JPEG2000 – Wavelets and Compression

Ben Keene

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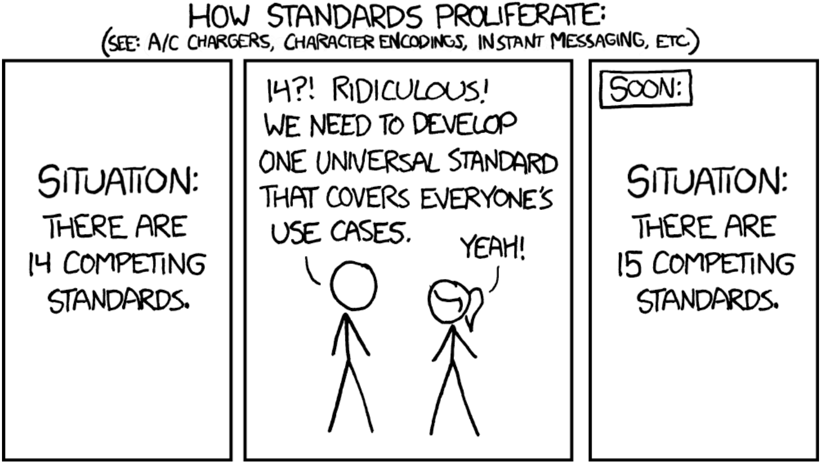


Standards. xkcd. https://xkcd.com/927/

Four Different Formats

GIF
Developed in 1987 by CompuServe to distributed color images over slow modems. Allowed multiple images in a stream. Largely replaced by mp4. Uses LZW, a dictionary based lossless compression algorithm.

PNG
Developed in 1996 to replace GIF, which required developers to pay royalties. PNG colloquially stands for "PNG's Not GIF". Uses a combination of pointers and encodings to achieve lossless compression.

JPEG
Developed in 1992 by the Joint Photographic Experts Group. Designed for lossy compression via the Discrete Cosine Transform (DCT). Lossless JPEG introduced in 1993.

JPEG2000
Developed in 2000 to replace JPEG. Uses the Discrete Wavelet Transform (DWT) instead of DCT. Supports both lossless and lossy compression. Lossless compression uses the Daubechies 5/3 wavelet, while lossy uses the Daubechies 9/7 wavelet.

1D DWT

Given a signal $(f_0, f_1, \ldots, f_{n-1})$, where $n = 2^J$, $(J \in \mathbb{N})$, then the function f can be represented in a wavelet basis with coefficients C_{00} and d_{jk} as follows:

$$f(x) = C_{00}\phi(x) + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} d_{jk} 2^{j/2} \psi(2^j x - k)$$

where

ϕ is the scaling function (Father wavelet).

ψ is the wavelet function (Mother wavelet).

C_{00} is the scaling coefficient (average value of the signal).

d_{jk} are the wavelet coefficients of level j at position k.

1D DWT

$$f(x) = \underbrace{C_{00}\phi(x)}_{(1)} + \underbrace{\sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1}}_{(3)} d_{jk}\, \underbrace{2^{j/2}\psi(2^j x - k)}_{(2)}$$

$$C_{00} = \frac{\langle f, \phi_{0,0}\rangle}{||\phi_{0,0}||^2} \qquad d_{jk} = \frac{\langle f, \psi_{j,k}\rangle}{||\psi_{j,k}||^2}$$

(1) represents the coarsest approximation of f. What does f look like from the perspective of the scaling function?

(2) is the tool with which we are measuring f at a very fine resolution.

(3) ensures we are measuring f over all positions k and resolutions j.

Color space

A color space is a way of representing colors as tuples of numbers. Some examples are RGB (red, green, blue), CMY (cyan, magenta, yellow), HSL (hue, saturation, lightness), and HSV (hue, saturation, value).

JPEG2000 relies upon YCbCr (luminance, blue chrominance, red chrominance) for ossy compression.

There doesn't exist a bijection between the two color spaces, but since YCbCr offers better compression, images are converted before the DWT.

Lossless compression, outside the scope of this presentation, uses the YUV color space.

CDF 9-tap/7-tap wavelet

The Cohen-Daubechies-Feauveau (CDF) 9-tap/7-tap wavelet (also called the Biorthogonal 4.4 wavelet) is used for lossy compression in JPEG2000.

The numbers 9 and 7 in CDF 9/7 refer to the number of length of the low-pass and high-pass analysis filters.

The numbers 4 and 4 in "Biorthogonal 4.4" refer to the number of vanishing moments of the filters.

k	$\mathrm{dec}_{\mathrm{lo}}$	$\mathrm{dec}_{\mathrm{hi}}$	$\mathrm{rec}_{\mathrm{lo}}$	$\mathrm{rec}_{\mathrm{hi}}$
-4	0.026748757411	0	0	0.026748757411
-3	-0.016864118443	0.091271763114	-0.091271763114	0.016864118443
-2	-0.078223266529	-0.057543526229	-0.057543526229	-0.078223266529
-1	0.266864118443	-0.591271763114	0.591271763114	-0.266864118443
0	0.602949018236	1.11508705	1.11508705	0.602949018236
1	0.266864118443	-0.591271763114	0.591271763114	-0.266864118443
2	-0.078223266529	-0.057543526229	-0.057543526229	-0.078223266529
3	-0.016864118443	0.091271763114	-0.091271763114	0.016864118443
4	0.026748757411	0	0	0.026748757411

CDF 9/7 Plots

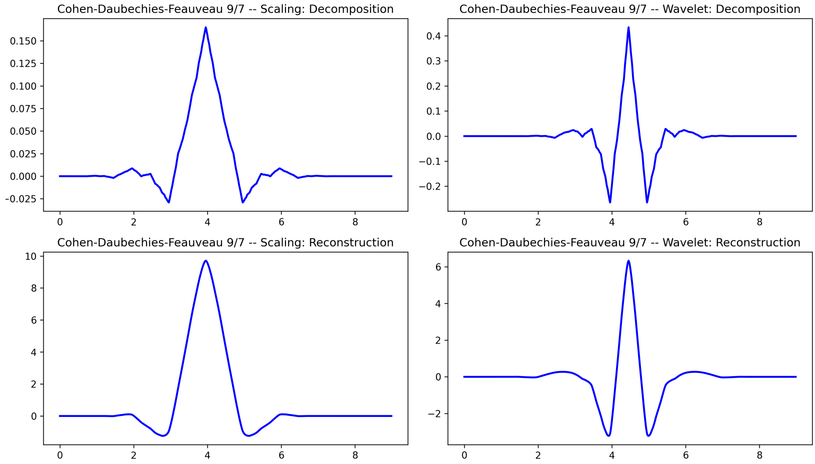
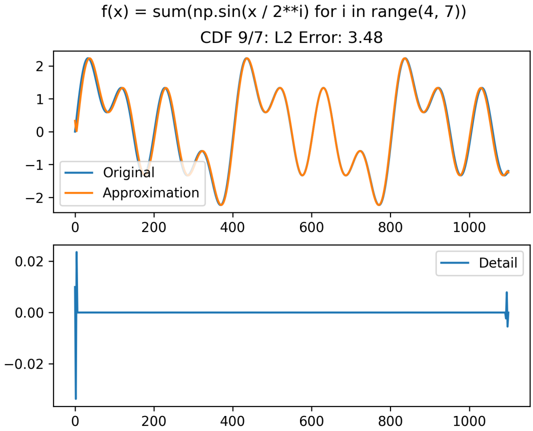


Figure: Scaling and Wavelet functions for CDF 9/7

Approximation

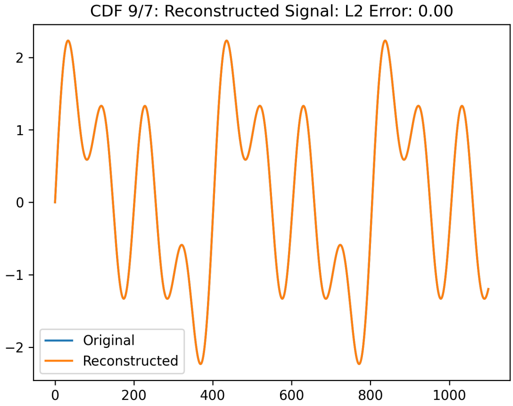
$$\text{Consider: } f(x) = \sin\left(\frac{x}{16}\right) + \sin\left(\frac{x}{32}\right) + \sin\left(\frac{x}{64}\right)$$

The approximation of f via the DWT with CDF $9/7$ wavelet is not too bad.



Reconstruction

As expected, we get perfect reconstruction.



2D DWT: Notation

$\Phi(x, y) = \phi(x)\phi(y)$: 2D scaling function.

$\Psi^H(x, y) = \psi(x)\phi(y)$: Horizontal wavelet function.

$\Psi^V(x, y) = \phi(x)\psi(y)$: Vertical wavelet function.

$\Psi^D(x, y) = \psi(x)\psi(y)$: Diagonal wavelet function.

$\Phi_{jmn}(x, y) = 2^{j/2}\Phi(2^j x - m, 2^j y - n)$: 2D scaling function at level j and position (m, n).

$\Psi^{\alpha}_{jmn}(x, y) = 2^{j/2}\Psi^{\alpha}(2^j x - m, 2^j y - n)$: 2D wavelet function at level j and position (m, n) for orientation $\alpha \in \{H, V, D\}$.

2D DWT: Notation

Given a 2D signal $f(x,y)$ of size $M \times N$, the 2D Discrete Wavelet Transform relates to f via:
For simplicity, assume $M = N = 2^J$ for some $J \in \mathbb{N}$. Then, we can construct a representation of f in a wavelet basis as follows:

$$f(x,y) = C_{000}\Phi(x,y) + \sum_{\alpha \in H,V,D} \sum_{j=0}^{J-1} \sum_{m=0}^{2^j-1} \sum_{n=0}^{2^j-1} d^{\alpha}_{jmn} 2^{j/2} \Psi^{\alpha}(2^j x - m, 2^j y - n)$$

where

C_{000} is the 2D scaling coefficient (average value of the signal).

d^{α}_{jmn} are the wavelet coefficients for orientation α at level j and position (m,n).

2D DWT

$$f(x,y) = \underbrace{C_{000}\Phi(x,y)}_{(1)} + \underbrace{\sum_{\alpha \in H,V,D} \sum_{j=0}^{J-1} \sum_{m=0}^{2^j-1} \sum_{n=0}^{2^j-1}}_{(3)} d^{\alpha}_{jmn} \underbrace{2^{j/2}\Psi^{\alpha}(2^j x - m, 2^j y - n)}_{(2)}$$

$$C_{000} = \frac{\langle f, \Phi_{0,0,0} \rangle}{||\Phi_{0,0,0}||^2} \qquad d^{\alpha}_{jmn} = \frac{\langle f, \Psi^{\alpha}_{j,m,n} \rangle}{||\Psi^{\alpha}_{j,m,n}||^2}$$

(1) represents the coarsest approximation of f. What does f look like from the perspective of the scaling function?

(2) is the tool with which we are measuring f at a very fine resolution.

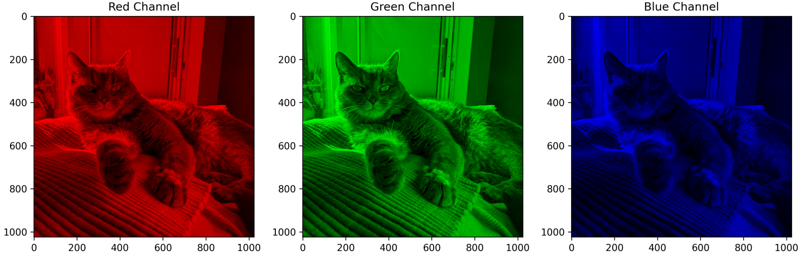
(3) ensures we are measuring f over all positions (m, n) and resolutions j for each orientation α.

Our Subject

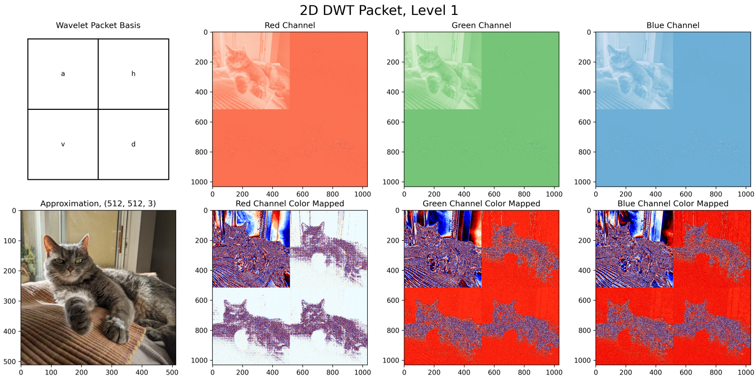
Figure: Ava, short for Avocado

Next, let's split into RGB as a toy example.

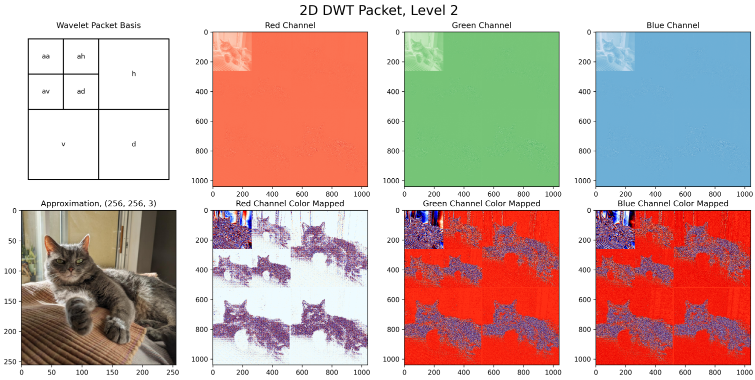
RGB


Red Channel
Green Channel
Blue Channel
0
200
400
600
800
1000

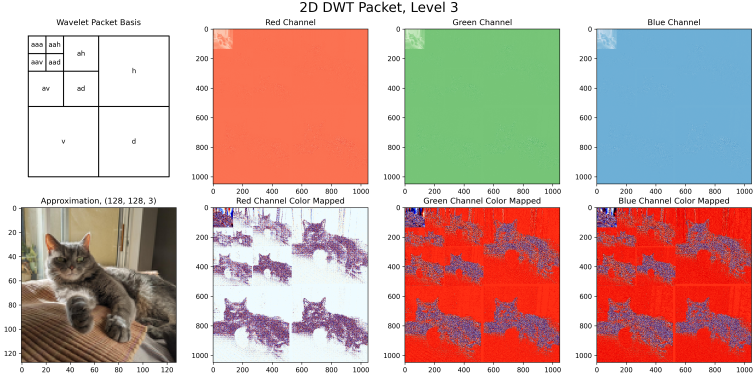
Level 1


2D DWT Packet, Level 1
Wavelet Packet Basis
a
h
v
d
Red Channel
Green Channel
Blue Channel
Approximation, (512, 512, 3)
Red Channel Color Mapped
Green Channel Color Mapped
Blue Channel Color Mapped

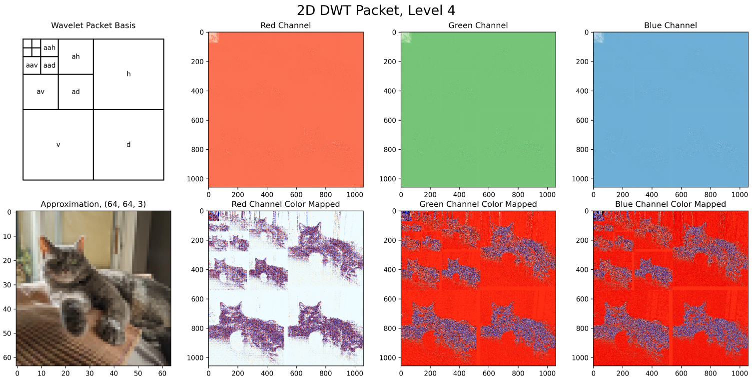
Level 2


2D DWT Packet, Level 2
Wavelet Packet Basis
aa
ah
h
av
ad
v
d
Red Channel
Green Channel
Blue Channel
Approximation, (256, 256, 3)
Red Channel Color Mapped
Green Channel Color Mapped
Blue Channel Color Mapped

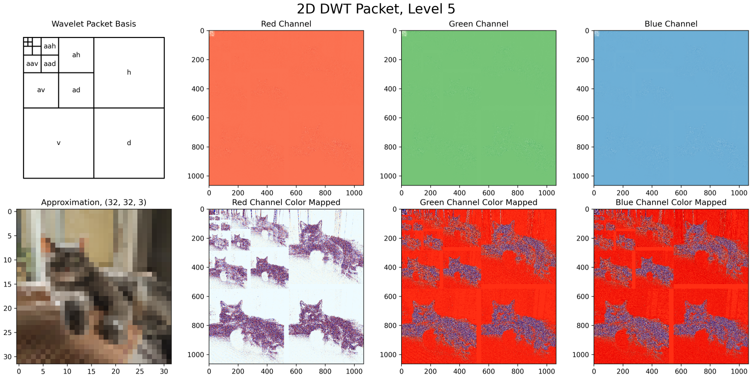
Level 3


2D DWT Packet, Level 3
Wavelet Packet Basis
aaa
aah
aav
aad
ah
av
ad
h
v
d
Red Channel
Green Channel
Blue Channel
Approximation, (128, 128, 3)
Red Channel Color Mapped
Green Channel Color Mapped
Blue Channel Color Mapped

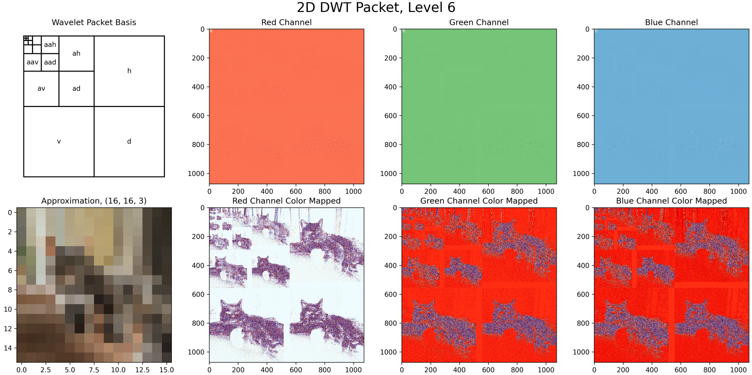
Level 4


2D DWT Packet, Level 4
Wavelet Packet Basis
aah
ah
aav
aad
h
av
ad
v
d
Red Channel
Green Channel
Blue Channel
Approximation, (64, 64, 3)
Red Channel Color Mapped
Green Channel Color Mapped
Blue Channel Color Mapped

Level 5


2D DWT Packet, Level 5
Wavelet Packet Basis
aah
ah
aav
aad
h
av
ad
v
d
Red Channel
Green Channel
Blue Channel
Approximation, (32, 32, 3)
Red Channel Color Mapped
Green Channel Color Mapped
Blue Channel Color Mapped

Level 6


2D DWT Packet, Level 6
Wavelet Packet Basis
aah
ah
aav
aad
h
av
ad
v
d
Red Channel
Green Channel
Blue Channel
Approximation, (16, 16, 3)
Red Channel Color Mapped
Green Channel Color Mapped
Blue Channel Color Mapped

What's left?

There are two **major** steps left to the JPEG2000 algorithm that are outside the scope of this presentation. Quantization and encoding.

This project used the CDF 9/7 wavelet, which is intended to be used for lossy compression. Consider applying a pointwise threshold to each pixel in the detail coefficients. This is quantization.

Encoding is the method through which the quantized coefficients are stored in a file. In common practice, the Embedded Block Coding with Optimized Truncation (EBCOT) algorithm is used.

References

- I Daubechies (1992). Ten lectures on wavelets. Philadelphia, Pa.: Society For Industrial And Applied Mathematics.
- International Organization for Standardization. (2000). Information technology JPEG 2000 image coding system Part 1: Core coding system (ISO Standard No. 15444-1:2000). https://www.iso.org/standard/27687.html
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