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Reflection and Refraction of Plane Waves

22.1 Introduction

In reality, plane electromagnetic waves frequently encounter obstacles along their propagation paths: hills, buildings, metallic antennas aimed at receiving the messages the waves carry, objects from which they are supposed to partly reflect (as when the wave is a radar beam), and so on. In such cases, the wave induces conduction currents in the object (if the object is metallic), or polarization currents (if the object is made of an insulator). These currents are, of course, sources of a secondary electromagnetic field. This field is known as the *scattered field*, and the process that creates it is known as *scattering of electromagnetic waves*. The objects, or obstacles, are called *scatterers*.

The determination of scattered fields is a difficult problem even in the case of simple scatterers, and can rarely be solved analytically. Numerical analysis offers various solutions. There is one class of problems, however, for which the determination of the scattered field is remarkably simple. When a *plane* electromagnetic wave is incident on a *planar* boundary between two *homogeneous* media, the scattered waves are also plane waves. One of these waves is radiated back into the half-space of the incident wave: this wave is known as the *reflected wave*. There is also a wave in the other half-space (except in the case of a perfect conductor), propagating generally

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in a different direction from the incident wave; it is therefore called the *refracted or transmitted wave.*

Naturally, the described geometry is an idealized one. Nevertheless, the results we will arrive at have great practical importance because many real problems can be solved in this manner with sufficient accuracy.

22.2   Plane Waves Normally Incident on a Perfectly Conducting Plane

The simplest case of wave reflection is when a uniform plane wave is incident normally on the planar interface between a perfect dielectric, of parameters ϵ and μ, and a perfect conductor. Let the interface be at $z = 0$, and let the wave of angular frequency ω have E_x and H_y components, as indicated in Fig. 22.1. We wish to determine the resulting wave for $z \leq 0$. (We know that inside the perfect conductor there is no field.)

The physics of wave scattering in this case is fairly obvious. The incident wave induces currents and charges only on the surface of the perfect conductor. (For a perfect conductor, the skin depth is infinitely small.) Since inside the conductor there is no field, we can consider this layer of currents and charges to exist in a homogeneous dielectric of parameters ϵ and μ. The distribution of these sources must be such that their field exactly cancels the incident field inside the conductor (that is, for $z > 0$). So we know that the scattered field for $z > 0$ is exactly the same in amplitude as the incident field, but is π out of phase. The current sheet obviously produces a symmetrical field in the half-space $z < 0$, that is, a plane wave propagating back in the $-z$ direction. This reradiated wave is the reflected wave. From this reasoning, the reflected wave has the same amplitude as the incident wave. At $z = 0$, its E-field vector is the same as that of the incident field, but in the opposite direction.

To put these conclusions into equations, let the incident wave (in phasor form) be represented by

$$\mathbf{E}_i(z) = Ee^{-j\beta z}\mathbf{u}_x \qquad \mathbf{H}_i(z) = He^{-j\beta z}\mathbf{u}_y, \tag{22.1}$$
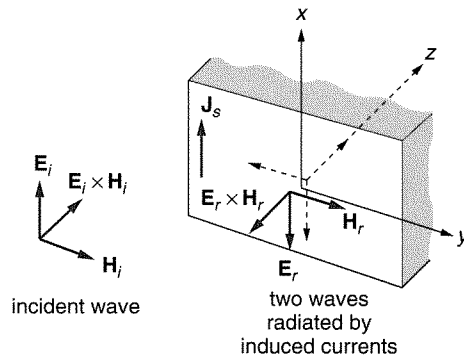

Figure 22.1 Uniform plane wave normally incident on the planar interface between a perfect dielectric and perfect conductor

where $E/H = \eta$ ($\eta = \sqrt{\mu/\epsilon}$, the intrinsic impedance of the medium). The reflected wave is then of the form

$$\mathbf{E}_r(z) = -Ee^{+j\beta z}\mathbf{u}_x \qquad \mathbf{H}_r(z) = He^{+j\beta z}\mathbf{u}_y. \qquad (22.2)$$

The Poynting vector (which represents power flow) for the reflected wave is $-z$ oriented, which determines the sign of $\mathbf{H}_r(z)$.

The total field for $z < 0$ is obtained as a superposition of the waves in Eqs. (22.1) and (22.2):

$$\mathbf{E}_{\text{tot}}(z) = \mathbf{E}_i(z) + \mathbf{E}_r(z) = E\left(e^{-j\beta z} - e^{+j\beta z}\right)\mathbf{u}_x = -2jE\sin\beta z\,\mathbf{u}_x, \qquad (22.3)$$

$$\mathbf{H}_{\text{tot}}(z) = \mathbf{H}_i(z) + \mathbf{H}_r(z) = H\left(e^{-j\beta z} + e^{+j\beta z}\right)\mathbf{u}_y = 2H\cos\beta z\,\mathbf{u}_y. \qquad (22.4)$$

The instantaneous values of the two vectors are therefore

$$\mathbf{E}_{\text{tot}}(z, t) = 2E\sqrt{2}\sin\beta z\cos(\omega t - \pi/2)\,\mathbf{u}_x = 2E\sqrt{2}\sin\beta z\sin\omega t\,\mathbf{u}_x, \qquad (22.5)$$

$$\mathbf{H}_{\text{tot}}(z, t) = 2H\sqrt{2}\cos\beta z\cos\omega t\,\mathbf{u}_y. \qquad (22.6)$$

The total wave does *not* contain the factor $e^{\pm j\beta z}$. We already had such a case for voltage and current along open and shorted transmission lines. We know that it is not a progressive, traveling wave in either direction, but a *standing wave*. As along such transmission lines, there are planes in which $\mathbf{E}_{\text{tot}}(z, t)$ is zero at all times. These planes are defined by $\beta z = -n\pi$, $n = 0, 1, 2, \ldots$. Similarly, the magnetic field is zero at all times in planes defined by $\beta z = -(2n + 1)\pi/2$, $n = 0, 1, 2, \ldots$. Thus, the total wave actually stays where it is, only pulsating in time according to the sine law (the E field) or the cosine law (the H field). The expressions describing a wave in which the time and space coordinates are as in Eqs. (22.3) and (22.4) in complex notation, or as in Eqs. (22.5) and (22.6) in the time domain, always represent standing waves. A sketch of instantaneous values of the total E and H field in front of the interface, for $\omega t = 0, \pi/4, 2\pi/4$, and $3\pi/4$, is shown in Fig. 22.2.

Example 22.1—The Fabry-Perot resonator. Consider again the case of a plane wave reflecting off a perfectly conducting plane, which behaves like a mirror. Note that according to Eq. (22.5), $\mathbf{E}_{\text{tot}}(-n\lambda/2, t) = 0$ *at all times* in the planes $z = -n\lambda/2$, $n = 1, 2, \ldots$. The electric field vector is tangential to these planes. Therefore, if we insert a perfectly conducting sheet (also a mirror) in any of these planes (i.e., for any n), nothing will change, since the boundary condition on the plane for vector $\mathbf{E}$ is satisfied automatically. In this manner, we obtained a semi-infinite region to the left of the sheet with the standing wave, and a region between the original mirror and the sheet in which *the electric and magnetic fields oscillate* as in Fig. 22.2. When the electric field is maximum, the magnetic field is zero, and conversely. This means that in the region between the two mirrors, the electric energy is being converted into magnetic energy, and vice versa. This is typical behavior for resonant electric circuits. We can conclude that from the energy point of view, just like in an LC circuit, this is a resonator, but a *spatial resonator*. This particular type of spatial resonator is known as the *Fabry-Perot resonator*, and is used extensively in optics and at millimeter-wave and infrared frequencies.

The Fabry-Perot resonator has a very useful property. Note that losses exist only in the original plane and in the sheet, due to the skin effect and finite conductivity of the metal. The
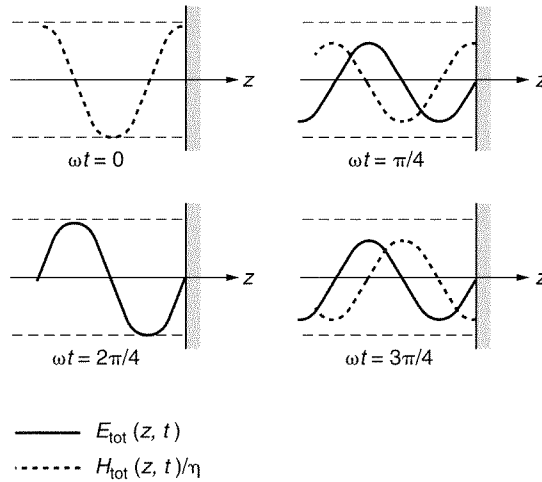
$$\qquad E_{tot}(z, t)$$
$$\cdots\cdots H_{tot}(z, t)/\eta$$

Figure 22.2 Sketch of instantaneous values of $E_{tot}(z, t)$ and $H_{tot}(z, t)$ in Eqs. (22.5) and (22.6) for a standing wave

electromagnetic energy located in the resonator, however, increases with the number n, that is, with the number of half wavelengths of the wave contained in the region between the two mirrors. At high frequencies (e.g., in the microwave region or in optics), n can be made very large with reasonable dimensions of the resonator. (For example, at 30 GHz, the wavelength is 1 cm, and a 10-cm-long resonator has $n = 20$.) Therefore, the Fabry-Perot resonator can have an arbitrarily large ratio between the energy stored in the resonator and losses in one cycle. We know that this ratio is proportional to the quality factor of the resonator (the Q factor). Therefore, the Q factor of a Fabry-Perot resonator can be extremely large (on the order of tens of thousands) when compared with that of a resonant circuit (which has a maximum Q factor of about one hundred). Of course, due to the finite size of the plates in reality, there will always be some leakage of electromagnetic energy from the resonator, which we do not take into account in this simplified analysis. Also, usually energy is purposely taken out of the resonator: for example, one of the mirrors may be partially transparent so that not all of the energy is reflected back into the cavity. This also is not taken into account in our simplified analysis.

Questions and problems: Q22.1 to Q22.4, P22.1 to P22.6

22.3   Reflection and Transmission of Plane Waves Normally Incident on a Planar Boundary Surface Between Two Dielectric Media

Let us consider two lossless dielectric media, 1 and 2, of parameters ϵ_1 and μ_1, and ϵ_2 and μ_2, respectively, separated by a planar interface, as in Fig. 22.3. Let the incident wave, with an electric field E_{1i} and of angular frequency ω, propagate in medium 1 toward the interface, normal to it, with the vector $\mathbf{E}$ parallel to the x axis (Fig. 22.3).
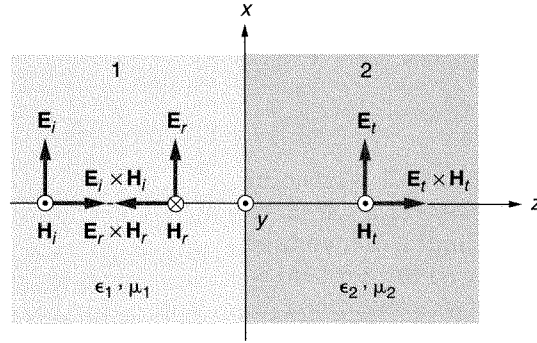
Figure 22.3 Two lossless dielectric media divided by a plane interface. The incident wave from medium 1 is partly reflected, and partly transmitted to medium 2.

A part of the incident electromagnetic energy will be reflected from the interface, and a part will be transmitted into medium 2. Assume the reference directions of the E field for the reflected and transmitted waves as indicated. The reference directions of the H field for the three waves are then as shown in the figure.

We wish to determine the relative intensities E_{1r} and E_2 at $z = 0$ of the E field for the reflected and transmitted waves. For this, we first need the expressions for these fields. With the adopted reference directions of the vectors in Fig. 22.3, they have the forms

$$\mathbf{E}_i(z) = E_{1i}e^{-j\beta_1 z}\mathbf{u}_x, \quad \mathbf{H}_i(z) = \frac{E_{1i}}{\eta_1}e^{-j\beta_1 z}\mathbf{u}_y, \tag{22.7}$$

$$\mathbf{E}_r(z) = E_{1r}e^{+j\beta_1 z}\mathbf{u}_x, \quad \mathbf{H}_r(z) = -\frac{E_{1r}}{\eta_1}e^{+j\beta_1 z}\mathbf{u}_y, \tag{22.8}$$

$$\mathbf{E}_t(z) = E_2 e^{-j\beta_2 z}\mathbf{u}_x, \quad \mathbf{H}_t(z) = \frac{E_2}{\eta_2}e^{-j\beta_2 z}\mathbf{u}_y. \tag{22.9}$$

We can now write the boundary conditions, i.e., the requirements that the tangential components of the total vectors $\mathbf{E}$ and $\mathbf{H}$ on two sides of the interface be the same:

$$E_{1i} + E_{1r} = E_2, \quad \frac{E_{1i}}{\eta_1} - \frac{E_{1r}}{\eta_1} = \frac{E_2}{\eta_2}. \tag{22.10}$$

By solving these two equations for E_{1r} and E_2, we obtain

$$E_{1r} = \frac{\eta_2 - \eta_1}{\eta_1 + \eta_2}E_{1i}, \quad E_2 = \frac{2\eta_2}{\eta_1 + \eta_2}E_{1i}. \tag{22.11}$$

Note that these are the same expressions we found in Chapter 18 for the incident (forward) and reflected (backward) voltages along a line of characteristic impedance Z_1 terminated in an infinite line of characteristic impedance Z_2 (Example 18.8). As

in the case of transmission lines, the ratio E_{1r}/E_{1i} is known as the *reflection coefficient*, and the ratio E_2/E_{1i} the *transmission coefficient*:

$$\rho = \frac{\eta_2 - \eta_1}{\eta_1 + \eta_2} \qquad \text{(the reflection coefficient, dimensionless)} \qquad (22.12)$$

$$\tau = \frac{2\eta_2}{\eta_1 + \eta_2} \qquad \text{(the transmission coefficient, dimensionless).} \qquad (22.13)$$

Note also that ρ and τ as just derived are defined with respect to the same reference directions of all three components of the *electric field* of the three waves.

In medium 2 there is only the progressive transmitted wave. In medium 1, however, we have the incident wave and the reflected wave, the total field being the sum of the two:

$$\mathbf{E}_1(z) = \mathbf{E}_i(z) + \mathbf{E}_r(z) = E_{1i}e^{-j\beta_1 z}\left(1 + \rho e^{+j2\beta_1 z}\right)\mathbf{u}_x. \qquad (22.14)$$

This is the same expression as for the voltage along a transmission line terminated in a load, Eq. (18.22*a*). The electric field in medium 1 is therefore of the same form as the voltage in the analogous transmission-line case. The following analysis, which parallels that from Chapter 18, shows this clearly.

If $\rho > 0$ (that is, if $\eta_2 > \eta_1$), the expression in parentheses is the largest, equal to $(1 + \rho)$, in planes defined by the following equation (note that medium 1 occupies the half-space $z < 0$):

$$2\beta_1 z_{\max} = -2n\pi, \qquad \text{or} \qquad z_{\max} = -\frac{n\pi}{\beta_1} = -\frac{n\lambda_1}{2} \qquad n = 0, 1, \ldots. \qquad (22.15)$$

This expression is minimal, equal to $(1 - \rho)$, in planes

$$z_{\min} = -(2n + 1)\frac{\lambda_1}{4} \qquad n = 0, 1, \ldots. \qquad (22.16)$$

If $\rho < 0$ (that is, if $\eta_2 < \eta_1$), $z_{\max}$ and $z_{\min}$ simply exchange places.

The resultant wave in medium 1 can be visualized as a sum of a progressive wave of rms value $(1 - |\rho|)E_{1i}$ and a standing wave of rms value (in the maximum of the standing wave) $2|\rho|E_{1i}$. This becomes evident if Eq. (22.14) is rewritten in the form

$$\mathbf{E}_1(z) = (1 - \rho)E_{1i}e^{-j\beta_1 z}\mathbf{u}_x + 2\rho E_{1i}\cos\beta_1 z\,\mathbf{u}_x \qquad (\rho > 0), \qquad (22.17)$$

namely,

$$\mathbf{E}_1(z) = (1 + \rho)E_{1i}e^{-j\beta_1 z}\mathbf{u}_x + 2j\rho E_{1i}\sin\beta_1 z\,\mathbf{u}_x \qquad (\rho < 0). \qquad (22.18)$$

The ratio analogous to the voltage standing-wave ratio, VSWR, from transmission lines,

$$\text{SWR} = \frac{|E_1(z)|_{\max}}{|E_1(z)|_{\min}} = \frac{1 + |\rho|}{1 - |\rho|} \qquad \text{(standing-wave ratio, dimensionless)} \quad (22.19)$$

is known as the *standing-wave ratio*. Since $|\rho| < 1$, it increases with the increase of $|\rho|$.

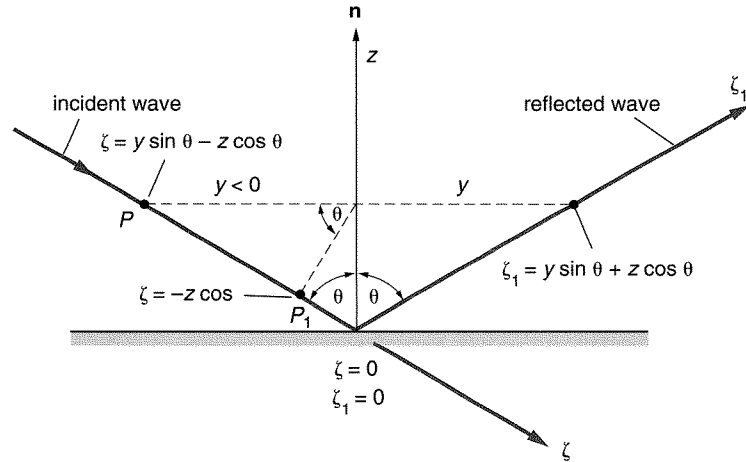*Questions and problems:*   Q22.5, P22.7 to P22.11

Figure 22.4 The direction of propagation of a wave reflected off a perfectly conducting plane is at the same angle with respect to the normal to the plane as the incident wave direction.

22.4 Plane Waves Obliquely Incident on a Perfectly Conducting Plane

Assume that a uniform plane wave propagating in a perfect dielectric of parameters ϵ and μ is obliquely incident on a perfectly conducting plane. As in the case of normal incidence, the scattered field due to the currents and charges induced on the plane must be such that it cancels the incident field in the perfect conductor. Thus, these currents and charges will produce a wave inside the conductor. This wave will be exactly the same as the incident wave, propagating in the same direction, but of opposite phase. The same field will be produced on the other side of the plane, resulting in a "reflected wave." Therefore, the direction of propagation of the reflected wave will be at the same angle θ (with respect to the normal to the plane) as the incident wave (Fig. 22.4).

The plane containing the vector $\mathbf{n}$ and the directions of propagation of the incident and reflected waves is known as the *plane of incidence*. Any incident plane wave can be represented as a superposition of two plane waves, one with the vector $\mathbf{E}$ normal to the plane of incidence, and the other with the vector $\mathbf{E}$ parallel to it. These two cases are simpler to analyze than any other. Therefore, we consider these two special cases only, knowing that any other case can be obtained by superposition.

22.4.1   VECTOR E NORMAL TO THE PLANE OF INCIDENCE

It is customary to say that this wave has *normal* or *horizontal polarization*. (The term "normal" refers to the plane of incidence, and "horizontal" refers to the fact that frequently the reflection plane is the earth's surface, in which case the vector $\mathbf{E}$ of this wave is horizontal.) The case of a horizontally polarized incident wave is sketched in Fig. 22.5, with the adopted reference directions for vectors $\mathbf{E}$ and $\mathbf{H}$.
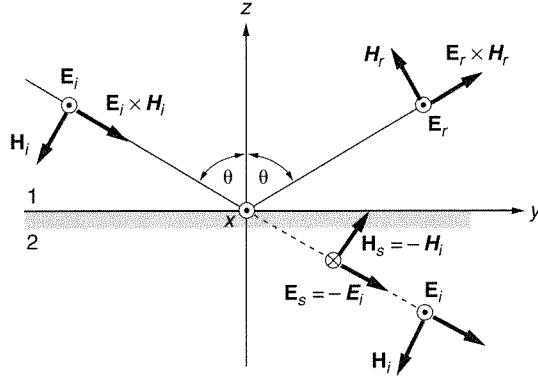
Figure 22.5 A horizontally polarized plane wave
reflected from a perfectly conducting plane

The wave propagates along an axis (ζ axis, Fig. 22.4) not coinciding with a coordinate axis x, y, or z. To write the expression for the wave in terms of the rectangular coordinates, we need to determine the distance of a point P (Fig. 22.4) from the origin of the ζ axis ($\zeta = 0$) in terms of x, y, and z. From Fig. 22.4 it is seen that for the incident wave this distance equals $\zeta = y \sin\theta - z \cos\theta$. For P note that $y < 0$ and $z > 0$, and that P is in the negative part of the ζ axis. So the factor $e^{-j\beta\zeta}$ (the factor for the wave propagating in the direction of the ζ axis) becomes $e^{-j\beta(y\sin\theta - z\cos\theta)}$. The expression for the complex electric field of the incident wave is thus

$$\mathbf{E}_i(y, z) = E e^{-j\beta(y \sin\theta - z \cos\theta)} \mathbf{u}_x. \tag{22.20}$$

In the same wave we conclude that the E field of the reflected wave is given by

$$\mathbf{E}_r(y, z) = -E e^{-j\beta(y \sin\theta + z \cos\theta)} \mathbf{u}_x. \tag{22.21}$$

The minus sign comes from the requirement that the total tangential E field on the plane $z = 0$ must be zero for any y.

The total electric field has only an x component, given by

$$E_{\text{tot}}(y, z) = E_i(y, z) + E_r(y, z) = E e^{-j\beta y \sin\theta} \left(e^{j\beta z \cos\theta} - e^{-j\beta z \cos\theta} \right),$$

from which

$$E_{\text{tot}}(y, z) = 2jE \sin(\beta z \cos\theta)\, e^{-j\beta y \sin\theta}. \tag{22.22}$$

We see that the total electric field is a standing wave in the z direction, and a traveling wave in the y direction. The wavelength in the z direction is given by

$$\lambda_z = \frac{2\pi}{\beta \cos\theta} = \frac{\lambda}{\cos\theta}, \tag{22.23}$$

where λ is the wavelength of the incident (and reflected) wave. The vector $\mathbf{E}$ is zero in the planes in which $\beta z \cos\theta = n\pi$, $n = 0, 1, 2, \ldots$, or

$$z_{E=0} = \frac{n\lambda_z}{2} = \frac{n\lambda}{2\cos\theta}, \quad n = 0, 1, 2, \ldots. \tag{22.24}$$

In the direction of the y axis, the total field behaves as a traveling wave, with a phase velocity along the y axis

$$v_{ph} = \frac{\omega}{\beta_y} = \frac{\omega}{\beta \sin\theta} = \frac{c}{\sin\theta}, \qquad c = \frac{1}{\sqrt{\epsilon\mu}} \tag{22.25}$$

(note that the phase coefficient with respect to the y axis is the entire factor of jy in the exponent, that is, $\beta_y = \beta \sin\theta$), and with a wavelength along the y axis

$$\lambda_y = \frac{2\pi}{\beta_y} = \frac{\lambda}{\sin\theta}. \tag{22.26}$$

Example 22.2—The rectangular waveguide. Because in the planes $z_{E=0}$ the magnitude of the E field is zero at all times, we can insert into any one of these planes a perfectly conducting sheet. Assume that in some way we switch the field above the sheet off. What remains is a system of two perfectly conducting planes guiding a specific wave propagating in the y direction.

We can go a step further. The vector **E** is normal to the planes defined by $x =$ constant. Introducing a perfectly conducting sheet in one or more such planes will not change the field— it will only induce surface charges of opposite signs on the two faces of the sheet. However, if we imagine two such sheets together with the first sheet parallel to the plane $z = 0$, we obtain a rectangular tube through which an electromagnetic wave propagates just like water flows through a pipe. Such a rectangular metallic tube is known as the *rectangular waveguide*. It is used extensively for guiding electromagnetic energy at microwave and millimeter-wave frequencies.

We will learn in the next chapter that this type of electromagnetic wave is only one of an infinite number of wave types that can propagate through such rectangular metallic tubes.

Example 22.3—Determination of the total H field. With reference to Fig. 22.5, the total H field has two components, H_y and H_z. Let us determine them as an exercise. The two components of the total H field are obtained as the sum of these components for the incident and the reflected waves:

$$H_{\text{tot } y}(y, z) = H_{iy}(y, z) + H_{ry}(y, z)$$

$$= -\frac{E}{\eta} e^{-j\beta(y\sin\theta - z\cos\theta)} \cos\theta - \frac{E}{\eta} e^{-j\beta(y\sin\theta + z\cos\theta)} \cos\theta.$$

After simple rearrangements similar to those in deriving Eq. (22.22), we obtain

$$H_{\text{tot } y}(y, z) = -2\frac{E}{\eta} \cos\theta \cos(\beta z \cos\theta) e^{-j\beta y \sin\theta}.$$

The H_z component is obtained in a similar way, which is left as an exercise for the reader. The result is

$$H_{\text{tot } z}(y, z) = 2j\frac{E}{\eta} \sin\theta \sin(\beta z \cos\theta) e^{-j\beta y \sin\theta}.$$

Note that H_z is zero on the perfectly conducting plane, as it should be (the magnetic field can have no normal component on a perfect conductor—see Example 20.2).

22.4.2 VECTOR E PARALLEL TO THE PLANE OF INCIDENCE

Assume now that vector **E** is parallel to the plane of incidence, as sketched in Fig. 22.6. Because the tangential component (y component) at the plane must be zero, again the reflected wave has the same amplitude. The directions of the E and H vectors indicated in the figure represent their reference directions.

We now have two E-field components of the incident and the reflected waves, the y and the z components. Both must be of the form in Eqs. (22.20) and (22.21):

$$E_{iy}(y, z) = E \cos \theta e^{-j\beta(y \sin \theta - z \cos \theta)}, \tag{22.27}$$

$$E_{iz}(y, z) = E \sin \theta e^{-j\beta(y \sin \theta - z \cos \theta)}, \tag{22.28}$$

and

$$E_{ry}(y, z) = -E \cos \theta e^{-j\beta(y \sin \theta + z \cos \theta)}, \tag{22.29}$$

$$E_{rz}(y, z) = E \sin \theta e^{-j\beta(y \sin \theta + z \cos \theta)}. \tag{22.30}$$

The total components are the sum of these:

$$E_{\text{tot } y}(y, z) = E_{iy}(y, z) + E_{ry}(y, z) = 2jE \cos \theta \sin(\beta z \cos \theta) e^{-j\beta y \sin \theta}, \tag{22.31}$$

$$E_{\text{tot } z}(y, z) = E_{iz}(y, z) + E_{rz}(y, z) = 2E \sin \theta \cos(\beta z \cos \theta) e^{-j\beta y \sin \theta}. \tag{22.32}$$

Example 22.4—Determination of the total H field. With reference to Fig. 22.6, the total H field in this case has the x component only. Because we know that $H = E/\eta$, the expressions
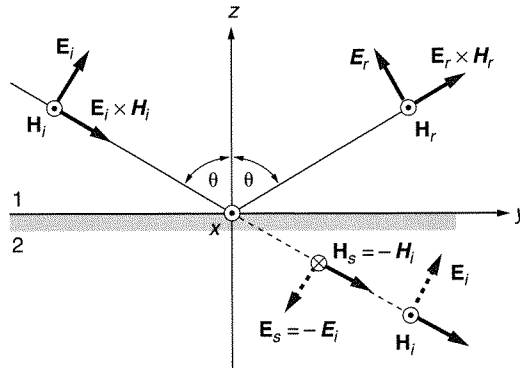

Figure 22.6 Reflection of a vertically polarized plane wave from a perfectly conducting plane

for the H field of the incident and reflected waves are

$$H_{ix}(y, z) = \frac{E}{\eta} e^{-j\beta(y\sin\theta - z\cos\theta)} \quad \text{and} \quad H_{rx}(y, z) = \frac{E}{\eta} e^{-j\beta(y\sin\theta + z\cos\theta)}.$$

The total H field is hence

$$H_{\text{tot } x}(y, z) = 2\frac{E}{\eta}\cos(\beta z\cos\theta)e^{-j\beta y\sin\theta}.$$

Example 22.5—Maximal emf induced in a small loop above a perfectly conducting plane. Assume that we wish to receive a signal contained in the incident wave. One way, which is quite easy to understand, is to use a loop of wire (e.g., a circular one of radius a) much smaller than the wavelength of the wave. The emf induced in the loop is then obtained according to Faraday's law. So to obtain a maximal emf, the principal thing we have to determine is where the magnetic field is maximal, and what its direction is at that point.

The magnetic field has a maximum at $z = 0$, with a value equal to

$$H_{\text{tot } x}(y, 0) = 2\frac{E}{\eta} e^{-j\beta y\sin\theta}.$$

The last factor in this expression determines just the initial phase of the field along the y axis. To simplify, let $y = 0$. The maximal possible complex emf [note that the complex counterpart of the expression $e(t) = -d\Phi(t)/dt$ is $\underline{e} = -j\omega\underline{\Phi}$] induced in the loop is thus

$$\mathcal{E}_{\max} = -2j\frac{E}{\eta}\omega\mu_0 a^2\pi.$$

In this case, we have used the small loop as a receiving antenna. This type of antenna, which develops a voltage between its terminals due to a time-variable flux of the magnetic field through its contour, is called a *loop antenna.* We could also have used two short straight wires connected to a voltmeter that measures the emf. In this case, there is an induced emf due to the integral of the induced electric field along the wires. This type of antenna is called a *short dipole antenna.*

Questions and problems: Q22.6 and Q22.7, P22.12 and P22.13

22.5   Reflection and Transmission of Plane Waves Obliquely Incident on a Planar Boundary Surface Between Two Dielectric Media

When a plane wave is obliquely (at an angle) incident on a plane interface between two media, the formulation of boundary conditions becomes more complex than when incidence is normal. Of course, again a part of the incident energy is reflected back into medium 1 (of parameters ϵ_1 and μ_1), and a part is transmitted into medium 2 (of parameters ϵ_2 and μ_2). We shall see that the direction of propagation of the reflected wave makes the same angle with the normal to the interface as the incident wave, as before. However, the transmitted wave is deflected with respect to this normal. The transmitted wave in this case is therefore frequently termed the *refracted wave.*

The amplitudes of the reflected and refracted waves depend, among other things, on the polarization of the wave (i.e., on the electric field vector being parallel or normal to the plane of incidence). However, the angles that the direction of propagation of the two secondary waves make with the normal to the interface are the same for any polarization.

Figure 22.7 shows equiphase planes (planes of constant phase) and the directions of propagation of the incident, reflected, and refracted waves. These planes in medium 1 are moving with a velocity $c_1 = 1/\sqrt{\epsilon_1\mu_1}$, and in medium 2 with a velocity $c_2 = 1/\sqrt{\epsilon_2\mu_2}$. Indicated in the figure are a few equiphase planes of the three waves. Let the boundary conditions be satisfied at the instant for which Fig. 22.7 is valid. In order that they remain satisfied at all times, it is necessary that the relative amplitudes and phases of the three waves at the interface remain unchanged. This is possible only if the intersections of the equiphase planes with the interface move *along the interface* at the same speed.

From Fig. 22.7, this velocity for the incident and reflected wave is $c_1/\sin\theta_i$ and $c_1/\sin\theta_r$, and for the refracted wave $c_2/\sin\theta_2$. To satisfy this condition we conclude that, first, $\theta_r = \theta_i$. This angle we shall therefore denote as θ_1. Second, the condition $c_1/\sin\theta_1 = c_2/\sin\theta_2$ must also hold, so

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{c_1}{c_2} = \sqrt{\frac{\epsilon_2\mu_2}{\epsilon_1\mu_1}}. \tag{22.33}$$

(Snell's law)

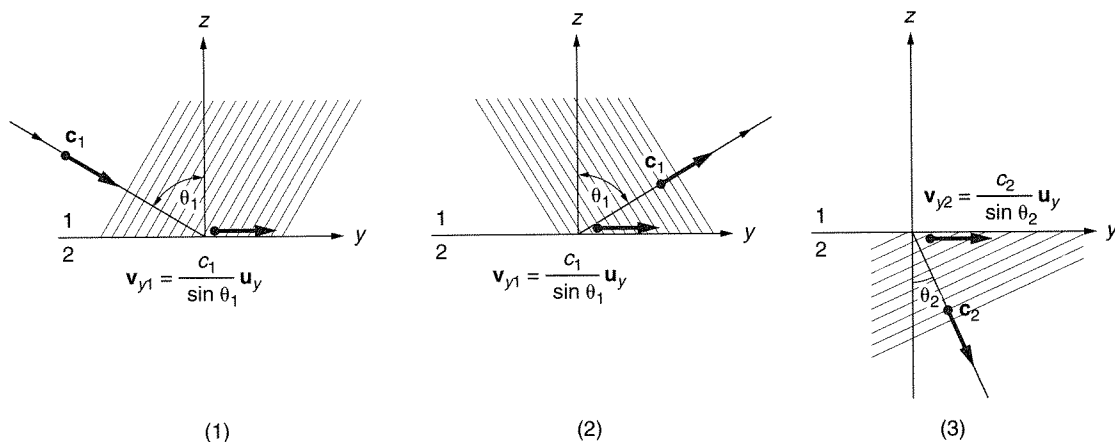(1)   (2)   (3)

Figure 22.7 Equiphase planes and the directions of propagation of the incident, reflected, and refracted waves

This relation is known as *Snell's law*. The ratio c_1/c_2 is termed the *index of refraction* for media 1 and 2, and is often denoted as n_{12}, especially in optics. If $\mu_1 = \mu_2 = \mu_0$ (which is most often the case),

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{c_1}{c_2} = \sqrt{\frac{\epsilon_2}{\epsilon_1}} \qquad (\mu_1 = \mu_2). \tag{22.34}$$

We know that if $0 \leq \beta < \alpha \leq \pi/2$, then $\sin\alpha > \sin\beta$. Snell's law and Eq. (22.34) therefore tell us that for $\epsilon_1 < \epsilon_2$, $\theta_1 > \theta_2$. This means that the wave is refracted toward the normal. The refracted wave exists for any θ_1.

If $\epsilon_1 > \epsilon_2$, however, $\theta_1 < \theta_2$. This means that the direction of propagation of the refracted wave makes a greater angle with the normal than that of the incident wave. So for a certain angle θ_1 the angle θ_2 will become $\pi/2$, and cannot increase further. From Eq. (22.34), this limiting angle $\theta_1 = \theta_t$ is defined by

$$\frac{\sin\theta_t}{\sin(\pi/2)} = \sin\theta_t = \sqrt{\frac{\epsilon_2}{\epsilon_1}} \qquad (\epsilon_1 > \epsilon_2, \mu_1 = \mu_2). \tag{22.35}$$

[Sine of critical angle (angle of total reflection)]

This particular angle $\theta_1 = \theta_t$ is known as the *critical angle*, or the *angle of total reflection*.

For $\theta_1 > \theta_t$, the sine of θ_2 must be *greater than one* in order for the boundary conditions to be satisfied. At first glance it might seem as if we made a mistake. The sine of a *real* angle cannot be greater than one. However, the sine of a complex angle can be larger than unity. This is easily understood if we set $\theta_2 = \pi/2 - jx$ and recall the expression for the sine in terms of the exponential function:

$$\sin(\pi/2 - jx) = \frac{1}{2j}\left[e^{j(\pi/2 - jx)} - e^{-j(\pi/2 - jx)}\right] = \frac{1}{2j}\left(je^x + je^{-x}\right) = \cosh x,$$

since $e^{\pm j\pi/2} = \pm j$. The *hyperbolic consine* function of x, $\cosh x = (e^x + e^{-x})/2$ can have any positive value between one and infinity.

What happens then if $\theta_1 > \theta_t$? Obviously, there can be no refracted wave in medium 2, so all of the energy of the incident wave is reflected back into medium 1. Example 22.9 will show that indeed, the magnitude of the reflection coefficient is then equal to one. This is known as *total reflection*. It has many applications and is encountered on many occasions.

Example 22.6—Apparent shape of an oar observed from a rowboat. If you are in a rowboat on clear, calm water, and observe the oar immersed in the water, the oar looks as if it is broken at the water level: the immersed part of the oar appears higher than expected. This is easy to explain using Snell's law. You see the immersed part of the oar because light rays, i.e., electromagnetic waves, are reflected from the oar toward your eyes. They pass the water-air interface and are refracted in the air away from the normal because the permittivity of water is greater than that of air. Therefore, the oar looks broken.

If you observe the oar from a distant point, you will not be able to see the submerged part of the oar. This is because the rays from the submerged part of the oar in that case are incident on the water-air interface at an angle greater than the critical angle, and there are no transmitted rays in the air in your direction anymore.

Questions and problems: Q22.8, P22.14

22.6   Fresnel Coefficients

Snell's law and the phenomenon of total reflection are valid for any polarization of the incident wave. The reflection and transmission coefficients, which are defined in the same way as for normal incidence, are different for normal and parallel polarization. In this section we derive the so-called *Fresnel coefficients*, which are reflection and transmission coefficients written in terms of the angle of incidence and the material properties (wave impedances) of the two media.

For waves obliquely incident on the interface between two dielectrics, we need to consider the two polarizations separately, similarly to the conductor case.

22.6.1   VECTOR E NORMAL TO THE PLANE OF INCIDENCE (TRANSVERSE ELECTRIC CASE)

Let the reference directions of the field vectors of the incident, reflected, and refracted waves be as in Fig. 22.8. Let E_{1i}, E_{1r}, and E_2 and H_{1i}, H_{1r}, and H_2 be the complex rms values of the vectors of the three waves at the interface ($z = 0$). The boundary conditions require that the tangential components of the total E field and the total
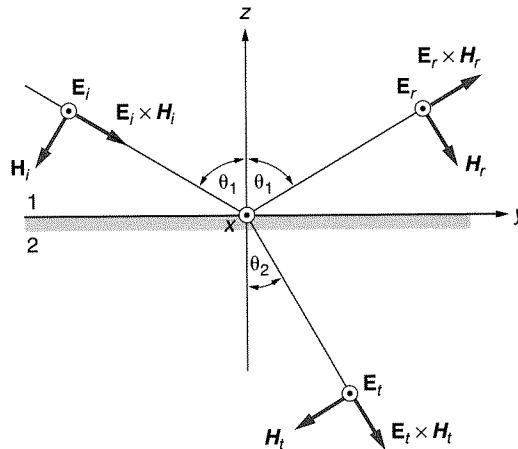

Figure 22.8 Reference directions of the field vectors of the incident, reflected, and refracted waves for a normally polarized incident wave

H field on the two sides of the interface be equal. This results in the following two equations:

$$E_{1i} + E_{1r} = E_2 \qquad (H_{1i} - H_{1r}) \cos\theta_1 = H_2 \cos\theta_2. \qquad (22.36)$$

Since $H_{1i} = E_{1i}/\eta_1$, $H_{1r} = E_{1r}/\eta_1$, and $H_2 = E_2/\eta_2$, we have two linear equations in two unknowns, E_{1r} and E_2. Solving these equations we obtain

$$\rho_n = \left(\frac{E_{1r}}{E_{1i}}\right)_n = \frac{\eta_2 \cos\theta_1 - \eta_1 \cos\theta_2}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}, \qquad (22.37)$$

$$\tau_n = \left(\frac{E_2}{E_{1i}}\right)_n = \frac{2\eta_2 \cos\theta_1}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}. \qquad (22.38)$$

[Fresnel's coefficients for normal (TE) polarization]

The reflection and transmission coefficients, ρ_n and τ_n, are known as the *Fresnel coefficients* for normal polarization. They are also sometimes called the *transverse electric (TE)* Fresnel coefficients. In these expressions, according to Snell's law in Eq. (22.33), $\cos\theta_2$ must be calculated as

$$\cos\theta_2 = \sqrt{1 - \sin^2\theta_2} = \frac{c_2}{c_1}\sqrt{\left(\frac{c_1}{c_2}\right)^2 - \sin^2\theta_1}. \qquad (22.39)$$

The expressions for the ρ and τ coefficients are general. For perfect dielectrics, having real intrinsic impedances, they are real. As a consequence, the reflected wave on the interface is either in phase (if $\rho > 0$) or in counterphase (if $\rho < 0$) with respect to the incident wave. If either of the two media is not a perfect dielectric, the intrinsic impedance of that medium is complex, so that both ρ and τ are complex as well, and the phase difference between the field vectors on the interface is different from π or zero.

22.6.2 VECTOR E PARALLEL TO THE PLANE OF INCIDENCE (TRANSVERSE MAGNETIC CASE)

Assume that the reference directions of the field vectors of the incident, reflected, and refracted waves in this case is as in Fig. 22.9. Again let E_{1i}, E_{1r}, and E_2 and H_{1i}, H_{1r}, and H_2 be the complex rms values of the field vectors of the three waves at $z = 0$. The boundary conditions in this case are

$$(E_{1i} - E_{1r}) \cos\theta_1 = E_2 \cos\theta_2 \qquad H_{1i} + H_{1r} = H_2. \qquad (22.40)$$

Expressing the magnetic field intensities as E/η with appropriate subscripts, we again obtain two linear equations in unknowns E_{1r} and E_2. The solution of these equations is straightforward. The reflection and transmission coefficients are found to be
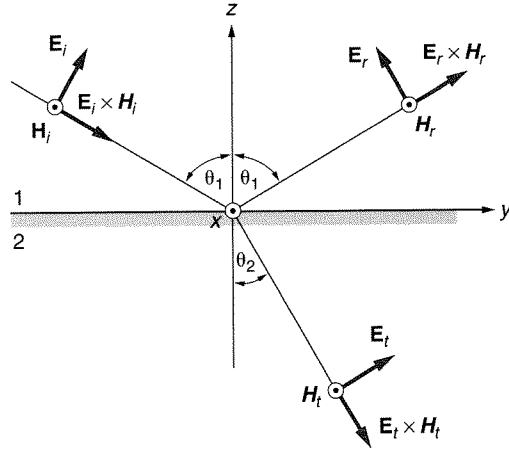
Figure 22.9 Reference directions of the field vectors of the incident, reflected, and refracted waves for the parallel polarization of the incident wave

$$\rho_p = \left(\frac{E_{1r}}{E_{1i}}\right)_p = \frac{\eta_1 \cos\theta_1 - \eta_2 \cos\theta_2}{\eta_1 \cos\theta_1 + \eta_2 \cos\theta_2}, \tag{22.41}$$

$$\tau_p = \left(\frac{E_2}{E_{1i}}\right)_p = \frac{2\eta_2 \cos\theta_1}{\eta_1 \cos\theta_1 + \eta_2 \cos\theta_2}. \tag{22.42}$$

[Fresnel's coefficients for parallel (TM) polarization]

Of course, in these two expressions $\cos\theta_2$ must also be calculated as in Eq. (22.39). The coefficients ρ_p and τ_p in Eqs. (22.41) and (22.42) are the parallel polarization Fresnel coefficients, sometimes also called the *transverse magnetic (TM)* Fresnel coefficients.

Example 22.7—Transmission-line models for oblique incidence of plane waves. We mentioned earlier that transmission-line theory can be used for plane waves incident normally to any interface. It turns out that with a slight modification, we can also use transmission-line theory for oblique incidence. This can be seen if we rewrite the Fresnel coefficients, Eqs. (22.37) and (22.42), as
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$$\rho_n = \frac{\dfrac{\eta_2}{\cos\theta_2} - \dfrac{\eta_1}{\cos\theta_1}}{\dfrac{\eta_2}{\cos\theta_2} + \dfrac{\eta_1}{\cos\theta_1}} = \frac{\eta_{2n} - \eta_{1n}}{\eta_{2n} + \eta_{1n}}, \tag{22.43}$$

$$\rho_p = -\frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1} = -\frac{\eta_{2p} - \eta_{1p}}{\eta_{2p} + \eta_{1p}}, \tag{22.44}$$

where we have now defined the normal and parallel wave impedances as $\eta_{in} = \eta_i / \cos \theta_i$ and $\eta_{ip} = \eta_i \cos \theta_i$. (The minus sign in ρ_p results from the adopted reference directions, and is of no importance.) The transmission coefficients can, obviously, be written in the same way. As an exercise, it is suggested that the reader determine ρ for a normally polarized wave incident at a 45-degree angle from air on a dielectric with $\epsilon_r = 4$ and $\mu = \mu_0$.

Example 22.8—Fresnel coefficients for perfect dielectrics with equal permeabilities. In practice the most common case is actually the special case of the two media being perfect dielectrics of equal permeabilities. Then $\eta_1/\eta_2 = \sqrt{\epsilon_2/\epsilon_1}$, and the reflection and transmission coefficients for the normal polarization in Eqs. (22.37) and (22.38) become

$$\rho_n = \frac{\cos \theta_1 - \sqrt{\epsilon_2/\epsilon_1} \cos \theta_2}{\cos \theta_1 + \sqrt{\epsilon_2/\epsilon_1} \cos \theta_2}, \qquad \tau_n = \frac{2 \cos \theta_1}{\cos \theta_1 + \sqrt{\epsilon_2/\epsilon_1} \cos \theta_2} \qquad (\mu_1 = \mu_2). \tag{22.45}$$

For the parallel polarization, the reflection and transmission coefficients in this case become

$$\rho_p = \frac{\sqrt{\epsilon_2/\epsilon_1} \cos \theta_1 - \cos \theta_2}{\sqrt{\epsilon_2/\epsilon_1} \cos \theta_1 + \cos \theta_2}, \qquad \tau_p = \frac{2 \cos \theta_1}{\sqrt{\epsilon_2/\epsilon_1} \cos \theta_1 + \cos \theta_2} \qquad (\mu_1 = \mu_2). \tag{22.46}$$

Example 22.9—The Brewster angle. From Example 22.8, a few simple conclusions can be drawn:

1. It is not difficult to understand that ρ_n can never be zero. This would require that, simultaneously, $\sin \theta_1 / \sin \theta_2 = \sqrt{\epsilon_2/\epsilon_1}$ (Snell's law) and $\cos \theta_1 / \cos \theta_2 = \sqrt{\epsilon_2/\epsilon_1}$ (the equation resulting from $\rho_n = 0$), which is not possible.

2. If θ_1 is greater than the critical angle for total reflection, we know that $\sin^2 \theta_1 > \epsilon_2/\epsilon_1$, so that $\cos \theta_2$ is purely imaginary. We see from Eq. (22.45) that ρ_n is then in the form $(a - jb)/(a + jb)$. This means that the magnitude of ρ_n is equal to one, that is, that the entire energy of the incident wave is reflected back into medium 1. The same conclusion can be reached for ρ_p.

3. The reflection coefficient in the parallel polarization case *can* be zero. For that to happen, it is necessary that $\cos \theta_1 / \cos \theta_2 = \sqrt{\epsilon_1/\epsilon_2}$. This is now not in contradiction with Snell's law, but both equations must simultaneously be satisfied. If we multiply the two equations, we obtain that the reflected wave does not exist if

$$\sin \theta_1 \cos \theta_1 = \sin \theta_2 \cos \theta_2, \qquad \text{or} \qquad \sin 2\theta_1 = \sin 2\theta_2. \tag{22.47}$$

This equation is satisfied if $2\theta_1 = (\pi - 2\theta_2)$, that is, if $(\theta_1 + \theta_2) = \pi/2$. But for two angles adding to $\pi/2$ the sine of one equals the cosine of the other, so that from Snell's law,

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{\sin \theta_1}{\cos \theta_1} = \tan \theta_1 = n_{12}. \tag{22.48}$$

(The Brewster angle, parallel polarization only)

This particular angle of incidence of a wave with parallel polarization, for which the reflected wave disappears, is known as the *Brewster angle* or the *polarization angle*.

Example 22.10—Polarization of reflected waves incident at the Brewster angle. We know that an arbitrarily polarized wave can always be represented as a superposition of two

linearly polarized waves. Therefore, if we have, for example, an elliptically polarized wave incident at the Brewster angle, the reflected wave will not contain the component with parallel polarization, i.e., it will have only a normally polarized electric field. In other words, any wave incident on a plane interface of two dielectric media at the Brewster angle will be reflected as a *linearly polarized wave*.

Example 22.11—Elimination of the reflected wave in the case of an arbitrarily polarized incident wave. Suppose that we introduce in the path of the reflected wave from the preceding example a dielectric slab oriented so that the wave is incident on it at the Brewster angle, and that the polarization of the wave (recall that it is defined with respect to the plane of incidence) is parallel. The reflected wave is then going to disappear completely. This is exactly how this phenomenon was discovered experimentally by Brewster, using electromagnetic waves in the visible light frequency region.

Questions and problems: Q22.9 and Q22.10, P22.15 to P22.19

22.7   Chapter Summary

1. If a plane wave is incident on a plane boundary surface between two media, boundary conditions can be satisfied by assuming that the wave resulting from the discontinuity (the scattered wave) consists of a reflected plane wave, and (if the other medium is not perfectly conducting) of a transmitted, or refracted, plane wave. This enables relatively simple analysis of electromagnetic scattering of plane waves, similar to transmission-line analysis.

2. If a plane wave is normally incident on a perfectly conducting plane, a standing wave in front of the plane results. If incidence is not normal, there is a standing wave in the direction normal to the plane, and a traveling wave parallel to it.

3. If a plane wave is incident on a plane boundary surface between two dielectric media, a plane wave is reflected from the interface, and a plane wave is transmitted into the other medium.

4. For an arbitrary angle of the incident wave, the plane of incidence is defined as the plane normal to the boundary and containing the direction of propagation of the incident wave.

5. The incident wave is said to have normal polarization if **E** is normal to the plane of incidence, and to have parallel polarization if **E** is parallel to that plane.

6. The ratios of the amplitudes of the reflected and incident waves, and of the transmitted and incident waves, are known as the Fresnel coefficients, with one set for normal polarization and one for parallel polarization.

QUESTIONS

Q22.1. For what orientation and position of a small wire loop, Fig. 22.1, is the emf induced in it maximal?

Q22.2. Prove that the time-average value of the Poynting vector at any point in Fig. 22.1 is zero.

Q22.3. If waves are represented in phasor form, how can you distinguish a standing wave from a traveling wave?

Q22.4. If waves are represented in the time domain, how can you distinguish a standing wave from a traveling wave?

Q22.5. Does the emf induced in a small loop of area S placed in Fig. 22.3 at a coordinate $z > 0$ depend on z? Does it depend on z if $z < 0$? Explain.

Q22.6. Can a small wire loop be placed in Fig. 22.5 so that the emf induced in it is practically zero irrespective of the orientation of the loop?

Q22.7. Repeat question Q22.6 for Fig. 22.6.

Q22.8. Is total reflection possible if a wave is incident from air onto a dielectric surface? Explain.

Q22.9. Why is there no counterpart of the Brewster angle for a wave with vector **E** normal to the plane of incidence?

Q22.10. A linearly polarized plane wave is incident from air onto a dielectric half-space, with the vector **E** at an angle $\alpha(0 < \alpha < \pi/2)$ with respect to the plane of incidence. Is the polarization of the transmitted and reflected wave linear? If not, what is the polarization of the two waves? Does it depend, for a given α, on the properties of the dielectric medium?

PROBLEMS

P22.1. A linearly polarized plane wave of rms electric field strength E and angular frequency ω is normally incident from a vacuum on a large, perfectly conducting flat sheet. Determine the induced surface charges and currents on the sheet.

P22.2. Note that the induced surface currents in problem P22.1 are situated in the magnetic field of the incident wave. Determine the time-average force per unit area (the pressure) on the sheet. (This is known as *radiation pressure*.)

P22.3. Repeat problems P22.1 and P22.2 assuming the wave is polarized circularly.

P22.4. If the conductivity σ of the sheet in problem P22.1 is large, but finite, its permeability is μ, and the frequency of the wave is ω, find the time-average power losses in the sheet per unit area. Specifically, find these losses if $f = 1\,\text{MHz}$, $E = 1\,\text{V/m}$, $\sigma = 56 \cdot 10^6\,\text{S/m}$ (copper), and $\mu = \mu_0$.

P22.5. A plane wave, of wavelength λ, is normally incident from a vacuum on a large, perfectly conducting sheet. A circular loop of radius $a(a \ll \lambda)$ should be at a location at which the induced emf is maximal, as near as possible to the sheet. If the electric field of the incident wave is E, calculate this maximal emf.

P22.6. Assume that a time-harmonic surface current of density $J_{sx} = J_{s0} \cos \omega t$ exists over an infinitely large plane sheet. Write the integral expression for the electric field strength vector at a distance z from the sheet. Do not evaluate the integral, but reconsider problem P22.1 to see if you know what the result must be.

P22.7. A linearly polarized plane wave, of frequency $f = 1\,\text{MHz}$, is normally incident from a vacuum on the planar surface of distilled water ($\mu = \mu_0$, $\epsilon = 81\epsilon_0$, $\sigma \simeq 0$). The rms value of the electric field strength of the incident wave is $E = 100\,\text{mV/m}$. A loop of area $S = 100\,\text{cm}^2$ wound with $N = 5$ turns is situated in water so that the emf induced in it is maximal. Determine the rms value of the emf.

P22.8. A plane wave propagating in dielectric 1, of permittivity ϵ_1 and permeability μ_1, impinges normally on a dielectric slab 2, of permittivity ϵ_2, permeability μ_2, and thickness d. To the right of the slab there is a semi-infinite medium of permittivity ϵ_3 and permeability μ_3. Determine the reflection coefficient at the interface between media 1 and 2. Plot the reflection coefficient as a function of the slab thickness, d, for given permittivities. Consider cases when (1) $\epsilon_1 > \epsilon_2 > \epsilon_3$, (2) $\epsilon_3 > \epsilon_2 > \epsilon_1$, (3) $\epsilon_2 > \epsilon_1 > \epsilon_3$, and (4) $\epsilon_2 > \epsilon_3 > \epsilon_1$.

P22.9. Assume that in the preceding problem the thickness of the slab is (1) half a wavelength, and (2) a quarter of a wavelength in the slab. Determine the relationship between the intrinsic impedances of the three media for which in the two cases there will be no reflected wave into medium 1. (The first of these conditions is used in antenna covers, called radomes. The second is used in optics, for so-called anti-reflection, or AR, coatings. The thickness and relative permittivity of a thin transparent layer over lenses can be designed in this way so that the reflection of light from the lens is minimized.)

P22.10. Find the reflection and transmission coefficients for the interface between air and fresh water ($\epsilon = 81\epsilon_0, \sigma \simeq 0$), in the case of perpendicular incidence.

P22.11. A plane wave is normally incident on the interface between air and a dielectric having a permeability $\mu = \mu_0$, and an unknown permittivity ϵ. The measured standing-wave ratio in air is 1.8. Determine ϵ.

P22.12. What is the position of a small loop of area S in Fig. 22.6 in order that the emf induced in it be maximal? If the electric field of the wave is E and its frequency f, calculate this maximal emf.

P22.13. Repeat problem P22.12 for a small loop placed in the wave in Fig. 22.5.

P22.14. Determine the minimal relative permittivity of a dielectric medium for which the critical angle of total reflection from the dielectric into air is less than 45 degrees. Is it possible to make from such a dielectric a right-angled isosceles triangular prism that returns the light wave as in Fig. P22.14? Is there reflection of the light wave when it enters the prism?
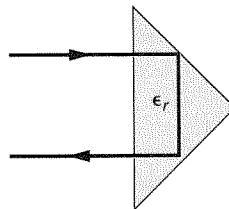

Figure P22.14 Reflection of a light wave by a prism

P22.15. A plane wave with parallel polarization is incident at an angle of $\pi/4$ from air on a perfect dielectric with $\epsilon_r = 4$ and $\mu = \mu_0$. Find the Fresnel coefficients. What fraction of the incident power is reflected, and what is transmitted into the dielectric? More generally, plot the Fresnel coefficients and the reflected and transmitted power as a function of ϵ_r, assuming its value is between 1 and 80.

P22.16. Repeat problem P22.15 for a normally polarized wave.

P22.17. A plane wave with normal polarization is incident at an angle of $60°$ from air onto deep fresh water with $\epsilon_r = 81$ ($\sigma = 0$). The rms value of the incident electric field is $1\,\text{V/m}$. Find the rms value of the reflected and transmitted electric field.

P22.18. Repeat problem P22.17 for parallel polarization.

P22.19. Is there an incident angle in problems P22.17 and P22.18 for which the reflected wave is eliminated? If so, calculate this angle for the two polarizations.