PRACTICAL-01

BAYES ESTIMATORS

QUESTION-01

The squared error loss function is given by $(\hat{\theta}-\theta)^2$

The density function is given by

$$f(x/\theta) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(x-\theta)^2\right], \quad -\infty < x < \infty, \; -\infty < \theta < \infty$$

We need to estimate the parameter θ using Bayesian estimation under squared error and absolute error loss functions.

The posterior distribution of θ can be obtained by

$$f(\theta/x) = \frac{\prod_{i=1}^{n} f(x_i,\theta)\, g(\theta)}{\int \prod_{i=1}^{n} f(x_i,\theta)\, g(\theta)\, d\theta}$$

$$= \frac{\prod_{i=1}^{n}\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_i-\theta)^2}\right) \times \frac{1}{\sqrt{2\pi}} e^{-1/2(\theta-\mu_0)^2}}{\int_{-\infty}^{\infty} \prod \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_i-\theta)^2} \times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\theta-\mu_0)^2}\, d\theta}$$

The base estimator of θ under squared error loss function is the posterior mean or mean of the posterior distribution

Here; the posterior distribution is again normal distribution ~~with~~ with mean $\frac{\mu_0 + \sum_{i=1}^{n} x_i}{n+1}$.

Therefore the posterior mean is given by

$$E[\theta|\underline{x}] = \frac{\mu_0 + \sum_{i=1}^{n} x_i}{n+1}$$

(i) $\mu_0 = 3$

$\sum_{i=1}^{n} x_i = 15.27$

$$E[\theta|\underline{x}] = \frac{3+15.27}{11} = 1.6609$$ //

(II) $\mu_0 = 1.5$

$$E[\theta|\underline{x}] = \frac{1.5+15.27}{11} = 1.5245$$ //

Since normal distribution is symmetric distribution mean = median = mode.

Here, the posterior distribution is normal therefore under the absolute error loss function the posterior median is the base estimator for 'θ' therefore posterior mean = posterior median.

$\Rightarrow$ posterior median $= \frac{\mu_0 + \sum_{i=1}^{n} x_i}{n+1}$ = posterior mean

QUESTION 02:

The squared error loss function is given by $(\hat{\theta}-\theta)^2$.
The density function is given by

$$f(x|\theta) = \theta \exp(-\theta x), \; x>0, \; \theta>0$$

- The posterior distribution of θ can be obtained by

$$f(\theta|x) = \frac{\prod_{i=1}^{n} f(x_i,\theta)\, g(\theta)}{\int \prod_{i=1}^{n} f(x_i,\theta)\, g(\theta)\, d\theta}$$

$$= \frac{\theta^n e^{-\theta\sum_{i=1}^{n} x_i} \times k e^{-\theta k}}{\int_0^\infty \theta^n e^{-\theta\sum_{i=1}^{n} x_i} \times k e^{-\theta k}\, d\theta}$$

$$= \frac{\theta^n k e^{-\theta(\sum x_i + k)}}{\int_0^\infty \theta^n k e^{-\theta(\sum x_i+k)}\, d\theta}$$

$$= \frac{\theta^{n+1-1} e^{-\theta(\sum x_i + k)}}{\frac{\Gamma(n+1)}{(\sum x_i + k)^{n+1}}}$$

$$= \frac{(\sum x_i + k)^{n+1}}{\Gamma(n+1)} \times \theta^{n+1-1} e^{-\theta(\sum x_i + k)}$$

$f(\theta|x) \sim$ Gamma dist^n $(n+1, \sum x_i + k)$

$\therefore$ The base estimator for θ under loss function is

$$\hat{\theta}_{SELF} = \frac{n+1}{\sum_{i=1}^{n} x_i + k}$$

i) $k = 0.5$

$$\hat{\theta} = \frac{30+1}{1788+0.5} = 0.017$$

ii) $k = 3.5$

$$\hat{\theta} = \frac{31}{1788+3.5} = 0.017$$

The LINEX function is given by

$$L(\hat{\theta}-\theta) = e^{a(\hat{\theta}-\theta)} - a(\hat{\theta}-\theta) - 1, \quad \theta > 0, \; a \neq 0$$

The Bayes estimator under Linex loss function is given by

$$\hat{\theta}_{LLF} = \frac{-1}{a} \ln\left[E(e^{-a\theta}) \right]$$

consider,

$$E\left[e^{-a\theta}\right] = \int_0^\infty e^{-a\theta} \times \frac{(\Sigma x_i + k)^{n+1}}{\Gamma n+1} \theta^{n+1-1} e^{-\theta(\Sigma x_i + k)} \, d\theta$$

$$= \int_0^\infty \frac{(\Sigma x_i + k)^{n+1}}{\Gamma n+1} e^{-\theta(\Sigma x_i + k + a)} \theta^{n+1-1} \, d\theta$$

$$E\left[e^{-a\theta}\right] = \frac{\Gamma n+1}{(\Sigma x_i + K + a)^{n+1}} \times \frac{(\Sigma x_i + K)^{n+1}}{\Gamma n+1}$$

$$= \left(\frac{(\Sigma x_i + K)}{(\Sigma x_i + K + a)}\right)^{n+1} //$$

$$\hat{\theta}_{LLF} = \frac{-1}{a} \ln \left(\frac{\Sigma x_i + K}{\Sigma x_i + K + a}\right)^{n+1}$$

i) $a = 0.5$, $k = 0.5$, $n = 30$

$$\hat{\theta}_{LLF} = \frac{-1}{0.5} \ln \left(\frac{1788 + 0.5}{1788 + 0.5 + 0.5}\right)^{31}$$

$$= \frac{-1}{0.5} \ln (0.9913)$$

$$= 0.01733 //$$

ii) $a = -0.5$, $k = 0.5$, $n = 30$

$$\hat{\theta}_{LLF} = \frac{-1}{-0.5} \ln \left(\frac{1788 + 0.5}{1788 + 0.5 - 0.5}\right)^{31}$$

$$= 0.1733 //$$

iii) $a = 2.5$, $K = 0.5$, $n = 30$

$$\hat{\theta}_{LLF} = \frac{-1}{2.5} \ln \left(\frac{1788 + 0.5}{1788 + 0.5 + 2.5}\right)^{31}$$

$$= 0.01732 //$$

iv) $a = 0.5$, $k = 3.5$

$$\hat{\theta}_{LLF} = \frac{-1}{0.5} \ln\left(\frac{1788 + 3.5}{1788 + 3.5 + 0.5}\right)^{31}$$

$$= 0.017$$ //

v) $a = -0.5$, $k = 3.5$

$$\hat{\theta}_{LLF} = 0.017$$

vi) $a = 2.5$, $k = 3.5$

$$\hat{\theta}_{LLF} = 0.017 .$$

QUESTION 03:

The p.m.f of Bernoulli distribution is

$$P(x|p) = p^x (1-p)^{1-x} , \quad x = 0, 1, \quad 0 < p < 1$$

The prior distribution of p is $U(0,1)$

$$g(p) = 1 , \quad 0 < p < 1$$

The posterior distribution can be obtained by

$$f(p|x) = \frac{\prod_{i=1}^{n} f(x_i|p)\, g(p)}{\int_0^1 \prod_{i=1}^{n} f(x_i|p)\, g(p)\, dp} .$$

$$= \frac{p^{\Sigma x_i} (1-p)^{n-\Sigma x_i} \times 1}{\int_0^1 p^{\Sigma x_i + 1 - 1} (1-p)^{n - \Sigma x_i + 1 - 1} dp}$$

$$= \frac{p^{\Sigma x_i + 1 - 1} (1-p)^{n - \Sigma x_i + 1 - 1}}{\beta(\Sigma x_i + 1, n - \Sigma x_i + 1)}$$

$$\Rightarrow f(\theta|x) \sim \beta(\Sigma x_i + 1, n - \Sigma x_i + 1)$$

$$= \frac{\Sigma x_i + 1}{\Sigma x_i + 1 + n - \Sigma x_i}$$

$$= \frac{\Sigma x_i + 1}{n+2}$$

$$\therefore \hat{\theta}_{SELF} = \frac{\Sigma x_i + 1}{n+2}$$

$$= \frac{11}{17} = 0.647 \;//$$

under absolute error loss function the posterior median is the base estimator Therefore, the median of the $\beta(\Sigma x_i + 1, n - \Sigma x_i + 1)$

$$\Rightarrow \text{median} \simeq \frac{m - 1/3}{m + n - 2/3}$$

$$\Rightarrow \hat{\theta}_{AELF} = \frac{\Sigma x_i + 1 - 1/3}{\Sigma x_i + 1 + n - \Sigma x_i + 1 - 2/3}$$

$$= \frac{\Sigma x_i + 2/3}{n+2-2/3}$$

We have; $m = \Sigma x_i + 1$, $n = n - \Sigma x_i + 1$

$$\hat{\theta}_{AELF} = \frac{10 + 0.6666}{15+2-0.666}$$

$$= 0.65$$

QUESTION: 04.

The pmf of poisson distribution is

$$f(x|\theta) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$$

The prior distribution of λ is given by

$$g(\lambda) = k e^{-k\lambda}, \quad k > 0, \lambda > 0.$$

The posterior distribution can be obtained by

$$f(\lambda|x) = \frac{\prod_{i=1}^{n} f(x_i/\lambda)\, g(\lambda)}{\int_0^\infty \prod_{i=1}^{n} f(x_i/\lambda)\, g(\lambda)\, d\lambda}$$

$$= \frac{\prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^x}{x_i!} \times k e^{-k\lambda}}{\int_0^\infty \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^x}{x_i!} \times k e^{-k\lambda}\, d\lambda}$$

$$= \frac{\dfrac{e^{-n\lambda} \cdot \lambda^{\Sigma x_i}}{\prod_{i=1}^{n} x_i!} \times K e^{-K\lambda}}{\displaystyle\int_0^\infty \dfrac{e^{-n\lambda} \lambda^{\Sigma x_i}}{\prod_{i=1}^{n} x_i!} \times K e^{-K\lambda} \, d\lambda}$$

$$= \frac{e^{-\lambda(n+K)} \times \lambda^{\Sigma x_i + 1 - 1}}{\displaystyle\int_0^\infty e^{-(n+K)\lambda} \times \lambda^{\Sigma x_i + 1 - 1} \, d\lambda}$$

$$= \frac{e^{-\lambda(n+K)} \lambda^{\Sigma x_i + 1 - 1}}{\dfrac{\Gamma(\Sigma x_i + 1)}{(n+K)^{\Sigma x_i + 1}}}$$

$$= \frac{(n+K)^{\Sigma x_i + 1}}{\Gamma(\Sigma x_i + 1)} \times e^{-\lambda(n+K)} \times \lambda^{\Sigma x_i + 1 - 1}$$

$f(\theta|x) \sim \text{gamma}(\Sigma x_i + 1, n+K)$

Mean of the gamma distⁿ $= \dfrac{\Sigma x_i + 1}{n+K}$

$$\therefore \hat{\theta}_{SELF} = \frac{\Sigma x_i + 1}{n+K}$$

i) $K = 10 \Rightarrow \dfrac{104+1}{15+10} = 4.2$

11) $k = 2$

$$\hat{\theta}_{SELF} = \frac{104+1}{15+2} = 6.176$$

The linex function ~~of~~ is given by

$$\hat{\lambda}_{LLF} = \frac{-1}{a} \ln \left[E\left(e^{-a\lambda} \right) \right]$$

consider;

$$E[e^{-a\lambda}] = \int_0^\infty e^{-a\lambda} \frac{(n+k)^{\sum x_i+1}}{\Gamma \sum x_i+1} e^{-\lambda(n+k)} \lambda^{\sum x_i+1-1} \, d\lambda$$

$$= \int_0^\infty \frac{(n+k)^{\sum x_i+1}}{\Gamma \sum x_i+1} e^{-\lambda(n+k+a)} \lambda^{\sum x_i+1-1} \, d\lambda$$

$$= \frac{\Gamma \sum x_i+1}{(n+k+a)^{\sum x_i+1}} \times \frac{(n+k)^{\sum x_i+1}}{\Gamma \sum x_i+1}$$

$$= \left(\frac{n+k}{n+k+a} \right)^{\sum x_i+1}$$

i) $a = 0.5$, $k = 10$, $n = 15$

$$\hat{\theta}_{LLF} = \frac{-1}{a} \ln \left\{ \left(\frac{n+k}{n+k+a} \right)^{\sum x_i+1} \right\}$$

$$= \frac{-1}{0.5} \ln \left(\frac{15+10}{15+10+0.5} \right)^{104+1}$$

$$= 4.1585$$

ii) $a=5, k=10, n=15$

$$\hat{\theta}_{LLF} = -\frac{1}{5} \ln\left(\frac{15+10}{15+10+5}\right)^{105}$$

$$= 3.828$$

iii) $a=10, k=10, n=15$

$$\hat{\theta}_{LLF} = \frac{-1}{10} \ln\left(\frac{15+10}{15+10+10}\right)^{105}$$

$$= 3.5329$$

iv) $a=0.5, k=2$

$$\hat{\theta}_{LLF} = \frac{-1}{0.5} \ln\left(\frac{15+2}{15+2+0.5}\right)^{105}$$

$$= 6.0873$$

v) $a=5, k=2$.

$$\hat{\theta}_{LLF} = \frac{-1}{0.5} \ln\left(\frac{15+2}{15+2+5}\right)^{105}$$

$$= 5.414$$

vi) $a=10, k=2$

$$\hat{\theta}_{LLF} = \frac{1}{10} \ln\left(\frac{15+2}{15+2+10}\right)^{105}$$

$$= 4.87$$