Notes on Lyapunov Exponents

Benjamin Wolba

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1 Introduction

Lyapunov exponents aim to quantify the term "**sensitive dependence of initial conditions**". Lets imagine two trajectories in phase space $\vec{x}(t)$ and $\vec{x}(t) + \vec{\delta}(t)$ that are initially separated by a tiny amount $|\vec{\delta}(0)| = 10^{-15}$, which is of the order of floating-point precision.

Sensitive dependence on initial conditions means, that neighboring trajectories separate exponentially fast

$$|\vec{\delta}(t)| \approx |\vec{\delta}(0)| \, e^{\lambda t} \tag{1}$$

given a positive **Lyapunov exponent** λ. The Lyapunov exponents sets the **time horizon** beyond which our predictions break down: Suppose the distance a indicates our tolerance, i.e. the accuracy of our measurement equipment, so that if a measurement is within a, we consider it acceptable. The prediction becomes inaccurate for $|\vec{\delta}(t)| > a$, which happens after the time

$$t = \frac{1}{\lambda} \ln \left(\frac{a}{|\vec{\delta}(0)|} \right) \tag{2}$$

So even if we increase our accuracy and lower $|\vec{\delta}(0)|$ tremendously, our time horizon expands only by a couple of $1/\lambda$.

This treatment so far has closely followed Strogatz book ([1, chapter 9.3]), which also points out a more rigorous definition of Lyapunov exponents. On the one hand the strength of exponential divergence varies somewhat along a trajectory, thus λ should be average over various points. On the other hand for a n-dimensional phase space there are actually n **different Lyapunov exponents**: Every point in phase space can be decomposed into n different expanding or contracting directions featuring a distinct Lyapunov exponent λ_k with $k = 1, \ldots, n$.

2 Intuitive Example

As an intuitive example let's take a look at the **double pendulum**, parameterized by the two polar angle φ_1 and φ_2, whose equations of motion are given by

$$\ddot{\varphi}_1 = -\frac{m_2 l_2}{(m_1+m_2)l_1}\ddot{\varphi}_2\cos(\varphi_1-\varphi_2) - \frac{m_2 l_2}{(m_1+m_2)l_1}\dot{\varphi}_2^2\sin(\varphi_1-\varphi_2) - \frac{g}{l_1}\sin(\varphi_1)$$
$$\ddot{\varphi}_2 = -\frac{l_1}{l_2}(\ddot{\varphi}_1\cos(\varphi_1-\varphi_2) - \dot{\varphi}_1^2\sin(\varphi_1-\varphi_2)) - \frac{g}{l_2}\sin(\varphi_2)$$

where m_1, m_2 and l_1, l_2 are the respective masses and pendulum lengths and g represents the gravitational acceleration constant g. Setting all of these quantities to unity for our example, we calculate the time evolution of the double pendulum starting at

$$\varphi_1 = \frac{\pi}{2}, \quad \dot{\varphi}_1 = 0, \quad \varphi_2 = 0.1+\delta, \quad \dot{\varphi}_2 = 0$$

where the first run $\delta = 0$ and the second run $\delta = 10^{-12}$. As a **measure of separation** we calculate the Euclidean distance between the position of the second bob at x_2, y_2 in the first and in the second run. Transforming to Cartesian coordinates yields

$$x_1 = l_1\sin(\varphi_1) \quad y_1 = l_1\cos(\varphi_1)$$
$$x_2 = x_1 + l_2\sin(\varphi_2) \quad y_2 = y_1 + l_2\cos(\varphi_2)$$

and the Euclidean distance is given by

$$d = \sqrt{(x_2(\text{1st run}) - x_2(\text{2nd run}))^2 + (y_2(\text{1st run}) - y_2(\text{2nd run}))^2}$$

The distance d should increase exponentially according to some Lyapunov Exponent λ, at least at the beginning. Thus, a rough estimate for λ is obtained by plotting $\ln(d)$ versus t and applying a linear fit, where the slope represents λ. As can be seen from Fig. 1, this linear fit is a crude approximation, as $\ln(d)$ shows some wiggles as the exponential increase varies at different points of the trajectory. So this example is tailored more towards visualizing the intuition behind Lyapunov exponents rather then a rigorous calculation.

After some time, the distance d saturates, as the two runs cannot be farther apart then being opposite to each other. Thus, the maximum value of d in our example is $d = 2\cdot(l_1+l_2) = 4$ and so $\ln(d) = 1.39$. One can see, that the trajectories of both runs are not correlated anymore.

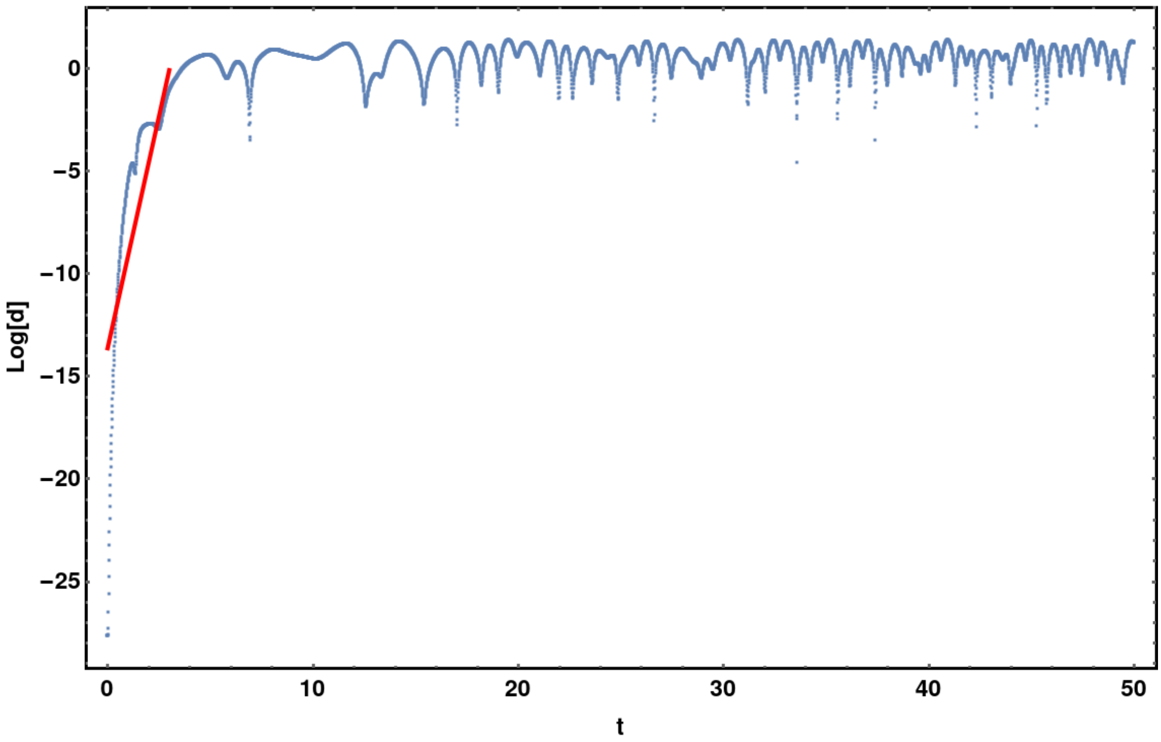

Figure 1: Exponential deviation between the trajectories of a double pendulum for slightly different initial conditions.

References

[1] S. H. Strogatz, *Nonlinear dynamics and chaos* (Perseus Books Publishing, 1994).

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