Chapter 3

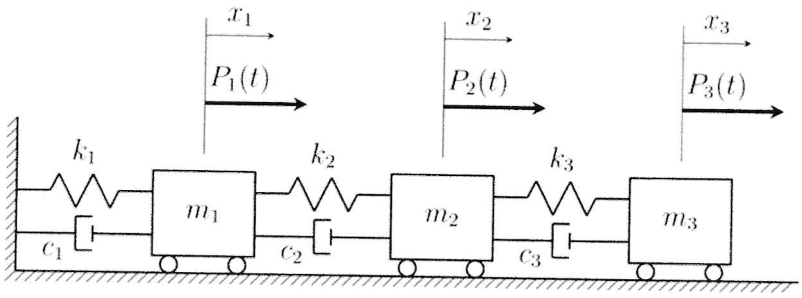
Multi-Degree-of-Freedom Systems

For an accurate description of the displacement configuration of a structure subjected to a dynamic loading, often displacements along more than one coordinate are necessary. Such a system is known as a *multi-degree-of-freedom system*.

We begin by discussing the formulation of the equations of motion for multi-degree-of-freedom (MDOF) systems.

3.1 Formulation of the Equation of Motion for an MDOF System

We begin with an example of a MDOF system with three degrees-of-freedom, as shown in Figure 3.1. Note that the displacements are time-varying, but are simply designated as x rather than $x(t)$ for succinctness.



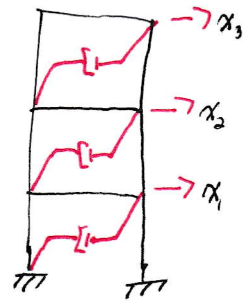
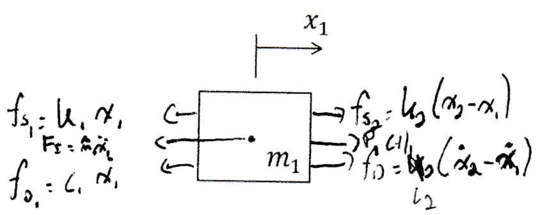


Figure 3.1 Three degree-of-freedom spring-mass-damper model

Example 3.1 Derive the equations of motion for the 3DOF system shown in Figure 3.1.

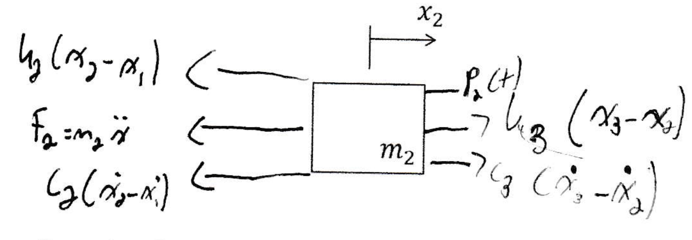
Solution: First, consider the free body diagram of m_1.



The equation of motion is

$$m_1\ddot{x}_1 + k_1 x_1 + c_1\dot{x}_1 + k_2(x_2-x_1) - c_2(\dot{x}_2-\dot{x}_1) = P_1(t) \tag{3.1}$$

Next, consider the free body diagram of m_2.

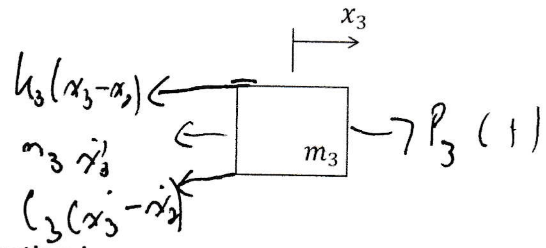


The equation of motion is

$$x_2\ddot{x}_2 + k_2(x_2-x_1) + c_2(\dot{x}_2-\dot{x}_1) - k_3(x_3-x_2) - c_3(\dot{x}_3-\dot{x}_2) = P_2 \tag{3.2}$$

$$m_2\ddot{x}_2 + [c_2+c_3]\dot{x}_2 + [k_2+k_3]x_2 - c_2\dot{x}_1 - k_2x_1 - k_3x_3 - c_3\dot{x}_3 = P_2$$

Finally, consider the free body diagram of m_3.



The equation of motion is

$$m_3\ddot{x}_3 + c_3\dot{x}_3 - c_3\dot{x}_2 + k_3x_3 - k_3x_2 = P_3 \tag{3.3}$$

Equations 3.1 through 3.3 can be written in matrix form as

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix}\ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3\end{bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3\end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}\begin{bmatrix}x_1 \\ x_2 \\ x_3\end{bmatrix} \tag{3.4}$$

$$= \begin{bmatrix} P_1(t) \\ P_2(t) \\ P_3(t) \end{bmatrix} \tag{3.5}$$

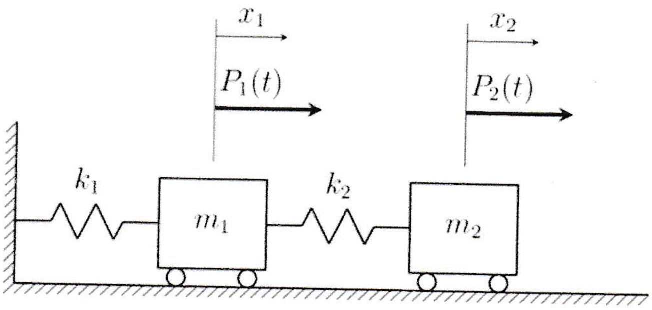
where $\boldsymbol{M}$ is the *mass matrix*, $\boldsymbol{C}$ is the *damping matrix*, and $\boldsymbol{K}$ is the *stiffness matrix*.

$$M\ddot{x} + C\dot{x} + Kx = P(t)$$

We can also formulate MDOF equations of motion using an analytical mechanics approach. *Lagrange's Equation* is a powerful method but will not be covered in this course.

3.2 Solution for a 2-DOF System

We will now consider the solution for a 2-DOF system. We begin by finding the equation of motion for a 2-DOF system with no damping.



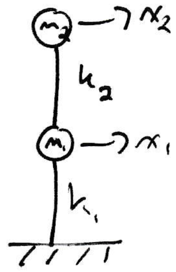
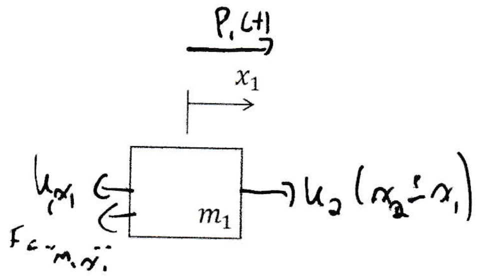


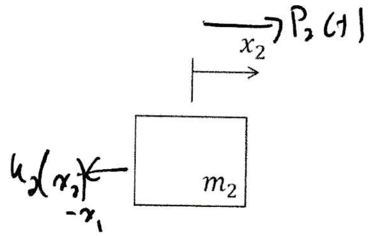
Figure 3.2 Two degree-of-freedom spring-mass model

Using Newtonian Mechanics, we can derive the equation of motion. For mass m_1, the free body diagram is:

using Blueplan



Similarly, for the mass m_2, the free body diagram is



The equations of motion are

$$P_1(t) = k_1x_1 - k_2(x_2-x_1) + m_1\ddot{x}_1 \quad , \quad m_1\ddot{x}_1 + (k_1+k_2)x_2 - k_2x_2 = P_1 \tag{3.6a}$$

$$P_2(t) = k_2x_2 + m_2\ddot{x}_2 \tag{3.6b}$$

which can be written in matrix form as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix} \tag{3.7}$$

$$M\ddot{x} + Kx = P$$

Several observations can be made from this result.

- Equation 3.7 is uncoupled in the mass terms, but coupled in the stiffness terms.
- Coupling is not an inherent property of the system; rather, it is a consequence of the coordinate system used.
- Both the mass and stiffness matrices are symmetric.

We will now turn to solving this system. Assuming the solution is of the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sin(\omega_n t + \alpha) \quad , \tag{3.8}$$

where X_1 and X_2 are the amplitudes of x_1 and x_2, respectively. As before, ω_n is the circular natural frequency, and α is the phase angle. Substituting Equation 3.8 into Equations 3.6 gives,

$$-m_1\omega_n^2 X_1 \sin(\omega_n t + \alpha) + (k_1 + k_2)X_1 \sin(\omega_n t + \alpha) - k_2 X_2 \sin(\omega_n t + \alpha) = P_1(t) \tag{3.9a}$$

$$-m_2\omega_n^2 X_2 \sin(\omega_n t + \alpha) - k_2 X_1 \sin(\omega_n t + \alpha) + k_2 X_2 \sin(\omega_n t + \alpha) = P_2(t) \tag{3.9b}$$

3.2.1 Free Vibration Response

For the free vibration response ($P_1(t) = P_2(t) = 0$), the solution becomes

$$[-m_1\omega_n^2 X_1 + (k_1 + k_2)X_1 - k_2 X_2] \sin(\omega_n t + \alpha) = 0 \tag{3.10a}$$

Homogeneous system →

$$[-m_2\omega_n^2 X_2 - k_2 X_1 + k_2 X_2] \sin(\omega_n t + \alpha) = 0 \tag{3.10b}$$

For a nontrivial solution, X_1 and $X_2 \neq 0$ @ same time

∴ Non invertible, determinate = 0. $\begin{bmatrix} (k_1+k_2)-m_1\omega_n^2 & -k_2 \\ -k_2 & k_2-m_2\omega_n^2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (3.11)

Therefore,

AD - BC = 0

$\begin{vmatrix} A & B \\ C & D \end{vmatrix}$ (3.12)

Expanding the determinant gives

$$\omega_n^4 - \left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right)\omega_n^2 + \frac{k_1 k_2}{m_1 m_2} = 0 \tag{3.13}$$

Equation 3.13 is known as the characteristic equation of the system. The roots are

$$\omega_n^2 = \frac{1}{2}\left[\left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right) \pm \sqrt{\left(\left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right)^2 - 4\frac{k_1 k_2}{m_1 m_2}\right)}\right] \tag{3.14}$$

The roots $\omega_{n,1}$ and $\omega_{n,2}$ are real and positive. The lower of $\omega_{n,1}$ and $\omega_{n,2}$ is known as the first or **fundamental** *circular natural frequency* of the system.

∴ has 2 undamped natural frequencies; can never exactly determine amplitudes

From Equation 3.14, there are two roots. Corresponding to each value of ω_n^2, we can determine a ratio of X_2/X_1 in Equation 3.11. However, it is not possible to determine X_1 and X_2 exactly, since Equation 3.11 is homogeneous. That is, if $\boldsymbol{X} = [X_1, X_2]$ is a solution, then $\alpha[X_1, X_2]$ is also a solution, where α is any scalar.

Example 3.2 For $k_1 = k$, $k_2 = 3k/4$, $m_1 = m$, and $m_2 = m/2$, determine the circular natural frequencies and the modes of vibration for the 2-DOF system.

Solution: From Equation 3.14, the circular natural frequencies are computed from

$$\omega_n^2 = \frac{1}{2}\left[\left(\frac{k_1+k_2}{m_1} + \frac{k_2}{m_2}\right) \pm \sqrt{\left(\left(\frac{k_1+k_2}{m_1} + \frac{k_2}{m_2}\right)^2 - 4\frac{k_1 k_2}{m_1 m_2}\right)}\right]$$

$$= \frac{1}{2}\frac{k}{m}\left[3.25 \pm \sqrt{(3.25)^2 - 6}\right] \tag{3.15}$$

$$= \frac{1}{2}\frac{k}{m}\left(3.25 \pm \sqrt{4.563}\right)$$

Therefore, the first and second circular natural frequencies are

$$\omega_{n,1} = 0.746\sqrt{k/m} \quad rad/s \tag{3.16}$$

$$\omega_{n,2} = 1.64\sqrt{k/m} \quad rad/s \tag{3.17}$$

Recall, $[(k_1+k_2)-m_1\omega_n^2]X_1 - k_2 X_2 = 0$

$-k_2 X_1 + (k_2 - m_2\omega_n^2)X_2 = 0$

(3.18a)

(3.18b)

Substituting $\omega_{n,1} = 0.746\sqrt{k/m}$ along with the other relevant quantities gives

$$\beta_1 = \frac{X_2}{X_1} = 1.591 \tag{3.19}$$

Similarly, substituting $\omega_{n,2} = 1.64\sqrt{k/m}$ into Equation 3.18

$$\beta_2 = \frac{X_2}{X_1} = -1.257 \tag{3.20}$$

We now have the mode shape vector for the first and second modes,

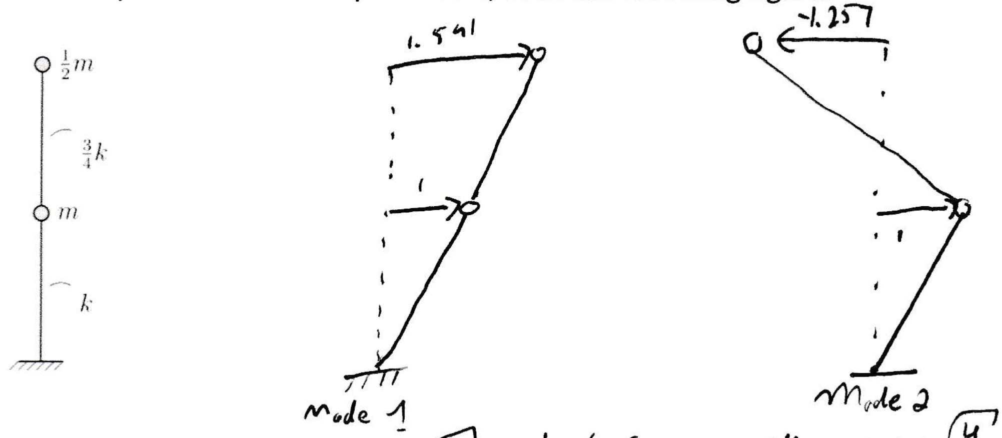
$\phi_1 = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.591 \end{bmatrix}$ (3.21a)

Corresponds to $\omega_{n,1}$

$\phi_2 = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}_2 = \begin{bmatrix} 1 \\ -1.257 \end{bmatrix}$ (3.21b)

Corresponds to $\omega_{n,2}$

We can plot the mode shape vectors, as in the following figure.



$\omega_{n,1} = 0.746\sqrt{\frac{k}{m}}$ rads/sec $\qquad \omega_{n,2} = 1.64\sqrt{\frac{k}{m}}$ rads/sec

We can now write the solution for the 2-DOF system as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A_1 \phi_1 \sin(\omega_{n,1} t + \alpha_1) + A_2 \phi_2 \sin(\omega_{n,2} t + \alpha_2) \tag{3.22}$$

where A_1, A_2, α_1, and α_2 are constants which can be determined from initial conditions. If $X_1 = 1$, then Equation 3.22 can be rewritten as

$$x_1 = A_1 \sin(\omega_{n,1} t + \alpha_1) + A_2 \sin(\omega_{n,2} t + \alpha_2) \tag{3.23a}$$

$$x_2 = A_2 \beta_1 \sin(\omega_{n,1} t + \alpha_1) + A_2 \beta_2 \sin(\omega_{n,2} t + \alpha_2) \tag{3.23b}$$

or in matrix form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A_1 \begin{bmatrix} 1 \\ \beta_1 \end{bmatrix} \sin(\omega_{n,1} t + \alpha_1) + A_2 \begin{bmatrix} 1 \\ \beta_2 \end{bmatrix} \sin(\omega_{n,2} t + \alpha_2) \tag{3.24}$$

We can write Equation 3.24 in a slightly different form.

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = a_1 \begin{Bmatrix} 1 \\ \beta_1 \end{Bmatrix} \cos \omega_{n,1} t + b_2 \begin{Bmatrix} 1 \\ \beta_2 \end{Bmatrix} \sin \omega_{n,2} t + b_1 \begin{Bmatrix} 1 \\ \beta_1 \end{Bmatrix} \sin \omega_{n,1} t + a_2 \begin{Bmatrix} 1 \\ \beta_2 \end{Bmatrix} \cos \omega_{n,2} t \tag{3.25}$$

where $A_1 \sin \alpha_1 = a_1$, $A_1 \cos \alpha_1 = b_1$, $A_2 \sin \alpha_2 = a_2$, and $A_2 \cos \alpha_2 = b_2$ are constants. Equation 3.25 can be rewritten as

$$\boldsymbol{x} = \sum_{i=1}^{2} (a_i \boldsymbol{\phi}_i \cos \omega_{n,i} t + b_i \boldsymbol{\phi}_i \sin \omega_{n,i} t) = \sum_{i=1}^{2} \phi_i A_i \sin(\omega_{n,i} t + \alpha_i) \tag{3.26}$$

where

$$\boldsymbol{\phi}_i = \begin{Bmatrix} 1 \\ \beta_i \end{Bmatrix} \tag{3.27}$$

and the constants a_i and b_i can be determined from initial conditions.

- The natural modes and frequencies are intrinsic properties of the system and depend only on the mass and stiffness.
- The modes of a system are the simplest possible cases of undamped free vibration response – one exactly described by simple harmonic motion.
- When an undamped system is vibrating in one of its natural modes, all DOFs are oscillating in harmonic motion with the same natural frequency.
- As we can see from the results in Example 3.2, the general response of an MDOF system is given by a combination of the natural modes. This is a very important observation that will become clear later.

Before proceeding further, let's investigate some important properties of the mass matrix, $\boldsymbol{M}$, and the stiffness matrix, $\boldsymbol{K}$.

Note :

• m and k are symmetric : ∴ $M^T = M$, $K^T = K$

• for system w/o rigid body modes. m & k are positive, definite matrices

$$V = \frac{1}{2}x^T\boldsymbol{K}x > 0 \qquad (3.28a)$$

← Kx^2 = potential energy. If system able to store potential energy.

$$T = \frac{1}{2}\dot{x}^T\boldsymbol{M}\dot{x} > 0 \qquad (3.28b)$$

← $M\dot{x}^2$ = kinetic energy. If system able to store kinetic energy

3.3 Modes of the System

3.3.1 Background on Eigenvalues and Eigenvectors

- Eigenvalues and eigenvectors play a central role in structural dynamics problems. We discuss the standard eigen problem of the form:

$$Ax = \lambda x \quad , A \text{ is vector, } \lambda \text{ is scalar} \qquad (3.29)$$

  where, $\boldsymbol{x}$ is a vector of length n and $\boldsymbol{A}$ is a matrix of size $n \times n$. λ is some scalar.
- Let's consider an example[1]. Let:

$$\boldsymbol{A} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \qquad \boldsymbol{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

  The products:

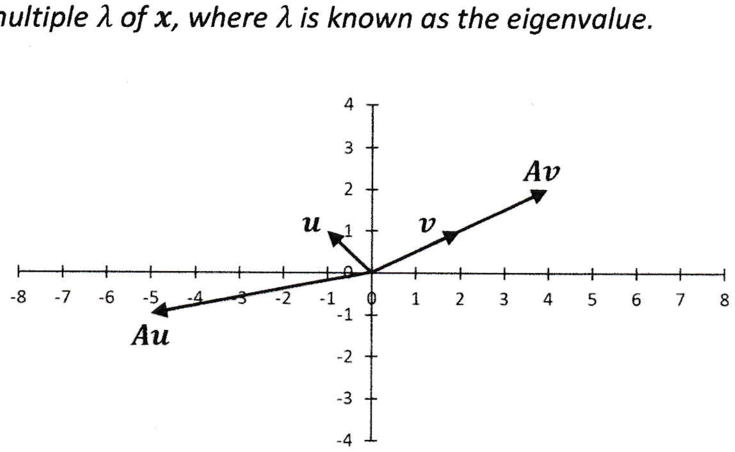
$$Au = \begin{bmatrix} -5 \\ -1 \end{bmatrix}, \quad Av = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

  . v is eigenvector, u is not, w/ $\lambda = 2$

- Clearly, the product $\boldsymbol{Av}$ results in a scalar multiple (of 2) of the original vector (see the figure), $\boldsymbol{v}$, while the product $\boldsymbol{Au}$ does not. In this case $\boldsymbol{v}$ is an *eigenvector* of $\boldsymbol{A}$, and the corresponding *eigenvalue* is 2.

[1]Lay, D. (1994) *Linear Algebra and its Applications*, Addison-Wesley, USA.

Def.: An eigenvector of a matrix $\boldsymbol{A}$ is a non-zero vector $\boldsymbol{x}$ such that the product $\boldsymbol{Ax}$ results in a scalar multiple λ of $\boldsymbol{x}$, where λ is known as the eigenvalue.



- Now, let's examine some important properties of eigenvectors. In doing so, the eigen problem for the transpose of $\boldsymbol{A}$ can be written as:

$$A^T y = \bar{\lambda} y \tag{3.30}$$

Eq. 3.30 corresponds to:

$$\left[A^T - \bar{\lambda} I\right] y = 0, \quad y = \left[\,\vdots\,\right] \tag{3.31}$$

The characteristic equation to be solved for a non-trivial solution to exist for Eq. 3.31 is:

$$\det\left(A^T - \bar{\lambda} I\right) = 0 \tag{3.32}$$

But, we know from the properties of determinants that $|\boldsymbol{A}| = |\boldsymbol{A}^T|$. Hence, Eq. 3.32 is the same as: $\det(A^T) = \det(A)$

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0 \tag{3.33}$$

and we can conclude that, $\bar{\lambda} = \lambda$.

Now, we can write Eq. 3.30 as:

$$\boldsymbol{A}^T \boldsymbol{y} = \lambda \boldsymbol{y} \tag{3.34}$$

Taking the transpose on both sides of Eq. 3.34, we get:

$$y^T A = \lambda y^T \tag{3.35}$$

Now, let's write Eq. 3.35 and Eq. 3.29 for the i^{th} and j^{th} eigenvectors as follows:

$$y_i^T A = \lambda_i y_i^T \tag{3.36}$$

$$A x_j = \lambda_j x_j \tag{3.37}$$

Post-multiplying Eq. 3.36 by $\boldsymbol{x}_j$ and pre-multiplying Eq. 3.37 by $\boldsymbol{y}_i^T$,

$$y_i^T A x_j = \lambda_i y_i^T x_j \quad (3.38) \qquad\qquad y_i^T A x_j = \lambda_j y_i^T x_j \quad (3.39)$$

Subtracting them, we get:

$$(\lambda_i - \lambda_j) y_i^T x_j = 0 \tag{3.40}$$

$\neq 0$

This leads to the conclusion that:

$$y_i^T x_j = 0, \quad \lambda_i \neq \lambda_j \tag{3.41}$$

since y and x are vectors, this is equivalent to $y \cdot x = 0$

where, $\boldsymbol{y}$ are called the *left* eigenvectors and $\boldsymbol{x}$ are the *right* eigenvectors.

- The results show that the left and right eigenvectors corresponding to two distinct eigenvalues are orthogonal to each other. If $\boldsymbol{A}$ is symmetric, we do not need to differentiate between the left and right eigenvectors; they are the same.
- It can also be shown that the eigenvectors corresponding to distinct eigenvalues of $\boldsymbol{A}$ are linearly independent. $x_1, x_2, x_3 \ldots \; x_1 \neq a_1 x_2 + a_2 x_3 \ldots$ since distinct

  Hence, any arbitrary vector $\boldsymbol{u}$ can be written as a linear combination of N eigenvectors of $\boldsymbol{A}$. That is:

$$u_{n\times 1} = c_1 x_1 + c_2 x_2 + \cdots + c_N x_N \tag{3.42}$$

  To calculate the coefficients, we simply pre-multiply the equation by $\boldsymbol{y}_i^T$ to get:

$$y_i^T u = c_i y_i^T x_i \tag{3.43}$$

  This results in:

$$c_i = \frac{y_i^T u}{y_i^T x_i} \tag{3.44}$$

  Of course, if $\boldsymbol{A}$ is symmetric, then $\boldsymbol{y}$ and $\boldsymbol{x}$ are the same.

- The solution to the eigen problem (i.e. determining the eigenvalues and eigenvectors) is usually accomplished numerically in practice.

3.3.2 Natural Frequencies and Modes

Now, let's revisit the idea of the natural frequencies and modes of an MDOF system. We will look at the similarities between the standard eigenproblem described earlier, and free vibration problem we encounter in structural dynamics. Consider an arbitrary N-DOF system without damping.

(3.45)

- We assume that the free vibration of the system in one of its natural modes takes the form:

$$x = \phi_i y_i(t)$$

$$\underbrace{M}_{N\times N}\underbrace{\ddot{x}}_{N\cdot 1} + \underbrace{k}_{N\times N}\underbrace{x}_{N\cdot 1} = 0 \qquad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{bmatrix} A\sin(\omega_n t + \alpha) = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{bmatrix} \left(B\cos\omega_n t + C\cos\omega_n t\right) \tag{3.46}$$

where $\boldsymbol{\phi}_n$ is a vector containing the amplitudes of $\boldsymbol{x}$, and $y_i(t)$ is nothing more than undamped free vibration response in mode i.

- Substituting Eq. 3.46 into Eq. 3.45 yields

$$\left[-\omega_n^2 M \phi_n + K\phi_n\right] y_i(t) = 0 \tag{3.47}$$

- For a nontrivial solution,

$$K\phi_n = \omega_n^2 M \phi_n \tag{3.48}$$

- If we pre-multiply both sides of Eq. 3.45 by $\boldsymbol{M}^{-1}$, we get:

$$M^{-1}K\phi_n = \omega_n^2 \phi_n \tag{3.49}$$

We can easily see that Eq. 3.46 is the same as the standard eigenproblem in Eq. 3.29 with $\boldsymbol{M}^{-1}\boldsymbol{K}$ being analogous to $\boldsymbol{A}$ and ω_n^2 having the same interpretation as λ.

- The matrix eigenvalue problem in Eq. 3.48 in which ω_n^2 are the eigenvalues and the amplitude vector $\boldsymbol{\phi}_n$ are the corresponding eigenvectors can also be posed as

$$(-\omega_n^2 M + K)\phi_n = 0 \tag{3.50}$$

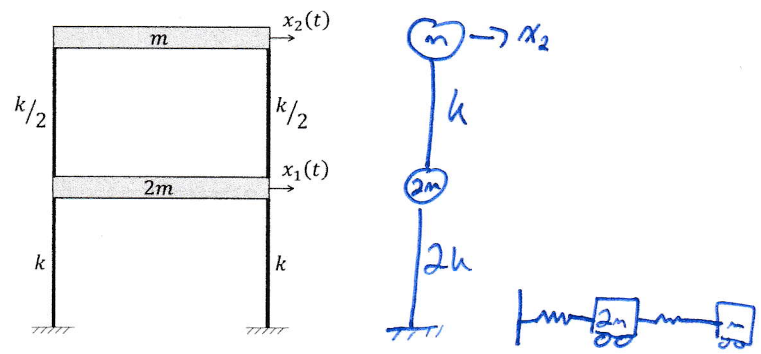
- Which has a nontrivial solution if

$$\det\left[-\omega_n^2 M + k\right] = 0 \tag{3.51}$$

- If $\boldsymbol{M}$ and $\boldsymbol{K}$ are real and positive definite, the eigenvalues ω_n^2 will be real and positive, and the eigenvectors will be real.
- For an N-DOF system, solving Eq. 3.51 will yield N real and positive roots for ω_n^2.
- For each ω_n, the corresponding eigenvector $\boldsymbol{\phi}_n$ can be obtained from Eq. 3.50. However, as we saw in Section 3.2, we cannot solve for exact values. It is only possible to know the relative values in $\boldsymbol{\phi}_n$.
- The eigenvectors $\boldsymbol{\phi}_n$ correspond to the response amplitude and are called the *mode shape vectors*. As we saw in Example 3.2, mode shape vectors give us a general sense of what the free vibration response looks like in the natural modes.

- The natural frequencies and associated mode shape vectors are numbered from the lowest frequency to highest from 1 to N. $\omega_{n,1}$ is called the *fundamental* circular natural frequency.

Example 3.3 Determine the natural periods and sketch the mode shapes for the two-storey frame shown below.



Solution: The equation of motion is

$$\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.52}$$

$$\boldsymbol{M\ddot{x}} + \boldsymbol{Kx} = \boldsymbol{0}$$

Assume
$x = \phi_n A \sin(\omega_n t + \alpha)$

To determine the natural frequencies and modes, we solve the eigenvalue problem:

$$(-\omega_n^2 \boldsymbol{M} + \boldsymbol{K})\boldsymbol{\phi}_n = \boldsymbol{0}$$

$$\begin{bmatrix} 3k - 2\omega_n^2 m & -k \\ -k & k - \omega_n^2 m \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}_n = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.53}$$

Which has nontrivial solutions if

$$\begin{vmatrix} 3k - 2\omega_n^2 m & -k \\ -k & k - \omega_n^2 m \end{vmatrix} = 0$$

$$(3k - 2\omega_n^2 m)(k - \omega_n^2 m) - k^2 = 0 \tag{3.54}$$

$$\omega_n^4 - \frac{5k}{2m}\omega_n^2 + \frac{k^2}{m^2} = 0$$

The 2-DOF system yields two roots,

$$\omega_{n,1}^2 = \frac{k}{2m}, \quad \omega_{n,1} = 0.707\sqrt{\frac{k}{m}} \text{ rads/sec} \tag{3.55a}$$

$$\omega_{n,1}^2 = \frac{2k}{m}, \quad \omega_{n,2} = 1.414\sqrt{\frac{k}{m}} \text{ rads/sec} \tag{3.55b}$$

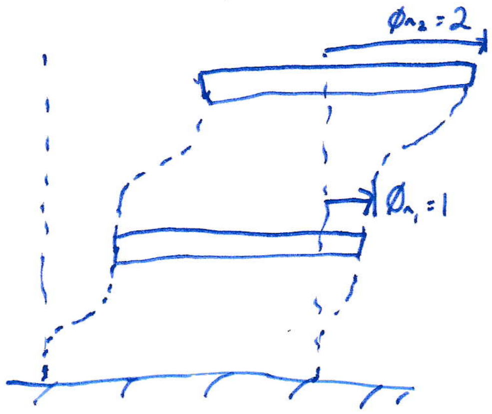
We can now solve for the mode shape vectors. For simplicity take ϕ_1 = 1.

$$\begin{bmatrix} 3k - 2\omega_n^2 m & -k \\ -k & k - \omega_n^2 m \end{bmatrix} \begin{bmatrix} 1 \\ \phi_2 \end{bmatrix}_n = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.56}$$

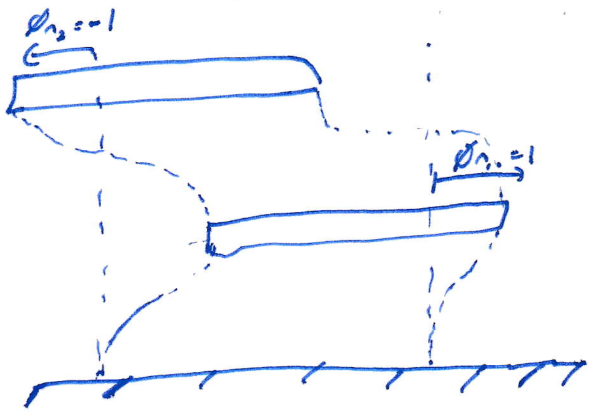
Subbing $\omega_{n,1}$ in, $3k - 2\omega_n^2 m - \phi_2 k = 0$, $\phi_2 = 2$, Mode shape $1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\omega_{n,2} \rightarrow 3k - 2\omega_{n,2}^2 m - \phi_2 k = 0$, $\phi_2 = -1$ $= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

MODE 1 ($\omega_{n,1} = 0.707\sqrt{k/m}$ rad/s)



MODE 2 ($\omega_{n,2} = 1.414\sqrt{k/m}$ rad/s)



3.3.3 Orthogonality Property

Recall the matrix eigenvalue problem

$$\boldsymbol{K}\boldsymbol{\phi}_n = \omega_n^2 \boldsymbol{M}\boldsymbol{\phi}_n \tag{3.57}$$

Equation 3.57 can be written for each frequency and mode as

Mode i

$$K\phi_{n,i} = \omega_{n,i}^2 M \phi_{n,i} \tag{3.58a}$$

Mode j

$$K\phi_{n,j} = \omega_{n,j}^2 M \phi_{n,j} \tag{3.58b}$$

Pre-multiplying Equation 3.58a by $\boldsymbol{\phi}_{n,j}^T$

$$\phi_{n,j}^T K \phi_{n,i} = \omega_{n,i}^2 \phi_{n,j}^T M \phi_{n,i} \tag{3.59}$$

Pre-multiplying Equation 3.58b by $\boldsymbol{\phi}_{n,i}^T$

$$\boldsymbol{\phi}_{n,i}^T \boldsymbol{K}\boldsymbol{\phi}_{n,j} = \omega_{n,j}^2 \boldsymbol{\phi}_{n,i}^T \boldsymbol{M}\boldsymbol{\phi}_{n,j} \tag{3.60}$$

Taking the transpose of both sides in Equation 3.59 gives

$$[LS]^T = [RS]^T, \quad [\phi_{n,i}^T K \phi_{n,j}]^T = \omega_{n,i}^2 [\phi_{n,i}^T M \phi_{n,j}]^T$$
$$\phi_{n,j}^T K \phi_{n,i} = \omega_{n,i}^2 \phi_{n,j}^T M \phi_{n,i} \tag{3.61}$$

Subtracting Equation 3.61 from Equation 3.60,

$$(\omega_{n,j}^2 - \omega_{n,i}^2)\, \phi_{n,i}^T M \phi_{n,j} = 0 \tag{3.62}$$

Since $\omega_{n,i}^2 \neq \omega_{n,j}^2 \neq 0$ (assuming $i \neq j$), it follows that

$$\phi_{n,i}^T M \phi_{n,j} = 0 \tag{3.63}$$

which is known as the mass orthogonality property of the modes. Substituting this result back into Equation 3.60, we find

$$\phi_{n,i}^T K \phi_{n,j} = 0 \quad \text{for } i \neq j \tag{3.64}$$

demonstrating the orthogonality of the stiffness matrix and the modes. Finally, pre-multiplying Equation 3.58a by $\boldsymbol{\phi}_{n,i}^T$ (or Equation 3.58b by $\boldsymbol{\phi}_{n,j}^T$, we get

$$\frac{\phi_{n,i}^T K \phi_{n,i}}{\phi_{n,i}^T M \phi_{n,i}} = \omega_{n,i}^2 \qquad \underline{K \phi_{n,i} = \omega_{n,i}^2 M \phi_{n,i}} \tag{3.65}$$

We now introduce the *modal matrix*, $\boldsymbol{\Phi}$.

$$\boldsymbol{\Phi} = \begin{array}{c} \begin{array}{cccc} \phi_{n,1} & \phi_{n,2} & & \phi_{n,N} \end{array} \\ \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N1} & \phi_{N2} & \cdots & \phi_{NN} \end{bmatrix} \end{array} = \begin{bmatrix} \phi_{n,1} & \phi_{n,2} & \cdots & \phi_{n,N} \end{bmatrix} \tag{3.66}$$

The orthogonality of natural modes implies the following:

$$\Phi^T M \Phi = \begin{bmatrix} \phi_{n,1}^T M \phi_{n,1} & 0 \\ 0 & \phi_{n,2}^T M \phi_{n,2} \end{bmatrix} \tag{3.67a}$$

$$\Phi^T K \Phi = \begin{bmatrix} \phi_{n,1}^T K \phi_{n,1} & 0 \\ 0 & \ddots \end{bmatrix} \tag{3.67b}$$

It follows that the following matrices are diagonal.

$$\phi_1^T M \phi_2 = 0 \tag{3.68a}$$

SWITCH

$$\phi_1^T K \phi_2 = 0 \tag{3.68b}$$

Let's return to the solution of a undamped 2-DOF system under free vibration, given in Equation 3.26.

$x_0 = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix}$

$$\boldsymbol{x} = \sum_{i=1}^{2} \boldsymbol{\phi}_i \left(a_i \cos \omega_{n,i} t + b_i \sin \omega_{n,i} t\right) \tag{3.67}$$

If $\boldsymbol{x}(0) = \boldsymbol{x}_0$ and $\dot{\boldsymbol{x}}(0) = \boldsymbol{0}$ are the displacement and velocity initial conditions, respectively, we can write the initial displacement of the system as

$\dot{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$x(0) = x_0 = a_1 \phi_1 + a_2 \phi_2 \tag{3.68}$$

Let us specify a particular configuration, or "shape", for the displacement initial condition. Furthermore, let us specify the initial displacement condition in the form of the first mode of vibration, $\boldsymbol{\phi}_1$. Therefore,

$\begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix} = \begin{bmatrix} \phi_1' \\ \phi_2' \end{bmatrix}$

$$\boldsymbol{x}_0 = \boldsymbol{\phi}_1 = a_1 \boldsymbol{\phi}_1 + a_2 \boldsymbol{\phi}_2 \tag{3.69}$$

Pre-multiplying each term in Equation 3.69 by $\boldsymbol{\phi}_1^T \boldsymbol{M}$, we get

$$\phi_1^T M \phi_1 = a_1 \phi_1^T M \phi_1 + a_2 \cancel{\phi_1^T M \phi_2}^{\,0} \tag{3.70}$$

(if initial excitement only MODE 2), will only be MODE 2

$= a_1 = 1$

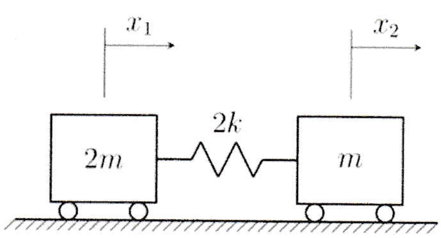
This implies $a_1 = 1$, which is the only non-zero coefficient. This means the structure continues to vibrate in the first mode.

- The coefficients a_1 and a_2 represent the coefficients in an expansion of the initial deflection configuration in terms of the natural modes.
- In general, all modes of the system are excited. However, the extent to which each mode participates in the total response depends on the initial conditions.

3.3.4 Rigid Body Modes

Let us investigate what happens when a system is not properly constrained by considering the following example.

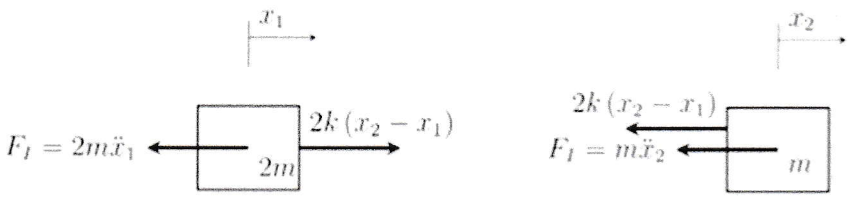
Example 3.4 Find the fundamental circular natural frequency and the corresponding natural mode for the system shown in Figure 3.3.



(Not stable

Figure 3.3 System exhibiting a rigid body natural mode

<u>Solution</u>: Using Newtonian Mechanics, we can derive the equations of motion. From the free body diagrams



$-m\ddot{x}_2 - 2k(x_2 - x_1)$

The equations of motion are

$-2m\ddot{x}_1 + 2k(x_2 - x_1)$

$$\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -2k \\ -2k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3.71a) \; (3.71b)$$

Assuming the solution is of the form

lower uppercase

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sin(\omega_n t + \alpha) \quad (3.72)$$

For the non-trivial solution,

$$\begin{bmatrix} (2k - 2m\omega_n^2) & -2k \\ -2k & (k - m\omega_n^2) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (3.73)$$

We can find the roots by setting the following determinant to zero:

$$\begin{vmatrix} 2k - 2m\omega_n^2 & -2k \\ -2k & k - m\omega_n^2 \end{vmatrix} = 0 \quad (3.74)$$

Expanding the determinant to find the characteristic equation and solving for ω_n^2, we find the following roots:

$$\omega_{n,1}^2 = 0 \quad (3.75a) \qquad \omega_{n,2}^2 = \frac{3k}{m} \quad (3.75b)$$

zero stiffness

The fact that $\omega_{n,1}^2 = 0$ indicates the presence of a rigid body mode. Substituting this result into Equation 3.73 and selecting $X_2 = 1$, we find the corresponding natural mode.

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad (3.76)$$

which is also indicative of a rigid body mode. For the second natural mode,

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_2 = \begin{Bmatrix} 1 \\ -2 \end{Bmatrix} \quad (3.77)$$

3.3.5 Mode Normalization / Scaling

We have already seen that the mode shape vectors (the eigenvectors) can only be determined to a scaling factor. That is, for a 2-DOF system, only the relative values can be obtained for $\{X_1\ \ X_2\}^T$. We can set $X_1 = 1$ and find $\beta_1 = (X_2/X_1)_1$ to obtain $\boldsymbol{\phi}_1 = \{1\ \ \beta_1\}^T$. Similarly, $\boldsymbol{\phi}_2 = \{1\ \ \beta_2\}^T$, where $\beta_2 = (X_2/X_1)_2$. In other words, the elements of a modal matrix are only known in a relative sense; their absolute magnitudes are indeterminate. We will now investigate other ways to scale the natural modes.

3.3.6 Mass Normalization

As discussed above, we can only ever determine relative values for the mode shape vectors. Any vector proportional to $\boldsymbol{\phi}_i$ yields the same mode shape and satisfies the eigenvalue problem. As a result, it is typical to normalize the mode shape vectors in some way. It may be done such that the largest element in $\boldsymbol{\phi}_i$ is unity or such that the element corresponding to a certain DOF is unity. It is also common to normalize modes such that

$$M_{ii} = \phi_i^T M \phi_i = 1 \tag{3.78}$$

This is known as *mass normalization*. Note that due to mode orthogonality, $\boldsymbol{\phi}_i^T \boldsymbol{M} \boldsymbol{\phi}_j = 0$ for $i \neq j$. Equation 3.78 can also be written as

$$\Phi^T M \Phi = I \leftarrow \text{identity matrix} \tag{3.79}$$

where $\boldsymbol{\Phi} = [\boldsymbol{\phi}_1\ \ \boldsymbol{\phi}_2\ \ \cdots\ \ \boldsymbol{\phi}_N]$. Hence, from Equation 3.65,

$$\frac{\phi_{n,i}^T K \phi_{n,i}}{\phi_{n,i}^T M \phi_{n,i}}$$

$$\Phi^T K \Phi = \begin{bmatrix} \omega_{n,1}^2 & 0 & 0 \\ 0 & \omega_{n,2} & 0 \\ 0 & 0 & \ddots\ \omega_{n,N}^2 \end{bmatrix} = \Lambda \leftarrow \text{big lambda} \tag{3.80}$$

Let us consider an illustrative example.

Example 3.5 Determine the mass normalized modes, $\boldsymbol{\Phi} = [\boldsymbol{\phi}_1\ \ \boldsymbol{\phi}_2]$ for $\boldsymbol{M} = \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix}$, assuming we have already calculated the natural modes $\boldsymbol{\Phi} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$.
$\Phi \leftarrow$ phi

Solution: Begin by letting $\boldsymbol{\phi}_1 = \alpha \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$ and $\boldsymbol{\phi}_2 = \beta \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$, where α and β are to be determined. We would like to normalize the modes such that $\boldsymbol{\phi}_1^T \boldsymbol{M} \boldsymbol{\phi}_1 = 1$.

$$[\alpha\ \ 2\alpha] \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \alpha \\ 2\alpha \end{bmatrix} = 1 \tag{3.81}$$

$$2\alpha^2 m + 4\alpha^2 m = 1, \quad \alpha = \frac{1}{\sqrt{6m}}, \quad \phi_1 = \frac{1}{\sqrt{6m}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Solving for α, we find that

$$\alpha = \frac{1}{\sqrt{6m}} \tag{3.82}$$

Similarly, for $\boldsymbol{\phi}_2^T \boldsymbol{M} \boldsymbol{\phi}_2 = 1$, we have

$$\begin{bmatrix} \beta & -\beta \end{bmatrix} \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \beta \\ -\beta \end{bmatrix} = 1 \quad , \tag{3.83}$$

Solving for β, we find that

$$\beta = \frac{1}{\sqrt{3m}} \rightarrow \phi_2 = \frac{1}{\sqrt{3m}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{3.84}$$

Therefore, the mass normalized modal matrix is

$$\boldsymbol{\Phi} = \begin{bmatrix} \frac{1}{\sqrt{6m}} & \frac{1}{\sqrt{3m}} \\ \frac{2}{\sqrt{6m}} & -\frac{1}{\sqrt{3m}} \end{bmatrix} \quad \text{Verify } \Phi^T M \Phi = I \tag{3.85}$$

$$\Phi^T K \Phi = \Lambda$$

3.3.7 Mode Superposition

Recall the solution for the undamped MDOF system under free vibration.

(3.86)

Let

$$\boldsymbol{\Phi} = [\boldsymbol{\phi}_1 \;\; \boldsymbol{\phi}_2 \;\; \cdots \;\; \boldsymbol{\phi}_N] = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N1} & \phi_{N2} & \cdots & \phi_{NN} \end{bmatrix} \tag{3.87}$$

be the modal matrix. Expanding Equation 3.86, we have

$$\begin{aligned} x_1 = a_1\phi_{11}\cos\omega_{n,1}t + b_1\phi_{11}\sin\omega_{n,1}t + a_2\phi_{12}\cos\omega_{n,2}t + b_2\phi_{12}\sin\omega_{n,2}t + \cdots \\ + a_N\phi_{1N}\cos\omega_{n,N}t + b_N\phi_{1N}\sin\omega_{n,N}t \end{aligned} \tag{3.88a}$$

$$\begin{aligned} x_2 = a_1\phi_{21}\cos\omega_{n,1}t + b_1\phi_{21}\sin\omega_{n,1}t + a_2\phi_{22}\cos\omega_{n,2}t + b_2\phi_{22}\sin\omega_{n,2}t + \cdots \\ + a_N\phi_{2N}\cos\omega_{n,N}t + b_N\phi_{2N}\sin\omega_{n,N}t \end{aligned} \tag{3.88b}$$

$$\vdots$$

$$x_N = a_1\phi_{N1}\cos\omega_{n,1}t + b_1\phi_{N1}\sin\omega_{n,1}t + a_2\phi_{N2}\cos\omega_{n,2}t + b_2\phi_{N2}\sin\omega_{n,2}t + \cdots + a_N\phi_{NN}\cos\omega_{n,N}t + b_N\phi_{NN}\sin\omega_{n,N}t \quad (3.88c)$$

In matrix form, Equations 3.88 can be written as

$$\boldsymbol{x} = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N1} & \phi_{N2} & \cdots & \phi_{NN} \end{bmatrix} \begin{Bmatrix} a_1\cos\omega_{n,1}t + b_1\sin\omega_{n,1}t \\ a_2\cos\omega_{n,2}t + b_2\sin\omega_{n,2}t \\ \vdots \\ a_N\cos\omega_{n,N}t + b_N\sin\omega_{n,N}t \end{Bmatrix} = \boldsymbol{\Phi y} \quad (3.89)$$

The modal matrix, $\boldsymbol{\Phi}$, we have calculated can be very powerful to solve the equations of motion for MDOF system. Begin with the equations of motion for an undamped MDOF system under free vibration.

$$\boldsymbol{M\ddot{x}} + \boldsymbol{Kx} = \boldsymbol{0} \quad (3.90)$$

Substituting the coordinate transformation found in Equation 3.89 into Equation 3.90

(3.91)

we get

(3.92)

Pre-multiplying throughout by $\boldsymbol{\Phi}^T$,

(3.93)

Using Equations 3.79 and 3.80, Equation 3.93 can be written as

(3.94)

Since $\mathbf{I}$ and $\boldsymbol{\Lambda}$ are diagonal, we can write Equation 3.94 as N uncoupled SDOF equations using the *modal coordinate* y.

$$\ddot{y}_1 + \omega_{n,1}^2 y_1 = 0 \quad (3.95a)$$

$$\ddot{y}_2 + \omega_{n,2}^2 y_2 = 0 \quad (3.95b)$$

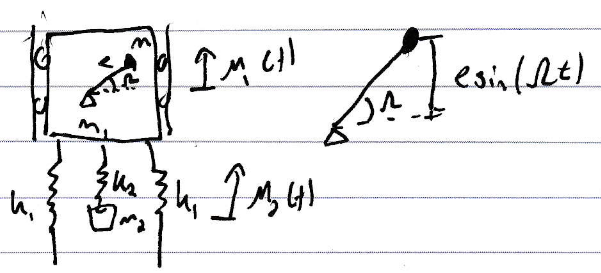
$$\vdots$$

$$\ddot{y}_N + \omega_{n,N}^2 y_N = 0 \quad (3.95c)$$

Recall from our study of undamped SDOF systems under free vibration, the solutions are

(3.96a)

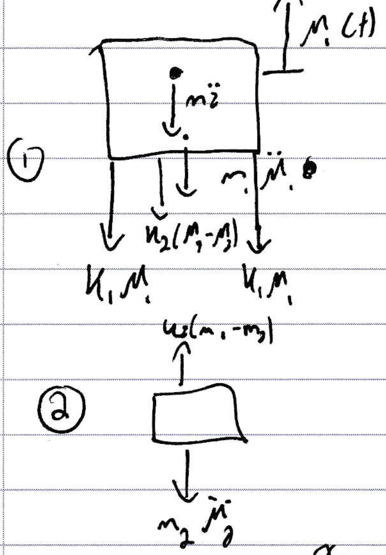
1

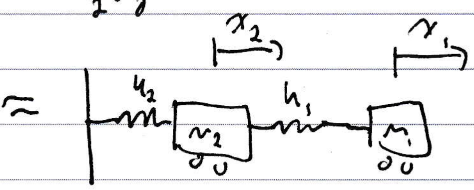


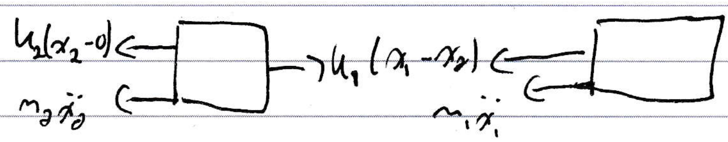
for rotating mass m_1 $\quad z(t) = y_1(t) + e\sin\Omega t$

$$\ddot{z}(t) = \ddot{y}_1(t) - e\Omega^2\sin(\Omega t)$$

FBD







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① $\sum F_y = 0$, $-m_1\ddot{x}_1 - 2k_1 x_1 - k_2(x_1 - x_2) - m\ddot{z} = 0$

$= -m_1\ddot{x}_1 - 2k_1 x_1 - k_2 x_1 + k_2 x_2 - m(\ddot{x}_1(t) - e\Omega^2 \sin\Omega t)$

$= (m_1+m)\ddot{x}_1 + 2(k_1+k_2)x_1 - k_2 x_2 = me\Omega^2 \sin\Omega t$

② $\sum F_y = 0$

$m_2\ddot{x}_2 - k_2(x_1 - x_2) = 0$, $m_2\ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0$

for ①,

$$\begin{bmatrix} m_1+m & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} me\Omega^2 \sin\Omega t \\ 0 \end{bmatrix}$$

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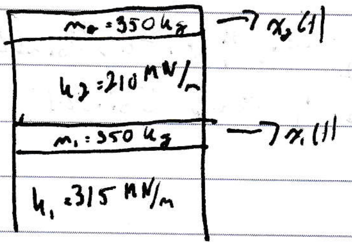
Question 2 - 2 storey shear building



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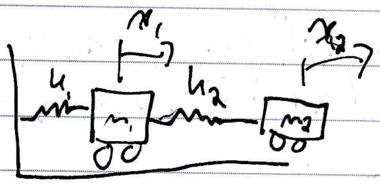
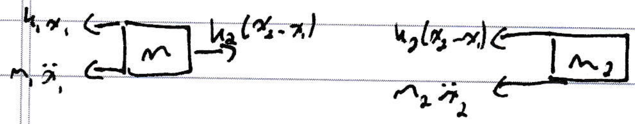


a) Equation of motion



, $\sum F_x = 0$

$m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0$, $m_2\ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0$

$m_1\ddot{x}_1 + k_1 x_1 - k_2(x_2 - x_1) = 0$

$m_1\ddot{x}_1 + (k_2 + k_1)x_1 - k_2 x_2 = 0$

$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{M} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}}_{K} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

b) Natural frequencies and mode shapes

$\omega_n^2 = k/m$ (SDOF)

$(-\omega_n^2 M + K)\,\phi_n = 0$

$\uparrow = 0$, i.e. determinate

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$$\det|K - \omega_n^2 M| = 0$$

$$10^3 \begin{bmatrix} 350 & 0 \\ 0 & 350 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + 10^6 \begin{bmatrix} 525 & -210 \\ -210 & 210 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} 525000 - 350\omega_n^2 & -210000 \\ -210000 & 210000 - 350\omega_n^2 \end{vmatrix}$$

$$\therefore \omega_n^2 = 300, \; 1800$$

@ 300, $\therefore \omega_{n,1} = 17.321$ rads/sec, $T_{n,1} = 0.363$ (Fundamental Period)

@ 1800, $\omega_n = 42.426$ rads/sec, $T_{n,2} = 0.148$

For Mode 1, plug in 300

$$\begin{vmatrix} 525000 - 350(300) & -210000 \\ -210000 & 210000 - 350(300) \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0, \text{ letting } x_1 = 1$$

$$\phi_{n,1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

For Mode 2, letting $\omega_n^2 = 1800$, $\phi_{n,2} = \begin{bmatrix} 1 \\ -.5 \end{bmatrix}$

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c) Mass normalize the shape modes

$$(\alpha \phi_{n,:}^T) M (\alpha \phi_{n,:}) = 1$$

$$\alpha^2 \phi_{n,:}^T M \phi_{n,:} = 1$$

Mode 1, $\alpha_1^2 \phi_{n,1}^T M \phi_{n,1} = 1$

$$\alpha_1^2 [1 \ 2]\, 10^3 \begin{bmatrix} 350 & 0 \\ 0 & 350 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1$$

$$1750\, \alpha_1^2 = 1, \quad \alpha_1 = 0.0239$$

Mode 2, $\alpha_2^2 [1 \ -0.5]\, 10^3 \begin{bmatrix} 350 & 0 \\ 0 & 350 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = 1$

$$437.5\, \alpha_2^2 = 1, \quad \alpha_2 = 0.0478$$

$$\therefore \phi_n = \begin{array}{cc} \phi_{n,1} & \phi_{n,2} \\ \end{array}$$
$$\phi_n = \begin{bmatrix} \alpha_1 (1) & \alpha_2 (1) \\ \alpha_1 (2) & \alpha_2 (-0.5) \end{bmatrix}$$

$$= \begin{bmatrix} 0.0239 & 0.0478 \\ 0.0478 & -0.0239 \end{bmatrix}$$