Gaussian Filters

3.8 Exercises

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1 Kalman Filter

A A minimal state vector allowing recursive estimation contains the current position and the velocity of the car. Acceleration is not needed as a state variable since the task description states that

> "acceleration is set randomly at each point in time",

thus the acceleration of the previous time step does not influence does not influence the current state.

$$x_t = \begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix} \tag{1.1}$$

B The state transition probability of a Kalman filter takes the general form:

$$\begin{aligned} &p(x_t|u_t, x_{t-1}) \\ &= \frac{1}{\sqrt{(2\pi)^k \det(R)}} \exp\left(-\frac{1}{2}(x_t - (Ax_{t-1} + Bu_t))^\top R^{-1}(x - (Ax_{t-1} + Bu_t))\right) \end{aligned} \tag{1.2}$$

Model assumption of the Kalman filter is a state transition function that is linear in u_t and x_{t-1}:

$$x_t = Ax_{t-1} + Bu_t + \epsilon_t \tag{1.3}$$

In our case, there is no control input. I.e.

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{1.4}$$

Assuming constant acceleration during each time step, the position can be recursively calculated using:

$$x_t = x_{t-1} + \dot{x}_{t-1} + 0.5\ddot{x}_t \tag{1.5}$$

and the velocity can be calculated using:

$$\dot{x}_t = \dot{x}_{t-1} + \ddot{x}_t \tag{1.6}$$

From these two equations, we infer that

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{1.7}$$

and

$$\epsilon_t \sim \mathcal{N}\left(0, \begin{bmatrix} 0.25 & 0 \\ 0 & 1 \end{bmatrix}\right) \tag{1.8}$$

C The progression of 6 Gaussians is given by table 1.1.

Gaussian
$\mu = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad \text{cov} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
$\mu = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad \text{cov} = \begin{bmatrix} 0.25 & 0.5 \\ 0.5 & 1 \end{bmatrix}$
$\mu = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad \text{cov} = \begin{bmatrix} 2.5 & 2 \\ 2 & 2 \end{bmatrix}$
$\mu = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad \text{cov} = \begin{bmatrix} 8.75 & 4.5 \\ 4.5 & 3 \end{bmatrix}$
$\mu = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad \text{cov} = \begin{bmatrix} 21 & 8 \\ 8 & 4 \end{bmatrix}$
$\mu = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad \text{cov} = \begin{bmatrix} 41.25 & 12.5 \\ 12.5 & 5 \end{bmatrix}$

Table 1.1: Progression of the Kalman filter belief from time step 0 to 5.

It's interesting to observe that without measurements and control input, our system's best guess of the robot's state remains 0 position and 0 velocity.

D One observes the positive correlation between velocity and position in figure 1.1. At t=0, the uncertainty ellipse is simply the dot at the car's initial state with 0 position and velocity. At t=1 velocity and position are perfectly correlated due to the deterministic way the acceleration is affecting both position and velocity.

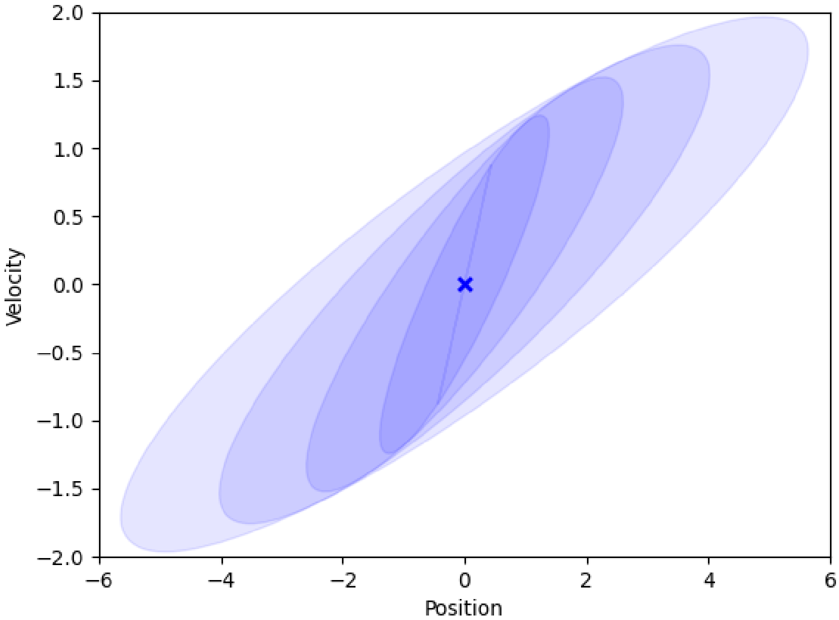

Figure 1.1: Uncertainty ellipses for the time steps 0 to 5. Each uncertainty ellipse contains 68% of the probability mass.

E The linear correlation coefficient between the velocity and position converges to 0.87, see figure 1.2.

2 Measurement Update

A The measurement model of a Kalman filter is a linear function of the state:

$$z_t = Cx_t + \delta \tag{2.1}$$

In this case $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $\delta \sim \mathcal{N}(0, 10)$.

B Thanks to the measurement $z_5 = 5$, the uncertainty of the state at time step 5 could be reduced, which can be observed by the smaller uncertainty ellipse in figure 2.1 and the smaller variances in the covariance matrices of table 2.1.

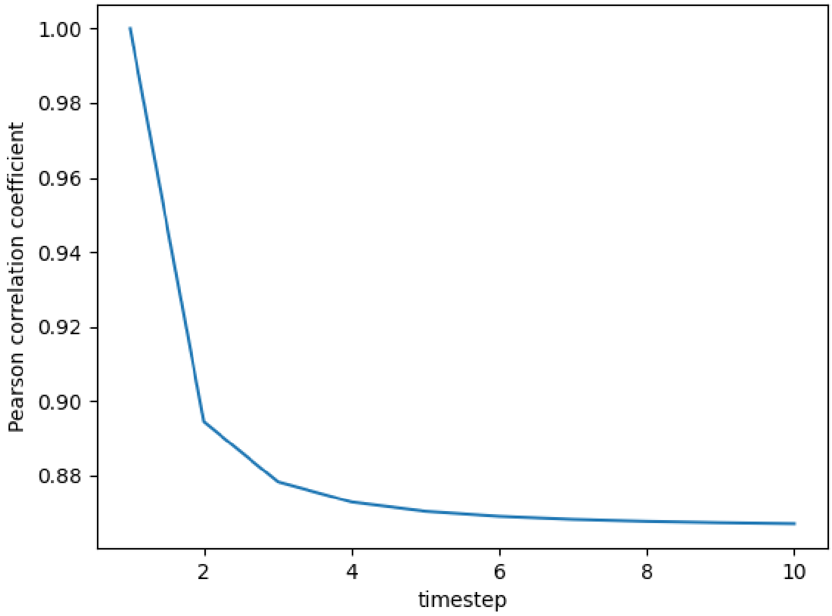

Figure 1.2: Plot of the correlation coefficient over 10 time steps.

State	Gaussian
Before	$\mu = \begin{bmatrix} 0 & 0 \end{bmatrix}$, $\quad \text{cov} = \begin{bmatrix} 41.25 & 12.5 \\ 12.5 & 5 \end{bmatrix}$
After	$\mu = \begin{bmatrix} 4.02 & 1.22 \end{bmatrix}$, $\quad \text{cov} = \begin{bmatrix} 8.05 & 2.44 \\ 2.44 & 1.95 \end{bmatrix}$

Table 2.1: Belief from time step 5 before and after incorporating the measurement z_5.

3 Prediction Step Derivation using Transforms

This exercise is about specializing the prediction step of the Bayes filter for Gaussian beliefs and linear state transition functions. It is to show that the equation 3.1 is Gaussian and that the update rules of equation 3.2 hold. Z-/Fourier-Transforms should be used to solve the integral.

$$p(x_t|z_{1:t-1}, u_{1:t}) = \int p(x_t|x_{t-1}, u_t)\, p(x_{t-1}|z_{1:t-1}, u_{1:t-1})\, dx_{t-1} \tag{3.1}$$

$$\begin{aligned} \mu_t &= A\mu_{t-1} + Bu_t \\ \Sigma_t &= A\Sigma_{t-1}A^\top + R \end{aligned} \tag{3.2}$$

For a linear state transition function in x_{t-1} and u_t with additive Gaussian noise $\epsilon \sim$

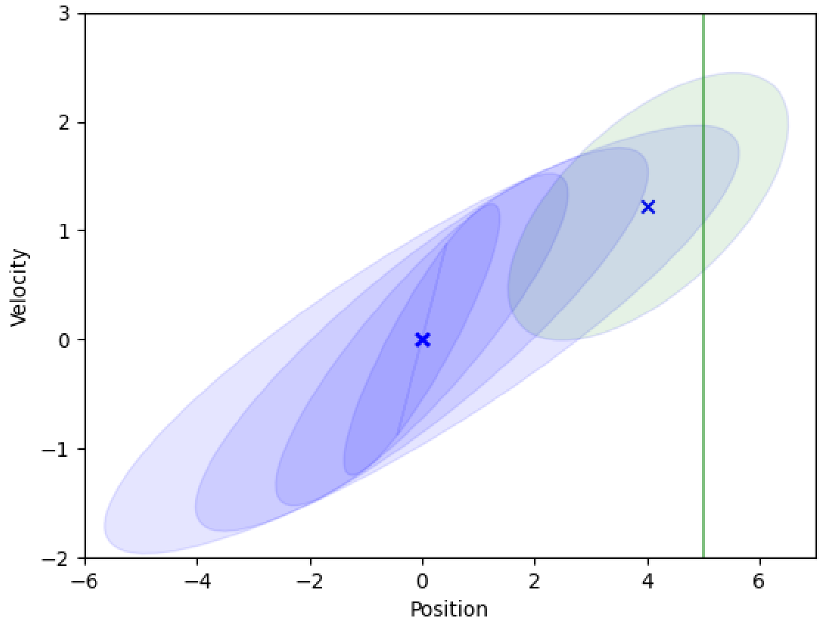

Figure 2.1: Uncertainty ellipses for the time steps 0 to 5. The green ellipse is the uncertainty ellipse at time step 5 after incorporating the measurement $z_5 = 5$, which is represented by the green line. Each uncertainty ellipse contains 68% of the probability mass.

$\mathcal{N}(0, R)$, $x_t = Ax_{t-1} + Bu_t + \epsilon$, the state transition probability $p(x_t|x_{t-1}, u_t)$ is a Gaussian with the following parameters $x_t \sim \mathcal{N}(Ax_{t-1} + Bu_t, R)$. The integral can be solved using the convolution theorem 3.3.

$$\{u * v\}(x) = \mathcal{F}^{-1}\{U \cdot V\} \tag{3.3}$$

In our case, we have the following time domain functions 3.4.

$$\begin{aligned} u(x) &= \eta_1 \exp\left(-\frac{1}{2}(x - Bu_t)^\top R^{-1}(x - Bu_t)\right) \\ v(x) &= \eta_2 \exp\left(-\frac{1}{2}(x - A\mu_{t-1})^\top (A\Sigma_{t-1}A^\top)^{-1}(x - A\mu_{t-1})\right) \end{aligned} \tag{3.4}$$

Without proof, I am claiming the Fourier transform of a multivariate Gaussian to be $F(k) = \exp\left(-2\pi i k^\top \mu - \frac{1}{2}k^\top \Sigma k\right)$. Using the convolution theorem, we confirm that $p(x_t|z_{1:t-1}, u_{1:t})$ is indeed a Gaussian with the aforementioned assumptions of the Kalman filter and that the update rules in equations 3.2 are correct:

$$
\begin{aligned}
&\mathcal{F}^{-1}(U \cdot V) \\
=&\mathcal{F}^{-1}\left(\exp\left(-2\pi i k^\top B u_t - \frac{1}{2}k^\top R k\right)\exp\left(-2\pi i k^\top A\mu_{t-1} - \frac{1}{2}k^\top A\Sigma_{t-1}A^\top k\right)\right) \\
=&\mathcal{F}^{-1}\left(\exp\left(-2\pi i k^\top (A\mu_{t-1} + Bu_t) - \frac{1}{2}k^\top (A\Sigma_{t-1}A^\top + R)k\right)\right) \\
=&\frac{1}{2\pi|A\Sigma_{t-1}A^\top + R|^{\frac{1}{2}}} \\
&\exp\left(-\frac{1}{2}(x_t - A\mu_{t-1} - Bu_t)^\top (A\Sigma_{t-1}A^\top + R)^{-1}(x_t - A\mu_{t-1} - Bu_t)\right)
\end{aligned}
\tag{3.5}
$$

4 Extended Kalman Filter

A Given the high uncertainty regarding the initial heading and that the robot moves one unit at the initial time step, the robot should likely end up somewhere close to the unit circle. Gaussian filters are however restricted to representing the belief as a Gaussian. A reasonable Gaussian belief after one prediction step should have a mean close to the origin, and a high position uncertainty while keeping the original heading uncertainty. The EKF prediction step updates the mean of the belief by propagating it through the nonlinear state transition function g: $\mu_t = g(\mu_{t-1}, u_t)$. This approach makes sure that the likeliest state is being taken as the new mean. It however disregards all other less likely outcomes of the state x_t resulting in a poor estimation of μ_t. The estimation of the covariance matrix is problematic as well. Given the system's nature, the uncertainty of the x and y coordinates should rise by a similar amount. The EKF prediction step, however, by only considering the likeliest state $x_{t-1} = \mu_{t-1}$, assigns the amount of uncertainty induced by the unknown heading solely to the y coordinate. This case exemplifies the EKF's weakness in dealing with high uncertainty and highly non-linear state transition functions. Figure 4.1 visualizes the initial uncertainty ellipse of the position and the uncertainty ellipse after the prediction step. The yellow unit circle visualizes the realistic locations of the robot.

B To perform the prediction step the Jacobian of the state transition function g needs to be calculated:

$$
\frac{\delta g(x,u)}{\delta x} = G(x,u) = \begin{bmatrix} 1 & 0 & -sin(\theta) \\ 0 & 1 & cos(\theta) \\ 0 & 0 & 1 \end{bmatrix} \tag{4.1}
$$

The initial belief and the belief after the prediction step are depicted in table 4.1.

C Refer to the previous task in paragraph A of this section.

D Intuitively, after considering the quality measurement of the x coordinate of the robot, the y coordinate can be determined to be $\pm a$ ensuring that (x,y) lies on the unit circle.

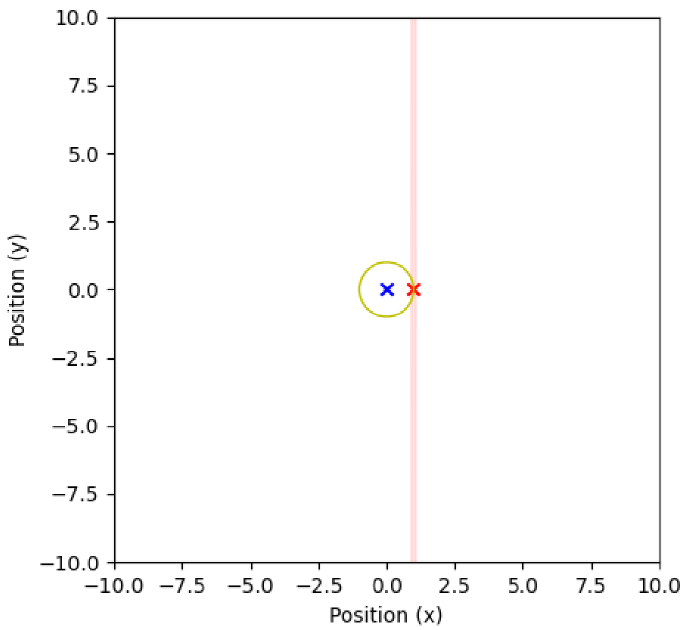

Figure 4.1: Uncertainty ellipses for the initial state and the state after performing one prediction step. The yellow unit circle visualizes the realistic locations of the robot after the prediction step. Each uncertainty ellipse contains 68% of the probability mass.

The EKF filter with its unimodal belief representation fails to estimate a realistic dual-modal state description. Additionally, since the uncertainty of the x coordinate after the prediction step is low, the measurement step could not move the mean of the final belief close to the realistic position of the robot. The measurement, intuitive realistic locations of the robot and the uncertainty ellipses of the beliefs predicted by the EKF are represented in figure 4.2. Table 4.2 shows the progression of the Gaussian belief parameters.

E The difference between the intuitive estimate of the posterior and the posterior estimated by the EKF is significant, see 4.2. The EKF's belief after the prediction step is poor mainly because of two reasons. First, the linearization of g turned out to be a poor estimate of g since this function is highly nonlinear with respect to θ. Second, a high uncertainty was associated with the state variable θ. The prediction step resulted in a state that could be deemed unrecoverable even with the quality measurements of the x value. The y and θ value will forever be associated with a high uncertainty since the covariance between x and y/θ remains 0 for all future update steps.

In the case that in the initial state y is unknown but θ is not, the EKF performs reasonably since g is linear with respect to y. The uncertainty around the y state will forever remain as the measurement don't provide any additional information about the absolute state.

State	Gaussian
Init	$\mu = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \quad \text{cov} = \begin{bmatrix} 0.01 & 0.00 & 0.00 \\ 0.00 & 0.01 & 0.00 \\ 0.00 & 0.00 & 10000.00 \end{bmatrix}$
Predicted	$\mu = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad \text{cov} = \begin{bmatrix} 0.01 & 0.00 & 0.00 \\ 0.00 & 10000.01 & 10000.00 \\ 0.00 & 10000.00 & 10000.00 \end{bmatrix}$

Table 4.1: Initial belief and belief after the prediction step.

State	Gaussian
Init	$\mu = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \quad \text{cov} = \begin{bmatrix} 0.01 & 0.00 & 0.00 \\ 0.00 & 0.01 & 0.00 \\ 0.00 & 0.00 & 10000.00 \end{bmatrix}$
Predicted	$\mu = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad \text{cov} = \begin{bmatrix} 0.01 & 0.00 & 0.00 \\ 0.00 & 10000.01 & 10000.00 \\ 0.00 & 10000.00 & 10000.00 \end{bmatrix}$
Corrected	$\mu = \begin{bmatrix} 0.15 & 0.00 & 0.00 \end{bmatrix}, \quad \text{cov} = \begin{bmatrix} 0.005 & 0.00 & 0.00 \\ 0.00 & 10000.01 & 10000.00 \\ 0.00 & 10000.00 & 10000.00 \end{bmatrix}$

Table 4.2: Initial belief, belief after the prediction step, and belief after the correction step.

5 Adding constant additive term to Kalman Filter

Constant additive terms in the motion and measurement models don't affect the calculation of the uncertainty. Line 2 and 5 in table 3.1 of the book change to:

$$\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t + c_{motion,t} \tag{5.1}$$

$$\mu_t = \bar{\mu}_t + K_t(z_t - c_{measurement,t} - C_t \bar{\mu}_t) \tag{5.2}$$

6 Information Matrix

Don't get this task. Other people on the internet don't seem to get this task either.

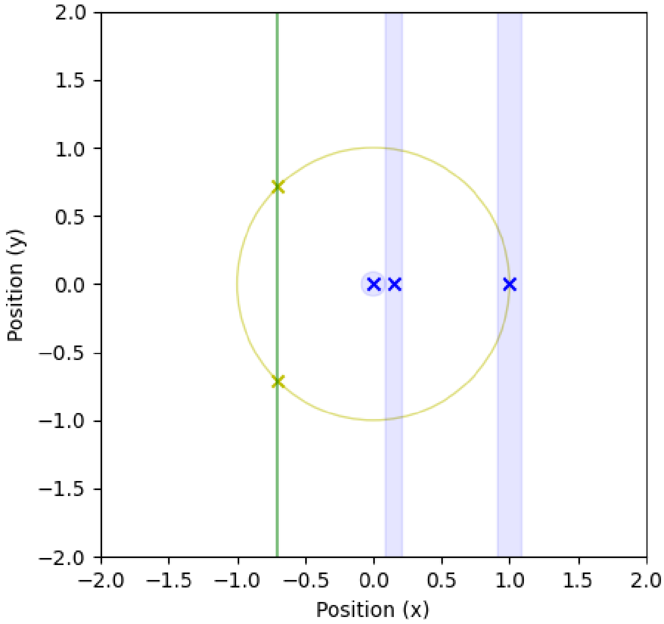

Figure 4.2: Uncertainty ellipses of the initial state (blue, leftmost), the uncertainty ellipse of the state after performing one prediction step (blue, rightmost), and the uncertainty ellipse of the state after performing the correction step with a measurement of $z_1 = -0.7$ (blue, middle). The yellow unit circle visualizes the realistic locations of the robot after the prediction step. The tow crosses indicate where the robot should intuitively be after considering the measurement. Each uncertainty ellipse contains 68% of the probability mass. In this example, the EKF has poor estimation performance.