Triangulation of the Cube

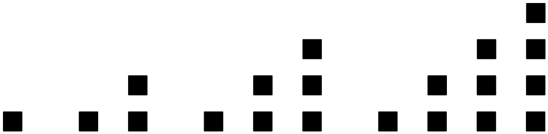
Ben Storlie

November 9, 2012

Triangular Numbers

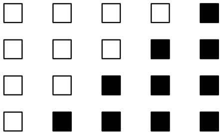
Triangular numbers are numbers in the form $\sum_{i=1}^{n} i$.
If $T_2(n)$ is the nth triangular number, then

$$T_2(1) = 1$$
$$T_2(2) = 1 + 2 = 3$$
$$T_2(3) = 1 + 2 + 3 = 6$$
$$T_2(4) = 1 + 2 + 3 + 4 = 10$$
$$T_2(5) = 1 + 2 + 3 + 4 + 5 = 15$$



Triangular Numbers

Recall that $T_2(n) = \frac{n(n+1)}{2}$. This gives us a familiar geometric interpretation of a rectangle sliced in half.

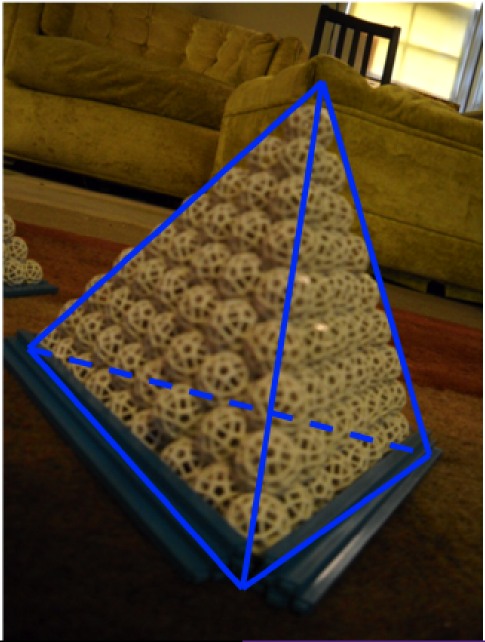


Tetrahedral Numbers

Since $\left(\!\binom{n}{2}\!\right) = \frac{n(n+1)}{2}$, $T_2(n) = \left(\!\binom{n}{2}\!\right)$.
Tetrahedral numbers are the sum of consecutive triangular numbers.

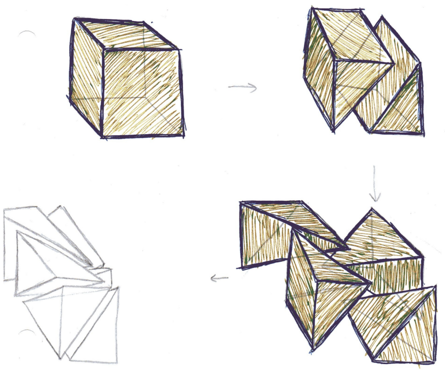
$$T_3(n) = \sum_{i=1}^{n} T_2(i)$$

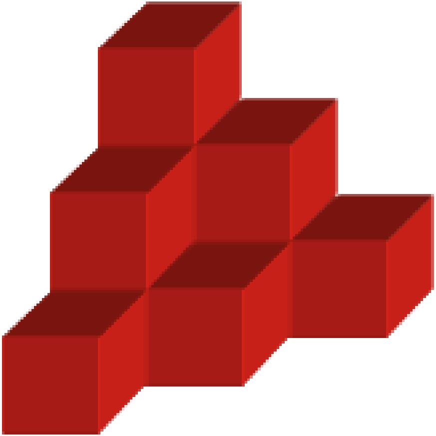
Since $\sum_{i=1}^{n} \left(\!\binom{i}{2}\!\right) = \left(\!\binom{n}{3}\!\right)$, we know that $T_3(n) = \left(\!\binom{n}{3}\!\right)$.

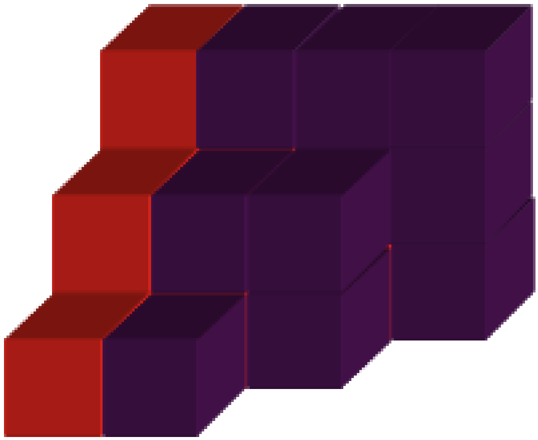


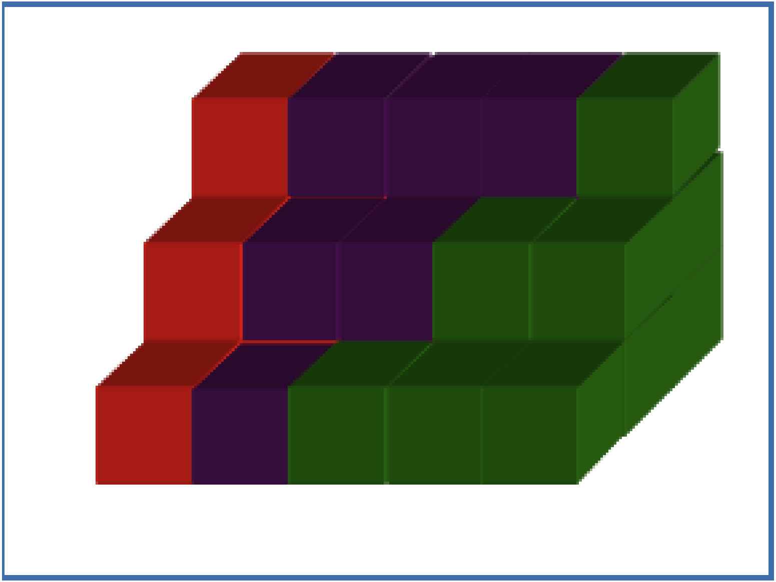
$$\left(\binom{n}{3}\right) = \frac{n(n+1)(n+2)}{6}$$

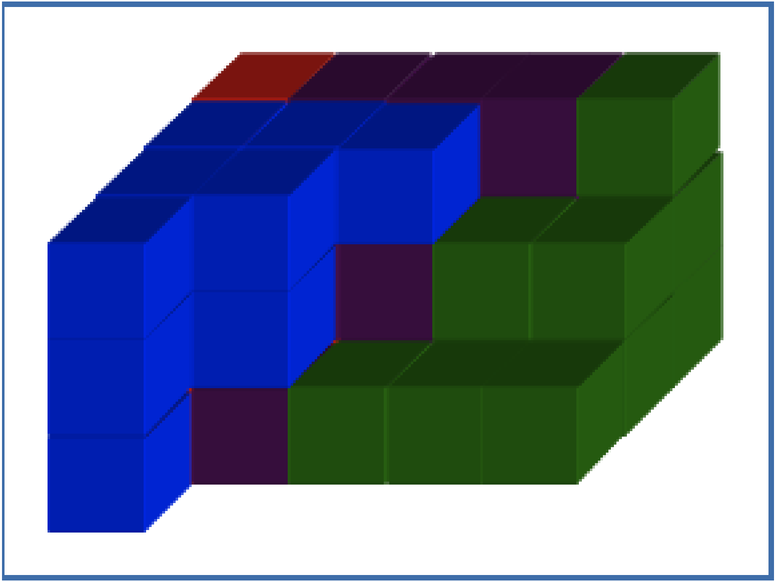
This implies that there is an analogous geometric interpretation to triangular numbers. That is, a $n \times n+1 \times n+2$ box can be divided into 6 tetrahedra.

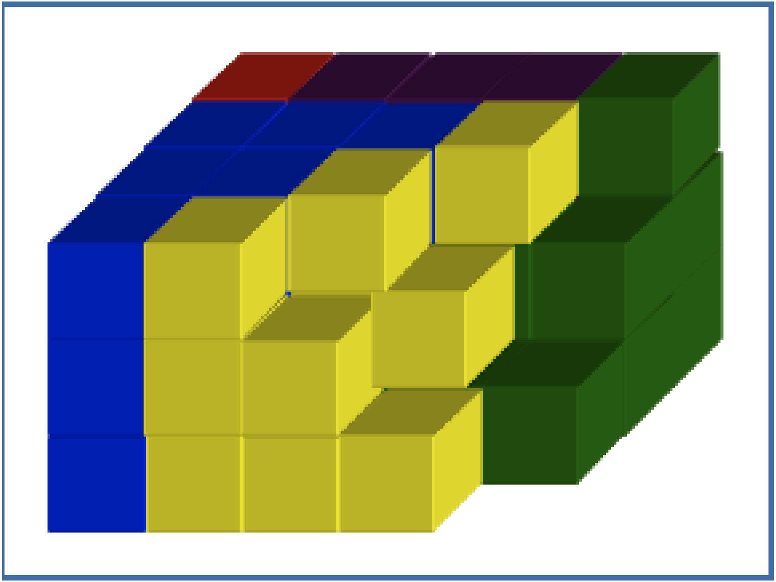


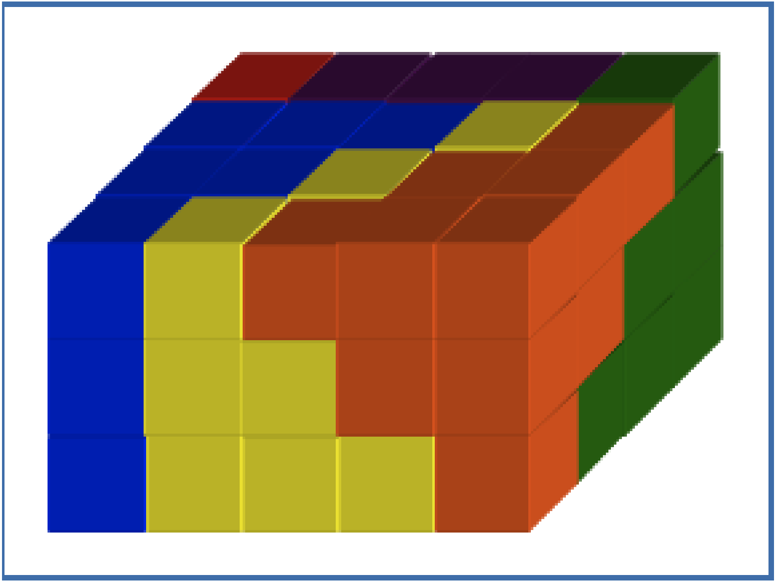












"Simplexal Numbers"

We can prove by induction that

$$T_k(n) = \left(\binom{n}{k}\right) = \frac{n(n+1)(n+2)\cdots(n+k-1)}{k!}$$

Which means that geometrically, a k-cube can be divided into $k!$ simplices.
How many ways are there to do that?

Parameters

- How many ways can an n-cube be divided into n-simplices?

Parameters

- How many ways can an *n*-cube be divided into *n*-simplices?
  - where each simplex has the same volume,

Parameters

- How many ways can an n-cube be divided into n-simplices?
  - where each simplex has the same volume,
  - and no new vertices are made.

Parameters

- How many ways can an n-cube be divided into n-simplices?
  - where each simplex has the same volume,
  - and no new vertices are made.
    - (i.e. The vertices of each simplex are chosen from the vertices of the original cube.)

How do Simplices Fit into a Cube?

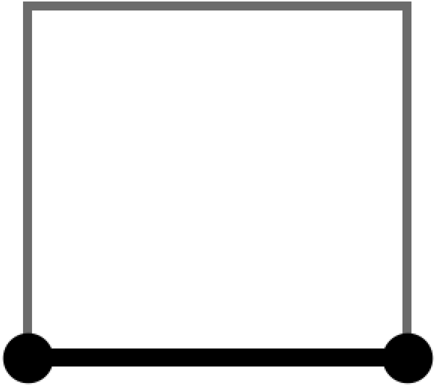
Zero and One Dimensions

A point in a point

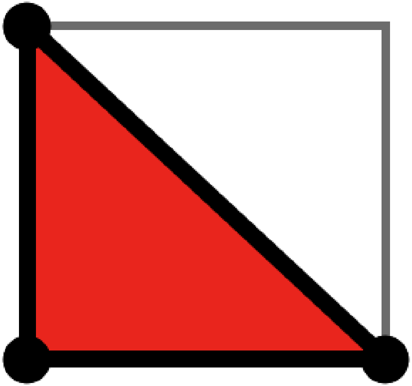
A line segment in a line segment

Two Dimensions

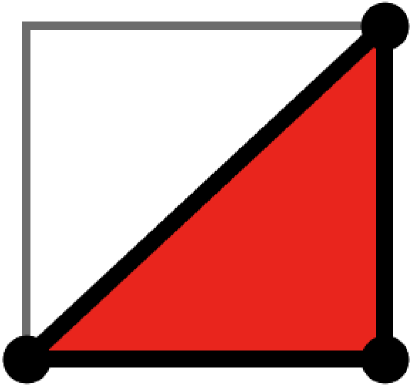
Two Dimensions



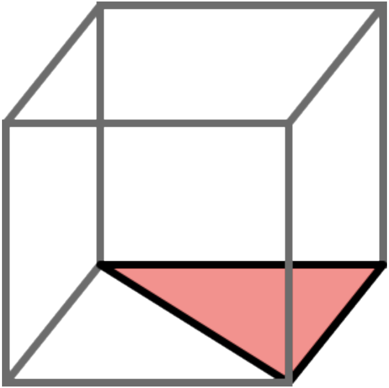
Two Dimensions



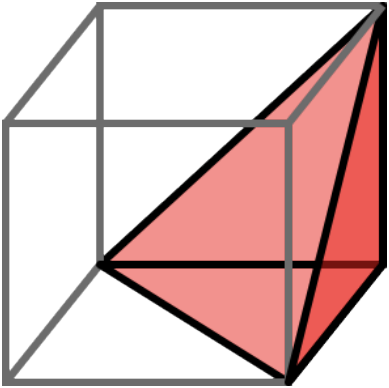
Two Dimensions



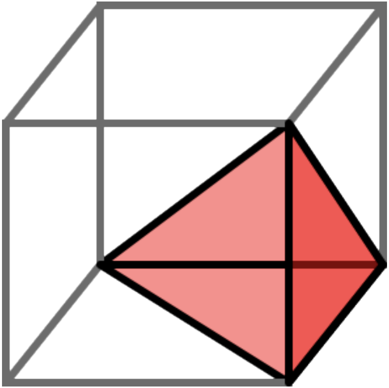
Three Dimensions



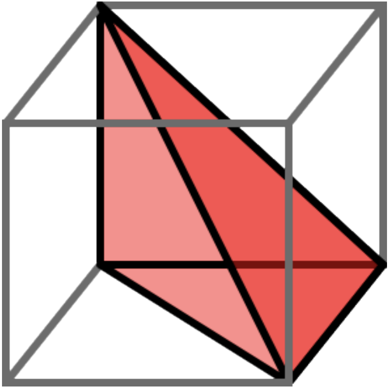
Three Dimensions



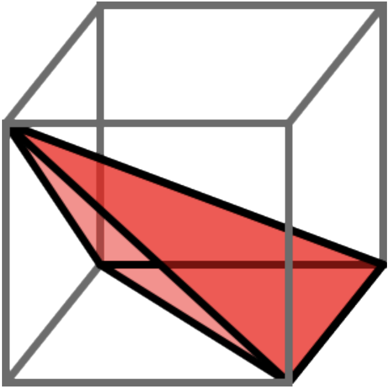
Three Dimensions



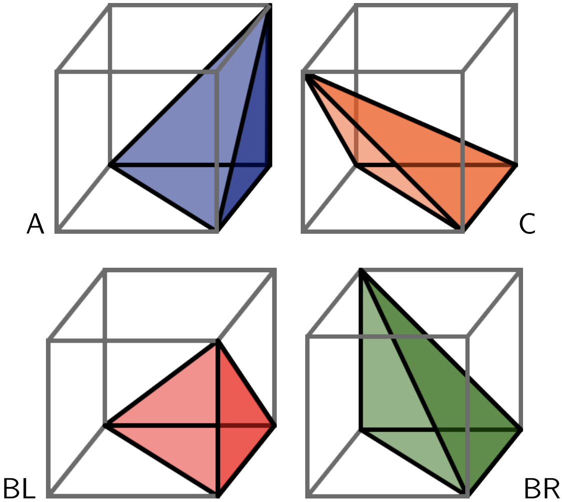
Three Dimensions



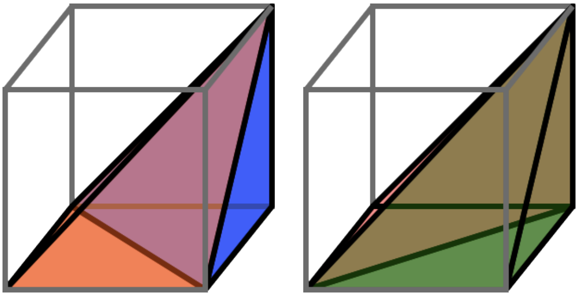
Three Dimensions



Three Dimensions



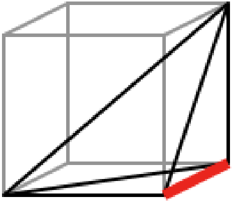
Pyramids

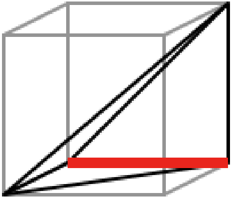


With linear algebra, it can be shown that **A**s and **C**s always come together in pairs like that. Which means that every tiling with **A** and **C**s in it has a corresponding tiling with only **B**s.

Tilings with only **B** tetrahedra

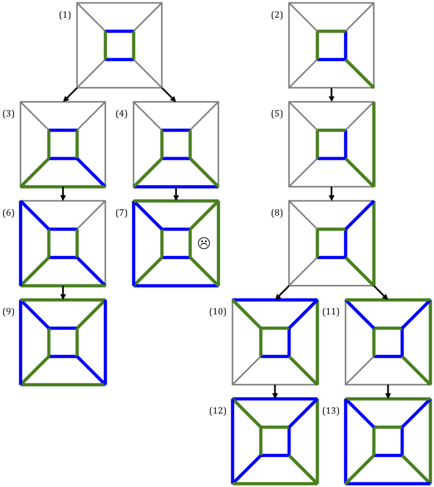
Since every **B** tetrahedron has exactly one edge with a 90° angle, the problem can be simplified to choosing six edges from the twelve edges of the cube. Let these chosen edges be called "B edges."





The following rules must be followed when choosing B edges.

1. Six B edges must be chosen.
   - Since $b = 6 - 2a$, and $a = 0$, there are exactly six **B** tetrahedra.
2. Each square face must have exactly two B edges.
   - Since each square face is made of exactly two triangular faces of **B** tetrahedra.
3. Each vertex can have at most two B edges.


(1)
(2)
(3)
(4)
(5)
(6)
(7)
☹
(8)
(9)
(10)
(11)
(12)
(13)

The above figure shows a two-dimensional projection of the cube. The edges highlighted blue are where B edges are, and the the edges highlighted green are where B edges *aren't*.
A particular face could have the two B edges being opposite from each other (1) or adjacent (2). (2) includes an extra green edge because of rule 3.

- Cube (1)
  A second face adjacent to the first face can either have the B edges
  adjacent (3) or opposite (4).
  - Cube (3)
    Rule 2 adds two B edges (6).
    Rule 2 applied again completes the cube (9). Cube (9) follows every rule, and so it is valid. ✓
  - Cube (4)
    This cube is impossible, by rule 2.
- Cube (4)
  A green edge is added to make cube (5) distinct from cube (3)