1 Empirical CDF

Let $X_1, ..., X_n \sim F \in \mathcal{F}$, and let $\hat{F}_n$ be the empirical CDF. Let us find the covariance between two random variables $\hat{F}_n(x)$ and $\hat{F}_n(y)$ for $x \neq y$:
From lecture 6a (Thereom 1), we have that $\mathbb{E}[\hat{F}_n(x)] = F(x)$. With the definition of covariance, we can define this case of covariance as

$$\implies Cov(\hat{F}_n(x), \hat{F}_n(y)) = \mathbb{E}[\hat{F}_n(x) \cdot \hat{F}_n(y)] - \mathbb{E}[\hat{F}_n(x)] \cdot \mathbb{E}[\hat{F}_n(y)] = \mathbb{E}[\hat{F}_n(x) \cdot \hat{F}_n(y)] - F(x) \cdot F(y)$$

With $\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} H(x - X_i)$, we reach the following implications:

$$\implies \mathbb{E}[\hat{F}_n(x) \cdot \hat{F}_n(y)] = \mathbb{E}[\frac{1}{n}\sum_{i=1}^{n} H(x - X_i) \cdot \frac{1}{n}\sum_{j=1}^{n} H(y - X_j)]$$

$$\implies \mathbb{E}[\hat{F}_n(x) \cdot \hat{F}_n(y)] = \mathbb{E}[\frac{1}{n^2}(\sum_{i=1}^{n}\sum_{j\neq i}^{n} H(x - X_i)H(y - X_j) + \sum_{i=1}^{n} H(x - X_i)H(y - X_i))]$$

$$\implies \mathbb{E}[\hat{F}_n(x) \cdot \hat{F}_n(y)] = \mathbb{E}[\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j\neq i}^{n} H(x - X_i)H(y - X_j)] + \mathbb{E}[\frac{1}{n^2}\sum_{i=1}^{n} H(x - X_i)H(y - X_i)]$$

In order to proceed further, we must handle two cases of $i = j$ and $i \neq j$. Let us also note that we assume that all X_i are iid. So, for $i = j$, if $x < y$ then we have $F(x)$ (and $x > y \implies F(y)$). Considering the behavior of the Heaviside function, we can determine the following:

$$\implies \mathbb{E}[\frac{1}{n^2}\sum_{i=1}^{n} H(x - X_i)H(y - X_i)] = \frac{1}{n^2}\sum_{i=1}^{n}(0 \cdot \mathbb{P}(X_i > min(x, y) + 1 \cdot \mathbb{P}(X \leq min(x, y)))$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{P}(X \leq min(x, y)) = \frac{1}{n^2}(n)F(min(x, y)) = \frac{1}{n}F(min(x, y))$$

For $i \neq j$, we have that X_i and X_j are independent:

$$\implies \mathbb{E}[\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j\neq i}^{n} H(x - X_i)H(y - X_j)] = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j\neq i}^{n} F(x)F(y) = \frac{1}{n^2}(n)(n-1)F(x)F(y)$$

Thus, we plug back in and solve for the covariance:

$$\implies Cov(\hat{F}_n(x), \hat{F}_n(y)) = \frac{1}{n^2}(n)(n-1)F(x)F(y) + \frac{1}{n}F(min(x, y)) - F(x)F(y)$$

$$\implies Cov(\hat{F}_n(x), \hat{F}_n(y)) = \frac{n-1}{n}F(x)F(y) + \frac{1}{n}F(min(x, y)) - \frac{1}{n}(n)F(x)F(y)$$

$$\implies Cov(\hat{F}_n(x), \hat{F}_n(y)) = \frac{1}{n}(F(min(x, y)) - F(x)F(y))$$

2 Measuring Symmetry

Let us find the plug-in estimate of κ_F *where it is defined as follows:*

$$\kappa_F = \frac{\int (x-\mu_F)^3 dF(x)}{(\int (x-\mu_F)^2 dF(x))^{\frac{3}{2}}}$$

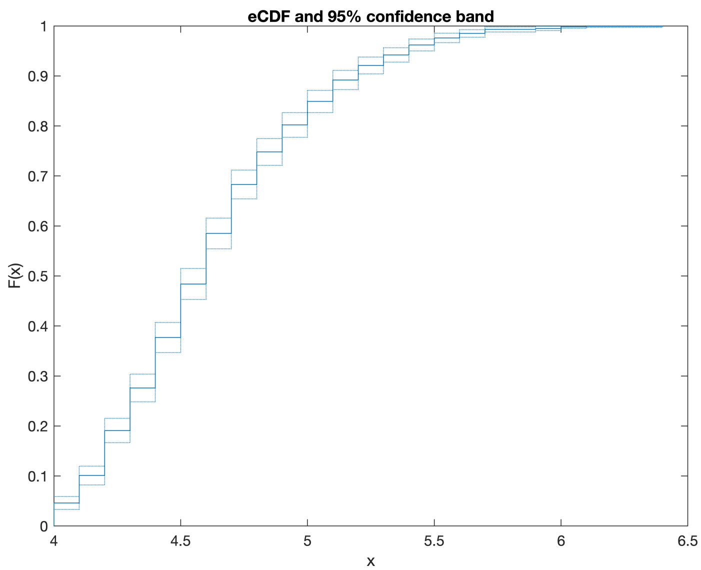
From lecture 6a (Plug-In Principle), we can solve for this estimate as follows:

$$\implies \hat{\kappa}_F = \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X}_n)^3}{(\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2)^{\frac{3}{2}}} = \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X}_n)^3}{(\hat{\sigma}_n^2)^{\frac{3}{2}}} = \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X}_n)^3}{\hat{\sigma}_n^3}$$

3 Nonparametric Confidence Band for the CDF

Part (a)

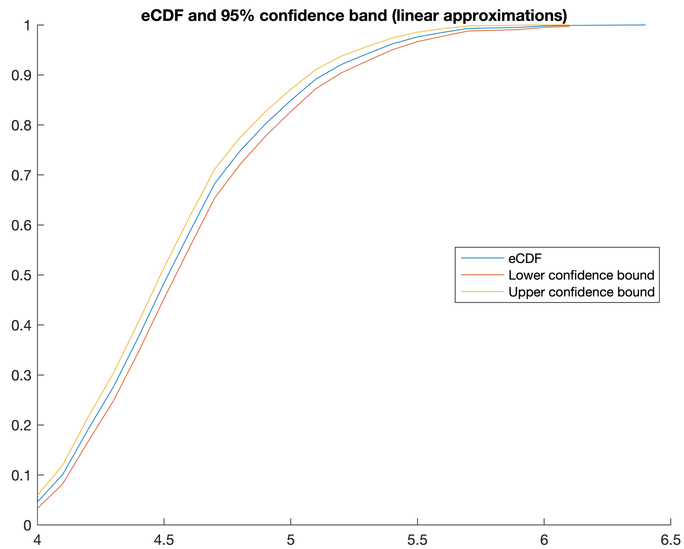
```
fiji = importdata("fiji.txt");
magnitudes = fiji(:,5);

figure
ecdf(magnitudes, "Alpha",0.05,"Bounds","on")
title("eCDF and 95% confidence band")
```



Part (b)

```
[f,x,flo,fup] = ecdf(magnitudes,"Alpha",0.05);
figure
hold on
plot(x,f)
plot(x,flo)
plot(x,fup)
legend("eCDF", "Lower confidence bound", "Upper confidence bound", "Location","east")
title("eCDF and 95% confidence band (linear approximations)")
```

1


eCDF and 95% confidence band (linear approximations)
eCDF
Lower confidence bound
Upper confidence bound
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
4
4.5
5
5.5
6
6.5

2

4 Jackknife: Theoretical Analysis

We have that $X_1, ..., X_n$ is a sample from the uniform distribution $\mathcal{U}[0, \theta]$ (unknown θ):

4.1 Part a

Let us find the plug-in estimate $\hat{\theta}_n$ of θ using the following representation of θ:

$$\theta = min\{x : F(x) = 1\}$$

With $\theta = t(F)$ and $\hat{\theta}_n = t(\hat{F}_n)$, we consider how to make $\hat{F}_n$ close to 1 since the maximum value that $F(x)$ can be is 1. So, with the given representation of θ, we can simply define the plug-in estimate as the maximum sample:

$$\implies \hat{\theta}_n = max\{X_1, ..., X_n\} = X_{(n)}$$

4.2 Part b

Let us find the bias of $\hat{\theta}_n$. From lecture 6b, the bias is defined as $\mathbb{B}[\hat{\theta}_n] = \mathbb{E}[\hat{\theta}_n] - \theta$. Following the given hint, we can arrive at the solution as follows:

$$X_1, ..., X_n \sim \mathcal{U}[0, \theta] \implies \mathbb{E}[X_{(n)}] = \frac{n\theta}{n+1} \quad \text{(where } X_{(n)} \text{ is the } n\text{th order statistic)}$$

$$\implies \mathbb{B}[\hat{\theta}_n] = \mathbb{B}[X_{(n)}] = \mathbb{E}[X_{(n)}] - \theta = \frac{n\theta}{n+1} - \theta = \frac{n\theta}{n+1} - \frac{n\theta + \theta}{n+1} = \frac{n\theta - n\theta - \theta}{n+1}$$

$$\implies \mathbb{B}[\hat{\theta}_n] = -\frac{\theta}{n+1}$$

4.3 Part c

Let us find the bias-corrected estimate $\hat{\theta}_n^J$ using the jackknife method. Following lecture 6b, let us define $\hat{\theta}_n^J$ as follows:

$$\implies \hat{\theta}_n^J = n\hat{\theta}_n - (n-1)\bar{\theta}_n^J$$

where

$$\bar{\theta}_n^J = \frac{1}{n}\sum_{i=1}^{n} \hat{\theta}_n^{(-1)} = \frac{1}{n}\sum_{i=1}^{n} s(X_1, ..., X_{i-1}, X_{i+1}, ..., X_n)$$

In other words, we have n samples of size $n-1$ such that one data point (X_i) is left out in each sample. Considering our definition of $\hat{\theta}_n$ (as $max\{X_1, ..., X_n\} = X_{(n)}$) with these jackknife replications, we can see that the n jackknife samples will be composed of $n-1$

samples of $X_{(n)}$ and 1 sample of $X_{(n-1)}$ since the maximum of $X_1, ..., X_n$ will be removed in one of the jackknife samples. Thus, we arrive at the following implications:

$$\implies \bar{\theta}_n^J = \frac{1}{n}\sum_{i=1}^{n} \hat{\theta}_n^{(-1)} = \frac{1}{n}((n-1)X_{(n)} + X_{(n-1)}) = \frac{nX_{(n)} - X_{(n)} + X_{(n-1)}}{n}$$

At this point, we can plug in for $\bar{\theta}_n^J$ with our definition of $\hat{\theta}_n^J$:

$$\implies \hat{\theta}_n^J = nX_{(n)} - (n-1) \cdot \frac{nX_{(n)} - X_{(n)} + X_{(n-1)}}{n}$$

$$\implies \hat{\theta}_n^J = nX_{(n)} - nX_{(n)} + X_{(n)} - X_{(n-1)} + \frac{nX_{(n)} - X_{(n)} + X_{(n-1)}}{n}$$

$$\implies \hat{\theta}_n^J = \frac{nX_{(n)}}{n} - \frac{nX_{(n-1)}}{n} + \frac{nX_{(n)} - X_{(n)} + X_{(n-1)}}{n}$$

$$\implies \hat{\theta}_n^J = \frac{nX_{(n)} - nX_{(n-1)} + nX_{(n)} - X_{(n)} + X_{(n-1)}}{n}$$

$$\implies \hat{\theta}_n^J = \frac{2nX_{(n)} - X_{(n)} - nX_{(n-1)} + X_{(n-1)}}{n}$$

$$\implies \hat{\theta}_n^J = \frac{(2n-1)X_{(n)} - (n-1)X_{(n-1)}}{n}$$

4.4 Part d

Let us find the bias of $\hat{\theta}_n^J$. From lecture 6a, we will use

$$\mathbb{B}[\hat{\theta}_n^J] = \mathbb{E}[\hat{\theta}_n] - \mathbb{E}[\mathbb{B}_J[\hat{\theta}_n]] - \theta$$

Let us solve for each component, starting with $\mathbb{E}[\hat{\theta}_n]$:

$$\implies \mathbb{E}[\hat{\theta}_n] = \mathbb{E}[X_{(n)}] = \frac{n\theta}{n+1} \quad \text{(using hint)}$$

For $\mathbb{E}[\mathbb{B}_J[\hat{\theta}_n]]$:

$$\implies \mathbb{B}_J[\hat{\theta}_n] = (n-1)(\bar{\theta}_n^J - \hat{\theta}_n)$$

$$\implies \mathbb{E}[(n-1)(\bar{\theta}_n^J - \hat{\theta}_n)] = (n-1)\mathbb{E}[\bar{\theta}_n^J - \hat{\theta}_n] = (n-1)(\mathbb{E}[\bar{\theta}_n^J] - \mathbb{E}[\hat{\theta}_n]) \quad (*)$$

$$\implies \mathbb{E}[\hat{\theta}_n] = \frac{n\theta}{n+1}$$

$$\implies \mathbb{E}[\bar{\theta}_n^J] = \mathbb{E}[\frac{1}{n}((n-1)X_{(n)} + X_{(n-1)})] = \frac{1}{n}\mathbb{E}[(n-1)X_{(n)} + X_{(n-1)}]$$

$$\implies \mathbb{E}[\bar{\theta}_n^J] = \frac{1}{n}((n-1)\mathbb{E}[X_{(n)}] + \mathbb{E}[X_{(n-1)}]) = \frac{1}{n}(\frac{(n-1)n\theta}{n+1} + \frac{(n-1)\theta}{n+1})$$

$$\implies \mathbb{E}[\bar{\theta}_n^J] = \frac{(n-1)n\theta + (n-1)\theta}{n(n+1)} = \frac{n^2\theta - n\theta + n\theta - \theta}{n(n+1)} = \frac{n^2\theta - \theta}{n(n+1)}$$

Plugging back into $(*)$:

$$\implies \mathbb{E}[\mathbb{B}_J[\hat{\theta}_n]] = (n-1)(\frac{n^2\theta - \theta}{n(n+1)} - \frac{n\theta}{n+1}) = (n-1)(\frac{n^2\theta - \theta - n^2\theta}{n(n+1)})$$

$$\implies \mathbb{E}[\mathbb{B}_J[\hat{\theta}_n]] = -\frac{\theta(n-1)}{n(n+1)}$$

Putting it all together:

$$\implies \mathbb{B}[\hat{\theta}_n^J] = \frac{n\theta}{n+1} + \frac{\theta(n-1)}{n(n+1)} - \theta = \frac{n^2\theta + n\theta - \theta - n^2\theta - n\theta}{n(n+1)}$$

$$\implies \mathbb{B}[\hat{\theta}_n^J] = -\frac{\theta}{n(n+1)}$$

5 Jackknife: Implementation

Let us now implement the jackknife method. Let $X_1, ..., X_n \sim \mathcal{N}(\mu, \sigma^2)$ with $\sigma^2 = 1$. We suppose that the parameter of interest is $\theta = e^u$. The plug-in estimate of θ is $\hat{\theta}_n = e^{\bar{X}_n}$, and it is biased. We want to reduce the bias by jackknifing $\hat{\theta}_n$:

5.1 Part a

We recall that the jackknife assumes that

$$\mathbb{B}[\hat{\theta}_n] = \frac{a}{n} + \frac{b}{n^2} + O(\frac{1}{n^3}) \quad \text{as } n \to \infty$$

Let us check this assumption for $\hat{\theta}_n$:

$$\mathbb{B}[\hat{\theta}_n] = \mathbb{E}[\hat{\theta}_n] - \theta = \mathbb{E}[e^{\bar{X}_n}] - e^{\mu} \quad (*)$$

$$\implies \mathbb{E}[e^{\bar{X}_n}] = \mathbb{E}[e^{\frac{1}{n}\sum_{i=1}^n X_i}] = \mathbb{E}[e^{\frac{1}{n}(X_1, X_2, ..., X_n)}] = \mathbb{E}[e^{\frac{1}{n}X_1} \cdot e^{\frac{1}{n}X_2} \cdot ... \cdot e^{\frac{1}{n}X_n}]$$

$$\implies \mathbb{E}[e^{\bar{X}_n}] = e^{n(\frac{\mu}{n} + \frac{\sigma^2}{2n^2})} = e^{\mu + \frac{\sigma^2}{2n}} = e^{\mu + \frac{1}{2n}} \quad \text{(using hint)}$$

Plugging back into $(*)$:

$$\implies \mathbb{B}[\hat{\theta}_n] = e^{\mu + \frac{1}{2n}} - e^{\mu} = e^{\mu} \cdot e^{\frac{1}{2n}} - e^{\mu} = e^{\mu}(e^{\frac{1}{2n}} - 1)$$

With the Taylor expansion of $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$, we can verify the initial jackknife assumption as follows:

$$\implies \mathbb{B}[\hat{\theta}_n] = e^{\mu}(1 + \frac{1}{2n} + \frac{1}{8n^2} + O(\frac{1}{n^3}) - 1) = e^{\mu}(\frac{1}{2n} + \frac{1}{8n^2} + O(\frac{1}{n^3}))$$

$$\implies \mathbb{B}[\hat{\theta}_n] = \frac{e^{\mu}}{2n} + \frac{e^{\mu}}{8n^2} + O(\frac{1}{n^3}) \implies a = \frac{e^{\mu}}{2}, b = \frac{e^{\mu}}{8}$$

Therefore, this assumption holds when $a = \frac{e^{\mu}}{2}$ and $b = \frac{e^{\mu}}{8}$.

5.2 Part b

```
mu = 5;
n = 100;
sigma = 1;
X = normrnd(mu,sigma,n,1);
actual_bias = exp(mu) * (exp(1/(2*n)) - 1);
theta = zeros(n,1);
for i = 1:n
    X_jk = X([1:i-1,i+1:end]);
    theta(i) = exp(mean(X_jk));
end
jk_theta_bar = mean(theta);
jk_bias_est = (n-1) * (jk_theta_bar - exp(mean(X)));

disp("Bias (actual):"); disp(actual_bias);
```

```
Bias (actual):
    0.7439
```

```
disp("Bias (jackknife estimate):"); disp(jk_bias_est);
```

```
Bias (jackknife estimate):
    0.6677
```

We see that the estimated bias value (using the jackknife method) is slightly lower than the actual bias value 0.7439, but the two values are generally very similar.

5.3 Part c

```
mu = 5;
n = 100;
sigma = 1;
r = 10^4;
theta = exp(mu);
theta_hat = zeros(r,1);
theta_hat_J = zeros(r,1);
for j = 1:r
    X = normrnd(mu,sigma,n,1);
    theta_hat(j) = exp(mean(X));
    theta_hat_J_j = zeros(n,1);
    for i = 1:n
        X_jk = X([1:i-1,i+1:end]);
        theta_hat_J_j(i) = exp(mean(X_jk));
    end
    theta_hat_J(j) = n*exp(mean(X)) - (n-1)*mean(theta_hat_J_j);
end
B1 = mean(theta_hat) - theta
```

```
B1 = 0.7372
```

```
B2 = mean(theta_hat_J) - theta
```

```
B2 = -0.0086
```

We see that B1 is approximately equal to the exact bias value seen in part (b). We also see that B1 is larger (in absolute value) than B2.