n | s_n = i\} + \beta\{\Re_{n-1} + \beta\Re_{n-2} + \cdots + \beta^{n-2}\Re_1 \\ &\quad + \beta^{n-1}\Re_0\} | s_n = i] \\ &= E[\Re_n | s_n = i] + \beta E[\Re_{n-1} + \beta\Re_{n-2} + \cdots + \beta^{n-2}\Re_1 \\ &\quad + \beta^{n-1}\Re_0 | s_n = i] \\ &= \sum_{j=1}^{N} p_{ij}(d) r_{ij}(d) + \beta \sum_{j=1}^{N} p_{ij}(d) v_{n-1}(j,d,\beta) \end{aligned}$$

$$= q_i(d) + \beta \sum_{j=1}^{N} p_{ij}(d) v_{n-1}(j, d, \beta)$$

Thus, the only difference is the introduction of the discount factor to the future returns.

10.2.5 Solution Methods

The solution to any decision process is a sequence of actions that optimizes a given value function. There are three general methods of solving MDP problems. These are the *value-iteration* method, which is used for finite-horizon problems; the *policy-iteration* method, which is used for infinite-horizon problems; and the *linear programming* method, which is also used for infinite-horizon problems but will not be discussed here. The value-iteration method is sometimes called the method of *successive approximations*.

The policy-iteration method will find the stationary policy that optimizes the value function both when no discounting is used and when discounting is used. The value-iteration method might not give the optimal policy using a finite number of iterations. However, compared to the policy-iteration method, it has the advantage that it does not require the solution of a system of simultaneous equations, as the policy-iteration and linear programming methods do. Thus, with the value-iteration method, each iteration can be performed simply and quickly.

Value-Iteration Method

The value-iteration method computes recursively a sequence of value functions approximating the optimal cost per unit time. It is essentially an extension of the technique used to solve the deterministic dynamic programming problem and thus utilizes a backward recursive relationship as follows:

$$v_n(i, d, \beta) = \sum_{j=1}^{N} p_{ij}(d) r_{ij}(d) + \sum_{j=1}^{N} p_{ij}(d) \beta v_{n-1}(j, d, \beta)$$

$$= q_i(d) + \beta \sum_{j=1}^{N} p_{ij}(d) v_{n-1}(j, d, \beta)$$

$$v_n(i, \beta) = \max_d \left\{ q_i(d) \sum_{j=1}^{N} p_{ij}(d) v_{n-1}(j, \beta) \right\} \quad i = 1, 2, \ldots, N$$

It starts by choosing a set of values of $v_0(i, d, \beta)$ as follows:

$$v_0(1, d, \beta) = v_0(2, d, \beta) = \ldots = v_0(N, d, \beta) = 0$$

Thus, we can obtain $v_1(i, d, \beta)$ as follows:

$$v_1(i, \beta) = \max_d \{q_i(d)\} \quad i = 1, 2, \ldots, N$$

This gives the expected total return in stage 1, given that the process is in state i at that stage when the optimal policy is used. Using this set of $v_1(i)$ we obtain $v_2(i)$, the expected total return in stage 2, as follows:

$$v_2(i, \beta) = \max_d \left\{ q_i(d) + \sum_{j=1}^{N} p_{ij}(d) v_1(j, \beta) \right\}$$

This process continues backward until we reach the first stage. The solution to the problem is $v_T(i)$, where T is the number of decision epochs, which corresponds to the number of stages. Thus, the process can be summarized as shown in Figure 10.2.

Example 10.2 Consider equipment that is inspected daily at the end of each work day. It can be in one of three states at the inspection time, which are Good, Acceptable, and Bad. It has been found that the condition at the time of inspection on a given day depends probabilistically on the condition at the time of inspection on the previous day as follows. Given that it is Good on a given day, it will be Good the following day with probability 0.6, Acceptable with probability 0.3, and Bad with probability 0.1. Similarly, given that it is Acceptable on a given day, it will be Good the following day with probability 0, Acceptable with probability 0.6, and Bad with probability 0.4. Finally, given that it is Bad on a given day, it will be Good the following day with probability 0, Acceptable with probability 0, and Bad with probability 1.0. The possible maintenance actions are as follows:

1. Do nothing, and thus follow the transition probabilities defined above.
2. Overhaul, which is equally likely to bring it to the Good condition or the Acceptable condition; the cost of an overhaul is \$500.
3. Replace the equipment, which automatically brings it to the Good condition; the cost of new equipment is \$2000.

Assume that when the equipment is operating in Good condition, the company makes \$1000. When it is operating in Acceptable condition, the company makes

$$v_T \leftarrow v_{T-1} \leftarrow \ldots \leftarrow v_2 \leftarrow v_1 \leftarrow v_0$$

Figure 10.2. Recursive procedure for the expected return.

$500; and when it is operating in Bad condition, the company loses $500. We consider the following policies:

1. Replace the equipment only when it is in Bad condition and nothing in other states.
2. Replace the equipment when it is in Bad condition and overhaul when it is in Acceptable condition.
3. Replace the equipment when it is in Acceptable condition and when it is in Bad condition.

Determine the optimal operating cost of the equipment when $T = 4$. ■

Solution: Let the states be defined as follows:

$$1 \equiv \text{Good}$$

$$2 \equiv \text{Acceptable}$$

$$3 \equiv \text{Bad}$$

The Markov chains for the different policies are shown in Figure 10.3.

We transform the problem into a maximization problem by assigning negative values to the costs incurred by the company. We take $500 as the baseline cost; that is, we assume that $500 is the unit cost. If we denote the state-transition matrix, reward matrix, and immediate expected return matrix

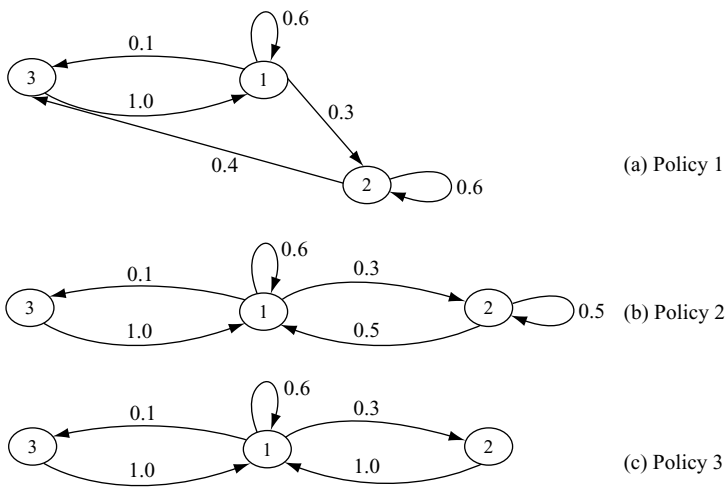


Figure 10.3. Markov chains for the policies of Example 10.2.

for policy d by $p(d)$, $R(d)$ and $Q(d)$, respectively, then the state-transition, reward, and immediate expected return matrices for the different policies are as follows:

$$P(1) = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0 & 0.6 & 0.4 \\ 1 & 0 & 0 \end{bmatrix} \quad R(1) = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & -1 \\ -2 & 0 & 0 \end{bmatrix} \quad Q(1) = \begin{bmatrix} 1.4 \\ 0.2 \\ -2 \end{bmatrix}$$

$$P(2) = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.5 & 0.5 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad R(2) = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \quad Q(2) = \begin{bmatrix} 1.4 \\ 0.5 \\ -2 \end{bmatrix}$$

$$P(3) = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad R(3) = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \quad Q(3) = \begin{bmatrix} 1.4 \\ 1 \\ -2 \end{bmatrix}$$

Note that the elements of $R(1)$, $R(2)$, and $R(3)$ take into account the cost of replacing and overhauling the equipment. For example, when the process is in state 2, under policy 2 the equipment is to be overhauled at a cost of 1 unit (or \$500), which will be subtracted from the gain made in the state into which the next transition takes place. Thus, the entries indicate the net rewards.

To continue the solution, we proceed in the following stages. We start with $v_0(i, d) = 0$, which gives $v_1(i, d) = q_i(d)$.

Stage 1:

	$v_1(i,d) = q_i(d)$			Optimal Solution	
i	$d=1$	$d=2$	$d=3$	$v_1(i)$	d^*
1	1.4	1.4	1.4	1.4	1, 2, 3
2	0.2	0.5	1	1	3
3	−2	−2	−2	−2	1, 2, 3

d^* is the optimal policy

Stage 2:

	$v_2(i,d) = q_i(d) + p_{i1}(d)v_1(1) + p_{i2}(d)v_1(2) + p_{i3}(d)v_1(3)$			Optimal Solution	
i	$d=1$	$d=2$	$d=3$	$v_2(i)$	d^*
1	2.34	2.34	2.34	2.34	1, 2, 3
2	0	1.7	2.4	2.4	3
3	−0.6	−0.6	−0.6	−0.6	1, 2, 3

d^* is the optimal policy

Stage 3:

	$v_3(i,d) = q_i(d) + p_{i1}(d)v_2(1) + p_{i2}(d)v_2(2) + p_{i3}(d)v_2(3)$			Optimal Solution	
i	$d=1$	$d=2$	$d=3$	$v_3(i)$	d^*
1	3.464	3.464	3.464	3.464	1, 2, 3
2	0.80	2.87	3.34	3.34	3
3	0.34	0.34	0.34	0.34	1, 2, 3

d^* is the optimal policy

Stage 4:

	$v_4(i,d) = q_i(d) + p_{i1}(d)v_3(1) + p_{i2}(d)v_3(2) + p_{i3}(d)v_3(3)$			Optimal Solution	
i	$d=1$	$d=2$	$d=3$	$v_4(i)$	d^*
1	4.5144	4.5144	4.5144	4.5144	1, 2, 3
2	2.340	3.902	4.464	4.464	3
3	1.464	1.464	1.464	1.464	1, 2, 3

d^* is the optimal policy

The optimal solution shows that the user should do nothing when the equipment is Good in any year, replace the equipment if it is Acceptable in any year, and replace the equipment if it is Bad in any year. The total expected return after 4 years is $v_4(1) = 4.5144$ units if the equipment is Good in the first year, $v_4(2) = 4.464$ units if it is Acceptable in the first year, and $v_4(3) = 1.464$ units if it is Bad in the first year.

Policy-Iteration Method

As discussed earlier, the policy-iteration method is used for infinite-horizon problems. The method requires the iteration of two steps: a *value determination* step followed by a *policy improvement* step. The value determination step is achieved by arbitrarily selecting an initial policy d and then solving the equation to obtain the long-run value function per unit time. Then using the policy improvement method a better policy is selected, and the value determination step is repeated. This step is continued until two successive iterations that lead to identical policies are reached at which time the procedure stops because an optimal policy has been obtained. After each step of the policy improvement scheme, an optimality test is carried out. If the test result is negative, an improvement is produced by another step; otherwise, the procedure stops and an optimal solution is obtained.

Recall that the value function at the the nth decision epoch given that the process is in state i and policy d is used is given by

$$v_n(i,d) = q_i(d) + \sum_{j=1}^{N} p_{ij}(d) v_{n-1}(j,d)$$

Howard (1960) has shown that for an ergodic Markov process with a stationary policy, $v_n(i,d)$ has the asymptotic form

$$v_n(i,d) = ng(d) + v_i(d) \quad i = 1, 2, \ldots, N$$

where $g(d)$ and $v_i d$ depend on the policy used. $g(d)$ is called the *gain of the system* under policy d and is given by

$$g(d) = \sum_{i=1}^{N} \pi_i \sum_{j=1}^{N} p_{ij}(d) r_{ij}(d) = \sum_{i=1}^{N} \pi_i q_i(d)$$

where π_i are the limiting-state probabilities of the Markov chain. Thus, substituting for $v_n(i)$ we obtain

$$\begin{aligned} v_n(i,d) = ng(d) + v_i(d) &= q_i(d) + \sum_{j=1}^{N} p_{ij}(d)\{(n-1)g(d) + v_j(d)\} \\ &= q_i(d) + (n-1)g(d) \sum_{j=1}^{N} p_{ij}(d) + \sum_{j=1}^{N} p_{ij}(d) v_j(d) \\ &= q_i(d) + (n-1)g(d) + \sum_{j=1}^{N} p_{ij}(d) v_j(d) \end{aligned}$$

where the last equality follows from the fact that

$$\sum_{j=1}^{N} p_{ij}(d) = 1$$

Thus, we obtain

$$ng(d) + v_i(d) = q_i(d) + (n-1)g(d) + \sum_{j=1}^{N} p_{ij}(d) v_j(d)$$

From this we obtain

$$g(d) + v_i(d) = q_i(d) + \sum_{j=1}^{N} p_{ij}(d)v_j(d) \quad i = 1, 2, \ldots, N$$

which is a set of N linear simultaneous equations. Thus, using the model in Howard (1960), we can summarize the solution algorithm as follows:

1. ***Value-determination operation*:** Use p_{ij} and q_i for a given policy to solve

$$g + v_i = q_i + \sum_{j=1}^{N} p_{ij}v_j \quad i = 1, 2, \ldots, N$$

   for all relative values v_i and q_i by setting $v_N = 0$.
2. ***Policy-improvement routine*:** For each state i, find the alternative policy d^* that maximizes

$$q_i(d) + \sum_{j=1}^{N} p_{ij}(d)v_j$$

   using the relative values v_i of the previous policy. Then d^* becomes the new policy in state i, $q_i(d^*)$ becomes q_i, and $p_{ij}(d^*)$ becomes p_{ij}.
3. ***Stopping rule*:** The optimal policy is reached (that is, g is maximized) when the policies on two successive iterations are identical. Thus, if the current value of d is not the same as the previous value of d, go back to step 1; otherwise, stop.

The policy-iteration method has the following properties:

- The problem reduces to solving sets of linear simultaneous equations and subsequent comparisons.
- Each succeeding policy has a higher gain than the previous one.
- The iteration cycle terminates on the policy that has the largest gain.

Example 10.3 Consider an MDP with the following parameters for two policies:

$$P(1) = \begin{bmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix} \qquad R(1) = \begin{bmatrix} 4 & 8 \\ -2 & 0 \end{bmatrix} \qquad Q(1) = \begin{bmatrix} 6.4 \\ -0.4 \end{bmatrix}$$

$$P(2) = \begin{bmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{bmatrix} \qquad R(1) = \begin{bmatrix} 4 & 6 \\ -1 & -1 \end{bmatrix} \qquad Q(1) = \begin{bmatrix} 4.4 \\ -1 \end{bmatrix}$$

We are required to obtain the long-run optimal operating policy for the problem. ■

Solution: From the values of $q_i(d)$, it seems reasonable to start with $d_1 = 1$ for both states. Thus, the system of equations to be solved is as follows:

$$g + v_1 = q_1(1) + p_{11}(1)v_1 + p_{12}(1)v_2$$
$$g + v_2 = q_2(1) + p_{21}(1)v_1 + p_{22}(1)v_2$$

Because there are 3 unknowns and 2 equations, we set $v_2 = 0$ and obtain the following system of equations:

$$g + v_1 = q_1(1) + p_{11}(1)v_1 = 6.4 + 0.4v_1$$
$$g = q_2(1) + p_{21}(1)v_1 = -0.4 + 0.2v_1$$

From these we obtain the relative values $v_1 = 8.5$, $v_2 = 0$, $g = 1.3$. Applying these values to the policy improvement routine we obtain

	$q_i(d) + p_{i1}(d)v_1 + p_{i2}(d)v_2 = q_i(d) + p_{i1}(d)v_1$		Improved Policy
i	$d = 1$	$d = 2$	d^*
1	$6.4 + (0.4)(8.5) = 9.8$	$4.4 + (0.8)(8.5) = 11.2$	2
2	$-0.4 + 0.2(8.5) = 1.3$	$-1 + 0.5(8.5) = 3.25$	2

This means that the improved policy used in the next round of the policy determination operation is policy 2 for both states 1 and 2. Thus, the system of equations to be solved is as follows:

$$g + v_1 = q_1(2) + p_{11}(2)v_1 + p_{12}(2)v_2$$
$$g + v_2 = q_2(2) + p_{21}(1)v_1 + p_{22}(2)v_2$$

As before, we set $v_2 = 0$ and obtain the following system of equations:

$$g + v_1 = q_1(2) + p_{11}(2)v_1 = 4.4 + 0.8v_1$$
$$g = q_2(2) + p_{21}(2)v_1 = -1 + 0.5v_1$$

From these equations we obtain the relative values $v_1 = 7.71$, $v_2 = 0$, and $g = 2.85$. Applying these values to the policy improvement routine we obtain

	$q_i(d) + p_{i1}(d)v_1 + p_{i2}(d)v_2 = q_i(d) + p_{i1}(d)v_1$		Improved Policy
i	$d = 1$	$d = 2$	d^*
1	$6.4 + (0.4)(7.71) = 9.484$	$4.4 + (0.8)(7.71) = 10.568$	2
2	$-0.4 + 0.2(7.71) = 1.142$	$-1 + 0.5(7.71) = 2.855$	2

The improved policy is policy 2 for both states 1 and 2, which is the same as the current policy. Thus, it is the optimal policy, and the procedure is terminated.

As stated earlier, the asymptotic solution under a fixed policy is

$$v_n(i) = ng + v_i \quad i = 1, 2, \ldots, N$$

For the preceding example, $g = 2.85$ and $v_1 - v_2 = 7.71$.

Policy-Iteration Method with Discounting

The solution procedure for the policy-iteration method with discounting is similar to that without discounting. However, there are two areas of difference between the two. First, a discount factor, β, is introduced. Second, as shown in Howard (1960), in the value determination operation we use the p_{ij} and q_i for the selected policy to solve the set of equations

$$v_i(\beta) = q_i + \beta \sum_{j=1}^{N} p_{ij} v_j(\beta) \quad i = 1, 2, \ldots, N$$

Note that the gain g does not appear in the equation. Thus, we have N equations in N unknowns, which means that values of the $v_i(\beta)$ are not relative values but exact values.

The policy improvement routine is different from that of the system without discount by the introduction of the discount factor. That is, for each state i, we find the policy d^* that maximizes

$$q_i(d) + \beta \sum_{j=1}^{N} p_{ij}(d) v_j(\beta)$$

using the current values of $v_i(\beta)$ from the previous policy.

Example 10.4 Consider the problem of Example 10.3, and assume that $\beta = 0.9$. That is, we consider an MDP with the following parameters for two policies:

$$P(1) = \begin{bmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix} \qquad R(1) = \begin{bmatrix} 4 & 8 \\ -2 & 0 \end{bmatrix} \qquad Q(1) = \begin{bmatrix} 6.4 \\ -0.4 \end{bmatrix}$$

$$P(2) = \begin{bmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{bmatrix} \qquad R(2) = \begin{bmatrix} 4 & 6 \\ -1 & -1 \end{bmatrix} \qquad Q(2) = \begin{bmatrix} 4.4 \\ -1 \end{bmatrix}$$

As in Example 10.3, we are required to obtain the long-run optimal operating policy for the problem. ■

Solution: As in Example 10.3, we start with $d_1 = 1$ for both states. Thus, the system of equations to be solved is as follows:

$$v_1 = q_1(1) + 0.9p_{11}(1)v_1 + 0.9p_{12}(1)v_2 = 6.4 + 0.9(0.4)v_1 + 0.9(0.6)v_2$$
$$= 6.4 + 0.36v_1 + 0.54v_2$$
$$v_2 = q_2(1) + 0.9p_{21}(1)v_1 + 0.9p_{22}(1)v_2 = -0.4 + 0.9(0.2)v_1 + 0.9(0.8)v_2$$
$$= -0.4 + 0.18v_1 + 0.72v_2$$

From these we obtain the solution $v_1 = 19.22$, $v_2 = 10.93$. Applying these values to the policy improvement routine we obtain

	$q_i(d) + \beta p_{i1}(d)v_1 + \beta p_{i2}(d)v_2$		Improved Policy
i	$d = 1$	$d = 2$	d^*
1	$6.4 + 0.9(0.4)(19.22)$ $+ 0.9(0.6)(10.93) = 19.22$	$4.4 + 0.9(0.8)(19.22)$ $+ 0.9(0.2)(10.93) = 20.21$	2
2	$-0.4 + 0.9(0.2)(19.22)$ $+ 0.9(0.8)(10.93) = 10.93$	$-1 + 0.9(0.5)(19.22)$ $+ 0.9(0.5)(10.93) = 12.57$	2

This means that the improved policy used in the next round of the policy determination operation is policy 2 for both states 1 and 2. Thus, the system of equations to be solved is as follows:

$$v_1 = q_1(2) + \beta p_{11}(2)v_1 + \beta p_{12}(2)v_2 = 4.4 + 0.9(0.8)v_1 + 0.9(0.2)v_2$$
$$= 4.4 + 0.72v_1 + 0.18v_2$$
$$v_2 = q_2(2) + \beta p_{21}(1)v_1 + \beta p_{22}(2)v_2 = -1 + 0.9(0.5)v_1 + 0.9(0.5)v_2$$
$$= -1 + 0.45v_1 + 0.45v_2$$

The solution to these equations is $v_1 = 30.69$, $v_2 = 23.29$. Applying these values to the policy improvement routine we obtain

	$q_i(d) + \beta p_{i1}(d)v_1 + \beta p_{i2}(d)v_2$		Improved Policy
i	$d = 1$	$d = 2$	d^*
1	$6.4 + (0.36)(30.69)$ $+ (0.54)(23.29) = 30.025$	$4.4 + (0.72)(30.69)$ $+ (0.18)(23.29) = 30.689$	2
2	$-0.4 + (0.18)(30.69)$ $+ (0.72)(23.29) = 21.893$	$-1 + (0.45)(30.69)$ $+ (0.45)(23.29) = 23.291$	2

This shows that the improved policy is policy 2 for both states 1 and 2. Because the improved policy is the same as the current policy, the procedure is terminated as the optimal policy has been obtained. The present values of the states 1 and 2 under the optimal policy are 30.689 and 23.291, respectively.

10.3 Semi-Markov Decision Processes

In Chapter 6 we defined a semi-Markov process as a process that makes transitions from state to state like a Markov process but in which the amount of time spent in each state before a transition to the next state occurs is an arbitrary random variable that depends on the next state the process will enter. This means that at transition instants a semi-Markov process behaves like a Markov process. Recall that when the process enters state i it chooses its next state as state j with probability p_{ij}. In the case of a discrete-time semi-Markov process, after choosing j, the process spends a time H_{ij}, called the holding time, before making the transition, where H_{ij} is a positive, integer-valued random variable with the PMF $p_{H_{ij}}(m) = P[H_{ij} = m]$, $m = 1, 2, \ldots$. Similarly, in the case of a continuous-time semi-Markov process, after choosing state j, the time H_{ij} that the process spends in state i until the next transition, which is the holding time for a transition from i to j, has the PDF $f_{H_{ij}}(t)$, $t \geq 0$.

The semi-Markov decision process is an extension of the semi-Markov process in which we control the state transitions through a set of actions and associate each action with a set of rewards. Thus, to describe a semi-Markov process we augment the description of a semi-Markov process by stating that whenever the system enters a state a set of possible actions will be taken, and associated with each action are the transition probability to the next state, the holding time, and the reward accruing from the transition to the next state.

The semi-Markov decision process (SMDP) is to the semi-Markov process what MDP is to the Markov chain. The main difference between MDP and SMDP can be explained as follows. In MDP it is assumed that decisions are made at specific epochs, and an action taken at epoch t affects only the state where the action was taken and the reward at epoch t. In SMDP the intervals between decisions are usually random. Alternatively, actions can take variable amounts of time to complete. In practice decisions are made when a change of state occurs. For example, in a queueing system, decisions can be made when a customer arrives at the system or when a customer leaves the system. Thus, an SMDP model includes an additional parameter that defines the duration of an action or the interval between actions. For this reason, an SMDP is a discrete-time probabilistic system that can be represented by the 5-tuple (S, A, R, P, H), where

- S is a finite set of N states, as in the MDP; that is $S = \{1, 2, \ldots, N\}$.
- A is a finite set of K actions that can be taken at any state; that is, $A = \{a_1, a_2, \ldots, a_K\}$.
- R is the reward matrix, which can vary with the action taken. Thus, for action $a \in A$ we denote the reward associated with a transition from state i to state j when action a is taken by $r_{ij}(a)$.

- P is the transition probability matrix, which can be different for each action. Thus, for action $a \in A$ we denote the probability that the system moves from state i to state j when action a is taken by $p_{ij}(a)$, which is independent of the history of the process up to the time the action was taken.
- H is the holding time distribution. $H_{ij}(a)$ is the holding time for a transition from state i to state j, which is the time the system spends in state i, given that upon leaving i the next state it goes to is state j when action a is taken in state i. The PDF of $H_{ij}(a)$ is $f_{H_{ij}}(t, a), t \geq 0$, where $E[H_{ij}(a)] = h_{ij}(a) < \infty$.

Thus, while the only relevant feature in MDP is the sequential nature of the decision process, not the time that elapses between decision epochs, in SMDP the time beween one decision epoch and the next is a random variable that can be real or integer-valued. For this reason, there are different types of SMDP that depend on whether the system state and the intervals between decision epochs are discrete or continuous. Specifically, there can be discrete-state SMDPs and continuous-state SMDPs, and discrete-decision-interval SMDPs and continuous-decision-interval SMDPs. From this classification we observe that an MDP is an SMDP with discrete-decision intervals, where the interval is constant at one time unit. In a discrete-decision-interval system the times between decisions are governed by a discrete random variable with a specified PMF. In a continuous-decision-interval system the time between decisions is a random variable with a specified PDF. In the remainder of this chapter we consider both discrete-state continuous-decision-interval SMDPs and discrete-state discrete-decision-interval SMDPs.

Sometimes a parameter T is listed as a component of the definition of the semi-Markov decision process. T is defined to be the probability distribution of the times between actions. Thus, $T_i(a)$ denotes the time until the next decision epoch given that action a is taken in the current state i; its PDF is $f_{T_i}(x, a)$, and mean $E[T_i(a)] > 0$. However, the holding time $H_{ij}(a)$ is related to $T_i(a)$ as follows:

$$T_i(a) = \sum_{j=1}^{N} p_{ij} H_{ij}(a)$$

Thus, there is no need for an additional parameter in the specification of the SMDP. Finally, we assume that stationary policies are used, which means that the same action a is taken whenever the system is in a given state i.

10.3.1 Semi-Markov Reward Model

In this section we consider continuous-time semi-Markov reward models. Using the method proposed by Ross (1970), we assume the following reward structure.

When the system enters state i, it incurs an immediate reward B_{ij}, and an extra reward accumulates at the rate of $b_{ij}(t)$ per unit time until the transition to state j occurs, where $B_{ij} < \infty$ and $b_{ij}(t) < \infty$. Howard (1971b) refers to B_{ij} as a bonus and $b_{ij}(t)$ as the "yield rate" of state i when the next state is j. Thus, if the system spends a time τ in state i before making a transition out of the state, the total expected accrued reward is

$$\begin{aligned} r_i(\tau) &= \sum_{j=1}^{N} p_{ij} \left\{ B_{ij} + \int_{t=0}^{\tau} b_{ij}(t)dt \right\} \int_{x=0}^{\tau} f_{H_{ij}}(x)dx \\ &= \sum_{j=1}^{N} p_{ij} B_{ij} + \sum_{j=1}^{N} p_{ij} \int_{x=0}^{\tau} \int_{t=0}^{\tau} b_{ij}(t) f_{H_{ij}}(x)dtdx \\ &= B_i + \sum_{j=1}^{N} p_{ij} \int_{x=0}^{\tau} \int_{t=0}^{\tau} b_{ij}(t) f_{H_{ij}}(x)dtdx \end{aligned}$$

where

$$B_i = \sum_{j=1}^{N} p_{ij} B_{ij}$$

Let $v_i(t)$ denote the total expected reward that will accrue by time t given that the process enters state i at time 0. Then, to compute $v_i(t)$ we observe that either by time t the process is still in state i or it has moved out of state i. In the first case the holding time in state i is greater than t, while in the second case it is less than or equal to t. If the holding time in state i is less than t, then we assume that the process made a transition from state i to some state j at time τ and spent the remaining time $t - \tau$ in state j. Thus, we have that

$$\begin{aligned} v_i(t) &= B_i + \sum_{j=1}^{N} p_{ij} \int_{\tau=t}^{\infty} \int_{x=0}^{t} b_{ij}(x) f_{H_{ij}}(\tau)dxd\tau \\ &\quad + \sum_{j=1}^{N} p_{ij} \int_{\tau=0}^{t} \left\{ \int_{x=0}^{\tau} b_{ij}(x)dx + v_j(t-\tau) \right\} f_{H_{ij}}(\tau)d\tau \\ &= r_i(t) + \sum_{j=1}^{N} p_{ij} \int_{\tau=0}^{t} v_j(t-\tau) f_{H_{ij}}(\tau)d\tau \quad i = 1, 2, \ldots, N, t \geq 0 \end{aligned}$$

where

$$r_i(t) = B_i + \sum_{j=1}^{N} p_{ij} \int_{\tau=t}^{\infty} \int_{x=0}^{t} b_{ij}(x) f_{H_{ij}}(\tau) dx d\tau$$
$$+ \sum_{j=1}^{N} p_{ij} \int_{\tau=0}^{t} \int_{x=0}^{\tau} b_{ij}(x) f_{H_{ij}}(\tau) dx d\tau$$

Thus, $r_i(t)$ is the total expected reward that will accrue by time t given that the process enters state i at time 0. Note that the expected total reward at state i from the instant the process enters the state until it leaves the state is given by

$$r_i(\infty) = B_i + \sum_{j=1}^{N} p_{ij} \int_{\tau=0}^{\infty} \int_{x=0}^{\tau} b_{ij}(x) f_{H_{ij}}(\tau) dx d\tau$$

In some applications, such as in the insurance industry, there is a fixed additional reward that is paid at the end of the process' holding time in a state. We will refer to this reward as the *terminal reward*. If we denote the terminal reward in state i by $v_i(0)$, then we have that

$$v_i(t) = B_i + \sum_{j=1}^{N} p_{ij} \int_{\tau=t}^{\infty} \left\{ \int_{x=0}^{t} b_{ij}(x) dx + v_i(0) \right\} f_{H_{ij}}(\tau) d\tau$$
$$+ \sum_{j=1}^{N} p_{ij} \int_{\tau=0}^{t} \left\{ \int_{x=0}^{\tau} b_{ij}(x) dx + v_j(t-\tau) \right\} f_{H_{ij}}(\tau) d\tau$$
$$= r_i(t) + \sum_{j=1}^{N} p_{ij} \int_{\tau=0}^{t} v_j(t-\tau) f_{H_{ij}}(\tau) d\tau + v_i(0) \sum_{j=1}^{N} p_{ij} \{1 - F_{H_{ij}}(t)\}$$

Note that the terminal reward in state i is only associated with the case where the system still occupies state i at time t. Thus, if we are interested in terminating the process after time t, then $v_i(0)$ will be the additional reward. Note also that the preceding model includes many types of reward that a real system might not include at the same time. For example, while $b_{ij}(t)$ might be permitted in many models, we are likely to have either B_{ij} or $v_i(0)$, but not both, in a model.

10.3.2 Discounted Reward

Let $u(t)$ denote the rate at which rewards will be accumulated t units from now, $t \geq 0$. The present value of this reward is defined by

$$\int_0^{\infty} u(t) e^{-\beta t} dt = U^*(\beta)$$

where $U^*(s)$ is the Laplace transform of $u(t)$ and β is the continuous-time discount rate. Thus, when discounting is used, the parameters $r_i(\tau)$ and $v_i(t)$ will be denoted by $r_i(\tau, \beta)$ and $v_i(t, \beta)$ and are given by

$$r_i(t, \beta) = B_i + \sum_{j=1}^{N} p_{ij} \int_{\tau=t}^{\infty} \int_{x=0}^{t} e^{-\beta x} b_{ij}(x) f_{H_{ij}}(\tau) dx d\tau$$

$$+ \sum_{j=1}^{N} p_{ij} \int_{\tau=0}^{t} \int_{x=0}^{\tau} e^{-\beta x} b_{ij}(x) f_{H_{ij}}(\tau) dx d\tau$$

$$r_i(\infty, \beta) = B_i + \sum_{j=1}^{N} p_{ij} \int_{\tau=0}^{\infty} \int_{x=0}^{\tau} e^{-\beta x} b_{ij}(x) f_{H_{ij}}(\tau) dx d\tau$$

$$v_i(t, \beta) = r_i(t, \beta) + \sum_{j=1}^{N} p_{ij} \int_{\tau=0}^{t} e^{-\beta \tau} v_j(t - \tau) f_{H_{ij}}(\tau) d\tau$$

where B_i is as previously defined. For the case where a terminal reward is paid out at the end of time t, we have that

$$v_i(t, \beta) = r_i(t, \beta) + \sum_{j=1}^{N} p_{ij} \int_{\tau=0}^{t} e^{-\beta \tau} v_j(t - \tau, \beta) f_{H_{ij}}(\tau) d\tau$$

$$+ v_i(0) \sum_{j=1}^{N} p_{ij} \int_{\tau=t}^{\infty} e^{-\beta \tau} f_{H_{ij}}(\tau) d\tau$$

10.3.3 Analysis of the Continuous-Decision-Interval SMDPs

We first consider a semi-Markov decision process without discounting. Following the model developed in the previous section, we assume that if the action a is taken when the system enters state i, it incurs an immediate cost or reward $B_{ij}(a)$, and an extra reward (or cost) accumulates at the rate of $b_{ij}(t, a)$ per unit time until the transition to state j occurs, where $B_{ij}(a) < \infty$ and $b_{ij}(t, a) < \infty$. In the remainder of the chapter we assume that there is no terminal reward; that is, $v_i(0) = 0$ for all i. We modify the results obtained earlier and obtain $r_i(t, a)$, the total expected reward that will accrue by time t given that the process enters state i at time 0 and

action a is taken, as follows:

$$r_i(t,a) = B_i(a) + \sum_{j=1}^{N} p_{ij}(a) \int_{\tau=t}^{\infty} \int_{x=0}^{t} b_{ij}(x,a) f_{H_{ij}}(\tau,a) dx d\tau$$

$$+ \sum_{j=1}^{N} p_{ij}(a) \int_{\tau=0}^{t} \int_{x=0}^{\tau} b_{ij}(x,a) f_{H_{ij}}(\tau,a) dx d\tau$$

where

$$B_i(a) = \sum_{j=1}^{N} p_{ij}(a) B_{ij}(a) \quad i = 1, \ldots, N$$

Therefore, the expected total reward between two decision epochs, given that the system occupies state i at the first epoch and action a was chosen at state i, is given by

$$r_i(\infty,a) = r_i(a) = B_i(a) + \sum_{j=1}^{N} p_{ij}(a) \int_{\tau=0}^{\infty} \int_{x=0}^{\tau} b_{ij}(x,a) f_{H_{ij}}(\tau,a) dx d\tau$$

Similarly, let $v_i(t,a)$ denote the total expected reward that will accrue by time t given that the process enters state i at time 0 and action a is used. Modifying the results obtained earlier, we have that

$$v_i(t,a) = r_i(t,a) + \sum_{j=1}^{N} p_{ij}(a) \int_{\tau=0}^{t} v_j(t-\tau,a) f_{H_{ij}}(\tau,a) d\tau \quad i = 1, \ldots, N$$

Thus, if we denote the long-run expected total return when the initial state is i and action a is taken by $v_i(a)$, we obtain

$$v_i(a) = r_i(a) + \sum_{j=1}^{N} p_{ij}(a) v_j(a) \quad i = 1, \ldots, N$$

The result can be expressed in a matrix form as follows. Let

$$V(a) = [v_1(a) \;\; v_2(a) \;\; \ldots \;\; v_N(a)]^T$$

$$R(a) = [r_1(a) \;\; r_2(a) \;\; \ldots \;\; r_N(a)]^T$$

$$P(a) = \begin{bmatrix} p_{11}(a) & p_{12}(a) & \ldots & p_{1N}(a) \\ p_{21}(a) & p_{22}(a) & \ldots & p_{2N}(a) \\ \ldots & \ldots & \ldots & \ldots \\ p_{N1}(a) & p_{N2}(a) & \ldots & p_{NN}(a) \end{bmatrix}$$

Then we obtain

$$V(a) = R(a) + P(a)V(a)$$
$$= [I - P(a)]^{-1}R(a)$$

Finally, the optimal long-run expected total return, which is given by

$$v_i^* = \max_a \left\{ r_i(a) + \sum_{j=1}^{N} p_{ij}(a)v_j^* \right\} \quad i = 1, \ldots, N$$

can be obtained by exhaustively solving for all the $v_i(a)$ and choosing a policy with the maximum expected return. However, this is usually an inefficient method particularly when either the state space or action space or both are large. A more efficient solution method is the policy-iteration method.

10.3.4 Solution by Policy-Iteration

The continuous-decision-interval SMDP is not easily amenable to analysis via the value-iteration method. Thus, we discuss only the policy-iteration method. But we must first define the average gain for the SMDP. Let $h_i(a)$ denote the expected time until the next decision epoch given that action a is chosen in the present state i. That is,

$$h_i(a) = \sum_{j=1}^{N} p_{ij}(a)h_{ij}(a) \quad i = 1, \ldots, N$$

Let $Z(t)$ denote the total reward that accrues up to time t, $t \geq 0$. We state the following theorem:

Theorem 10.1 Let the embedded Markov chain associated with a policy d be denoted by $\{X_n | (n = 0, 1, \ldots)\}$, where X_n is the state of the system at the nth decision epoch. If $\{X_n\}$ has no two disjoint closed communicating classes, then for each initial state $X_0 = i$, the limit

$$\lim_{t \to \infty} \frac{Z(t)}{t} = g(d)$$

exists and is independent of the initial state i. Moreover, if $\{\pi_j(d)\}$ denotes the set of limiting-state probabilities of $\{X_n\}$, then

$$g(d) = \frac{\sum_j r_j(d)\pi_j(d)}{\sum_j h_j(d)\pi_j(d)} = \frac{E[r(d)]}{E[h(d)]}$$

The proof of this theorem can be found in Ross (1970) and Tijms (2003). $g(d)$ is the long-run average reward per unit time when policy d is used and can also

be defined as the ratio of the expected reward per occupancy to the expected time between transitions in the steady state.

Recall from our discussion on MDP that $v_i(a,t)$ can be expressed in terms of $g(a)$ as follows:

$$v_i(a,t) = tg(a) + v_i(a)$$

When t is large, $v_i(a,t)$ can be expressed as follows:

$$v_i(t,a) = r_i(a) + \sum_{j=1}^{N} p_{ij}(a) \int_{\tau=0}^{\infty} v_j(t-\tau,a) f_{H_{ij}}(\tau,a) d\tau$$

Substituting for $v_i(t,a)$ and $v_j(t-\tau,a)$ in the limiting state gives

$$\begin{aligned} tg(a) + v_i(a) &= r_i(a) + \sum_{j=1}^{N} p_{ij}(a) \int_{\tau=0}^{\infty} \{g(a)(t-\tau) + v_j(a)\} f_{H_{ij}}(\tau,a) d\tau \\ &= r_i(a) + g(a)t - g(a) \sum_{j=1}^{N} p_{ij}(a) \int_{\tau=0}^{\infty} \tau f_{H_{ij}}(\tau) d\tau + \sum_{j=1}^{N} p_{ij}(a) v_j(a) \\ &= r_i(a) + tg(a) - g(a) \sum_{j=1}^{N} p_{ij}(a) h_{ij}(a) + \sum_{j=1}^{N} p_{ij}(a) v_j(a) \end{aligned}$$

This gives

$$v_i(a) = r_i(a) - g(a) \sum_{j=1}^{N} p_{ij}(a) h_{ij}(a) + \sum_{j=1}^{N} p_{ij}(a) v_j(a) \quad i = 1, \ldots, N$$

If we define the mean holding time in state i, $h_i(a)$, by

$$h_i(a) = \sum_{j=1}^{N} p_{ij}(a) h_{ij}(a),$$

we obtain the result

$$v_i(a) + g(a) h_i(a) = r_i(a) + \sum_{j=1}^{N} p_{ij}(a) v_j(a) \quad i = 1, \ldots, N$$

That is,

$$g(a) = \frac{1}{h_i(a)} \left\{ r_i(a) + \sum_{j=1}^{N} p_{ij}(a) v_j(a) - v_i(a) \right\} \quad i = 1, \ldots, N$$

Because there are $N + 1$ unknowns (N $v_i(a)$ and 1 $g(a)$), we set one of the unknowns to zero and thus obtain only relative values.

As in MDP we can summarize the solution algorithm as follows:

1. ***Value-determination operation***: Use p_{ij} and r_i for the present policy to solve

$$g(d)h_i(d) + v_i(d) = r_i(d) + \sum_{j=1}^{N} p_{ij}(d) v_j(d) \quad i = 1, 2, \ldots, N$$

   for all relative values v_i and r_i by setting $v_N = 0$.
2. ***Policy-improvement routine***: For each state i, find the alternative policy d^* that maximizes

$$r_i(d) + \frac{1}{h_i(d)} \left\{ \sum_{j=1}^{N} p_{ij}(d) v_j - v_i(d) \right\}$$

   using the relative values v_i of the previous policy. Then d^* becomes the new policy in state i, $r_i(d^*)$ becomes r_i, and $p_{ij}(d^*)$ becomes p_{ij}.
3. ***Stopping rule***: The optimal policy is reached (that is, g is maximized) when the policies on two successive iterations are identical. Thus, if the current value of d is not the same as the previous value of d, go back to step 1; otherwise, stop.

10.3.5 SMDP with Discounting

In many applications the expected discounted reward that the process will generate in some time interval might be important to the decision maker. For these applications we deal with the expected present values instead of the total expected reward. Let $r_i(t, a, \beta)$ be the equivalent of $r_i(t, a)$ when the discount factor is β. Similarly, let $v_i(t, a, \beta)$ be the equivalent of $v_i(t, a)$ when the discount factor is β.

Then from earlier results we have that

$$r_i(t,a,\beta) = B_i(a) + \sum_{j=1}^{N} p_{ij}(a) \int_{\tau=t}^{\infty} \int_{x=0}^{t} e^{-\beta x} b_{ij}(x,a) f_{H_{ij}}(\tau,a) dx d\tau$$

$$+ \sum_{j=1}^{N} p_{ij}(a) \int_{\tau=0}^{t} \int_{x=0}^{\tau} e^{-\beta x} b_{ij}(x,a) f_{H_{ij}}(\tau,a) dx d\tau$$

$$v_i(t,a,\beta) = r_i(t,a) + \sum_{j=1}^{N} p_{ij}(a) \int_{\tau=0}^{t} e^{-\beta\tau} v_j(t-\tau,a) f_{H_{ij}}(\tau,a) d\tau$$

The long-run values are given by

$$r_i(a,\beta) = B_i(a) + \sum_{j=1}^{N} p_{ij}(a) \int_{\tau=0}^{\infty} \int_{x=0}^{\tau} e^{-\beta x} b_{ij}(x,a) f_{H_{ij}}(\tau,a) dx d\tau$$

$$v_i(a,\beta) = r_i(a,\beta) + \sum_{j=1}^{N} p_{ij}(a) \int_{\tau=0}^{\infty} v_j(a,\beta) e^{-\beta\tau} f_{H_{ij}}(\tau,a) d\tau$$

$$= r_i(a,\beta) + \sum_{j=1}^{N} p_{ij}(a) M_{ij}^{H}(a,\beta) v_j(a,\beta)$$

where $M_{ij}^{H}(a,\beta)$ is the s-transform of $f_{H_{ij}}(\tau,a)$ evaluated at $s=\beta$. If we define

$$\theta_{ij}(a,\beta) = p_{ij}(a) M_{ij}^{H}(a,\beta)$$

we obtain the result

$$v_i(a,\beta) = r_i(a,\beta) + \sum_{j=1}^{N} \theta_{ij}(a,\beta) v_j(a,\beta)$$

This result can be expressed in a matrix form as follows. Let

$$V(a,\beta) = [v_1(a,\beta) \;\; v_2(a,\beta) \;\; \ldots \;\; v_N(a,\beta)]^T$$

$$R(a,\beta) = [r_1(a,\beta) \;\; r_2(a,\beta) \;\; \ldots \;\; r_N(a,\beta)]^T$$

$$\Theta(a,\beta) = \begin{bmatrix} \theta_{11}(a,\beta) & \theta_{12}(a,\beta) & \ldots & \theta_{1N}(a,\beta) \\ \theta_{21}(a,\beta) & \theta_{22}(a,\beta) & \ldots & \theta_{2N}(a,\beta) \\ \ldots & \ldots & \ldots & \ldots \\ \theta_{N1}(a,\beta) & \theta_{N2}(a,\beta) & \ldots & \theta_{NN}(a,\beta) \end{bmatrix}$$

Then we have that

$$V(a,\beta) = R(a,\beta) + \Theta(a,\beta)V(a,\beta)$$
$$= [I - \Theta(a,\beta)]^{-1} R(a,\beta)$$

Thus, for a given discount factor and a given action, we can obtain the expected discounted reward from the preceding matrix form. The optimal expected total long-run return is given by

$$v_i^* = \max_a \left\{ r_i(a,\beta) + \sum_{j=1}^{N} \theta_{ij}(a,\beta) v_j^*(\beta) \right\} \quad i = 1, \ldots, N$$

As discussed earlier, we can use the matrix method to obtain the result by an exhaustive search of all the possible solutions, which is not an efficient method of solving the problem particularly when either the state space or the action space or both are large. However, a more efficient solution is the policy iteration method.

10.3.6 Solution by Policy Iteration when Discounting Is Used

When t is large we have that

$$v_i(t,a,\beta) = r_i(a,\beta) + \sum_{j=1}^{N} p_{ij}(a) \int_{\tau=0}^{\infty} v_j(t-\tau,a,\beta) e^{-\beta\tau} f_{H_{ij}}(\tau,a) d\tau$$

Substituting for $v_i(t,a,\beta)$ and $v_j(t-\tau,a,\beta)$ in the limiting state gives

$$\begin{aligned} tg(a) + v_i(a,\beta) &= r_i(a,\beta) + \sum_{j=1}^{N} p_{ij}(a) \int_{\tau=0}^{\infty} \{g(a)(t-\tau) \\ &\quad + v_j(a,\beta)\} e^{-\beta\tau} f_{H_{ij}}(\tau,a) d\tau \\ &= r_i(a,\beta) + tg(a) - g(a) \sum_{j=1}^{N} p_{ij}(a) \int_{\tau=0}^{\infty} \tau e^{-\beta\tau} f_{H_{ij}}(\tau,a) d\tau \\ &\quad + \sum_{j=1}^{N} p_{ij}(a) \left\{ \int_{\tau=0}^{\infty} e^{-\beta\tau} f_{H_{ij}}(\tau,a) d\tau \right\} v_j(a,\beta) \\ &= r_i(a,\beta) + g(a)t + g(a) \sum_{j=1}^{N} p_{ij}(a) \frac{d}{d\beta} M_{ij}^H(a,\beta) \\ &\quad + \sum_{j=1}^{N} p_{ij}(a) M_{ij}^H(a,\beta) v_j(a,\beta) \end{aligned}$$

This gives

$$v_i(a,\beta) = r_i(a,\beta) + g(a)\sum_{j=1}^{N} p_{ij}(a)\frac{d}{d\beta}M_{ij}^{H}(a,\beta) + \sum_{j=1}^{N} p_{ij}v_j(a,\beta)M_{ij}^{H}(a,\beta)$$

That is,

$$g(a) = \frac{1}{\sum_{j=1}^{N} p_{ij}(a)\frac{d}{d\beta}M_{ij}^{H}(a,\beta)}\left\{ v_i(a,\beta) - r_i(a,\beta) - \sum_{j=1}^{N} p_{ij}v_j(a,\beta)M_{ij}^{H}(a,\beta) \right\}$$

Note that the calculation involves $g(d_k)$, which makes it different from the MDP with discounting where $g(a)$ does not appear in the calculations thereby enabling us to obtain the absolute values of $v_i(a,\beta)$ because we have N equations and N unknowns. In the SMDP with discounting, we can obtain only the relative values of the $v_i(a,\beta)$ because we have N equations and $N+1$ unknowns. Thus, the solution method is similar to that of the undiscounted system, which is the reason why we have to set one of the $v_i(a,\beta)$ to 0.

10.3.7 Analysis of the Discrete-Decision-Interval SMDPs with Discounting

The development of the discrete-decision-interval SMDP parallels that of its continuous counterpart. The primary difference is the fact that the holding times have PMFs instead of PDFs; consequently, summations replace integrations. Also, the discounting process is defined slightly differently. The present value of a reward r_n received n time units from now is $\beta^n r_n$.

Assume that the system has just entered state i with n time periods remaining. Let $v_i(n,a,\beta)$ denote the total expected reward that will accrue in the next n time periods after the process enters state i, action a is taken, and the discount factor is β. Then, as in the discussion on SMP rewards, to compute $v_i(n,a,\beta)$ we observe that either by the next n time periods the process is still in state i or it has moved out of state i. In the first case the holding time in state i is greater than n, while in the second case it is less than or equal to n. If the holding time in state i is less than n, then we assume that the process made a transition from state i to some state j at time $m < n$ and spent the remaining time $n - m$ in state j. Thus, if we denote the contribution of the first case by $v_G(n,a,\beta)$ and the contribution of the second case by $v_L(n,a,\beta)$, we have that

$$v_G(n,a,\beta) = \sum_{j=1}^{N} p_{ij}(a) \sum_{m=n+1}^{\infty} p_{H_{ij}}(m,a) \sum_{k=1}^{n} \beta^k b_{ij}(k,a)$$

$$v_L(n,a,\beta)=\sum_{j=1}^{N}p_{ij}(a)\sum_{m=1}^{n}p_{H_{ij}}(m,a)\left\{\sum_{k=1}^{m}\beta^k b_{ij}(k,a)+\beta^m v_j(n-m,a,\beta)\right\}$$

$$v_i(n,a,\beta)=B_i(a)+v_G(n,a,\beta)+v_L(n,a,\beta)$$

where $i=1,2,\ldots,N$, and $n=1,2,\ldots$. If we define $v_i(n,\beta)$ as the maximum expected present value of the future rewards that can accrue during an interval of length n given that the process has just entered state i and the discount factor is β, we obtain

$$v_i(n,\beta)=\max_a\{B_i(a)+v_G(n,a,\beta)+v_L(n,a,\beta)\}\quad i=1,\ldots,N$$

10.3.8 Continuous-Time Markov Decision Processes

A continuous-time MDP is a semi-Markov MDP in which the times between decision epochs are exponentially distributed. This means that the PDFs $f_{H_{ij}}(a,\tau)$ are exponential functions:

$$f_{H_{ij}}(a,\tau)=\lambda_{ij}^a e^{-\lambda_{ij}^a\tau}\quad \tau\geq 0$$

Thus,

$$E[H_{ij}(a)]=h_{ij}(a)=\frac{1}{\lambda_{ij}^a}$$

$$h_i(a)=\sum_{j=1}^{N}p_{ij}(a)h_{ij}(a)=\sum_{j=1}^{N}\frac{p_{ij}(a)}{\lambda_{ij}^a}$$

$$M_{H_{ij}}(a,\beta)=\frac{\lambda_{ij}^a}{\beta+\lambda_{ij}^a}$$

$$\frac{d}{d\beta}M_{H_{ij}}(a,\beta)=-\frac{\lambda_{ij}^a}{(\beta+\lambda_{ij}^a)^2}$$

$$\Theta(a,\beta)=\begin{bmatrix}\frac{p_{11}(a)\lambda_{11}^a}{\beta+\lambda_{11}^a} & \frac{p_{12}(a)\lambda_{12}^a}{\beta+\lambda_{12}^a} & \cdots & \frac{p_{1N}(a)\lambda_{1N}^a}{\beta+\lambda_{1N}^a}\\ \frac{p_{21}(a)\lambda_{21}^a}{\beta+\lambda_{21}^a} & \frac{p_{22}(a)\lambda_{22}^a}{\beta+\lambda_{22}^a} & \cdots & \frac{p_{2N}(a)\lambda_{2N}^a}{\beta+\lambda_{2N}^a}\\ \cdots & \cdots & \cdots & \cdots\\ \frac{p_{N1}(a)\lambda_{N1}^a}{\beta+\lambda_{N1}^a} & \frac{p_{N2}(a)\lambda_{N2}^a}{\beta+\lambda_{N2}^a} & \cdots & \frac{p_{NN}(a)\lambda_{NN}^a}{\beta+\lambda_{NN}^a}\end{bmatrix}$$

10.3.9 Applications of Semi-Markov Decision Processes

Semi-Markov decision processes are used to model admission control in a G/M/1 queueing system. In this system an agent regulates the system load by accepting or rejecting arriving customers. A rejected customer leaves the system without receiving service. An example is call arrival at a switchboard. Service times are assumed to be exponentially distributed, and each arriving customer brings a reward R. The system incurs a holding cost $c(n)$ when there are n customers in the system.

Another application is the service rate control in an M/G/1 queueing system. Here, an agent regulates the system load by varying the service rate; faster service rate is more expensive than a slower service rate. An agent changes the service rate upon completion of a service or upon the arrival of a customer to an empty system. There is a fixed cost K associated with switching the service rate and a holding cost $c(n)$ when there are n customers.

Semi-Markov decision processes are also used to model optimal control of queues. In this case, a service system has c identical servers, and the number of servers active at any time is controlled by an agent. Thus, it can be likened to a system of c service channels where a channel can be activated by turning it on and deactivated by turning it off. Examples include supermarket checkout counters, bank tellers, and production facilities. An active channel can service only one request at a time. A nonnegative cost $K(m, n)$ is incurred when the number of active channels is changed from m to n. There is an operation cost rate $r \geq 0$ per unit time associated with an active channel. Similarly, a holding cost $h \geq 0$ is incurred for each time unit a customer is in the system until its service is completed. The objective is to find a rule for controlling the number of active service channels so as to minimize the long-run average cost per unit time. Decision epochs are instants of service completion or epochs of a new arrival.

10.4 Partially Observable Markov Decision Processes

In Markov decision processes, the sequence of actions taken to make decisions assumes that the environment is completely observable and the effects of actions taken are deterministic. That is, in MDP it is assumed that at the decision epoch the state i, transition probabilities $\{p_{ij}(a)\}$, and immediate returns $\{r_i(a)\}$ are all known. However, the real world is not always completely observable, which means that the effects of actions taken are often nondeterministic. Decision making in such environments can be modeled by a partially observable Markov decision process (POMDP). That is, in POMDP, $\{p_{ij}(a)\}$ and $\{r_i(a)\}$ are all known at the decision epoch, but the state is not known precisely. Instead the agent has some observations from which to infer the probability of the system being in some state. From these

observations the agent takes an action that results in a reward. The reward received after an action is taken provides information on how good the action was.

As stated earlier, in a POMDP the agent chooses and executes an action at the decision epoch based on information from past observations and past actions and the current observation. Unfortunately the amount of memory required to store past observations and past actions can be large thereby making it difficult to maintain past information after a long period of time. This difficulty is usually overcome by maintaining the agent's *belief state* instead of its past information, where a belief state is the probability distribution over the hidden states of the Markov process given the past history of observations and actions. Thus, the belief state captures all the information contained in past information and current observation that is useful for selecting an action. Because the number of possible states of the environment is finite, maintaining the belief state is simpler than keeping track of all past information. Note that the fact that the belief state is defined in terms of probability distribution implies that the agent's knowledge is incomplete. Also, using the concept of belief state allows the POMDP to satisfy the Markov property because knowing the current belief state is all we need to predict the future. When the agent observes the current state of the environment it updates its belief state.

POMDP had it origin in the field of operations research but has now been applied in many other disciplines. For example, it has been applied in machine maintenance and quality control, medical diagnosis, and the search for moving objects, which are areas where the observation of the state of the underlying Markov process is not always possible. Drake (1962) formulated the first explicit POMDP problem. According to this model, the decision maker gains some probabilistic information about the state after every decision he makes. For a given probability distribution on the initial state, the decision maker can revise the distribution according to Bayes' rule. Dynkin (1965) and Aoki (1967) have also shown that using this Bayesian approach to estimate the state the process occupies enables the problem to be transformed to a Markov decision process with complete information. In Hauskrecht and Fraser (2000), POMDP is applied to the planning of the treatment of ischemic heart disease. Cassandra (1994), Cassandra (1998), Dean (1995), Shatkay (1999), and Simmons (1995) have used it to model robotic navigation within buildings. Jonas (2003) has also used it to model speech recognition. Different algorithms and applications of POMDPs are discussed in Zhang (2001).

One important application of POMDP is the *reinforcement learning* problem. As defined in Sutton and Barto (1998), reinforcement learning is learning what to do, without the intervention of a supervisor, to maximize a numerical reward. Thus, the agent is not told what to do, as in *supervised learning*, but discovers which actions yield the most reward by trying them. We begin our discussion on POMDP by discussing the partially observable Markov process.

10.4.1 Partially Observable Markov Processes

Consider a discrete-state Markov process that can be in one of two states S_1 and S_2. Given that it is currently in state S_1, it will enter state S_1 again with probability p_{11} and state S_2 next with probability $p_{12} = 1 - p_{11}$. Similarly, given that it is currently in state S_2, it will enter state S_1 next with probability p_{21} and state S_2 next with probability $p_{22} = 1 - p_{21}$. Assume that the dynamics of the process are being observed through an imperfect medium that allows us to observe two states Ω_1 and Ω_2. Let the conditional probability ϕ_{ij} be the probability that the process is actually in state S_i given that the observable state is Ω_j, $i, j = 1, 2$, where $\phi_{i1} + \phi_{i2} = 1$, $i = 1, 2$. Figure 10.4 represents the state-transition diagram of the Markov process with partial observability, which is called the partially observable Markov process (POMP).

There are two processes involved in POMP: the *core* process and the *observation* process. The core process is the underlying Markov process whose states are the S_i and the transition probabilities are the p_{ij}. The observation process is the process whose states Ω_i are in the observation space. In the preceding example, one can interprete Ω_i by the statement "the core process seems to be in state S_i." Note that the preceding example assumes that there is a one-to-one correspondence between the core process and the observation process, even though we are not able to link Ω_i with certainty to S_i. In a more general case the core process can have n states while the observation process has m states, where $m \neq n$.

The goal in the analysis of the POMP is to estimate the state of the Markov process given an observation or a set of observations. We assume that the observation space has no memory; that is, the observations are not correlated and are thus made independent of one another. The estimation can be based on considering each observation independent using the Bayes' rule. Assume that the steady-state

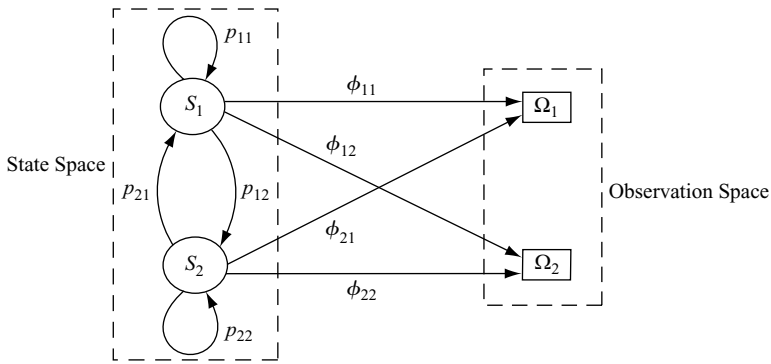


Figure 10.4. State-transition diagram for the partially observable process.

probability that the underlying Markov process is in state S_i at a random time is $P[S_1(n)] = P_1$ and $P[S_2(n)] = P_2 = 1 - P_1$. Let the function

$$\arg \max_y \{z\}$$

denote the argument y that corresponds to the maximum of the expression z. Also, let $\Omega_j(n)$ denote the event that the nth observation is Ω_j, and let $\hat{S}(\Omega_j(n))$ denote our estimate of the state as a result of $\Omega_j(n)$. Then the decision criterion becomes

$$\begin{aligned}
\hat{S}(\Omega_j(n)) &= \arg \max_{S_i} \{P[S_i(n)|\Omega_j(n)]\} \\
&= \arg \max_{S_i} \left\{ \frac{P[S_i(n)]P[\Omega_j(n)|S_i(n)]}{P[\Omega_j(n)]} \right\} \\
&= \arg \max_{S_i} \left\{ \frac{P[S_i(n)]P[\Omega_j(n)]|S_i(n)]}{P[S_i(n)]P[\Omega_j(n)|S_i(n)] + P[\bar{S}_i(n)]P[\Omega_j(n)|\bar{S}_i(n)]} \right\} \\
&= \arg \max_{S_i} \left\{ \frac{P_i\phi_{ij}}{P_i\phi_{ij} + (1 - P_i)\phi_{kj}} \right\}
\end{aligned}$$

where S_k is the state of the underlying Markov process with steady-state probability $P_k = 1 - P_i$. Applying this to the preceding example we obtain the sample space shown in Figure 10.5.

If we assume that $\phi_{11} > \phi_{12}$, $\phi_{22} > \phi_{21}$, and $P_1 = P_2$, then the decoding rule becomes that when the observed state is Ω_1 we consider the state of the core process to be S_1, and when the observed state is Ω_2 we assume that the state of the core process is S_2. Note that partially observable Markov processes are sometimes referred to as *hidden Markov models*, which will be discussed in the next chapter.

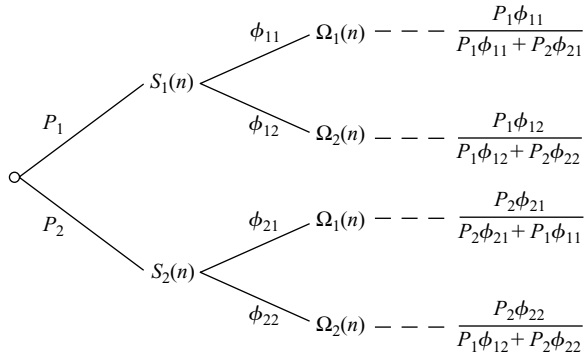


Figure 10.5. Sample space of state estimation process.

10.4.2 POMDP Basics

In POMDP the decision maker or agent has to solve two problems simultaneously; namely, a control problem like that of a standard MDP, and an identification problem for the unobserved states. Each time the agent takes an action, the transition to a new state implicitly provides new information about the underlying state of the process. This new knowledge can enable the agent to make the next decision. Thus, starting with an initial probability distribution, the agent revises the distribution after every transition to take into consideration the new information provided by the observation resulting from the transition. This revised distribution is called the *posterior distribution*, which is used to identify the unobserved state and to control the system at the next decision epoch.

More formally, a POMDP is a probabilistic model that can be represented by the 6-tuple $(S, A, \Omega, P, \Phi, R)$, where

- S is a finite set of N states of the core process; that is, $S = \{s_1, s_2, \ldots, s_N\}$. The state at time t is denoted by S_t.
- A is a finite set of K actions that can be taken at any state; that is, $A = \{a_1, a_2, \ldots, a_K\}$. The action taken at time t is denoted by A_t.
- Ω is the a finite set of M observations that can be made; that is, $\Omega = \{o_1, o_2, \ldots, o_M\}$. The observation at time t is denoted by Ω_t.
- P is the transition probability matrix, which can be different for each action. As in the case of MDP, for action $a \in A$ we denote the probability that the system moves from state s_i to state s_j when action a is taken by $p_{ij}(a)$, which is independent of the history of the process up to the time the action was taken. That is,

$$p_{ij}(a) = P[S_{t+1} = s_j | S_t = s_i, A_t = a]$$

  As stated earlier, it is assumed that the $p_{ij}(a)$ are known, but state s_i is not known at the decision epoch; it is inferred from the observation.
- Φ is the set of observation probabilities that describe the relationship between the observations, states of the core process, and actions. We let $\phi_{ij}(a)$ denote the probability of observing the state $o_j \in \Omega$ after action a is taken and the core process enters state s_i. That is,

$$\phi_{ij}(a) = P[\Omega_t = o_j | S_t = s_i, A_{t-1} = a]$$

- R is the reward function, which can vary with the action taken. The reward at time t is denoted by R_t. The reward that the agent receives by taking action $a \in A$ in state s_i that results in a transition to state s_j is denoted by $r_{ij}(a)$. The

total reward associated with action a in state S_i is

$$r_i(a) = \sum_{sj \in S} r_{ij}(a)\, p_{ij}(a)$$

Assume that the core process is in state S_t at time t. Because POMDP is based on a core process that is a Markov process, the current state S_t is sufficient to predict the future independent of the past states $\{S_0, S_1, \ldots, S_{t-1}\}$. As stated earlier, the state S_t is not directly observable but can only be inferred from the observations $\{\Omega_1, \Omega_2, \ldots, \Omega_t\}$. To help in making the determination of the state of the system, the agent keeps a complete trace of all observations and all actions it has taken and uses this information to choose actions to take next. The joint trace of actions and observations constitutes a *history* at time t, which is denoted by H_t and defined by

$$H_t = \{A_0, \Omega_1, A_1, \Omega_2, \ldots, \Omega_{t-1}, A_{t-1}, \Omega_t\}$$

Fortunately this history does not need to be represented explicitly but can be summarized via a *belief distribution* $b_t(s)$, which is defined by

$$b_t(s) = P[S_t = s | \Omega_t, A_{t-1}, \Omega_{t-1}, \ldots, A_0, b_0]$$

Thus, $0 \leq b_t(s) \leq 1$ is the probability that the process is in state $S_t = s$ given belief distribution b. That is, $b_t(s_j)$ is the agent's estimate that the core process is in state $S_t = s_j$. Therefore, based on the current belief state the agent chooses an action a and receives the reward $r_{jk}(a)$, and the core process makes a transition to state s_k that leads to the observation o_m. This is illustrated in Figure 10.6 where the component labeled SE is the state estimator that takes as input the last belief state, the most recent action, and the most recent observation and returns an updated belief state. The component labeled "d" represents the policy.

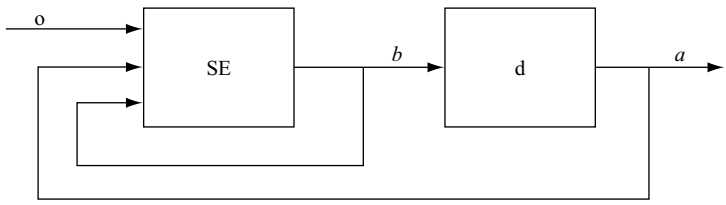


Figure 10.6. Structure of the state estimation process.

The initial state probability distribution, which defines the probability that the system is in state s at time $t = 0$, is given by

$$b_0(s) = P[S_0 = s]$$

Methods used to solve POMDPs are sometimes called reinforcement learning algorithms because the only feedback provided to the agent is a scalar reward signal at each step. One important feature of b_t is the fact that because it is a sufficient statistic for the history of the process, we can use it as a criterion for selecting actions. Also, it is computed recursively, using only the immediate past value, b_{t-1}, together with the most recent action A_{t-1} and the observation Ω_t. If we denote the belief state for state $S_t = s_k$ at time t by $b_t(s_k)$, then based on A_{t-1} and Ω_t the belief distribution is updated via the following Bayes' rule:

$$\begin{aligned}
b_t(s_k) &= P[S_t = s_k | \Omega_t = o_m, A_{t-1} = a, \Omega_{t-1}, \ldots, A_0] = P[s_k | o_m, a, b_{t-1}(s_j)] \\
&= \frac{P[s_k, o_m, a, b_{t-1}(s_j)]}{P[o_m, a, b_{t-1}(s_j)]} = \frac{P[o_m | s_k, a, b_{t-1}(s_j)] P[s_k, a, b_{t-1}(s_j)]}{P[o_m | a, b_{t-1}(s_j)] P[a, b_{t-1}(s_j)]} \\
&= \frac{P[o_m | s_k, a, b_{t-1}(s_j)] P[s_k | a, b_{t-1}(s_j)] P[a, b_{t-1}(s_j)]}{P[o_m | a, b_{t-1}(s_j)] P[a, b_{t-1}(s_j)]} \\
&= \frac{P[o_m | s_k, a, b_{t-1}(s_j)] P[s_k | a, b_{t-1}(s_j)]}{P[o_m | a, b_{t-1}(s_j)]} \\
&= \frac{P[o_m | s_k, a] \sum_{s \in S} P[s_k | a, b_{t-1}(s), s] P[s | a, b_{t-1}(s)]}{P[o_m | a, b_{t-1}(s_j)]} \\
&= \frac{P[o_m | s_k, a] \sum_{s_j \in S} P[s_k | a, s_j] b_{t-1}(s_j)}{P[o_m | a, b_{t-1}(s_j)]} = \frac{\phi_{km}(a) \sum_{s_j \in S} p_{jk}(a) b_{t-1}(s_j)}{P[o_m | a, b_{t-1}(s_j)]}
\end{aligned}$$

The denominator is independent of s_k and can be regarded as a normalizing factor. The numerator contains the observation function, the transition probability, and the current belief state. Thus, we can write

$$b_t(s_k) = \gamma \phi_{km}(a) \sum_{s_j \in S} p_{jk}(a) b_{t-1}(s_j)$$

where γ is a normalizing constant. Because the belief b_t at time t is computed recursively using the belief b_{t-1} as well as the most recent observation Ω_t and the

most recent action A_{t-1}, we can define the belief update procedure by the following operation:

$$b_t(s) = \tau(b_{t-1}, A_{t-1}, \Omega_t)$$

where $\tau(b_{t-1}, A_{t-1}, \Omega_t)$ is called the *belief updating function.* This shows that given a belief state, its successor belief state is determined by the action and observation.

10.4.3 Solving POMDPs

POMDP can be considered an MDP that is defined over the belief state space. Thus, when the belief state is updated via the action and observation, the solution methods used in MDP can then be used to obtain the optimal solution. In particular, the immediate reward associated with action a and belief state b is given by

$$r(b, a) = \sum_{s_i \in S} \sum_{s_j \in S} r_{ij}(a)\, p_{ij}(a) b(s_i) = \sum_{s_i \in S} r_i(a) b(s_i)$$

Let $v_t(b)$ denote the value function associated with belief b at time $t = 0, 1, \ldots$. If β is the discount factor, then using the method described in Sondik (1971) we apply multiple iterations of dynamic programming to compute more accurate values of the value function for each belief state. Thus, we have that

$$v_0^*(b) = \max_{a \in A} r(b, a)$$

$$v_t^*(b) = \max_{a \in A} \left\{ r(b, a) + \beta \sum_{o \in \Omega} P[o|b, a] v_{t-1}^*(\tau(b, a, o)) \right\}$$

$$= \max_{a \in A} \left\{ r(b, a) + \beta \sum_{s_k \in S} P[s_k|s_i, a] \sum_{o_j \in \Omega} P[o_j|s_k, a] b(s_i) v_{t-1}^*(\tau(b, a, o)) \right\}$$

$$= \max_{a \in A} \left\{ r(b, a) + \beta \sum_{s_k \in S} p_{ik}(a) \sum_{o_j \in \Omega} \phi_{kj}(a) b(s_i) v_{t-1}^*(\tau(b, a, o)) \right\}$$

where $\tau(b, a, o)$ is the belief updating function. Unfortunately, the state space for this MDP is an $|S|$-dimensional continuous space that is more complex than traditional MDPs with discrete state space.

10.4.4 Computing the Optimal Policy

As defined earlier, a policy is a sequence of decisions, and an optimal policy is a policy that maximizes the expected discounted return. Recall that we denote a policy by d, and for a given belief state b, a policy is of the form $d(b) \rightarrow a \in A$. The optimal policy is given by

$$d^*(b_t) = \arg \max_{a \in A} \left[r(b,a) + \beta \sum_{o \in \Omega} P[o|b,a] v^*_{t-1}(\tau(b,a,o)) \right]$$

$$= \arg \max_{a \in A} \left[r(b,a) + \beta \sum_{s_k \in S} p_{ik}(a) \sum_{o_j \in \Omega} \phi_{kj}(a) b(s_i) v^*_{t-1}(\tau(b,a,o)) \right]$$

where β is the discount factor.

10.4.5 Approximate Solutions of POMDP

As pointed out in Papadimitrios and Tsitsiklis (1987) and Madani (2003), the exact solution of POMDP is usually hard except for very simple cases. The reasons POMDP problems are more difficult than fully observable MDP problems are as follows. First, in fully observable MDP an agent knows exactly the current state of its environment, which means that information from the past (i.e., its past observations and actions) is irrelevant to the current decision. This is precisely the Markov property. Because the agent does not fully observe the state of its environment in POMDP, past information becomes very relevant as it can help the agent to better estimate the current state of its environment. Unfortunately the number of possible states of past information increases with time, and this presents computational difficulties.

Second, the effects of actions in MDP are fully observable at the next decision epoch. In POMDP, the effects of an action are not fully observable at the next decision epoch, which means that we cannot clearly tell the effects of the current action from those of the agent's future behavior. The ability to make this distinction requires looking into the future and considering the combination of each action with each of the agent's possible behaviors in a potentially large number of steps. This again becomes computationally involved because the number of ways that the agent can behave can be exponential in the number of future steps considered.

A variety of algorithms have been developed for solving POMDPs. Unfortunately most of these techniques do not scale well to problems involving more than a few states due to their computational complexity. As a result of this problem

different approximate solutions have been proposed. These approximate solutions can be grouped into two classes, namely:

- Those solutions where approximation takes place in the process of solving the POMDP.
- Those solutions that use model approximation. Such solutions approximate POMDP itself by another problem that is easier to solve and use the solution of the latter to construct an approximate solution to the original POMPD. The approximation can be in different forms, including developing a more informative observation model, a more deterministic action model, a simpler state space, or a combination of two or all of the three alternatives.

More detailed information on these approximate solutions can be found in Monahan (1982), Lovejoy (1991), Zhang and Liu (1997), Yu (2006), and Wang (2007).

10.5 Problems

10.1 A recent college graduate is presented with N job offers, one after another. After looking at an offer, she must either accept it and thus terminate the process or reject it. A rejected offer is lost forever. The only information she has at any time is the relative rank of the current offer compared to the previous one. Assuming that the N jobs are offered to her in a random order, which means that $N!$ orderings of the offers are equally likely, define the problem as a sequential decision problem where the objective is to maximize the probability of selecting the best offer.

10.2 The price of a certain stock is fluctuating among \$10, \$20, and \$30 from month to month. Market analysis indicates that given that the stock is at \$10 in the current month, then in the following month it will be at \$10 with probability $4/5$ and at \$20 with probability $1/5$. Similarly, given that the stock is at \$20 in the current month, then in the following month it will be at \$10 with probability $1/4$, at \$20 with probability $1/2$, and at \$30 with probability $1/4$. Finally, given that the stock is at \$30 in the current month, then in the following month it will be at \$20 with probability $5/8$ and at \$30 with probability $3/8$. Given a discount factor of 0.9, use the policy improvement method to determine when to sell and when to hold the stock to maximize the expected long-run total discounted profit.

10.3 A farmer is considering the optimal course of action for his farm each year. The two options are to fertilize the farm and not to fertilize the farm. Optimality is defined such that the farmer will realize the highest expected

revenue at the end of 4 years. The conditions (or states) of the farm are good (state 1), fair (state 2), and poor (state 3). If P_k and R_k $(k = 1, 2)$, which represent the transition probability matrix and the reward function matrix respectively, are given by

$$P_1 = \begin{bmatrix} 0.2 & 0.5 & 0.3 \\ 0.0 & 0.5 & 0.5 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \qquad R_1 = \begin{bmatrix} 7 & 6 & 3 \\ 0 & 5 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 0.3 & 0.6 & 0.1 \\ 0.1 & 0.6 & 0.3 \\ 0.1 & 0.4 & 0.5 \end{bmatrix} \qquad R_2 = \begin{bmatrix} 6 & 5 & -1 \\ 7 & 4 & 0 \\ 6 & 3 & -2 \end{bmatrix}$$

determine the optimal expected revenue and the optimal decision in each of the 4 years.

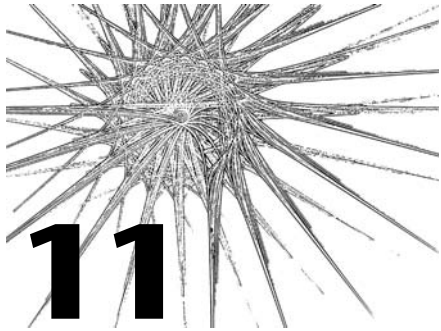
10.4 Solve Problem 10.3, assuming a discount factor of 0.9.

10.5 Consider a salesman who has offices in two towns called town 1 and town 2. He can be in one of these towns on any particular day but cannot split his time on any day between the two. On any day that he works in town 1, the probability of making a sale is 0.4; similarly, on any day that he is in town 2 the probability that he makes a sale is 0.25. The reward for making any sale is \$100, and the cost of switching towns is \$50. The salesman is considering two operating alternatives:

a. Stay in each town until he makes a sale and then go to the next town.
b. Work in one town one day and then go to the next town the next day whether or not a sale is made.

Define the four possible policies for the problem and find the salesman's long-run expected profit per day.

Hidden Markov Models



11.1 Introduction

With the exception of partially observable Markov processes, all the Markov models we have considered up until now have visible states in the sense that the state sequence of the processes is known. Thus, we can refer to these models as *visible Markov models*. In this chapter we consider a process in which the state sequence that the process passes through is not known but can only be guessed through a sequence of observations of the dynamics of the process. In the previous chapter we referred to this model as a partially observable Markov process. We devote this chapter to discussing this model, which is also called the *hidden Markov model* (HMM).

A hidden Markov model assumes that the underlying process is a Markov chain whose internal states are hidden from the observer. It is usually assumed that the number of states of the system and the state-transition probabilities are known. Thus, there are two parameters associated with each state of the Markov chain:

- Symbol emission probabilities that describe the probabilities of the different possible outputs from the state
- Transition probabilities that describe the probability of entering a new state from the current state

The visible Markov models have limited power in modeling many applications. Their limitation arises from the fact that they assume perfect knowledge of the system's internal dynamics and/or that a decision maker can control the system

evolution through some well-defined policy. Unfortunately, many applications do not conform to either of these two assumptions. For such applications the hidden Markov model can be used. HMMs are used in a variety of applications, but the two most important application areas are speech recognition and biological sequence analysis such as DNA sequence modeling.

HMMs were first applied to speech recognition in the early 1970s. The use of HMMs in speech recognition is reported in Levinson, Rabiner, and Sondhi (1983); and Juang and Rabiner (1991). In many languages, when the same word is pronounced in different contexts, at different times, and by different people, the sound can be extremely variable. In speech recognition, HMMs are used to characterize the sound signal in a word in probabilistic terms. A speech signal can be represented as a long sequence of about 256 category labels, such as the phonemes, that are valid for a particular language. From this set a speech recognition system has to determine what word was spoken. A well-trained speech recognition system assigns high probability to all sound sequences that are likely utterances of the word it models and low probability to any other sequence.

Applications of HMMs in bioinformatics have been reported in Thompson (1983); Churchill (1989, 1992); Guttorp, Newton, and Abkowitz (1990); Baldi (1994); and Krogh (1994). Another area of application of HMMs is financial time series modeling such as the stock market. Ryden (1998) used them to model temporal and distributional properties of daily data from speculative markets. Elliott and van der Hoek (1997) applied them to asset allocation problems.

Also, Albert (1991) and Le, Leroux, and Puterman (1992) have used HMMs to model time series of epileptic seizure counts. Similarly, Leroux and Puterman (1992) apply them to the pattern movement of a fetal lamb. HMMs have also been applied to hydroclimatology in Zucchini and Guttorp (1991).

HMMs have been used to model different communication environments. For example, they have been used to model fading communication channels in Turin and Sondhi (1993), Turin and van Nobelen (1998), Turin (2000), and Chao and Yao (1996). They are also used to model Internet traffic in Costamagna (2003).

The theory and methodology for HMMs are described in many sources. Tutorials on HMMs are given by Rabiner (1989) and Ephraim and Merhav (2002). Books on HMMs include Rabiner and Juang (1993); Elliott, Aggoun, and Moore (1995); MacDonald and Zucchini (1997); Durbin (1997); Koski (2001); and Cappe, Moulines, and Ryden (2005).

The relationship between the HMM and other Markov models is summarized in Table 11.1. The difference lies in whether or not the states are completely observable and whether or not the process can proceed without the intervention of an agent. Specifically, pure Markov chains have completely observable states, and the transitions are not under the control of an agent. Markov decision processes have completely observable states and the transitions are under the control of an agent. Partially observable Markov decision processes have partially observable

Table 11.1 How HMM is related to other Markov models

Markov Models	Are the states completely observable?	Do we have control over state transitions?
Markov Chains	Yes	No
Markov Decision Processes	Yes	Yes
POMDP	No	Yes
Hidden Markov Models	No	No

states, and the transitions are under the control of an agent. Finally, the HMM has partially observable states, and the states are not under the control of an agent.

11.2 HMM Basics

A hidden Markov model is a doubly stochastic process in which an underlying stochastic process that is not observable (i.e., it is hidden) can only be observed through another stochastic process that produces a sequence of observations. Thus, if $S = \{S_n, n = 1, 2, \ldots\}$ is a Markov process and $\Omega = \{\Omega_k, k = 1, 2, \ldots\}$ is a function of S, then S is a hidden Markov process (or hidden Markov model) that is observed through Ω, and we can write $\Omega_k = f(S_k)$ for some function f. In this way we can regard S as the *state process* that is hidden and Ω as the *observation process* that can be observed.

A hidden Markov model is usually defined as a 5-tuple $(S, \Omega, P, \Phi, \pi)$, where

- $S = \{s_1, s_2, \ldots, s_N\}$ is a finite set of N states
- $\Omega = \{o_1, o_2, \ldots, o_M\}$ is a finite set of M possible symbols
- $P = \{p_{ij}\}$ is the set of state-transition probabilities, where p_{ij} is the probability that the system goes from state s_i to state s_j
- $\Phi = \{\phi_i(o_k)\}$ are the observation probabilities, where $\phi_i(o_k)$ is the probability that the symbol o_k is emitted when the system is in state s_i
- $\pi = \{\pi_i\}$ are the initial state probabilities; that is, π_i is the probability that the system starts in state s_i

Because the states and output sequence are understood, it is customary to denote the parameters of an HMM by $\lambda = (P, \Phi, \pi)$.

As an illustration of HMMs, consider Bill whose mood changes with the weather in the fall. Assume that the fall weather can be in one of three states: Sunny, Cloudy, and Rainy. Given that it is Sunny on a given day, then the next day it will be Sunny with probability 0.5, Cloudy with probability 0.3, and Rainy with probability 0.2.

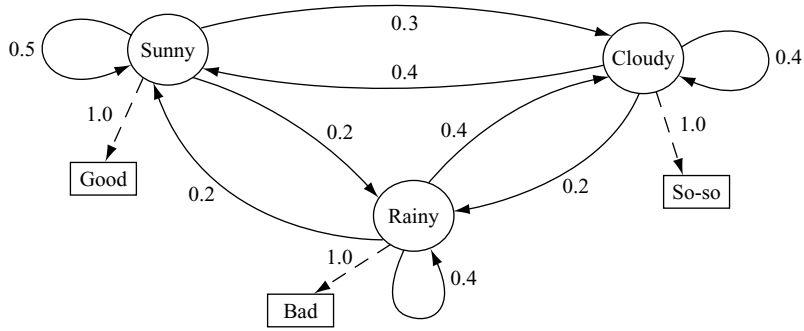


Figure 11.1. State-transition diagram for the fall weather.

Similarly, given it is Cloudy on a given day, then the next day it will be Sunny with probability 0.4, Cloudy with probability 0.4, and Rainy with probability 0.2. Finally, given that it is Rainy on a given day, then the next day it will be Sunny with probability 0.2, Cloudy with probability 0.4, and Rainy with probability 0.4. A study of Bill's mood swings shows that he is in a Good mood on a Sunny day, a So-so mood on a Cloudy day, and a Bad mood on a Rainy day. Thus, we can model the fall weather conditions and hence Bill's mood in the fall by a discrete-time Markov chain whose state-transition diagram is shown in Figure 11.1 along with the associated moods of Bill.

We can convert the process into a hidden Markov model as follows. Assume now that the weather still follows the probabilistic rules described earlier. However, Bill's mood can change with any weather condition. Specifically, when the weather is Sunny, he will be in a Good mood with probability 0.6, in a So-so mood with probability 0.3, and in a Bad mood with probability 0.1. Similarly, when the weather is Cloudy, he will be in a Good mood with probability 0.3, in a So-so mood with probability 0.5, and in a Bad mood with probability 0.2. Finally, when the weather is Rainy, he will be in a Good mood with probability 0.1, in a So-so mood with probability 0.3, and in a Bad mood with probability 0.6. The transition diagram for the new scheme is shown in Figure 11.2.

The problem with the new scheme is that when Bill is in, say, a So-so mood, we cannot know with certainty what the weather condition is. That is, we can no longer uniquely identify the state that a given mood was emitted from. Thus, if we observe Bill in the sequence of moods *Good-Good-Bad-Bad-So-so*, we cannot say exactly what weather state sequence produced the observed sequence of Bill's moods. For this reason, we say that the state sequence is "hidden" from us. However, we can calculate certain attributes of the model, such as the most likely state sequence that produced the observed sequence of Bill's moods. We will use this example to illustrate the analysis of HMMs in the remainder of this chapter. We can more formally represent an HMM as shown in Figure 11.3 where the S_i are the

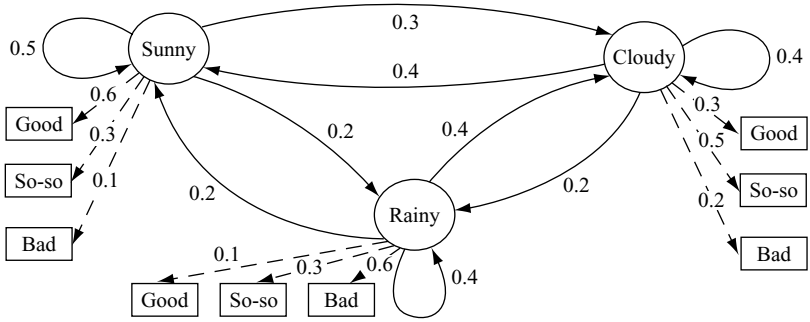


Figure 11.2. Hidden Markov model example.

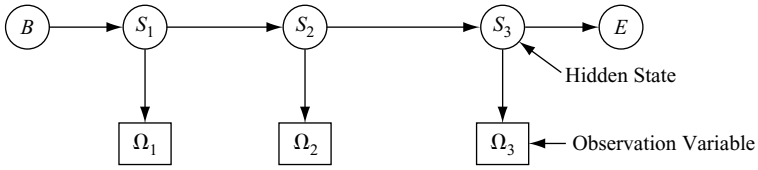


Figure 11.3. General structure of a hidden Markov model.

hidden states that we would like to estimate and the Ω_i are the observation random variables from which the S_i are to be estimated. The letters B and E indicate the *Beginning* and *End* of the sequence of states.

11.3 HMM Assumptions

Let $Q = \{q_t\}_{t=0}^{T}$ be the hidden state sequence in the interval $0 \leq t \leq T$, where $q_t \in S$. There are three major assumptions made in the analysis of HMM problems, which are as follows:

1. The *Markov assumption*: This assumption states that the next state depends only on the current state, which means that the transition probabilities are defined by

$$P(q_{t+1} = j | q_t = i, q_{t-1} = l, q_{t-2} = m, \ldots, q_0 = n)$$
$$= P(q_{t+1} = j | q_t = i) = p_{ij}$$

   In practice, the next state might depend on the past k states thereby giving rise to a kth order HMM. However, such models are more difficult to analyze than the preceding first-order HMMs.

2. The *stationarity assumption*: This assumption states that the state-transition probabilities are independent of the actual time the transitions take place. Thus, for any two times t_1 and t_2,

$$P(q_{t_1+1} = j | q_{t_1} = i) = P(q_{t_2+1} = j | q_{t_2} = i) = p_{ij}$$

3. The *observation independence assumption*: This assumption states that the current observation or output is statistically independent of previous observations. Thus, if we have the observation sequence $O = v_1, v_2, \ldots, v_T$, then

$$P(O|q_1, q_2, \ldots, q_T, \lambda) = \prod_{t=1}^{T} P(v_t|q_t, \lambda)$$

With these assumptions we can obtain the joint probability distribution $P(Q, O)$ by

$$P(Q, O) = \prod_{t=1}^{T} P(q_t|q_{t-1}) P(v_t|q_t)$$

where it is understood that $P(q_1|q_0) \equiv P(q_1)$.

11.4 Three Fundamental Problems

There are three fundamental problems in HMM:

1. The *evaluation problem*: Given a model $\lambda = (P, \Phi, \pi)$ and an observation sequence $O = v_1, v_2, \ldots, v_T$ of length T, where $v_i \in \Omega$, how do we efficiently compute the probability that the model generated the observation sequence; that is, what is $P[O|\lambda]$?
2. The *decoding problem*: Given a model $\lambda = (P, \Phi, \pi)$, what is the most likely sequence of hidden states that could have generated a given observation sequence? Thus, we would like to find $Q^* = \arg\max_Q P[Q, O|\lambda]$, where Q is the hidden state sequence, as defined earlier.
3. The *learning problem*: Given a set of observation sequences, find the HMM that best explains the observation sequences; that is, find the values for λ that maximize $P[O|\lambda]$, or $\lambda^* = \arg\max_\lambda P[O|\lambda]$. Stated differently, the problem is to estimate the most likely HMM parameters for a given observation sequence.

11.5 Solution Methods

There are different methods of solving HMM problems, depending on which of the three fundamental problems we would like to solve. The evaluation problem is usually solved by the *forward algorithm* and the *backward algorithm*. The decoding problem is usually solved by the *Viterbi algorithm*. Finally, the learning problem is solved by the *Baum-Welch algorithm*. These algorithms are described in the remainder of this section.

11.5.1 The Evaluation Problem

Consider a model $\lambda = (P, \Phi, \pi)$ and a given observation sequence $O = o_1, o_2, \ldots, o_T$. We would like to compute $P[O|\lambda]$, the probability of the observation sequence given a model. $P[O|\lambda]$ is given by

$$P[O|\lambda] = \sum_{Q} P[O|Q, \lambda] P[Q|\lambda]$$

where $Q = q_1, \ldots, q_T$ is a fixed sequence, $P[O|Q, \lambda]$ is the probability of the observation sequence O for the specific state sequence Q, and $P[Q|\lambda]$ is the probability of the sequence Q for a given model. Because we assume that the observations are independent, the two probabilities are given by

$$P[O|Q, \lambda] = \prod_{t=1}^{T} P[o_t|q_t, \lambda] = \phi_{q_1}(o_1)\phi_{q_2}(o_2)\ldots\phi_{q_T}(o_T)$$

$$P[Q|\lambda] = \pi_{q_1} p_{q_1 q_2} p_{q_2 q_3} \cdots p_{q_{T-1} q_T}$$

Thus, we obtain

$$\begin{aligned} P[O|\lambda] &= \sum_{Q} P[O|Q, \lambda] P[Q|\lambda] \\ &= \sum_{q_1, \ldots, q_T} \pi_{q_1} \phi_{q_1}(o_1) p_{q_1 q_2} \phi_{q_2}(o_2) p_{q_2 q_3} \cdots p_{q_{T-1} q_T} \phi_{q_T}(o_T) \end{aligned}$$

We make the following observations on the preceding result. First, the number of possible paths of length T is N^T, which means that the number of equations required to obtain the solution is exponential in T. Also, using this direct method to obtain $P[O|\lambda]$ requires on the order of $2TN^T$ calculations. Thus, even for small values of N and T the number of calculations is computationally large. For example, if we assume that $N = 3$ and $T = 100$, which can be associated with the problem with Bill's mood changes, the number of required calculations is on the order of $2 \times 100 \times 3^{100} \approx 10^{50}$. For this reason we need a more efficient algorithm to solve the evaluation problem. One such algorithm is the forward algorithm, which is discussed next.

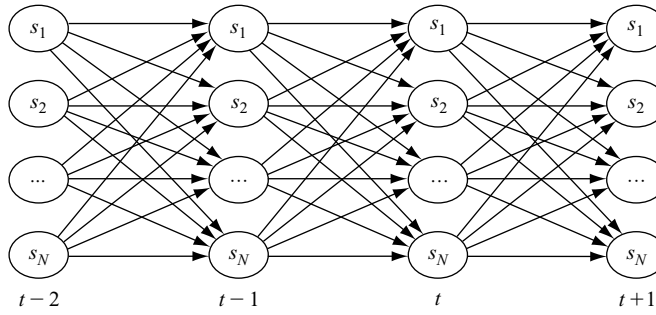


Figure 11.4. Trellis for forward algorithm.

The Forward Algorithm

One important observation in the calculation of $P[O|\lambda]$ by the direct method is that it requires many redundant calculations that are not saved and reused. To reduce the computational complexity we cache calculations. The caching is implemented as a trellis of states at each time step, as illustrated in Figure 11.4. A trellis can record the probability of all initial subpaths of the HMM that end in a certain state at a certain time. This allows the probability of longer subpaths to be worked out in terms of shorter subpaths.

A *forward probability variable* $\alpha_t(i)$ is defined as follows:

$$\alpha_t(i) = P[o_1, o_2, \ldots, o_t, q_t = s_i|\lambda] \quad t = 1, \ldots, T; i = 1, \ldots, N$$

That is, $\alpha_t(i)$ is the probability of being in state s_i at time t after having observed the sequence $\{o_1, o_2, \ldots, o_t\}$. It is calculated by summing probabilities for all incoming arcs at a trellis node. This follows from the fact that

$$\begin{aligned}
\alpha_t(i) &= P\big[o_1, o_2, \ldots, o_{t-1}, o_t, q_t = s_i|\lambda\big] \\
&= P\big[o_t|o_1, o_2, \ldots, o_{t-1}, q_t = s_i, \lambda\big]P\big[o_1, o_2, \ldots, o_{t-1}, q_t = s_i|\lambda\big] \\
&= P\big[o_t|q_t = s_i, \lambda\big]P\big[o_1, o_2, \ldots, o_{t-1}, q_t = s_i|\lambda\big] \\
&= P\big[o_t|q_t = s_i, \lambda\big]\sum_{s_j \in S} P\big[q_t = s_i|q_{t-1} = s_j, \lambda\big]P\big[o_1, o_2, \ldots, o_{t-1}, q_{t-1} = s_j|\lambda\big] \\
&= \phi_i(o_t)\sum_{j=1}^{N} p_{ji}\alpha_{t-1}(j) \quad 1 \le t \le T
\end{aligned}$$

where we have assumed that the observations are independent. Thus, if we work through the trellis filling in the values of the $\alpha_t(i)$, the sum of the final column of

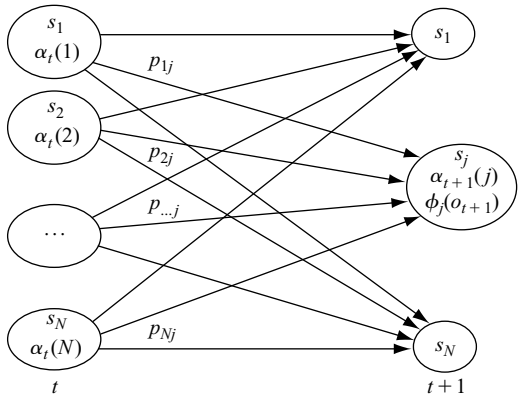


Figure 11.5. The induction step of the forward algorithm.

the trellis is the probability of the observation sequence. The forward algorithm works as follows:

1. Initialization:

$$\alpha_1(i) = \pi_i \phi_i(o_1) \quad 1 \leq i \leq N$$

2. Induction:

$$\alpha_{t+1}(j) = \left\{ \sum_{i=1}^{N} p_{ij} \alpha_t(i) \right\} \phi_j(o_{t+1}) \quad 1 \leq t \leq T-1,\ 1 \leq j \leq N$$

   This step is the key to the algorithm and can be represented as shown in Figure 11.5.

3. Update time: Set $t = t + 1$. If $t < T$, go to step 2; otherwise go to step 4.
4. Termination:

$$P[O|\lambda] = \sum_{i=1}^{N} \alpha_T(i) = \sum_{i=1}^{N} P[O, q_T = s_i|\lambda]$$

The forward algorithm requires $N(N+1)(T-1) + N$ multiplications and $N(N-1)(T-1)$ additions, giving a complexity on the order of N^2T rather than $2TN^T$. For example, in the Bill's mood changes example we considered the case when $N = 3$ and $T = 100$, which requires on the order of 900 calculations with the forward algorithm compared to the order of 10^{50} calculations required for the direct method.

Example 11.1 Consider Bill's mood change problem illustrated in Figure 11.2. Assume that we observed Bill in the following sequence of moods: *Good, Good,*

So-so, Bad, Bad. We are required to find the probability that the model generated such a sequence of moods using the forward algorithm. ■

Solution: Let S denote Sunny state, C denote Cloudy, and R denote Rainy. Similarly, let G denote Good mood, SS denote So-so mood, and B denote Bad mood. We assume that the process is equally likely to start in any state. That is, we assume that $\pi_S = \pi_C = \pi_R = 1/3$. Also, we have that $T = 5$. Then the initialization step becomes

$$\alpha_1(S) = \pi_S \phi_S(o_1) = \pi_S \phi_S(G) = \frac{1}{3}(0.6) = 0.2$$

$$\alpha_1(C) = \pi_C \phi_C(o_1) = \pi_C \phi_C(G) = \frac{1}{3}(0.3) = 0.1$$

$$\alpha_1(R) = \pi_R \phi_R(o_1) = \pi_R \phi_R(G) = \frac{1}{3}(0.1) = 0.033$$

The induction step for $t = 2$ is given by

$$\alpha_2(j) = \left\{ \sum_{i=1}^{N} p_{ij} \alpha_1(i) \right\} \phi_j(o_2) = \left\{ \sum_{i=1}^{N} p_{ij} \alpha_1(i) \right\} \phi_j(G)$$

$$\begin{aligned}
\alpha_2(S) &= \{p_{SS}\alpha_1(S) + p_{CS}\alpha_1(C) + p_{RS}\alpha_1(R)\}\phi_S(G) \\
&= \{(0.5)(0.2) + (0.4)(0.1) + (0.2)(0.033)\}(0.6) \\
&= 0.088 \\
\alpha_2(C) &= \{p_{SC}\alpha_1(S) + p_{CC}\alpha_1(C) + p_{RC}\alpha_1(R)\}\phi_C(G) \\
&= \{(0.3)(0.2) + (0.4)(0.1) + (0.4)(0.033)\}(0.3) \\
&= 0.034 \\
\alpha_2(R) &= \{p_{SR}\alpha_1(S) + p_{CR}\alpha_1(C) + p_{RR}\alpha_1(R)\}\phi_R(G) \\
&= \{(0.2)(0.2) + (0.2)(0.1) + (0.4)(0.033)\}(0.1) \\
&= 0.007
\end{aligned}$$

The induction step for $t = 3$ is given by

$$\alpha_3(j) = \left\{ \sum_{i=1}^{N} p_{ij} \alpha_2(i) \right\} \phi_j(o_3) = \left\{ \sum_{i=1}^{N} p_{ij} \alpha_2(i) \right\} \phi_j(SS)$$

$$\begin{aligned}
\alpha_3(S) &= \{p_{SS}\alpha_2(S) + p_{CS}\alpha_2(C) + p_{RS}\alpha_2(R)\}\phi_S(SS) \\
&= \{(0.5)(0.088) + (0.4)(0.034) + (0.2)(0.007)\}(0.3) \\
&= 0.018
\end{aligned}$$

$$\begin{aligned}
\alpha_3(C) &= \{p_{SC}\alpha_2(S) + p_{CC}\alpha_2(C) + p_{RC}\alpha_2(R)\}\,\phi_C(SS) \\
&= \{(0.3)(0.088) + (0.4)(0.034) + (0.4)(0.007)\}\,(0.5) \\
&= 0.021 \\
\alpha_3(R) &= \{p_{SR}\alpha_2(S) + p_{CR}\alpha_2(C) + p_{RR}\alpha_2(R)\}\,\phi_R(SS) \\
&= \{(0.2)(0.088) + (0.2)(0.034) + (0.4)(0.007)\}\,(0.3) \\
&= 0.008
\end{aligned}$$

The induction step for $t = 4$ is given by

$$\alpha_4(j) = \left\{\sum_{i=1}^{N} p_{ij}\alpha_3(i)\right\}\phi_j(o_4) = \left\{\sum_{i=1}^{N} p_{ij}\alpha_3(i)\right\}\phi_j(B)$$

$$\begin{aligned}
\alpha_4(S) &= \{p_{SS}\alpha_3(S) + p_{CS}\alpha_3(C) + p_{RS}\alpha_3(R)\}\,\phi_S(B) \\
&= \{(0.5)(0.018) + (0.4)(0.021) + (0.2)(0.008)\}\,(0.1) \\
&= 0.002 \\
\alpha_4(C) &= \{p_{SC}\alpha_3(S) + p_{CC}\alpha_3(C) + p_{RC}\alpha_3(R)\}\,\phi_C(B) \\
&= \{(0.3)(0.018) + (0.4)(0.021) + (0.4)(0.008)\}\,(0.2) \\
&= 0.003 \\
\alpha_4(R) &= \{p_{SR}\alpha_3(S) + p_{CR}\alpha_3(C) + p_{RR}\alpha_3(R)\}\,\phi_R(B) \\
&= \{(0.2)(0.018) + (0.2)(0.021) + 0.4(0.008)\}\,(0.6) \\
&= 0.007
\end{aligned}$$

The final induction step for $t = 5$ is given by

$$\alpha_5(j) = \left\{\sum_{i=1}^{N} p_{ij}\alpha_4(i)\right\}\phi_j(o_5) = \left\{\sum_{i=1}^{N} p_{ij}\alpha_4(i)\right\}\phi_j(B)$$

$$\begin{aligned}
\alpha_5(S) &= \{p_{SS}\alpha_4(S) + p_{CS}\alpha_4(C) + p_{RS}\alpha_4(R)\}\,\phi_S(B) \\
&= \{(0.5)(0.002) + (0.4)(0.003) + (0.2)(0.007)\}\,(0.1) \\
&= 0.0004 \\
\alpha_5(C) &= \{p_{SC}\alpha_4(S) + p_{CC}\alpha_4(C) + p_{RC}\alpha_4(R)\}\,\phi_C(B) \\
&= \{(0.3)(0.002) + (0.4)(0.003) + (0.4)(0.007)\}\,(0.2) \\
&= 0.0009
\end{aligned}$$

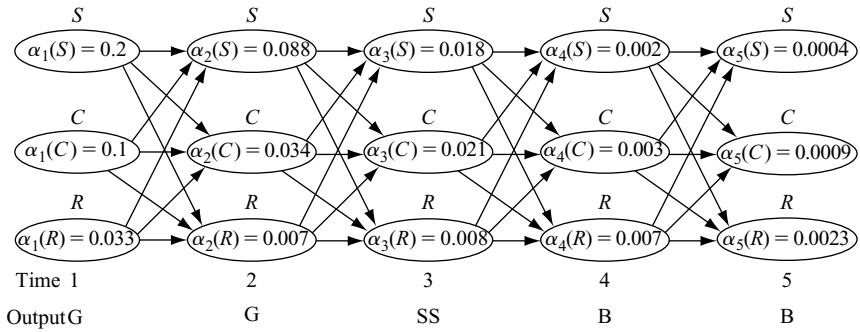


Figure 11.6. Trellis for Example 11.1.

$$\begin{aligned}\alpha_5(R) &= \{p_{SR}\alpha_4(S) + p_{CR}\alpha_4(C) + p_{RR}\alpha_4(R)\}\,\phi_R(B) \\ &= \{(0.2)(0.002) + (0.2)(0.003) + (0.4)(0.007)\}\,(0.6) \\ &= 0.0023\end{aligned}$$

Thus, at the termination of the algorithm we obtain the solution as

$$P[O = G, G, SS, B, B|\lambda] = \sum_{i=1}^{N} \alpha_T(i) = \alpha_5(S) + \alpha_5(C) + \alpha_5(R) = 0.0036$$

The trellis for the problem is shown in Figure 11.6. From the figure we can see one of the advantages of the trellis: It enables us to obtain such intermediate results as the probability that the model generated the sequence *Good, Good, So-so, Bad,* with the intermediate result $\alpha_4(S) + \alpha_4(C) + \alpha_4(R) = 0.012$; the probability that the model generated the sequence *Good*, *Good*, *So-so*, which is $\alpha_3(S) + \alpha_3(C) + \alpha_3(R) = 0.047$; and the probability that the model generated the sequence *Good*, *Good*, which is $\alpha_2(S) + \alpha_2(C) + \alpha_2(R) = 0.129$. All these results are valid for the assumption that the hidden process is equally likely to start from any of the three states.

The Backward Algorithm

The backward algorithm is a dual method to solve the evaluation problem. It starts by defining a *backward probability variable* $\beta_t(i)$ as follows:

$$\beta_t(i) = P[o_{t+1}, o_{t+2}, \ldots, o_T|q_t = s_i, \lambda] \qquad t = 1, \ldots, T;\ s_i \in S$$

That is, $\beta_t(i)$ is the conditional probability of the partial observation $o_{t+1}, o_{t+2}, \ldots, o_T$ given that the model is in state s_i at time t. Note that $\beta_t(i)$ is given by

$$\begin{aligned}
\beta_t(i) &= P[o_{t+1}, o_{t+2}, \ldots, o_T | q_t = s_i, \lambda] \\
&= \sum_{s_j \in S} P[o_{t+1}, o_{t+2}, \ldots, o_T, q_{t+1} = s_j | q_t = s_i, \lambda] \\
&= \sum_{s_j \in S} P[o_{t+1} | q_{t+1} = s_j] P[o_{t+2}, \ldots, o_T, q_{t+1} = s_j | q_t = s_i, \lambda] \\
&= \sum_{s_j \in S} P[o_{t+1} | q_{t+1} = s_j] P[o_{t+2}, \ldots, o_T | q_{t+1} = s_j] P[q_{t+1} = s_j | q_t = s_i, \lambda] \\
&= \sum_{j=1}^{N} \phi_j(o_{t+1}) \beta_{t+1}(j) p_{ij} \quad t = 1, \ldots, T; i = 1, \ldots, N
\end{aligned}$$

The backward algorithm works from right to left through the same trellis as follows:

1. Initialization:

$$\beta_T(i) = 1 \qquad 1 \le i \le N$$

2. Induction:

$$\beta_t(i) = \sum_{j=1}^{N} p_{ij} \beta_{t+1}(j) \phi_j(o_{t+1}) \qquad 1 \le t \le T-1, 1 \le i \le N$$

The induction step is illustrated in Figure 11.7.

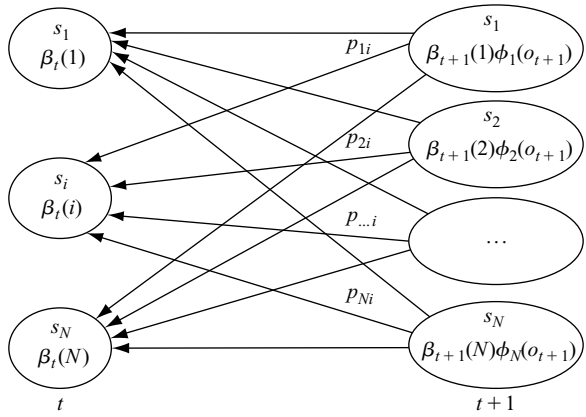


Figure 11.7. The induction step of the backward algorithm.

3. Update time: Set $t = t - 1$. If $t > 0$, go to step 2; otherwise go to step 4.
4. Termination:

$$P[O|\lambda] = \sum_{i=1}^{N} \beta_1(i)\alpha_1(i) = \sum_{i=1}^{N} \beta_1(i)\pi_i\phi_i(o_1)$$

The so-called forward-backward algorithm is obtained from the observation that for any t, $1 \leq t \leq T$, it can be shown that

$$P[O|\lambda] = \sum_{i=1}^{N} \beta_t(i)\alpha_t(i)$$

Example 11.2 Consider Bill's mood change problem illustrated in Figure 11.2. Assume that we observed Bill in the following sequence of moods: *Good, Good, So-so, Bad, Bad*. We are required to find the probability that the model generated such a sequence of moods using the backward algorithm. ■

Solution: We use the same notation as in Example 11.1. Because $T = 5$, the initialization step is as follows:

$$\beta_5(S) = \beta_5(C) = \beta_5(R) = 1$$

The induction step for $t = 4$ is given by

$$\begin{aligned}
\beta_4(S) &= p_{SS}\beta_5(S)\phi_S(o_5) + p_{SC}\beta_5(C)\phi_C(o_5) + p_{SR}\beta_5(R)\phi_R(o_5) \\
&= p_{SS}\phi_S(B) + p_{SC}\phi_C(B) + p_{SR}\phi_R(B) \\
&= (0.5)(0.1) + (0.3)(0.2) + (0.2)(0.6) = 0.23 \\
\beta_4(C) &= p_{CS}\beta_5(S)\phi_S(o_5) + p_{CC}\beta_5(C)\phi_C(o_5) + p_{CR}\beta_5(R)\phi_R(o_5) \\
&= p_{CS}\phi_S(B) + p_{CC}\phi_C(B) + p_{CR}\phi_R(B) \\
&= (0.4)(0.1) + (0.4)(0.2) + (0.2)(0.6) = 0.24 \\
\beta_4(R) &= p_{RS}\beta_5(S)\phi_S(o_5) + p_{RC}\beta_5(C)\phi_C(o_5) + p_{RR}\beta_5(R)\phi_R(o_5) \\
&= p_{RS}\phi_S(B) + p_{RC}\phi_C(B) + p_{RR}\phi_R(B) \\
&= (0.2)(0.1) + (0.4)(0.2) + (0.4)(0.6) = 0.34
\end{aligned}$$

The induction step for $t = 3$ is given by

$$\begin{aligned}
\beta_3(S) &= p_{SS}\beta_4(S)\phi_S(o_4) + p_{SC}\beta_4(C)\phi_C(o_4) + p_{SR}\beta_4(R)\phi_R(o_4) \\
&= p_{SS}\beta_4(S)\phi_S(B) + p_{SC}\beta_4(C)\phi_C(B) + p_{SR}\beta_4(R)\phi_R(B) \\
&= (0.5)(0.23)(0.1) + (0.3)(0.24)(0.2) + (0.2)(0.34)(0.6) \\
&= 0.0667
\end{aligned}$$

$$\begin{aligned}
\beta_3(C) &= p_{CS}\beta_4(S)\phi_S(o_4) + p_{CC}\beta_4(C)\phi_C(o_4) + p_{CR}\beta_4(R)\phi_R(o_4) \\
&= p_{CS}\beta_4(S)\phi_S(B) + p_{CC}\beta_4(C)\phi_C(B) + p_{CR}\beta_4(R)\phi_R(B) \\
&= (0.4)(0.23)(0.1) + (0.4)(0.24)(0.2) + (0.2)(0.34)(0.6) \\
&= 0.0692 \\
\beta_3(R) &= p_{RS}\beta_4(S)\phi_S(o_4) + p_{RC}\beta_4(C)\phi_C(o_4) + p_{RR}\beta_4(R)\phi_R(o_4) \\
&= p_{RS}\beta_4(S)\phi_S(B) + p_{RC}\beta_4(C)\phi_C(B) + p_{RR}\beta_4(R)\phi_R(B) \\
&= (0.2)(0.23)(0.1) + (0.4)(0.24)(0.2) + (0.4)(0.34)(0.6) \\
&= 0.1054
\end{aligned}$$

The induction step for $t = 2$ is given by

$$\begin{aligned}
\beta_2(S) &= p_{SS}\beta_3(S)\phi_S(o_3) + p_{SC}\beta_3(C)\phi_C(o_3) + p_{SR}\beta_3(R)\phi_R(o_3) \\
&= p_{SS}\beta_3(S)\phi_S(SS) + p_{SC}\beta_3(C)\phi_C(SS) + p_{SR}\beta_3(R)\phi_R(SS) \\
&= (0.5)(0.0667)(0.3) + (0.3)(0.0692)(0.5) + (0.2)(0.1054)(0.3) \\
&= 0.0267 \\
\beta_2(C) &= p_{CS}\beta_3(S)\phi_S(o_3) + p_{CC}\beta_3(C)\phi_C(o_3) + p_{CR}\beta_3(R)\phi_R(o_3) \\
&= p_{CS}\beta_3(S)\phi_S(SS) + p_{CC}\beta_3(C)\phi_C(SS) + p_{CR}\beta_3(R)\phi_R(SS) \\
&= (0.4)(0.0667)(0.3) + (0.4)(0.0692)(0.5) + (0.2)(0.1054)(0.3) \\
&= 0.0282 \\
\beta_2(R) &= p_{RS}\beta_3(S)\phi_S(o_3) + p_{RC}\beta_3(C)\phi_C(o_3) + p_{RR}\beta_3(R)\phi_R(o_3) \\
&= p_{RS}\beta_3(S)\phi_S(SS) + p_{RC}\beta_3(C)\phi_C(SS) + p_{RR}\beta_3(R)\phi_R(SS) \\
&= (0.2)(0.0667)(0.3) + (0.4)(0.0692)(0.5) + (0.4)(0.1054)(0.3) \\
&= 0.0305
\end{aligned}$$

The induction step for $t = 1$ is given by

$$\begin{aligned}
\beta_1(S) &= p_{SS}\beta_2(S)\phi_S(o_2) + p_{SC}\beta_2(C)\phi_C(o_2) + p_{SR}\beta_2(R)\phi_R(o_2) \\
&= p_{SS}\beta_2(S)\phi_S(G) + p_{SC}\beta_2(C)\phi_C(G) + p_{SR}\beta_2(R)\phi_R(G) \\
&= (0.5)(0.0267)(0.6) + (0.3)(0.0282)(0.3) + (0.2)(0.0305)(0.1) \\
&= 0.0112 \\
\beta_1(C) &= p_{CS}\beta_2(S)\phi_S(o_2) + p_{CC}\beta_2(C)\phi_C(o_2) + p_{CR}\beta_2(R)\phi_R(o_2) \\
&= p_{CS}\beta_2(S)\phi_S(G) + p_{CC}\beta_2(C)\phi_C(G) + p_{CR}\beta_2(R)\phi_R(G)
\end{aligned}$$

$$= (0.4)(0.0267)(0.6) + (0.4)(0.0282)(0.3) + (0.2)(0.0305)(0.1)$$
$$= 0.0104$$
$$\beta_1(R) = p_{RS}\beta_2(S)\phi_S(o_2) + p_{RC}\beta_2(C)\phi_C(o_2) + p_{RR}\beta_2(R)\phi_R(o_2)$$
$$= p_{RS}\beta_2(S)\phi_S(G) + p_{RC}\beta_2(C)\phi_C(G) + p_{RR}\beta_2(R)\phi_R(G)$$
$$= (0.2)(0.0267)(0.6) + (0.4)(0.0282)(0.3) + (0.4)(0.0305)(0.1)$$
$$= 0.0078$$

Thus, at the termination of the algorithm we obtain the solution as

$$P(O = G, G, SS, B, B|\lambda) = \sum_{i=1}^{N} \beta_1(i)\alpha_1(i) = \sum_{i=1}^{N} \beta_1(i)\pi_i\phi_i(o_1)$$
$$= \beta_1(S)\pi_S\phi_S(G) + \beta_1(C)\pi_C\phi_C(G) + \beta_1(R)\pi_R\phi_R(G)$$
$$= \frac{1}{3}\{(0.0112)(0.6) + (0.0104)(0.3) + (0.0078)(0.1)\}$$
$$= 0.00354$$

This result is consistent with the result obtained using the forward algorithm.

11.5.2 The Decoding Problem and the Viterbi Algorithm

The second HMM problem is the decoding problem, which seeks to find the best (or optimal) state sequence associated with a given observation sequence O of a given model λ. The first step is to define what we mean by an optimal state sequence because there are several possible optimality criteria. One possible definition is the state sequence that has the highest probability of producing the given observation sequence. Thus, we find the state sequence Q that maximizes $P[Q|O, \lambda]$. Unfortunately, for an observation sequence of T symbols and a system with N states, there are N^T possible sequences for Q. For our example of Bill's mood swings with $N = 3$ and $T = 100$, there are 3^{100} possible sequences.

Consider the case where we find the most likely states individually rather than as a whole sequence. For each time t, $1 \leq t \leq T$, we define the variable $\gamma_t(i)$ as follows:

$$\gamma_t(i) = P[q_t = s_i|O, \lambda] = \frac{P[q_t = s_i, O|\lambda]}{P[O|\lambda]}$$
$$= \frac{P[q_t = s_i, o_1, o_2, \ldots, o_T|\lambda]}{P[O|\lambda]}$$

$$= \frac{P[q_t = s_i, o_1, o_2, \ldots, o_t, o_{t+1}, \ldots, o_T|\lambda]}{P[O|\lambda]}$$

$$= \frac{P[o_1, o_2, \ldots, o_t, o_{t+1}, \ldots, o_T|q_t = s_i, \lambda]P[q_t = s_i|\lambda]}{P[O|\lambda]}$$

$$= \frac{P[o_1, o_2, \ldots, o_t|o_{t+1}, \ldots, o_T, q_t = s_i, \lambda]P[o_{t+1}, \ldots, o_T|q_t = s_i, \lambda]P[q_t = s_i|\lambda]}{P[O|\lambda]}$$

$$= \frac{P[o_1, o_2, \ldots, o_t|q_t = s_i, \lambda]P[q_t = s_i|\lambda]P[o_{t+1}, \ldots, o_T|q_t = s_i, \lambda]}{P[O|\lambda]}$$

$$= \frac{P[o_1, o_2, \ldots, o_t, q_t = s_i|\lambda]P[o_{t+1}, \ldots, o_T|q_t = s_i, \lambda]}{P[O|\lambda]}$$

$$= \frac{\alpha_t(i)\beta_t(i)}{\sum_{i=1}^{N} \beta_t(i)\alpha_t(i)}$$

where the last equality follows from our earlier definitions of $\alpha_t(i)$ and $\beta_t(i)$ and the statement that

$$P[O|\lambda] = \sum_{i=1}^{N} \beta_t(i)\alpha_t(i)$$

Note that

$$\sum_{i=1}^{N} \gamma_t(i) = 1$$

which makes $\gamma_t(i)$ a true conditional probability. Thus, the individual most likely state at time t is

$$q_t^* = \arg \max_{1 \le i \le N} \{\gamma_t(i)\} \qquad 1 \le t \le T$$

Thus the method generates the most likely state sequence $Q^* = \{q_1^*, q_2^*, \ldots, q_T^*\}$ for the given observation sequence $O = \{o_1, o_2, \ldots, o_T\}$. Unfortunately, the scheme might generate an unlikely state sequence because it does not take state-transition probabilities into consideration. For example, if we have a sequence that includes two neighboring states s_i and s_j in the sequence whose transition probability $p_{ij} = 0$, then the result is an invalid state sequence. An efficient method that avoids such unlikely sequences is the Viterbi algorithm, which is based on dynamic programming.

The Viterbi Algorithm

The Viterbi algorithm was originally designed for decoding convolutional codes and is now applied in many other areas. In HMMs it is used to find the most likely state sequence $Q^* = \{q_1^*, q_2^*, \ldots, q_T^*\}$ for a given observation sequence $O = \{o_1, o_2, \ldots, o_T\}$. As defined earlier, let the function

$$\arg \max_y \{z\}$$

denote the argument y that corresponds to the maximum of the expression z. The Viterbi algorithm simultaneously maximizes both the joint probability $P[q, O]$ and the conditional probability $P[q|O]$ due to the fact that

$$\arg \max_Q \{P[Q|O, \lambda]\} = \arg \max_Q \left\{ \frac{P[Q, O|\lambda]}{P[O|\lambda]} \right\} = \arg \max_Q \{P[Q, O|\lambda]\}$$

The algorithm defines the variable $\delta_t(i)$ as follows:

$$\delta_t(i) = \max_{q_1, q_2, \ldots, q_{t-1}} P[q_1, q_2, \ldots, q_{t-1}, q_t = s_i, o_1, o_2, \ldots, o_{t-1}, o_t|\lambda]$$

That is, $\delta_t(i)$ is the largest probability along a single path that accounts for the first t observations and ends in state s_i. Thus, it is the probability of the most likely state path for the partial observation sequence. Another variable $\psi_t(j)$ stores the node of the incoming arc that leads to this most probable path. That is,

$$\psi_t(j) = \arg \max_{1 \le i \le N} \{\delta_{t-1}(i) p_{ij}\}$$

The details of the algorithm are as follows:

1. Initialization:

$$\begin{aligned} \delta_1(i) &= \pi_i b_i(o_1) \\ \psi_1(i) &= 0 \end{aligned} \qquad 1 \le i \le N$$

2. Recursion:

$$\begin{aligned} \delta_t(j) &= \max_{1 \le i \le N} \{\delta_{t-1}(i) p_{ij}\} b_j(o_t) \\ \psi_t(j) &= \arg \max_{1 \le i \le N} \{\delta_{t-1}(i) p_{ij}\} \end{aligned} \qquad 1 \le j \le N, 2 \le t \le T$$

Note that this step is similar to the induction step of the forward algorithm. The main difference between the two is that the forward algorithm uses summation over previous states while the Viterbi algorithm uses minimization.

3. Update time: Set $t = t + 1$. If $t < T$, go to step 2; otherwise go to step 4.
4. Termination:

$$P^* = \max_{1 \le i \le N} \{\delta_T(i)\}$$

$$q_T^* = \arg \max_{1 \le i \le N} \{\delta_T(i)\}$$

5. Path (or state sequence) backtracking:

$$q_t^* = \psi_{t+1}(q_{t+1}^*) \qquad t = T-1, T-2, \ldots, 1$$

The backtracking step allows the best state sequence to be found from the back pointers stored in the recursion step.

Example 11.3 Consider Bill's mood change problem illustrated in Figure 11.2. Assume that we observed Bill in the following sequence of moods: *Good, Good, So-so, Bad, Bad*. We are required to find the most likely state sequence that generated such a sequence of moods using the Viterbi algorithm. ■

Solution: We use the same notation and assumptions of initial distribution as in Example 11.1. The initialization step is as follows:

$$\delta_1(S) = \pi_S \phi_S(o_1) = \pi_S \phi_S(G) = \frac{1}{3}(0.6) = 0.2$$

$$\delta_1(C) = \pi_C \phi_C(o_1) = \pi_C \phi_C(G) = \frac{1}{3}(0.3) = 0.1$$

$$\delta_1(R) = \pi_R \phi_R(o_1) = \pi_R \phi_R(G) = \frac{1}{3}(0.1) = 0.033$$

$$\psi_1(S) = \psi_1(C) = \psi_1(R) = 0$$

The recursion step for $t = 2$ is given by

$$\begin{aligned}
\delta_2(S) &= \max\{\delta_1(S)p_{SS}, \delta_1(C)p_{CS}, \delta_1(R)p_{RS}\}\phi_S(o_2) \\
&= \max\{\delta_1(S)p_{SS}, \delta_1(C)p_{CS}, \delta_1(R)p_{RS}\}\phi_S(G) \\
&= \max\{(0.2)(0.5), (0.1)(0.4), (0.033)(0.2)\}(0.6) \\
&= \max\{0.1, 0.04, 0.066\}(0.6) = 0.06
\end{aligned}$$

$$\psi_2(S) = S$$

$$\begin{aligned}
\delta_2(C) &= \max\{\delta_1(S)p_{SC}, \delta_1(C)p_{CC}, \delta_1(R)p_{RC}\}\phi_C(o_2) \\
&= \max\{\delta_1(S)p_{SC}, \delta_1(C)p_{CC}, \delta_1(R)p_{RC}\}\phi_C(G)
\end{aligned}$$

$$
\begin{aligned}
&= \max\{(0.2)(0.3), (0.1)(0.4), (0.033)(0.4)\}(0.3) \\
&= \max\{0.06, 0.04, 0.0132\}(0.6) = 0.036 \\
\psi_2(C) &= S \\
\delta_2(R) &= \max\{\delta_1(S)p_{SR}, \delta_1(C)p_{CR}, \delta_1(R)p_{RR}\}\phi_R(o_2) \\
&= \max\{\delta_1(S)p_{SR}, \delta_1(C)p_{CR}, \delta_1(R)p_{RR}\}\phi_R(G) \\
&= \max\{(0.2)(0.2), (0.1)(0.2), (0.033)(0.4)\}(0.1) \\
&= \max\{0.04, 0.02, 0.0132\}(0.1) = 0.004 \\
\psi_2(R) &= S
\end{aligned}
$$

The recursion step for $t = 3$ is given by

$$
\begin{aligned}
\delta_3(S) &= \max\{\delta_2(S)p_{SS}, \delta_2(C)p_{CS}, \delta_2(R)p_{RS}\}\phi_S(o_3) \\
&= \max\{\delta_2(S)p_{SS}, \delta_2(C)p_{CS}, \delta_2(R)p_{RS}\}\phi_S(SS) \\
&= \max\{(0.06)(0.5), (0.036)(0.4), (0.004)(0.2)\}(0.3) \\
&= \max\{0.03, 0.0144, 0.0008\}(0.3) = 0.009 \\
\psi_3(S) &= S \\
\delta_3(C) &= \max\{\delta_2(S)p_{SC}, \delta_2(C)p_{CC}, \delta_2(R)p_{RC}\}\phi_C(o_3) \\
&= \max\{\delta_2(S)p_{SC}, \delta_2(C)p_{CC}, \delta_2(R)p_{RC}\}\phi_C(SS) \\
&= \max\{(0.06)(0.3), (0.036)(0.4), (0.004)(0.4)\}(0.5) \\
&= \max\{0.018, 0.0144, 0.0016\}(0.5) = 0.009 \\
\psi_3(C) &= S \\
\delta_3(R) &= \max\{\delta_2(S)p_{SR}, \delta_2(C)p_{CR}, \delta_2(R)p_{RR}\}\phi_R(o_3) \\
&= \max\{\delta_2(S)p_{SR}, \delta_2(C)p_{CR}, \delta_2(R)p_{RR}\}\phi_R(SS) \\
&= \max\{(0.06)(0.2), (0.036)(0.2), (0.004)(0.4)\}(0.3) \\
&= \max\{0.012, 0.0072, 0.0016\}(0.3) = 0.0036 \\
\psi_3(R) &= S
\end{aligned}
$$

The recursion step for $t = 4$ is given by

$$
\begin{aligned}
\delta_4(S) &= \max\{\delta_3(S)p_{SS}, \delta_3(C)p_{CS}, \delta_3(R)p_{RS}\}\phi_S(o_4) \\
&= \max\{\delta_3(S)p_{SS}, \delta_3(C)p_{CS}, \delta_3(R)p_{RS}\}\phi_S(B)
\end{aligned}
$$

$$
\begin{aligned}
&= \max\{(0.009)(0.5), (0.009)(0.4), (0.0036)(0.2)\}(0.1) \\
&= \max\{0.0045, 0.0036, 0.00072\}(0.1) = 0.00045 \\
\psi_4(S) &= S \\
\delta_4(C) &= \max\{\delta_3(S)p_{SC}, \delta_3(C)p_{CC}, \delta_3(R)p_{RC}\}\phi_C(o_4) \\
&= \max\{\delta_3(S)p_{SC}, \delta_3(C)p_{CC}, \delta_3(R)p_{RC}\}\phi_C(B) \\
&= \max\{(0.009)(0.3), (0.009)(0.4), (0.0036)(0.4)\}(0.2) \\
&= \max\{0.0027, 0.0036, 0.00144\}(0.2) = 0.00072 \\
\psi_4(C) &= C \\
\delta_4(R) &= \max\{\delta_3(S)p_{SR}, \delta_3(C)p_{CR}, \delta_3(R)p_{RR}\}\phi_R(o_4) \\
&= \max\{\delta_3(S)p_{SR}, \delta_3(C)p_{CR}, \delta_3(R)p_{RR}\}\phi_R(B) \\
&= \max\{(0.009)(0.2), (0.009)(0.2), (0.0036)(0.4)\}(0.6) \\
&= \max\{0.0018, 0.0018, 0.00144\}(0.6) = 0.00108 \\
\psi_4(R) &= S, C
\end{aligned}
$$

The recursion step for $t = 5$ is given by

$$
\begin{aligned}
\delta_5(S) &= \max\{\delta_4(S)p_{SS}, \delta_4(C)p_{CS}, \delta_4(R)p_{RS}\}\phi_S(o_5) \\
&= \max\{\delta_4(S)p_{SS}, \delta_4(C)p_{CS}, \delta_4(R)p_{RS}\}\phi_S(B) \\
&= \max\{(0.00045)(0.5), (0.00072)(0.4), (0.00108)(0.2)\}(0.1) \\
&= \max\{0.000225, 0.000288, 0.000216\}(0.1) = 0.0000288 \\
\psi_5(S) &= C \\
\delta_5(C) &= \max\{\delta_4(S)p_{SC}, \delta_4(C)p_{CC}, \delta_4(R)p_{RC}\}\phi_C(o_5) \\
&= \max\{\delta_4(S)p_{SC}, \delta_4(C)p_{CC}, \delta_4(R)p_{RC}\}\phi_C(B) \\
&= \max\{(0.00045)(0.3), (0.00072)(0.4), (0.00108)(0.4)\}(0.2) \\
&= \max\{0.000135, 0.000288, 0.000432\}(0.2) = 0.0000864 \\
\psi_5(C) &= R \\
\delta_5(R) &= \max\{\delta_4(S)p_{SR}, \delta_4(C)p_{CR}, \delta_4(R)p_{RR}\}\phi_R(o_5) \\
&= \max\{\delta_4(S)p_{SR}, \delta_4(C)p_{CR}, \delta_4(R)p_{RR}\}\phi_R(B) \\
&= \max\{(0.00045)(0.2), (0.00072)(0.2), (0.00108)(0.4)\}(0.6) \\
&= \max\{0.00009, 0.000144, 0.000432\}(0.6) = 0.0002592 \\
\psi_5(R) &= R
\end{aligned}
$$

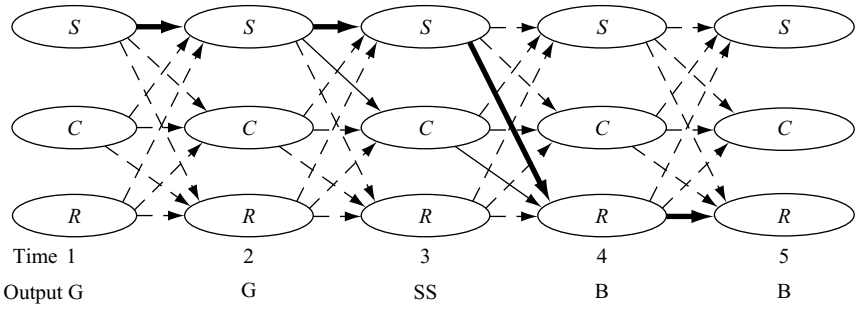


Figure 11.8. Trellis for the Viterbi algorithm.

The termination step is given by

$$\begin{aligned} P^* &= \max\{\delta_5(S), \delta_5(C), \delta_5(R)\} \\ &= \max\{0.0000288, 0.0000864, 0.0002592\} = 0.0002592 \\ {q_T}^* &= \arg\max\{\delta_5(S), \delta_5(C), \delta_5(R)\} = R \end{aligned}$$

The path backtracking step is as follows:

$$\begin{aligned} {q_t}^* &= \psi_{t+1}({q_{t+1}}^*) \\ {q_4}^* &= \psi_5({q_5}^*) = \psi_5(R) = R \\ {q_3}^* &= \psi_4({q_4}^*) = \psi_4(R) = S, C \\ {q_2}^* &= \psi_3({q_3}^*) = \psi_3(S) = \psi_3(C) = S \\ {q_1}^* &= \psi_2({q_2}^*) = \psi_2(S) = S \end{aligned}$$

Thus, one of the most likely state sequences is $Q^* = \{S, S, S, R, R\}$. This path is illustrated in Figure 11.8.

The other most likely state sequence is $Q^* = \{S, S, C, R, R\}$, which is differentiated from the previous sequence by the thin solid lines between times 2 and 3 and times 3 and 4 in Figure 11.8.

11.5.3 The Learning Problem and the Baum-Welch Algorithm

The learning problem deals with how we can adjust the HMM parameters so that the given set of observations, which is usually referred to as the *training set*, is represented by the model in the best way for the intended application. Because we are looking for the "best" way to represent the observation, we are solving an optimization problem, and we must define the criterion for optimization. The most

commonly used optimization criterion is the maximum likelihood criterion, which seeks to find the parameters of the HMM that maximize the probability of a given observation sequence. That is, we find the following solution:

$$\lambda^* = \arg\max_{\lambda}\{P[O|\lambda]\}$$

Unfortunately, this problem is so complex that there is no known method to analytically obtain λ that maximizes $P[O|\lambda]$, but we can choose the model parameters in such a way that $P[O|\lambda]$ is locally maximized. This method is an iterative solution called the *Baum-Welch algorithm*, which is sometimes called the *forward-backward algorithm* and is a special case of the *Expectation Maximization* (EM) method.

The Baum-Welch Algorithm

The algorithm starts by setting the parameters P, Φ, and π to some initial values that can be chosen from some prior knowledge or from some uniform distribution. Then using the current model, all possible paths for each training set are considered to get new estimates $\hat{P}$, $\hat{\Phi}$, and $\hat{\pi}$. The procedure is repeated until there are insignificant changes in the parameters of the current model.

As a forward-backward algorithm the Baum-Welch algorithm uses the same forward probability variable $\alpha_t(i)$ and backward probability variable $\beta_t(i)$ used in the evaluation problem that were defined earlier as follows:

$$\alpha_t(i) = P[o_1, o_2, \ldots, o_t, q_t = s_i|\lambda]$$
$$\beta_t(i) = P[o_{t+1}, o_{t+2}, \ldots, o_T|q_t = s_i, \lambda]$$

where $t = 1, \ldots, T$; $i = 1, \ldots, N$. Recall that $\alpha_t(i)$ is the probability of being in state s_i at time t after having observed the sequence $\{o_1, o_2, \ldots, o_t\}$, and $\beta_t(i)$ is the conditional probability of the partial observation $o_{t+1}, o_{t+2}, \ldots, o_T$ given that the model is in state s_i at time t. Also, recall that these variables are computed inductively as follows:

$$\alpha_t(i) = \pi_i \phi_i(o_1) \qquad 1 \le i \le N$$
$$\alpha_{t+1}(j) = \left\{\sum_{i=1}^{N} p_{ij}\alpha_t(i)\right\}\phi_j(o_{t+1}) \qquad 1 \le t \le T-1, 1 \le j \le N$$
$$\beta_T(i) = 1 \qquad 1 \le i \le N$$
$$\beta_t(i) = \sum_{j=1}^{N} p_{ij}\beta_{t+1}(j)\phi_j(o_{t+1}) \qquad 1 \le t \le T-1, 1 \le i \le N$$

As in the Viterbi algorithm, we define the probability variable $\gamma_t(i)$ as follows:

$$\gamma_t(i) = \frac{\alpha_t(i)\beta_t(i)}{P[O|\lambda]} = \frac{\alpha_t(i)\beta_t(i)}{\sum_{i=1}^{N} \beta_t(i)\alpha_t(i)}$$

This is the probability of being in state s_i at time t given the entire observation sequence and the model. Summing $\gamma_t(i)$ over t gives the expected number of transitions made from state s_i. Finally, we define the variable $\xi_t(i,j)$ as the probability of being in state s_i at time t and in state s_j at time $t+1$ given the observation sequence and the model; that is,

$$\xi_t(i,j) = P(q_t = s_i, q_{t+1} = s_j | O, \lambda) = \frac{P[q_t = s_i, q_{t+1} = s_j, O|\lambda]}{P[O|\lambda]}$$

$$= \frac{\alpha_t(i)\, p_{ij}\phi_j(o_{t+1})\beta_{t+1}(j)}{\sum_{i=1}^{N} \beta_t(i)\alpha_t(i)} = \frac{\alpha_t(i)\, p_{ij}\phi_j(o_{t+1})\beta_{t+1}(j)}{\sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_t(i)\, p_{ij}\phi_j(o_{t+1})\beta_{t+1}(j)}$$

Note that $\gamma_t(i)$ and $\xi_t(i,j)$ are related as follows:

$$\gamma_t(i) = \sum_{j=1}^{N} \xi_t(i,j)$$

Summing $\xi t(i,j)$ over t gives a value that can be interpreted as the expected number of transitions from state s_i to s_j. Now, we can estimate p_{ij} as the expected number of transitions from state s_i to s_j normalized by the expected number of transitions from state s_i; that is,

$$\bar{p}_{ij} = \frac{\sum_{t=1}^{T-1} \xi_t(i,j)}{\sum_{t=1}^{T-1} \gamma_t(i)}$$

Similarly, we can estimate the probability that the output symbol $o_t = k$ is emitted at time t when the system is in state s_j as the ratio of the expected number of times the system is in state s_j and observing the symbol k to the expected number of times it is in state s_j; that is,

$$\bar{\phi}_j(k) = \frac{\sum_{t=1, o_t=k}^{T} \gamma_t(j)}{\sum_{t=1}^{T} \gamma_t(j)}$$

The details of the algorithm are as follows:

1. Obtain the estimate of the initial state distribution for state s_i as the expected frequency with which state s_i is visited at time $t = 1$; that is

$$\bar{\pi}_i = \gamma_1(i)$$

2. Obtain the estimates $\bar{p}_{ij}$ and $\bar{\phi}_j(k)$ as defined earlier.
3. Let the current model be $\lambda = (P, \Phi, \pi)$ that is used to compute the values of $\bar{p}_{ij}$ and $\bar{\phi}_j(k)$. Let the reestimated model be $\bar{\lambda} = (\bar{P}, \bar{\Phi}, \bar{\pi})$. Using the updated model $\bar{\lambda} = (\bar{P}, \bar{\Phi}, \bar{\pi})$ we perform a new iteration until convergence.
4. If $P[O|\bar{\lambda}] - P[O|\lambda] < \delta$, stop, where δ is a predefined threshold value.

The EM theory states that after each iteration one of two things can happen:

a. $\bar{\lambda}$ is more likely than λ in the sense that $P[O|\bar{\lambda}] > P[O|\lambda]$, or
b. we have reached a stationary point of the likelihood function at which $\bar{\lambda} = \lambda$.

It must be emphasized that the algorithm is not guaranteed to converge at the global maximum, which is the main problem with the algorithm. This is because many local maxima of the target function might exist. One way to deal with this problem is to run the algorithm several times, each time with different initial values for λ. This problem notwithstanding, the algorithm has been found to yield good results in practice.

11.6 Types of Hidden Markov Models

HMMs can be classified according to the nature of the distribution of the output probabilities $\phi_i(v_k)$. If the observations v_k are discrete quantities, as we have assumed up until now, then $\phi_i(v_k)$ are PMFs, and the HMM is called a *discrete* HMM. If the observations are continuous random variables, then the HMM is called a *continuous* HMM. In this case, $\phi_i(v_k)$ are PDFs and we have a continuous observation space.

Another popular model is the *left-to-right HMM*. A left-to-right HMM has a left-to-right transition to the next state as well as a self-transition. The self-transition is used to model contiguous features in the same state. It is popularly used to model speech as a time sequence of distinct events that start at an initial state, which is usually labeled *Begin*, and end at a final state, which is usually labeled *End*. The model is also used in *profile HMMs* that will be discussed later. An example of a left-to-right HMM is illustrated in Figure 11.9 where the states labeled B and E denote Begin and End respectively of a sequence.

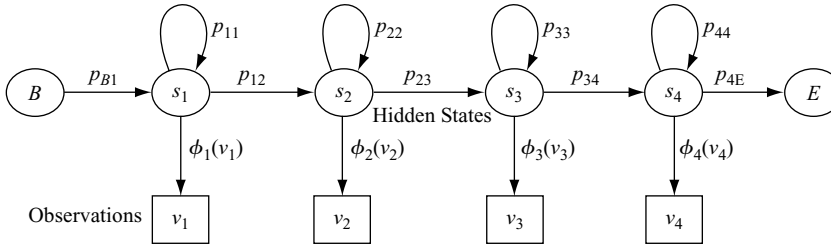
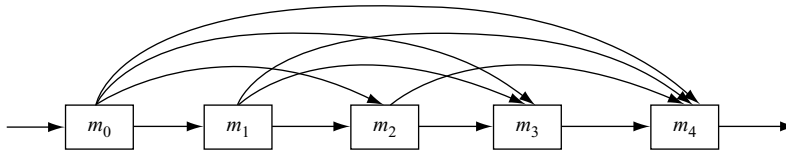
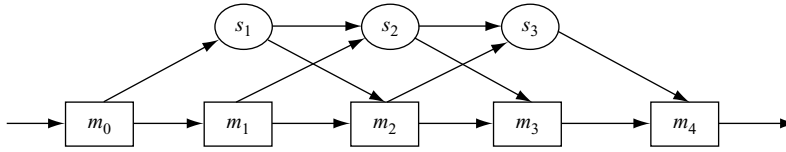


Figure 11.9. Example of left-to-right HMM.



(a) Without Silent States



(b) With Silent States

Figure 11.10. The use of silent states.

11.7 Hidden Markov Models with Silent States

Silent states are special states that do not emit any symbols. They are usually introduced to enhance the clarity of the HMM. In particular they are used to reduce the number of transitions in a model. For example, if every state is connected to many other states, silent states can be used to skip any state that emits symbols, as shown in Figure 11.10 where m denotes an emitting state (or state that emits symbols) and s denotes a silent state. An emitting state is also called a *match* state.

The silent states enable any or all match states to be skipped.

11.8 Extensions of Hidden Markov Models

Different extensions of the HMM have been proposed by adding flexibility to the model, either through introducing additional sets of new features, developing

dependencies among existing feature sets, or creating additional relationships between existing features. In this section we provide a brief description of five of these extensions, which are the hierarchical HMM, factorial HMM, coupled HMM, hidden semi-Markov model, and profile HMM.

11.8.1 Hierarchical Hidden Markov Model

Hierarchical HMM (HHMM) was proposed by Fine (1998) to extend the standard HMM in a hierarchical manner to a hierarchy of hidden states. Alternatively, it can be considered a structured multilevel model that makes each hidden state in the standard HMM an HHMM as well. This means that each state can emit sequences rather than single symbols. There are two types of states: the "normal" HMM states $S = \{s_1, s_2, \ldots, s_N\}$, which are called *production states*, and *internal* states $I = \{i_1, i_2, \ldots, i_M\}$ that can connect to other states but cannot produce observations. Only the production states can produce observations. There are *end states* at every level from where control is returned to the immediate upper level internal state from where the transition to the sub-HMM originated. That is, entering an end state causes a sub-HMM to terminate, and a transition to an end state could be triggered by some environmental condition.

An example of the HHMM is illustrated in Figure 11.11 where i_{kl} is an internal state $l, l = 1, 2, \ldots$, at level $k, k = 0, 1, \ldots$; q_{kl} is a production state $l, l = 1, 2, \ldots$, at level k, $k = 1, 2, \ldots$; and e_{kl} is an end state k at level l. The output states are $v_k, k = 1, 2, \ldots$.

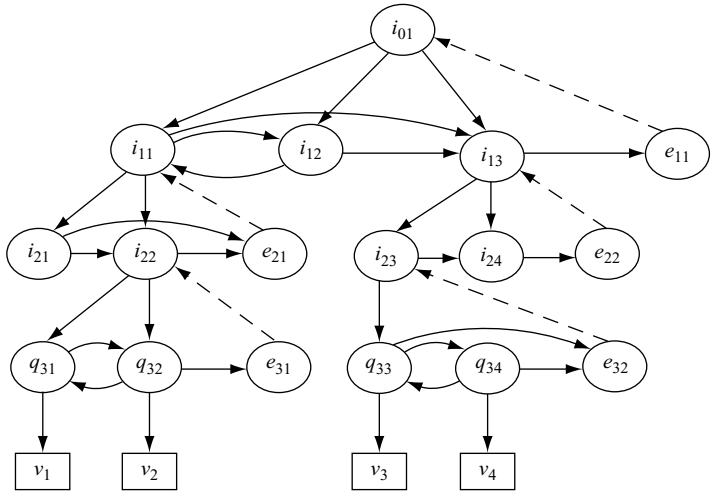


Figure 11.11. Example of a four-level HHMM.

HHMM is useful in modeling domains with hierarchical structures. For example, it has been used by Fine (1998) to model handwriting, by Ivanov and Bobick (2000) to model visual action recognition, and by Bui (2001) to model spatial navigation.

One of the limitations of HHMM is its computational complexity, which is known to be $O(T^3Nb^2)$, where T is the length of the observation sequence, N is the total number of hidden states, and b is the maximum number of substates of each state. Also, the state hierarchy in the original HHMM is restricted to the tree structure. However, Bui (2004) presents a general HHMM in which the state hierarchy can be a lattice that permits arbitrary sharing of substructures.

11.8.2 Factorial Hidden Markov Model

Factorial hidden Markov model (FHMM) was proposed by Ghahramani and Jordan (1997). In a regular HMM, information about the past is conveyed through a single discrete variable, which is the hidden state. FHMM permits the state to be factored into multiple state variables and is therefore represented in a distributed manner. Thus, FHMM can be used to represent a combination of multiple signals produced independently where the characteristics of each signal are described by a distinct Markov chain. For example, Kadirkamanathan and Varga (1991) used one chain to represent speech and another chain to represent some dynamic noise source. Similarly, Logan and Moreno (1998) used two chains to represent two underlying concurrent subprocesses governing the realization of an observation vector in speech processing. Jacobs (2002) developed a generalized backfitting algorithm that computes customized error signals for each hidden Markov chain of an FHMM and then trains each chain one at a time using conventional techniques. Figure 11.12 represents a factorial HMM with two underlying Markov chains governing two subprocesses.

While FHMM enhances the representation power of hidden states by using multiple hidden state chains for one HMM, it also makes the model training difficult

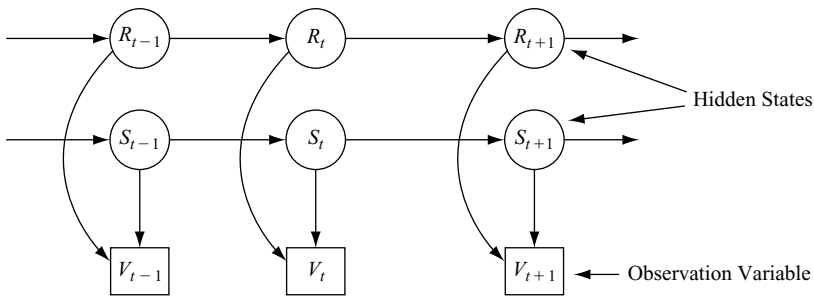


Figure 11.12. FHMM with two underlying Markov chains.

and sometimes impossible when the number of hidden state chains is large. Thus, the combinatorial nature of the model renders the exact algorithm intractable. Consequently, only approximate inference can be obtained using Gibbs sampling or variational methods.

11.8.3 Coupled Hidden Markov Model

Coupled hidden Markov model (CHMM) was introduced by Brand (1996, 1997) to solve one of the limitations of regular HMM, which is its strong restrictive assumptions about the system generating the signal. HMM essentially assumes that there is a single process with a small number of states and an extremely limited state memory. The single-process model is often inappropriate for vision, speech, and other applications that are composed of multiple interacting processes. Coupled HMM provides an efficient way to resolve many of these problems by coupling HMMs to model interacting processes. It is particularly useful for modeling multimedia applications that integrate multiple streams of data. In this case, one HMM can be used to model one data stream and the model becomes a collection of HMMs.

The simplest type of CHMM consists of two HMM chains with separate observation alphabets, say A and B. Each state has two parents, one from each chain, and the state variable at time t depends on the states of both chains at time $t-1$. In this way we are able to capture the temporal relationship between the two chains. An example of a two-chain CHMM is illustrated in Figure 11.13.

CHMMs have been applied in several areas. Brand (1997) demonstrated their superiority to regular HMMs in a vision task classifying two-handed actions. Rezek (2002) derived the maximum *a posteriori* equations for the Expectation

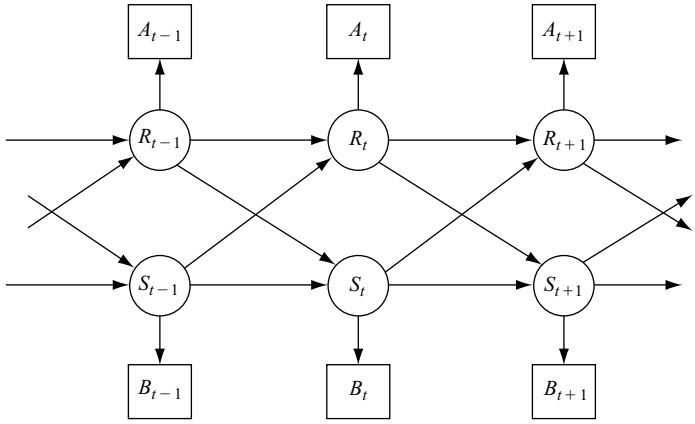


Figure 11.13. CHMM with two HMM chains.

Maximization algorithm for CHMM and applied the model to a variety of biomedical signal analysis problems. Kwon and Murphy (2000) used CHMM to model freeway traffic. Xie and Liu (2006) used CHMM for speech animation approach. CHMM permitted the authors to model the asynchrony, different discriminative abilities, and temporal coupling between the audio speech and the visual speech, which are important factors for animations to look natural.

11.8.4 Hidden Semi-Markov Models

Just as the semi-Markov process attempts to generalize the Markov process by permitting a generally distributed holding time at each state instead of the exponential or geometric holding time, the hidden semi-Markov model (HSMM) is an HMM in which the number of symbols emitted when the process is at a given state before it moves to a new state is a random variable with some mean and variance. Thus, each state can emit a sequence of observations. In Ferguson (1980) and Levinson (1986), HSMM is called "HMM with variable duration," while in Mitchell (1995) it is called "HMM with explicit duration." The model was first investigated by Ferguson (1980).

As stated earlier, a hidden state can emit a string of symbols rather than a single symbol. A hidden state does not have a self-transition because a self-transition defines a geometric distribution over the holding time at the state. A good graphical representation of the model, which is given in Murphy (2002), is shown in Figure 11.14. The states s_k are the regular states that emit symbols, and the states d_k are used to capture the remaining duration of the process in state s_k. When the process enters state s_k, the value of the duration in d_k is chosen according to the probability distribution associated with s_k. When the time in d_k counts down to zero, the state is free to change. Details of the model are given in Murphy (2002). Note that the Ω_k are strings of symbols; that is, $\Omega = \{v_1, \ldots, v_m\}$.

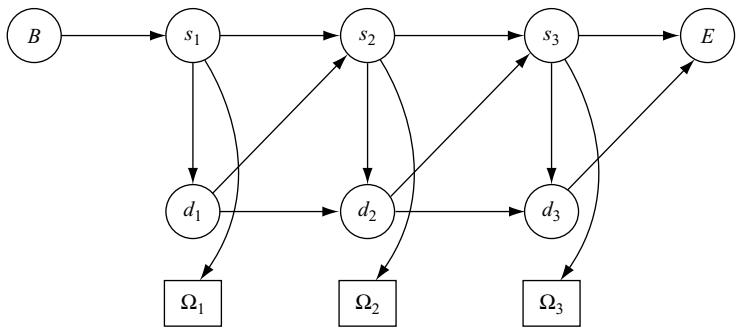


Figure 11.14. Example of HSMM.

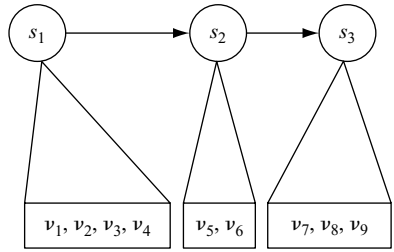


Figure 11.15. Example of output sequence of model in Figure 11.14.

Because a string of symbols can be emitted from a hidden state, one of the problems that needs to be solved in addition to the standard HMM problems is to calculate the duration distribution of a given state. For the model in Figure 11.14, an example of the time-series output sequence is illustrated in Figure 11.15, where $\Omega_1 = \{v_1, v_2, v_3, v_4\}$, $\Omega_2 = \{v_5, v_6\}$, and $\Omega_3 = \{v_7, v_8, v_9\}$.

11.8.5 Profile HMMs for Biological Sequence Analysis

Deoxyribonucleic acid (DNA), ribonucleic acid (RNA), and proteins are the fundamental building blocks of life. The three are large molecules. DNA is composed of four bases: *adenine* (A), *cytosine* (C), *guanine* (G), and *thymine* (T). Similarly, RNA has four bases: *adenine* (A), *cytosine* (C), *guanine* (G), and *uracil* (U). Thus, one major difference between DNA and RNA is that RNA has uracil instead of thymine. Proteins are more diverse in structure and function than the other kinds of molecules and are built from an alphabet of 20 smaller molecules known as *amino acids* whose single letter representations are A, V, L, I, F, P, M, D, E, K, R, S, T, C, N, Q, H, Y, W, and G. The molecules are usually connected in a linear sequence such that a DNA molecule, RNA molecule, or protein molecule is represented as a sequence of letters. Such sequences are called *biological sequences*.

This simple sequence representation of the molecules enables them to be compared in a simple way. Thus, it is possible to match or align two sequences letter by letter to see how they pair up. One of the reasons for making such a comparison is to find the evolutionary relation between species on a molecular level. The use of computers has enabled efficient *sequence alignment* methods that are now commonly used in bioinformatics and molecular biology.

Early research in molecular biology and bioinformatics was motivated by protein sequence analysis. However, due to the human genome project and other high-throughput projects, there is a dramatic increase in many types of biological data available. This has extended the scope of bioinformatics research to include such topics as protein classification, RNA analysis, structural and functional predictions, and gene prediction.

We can make an analogy between speech recognition and protein sequence analysis. Both attempt to determine what a sequence represents based on a set of symbols from some alphabet. The alphabet in speech recognition can be a set of valid phonemes for a particular language, while in protein sequence analysis the alphabet is the set of 20 amino acids from which protein molecules are constructed. As in speech recognition, a good stochastic model for a set of proteins is one that assigns high probability to sequences in that particular set and low probability to any other sequence.

Hidden Markov models have become one of the most statistically powerful methods used to model sequence alignment. A special type of left-to-right HMMs called *profile HMM* (PHMM) is commonly used to model multiple alignments. The architecture of PHMM was introduced by Krogh (1994). PHMM is well-suited to the popular "profile" methods for searching databases using multiple sequence alignments instead of single query sequences. It has three types of states: **match states** that are represented by squares labeled m, **insert states** that are represented by diamonds labeled i, and **delete states** that are represented by circles labeled d.

Match states generate amino acids according to a probability distribution for the 20 amino acids, and different probability distributions apply to different match states. They thus correspond to positions in a protein or columns in multiple alignment. The amino acids emitted in these states are the same as those in the common ancestor, and if not, then they are the result of substitutions. We assume that there are M match states and match state m_k generates amino acid x with probability $P[x|m_k], k = 1, \ldots, M$.

Delete states are silent states that do not generate amino acids and are used for sequences from the family in which the amino acid from such a column has been deleted. They are "dummy" states that are used to skip the match states. For each match state m_k there is a corresponding delete state d_k used to skip m_k. The match-delete pair is sometimes called a *fat state* that is visited exactly once on every path from *Begin* to *End*.

Insert states also generate amino acids according to some probability distribution and represent sequences with one or more inserted amino acids between columns of multiple sequences. That is, insert states are used to represent possible amino acids that are not found in most of the sequences in the family being modeled and are thus the result of insertion. There are $M + 1$ insert states that generate amino acid x with probability $P[x|i_k], k = 1, \ldots, M + 1$, and they permit self-transitions. Figure 11.16 illustrates the architecture of the profile HMM.

Note that there are at most three transitions into each state and three transitions out of each state. Thus, when the Viterbi algorithm is used to analyze the model the computational complexity is $O(nt)$, where n is the number of states and t

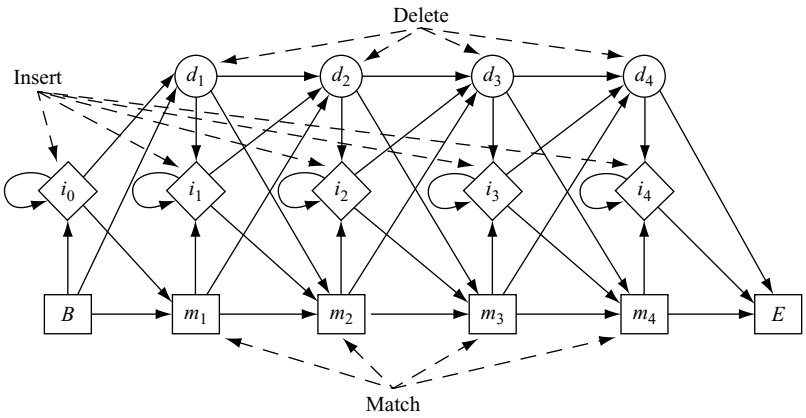


Figure 11.16. Architecture of profile HMM.

is the observation sequence length. For a traditional HMM, the computational complexity is $O(n^2t)$.

PHMM Training

HMM training is the estimation of the emission and transition probabilities. For PHMM, these parameters are obtained from multiple alignment sequences in a protein, DNA, or RNA sequence family. If there is any sequence whose components are known, then it can be used for the training. In general the emission probabilities are the maximum likelihood estimates of the letters in each column. Similarly, the transition probabilities are obtained by counting the number of times each transition would be taken.

Multiple alignment means taking a group of three or more sequences and identifying the amino acids that are homologous (or structurally and functionally similar). Proteins and nucleic sequences and their interrelationships can be demonstrated by multiple alignment of the sequences. The information from the multiple alignment is usually condensed into a single sequence called a *consensus sequence* that shows which symbols are conserved (are always the same) and which symbols are variable. Multiple alignment is performed by arranging the sequences in a matrix such that each row of the matrix consists of one sequence padded by gaps, and the individual columns represent homologous characters. The columns of the matrix highlight similarity (or *residue conservation*) between positions of each sequence. An optimal multiple alignment is one that has the highest degree of similarity.

Consider the following sequences:

$$ACAATC$$
$$TCAACTATC$$
$$ACACAGC$$
$$AGAATG$$
$$ACCGATC$$

Because the sequences are of different lengths, the first step is to introduce gaps to make them of the same length as follows:

A C A - - - A T C
T C A A C T A T C
A C A C - - A G C
A G A - - - A T G
A C C G - - A T C

Thus, we can create a PHMM for this multiple sequence alignment as follows. Three columns were introduced to equalize the number of columns, and these constitute the insert state. There are six other columns that constitute the match states. The first column consists of two distinct letters with the following frequencies of occurrence: A (4) and T (1), which means that the emission probabilities in the first state are $4/5 = 0.8$ for A and $1/5 = 0.2$ for T. These probabilities are used to populate match state m_1. Similarly, in column 2, the emission probability is $4/5 = 0.8$ for C and $1/5 = 0.2$ for G. These probabilities are used to populate match state m_2 and so on for the other match states. The transition probability from m_1 to m_2 is 1 and from m_2 to m_3 is also 1. Two of the insertion rows contain only gaps, which means that the probability of a direct transition from m_3 to m_4 is $2/5 = 0.4$, and the probability of a transition from m_2 to the insert state i_2 is $3/5 = 0.6$. Also, the transition probability from m_4 to m_5 is 1 and from m_5 to m_6 is also 1. Five letters can be emitted in the insert state, and their emission probabilities are as follows: $1/5 = 0.2$ each for A, G, and T, and $2/5 = 0.4$ for C. Finally, the insert state requires two self-transitions: A to C and C to T. Thus, since two of the five letters emitted in this state are from self-transition action, the probability of a self-transition is $2/5 = 0.4$. With all the necessary parameters defined, we can construct the PHMM as shown in Figure 11.17.

Note that sometimes a match state is defined as a column in which the number of gaps is no more than half the number of elements in the column. Thus, in this case, column 4 of the preceding example would be a match state, and we would have an insert state i_4 and a delete state d_4 associated with it. The delete state would permit a transition from column 3, which is m_3, to column 7, which would be m_5.

While PHMM enhances the modeling capability of the standard HMM, it utilizes the solution methodologies of the standard HMM.

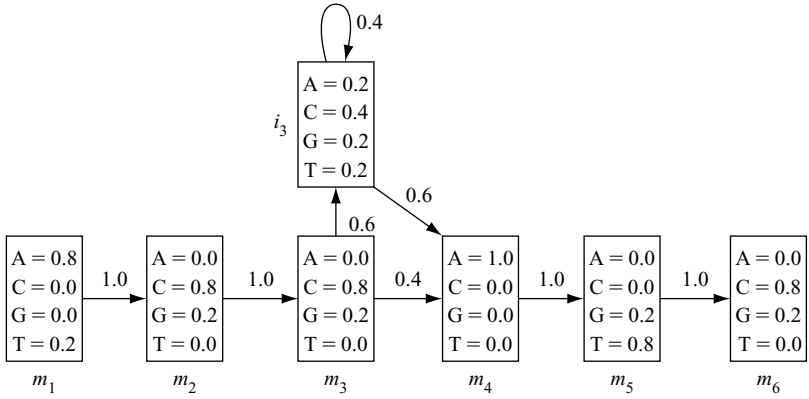


Figure 11.17. Example of PHMM construction.

Scoring a Sequence with PHMM

Any sequence can be represented by a path through the model. The probability of any sequence, given the model, is computed by multiplying the emission and transition probabilities along the path. Given a profile HMM, the probability of a sequence is the product of the emission and transition probabilities along the path of the sequence. For example, the probability of AGCATG, given the PHMM in Figure 11.17, is

$$0.8 \times 1.0 \times 0.2 \times 1.0 \times 0.8 \times 0.4 \times 1.0 \times 1.0 \times 0.8 \times 1.0 \times 0.2 = 0.008192$$

The probability of a sequence is used to calculate a *score* for the sequence. Because multiplication of fractions is computationally expensive and prone to floating point errors such as underflow, the calculation is simplified by taking the logarithm of the score, thereby replacing multiplication by addition. The resulting number is the *log score* of a sequence. Applying this method to the previous calculation we obtain the log score as follows:

$$3\log_e(0.8) + 4\log_e(1.0) + 2\log_e(0.2) + \log_e(0.4) = -4.8046$$

Because a score measures the probability that a sequence belongs to a given family, a high score implies that the sequence of interest is probably a member of the class while a low score implies it is probably not a member.

11.9 Other Extensions of HMM

We have discussed five extensions of the basic hidden Markov model, which are the hierarchical HMM, factorial HMM, coupled HMM, hidden semi-Markov

model, and profile HMM. However, many other extensions of HMM have also been proposed but are not discussed in this book. These include the *buried Markov model*, which was introduced by Bilmes (2003), and the *partially hidden Markov model*, which was introduced by Forchhammer and Rissanen (1996). The *partly hidden Markov model* discussed by Kobayashi (1999) and Ogawa and Kobayashi (2005) is an extension of the partially hidden Markov model.

11.10 Problems

11.1 Consider an HMM with two states 1 and 2 and that emits two symbols: A and B. The state-transition diagram is shown in Figure 11.18.

- **a.** Use the Viterbi algorithm to obtain the most likely state sequence that produced the observation sequence {ABBAB}.
- **b.** Estimate the probability that the sequence {BAABA} was emitted by the preceding system.

11.2 Consider the HMM shown in Figure 11.19 that has three hidden states 1, 2, and 3, and emits two output symbols: U and V. When it is in state 1, it is equally likely to emit either symbol. When it is in state 2, the probability that it emits the symbol U is 0.1, and the probability that it emits the symbol V is 0.9. Finally, when it is in state 3, the probability of emitting the symbol U is 0.9 and the probability of emitting the symbol V is 0.1.

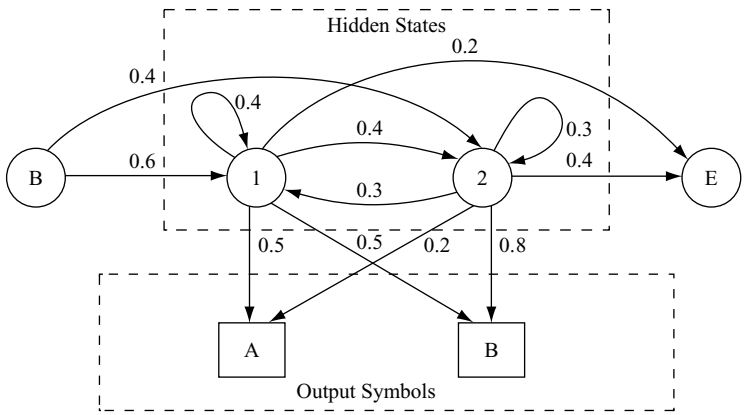


Figure 11.18. Figure for Problem 11.1.

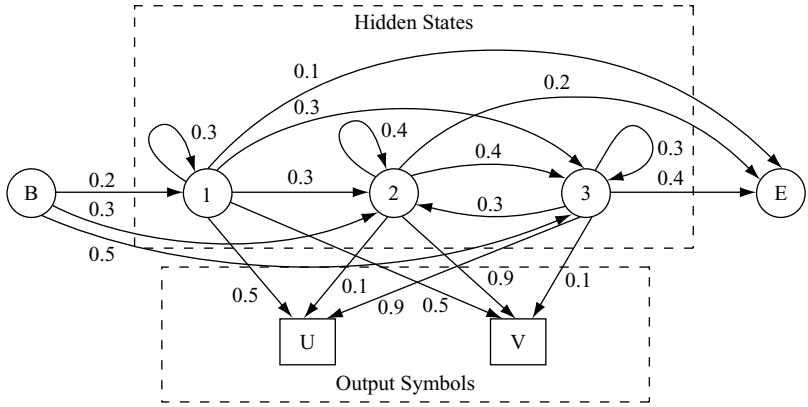


Figure 11.19. Figure for Problem 11.2.

a. If the output symbol is $\{UUV\}$, estimate the most likely transition path through the system.

b. Convert the HMM into an HMM with silent states.

11.3 Construct the profile HMM for the following variable length sequences DOR, DM, DAP, VGBLM. (Hint: Use the following alignment to identify the match, insert, and delete states.)

D	O	-	-	R
D	-	-	-	M
D	A	-	-	P
V	G	B	L	M

11.4 Consider three coins labeled 1, 2, and 3. When coin 1 is tossed, the probability that it comes up heads is 0.75 and the probability that it comes up tails is 0.25. Similarly, when coin 2 is tossed, the probability that it comes up heads is 0.25 and the probability that it comes up tails is 0.75. Finally, when coin 3 is tossed, the probability that it comes up heads is 0.5 and the probability that it comes up tails is 0.5. Assume that in an experiment that involves a sequence of tosses of these coins, the experimenter is equally likely to choose any coin during the next toss. Thus, if we denote "heads" by H and "tails" by T, the experiment can be modeled by the Markov chain shown in Figure 11.20. Assume that the experimenter is equally likely to start a sequence of tosses with any coin.

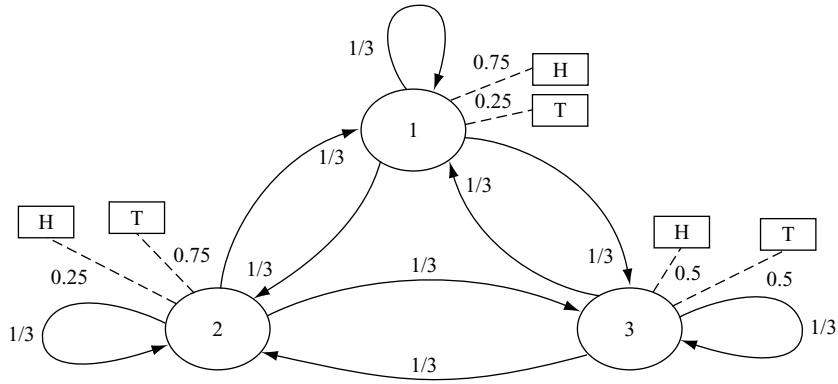


Figure 11.20. Figure for Problem 11.4.

a. What is the probability that the observation sequence {HTTHT} was emitted by the model?

b. Use the Viterbi algorithm to obtain the most likely state sequence that produced the observation sequence {HTTHT}.

11.5 Consider a system that can be modeled by an array of 6 states labeled $1, 2, \ldots, 6$. Apart from state 4, which makes a transition to itself with probability p, every other state is visited only once in each experiment that starts in the Begin state and ends when the End state is reached. The model is illustrated in Figure 11.21.

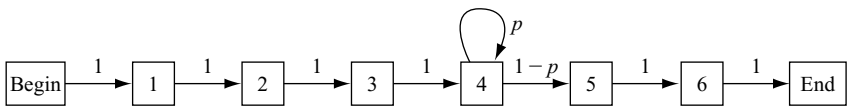


Figure 11.21. Figure for Problem 11.5.

Let L be a random variable that denotes the length of time that the process spends in state 4 when the process reaches that state in an experiment.

a. What is the PMF of L?

b. What is the expected value of L?

11.6 Consider a system that can be modeled by an array of 6 states labeled $1, 2, \ldots, 6$. Every state makes a transition to itself with probability p and makes a transition to the next higher state with probability $1 - p$. An experiment starts in Begin state and ends when the End state is reached. The model is illustrated in Figure 11.22.

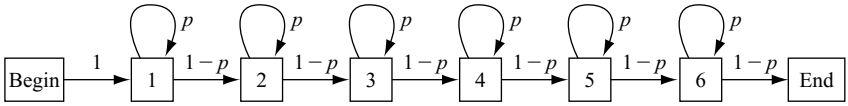


Figure 11.22. Figure for Problem 11.6.

Let L be a random variable that denotes the length of time that the process takes to pass from state 1 to the End state in an experiment.

a. What is the PMF of L?
b. What is the expected value of L?

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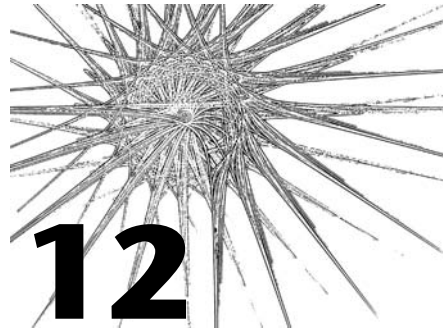
Markov Random Fields



12.1 Introduction

A random field is essentially a stochastic process defined on set of spatial nodes (or sites). Specifically, let $S = \{1, \ldots, N\}$ be a finite set and let $\{X(s), s \in S\}$ be a collection of random variables on the sample space Ω. Let $X(s_1) = x_1, \ldots, X(s_m) = x_m$, where $x_i \in \Omega$. Then the joint event $x = \{x_1, \ldots, x_m\}$ is called a *configuration* of $X(s)$, which corresponds to a realization of the random field. A good introduction to random fields is given in Griffeath (1976), Vanmarcke (1988), Christakos (1992), and Bremaud (1999).

There are several ways to classify random fields. If all finite dimensional PDFs are multivariate Gaussian, the random field is defined to be a *Gaussian random field*; otherwise, it is a non-Gaussian random field. Also, the random field can be a *discrete-space random field* if the space is discrete, which means that $S = \{0, 1, \ldots\}$; otherwise it is a *continuous-space random field* with $S = [0, \infty)$. Similarly, the random field can be a *discrete-state random field* if the state space is discrete, $\Omega = \{1, 2, \ldots\}$; otherwise it is a *continuous-state random field* with $\Omega = \Re = (-\infty, \infty)$. Random fields can also be classified according to their spatial variability. The term *homogeneous random field* is used to denote a random field in which the statistical values of the point properties are constant and the statistical value of the cross-point properties, namely autocorrelation and autocovariance, depends only on the separation between the points. For nonhomogeneous random fields, the statistical properties depend on the space origin. Finally, random fields can be classified by their memory property. In this case, we have

Markov random fields and *non-Markov random fields*. If for $s_1 < s_2 < \ldots < s_m$ we have that the joint PDF

$$f_{X_1 X_2 \ldots X_m}(s_1, s_2, \ldots, s_m) = f_{X_1}(s_1) f_{X_2|X_1}(s_2|s_1) \ldots f_{X_m|X_{m-1}}(s_m|s_{m-1})$$

the random field is defined to be a Markov random field. In the case of the non-Markov random field, we have that for $s_1 < s_2 < \ldots < s_m$, the random variables $X_1(s_1), X_2(s_2), \ldots, X_m(s_m)$ are independent; that is,

$$f_{X_1 X_2 \ldots X_m}(s_1, s_2, \ldots, s_m) = \prod_{i=1}^{m} f_{X_i}(s_i)$$

These classifications are independent of each other, which means, for example, that a random field can be a homogeneous Gaussian-Markov random field.

A random field is defined to be strictly stationary if for any finite set of sites $s_1, \ldots, s_m$ and any $v \in S$, the joint distribution of $\{X(s_1), \ldots, X(s_m)\}$ and that of $\{X(s_1 + v), \ldots, X(s_m + v)\}$ are the same. A stationary random field in which the covariance function depends only on the absolute distance between the points is said to be *isotropic*, otherwise it is said to be *anisotropic*. That is, $\{X(s), s \in S\}$ is defined to be isotropic if

$$C_{XX}(s, u) = E[\{X(s) - \overline{X(s)}\}\{X(u) - \overline{X(u)}\}] = C_{XX}(\tau)$$

where $\tau = ||s - u||$. Sometimes the sites are points on a lattice and are, therefore, spatially regular. For such cases we consider an $n \times m$ lattice where $S = \{(i, j) | 1 \leq i \leq n, 1 \leq j \leq m\}$. Also, there might be an interrelationship between sites, which is captured by the concept of a *neighborhood system* that is discussed in the next section.

12.2 Markov Random Fields

Markov random fields were originally used in statistical mechanics to model systems of particles interacting in a two-dimensional or three-dimensional lattice. More recently they have been widely used in statistics and image analysis. They were introduced into image segmentation by Geman and Geman (1984). Since then the Markov random field (MRF) theory has become the basic framework for statistical image analysis where images are modeled as data organized in lattices and represented as pixels or voxels. Thus, pixels and voxels play the role of particles in the physical system.

The idea behind the concept of Markov random fields is that when a process is at a particular location, it is more likely to be influenced by events at other points that are relatively nearer the location than events at points that are farther away. Thus, the process attempts to define the concept of a neighborhood within which it is affected by the points of the process and outside of which the impact of points is considered to be negligible and hence ignored. For example, in the case of image analysis, regions in real images are often homogeneous in the sense that neighboring pixels usually have similar properties such as intensity, color, texture, etc. These contextual constraints are captured by the Markov property of MRF.

More formally, in a Markov random field, the sites in S are related to one another through a neighborhood system denoted by $\aleph = \{\aleph(i), i \in S\}$, where $\aleph(i)$ is the set of sites that are neighbors of i, $i \notin \aleph(i)$. The neighborhood relation is symmetrical, which means that $i \in \aleph(j) \Leftrightarrow j \in \aleph(i)$. Thus, for a finite set of sites $S = \{1, \ldots, N\}$, a Markov random field is a family of random variables $X_i, i \in S$ with probability functions that satisfy the following conditions relative to the neighborhood system $\aleph$:

$$P[X = x] > 0$$

$$P[X_i = x_i | X_j = x_j, j \neq i] = P[X_i = x_i | X_j = x_j, j \in \aleph(i)]$$

The first condition is called the *positivity property*, which ensures that all configurations (or possible realizations of X) have a chance of occurring. The second is usually called the *Markovianity property* that establishes the local characteristics of X; that is, only neighboring sites have direct interactions on each other. This property is sometimes expressed as follows:

$$P[X_i | X_{s-\{i\}}] = P[X_i | X_{\aleph(i)}]$$

It is this ability to describe local properties that makes MRFs useful for image processing because, as we stated earlier, regions in real images are often homogeneous in the sense that neighboring pixels tend to have similar properties such as intensity, color, and texture. MRFs are also widely used in speech recognition, neural networks, and coding.

However, the specification of MRFs via local conditional probabilities has some disadvantages. First, there is no direct method for deducing the joint probability distribution $P[X_1, \ldots, X_N]$ from the conditional probabilities $P[X_i | X_j, j \in \aleph(i)$. This is an important issue because it is the joint probability mass function and not the conditional PMFs that contains the complete image representation. Also, the equilibrium conditions of a random process are usually specified in terms of the joint probability function rather than the conditional probabilities. Later in this section we will see how this problem is resolved through the Hammersley-Clifford theorem, which is proved in Besag (1974). Note that the probability function is

the probability mass function when the X_i are discrete random variables and the probability density function when they are continuous random variables. In the remainder of the discussion we use the generic term *probability function* except where there is an explicit need to specify a PMF or PDF.

Another way to define a Markov random field is through the concept of *conditional independence*. Two random variables X and Y are conditionally independent given the random variable Z, written $X \perp Y|Z$, if and only if

$$P[X, Y|Z] = P[X|Z]P[Y|Z]$$

A random field $\{X(s), s \in S\}$ is defined to be a Markov random field if the random variable $X(s)$ is conditionally independent of all other sites in S, given its values in $\aleph(s)$; that is,

$$X(s) \perp X(S - \{s \cup \aleph(s)\})|X(\aleph(s))$$

As long as $X(s)$ satisfies the positivity condition, the joint probability distribution can be obtained from the conditional functions as follows. Consider two configurations $x = \{x_1, x_2, \ldots, x_n\} \in S^n$ and $y = \{y_1, y_2, \ldots, y_n\} \in S^n$. Then we have that

$$P[x_1, x_2, \ldots, x_{n-1}, x_n] = P[x_n|x_1, x_2, \ldots, x_{n-1}]P[x_1, x_2, \ldots, x_{n-1}]$$
$$P[x_1, x_2, \ldots, x_{n-1}, y_n] = P[y_n|x_1, x_2, \ldots, x_{n-1}]P[x_1, x_2, \ldots, x_{n-1}]$$

From this we obtain

$$P[x_1, x_2, \ldots, x_{n-1}] = \frac{P[x_1, x_2, \ldots, x_{n-1}, x_n]}{P[x_n|x_1, x_2, \ldots, x_{n-1}]} = \frac{P[x_1, x_2, \ldots, x_{n-1}, y_n]}{P[y_n|x_1, x_2, \ldots, x_{n-1}]}$$

Thus,

$$P[x_1, x_2, \ldots, x_{n-1}, x_n] = \frac{P[x_n|x_1, x_2, \ldots, x_{n-1}]}{P[y_n|x_1, x_2, \ldots, x_{n-1}]} P[x_1, x_2, \ldots, x_{n-1}, y_n]$$

Similarly,

$$P[x_1, x_2, \ldots, x_{n-1}, y_n] = P[x_{n-1}|x_1, x_2, \ldots, x_{n-2}, y_n]P[x_1, x_2, \ldots x_{n-2}, y_n]$$
$$P[x_1, x_2, \ldots, x_{n-2}, y_{n-1}, y_n] = P[y_{n-1}|x_1, x_2, \ldots, x_{n-2}, y_n]P[x_1, x_2, \ldots x_{n-2}, y_n]$$

From this we obtain

$$P[x_1, x_2, \ldots, x_{n-2}, y_n] = \frac{P[x_1, x_2, \ldots, x_{n-1}, y_n]}{P[x_{n-1}|x_1, x_2, \ldots, x_{n-2}, y_n]} = \frac{P[x_1, x_2, \ldots, x_{n-2}, y_{n-1}, y_n]}{P[y_{n-1}|x_1, x_2, \ldots, x_{n-2}, y_n]},$$

which gives

$$P[x_1, x_2, \ldots, x_{n-1}, y_n] = \frac{P[x_{n-1}|x_1, x_2, \ldots, x_{n-2}, y_n]}{P[y_{n-1}|x_1, x_2, \ldots, x_{n-2}, y_n]} P[x_1, x_2, \ldots, x_{n-2}, y_{n-1}, y_n]$$

Combining the two results we obtain

$$P[x_1, x_2, \ldots, x_{n-1}, x_n] = \frac{P[x_n|x_1, x_2, \ldots, x_{n-1}]}{P[y_n|x_1, x_2, \ldots, x_{n-1}]} \frac{P[x_{n-1}|x_1, x_2, \ldots, x_{n-2}, y_n]}{P[y_{n-1}|x_1, x_2, \ldots, x_{n-2}, y_n]} P[x_1, x_2, \ldots, x_{n-2}, y_{n-1}, y_n]$$

Proceeding inductively, we obtain

$$\frac{P[x_1, x_2, \ldots, x_n]}{P[y_1, y_2, \ldots, y_n]} = \prod_{i=1}^{n} \frac{P[x_i|x_1, x_2, \ldots, x_{i-1}, y_{i+1}, \ldots, y_n]}{P[y_i|x_1, x_2, \ldots, x_{i-1}, y_{i+1}, \ldots, y_n]}$$

Thus, the ratio $P[x]/P[y]$ is determined by the conditional probabilities. Assume now that $X(s)$ is a Markov random field relative to the neighborhood $\aleph$, and let $\aleph(i)^+$ denote the set of neighbors X_j of site X_i such that $i < j$. Similarly, let $\aleph(i)^-$ denote the set of neighbors X_j of site X_i such that $i > j$. Then we obtain

$$\frac{P[x_1, x_2, \ldots, x_n]}{P[y_1, y_2, \ldots, y_n]} = \prod_{i=1}^{n} \frac{P\left[x_i|x_j \text{ for all } X_j \in \aleph(i)^-, y_j \text{ for all } Y_j \in \aleph(i)^+\right]}{P\left[y_i|x_j \text{ for all } X_j \in \aleph(i)^-, y_j \text{ for all } Y_j \in \aleph(i)^+\right]}$$

Thus, we obtain the ratio $P[x]/P[y]$ in terms of the conditional probabilities over the neighborhood system. Alternatively, we can write

$$P[x] = P[y] \prod_{i=1}^{n} \frac{P\left[x_i|x_j \text{ for all } X_j \in \aleph(i)^-, y_j \text{ for all } Y_j \in \aleph(i)^+\right]}{P\left[y_i|x_j \text{ for all } X_j \in \aleph(i)^-, y_j \text{ for all } Y_j \in \aleph(i)^+\right]}$$

12.2.1 Graphical Representation

An important characteristic of image data is the special nature of the statistical dependence of the grey level at a lattice site on those of its neighbors. For this reason it is important to understand the neighborhood structure in Markov random fields. This neighborhood structure can be represented by a graph with sites as the *nodes*, and two sites are connected by an *edge* if and only if they are neighbors. Let $\aleph(i)$ and $\aleph$ be as defined earlier. Then we consider two types of neighborhood systems on the regular rectangular lattice: *first-order neighborhood* and *second-order neighborhood*. Relative to node (i, j), the diagonal nodes are not considered its first-order neighbors. Thus, the first-order neighborhood system of node (i, j) is given by

$$\aleph_1(i, j) = \{(i-1, j), (i, j-1), (i+1, j), (i, j+1)\}$$

Therefore, if node (i, j) is not at the boundary of the lattice, it has four neighbors, as illustrated in Figure 12.1.

An example of a first-order neighborhood for a linear graph is illustrated in Figure 12.2 and is characterized by the relationship:

$$P[X_i|X_j, j \notin i] = P[X_i|X_{i-1}, X_{i+1}] \Rightarrow X_{\aleph(i)} = \{X_{i-1}, X_{i+1}\}$$

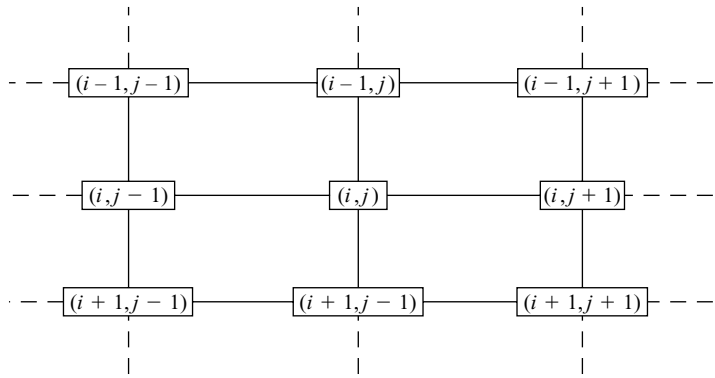


Figure 12.1. Example of a first-order neighborhood system.

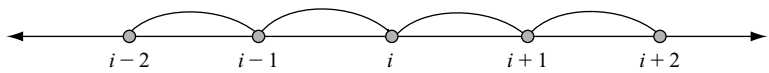


Figure 12.2. Example of a first-order neighborhood system on a linear graph.

In the case of a second-order neighborhood, the diagonal nodes are considered to be neighbors, which means that the second-order neighborhood of (i, j) is given by

$$\aleph_2(i, j) = \{(i-1, j), (i-1, j-1), (i, j-1), (i+1, j-1), (i+1, j),$$
$$(i+1, j+1), (i, j+1), (i-1, j+1)\}$$

Thus, if node (i, j) is not at the boundary of the lattice, it has eight neighbors, as illustrated in Figure 12.3.

In general, the cth-order neighborhood system for node (i, j) is given by

$$\aleph_c(i, j) = \{(k, l) | 0 < (k-i)^2 + (l-j)^2 \leq c\}$$

Recall from Chapter 8 that $G = (V, E)$ is a graph in which V is the set of nodes or vertices and E is the set of edges. An edge exists between nodes i and j if and only if they are neighbors; that is, $i \in \Gamma(j)$. We consider undirected graphs in which $i \in \Gamma(j) \Leftrightarrow j \in \Gamma(i)$. Recall also that in a complete graph all the nodes are neighbors of all other nodes. Figure 12.4 illustrates examples of complete graphs.

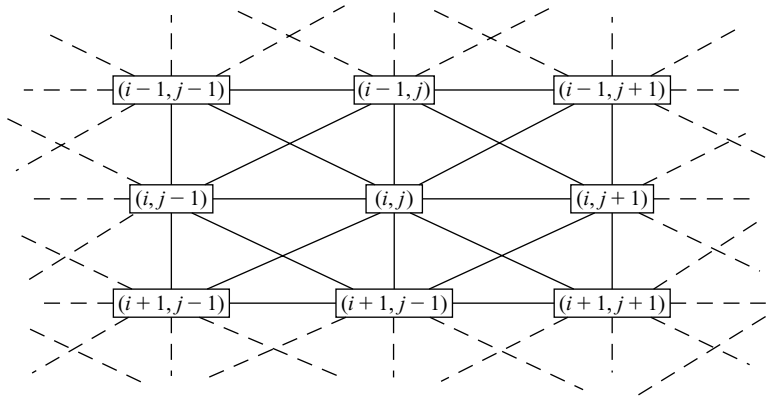


Figure 12.3. Examples of second-order neighborhood system.

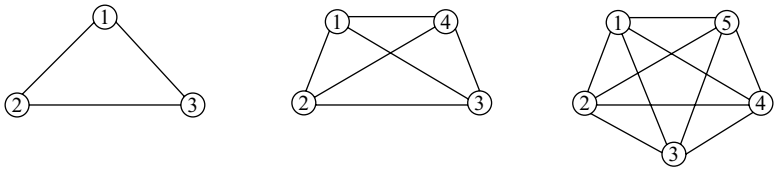


Figure 12.4. Examples of complete graphs.

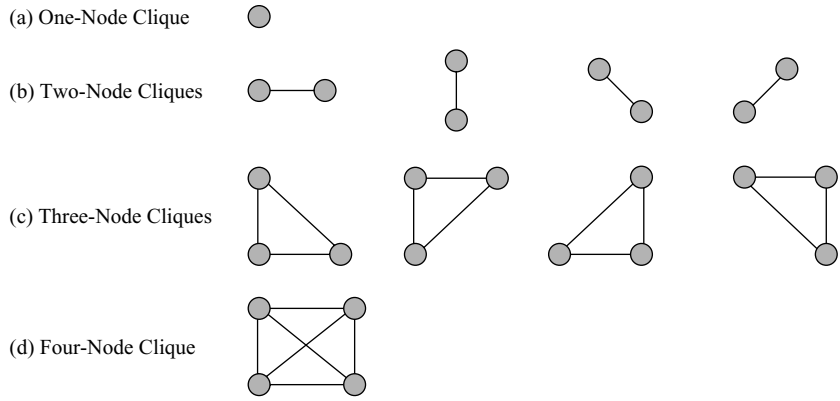


Figure 12.5. Examples of cliques.

A *clique* of a graph G is a single node or a complete subgraph of G. That is, a clique is a subgraph of G in which every site is a neighbor of all other sites. Figure 12.5 shows examples of cliques.

The local conditional probability $P[X_i|X_j, j \in \Gamma(i)]$ is represented through cliques because every node in a clique is a neighbor to all other nodes in the clique. Thus, the MRF model consists of a set of cliques.

12.2.2 Gibbs Random Fields and the Hammersley-Clifford Theorem

As stated earlier, the specification of MRFs via local conditional probabilities has the disadvantage that it does not provide a direct method for deducing the joint probability distribution $P[X_1, \ldots, X_N]$ from the conditional probabilities $P[X_i|X_j, j \in \aleph(i)]$. Fortunately, this problem is resolved via the *Hammersley-Clifford theorem* that states as follows:

Theorem 12.1 The random field X is a Markov random field if and only if X is a Gibbs random field.

The proof of this theorem is given in Besag (1974). The implication of the theorem is that any conditional MRF distribution has a joint distribution that is a Gibbs distribution. The theorem thus establishes a connection between the local and global specifications of a Markov random field.

The Gibbs distribution originated in statistical mechanics where the large-scale properties of a lattice system are to be deduced from local models. The approach was pioneered by Ising (1925) who modeled the behavior of ferromagnetic material by considering only the interactions of spins of neighboring atoms. Two

neighboring atoms of opposite spin were considered to have a positive *potential*, and the state of the lattice was characterized by its *energy*, which is computed as the sum of the potentials. In this system, configurations of low energy are defined to be more stable than those of high energy.

A random variable X is defined to have the Gibbs distribution if its distribution function is of the following form:

$$P[X = x] = \frac{1}{Z} \exp\{-\beta U(x)\}$$

$U(x)$ is called the *energy function*, which is such that the higher the energy of the configuration, the smaller the probability, and β is a nonnegative parameter called the *spatial smoothness parameter* that characterizes the label scale variability in an image. The value $\beta = 0$ corresponds to a uniform distribution on the configuration space; small values of β indicate small and broken structures, while large values imply large and bloblike structures that make it more likely to observe largely clustered configurations corresponding to large $U(x)$. The denominator, Z, is a normalizing constant called the *partition function* and is given by

$$Z = \sum_{x} \exp\{-\beta U(x)\}$$

Note that $\beta = 1/kT$ in the definition of the Gibbs distribution used in statistical mechanics, where T is the *temperature* (in absolute degrees) and k is the *Boltzmann's constant*. $U(x)$ is usually defined in terms of the cliques and given by

$$U(x) = \sum_{c \in C} V_c(x) = \sum_{i \in \aleph_1} V_{\aleph_1}(x_i) + \sum_{(i,j) \in \aleph_2} V_{\aleph_2}(x_i, x_j) + \cdots$$

where the sum is over all cliques, and $V_c(x)$ is called the *potential function* that is associated with clique $c \in C$, and C is the family of cliques. In image analysis, $V_c(x)$ is a function of the values of the pixels in the clique $c \in C$.

A random field $X(s)$ is defined to be a Gibbs random field on S with respect to $\aleph$ if and only if its joint probability distribution is the Gibbs distribution; that is, a Gibbs random field is a family of random variables that have the Gibbs distribution.

As stated earlier, a Markov random field is characterized by its local property, according to the Markovianity property, while the Gibbs random field is characterized by its global property, which is the Gibbs distribution. The equivalence between the two fields is established by the Hammersley-Clifford theorem, which fundamentally states that a random field X is an MRF if and only if X is a Gibbs random field; that is, X is an MRF if and only if it has a Gibbs distribution with potentials defined on the cliques of the neighborhood system $\aleph$. The importance of the theorem lies in the fact that it provides a simple way to specify the joint

probability distribution, which is by defining the clique potential functions. That is, a Markov random field can be specified via the clique potentials. The conditional probability $P[X_i|X_j,\, j \in \aleph(i)]$ is given by

$$P[X_i|X_j,\, j \in \aleph(i)] = \frac{1}{Z_i} \exp\left\{-\beta \sum_{c:i\in c} V_c(x)\right\}$$

where

$$Z_i = \sum_{w\in\Omega} \exp\left\{-\beta \sum_{c:i\in c} V_c(x|X_i = w)\right\}$$

$$x = \{x_1, \ldots, x_i, \ldots, x_N\}$$

$$x|\{X_i = w\} = \{x_1, \ldots, w, \ldots, x_N\}$$

One important feature of the Gibbs random field is that its definition is rather general in the sense that the only restriction on the potential field $V_c(x)$ is its dependence on the values associated with the clique c. There is no consistency condition among different values of the potential functions with the result that their choice represents a basic modeling decision that needs to be made when using Markov random fields.

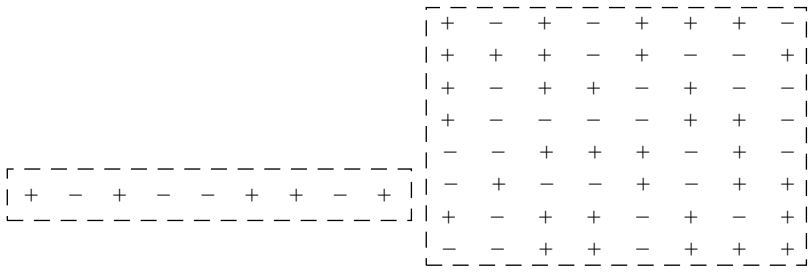
12.3 Examples of Markov Random Fields

In this section we consider three examples of the Markov random field. These are the Ising model, the Potts model, and the Gauss-Markov random field. We discuss them from the point of view of their use in image analysis.

12.3.1 The Ising Model

The Ising model has its origin in statistical mechanics and is the simplest model for ferromagnetism that predicts phase transitions and critical phenomena. Specifically, when a piece of iron is placed in a magnetic field it becomes magnetized. If the magnet is then heated, the strength of the iron's magnetic field weakens until it disappears. The temperature at which this happens is called the *Curie temperature*, and this change in behavior is known as a *phase transition*. A good introduction to the Ising model is given in Kindermann and Snell (1980) and Cipra (1987).

In this model, each site can have two values, such as 0 and 1, or $+1$ and -1. Examples of configurations in one-dimensional and two-dimensional lattices are given in Figure 12.6 where it is assumed that $\Omega = \{+1, -1\}$.



(a) Ising Model on 1D Lattice (b) Ising Model on 2D Lattice

Figure 12.6. Examples of the Ising model.

Assume that there are N lattice sites labeled $i = 1, \ldots, N$, and let the random variable $\varphi_i(x)$ be assigned to site i. The random variables $\varphi_i(x)$ are usually called "spins" and take on only two values, namely $\varphi_i(x) = \pm 1$, which are the states of the sites, where we assume that for the configuration $x = \{x_1, x_2, \ldots, x_N\}$ the values of the random variable are as follows:

$$\varphi_i(x) = \begin{cases} 1 & x_i = + \\ -1 & x_i = - \end{cases}$$

Thus, there are 2^N possible configurations, $\{\varphi_1(x), \varphi_2(x), \ldots, \varphi_N(x)\}$, of the system. The energy function of the system, which is also called the *Hamiltonian* of the system, is defined as follows:

$$U(x) = -J \sum_{i,j} \varphi_i(x)\varphi_j(x) - H \sum_i \varphi_i(x)$$

where the first sum is taken over all pairs of nearest-neighbor spins (i.e., sites i and j such that $|i - j| = 1$), J is the "coupling constant" that defines the strength of the interaction between spins, and H represents the interaction between the spins and the external magnetic field (assuming that one exists).

Example 12.1 Consider the energy function of the two-dimensional lattice shown in Figure 12.7. The labels on the edges indicate the value of $\varphi_i(x)\varphi_j(x)$ for the nearest neighbor spins. ■

If we assume that $H = 0$, then the value of the energy function is given by adding the values on the edges. That is, $U(x) = 2J$.

The one-dimensional Ising model was solved in Ising (1925), while the two-dimensional case was solved by Onsager (1944). Unfortunately, there is no known analytical solution for the three-dimensional case, which is usually solved numerically via Monte Carlo simulation methods.

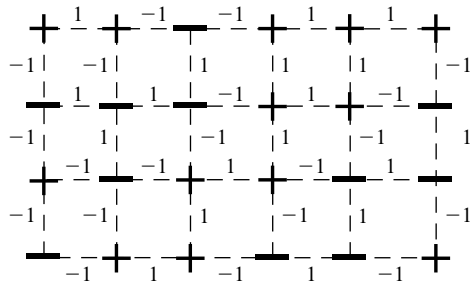


Figure 12.7. Figure for Example 12.1.

12.3.2 The Potts Model

The Ising model assumes that each site in a lattice can have one of two spins. The Potts model is a generalization of the Ising model where each site of the lattice can be in one of $q \geq 2$ distinct states. Nearest-neighbor sites interact with energy e_0 if they are in the same state and with energy e_1 if they are in different states. The Ising model corresponds to the Potts model with $q = 2$.

For a Potts model on a lattice of N sites, the energy function is of the form

$$U(x) = -H \sum_i \delta(\varphi_i(x)) - J_0 \sum_{i,j} \delta(\varphi_i(x), \varphi_j(x)) - J_1 \sum_{i,j,k} \delta(\varphi_i(x), \varphi_j(x), \varphi_k(x)) - \cdots,$$

where

$$\delta(\varphi_i(x), \ldots, \varphi_k(x)) = \begin{cases} 1 & \varphi_i(x) = \ldots = \varphi_k(x) \\ 0 & \text{otherwise} \end{cases}$$

H represents the interaction between the spins and the external magnetic field, J_0 represents the strength of the interaction between two nearest-neighbor sites, J_1 represents the strength of the interaction between three sites, etc. In the special case that there is no interaction between spins and the external magnetic field (i.e., $H = 0$) and $J_k = 0$ for $k > 0$, the energy function represents the number of neighboring pairs with the same spin. In this case, large values of $U(x)$ correspond to spatial patterns with the large patches of individual sites belonging to the same cluster while small values correspond to patterns with no well-defined spatial organization.

Example 12.2 Figure 12.8 illustrates a Potts model with $q = 3$. Three symbols represent the three states: square, triangle, and circle. We assume that $H = 0$ and

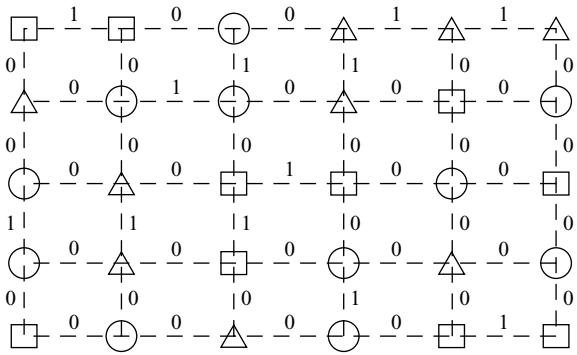


Figure 12.8. Figure for Example 12.2.

that there are no three-site interactions. The numbers on the edges of the lattice indicate the values of $\delta(\varphi_i(x), \varphi_j(x))$. Thus, the energy function for this example is $U(x) = -12J_0$. ■

12.3.3 Gauss-Markov Random Fields

If the individual random variables $\{X(s), s \in S\}$ are Gaussian random variables, the Markov random field is called a Gauss-Markov random field (GMRF). That is, a random field $x = (x_1, \ldots, x_N)$ is a GMRF relative to the neighborhood system $\aleph$ if $x \sim N(\mu, \Sigma)$ and

$$\sum_{ij} = E[(x_i - \mu_i)(x_j - \mu_j)] \neq 0 \Leftrightarrow j \in \aleph(i)$$

where μ is the vector of the means and $\sum$ is the covariance matrix. Thus, if b^T denotes the transpose of b, then the probability distribution of x is given by

$$P[x] = \frac{1}{(2\pi)^{N/2} |\sum|^{1/2}} \exp\left\{-\frac{1}{2}(x-\mu)^T \sum^{-1} (x-\mu)\right\}$$

In this field, only cliques containing up to two nodes (i.e., single-node and two-node cliques) are used so that the Gibbs energy is of the form

$$U(x) = \sum_{c \in C} V_c(x) = \sum_{i \in \aleph_1} V_{\aleph_1}(x_i) + \sum_{(i,j) \in \aleph_2} V_{\aleph_2}(x_i, x_j)$$

where

$$V_{\aleph_1}(x_i) = \frac{(x_i - \mu_i)^2}{2\sigma^2}$$

$$V_{\aleph_2}(x_i, x_j) = -\rho_{ij}\frac{(x_i - \mu_i)(x_j - \mu_j)}{\sigma^2}, \quad j \in \aleph(i)$$

The parameter μ is the mean grey level value, σ^2 is the grey level variance, and ρ_{ij} is the correlation between sites i and j.

12.4 Hidden Markov Random Fields

The fundamental premise of the hidden Markov model is that there are noisy observations from an underlying process that is assumed to possess the Markov property. As discussed in Chapter 11, most of the hidden Markov models that have been studied deal with temporal observations. That is, HMM is essentially a one-dimensional Markov chain that cannot be used to model higher dimensional systems. The hidden Markov random field is an extension of the hidden Markov model that is designed to handle spatially distributed data instead of temporally distributed data. That is, a hidden Markov random field (HMRF) is a stochastic process generated by a Markov random field whose state sequence cannot be observed directly but must be observed through an observation field. Alternatively, it can be defined as a process that is obtained when a Markov random field is corrupted by some observational noise process.

Hidden Markov random fields are discussed in Kunsch, Geman, and Kehagias (1995) and Elliott (1995, Chapter 9). Consider a hidden random field $\{X(s), s \in S\}$ with values in a finite-state space Ω, where the state of X is not observable. Consider another random field $\{Y(s), s \in S\}$ with values in a finite-state space Ψ. Y is the *emitted random field* that is observable such that for any given configuration $x \in X$ of the hidden random field there is a known conditional probability distribution $p(y_i|x_i)$ called the *emission probability distribution*. For any given configuration $x \in X$, the random variables Y_i are conditionally independent; that is,

$$P[y|x] = \prod_{i \in S} P[y_i|x_i]$$

From this we obtain the joint probability of X and Y as follows:

$$\begin{aligned} P[y, x] &= P[y|x]P[x] \\ &= P[x]\prod_{i\in S} P[y_i|x_i] \\ &= P[x]\exp\left\{\sum_{i\in S}\log P[y_i|x_i]\right\} \end{aligned}$$

Based on the local characteristics of the Markov random field, the joint probability of any pair of (X_i, Y_i), given the neighborhood configuration X_{η_i} of X_i, is given by

$$P[y_i, x_i|x_{\aleph(i)}] = P[y_i|x_i]P[x_i|x_{\aleph_{(j)}}]$$

Using Bayes' rule and the conditional independence property we obtain the *a posteriori* probability $P[X|Y]$, which is the conditional distribution of the parameter of interest X after the data Y has been observed, as follows:

$$\begin{aligned} P[X = x|Y = y] &= \frac{P[Y = y|X = x]P[X = x]}{P[Y = y]} \propto P[Y = y|X = x]P[X = x] \\ &= P[x]\prod_{i\in S} P[y_i|x_i] = P[x]\exp\left\{\sum_{i\in S}\log P[y_i|x_i]\right\} \\ &= \frac{1}{Z}\exp\{-\beta U(x)\}\exp\left\{\sum_{i\in S}\log P[y_i|x_i]\right\} \\ &= \frac{1}{Z}\exp\left\{-\beta U(x) + \sum_{i\in S}\log P[y_i|x_i]\right\}, \end{aligned}$$

where the proportionality in the first line follows from the fact that the denominator does not depend on X. Thus, we see that

$$P[X = x|Y = y] \propto \exp\left\{-\beta U(x) + \sum_{i\in S}\log P[y_i|x_i]\right\}$$

By defining an appropriate constant of proportionality $Z(\beta)$, we can obtain the *a posteriori* probability as

$$P[X = x|Y = y] = \frac{1}{Z(\beta)}\exp\left\{-\beta U(x) + \sum_{i\in S}\log P[y_i|x_i]\right\}$$

Unfortunately, it is not generally easy to obtain the preceding *a posteriori* probability directly. It is usually obtained iteratively using, for example, the Gibbs

sampler that was introduced in the context of image processing by Geman and Geman (1984). Also, most of the cited references contain information on methods of estimating the parameters of the HMRF.

A special case of the hidden Markov random field is the hidden Gauss-Markov random field in which the emission probability distribution is a Gaussian distribution. That is,

$$P\left[y_i|x_{\aleph(i)},\theta\right]=\sum_{x_k\in\aleph(i)} g(y_i,\theta_k)P\left[x_k|x_{\aleph(i)}\right]$$

where

$$g(y_i,\theta_k)=P[y_i|x_i,\theta]=\frac{1}{\sqrt{2\pi\sigma_k^2}}\exp\left\{-\frac{(y_i-\mu_k)^2}{2\sigma_k^2}\right\}$$

$$\theta=\{\theta_1,\ldots,\theta_n\}$$

$$\theta_k=\{\mu_k,\sigma_k\}$$

Because

$$P[x|y]=\frac{P[y|x]\,P[x]}{P[y]}\propto P[y|x]P[x]$$

the maximum *a posteriori* (MAP) estimate is the $\hat{x}$ that satisfies the condition

$$\hat{x}=\arg\max_{x\in X}\{P[y|x]P[x]\}$$

Because of conditional independence,

$$P[y|x]=\prod_{i\in S}P[y_i|x_i,\theta]=\prod_{i\in S}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(y_i-\mu_i)^2}{2\sigma_i^2}-\log(\sigma_i)\right\}$$

where $\sigma_i=\sigma_{x_i}$ and $\mu_i=\mu_{x_i}$. The preceding result can written as follows:

$$P[y|x]=\frac{1}{Z'}\exp\{-U(y|x)\}$$

where $U(y|x)$ is defined as the *likelihood energy function* that is given by

$$U(y|x)=\sum_{i\in S}U(y_i|x_i)=\sum_{i\in S}\left[\frac{(y_i-\mu_i)^2}{2\sigma_i^2}+\log(\sigma_i)\right]$$

and the normalizing constant $Z' = (2\pi)^{N/2}$. Now,

$$P[x|y] = \frac{P[y|x]\,P[x]}{P[y]} \propto P[y|x]P[x] = \frac{1}{\tilde{Z}} \exp\{-U(y|x) - \beta U(x)\}$$

Thus,

$$\log\left(P[x|y]\right) \propto -\{U\left(y|x\right) + \beta U(x)\}$$

and the MAP estimation is equivalent to the following criterion:

$$\hat{x} = \arg\min_{x \in X} \{U(y|x) + \beta U(x)\}$$

12.5 Applications of Markov Random Fields

Markov random field models are popularly used in several areas of image analysis because of their flexibility. Image analysis seeks to find an adequate representation of the intensity distribution of a given image. An image is typically represented by two-dimensional scalar data whose grey level variations are defined over a rectangular or square lattice. One important characteristic of image data is the statistical dependence of the grey level at a lattice point on those of its neighbors. By placing specific restrictions on the membership of the neighbor set, different representations are obtained.

Markov random field clustering has been used for remote sensing in Solberg (1996), Smits (1997), Jackson (2002), and Sarkar (2002). Unfortunately, as observed in Besag (1986), classification results can be poor when the estimation of initial parameters fails. This is due to the fact that MRF clustering methods tend to converge very rapidly, and a locally optimal solution is often obtained instead of the global solution. Thus, developing initialization schemes for the MRF is an important area of research. One example of this effort is reported in Tran (2005).

MRF has been used for supervised segmentation of remote sensing images in Poggi (2005). It has also been used by Melgani and Serpico (2003) for spatio-temporal contextual image classification. A combination of wavelet denoising and MRF is used in synthetic aperture radar speckle reduction by Xie (2002). Hazel (2000) used the Gauss-Markov random field to model multispectral scene segmentation and anomaly detection. Also, Tso and Mather (1999) used the MRF for classification of multisource remote sensing imagery.

Markov random fields have also been used in medicine. Held (1997) used MRF for segmentation of brain magnetic resonance images. Similarly, Descombes (1998) used MRF for the spatio-temporal functional magnetic resonance image analysis.

Chellappa and Chatterjee (1985) used the Gaussian-Markov random fields for the classification of textures. Onural (1991) used the MRF to model fractals. Manjunath and Chellappa (1991) used MRF for unsupervised texture segmentation. Wheeler and Ikeuchi (1995) used MRF for sensor modeling. It has also been used by Panjwani and Healey (1995) to model unsupervised representation of textured color images and by Andrey and Taroux (1998) to model textured images. Kliewer (2006) used MRF for iterative source-channel decoding, and Dogandzic and Zhang (2006) used it for distributed estimation and detection for sensor networks.

Several applications of Gaussian-Markov random fields are given in Rue and Held (2005). These applications include structural time-series analysis, analysis of longitudinal and survival data, graphical models, semiparametric regression and splines, image analysis, and spatial statistics.

Hidden Markov random fields have been used to model segmentation of magnetic resonance imaging in Zhang, Brady, and Smith (2001), and an approximation method is developed in Forbes and Peyrard (2003) for parameter estimation and analysis in hidden Markov random fields. They have also been used in unsupervised classification of radar images by Fjortoft (2003). Francois, Ancelet, and Guillot (2006) used them to develop a Bayesian clustering algorithm for studying population structure.

It must be noted that most of these problems are usually solved via the Monte Carlo simulation because they are generally very complex problems that cannot be easily solved via analytical methods.

12.6 Problems

12.1 Show that the Markov chain $\{X_n, n = 0, 1, \ldots, N\}$ is a Markov random field.
Hint: show that $P[X] = P[X_0, X_1, \ldots, X_N]$ can be expressed in the form $P[X = x] = Z^{-1} \exp\{-\beta U(x)\}$ and identify the parameters Z and $V(x) = \beta U(x)$.

12.2 Consider the three 4×4 binary image data shown in Figure 12.9 that correspond to the Gibbs random field realizations denoted by $X = x_1$, $X = x_2$, and $X = x_3$, respectively. Let the clique potential be defined as follows:

$$V_c(x) = \begin{cases} \beta & \text{if both elements in the horizontal pair clique are identical} \\ -\beta & \text{otherwise} \end{cases}$$

Obtain the following ratios:

a. $P[x_2]/P[x_1]$

1	0	1	0
1	0	1	0
1	0	1	0
1	0	1	0

(a) $X = x_1$

1	1	1	1
0	0	0	0
1	1	1	1
0	0	0	0

(b) $X = x_2$

1	0	0	0
1	1	1	0
0	0	1	0
1	0	1	0

(c) $X = x_3$

Figure 12.9. Figure for Problem 12.2.

b. $P[x_3]/P[x_1]$

12.3 For the three 4×4 binary image data shown in Figure 12.9 that correspond to the Gibbs random field realizations denoted by $X = x_1$, $X = x_2$, and $X = x_3$, respectively, assume that the clique potential is defined as follows:

$$V_c(x) = \begin{cases} \beta & \text{if both horizontal or vertical pair elements are identical} \\ -\beta & \text{otherwise} \end{cases}$$

Obtain the following ratios:

a. $P[x_2]/P[x_1]$

b. $P[x_3]/P[x_1]$

12.4 Assume that the definition of the clique potential for Problem 12.3 is expanded to include horizontal pair neighbors, vertical pair neighbors, and diagonal pair neighbors. Obtain the following ratios:

a. $P[x_2]/P[x_1]$

b. $P[x_3]/P[x_1]$

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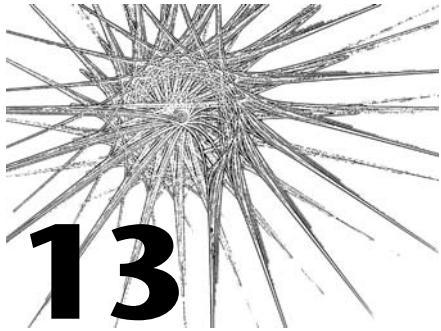
Markov Point Processes



13

13.1 Introduction

Point processes are stochastic processes that are used to model events that occur at random intervals relative to the time axis or the space axis. Thus, there are two types of point processes: *temporal point processes* and *spatial point processes*. The representation of physical events as point processes is based on two major assumptions. The first is that the physical events must be pointlike in the sense of occupying a small area in the relevant domain. The second is that the events must be discrete entities so that there will be no ambiguity when they occur. For this reason, a point process can be considered a set of discrete events that occur at well-defined but random points in time or space.

Temporal point processes deal with events that are observed over time as a time series while spatial point processes describe the locations of objects in a d-dimensional space R^d, where $d=2$ or $d=3$ in many applications of interest. Spatial point processes are used in modeling in a variety of scientific disciplines, including agriculture, astronomy, bacteriology, biology, climatology, ecology, epidemiology, forestry, geography, geology, and seismology.

Let X be a point process defined in a given bounded set S, and let $x=\{x_1,\ldots,x_n\}$ be a configuration of X. Let $N_X(B)$ denote the random variable that represents the number of points of X in the finite set (or region) $B\subseteq S$; that is,

$$N_X(B)=\#\{x_i\in B\}$$

where the symbol # is used to denote the number of points in the set following it. For temporal point processes, $B \subseteq S \subseteq R$, where R is the real line; and for spatial processes, $B \subseteq S \subseteq R^d$, where typically $d = 2$ or $d = 3$.

A point process is called a *simple point process* if, with probability one, all points are distinct. It is called an *orderly point process* if for any t,

$$\lim_{\Delta t \to 0} \left\{ \frac{P[N_X(t, t + \Delta t) > 1]}{\Delta t} \right\} = 0$$

Thus, an orderly point process is one that does not allow multiple simultaneous event occurrences. Another way to mathematically define an orderly point process is as follows:

$$P[N_X(t, t + \Delta t) > 1] = o(\Delta t) \qquad \forall t \in R$$

A *self-exciting point process* is one in which $\text{Cov}\{N_X(A),\ N_X(B)\} > 0$ for any two adjacent sets A and B. A *self-correcting point process* is one in which $\text{Cov}\{N_X(A),\ N_X(B)\} < 0$. Thus, in a self-exciting point process, the occurrence of a point (or event) enables other events to occur, whereas events in a self-correcting point process tend to inhibit the occurrence of other events.

13.2 Temporal Point Processes

A temporal point process is a stochastic process where the time points of occurrence of events consist of the times $\{T_k\}$ of isolated events scattered in time, where $0 \leq T_1 \leq T_2 \leq \cdots$. There are two parts to the process: a *counting process* that deals with the number of events in a fixed time interval and an *interval process* that deals with the time intervals between subsequent events. Thus, the process X can be represented in a number of ways including

- Counting measure $N_X(0, t)$, which denotes the number of events over the interval $(0, t)$. Then we can write

$$N_X(0, t) = \sum_{k=1}^{\infty} \text{I}(T_k \leq t) = \#\{0 < T_k \leq t\}$$

  where $\text{I}(a)$ is an indicator function that is 1 if the statement a is true and 0 otherwise.
- Interevent intervals $\{W_k\}$, which are given by $W_k = T_k - T_{k-1} \geq 0$. The random variables W_k represent distances between points.

Thus, we can alternatively define a temporal point process on R_+, where R_+ is the set of positive real numbers, as a sequence $\{T_k, k \geq 0\}$ of nonnegative random variables with the following properties:

a. $T_0 = 0$
b. $0 < T_1 < T_2 < \cdots$
c. $\lim_{k\to\infty} T_k = \infty$

Thus, the distances between points, W_k, are given by $W_1 = T_1$ and $W_k = T_k - T_{k-1}, k = 2, 3, \ldots$.

A temporal point process $\{X\}$ is defined to be a *time-stationary point process* if $N_X(B)$ has the same distribution as $N_X(B + t)$ for some $t \in R$, where R is the set of real numbers, and the sequence $\{W_k\}$ corresponding to $N_X(B)$ has the same distribution as $\{W_{k+l}\}$ for some $l \in Z$, where Z is the set of integers. Thus, for a stationary temporal point process X the time origin can be any arbitrary point, and the expected value of the counting measure $N_X(B)$ is directly proportional to $|B|$ and is given by

$$E[N_X(B)] = \Lambda|B|$$

where $|B|$ is the Lebesgue measure of B and Λ is a constant of proportionality called the *intensity measure* or the *rate* of the process.

Assume that the intervals, W_k, between events in a stationary temporal point process are identically distributed, and let $f_W(w)$ and $F_W(w)$ denote the PDF and CDF respectively of W. Assume also that $E[W] = \eta$. Let the random variable R denote the forward-recurrence time, which is the time from now until the next event occurs, and let $f_R(r)$ and $F_R(r)$ denote the PDF and CDF, respectively, of R. These random variables are shown in Figure 13.1, which is the same as Figure 6.2.

Suppose we select an interval between events of length V at random and choose the point indicated in Figure 13.1 at random also. As discussed in Chapter 6,

$$f_V(v) = \frac{v f_W(v)}{\eta}$$

$$f_{RV}(r, v) = \frac{f_W(v)}{\eta} \qquad 0 \leq r \leq v \leq \infty$$

$$f_R(r) = \frac{1 - F_W(r)}{\eta} \qquad r \geq 0$$

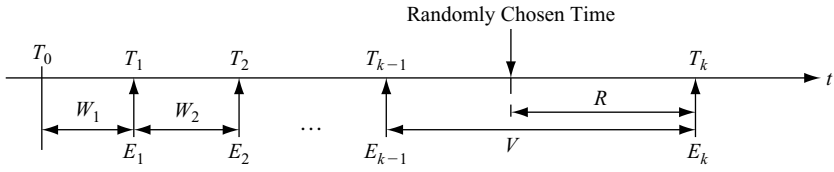


Figure 13.1. Relationship between W and R.

13.2.1 Specific Temporal Point Processes

There are many examples of temporal point processes, which include the Poisson point process, the renewal point process, and the Cox point process, all of which we briefly discuss in the following sections. The Poisson process is the most important point process and is widely used as a building block for more complex models.

Poisson Point Processes

The general characteristics of the Poisson process are discussed in Chapter 1. In this section we summarize those characteristics that are pertinent to the theory of temporal point processes. A Poisson process X with rate $\Lambda(B)$ is a stochastic process whose PMF for the number of events in the interval $(t_1, t_2]$ is given by

$$p_{N_X(t_1,t_2)}(n) = P[N_X(t_1, t_2) = n] = e^{-\Lambda(\tau)} \frac{\{\Lambda(\tau)\}^n}{n!} \quad n = 0, 1, \ldots$$

where $\tau = t_2 - t_1$. For a stationary Poisson process, $\Lambda(\tau) = \lambda\tau$ and $E[N_X(t_1, t_2)] = \lambda\tau$. Also, the PDF of the times between events (or the waiting times) for a stationary Poisson process with rate λ is given by

$$f_W(w) = \lambda\, e^{-\lambda w} \quad w \geq 0$$

That is, the times between events are independent and exponentially distributed with a mean of $1/\lambda$.

A nonhomogeneous Poisson point process is a Poisson process with a variable rate $\lambda(t)$. It is used to model Poisson arrival processes where arrival occurrence epochs depend on time, such as time of the day or time of the year.

Cox Point Processes

A Cox process is sometimes called the *doubly stochastic Poisson process* because it is an extension of the Poisson process. It can be obtained by first randomly generating the intensity measure $\Lambda = \lambda$, then generating a Poisson process with the intensity measure λ. Thus, a point process X is a Cox process if, conditional on the intensity measure $\Lambda = \lambda$, X is a Poisson process with rate λ. This means that for a single realization of X, it is not possible to distinguish a Cox process from its corresponding Poisson process.

Note that the intensity measure $\Lambda(t)$ is a random variable with a predefined probability distribution. This means that we can also define a Cox process as a Poisson process with a variable intensity measure that is itself a stochastic process, which we refer to as the intensity process. The Cox point process is stationary if and only if the intensity process is stationary.

13.3 Spatial Point Processes

Spatial point processes are used to model events that occur in space (or on a plane). Thus, a spatial point process is a finite subset S of a d-dimensional space R^d or the entire R^d; that is, $S \subseteq R^d$. One important class of spatial point processes is the *stationary* or *homogeneous* point process. A stationary (or homogeneous) spatial point process is a spatial point process whose distribution is invariant under translation; that is, for integer k and regions B_i, $i = 1, \ldots, k$, the joint distribution of $N_X(B_1), \ldots, N_X(B_k)$ is equal to the distribution of $N_X(B_1 + y), \ldots, N_X(B_k + y)$ for an arbitrary y. Another class of spatial point processes is the *isotropic* point process, which is a spatial point process whose distribution is invariant under rotation through an arbitrary angle; that is, there is no directional effect.

Spatial point processes are used in many applications. They are used to model multihop radio networks in Cheng and Robertazzi (1990). They have also been used to model defensive strategies in Kornak (2006). In Ayala (2006), they are used in clustering of spatial point patterns where the interest is in finding groups of images corresponding with groups of spatial point patterns.

Spatial point processes are commonly characterized by their moment measures. For a spatial point process X, the intensity measure is given by

$$\Lambda(B) = E[N_X(B)]$$

The first- and second-order intensities are used to determine the mean and dependency structure of the data. The intensity measure is related to the *first-order intensity function* $\lambda(x)$ as follows:

$$\Lambda(B) = \int_B \lambda(x)dx$$

For a spatial point process that is stationary and isotropic, the intensity function is a constant, λ. In the case where $S \subseteq R^2$, the first-order intensity function is defined as the number of events per unit area. That is,

$$\lambda(x) = \lim_{|\Delta x| \to 0} \left\{ \frac{E[N_X(\Delta x)]}{|\Delta x|} \right\}$$

The *second-order intensity function* of a spatial point process is defined by

$$\lambda_2(x_1, x_2) = \lim_{\substack{|\Delta x_1| \to 0 \\ |\Delta x_2| \to 0}} \left\{ \frac{E[N_X(\Delta x_1) N_X(\Delta x_2)]}{|\Delta x_1||\Delta x_2|} \right\}$$

The quantity $\lambda_2(x_1, x_2)\,dx_1dx_2$ is the approximate probability that there is at least one point of X in each of the regions dx_1 and dx_2. In a homogeneous case, we have that

$$\lambda_2(x_1, x_2) = \lambda_2(x_1 - x_2)$$

The second-order intensity function is also called the *second-order product density*. For a stationary and isotropic spatial process, we have that

$$\lambda_2(x_1, x_2) = \lambda_2(u)$$

where $u = |x_1 - x_2|$. The covariance density of a spatial point process is given by

$$\gamma(x_1, x_2) = \lambda_2(x_1, x_2) - \lambda(x_1)\lambda(x_2)$$

The *pair correlation function* is the normalized second-order intensity function and is defined by

$$g(x_1, x_2) = \frac{\lambda_2(x_1, x_2)}{\lambda(x_1)\lambda(x_2)}$$

For a spatial Poisson point process, which is a completely random process, the pair correlation function $g(x_1, x_2) = 1$. If $g(x_1, x_2) > 1$, it means that for a relatively small interpoint distance $u = |x_1 - x_2|$ the interdistance is more frequent than in a random point pattern, which implies that the points in X tend to cluster relative to a Poisson process with the same intensity function as the process X. Similarly, when $g(x_1, x_2) < 1$, the points tend to repel relative to a Poisson process with the same intensity function.

Another common second-order characteristic of a stationary isotropic spatial point process is the K function (or *reduced second-moment measure*), which is defined as follows:

$$K(r) = \frac{1}{\lambda}\int_{\theta=0}^{2\pi}\int_{x=0}^{r} \lambda_2(x, \theta)\, x dx d\theta$$

where λ is the rate of the process. $K(r)$ measures the expected relative rate of events within a distance r of an arbitrary event. If a point process is clustered, then each event is likely to be surrounded by more events from the same cluster, which means that $K(r)$ will be relatively large for small r. Similarly, if a process is randomly distributed in space, then each event is likely to be surrounded by an empty space, which means that $K(r)$ will be relatively small for small r. For a stationary Poisson process, $K(r) = \pi r^2$. When a process has $K(r) > \pi r^2$, it means that the points tend to cluster, and when $K(r) < \pi r^2$, it means that the points tend to repel each other.

Several examples of applications of spatial point processes in urban public systems are discussed in Larson and Odoni (1981).

13.3.1 Specific Spatial Point Processes

As in the case of temporal point processes, examples of temporal point processes include the Poisson process, the renewal process, the Cox point process, and the Gibbs process.

Spatial Poisson Point Processes

A spatial point process X defined in a given bounded region S is a Poisson process with rate λ if the number $N_X(B)$ of points of X in the region $B \subseteq S$ has a Poisson distribution with mean $\lambda|B|$. Thus, the PMF of $N_X(B)$ is given by

$$p_{N_X(B)}(n) = P[N_X(B) = n] = \frac{\{\lambda|B|\}^n}{n!} \exp\{-\lambda|B|\} \quad \lambda|B| \geq 0,\ n = 0, 1, \ldots$$

The homogeneous spatial Poisson point process exhibits what is known as *complete spatial randomness* (CSR), which means that the events are independent and uniformly distributed over the region B.

For a nonhomogeneous Poisson process, the rate λ will not be a constant but some function of position, $\lambda(B)$. Also, in the case where $S \subseteq R^2$, the Lebesgue measure $|B|$ will be the area of the region B. In this case, if we denote the probability that a point occurs in the region $\{(x, x + dx),\ (y, y + dy)\}$ by $\lambda(x, y)\, dxdy$, then we have that

$$\lambda|B| = \int_B \int \lambda(x, y)\, dxdy$$

As stated earlier, because of the CSR property of the spatial Poisson point process, given that $N_X(B) = n$, the locations of the n points in B are independent and uniformly distributed random variables. Thus, one realization of the process is shown in Figure 13.2.

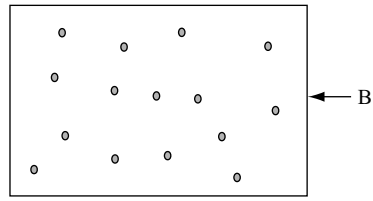


Figure 13.2. Realization of two-dimensional Poisson process.

For a stationary and isotropic Poisson point process, the first-order intensity function is given by

$$\lambda(x) = \frac{E[N_X(B)]}{|B|} = \frac{\lambda|B|}{|B|} = \lambda$$

which is a constant for all B. Similarly, the second-order intensity function depends only on the distance between locations x_1 and x_2; that is,

$$\lambda_2(x_1, x_2) = \lambda_2(||x_1 - x_2||)$$

Thus, the covariance density of the process is given by

$$\gamma(x_1, x_2) = \lambda_2(||x_1 - x_2||) - \lambda^2 = \gamma(||x_1 - x_2||)$$

Example 13.1 Consider a city in which police cars are distributed according to a Poisson process with a rate of η cars per square mile. Assume that an incident requiring police presence occurs somewhere in the city. What is the PDF of the distance L between the location of the incident and the nearest police car, assuming a Euclidean travel distance? ■

Solution: This is an example of a class of problems called the *nearest-neighbor* problems. Let the point of the incident be (x, y). For a Euclidean travel distance, we construct a circle of radius r centered at (x, y) so that the Lebesgue measure is the area of the circle; that is, $|B| = \pi r^2$. Let $M(r)$ denote the number of police cars within a circle of radius r. Then the PMF of $M(r)$ is

$$p_{M(r)}(m) = P[M(r) = m] = \frac{(\eta\pi r^2)^n \exp(-\eta\pi r^2)}{m!} \quad m = 0, 1, \ldots$$

The CDF of L is given by

$$F_L(l) = P[L \le l] = 1 - P[L > l] = 1 - P[M(l) = 0] = 1 - \exp(-\eta\pi l^2) \quad l \ge 0$$

Thus, the PDF of L is

$$f_L(l) = \frac{d}{dl} F_L(l) = 2l\eta\pi\, e^{-\eta\pi l^2} \quad l \ge 0$$

Spatial Cox Point Processes

As discussed in an earlier section, the Cox process X is a doubly stochastic Poisson process. Thus, conditional on $\Lambda = \lambda$, the process becomes a spatial Poisson process with mean $\lambda|B|$. One property of Cox processes is that their variances always exceed those of the stationary Poisson processes with the same intensity. This can be demonstrated in the following manner using the method used in Kingman (1993), as follows:

$$\begin{aligned} E[\{N_X(B)\}^2] &= E[E[\{N_X(B)\}^2|\lambda]] = E[\text{Var}(\text{Poi}(\lambda(B))) + \{E[\text{Poi}(\lambda(B))]\}^2] \\ &= E[\lambda(B) + \{\lambda(B)\}^2] = E[\lambda(B)] + E[\{\lambda(B)\}^2] \\ &= E[\lambda(B)] + \text{Var}(\lambda(B)) + \{E[\lambda(B)]\}^2 \end{aligned}$$

where $\text{Var}(\text{Poi}(\lambda(B)))$ is the variance of a Poisson process with mean $\lambda(B)$. From this we obtain

$$E[\{N_X(B)\}^2] - \{E[\lambda(B)]\}^2 = \text{Var}(N_X(B)) = E[\lambda(B)] + \text{Var}(\lambda(B))$$

Because $E[\lambda(B)] \geq 0$, we have that $\text{Var}(N_X(B)) \geq \text{Var}(\lambda(B))$ and also $\text{Var}(N_X(B)) \geq E[\lambda(B)]$. Thus, the count measure $N_X(B)$ has a greater variance than a Poisson random variable with the same mean. For this reason, all Cox processes are said to be "over-dispersed" relative to the Poisson processes.

Spatial Gibbs Processes

The Gibbs process was discussed in Chapter 12 in connection with Markov random fields. As we discussed in that chapter, the process originated in statistical physics and is described in terms of forces acting on and between particles. It enables the total potential energy associated with a given configuration of particles to be decomposed into terms representing the external force field on individual particles and terms representing interactions between particles taken in pairs, triplets, etc.

Gibbs processes are not universal models that apply to all situations. Instead, their distributions are defined according to the application of interest. More importantly, they do not perform well in applications with strong regularity; they are good for applications with some degree of regularity. Also, they are more applicable when the system to be modeled contains only a finite number of points in a bounded region B.

One advantage of the Gibbs point process over the Poisson point process is that the Poisson point process is not able to account for interactions between points, while the Gibbs point process can. In fact, the Gibbs point process can be regarded as a pairwise interacting process.

From Chapter 12, the PDF of the Gibbs process X is given by

$$f_X(x) = f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \frac{1}{Z}\exp\{-\beta U(x)\} = \frac{1}{Z}\exp\{-\beta U(x_1,\ldots,x_n)\}$$

where $U(x)$ is called the *energy function*; $\beta = 1/kT$, where T is the *temperature* (in absolute degrees) and k is the *Boltzmann's constant*; and Z is a normalizing constant called the *partition function* and is given by

$$Z = \int_B \exp\{-\beta U(x_1,\ldots,x_n)\}\,dx_1\cdots dx_n$$

$U(x)$ is usually defined in such a manner as to match the application. However, it is generally in the form of a series as follows:

$$U(x) = \sum_{i=1}^{n} V_1(x_i) + \sum_{i_1>i_2} V_2(x_{i_1} - x_{i_2}) + \sum_{i_1>i_2>i_3} V_3(x_{i_1} - x_{i_2}, x_{i_1} - x_{i_3})$$
$$+ \cdots + \sum_{i_1>\ldots>i_k} V_k(x_{i_1} - x_{i_2},\ldots,x_{i_1} - x_{i_k})$$

where the $V_1(\cdot)$ are called the *potential functions* and $V_i(\cdot)$, $i > 1$, is the ith-order interaction potential function. In many applications it is assumed that only the first-order and second-order interactions are significant; that is, $k = 2$. In this case we obtain

$$U(x_1,\ldots,x_n) = \sum_{i=1}^{n} V_1(x_i) + \sum_{i_1>i_2} V_2(x_{i_1} - x_{i_2})$$
$$= \sum_{i=1}^{n} V_1(x_i) + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} V_2(x_i - x_j)$$

Discussions on the different restrictions on the values of the potential functions can be found in Cox and Isham (2000) and Daley and Vere-Jones (2003).

13.4 Spatial-Temporal Point Processes

A spatial-temporal (or spatio-temporal, or space-time) point process is a random collection of points whose coordinates represent the time and location of an event. If the space component describes the locations of objects in a d-dimensional space

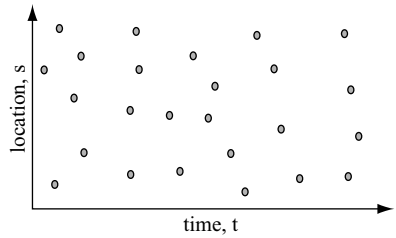


Figure 13.3. Illustration of spatial-temporal point process.

R^d, then $S \subseteq R \times R^d$. For the case of $d = 2$, the points in a spatial-temporal process are represented in the form (t, x, y), where t denotes the time of occurrence of the event, x denotes the location on the x-axis, and y denotes the location on the y-axis. Figure 13.3 illustrates a spatial-temporal point process for a one-dimensional space.

Spatial-temporal processes are used in many disciplines including the study of earthquakes, epidemiology, and occurrence of fire and lightning strikes. One important issue in the analysis of spatial-temporal processes is the interaction between the space and time components. They may be noninteracting or interact in any number of ways.

A simple illustration of the spatial-temporal process X is the following. Consider a region $B \subseteq S \subseteq R^d$ where events occur according to a Poisson process with rate $\lambda|B|$ events per unit time. Let $N_X(B, t)$ denote the number of events in the region B over a time interval t. Then the PMF of $N_X(B, t)$ is given by

$$\begin{aligned} p_{N_X(B,t)}(n, t) &= P[N_X(B, t) = n] \\ &= \frac{\{\lambda t|B|\}^n}{n!} \exp\{-\lambda t|B|\} \quad \lambda|B| \geq 0, \quad t \geq 0, \quad n = 0, 1, \ldots \end{aligned}$$

An application of the spatial-temporal point process to model Poisson processes with birth, death and movement is discussed in Cox and Isham (2000). The process is also used to model earthquakes in Rathbun (1996), Ogata (1999), and Choi and Hall (1999). It is used in Rathbun and Cressie (1994) to analyze longleaf pines in southern Georgia.

The intensity measures are functions of both time and space. Thus, the first-order intensity function is given by

$$\lambda(x, t) = \lim_{\substack{|\Delta x| \to 0 \\ |\Delta t| \to 0}} \left\{ \frac{E[N_X(\Delta x, \Delta t)]}{|\Delta x||\Delta t|} \right\}$$

where Δx is an infinitesimal disc containing the location x, and Δt is an infinitesimal interval containing the time t. The *marginal first-order spatial intensity function* is given

$$\lambda(x, -) = \int_T \lambda(x, t)\, dt$$

where integration is over all time T. Similarly, the *marginal first-order temporal intensity function* is given by

$$\lambda(-, t) = \int_A \lambda(x, t)\, dx$$

where integration is over the region A. These marginal intensity functions permit us to view one component while ignoring the other. The *conditional first-order spatial intensity function* is given by

$$\lambda(x|t = t_0) = \lim_{|\Delta x| \to 0} \left\{ \frac{E[N_X(\Delta x, t_0)]}{|\Delta x|} \right\}$$

Similarly, the *conditional first-order temporal intensity function* is given by

$$\lambda(t|x = x_0) = \lim_{|\Delta t| \to 0} \left\{ \frac{E[N_X(x_0, \Delta t)]}{|\Delta t|} \right\}$$

A spatial-temporal point process is defined to be *first-order stationary in space* if

$$\lambda(t|x) = \lambda(t)$$

Similarly, a spatial-temporal point process is defined to be *first-order stationary in time* if

$$\lambda(x|t) = \lambda(x)$$

Thus, the conditional first-order temporal intensity function of a stationary temporal-spatial point process is independent of location, and the conditional first-order spatial intensity function is independent of time.

We can also obtain the second-order intensity function by

$$\lambda_2(x_1, x_2, t_1, t_2) = \lim_{\substack{|\Delta x_1|\to 0,\ |\Delta t_1|\to 0 \\ |\Delta x_2|\to 0,\ |\Delta t_2|\to 0}} \left\{ \frac{E[N_X(\Delta x_1, \Delta t_1) N_X(\Delta x_2, \Delta t_2)]}{|\Delta x_1||\Delta x_2||\Delta t_1||\Delta t_2|} \right\}$$

The *marginal second-order spatial intensity function* and the *marginal second-order temporal intensity function* are given, respectively, by

$$\lambda_2(x_1, x_2, -, -) = \int_T \int_T \lambda_2(x_1, x_2, t_1, t_2)\, dt_1 dt_2$$

$$\lambda_2(-, -, t_1, t_2) = \int_{A_1} \int_{A_2} \lambda_2(x_1, x_2, t_1, t_2)\, dx_1 dx_2$$

Details on how to compute other statistics of the spatial-temporal point process can be found in Dorai-Raj (2001).

13.5 Operations on Point Processes

Sometimes new point processes can be constructed from old ones to fit the environment of interest. There are several methods used to generate new processes from old ones, but we consider only three of them, which are:

- Thinning
- Superposition
- Clustering

13.5.1 Thinning

Thinning is an operation on a point process that essentially reduces the average density but leaves the correlation function intact. The operation is similar to the filtering operation used to generate the filtered Poisson process discussed in Chapter 1 and works as follows. Given a point process X with the intensity $\Lambda(B)$, obtain a new process in the following manner. For each point x_k in the configuration $x = \{x_1, \ldots, x_n\}$, independently of other points, either retain the point with probability q_k or delete it with probability $1 - q_k$, where $0 \le q_k \le 1$. The new process is a point process with intensity $q\Lambda(B)$, where $q = \{q_1, \ldots, q_n\}$.

Thinning can be used to generate a *hard-core point process*, which is a point process in which the points are not allowed to lie closer than a predefined minimum distance. One type of hard-core process, called the *Matern hard-core process*, is obtained by applying thinning to a stationary Poisson process. In this case, the

points in the Poisson process are randomly marked, and points that are within a distance less than $R/2$ from a marked point are deleted, where R is the predefined minimum distance between points.

13.5.2 Superposition

The superposition of independent point processes $X_1, \ldots, X_K$ is the union of these processes. That is, if the process X is generated from the superposition of these processes, then

$$X = \bigcup_{k=1}^{K} X_k$$

We refer to X as the *pooled process*. Let $N_{X_k}(B)$ denote the counting measure of the process X_k, and let $\Lambda_k(B)$ denote its rate. Then the rate of the pooled process is given by

$$N_X(B) = \sum_{k=1}^{K} N_{X_k}(B)$$

$$\Lambda(B) = \sum_{k=1}^{K} \Lambda_k(B)$$

The probability mass function of $N_X(B)$ is the K-fold convolution of the probability mass functions of the $N_{X_k}(B)$. That is,

$$p_{N_X}(x) = p_{N_{X_1}}(x) * p_{N_{X_2}}(x) * \cdots * p_{N_{X_K}}(x)$$

where the symbol $*$ denotes convolution operation.

13.5.3 Clustering

In a clustering operation, every point in a given point process, called the *parent point process*, is used to generate a cluster of points, called *child points*. Each cluster is generated independently of other clusters; however, the same construction rules apply to all clusters. Thus, within a cluster the child points are placed independently according to the density function of the cluster points.

Each cluster can be regarded as being within a disc of radius $R > 0$ with the parent point as the center of the disc. Thus, there are three parameters that characterize a clustering operation: the intensity $\Lambda(B)$ of the parent point process, which defines the locations of the centers of the clusters; the disc radius R; and the cluster intensity $\Lambda_1(B_R)$, where B_R is the region within a cluster.

For most of the commonly used clusters, the parent point process is a homogeneous (stationary) Poisson point process. These clusters differ primarily in the way the child points are generated and placed within the cluster. Examples of cluster processes whose parent point processes are homogeneous Poisson point processes include the following:

- *Matern cluster process*, where the number of points per cluster is a Poisson process, and these child points are uniformly placed within a disc of radius R centered about the cluster, where R is the same for all clusters.
- *Thomas cluster process*, where the number of points per cluster is a Poisson process, and the child points in each cluster are distributed independently according to a symmetric normal distribution around the cluster origin.
- *Neyman-Scott cluster process*, where the number of points in a cluster is an independent and identically distributed random variable. The points are also placed uniformly and independently within a disc of radius R around each cluster's center.
- *Hawkes (or self-exciting) cluster process*, where a parent point produces a cluster of child points, and each child point further produces its own cluster of child points, and so on.

13.6 Marked Point Processes

As stated earlier, a point process is a stochastic system that places points in the plane. If each point has a mark (generally, a real number or a set of real numbers) associated with it, the process is called a marked point process. Let X be a point process on $S \subseteq R^d$. Given some space L, if a random mark $m_k \in M$ is attached to each point $x_k \in X$, where M is a set defined on L, then the process

$$Y = \{(x_k, m_k) | x_k \in X\}$$

is called a marked point process with points in S and mark space L, which can be a finite set or $L \subseteq R^p$, $p \geq 1$. The process X is called the *ground process*, and S is the *ground space*.

Marked point processes are useful for describing many physical systems. In general, they are commonly used for representing a finite number of events located in space and time. For example, consider a queueing system in which the nth customer arrives at time x_n and brings with it an amount of service m_n. The process $\{(x_n, m_n), n \geq 1\}$ is a marked point process that can be used in the performance analysis of the system.

Marked point processes have been used in Vere-Jones (1995), Ogata (1998), and Holden (2003) to model earthquakes. They have also been used in Smith (1993)

to model raindrop-size distributions. In Descombes and Zerubia (2002) they are used in image analysis and in Prigent (2001) to model option pricing. They have also been used in ecological and forestry studies in Gavrikov and Stoyan (1995), and Stoyan and Penttinen (2000) present a summary of the applications of marked point processes in forestry. They are used by McBride (2002) to model the source proximity effect in the indoor environment. In Stoica (2000) and van Lieshout and Stoica (2003), a marked point process model for line segments called the Candy model is presented as a prior distribution for the image analysis problem of extracting linear networks, such as roads and rivers, from images.

Because M is a process on a bounded set L, we can interpret the marked point process Y as an ordinary point process in $R^d \times L$. If X is defined in the finite region $B \subseteq S$ and $N_Y(B \times L)$ denotes the random variable that represents the number of points of X with marks in M, then the intensity measure of N_Y is given by

$$\Lambda(B \times L) = E[N_Y(B \times L)]$$

There are two types of marks. In one case the marks are independent and identically distributed random variables that are independent of the point process. In another case the marks depend on the point process. We first consider the case where the marks are independent of the ground process. Let the random variable $M(B)$ denote the number of marks in the region B. Then given that $N_X(B) = n$, $M(B)$ is the sum of n independent and identically distributed random variables, and as shown in Ibe (2005) we have that

$$\begin{aligned} E[M(B)] &= E[N_X(B)]E[M] = \Lambda|B|E[M] \\ \text{Var}\{M(B)\} &= \Lambda|B|\sigma_M^2 + (E[M])^2\text{Var}\{N_X(B)\} \\ \text{Cov}\{N_X(B),\, M(B)\} &= E[M]\text{Var}\{N_X(B)\} \end{aligned}$$

The case when the marks depend on the point process can only be analyzed on a case-by-case basis because there is no general solution.

13.7 Markov Point Processes

Markov point processes were introduced in Ripley and Kelly (1977) and are popular models for point patterns with interaction between the points. A good discussion of Markov point processes is given in Baddeley and Moller (1989), van Lieshout (2000), and in Moller and Waagepetersen (2004). The interactions between points are usually local with respect to a neighborhood system, in a manner similar to the Markov random fields discussed in Chapter 12.

Since the introduction of Markov point processes, attention has focused on a special class called the *pairwise interaction point process*. In the pairwise interaction process, the interaction between points is local with respect to a neighborhood system. Thus, for a point process X with a configuration $x = \{x_1, \ldots, x_n\}$, where $x_i \in X,\ i = 1, \ldots, n$, the configuration interacts only via pairs of points from this configuration. Thus, the analysis of this special class of Markov point processes is similar to that of the Markov random field. We denote the neighborhood relation by $\diamondsuit$ and define two points $x_i,\ x_j \in S$ to be neighbors, written $x_i \sim x_j$, if for a number $r > 0$, $x_i \diamondsuit x_j < r$. Then, both the positivity and Markovianity properties hold; that is,

$$P[X = x] > 0$$
$$P[X_i = x_i | X_j = x_j,\ j \neq i] = P[X_i = x_i | X_j = x_j,\ \ j \in \aleph(i)]$$

where $X = \{X_1, \ldots, X_n\}$ and $\aleph(i)$ is the set of neighbors of point x_i. Another important general property of Markov point processes is the *hereditary* property, which states that for any finite configuration x

$$f_X(x) > 0 \Rightarrow f_X(y) > 0 \quad \forall\, y \subset x$$

For a pairwise interaction process, the PDF of the configuration X is of the form

$$f_X(x) = \alpha \prod_{i=1}^{n} \lambda(x_i) \prod_{x_i \sim x_j} h(x_i \diamondsuit x_j)$$

where h is the interaction function, n is the cardinality of x, $\lambda(x)$ is the intensity function, and α is a normalizing constant. A pairwise interaction function is defined to be homogeneous if $\lambda(x) = \lambda$, a constant, and $h(x_i \diamondsuit x_j)$ is invariant under motions, which permits us to define the neighborhood relation by the Euclidean distance; that is, $x_i \diamondsuit x_j = ||x_i - x_j||$. Thus, in this case the set of neighbors of point x_i is given by

$$\aleph(i) = \{x_j | x_i \diamondsuit x_j < r,\ j \neq i\} = \{x_j : ||x_i - x_j|| < r,\ j \neq i\}$$

We denote the neighborhood system by $\aleph = \{\aleph(i)\}$. The neighborhood relation is symmetrical in the sense that $x_i \in \aleph(j) \Longleftrightarrow x_j \in \aleph(i)$; that is, $x_i \sim x_j \Longleftrightarrow x_j \sim x_i$. Also, the relation is reflexive in the sense that $x_i \sim x_i$. In the remainder of the

discussion, we assume that the pairwise interaction process, and hence the Markov point process, is homogeneous; that is,

$$f_X(x) = \alpha\lambda^n \prod_{x_i \sim x_j} h(||x_i - x_j||)$$

The PDF $f_X(x)$ can be expressed in the form of the Gibbs distribution using a version of the Hammersley-Clifford theorem as follows:

$$f_X(x) = \frac{1}{Z} \exp\{-\beta U(x)\}$$

where $U(x)$ is the *energy function* of the configuration x, Z is the partition function (or the normalizing constant), $\beta = 1/kT$, T is the absolute temperature, and k is the Boltzmann's constant. The simplest form of the energy function for pairwise interaction processes is given by

$$U(x) = V_0 + \sum_{i=1}^{n} V_1(x_i) + \sum_{1 \le i < j \le n} V_2(x_i, x_j)$$

where the function V_k is called the *potential of order k.* Thus, the energy is computed by taking one point at a time or two points at a time. For the process to be homogeneous, we need $V_1(u)$ to be a constant, such as $V_1(u) = a_1$; similarly, we need $V_2(x_i, x_j) = V_2(||x_i - x_j||)$. If we let $V_0 = a_0$, we obtain

$$\begin{aligned} f_X(x) &= \frac{1}{Z} \exp\{-\beta U(x)\} \\ &= \frac{1}{Z} e^{-\beta a_0} e^{-n\beta a_1} \exp\left\{-\beta \sum_{1 \le i < j \le n} V_2(||x_i - x_j||)\right\} \\ &\equiv ab^n \exp\left\{-\beta \sum_{1 \le i < j \le n} V_2(||x_i - x_j||)\right\} \end{aligned}$$

Because we earlier obtained $f_X(x)$ as

$$f_X(x) = \alpha\lambda^n \prod_{x_i \sim x_j} h(||x_i - x_j||) = \alpha\lambda^n \exp\left\{\sum_{x_i \sim x_j} \log[h(||x_i - x_j||)]\right\}$$

which is similar to the result obtained via the Gibbs distribution, it means that every homogeneous Markov point process is a Gibbs point process. We consider the case where the second-order (or pair) potential is of the form

$$V_2(u) = \begin{cases} \infty & u = 0 \\ q & 0 < u \leq \mu \\ 0 & u > \mu \end{cases}$$

where $q > 0$ and $\mu > 0$. We obtain

$$f_X(x) = ab^n \exp\{-\beta\mu n_\mu(x)\} = ab^n \gamma^{n_\mu(x)}$$

where $n_\mu(x)$ is the number of point pairs with a Euclidean distance of less than μ and $\gamma = \exp\{-\beta\mu\}$. This is the PDF of a point process called the *Strauss point process*.

13.8 Markov Marked Point Processes

As discussed earlier, if X is a point process on $S \subseteq R^d$, and a random mark $m_k \in M$ is attached to each point $x_k \in X$, where M is a set defined on some space L, then the process

$$Y = \{(x_k, m_k) | x_k \in X\}$$

is called a marked point process with points in S and mark space L. In the remainder of this section we assume that the marks are independent and identically distributed random variables that are independent of the point process.

A marked point process Y is defined to be a Markov marked point process if the following conditions hold:

$$P[Y = y] > 0$$
$$P[y] > 0 \Rightarrow P[z > 0] \quad \forall z \subset y$$
$$P[Y_i = y_i | Y_j = y_j, j \neq i] = P[Y_i = y_i | Y_j = y_j, j \in \aleph(i)]$$

These are the positivity, hereditary, and Markovianity properties, respectively. The discussion on Markov point processes can be extended to the Markov marked point process fairly easily. In particular, using the pairwise interaction point process

model, we have that if $y = \{(y_1, m_1), \ldots, (y_n, m_n)\}$ is a configuration of Y, then the PDF of Y is given by

$$f_Y(y) = \alpha \prod_{i=1}^{n} \lambda(y_i, m_i) \prod_{(y_i, m_i) \sim (y_j, m_j)} h(||(y_i, m_i) - (y_j, m_j)||)$$

where the neighborhood relation has to be appropriately defined. It must also be stated that the process is also a Markov point process with respect to the neighborhood relation

$$(y_i, m_i) \sim (y_j, m_j) \quad \text{iff} \quad ||y_i - y_j|| < r$$

for some $r > 0$.

13.9 Applications of Markov Point Processes

Point processes, marked point processes, and Markov point processes are used to model many applications that include earthquakes, raindrop-size distributions, image analysis, option pricing, and ecological and forestry studies. The fundamental principles of these processes have been discussed in this chapter. More comprehensive discussion on the principles and applications of point processes and their derivatives are discussed in Bremaud (1981), Ripley (1988), Reiss (1993), Stoyan (1995), Cox and Isham (2000), Daley and Vere-Jones (2003), and Jacobsen (2006).

The analysis of point processes, marked point processes, and Markov point processes is usually complex and is generally done via Monte Carlo simulation. Also, a major aspect of the analysis deals with parameter estimation, which is not covered in this chapter and can be found in most of the references cited. Even in the case of parameter estimation, the preferred method is the Monte Carlo maximum likelihood estimate method for missing data models proposed in Gelfand and Carlin (1993).

13.10 Problems

13.1 Two classes of customers arrive at Paul's barber shop: class 1 and class 2. Class 1 customers arrive according to a Poisson process with rate λ_1 customers per hour, and class 2 customers arrive according to a Poisson process with rate λ_2 customers per hour.

a. What is the probability that no customer arrives over a two-hour period?

b. What is the mean time between customer arrivals at Paul's shop?

c. Given that a customer has just arrived at the shop, what is the probability that the next customer to arrive at the shop is a class 2 customer?

13.2 A hard-core point process is produced from a Poisson process of rate λ by deleting any point within distance v_0 of another point, regardless of whether that point has itself already been deleted. Prove that the rate of the hard-core process is $\lambda \exp(-\lambda \pi v_0^2)$.

13.3 Let $\{X_1, X_2, \ldots\}$ be a sequence of independent and identically distributed random variables with PDF $f_X(x)$, and let N be an integer-valued random variable with PMF $p_N(n)$, where N and the X_i are independent. Consider a process in which events occur at times $X_1, X_1 + X_2, \ldots, X_1 + X_2 + \cdots + X_N$.

a. Calculate the mean, variance, and s-transform of the PDF of the time of the last event.

b. What is the expected number of events in the interval $(0, t)$?

13.4 A restaurant has two entrances, A and B. Customers arrive at the restaurant through entrance A according to a Poisson process with a rate of five customers per hour, and customers arrive through entrance B according to a Poission process with a rate of seven customers per hour.

a. What is the probability that no new customer arrives over a 2-hour period?

b. What is the mean time between arrivals of new customers at the restaurant?

c. What is the probability that a given customer arrived through entrance B?

13.5 Passengers arrive at a train station according to a Poisson process with a rate of 25 customers per hour. It has been found that 60% of the passengers are female. What is the probability that no male customer arrives at the station over a 15-minute period?

13.6 One important parameter in stationary point processes is the *autointensity function*. The autointensity function, $h_{NN}(t, v)$, of the process $N(t)$ is the conditional probability that a point occurs at time $t + v$, given that a point occurs at time t. Specifically,

$$h_{NN}(t, v) = \frac{P[dN(t+v) = 1 | dN(t) = 1]}{dv}$$

where

$$dN(t) = \begin{cases} 1 & \text{if a point in } (0, 0 + dt) \\ 0 & \text{otherwise} \end{cases}$$

Show that for a Poisson process the autointensity function is a constant. (Note: This demonstrates the nonpredictability of the Poisson process.)

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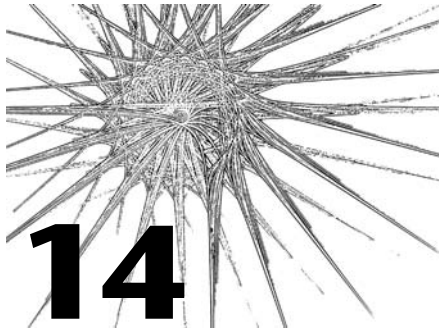
Markov Chain Monte Carlo



14.1 Introduction

It is not always possible for us to develop analytical models of many of the systems that we build. These systems, which include communication systems, remote sensing systems, expert systems, and systems associated with molecular biology, are generally too complex to be analytically modeled with reasonable accuracy. Although these systems can be analyzed by performing measurements on an actual system, very often these measurements can be either impossible or impractical to make. For such complex and thus analytically intractable systems, the simulation method has become the only available tool.

A simulation model attempts to imitate the behavior of the system under investigation by studying the interactions among its components. Thus, it enables us to build an artificial model of a real system to study and understand it. It can also be viewed as the technique whereby we perform sampling experiments on the model of the system. Thus, rather than performing the experiment on the real system, we do it on a model of the system because performing it on the real system will be either too expensive, inconvenient, or time consuming. This implies that a simulation can be regarded as a statistical experiment.

Simulation is usually performed on a computer because it involves the generation and processing of a large amount of data. A simulation model consists of three phases:

- Data generation
- Bookkeeping
- Output analysis

The data generation phase produces data that can be used to represent the sequence of events that govern the dynamic behavior of the system under investigation. The bookkeeping phase deals with updating the system when new events occur, monitoring and recording changes in system states, and keeping track of parameters of interest. The output phase provides analysis of the experiment and ensures that there is enough confidence in the results.

A simulation model requires a mechanism for generating the sequence of events associated with the data phase. Because of the random nature of these events, the interevent intervals are usually characterized by probability distributions. Thus, the simulation model must be capable of generating the random variables associated with these interevent intervals. The most common method of generating these random variables is via random numbers. Because of the use of random numbers to perform simulations, they are sometimes called *statistical simulations* or *Monte Carlo* simulations. Note that generally the term *Monte Carlo method* refers to any technique used for obtaining solutions to deterministic problems using random numbers. However, in the remainder of this chapter, we use the term Monte Carlo simulation and statistical simulation interchangeably.

This chapter deals with a class of Monte Carlo simulations called the Markov chain Monte Carlo (MCMC). MCMC is a stochastic simulation technique that was first described by Metropolis et al. (1953) and later refined by other people including Hastings (1970), Geman and Geman (1984), Gelfand and Smith (1990), Gelman and Rubin (1992), and Green (1995). A comprehensive review of the technique is given in Brooks (1998).

The original technique was developed as a way to analyze crystal lattices. However, since then it has proved to be a powerful technique for numerical simulation in several fields, particularly where the problem being studied is too complicated to be solved analytically. It adopts the probabilistic methodology of Monte Carlo simulations and extends it through the use of Markov chains as the algorithm for successive model selection.

Before discussing MCMC we first discuss the basic principles of Monte Carlo simulation. This is followed by a discussion on MCMC principles and the two commonly used MCMC methods, which are the Metropolis-Hastings algorithm and the Gibbs sampling algorithm. Finally, we discuss applications of MCMC.

14.2 Monte Carlo Simulation Basics

A fundamental issue in Monte Carlo simulation is the technique for generating random variables. Because these variables are derived from random numbers that are generated by computers, we deal with computer-generated integers Z_i

that lie between 0 and some positive integer m; that is, $0 \leq Z_i \leq m$. Thus, the quantity

$$Y_i = Z_i/m$$

is a fraction that lies between 0 and 1 and, therefore, has the potential to represent the probability of an event.

The most commonly used method of generating the Z_i is the *multiplicative congruential method*, which is also called the *power residue method*. The numbers are generated according to the following relationship:

$$Z_i = aZ_{i-1} \text{mod}(m)$$

The sequence $\{Z_i\}$ generated by the preceding method is completely determined when the parameters a, m, and Z_0 are known. The initial value Z_0 is called the *seed*. Observe that the sequence $\{Z_i\}$ is not truly random because it can always be replicated when a, m, and Z_0 are known. However, we can accept the sequence as being sufficiently random for all practical purposes, and thus such a sequence is called a *pseudo-random sequence*.

The parameters a and m are usually chosen to satisfy certain number-theoretic properties. Specifically, it can be shown that when a modulus m is chosen such that

$$m = 1, 2, 4, p^n, \text{ or } 2p^n$$

where p is an odd prime number and n is a positive integer, then there exists an element a such that

$$a^{\phi(m)} = 1 \text{mod}(m)$$

Such an element is called a *primitive root* of m. The quantity $\phi(m)$ is called the *Euler's totient function* of m, which is the number of values between 1 and $m-1$ that are relatively prime to m. Obviously, if m is prime, $\phi(m) = m - 1$. If we choose a to be a primitive root and require Z_0 to be relatively prime to m, then the random sequence $\{Z_i\}$ has a period of $m - 1$.

Studies have been carried out to determine good numerical values of a, m, and Z_0, where a good set of values is the set that permits the generation of a random number sequence with a full cycle period of $m - 1$. In this type of sequence, all integers in the set $\{1, 2, \ldots, m-1\}$ appear only once within a period. Lewis, Goodman, and Miller (1969) discuss a random-number generator that produces a sequence with satisfactory statistical properties. The generator uses the following values:

$$m = 2^{31} - 1 = 2{,}147{,}483{,}647$$

$$a = 7^5 = 16{,}807$$

Thus, the sequence becomes

$$Z_{i+1} = 7^5 Z_i \text{ mod}(2^{31} - 1)$$

14.2.1 Generating Random Variables from the Random Numbers

After the random number sequence $\{Z_i\}$ has been generated, it is converted into the random fraction sequence $\{Y_i\}$, where $0 < Y_i = Z_i/m \leq 1$. The next step is to translate these fractions into random variables. Because the Z_i lie between 0 and $m-1$ with a uniform frequency, the fractions Y_i are uniformly distributed between 0 and 1. There are different ways to generate a sequence of random variables with the above probability distribution. These include the inverse transform method, the composition method, and the acceptance-rejection method.

The Inverse Transform Method

Consider a random variable X with the cumulative distribution function $F_X(x) = P[X \leq x]$. We know that $0 \leq F_X(x) \leq 1$ and $F_X(x)$ is nondecreasing; that is, if $x_1 < x_2$, then $F(x_1) \leq F(x_2)$. Recall that the elements of the sequence $\{Y_i\}$ are such that if $Y \in \{Y_i\}$, then Y is uniformly distributed between 0 and 1, which we denote by $Y \sim \mathrm{U}(0, 1)$. Let $F_X^{-1}(x)$ denote the inverse of the CDF $F_X(x)$. The inverse transform method assigns the value

$$F_X(x) = Y$$

If $F_X(x)$ is a strictly increasing function, we can obtain the value $X = x$ through the inverse function

$$x = F_X^{-1}(Y)$$

The inverse mapping $F_X^{-1}(Y)$ can be done by writing the equation for this function. It can also be done by developing a table that gives the values of X for the sets of values of Y from 0 to 1.

Example 14.1 Assume that X is an exponentially distributed random variable with a mean of $1/\lambda$. This means that the PDF of X is $f_X(x) = \lambda e^{-\lambda x}$, $x \geq 0$, and the CDF is given by

$$F_X(x) = \int_{u=0}^{x} \lambda e^{-\lambda u} du = 1 - e^{x-\lambda x} \qquad x \geq 0$$

Thus, we obtain x from the equation

$$Y = 1 - e^{-\lambda x}$$

$$x = F_X^{-1}(Y) = -\left\{\frac{\ln(1-Y)}{\lambda}\right\}$$

Because $Y \sim \mathrm{U}(0, 1)$, we note that $1 - Y \sim \mathrm{U}(0, 1)$ also. Therefore, the preceding result becomes

$$x = -\left\{\frac{\ln(Y)}{\lambda}\right\}$$

■

Example 14.2 Assume that X is a geometrically distributed random variable with success probability p. Then the PMF of X is $p_X(x) = p(1-p)^{x-1}$, $x = 1, 2, \ldots$, and the CDF is given by

$$F_X(x) = \sum_{k=1}^{x} p(1-p)^{k-1} = 1 - (1-p)^x \qquad x \geq 1$$

Thus, we obtain x from the equation

$$\begin{aligned} Y &= 1 - (1-p)^x \\ x &= F_X^{-1}(Y) = \tfrac{\ln(1-Y)}{\ln(1-p)} \end{aligned}$$

As discussed earlier, because $Y \sim \mathrm{U}(0, 1)$, we note that $1 - Y \sim \mathrm{U}(0, 1)$ also. Therefore, we obtain

$$x = \frac{\ln(Y)}{\ln(1-p)}$$

Finally, because X is a discrete random variable, we use the ceiling operation to obtain

$$x = \left\lceil \frac{\ln(Y)}{\ln(1-p)} \right\rceil$$

where $\lceil b \rceil$ is the smallest integer greater than b. ■

Example 14.3 Assume that X is uniformly distributed between a and b. The CDF of X is given by

$$F_X(x) = \frac{1}{b-a} \int_{u=0}^{x} du = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \leq x < b \\ 1 & x \geq b \end{cases}$$

Thus, we obtain x from the equation $Y = (x-a)/(b-a)$ as follows:

$$x = F_X^{-1}(Y) = a + Y(b-a)$$ ■

The Composition Method

The composition method is used when the CDF of the random variable X is a convex combination of the CDFs of other random variables $X_1, X_2, \ldots, X_k$ with CDFs $F_{X_1}(x), F_{X_2}(x), \ldots, F_{X_k}(x)$, respectively. That is, we assume that

$$F_X(x) = \sum_{j=1}^{k} p_j F_{X_j}(x)$$

$$1 = \sum_{j=1}^{k} p_j$$

where $p_j \geq 0$. The algorithm operates as follows: First, we simulate a random variable J that is equal to j with probability p_j, $j = 1, \ldots, k$, and then we simulate a second random variable for the distribution of $F_{X_j}(x)$. If the simulated value of J is $j = i$, the second simulation is from $F_{X_i}(x)$. It is easy to see that the distribution of the returned value of X is given by

$$P[X \leq x] = \sum_{j=1}^{k} P[X \leq x | J = j] P[J = j]$$

$$= \sum_{j=1}^{k} F_{X_j}(x) p_j = F_X(x)$$

Example 14.4 Assume that X is a two-stage hyperexponential random variable with the PDF

$$f_X(x) = 0.6\lambda e^{-\lambda x} + 0.4\mu e^{-\mu x} \equiv 0.6 f_{X_1}(x) + 0.4 f_{X_2}(x) \quad x \geq 0$$

The CDF of X is given by

$$F_X(x) = 0.6\{1 - e^{-\lambda x}\} + 0.4\{1 - e^{-\mu x}\} \equiv 0.6 F_{X_1}(x) + 0.4 F_{X_2}(x) \quad x \geq 0$$

We generate two random fractions Y_1 and Y_2. Then we use the following assignment rule:

$$x = \begin{cases} F_{X_1}^{-1}(Y_2) = -\left\{\dfrac{\ln(1 - Y_2)}{\lambda}\right\} & Y_1 \leq 0.6 \\ F_{X_2}^{-1}(Y_2) = -\left\{\dfrac{\ln(1 - Y_2)}{\mu}\right\} & Y_1 > 0.6 \end{cases}$$

■

The Acceptance-Rejection Method

Sometimes it is difficult to evaluate the integral $F_X(x) = Y$ or its inverse $F_X^{-1}(Y)$. This makes it difficult for the inverse transform method to be used. A technique called the acceptance-rejection method has been developed to deal with this situation. There are different forms of this method. We explain the method for continuous distributions. Assume that the random variable X has a PDF $f_X(x)$, but it is difficult to simulate X via the inverse transform method. Assume also that $f_X(x)$ has a finite range $a \leq x \leq b$. Finally, assume that there exists a function $g(x)$ from which we can generate random samples and $f_X(x) \leq kg(x), a \leq x \leq b$, where $k \geq 1$. Then $f_X(x)/kg(x) \leq 1$.

The algorithm for the acceptance-rejection method is as follows:

1. Sample x from $g(x)$ and Y from U(0, 1) thereby obtaining a random pair.
2. Check to see if $Y \leq f_X(x)/kg(x)$.

   - If this holds, set $x = Y$ thereby accepting x as a sample of $f_X(x)$.
   - Otherwise, reject the value of x and repeat step 1.

On the average we need to simulate k pairs of values (x, Y) to get one accepted, so it is best to make k as small as possible. Thus, unless k is small, the method is not an efficient one.

14.2.2 Monte Carlo Integration

As stated earlier, the Monte Carlo method is any technique that is used for obtaining solutions to deterministic problems using random numbers. The method has often been used to evaluate integrals. Consider a function $g(x)$ for which we are required to evaluate the integral

$$\int_a^b g(x)dx$$

Assume that it is not easy to evaluate the integral but that we can decompose $g(x)$ into a function $h(x)$ and a PDF $f_X(x)$ defined over the interval (a, b). Then we obtain the following:

$$\int_a^b g(x)dx = \int_a^b h(x) f_X(x)dx = E[h(x)]$$

Thus, we have transformed the integral into an expectation of the function $h(x)$ over the PDF $f_X(x)$. Therefore, if we draw a number of samples $x_1, x_2, \ldots, x_n$

from the PDF $f_X(x)$, then according to the law of large numbers we can approximate the integral for a sufficiently large value of n by

$$\int_a^b g(x)dx = E[h(x)] \cong \frac{1}{n}\sum_{k=1}^{n} h(x_k)$$

This result is called the Monte Carlo integration.

Importance Sampling

An extension of the Monte Carlo integration technique is the so-called *importance sampling*, which is useful when we want to find the expected value of $h(x)$ but cannot easily generate random variables from the PDF $f_X(x)$. The method is based on the idea of "distorting" the PDF $f_X(x)$ in order to make the sampling easier. Suppose $f_X(x)$ is too complex to sample, but we know another PDF $p(x)$, called the *sampler PDF*, that roughly approximates $f_X(x)$. In particular, $p(x) = 0$ whenever $f_X(x) = 0$. Then we can write

$$\begin{aligned}\int_a^b g(x)dx &= \int_a^b h(x) f_X(x)dx = \int_a^b h(x)\left\{\frac{f_X(x)}{p(x)}\right\} p(x)dx \\ &= E\left[\frac{h(x) f_X(x)}{p(x)}\right] \cong \frac{1}{n}\sum_{k=1}^{n}\left\{\frac{f_X(x_k)}{p(x_k)}\right\} h(x) \\ &= \frac{1}{n}\sum_{k=1}^{n} w_k h(x)\end{aligned}$$

where the x_k are drawn from the PDF $p(x)$ and the weights $w_k = f_X(x_k)/p(x_k)$ essentially adjust the importance of each sample.

Monte Carlo Integration and Bayesian Estimation

Monte Carlo integration is an important technique used by Bayesian practitioners to sample from the posterior distribution. According to Morgenthal (1961), the term "Monte Carlo" was coined by von Neumann and Ulam in the 1940s in the context of such problems. In Bayesian statistics, we are interested in a probability model that relates observed data $y = \{y_1, y_2, \ldots, y_n\}$ to a set of unknown parameters $\theta = \{\theta_1, \theta_2, \ldots, \theta_k\}$. The unknown parameters θ are treated as random variables, while the observed data y is treated as being deterministic. A major quantity of interest is the distribution of θ after y is observed, which

is called the *posterior distribution*, $f(\theta|Y)$, that can be can be obtained using Bayes' rule:

$$f(\theta|y) = \frac{f(y|\theta)\, f(\theta)}{f(y)} = \frac{f(y|\theta)\, f(\theta)}{\int_\theta f(y|\theta)\, f(\theta) d\theta}$$

The posterior distribution is the conditional distribution of the parameters after the data has been observed. The quantity $f(y|\theta)$ is called the *likelihood function*, while $f(\theta)$ is called the *prior distribution* because it contains information about the parameter values available to the observer before he or she observes the data. In many cases the integral of the denominator cannot be easily evaluated. For such cases the Monte Carlo method can be used to approximate the integral as follows:

$$\int_\theta f(y|\theta)\, f(\theta) d\theta = E[f(y|\theta)] \cong \frac{1}{n} \sum_{k=1}^{n} f(y|\theta_k)$$

where the θ_k are samples from the PDF $f(\theta)$.

14.3 Markov Chains Revisited

Because this chapter deals with MCMC and we have briefly introduced the Monte Carlo integration, we make the chapter self-contained by reviewing the aspects of Markov chains that will be used in the remainder of the chapter. Recall from Chapter 3 that a Markov chain X is a sequence $X_0, X_1, X_2, \ldots$ of random variables that, for $n > 0$ and for all values $s_0, s_1, \ldots, s_n$, satisfies the following condition:

$$\begin{aligned} P[X_n = s_n | X_{n-1} = s_{n-1}, X_{n-2} = s_{n-2}, \ldots, X_0 = s_0] \\ = P[X_n = s_n | X_{n-1} = s_{n-1}] \end{aligned}$$

which is the so-called Markov property. A Markov chain is defined by its transition probabilities (or the transition kernel) p_{ij}, which are given by

$$p_{ij} = P[X_n = s_j | X_{n-1} = s_i]$$

where

$$\sum_j p_{ij} = 1$$

The k-step transition probability $p_{ij}(k)$ is defined as the probability that the chain is in state s_j given that it was in state s_i k steps ago. Alternatively, $p_{ij}(k)$ is

the probability that the process will be in state s_j k steps from now, given that it is currently in state s_i. That is,

$$p_{ij}(k) = P[X_{n+k} = s_j | X_n = s_i]$$

A state s_j is defined to be a *recurrent* (or *persistent*) state if the probability that the process will ever return to the state after leaving it is 1; otherwise the state is defined to be a *transient* state. A recurrent state is defined to be *nonnull* if the mean recurrence time (or the mean time to return to the state after leaving it) is finite; otherwise it is defined to be *null*. A nonnull recurrent state is also called a *positive recurrent* state. A Markov chain is defined to be *irreducible* if there exists a positive integer k_{ij} such that $p_{ij}(k_{ij}) > 0$ for all i and j. This implies that all states can communicate with each other in an irreducible Markov chain; alternatively, every state is reachable from all other states in the chain. A Markov chain is defined to be *aperiodic* if the number of steps required to return to any state after leaving the state is not a multiple of some integer. A Markov chain that is both irreducible and aperiodic and in which every state is nonnull recurrent is defined to be an *ergodic* Markov chain.

Assume that $X_t \in X$ denotes the value of a random variable at time t, where $\{X\}$ is a Markov chain. Let $\pi_j(t)$ denote the probability that the Markov chain is in state s_j at time t; that is,

$$\pi_j(t) = P[X_t = s_j]$$

The $\pi_j(t)$ are related by the following Chapman-Kolmogorov equation:

$$\pi_j(t+1) = \sum_k p_{kj} \pi_k(t)$$

A Markov chain is defined to possess a *stationary distribution* if the limit

$$\lim_{t \to \infty} \pi_j(t) = \pi_j$$

exists for all j, where

$$\sum_j \pi_j = 1$$

π_j is called the limiting state (or steady-state) probability of the chain being in state s_j. Let $\pi = \{\pi_1, \pi_2, \ldots\}$ denote the vector of the steady-state probabilities (also called the stationary distribution). Then a sufficient condition for a unique stationary distribution is that the so-called *detailed balance equation* holds for all s_i and s_j, which is that

$$\pi_i p_{ij} = \pi_i p_{ji}$$

The preceding condition is also the reversibility condition. Note that the *ergodic theorem* also states that if a Markov chain is ergodic, then a unique steady-state probability distribution $\pi = \{\pi_1, \pi_2, \ldots\}$ exists and this distribution is independent of the initial state of the chain.

These results for the discrete-state Markov chain can be extended to the continuous-state Markov process that has the transition probabilities $p_{ij}(t) = P[s_t = j | s_0 = i]$, where

$$\sum_j p_{ij}(t) = 1$$

The continuous-time extension of the Chapman-Kolmogorov equation is given by

$$p_{ij}(t+s) = \sum_{k=0}^{\infty} p_{ik}(t)\, p_{kj}(s) = \sum_{k=0}^{\infty} p_{ik}(s)\, p_{kj}(t)$$

14.4 MCMC Simulation

One problem with using the Monte Carlo integration is in obtaining samples for some complex probability distributions. MCMC, which is an extension of the basic Monte Carlo integration method, addresses this problem. The method is called MCMC because it involves the use of the current sample value to randomly generate the next sample value, which is the fundamental idea behind a Markov chain. Thus, it can be considered a Monte Carlo integration procedure in which the random samples are produced by evolving an ergodic Markov chain.

MCMC generates successive states $X_1, X_2, \ldots$ of an ergodic Markov chain on a state space X with stationary distribution π such that if a sample x maps into the stationary probability $\pi(x)$, then the expected value of the function $h(x)$ defined on X can be approximated by

$$E[h(x)] = \int_x h(x)\pi(x)dx \cong \frac{1}{n}\sum_{k=1}^{n} h(x_k)$$

The usefulness of the method arises from the fact that it is often much easier to construct a Markov chain with a specified stationary distribution than it is to work with the distribution itself. In fact, MCMC permits simulations to be drawn from a wide range of distributions, including many for which simulation methods were previously very difficult to implement.

There are two basic methods of MCMC, which are the Metropolis-Hastings method and the Gibbs sampler (also called the Gibbs sampling method). The Gibbs sampler is a special case of the Metropolis-Hastings method.

14.4.1 The Metropolis-Hastings Algorithm

The Metropolis-Hastings algorithm, which was developed by Metropolis (1953) and further expanded and introduced to a mathematical audience by Hastings (1970), is the most general method of implementing the MCMC. A good summary of the algorithm is described in Chib and Greenberg (1995).

The algorithm generates a set of states $\{X_t\}$ that is a Markov chain because each state X_t depends only on the previous state X_{t-1}. Suppose we have a probability density $\pi(x)$, called the *target density*, from which we want to simulate. To do this we use a function $q(y|x)$ called the *proposal* (or *candidate*) distribution, which can be any probability density that creates an ergodic Markov chain. The proposal distribution depends on the current state x and can generate a new proposed sample y. To ensure reversibility the proposed sample is accepted with probability $\alpha(x, y)$ and rejected otherwise. The rationale for this is that because the proposal distribution is usually an arbitrary distribution, we cannot expect it to satisfy the equation for the target density $\pi(x)$. That is, we expect that

$$q(y|x)\pi(x) \neq q(x|y)\pi(y)$$

Suppose we have that $q(y|x)\pi(x) > q(x|y)\pi(y)$. Then the factor $\alpha(x, y) < 1$ is defined such that it balances the inequality; that is,

$$q(y|x)\pi(x)\alpha(x, y) = q(x|y)\pi(y)$$

From this we obtain the value of $\alpha(x, y)$, which is referred to as the *acceptance probability*, as

$$\alpha(x, y) = \min\left(1, \frac{q(x|y)\pi(y)}{q(y|x)\pi(x)}\right)$$

Note that Metropolis-Hastings method requires only that $\pi(x)$ be defined up to a normalizing constant because the constant drops out in the ratio $\pi(y)/\pi(x)$. The algorithm can then be formally described as follows:

1. Start with any initial value x_0 satisfying $\pi(x_0) > 0$ and set $k = 0$.
2. Let x_k be the current state of the chain. Sample a point Y_k from the proposal distribution $q(y|x)$.
3. Generate a random variable U from the distribution $U(0, 1)$.
4. Update as follows:
   - If $U \leq \alpha(x_k, Y_k)$, set $x_{k+1} = Y_k$
   - Otherwise set $x_{k+1} = x_k$
5. Increment k.
6. If $k < N$ go to step 2; otherwise return $\{x_1, x_2, \ldots, x_N\}$.

Some Common Choices for the Proposal Distribution

The fundamental issue in MCMC is that the target density $\pi(x)$ can be very complex and impossible to sample directly. The Metropolis-Hastings algorithm requires us to sample only the proposal distribution whose only requirement is that the resulting chain be ergodic. The acceptance rate of the algorithm will depend on the proposal distribution.

One desirable property is for the acceptance rate to be sufficiently high. However, Robert and Casella (1999) state that it is not necessary that a high acceptance rate indicate that the algorithm is moving correctly. It might rather indicate that the chain is moving too slowly on the surface of $\pi(x)$. Both Metropolis (1953) and Hastings (1970) give families of target distributions used in their work. Another choice reported in Chib and Greenberg (1995) is obtained by exploiting the known form of $\pi(x)$. For example, if we can express $\pi(u) \propto \varphi(u)h(u)$, where $h(u)$ is a probability density that can be sampled and $\varphi(u)$ is uniformly bounded, then we can set $q(y|x) = h(y)$. This gives the result

$$\alpha(x, y) = \min\left(1, \frac{\varphi(y)}{\varphi(x)}\right)$$

Metropolis-Hastings Sampling as a Markov Chain

We show that the Metropolis-Hastings algorithm produces a Markov chain whose equilibrium distribution is the target density $\pi(x)$. To demonstrate this, we need to prove that the proposal distribution satisfies the detailed balance equation with $\pi(x)$. Let p_{xy} denote the probability of making a transition from x to y. Then

$$p_{xy} = q(y|x)\alpha(x, y) = q(y|x) \times \min\left(1, \frac{q(x|y)\pi(y)}{q(y|x)\pi(x)}\right)$$

This means that if the algorithm satisfies the detailed balance

$$p_{xy}\pi(x) = p_{yx}\pi(y) \Rightarrow q(y|x)\alpha(x, y)\pi(x) = q(x|y)\alpha(y, x)\pi(y) \qquad \text{for all } x, y$$

then the stationary distribution of the transition probability corresponds to samples from the target distribution. Consider the following three cases:

1. $q(y|x)\pi(x) = q(x|y)\pi(y)$. In this case $\alpha(x, y) = \alpha(y, x) = 1$, which means that $p_{xy}\pi(x) = q(y|x)\pi(x)$ and $p_{yx}\pi(y) = q(x|y)\pi(y)$. This in turn means that $p_{xy}\pi(x) = p_{yx}\pi(y)$ and the detailed balance equation holds.

2. $q(y|x)\pi(x) > q(x|y)\pi(y)$. In this case

$$\alpha(x, y) = \frac{q(x|y)\pi(y)}{q(y|x)\pi(x)}$$
$$\alpha(y, x) = 1$$

Thus we have that

$$p_{xy}\pi(x) = q(y|x)\alpha(x, y)\pi(x) = q(y|x)\left\{\frac{q(x|y)\pi(y)}{q(y|x)\pi(x)}\right\}\pi(x)$$
$$= q(x|y)\pi(y) = q(x|y)\alpha(y, x)\pi(y)$$
$$= p_{yx}\pi(y)$$

As in the first case, the detailed balance equations also hold.

3. $q(y|x)\pi(x) < q(x|y)\pi(y)$. In this case

$$\alpha(x, y) = 1$$
$$\alpha(y, x) = \frac{q(y|x)\pi(x)}{q(x|y)\pi(y)}$$

Thus,

$$p_{yx}\pi(y) = q(x|y)\alpha(y, x)\pi(y) = q(x|y)\left\{\frac{q(y|x)\pi(x)}{q(x|y)\pi(y)}\right\}\pi(y)$$
$$= q(y|x)\pi(x) = q(x|y)\alpha(x, y)\pi(x)$$
$$= p_{xy}\pi(x)$$

Because the detailed balance equation holds in all three cases, we conclude that the algorithm produces a Markov chain.

14.4.2 Gibbs Sampling

The Gibbs sampling method (or the Gibbs sampler) is an algorithm used to generate a sequence of samples from the joint probability distribution of two or more random variables. It is a special case of the Metropolis-Hastings algorithm and was introduced in the context of image analysis by Geman and Geman (1984). Gelfand and Smith (1990) showed how it can be applied to a wide variety of Bayesian inference problems. Casella and George (1992) and Gelfand (2000) provide good tutorials on the method.

The algorithm is used when the joint distribution is not explicitly known but the conditional distribution of each random variable is known. It generates an instance from the distribution of each random variable in turn, conditional on the

current values of the other random variables. It can be shown that the sequence of samples constitutes a Markov chain whose stationary distribution is the desired joint distribution.

To illustrate the technique, assume that we have the joint PDF $f_{XY_1 \ldots Y_n}(x, y_1, \ldots, y_n)$ and we want to obtain the marginal PDF $f_X(x)$. Traditionally we would approach the problem by performing the integration

$$f_X(x) = \int \cdots \int f_{XY_1 \ldots Y_n}(x, y_1, \ldots, y_n) dy_1 \ldots dy_n$$

Unfortunately there are many cases where the preceding integration is very difficult to perform either numerically or analytically. The Gibbs sampler provides an alternative method for obtaining $f_X(x)$.

Gibbs Sampler for Two Random Variables

As an MCMC method, the Gibbs sampler generates samples from $f_X(x)$ that can be used for the desired statistic of $f_X(x)$ without computing the PDF directly. Consider the bivariate random variable (X, Y) and assume that we are required to obtain the marginal PDF $f_X(x)$ or $f_Y(y)$ or both. The principle of the method is that it is easier to consider a sequence of conditional PDFs $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$ than to obtain the marginal PDFs directly. Note that

$$f_X(x) = \int f_{XY}(x, y) dy = \int f_{X|Y}(x|y) f_y(y) dy$$

$$= E_Y[f_{X|Y}(x/y)] \cong \frac{1}{m} \sum_{k=1}^{m} f_{X|Y}(x|y = y_k)$$

where $E_Y[U]$ is the expected value of U with respect to the PDF of Y. Thus, the algorithm can be used to approximate the marginal distribution.

The algorithm can be summarized as follows. It starts with some intial value y_0 and obtains x_0 by generating a random variable from the conditional PDF $f_{X|Y}(x|y = y_0)$. Then it uses x_0 to generate a new value y_1 from the conditional PDF $f_{Y|X}(y|x = x_0)$, and so on until a Gibbs sequence of length k is generated.

One iteration of the conditional PDFs is called a *scan* of the sampler. Observe that each sample point is essentially a vector of two parameters associated with X and Y. As in any simulation, data collection is started after a burn-in period to remove the effects of the initial sampling values. After the burn-in period is over, the samples actually used for the computation are those obtained at predefined points, such as those obtained at intervals of n scans. Thus, from the k sample points, we choose m samples obtained according to this formula for computing $f_X(x)$.

In the preceding example we used the conditional PDF $f_{X|Y}(x|y)$ to approximate the marginal PDF $f_X(x)$. Assume that the two conditional PDFs $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$ are given. We can obtain the marginal $f_X(x)$ as follows:

$$
\begin{aligned}
f_Y(y) &= \int f_{Y|X}(y|x) f_X(x) dx \\
f_X(x) &= \int f_{X|Y}(x|y) f_Y(y) dy = \int f_{X|Y}(x|y) \int f_{Y|X}(y|u) f_X(u) du dy \\
&= \int \left\{ \int f_{X|Y}(x|y) f_{Y|X}(y|u) dy \right\} f_X(u) du \\
&= \int h(x,u) f_X(u) du
\end{aligned}
$$

where

$$
h(x,u) = \int f_{X|Y}(x|y) f_{Y|X}(y|u) dy
$$

The equation

$$
f_X(x) = \int h(x,u) f_X(u) du
$$

is a *fixed point integral equation* whose unique solution is $f_X(x)$. Unfortunately, while the joint PDF of X and Y determines all the conditional and marginal PDFs, a set of proper conditional PDFs does not always determine a proper marginal PDF.

Note that while the preceding discussion has focused on using continuous random variables and has PDFs, the results hold for the case of discrete random variables where we use PMFs instead of PDFs.

Gibbs Sampler for More Than Two Random Variables

When we have more than two random variables, the Gibbs sampler uses the concept of full conditional distributions (or full conditionals), which can be defined as follows. Suppose we have three random variables X, Y, and Z with the joint PDF $f_{XYZ}(x, y, z)$. The full conditional $f_{X|YZ}(x|y, z)$ is the conditional PDF of X given Y and Z; similarly, the full conditional $f_{Y|XZ}(y|x, z)$ is the conditional PDF of Y given X and Z, and so on. Thus, for example, assume that we have

three random variables X, Y, and Z, and we want to obtain $f_X(x)$. The algorithm operates as follows. In the kth iteration,

- Draw $X^{(k)}$ from $f_{X|YZ}(x|y^{(k-1)}, z^{(k-1)})$
- Draw $Y^{(k)}$ from $f_{Y|XZ}(y|x^{(k)}, z^{(k-1)})$
- Draw $Z^{(k)}$ from $f_{Z|XY}(z|x^{(k)}, y^{(k)})$

Thus, if we define the vector $\Theta = (X, Y, Z)$, the entire vector is updated by updating each element of the vector as indicated. The procedure can be extended to vectors with more than three elements as follows. If we define the vector $X = (X_1, X_2, \ldots, X_n)$ and denote by $X^{(t)}$ the sample at step t, then

- Draw $X_1^{(t)}$ from the distribution of X_1 given $x_2^{(t-1)}, x_3^{(t-1)}, \ldots, x_n^{(t-1)}$
- Draw $X_2^{(t)}$ from the distribution of X_2 given $x_1^{(t)}, x_3^{(t-1)}, \ldots, x_n^{(t-1)}$
- $\ldots$
- Draw $X_n^{(t)}$ from the distribution of X_n given $x_1^{(t)}, x_2^{(t)}, \ldots, x_{n-1}^{(t)}$

Thus, a new value for X_{k-1} is used immediately when picking the next value for X_k.

Relationship with the Metropolis-Hastings Algorithm

Let $\theta_x = (\theta_1^{(t)}, \theta_2^{(t)}, \ldots, \theta_n^{(t)})$ denote the current point in the Gibbs sampler, and let the point $\theta_y = (\theta_1^{(t+1)}, \theta_2^{(t)}, \ldots, \theta_n^{(t)})$ be the candidate point for updating θ_1. The Gibbs sampler is a special case of the Metropolis-Hastings algorithm where the proposal distribution is equal to the transition probability. That is, $q(\theta_i|\theta_{-i}) = \pi_i(\theta_i|\theta_{-i})$, where θ_{-i} is the vector θ without the element θ_i. Then the ratio that defines the acceptance probability is given by

$$\begin{aligned}
\frac{q(\theta_x|\theta_y)\pi(\theta_y)}{q(\theta_y|\theta_x)\pi(\theta_x)} &= \frac{\pi_1(\theta_1^{(t)}|\theta_{-1}^{(t)})\pi(\theta_1^{(t+1)}, \theta_2^{(t)}, \ldots, \theta_n^{(t)})}{\pi_1(\theta_1^{(t+1)}|\theta_{-1}^{(t+1)})\pi(\theta_1^{(t)}, \theta_2^{(t)}, \ldots, \theta_n^{(t)})} \\
&= \frac{\pi_1(\theta_1^{(t)}|\theta_2^{(t)}, \ldots, \theta_n^{(t)})\pi(\theta_1^{(t+1)}, \theta_2^{(t)}, \ldots, \theta_n^{(t)})}{\pi_1(\theta_1^{(t+1)}|\theta_2^{(t)}, \ldots, \theta_n^{(t)})\pi(\theta_1^{(t)}, \theta_2^{(t)}, \ldots, \theta_n^{(t)})} \\
&= \frac{\pi(\theta_1^{(t)}, \theta_2^{(t)}, \ldots, \theta_n^{(t)})\pi(\theta_1^{(t+1)}, \theta_2^{(t)}, \ldots, \theta_n^{(t)})\pi(\theta_2^{(t)}, \ldots, \theta_n^{(t)})}{\pi(\theta_1^{(t+1)}, \theta_2^{(t)}, \ldots, \theta_n^{(t)})\pi(\theta_1^{(t)}, \theta_2^{(t)}, \ldots, \theta_n^{(t)})\pi(\theta_2^{(t)}, \ldots, \theta_n^{(t)})} \\
&= 1
\end{aligned}$$

Thus, the Gibbs sampler is an MCMC algorithm with acceptance probability of 1, which means that every sample is always accepted.

Spatial Birth-and-Death Processes

A spatial birth-and-death process is a continuous-time Markov jump point process $\{Y(x,t): x \in \chi, t \in [0,\infty)\}$ whose state space is a family χ of point patterns and whose transitions over time consist of two types of jumps: "births" and "deaths." During a birth a single point is added, and during a death one of the points is deleted. The process was introduced by Preston (1977) and discussed in Ripley (1977), Stoyan (1995), van Lieshout (2000), and Moller and Waagepetersen (2004).

Thus, the process is characterized by two transition functions: b for births and d for deaths, both of which are positive parameters. Specifically, given that the process is state x at time t, the probability that a point within the set A is added to x in the short interval $(t, t+s)$ is

$$s\int_A b(x,\theta)dv(\theta) + o(s)$$

where $v(\cdot)$ is the intensity measure on A. Similarly, the probability that the point θ is deleted from $x \cup \theta$ in the interval $(t, t+s)$ is $sd(x,\theta)$. The total transition rate $\lambda(x)$ at x is the sum of the total birth rate and the total death rate. That is,

$$\lambda(x) = B(x) + D(x)$$

where

$$B(x) = \int_\chi b(x,\theta)dv(\theta)$$

$$D(x) = \sum_{\theta \in x} d(x,\theta)$$

Thus, the sojourn time in state x is $1/\lambda(x)$. The next transition is a birth with probability $B(x)/\lambda(x)$ and a death with probability $D(x)/\lambda(x)$. Suppose that the probability $p(x \cup \{\theta\}) > 0$ exists. Then the detailed balance equations are given by

$$b(x,\theta)\,p(x) = d(x,\theta)\,p(x \cup \{\theta\})$$

This equation ensures that births from x to $x \cup \{\theta\}$ match deaths from $x \cup \{\theta\}$ to x. The equation thus shows that the stochastic process is time reversible, which implies that the point process specified by $p(\cdot)$ is its unique equilibrium distribution. According to Ripley (1977), the ratio

$$\frac{b(x,\theta)}{d(x,\theta)} = \frac{p(x \cup \{\theta\})}{p(x)} = \rho(x,\theta)$$

where $d(x,\theta) > 0$ is a measure of the viability of a point at θ given the rest of the points x. The quantity $-\ln\{\rho(x,\theta)\}$ is usually considered the local energy for x

and θ and designated $U(x, \theta)$. That is,

$$U(x, \theta) = -\ln\left\{\frac{b(x, \theta)}{d(x, \theta)}\right\} \Rightarrow b(x, \theta) = d(x, \theta)\exp\{-U(x, \theta)\}$$

Recall that a random variable X is defined to have the Gibbs distribution if its distribution function is of the following form:

$$P[X = x] = \frac{1}{Z}\exp\{-\beta U(x)\}$$

where $U(x)$ is the energy function, β is a nonnegative parameter, and Z is a normalizing constant. Thus, the preceding discussion suggests that realizations of Gibbs processes can be obtained by simulating a corresponding spatial birth-and-death process until it reaches equilibrium. This means that the simulation of Gibbs processes can be summarized as follows:

a. Choose the rates b and d. A good starting point is $b(x, \theta) = 1$ for $p(x \cup \theta) > 0$, and the death rate is scaled accordingly: $d(x, \theta) = 1/\rho(x, \theta) = \exp\{U(x, \theta)\}$.
b. Choose a starting configuration, which can be an empirical pattern under investigation or n points generated by a uniform distribution in the space of interest.
c. Simulate the birth-and-death process with the preceding rates until the process is considered to have reached equilibrium, in which case the final configuration is considered to behave approximately as a Gibbs process. The procedure is as follows:
    1. Assume that the process $\{Y(x, t)\}$ has jumps at times $T_0 = 0 < T_1 < T_2 < \ldots$. Let $Y(x, T_n) = K_n$. Then the jump chain $K_0, K_1, K_2, \ldots$ is a Markov chain, and the intervals $T_{n+1} - T_n$ are exponentially distributed with mean $1/\lambda(x)$, where $\lambda(x) > 0$.
    2. Define the PDF $\bar{b}(x, \theta) = b(x, \theta)/B(x)$ and the PMF $\bar{d}(x, \theta) = d(x, \theta)/D(x)$.
    3. Given that $(K_n, T_n) = (x, t)$ and $\lambda(x) > 0$, generate two random numbers that will be used to update K and T: $u_1 \sim$ Uniform $[0, 1]$ and $u_2 \sim$ Uniform $[0, 1]$.
    4. Set $T_{n+1} = t - \{\ln(u_1)\}/\lambda(x)$, which follows from the fact that for an exponentially distributed random variable X with mean $1/\mu$, the CDF and the random number u are related by

$$F_X(x) = 1 - \exp(-\mu x) = u \Rightarrow x = -\frac{1}{\mu}\ln(1 - u) \equiv -\frac{1}{\mu}\ln(u)$$

    where the equivalence follows from the fact that because u is a random quantity, $1 - u$ is also a random quantity. Thus, to avoid an extra subtraction operation, we work directly with u.

5. If $u_2 \leq B(x)/\lambda(x)$, generate the point θ_n from the PDF $\bar{b}(x, \theta)$ and set $K_{n+1} = x \cup \theta_n$; otherwise generate the point ζ_n from the PMF $\bar{d}(x, \theta)$ and set $K_{n+1} = x/\xi_n$.

It is observed in Ripley (1977) that the preceding procedure is slow to converge because newborn points are often removed almost immediately. An alternative procedure that assumes a constant death rate $d(x, \theta) = 1$ and hence the birth rate $b(x, \theta) = \exp\{-U(x, \theta)\}$ is discussed in Stoyan (1995). Additional algorithms are discussed in Moller and Waagepetersen (2004).

14.5 Applications of MCMC

MCMC is widely used in several disciplines. As illustrated earlier, it is used in estimating PDFs. We conclude the chapter by discussing its use in simulated annealing and inference in belief networks.

14.5.1 Simulated Annealing

Annealing is a process in which a solid, usually a metal, is first heated and then allowed to cool slowly to the ambient temperature. The heating enables the atoms to move around freely, and as the temperature is very slowly decreased the atoms start settling into favorable arrangements. As the temperature continues to decrease, the mobility of the atoms is further reduced, and they eventually settle into a regular array corresponding to a minimum energy arrangement by the time room temperature is reached. Such an arrangement cannot easily propagate throughout the solid if the cooling occurs quickly, which is a process that is referred to as *quenching*. When quenching is used, boundaries between different "domains of regularity" occur. Such boundaries introduce potential "fault lines" along which a fracture is most likely to occur when the material is stressed. Annealing prevents the occurrence of such fault lines.

Simulated annealing is a numerical optimization method that was developed independently by Kirkpatrick (1983) and Cerny (1985). It uses an analogy between the process of physical annealing and the mathematical process of obtaining the global minimum of a function that might have local minima. Simulated annealing has at its core the Metropolis-Hastings algorithm. Its major advantage over other methods is an ability to avoid becoming trapped at local minima.

The original work of Metropolis (1953) was based on an attempt to simulate a thermodynamic system. A potential state change from one with energy E_1 to one with energy E_2 was accepted with a probability

$$P_A = \min(1, e^{-(E_2 - E_1)/kT})$$

where T is the "temperature" and k is the Boltzmann's constant. Thus, this always accepts a change if it moves to a state of lower energy (i.e., the change in energy is negative), but sometimes accepts the change even though the system moves to a state with a higher energy (i.e., the change in energy is positive). Note that for small T there is a very small probability of accepting an unfavorable move, while for large T the probability of acceptance can be quite high.

The simulated annealing algorithm is a Metropolis Monte Carlo simulation that starts at a high temperature. The temperature is slowly reduced so that the search space becomes smaller for the Metropolis simulation, and when the temperature is low enough the system will hopefully have settled into the most favorable state. Thus, in the spirit of the Metropolis-Hastings algorithm, simulated annealing always accepts moves that decrease the given objective function. Moves that increase the value of the objective function are accepted with probability p that will be defined shortly.

Before using the simulated annealing algorithm we must first specify the following parameters:

- The possible system states
- The objective function that we are trying to minimize, where it must be emphasized that this is a minimization problem; therefore, to handle a maximization problem the latter must be transformed into an appropriate minimization problem
- State transition probabilities
- A control parameter T, analogous to the temperature above, which governs the probability of acceptance of the proposed change, together with an annealing schedule specifying how the temperature is to be lowered and the length of time over which the simulation would evolve

The basic iteration proceeds as follows. As a search algorithm, each point s in the search space is analogous to a state of some physical system. Similarly, the function $E(s)$ to be minimized is analogous to the internal energy of the system at that state. At each step the algorithm considers some neighbors of the current state s_i and probabilistically decides between moving to state s_j or remaining in s_i. The goal is to choose the probability of moving to be such that such a move will ultimately lead to states of lower energy from an arbitrary initial state.

While the states are defined in an application-specific manner, most problems using simulated annealing define states by the vertices (or nodes) of a graph. In this case, two states are neighbors if an edge exists between them. The transition probability between the current state s_i and a new state s_j is given by $P[\delta E, T]$, which is a function of the energy difference $\delta E = E(s_j) - E(s_i)$ between the

two states and a time-varying parameter T called the temperature. The classical definition of $P[\delta E, T]$ used by Kirkpatrick (1983) is as follows:

$$P[\delta E, T] = \begin{cases} 1 & \delta E < 1 \\ e^{-\delta E/T} & \text{otherwise} \end{cases}$$

This definition is from the Metropolis-Hastings algorithm. Thus, the algorithm takes random walks through the problem space looking for points with low energies. Initially the temperature T is set to a high value, and a random walk is carried out at that temperature. Then the temperature is lowered very slightly (according to a *cooling schedule*) and another random walk is taken. The initial value of T, denoted by T_0, is chosen to make the uphill and downhill transition probabilities about the same. At the end of the allotted time, the value of T should be zero or nearly zero. Thus, the cooling schedule could be to exponentially decrease T by a factor of $\alpha < 1$ in every step. That is, in the kth iteration,

$$T_k = \alpha T_{k-1} \quad k = 1, 2, \ldots$$

Numerical studies reported by Kirkpatrick (1983) indicate that good results have been obtained with simulated annealing with computational complexity that scales with N or with a small power of N. We illustrate the algorithm with the traveling salesman problem.

The Traveling Salesman Problem

The traveling salesman problem is a graph-theoretic problem that seeks to find the most efficient (or least total distance) tour through a graph such that each of the N nodes of the graph is visited exactly once. The cost of each edge is usually the distance between the two nodes interconnected by the edge. The problem belongs to a class of problems known as NP-complete problems that require a computation time that is exponential in N for an exact solution. The most direct solution would be a brute force method that tries all the circuit permutations (i.e., every possible circuit) and see which one is cheapest. However, given that the number of permutations is $N!$, this solution rapidly becomes impractical. With dynamic programming techniques, the problem can be solved exactly in time $O(N^2 2^N)$, which is also exponential, though it is much better than $O(N!)$. Unfortunately, the problem has applications in many areas of science and engineering. For example, in the manufacture of a circuit board, it is important to determine the best order in which a laser will drill thousands of holes. An efficient solution to this problem reduces production costs.

The traveling salesman problem is an example of where simulated annealing has been applied. Typically the nodes of the graph represent the cities, and an edge

exists between two nodes if there is a road that directly interconnects the cities represented by those nodes. If the salesman starts with a random itinerary, he can then pairwise trade the order of visits to cities, hoping to reduce the distance with each exchange. The difficulty with this approach is that while it rapidly finds a local minimum, it cannot get from there to the global minimum.

Simulated annealing improves this strategy as follows. Let the cities be numbered $i = 1, 2, \ldots, N$ and let the coordinates of city i be (x_i, y_i). A *configuration* is a permutation of the numbers $1, \ldots, N$, which denotes the order in which the cities are to be visited. A simple objective function, $E(N)$, is the length of the tour of a configuration; that is,

$$E(N) = \sum_{i=1}^{N} \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2}$$

The solution starts with a random configuration that is improved on through a series of moves, where a move consists of creating a new configuration and then accepting or rejecting it based on the probability equation. If a move is accepted, the temperature is changed and the process is repeated. One example of a move is to remove a section of the path and replace it with the same cities running in the opposite order. Another example is to remove a section of a path and place it between two cities on another randomly selected part of the path.

When simulated annealing is used to solve the traveling salesman problem, the solution is not guaranteed to be globally optimal. However, the solution will usually be better than that obtained via any standard optimization method. Thus, although it is proven to converge to the optimum, it converges in infinite time. Also, the algorithm is usually not faster than other combinatorial algorithms.

These drawbacks notwithstanding, given the right initial parameters, the method has been known to converge to the optimum solution of a problem.

14.5.2 Inference in Belief Networks

A belief network, which is also called a *Bayesian network*, is a graphical model that can model domains with uncertainty. It consists of nodes representing random variables and directed edges between pairs of nodes. The graph is a directed acyclic graph in the sense that it contains no directed cycles. It is used to encode the dependencies among a set of variables. Specifically, edges connect random variables that are directly dependent on each other. A good tutorial on Bayesian networks in given in Charniak (1991), while Pearl (1988) and Jensen (2001) are two of the many books on the subject.

The most common use of belief networks is within the framework of Bayesian analysis, which is one of the reasons why they are also called Bayesian networks.

That is, they use the Bayes' rule for probabilistic inference. Typical applications of belief networks include:

- To model and explain a domain
- To update beliefs about states of certain variables when some other variables were observed; that is, computing conditional probability distributions
- To find the most probable configurations of variables
- To support decision making under uncertainty
- To find good strategies for solving tasks in a domain with uncertainty

More formally, a belief network consists of two components:

- A directed acyclic graph $G = (V, E)$ such that
  - Each node $i \in V$ corresponds to a random variable
  - Directed edges between pairs of nodes, where an edge points from one node called the *parent* to another node called the *child*. Because each node represents a random variable, each directed edge represents an immediate dependence or direct influence.
  - $Pa(i)$ denotes the set of parents of node $i \in V$
- Conditional probability of each variable given its parents in the directed acyclic graph; that is,
  - To each node $i \in V$ corresponds a conditional probability table $P[X_i|X_j]$, where $j \in Pa(i)$

As stated earlier, when constructed, a belief network can be used to deduce inferences. That is, it uses general knowledge to try to answer some questions related to specific situtations. Due to the probabilistic nature of the Bayesian method on which the belief network is based, this becomes the problem of finding the conditional probability $P[A|B]$ that question A has answers given the data B. Thus, by inference we mean the computation of conditional probabilities. In terms of belief networks, we would like to know the probability distribution of a certain set of nodes given that we know the values that another set takes. As stated earlier, the main idea behind belief networks is to represent dependencies among random variables via a graph.

As an example, consider the belief network shown in Figure 14.1 that illustrates the dependency relationships among a set of 10 random variables $X_1, X_2, \ldots, X_{10}$. In this example, the joint distribution $P[X_1, X_2, \ldots, X_{10}]$ is obtained by using the following chain rule of probability:

$$P[X_1, X_2, \ldots, X_{10}] = P[X_{10}|X_9, \ldots, X_1] \\ * P[X_9|X_8, \ldots, X_1] * \ldots * P[X_2|X_1] * P[X_1]$$

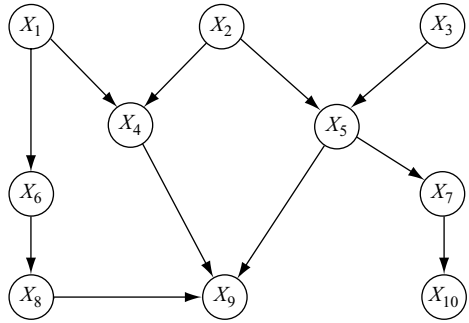


Figure 14.1. Example of a belief network.

However, the dependency relationships enable us to obtain the joint distribution as follows:

$$\begin{aligned} P[X_1, X_2, \ldots, X_{10}] &= \prod_{i=1}^{10} P[X_i | Pa(X_i)] \\ &= P[X_{10}|X_7] * P[X_9|X_4, X_5, X_8] * P[X_8|X_6] * P[X_7|X_5] \\ &\quad * P[X_6|X_1] * P[X_5|X_2, X_3] * P[X_4|X_1, X_2] * P[X_3] \\ &\quad * P[X_2] * P[X_1] \end{aligned}$$

Note that if a node has no parents (i.e., $Pa(X_i) = \Phi$, where Φ is the null set), we have that $P[X_i|Pa(X_i)] = P[X_i]$.

Unfortunately, in many cases the computation of these probabilities is very difficult because the computation time is exponential in the number of nodes. In a singly connected network where there is only one undirected path from any node to any other node the belief network inference problem is straightforward because it is solved via *belief propagation*, which was introduced by Pearl (1988). However, when the network is not a singly connected network, it is often necessary to resort to approximation methods that include the Gibbs sampling method. The Gibbs sampling method starts by fixing the observed variables to their known values and setting the unobserved variables arbitrarily. Each unobserved variable is repeatedly visited in turn, each time a new value of the variable is randomly selected from its conditional distribution given the current values of the other variables.

14.6 Choice of Method

There is no established formula for choosing between the Metropolis-Hastings algorithm and the Gibbs sampler. The Gibbs sampler has some clear advantages

when the full conditional $\pi(\theta_i|\theta_{-i})$ can be efficiently generated. It is not necessary to calculate the acceptance probability, which simplifies programming. When sampling from $\pi(\theta_i|\theta_{-i})$ is difficult, it is a common practice to use the Metropolis-Hastings algorithm.

14.7 Problems

14.1 Consider the Weibull random variable X with the CDF $F_X(x) = 1 - \exp\{-[(x - x_0)/a]^c\}$ where $x > x_0$ and $a, c > 0$ are parameters. Obtain $x = F_X^{-1}(Y)$, where $Y = F_X(x)$.

14.2 Consider a random variable X whose PDF is given by $f_X(x) = 1/(1+x)^2$, where $x > 0$. If we define $Y = F_X(x)$, obtain $x = F_X^{-1}(Y)$.

14.3 Consider the PDF given by

$$f_X(x) = \begin{cases} \dfrac{3x^2}{2} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

If we define $Y = F_X(x)$, use the inverse transform method to obtain $x = F_X^{-1}(Y)$.

14.4 Consider the function

$$f_{(x)} = \begin{cases} \dfrac{3x^3}{2} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Suggest a probability density function $\pi(x)$, where $-1 \le x \le 1$, and a function $h(x)$ such that we can obtain the following Monte Carlo integration:

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} h(x)\pi(x)dx = E_\pi[h(x)] \cong \frac{1}{n}\sum_{k=1}^{n} h(x_k)$$

14.5 Write a simulation program that generates random samples from the standard normal distribution, $N(0, 1)$. Generate at least 2000 samples and record the number of samples that fall into each of the following intervals: $(-\infty, -2)$, $(-2, -1)$, $(-1, 0)$, $(0, 1)$, $(1, 2)$, $(2, \infty)$. Use these numbers to estimate the probability that a random sample will lie in each interval.

14.6 A *self-avoiding random walk (SAW)* is a special random walk along an m-dimensional lattice such that no vertex (or site) is visited more than once, Hayes (1998). In this way the trajectory never intersects itself. Self-avoiding random walks arise in modeling physical processes like the

folding of polymer molecules. Such walks are difficult to model using traditional analytical methods, so they are best studied by simulation. In this problem we simulate a self-avoiding walk of length N on the lattice using the Metropolis-Hastings algorithm. Specifically, consider a two-dimensional square lattice with unit spacing between sites. Starting at site 1, which is the origin, we choose site 2 at random from its four neighbors. Then for each site i, $i = 2, \ldots, N-1$, generate the next site $i+1$ from the three neighbors of i after excluding $i-1$. If a site is revisited at any stage of the process, the walk is abandoned and restarted.

a. Run the experiment 500 times. What is the expected length of a walk?

b. Consider a refinement of the process, which is that for each site i, $i = 2, \ldots, N-1$, generate the next site $i+1$ from the neighbors of i that have not been visited before. Continue until a walk of length N is obtained or the process is trapped. If we run the experiment 100 times, how many full-length SAWs (i.e., SAWs of length N) can you get for $N = 20$ and $N = 50$?

14.7 Consider a 50×50 square array of spins. Initially assign all sites the spin value of $+1$ and select one spin randomly. The simulation involves flipping a random lattice point and determining if the net energy decreases. If the energy decreases because of the flip, then the new state is allowed, and the simulation continues. If the new lattice configuration causes the overall energy to increase, then a random number between 0 and 1 is generated. If the exponential of the temperature and change in energy is less than this randomly generated number, then the state is allowed, and the simulation continues. If this exponential is less than the randomly generated value, the flip is not allowed, the flipped lattice point is returned to its previous state, a different random lattice point is chosen, and the simulation continues. Thus, we summarize the algorithm in the following step form:

a. Pick a site x on the lattice at random and flip its spin, changing its state from x to x'.

b. Compute the trial change in energy $\Delta U = U(x') - E(x)$, where

$$U(x) = -\sum_{k \in \aleph(x)} s_x s_k$$

s_k is the spin of site k and $\aleph(x)$ is the set of neighbors of site x.

c. If $\Delta U < 0$, accept the flip.

d. If $\Delta U > 0$, generate a random number r, where $r \in (0, 1)$. If $r < \exp(-\beta \Delta U)$, accept the change in sign, where we assume that $\beta = 0.5$; otherwise, reject the flip and return the spin to the previous value.

e. Repeat this process 2499 more times (for the 50×50 sites), choosing a new site at random. This constitutes one sweep of the lattice.

After several sweeps to flush out the initial spin assignments, this procedure generates spin configurations that naturally follow the real probability function.

14.8 Consider the following acceptance probability for the Metropolis-Hastings algorithm:

$$\alpha(x, y) = \frac{\pi(y)}{\pi(x) + \pi(y)}$$

Show that this definition of $\alpha(x, y)$ produces a reversible Markov chain and has the stationary distribution π if the transition probability $q(x|y)$ is symmetric.

14.9 Simulate a standard Brownian motion by repeatedly generating independent normal random variables with mean 0 and standard deviation $\sqrt{(1/N)}$, where N is the number of experiments. Specifically, assume an interval from 0 to 1 that is divided into N equally spaced points. Let the points $T_1, T_2, \ldots, T_N$ be the sample points, where $T_{n+1} - T_n = 1/N$, $1 \leq n \leq N - 1$. The value of the Brownian motion at time T_n is the sum of the first n simulated values. Assume that $N = 1000$.

14.10 Consider the geometric Brownian motion $Y(t)$, which is used to model stock prices. It is characterized by the drift rate $\mu > 0$ and the standard deviation parameter σ that is usually referred to as the volatility in the field of finance. The stochastic differential equation (SDE) is given by

$$dY(t) = \mu Y(t)dt + \sigma Y(t)dB(t) \quad t \geq 0$$

where $B(t)$ is the standard Brownian motion; that is, $B(t) \sim N(0, 1)$. The solution to the SDE is

$$Y(t) = Y(0) \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B(t)\right\}$$

Assume that $Y(0) = 50$, $\mu = 0.1$, and $\sigma = 0.04$. Simulate the geometric Brownian motion from time $t = 0$ to $t = T$ as follows. Assume that $T = 2$ and $N = 1000$. Define $\Delta t = T/N$ and assume that the sample points are $t_1 = \Delta t$, $t_2 = 2\Delta t, \ldots, t_N = T$. Generate the standard normal random variable B with mean 0 and standard deviation $\sqrt{(1/N)}$ at point t_1, and obtain the current value of $Y(t_1)$ by $Y(0)(1 + \mu\Delta t + \sigma B)$. For each sample point t_n, $2 \leq n \leq N$, generate the standard normal random variable with mean 0 and standard deviation $\sqrt{(1/N)}$ and obtain the current value of the process by multiplying the previous value by $(1 + \mu\Delta t + \sigma B)$. The value after the Nth operation is the required result.

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