Notes on Lyapunov Exponents

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1 Introduction

Lyapunov exponents aim to quantify the term "'**sensitive dependence of initial conditions**"'. Lets imagine two trajectories in phase space $\vec{x}(t)$ and $\vec{x}(t) + \vec{\delta}(t)$ that are initially separated by a tiny amount $|\vec{\delta}(0)| = 10^{-15}$, which is of the order of floating-point precision.

Sensitive dependence on initial conditions means, that neighboring trajectories separate exponentially fast

$$|\vec{\delta}(t)| \approx |\vec{\delta}(0)|\, e^{\lambda t} \tag{1}$$

given a positive **Lyapunov exponent** λ. The Lyapunov exponents sets the **time horizon** beyond which our predictions break down: Suppose the distance a indicates our tolerance, i.e. the accuracy of our measurement equipment, so that if a measurement is within a, we consider it acceptable. The prediction becomes inaccurate for $|\vec{\delta}(t)| > a$, which happens after the time

$$t = \frac{1}{\lambda} \ln \left(\frac{a}{|\vec{\delta}(0)|} \right) \tag{2}$$

So even if we increase our accuracy and lower $|\vec{\delta}(0)|$ tremendously, our time horizon expands only by a couple of $1/\lambda$.

This treatment so far has closely followed Strogatz book ([1, chapter 9.3]), which also points out a more rigorous definition of Lyapunov exponents. On the one hand the strength of exponential divergence varies somewhat along a trajectory, thus λ should be average over various points. On the other hand for a n-dimensional phase space there are actually n **different Lyapunov exponents**: Every point in phase space can be decomposed into n different expanding or contracting directions featuring a distinct Lyapunov exponent λ_k with $k = 1, \ldots, n$.

This can be used to determine the **Lyapunov spectrum** in a rigorous way: Consider the evolution of an infinitesimal sphere of initial conditions at a give point in phase space. After some time, the infinitesimal sphere will evolve into an infinitesimal ellipsoid, being elongated along the expanding directions and shortened along the contracting directions. So the length of the k-th principal axis of this ellipsoid grows according to $\delta_k(t) \sim \delta_0 \, e^{\lambda_k t}$ and for large t the ellipsoids diameter is governed by the largest λ_k.

Now lets turn this thoughts into an algorithm using Gram-Schmidt orthogonalization as originally proposed by Wolf et al [**Wolf1994**]. For example in $n = 3$ dimensionas the infinitesimal sphere is captured by an orthonormal system of unit vectors $\vec{u}^0 = [\vec{e}^0_1, \vec{e}^0_2, \vec{e}^0_3]$.

The time evolution of the dynamical system is governed by

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \vec{f}(\vec{x}) \tag{3}$$

and the time evolution of $\vec{u}$ is governed by the linearization

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{u} = \hat{J}\vec{u} \tag{4}$$

These $3+9$ equations are evolved simultaneously in time: During every time step the Jacobian $J_{i,j} = \frac{\partial f_i}{\partial x_j}$ is updated according to the evolution of the dynamical system, which in turn determines the evolution of the system $\vec{u}$.

After an iteration period of Δt, a new vector $\vec{u}^k = \vec{e}^k_1, \vec{e}^k_2, \vec{e}^k_3$ is obtained, which represents the distorted infinitesimal ellipsoid. Using Gram-Schmidt orthogonalization one obtained the system $\vec{v}^k = \vec{v}^k_1, \vec{v}^k_2, \vec{v}^k_3$.

This can be normalized for the next round if iteration, which yields $\vec{u}^k_0$. Repeating the process eventually leads to the Lyapunov exponents being given by

$$\lambda_i = \frac{1}{k\Delta t} \sum_k \ln\big(|v^k_i|\big) \tag{5}$$

2 The Intuitive Approach

3 Gram-Schmidt Method

References

[1] S. H. Strogatz, *Nonlinear dynamics and chaos* (Perseus Books Publishing, 1994).