Ben Juarez

ACM 116 - PS5

(1) $n = 10$, 1:30 - 5:30 pm, $\Delta t = 30$ min, $T_1, \dots, T_n$ independent exponential

(a) Find $\mathbb{E}[W_2]$

$$W_2 = \begin{cases} 0 & \text{if } T_1 \le 30 \\ T_1 - 30 & \text{if } T_1 > 30 \end{cases} \implies \mathbb{E}[W_2] = \mathbb{P}(T_1 \le 30) \cdot 0 + \mathbb{P}(T_1 > 30) \cdot \mathbb{E}[W_2 | T_1 > 30]$$

$$= \mathbb{P}(T_1 > 30) \cdot \mathbb{E}[W_2 | T_1 > 30]$$

$$= \int_{30}^{\infty} (T-30) \frac{1}{30} \cdot e^{-T/30} \, dT$$

$$= 30 e^{-1}$$

$$\mathbb{E}[W_2] = \frac{30}{e}$$

(b) See attached MATLAB code

(c) See attached MATLAB code

② · Let $\{N_t, t \geq 0\}$ be a Poisson process w/ intensity λ

ⓐ Find prob. of no events in $(t_1, t_2]$

$$\Longrightarrow \mathbb{P}(N_{t_2} - N_{t_1} = 0) = e^{-\lambda(t_2 - t_1)} \cdot \frac{(\lambda(t_2 - t_1))^0}{0!} = e^{-\lambda(t_2 - t_1)}$$

ⓑ Find probability that there is exactly 1 event in each of given intervals

· intervals $(0,1], (1,2], \ldots, (n-1, n]$

$$\Longrightarrow \Delta t = 1 \text{ for each increment} \Longrightarrow \mathbb{P}(N_{t_1} - N_{t_0} = 1) = \mathbb{P}(N_{t_2} - N_{t_1} = 1) = \ldots = \mathbb{P}(N_{t_n} - N_{t_{n-1}} = 1)$$

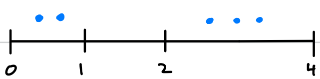
*independent stationary increments

$$\Longrightarrow \mathbb{P}(N_{t_n} - N_{t_{n-1}} = 1) \text{ for each } n \text{ is equivalent to } (\mathbb{P}(N_1 - N_0 = 1))^n$$

$$\Longrightarrow (\mathbb{P}(N_1 - N_0 = 1))^n = \left(e^{-\lambda} \frac{\lambda^1}{1!}\right)^n = (\lambda e^{-\lambda})^n$$

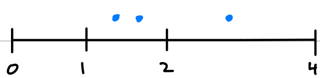
ⓒ Find probability that there are two events in $(0,2]$ and three in $(1,4]$

Ⓐ



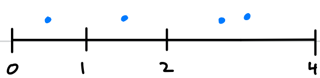
$$\Longrightarrow \mathbb{P}(N_1 - N_0 = 2) \cdot \mathbb{P}(N_2 - N_1 = 0)\, \mathbb{P}(N_4 - N_2 = 3) = \left(e^{-\lambda} \cdot \frac{\lambda^2}{2}\right)\left(e^{-\lambda}\right)\left(e^{-2\lambda} \cdot \frac{(2\lambda)^3}{6}\right)$$
$$= e^{-4\lambda}\left(\frac{2\lambda^2\lambda^3}{3}\right)$$

Ⓑ



$$\Longrightarrow \mathbb{P}(N_1 - N_0 = 0)\, \mathbb{P}(N_2 - N_1 = 2)\, \mathbb{P}(N_4 - N_2 = 1) = \left(e^{-\lambda}\right)\left(e^{-\lambda} \cdot \frac{\lambda^2}{2}\right)\left(e^{-2\lambda} \cdot 2\lambda\right)$$
$$= e^{-4\lambda}(\lambda^3)$$

Ⓒ



$$\Longrightarrow \mathbb{P}(N_1 - N_0 = 1)\, \mathbb{P}(N_2 - N_1 = 1)\, \mathbb{P}(N_4 - N_2 = 2) = \left(e^{-\lambda} \cdot \lambda\right)\left(e^{-\lambda} \cdot \lambda\right)\left(e^{-2\lambda} \cdot \frac{(2\lambda)^2}{2}\right)$$
$$= e^{-4\lambda}(2\lambda^4)$$

$$\Longrightarrow \mathbb{P}(N_2 - N_0 = 2,\ N_4 - N_1 = 3) = \mathbb{P}_A + \mathbb{P}_B + \mathbb{P}_C$$
$$= e^{-4\lambda}\left(\frac{2\lambda^5}{3}\right) + e^{-4\lambda}(\lambda^3) + e^{-4\lambda}(2\lambda^4)$$
$$= \lambda^3 e^{-4\lambda}\left(\frac{2}{3}\lambda^2 + 1 + 2\lambda\right)$$

(3) · Let $\{N_t, t \geq 0\}$ be Poisson process w/ rate λ w/ $T_1 \sim Exp(\lambda)$

Find cond. distribution of T_1 given that $N_t = 1$

$$\Rightarrow f_{T_1|N_t=1}(t_1|1) = \frac{f_{T_1,N_t}(t_1,1)}{\mathbb{P}(N_t=1)} = \frac{\mathbb{P}(T_1 \leq t_1, N_t=1)}{\mathbb{P}(N_t=1)} \quad \text{for } t_1 \leq t$$

$$\Rightarrow \frac{\mathbb{P}(T_1 \leq t_1, N_t=1)}{\mathbb{P}(N_t=1)} = \frac{\mathbb{P}(T_1 \leq t_1, N_t=1)}{(\lambda t)e^{-\lambda t}}$$

$$= \frac{\mathbb{P}(N_{t_1} - N_0 = 1,\ N_t - N_{t_1} = 0)}{(\lambda t)e^{-\lambda t}}$$

$$= \frac{\mathbb{P}(N_{t_1} - N_0 = 1) \cdot \mathbb{P}(N_t - N_{t_1} = 0)}{(\lambda t)e^{-\lambda t}}$$

$$= \frac{(\lambda t_1)e^{-\lambda t_1} \cdot e^{-\lambda(t-t_1)}}{(\lambda t)e^{-\lambda t}}$$

$$= \frac{(\lambda t_1)e^{-\lambda t_1} \cdot e^{-\lambda t} \cdot e^{\lambda t_1}}{(\lambda t)e^{-\lambda t}}$$

$$\Rightarrow f_{T_1|N_t=1}(t_1|1) = \frac{t_1}{t}$$

$\Rightarrow$ uniform distribution $U[\alpha=0, \beta=t]$

4 (a) Find probability that student's wait time is 0

$\mathbb{P}(N_{\tau+t} - N_t = 0) = e^{-\tau\lambda} \cdot \frac{(\lambda\tau)^0}{0!} = e^{-\tau\lambda}$

(b) Find expected waiting time

- Let W denote waiting time
- Let car i be next car
- Let t_i denote time until car i passes

$\Longrightarrow W = \begin{cases} 0 & \text{if } t_i > \tau \\ t_i + W & \text{if } t_i \le \tau \end{cases}$

$$\Longrightarrow \mathbb{E}[W] = \mathbb{P}(t_i > \tau)\cdot 0 + \mathbb{P}(t_i \le \tau)\cdot\left(\mathbb{E}[t_i \mid t_i \le \tau] + \mathbb{E}[W]\right)$$

$$\mathbb{E}[W] = \mathbb{P}(t_i \le \tau)\cdot\left(\mathbb{E}[t_i \mid t_i \le \tau] + \mathbb{E}[W]\right)$$

$$\mathbb{E}[W] = \mathbb{P}(t_i \le \tau)\cdot\mathbb{E}[t_i \mid t_i \le \tau] + \mathbb{P}(t_i \le \tau)\cdot\mathbb{E}[W]$$

$$\mathbb{E}[W] - \mathbb{P}(t_i \le \tau)\cdot\mathbb{E}[W] = \mathbb{P}(t_i \le \tau)\cdot\mathbb{E}[t_i \mid t_i \le \tau]$$

$$\mathbb{E}[W] = \frac{\mathbb{P}(t_i \le \tau)\cdot\mathbb{E}[t_i \mid t_i \le \tau]}{1 - \mathbb{P}(t_i \le \tau)}$$

$$= \frac{\mathbb{P}(t_i \le \tau)\,\mathbb{E}[t_i \mid t_i \le \tau]}{1 - (1 - \mathbb{P}(t_i > \tau))}$$

* $\mathbb{P}(t_i \le \tau) = 1 - \mathbb{P}(t_i > \tau)$ ≈ at least one car comes before next τ seconds

$$= \frac{\int_0^\tau t\lambda e^{-\lambda t}\,dt}{e^{-\lambda\tau}}$$

* $\mathbb{P}(t_i > \tau) = e^{-\lambda\tau}$

$$= \frac{\frac{1}{\lambda}\left(1 - (\lambda\tau + 1)e^{-\lambda\tau}\right)}{e^{-\lambda\tau}}$$

$$= \frac{\frac{1}{\lambda}\left(1 - \lambda\tau e^{-\lambda\tau} - e^{-\lambda\tau}\right)}{e^{-\lambda\tau}}$$

$$\Rightarrow \mathbb{E}[W] = \frac{1}{\lambda}\left(e^{\lambda\tau} - \lambda\tau - 1\right)$$

5 (a) Find probability of winning under τ-strategy

- stop after 1st customer after $\tau \le \tau^*$ (assume $\tau^* > \frac{1}{\lambda}$)

$\Rightarrow \mathbb{P}(N_{\tau^*} - N_{\tau} = 1) = \lambda(\tau^* - \tau) \cdot e^{-\lambda(\tau^* - \tau)}$

(b) Find optimal value τ_{opt} of τ that maximizes probability of winning

$\Rightarrow$ Let P_w denote $\mathbb{P}(N_{\tau^*} - N_{\tau} = 1)$

$$\Rightarrow \frac{dP_w}{d\tau} = \frac{d}{d\tau}\left(\lambda(\tau^* - \tau) \cdot e^{-\lambda(\tau^* - \tau)}\right) = 0 = \lambda(\tau^* - \tau) \cdot \lambda e^{-\lambda(\tau^* - \tau)} - \lambda e^{-\lambda(\tau^* - \tau)}$$

$$= \lambda e^{-\lambda(\tau^* - \tau)}\left(\lambda(\tau^* - \tau) - 1\right)$$

$$= \lambda(\tau^* - \tau) - 1$$

$$1 = \lambda\tau^* - \lambda\tau \Rightarrow \tau_{opt} = \tau^* - \frac{1}{\lambda} \qquad * \tau^* > \frac{1}{\lambda}$$

(c) What is probability of winning under optimal strategy?

$$\Rightarrow \mathbb{P}(N_{\tau^*} - N_{\tau} = 1 \mid \tau = \tau_{opt}) = \lambda\left(\tau^* - \left(\tau^* - \frac{1}{\lambda}\right)\right) \cdot e^{-\lambda\left(\tau^* - \left(\tau^* - \frac{1}{\lambda}\right)\right)} = e^{-1} = \frac{1}{e}$$

(d) What is expected # of crossants under optimal strogedy?

$$\Rightarrow \mathbb{E}[c] = \mathbb{E}[N_{\tau_{opt}} - N_0] + \mathbb{P}(N_{\tau^*} - N_{\tau_{opt}} \ge 1)$$

$$= \lambda\tau_{opt} + \left(1 - \mathbb{P}(N_{\tau^*} - N_{\tau_{opt}} < 1)\right)$$

$$= \lambda\tau_{opt} + \left(1 - \mathbb{P}(N_{\tau^*} - N_{\tau_{opt}} = 0)\right)$$

$$= \lambda\tau_{opt} + \left(1 - e^{-\lambda(\tau^* - \tau_{opt})}\right)$$

$$= \lambda\left(\tau^* - \frac{1}{\lambda}\right) + \left(1 - e^{-\lambda\left(\tau^* - \tau^* + \frac{1}{\lambda}\right)}\right)$$

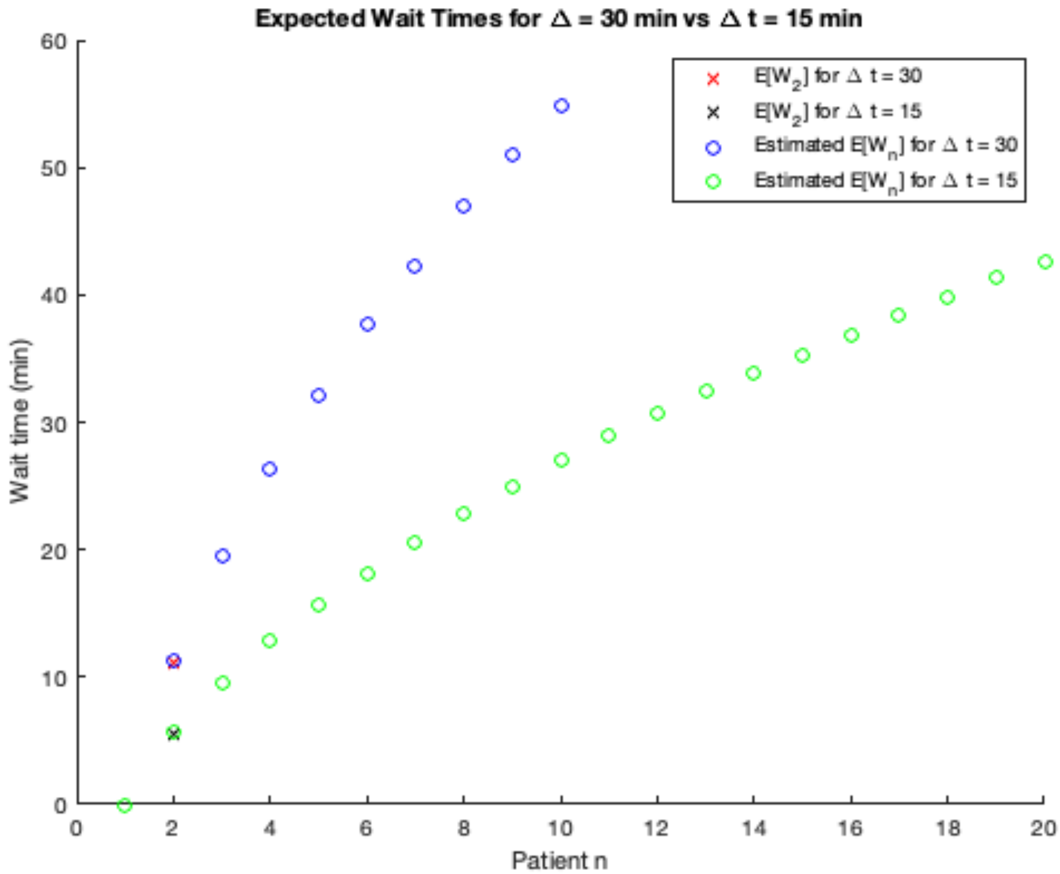
$$= \lambda\tau^* - 1 + 1 - e^{-1}$$

$$\Rightarrow \mathbb{E}[c] = \lambda\tau^* - \frac{1}{e} \quad \text{croissants}$$

```matlab
% Ben Juarez - PS4Q1bc

% Part b
delta_t1 = 30;
delta_t2 = 15;
n1 = 10;
n2 = 20;
N=10^4;
E_W1 = zeros(0,n1);
E_W2 = zeros(0,n2);
for i = 1:N
    W_n1 = zeros(1,n1);
    W_n2 = zeros(1,n2);
    for j = 1:n1
        if j == 1
            W_n1(j) = 0;
        else
            W_n1(j) = max(0, W_n1(j-1) + exprnd(delta_t1) - delta_t1);
        end
    end
    for j = 1:n2
        if j == 1
            W_n2(j) = 0;
        else
            W_n2(j) = max(0, W_n2(j-1) + exprnd(delta_t2) - delta_t2);
        end
    end
    E_W1 = [E_W1; W_n1];
    E_W2 = [E_W2; W_n2];
end
E_W1 = mean(E_W1);
E_W2 = mean(E_W2);

hold on
scatter(2, delta_t1 / exp(1), "X", "red")
scatter(2, delta_t2 / exp(1), "X", "black")
scatter(1:1:n1, E_W1, "o", "blue")
scatter(1:1:n2, E_W2, "o", "green")
legend("E[W_2] for \Delta t = 30", ...
    "E[W_2] for \Delta t = 15", ...
    "Estimated E[W_n] for \Delta t = 30", ...
    "Estimated E[W_n] for \Delta t = 15")
title("Expected Wait Times for \Delta = 30 min vs \Delta t = 15 min")
xlabel("Patient n")
ylabel("Wait time (min)")
hold off
% Part c
% We can see that the wait time is estimated to be lower for the 20th
% patient with delta t = 15 min (approx 43-44 min) compared to the 10th
% patient with delta t = 30 min (approx 53-54 min).  It is better to be
% be the 20th patient with delta t = 15 min!
```



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