An Exploration of The Polar Wave Equation

Zach Homans, Ben Kahle, Ruby Spring

December 2014

1 The Wave Equation

The general form of the wave equation is as follows, where $u = u(x_1, x_2, \dots x_n, t)$:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

where u is the displacement of the wave in the vertical direction and c is the waves phase velocity, or the velocity at which the wave propagates through space.

This second order partial differential equation is derived from equating Hooke's law with Newton's Second Law of Motion, and can be applied to any real coordinate system by taking the Laplacian of u.

Applying the wave equation to Cartesian coordinates, for example, yields the following equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

In this two coordinate system we have $u = u(x, y, t)$. The boundary condition is $u = 0$ for all $t \geq 0$ and the initial conditions are a given displacement and phase velocity.

Taking the Laplacian in Polar coordinates (so $u = u(r, \theta, t)$) yields the variant of the wave equation that we care about:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{r \partial r} + \frac{\partial^2 u}{r^2 \partial \theta^2} \right)$$

In this case, given an outer radius of $r = R$ where $u(R, \theta, t)$ for all θ and all t is fixed at zero (meaning a wave will bounce back from the edge as apposed to continuing infinitely), we have the boundary condition $u(R, \theta, t) = 0$ for $t \geq 0$. Once again the initial conditions are a given displacement and phase velocity.

2 Discretization of the Wave Equation in Polar Coordinates

The following equations result from the discretization of the continuous Polar Wave Equation. The left side of each of the following equations corresponds to one of the right-side terms in the continuous equation. These are the discrete equations we used to generate our matlab simulation.

$$\frac{\partial^2 u}{\partial r^2} \approx \frac{u_{n+1,m} - 2u_{n,m} + u_{n-1,m}}{(\Delta r)^2}$$

$$\frac{\partial u}{\partial r} \approx \frac{u_{n+1,m} - u_{n,m}}{\Delta r}$$

$$\frac{\partial^2 u}{\partial \theta^2} \approx \frac{u_{n,m+1} - 2u_{n,m} + u_{n,m-1}}{(\Delta \theta)^2}$$

You might be skeptical, with good reason, so we present you with Fig. 1 (below), which attempts to illustrate this discretization.

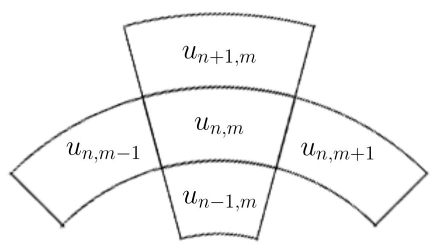

Figure 1: A visualization of how discrete equations can be derived from the Polar-coordinate Wave Equation.

And, finally, Fig. 2 is a screen-shot of the simulation with initial displacement at the origin.

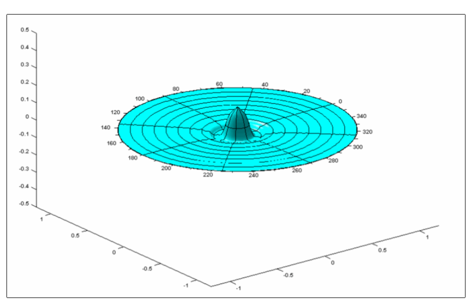

Figure 2: A visualization of how discrete equations can be derived from the Polar-coordinate Wave Equation.