Ben Juarez

ACM 116 - PS4

(1) Let X, Y, Z be jointly continuous r.v. Consider $W = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$ where $X \sim U[1,2]$, $Y|X=x \sim \mathrm{Exp}(\frac{1}{x})$, $Z|X=x, Y=y \sim N(x,1)$

(a) Find mean vector and covariance matrix of W

We can first determine: $\mathbb{E}[X] = \frac{1+2}{2} = \frac{3}{2}$, $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X=x \sim \mathrm{Exp}(\frac{1}{x})]] = \mathbb{E}[X]$, $\mathbb{E}[Z] = \mathbb{E}[\mathbb{E}[Z|X=x, Y=y]] = \mathbb{E}[X]$

So, we can plug into μ_w:

$$\Rightarrow \mu_w = \begin{bmatrix} \mathbb{E}[X] \\ \mathbb{E}[Y] \\ \mathbb{E}[Z] \end{bmatrix} = \begin{bmatrix} \mathbb{E}[X] \\ \mathbb{E}[\mathbb{E}[Y|X=x]] \\ \mathbb{E}[\mathbb{E}[Z|X=x]] \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ \frac{3}{2} \end{bmatrix} \qquad \Sigma_w = \begin{bmatrix} \mathbb{V}[X] & \mathrm{Cov}(X,Y) & \mathrm{Cov}(X,Z) \\ \mathrm{Cov}(X,Y) & \mathbb{V}[Y] & \mathrm{Cov}(Y,Z) \\ \mathrm{Cov}(X,Z) & \mathrm{Cov}(Y,Z) & \mathbb{V}[Z] \end{bmatrix}$$

Let us now solve for terms of $\mathrm{Cov}(W)$ (or Σ_w):

$$\Rightarrow \mathbb{V}[X] = \frac{(2-1)^2}{12} = \frac{1}{12}$$

$$\Rightarrow \mathbb{V}[Y] = \mathbb{E}[\mathbb{V}[Y|X]] + \mathbb{V}[\mathbb{E}[Y|X]] = \mathbb{E}[X^2] + \mathbb{V}[X] = \mathbb{V}[X] + \mathbb{E}[X]^2 + \mathbb{V}[X] = \frac{1}{12} + \frac{9}{4} + \frac{1}{12} = \frac{27}{12} + \frac{2}{12} = \frac{29}{12}$$

$$\Rightarrow \mathbb{V}[Z] = \mathbb{E}[\mathbb{V}[Z|X]] + \mathbb{V}[\mathbb{E}[Z|X]] = \mathbb{E}[1] + \mathbb{V}[X] = 1 + \frac{1}{12} = \frac{12}{12} + \frac{1}{12} = \frac{13}{12}$$

$* \mathbb{E}[X^2] = \mathbb{V}[X] + \mathbb{E}[X]^2$

$$\Rightarrow \mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[XY|X=x]] - \mathbb{E}[X]\mathbb{E}[Y] = \mathbb{E}[X^2] - \mathbb{E}[X]\mathbb{E}[Y] = \frac{28}{12} - \frac{9}{4} = \frac{28}{12} - \frac{27}{12} = \frac{1}{12}$$

$$\Rightarrow \mathrm{Cov}(X,Z) = \mathbb{E}[XZ] - \mathbb{E}[X]\mathbb{E}[Z] = \mathbb{E}[\mathbb{E}[XZ|X=x]] - \mathbb{E}[X]\mathbb{E}[Z] = \mathbb{E}[X^2] - \mathbb{E}[X]\mathbb{E}[Z] = \frac{28}{12} - \frac{9}{4} = \frac{1}{12}$$

$$\Rightarrow \mathrm{Cov}(Y,Z) = \mathbb{E}[YZ] - \mathbb{E}[Y]\mathbb{E}[Z] = \mathbb{E}[\mathbb{E}[YZ|X=x]] - \mathbb{E}[Y]\mathbb{E}[Z] = \mathbb{E}[X^2] - \mathbb{E}[Y]\mathbb{E}[Z] = \frac{28}{12} - \frac{9}{4} = \frac{1}{12}$$

Thus, we have $\Sigma_w = \begin{bmatrix} \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{29}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{12} & \frac{13}{12} \end{bmatrix}$

(b) See corresponding MATLAB section

② Let X be a random 2-vector whose components are independently uniformly distributed on $[0,1]$.
Let $Y=(Y_1,Y_2)^T$ where

$$Y_1=\sqrt{-2\log X_1}\ \cos(2\pi X_2)$$
$$Y_2=\sqrt{-2\log X_1}\ \sin(2\pi X_2)$$

ⓐ <u>Are components of Y independent?</u>

Joint PDF of Y: $f_Y(y)=f_X(H(y))\,|\det(J_H(y))|$

$\Rightarrow X_1 \perp\!\!\!\perp X_2 \Rightarrow f_X=f_{X_1}\cdot f_{X_2}=\frac{1}{1-0}\cdot\frac{1}{1-0}=1$ We know $J_H(y)=\begin{bmatrix}\frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2}\\ \frac{\partial h_2}{\partial y_2} & \frac{\partial h_2}{\partial y_2}\end{bmatrix}$

$\Rightarrow f_Y(y)=|\det(J_H(y))|$

$\Rightarrow Y_2=\sqrt{-2\log X_1}\ \sin(2\pi X_2)\Rightarrow X_2=\frac{1}{2\pi}\sin^{-1}\left(\frac{Y_2}{\sqrt{-2\log X_1}}\right)$

Plug X_2 into $Y_1 \Rightarrow Y_1=\sqrt{-2\log X_1}\ \cos\left(2\pi\frac{1}{2\pi}\sin^{-1}\left(\frac{Y_2}{\sqrt{-2\log X_1}}\right)\right)$

$$Y_1=\sqrt{-2\log X_1}\cdot\cos\left(\sin^{-1}\left(\frac{Y_2}{\sqrt{-2\log X_1}}\right)\right)$$

$$\Rightarrow X_1=e^{-\frac{1}{2}(Y_1^2+Y_2^2)}$$

Solve for $X_2 \Rightarrow Y_1=\sqrt{-2\log\left(e^{-\frac{1}{2}(Y_1^2+Y_2^2)}\right)}\cdot\cos(2\pi X_2)$

$$\Rightarrow X_2=\frac{1}{2\pi}\tan^{-1}\left(\frac{Y_2}{Y_1}\right)$$

$\Rightarrow h_1=X_1,\ h_2=X_2$

$\Rightarrow \frac{\partial h_1}{\partial y_1}=-Y_1e^{-\frac{1}{2}(Y_1^2+Y_2^2)},\ \frac{\partial h_1}{\partial y_2}=-Y_2e^{-\frac{1}{2}(Y_1^2+Y_2^2)}$ $\Longrightarrow J_H(y)=\begin{bmatrix}-Y_1e^{-\frac{1}{2}(Y_1^2+Y_2^2)} & -Y_2e^{-\frac{1}{2}(Y_1^2+Y_2^2)}\\ \frac{-Y_2}{2\pi(Y_1^2+Y_2^2)} & \frac{Y_1}{2\pi(Y_1^2+Y_2^2)}\end{bmatrix}$

$\Rightarrow \frac{\partial h_2}{\partial y_1}=\frac{-Y_2}{2\pi(Y_1^2+Y_2^2)},\ \frac{\partial h_2}{\partial y_2}=\frac{Y_1}{2\pi(Y_1^2+Y_2^2)}$

$$|J_H(y)|=\left|\frac{-Y_1^2e^{-\frac{1}{2}(Y_1^2+Y_2^2)}}{2\pi(Y_1^2+Y_2^2)}-\frac{Y_2^2e^{-\frac{1}{2}(Y_1^2+Y_2^2)}}{2\pi(Y_1^2+Y_2^2)}\right|=\left|\frac{-(Y_1^2+Y_2^2)e^{-\frac{1}{2}(Y_1^2+Y_2^2)}}{2\pi(Y_1^2+Y_2^2)}\right|=\frac{1}{2\pi}e^{-\frac{1}{2}(Y_1^2+Y_2^2)}$$

$$\Rightarrow f_Y(y)=\frac{1}{2\pi}e^{-\frac{1}{2}(Y_1^2+Y_2^2)}=\frac{1}{2\pi}\left(e^{-\frac{1}{2}(Y_1^2)}\cdot e^{-\frac{1}{2}(Y_2^2)}\right)=\frac{1}{\sqrt{2\pi}}e^{-\frac{Y_1^2}{2}}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{Y_2^2}{2}}=f_{Y_1}(y)\cdot f_{Y_2}(y)\Rightarrow Y_1\perp\!\!\!\perp Y_2$$

ⓑ <u>Find marginal distributions of Y_1 and Y_2</u>

$$f_{Y_1}(y)=\int_{-\infty}^{\infty}f_Y(y)\,dY_2=\int\frac{1}{2\pi}e^{-\frac{1}{2}(Y_1^2+Y_2^2)}dY_2=\frac{1}{\sqrt{2\pi}}e^{-\frac{Y_1^2}{2}}\Rightarrow Y_1\sim N(0,1)$$

$$f_{Y_2}(y)=\int_{-\infty}^{\infty}f_Y(y)\,dY_1\ \int\frac{1}{2\pi}e^{-\frac{1}{2}(Y_1^2+Y_2^2)}dY_1=\frac{1}{\sqrt{2\pi}}e^{-\frac{Y_2^2}{2}}\Rightarrow Y_2\sim N(0,1)$$

③ · Let X be unobserved rand. vector w/ mean μ_X and covariance matrix Σ_X

- Noisy channel input X produces output $Y = GX + W$ s.t. G is known "gain" matrix, W is r.n.v. w/ mean $\mu_W = 0$, Σ_W
- Assume input X and noise W are uncorrelated, $\Sigma_{XW} = 0$, Σ_Y is nonsingular

ⓐ Find Wiener filter $g(Y)$ for X based on Y, i.e. $g(Y) = AY + b$ that minimizes mean-squared error $\mathbb{E}[\|X - g(Y)\|^2]$

Σ_Y is nonsingular $\Rightarrow g(Y) = \Sigma_{XY}\Sigma_Y^{-1}(Y \ \mu_Y) + \mu_X$

$$\Rightarrow \Sigma_{XY} = \Sigma_{X,GX+W} = \Sigma_{X,GX} + \Sigma_{X,W} = \Sigma_{X,GX} = \mathbb{E}[X(GX)^T] - \mathbb{E}[X]\,\mathbb{E}[GX]^T$$

$* X \perp W$

$$= \left(\mathbb{E}[XX^T] - \mathbb{E}[X]\,\mathbb{E}[X]^T\right) G^T \quad * G \text{ is deterministic}$$

$$\Rightarrow \Sigma_{XY} = \Sigma_X G^T$$

$$\Rightarrow \Sigma_Y = \Sigma_{GX+W} = \Sigma_{GX} + \Sigma_{GX,W} + \Sigma_{W,XG} + \Sigma_W$$

$$\Rightarrow \Sigma_{GX} = G\Sigma_X G^T \qquad \Rightarrow \Sigma_{GX,W} = \mathbb{E}[GXW^T] - \mathbb{E}[GX]\,\mathbb{E}[W]^T \qquad \Rightarrow \Sigma_{W,XG} = (\Sigma_{G,XW})^T = 0$$

$$= \left(\mathbb{E}[XW^T - \mathbb{E}[X]\mathbb{E}[W]^T\right) G$$

$$= \Sigma_{XW} G = 0 \quad * X \perp W$$

$$\Rightarrow \Sigma_Y = G\Sigma_X G^T + \Sigma_W$$

$$\Rightarrow \mu_Y = \mathbb{E}[Y] = \mathbb{E}[GX + W] = \mathbb{E}[GX] + \mathbb{E}[W] = G\mathbb{E}[X] + \mu_W = G\mu_X$$

Thus, we have: $g(Y) = \Sigma_X G^T (G\Sigma_X G^T + \Sigma_W)^{-1}(GX + W - G\mu_X) + \mu_X$

ⓑ See corresponding MATLAB section

(4) Let $X_1 \sim N(0,1)$, $X_2 = 3X_1$, and $X = (X_1, X_2)^T$

(a) Is X Gaussian?

$\Rightarrow X = (X_1, X_2)^T = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ 3X_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}[X_1] + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow X = AZ + \mu$ where $X_1 \sim N(0,1) \Rightarrow X$ is jointly normally distributed.

Thus, X must Gaussian.

(b) Find Σ_X

If $X = AZ + \mu$, then $Cov(X) = \Sigma_X = AA^T = \begin{bmatrix} 1 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$

(c) Does X have a density?

$\det(\Sigma_X) = 1(9) - 3(3) = 0 \Rightarrow \Sigma_X$ is singular $\Rightarrow X$ does not have density

5. Let X and Y be two random vectors s.t. $Y = AX$ where A is nonzero deterministic matrix

(a) Prove/disprove: If X is a Gaussian vector, then so is Y

If X is a Gaussian vector, then we know its components are jointly normally distributed

$\Rightarrow X = BZ + \mu_x$ s.t. $Z_i, \dots, Z_m \overset{iid}{\sim} N(0,1)$

So, with $Y = AX$, let us plug in for X as follows:

$\Rightarrow Y = A(BZ + \mu_x) = ABZ + A\mu_x$

Let $C = AB$ and $\mu_y = A\mu_x$

$\Rightarrow Y = CZ + \mu_y$ s.t. $Z_i, \dots, Z_m \overset{iid}{\sim} N(0,1) \Rightarrow$ components of Y are jointly normally distributed $\Rightarrow Y$ is Gaussian vector

Therefore, we have that Y is a Gaussian vector if X is a Gaussian vector, proving the claim.

(b) Prove/disprove: If Y is a Gaussian vector, then so is X.

Let us disprove this claim by counterexample:

- Let X NOT be a Gaussian vector s.t. $X = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$ where $Z_1 \sim N(0,1)$, $Z_2 \not\sim N(0,1)$
- Let Y be a Gaussian vector s.t. $Y = AX$ and $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

$$\Rightarrow Y = AX = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 1(Z_1) + 0(Z_2) \\ 1(Z_1) + 0(Z_2) \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_1 \end{bmatrix}$$

Since we know $Z_1 \sim N(0,1)$, we have that the components of Y are jointly normally distributed.

So, we have a case where Y is a Gaussian vector, but X is not, disproving the claim.

```
% Ben Juarez - PS4Q1b
CV = [1/12 1/12 1/12;
    1/12 29/12 1/12;
    1/12 1/12 13/12];

[Q,D]=eig(CV); % calculating eigenvectors
transpose(Q) % tranposing Q to get decorrelating transformation


ans =

    0.9963   -0.0326   -0.0796
    0.0772   -0.0670    0.9948
    0.0378    0.9972    0.0643
```

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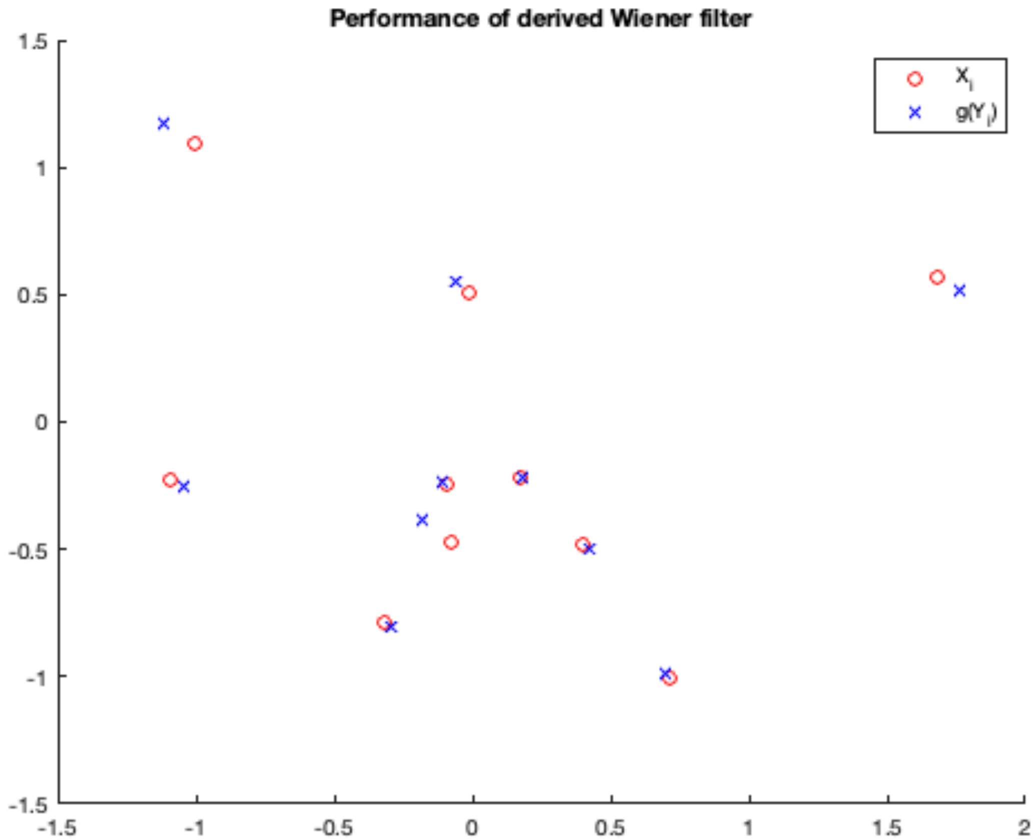
```
% Ben Juarez - PS4Q3b
n = 10;
G = [1 2;3, 4]; % gain matrix
e = 0.03;
X = zeros(0,2);
X_hat = zeros(0,2);

for i = 1:n
     x = normrnd(0,1,2,1); % standard normal
     X = [X;x']; % inputs X_i

     Y = G*x + mvnrnd([0;0],[e^2 0;0 e^2])'; % Y = GX + W
     u_x = [0;0]; % mean
     sig_x = [1 0; 0 1]; % covariance matrix X
     sig_w = [e^2 0; 0 e^2]; % covariance matrix W
     g = sig_x*G'*inv(G*sig_x*G'+sig_w)*(Y-G*u_x)+u_x; % g(Y)
     X_hat = [X_hat; g']; % estimates g(Y_i)
end

hold on
scatter(X(:,1), X(:,2), "red")
scatter(X_hat(:,1), X_hat(:,2), "blue", "x")
legend("X_i", "g(Y_i)")
title("Performance of derived Wiener filter")
hold off
```



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