1 Testing the Uniform Distribution

Let $X_1, ..., X_n \sim \mathcal{U}[0, \theta]$ and suppose that we want to test

$$H_0 : \theta = \frac{1}{2} \quad vs. \quad H_1 : \theta > \frac{1}{2}$$

We note that it seems natural to reject H_0 if $X_{(n)} = max\{X_1, ...X_n\}$ is large. So, we use a test with the rejection region of the following form: $\mathcal{R} = \{X : X_{(n)} > c\}$:

1.1 Part a:

Let us find the power function $\beta(\theta)$ of this test:

$$\implies \beta(\theta) = \mathbb{P}(X_{(n)} > c|\theta) \quad \text{*Lecture 11}$$

$$\implies \mathbb{P}(\text{Type I error}) = \mathbb{P}(X_{(n)} > c|\theta = \frac{1}{2}), \quad \mathbb{P}(\text{Type II error}) = 1 - \mathbb{P}(X_{(n)} > c|\theta > \frac{1}{2})$$

Let us consider three cases for c: **(i)** $c < 0$, **(ii)** $c \in [0, \theta]$, **(iii)** $c > \theta$

(i) If $c < 0$, then $\mathbb{P}(X_{(n)} > c|\theta) = 1$ since each $X_i \geq 0$.

(ii) If $c \in [0, \theta]$, then let us consider $\mathbb{P}(X_{(n)} > c|\theta)$:

$$\implies \mathbb{P}(X_{(n)} > c|\theta) = 1 - \mathbb{P}(X_{(n)} \leq c|\theta)$$

$$\implies \mathbb{P}(X_{(n)} \leq c|\theta) = \mathbb{P}(X_1 \leq c, X_2 \leq c, ..., X_n \leq c|\theta) = \prod_{i=1}^{n} \frac{c-0}{\theta - 0} = \left(\frac{c}{\theta}\right)^n$$

$$\implies \mathbb{P}(X_{(n)} > c|\theta) = 1 - \left(\frac{c}{\theta}\right)^n$$

(iii) If $c > \theta$, then $\mathbb{P}(X_{(n)} > c|\theta) = 0$ since each $X_i \leq \theta$.

Therefore, we define the power function as follows:

$$\beta(\theta) = \mathbb{P}(X_{(n)} > c|\theta) = \begin{cases} 1 & \text{if } c \leq 0 \\ 1 - (\frac{c}{\theta})^n & \text{if } c \in [0, \theta] \\ 0 & \text{if } c \geq \theta \end{cases}$$

1.2 Part b

Let us determine what choice of c will make the size of the test α. From Lecture 11 (Eq. 14), we know that the size of the test is the largest possible probability of the type I error:

$$\alpha = \sup_{\theta \in \Theta_0} \beta(\theta) \implies \alpha = \beta\left(\frac{1}{2}\right) = 1 - \left(\frac{c}{\frac{1}{2}}\right)^n = 1 - (2c)^n \implies c = \frac{1}{2}(1 - \alpha)^{\frac{1}{n}}$$

1.3 Part c

Let us find the p-value if $n = 20$ and $X_{(n)} = 0.48$. Let us plug in accordingly:

$$\implies p(X) = \alpha^* = 1 - (2 \cdot 0.48)^{20} \approx 0.558$$

2 Testing the Normal Distribution

Let $X_1, ..., X_n \sim \mathcal{N}(\mu, \sigma^2)$ such that $\sigma^2 = 1$. Suppose we want to test

$$H_0 : \mu = 0 \quad \textit{vs.} \quad H_1 : \mu = 1$$

We consider a test with the rejection region $\mathcal{R} : \{X : \bar{X}_n > c\}$:

2.1 Part a

Let us construct a test of size α. We follow a related example from Lecture 11:

$$\beta(\mu) = \mathbb{P}(\bar{X}_n > c|\mu)$$

$$\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n}) \implies \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \sim \mathcal{N}(0, 1)$$

With $\sigma^2 = 1$:

$$\implies \beta(\mu) = \mathbb{P}\left(\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} = \sqrt{n}(\bar{X} - \mu) > \frac{\sqrt{n}(c - \mu)}{\sigma} = \sqrt{n}(c - \mu)\right)$$

$$\implies \beta(\mu) = 1 - \Phi(\sqrt{n}(c - \mu))$$

$$\implies \alpha = \beta(\mu = 0) = 1 - \Phi(\sqrt{n}(c - 0)) = 1 - \Phi(\sqrt{n} \cdot c)$$

$$\implies c = \frac{\Phi^{-1}(1 - \alpha)}{\sqrt{n}}$$

2.2 Part b

Let us find its power under H_1. So, let us plug into $\beta(\mu)$ for $\mu = 1$:

$$\implies \beta(1) = 1 - \Phi(\sqrt{n}(c - 1)) = 1 - \Phi\left(\sqrt{n}\left(\frac{\Phi^{-1}(1 - \alpha)}{\sqrt{n}} - 1\right)\right)$$

2.3 Part c

Let us find the limit of the power function as $n \to \infty$. Let us find the limits separately for H_0 and H_1:

$$H_0 \implies \lim_{n\to\infty}\left(1 - \Phi\left(\sqrt{n} \cdot \frac{\Phi^{-1}(1 - \alpha)}{\sqrt{n}}\right)\right) = \lim_{n\to\infty}\left(1 - \Phi\left(\Phi^{-1}(1 - \alpha)\right)\right) = 1 - 1 + \alpha = \alpha$$

$$H_1 \implies \lim_{n\to\infty}\left(1 - \Phi\left(\sqrt{n}\left(\frac{\Phi^{-1}(1 - \alpha)}{\sqrt{n}} - 1\right)\right)\right) = \lim_{n\to\infty}\left(1 - \Phi\left(\Phi^{-1}(1 - \alpha) - \sqrt{n}\right)\right) = 1$$

3 Wald Meets Poisson

Let $X_1, ..., X_n \sim$ Poisson(λ). Suppose we want to test $H_0 : \lambda = \lambda_0$ vs. $H_1 : \lambda \neq \lambda_0$ where $\lambda_0 > 0$ is some constant:

3.1 Part a

Let us construct the size α Wald test with the estimate of λ using the maximum likelihood method. From Lecture 12 we know that the size α Wald test rejects H_0 when

$$W = \left| \frac{\hat{\lambda} - \lambda_0}{\hat{se}} \right| > z_{1-\frac{\alpha}{2}}$$

Let us estimate λ:

$$\mathcal{L}(\lambda | X_1, ..., X_n) = \prod_{i=1}^{n} f(X_i; \lambda)$$

Since we have $X_1, ..., X_n \sim$ Poisson(λ), we know the PDF f:

$$\implies \mathcal{L}(\lambda | X_1, ..., X_n) = \prod_{i=1}^{n} \frac{\lambda^{X_i}}{X_i!} e^{-\lambda} = e^{-n\lambda} \prod_{i=1}^{n} \frac{\lambda^{X_i}}{X_i!}$$

$$\implies \ln(\mathcal{L}) = \ln\left(e^{-n\lambda} \prod_{i=1}^{n} \frac{\lambda^{X_i}}{X_i!} \right) = \ln(e^{-n\lambda}) + \ln\left(\prod_{i=1}^{n} \frac{\lambda^{X_i}}{X_i!} \right)$$

$$\implies \ln(\mathcal{L}) = \ln(e^{-n\lambda}) + \ln\left(\prod_{i=1}^{n} \lambda^{X_i} \right) - \ln\left(\prod_{i=1}^{n} X_i! \right)$$

$$\implies \ln(\mathcal{L}) = -n\lambda + \ln(\lambda) \cdot \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \ln(X_i!)$$

Now, let us take the first derivative of $\ln(\mathcal{L})$ with respect to λ:

$$\implies \frac{d}{d\lambda}(\ln(\mathcal{L})) = -n + \frac{1}{\lambda} \sum_{i=1}^{n} X_i = 0 \implies \hat{\lambda}_{MLE} = \frac{1}{n} \sum_{i=1}^{n} X_i = \bar{X}_n$$

Let us note that $\hat{\lambda}_{MLE}$ is a maximum since the second derivative is negative:

$$\implies \frac{d^2}{d\lambda^2}(\ln(\mathcal{L})) = -\frac{1}{\lambda^2} \sum_{i=1}^{n} X_i < 0 \quad \checkmark$$

At this point, we have $W = \left|\frac{\hat{\lambda}_{MLE} - \lambda_0}{\hat{se}}\right| = \left|\frac{\bar{X}_n - \lambda_0}{\hat{se}}\right| > z_{1-\frac{\alpha}{2}}$. Let us solve for the standard error of a MLE (following asymptotic normality from Lecture 10):

$$\implies \hat{se} = \frac{1}{\sqrt{n \cdot I(\hat{\lambda}_{MLE})}} = \frac{1}{\sqrt{n \cdot I(\bar{X}_n)}}$$

We have that the Fisher information for the Poisson(λ) is $\frac{1}{\lambda}$. With this, we can further solve for the standard error as follows:

$$\implies \hat{se} = \frac{1}{\sqrt{n \cdot \frac{1}{\hat{\lambda}_{MLE}}}} = \frac{1}{\sqrt{n \cdot \frac{1}{\bar{X}_n}}} = \sqrt{\frac{\bar{X}_n}{n}}$$

Thus, we reject the null hypothesis H_0 when

$$W = \left|\frac{\bar{X}_n - \lambda_0}{\sqrt{\frac{\bar{X}_n}{n}}}\right| > z_{1-\frac{\alpha}{2}}$$

3.2 Part b

```
l0 = 1;
n = 20;
a = 0.05;
m = 10^4;

X = poissrnd(l0, 1, n);
X_bar = mean(X);
W = abs((X_bar - l0) / sqrt(X_bar / n));
disp("W: "); disp(W);
```

```
W:
    1.2910
```

Since W < 1.96, we do not reject the null hypothesis H_0 in this case. Let us now repeat this process $m = 10^4$ times and report the estimated type I error rate.

```
reject = 0;
for i = 1:m
    data = poissrnd(l0, 1, n);
    meanX = mean(data);
    Wald = abs((meanX - l0) / sqrt(meanX / n));
    if Wald > 1.96
        reject = reject + 1;
    end
end
typeI_error_rate = reject / m;
disp("Estimated type I error rate: "); disp(typeI_error_rate);
```

```
Estimated type I error rate:
    0.0544
```

We see that the estimated type I error rate is typically quite low which suggests that the null hypothesis is generally not rejected.

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4 Application to Geology

Given the soild pH level data, let us determine whether or not the data suggests that the true mean soil pH values differ for the two locations.

```
locA = [7.58 8.52 8.01 7.99 7.93 7.89 7.85 7.82 7.80];
locB = [7.85 7.73 8.53 7.40 7.35 7.30 7.27 7.27 7.23];
n = length(locA);
m = length(locB);

K = 10^5;
s_obs = abs(mean(locA) - mean(locB));
Z = [locA locB];

count = 0;
for i = 1:K
    Z_pi = Z(:,randperm(n + m));
    s_pi = abs(mean(Z_pi(1:n)) - mean(Z_pi(n+1:2*n)));
    if s_pi > s_obs
        count = count + 1;
    end
end
p = count / K;
disp("Estimate p-value: "); disp(p);
```

```
Estimate p-value:
    0.0339
```

With an estimated p-value of approx. 0.03, there is strong evidence to reject the null hypothesis. This implies that we have strong evidence that the true mean soil pH values differ for the two locations.

5 Application to Linguistics

Given the proportions of the three-letter words found in 8 Twain essays vs. 10 Snodgrass essays, let us investigate whether or not the Snodgrass essays were written by Mark Twain.

Part a

Let us first perform the Wald test for equality of the means. In other words, so solve for the p-value, we will solve $W(X) = z_{1-\frac{\alpha}{2}}$ such that $p(X) = 2\Phi(-W(X))$.

```
twain = [0.225, 0.262, 0.217, 0.240, 0.230, 0.229, 0.235, 0.217];
snodgrass = [0.209, 0.205, 0.196, 0.210, 0.202, 0.207, 0.224, 0.223, 0.220, 0.201];
n = length(twain);
m = length(snodgrass);

W = abs((mean(twain) - mean(snodgrass)) / sqrt((var(twain)/n) + (var(snodgrass)/m)));
p_value = 2*normcdf(-1*W);
disp("p-value: "); disp(p_value);
```

```
p-value:
   2.1260e-04
```

Since the p-value is very small and close to 0 (less than 0.01), we have very strong evidence against the null hypothesis. Thus, we conclude that it is very likely that Twain did not write the Snodgrass essays.

Part b

Let us now perform the permutation test in order to avoid large sample normality assumptions.

```
K = 10^5;
s_obs = abs(mean(twain) - mean(snodgrass));
Z = [twain snodgrass];

count = 0;
for i = 1:K
    Z_pi = Z(:,randperm(n + m));
    s_pi = abs(mean(Z_pi(1:n)) - mean(Z_pi(n+1:n+m)));
    if s_pi > s_obs
        count = count + 1;
    end
end
p = count / K;
disp("Estimated p-value: "); disp(p);
```

```
Estimated p-value:
   7.5000e-04
```

Furthermore, since this estimated p-value is also very small and close to zero, we make the same conclusions as in part a such that we have very strong evidence against the null hypothesis. Thus, it is very strongly suggested that Twain did not write the Snodgrass essays.

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