Project 1

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$$\boxed{\cosh t = \frac{e^t + e^{-t}}{2},\ \sinh t = \frac{e^t - e^{-t}}{2}}$$

1. (a)

$$\begin{aligned}
1 &= \cosh^2 t - \sinh^2 t \\
1 &= \left(\frac{e^t + e^{-t}}{2}\right)^2 - \left(\frac{e^t - e^{-t}}{2}\right)^2 \\
1 &= \frac{\cancel{e^{2t}} + 2 + \cancel{e^{-2t}}}{4} - \frac{\cancel{e^{2t}} - 2 + \cancel{e^{-2t}}}{4} \\
1 &= \frac{2 - -2}{4} = \frac{4}{4} \\
1 &= 1
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{1 + \cosh 2t}{2} &= \cosh^2 t \\
\frac{1}{2} + \frac{\cosh 2t}{2} &= \left(\frac{e^t + e^{-t}}{2}\right)^2 \\
\cancel{\frac{1}{2}} + \frac{\cosh 2t}{2} &= \frac{e^{2t} + \cancel{2} + e^{-2t}}{4} \\
\frac{e^{2t} + e^{-2t}}{4} &= \frac{e^{2t} + e^{-2t}}{4}
\end{aligned}$$

(c)

$$\begin{aligned}
\sinh t &= \frac{d}{dt} \cosh t \\
&= \frac{d}{dt}\left(\frac{e^t + e^{-t}}{2}\right) \\
&= \frac{d}{dt}\left(\frac{e^t}{2} + \frac{e^{-t}}{2}\right) \\
&= \frac{e^t}{2} - \frac{e^{-t}}{2} \\
&= \frac{e^t - e^{-t}}{2} \\
\sinh t &= \sinh t
\end{aligned}$$

(d)

$$\begin{aligned}
\cosh t &= \frac{d}{dt}\sinh t \\
&= \frac{d}{dt}\left(\frac{e^t - e^{-t}}{2}\right) \\
&= \frac{d}{dt}\left(\frac{e^t}{2} - \frac{e^{-t}}{2}\right) \\
&= \frac{e^t}{2} + \frac{e^{-t}}{2} \\
&= \frac{e^t + e^{-t}}{2} \\
\cosh t &= \cosh t
\end{aligned}$$

2.

$$\boxed{\begin{aligned} \tanh^2 x + \operatorname{sech}^2 x = 1 \\ -\tanh x \operatorname{sech} x = \frac{d}{dt}\operatorname{sech} x \end{aligned}}$$

$$\begin{aligned}
&\int \tanh^3 t \operatorname{sech} t\, dt & u &= \operatorname{sech} t \\
=&-\int \tanh^2 t\, du & du &= -\tanh t \operatorname{sech} t\, dt \\
=&-\int 1 - u^2\, du \\
=&\frac{1}{3}u^3 - u \\
=&\frac{1}{3}\operatorname{sech}^3 t - \operatorname{sech} t
\end{aligned}$$

3.

$$\boxed{\begin{aligned} z = \sinh t \iff t = \ln\left(z + \sqrt{z^2+1}\right) \\ \frac{d}{dx}\tanh x = \operatorname{sech}^2 x \\ \sinh 2x =& 2\sinh x \cosh x \\ \cosh 2x =& 2\cosh^2 x - 1 \end{aligned}}$$

$$
\begin{aligned}
&\int \sqrt{\left(1+x^2\right)^3}\, dx && x = \sinh\theta \\
=&\int \cosh\theta \left(\sqrt{1+x^2}\right)^3 d\theta && dx = \cosh\theta d\theta \\
=&\int \cosh^4\theta\, d\theta && 1 \leq \cosh\theta < \infty \;\Rightarrow\; \sqrt{\cosh^2\theta} = \cosh\theta \\
=&\int \left(\frac{1+\cosh 2\theta}{2}\right)^2 d\theta && u = 2\theta \\
=&\int \frac{1+2\cosh 2\theta+\cosh^2 2\theta}{4}\, d\theta && du = 2d\theta \\
=&\frac{1}{8}\int 1+2\cosh u+\cosh^2 u\, du \\
=&\frac{1}{8}\left(u+2\sinh u+\int \cosh^2 u\, du\right) \\
=&\frac{1}{8}\left(u+2\sinh u+\int \frac{1+\cosh 2u}{2}\, du\right) && v = 2u \\
=&\frac{1}{8}\left(u+2\sinh u+\frac{1}{4}\int 1+\cosh v\, dv\right) && dv = 2du \\
=&\frac{1}{8}\left(u+2\sinh u+\frac{1}{4}\left(v+\sinh v\right)\right)+C && dv = 2du \\
=&\frac{\theta}{4}+\frac{\sinh 2\theta}{4}+\frac{\theta}{8}+\frac{\sinh 4\theta}{32}+C \\
=&\frac{3\theta}{8}+\frac{\sinh 2\theta}{4}+\frac{\sinh 4\theta}{32}+C \\
=&\frac{3\operatorname{arsinh} x}{8}+\frac{\sinh\theta\cosh\theta}{2}+\frac{\sinh 2\theta\cosh 2\theta}{16}+C \\
=&\frac{3\operatorname{arsinh} x}{8}+\frac{\sinh\theta\cosh\theta}{2}+\frac{\sinh\theta\cosh\theta\left(2\cosh^2\theta-1\right)}{8}+C \\
=&\frac{3\operatorname{arsinh} x}{8}+\frac{\sinh\theta\cosh\theta}{2}+\frac{\sinh\theta\cosh^3\theta}{4}-\frac{\sinh\theta\cosh\theta}{8}+C && 1 = \cosh^2\theta + x^2 \\
=&\frac{3\operatorname{arsinh} x}{8}+\frac{3x\sqrt{1+x^2}}{8}+\frac{x\sqrt{\left(1+x^2\right)^3}}{4}+C && \cosh\theta = \sqrt{1+x^2}
\end{aligned}
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