Random Resistor Networks

Benjamin Wolba, October 26, 2016

I Introduction

Random resistor Networks (RRN) consist of many resistors, which are arranged in serial and parallel connections to form large electrical networks. We will consider here two- and three-dimensional networks built on a quadratic, respectively cubic, lattice of nodes. The ohmic resistors in between of them are chosen randomly either to be metallic ρ_m or insulating ρ_i. From this setting the total specific network resistance can be calculated in certain ways, for example between two nodes or between two equipotential surfaces.
Some may now ask at this point, why do we need RRN? RRN are not only intrinsically interesting from the electrotechnical point of view. Moreover they are quite useful for modeling for instance manganite compounds in solid state physics, a class of materials exhibiting an insulator-metal-transition (IMT). For example if one decreases temperature, from room temperature to liquid nitrogen temperature, the compound $La_{0.7}Ca_{0.3}MnO_3$ shows a phase transition from an insulating paramagnetic to a metallic paramagnetic state [2, 3]. By using RRN one could simulate the temperature dependence of the specific resistance ρ, not only for manganites. Additionally there are connections to random walks, fractals and percolation theory.
Freshly motivated we will recap now at first some fundamental basics in terms of electrical networks, before we come to the special case of RRN. In such an electrical network, the electrical voltage U and the electrical current I are macroscopically connected through Ohm's law by the resistance R

$$U = R\,I \tag{1}$$

The inverse of R is the so-called electrical conductivity $G = \frac{1}{R}$. These currents and voltages are on the one hand macroscopically determined by the Kirchhoff's circuit laws:

1. Kirchhoff's current law (KCL): At any node in an electrical circuit, the sum of currents flowing into that node is equal to the sum of currents flowing out of it. This means algebraically $\sum_{k=1}^{n} I_k = 0$.

2. Kirchhoff's voltage law (KVL): The directed sum of the electrical potential differences (voltages) around any closed network is zero. Hence $\sum_{k=1}^{n} U_k = 0$.

On the other hand there is the equivalent microscopic description, given by the microscopic formulation of Ohms law

$$\vec{j}(\vec{r}) = \sigma(\vec{r})\,\vec{E} = \sigma(\vec{r})\,\vec{\nabla}V \tag{2}$$

and the electrical current conservation condition

$$\vec{\nabla}\cdot\vec{j}(\vec{r}) = 0 \tag{3}$$

These equations apply to the current density $\vec{j}(\vec{r})$, the local electric field $\vec{E}$, the microscopic electrical conductivity $\sigma(\vec{r})$ and the introduced local potential V (confusingly sometimes also called local voltage), which is just the negative electrostatic potential $V = -\phi_{\text{el}}$.
Depending on the physics question of interest either the macroscopic or the microscopic approach is useful. Here we will consider at first two-dimensional RRN on the microscopic level, which could be later on generalized to the three-dimensional case. To these networks an exact solution is available by a matrix approach, which is described in the following part. Despite that this method is very slow for extensive networks, it is still useful to compare with the numerically obtained values. The numerical approach presented here in the part after is based on a relaxation method (according to [1, S. 309/310]).

II Exact Matrix Solution

We will consider now a two-dimensional square lattice, which will be later also useful for understanding the numerical relaxation method. To get a formulation in terms of vectors and matrices, let us numerate the nodes within the RRN and put them all into one vector. This is shown for example for a two-dimensional square lattice network of 3×3 nodes:

$$\begin{pmatrix} 0 & 3 & 6 \\ 1 & 4 & 7 \\ 2 & 5 & 8 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 7 \\ 8 \end{pmatrix}$$

Caveat: the numeration will always start with the 0 to be the first element.
At each node i the local potential will take a certain value V_i and two nodes i and j are connected by the microscopic conductivity $\sigma_{i,j}$. Now consider each node connected to all the other $n-1$ nodes. As we know that there is on a square lattice only a connection between neighboring nodes, the conductivity of the other connections is taken to be zero. Let us define the total current density J_i (capital J in contrast to the single current density j) at each node through

$$J_i = \sum_{i \neq j} \sigma_{i,j}(V_j - V_i) \tag{4}$$

As most of the conductivities $\sigma_{i,j}$ are zero, effectively only four terms of the nearest neighbor nodes of node i are contributing to the sum. We are getting now a large vectors of $n \times n$ components for the local potential $\vec{V}$ and for the total currents $\vec{J}$.
Our next step is to construct a matrix $\hat{G}$, to connect both of them. We could obtain the matrix elements of $\hat{G}$ from equation (4) as follows

$$\vec{J} = \hat{G}\,\vec{V}, \quad G_{i,j} = \begin{cases} \sigma_{i,j} & i \neq j \\ -\sum_{i \neq k} \sigma_{i,k} & i = j \end{cases} \tag{5}$$

With the abbreviation $S_i = \sum_{i \neq k} \sigma_{i,k}$ the conductivity matrix $\hat{G}$ therefore reads

$$\hat{G} = \begin{pmatrix} -S_0 & G_{0,1} & G_{0,2} & \dots & G_{0,n-1} \\ G_{1,0} & -S_1 & G_{1,2} & \dots & G_{1,n-1} \\ G_{2,0} & G_{2,1} & -S_2 & \dots & G_{2,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G_{n-2,0} & G_{n-2,1} & G_{n-2,2} & \dots & G_{n-2,n-1} \\ G_{n-1,0} & G_{n-1,1} & G_{n-1,2} & \dots & -S_{n-1} \end{pmatrix}$$

Before we think of how to solve this system of linear equations (5), it is worth to regards the properties of the conductivity matrix $\hat{G}$:

1. It is a so-called tree matrix. Hence the sum of every row / column equals to zero.

2. It is symmetric by construction.

3. The matrix is singular: $\det \hat{G} = 0$.

Especially the last property is remarkable. Consider a constant potential vector. Hence the local potential takes a constant value at every node and there would be no current flows between the nodes. But this means also, that there is no total current at each node. We see therefore $\hat{G}\,\vec{c} = \vec{0}$, when $\hat{G}$ is applied to a constant vector $\vec{c}$. This shows, why the conductivity matrix $\hat{G}$ has to be singular: to every solution of equation (5), we could add a constant potential value at each node and it will not change the current flows.
In literature often the total current is set as boundary condition. Since it takes the value zero in the interior of the network (current conservation), the total current vector is completely determined by the total currents at the boundary of the RRN. If the conductivity matrix $\hat{G}$ would not be singular, we would be able to simply invert it, to obtain the potential distribution. But this is not possible directly, so one would have to think of a different approach, for example LU-decomposition of $\hat{G}$ and making use of the Gaussian elimination method. Additionally it would be more convenient to set the potential as boundary condition and calculate the total currents, since the latter can be easily added up. That is not straight forward for the potential values. Since the total network resistance is independent of both ways, we will chose the latter one and fix the potential.

In the following part the solution of this system of linear equations is demonstrated for the example $n = 3$, where the total conductivity matrix $\hat{G}$ reads

$$\begin{pmatrix} J_0 \\ J_1 \\ J_2 \\ 0 \\ 0 \\ 0 \\ J_6 \\ J_7 \\ J_8 \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix} -S_0 & G_{0,1} & G_{0,2} \\ G_{1,0} & -S_1 & G_{1,2} \\ G_{2,0} & G_{2,1} & -S_2 \end{matrix}} & \boxed{\begin{matrix} G_{0,3} & 0 & 0 \\ 0 & G_{1,4} & 0 \\ 0 & 0 & G_{2,5} \end{matrix}} & \boxed{\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}} \\ \boxed{\begin{matrix} G_{3,0} & 0 & 0 \\ 0 & G_{4,1} & 0 \\ 0 & 0 & G_{5,2} \end{matrix}} & \boxed{\begin{matrix} -S_3 & G_{3,4} & G_{3,5} \\ G_{4,3} & -S_4 & G_{4,5} \\ G_{5,3} & G_{5,4} & -S_5 \end{matrix}} & \boxed{\begin{matrix} G_{3,6} & 0 & 0 \\ 0 & G_{4,7} & 0 \\ 0 & 0 & G_{5,8} \end{matrix}} \\ \boxed{\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}} & \boxed{\begin{matrix} G_{6,3} & 0 & 0 \\ 0 & G_{7,4} & 0 \\ 0 & 0 & G_{8,5} \end{matrix}} & \boxed{\begin{matrix} -S_6 & G_{6,7} & G_{6,8} \\ G_{7,6} & -S_7 & G_{7,8} \\ G_{8,6} & G_{8,7} & -S_8 \end{matrix}} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ V_3 \\ V_4 \\ V_5 \\ V_c \\ V_c \\ V_c \end{pmatrix} \tag{6}$$

From that we could see, that this systems grows impressively large for higher n. One observes, that for example node 0 is connected to node 1, 2 and 3, but not to node 6, as there are no periodic boundary conditions in x- but in y-direction assumed. The total conductivity matrix $\hat{G}$ consists of smaller symmetric, diagonal or zero matrices (as shown through red boxes). Furthermore one could see, that the potential values V_3, V_4 and V_5 in the middle do not depend on the unknown total currents at the boundary of the network. Hence we have only to solve the following part of the system

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix} G_{3,0} & 0 & 0 \\ 0 & G_{4,1} & 0 \\ 0 & 0 & G_{5,2} \end{matrix}} & \boxed{\begin{matrix} -S_3 & G_{3,4} & G_{3,5} \\ G_{4,3} & -S_4 & G_{4,5} \\ G_{5,3} & G_{5,4} & -S_5 \end{matrix}} & \boxed{\begin{matrix} G_{3,6} & 0 & 0 \\ 0 & G_{4,7} & 0 \\ 0 & 0 & G_{5,8} \end{matrix}} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ V_3 \\ V_4 \\ V_5 \\ V_c \\ V_c \\ V_c \end{pmatrix} \tag{7}$$

This is nice, but the matrix in between is not diagonal any more. This can be fixed by doing the following matrix products, resulting in a constant vector

$$\begin{pmatrix} G_{3,0} & 0 & 0 \\ 0 & G_{4,1} & 0 \\ 0 & 0 & G_{5,2} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \vec{0} \quad \text{and} \quad \begin{pmatrix} G_{3,6} & 0 & 0 \\ 0 & G_{4,7} & 0 \\ 0 & 0 & G_{5,8} \end{pmatrix} \begin{pmatrix} V_c \\ V_c \\ V_c \end{pmatrix} = \vec{c} \tag{8}$$

We end up with the following reduced system of $n^2 - 2 \cdot n = 3$ equations

$$\vec{0} = \vec{c} + \begin{pmatrix} -S_3 & G_{3,4} & G_{3,5} \\ G_{4,3} & -S_4 & G_{4,5} \\ G_{5,3} & G_{5,4} & -S_5 \end{pmatrix} \begin{pmatrix} V_3 \\ V_4 \\ V_5 \end{pmatrix} = \vec{c} + \hat{G}_{\text{red}} \vec{V}_{\text{red}} \tag{9}$$

The reduced conductivity matrix $\hat{G}_{\text{red}}$ and the reduced problem could be solved for instance by inverting it. For the numerics we will choose a more efficient standard solution method.

Once obtained the reduced potential vector $\vec{V}_{\text{red}}$ we know also the total potential vector $\vec{V}$. Now we are able to calculate the total current vector directly by applying $\hat{G}$ to $\vec{V}$. The resulting vector $\vec{J}$ has to satisfy according to its components J_i

$$\sum_i J_i = 0 \tag{10}$$

This means that the current conservation law is fulfilled and that hence the total entering current equals the total leaving current

$$J_{\text{in}} = \sum_{i=0}^{n-1} J_i = -\sum_{i=n^2-n}^{n^2-1} J_i = J_{\text{out}} \tag{11}$$

Finally we could calculated to total network resistance through

$$\rho_{\text{tot}} = \frac{V_c}{J_{\text{in}}} \tag{12}$$

III Relaxation Method

The matrix method yields the exact solution for a RRN, but it is incredibly slow for large n. Therefore we will consider in this part a numerical solution, based on a relaxation method described in [1, S. 309/310]. Again we start with the microscopic equations (2) and (3). Discretization on the resistor lattice is leading to

$$j_x(\vec{r}) = \sigma_x(\vec{r}) \cdot [V(\vec{r}+\vec{e}_x) - V(\vec{r})] \tag{13}$$

$$\begin{aligned} \partial_x j_x &= j_x(\vec{r}+\vec{e}_x) - j_x(\vec{r}) \\ &= \sigma_x(\vec{r}+\vec{e}_x) \cdot [V(\vec{r}+\vec{e}_x) - V(\vec{r})] \\ &\quad -\sigma_x(\vec{r}) \cdot [V(\vec{r}) - V(\vec{r}-\vec{e}_x)] \end{aligned} \tag{14}$$

as shown for the x-direction. Here the vector $\vec{r} = \binom{x}{y}$ denotes the position of the nodes of the RRN and $\vec{e}_x$ is the unit vector in x-direction. Hence we get a potential matrix $\hat{V}$ describing the local potential at each node.

Combining these equations with the analogous equations for the y-direction one obtains the following beautiful expression

$$V(\vec{r}) = \frac{\sum_{\text{NN}} \sigma_{\vec{r}\vec{r}'} V(\vec{r}')}{\sum_{\text{NN}} \sigma_{\vec{r}\vec{r}'}} \tag{15}$$

This means that the local potential at the node $\vec{r}$ is a suitably-weighted, harmonic average of the local potentials at neighboring nodes (nearest neighbors, NN).
This relation for the local potential is used to determine the local potential distribution $V(\vec{r})$ iteratively. Let the local potential be fixed to zero at the left side of the square lattice of the RRN and let it take the value V_c at the right site (in x-direction). Therefore one can assume an initial potential distribution, which is given by

$$V_0(\vec{r}) = V_c \frac{x}{L} \tag{16}$$

It depends only on the x-position of the nodes and on the extension of the RRN in x-direction, so that the local potential is increasing linearly between $V_0(\vec{r} = \binom{0}{y}) = 0.0$ and $V_0(\vec{r} = \binom{L}{y}) = V_c$. Now relation (15) could be applied to each node to calculate the next, updated local potential matrix. After some hundred or thousand iterations the potential distribution has relaxed. One could control that by requiring the change of each matrix element of the local potential distribution to be lower then some value

$$|V_{n+1}(\vec{r}) - V_n(\vec{r})| < \text{eps} \tag{17}$$

for two consecutive iterations. Now we have the local potential $V(\vec{r})$ given at each intersection point. Therefore one could calculate the current density $\vec{j}$ between every two intersection points (i.e. through every resistor). The sum of all current densities in one column of the RRN is the same for all columns and equals the total network current due to electrical charge conservation (3). By using Ohms law (2) one could calculate the total specific network resistance

$$\rho_{\text{tot}} = \frac{1}{\sigma_{\text{tot}}} = \frac{V_c}{j_{\text{tot}}} \tag{18}$$

In doing so the potential difference (the actual voltage) $\Delta V = V_c - 0$ is forced onto the system in advance. The systems reacts to that in the way we figure out, when we calculate the total current density j_{tot} per column numerically. The total specific network resistance ρ_{tot} is independent of the applied voltage.

III Numerical Implementation

At first a matrix for the random resistors is generated. As we consider a $n \cdot n$ lattice of nodes, we will need $2 \cdot n \cdot n$ resistors. The $2 \cdot n$ resistors of a column are stored in a column of the resistor matrix, as illustrated in Fig. 2. There the array indices are quoted as indices of the resistances $R_{i,j}$. The nodes take themselves the coordinates (i, j) as shown in Fig. 1, which are taken also as indices for the matrix of the local potential V.
As a next step this matrix for the local potential distribution is set up according to the initial configuration given by (16), so that it reads

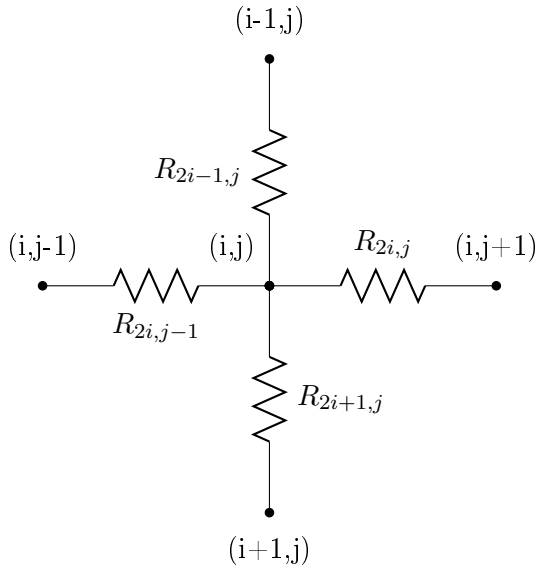

Figure 1: Indices for intersection point (i,j).

$$\hat{V_0} = \begin{pmatrix} 0.0 & V_1 & \dots & V_{n-2} & V_c \\ 0.0 & V_1 & \dots & V_{n-2} & V_c \\ \vdots & \vdots & & \vdots & \vdots \\ 0.0 & V_1 & \dots & V_{n-2} & V_c \\ 0.0 & V_1 & \dots & V_{n-2} & V_c \end{pmatrix} \tag{19}$$

This matrix is iteratively updated as described in the previous section, until the relaxation condition

$$\max(|\hat{V}_{n+1} - \hat{V}_n|) < \text{eps} \tag{20}$$

for the maximum change is fulfilled. For getting better statistics this is done for several RRN-configurations and then the average values of j_{tot}, σ_{tot} and ρ_{tot} are determined. Additionally we have to iterate over the temperature and the value of the metallic fraction p yielding two more for-loops.

References

[1] Elbio Dagotto. *Nanoscale Phase Separation and Colossal Magnetoresistance*. Springer Verlag, 2002.

[2] M. Fäth et al. "Spatially Inhomogeneous Metal-Insulator Transition in Doped Manganites". In: *Science* 285 (1999), pp. 1540–1542.

[3] C. Israel et al. "Translating reproducible phase-separated texture in manganites into reproducible two-state low-field magnetoresistance: An imaging and transport study". In: *Phys. Rev. B: Condens. Matter Mater. Phys.* 78 (2008), p. 054409.

Appendix

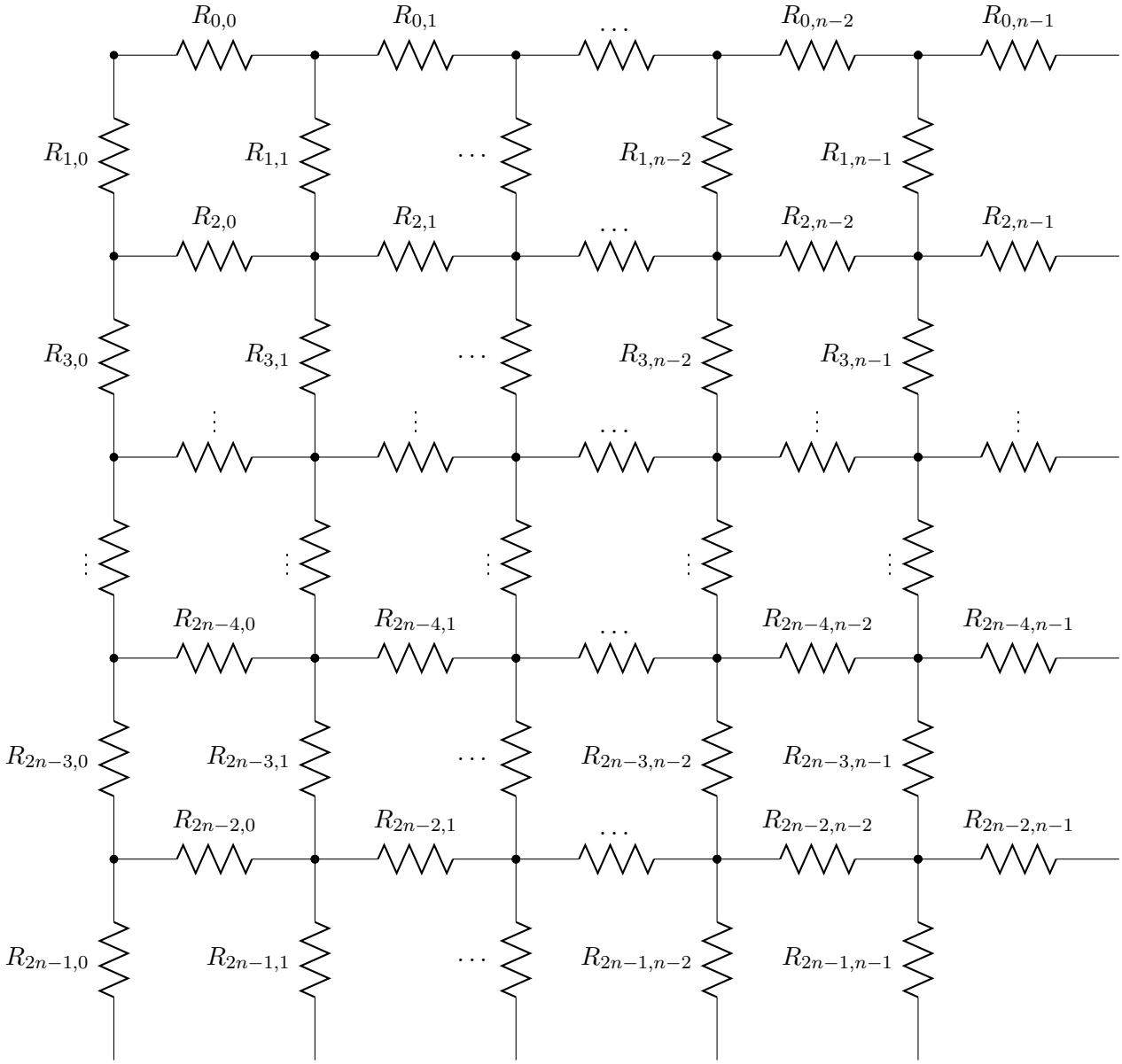

Figure 2: Illustration of a RRN and indexing of the resistors $R_{i,j}$ for storage in a resistor matrix.