Project 1

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$$\boxed{\cosh t = \frac{e^t + e^{-t}}{2},\ \sinh t = \frac{e^t - e^{-t}}{2}}$$

1. (a)

$$\begin{aligned}
1 &= \cosh^2 t - \sinh^2 t \\
1 &= \left(\frac{e^t + e^{-t}}{2}\right)^2 - \left(\frac{e^t - e^{-t}}{2}\right)^2 \\
1 &= \frac{\cancel{e^{2t}} + 2 + \cancel{e^{-2t}}}{4} - \frac{\cancel{e^{2t}} - 2 + \cancel{e^{-2t}}}{4} \\
1 &= \frac{2 - -2}{4} = \frac{4}{4} \\
1 &= 1
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{1 + \cosh 2t}{2} &= \cosh^2 t \\
\frac{1}{2} + \frac{\cosh 2t}{2} &= \left(\frac{e^t + e^{-t}}{2}\right)^2 \\
\cancel{\frac{1}{2}} + \frac{\cosh 2t}{2} &= \frac{e^{2t} + \cancel{2} + e^{-2t}}{4} \\
\frac{e^{2t} + e^{-2t}}{4} &= \frac{e^{2t} + e^{-2t}}{4}
\end{aligned}$$

(c)

$$\begin{aligned}
\sinh t &= \frac{d}{dt} \cosh t \\
&= \frac{d}{dt}\left(\frac{e^t + e^{-t}}{2}\right) \\
&= \frac{d}{dt}\left(\frac{e^t}{2} + \frac{e^{-t}}{2}\right) \\
&= \frac{e^t}{2} - \frac{e^{-t}}{2} \\
&= \frac{e^t - e^{-t}}{2} \\
\sinh t &= \sinh t
\end{aligned}$$

(d)

$$\begin{aligned}
\cosh t &= \frac{d}{dt}\sinh t \\
&= \frac{d}{dt}\left(\frac{e^t - e^{-t}}{2}\right) \\
&= \frac{d}{dt}\left(\frac{e^t}{2} - \frac{e^{-t}}{2}\right) \\
&= \frac{e^t}{2} + \frac{e^{-t}}{2} \\
&= \frac{e^t + e^{-t}}{2} \\
\cosh t &= \cosh t
\end{aligned}$$

2.

$$\boxed{\begin{aligned} \tanh^2 x + \operatorname{sech}^2 x = 1 \\ -\tanh x \operatorname{sech} x = \frac{d}{dt}\operatorname{sech} x \end{aligned}}$$

$$\begin{aligned}
&\int \tanh^3 t \operatorname{sech} t \, dt && u = \operatorname{sech} t \\
=& -\int \tanh^2 t \, du && du = -\tanh t \operatorname{sech} t \, dt \\
=& -\int 1 - u^2 \, du \\
=& \frac{1}{3}u^3 - u \\
=& \frac{1}{3}\operatorname{sech}^3 t - \operatorname{sech} t
\end{aligned}$$

3.

$$\boxed{\begin{aligned} & z = \sinh t \iff t = \ln\left(z + \sqrt{z^2+1}\right) \\ & \frac{d}{dx}\tanh x = \operatorname{sech}^2 x \end{aligned}}$$

$$
\begin{aligned}
&\int \sqrt{\left(1+x^2\right)^3}\, dx && x = \sinh\theta \\
=&\int \cosh\theta \left(\sqrt{1+x^2}\right)^3 d\theta && dx = \cosh\theta d\theta \\
=&\int \cosh^4\theta\, d\theta && 1 \le \cosh\theta < \infty \;\Rightarrow\; \sqrt{\cosh^2\theta} = \cosh\theta \\
=&\int \operatorname{sech}^{-4}\theta\, d\theta && u = \tanh\theta \\
=&\int \left(1-u^2\right)^{-3} du && du = \operatorname{sech}^2\theta d\theta \\
=&\int \left(u^4-2u^2+1\right)\left(1-u^2\right) du \\
=&\int -u^6+3u^4-3u^2+1\, du \\
=&\frac{-1}{7}u^7+\frac{3}{5}u^5-u^3+u+C \\
=&\frac{-1}{7}\tanh^7\theta+\frac{3}{5}\tanh^5\theta-\tanh^3\theta+\tanh\theta+C \\
=&\frac{-1}{7}\frac{\sinh^7\theta}{\cosh^7\theta}+\frac{3}{5}\frac{\sinh^5\theta}{\cosh^5\theta}-\frac{\sinh^3\theta}{\cosh^3\theta}+\frac{\sinh\theta}{\cosh\theta}+C \\
=&\frac{-x^7}{7\cosh^7\theta}+\frac{3x^5}{5\cosh^5\theta}-\frac{x^3}{\cosh^3\theta}+\frac{x}{\cosh\theta}+C && \cosh^2\theta = 1+\sinh^2\theta \\
=&\frac{-x^7}{7(1+x^2)^{\frac{7}{2}}}+\frac{3x^5}{5(1+x^2)^{\frac{5}{2}}}-\frac{x^3}{(1+x^2)^{\frac{3}{2}}}+\frac{x}{(1+x^2)^{\frac{1}{2}}}+C && \cosh\theta \ge 1 \Rightarrow \cosh\theta = \sqrt{1+x^2}
\end{aligned}
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