MATH 118 Final Exam Review W19

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Final Exam Review: Sunday, April 7th, 2019 from 2:00PM - 6:00PM in STC 1012

This final exam review package is meant to help prepare MATH 118 students for the upcoming final examination. The problems have been selected by the instructor to provide the best representation of the scope of the entire course and final examination.

Here are a few notes regarding the final exam:

1. A formula sheet will be provided during the exam and has been provided at the end of this review package. Please use this formula sheet during your studies to familiarize yourself with its use and its contents. Any concepts, formulae or equations that are not on the formula sheet will not be provided to you.

2. When you write an exam, please be sure to read the front matter! Please read it during the midterm so that you're not missing out on any important details. Take note that (a) no calculators are allowed and (b) there is an extra blank page for rough work.

3. Remember that "blank page for rough work"? Read the instructions on this extra page as well! Note that if you want to use this page for rough work...then it's just that. BUT, **If you run out of room and wish to continue your answer on the extra page, you MUST indicate in the original question page that you have chosen to continued the solution on the extra page!**.

It is important to note that although these questions provide a very clear understanding of the concepts that have been covered throughout the course, *they do not cover every single type of question* that will appear on the final exam. As with every test, there must be some sort of unseen element that truly tests your knowledge of the concepts.

Some parting words... I've had a wonderful time being a part of your first year at UofWaterloo, and thank you for making it such an amazing experience for me as well. Best of luck with your final exams, and always try your best in whatever you do, now and in the future. I look forward to seeing you all make your own dreams come true, and to see how you impact the lives around you as you pursue your careers. :-) - Brenda

1 Integration Techniques

1. Evaluate the given integral using any method or method(s) you deem suitable.

   (a) $\displaystyle\int \frac{\cos x}{1-\sin x}\,dx$

   (b) $\displaystyle\int \frac{x^3}{x^4+2}\,dx$

   (c) $\displaystyle\int \frac{x+2}{x^2+3x-4}\,dx$

   (d) $\displaystyle\int \frac{1}{x^3\sqrt{x^2-1}}\,dx$

   (e) $\displaystyle\int \sin^5 x\cos^4 x dx$

   (f) $\displaystyle\int \frac{1}{x+2\sqrt{1-x}+2}\,dx$

(g) $\displaystyle\int \frac{x-2}{x-3\sqrt{2x-4}+2}\,dx$

(h) $\displaystyle\int \frac{4}{x^2+5x-14}\,dx$

(i) $\displaystyle\int \sin^2 x \sin(2x)\,dx$

(j) $\displaystyle\int \sqrt{4-9x^2}\,dx$

(k) $\displaystyle\int \frac{8-3x}{10x^2+13x-3}\,dx$

(l) $\displaystyle\int 4x\cos(2-3x)\,dx$

(m) $\displaystyle\int \frac{x^2+7x}{(x+2)(x-1)(x-4)}\,dx$

(n) $\displaystyle\int \sqrt{\cos x}\sin^3 x\,dx$

(o) $\displaystyle\int \frac{\sqrt{x^2+16}}{x^4}\,dx$

(p) $\displaystyle\int \frac{3x^2+1}{(x+1)(x-5)^2}\,dx$

(q) $\displaystyle\int \sin^2 x\cos^3 x\,dx$

(r) $\displaystyle\int \sin^3 x\cos^2 x\,dx$

2 Improper Integrals

1. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

   (a) $\displaystyle\int_0^{\infty} (1_2 x)e^{-x}\,dx$

   (b) $\displaystyle\int_{-5}^{1} \frac{1}{10+2x}\,dx$

   (c) $\displaystyle\int_0^{4} \frac{x}{x^2-9}\,dx$

3 Numerical Integration

1. For the following integral $I = \int_0^{\pi} \sin x\,dx$:

   (a) Find an approximation for I for $n = 5$ using Simpson's Rule, and then the error involved in this approximation.

   (b) How large do we have to choose n so that the approximation from (a) is accurate to within 0.0001?

   (c) Find an approximation for I for $n = 2$ using the Trapezoidal Rule, and then the error involved in this approximation.

   (d) How large do we have to choose n so that the approximation from (c) is accurate to within 0.0001?

2. For the following integral $I = \int_1^2 e^{1/x}\ dx$:

   (a) Find an approximation for I for $n = 5$ using Simpson's Rule, and then the error involved in this approximation.

   (b) How large do we have to choose n so that the approximation from (a) is accurate to within 0.0001?

   (c) Find an approximation for I for $n = 2$ using the Trapezoidal Rule, and then the error involved in this approximation.

   (d) How large do we have to choose n so that the approximation from (c) is accurate to within 0.0001?

4 Differential Equations

1. Solve the following differential equation, and if required, solve the initial value problem.

   (a) $\dfrac{dy}{dx} = 3x^2y^2$

   (b) $xy'' + y' = 4x$

   (c) $\dfrac{dy}{dx} = x\sqrt{y}$

   (d) $y'' = yy'$

   (e) $y'' = y' + 2x$

   (f) $\dfrac{dy}{dx} = xe^y, \qquad y(0) = 0$

   (g) $x^2y'' = (y')^2$

   (h) $x^2y' + 2xy = \ln x, \qquad y(1) = 2$

   (i) $y'' + 4y = 0$

   (j) $y' = x - y$

5 Sequences

1. Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

   (a) $\left\{\dfrac{1}{2},\ \dfrac{1}{4},\ \dfrac{1}{6},\ \dfrac{1}{8}, \ldots\right\}$

   (b) $\left\{5,\ 8,\ 11,\ 14,\ 17, \ldots\right\}$

2. Determine whether the sequence converges or diverges. If it converges, find the limit.

   (a) $a_n = \dfrac{3 + 5n^2}{n + n^2}$

   (b) $a_n = \dfrac{n^4}{n^3 - 2n}$

3. A sequence a_n is given by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$.

(a) By induction, show that a_n is increasing and bounded above by 3. Apply the Monotonic Sequence Theorem to show that the $\lim_{n\to\infty} a_n$ exists.

(b) Find $\lim_{n\to\infty} a_n$

4. Use the formal definition of the limit of a sequence to prove the following is true.

   (a) $\displaystyle\lim_{n\to\infty} \frac{1}{2^{n-1}} = 0$

   (b) $\displaystyle\lim_{n\to\infty} \frac{5n}{2n+8} = \frac{5}{2}$

6 Series and Convergence Tests

1. Determine whether the series is convergent (absolutely or conditionally) or divergent using a suitable test of your choice. Be sure to state the names of the tests used.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{3ne^n}{n^2+1}$

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n}$

   (c) $\displaystyle\sum_{n=4}^{\infty} \frac{n^2}{n^3-3}$

   (d) $\displaystyle\sum_{n=2}^{\infty} \frac{(-2)^{1+3n}(n+1)}{n^2 5^{1+n}}$

   (e) $\displaystyle\sum_{n=3}^{\infty} \frac{e^{4n}}{(n-2)!}$

   (f) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n\sqrt{n}}$

   (g) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$

   (h) $\displaystyle\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1}$

   (i) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln n}$

   (j) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{n^3+1}$

   (k) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{2n^2+4n+1}}{n^3+9}$

   (l) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{n^2}+1\right)^2$

   (m) $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$

   (n) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$

(o) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+3}}{n^3+4n+1}$

(p) $\displaystyle\sum_{n=1}^{\infty} \cos\frac{1}{n}$

(q) $\displaystyle\sum_{n=1}^{\infty} \left(2^{1/n}-1\right)^n$

(r) $\displaystyle\sum_{n=2}^{\infty} \frac{7}{n(n+1)}$

(s) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{7+2n}$

(t) $\displaystyle\sum_{n=1}^{\infty} \frac{5^{n+1}}{7^{n-2}}$

(u) $\displaystyle\sum_{n=0}^{\infty} \frac{3}{3+5n}$

(v) $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{5n+1}$

(w) $\displaystyle\sum_{n=1}^{\infty} \frac{n!(n+1)!}{(3n)!}$

(x) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n \sin^2(5n)}{4^n+\cos^2(n)}$

(y) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{1-2n}}{n^2+1}$

(z) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$

2. Determine the number of terms of the indicated series that are needed to compute the sum of the series with an error less than the stated error E.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+n}$ $\quad E < 0.001$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2(-1)^n}{n}$ $\quad E < 0.01$

7 Power Series, Taylor and Maclaurin Series

1. Find the radius of convergence and interval of convergence of the given series.

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n n x^n$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^4 4n}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{nx^n}{2^n(n^2+1)}$

2. Find a power series representation for the function and determine the interval of convergence.

   (a) $f(x) = \dfrac{1}{1+x}$

   (b) $f(x) = \dfrac{2}{3-x}$

   (c) $f(x) = \dfrac{4}{(2-x)^2}$

   (d) $f(x) = \dfrac{1}{(1-x)^2}$

   (e) $f(x) = \dfrac{5}{1-4x^2}$

   (f) $f'(x)$ if $f(x) = \dfrac{5x}{1-3x^5}$

   (g) $f'(x)$ if $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$

3. Find the Taylor series for $f(x)$ centered at the given value of a. If a is not stated, find the Maclaurin series for $f(x)$. Find their radius of convergence (optional).

   (a) $f(x) = xe^x, \qquad a = 0$

   (b) $f(x) = \sin x, \qquad a = \pi/6$

   (c) $f(x) = \arctan x$

   (d) $f(x) = e^{-2x}$

4. Approximate $f(x)$ by a Taylor polynomial with degree n at the number a. Then, use Taylor's Inequality to estimate the upper bound of error for the approximation.

   (a) $f(x) = 1/x, \qquad a = 1, \qquad n = 2, \qquad 0.7 \le x \le 1.3$

   (b) $f(x) = x^{2/3}, \qquad a = 1, \qquad n = 3, \qquad 0.8 \le x \le 1.2$

   (c) $f(x) = \sin x, \qquad a = \pi/6, \qquad n = 4, \qquad 0 \le x \le \pi/3$

   (d) $f(x) = x \ln x, \qquad a = 1, \qquad n = 3, \qquad 0.5 \le x \le 1.5$

5. Use a known series to evaluate the given limit.

   (a) $\displaystyle\lim_{x\to 0} \frac{\tan x}{x}$

   (b) $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2}$

   (c) $\displaystyle\lim_{x\to 0} \frac{(1-\cos x)^2}{3x^4}$

8 Parametric Equations

1. Sketch the curve by using the parametric equatons to plot points. Indicate with an arrow the direction in which the curve is traced as t increases.

(a) $x = 2t - 1, \qquad y = 0.5t + 1$

(b) $x = t^2 - 3, \qquad y = t + 2, \qquad -3 \leq tl33$

(c) $x = t^2, \qquad y = t^3$

(d) $x = \sin t, \qquad y = 1 - \cos t, \qquad 0 \leq t \leq 2\pi$

2. Find the slope of the parametric curve at the point corresponding to the given value of the parameter.

(a) $x = t^3 + 1, \qquad y = t^4 + t; \qquad t = -1$

(b) $x = \sqrt{t}, \qquad y = t^2 - 2t; \qquad t = 4$

(c) $x = t^3 - 3t, \qquad y = t^2 - 3;$ find point(s) where tangent is horizontal or vertical

(d) $x = 1 + \ln t, \qquad y = t^2 + 2; \qquad (1, 3)$

(e) $x = t^3 - 3t, \qquad y = t^3 - 3t^2;$ find point(s) where tangent is horizontal or vertical

(f) $x = 1 + \sqrt{t}, \qquad y = e^{t^2}; \qquad (2, e)$

3. Set up an integral that represents the length of the curve.

(a) $x = t + e^{-t}, \qquad y = t - e^{-t}; \qquad 0 \leq t \leq 2$

(b) $x = t^2 - t, \qquad y = t^4; \qquad 1 \leq t \leq 4$

9 Polar Coordinates and Curves

1. Find the polar coordinates (r, θ) of the given point, where $r > 0$ and $0 \leq \theta < 2\pi$

(a) $(-4, 4)$

(b) $(3, 3\sqrt{3})$

(c) $(\sqrt{3}, -1)$

2. Sketch the polar curve with the given polar equation and be sure to show all of your steps.

(a) $r = -2\sin\theta$

(b) $r = 2(1 + \cos\theta)$

(c) $r = 3\cos 3\theta$

(d) $r = 1 + 3\cos\theta$

(e) $r = 2\sin 6\theta$

(f) $r^2 = 9\sin 2\theta$

3. Find the area of the region that represents the area of the region bounded by the given curve and lies in the specified sector.

(a) $r = \cos\theta, \qquad 0 \leq \theta \leq \pi/6$

(b) $r = e^{-\theta/4}, \qquad \pi/2 \leq \theta \leq \pi$

4. Set up an integral that represents the length of the polar curve.

   (a) $r = 2\cos\theta, \qquad 0 \le \theta \le \pi$

   (b) $r = 5^{\theta}, \qquad 0 \le \theta \le 2\pi$

10 FORMULA SHEET

Geometric Series
$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ if $|r| < 1$. If $|r| \ge 1$ then the series diverges.

The Divergence Test
If $\lim\limits_{n\to\infty} a_n$ does not exist or if $\lim\limits_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

The Integral Test
If $f(x)$ is a continuous, positive, decreasing function on $[N, \infty)$, for some constant $N > 0$, such that $a_n = f(n)$ for all $n \ge N$, then the series $\sum_{n=N}^{\infty} a_n$ converges if and only if the improper integral $\int_N^{\infty} f(x)\,dx$ converges.

Integral Estimation Error
Let S represent the sum of a convergent series satisfying the integral test. If we use the partial sum S_n to approximate S then the error satisfies $\int_{n+1}^{\infty} f(x)\,dx \le S - S_n \le \int_n^{\infty} f(x)\,dx$.

p-Series
The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$ and diverges when $p \le 1$.

Comparison Test
Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

(a) If $\sum_{n=1}^{\infty} b_n$ converges and $a_n \le b_n$ for all n then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\sum_{n=1}^{\infty} b_n$ diverges and $a_n \ge b_n$ for all n then $\sum_{n=1}^{\infty} a_n$ diverges.

Limit Comparison Test
Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with all positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = C$ where $0 < C < \infty$ then either both series converge or both diverge.

Alternating Series Test (A.S.T.)
If the alternating series $\sum_{n=1}^{\infty} (-1)^n b_n$ (where $b_n \ge 0$ for all n) satisfies the conditions

(a) $\lim_{n\to\infty} b_n = 0$

(b) $b_n \ge b_{n+1}$ for all n

then the series converges.

Alternating Series Estimation Theorem
Let S denote the sum of a convergent alternating series (satisfying A.S.T). If we use the partial sum S_n to approximate S then the error satisfies $|S - S_n| \le b_{n+1}$.

Ratio Test
Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$. Then

(a) If $L < 1$ then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(b) If $L > 1$ then the series $\sum_{n=1}^{\infty} a_n$ diverges. Also, if $L = \infty$ then the series diverges.

(c) If $L = 1$ then the test is inconclusive.

Root Test
Same as ratio test but with $L = \lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}}$.

Some Known Maclaurin Series

$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ for all $x \in \mathbb{R}$	$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ for all $x \in [-1, 1]$
$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ for all $x \in \mathbb{R}$	$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$ for all $x \in (-1, 1]$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x \in \mathbb{R}$	$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$ for all $x \in (-1, 1)$, $k \in \mathbb{R}$

Taylor's Inequality
Let $T_n(x)$ be the nth-degree Taylor polynomial of the function $f(x)$ centered at a. Let $R_n(x) = f(x) - T_n(x)$, and suppose that $|f^{(n+1)}(t)| \le M$ for all t between x and a. Then

$$|R_n(x)| \le \frac{M|x-a|^{n+1}}{(n+1)!}.$$