Modelling Tail Risk using

Empirical Copulas

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Contents

Motivation

Pearson Correlation Coefficient

Does everything have linear dependence?

$$\rho = \begin{pmatrix} 1 & \rho_{12} & \dots & \rho_{1d} \\ \rho_{21} & 1 & \dots & \rho_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \dots & 1 \end{pmatrix}, \ \rho_{ij} = \frac{Cov(X_i, X_j)}{\sqrt{Var(X_i)Var(X_j)}}$$

Pitfalls

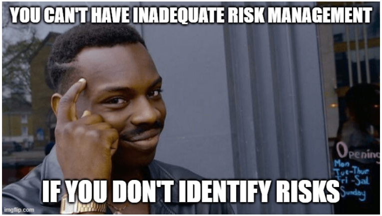
1. Non-existence if second moment is not defined
2. Violation of invariance property
3. Violates one-to-one correspondence between marginal distributions and correlation coefficient on the joint distribution
4. Uncorrelatedness $\nRightarrow$ Independence
5. ρ might not attain the maximal and minimal bound: $[-1, 1]$

Modelling Tail Risk – Underestimation?

1. Usage of the no-tail-dependent Gaussian copula indirectly led to the financial crisis in 2007 as securities were mispriced due to the underestimation of tail default risk.
2. "1-in-500" flood in Hong Kong: what does it actually mean?
3. Only in extreme events / scenarios, disaster occur: We are very unprepared for it!!

Empirical Copulas on upper and lower tail probabilities

1. Does empirical copulas systematically underestimates tail risk with respect to the true underlying copula?
2. Computing the up-tail and low-tail probability
3. **Small sample size** empirical copulas widely applied in modelling of dependent risk?



"True" Copulas Considered: A Brief Introduction

Copulas

Definition 1

Copula (Hofert et al., 2018)
A copula is a continuous d-dimensional distribution function $C(u_1, u_2, \dots, u_d)$ that satisfies the following properties:

1. Groundedness: $C(u_1, \dots, u_{i-1}, 0, u_{i+1}, \dots, u_d) = 0$ if any $u_i = 0$.
2. $\mathcal{U}(0,1)$ margins: $\forall i \in \{1, \dots, d\}$, $C(1, \dots, 1, u_i, 1, \dots, 1) = u_i$
3. d-increasing:
$$\boldsymbol{a}, \boldsymbol{b} \in [0,1]^d, \Delta_{(\boldsymbol{a},\boldsymbol{b}]}C = \sum_{i \in \{0,1\}^d} (-1)^{\sum_{j=1}^d i_j} C(a_1^{i_1} b_1^{1-i_1}, \dots, a_d^{i_d} b_d^{1-i_d}) \geq 0,$$
where $\Delta_{(\boldsymbol{a},\boldsymbol{b}]}C$ is the C-volume defined w.r.t hyperrectangle $(\boldsymbol{a}, \boldsymbol{b}]$.

Sklar's Theorem

Theorem 2

Sklar's Theorem (Sklar, 1959)

1. *Existence: For univariate margins* $F_1, F_2, \ldots, F_d \sim \mathcal{U}(0,1)$*, a d-dimensional distribution function* F*,* $\exists C : [0,1]^d \to [0,1]$ *such that* $F(\boldsymbol{x}) = C(F_1(x_1), \ldots, F_d(x_d))$*,* $\boldsymbol{x} \in \mathbb{R}^d$*, where* C *is uniquely defined on* $\prod_{j=1}^{d}$ *ran* F_j*.*
2. *Converse: Given any copula* $C : [0,1]^d \to [0,1]$ *and univariate margins* $F_1, F_2, \ldots, F_d$*,* $F(\boldsymbol{x}) = C(F_1(x_1), \ldots, F_d(x_d))$ *is a d-dimensional distribution function,* $\boldsymbol{x} \in \mathbb{R}^d$*.*

Invariance

Theorem 3

Invariance Property of Copulas
Let $\boldsymbol{X} \sim F$ with continuous univariate margins $F_1, F_2, \ldots, F_d$ and copula C. It follows that $T_1(X_1), T_2(X_2), \ldots, T_d(X_d)$ also has copula C, where $\forall j \in$\{1, ..., d\}, T_j is a strictly increasing function on ran X_j.

Proof:

$$\begin{aligned} P(T_1(X_1) \leq x_1, \ldots, T_d(X_d) \leq x_d) &= P(X_1 \leq T_1^{-1}(x_1), \ldots, X_d \leq T_d^{-1}(x_d)). \\ &= C(F_1(T_1^{-1}(x_1)), \ldots, F_d(T_d^{-1}(x_d))) \end{aligned}$$

We can also see that $\forall j \in \{1, \ldots, d\}$, $P(T_j(X_j) \leq x_j) = P(X_j \leq T_j^{-1}(x_j)) = F_j(T_j^{-1}(x_j))$ and the result follows.

Archimedean Copulas

Definition 4

Archimedean copula (Hofert et al., 2018)
$C(\boldsymbol{u})$ is an archimedean copula if it exhibits the following form:

$$C(\boldsymbol{u}) = \psi(\psi^{-1}(u_1) + \psi^{-1}(u_2) + \cdots + \psi^{-1}(u_d)), \quad \boldsymbol{u} \in [0,1]^d$$

where ψ is a generator function.

Definition 5

Generator Function (Hofert et al., 2018)
ψ is a generator function for an archimedean copula if it exhibits the following properties:

1. $\psi : [0, \infty] \to [0, 1]$ is continuous, strictly decreasing; $\psi(0) = 1$, $\psi(\infty) \to 0$
2. $\psi \in \Psi$, where Ψ is the class of all generators
3. $\forall k \in \{0, \ldots, d-2\}, (-1)^k \psi^{(k)}(t) \geq 0$ and $(-1)^{d-2}\psi^{(d-2)}(t)$ is nonincreasing and convex on $t \in (0, \infty)$ (d-monotonicity)

Elliptical Copula

Definition 6

Elliptical copula (Hofert et al., 2018)
$C(\boldsymbol{u})$ is an elliptical copula if it exhibits the following form:

$$C(\boldsymbol{u}) = H(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)), \quad \boldsymbol{u} \in \prod_{j=1}^{d} \operatorname{ran} F_j$$

where H is a multivariate elliptical distribution function, $F_1, \ldots, F_d$ are the univariate margins.

Elliptical Distribution

Definition 7

Elliptical distribution (Hofert et al., 2018)
$\boldsymbol{X}$ is an elliptical distribution if it exhibits the following form:

$$\boldsymbol{X} = \boldsymbol{\mu} + R\boldsymbol{A}\boldsymbol{S}$$

where $\boldsymbol{S} \sim \mathcal{U}(\{\boldsymbol{x} \in \mathbb{R}^k : \|\boldsymbol{x}\| = 1\})$ is uniformly distributed across the unit sphere of order k, $\boldsymbol{A}$ is the lower triangular square matrix with nonnegative entries such that $\boldsymbol{A}\boldsymbol{A}^T = \Sigma$, where Σ is the covariance matrix (Cholesky decomposition of Σ), and R $\geq$ 0 having a density:

$$f_R(r) = \frac{2\pi^{\frac{d}{2}} r^{d-1} g(r^2)}{\Gamma(\frac{d}{2})}$$

Common Archimedean Copulas

1. **Clayton copula** (Clayton, 1978)

$$C_\theta^C(\boldsymbol{u}) =$$

$$\begin{cases} \max\{u_1^{-\theta} + u_2^{-\theta} - 1, 0\}^{-1/\theta}, \ d = 2, \ \theta \in [-1, \infty)\backslash\{0\}, \ \boldsymbol{u} \in [0,1]^2 \\ (1 - d + \sum_{j=1}^d u_j^{-\theta})^{-1/\theta}, \ d \geq 3, \ \theta \in (0, \infty), \ \boldsymbol{u} \in [0,1]^d \end{cases}$$

where $\psi(t) = (1+t)^{-1/\theta}$

2. **Gumbel-Hougaard copula** (Gumbel, 1961; Hougaard, 1986)

$$C_\theta^{GH}(\boldsymbol{u}) = \exp\left(-\left(\sum_{j=1}^d (-\ln u_j)^\theta\right)^{1/\theta}\right), \quad \theta \in [1, \infty), \ \boldsymbol{u} \in [0,1]^d$$

where $\psi(t) = \exp(-t^{1/\theta})$

Common Elliptical Copulas

1. **Student-t copula**
   For correlation matrix Σ,

$$C^t(\boldsymbol{u}) = \int_0^{u_d} \cdots \int_0^{u_1} c^t(x_1, \ldots, x_d)\, \mathrm{d}x_1 \ldots \mathrm{d}x_d, \text{ where}$$

$$c^t(\boldsymbol{u}) = \frac{\Gamma(\frac{v+d}{2})}{\Gamma(\frac{v}{2})\sqrt{\det \Sigma}} \left(\frac{\Gamma(\frac{v}{2})}{\Gamma(\frac{v+1}{2})} \right)^d \frac{(1 + \frac{\boldsymbol{x}^T \Sigma^{-1} \boldsymbol{x}}{v})^{-\frac{v+d}{2}}}{\prod_{j=1}^d (1 + \frac{x_j^2}{v})^{-\frac{v+1}{2}}}, \; \boldsymbol{u} \in (0,1)^d$$

2. **Gaussian Copula**
   For correlation matrix Σ,

$$C^N(\boldsymbol{u}) = \int_{-\infty}^{\Phi^{-1}(u_d)} \cdots \int_{-\infty}^{\Phi^{-1}(u_1)} \frac{\exp\left(-\frac{1}{2}\boldsymbol{x}^T \Sigma^{-1} \boldsymbol{x}\right)}{(2\pi)^{\frac{d}{2}} \sqrt{\det \Sigma}}\, \mathrm{d}x_1 \ldots \mathrm{d}x_d, \quad \boldsymbol{u} \in (0,1)^d$$

Survival Copulas

Definition 8

Survival Copula
The survival copula is a multivariate survival function: $\bar{F}(\boldsymbol{x}) = P(\boldsymbol{X} > \boldsymbol{x}), \boldsymbol{x} \in \mathbb{R}$ with marginal distributions $F_j\bar{(}x_j) = \bar{H}(\infty, \ldots, x_j, \ldots, \infty) = P(X_j > x_j)$.

Theorem 9

Sklar's Theorem for Survival Copulas (Hofert et al., 2018)

1. *Existence: For univariate margins $\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_d$, a d-dimensional survival function $\bar{F}$, $\exists\, \bar{C} : [0,1]^d \to [0,1]$ such that $\bar{F}(\boldsymbol{x}) = \bar{C}(\bar{F}_1(x_1), \ldots, \bar{F}_d(x_d))$, $\boldsymbol{x} \in \mathbb{R}^d$, where C is uniquely defined on $\prod_{j=1}^{d} \operatorname{ran} \bar{F}_j$.*
2. *Converse: Given any survival copula $\bar{C} : [0,1]^d \to [0,1]$ and univariate margins $\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_d$, $\bar{F}(\boldsymbol{x}) = \bar{C}(\bar{F}_1(x_1), \ldots, \bar{F}_d(x_d))$ is a d-dimensional survival function, $\boldsymbol{x} \in \mathbb{R}^d$.*

Recovering Survival Copulas from the original Copula

Theorem 10

Properties of Survival Copulas (Hofert et al., 2018)
For $\boldsymbol{U} \sim C$*:*

1. $\mathbf{1} - \boldsymbol{U} \sim \bar{C}$
2. C *can be recovered by* $\bar{C}$ *by:*

$$\bar{C}(\boldsymbol{u}) = \sum_{J \subseteq \{1,\ldots,d\}} (-1)^{|J|} C\left((1-u_1)^{\mathbb{1}(1 \in J)}, \ldots, (1-u_d)^{\mathbb{1}(d \in J)}\right), \ \ \boldsymbol{u} \in [0,1]^d$$

Definition 11

Rotated Copulas (Hofert et al., 2018)
If $\boldsymbol{U} \sim C$, then we can define rotated copula of C $\mathrm{rot}_{\boldsymbol{r}}(C)$ by:

$$((1-r_1)U_1 + r_1(1-U_1), \ldots, (1-r_d)U_d + r_d(1-U_d)) \sim \mathrm{rot}_{\boldsymbol{r}}(C)$$

It can be observed that $\bar{C} = \mathrm{rot}_{\mathbf{1}}(C)$, where $\mathbf{1} \in 1^d$.

Empirical Copulas

Empirical copula

Definition 12

Empirical copula (Deheuvels, 1979; Segers et al., 2017)
Consider i.i.d random vectors $\boldsymbol{X}_i = (X_{i,1}, \ldots, X_{i,d})$ for $i = \{1, \ldots, n\}$. Denote $R_{i,j}$ be the rank of $X_{i,j}$ within $X_{1,j}, \ldots, X_{n,j}$.
Consider an empirical distributions:

$$\hat{F}_{n,j}(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{X_{i,j} \leq x\}.$$

Substitute x with $X_{i,j}$ with the reasoning that $\hat{F}_{n,j}(X_{i,j}) \sim \mathcal{U}(0,1)$ asymptotically then

$$\hat{F}_{n,j}(X_{k,j}) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{X_{i,j} \leq X_{k,j}\} = \frac{R_{k,j}}{n}.$$

Empirical copula (continued)

Definition 13

Empirical copula (continued) (Deheuvels, 1979; Segers et al., 2017)
As $X_{i,1}, \ldots, X_{i,d}$ are i.i.d random vectors, the empirical copula is the empirical distribution function of $(\hat{F}_{n,1}(X_{i,1}), \ldots, \hat{F}_{n,d}(X_{i,d}))$ given by

$$C_n(\boldsymbol{u}) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\left\{\frac{\boldsymbol{R}_i}{n} \leq \boldsymbol{u}\right\} = \frac{1}{n}\sum_{i=1}^{n}\prod_{j=1}^{d} \mathbb{1}\left\{\frac{R_{i,j}}{n} \leq u_j\right\}.$$

Definition 14

Survival function of the empirical copula
The proxy survival function for the empirical copula is defined as:

$$\bar{H}(\boldsymbol{u}) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\left\{\frac{\boldsymbol{R}_i}{n} \geq \boldsymbol{u}\right\} = \frac{1}{n}\sum_{i=1}^{n}\prod_{j=1}^{d} \mathbb{1}\left\{\frac{R_{i,j}}{n} \geq u_j\right\}.$$

Empirical beta copula

Definition 15

(Segers et al., 2017) **Empirical Beta Copula**
The empirical beta copula is defined by:

$$C_m^{\beta}(\boldsymbol{u}) = \frac{1}{n}\sum_{i=1}^{n}\prod_{j=1}^{d} F_{n,R_{i,j}}(u_j), \quad \boldsymbol{u} \in [0,1]^d$$

where $F_{n,r}$ is the cumulative distribution function of the beta distribution with shape parameters r and $n+1-r$ respectively, i.e. $\mathcal{B}(r,\, n+1-r)$.

Definition 16

Survival function of the empirical beta copula
The survival function of the empirical beta copula is defined by:

$$\bar{H}^{\beta}(\boldsymbol{u}) = \frac{1}{n}\sum_{i=1}^{n}\prod_{j=1}^{d} P(V_{R_{i,j},j} \geq u_j) = \frac{1}{n}\sum_{i=1}^{n}\prod_{j=1}^{d}(1 - F_{n,R_{i,j}}(u_j)),\ \boldsymbol{u} \in [0,1]^d$$

Empirical checkerboard copula

Definition 17

Empirical checkerboard copula
The empirical checkerboard copula is defined by

$$C_n^{\boxplus}(\boldsymbol{u}) = \frac{1}{n}\sum_{i=1}^{n}\prod_{j=1}^{d}\min\{\max\{nu_j - R_{i,j} + 1, 0\}, 1\}.$$

EBC-adapted empirical copulas with smoothed survival margins

1. Binomial survival margins

$$C_n^v(\boldsymbol{u}) = \frac{1}{n}\sum_{i=1}^{n} C_n^\beta \left[\bar{B}_{n,u_1}^{\boldsymbol{X}}\left(R_{i,1}-1\right), \ldots, \bar{B}_{n,u_d}^{\boldsymbol{X}}\left(R_{i,d}-1\right)\right], \quad \boldsymbol{u} \in [0,1]^d$$

2. Beta-binomial survival margins with $\rho = 4$

$$C_n^v(\boldsymbol{u}) = \frac{1}{n}\sum_{i=1}^{n} C_n^\beta \left[\overline{BB}_{u_1}^{\boldsymbol{X}}\left(R_{i,1}-1\right), \ldots, \overline{BB}_{u_d}^{\boldsymbol{X}}\left(R_{i,d}-1\right)\right], \quad \boldsymbol{u} \in [0,1]^d$$

3. Beta survival margins with $\rho = 4$

$$C_n^v(\boldsymbol{u}) = \frac{1}{n}\sum_{i=1}^{n} C_n^\beta \left[\bar{\mathcal{B}}_{u_1}^{\boldsymbol{X}}\left(\frac{R_{i,1}-0.5}{n}\right), \ldots, \bar{\mathcal{B}}_{u_d}^{\boldsymbol{X}}\left(\frac{R_{i,d}-0.5}{n}\right)\right], \quad \boldsymbol{u} \in [0,1]^d$$

Rank Correlation Coefficient

Kendall's Tau

Definition 18

Kendall's Tau (Kendall, 1938; Hofert et al., 2018)
For a bivariate random vector (X, Y), where $X \sim F$ and $Y \sim G$, and a vector $(X^{'}, Y^{'})$ which is i.i.d. with (X, Y),

$$\tau = \tau(X, Y) = \mathbb{E}[\mathrm{sign}((X - X^{'})(Y - Y^{'}))]$$

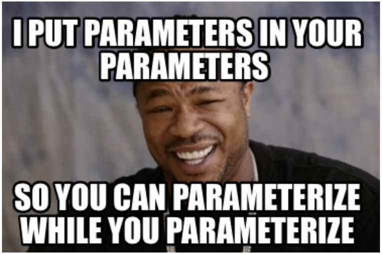
$$\text{Where } \mathrm{sign}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

We will do Kendall's tau matching (to ensure comparability).

Simulation

Parameters

1. $n = 125$, $B = 30$
2. $d \in \{2, 3, 4, 5\}$
3. $\tau \in \{-0.75, -0.5, -0.25, 0, 0.25, 0.5, 0.75\}$
4. $u \in [0.95, \ldots, 1]$ for upper tail, $u \in [0, \ldots, 0.05]$ for lower tail
5. Some copulas do not have certain combinations of parameters (sadly...)



Algorithm

Algorithm Probabilities of empirical copulas

procedure PROBABILITY(copula) ▷ Exogenously inputted.
    $n \leftarrow 125$
    $B \leftarrow 30$
    $d \leftarrow$ dim(copula)
    $\boldsymbol{u\_mat} \leftarrow \boldsymbol{u} = (\boldsymbol{u}_{\cdot,1}, \boldsymbol{u}_{\cdot,2}, \ldots, \boldsymbol{u}_{\cdot,d})$ ▷ $\boldsymbol{u_i}$: identical evaluation vector
    true_res_mat $\leftarrow C(\boldsymbol{u\_mat})$ ▷ obtain E / C probability for true copula
    for b in 1:B **do**
        ec_res_mat(b) $\leftarrow$ FunctionEC($\boldsymbol{u\_mat}$, $\boldsymbol{U}^{n,d}$)
    end for
    mean_ec_res_mat $\leftarrow \frac{1}{b}\sum_{i=1}^{b}$ ec_res_mat(i)
    CI_ec_res_mat $\leftarrow$ 2.5% and 97.5% quantile of ec_res_mat
    return true_res_mat, mean_ec_res_mat, CI_ec_res_mat,
end procedure

Cramer-von-Mises Statistic

Definition 19

Cramer-von-Mises statistic (Hofert et al., 2018; Genest et al., 2009)

$$S_n^{CvM} = \int_{[0,1]^d} n(C_n(\boldsymbol{u}) - C(\boldsymbol{u}))^2 \, \mathrm{d}C_n(\boldsymbol{u}).$$

Definition 20

Discretized Cramer-von-Mises statistic (Hofert et al., 2018; Genest et al., 2009)

$$S_n^{CvM} = \sum_{i=1}^{n} (C_n(\boldsymbol{u}_i) - C(\boldsymbol{u}_i))^2$$

Exceedance Probability (1)

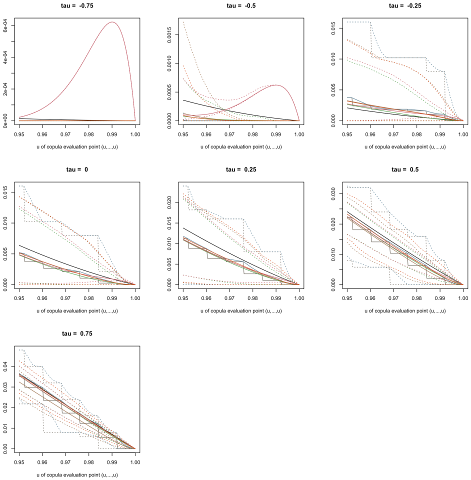


Figure: Student-t copula with $d = 2$

Exceedance Probability (2)

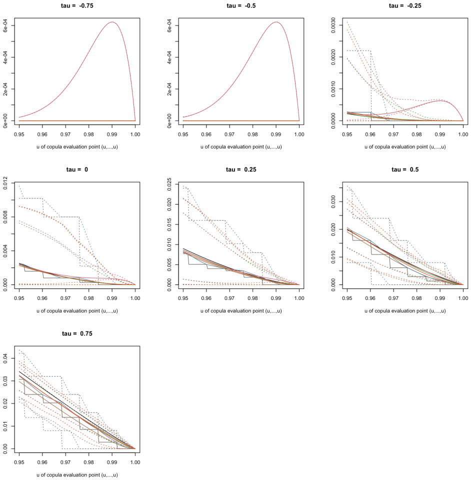


Figure: Gaussian copula with $d = 2$

Exceedance Probability (3)

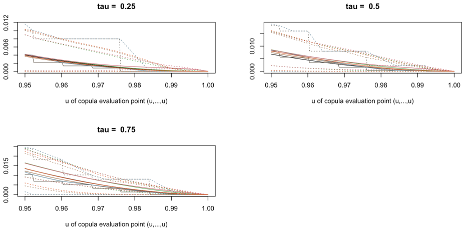
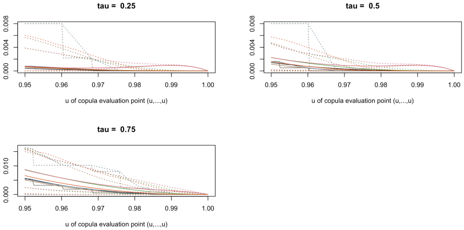


Figure: Clayton copula with $d = 2$, 3



Exceedance Probability (4)

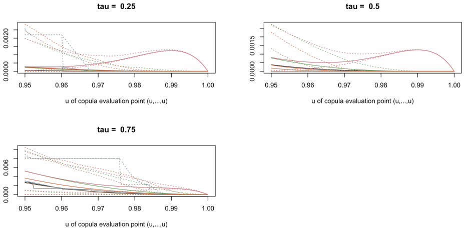
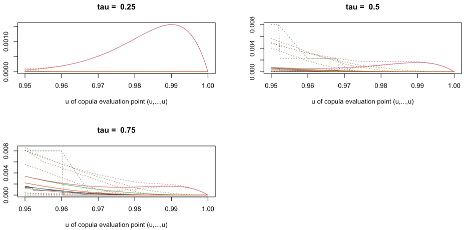


Figure: Clayton copula with $d = 4, 5$



Exceedance Probability (5)

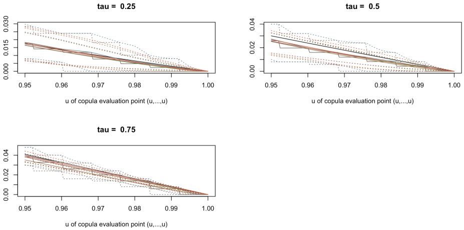
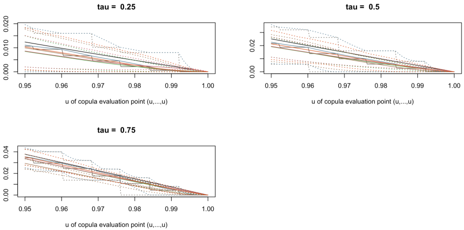


Figure: Gumbel-Hougaard copula with $d = 2, 3$



Exceedance Probability (6)

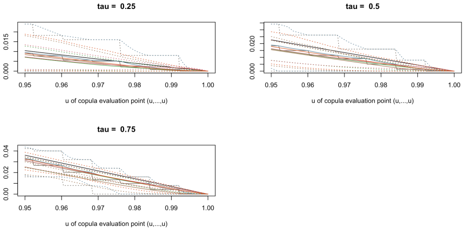
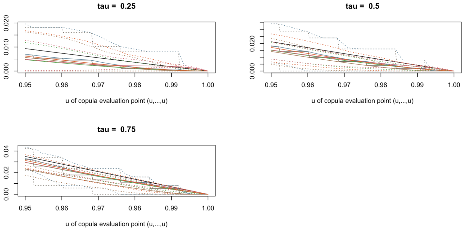


Figure: Gumbel-Hougaard copula with $d = 4$, 5



Cumulative Probability (1)

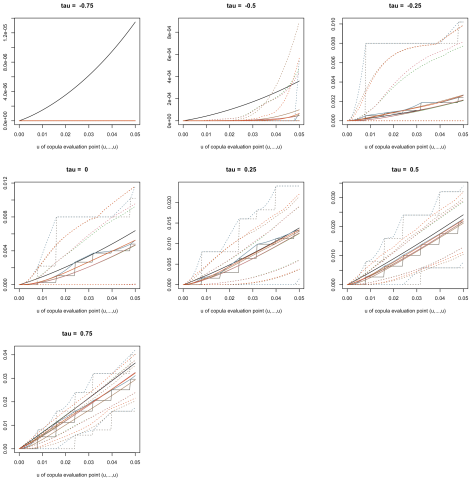


Figure: Student-t copula with $d = 2$

Cumulative Probability (2)

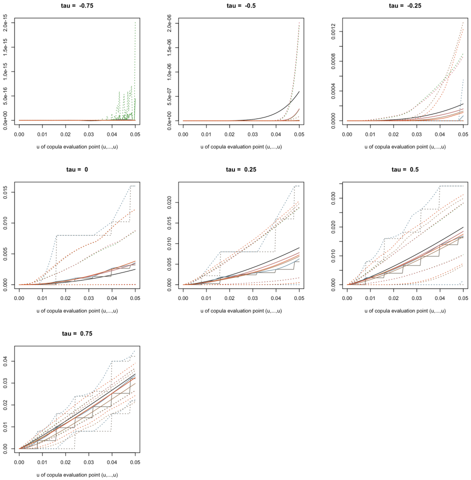


Figure: Gaussian copula with $d = 2$

Cumulative Probability (3)

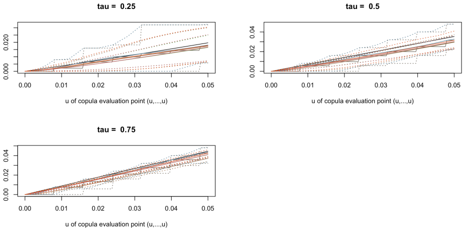
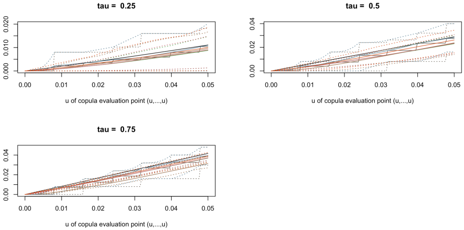


Figure: Clayton copula with $d = 2$, 3



Cumulative Probability (4)

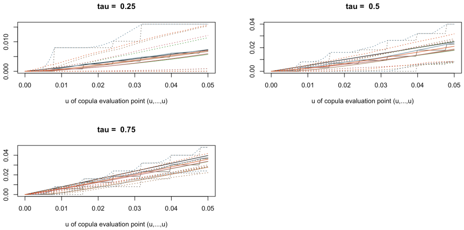
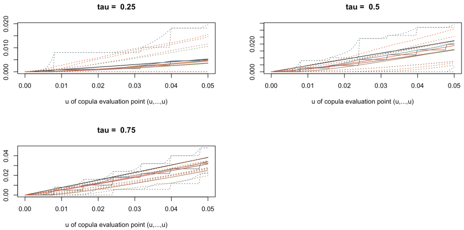


Figure: Clayton copula with $d = 4$, 5



Cumulative Probability (5)

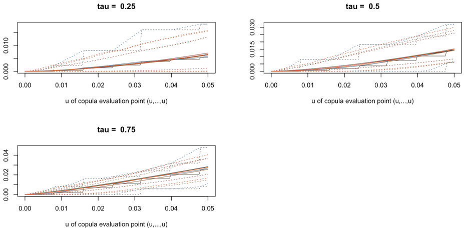
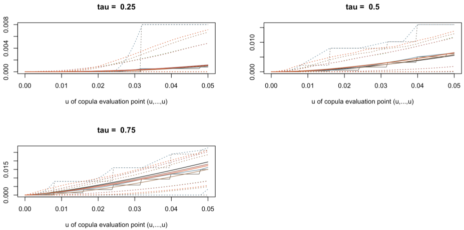


Figure: Gumbel-Hougaard copula with $d = 2, 3$



Cumulative Probability (6)

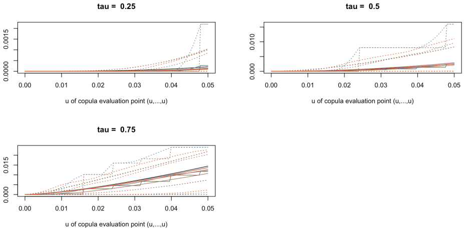
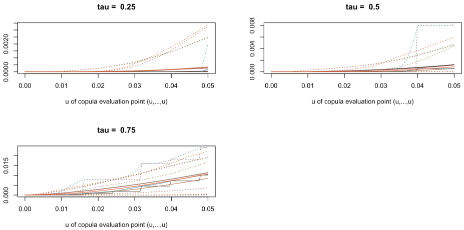


Figure: Gumbel-Hougaard copula with $d = 4$, 5



Exceedance CvM (1)

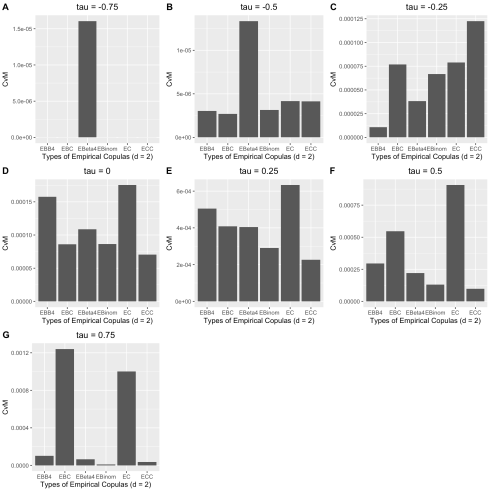


Figure: Student-t copula with $d = 2$

Exceedance CvM (2)

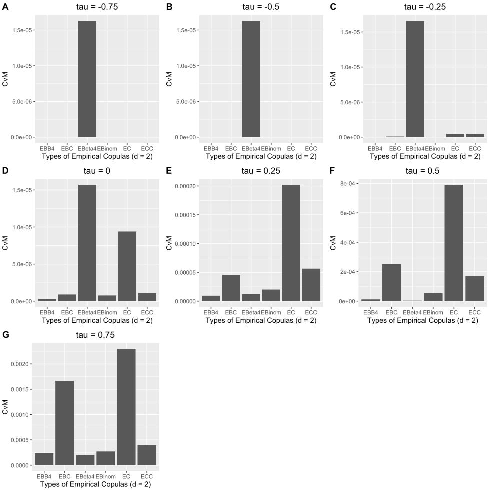


Figure: Gaussian copula with $d = 2$

Exceedance CvM (3)

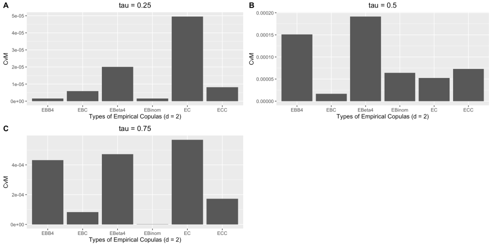
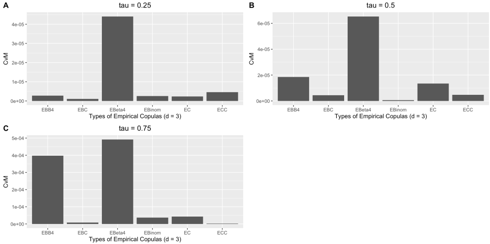


Figure: Clayton copula with $d = 2, 3$



Exceedance CvM (4)

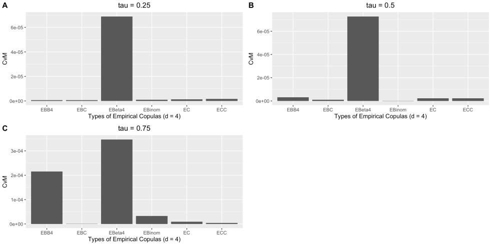
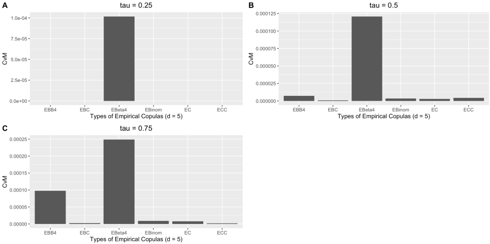


Figure: Clayton copula with $d = 4, 5$



Exceedance CvM (5)

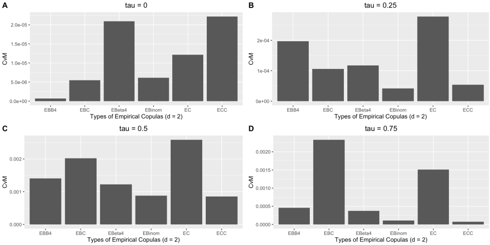
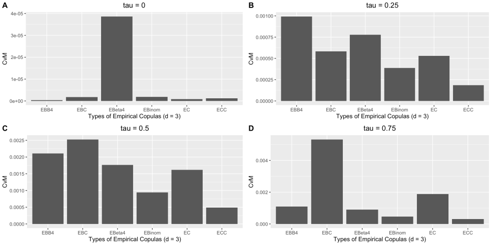


Figure: Gumbel-Hougaard copula with $d = 2, 3$



Exceedance CvM (6)

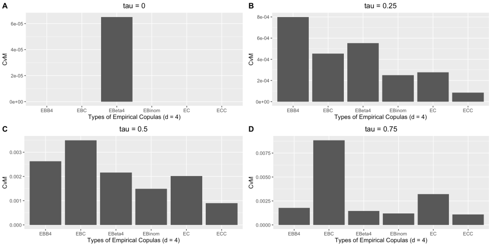
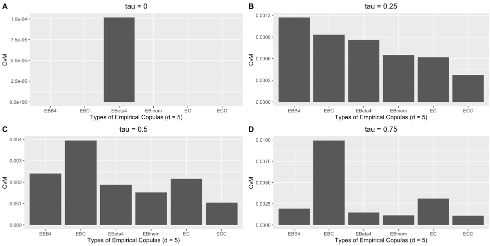


Figure: Gumbel-Hougaard copula with $d = 4$, 5



Cumulative CvM (1)

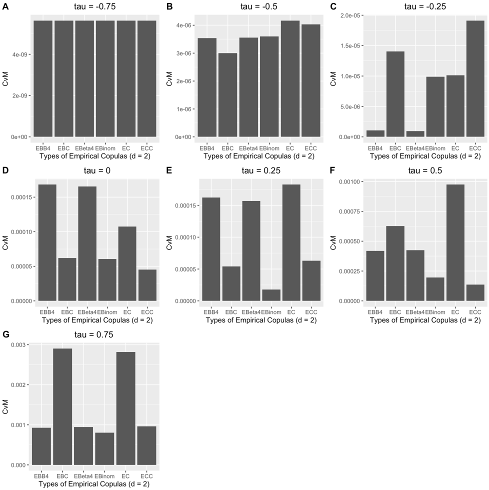


Figure: Student-t copula with $d = 2$

Cumulative CvM (2)

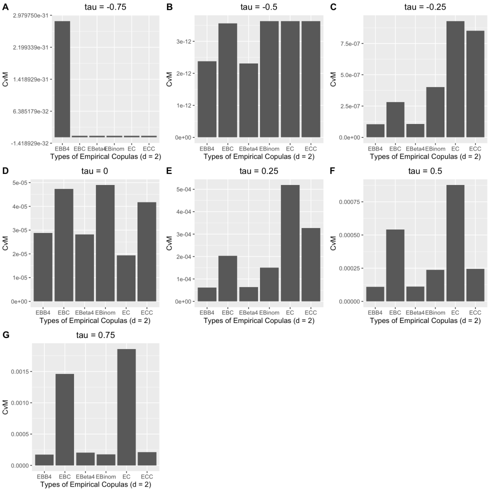


Figure: Gaussian copula with $d = 2$

Cumulative CvM (3)

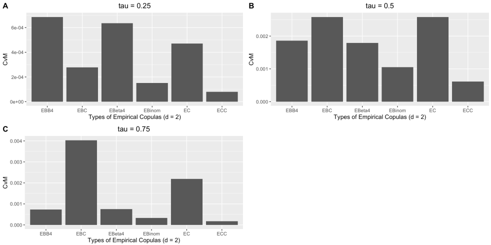
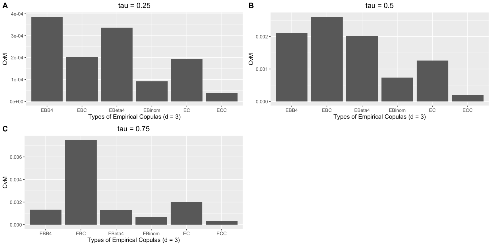


Figure: Clayton copula with $d = 2$, 3



Cumulative CvM (4)

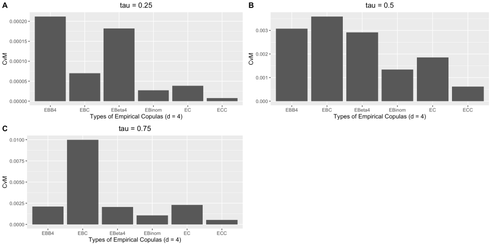
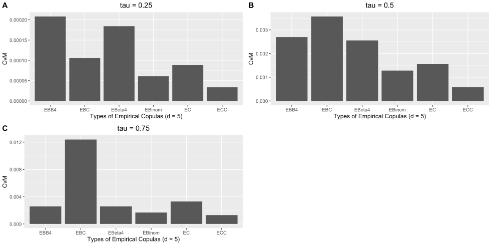


Figure: Clayton copula with $d = 4$, 5



Cumulative CvM (5)

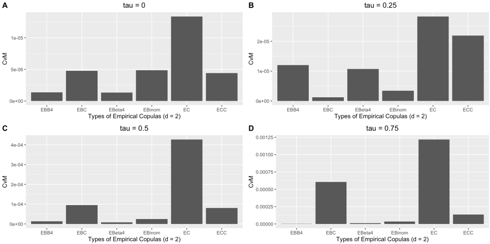
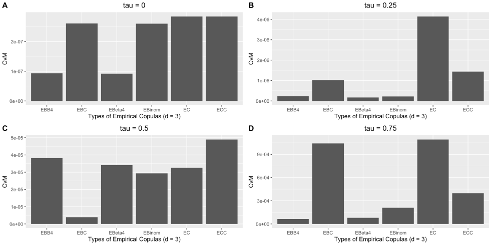


Figure: Gumbel-Hougaard copula with $d = 2$, 3



Cumulative CvM (6)

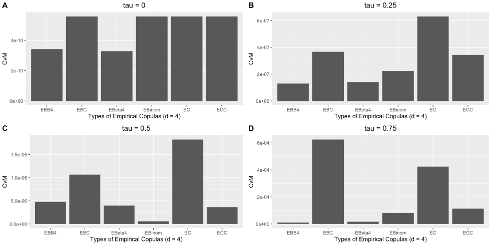
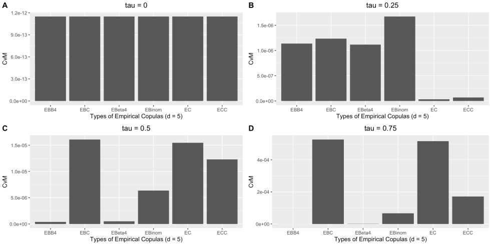


Figure: Gumbel-Hougaard copula with $d = 4$, 5



Understanding the result - **Exceedance probability**

1. **Lower Kendall's tau values: Extremely bad predictive performance**
   - Surprising predictive performance of EBC-adapted empirical copula with beta-binomial survival margins
2. **Erratic behaviour from smoothed EBC-adapted empirical copula with beta survival margins**
   - Due to beta distribution being heavy-right-tailed at small α and large β
3. **Systematic underestimation of the upper-tail at positive Kendall's τ**
   - Low bias: EBC-adapted empirical copula with binomial survival margins + Empirical checkerboard copula
   - Empirical checkerboard copula have a higher variance!
4. **Comparison between the smoothed empirical copulas proposed by Kojadinovic and Yi 2024 and the classical empirical copulas**
   - Mostly outperform the classical empirical copulas in higher values of Kendall's τ for bias, and strictly outperform the classical empirical copulas in terms of variance.

Understanding the result - **Cumulative probability**

1. **Lower Kendall's tau values: Extremely bad predictive performance**
   - One combination yields a joint probability lower than the double precision machine accuracy!
   - For closer-to-0 negative τ values, EBC-adapted empirical copula with beta + beta-binomial survival margins outperform the other empirical copulas
2. **Systematic underestimation of the lower-tail at positive Kendall's τ**
   - Low bias: EBC-adapted empirical copula with binomial survival margins + Empirical checkerboard copula
   - Empirical checkerboard copula have a higher variance! (similar to exceedance probabilities)
3. **Comparison between the smoothed empirical copulas proposed by Kojadinovic and Yi 2024 and the classical empirical copulas**
   - Similar to exceedance probabilities

Recommendations

Modelling exceedance probabilities at upper-tail:

1. $\tau \leq -0.5$: N/A
2. $\tau = -0.25$: EBC-adapted empirical copula with beta-binomial survival margins
3. $\tau \geq 0$: EBC-adapted empirical copula with binomial survival margins

Modelling cumulative probabilities at lower-tail:

1. $\tau \leq -0.5$: N/A
2. $\tau = -0.25$: EBC-adapted empirical copula with beta or beta-binomial survival margins
3. $\tau \geq 0$: EBC-adapted empirical copula with binomial survival margins

Conclusion

Possible improvements

Algorithm optimization:

1. More replications small sample size!
   - Data are usually not available, especially more extreme values!
2. Analytical form of survival functions of the empirical copulas
   - $d \to 2^d$ empirical copulas needed to create the survival function: computationally prohibitive!

Systematic underestimation:

1. Upward adjustments to counter the underestimations
   - Try to understand whether there is a systematic trend on the magnitude of underestimations

Thank you!

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