Decomposing the d-Cube into Simplices

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Scripps College

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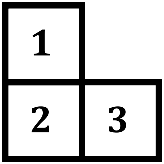
Triangular Numbers

$$1 + 2 + 3 + \cdots + n$$

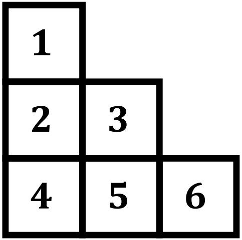
Triangular Numbers

$\boxed{1}$

Triangular Numbers



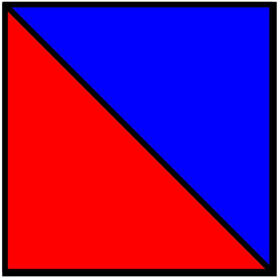
Triangular Numbers

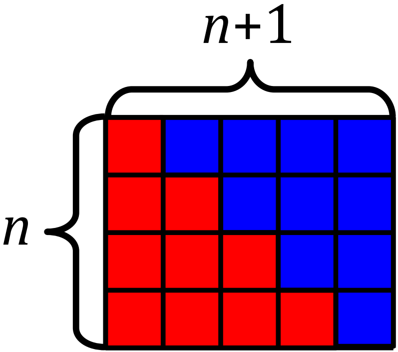


Triangular Numbers

1			
2	**3**		
4	**5**	**6**	
7	**8**	**9**	**10**

Triangular Numbers





$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Pascal's Triangle

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Pascal's Triangle

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Hockeystick Theorem

1

1 **1**

+

1 **2** 1

=

1 3 **3** 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Hockeystick Theorem

1

1 **1**

\+

1 **2** 1

\+

1 **3** 3 1

=

1 4 **6** 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Hockeystick Theorem

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & \mathbf{1} & & & & & \\
 & & & & & & + & & & & & & \\
 & & & & 1 & & \mathbf{2} & & 1 & & & & \\
 & & & & & + & & & & & & & \\
 & & & 1 & & \mathbf{3} & & 3 & & 1 & & & \\
 & & & & + & & & & & & & & \\
 & & 1 & & \mathbf{4} & & 6 & & 4 & & 1 & & \\
 & & & & = & & & & & & & & \\
 & 1 & & 5 & & \mathbf{10} & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Hockeystick Theorem

n: 1, 2, 3, 4, 5, 6, 7

k: 0, 1, 2, 3, 4, 5, 6

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

$$\frac{n(n+1)\cdots(n+k-1)}{k!} = \frac{(n+k-1)!}{(n-1)!\,k!}$$

Hockeystick Theorem

The triangular numbers are when $k = 2$,

so the nth triangular number is $\dfrac{n(n+1)}{2}$.

Hockeystick Theorem

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & \mathbf{1} & & & & \\
 & & & & & & & + & & & & & \\
 & & & 1 & & 3 & & \mathbf{3} & & 1 & & & \\
 & & & & & & + & & & & & & \\
 & & 1 & & 4 & & \mathbf{6} & & 4 & & 1 & & \\
 & & & & & + & & & & & & & \\
 & 1 & & 5 & & \mathbf{10} & & 10 & & 5 & & 1 & \\
 & & & & & = & & & & & & & \\
1 & & 6 & & 15 & & \mathbf{20} & & 15 & & 6 & & 1
\end{array}
$$

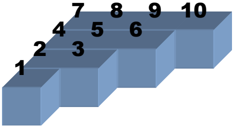
$$
\begin{aligned}
&1 + (1+2) + (1+2+3) + \cdots + (1+\cdots+n) \\
&= \frac{1\cdot 2}{2} + \frac{2\cdot 3}{2} + \frac{3\cdot 4}{2} + \cdots + \frac{n(n+1)}{2}
\end{aligned}
$$

Hockeystick Theorem

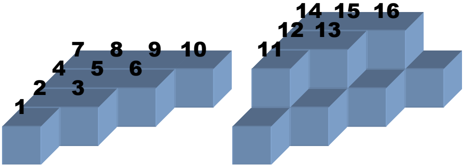
$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & \mathbf{1} & & & & \\
 & & & & & & & + & & & & & \\
 & & & 1 & & 3 & & \mathbf{3} & & 1 & & & \\
 & & & & & & + & & & & & & \\
 & & 1 & & 4 & & \mathbf{6} & & 4 & & 1 & & \\
 & & & & & + & & & & & & & \\
 & 1 & & 5 & & \mathbf{10} & & 10 & & 5 & & 1 & \\
 & & & & & = & & & & & & & \\
1 & & 6 & & 15 & & \mathbf{20} & & 15 & & 6 & & 1
\end{array}
$$

$$
\begin{aligned}
&1 + (1+2) + (1+2+3) + \cdots + (1 + \cdots + n) \\
&= \frac{1 \cdot 2}{2} + \frac{2 \cdot 3}{2} + \frac{3 \cdot 4}{2} + \cdots + \frac{n(n+1)}{2} \\
&= \frac{n(n+1)(n+2)}{6}
\end{aligned}
$$

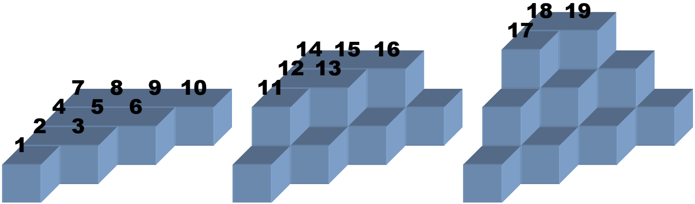
Tetrahedral Numbers



Tetrahedral Numbers



Tetrahedral Numbers



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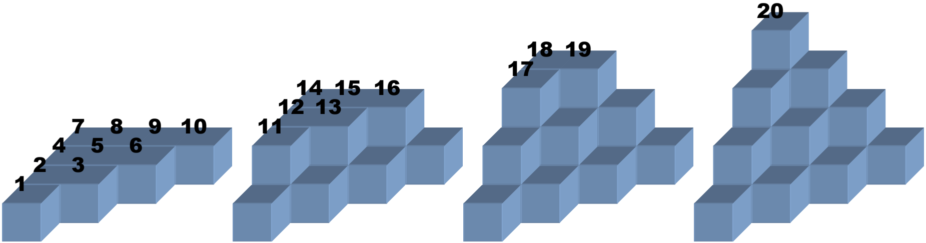
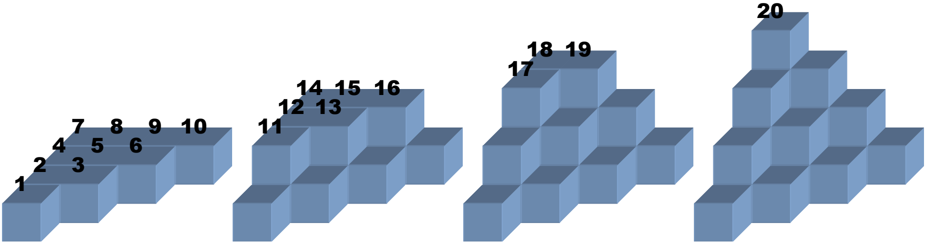


Tetrahedral Numbers



$$1 + 3 + 6 + 10 = 20$$

Tetrahedral Numbers

- $\text{Triangle}(n) = \dfrac{n(n+1)}{2}$.

Tetrahedral Numbers

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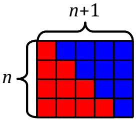
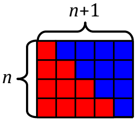


Tetrahedral Numbers

- $\text{Triangle}(n) = \dfrac{n(n+1)}{2}$.



- $\text{Tetrahedron}(n) = \dfrac{n(n+1)(n+2)}{6}$

Tetrahedral Numbers

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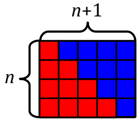
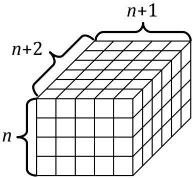
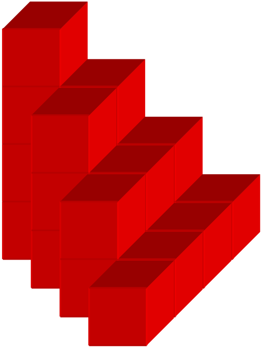


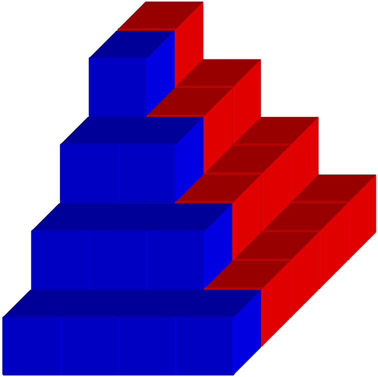
- $\text{Tetrahedron}(n) = \dfrac{n(n+1)(n+2)}{6}$



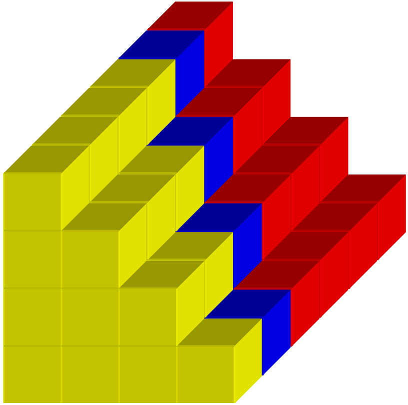
Tetrahedral Numbers



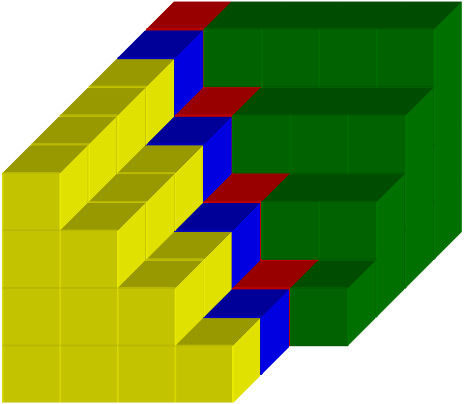
Tetrahedral Numbers



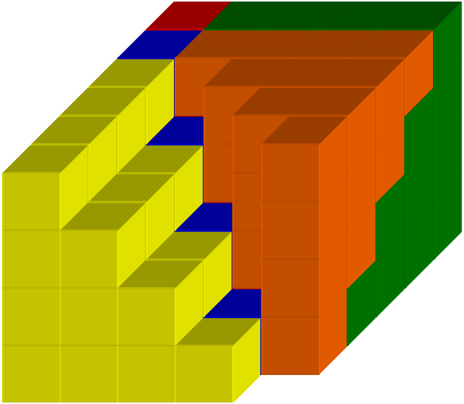
Tetrahedral Numbers



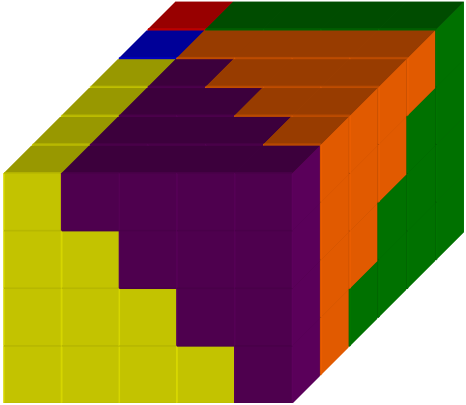
Tetrahedral Numbers



Tetrahedral Numbers

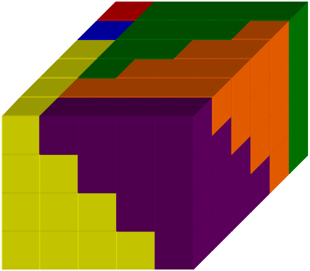


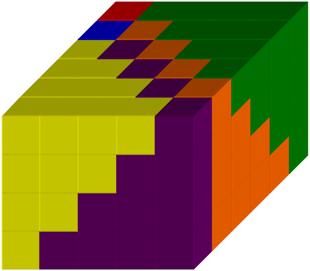
Tetrahedral Numbers



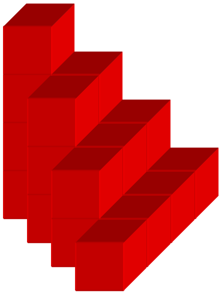
Tetrahedral Numbers

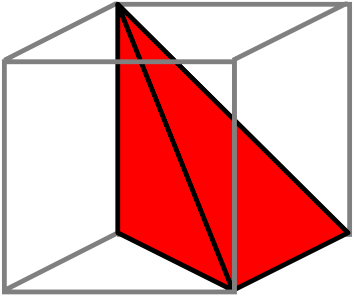
There is more than one way to divide up the rectangular prism.

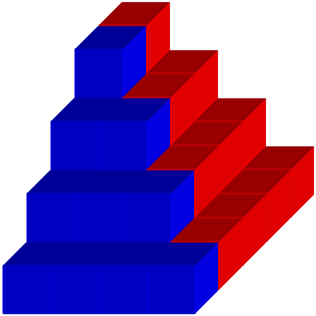


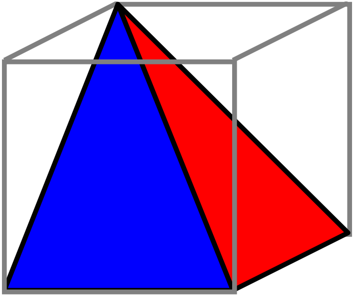


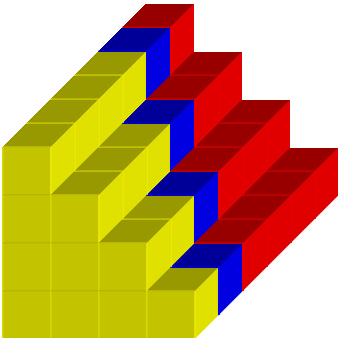
How many ways are there?

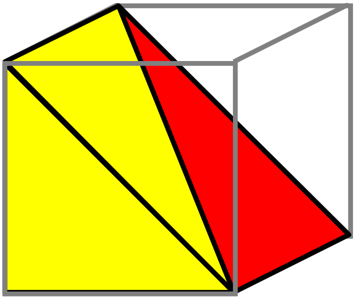


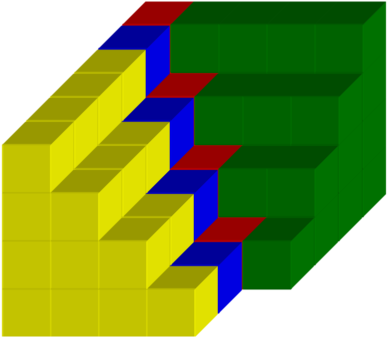


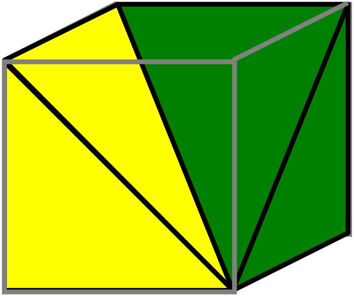


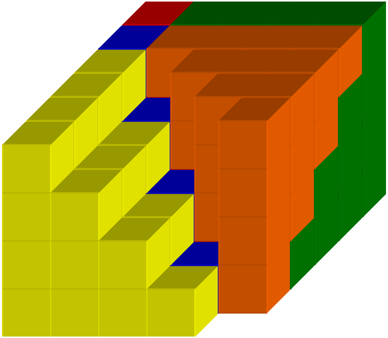


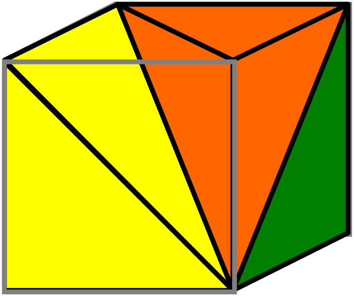


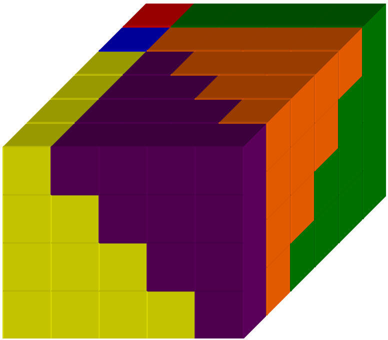


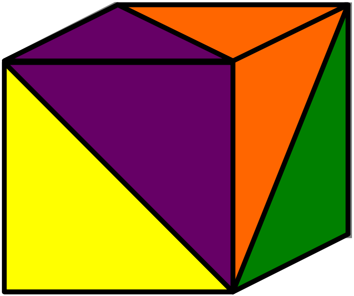




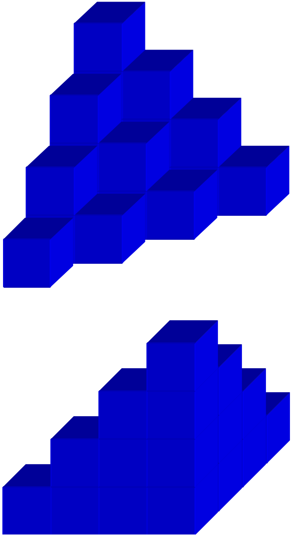


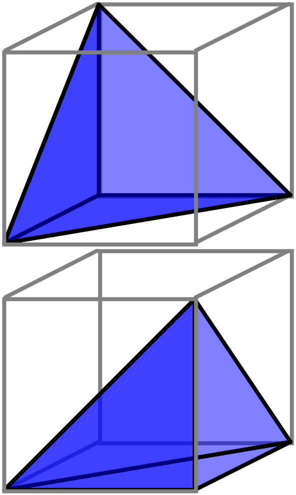




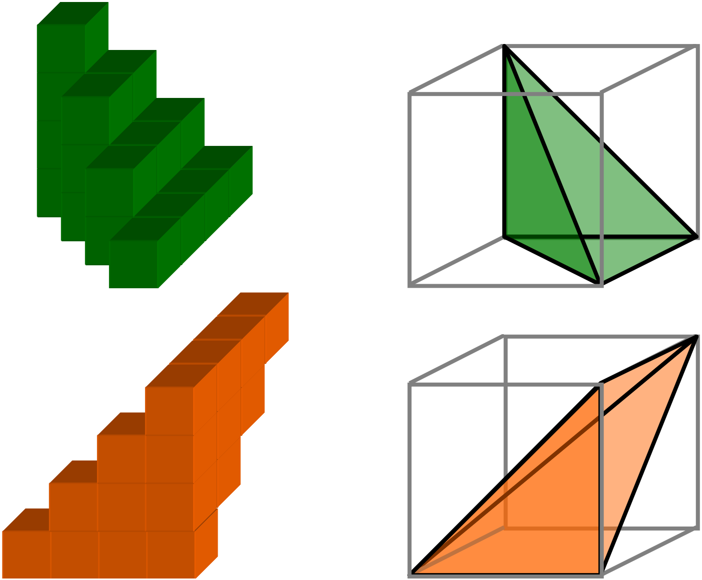


A Tetrahedron

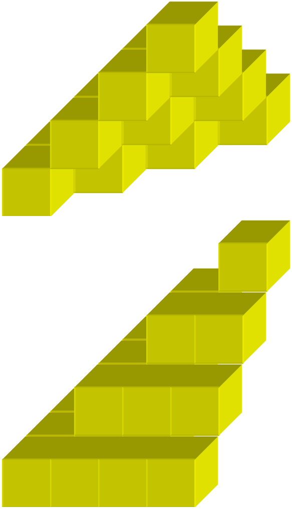


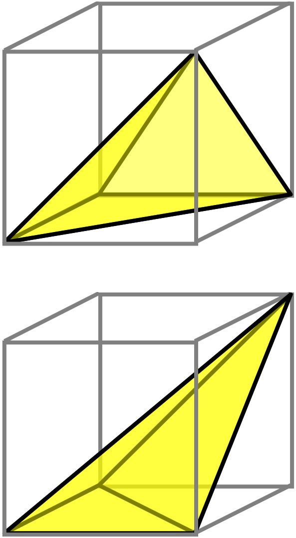


B Tetrahedron

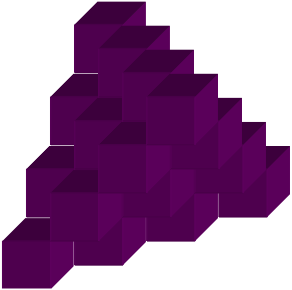


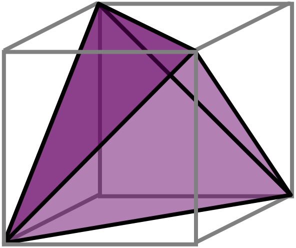
C Tetrahedron



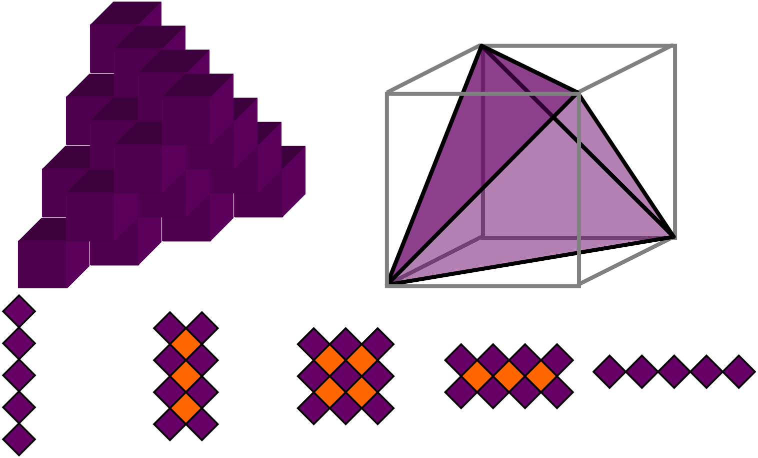


D Tetrahedron

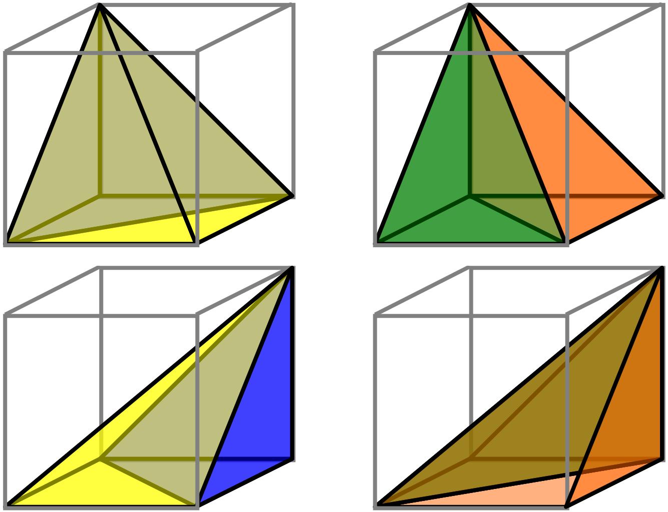




D Tetrahedron



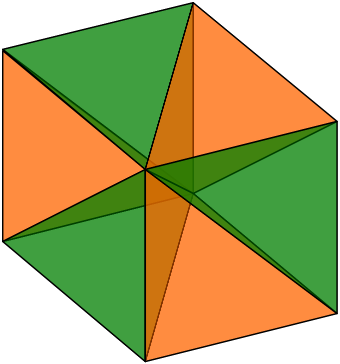
Lemma

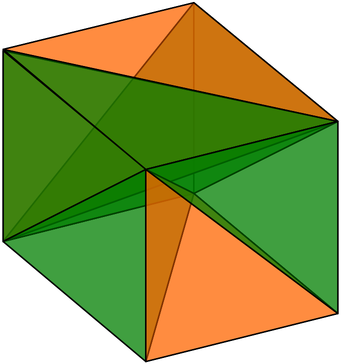


Lemma

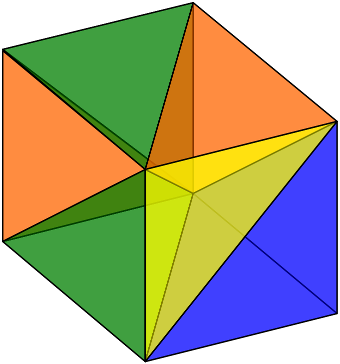
- **A**s and **C**s always come together in pairs in the shape of a square pyramid like this.
- For any given tiling, every **AC** pair can be replaced with a pair of **B**s, creating a tiling with only **B**s.
- So, every tiling with only **B**s can be used to generate a set of tilings made of **A**s, **B**s, and **C**s.

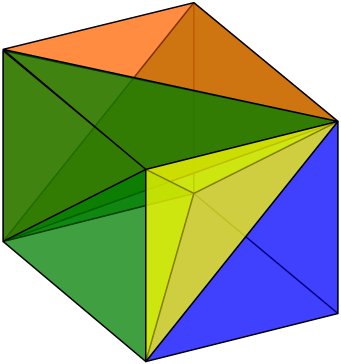
Decompositions With Only **B** Tetrahedra



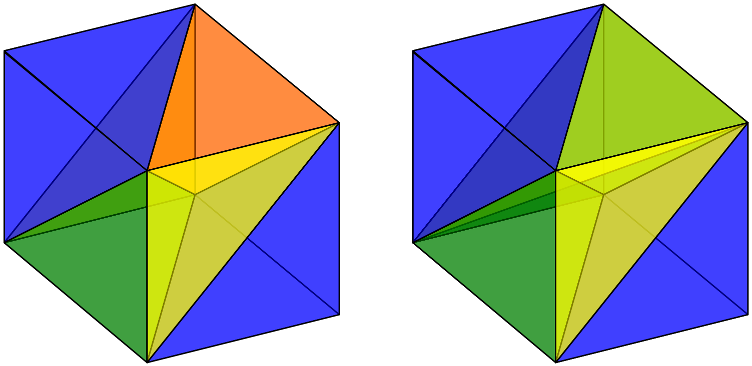


Decompositions of the Cube

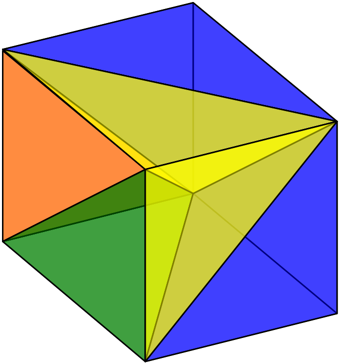


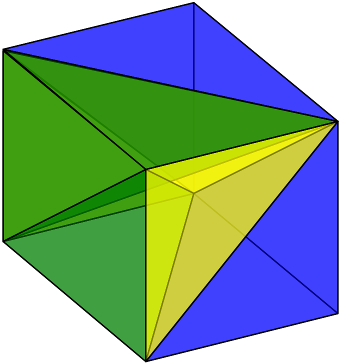


Decompositions of the Cube



Decompositions of the Cube





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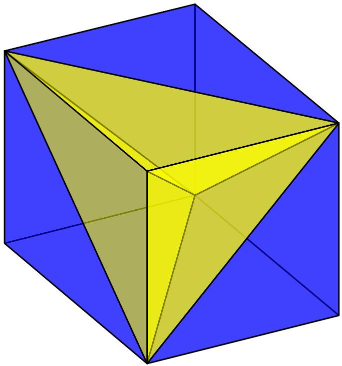

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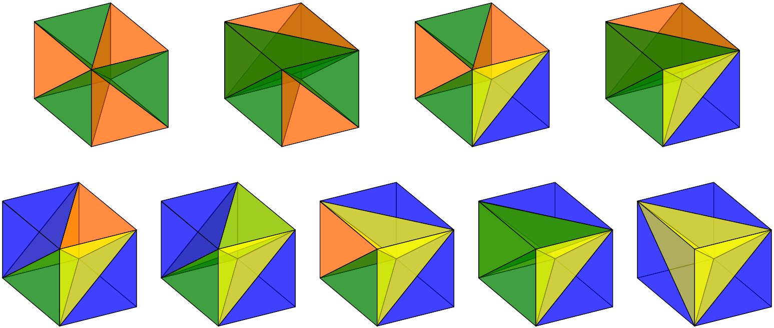


What's next?

What's next?

Four dimensions!

Hockeystick Theorem, Again

1

1 1

1 2 1

1 3 3 **1**

+

1 4 6 **4** 1

+

1 5 10 **10** 5 1

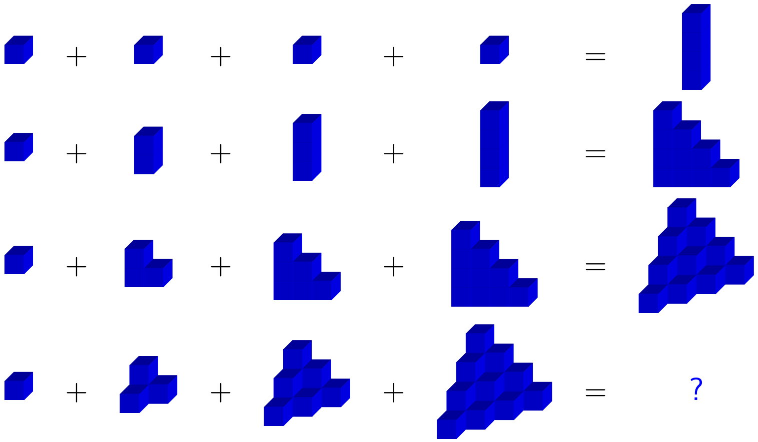
+

1 6 15 **20** 15 6 1

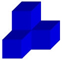
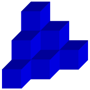
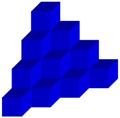
=

1 7 21 35 **35** 21 7 1

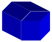
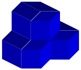
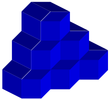
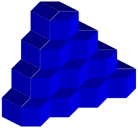
Hockeystick Theorem



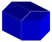
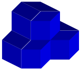
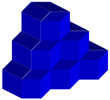
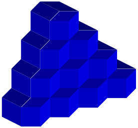
Hockeystick Theorem

 $+$  $+$  $+$  $=$

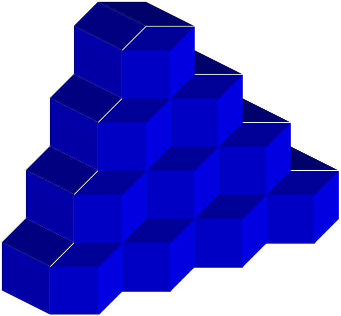
$$1 + 4 + 10 + 20 = 35$$

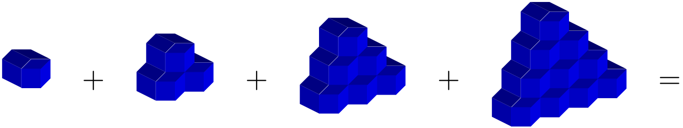
 $+$  $+$  $+$  $=$

$$1 + 4 + 10 + 20 = 35$$

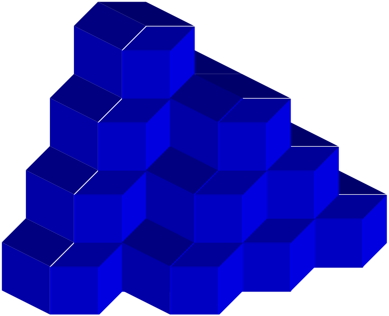
 $+$  $+$  $+$  $=$

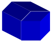
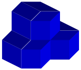
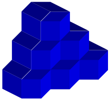
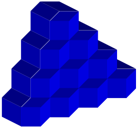
$$1 + 4 + 10 + 20 = 35$$



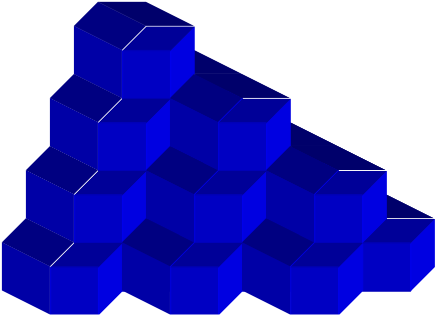


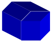
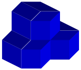
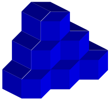
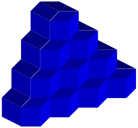
$$1 + 4 + 10 + 20 = 35$$



 +  +  +  =

1 + 4 + 10 + 20 = 35



 $+$  $+$  $+$  $=$

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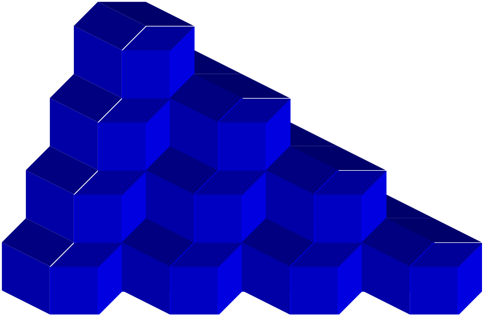

Simplices

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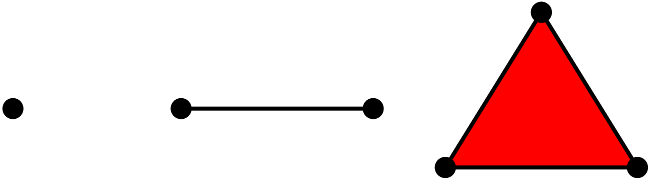

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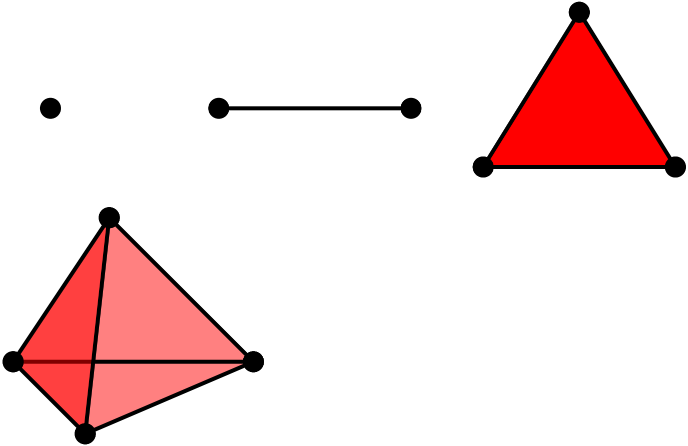

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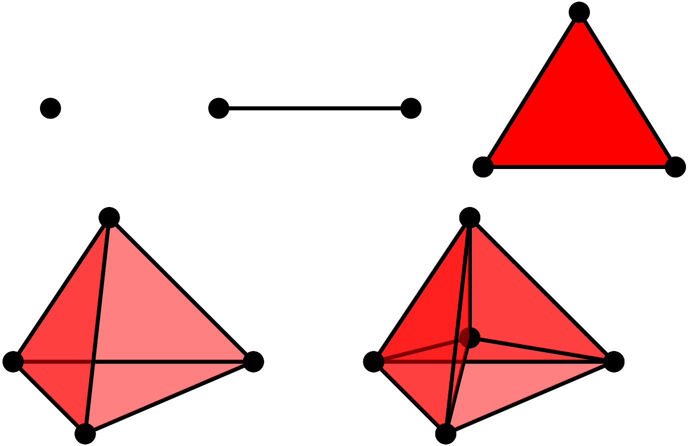
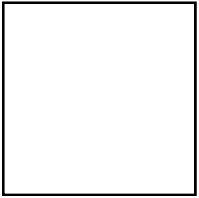


$$\frac{n(n+1)}{2!}$$

$$\frac{n(n+1)(n+2)}{3!}$$

$$\frac{n(n+1)(n+2)(n+3)}{4!}$$

Standard Decomposition



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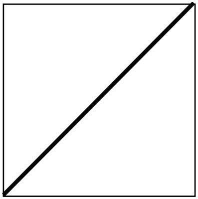

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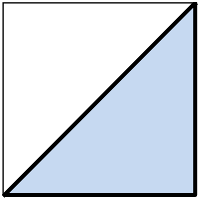

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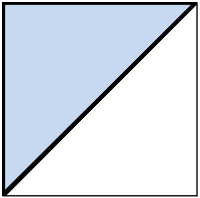

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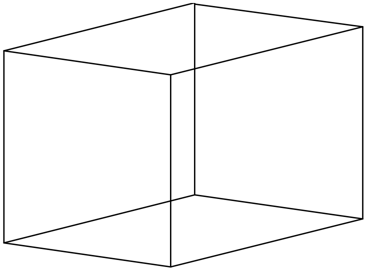

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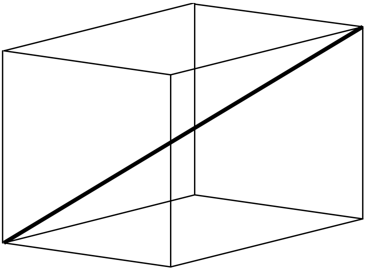

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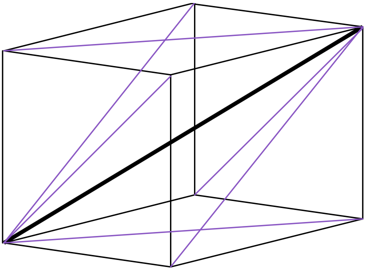

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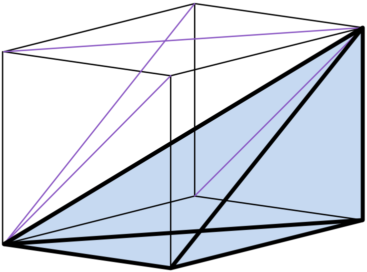

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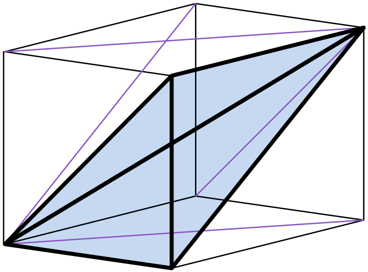

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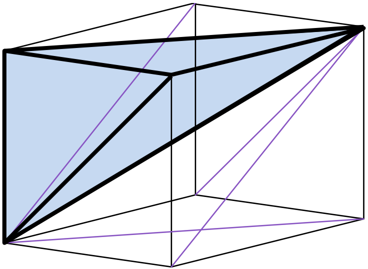

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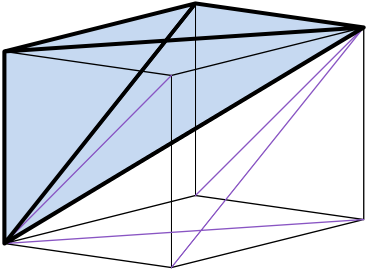

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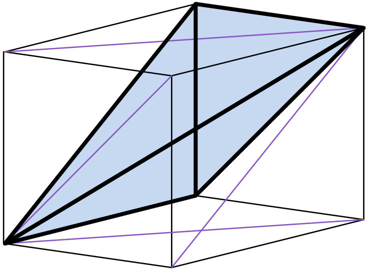

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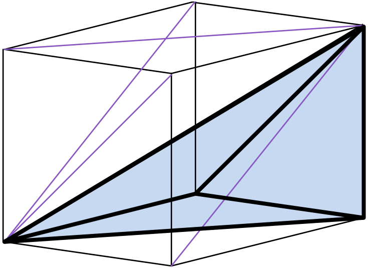

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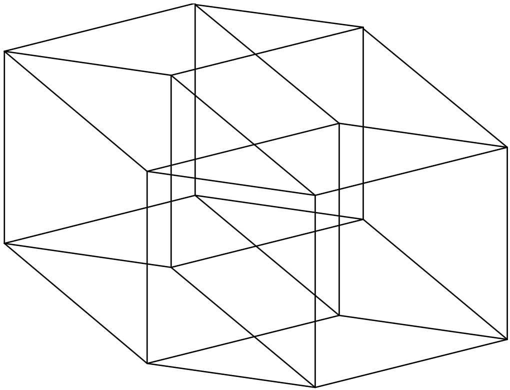

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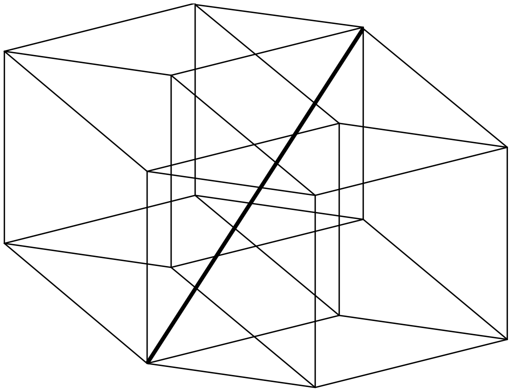

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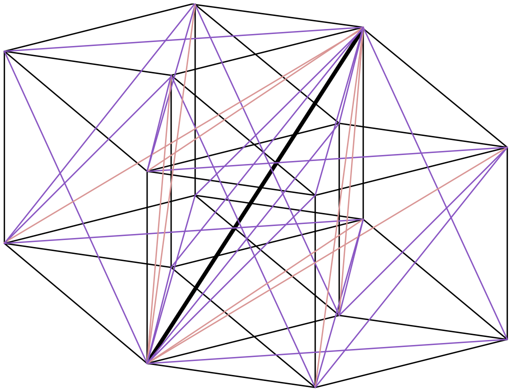

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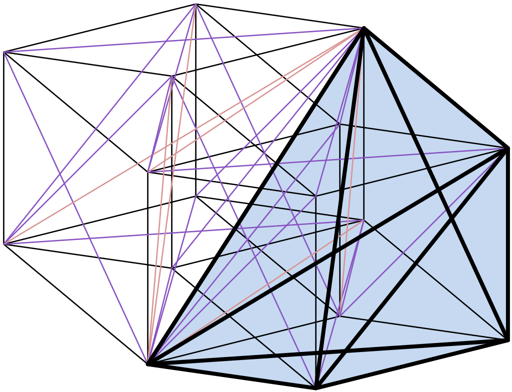

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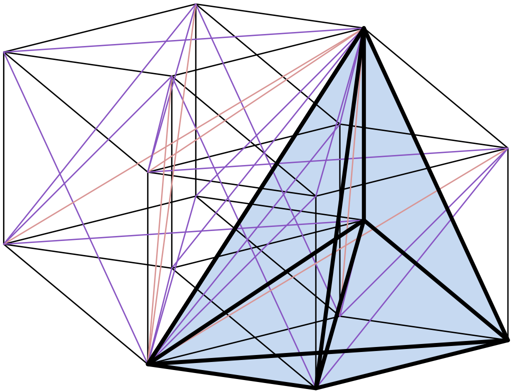

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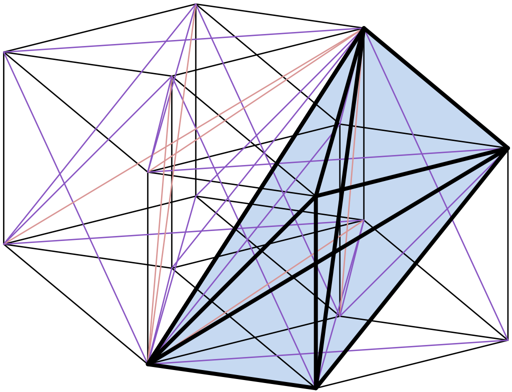

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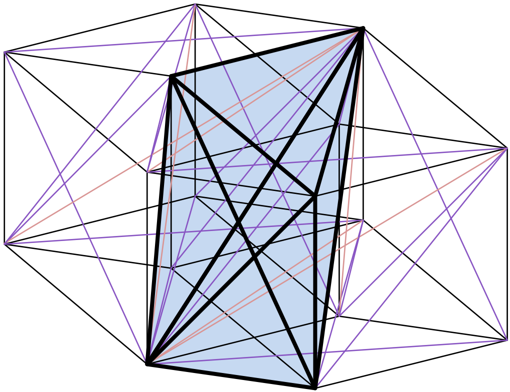

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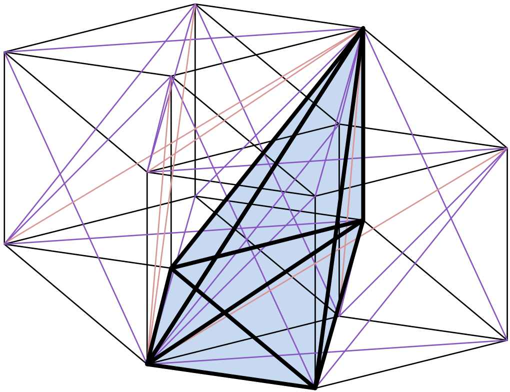

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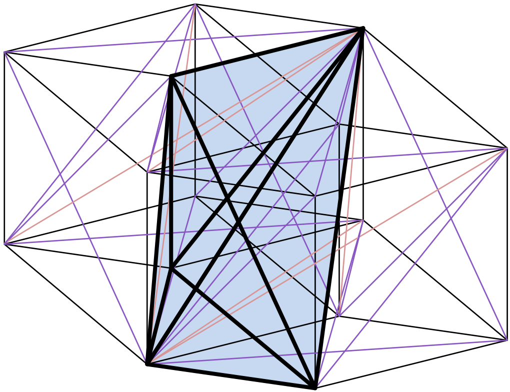

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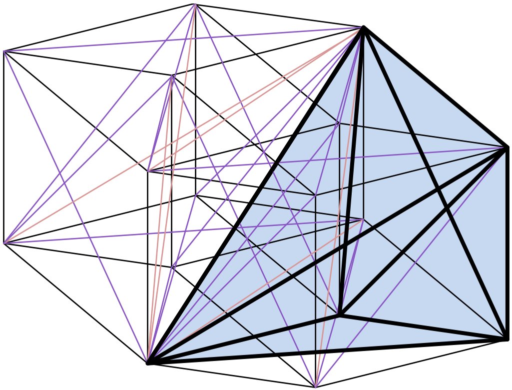

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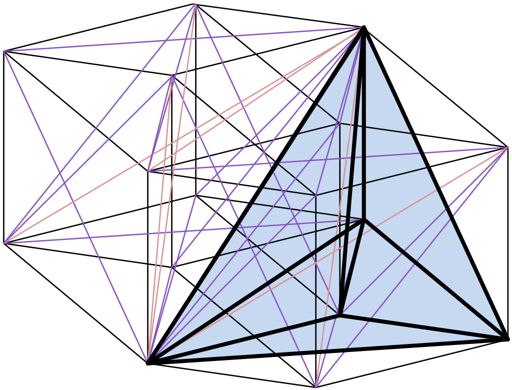

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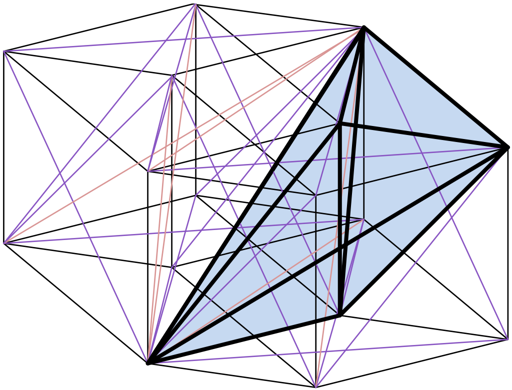

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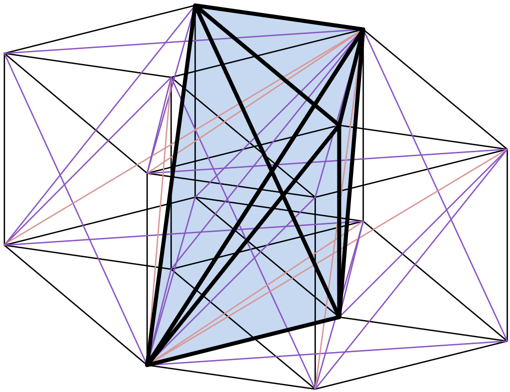

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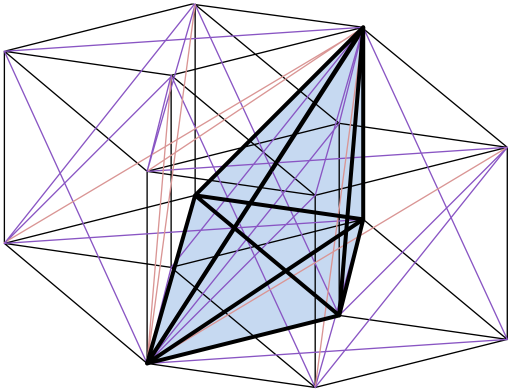

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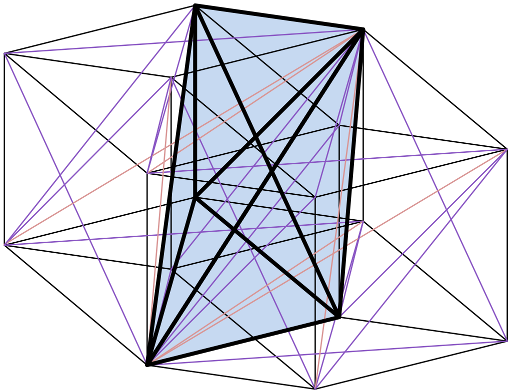

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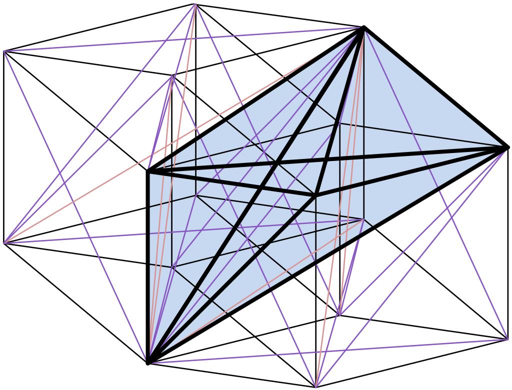

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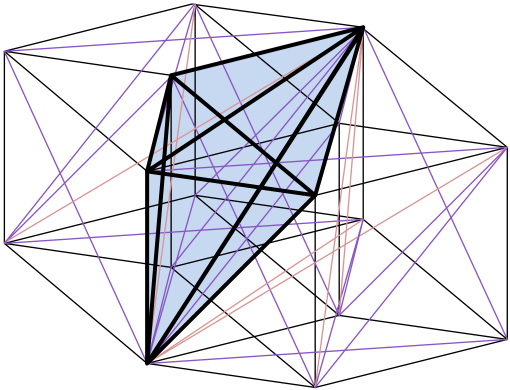

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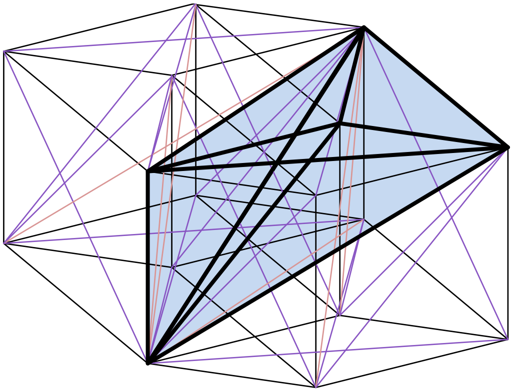

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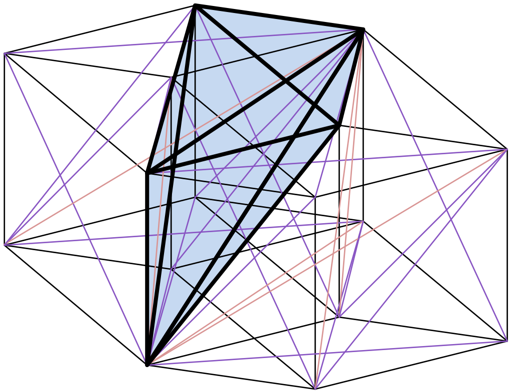

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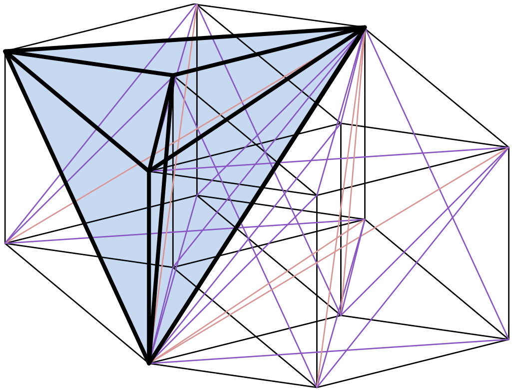

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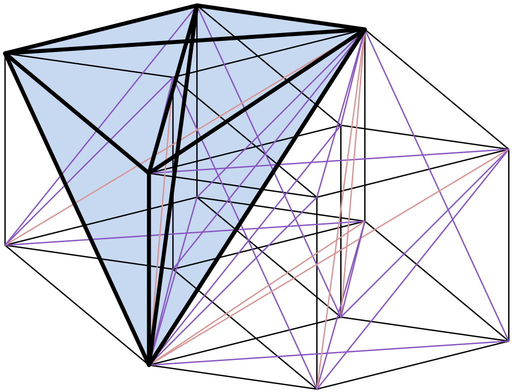

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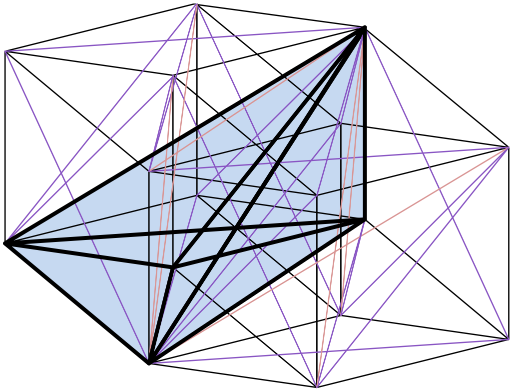

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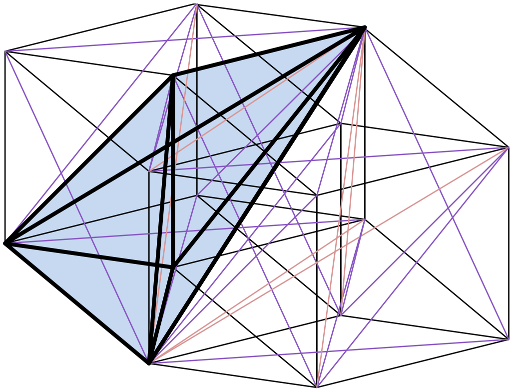

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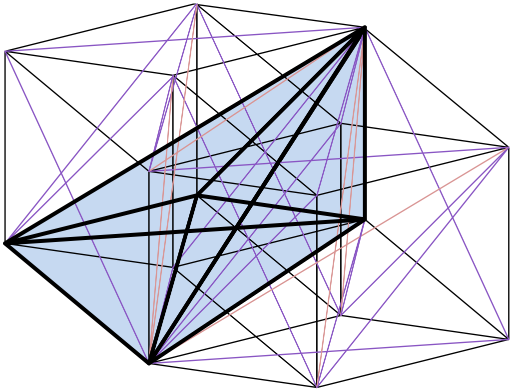

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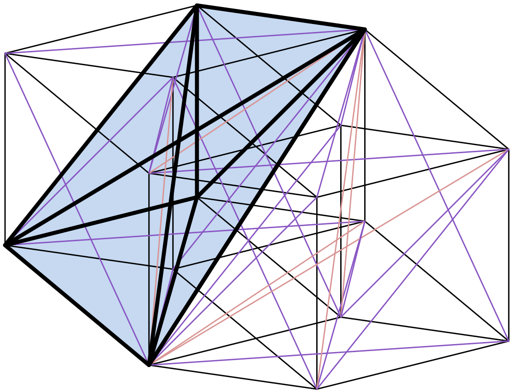

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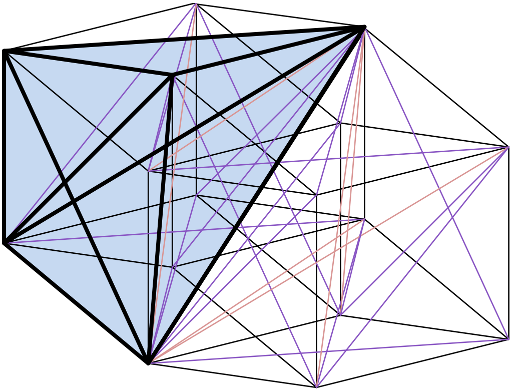

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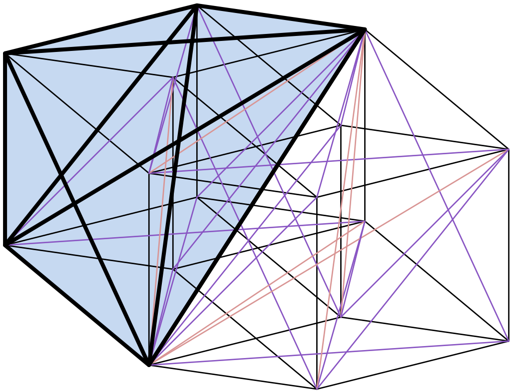

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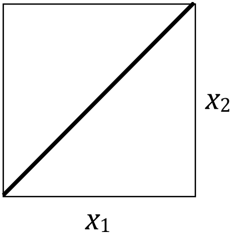
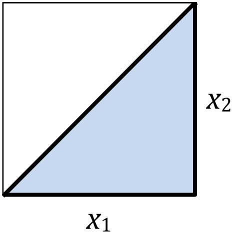
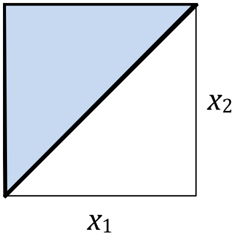


$x_1 \geq x_2$



$x_2 \geq x_1$



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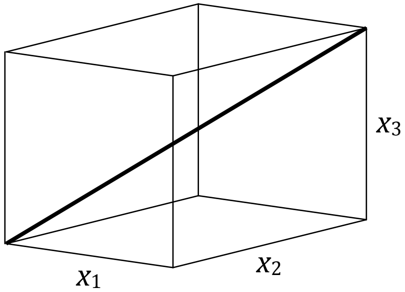

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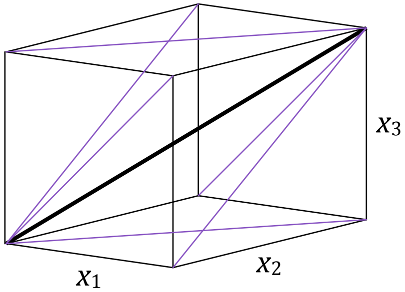
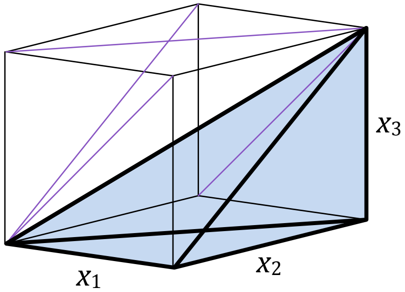
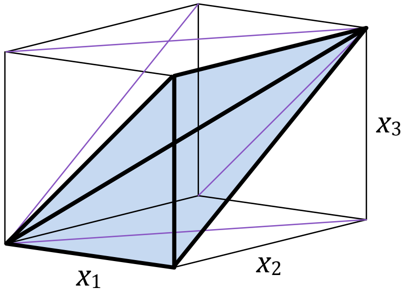


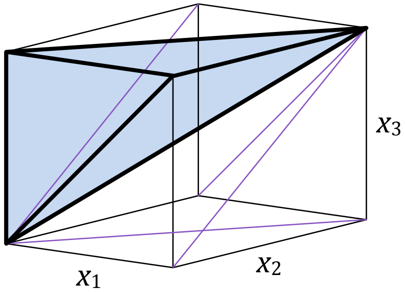
$x_1 \geq x_2 \geq x_3$



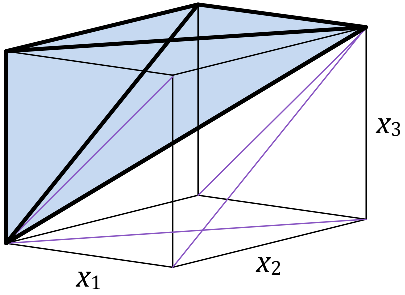
$x_1 \geq x_3 \geq x_2$



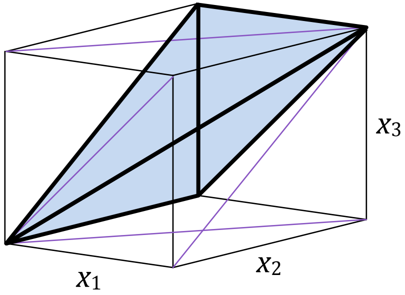
$x_3 \geq x_1 \geq x_2$



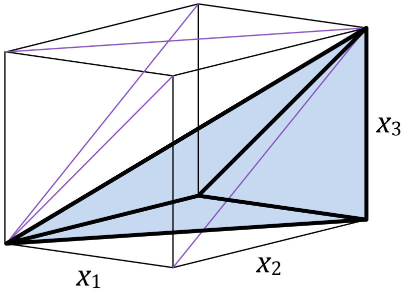
$x_3 \geq x_2 \geq x_1$

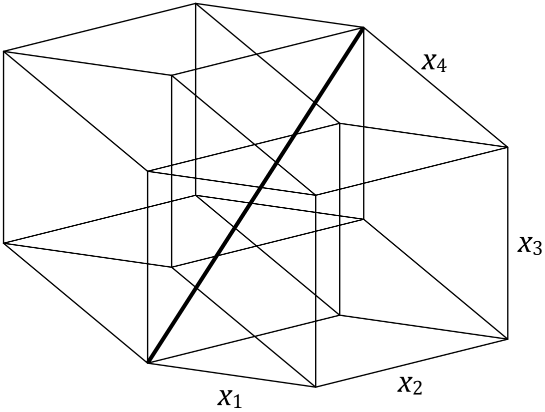


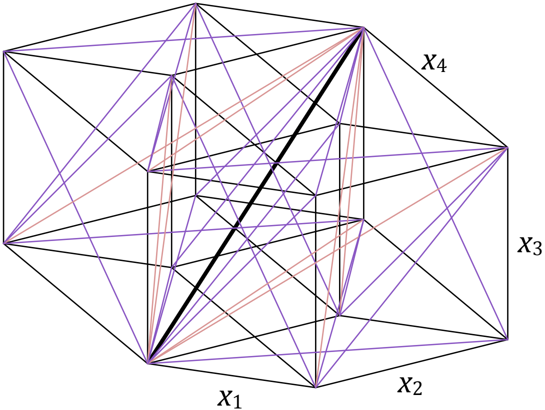
$x_2 \geq x_3 \geq x_1$



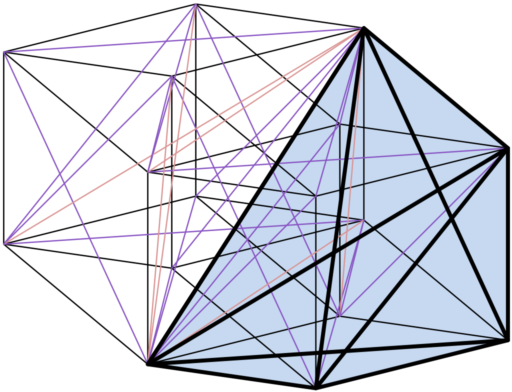
$x_2 \geq x_1 \geq x_3$



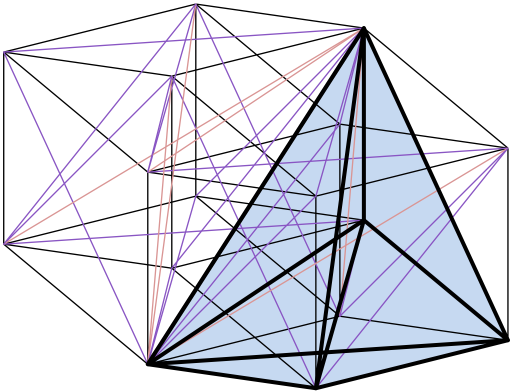

x_4
x_3
x_1
x_2


x_4
x_3
x_2
x_1

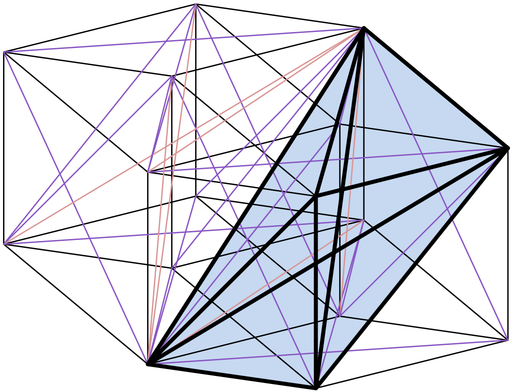
$x_1 \geq x_2 \geq x_3 \geq x_4$



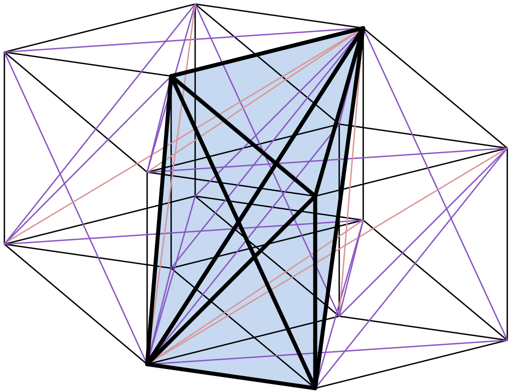
$x_1 \geq x_2 \geq x_4 \geq x_3$



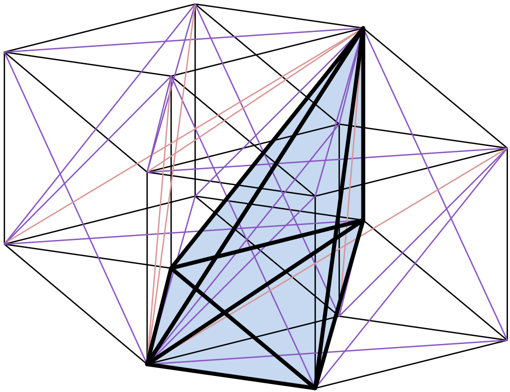
$x_1 \geq x_3 \geq x_2 \geq x_4$



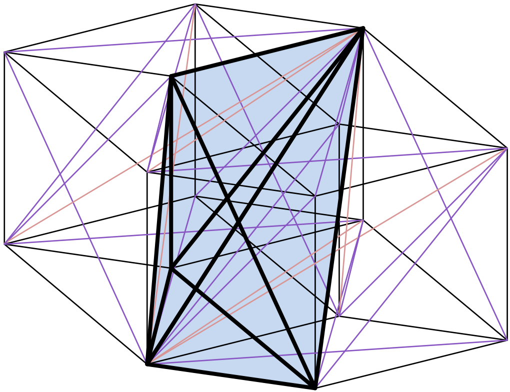
$x_1 \geq x_3 \geq x_4 \geq x_2$



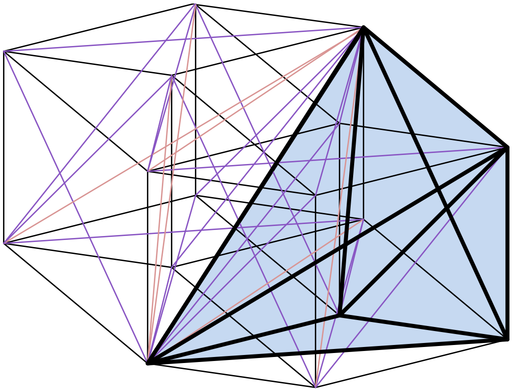
$x_1 \geq x_4 \geq x_2 \geq x_3$



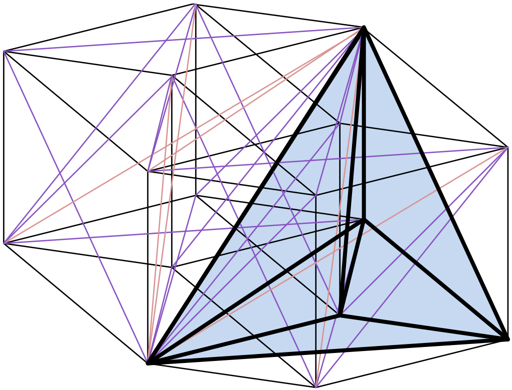
$x_1 \geq x_4 \geq x_3 \geq x_2$



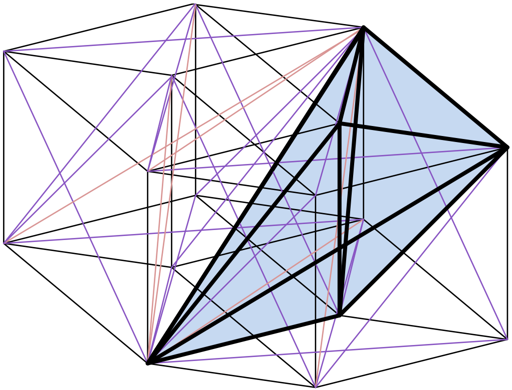
$x_2 \geq x_1 \geq x_3 \geq x_4$



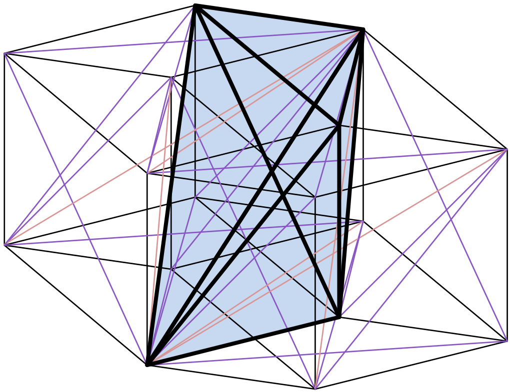
$x_2 \geq x_1 \geq x_4 \geq x_3$



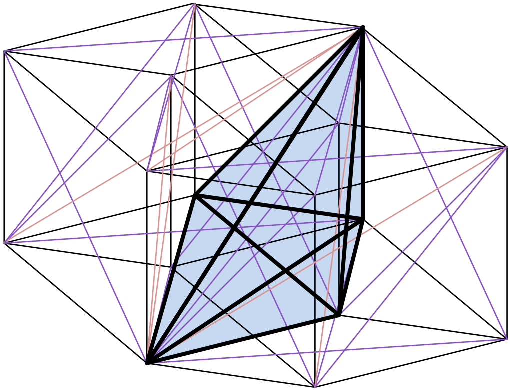
$x_2 \geq x_3 \geq x_1 \geq x_4$



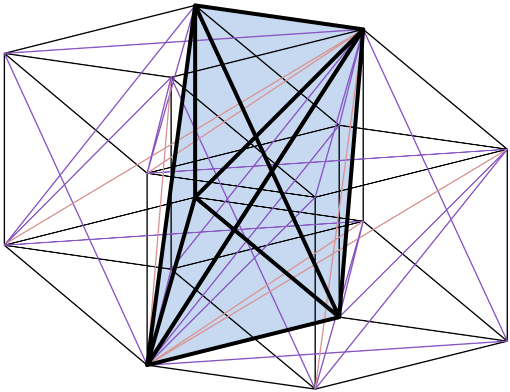
$x_2 \geq x_3 \geq x_4 \geq x_1$



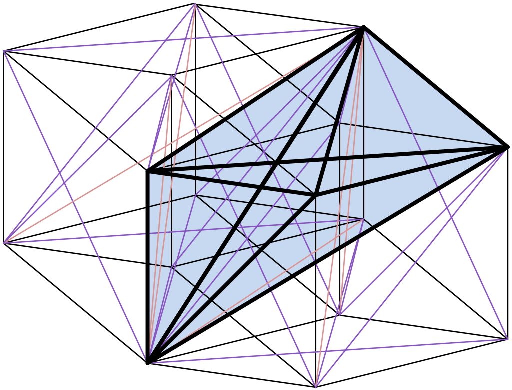
$x_2 \geq x_4 \geq x_1 \geq x_3$



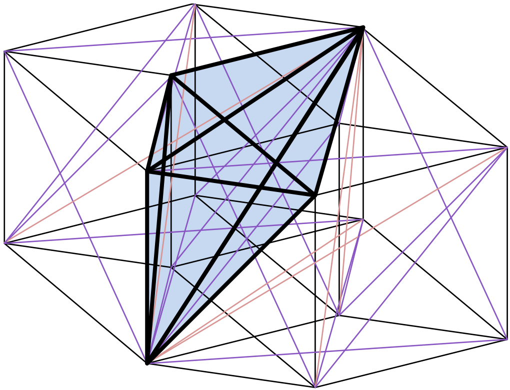
$x_2 \geq x_4 \geq x_3 \geq x_1$



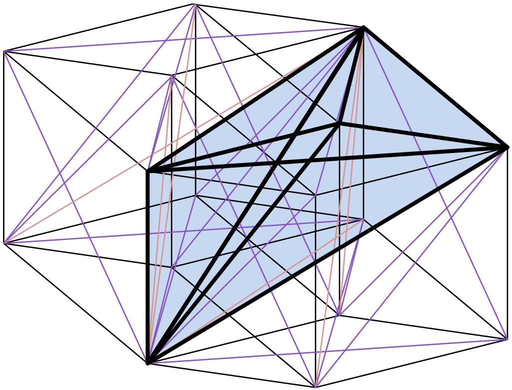
$x_3 \geq x_1 \geq x_2 \geq x_4$



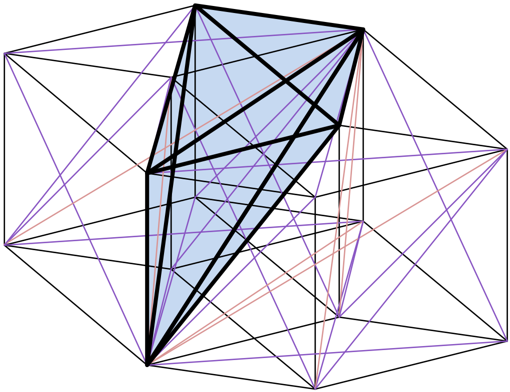
$x_3 \geq x_1 \geq x_4 \geq x_2$



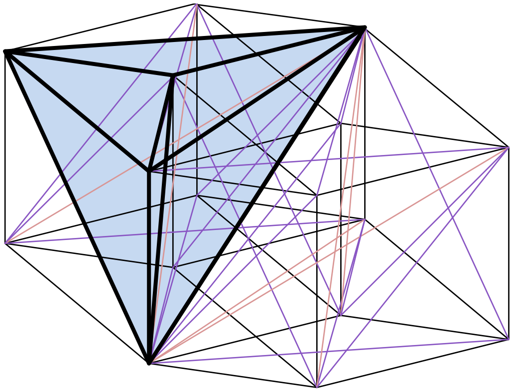
$x_3 \geq x_2 \geq x_1 \geq x_4$



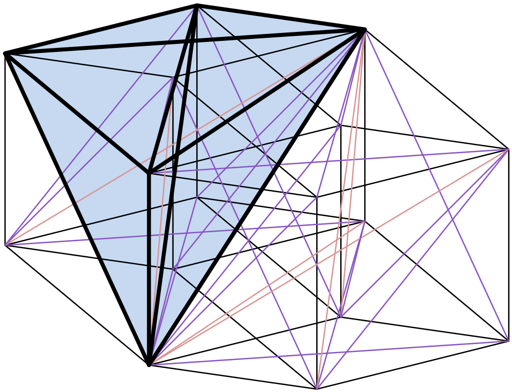
$x_3 \geq x_2 \geq x_4 \geq x_1$



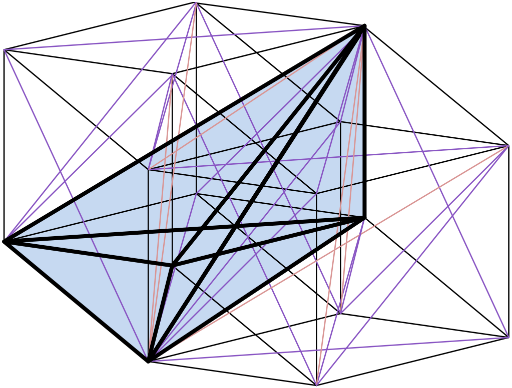
$x_3 \geq x_4 \geq x_1 \geq x_2$



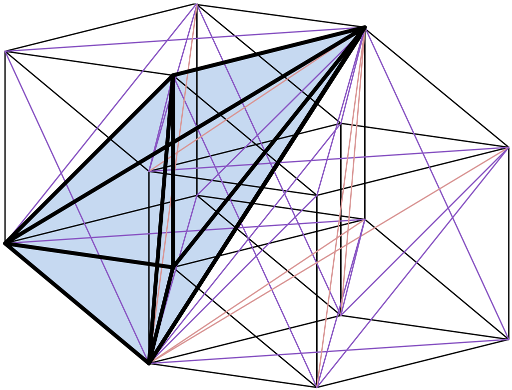
$x_3 \geq x_4 \geq x_2 \geq x_1$



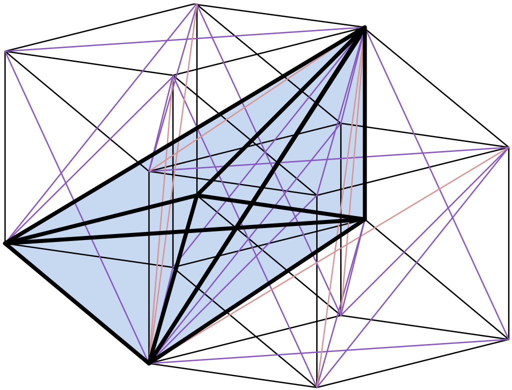
$x_4 \geq x_1 \geq x_2 \geq x_3$

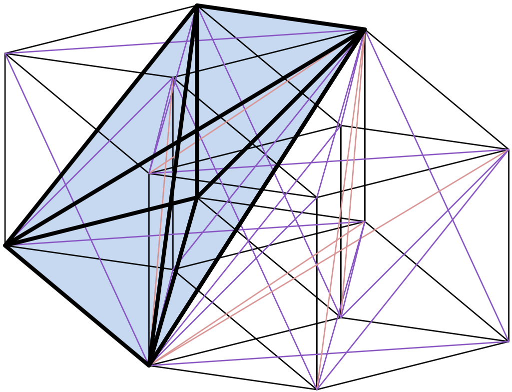


$x_4 \geq x_1 \geq x_3 \geq x_2$

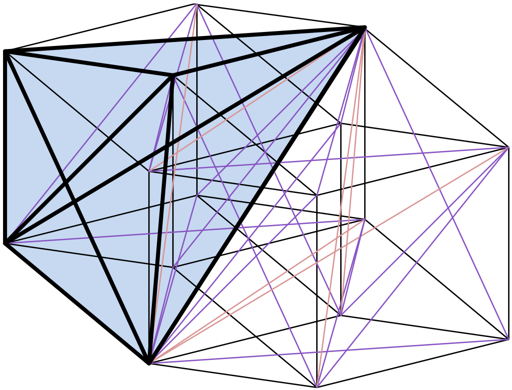


$x_4 \geq x_2 \geq x_1 \geq x_3$





$x_4 \geq x_3 \geq x_1 \geq x_2$



$x_4 \geq x_3 \geq x_2 \geq x_1$