CHAPTER 14

EXERCISES 14.1

1. Open, connected, simply-connected domain
2. Closed, connected
3. Open, connected, domain, but not a simply-connected domain
4. Connected
5. Open, connected, simply-connected domain
6. Closed, connected
7. Open, connected, simply-connected domain
8. Open
9. Open, connected, domain, but not a simply-connected domain
10. For interior, exterior, and boundary points replace circle with sphere in planar definitions. Open, closed, connected, and domain definitions are identical. A domain is simply-connected if every closed curve in the domain is the boundary of a surface that contains only points of the domain.
11. Open, connected, simply-connected domain
12. Closed, connected
13. Open, connected, simply-connected domain
14. Connected
15. Open, connected, domain, but not a simply connected domain
16. Open
17. Open
18. Open, connected, simply-connected domain
19. Open, connected, domain, but not a simply-connected domain
20. To be open a set must not contain any of its boundary points. To be closed it must contain all of its boundary points. The only way to satisfy both conditions is for the set to have no boundary points. The whole plane is the only nonempty set that has no boundary points.
21. $\nabla f = 6xy\hat{\mathbf{i}} + (3x^2 - 3y^2z^2)\hat{\mathbf{j}} - 2y^3z\hat{\mathbf{k}}$
22. $\nabla f = -\dfrac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}) = -(x^2+y^2+z^2)^{-3/2}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$
23. $\nabla f = \dfrac{1}{1+y^2/x^2}\left(-\dfrac{y}{x^2}\hat{\mathbf{i}} + \dfrac{1}{x}\hat{\mathbf{j}}\right) = \dfrac{-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}}{x^2+y^2}$
24. $\nabla f_{|(1,2)} = (3x^2y - 2\cos y)\hat{\mathbf{i}} + (x^3 + 2x\sin y)\hat{\mathbf{j}}_{|(1,2)} = (6 - 2\cos 2)\hat{\mathbf{i}} + (1 + 2\sin 2)\hat{\mathbf{j}}$
25. $\nabla f_{|(1,-1,4)} = e^{xyz}(yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}})_{|(1,-1,4)} = e^{-4}(-4\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - \hat{\mathbf{k}})$
26. $\nabla \cdot \mathbf{F} = 2e^y + 0 - 2x^2y = 2(e^y - x^2y)$
27. $\nabla \cdot \mathbf{F} = \ln y - 3y^2e^x$
28. $\nabla \cdot \mathbf{F} = 2x\cos(x^2+y^2+z^2) - \sin(y+z)$
29. $\nabla \cdot \mathbf{F} = e^x + e^y$
30. $\nabla \cdot \mathbf{F}_{(1,1,1)} = (2xy^3 - 3x + 2z)_{(1,1,1)} = 2 - 3 + 2 = 1.$
31. $\nabla \cdot \mathbf{F}_{|(-1,3)} = [2(x+y) + 2(x+y)]_{|(-1,3)} = 8$

32. $$\nabla\cdot\mathbf{F}=\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)+\frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)+\frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$
$$=\left[\frac{1}{\sqrt{x^2+y^2+z^2}}-\frac{x^2}{(x^2+y^2+z^2)^{3/2}}\right]+\left[\frac{1}{\sqrt{x^2+y^2+z^2}}-\frac{y^2}{(x^2+y^2+z^2)^{3/2}}\right]$$
$$+\left[\frac{1}{\sqrt{x^2+y^2+z^2}}-\frac{z^2}{(x^2+y^2+z^2)^{3/2}}\right]$$
$$=\frac{3(x^2+y^2+z^2)-(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}}=\frac{2}{\sqrt{x^2+y^2+z^2}}$$

33. $\nabla\cdot\mathbf{F}=\dfrac{-y}{1+x^2y^2}+\dfrac{x}{1+x^2y^2}=\dfrac{x-y}{1+x^2y^2}$

34. $$\nabla\times\mathbf{F}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ x^2z & 12xyz & 32y^2z^4\end{vmatrix}=(64yz^4-12xy)\hat{\mathbf{i}}+x^2\hat{\mathbf{j}}+12yz\hat{\mathbf{k}}$$

35. $$\nabla\times\mathbf{F}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ xe^y & -2xy^2 & 0\end{vmatrix}=(-2y^2-xe^y)\hat{\mathbf{k}}$$

36. $$\nabla\times\mathbf{F}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ x^2 & y^2 & z^2\end{vmatrix}=\mathbf{0}$$

37. $$\nabla\times\mathbf{F}_{|(1,-1,1)}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ xz^3 & -2x^2yz & 2yz^4\end{vmatrix}_{|(1,-1,1)}=[(2z^4+2x^2y)\hat{\mathbf{i}}+3xz^2\hat{\mathbf{j}}-4xyz\hat{\mathbf{k}}]_{|(1,-1,1)}=3\hat{\mathbf{j}}+4\hat{\mathbf{k}}$$

38. $$\nabla\times\mathbf{F}_{|(2,0)}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ y & -x & 0\end{vmatrix}_{|(2,0)}=(-2)\hat{\mathbf{k}}$$

39. $$\nabla\times\mathbf{F}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ \ln(x+y+z) & \ln(x+y+z) & \ln(x+y+z)\end{vmatrix}=\mathbf{0}$$

40. $$\nabla\times\mathbf{F}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ \mathrm{Sec}^{-1}(x+y) & \mathrm{Csc}^{-1}(y+x) & 0\end{vmatrix}$$
$$=\left(\frac{-1}{(y+x)\sqrt{(y+x)^2-1}}-\frac{1}{(x+y)\sqrt{(x+y)^2-1}}\right)\hat{\mathbf{k}}=\frac{-2}{(x+y)\sqrt{(x+y)^2-1}}\hat{\mathbf{k}}$$

41. Since $\nabla\times\mathbf{F}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ x^2y & -2xz & 2yz\end{vmatrix}=(2z+2x)\hat{\mathbf{i}}+(-2z-x^2)\hat{\mathbf{k}}$,

$$\nabla\times\nabla\times\mathbf{F}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 2z+2x & 0 & -2z-x^2\end{vmatrix}=(2+2x)\hat{\mathbf{j}}.$$

42. (a) For $f(x,y,z)$, the equation $\nabla\cdot\nabla f=0$ can be written in the form

$$0=\nabla\cdot\left(\frac{\partial f}{\partial x}\hat{\mathbf{i}}+\frac{\partial f}{\partial y}\hat{\mathbf{j}}+\frac{\partial f}{\partial z}\hat{\mathbf{k}}\right)=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2},$$

which is equation 12.12.

(b) Since $\dfrac{\partial f}{\partial x} = \dfrac{-x}{(x^2+y^2+z^2)^{3/2}}$, it follows that

$$\frac{\partial^2 f}{\partial x^2} = \frac{-1}{(x^2+y^2+z^2)^{3/2}} + \frac{3x^2}{(x^2+y^2+z^2)^{5/2}} = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}.$$

Similarly, $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}$ and $\dfrac{\partial^2 f}{\partial z^2} = \dfrac{2z^2-y^2-x^2}{(x^2+y^2+z^2)^{5/2}}$. When these second partial derivatives are added together, the result is zero.

43. 14.7:
$$\begin{aligned}\nabla(f+g) &= (f_x+g_x)\hat{\mathbf{i}} + (f_y+g_y)\hat{\mathbf{j}} + (f_z+g_z)\hat{\mathbf{k}}\\ &= (f_x\hat{\mathbf{i}} + f_y\hat{\mathbf{j}} + f_z\hat{\mathbf{k}}) + (g_x\hat{\mathbf{i}} + g_y\hat{\mathbf{j}} + g_z\hat{\mathbf{k}}) = \nabla f + \nabla g\end{aligned}$$

14.8:
$$\begin{aligned}\nabla\cdot(\mathbf{F}+\mathbf{G}) &= \frac{\partial}{\partial x}(F_x+G_x) + \frac{\partial}{\partial y}(F_y+G_y) + \frac{\partial}{\partial z}(F_z+G_z)\\ &= \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) + \left(\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z}\right) = \nabla\cdot\mathbf{F} + \nabla\cdot\mathbf{G}\end{aligned}$$

14.9:
$$\begin{aligned}\nabla\times(\mathbf{F}+\mathbf{G}) &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ F_x+G_x & F_y+G_y & F_z+G_z\end{vmatrix}\\ &= \left[\frac{\partial}{\partial y}(F_z+G_z) - \frac{\partial}{\partial z}(F_y+G_y)\right]\hat{\mathbf{i}} + \left[\frac{\partial}{\partial z}(F_x+G_x) - \frac{\partial}{\partial x}(F_z+G_z)\right]\hat{\mathbf{j}}\\ &\quad + \left[\frac{\partial}{\partial x}(F_y+G_y) - \frac{\partial}{\partial y}(F_x+G_x)\right]\hat{\mathbf{k}}\\ &= \left[\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\mathbf{k}}\right]\\ &\quad + \left[\left(\frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial G_x}{\partial z} - \frac{\partial G_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y}\right)\hat{\mathbf{k}}\right]\\ &= \nabla\times\mathbf{F} + \nabla\times\mathbf{G}\end{aligned}$$

14.10:
$$\begin{aligned}\nabla(fg) &= \frac{\partial}{\partial x}(fg)\hat{\mathbf{i}} + \frac{\partial}{\partial y}(fg)\hat{\mathbf{j}} + \frac{\partial}{\partial z}(fg)\hat{\mathbf{k}}\\ &= \left(g\frac{\partial f}{\partial x} + f\frac{\partial g}{\partial x}\right)\hat{\mathbf{i}} + \left(g\frac{\partial f}{\partial y} + f\frac{\partial g}{\partial y}\right)\hat{\mathbf{j}} + \left(g\frac{\partial f}{\partial z} + f\frac{\partial g}{\partial z}\right)\hat{\mathbf{k}}\\ &= g\left(\frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}\right) + f\left(\frac{\partial g}{\partial x}\hat{\mathbf{i}} + \frac{\partial g}{\partial y}\hat{\mathbf{j}} + \frac{\partial g}{\partial z}\hat{\mathbf{k}}\right) = g\nabla f + f\nabla g\end{aligned}$$

14.11:
$$\begin{aligned}\nabla\cdot(f\mathbf{F}) &= \frac{\partial}{\partial x}(fF_x) + \frac{\partial}{\partial y}(fF_y) + \frac{\partial}{\partial z}(fF_z)\\ &= \left(\frac{\partial f}{\partial x}F_x + f\frac{\partial F_x}{\partial x}\right) + \left(\frac{\partial f}{\partial y}F_y + f\frac{\partial F_y}{\partial y}\right) + \left(\frac{\partial f}{\partial z}F_z + f\frac{\partial F_z}{\partial z}\right)\\ &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\cdot(F_x, F_y, F_z) + f\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) = \nabla f\cdot\mathbf{F} + f\,\nabla\cdot\mathbf{F}\end{aligned}$$

14.12:
$$\begin{aligned}\nabla\times(f\mathbf{F}) &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ fF_x & fF_y & fF_z\end{vmatrix}\\ &= \left(\frac{\partial f}{\partial y}F_z + f\frac{\partial F_z}{\partial y} - \frac{\partial f}{\partial z}F_y - f\frac{\partial F_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial f}{\partial z}F_x + f\frac{\partial F_x}{\partial z} - \frac{\partial f}{\partial x}F_z - f\frac{\partial F_z}{\partial x}\right)\hat{\mathbf{j}}\\ &\quad + \left(\frac{\partial f}{\partial x}F_y + f\frac{\partial F_y}{\partial x} - \frac{\partial f}{\partial y}F_x - f\frac{\partial F_x}{\partial y}\right)\hat{\mathbf{k}}\\ &= \left[\left(\frac{\partial f}{\partial y}F_z - \frac{\partial f}{\partial z}F_y\right)\hat{\mathbf{i}} + \left(\frac{\partial f}{\partial z}F_x - \frac{\partial f}{\partial x}F_z\right)\hat{\mathbf{j}} + \left(\frac{\partial f}{\partial x}F_y - \frac{\partial f}{\partial y}F_x\right)\hat{\mathbf{k}}\right]\end{aligned}$$

$$+f\left[\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\hat{\mathbf{i}}+\left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\hat{\mathbf{j}}+\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\hat{\mathbf{k}}\right]$$

$$=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z\\ F_x & F_y & F_z\end{vmatrix}+f\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ F_x & F_y & F_z\end{vmatrix}$$

$$=\nabla f\times\mathbf{F}+f(\nabla\times\mathbf{F})$$

14.14: $$\nabla\times(\nabla f)=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z\end{vmatrix}$$

$$=\left(\frac{\partial^2 f}{\partial y\partial z}-\frac{\partial^2 f}{\partial z\partial y}\right)\hat{\mathbf{i}}+\left(\frac{\partial^2 f}{\partial z\partial x}-\frac{\partial^2 f}{\partial x\partial z}\right)\hat{\mathbf{j}}+\left(\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial^2 f}{\partial y\partial x}\right)\hat{\mathbf{k}}=\mathbf{0}$$

14.15: $$\nabla\cdot(\nabla\times\mathbf{F})=\nabla\cdot\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ F_x & F_y & F_z\end{vmatrix}$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)$$

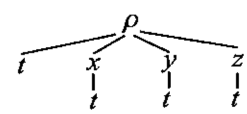
$$=\left(\frac{\partial^2 F_z}{\partial x\partial y}-\frac{\partial^2 F_z}{\partial y\partial x}\right)+\left(\frac{\partial^2 F_y}{\partial z\partial x}-\frac{\partial^2 F_y}{\partial x\partial z}\right)+\left(\frac{\partial^2 F_x}{\partial y\partial z}-\frac{\partial^2 F_x}{\partial z\partial y}\right)=0$$

44. From the schematic,

$$\frac{d\rho}{dt}=\frac{\partial\rho}{\partial t}+\frac{\partial\rho}{\partial x}\frac{dx}{dt}+\frac{\partial\rho}{\partial y}\frac{dy}{dt}+\frac{\partial\rho}{\partial z}\frac{dz}{dt}$$

$$=\frac{\partial\rho}{\partial t}+\left(\frac{\partial\rho}{\partial x}\hat{\mathbf{i}}+\frac{\partial\rho}{\partial y}\hat{\mathbf{j}}+\frac{\partial\rho}{\partial z}\hat{\mathbf{k}}\right)\cdot\left(\frac{dx}{dt}\hat{\mathbf{i}}+\frac{dy}{dt}\hat{\mathbf{j}}+\frac{dz}{dt}\hat{\mathbf{k}}\right)$$

$$=\frac{\partial\rho}{\partial t}+\nabla\rho\cdot\frac{d\mathbf{r}}{dt}.$$



45. If $\nabla f=2xy\hat{\mathbf{i}}+x^2\hat{\mathbf{j}}$, then $\dfrac{\partial f}{\partial x}=2xy$, $\dfrac{\partial f}{\partial y}=x^2$. Integrating the first gives $f(x,y)=x^2y+v(y)$. Substitution into the second requires $x^2+v'(y)=x^2\implies v(y)=C=\text{constant}$. Thus, $f(x,y)=x^2y+C$.

46. If $\nabla f=\mathbf{F}$, then $\dfrac{\partial f}{\partial x}=3x^2y^2+3$, $\dfrac{\partial f}{\partial y}=2x^3y+2$. From the first of these, $f(x,y)=x^3y^2+3x+v(y)$, which substituted into the second requires $2x^3y+v'(y)=2x^3y+2$. Thus, $v(y)=2y+C$, where C is a constant, and $f(x,y)=x^3y^2+3x+2y+C$.

47. If $\nabla f=e^y\hat{\mathbf{i}}+(xe^y+4y^2)\hat{\mathbf{j}}$, then $\dfrac{\partial f}{\partial x}=e^y$, $\dfrac{\partial f}{\partial y}=xe^y+4y^2$. Integrating the first gives $f(x,y)=xe^y+v(y)$. Substitution into the second requires $xe^y+v'(y)=xe^y+4y^2\implies v(y)=4y^3/3+C$, where C is a constant. Thus, $f(x,y)=xe^y+4y^3/3+C$.

48. If $\nabla f=\mathbf{F}$, then $\dfrac{\partial f}{\partial x}=\dfrac{1}{x+y}$, $\dfrac{\partial f}{\partial y}=\dfrac{1}{x+y}$. From the first, $f(x,y)=\ln|x+y|+v(y)$, which substituted into the second requires $1/(x+y)+v'(y)=1/(x+y)$. Thus, $v(y)=C$, where C is a constant, and $f(x,y)=\ln|x+y|+C$.

49. If $\nabla f=-xy(1-x^2y^2)^{-1/2}(y\hat{\mathbf{i}}+x\hat{\mathbf{j}})$, then $\dfrac{\partial f}{\partial x}=\dfrac{-xy^2}{\sqrt{1-x^2y^2}}$, $\dfrac{\partial f}{\partial y}=\dfrac{-x^2y}{\sqrt{1-x^2y^2}}$. Integration of the first gives $f(x,y)=\sqrt{1-x^2y^2}+v(y)$. Substitution into the second requires

$$\frac{-x^2y}{\sqrt{1-x^2y^2}}+v'(y)=\frac{-x^2y}{\sqrt{1-x^2y^2}}\quad\implies v(y)=C=\text{constant}.$$

Thus, $f(x,y)=\sqrt{1-x^2y^2}+C$.

50. If $\nabla f=x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}$, then $\dfrac{\partial f}{\partial x}=x$, $\dfrac{\partial f}{\partial y}=y$, $\dfrac{\partial f}{\partial z}=z$. From the first, $f(x,y,z)=x^2/2+v(y,z)$, which substituted into the second requires $\partial v/\partial y = y$. Thus, $v(y,z) = y^2/2 + w(z)$, and $f(x,y,z) = x^2/2 + y^2/2 + w(z)$. Substitution into the third equation gives $w'(z) = z$. Thus, $w(z) = z^2/2 + C$, where C is a constant, and $f(x,y,z) = x^2/2 + y^2/2 + z^2/2 + C$.

51. If $\nabla f = yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + (yx - 3)\hat{\mathbf{k}}$, then $\dfrac{\partial f}{\partial x} = yz$, $\dfrac{\partial f}{\partial y} = xz$, $\dfrac{\partial f}{\partial z} = yx - 3$. From the first, $f(x,y,z) = xyz + v(y,z)$, which substituted into the second requires $xz + \partial v/\partial y = xz$. Thus, $v(y,z) = w(z)$, and $f(x,y,z) = xyz + w(z)$. Substitution into the third equation gives $xy + w'(z) = yx - 3$. Thus, $w(z) = -3z + C$, where C is a constant, and $f(x,y,z) = xyz - 3z + C$.

52. If $\nabla f = (1 + x + y + z)^{-1}(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$, then

$$\frac{\partial f}{\partial x} = \frac{1}{1+x+y+z}, \quad \frac{\partial f}{\partial y} = \frac{1}{1+x+y+z}, \quad \frac{\partial f}{\partial z} = \frac{1}{1+x+y+z}.$$

From the first, $f(x,y,z) = \ln|1 + x + y + z| + v(y,z)$, which substituted into the second requires

$$\frac{1}{1+x+y+z} + \frac{\partial v}{\partial y} = \frac{1}{1+x+y+z}.$$

Thus, $v(y,z) = w(z)$, and $f(x,y,z) = \ln|1 + x + y + z| + w(z)$. Substitution into the third equation gives $\dfrac{1}{1+x+y+z} + w'(z) = \dfrac{1}{1+x+y+z}$. Thus, $w(z) = C$, and $f(x,y,z) = \ln|1 + x + y + z| + C$.

53. If $\nabla f = (2x/y^2 + 1)\hat{\mathbf{i}} - (2x^2/y^3)\hat{\mathbf{j}} - 2z\hat{\mathbf{k}}$, then $\dfrac{\partial f}{\partial x} = \dfrac{2x}{y^2} + 1$, $\dfrac{\partial f}{\partial y} = -\dfrac{2x^2}{y^3}$, $\dfrac{\partial f}{\partial z} = -2z$. From the first, $f(x,y,z) = x^2/y^2 + x + v(y,z)$, which substituted into the second requires $-\dfrac{2x^2}{y^3} + \dfrac{\partial v}{\partial y} = -\dfrac{2x^2}{y^3}$. Thus, $v(y,z) = w(z)$, and $f(x,y,z) = x^2/y^2 + x + w(z)$. Substitution into the third equation gives $w'(z) = -2z$, from which $w(z) = -z^2 + C$, where C is a constant. Thus, $f(x,y,z) = x^2/y^2 + x - z^2 + C$.

54. If $\nabla f = (1 + x^2y^2)^{-1}(y\hat{\mathbf{i}} + x\hat{\mathbf{j}}) + z\hat{\mathbf{k}}$, then $\dfrac{\partial f}{\partial x} = \dfrac{y}{1+x^2y^2}$, $\dfrac{\partial f}{\partial y} = \dfrac{x}{1+x^2y^2}$, $\dfrac{\partial f}{\partial z} = z$. From the first, $f(x,y,z) = \mathrm{Tan}^{-1}(xy) + v(y,z)$, which substituted into the second requires

$$\frac{x}{1+x^2y^2} + \frac{\partial v}{\partial y} = \frac{x}{1+x^2y^2}.$$

Thus, $v(y,z) = w(z)$, and $f(x,y,z) = \mathrm{Tan}^{-1}(xy) + w(z)$. Substitution into the third equation gives $w'(z) = z$. Thus, $w(z) = z^2/2 + C$, where C is a constant, and $f(x,y,z) = \mathrm{Tan}^{-1}(xy) + z^2/2 + C$.

55. If $\nabla f = \mathbf{F}$, then $\dfrac{\partial f}{\partial x} = 3x^2y + yz + 2xz^2$, $\dfrac{\partial f}{\partial y} = xz + x^3 + 3z^2 - 6y^2z$, $\dfrac{\partial f}{\partial z} = 2x^2z + 6yz - 2y^3 + xy$. Integrating the first gives $f(x,y,z) = x^3y + xyz + x^2z^2 + v(y,z)$. Substitution into the second requires

$$x^3 + xz + \frac{\partial v}{\partial y} = xz + x^3 + 3z^2 - 6y^2z \quad \Longrightarrow \quad v(y,z) = 3yz^2 - 2y^3z + w(z).$$

Thus, $f(x,y,z) = x^3y + xyz + x^2z^2 + 3yz^2 - 2y^3z + w(z)$. Substitution into the third equation gives

$$xy + 2x^2z + 6yz - 2y^3 + w'(z) = 2x^2z + 6yz - 2y^3 + xy \quad \Longrightarrow \quad w(z) = C, \text{ a constant.}$$

Thus, $f(x,y,z) = x^3y + xyz + x^2z^2 + 3yz^2 - 2y^3z + C$.

56. (a) $\mathbf{F}$ is irrotational if

$$\mathbf{0} = \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = (c+1)\hat{\mathbf{i}} + (a-4)\hat{\mathbf{j}} + (b-2)\hat{\mathbf{k}}.$$

Thus, $a = 4$, $b = 2$, $c = -1$.

(b) If $\nabla f = \mathbf{F}$, then $\dfrac{\partial f}{\partial x} = x^2 + 2y + 4z$, $\dfrac{\partial f}{\partial y} = 2x - 3y - z$, $\dfrac{\partial f}{\partial z} = 4x - y + 2z$. From the first, $f(x,y,z) = x^3/3 + 2xy + 4xz + v(y,z)$, which substituted into the second requires $2x + \dfrac{\partial v}{\partial y} = 2x - 3y - z$. Thus, $v(y,z) = -3y^2/2 - yz + w(z)$, and $f(x,y,z) = x^3/3 + 2xy + 4xz - 3y^2/2 - yz + w(z)$. Substitution into the third equation gives $4x - y + w'(z) = 4x - y + 2z$. Thus, $w(z) = z^2 + C$, where C is a constant, and $f(x,y,z) = x^3/3 + 2xy + 4xz - 3y^2/2 - yz + z^2 + C$.

57. (a) Since $\nabla \cdot \mathbf{F} = (4x + 8y^2 z) + (3x^3 - 3x) - (8y^2 z + 2x^3) = x + x^3 \neq 0$, this vector field is not solenoidal. For the second field,

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \nabla(xyz^2) \cdot \mathbf{F} + xyz^2(\nabla \cdot \mathbf{F})\\ &= (yz^2, xz^2, 2xyz) \cdot [(2x^2 + 8xy^2 z)\hat{\mathbf{i}} + (3x^3 y - 3xy)\hat{\mathbf{j}} - (4y^2 z^2 + 2x^3 z)\hat{\mathbf{k}}] + xyz^2(x + x^3) = 0.\end{aligned}$$

This vector field is solenoidal.

$$\begin{aligned}\text{(b)}\quad \nabla \cdot (\nabla f \times \nabla g) &= \nabla \cdot \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ f_x & f_y & f_z \\ g_x & g_y & g_z \end{vmatrix}\\ &= \nabla \cdot [(f_y g_z - f_z g_y)\hat{\mathbf{i}} + (f_z g_x - f_x g_z)\hat{\mathbf{j}} + (f_x g_y - f_y g_x)\hat{\mathbf{k}}]\\ &= (f_{yx} g_z + f_y g_{zx} - f_{zx} g_y - f_z g_{yx}) + (f_{zy} g_x + f_z g_{xy} - f_{xy} g_z - f_x g_{zy})\\ &\quad + (f_{xz} g_y + f_x g_{yz} - f_{yz} g_x - f_y g_{xz}) = 0\end{aligned}$$

58. (a) $\nabla V = -\mathbf{E} = -\dfrac{\mathbf{F}}{Q} = -\dfrac{q}{4\pi\epsilon_0 |\mathbf{r}|^3}\mathbf{r} = \dfrac{-q}{4\pi\epsilon_0} \dfrac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{(x^2 + y^2 + z^2)^{3/2}}$. This implies that

$$\frac{\partial V}{\partial x} = \frac{-q}{4\pi\epsilon_0} \frac{x}{(x^2 + y^2 + z^2)^{3/2}}, \quad \frac{\partial V}{\partial y} = \frac{-q}{4\pi\epsilon_0} \frac{y}{(x^2 + y^2 + z^2)^{3/2}}, \quad \frac{\partial V}{\partial z} = \frac{-q}{4\pi\epsilon_0} \frac{z}{(x^2 + y^2 + z^2)^{3/2}}.$$

From the first, $V(x,y,z) = \dfrac{q}{4\pi\epsilon_0 \sqrt{x^2 + y^2 + z^2}} + w(y,z)$, which substituted into the second requires

$$\frac{-q}{4\pi\epsilon_0} \frac{y}{(x^2 + y^2 + z^2)^{3/2}} + \frac{\partial w}{\partial y} = \frac{-q}{4\pi\epsilon_0} \frac{y}{(x^2 + y^2 + z^2)^{3/2}}.$$

Thus, $w(y,z) = k(z)$, and $V(x,y,z) = \dfrac{q}{4\pi\epsilon_0 \sqrt{x^2 + y^2 + z^2}} + k(z)$. Substitution into the third equation gives $\dfrac{-q}{4\pi\epsilon_0} \dfrac{z}{(x^2 + y^2 + z^2)^{3/2}} + k'(z) = \dfrac{-q}{4\pi\epsilon_0} \dfrac{z}{(x^2 + y^2 + z^2)^{3/2}}$. Hence, $k(z) = C$, where C is a constant, and $V(x,y,z) = \dfrac{q}{4\pi\epsilon_0 \sqrt{x^2 + y^2 + z^2}} + C$.

(b) $\nabla V = -\mathbf{E} = -\dfrac{\mathbf{F}}{Q} = -\dfrac{\sigma}{2\epsilon_0}\hat{\mathbf{k}}$. This implies that $\dfrac{\partial V}{\partial x} = 0$, $\dfrac{\partial V}{\partial y} = 0$, $\dfrac{\partial V}{\partial z} = -\dfrac{\sigma}{2\epsilon_0}$. These require $V(x,y,z) = -\dfrac{\sigma z}{2\epsilon_0} + C$.

59. If $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix} = (z\omega_y - y\omega_z)\hat{\mathbf{i}} + (x\omega_z - z\omega_x)\hat{\mathbf{j}} + (y\omega_x - x\omega_y)\hat{\mathbf{k}}$, then

$$\frac{1}{2}(\nabla \times \mathbf{v}) = \frac{1}{2}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z\omega_y - y\omega_z & x\omega_z - z\omega_x & y\omega_x - x\omega_y \end{vmatrix} = \frac{1}{2}[(\omega_x + \omega_x)\hat{\mathbf{i}} + (\omega_y + \omega_y)\hat{\mathbf{j}} + (\omega_z + \omega_z)\hat{\mathbf{k}}] = \boldsymbol{\omega}.$$

60. Exercise 42(a) indicates that if $f(x,y,z)$ satisfies Laplace's equation, then $\nabla \cdot (\nabla f) = 0$. In other words, ∇f is solenoidal. On the other hand, for any function whatsoever, equation 14.14 indicates that $\nabla \times (\nabla f) = \mathbf{0}$; i.e., ∇f is irrotational.

61. (a) If $\mathbf{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}} = \nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ L & M & N \end{vmatrix}$, then

$$P = \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z}, \qquad Q = \frac{\partial L}{\partial z} - \frac{\partial N}{\partial x}, \qquad R = \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}.$$

(b) Since $\mathbf{F} \times (x, y, z) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ P & Q & R \\ x & y & z \end{vmatrix} = (zQ - yR)\hat{\mathbf{i}} + (xR - zP)\hat{\mathbf{j}} + (yP - xQ)\hat{\mathbf{k}}$,

$$\mathbf{v} = \int_0^1 [t(zQ - yR)\hat{\mathbf{i}} + t(xR - zP)\hat{\mathbf{j}} + t(yP - xQ)\hat{\mathbf{k}}]\, dt,$$

where the arguments of P, Q and R are tx, ty, and tz. Consequently,

$$L = \int_0^1 t[zQ(tx, ty, tz) - yR(tx, ty, tz)]\, dt, \quad M = \int_0^1 t[xR(tx, ty, tz) - zP(tx, ty, tz)]\, dt,$$

$$N = \int_0^1 t[yP(tx, ty, tz) - xQ(tx, ty, tz)]\, dt.$$

We now show that $P = \partial N/\partial y - \partial M/\partial z$. To do this we set $a = tx$, $b = ty$, and $c = tz$. Then

$$\begin{aligned} \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} &= \int_0^1 t\left[y\frac{\partial P(a,b,c)}{\partial b}t + P(a,b,c) - x\frac{\partial Q(a,b,c)}{\partial b}t\right] dt \\ &\quad - \int_0^1 t\left[x\frac{\partial R(a,b,c)}{\partial c}t - z\frac{\partial P(a,b,c)}{\partial c}t - P(a,b,c)\right] dt \\ &= \int_0^1 t\left[2P + t\left(y\frac{\partial P}{\partial b} + z\frac{\partial P}{\partial c}\right) - xt\left(\frac{\partial Q}{\partial b} + \frac{\partial R}{\partial c}\right)\right] dt, \end{aligned}$$

where the arguments of P, Q, and R are a, b, and c. Now $\mathbf{F}$ is solenoidal so that $\dfrac{\partial P}{\partial a} + \dfrac{\partial Q}{\partial b} + \dfrac{\partial R}{\partial c} = 0$. Hence

$$\begin{aligned} \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} &= \int_0^1 t\left[2P + t\left(y\frac{\partial P}{\partial b} + z\frac{\partial P}{\partial c}\right) - xt\left(-\frac{\partial P}{\partial a}\right)\right] dt \\ &= \int_0^1 t\left[2P + t\left(x\frac{\partial P}{\partial a} + y\frac{\partial P}{\partial b} + z\frac{\partial P}{\partial c}\right)\right] dt. \end{aligned}$$

But $\dfrac{dP}{dt} = \dfrac{\partial P}{\partial a}\dfrac{da}{dt} + \dfrac{\partial P}{\partial b}\dfrac{db}{dt} + \dfrac{\partial P}{\partial c}\dfrac{dc}{dt} = x\dfrac{\partial P}{\partial a} + y\dfrac{\partial P}{\partial b} + z\dfrac{\partial P}{\partial c}$, and therefore

$$\frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} = \int_0^1 t\left(2P + t\frac{dP}{dt}\right) dt = \int_0^1 \frac{d}{dt}(t^2 P)\, dt = \left\{t^2 P\right\}_0^1 = P(x, y, z).$$

A similar proof shows that $Q = \dfrac{\partial L}{\partial z} - \dfrac{\partial N}{\partial x}$ and $R = \dfrac{\partial M}{\partial x} - \dfrac{\partial L}{\partial y}$.

The vector $\mathbf{v}$ defined by this equation is unique, but the vector $\mathbf{v}$ satisfying $\mathbf{F} = \nabla \times \mathbf{v}$ is not. Equation 14.14 indicates that for any function u, $\nabla \times (\nabla u) = \mathbf{0}$. This means that if to $\mathbf{v}$ we add any gradient ∇u, then

$$\nabla \times (\mathbf{v} + \nabla u) = \nabla \times \mathbf{v} + \nabla \times (\nabla u) = \nabla \times \mathbf{v} = \mathbf{F}.$$

Thus, we can add to $\mathbf{v}$ any gradient and still have a vector whose curl is $\mathbf{F}$.

(c) If we use part (b) to define $\mathbf{v}$,

$$\mathbf{v} = \int_0^1 t(t^n)\mathbf{F}(x, y, z) \times (x, y, z)\, dt = \mathbf{F}(x, y, z) \times (x, y, z)\left\{\frac{t^{n+2}}{n+2}\right\}_0^1 = \frac{1}{n+2}\mathbf{F} \times \mathbf{r}.$$

62. Since $\nabla\cdot\mathbf{F}=1+1-2=0$, $\mathbf{F}$ is solenoidal. Because $\mathbf{F}(tx,ty,tz)=t\mathbf{F}(x,y,z)$, we use the formula in Exercise 61(c) to obtain

$$\mathbf{v}=\frac{1}{3}\mathbf{F}\times\mathbf{r}=\frac{1}{3}\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ x & y & -2z\\ x & y & z\end{vmatrix}=\frac{1}{3}(3yz\hat{\mathbf{i}}-3xz\hat{\mathbf{j}})=yz\hat{\mathbf{i}}-xz\hat{\mathbf{j}}.$$

63. Since $\nabla\cdot\mathbf{F}=1-1=0$, the vector field is solenoidal. We use the formula in Exercise 61(b) to find $\mathbf{v}$. Since

$$\begin{aligned}\mathbf{F}(tx,ty,tz)\times(x,y,z)&=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1+tx & 0 & -tx-tz\\ x & y & z\end{vmatrix}\\ &=y(tx+tz)\hat{\mathbf{i}}+(-tx^2-txz-z-txz)\hat{\mathbf{j}}+y(1+tx)\hat{\mathbf{k}},\end{aligned}$$

$$\begin{aligned}\mathbf{v}&=\int_0^1[ty(tx+tz)\hat{\mathbf{i}}-t(tx^2+2txz+z)\hat{\mathbf{j}}+ty(1+tx)\hat{\mathbf{k}}]\,dt\\ &=\left\{\frac{t^3}{3}(xy+yz)\hat{\mathbf{i}}-\left(\frac{t^3x^2}{3}+\frac{2t^3xz}{3}+\frac{t^2z}{2}\right)\hat{\mathbf{j}}+\left(\frac{t^2y}{2}+\frac{t^3xy}{3}\right)\hat{\mathbf{k}}\right\}_0^1\\ &=\frac{1}{3}(xy+yz)\hat{\mathbf{i}}-\left(\frac{x^2}{3}+\frac{2xz}{3}+\frac{z}{2}\right)\hat{\mathbf{j}}+\left(\frac{y}{2}+\frac{xy}{3}\right)\hat{\mathbf{k}}.\end{aligned}$$

64. Since $\nabla\cdot\mathbf{F}=4x-2y+2y-4x=0$, $\mathbf{F}$ is solenoidal. Because $\mathbf{F}(tx,ty,tz)=t^2\mathbf{F}(x,y,z)$, we use the formula in Exercise 61(c) to obtain

$$\begin{aligned}\mathbf{v}=\frac{1}{4}\mathbf{F}\times\mathbf{r}&=\frac{1}{4}\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 2x^2 & -y^2 & 2yz-4xz\\ x & y & z\end{vmatrix}\\ &=\frac{1}{4}\big[(-y^2z-2y^2z+4xyz)\hat{\mathbf{i}}+(2xyz-4x^2z-2x^2z)\hat{\mathbf{j}}+(2x^2y+xy^2)\hat{\mathbf{k}}\big]\\ &=\frac{1}{4}\big[(4xyz-3y^2z)\hat{\mathbf{i}}+(2xyz-6x^2z)\hat{\mathbf{j}}+(2x^2y+xy^2)\hat{\mathbf{k}}\big].\end{aligned}$$

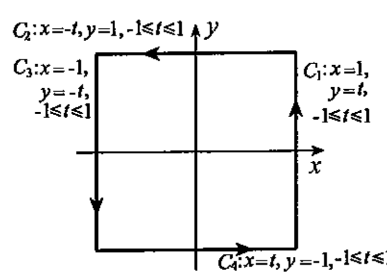
EXERCISES 14.2

1. Using x as the parameter along the curve,

$$\int_C x\,ds=\int_0^1 x\sqrt{1+(2x)^2}\,dx=\left\{\frac{1}{12}(1+4x^2)^{3/2}\right\}_0^1=\frac{1}{12}(5\sqrt{5}-1).$$

2.

$$\begin{aligned}\oint_C(x^2+y^2)\,ds&=\int_{C_1}(x^2+y^2)\,ds+\int_{C_2}(x^2+y^2)\,ds\\ &\quad+\int_{C_3}(x^2+y^2)\,ds+\int_{C_4}(x^2+y^2)\,ds\\ &=\int_{-1}^1(1+t^2)\,dt+\int_{-1}^1(t^2+1)\,dt\\ &\quad+\int_{-1}^1(1+t^2)\,dt+\int_{-1}^1(t^2+1)\,dt\\ &=4\int_{-1}^1(1+t^2)\,dt=4\left\{t+\frac{t^3}{3}\right\}_{-1}^1=\frac{32}{3}\end{aligned}$$



3. With parametric equations $C: \ x = 2\cos t, \ y = -2\sin t, \ -\pi < t \le \pi$,

$$\oint_C (2 + x - 2xy)\, ds = \int_{-\pi}^{\pi} (2 + 2\cos t + 8\cos t \sin t)\sqrt{(-2\sin t)^2 + (-2\cos t)^2}\, dt$$

$$= 4\int_{-\pi}^{\pi} (1 + \cos t + 4\cos t \sin t)\, dt = 4\Big\{t + \sin t + 2\sin^2 t\Big\}_{-\pi}^{\pi} = 8\pi$$

4. With parametric equations $C: \ x = 1 + 2t, \ y = 2, \ z = -1 + 6t, \ 0 \le t \le 1$,

$$\int_C (x^2 + yz)\, ds = \int_0^1 [(1+2t)^2 + 2(-1+6t)]\sqrt{2^2 + 0 + 6^2}\, dt$$

$$= 2\sqrt{10}\left\{\frac{1}{6}(1+2t)^3 + \frac{1}{6}(-1+6t)^2\right\}_0^1 = \frac{50\sqrt{10}}{3}$$

5. With parametric equations $x = \cos t$, $y = \sin t$, $z = \sin t$, $0 \le t \le \pi/2$,

$$\int_C xy\, ds = \int_0^{\pi/2} \cos t \sin t\sqrt{(-\sin t)^2 + (\cos t)^2 + (\cos t)^2}\, dt = \int_0^{\pi/2} \cos t \sin t\sqrt{1 + \cos^2 t}\, dt$$

$$= \left\{-\frac{1}{3}(1 + \cos^2 t)^{3/2}\right\}_0^{\pi/2} = \frac{2\sqrt{2} - 1}{3}.$$

6. With parametric equations $C: \ x = 1 - 4t, \ y = -1/2 + 4t, \ z = 1/2, \ 0 \le t \le 1$,

$$\int_C x^2yz\, ds = \int_0^1 (1-4t)^2\left(-\frac{1}{2} + 4t\right)\left(\frac{1}{2}\right)\sqrt{(-4)^2 + (4)^2}\, dt = \sqrt{2}\int_0^1 (1-4t)^2(8t-1)\, dt$$

$$= \sqrt{2}\int_0^1 (128t^3 - 80t^2 + 16t - 1)\, dt = \sqrt{2}\left\{32t^4 - \frac{80t^3}{3} + 8t^2 - t\right\}_0^1 = \frac{37\sqrt{2}}{3}$$

7. Parametric equations for $x^2 + y^2 = r^2$ are $x = r\cos t, \ y = r\sin t, \ -\pi < t \le \pi$. Its length is

$$L = \int_C ds = \int_{-\pi}^{\pi} \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt = r\int_{-\pi}^{\pi} dt = 2\pi r.$$

8. $$L = \int_C ds = \int_0^{12\pi} \sqrt{(-3\sin t)^2 + (3\cos t)^2 + [3/(4\pi)]^2}\, dt$$

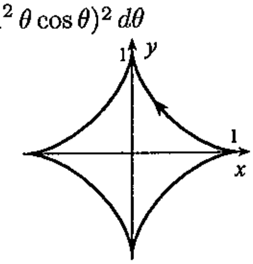
$$= \int_0^{12\pi} \sqrt{9 + \frac{9}{16\pi^2}}\,dt = \frac{3}{4\pi}\sqrt{16\pi^2 + 1}\Big\{t\Big\}_0^{12\pi} = 9\sqrt{1 + 16\pi^2} \text{ cm}$$

9. If C is the first quadrant part of the astroid,

$$A = 4\int_C (x^2 + y^2)\, ds = 4\int_0^{\pi/2} (\cos^6\theta + \sin^6\theta)\sqrt{(-3\cos^2\theta\sin\theta)^2 + (3\sin^2\theta\cos\theta)^2}\, d\theta$$

$$= 12\int_0^{\pi/2} (\cos^6\theta + \sin^6\theta)\cos\theta\sin\theta\, d\theta$$

$$= 12\left\{-\frac{1}{8}\cos^8\theta + \frac{1}{8}\sin^8\theta\right\}_0^{\pi/2} = 3$$



10. With parametric equations $C: \ y = x^2, \ z = 1 - x^2, \ 0 \le x \le 1$,

$$\int_C xz\, ds = \int_0^1 x(1-x^2)\sqrt{1^2 + (2x)^2 + (-2x)^2}\, dx = \int_0^1 x(1-x^2)\sqrt{1 + 8x^2}\, dx.$$

If we set $u = 1 + 8x^2$ and $du = 16x\, dx$, then

$$\int_C xz\, ds = \int_1^9 \left[1 - \left(\frac{u-1}{8}\right)\right]\sqrt{u}\left(\frac{du}{16}\right) = \frac{1}{128}\int_1^9 (9\sqrt{u} - u^{3/2})\, du = \frac{1}{128}\left\{6u^{3/2} - \frac{2}{5}u^{5/2}\right\}_1^9 = \frac{37}{80}.$$

11. $$\int_C (x+y)^5\,ds = \int_1^4 (2t)^5\sqrt{\left(1-\frac{1}{t^2}\right)^2+\left(1+\frac{1}{t^2}\right)^2}\,dt = 32\sqrt{2}\int_1^4 t^3\sqrt{1+t^4}\,dt$$
$$= 32\sqrt{2}\left\{\frac{1}{6}(1+t^4)^{3/2}\right\}_1^4 = \frac{16\sqrt{2}(257\sqrt{257}-2\sqrt{2})}{3}$$

12. With parametric equations $C:\ x=1-t,\ y=9t/14,\ z=1+4t/7,\ 0\le t\le 1$,

$$\int_C x\sqrt{y+z}\,ds = \int_0^1 (1-t)\sqrt{\frac{9t}{14}+1+\frac{4t}{7}}\sqrt{(-1)^2+\left(\frac{9}{14}\right)^2+\left(\frac{4}{7}\right)^2}\,dt = \frac{\sqrt{341}}{14\sqrt{14}}\int_0^1 (1-t)\sqrt{14+17t}\,dt.$$

If we set $u=14+17t$ and $du=17\,dt$, then

$$\int_C x\sqrt{y+z}\,ds = \frac{\sqrt{341}}{14\sqrt{14}}\int_{14}^{31}\left[1-\left(\frac{u-14}{17}\right)\right]\sqrt{u}\left(\frac{du}{17}\right) = \frac{\sqrt{341}}{4046\sqrt{14}}\int_{14}^{31}(31\sqrt{u}-u^{3/2})\,du$$
$$= \frac{\sqrt{341}}{4046\sqrt{14}}\left\{\frac{62u^{3/2}}{3}-\frac{2u^{5/2}}{5}\right\}_{14}^{31} = 0.78.$$

13. Using y as the parameter along the curve,

$$\int_C xy\,ds = \int_0^1 (1-y^2)y\sqrt{1+(-2y)^2}\,dy.$$

If we set $u=1+4y^2$ and $du=8y\,dy$, then

$$\int_C xy\,ds = \int_1^5\left(1-\frac{u-1}{4}\right)\sqrt{u}\left(\frac{du}{8}\right) = \frac{1}{32}\int_1^5(5\sqrt{u}-u^{3/2})\,du$$
$$= \frac{1}{32}\left\{\frac{10u^{3/2}}{3}-\frac{2u^{5/2}}{5}\right\}_1^5 = \frac{25\sqrt{5}-11}{120}$$

14. With parametric equations $C:\ y=x,\ z=1+x^4,\ -1\le x\le 1$,

$$\int_C (x+y)z\,ds = \int_{-1}^1 (x+x)(1+x^4)\sqrt{1+1+(4x^3)^2}\,dx = 2\int_{-1}^1 (x+x^5)\sqrt{2+16x^6}\,dx = 0,$$

since the integrand is an odd function of x.

15. Using x as the parameter along the curve,

$$\int_C \frac{1}{y+z}\,ds = \int_1^2 \frac{1}{x^2+x^2}\sqrt{1+(2x)^2+(2x)^2}\,dx = \frac{1}{2}\int_1^2\frac{\sqrt{1+8x^2}}{x^2}\,dx.$$

If we set $x=[1/(2\sqrt{2})]\tan\theta$ and $dx=[1/(2\sqrt{2})/\sec^2\theta\,d\theta$, then for $\theta_1=\mathrm{Tan}^{-1}(2\sqrt{2})$ and $\theta_2=\mathrm{Tan}^{-1}(4\sqrt{2})$,

$$\int_C\frac{1}{y+z}\,ds = \frac{1}{2}\int_{\theta_1}^{\theta_2}\frac{8\sec\theta}{\tan^2\theta}\frac{1}{2\sqrt{2}}\sec^2\theta\,d\theta = \sqrt{2}\int_{\theta_1}^{\theta_2}\frac{\sec\theta(1+\tan^2\theta)}{\tan^2\theta}\,d\theta$$
$$= \sqrt{2}\int_{\theta_1}^{\theta_2}(\csc\theta\cot\theta+\sec\theta)\,d\theta = \sqrt{2}\Big\{-\csc\theta+\ln|\sec\theta+\tan\theta|\Big\}_{\theta_1}^{\theta_2} = 1.013.$$

16. With parametric equations $C:\ x=t^2,\ y=-t,\ z=-t^3,\ 0\le t\le 2$,

$$\int_C (2y+9z)\,ds = \int_0^2(-2t-9t^3)\sqrt{(2t)^2+1+(-3t^2)^2}\,dt = -\int_0^2(2t+9t^3)\sqrt{1+4t^2+9t^4}\,dt.$$

If we set $u=1+4t^2+9t^4$ and $du=(8t+36t^3)\,dt=4(2t+9t^3)\,dt$, then

$$\int_C(2y+9z)\,ds = -\int_1^{161}\sqrt{u}\left(\frac{du}{4}\right) = -\left\{\frac{1}{6}u^{3/2}\right\}_1^{161} = \frac{1-161\sqrt{161}}{6}.$$

17. (a) When a small length ds at position (x, y) on C is rotated around the y-axis, it traces a ribbon with approximate area $2\pi x\, ds$. The total surface area is therefore the limit of the summation of all such ribbons, namely, $\int_C 2\pi x\, ds$.

(b) The surface area in this case is $\int_C 2\pi y\, ds$.

18. When we use x as parameter along the curve,

$$A = \int_C 2\pi y\, ds = 2\pi \int_1^2 x^3 \sqrt{1 + (3x^2)^2}\, dx = 2\pi \int_1^2 x^3 \sqrt{1 + 9x^4}\, dx = 2\pi \left\{ \frac{(1 + 9x^4)^{3/2}}{54} \right\}_1^2$$
$$= \frac{(145\sqrt{145} - 10\sqrt{10})\pi}{27}.$$

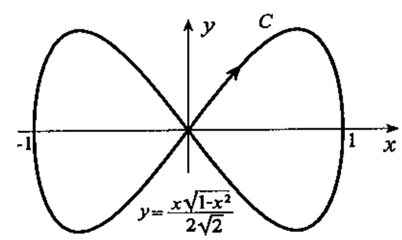
19. Since $y = x^3/24 + 2/x$, $(ds)^2 = \left[1 + \left(\frac{x^2}{8} - \frac{2}{x^2}\right)^2\right](dx)^2 = \left[1 + \frac{x^4}{64} - \frac{1}{2} + \frac{4}{x^4}\right](dx)^2$
$= \left(\frac{x^2}{8} + \frac{2}{x^2}\right)^2 (dx)^2$. The surface area is

$$A = \int_C 2\pi x\, ds = 2\pi \int_1^2 x\left(\frac{x^2}{8} + \frac{2}{x^2}\right) dx = 2\pi \left\{\frac{x^4}{32} + 2\ln x\right\}_1^2 = \frac{\pi(15 + 64\ln 2)}{16}.$$

20. We double the area obtained by rotating the first quadrant part of the curve.

$$A = 2\int_C 2\pi y\, ds \text{ where}$$

$$ds = \sqrt{1 + \left[\frac{1}{2\sqrt{2}}\left(\sqrt{1 - x^2} - \frac{x^2}{\sqrt{1 - x^2}}\right)\right]^2} dx$$
$$= \sqrt{1 + \frac{1}{8}\left(\frac{1 - 2x^2}{\sqrt{1 - x^2}}\right)^2} dx$$
$$= \sqrt{\frac{8(1 - x^2) + (1 - 2x^2)^2}{8(1 - x^2)}} dx = \frac{3 - 2x^2}{2\sqrt{2}\sqrt{1 - x^2}} dx$$



Thus, $A = 4\pi \int_0^1 \frac{x\sqrt{1 - x^2}}{2\sqrt{2}} \frac{3 - 2x^2}{2\sqrt{2}\sqrt{1 - x^2}} dx = \frac{\pi}{2}\int_0^1 (3x - 2x^3)\, dx = \frac{\pi}{2}\left\{\frac{3x^2}{2} - \frac{x^4}{2}\right\}_0^1 = \frac{\pi}{2}.$

21. $L = \int_0^1 \sqrt{1 + (2x)^2}\, dx = \int_0^1 \sqrt{1 + 4x^2}\, dx$ If we set $x = (1/2)\tan\theta$ and $dx = (1/2)\sec^2\theta\, d\theta$,

$$L = \int_0^{\mathrm{Tan}^{-1}2} \sec\theta (1/2)\sec^2\theta\, d\theta = \frac{1}{2}\int_0^{\mathrm{Tan}^{-1}2} \sec^3\theta\, d\theta$$
$$= \frac{1}{4}\Big\{\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big\}_0^{\mathrm{Tan}^{-1}2} \quad \text{(see Example 8.9)}$$
$$= \frac{1}{4}[2\sqrt{5} + \ln(\sqrt{5} + 2)].$$

22. Using x as the parameter along the curve,

$$\begin{aligned}\int_C xy\,ds &= \int_0^{1/2} x^4\sqrt{1+(3x^2)^2}\,dx = \int_0^{1/2} x^4\sqrt{1+9x^4}\,dx\\ &= \int_0^{1/2} x^4\left[1+\frac{1}{2}(9x^4)+\frac{(1/2)(-1/2)}{2}(9x^4)^2+\cdots\right]dx\\ &= \int_0^{1/2}\left(x^4+\frac{9}{2}x^8-\frac{9^2}{2^2 2!}x^{12}+\frac{9^3 3}{2^3 3!}x^{16}-\frac{9^4 3\cdot 5}{2^4 4!}x^{20}+\cdots\right]dx\\ &= \left\{\frac{x^5}{5}+\frac{x^9}{2}-\frac{9^2x^{13}}{2^2 2!13}+\frac{9^3 3x^{17}}{2^3 3!17}-\frac{9^4 3\cdot 5x^{21}}{2^4 4!21}+\cdots\right\}_0^{1/2}\\ &= \frac{1}{5\cdot 2^5}+\frac{1}{2^{10}}-\frac{9^2}{2^{15}2!13}+\frac{9^3 3}{2^{20}3!17}-\frac{9^4 3\cdot 5}{2^{25}4!21}+\frac{9^5 3\cdot 5\cdot 7}{2^{30}5!25}+\cdots.\end{aligned}$$

Because this series is alternating (after the first term), and absolute values of terms decrease and approach zero, the sum is between any two consecutive partial sums. Since the sum of the first two terms is 0.007 22 and the sum of the first three terms is 0.007 13, we can say that to three decimals, the value of the integral is 0.007.

23. Using parametric equations $x=-1+t,\ y=3-2t,\ z=t,\ 0\le t\le 1$,

$$\begin{aligned}\int_C e^{-(x+y-2)^2}ds &= \int_0^1 e^{-t^2}\sqrt{(1)^2+(-2)^2+(1)^2}\,dt = \sqrt{6}\int_0^1 e^{-t^2}dt\\ &= \sqrt{6}\int_0^1\left(1-t^2+\frac{t^4}{2!}-\frac{t^6}{3!}+\cdots\right)dt = \sqrt{6}\left\{t-\frac{t^3}{3}+\frac{t^5}{5\cdot 2!}-\frac{t^7}{7\cdot 3!}+\cdots\right\}_0^1\\ &= \sqrt{6}\left(1-\frac{1}{3}+\frac{1}{5\cdot 2!}-\frac{1}{7\cdot 3!}+\cdots\right).\end{aligned}$$

Because this series is alternating and absolute values of terms decrease and approach zero, the sum is between any two consecutive partial sums. Since the sum of the first six terms is 1.829 11 and the sum of the first seven terms is 1.829 38, we can say that to three decimals, the value of the integral is 1.829.

24. With parametric equations $C:\ x=2\cos t,\ y=2\sin t,\ -\pi<t\le\pi$,

$$\begin{aligned}\overline{f} &= \frac{1}{2\pi(2)}\int_C x^2y^2\,ds = \frac{1}{4\pi}\int_{-\pi}^{\pi}4\cos^2 t\,4\sin^2 t\sqrt{4\sin^2 t+4\cos^2 t}\,dt = \frac{8}{\pi}\int_{-\pi}^{\pi}\cos^2 t\,\sin^2 t\,dt\\ &= \frac{8}{\pi}\int_{-\pi}^{\pi}\left(\frac{\sin 2t}{2}\right)^2 dt = \frac{2}{\pi}\int_{-\pi}^{\pi}\left(\frac{1-\cos 4t}{2}\right)dt = \frac{1}{\pi}\left\{t-\frac{\sin 4t}{4}\right\}_{-\pi}^{\pi} = 2.\end{aligned}$$

25. Since $\quad L=\displaystyle\int_0^{\pi}\sqrt{(-\sin t)^2+(\cos t)^2+1}\,dt=\int_0^{\pi}\sqrt{2}\,dt=\sqrt{2}\pi,$

$$\begin{aligned}\overline{f} &= \frac{1}{\sqrt{2}\pi}\int_C(x^2+y^2+z^2)\,ds = \frac{1}{\sqrt{2}\pi}\int_0^{\pi}(\cos^2 t+\sin^2 t+t^2)\sqrt{2}\,dt\\ &= \frac{1}{\pi}\int_0^{\pi}(1+t^2)\,dt = \frac{1}{\pi}\left\{t+\frac{t^3}{3}\right\}_0^{\pi} = \frac{3+\pi^2}{3}.\end{aligned}$$

26. With parametric equations $C:\ y=x^2,\ z=x^2,\ 0\le x\le 1$, the length of the curve is $L=\displaystyle\int_0^1\sqrt{1+(2x)^2+(2x)^2}\,dx=\int_0^1\sqrt{1+8x^2}\,dx$. If we set $x=[1/(2\sqrt{2})]\tan\theta$, then

$$\begin{aligned}L &= \int_0^{\overline{\theta}}\sec\theta\frac{1}{2\sqrt{2}}\sec^2\theta\,d\theta \quad (\overline{\theta}=\mathrm{Tan}^{-1}(2\sqrt{2}))\\ &= \frac{1}{4\sqrt{2}}\Big\{\sec\theta\,\tan\theta+\ln|\sec\theta+\tan\theta|\Big\}_0^{\overline{\theta}} \quad \text{(see Example 8.9)}\\ &= \frac{1}{4\sqrt{2}}[6\sqrt{2}+\ln(3+2\sqrt{2})].\end{aligned}$$

The average value of the function is $\overline{f} = \dfrac{1}{L}\displaystyle\int_0^1 xyz\,ds = \dfrac{1}{L}\int_0^1 x^5\sqrt{1+8x^2}\,dx$. If we set $u = 1 + 8x^2$, then $du = 16x\,dx$, and

$$\overline{f} = \frac{1}{L}\int_1^9 \left(\frac{u-1}{8}\right)^2 \sqrt{u}\left(\frac{du}{16}\right) = \frac{1}{1024L}\int_1^9 (u^{5/2} - 2u^{3/2} + \sqrt{u})\,du$$
$$= \frac{1}{1024L}\left\{\frac{2u^{7/2}}{7} - \frac{4u^{5/2}}{5} + \frac{2u^{3/2}}{3}\right\}_1^9 = 0.242.$$

27. Since $ds^2 = \left[1 + \left(\dfrac{3x^2}{4} - \dfrac{1}{3x^2}\right)^2\right] dx^2 = \left(1 + \dfrac{9x^4}{16} - \dfrac{1}{2} + \dfrac{1}{9x^4}\right) dx^2 = \left(\dfrac{3x^2}{4} + \dfrac{1}{3x^2}\right)^2 dx^2$, the length of the curve is $L = \displaystyle\int_1^2 \left(\frac{3x^2}{4} + \frac{1}{3x^2}\right) dx = \left\{\frac{x^3}{4} - \frac{1}{3x}\right\}_1^2 = \frac{23}{12}$. The average value is

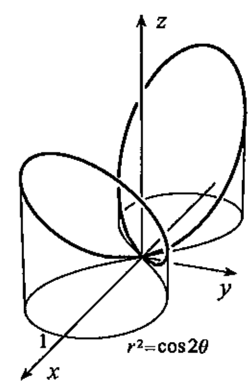
$$\overline{f} = \frac{12}{23}\int_C y\,ds = \frac{12}{23}\int_1^2 \left(\frac{x^3}{4} + \frac{1}{3x}\right)\left(\frac{3x^2}{4} + \frac{1}{3x^2}\right) dx$$
$$= \frac{12}{23}\int_1^2 \left(\frac{3x^5}{16} + \frac{x}{3} + \frac{1}{9x^3}\right) dx = \frac{12}{23}\left\{\frac{x^6}{32} + \frac{x^2}{6} - \frac{1}{18x^2}\right\}_1^2 = \frac{241}{184}.$$

28. $A = 4\displaystyle\int_C \sqrt{x^2+y^2}\,ds$ where C is that part of the lemniscate in the first quadrant. According to formula 9.14,

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

Hence,

$$A = 4\int_0^{\pi/4} r\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$
$$= 4\int_0^{\pi/4} \sqrt{\cos 2\theta}\sqrt{\cos 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta}}\,d\theta = 4\int_0^{\pi/4} d\theta = \pi.$$



29. $M = \displaystyle\int_C \left(1 - \frac{|x|}{40}\right) ds = 2\int_0^{20}\left(1 - \frac{x}{40}\right)\sqrt{1 + \sinh^2\left(\frac{x}{40}\right)}\,dx = 2\int_0^{20}\left(1 - \frac{x}{40}\right)\cosh\left(\frac{x}{40}\right) dx$

If we set $u = 1 - x/40$, $dv = \cosh(x/40)\,dx$, $du = -dx/40$, and $v = 40\sinh(x/40)$,

$$M = 2\left\{\left(1 - \frac{x}{40}\right) 40\sinh\left(\frac{x}{40}\right)\right\}_0^{20} - 2\int_0^{20} 40\sinh\left(\frac{x}{40}\right)\left(\frac{-dx}{40}\right)$$
$$= 40\sinh\left(\frac{1}{2}\right) + 2\left\{40\cosh\left(\frac{x}{40}\right)\right\}_0^{20} = 31.05 \text{ kg}.$$

30. With $ds = \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}\,d\theta = \sqrt{(2-\sin\theta)^2 + (-\cos\theta)^2}\,d\theta = \sqrt{5 - 4\sin\theta}\,d\theta$, we obtain

$$\int_C \frac{x}{\sqrt{x^2+y^2}}\,ds = \int_0^{\pi/2} \frac{r\cos\theta}{r}\sqrt{5 - 4\sin\theta}\,d\theta = \left\{-\frac{1}{6}(5 - 4\sin\theta)^{3/2}\right\}_0^{\pi/2} = \frac{5\sqrt{5} - 1}{6}.$$

31. $$\oint_C (x^2+y^2)\,ds = \int_{-\pi}^{\pi} r^2\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_{-\pi}^{\pi}(1+\cos\theta)^2\sqrt{(1+\cos\theta)^2+(-\sin\theta)^2}\,d\theta$$
$$= \int_{-\pi}^{\pi}(1+\cos\theta)^2\sqrt{2+2\cos\theta}\,d\theta = \sqrt{2}\int_{-\pi}^{\pi}(1+\cos\theta)^{5/2}\,d\theta$$
$$= \sqrt{2}\int_{-\pi}^{\pi}[1+2\sin^2(\theta/2)-1]^{5/2}\,d\theta = 8\int_{-\pi}^{\pi}\sin^5(\theta/2)\,d\theta$$
$$= 8\int_{-\pi}^{\pi}\sin(\theta/2)[1-\cos^2(\theta/2)]^2\,d\theta = 8\int_{-\pi}^{\pi}\sin(\theta/2)[1-2\cos^2(\theta/2)+\cos^4(\theta/2)]\,d\theta$$
$$= 8\left\{-2\cos(\theta/2)+\frac{4}{3}\cos^3(\theta/2)-\frac{2}{5}\cos^5(\theta/2)\right\}_{-\pi}^{\pi} = \frac{256}{15}$$

32. With $ds = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta = \sqrt{e^{2\theta}+e^{2\theta}}\,d\theta = \sqrt{2}e^{\theta}\,d\theta$,

$$\int_C xy\,ds = \int_0^{2\pi} r\cos\theta\, r\sin\theta\,\sqrt{2}e^{\theta}\,d\theta = \sqrt{2}\int_0^{2\pi}\sin\theta\,\cos\theta\,e^{3\theta}d\theta = \frac{1}{\sqrt{2}}\int_0^{2\pi}e^{3\theta}\sin 2\theta\,d\theta$$
$$= \frac{1}{\sqrt{2}}\left\{\frac{e^{3\theta}}{13}(3\sin 2\theta-2\cos 2\theta)\right\}_0^{2\pi} = \frac{\sqrt{2}(1-e^{6\pi})}{13}.$$

33. $$\oint_C \cos^3 2\theta\,ds = \int_0^{\pi/2}\cos^3 2\theta\sqrt{\sin 2\theta+\left(\frac{\cos 2\theta}{\sqrt{\sin 2\theta}}\right)^2}\,d\theta = \int_0^{\pi/2}\frac{\cos^3 2\theta}{\sqrt{\sin 2\theta}}d\theta$$
$$= \int_0^{\pi/2}\frac{(1-\sin^2 2\theta)\cos 2\theta}{\sqrt{\sin 2\theta}}d\theta = \left\{\sqrt{\sin 2\theta}-\frac{1}{5}\sin^{5/2}2\theta\right\}_0^{\pi/2} = 0$$

34. With parametric equations $C:\ y=1-x,\ z=2x^2-2x+1,\ -1\le x\le 1$,

$$I = \int_{-1}^{1}[x^2(1-x)+2x^2-2x+1]\sqrt{1+(-1)^2+(4x-2)^2}\,dx$$
$$= \int_{-1}^{1}(1-2x+3x^2-x^3)\sqrt{6-16x+16x^2}\,dx = \sqrt{2}\int_{-1}^{1}(1-2x+3x^2-x^3)\sqrt{3-8x+8x^2}\,dx.$$

If we denote the integrand by $f(x)$, then Simpson's rule with 10 equal subdivisions gives

$$I \approx \sqrt{2}\left(\frac{1/5}{3}\right)[f(-1)+4f(-0.8)+2f(-0.6)+\cdots+2f(0.6)+4f(0.8)+f(1)] = 17.08.$$

35. With parametric equations $C:\ x=3\cos t,\ y=2\sin t,\ -\pi<t\le\pi$,

$$\oint_C x^2y^2\,ds = 4\int_0^{\pi/2}(9\cos^2 t)(4\sin^2 t)\sqrt{(-3\sin t)^2+(2\cos t)^2}\,dt$$
$$= 144\int_0^{\pi/2}\cos^2 t\sin^2 t\sqrt{9\sin^2 t+4\cos^2 t}\,dt.$$

If we denote the integrand by $f(t)$, then Simpson's rule with ten equal subdivisions gives

$$\oint_C x^2y^2\,ds \approx \frac{144(\pi/2)}{30}[f(0)+4f(\pi/20)+2f(\pi/10)+\cdots+2f(2\pi/5)+4f(9\pi/20)+f(\pi/2)] = 71.74.$$

36. Using the result of Exercise 17, and parametric equations $C:\ x=a+b\cos t,\ y=b\sin t,\ -\pi<t\le\pi$,

$$A = \int_C 2\pi x\,ds = 2\pi\int_{-\pi}^{\pi}(a+b\cos t)\sqrt{(-b\sin t)^2+(b\cos t)^2}\,dt$$
$$= 2\pi b\int_{-\pi}^{\pi}(a+b\cos t)\,dt = 2\pi b\Big\{at+b\sin t\Big\}_{-\pi}^{\pi} = 4\pi^2 ab.$$

37. Suppose that a curve C has parametrization

$$C: \ x = x(t), \quad y = y(t), \quad z = z(t), \quad \alpha \le t \le \beta,$$

and we change parameters by setting $t = t(u)$ (or $u = u(t)$). Then

$$C: \ x = x[t(u)], \quad y = y[t(u)], \quad z = z[t(u)], \quad a = u(\alpha) \le u \le u(\beta) = b.$$

According to equation 14.20,

$$\int_C f(x,y,z)\,ds = \int_\alpha^\beta f[x(t),y(t),z(t)]\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt,$$

and if we change the variable of integration by $t = t(u)$, then $dt = t'(u)\,du$, and

$$\begin{aligned}\int_C f(x,y,z)\,ds &= \int_a^b f\{x[t(u)],y[t(u)],z[t(u)]\}\sqrt{\left(\frac{dx}{du}\frac{du}{dt}\right)^2 + \left(\frac{dy}{du}\frac{du}{dt}\right)^2 + \left(\frac{dz}{du}\frac{du}{dt}\right)^2}\,dt \\ &= \int_a^b f\{x[t(u)],y[t(u)],z[t(u)]\}\sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\,du.\end{aligned}$$

But this is equation 14.20 with $f(x,y,z)$ and ds expressed in term of parameter u instead of t. In other words, the value of the line integral is the same for any parametrization of the curve.

EXERCISES 14.3

1. Using x as parameter along the curve,

$$\int_C x\,dx + x^2y\,dy = \int_{-1}^2 x\,dx + x^2(x^3)(3x^2\,dx) = \int_{-1}^2 (x + 3x^7)\,dx = \left\{\frac{x^2}{2} + \frac{3x^8}{8}\right\}_{-1}^2 = \frac{777}{8}.$$

2. Using x as parameter along the curve,

$$\int_C x\,dx + yz\,dy + x^2\,dz = \int_{-1}^2 [x\,dx + xx^2\,dx + x^2(2x\,dx)] = \int_{-1}^2 (x + 3x^3)\,dx = \left\{\frac{x^2}{2} + \frac{3x^4}{4}\right\}_{-1}^2 = \frac{51}{4}.$$

3. Using parametric equations $x = 1 + t^2$, $y = -t$, $-1 \le t \le 1$,

$$\begin{aligned}\int_C x\,dx + (x+y)\,dy &= \int_{-1}^1 (1+t^2)(2t\,dt) + (1 + t^2 - t)(-dt) = \int_{-1}^1 (-1 + 3t - t^2 + 2t^3)\,dt \\ &= \left\{-t + \frac{3t^2}{2} - \frac{t^3}{3} + \frac{t^4}{2}\right\}_{-1}^1 = -\frac{8}{3}.\end{aligned}$$

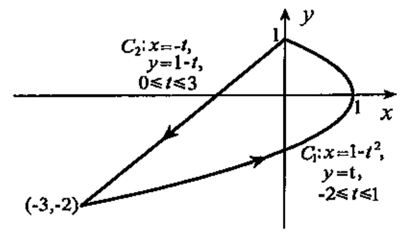
4. With parametric equations $C: \ x = -2 + 3t, \ y = 3 - 3t, \ z = 3 - 3t, \ 0 \le t \le 1$, for the straight line,

$$\begin{aligned}\int_C x^2\,dx + y^2\,dy + z^2\,dz &= \int_0^1 [(-2+3t)^2(3\,dt) + (3-3t)^2(-3\,dt) + (3-3t)^2(-3\,dt)] \\ &= 3\int_0^1 [(-2+3t)^2 - 2(3-3t)^2]\,dt = 3\left\{\frac{1}{9}(-2+3t)^3 + \frac{2}{9}(3-3t)^3\right\}_0^1 = -15.\end{aligned}$$

5. Using y as parameter along the curve,

$$\int_C (y + 2x^2z)\,dx = \int_{-2}^1 [y + 2(y^2)^2(y^4)](2y\,dy) = 2\int_{-2}^1 (y^2 + 2y^9)\,dy = 2\left\{\frac{y^3}{3} + \frac{y^{10}}{5}\right\}_{-2}^1 = -\frac{2016}{5}.$$

6. $$\oint_C x^2y\,dx + (x-y)\,dy = \int_{C_1} x^2y\,dx + (x-y)\,dy + \int_{C_2} x^2y\,dx + (x-y)\,dy$$
$$= \int_{-2}^{1} [(1-t^2)^2t(-2t\,dt) + (1-t^2-t)\,dt]$$
$$+ \int_0^3 [t^2(1-t)(-dt) + (-t-1+t)(-dt)]$$
$$= \int_{-2}^{1} (-2t^6 + 4t^4 - 3t^2 - t + 1)\,dt + \int_0^3 (t^3 - t^2 + 1)\,dt$$
$$= \left\{-\frac{2t^7}{7} + \frac{4t^5}{5} - t^3 - \frac{t^2}{2} + t\right\}_{-2}^{1} + \left\{\frac{t^4}{4} - \frac{t^3}{3} + t\right\}_0^3 = -\frac{99}{140}$$

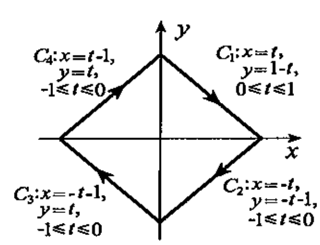


7. With parametric equations $C:\ x = \cos t,\ y = -\sin t,\ -\pi/2 \le t \le \pi/2$,
$$\int_C y^2\,dx + x^2\,dy = \int_{-\pi/2}^{\pi/2} \sin^2 t(-\sin t\,dt) + \cos^2 t(-\cos t\,dt) = \int_{-\pi/2}^{\pi/2} [-(1-\cos^2 t)\sin t - (1-\sin^2 t)\cos t]\,dt$$
$$= \left\{\cos t - \frac{1}{3}\cos^3 t - \sin t + \frac{1}{3}\sin^3 t\right\}_{-\pi/2}^{\pi/2} = -\frac{4}{3}.$$

8. With parametric equations $C:\ x = 1-y,\ z = 2y^2 - 2y + 1,\ 0 \le y \le 2$,
$$\int_C y\,dx + x\,dy + z\,dz = \int_0^2 [y(-dy) + (1-y)\,dy + (2y^2-2y+1)(4y-2)\,dy]$$
$$= \int_0^2 [-2y + 1 + (2y^2-2y+1)(4y-2)]\,dy = \left\{-y^2 + y + \frac{(2y^2-2y+1)^2}{2}\right\}_0^2 = 10.$$

9. With parametric equations $C:\ x = \cos t,\ y = \sin t,\ z = 1 - \cos t - \sin t,\ -\pi < t \le \pi$,
$$\oint_C x^2y\,dy + z\,dx = \int_{-\pi}^{\pi} \cos^2 t(\sin t)(\cos t\,dt) + (1-\cos t-\sin t)(-\sin t\,dt)$$
$$= \int_{-\pi}^{\pi} \left(\cos^3 t\sin t - \sin t + \cos t\sin t + \frac{1-\cos 2t}{2}\right) dt$$
$$= \left\{-\frac{1}{4}\cos^4 t + \cos t + \frac{1}{2}\sin^2 t + \frac{t}{2} - \frac{1}{4}\sin 2t\right\}_{-\pi}^{\pi} = \pi.$$

10. $$\oint_C y^2\,dx + x^2\,dy = \int_{C_1} y^2\,dx + x^2\,dy + \int_{C_2} y^2\,dx + x^2\,dy + \int_{C_3} y^2\,dx + x^2\,dy + \int_{C_4} y^2\,dx + x^2\,dy$$
$$= \int_0^1 [(1-t)^2\,dt + t^2(-dt)] + \int_{-1}^{0} [(t+1)^2(-dt) + t^2(-dt)]$$
$$+ \int_{-1}^{0} [t^2(-dt) + (t+1)^2\,dt] + \int_0^1 [t^2\,dt + (t-1)^2\,dt]$$
$$= \int_0^1 2(t-1)^2\,dt + \int_{-1}^{0} (-2t^2)\,dt$$
$$= \left\{\frac{2(t-1)^3}{3}\right\}_0^1 + \left\{-\frac{2t^3}{3}\right\}_{-1}^{0} = 0$$



11. With parametric equations $C: \ x = 1 + 5t, \ y = 5t, \ 0 \le t \le 1,$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C x^2 y\, dx + x\, dy = \int_0^1 (1+5t)^2(5t)(5\, dt) + (1+5t)(5\, dt)$$
$$= 5\int_0^1 (1 + 10t + 50t^2 + 125t^3)\, dt = 5\left\{t + 5t^2 + \frac{50t^3}{3} + \frac{125t^4}{4}\right\}_0^1 = \frac{3235}{12}.$$

12. (a) $$\int_C xy\, dx + x^2\, dy = \int_0^{\pi/2} [3\cos t(3\sin t)(-3\sin t\, dt) + 9\cos^2 t(3\cos t\, dt)]$$
$$= 27\int_0^{\pi/2} [-\sin^2 t\, \cos t + \cos t(1 - \sin^2 t)]\, dt = 27\left\{-\frac{2}{3}\sin^3 t + \sin t\right\}_0^{\pi/2} = 9$$

(b) $$\int_C xy\, dx + x^2\, dy = \int_0^3 \left[\sqrt{9-y^2}(y)\left(\frac{-y}{\sqrt{9-y^2}}dy\right) + (9-y^2)\, dy\right] = \int_0^3 (9 - 2y^2)\, dy$$
$$= \left\{9y - \frac{2y^3}{3}\right\}_0^3 = 9$$

13. (a) Along the straight line with parametric equations $C_1: \ x = -5 + 9t, \ y = 3 - 3t, \ 0 \le t \le 1,$

$$\int_{C_1} xy\, dx + x\, dy = \int_0^1 (-5+9t)(3-3t)(9\, dt) + (-5+9t)(-3\, dt)$$
$$= 3\int_0^1 (-40 + 117t - 81t^2)\, dt = 3\left\{-40t + \frac{117t^2}{2} - 27t^3\right\}_0^1 = -\frac{51}{2}.$$

(b) Along the parabola with parametric equations $C_2: \ x = 4 - t^2, \ y = -t, \ -3 \le t \le 0,$

$$\int_{C_2} xy\, dx + x\, dy = \int_{-3}^0 (4-t^2)(-t)(-2t\, dt) + (4-t^2)(-dt) = \int_{-3}^0 (-4 + 9t^2 - 2t^4)\, dt$$
$$= \left\{-4t + 3t^3 - \frac{2t^5}{5}\right\}_{-3}^0 = -\frac{141}{5}.$$

(c) Along the parabola with equation $C_3: \ y = (x^2 - 16)/3, \ -5 \le x \le 4,$

$$\int_{C_3} xy\, dx + x\, dy = \int_{-5}^4 x\left(\frac{x^2-16}{3}\right) dx + x\left(\frac{2x\, dx}{3}\right) = \frac{1}{3}\int_{-5}^4 (x^3 + 2x^2 - 16x)\, dx$$
$$= \frac{1}{3}\left\{\frac{x^4}{4} + \frac{2x^3}{3} - 8x^2\right\}_{-5}^4 = \frac{141}{4}.$$

14. With parametric equations $C: \ x = a\cos t, \ y = b\sin t, \ -\pi \le t < \pi,$

$$W = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C x\, dx + y\, dy = \int_{-\pi}^{\pi} [a\cos t(-a\sin t\, dt) + b\sin t(b\cos t\, dt)]$$
$$= (b^2 - a^2)\int_{-\pi}^{\pi} \sin t\, \cos t\, dt = (b^2 - a^2)\left\{\frac{\sin^2 t}{2}\right\}_{-\pi}^{\pi} = 0.$$

15. With parametric equations $C: \ x = \cos t, \ y = \sin t, \ z = \sin t, \ \pi/4 \le t \le 3\pi/4,$

$$\int_C \frac{1}{yz} dx = \int_{\pi/4}^{3\pi/4} \frac{1}{\sin^2 t}(-\sin t\, dt) = \int_{\pi/4}^{3\pi/4} -\csc t\, dt = -\Big\{\ln|\csc t - \cot t|\Big\}_{\pi/4}^{3\pi/4} = 0.$$

16. With parametric equations $C:\ x = 2 + \cos t,\ y = -\sin t,\ 0 \le t < 4\pi$,

$$\begin{aligned}\oint_C (x^2 + 2y^2)\,dy &= \int_0^{4\pi} [(2+\cos t)^2 + 2(-\sin t)^2](-\cos t\,dt)\\ &= -\int_0^{4\pi} (4 + 4\cos t + \cos^2 t + 2\sin^2 t)\cos t\,dt = -\int_0^{4\pi} (5 + 4\cos t + \sin^2 t)\cos t\,dt\\ &= -\int_0^{4\pi} [5\cos t + 2(1+\cos 2t) + \sin^2 t\,\cos t]\,dt = -\left\{5\sin t + 2t + \sin 2t + \frac{1}{3}\sin^3 t\right\}_0^{4\pi}\\ &= -8\pi.\end{aligned}$$

17. With parametric equations $C:\ x = 1 + \cos t,\ y = \sin t,\ z = \sqrt{2 - 2\cos t},\ 0 \le t \le \pi$,

$$\begin{aligned}\int_C y\,dx - y(x-1)\,dy + y^2 z\,dz &= \int_0^{\pi} \sin t(-\sin t\,dt) - \sin t(\cos t)(\cos\,dt) + \sin^2 t\sqrt{2-2\cos t}\left(\frac{\sin t\,dt}{\sqrt{2-2\cos t}}\right)\\ &= \int_0^{\pi}\left[\frac{\cos 2t - 1}{2} - \cos^2 t\sin t + (1 - \cos^2 t)\sin t\right]dt\\ &= \left\{\frac{1}{4}\sin 2t - \frac{t}{2} + \frac{2}{3}\cos^3 t - \cos t\right\}_0^{\pi} = \frac{2}{3} - \frac{\pi}{2}.\end{aligned}$$

18. With parametric equations $C:\ x = t - 1,\ y = 1 + 2t^2,\ z = t,\ 1 \le t \le 2$,

$$\begin{aligned}\int_C x^2 y\,dx + y\,dy + \sqrt{1-x^2}\,dz &= \int_1^2 [(t-1)^2(1+2t^2)\,dt + (1+2t^2)(4t\,dt) + \sqrt{1-(t-1)^2}\,dt]\\ &= \int_1^2 [2t^4 + 4t^3 + 3t^2 + 2t + 1 + \sqrt{1-(t-1)^2}]\,dt.\end{aligned}$$

In the last term we set $t - 1 = \sin\theta$ and $dt = \cos\theta\,d\theta$,

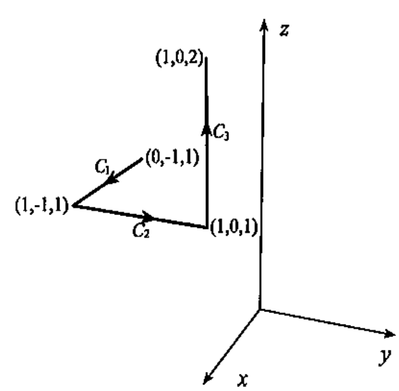
$$\begin{aligned}\int_C x^2 y\,dx + y\,dy + \sqrt{1-x^2}\,dz &= \left\{\frac{2t^5}{5} + t^4 + t^3 + t^2 + t\right\}_1^2 + \int_0^{\pi/2}\cos^2\theta\,d\theta\\ &= \frac{192}{5} + \int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta = \frac{192}{5} + \left\{\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right\}_0^{\pi/2} = \frac{192}{5} + \frac{\pi}{4}.\end{aligned}$$

19. With parametric equations $C:\ x = 2 + t,\ y = -1 + 2t,\ z = 4 - 5t,\ 0 \le t \le 1$,

$$\begin{aligned}\int_C x\,dx + xy\,dy + 2\,dz &= \int_0^1 (2+t)\,dt + (2+t)(-1+2t)(2\,dt) + 2(-5\,dt)\\ &= \int_0^1 (-12 + 7t + 4t^2)\,dt = \left\{-12t + \frac{7t^2}{2} + \frac{4t^3}{3}\right\}_0^1 = -\frac{43}{6}.\end{aligned}$$

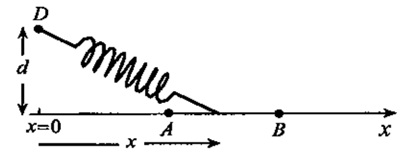
20. The line integral along C is equal to the sum of the line integrals along C_1, C_2, and C_3; that is,

$$\begin{aligned}&\int_C \frac{x^3}{(1+x^4)^3}dx + y^2 e^y\,dy + \frac{z}{\sqrt{1+z^2}}dz\\ &= \int_0^1 \frac{x^3}{(1+x^4)^3}dx + \int_{-1}^0 y^2 e^y\,dy\\ &\quad + \int_1^2 \frac{z}{\sqrt{1+z^2}}dz\\ &= \left\{\frac{-1}{8(1+x^4)^2}\right\}_0^1 + \left\{y^2 e^y - 2ye^y + 2e^y\right\}_{-1}^0\\ &\quad + \left\{\sqrt{1+z^2}\right\}_1^2\\ &= \frac{67}{32} + \sqrt{5} - \sqrt{2} - \frac{5}{e}.\end{aligned}$$



21. Since the spring is stretched an amount ℓ at A, its natural length is $L = \sqrt{a^2 + d^2} - \ell$.
Its stretch at x is therefore $\sqrt{x^2 + d^2} - L$.
The force necessary to counteract the spring at this position is

$$k(\sqrt{x^2 + d^2} - L)\frac{(x\hat{\mathbf{i}} - d\hat{\mathbf{j}})}{\sqrt{x^2 + d^2}}.$$



The work done by this force along that part C of the x-axis from A to B is

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b k(\sqrt{x^2 + d^2} - L)\left(\frac{x\hat{\mathbf{i}} - d\hat{\mathbf{j}}}{\sqrt{x^2 + d^2}}\right) \cdot (dx\,\hat{\mathbf{i}}) = k\int_a^b (\sqrt{x^2 + d^2} - L)\frac{x}{\sqrt{x^2 + d^2}}dx \\ &= k\int_a^b \left(x - \frac{Lx}{\sqrt{x^2 + d^2}}\right)dx = k\left\{\frac{x^2}{2} - L\sqrt{x^2 + d^2}\right\}_a^b \\ &= k\left[\left(\frac{b^2}{2} - L\sqrt{b^2 + d^2}\right) - \left(\frac{a^2}{2} - L\sqrt{a^2 + d^2}\right)\right]\end{aligned}$$

22. At position x, the magnitude of the force $\mathbf{F}_1$ of q_1 on q_3 is $|\mathbf{F}_1| = \dfrac{q_1 q_3}{4\pi\epsilon_0[(x-5)^2 + 25]}$. Since a unit vector in the direction of $\mathbf{F}_1$ is

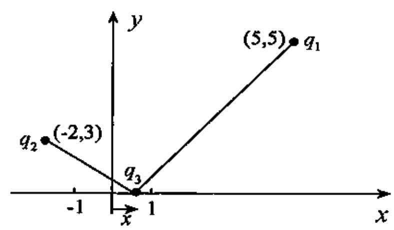
$$\hat{\mathbf{F}}_1 = \frac{(x-5, -5)}{\sqrt{(x-5)^2 + 25}},$$

it follows that

$$\mathbf{F}_1 = \frac{q_1 q_3}{4\pi\epsilon_0[(x-5)^2 + 25]^{3/2}}(x-5, -5).$$

Similarly, the force $\mathbf{F}_2$ of q_2 on q_3 is

$$\mathbf{F}_2 = \frac{q_2 q_3}{4\pi\epsilon_0[(x+2)^2 + 9]^{3/2}}(x+2, -3).$$



The work done by these forces is

$$\begin{aligned}W &= \int_C (\mathbf{F}_1 + \mathbf{F}_2) \cdot d\mathbf{r} = \int_C (\mathbf{F}_1 + \mathbf{F}_2) \cdot (dx\,\hat{\mathbf{i}}) \\ &= \int_C \left\{\frac{q_1 q_3}{4\pi\epsilon_0[(x-5)^2 + 25]^{3/2}}(x-5) + \frac{q_2 q_3}{4\pi\epsilon_0[(x+2)^2 + 9]^{3/2}}(x+2)\right\}dx \\ &= \int_{-1}^{1} \left\{\frac{q_1 q_3}{4\pi\epsilon_0[(-t-5)^2 + 25]^{3/2}}(-t-5) + \frac{q_2 q_3}{4\pi\epsilon_0[(2-t)^2 + 9]^{3/2}}(2-t)\right\}(-dt) \\ &= \left\{\frac{-q_1 q_3}{4\pi\epsilon_0\sqrt{(t+5)^2 + 25}} + \frac{-q_2 q_3}{4\pi\epsilon_0\sqrt{(2-t)^2 + 9}}\right\}_{-1}^{1} \\ &= \frac{1}{4\pi\epsilon_0}\left[q_1 q_3\left(\frac{1}{\sqrt{41}} - \frac{1}{\sqrt{61}}\right) + q_2 q_3\left(\frac{1}{3\sqrt{2}} - \frac{1}{\sqrt{10}}\right)\right].\end{aligned}$$

23. Using points A and D we obtain $k_2 = 12\,000$ and $k_1 = 20\,000$. Designating the four parts of the cycle starting at A by C_1, C_2, C_3, and C_4,

$$\begin{aligned}W &= \int_{C_1} P\,dV + \int_{C_2} P\,dV + \int_{C_3} P\,dV + \int_{C_4} P\,dV \\ &= \int_{3/5}^{1/5} \frac{k_2}{V}dV + 0 + \int_{1/5}^{3/5} \frac{k_1}{V}dV + 0 = (k_1 - k_2)\int_{1/5}^{3/5} \frac{1}{V}dV = (k_1 - k_2)\Big\{\ln V\Big\}_{1/5}^{3/5} = 8.8 \times 10^3 \text{ J}.\end{aligned}$$

24. Using points A and D we obtain $k_2 = 1000$ and $k_1 = 20\,000$. Designating the four parts of the cycle starting at A by C_1, C_2, C_3, and C_4,

$$W = \int_{C_1} P\,dV + \int_{C_2} P\,dV + \int_{C_3} P\,dV + \int_{C_4} P\,dV = \int_{1/10}^{1/100} \frac{k_2}{V}dV + \int_{1/100}^{1/5} 10^5\,dV + \int_{1/5}^{2} \frac{k_1}{V}dV + \int_{2}^{1/10} 10^4\,dV$$
$$= k_2\Big\{\ln V\Big\}_{1/10}^{1/100} + 10^5\left(\frac{1}{5} - \frac{1}{100}\right) + k_1\Big\{\ln V\Big\}_{1/5}^{2} + 10^4\left(\frac{1}{10} - 2\right) = 4.4 \times 10^4 \text{ J}.$$

25. Using points B and C we obtain $k_2 = 4.64$ and $k_1 = 6.89$. Designating the four parts of the cycle starting at A by C_1, C_2, C_3, and C_4,

$$W = \int_{C_1} P\,dV + \int_{C_2} P\,dV + \int_{C_3} P\,dV + \int_{C_4} P\,dV = \int_{8\times 10^{-4}}^{2\times 10^{-4}} \frac{k_2}{V^{1.4}}dV + 0 + \int_{2\times 10^{-4}}^{8\times 10^{-4}} \frac{k_1}{V^{1.4}}dV + 0$$
$$= (k_1 - k_2)\int_{2\times 10^{-4}}^{8\times 10^{-4}} \frac{1}{V^{1.4}}dV = (k_1 - k_2)\left\{\frac{-1}{0.4V^{0.4}}\right\}_{2\times 10^{-4}}^{8\times 10^{-4}} = 72 \text{ J}.$$

26. Using points A and D we obtain $k_2 = 15.0$ and $k_1 = 66.6$. Designating the four parts of the cycle starting at A by C_1, C_2, C_3, and C_4,

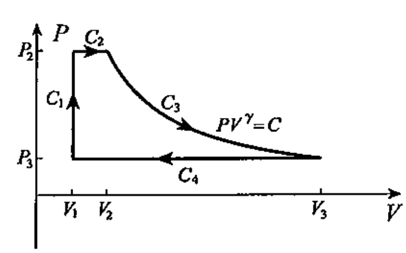
$$W = \int_{C_1} P\,dV + \int_{C_2} P\,dV + \int_{C_3} P\,dV + \int_{C_4} P\,dV$$
$$= \int_{20\times 10^{-4}}^{2\times 10^{-4}} \frac{k_2}{V^{1.4}}dV + \int_{2\times 10^{-4}}^{5.75\times 10^{-4}} (23 \times 10^5)\,dV + \int_{5.75\times 10^{-4}}^{20\times 10^{-4}} \frac{k_1}{V^{1.4}}dV + 0$$
$$= k_2\left\{\frac{-1}{0.4V^{0.4}}\right\}_{20\times 10^{-4}}^{2\times 10^{-4}} + 23 \times 10^5(5.75 \times 10^{-4} - 2 \times 10^{-4}) + k_1\left\{\frac{-1}{04.V^{0.4}}\right\}_{5.75\times 10^{-4}}^{20\times 10^{-4}} = 1.5 \times 10^3 \text{ J}.$$

27. Using points A and D we obtain $k_2 = 2.62$ and $k_1 = 32.2$. Designating the four parts of the cycle starting at A by C_1, C_2, C_3, and C_4,

$$W = \int_{C_1} P\,dV + \int_{C_2} P\,dV + \int_{C_3} P\,dV + \int_{C_4} P\,dV$$
$$= \int_{4\times 10^{-4}}^{10^{-4}} \frac{k_2}{V^{1.4}}dV + \int_{10^{-4}}^{6\times 10^{-4}} (10.4 \times 10^5)\,dV + \int_{6\times 10^{-4}}^{24\times 10^{-4}} \frac{k_1}{V^{1.4}}dV + \int_{24\times 10^{-4}}^{4\times 10^{-4}} (1.5 \times 10^5)\,dV$$
$$= k_2\left\{\frac{-1}{0.4V^{0.4}}\right\}_{4\times 10^{-4}}^{10^{-4}} + 10.4 \times 10^5(5 \times 10^{-4}) + k_1\left\{\frac{-1}{0.4V^{0.4}}\right\}_{6\times 10^{-4}}^{24\times 10^{-4}} + 1.5 \times 10^5(-20 \times 10^{-4})$$
$$= 7.8 \times 10^2 \text{ J}.$$

28. Designating the four parts of the cycle by C_1, C_2, C_3, and C_4,

$$W = \int_{C_1} P\,dV + \int_{C_2} P\,dV + \int_{C_3} P\,dV + \int_{C_4} P\,dV$$
$$= 0 + \int_{V_1}^{V_2} P_2\,dV + \int_{V_2}^{V_3} CV^{-\gamma}\,dV + \int_{V_3}^{V_2} P_3\,dV$$
$$= P_2(V_2 - V_1) + C\left\{\frac{V^{1-\gamma}}{1-\gamma}\right\}_{V_2}^{V_3} + P_3(V_2 - V_3)$$
$$= P_2(V_2 - V_1) + \frac{C}{1-\gamma}(V_3^{1-\gamma} - V_2^{1-\gamma}) + P_3(V_2 - V_3).$$



Since $P_2V_2^\gamma = C = P_3V_3^\gamma$, it follows that $P_3 = P_2(V_2/V_3)^\gamma$, and

$$W = P_2(V_2 - V_1) + \frac{P_2V_2^\gamma}{1-\gamma}(V_3^{1-\gamma} - V_2^{1-\gamma}) + P_2\left(\frac{V_2}{V_3}\right)^\gamma (V_2 - V_3).$$

29. $\displaystyle\int_C xy\,dx + xy^2\,dy = \int_0^2 \frac{x}{\sqrt{1+x^3}}dx + \frac{x}{1+x^3}\left[\frac{-3x^2}{2(1+x^3)^{3/2}}dx\right] = \int_0^2 \frac{2x(1+x^3)^2 - 3x^3}{2(1+x^3)^{5/2}}dx$

If we set $f(x) = [2x(1+x^3)^2 - 3x^3]/(1+x^3)^{5/2}$, and use Simpson's rule with 10 equal subdivisions,

$$\int_C xy\,dx + xy^2\,dy \approx \frac{1/5}{2(3)}[f(0) + 4f(1/5) + 2f(2/5) + \cdots + 2f(8/5) + 4f(9/5) + f(2)] = 0.8584.$$

30. When we use y as the parameter along the curve, the value of the line integral is

$$\begin{aligned} I &= \int_{-1}^1 y^2y^3(2y\,dy) + \tan y^2\,dy + e^{y^3}(3y^2\,dy) = \int_{-1}^1 (2y^6 + 3y^2e^{y^3} + \tan y^2)\,dy \\ &= \left\{\frac{2y^7}{7} + e^{y^3}\right\}_{-1}^1 + 2\int_0^1 \tan y^2\,dy. \end{aligned}$$

If we use Simpson's rule with 10 equal subdivisions on the remaining integral

$$\begin{aligned} I \approx \left(\frac{2}{7} + e\right) - \left(-\frac{2}{7} + e^{-1}\right) + \frac{1/5}{3}\big[&\tan(0) + 4\tan(0.01) + 2\tan(0.04) \\ &+ \cdots + 2\tan(0.64) + 4\tan(0.81) + \tan(1)\big] = 3.719. \end{aligned}$$

31. $\displaystyle\int_C \sqrt{1+y^2}\,dz + zy\,dy = \int_0^{\pi/2} \sqrt{1+\cos^6 t}(3\sin^2 t\cos t\,dt) + \cos^3 t\sin^3 t(-3\cos^2 t\sin t\,dt)$

$$= 3\int_0^{\pi/2} (\sin^2 t\cos t\sqrt{1+\cos^6 t} - \cos^5 t\sin^4 t)\,dt$$

If we set $f(t) = \sin^2 t\cos t\sqrt{1+\cos^6 t} - \cos^5 t\sin^4 t$, and use Simpson's rule with 10 equal subdivisions,

$$\begin{aligned}\int_C \sqrt{1+y^2}\,dz + zy\,dy &\approx \frac{3(\pi/20)}{3}[f(0) + 4f(\pi/20) + 2f(2\pi/20) + \cdots + 2f(8\pi/20) + 4f(9\pi/20) + f(\pi/2)] \\ &= 0.9934.\end{aligned}$$

32. $\displaystyle\int_C xyz\,dy = \int_{-1}^1 \left(\frac{1-t^2}{1+t^2}\right)\left[\frac{t(1-t^2)}{1+t^2}\right]t\left[\frac{(1+t^2)(1-3t^2) - (t-t^3)(2t)}{(1+t^2)^2}\right]dt$

$$= \int_{-1}^1 \frac{t^2(1-t^2)^2(1-4t^2-t^4)}{(1+t^2)^4}dt$$

If we denote the integrand by $f(t)$, then Simpson's rule with 10 equal subdivisions gives

$$\begin{aligned}\int_C xyz\,dy &\approx \frac{2/10}{3}[f(-1) + 4f(-0.8) + 2f(-0.6) + \cdots + 2f(0.6) + 4f(0.8) + f(1)] \\ &= -4.26 \times 10^{-4}.\end{aligned}$$

33. With $x = r\cos\theta = (1-\cos\theta)\cos\theta$ and $y = r\sin\theta = (1-\cos\theta)\sin\theta$,

$$\begin{aligned}\oint_C y\,dx &= \int_{-\pi}^{\pi} (1-\cos\theta)\sin\theta(-\sin\theta + 2\cos\theta\sin\theta)\,d\theta \\ &= \int_{-\pi}^{\pi} (-\sin^2\theta + 3\sin^2\theta\cos\theta - 2\sin^2\theta\cos^2\theta)\,d\theta \\ &= \int_{-\pi}^{\pi} \left(\frac{\cos 2\theta - 1}{2} + 3\sin^2\theta\cos\theta - \frac{\sin^2 2\theta}{2}\right)d\theta \\ &= \int_{-\pi}^{\pi} \left(\frac{\cos 2\theta - 1}{2} + 3\sin^2\theta\cos\theta - \frac{1-\cos 4\theta}{4}\right)d\theta \\ &= \left\{-\frac{3\theta}{4} + \frac{1}{4}\sin 2\theta + \sin^3\theta + \frac{1}{16}\sin 4\theta\right\}_{-\pi}^{\pi} = -\frac{3\pi}{2}\end{aligned}$$

34. Since $x = r\cos\theta$, $y = r\sin\theta$, and $r = \theta$, it follows that $x = \theta\cos\theta$ and $y = \theta\sin\theta$. Hence,

$$\begin{aligned}\int_C y\,dx + x\,dy &= \int_0^\pi \theta\sin\theta(\cos\theta - \theta\sin\theta)\,d\theta + \theta\cos\theta(\sin\theta + \theta\cos\theta)\,d\theta\\ &= \int_0^\pi [2\theta\,\sin\theta\,\cos\theta + \theta^2(\cos^2\theta - \sin^2\theta)]\,d\theta\\ &= \int_0^\pi [\theta\sin 2\theta + \theta^2\cos 2\theta]\,d\theta = \left\{\frac{\theta^2}{2}\sin 2\theta\right\}_0^\pi = 0.\end{aligned}$$

35. With parametric equations $x = r\cos t$, $y = r\sin t$, $z = 1$, $-\pi < t \le \pi$,

$$\text{(a) } \Gamma = \int_C \frac{x\,dx + y\,dy + z\,dz}{(x^2+y^2+z^2)^{3/2}} = \int_{-\pi}^{\pi} \frac{r\cos t(-r\sin t\,dt) + r\sin t(r\cos t\,dt)}{(r^2+1)^{3/2}} = 0$$

$$\text{(b) } \Gamma = \int_C -y\,dx + x\,dy = \int_{-\pi}^{\pi} -r\sin t(-r\sin t\,dt) + r\cos t(r\cos t\,dt) = r^2\int_{-\pi}^{\pi} dt = 2\pi r^2$$

36. When C: $x = x(t)$, $y = y(t)$, $z = z(t)$, $\alpha \le t \le \beta$ are parametric equations for C, then parametric equations for $-C$ are $x = x(t)$, $y = y(t)$, $z = z(t)$, $\beta \ge t \ge \alpha$. To obtain an increasing parameter along $-C$, we set $u = -t$, in which case

$$-C: \quad x = x(-u), \quad y = y(-u), \quad z = z(-u), \quad -\beta \le u \le -\alpha.$$

If $\mathbf{F} = P\hat{\mathbf{\i}} + Q\hat{\mathbf{\j}} + R\hat{\mathbf{k}}$, then the value of the line integral along $-C$ can be expressed as a definite integral with respect to u:

$$\begin{aligned}\int_{-C} \mathbf{F}\cdot d\mathbf{r} &= \int_{-C} P\,dx + Q\,dy + R\,dz\\ &= \int_{-\beta}^{-\alpha}\left\{P[x(-u),y(-u),z(-u)]\frac{dx}{du} + Q[x(-u),y(-u),z(-u)]\frac{dy}{du}\right.\\ &\qquad\left. + R[x(-u),y(-u),z(-u)]\frac{dz}{du}\right\}du.\end{aligned}$$

If we now change variables of integration by setting $t = -u$,

$$\frac{dx}{du} = \frac{dx}{dt}\frac{dt}{du} = -\frac{dx}{dt},$$

and similarly for dy/du and dz/du. Consequently,

$$\begin{aligned}\int_{-C} \mathbf{F}\cdot d\mathbf{r} &= \int_\beta^\alpha \left\{P[x(t),y(t),z(t)]\left(-\frac{dx}{dt}\right) + Q[x(t),y(t),z(t)]\left(-\frac{dy}{dt}\right)\right.\\ &\qquad\left. + R[x(t),y(t),z(t)]\left(-\frac{dz}{dt}\right)\right\}(-dt)\\ &= -\int_\alpha^\beta \left\{P[x(t),y(t),z(t)]\frac{dx}{dt} + Q[x(t),y(t),z(t)]\frac{dy}{dt}\right.\\ &\qquad\left. + R[x(t),y(t),z(t)]\frac{dz}{dt}\right\}dt\\ &= -\int_C \mathbf{F}\cdot d\mathbf{r}.\end{aligned}$$

37. For $\mathbf{F}$ to satisfy $f = \mathbf{F}\cdot\hat{\mathbf{T}}$, we must have $f = |\mathbf{F}|\cos\theta$. This leaves us the freedom to choose the magnitude $|\mathbf{F}|$ and the angle θ so that $f = |\mathbf{F}|\cos\theta$. This can be done in many ways so that $\mathbf{F}$ is not unique. For example, suppose C is that part of the x-axis from $x = a$ to $x = b$, and $f(x,y,z) > 0$ is given. Then for $f = |\mathbf{F}|\cos\theta$, one possible choice is $|\mathbf{F}| = f$ and $\theta = 0$. Another is $|\mathbf{F}| = \sqrt{2}f$ and $\theta = \pi/4$.

38. (a) Since the centre of the circle has coordinates $(R\theta, R)$, the direction of the unit force is

$$(R\theta - R\theta + R\sin\theta, R - R + R\cos\theta) = R(\sin\theta, \cos\theta).$$

Hence, the unit force has components $\mathbf{F} = (\sin\theta, \cos\theta)$. The work done during a half revolution is

$$\begin{aligned} W &= \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \sin\theta\,dx + \cos\theta\,dy = \int_0^\pi [\sin\theta(R - R\cos\theta)\,d\theta + \cos\theta(R\sin\theta)\,d\theta \\ &= R\int_0^\pi \sin\theta\,d\theta = R\Big\{-\cos\theta\Big\}_0^\pi = 2R. \end{aligned}$$

(b) The work done by the vertical component is

$$W = \int_C \cos\theta\hat{\mathbf{j}}\cdot d\mathbf{r} = \int_C \cos\theta\,dy = \int_0^\pi \cos\theta(R\sin\theta)\,d\theta = \left\{\frac{R}{2}\sin^2\theta\right\}_0^\pi = 0.$$

39. Suppose we divide the integral into three separate integrals

$$\oint_C f(x)\,dx + g(y)\,dy + h(z)\,dz = \oint_C f(x)\,dx + \oint_C g(y)\,dy + \oint_C h(z)\,dz.$$

To evaluate the first integral, we could evaluate an antiderivative $F(x)$ of $f(x)$ at initial and final points on the curve. But because the curve is closed, these are the same point, and the value is zero. The same is true for the second and third terms.

EXERCISES 14.4

1. Since $\nabla(x^2y^2/2) = xy^2\hat{\mathbf{i}} + x^2y\hat{\mathbf{j}}$, the line integral is independent of path in the xy-plane, and

$$\int_C xy^2\,dx + x^2y\,dy = \left\{\frac{x^2y^2}{2}\right\}_{(0,0,0)}^{(1,1,0)} = \frac{1}{2}.$$

2. Since $\nabla(x^3 + xy) = (3x^2 + y)\hat{\mathbf{i}} + x\hat{\mathbf{j}}$, the line integral is independent of path in space, and

$$\int_C (3x^2 + y)\,dx + x\,dy = \Big\{x^3 + xy\Big\}_{(2,1,5)}^{(-3,2,4)} = -43.$$

3. Since $\nabla(x^2e^y + 3y) = 2xe^y\hat{\mathbf{i}} + (x^2e^y + 3)\hat{\mathbf{j}}$, the line integral is independent of path in the xy-plane, and

$$\int_C 2xe^y\,dx + (x^2e^y + 3)\,dy = \Big\{x^2e^y + 3y\Big\}_{(1,0,0)}^{(-1,0,0)} = 1 - 1 = 0.$$

4. Since $\nabla(x^3yz - 2z^2) = 3x^2yz\hat{\mathbf{i}} + x^3z\hat{\mathbf{j}} + (x^3y - 4z)\hat{\mathbf{k}}$, the line integral is independent of path in space, and

$$\int_C 3x^2yz\,dx + x^3z\,dy + (x^3y - 4z)\,dz = \Big\{x^3yz - 2z^2\Big\}_{(-1,-1,1)}^{(1,1,-1)} = -2.$$

5. Since $\nabla\left(\frac{y}{z}\cos x\right) = \left(-\frac{y}{z}\sin x\right)\hat{\mathbf{i}} + \left(\frac{1}{z}\cos x\right)\hat{\mathbf{j}} - \left(\frac{y}{z^2}\cos x\right)\hat{\mathbf{k}}$, the line integral is independent of path in any domain not containing points in the xy-plane. Since C does not pass through the xy-plane,

$$\int_C -\frac{y}{z}\sin x\,dx + \frac{1}{z}\cos x\,dy - \frac{y}{z^2}\cos x\,dz = \left\{\frac{y}{z}\cos x\right\}_{(2,0,2\pi)}^{(2,0,4\pi)} = 0.$$

6. Since $\nabla(y\sin x) = y\cos x\,\hat{\mathbf{i}} + \sin x\,\hat{\mathbf{j}}$, the line integral is independent of path in the xy-plane, and

$$\oint_C y\cos x\,dx + \sin x\,dy = 0.$$

7. Since $\nabla(x^3/3 + y^3/3 + z^3/3) = x^2\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}$, the line integral is independent of path in space, and

$$\int_C x^2\,dx + y^2\,dy + z^2\,dz = \left\{\frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3}\right\}_{(-2,3,3)}^{(1,0,0)} = \left(\frac{1}{3}\right) - \left(-\frac{8}{3} + 9 + 9\right) = -15.$$

8. Since $\nabla(xy + z^2/2) = y\hat{\mathbf{i}} + x\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, the line integral is independent of path in space, and

$$\int_C y\,dx + x\,dy + z\,dz = \left\{xy + \frac{z^2}{2}\right\}_{(1,0,1)}^{(-1,2,5)} = 10.$$

9. Since $\nabla(x/y + z) = (1/y)\hat{\mathbf{i}} - (x/y^2)\hat{\mathbf{j}} + \hat{\mathbf{k}}$, the line integral is independent of path in any domain not containing points in the xz-plane. Since C does not pass through the xz-plane,

$$\int_C \frac{1}{y}dx - \frac{x}{y^2}dy + dz = \left\{\frac{x}{y} + z\right\}_{(0,1,1)}^{(3,10,-11)} = \left(\frac{3}{10} - 11\right) - 1 = -\frac{117}{10}.$$

10. Since $\nabla(x^3y^3) = 3x^2y^3\hat{\mathbf{i}} + 3x^3y^2\hat{\mathbf{j}}$, the line integral is independent of path in the xy-plane, and

$$\int_C 3x^2y^3\,dx + 3x^3y^2\,dy = \left\{x^3y^3\right\}_{(0,1)}^{(1,e)} = e^3.$$

11. Since $\nabla \times [f(x)\hat{\mathbf{i}} + g(y)\hat{\mathbf{j}} + h(z)\hat{\mathbf{k}}] = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f(x) & g(y) & h(z) \end{vmatrix} = \mathbf{0}$, the line integral is independent of path.

12. No. It may be independent of path. All we know is that Theorem 14.4 fails to imply independence of path.

13. Since $\nabla(ze^{xy} - z) = yze^{xy}\hat{\mathbf{i}} + xze^{xy}\hat{\mathbf{j}} + (e^{xy} - 1)\hat{\mathbf{k}}$, the line integral is independent of path in space, and

$$\int_C yze^{xy}\,dx + xze^{xy}\,dy + (e^{xy} - 1)\,dz = \left\{ze^{xy} - z\right\}_{(1,1,1)}^{(2,4,8)} = (8e^8 - 8) - (e - 1) = 8e^8 - e - 7.$$

14. Since $\nabla(xy\tan x + z) = y(\tan x + x\sec^2 x)\hat{\mathbf{i}} + x\tan x\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$, the line integral is certainly independent of path in the domain $-1.1 < x < 1.1$ containing the curve $x^2 + y^2 = 1$. Hence

$$\oint_C y(\tan x + x\sec^2 x)\,dx + x\tan x\,dy + dz = 0.$$

15. Since $\nabla\left(\frac{1 + y^2}{-2x^2} - \frac{y^2}{2} + \frac{z^2}{2}\right) = \left(\frac{1 + y^2}{x^3}\right)\hat{\mathbf{i}} + \left(-\frac{y}{x^2} - y\right)\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, the line integral is independent of path in any domain not containing points in the yz-plane. Since C does not pass through the yz-plane,

$$\int_C \left(\frac{1 + y^2}{x^3}\right)dx - \left(\frac{y + x^2y}{x^2}\right)dy + z\,dz = \left\{\frac{1 + y^2}{-2x^2} - \frac{y^2}{2} + \frac{z^2}{2}\right\}_{(1,0,0)}^{(5,2,1)} = -\frac{11}{10}.$$

16. Since $\nabla(xz/y) = (z/y)\hat{\mathbf{i}} - (xz/y^2)\hat{\mathbf{j}} + (x/y)\hat{\mathbf{k}}$, the line integral is independent of path in any domain which not containing points in the xz-plane. Hence

$$\oint_C \frac{zy\,dx - xz\,dy + xy\,dz}{y^2} = 0.$$

17. Since $\nabla\left(\frac{1}{x}\mathrm{Tan}^{-1}y\right) = \left(-\frac{1}{x^2}\mathrm{Tan}^{-1}y\right)\hat{\mathbf{i}} + \frac{1}{x(1 + y^2)}\hat{\mathbf{j}}$, the line integral is independent of path in any domain not containing points on the y-axis. Since C does not pass through this axis,

$$\int_C -\frac{1}{x^2}\mathrm{Tan}^{-1}y\,dx + \frac{1}{x + xy^2}dy = \left\{\frac{1}{x}\mathrm{Tan}^{-1}y\right\}_{(2,-1)}^{(10,3)} = \frac{1}{10}\mathrm{Tan}^{-1}3 + \frac{\pi}{8}.$$

18. Since $\nabla\left(\dfrac{-1}{(x-3)(y+5)}+\ln|z+4|\right)=\dfrac{1}{(x-3)^2(y+5)}\hat{\mathbf{i}}+\dfrac{1}{(x-3)(y+5)^2}\hat{\mathbf{j}}+\dfrac{1}{z+4}\hat{\mathbf{k}}$, the line integral is independent of path in any domain not containing points in the planes $x=3$, $y=-5$, and $z=-4$. Thus,

$$\int_C \frac{1}{(x-3)^2(y+5)}dx+\frac{1}{(x-3)(y+5)^2}dy+\frac{1}{z+4}dz=\left\{\frac{-1}{(x-3)(y+5)}+\ln|z+4|\right\}_{0,0,0)}^{(2,2,2)}$$
$$=\frac{8}{105}+\ln(3/2).$$

19. (a) With parametric equations $x=\cos t$, $y=\sin t$, $-\pi\le t\le\pi$,

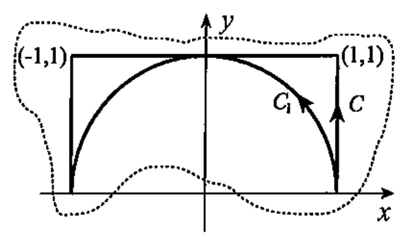
$$\oint_C \frac{-y\,dx+x\,dy}{x^2+y^2}=\int_{-\pi}^{\pi}-\sin t(-\sin t\,dt)+\cos t(\cos t\,dt)=\int_{-\pi}^{\pi}dt=2\pi.$$

(b) Since $\nabla\times\left(\dfrac{-y\hat{\mathbf{i}}+x\hat{\mathbf{j}}}{x^2+y^2}\right)=\mathbf{0}$ at every point except $(0,0)$, the line integral is independent of path in a domain that contains the circle (the circle does not contain $(0,0)$). The value of the line integral is therefore zero.

20. Since

$$\frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right)=\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$$

in the simply-connected domain shown, the line integral is independent of path therein. We may therefore replace C with the semicircle $C':\ x=\cos t,\ y=\sin t,\ 0\le t\le\pi$. Hence,



$$\int_C \frac{y}{x^2+y^2}dx-\frac{x}{x^2+y^2}dy=\int_{C'}\frac{y}{x^2+y^2}dx-\frac{x}{x^2+y^2}dy$$
$$=\int_0^{\pi}\frac{\sin t(-\sin t\,dt)-\cos t(\cos t\,dt)}{1}=-\int_0^{\pi}dt=-\pi.$$

21. Since $\nabla(\sqrt{x^2+y^2})=(x\hat{\mathbf{i}}+y\hat{\mathbf{j}})/\sqrt{x^2+y^2}$, the line integral is independent of path in any domain not containing the origin. A similar argument holds for the 3-space line integral.

22. A quick calculation shows that each of the partial derivatives $\dfrac{\partial}{\partial y}\left(\dfrac{y}{x^2+y^2}\right)$ and $\dfrac{\partial}{\partial x}\left(\dfrac{-x}{x^2+y^2}\right)$ is equal to $\dfrac{x^2-y^2}{(x^2+y^2)^2}$. These derivatives are equal in each of the domains specified. Since domains $x>0$, $x<0$, $y>0$, and $y<0$ are simply-connected, the line integral is independent of path therein.
The domain $x^2+y^2>0$ is not simply-connected. The line integral around $C:\ x=\cos t,\ y=\sin t,\ 0\le t\le 2\pi$ is

$$\oint_C \frac{y\,dx-x\,dy}{x^2+y^2}=\int_0^{2\pi}\sin t(-\sin t\,dt)-\cos t(\cos t\,dt)=-2\pi.$$

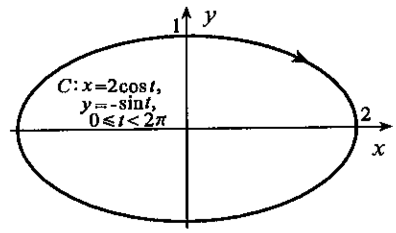
By Corollary 2 to Theorem 14.3, the line integral is not independent of path in $x^2+y^2>0$.

23. (a) With $dU=f'(T)\,dT$ and $P=nRT/V$, $\quad I=\displaystyle\int T^{-1}(dU+P\,dV)=\int\frac{f'(T)}{T}dT+\frac{nR}{V}dV.$

(b) When $f'(T)=k$, a constant, then in the TV-plane, $\quad\nabla(k\ln T+nR\ln V)=\dfrac{k}{T}\hat{\mathbf{T}}+\dfrac{nR}{V}\hat{\mathbf{V}}$, except when $T=0$ or $V=0$. Thus, $S=k\ln T+nR\ln V+S_0$, where S_0 is a constant.

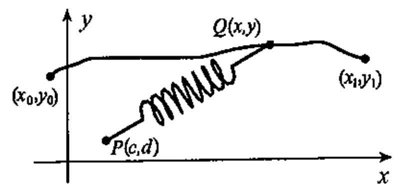
24. Since $\nabla(e^{x^2y}) = (2xye^{x^2y})\hat{\mathbf{i}} + x^2e^{x^2y}\hat{\mathbf{j}}$, the line integral $\int 2xye^{x^2y}\,dx + x^2e^{x^2y}dy$ is independent of path in space, and its value around the given curve is zero. The given line integral therefore reduces to

$$\begin{aligned}\oint_C x^2y\,dx &= \int_0^{2\pi} 4\cos^2 t(-\sin t)(-2\sin t\,dt)\\ &= 8\int_0^{2\pi}\left(\frac{\sin 2t}{2}\right)^2 dt\\ &= 2\int_0^{2\pi}\left(\frac{1-\cos 4t}{2}\right)dt = \left\{t - \frac{\sin 4t}{4}\right\}_0^{2\pi} = 2\pi.\end{aligned}$$



25. Suppose P has coordinates (c, d) and the unstretched length of the spring is L. At $Q(x, y)$, the force to counteract the spring is

$$\begin{aligned}\mathbf{F} &= k[\sqrt{(x-c)^2+(y-d)^2} - L]\widehat{\mathbf{PQ}}\\ &= k[\sqrt{(x-c)^2+(y-d)^2} - L]\left[\frac{(x-c)\hat{\mathbf{i}} + (y-d)\hat{\mathbf{j}}}{\sqrt{(x-c)^2+(y-d)^2}}\right].\end{aligned}$$



The work done by this force along C is

$$\begin{aligned}W = \int_C \mathbf{F}\cdot d\mathbf{r} &= k\int_C [\sqrt{(x-c)^2+(y-d)^2} - L]\frac{(x-c)\,dx + (y-d)\,dy}{\sqrt{(x-c)^2+(y-d)^2}}\\ &= k\int_C\left[1 - \frac{L}{\sqrt{(x-c)^2+(y-d)^2}}\right][(x-c)\,dx + (y-d)\,dy].\end{aligned}$$

Since

$$\begin{aligned}\nabla\left[\frac{1}{2}(x-c)^2 + \frac{1}{2}(y-d)^2 - L\sqrt{(x-c)^2+(y-d)^2}\right] &= \left[1 - \frac{L}{\sqrt{(x-c)^2+(y-d)^2}}\right](x-c)\hat{\mathbf{i}}\\ &\quad + \left[1 - \frac{L}{\sqrt{(x-c)^2+(y-d)^2}}\right](y-d)\hat{\mathbf{j}},\end{aligned}$$

the line integral is independent of path, and its value is

$$\begin{aligned}W &= k\left\{\frac{1}{2}(x-c)^2 + \frac{1}{2}(y-d)^2 - L\sqrt{(x-c)^2+(y-d)^2}\right\}_{(x_0,y_0)}^{(x_1,y_1)}\\ &= \frac{k}{2}\Big[(x_1-c)^2 + (y_1-d)^2 - 2L\sqrt{(x_1-c)^2+(y_1-d)^2} - (x_0-c)^2\\ &\qquad - (y_0-d)^2 + 2L\sqrt{(x_0-c)^2+(y_0-d)^2}\Big]\\ &= \frac{k}{2}\left[(x_1-c)^2 + (y_1-d)^2 - 2L\sqrt{(x_1-c)^2+(y_1-d)^2} + L^2\right]\\ &\qquad - \frac{k}{2}\left[(x_0-c)^2 + (y_0-d)^2 - 2L\sqrt{(x_0-c)^2+(y_0-d)^2} + L^2\right]\\ &= \frac{k}{2}[\sqrt{(x_1-c)^2+(y_1-d)^2} - L]^2 - \frac{k}{2}[(x_0-c)^2 + (y_0-d)^2 - L]^2\\ &= \frac{k}{2}(b^2 - a^2).\end{aligned}$$

26. (a) The curl of $\mathbf{F}$ is $\nabla \times \mathbf{F} = k \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \dfrac{x}{(x^2+y^2+z^2)^{3/2}} & \dfrac{y}{(x^2+y^2+z^2)^{3/2}} & \dfrac{z}{(x^2+y^2+z^2)^{3/2}} \end{vmatrix}$.

The x-component is k times $\dfrac{\partial}{\partial y}\left[\dfrac{z}{(x^2+y^2+z^2)^{3/2}}\right] - \dfrac{\partial}{\partial z}\left[\dfrac{y}{(x^2+y^2+z^2)^{3/2}}\right] = 0$. Similarly, the y- and z-components vanish. Hence, in any simply-connected domain that does not contain the origin, the line integral representing work done by $\mathbf{F}$ is independent of path.

(b) Since any two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ can be enclosed in a simply-connected domain not containing the origin, the line integral $\displaystyle\int_{P_1}^{P_2} \mathbf{F} \cdot d\mathbf{r}$ is independent of path, and its value is $\phi(P_2) - \phi(P_1)$ where $\phi(x, y, z)$ is any function satisfying $\nabla\phi = \mathbf{F}$. Since

$$\nabla\left(\frac{k}{|\mathbf{r}|}\right) = \frac{-k}{|\mathbf{r}|^2}\nabla|\mathbf{r}| = \frac{-k}{|\mathbf{r}|^2}\nabla\left(\sqrt{x^2+y^2+z^2}\right) = \frac{-k}{|\mathbf{r}|^2}\frac{1}{2\sqrt{x^2+y^2+z^2}}(2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}) = \frac{-k\hat{\mathbf{r}}}{|\mathbf{r}|^2},$$

it follows that $\displaystyle\int_{P_1}^{P_2} \mathbf{F} \cdot d\mathbf{r} = \left\{\frac{-k}{|\mathbf{r}|}\right\}_{P_1}^{P_2} = \frac{k}{d_1} - \frac{k}{d_2}$ where d_1 and d_2 are distances from P_1 and P_2 to the origin.

EXERCISES 14.5

1. Since $\nabla\left(\dfrac{-q_1 q_2}{4\pi\epsilon_0\sqrt{x^2+y^2+z^2}}\right) = \dfrac{q_1 q_2}{4\pi\epsilon_0(x^2+y^2+z^2)^{3/2}}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$, the force field is conservative in any domain not containing the origin. It is the electrostatic force between charges q_1 and q_2. A potential energy function is $V = \dfrac{q_1 q_2}{4\pi\epsilon_0\sqrt{x^2+y^2+z^2}}$.

2. Since $\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ mx & xy & 0 \end{vmatrix} = y\hat{\mathbf{k}}$, $\mathbf{F}$ is not conservative.

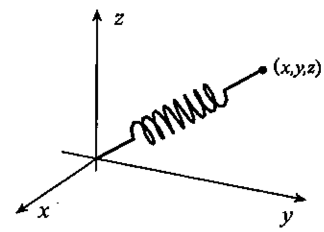
3. Since $\nabla(-kx^2/2) = -kx\hat{\mathbf{i}}$, the force field is conservative. It is that due to a spring with potential energy function $kx^2/2$.

4. Since $\nabla(-mgz) = \mathbf{F}$, $\mathbf{F}$ is conservative with potential energy function $U(z) = mgz$. $\mathbf{F}$ is the force of gravity on a mass m.

5. $\nabla\left(\dfrac{-GMm}{\sqrt{x^2+y^2+z^2}}\right) = \dfrac{GMm}{(x^2+y^2+z^2)^{3/2}}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$, the force field is conservative in any domain not containing the origin. It is the gravitational force between masses M and m. A potential energy function is $V = \dfrac{GMm}{\sqrt{x^2+y^2+z^2}}$.

6. A normal vector to the equipotential surface $U(x, y, z) = C$ at any point $P(x, y, z)$ on the surface is $\nabla(U - C) = \nabla U$. But $\mathbf{F} = -\nabla U$, and therefore $\mathbf{F}$ is normal to the surface at P.

7. For Exercises 1 and 5, they are spheres centred at the origin. In Exercise 4 they are planes parallel to the xy-plane.



8. The magnitude of the force is

$$|\mathbf{F}| = k\left(\sqrt{x^2+y^2+z^2} - L\right),$$

where k is the spring constant. Since $\mathbf{F}$ is directed toward the origin, it follows that

$$\mathbf{F} = -k\left(\sqrt{x^2+y^2+z^2} - L\right)\frac{(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})}{\sqrt{x^2+y^2+z^2}}.$$

Since $\nabla\left[-\frac{k}{2}\left(\sqrt{x^2+y^2+z^2} - L\right)^2\right] = \mathbf{F}$ in the domain $x^2+y^2+z^2 > 0$, it follows that the force is conservative.

9. Friction is not conservative because work done by (or against) friction depends on the path followed.

10. (a) Suppose we take the school at the origin. When a student is at position (x, y, z), the magnitude of the force is $|\mathbf{F}| = \dfrac{d}{x^2+y^2+z^2}$, where d a constant. Since the force is toward the origin,

$$\mathbf{F} = \frac{-d}{x^2+y^2+z^2}\frac{(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})}{\sqrt{x^2+y^2+z^2}} = \frac{-d(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})}{(x^2+y^2+z^2)^{3/2}}.$$

In the domain $x^2+y^2+z^2 > 10\,000$, $\nabla\left(\dfrac{d}{\sqrt{x^2+y^2+z^2}}\right) = \mathbf{F}$, and therefore $\mathbf{F}$ is conservative.

(b) Suppose the donut shop is at position (a, b, c). Then the force is

$$\mathbf{F} = \frac{-d}{[(x-a)^2+(y-b)^2+(z-c)^2]^{3/2}}[(x-a)\hat{\mathbf{i}} + (y-b)\hat{\mathbf{j}} + (z-c)\hat{\mathbf{k}}].$$

Since $\nabla\left(\dfrac{d}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}\right) = \mathbf{F}$, the force is still conservative.

11. (a) When a mass moves under the influence of a conservative force field, the sum of its potential energy $U(x)$ and its kinetic energy $K(x)$ is constant, $U + K = E$. Since $K = mv^2/2 = (1/2)m(dx/dt)^2$,

$$U + \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = E \quad \Longrightarrow \quad \frac{dx}{dt} = \sqrt{\frac{2}{m}(E-U)} \quad \Longrightarrow \quad \frac{1}{\sqrt{E-U(x)}}dx = \sqrt{\frac{2}{m}}dt.$$

This is a separated differential equation with solutions defined implicitly by

$$\int \frac{1}{\sqrt{E-U(x)}}dx = \sqrt{\frac{2}{m}}t + C.$$

In order to incorporate the initial condition $x(0) = x_0$, we rewrite the indefinite integral as a definite integral with a variable upper limit,

$$\int_0^x \frac{1}{\sqrt{E-U(x)}}dx = \sqrt{\frac{2}{m}}t + C.$$

The initial condition now implies that

$$\int_0^{x_0} \frac{1}{\sqrt{E-U(x)}}dx = C.$$

Thus,

$$\int_0^x \frac{1}{\sqrt{E-U(x)}}dx = \sqrt{\frac{2}{m}}t + \int_0^{x_0} \frac{1}{\sqrt{E-U(x)}}dx \quad \Longrightarrow \quad \int_{x_0}^x \frac{1}{\sqrt{E-U(x)}}dx = \sqrt{\frac{2}{m}}t.$$

(b) When $U(x) = kx^2/2$,

$$\sqrt{\frac{2}{m}}t = \int_{x_0}^{x} \frac{1}{\sqrt{E - kx^2/2}}dx = \sqrt{2}\int_{x_0}^{x} \frac{1}{\sqrt{2E - kx^2}}dx.$$

If we set $x = \sqrt{2E/k}\sin\theta$, and $dx = \sqrt{2E/k}\cos\theta\, d\theta$, then

$$\sqrt{2}\int \frac{1}{\sqrt{2E - kx^2}}dx = \sqrt{2}\int \frac{\sqrt{2E/k}\cos\theta}{\sqrt{2E}\cos\theta}d\theta = \sqrt{\frac{2}{k}}\theta + C = \sqrt{\frac{2}{k}}\mathrm{Sin}^{-1}\left(\sqrt{\frac{k}{2E}}x\right) + C.$$

Thus,

$$\sqrt{\frac{2}{m}}t = \left\{\sqrt{\frac{2}{k}}\mathrm{Sin}^{-1}\left(\sqrt{\frac{k}{2E}}x\right)\right\}_{x_0}^{x} = \sqrt{\frac{2}{k}}\left[\mathrm{Sin}^{-1}\left(\sqrt{\frac{k}{2E}}x\right) - \mathrm{Sin}^{-1}\left(\sqrt{\frac{k}{2E}}x_0\right)\right].$$

This implies that

$$\mathrm{Sin}^{-1}\left(\sqrt{\frac{k}{2E}}x\right) = \mathrm{Sin}^{-1}\left(\sqrt{\frac{k}{2E}}x_0\right) + \sqrt{\frac{k}{m}}t,$$

and if we take sines of both sides, we obtain

$$\begin{aligned} x &= \sqrt{\frac{2E}{k}}\sin\left[\mathrm{Sin}^{-1}\left(\sqrt{\frac{k}{2E}}x_0\right) + \sqrt{\frac{k}{m}}t\right] \\ &= \sqrt{\frac{2E}{k}}\left[\sqrt{\frac{k}{2E}}x_0\cos\sqrt{\frac{k}{m}}t + \sqrt{1 - \frac{kx_0^2}{2E}}\sin\sqrt{\frac{k}{m}}t\right] \\ &= x_0\cos\sqrt{\frac{k}{m}}t + \sqrt{\frac{2E}{k} - x_0^2}\sin\sqrt{\frac{k}{m}}t. \end{aligned}$$

This describes simple harmonic motion for the mass, as we would expect. When $v(0) = 0$, total energy is $E = kx_0^2/k$, and $x = x_0\cos\sqrt{\frac{k}{m}}t$.

12. (a) The curl of $\mathbf{F}$ is

$$\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f(r)x & f(r)y & f(r)z \end{vmatrix} = \left(z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z}\right)\hat{\mathbf{i}} + \left(x\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial x}\right)\hat{\mathbf{j}} + \left(y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}\right)\hat{\mathbf{k}}.$$

The x-component is

$$z\left(\frac{df}{dr}\frac{\partial r}{\partial y}\right) - y\left(\frac{df}{dr}\frac{\partial r}{\partial z}\right) = \frac{df}{dr}\left(\frac{zy}{\sqrt{x^2+y^2+z^2}} - \frac{yz}{\sqrt{x^2+y^2+z^2}}\right) = 0.$$

Similarly, the y- and z-components of $\nabla\times\mathbf{F}$ vanish, and $\nabla\times\mathbf{F} = \mathbf{0}$. According to Theorem 14.4, $\mathbf{F}$ is conservative in any simply-connected domain that does not contain the origin.

(b) $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C f(r)\mathbf{r}\cdot d\mathbf{r} = \int_C f(r)(x\,dx + y\,dy + z\,dz)$

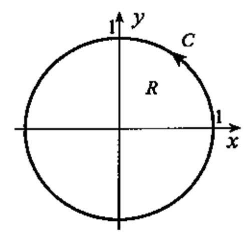
Since $r = \sqrt{x^2+y^2+z^2}$, it follows that

$$dr = \frac{x\,dx}{\sqrt{x^2+y^2+z^2}} + \frac{y\,dy}{\sqrt{x^2+y^2+z^2}} + \frac{z\,dz}{\sqrt{x^2+y^2+z^2}} = \frac{x\,dx + y\,dy + z\,dz}{r},$$

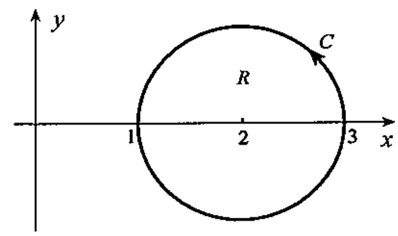
and therefore $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b f(r)r\,dr.$

(c) The electrostatic force in Example 14.14 and the gravitational force in Exercise 5 are radially symmetric.

EXERCISES 14.6

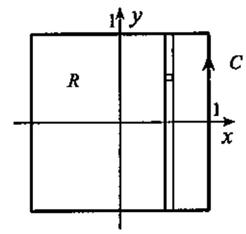


1. With Green's theorem, $\oint_C y^2\,dx + x^2\,dy = \iint_R (2x - 2y)\,dA = 0$ because x and y are odd functions, and the circle is symmetric about the x- and y-axes.



2. With Green's theorem,

$$\oint_C (x^2 + 2y^2)\,dy = \iint_R 2x\,dA$$
$$= 2(\text{First moment of } R \text{ about } y\text{-axis})$$
$$= 2(\text{Area of R})(\overline{x}) = 2(\pi)(2) = 4\pi.$$

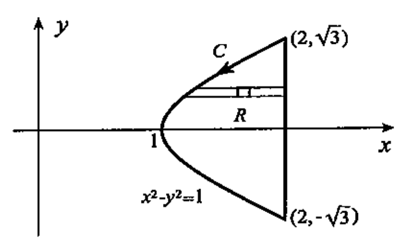


3. With Green's theorem,

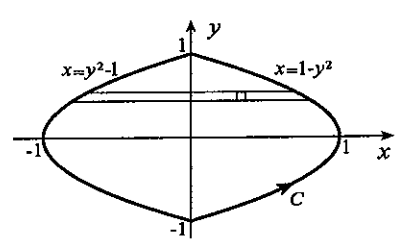
$$\oint_C x^2e^y\,dx + (x+y)\,dy = -\iint_R (1 - x^2e^y)\,dA$$
$$= -\int_{-1}^{1}\int_{-1}^{1} (1 - x^2e^y)\,dy\,dx = -\int_{-1}^{1} \left\{y - x^2e^y\right\}_{-1}^{1} dx$$
$$= -\int_{-1}^{1} [2 - (e - e^{-1})x^2]\,dx$$
$$= -\left\{2x - (e - e^{-1})\frac{x^3}{3}\right\}_{-1}^{1} = -4 + \frac{2}{3}(e - e^{-1}).$$

4. By Green's theorem,

$$\oint_C xy^3\,dx + x^2\,dy = \iint_R (2x - 3xy^2)\,dA = \int_{-\sqrt{3}}^{\sqrt{3}}\int_{\sqrt{1+y^2}}^{2} (2x - 3xy^2)\,dx\,dy$$
$$= \int_{-\sqrt{3}}^{\sqrt{3}} \left\{x^2 - \frac{3x^2y^2}{2}\right\}_{\sqrt{1+y^2}}^{2} dy$$
$$= \int_{-\sqrt{3}}^{\sqrt{3}} \left(3 - \frac{11y^2}{2} + \frac{3y^4}{2}\right) dy$$
$$= \left\{3y - \frac{11y^3}{6} + \frac{3y^5}{10}\right\}_{-\sqrt{3}}^{\sqrt{3}} = \frac{2\sqrt{3}}{5}.$$

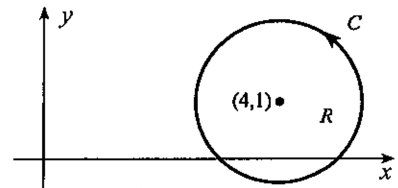


5. By Green's theorem,



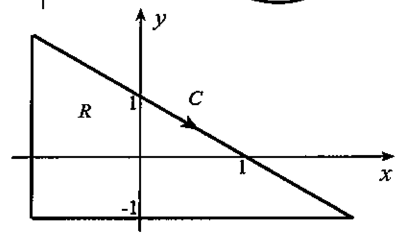
$$\oint_C (x^3 + y^3)\,dx + (x^3 - y^3)\,dy = \iint_R (3x^2 - 3y^2)\,dA$$
$$= 6\int_0^1 \int_{y^2-1}^{1-y^2} (x^2 - y^2)\,dx\,dy$$
$$= 6\int_0^1 \left\{\frac{x^3}{3} - xy^2\right\}_{y^2-1}^{1-y^2} dy$$
$$= 2\int_0^1 [(1-y^2)^3 - 3y^2(1-y^2) - (y^2-1)^3 + 3y^2(y^2-1)]\,dy$$
$$= 2\int_0^1 \{(2 - 12y^2 + 12y^4 - 2y^6)\,dy = 2\left\{2y - 4y^3 + \frac{12y^5}{5} - \frac{2y^7}{7}\right\}_0^1 = \frac{8}{35}$$

6. By Green's theorem,



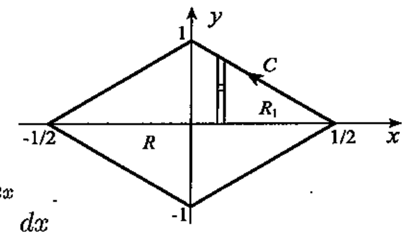
$$\oint_C 2\mathrm{Tan}^{-1}\left(\frac{y}{x}\right)dx + \ln(x^2+y^2)\,dy$$
$$= \iint_R \left[\frac{2x}{x^2+y^2} - \frac{2}{1+y^2/x^2}\left(\frac{1}{x}\right)\right] dA$$
$$= 0.$$

7. By Green's theorem,



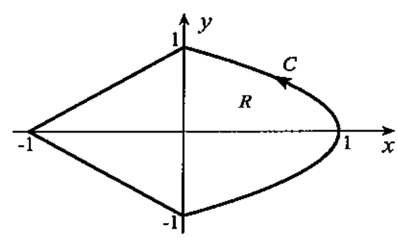
$$\oint_C (3x^2y^3 + y)\,dx + (3x^3y^2 + 2x)\,dy$$
$$= -\iint_R (9x^2y^2 + 2 - 9x^2y^2 - 1)\,dA$$
$$= -\iint_R dA = -\frac{1}{2}(3)(3) = -\frac{9}{2}$$

8. By Green's theorem,



$$\oint_C (x^3+y^3)\,dx + (x^3-y^3)\,dy = \iint_R (3x^2 - 3y^2)\,dA$$
$$= 12\iint_{R_1}(x^2-y^2)\,dA \quad (R_1 = \text{first quadrant part of } R)$$
$$= 12\int_0^{1/2}\int_0^{1-2x}(x^2-y^2)\,dy\,dx = 12\int_0^{1/2}\left\{x^2y - \frac{y^3}{3}\right\}_0^{1-2x} dx$$
$$= 4\int_0^{1/2}[3x^2 - 6x^3 - (1-2x)^3]\,dx = 4\left\{x^3 - \frac{3x^4}{2} + \frac{(1-2x)^4}{8}\right\}_0^{1/2} = -\frac{3}{8}.$$

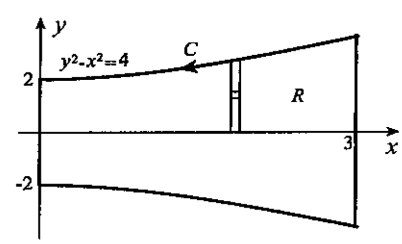
9. By Green's theorem,



$$\oint_C (x^2y^2 + 3x)\,dx + (2xy - y)\,dy = \iint_R (2y - 2x^2y)\,dA$$

The integral has value zero because $2y - 2x^2y$ is an odd function of y and R is symmetric about the x-axis.

10. By Green's theorem,



$$\oint_C (xy^2 + 2x)\,dx + (x^2y + y + x^2)\,dy$$
$$= \iint_R (2xy + 2x - 2xy)\,dA$$
$$= 2(2)\int_0^3\int_0^{\sqrt{4+x^2}} x\,dy\,dx = 4\int_0^3\left\{xy\right\}_0^{\sqrt{4+x^2}} dx$$
$$= 4\int_0^3 x\sqrt{4+x^2}\,dx = 4\left\{\frac{(4+x^2)^{3/2}}{3}\right\}_0^3 = \frac{4}{3}(13\sqrt{13} - 8).$$

11. Green's theorem cannot be used since $x/(x^2+y^2)$ and $y/(x^2+y^2)$ are not continuous at $(0,0)$. With the parametric equations, $x = \cos t$, $y = \sin t$, $-\pi \le t \le \pi$,

$$\oint_C \frac{-y\,dx + x\,dy}{x^2+y^2} = \int_{-\pi}^{\pi} -\sin t(-\sin t\,dt) + \cos t(\cos t\,dt) = \int_{-\pi}^{\pi} dt = 2\pi.$$

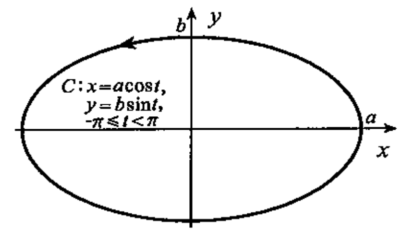
12. Since $\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}$,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_R (\nabla\times\mathbf{F})\cdot\hat{\mathbf{k}}\,dA.$$

13. If we set $\mathbf{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}}$ and $\hat{\mathbf{n}} = (dy, -dx)/ds$, then

$$\oint_C \mathbf{F} \cdot \hat{\mathbf{n}}\, ds = \oint_C (P\hat{\mathbf{i}} + Q\hat{\mathbf{j}}) \cdot \frac{(dy, -dx)}{ds} ds = \oint_C -Q\, dx + P\, dy = \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA = \iint_R \nabla \cdot \mathbf{F}\, dA.$$

14.
$$\begin{aligned} A &= \frac{1}{2}\oint_C x\, dy - y\, dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi} [a\cos t(b\cos t\, dt) - b\sin t(-a\sin t\, dt)] \\ &= \frac{ab}{2}\int_{-\pi}^{\pi} dt = \pi ab \end{aligned}$$

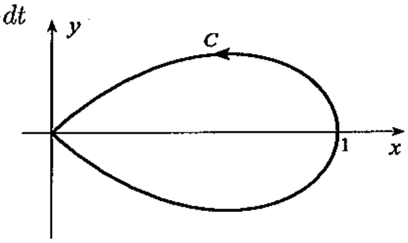


15. Since $\dfrac{dx}{dt} = \dfrac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2} = \dfrac{-4t}{(1+t^2)^2}$, the area enclosed by the strophoid is

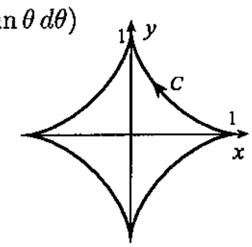
$$A = -\oint_C y\, dx = -\int_{-1}^{1} \left[\frac{t(1-t^2)}{1+t^2}\right]\left[\frac{-4t}{(1+t^2)^2}\right] dt = 4\int_{-1}^{1} \frac{t^2(1-t^2)}{(1+t^2)^3} dt$$

If we set $t = \tan\theta$ and $dt = \sec^2\theta\, d\theta$,

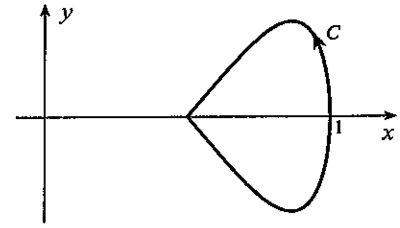
$$\begin{aligned} A &= 4\int_{-\pi/4}^{\pi/4} \frac{\tan^2\theta(1-\tan^2\theta)}{\sec^6\theta}(\sec^2\theta\, d\theta) \\ &= 4\int_{-\pi/4}^{\pi/4} (\sin^2\theta\cos^2\theta - \sin^4\theta)\, d\theta \\ &= 4\int_{-\pi/4}^{\pi/4} \sin^2\theta(\cos^2\theta - \sin^2\theta)\, d\theta = 4\int_{-\pi/4}^{\pi/4} \left(\frac{1-\cos 2\theta}{2}\right)\cos 2\theta\, d\theta \\ &= 2\int_{-\pi/4}^{\pi/4} \left(\cos 2\theta - \frac{1+\cos 4\theta}{2}\right) d\theta = 2\left\{\frac{1}{2}\sin 2\theta - \frac{\theta}{2} - \frac{1}{8}\sin 4\theta\right\}_{-\pi/4}^{\pi/4} = \frac{4-\pi}{2}. \end{aligned}$$



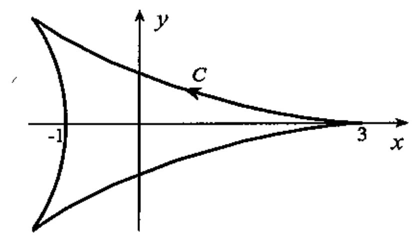
16.
$$\begin{aligned} A &= \frac{1}{2}\oint_C x\, dy - y\, dx = \frac{1}{2}\int_0^{2\pi} \cos^3\theta(3\sin^2\theta\,\cos\theta\, d\theta) - \sin^3\theta(-3\cos^2\theta\,\sin\theta\, d\theta) \\ &= \frac{3}{2}\int_0^{2\pi} (\cos^4\theta\,\sin^2\theta + \sin^4\theta\,\cos^2\theta)\, d\theta \\ &= \frac{3}{2}\int_0^{2\pi} \cos^2\theta\,\sin^2\theta\, d\theta = \frac{3}{2}\int_0^{2\pi} \frac{1}{4}\sin^2 2\theta\, d\theta \\ &= \frac{3}{8}\int_0^{2\pi} \left(\frac{1-\cos 4\theta}{2}\right) d\theta = \frac{3}{16}\left\{\theta - \frac{\sin 4\theta}{4}\right\}_0^{2\pi} = \frac{3\pi}{8} \end{aligned}$$



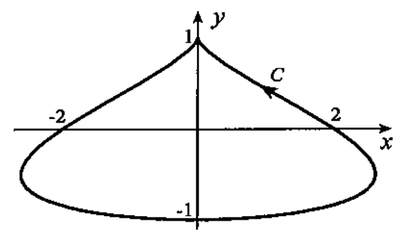
17.
$$\begin{aligned} A &= \oint_C x\, dy = \int_{-\pi/3}^{\pi/3} \cos\theta(3\cos 3\theta\, d\theta) \\ &= 3\int_{-\pi/3}^{\pi/3} \frac{1}{2}(\cos 4\theta + \cos 2\theta)\, d\theta \\ &= \frac{3}{2}\left\{\frac{1}{4}\sin 4\theta + \frac{1}{2}\sin 2\theta\right\}_{-\pi/3}^{\pi/3} = \frac{3\sqrt{3}}{8} \end{aligned}$$



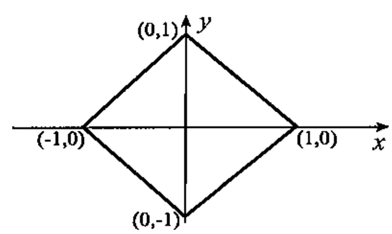
18. $$\begin{aligned} A &= \frac{1}{2}\oint_C x\,dy - y\,dx \\ &= \frac{1}{2}\int_0^{2\pi} (2\cos t + \cos 2t)(2\cos t - 2\cos 2t)\,dt \\ &\qquad -(2\sin t - \sin 2t)(-2\sin t - 2\sin 2t)\,dt \\ &= \int_0^{2\pi} (2\cos^2 t - \cos t\,\cos 2t - \cos^2 2t + 2\sin^2 t \\ &\qquad + \sin t\,\sin 2t - \sin^2 2t)\,dt \\ &= \int_0^{2\pi} (1 - \cos 3t)\,dt = \left\{ t - \frac{\sin 3t}{3} \right\}_0^{2\pi} = 2\pi \end{aligned}$$



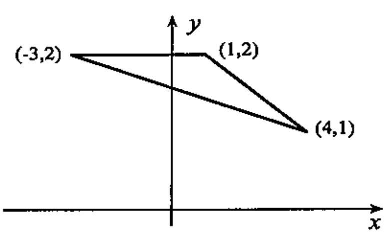
19. $$\begin{aligned} A &= \oint_C x\,dy = \int_{-\pi}^{\pi} (2\cos t - \sin 2t)(\cos t\,dt) \\ &= \int_{-\pi}^{\pi} (1 + \cos 2t - 2\sin t\cos^2 t)\,dt \\ &= \left\{ t + \frac{1}{2}\sin 2t + \frac{2}{3}\cos^3 t \right\}_{-\pi}^{\pi} = 2\pi \end{aligned}$$



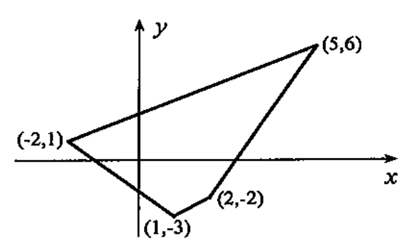
20. $$A = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$$
$$= \frac{1}{2}[(1+1) - (-1-1)] = 2$$



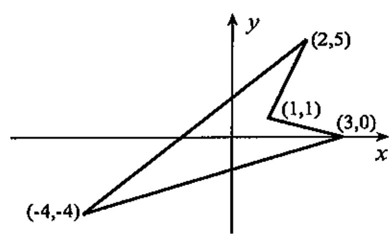
21. $$A = \frac{1}{2}\begin{bmatrix} 1 & 2 \\ -3 & 2 \\ 4 & 1 \\ 1 & 2 \end{bmatrix}$$
$$= \frac{1}{2}[(2 - 3 + 8) - (-6 + 8 + 1)] = 2$$



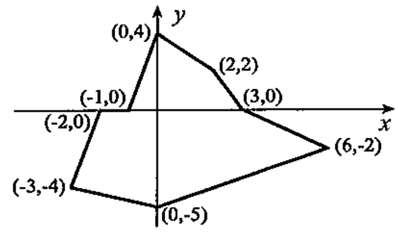
22. $$A = \frac{1}{2}\begin{bmatrix} 2 & -2 \\ 5 & 6 \\ -2 & 1 \\ 1 & -3 \\ 2 & -2 \end{bmatrix}$$
$$= \frac{1}{2}[(12 + 5 + 6 - 2) - (-10 - 12 + 1 - 6)] = 24$$



23. $$A = \frac{1}{2}\begin{bmatrix} 3 & 0 \\ 1 & 1 \\ 2 & 5 \\ -4 & -4 \\ 3 & 0 \end{bmatrix}$$
$$= \frac{1}{2}[(3 + 5 - 8) - (2 - 20 - 12)] = 15$$

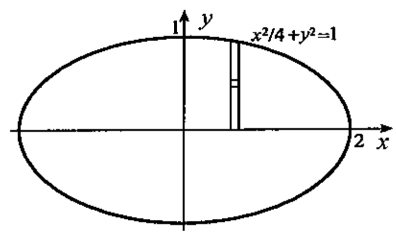


24. $A = \dfrac{1}{2}\begin{vmatrix} 0 & 4 \\ -1 & 0 \\ -2 & 0 \\ -3 & -4 \\ 0 & -5 \\ 6 & -2 \\ 3 & 0 \\ 2 & 2 \\ 0 & 4 \end{vmatrix} = \dfrac{1}{2}[(8+15+6+8)-(-4-30-6)] = 77/2$



25. By Green's theorem,

$$\begin{aligned}
&\oint_C (2xye^{x^2y} + 3x^2y)\,dx + x^2e^{x^2y}\,dy \\
&\quad = \iint_R (2xe^{x^2y} + 2x^3ye^{x^2y} - 2xe^{x^2y} - 2x^3ye^{x^2y} - 3x^2)\,dA \\
&\quad = -12\int_0^2\int_0^{(1/2)\sqrt{4-x^2}} x^2\,dy\,dx = -12\int_0^2 \left\{x^2y\right\}_0^{(1/2)\sqrt{4-x^2}} dx \\
&\quad = -6\int_0^2 x^2\sqrt{4-x^2}\,dx
\end{aligned}$$

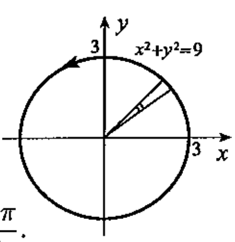


If we set $x = 2\sin\theta$ and $dx = 2\cos\theta\,d\theta$,

$$\begin{aligned}
\oint_C (2xye^{x^2y} + 3x^2y)\,dx + x^2e^{x^2y}\,dy &= -6\int_0^{\pi/2} 4\sin^2\theta(2\cos\theta)(2\cos\theta\,d\theta) = -96\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^2 d\theta \\
&= -24\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right)d\theta = -12\left\{\theta - \frac{1}{8}\sin 4\theta\right\}_0^{\pi/2} = -6\pi.
\end{aligned}$$

26. By Green's theorem,

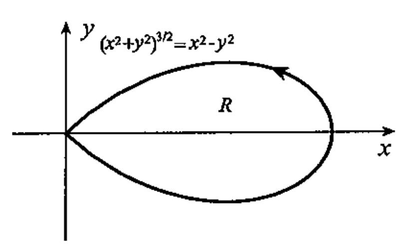
$$\begin{aligned}
&\oint_C (3x^2y^3 - x^2y)\,dx + (xy^2 + 3x^3y^2)\,dy \\
&\quad = \iint_R (y^2 + 9x^2y^2 - 9x^2y^2 + x^2)\,dA = \iint_R (x^2+y^2)\,dA \\
&\quad = 4\int_0^{\pi/2}\int_0^3 r^2\,r\,dr\,d\theta = 4\int_0^{\pi/2}\left\{\frac{r^4}{4}\right\}_0^3 d\theta = 81\left\{\theta\right\}_0^{\pi/2} = \frac{81\pi}{2}.
\end{aligned}$$



27. By Green's theorem,

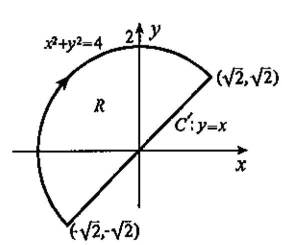
$$\oint_C -x^3y^2\,dx + x^2y^3\,dy = \iint_R (2xy^3 + 2x^3y)\,dA$$

Since the integrand is an odd function of y and R is symmetric about the x-axis, the integral has value zero.



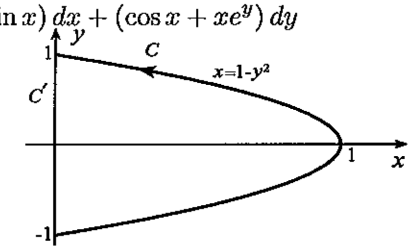
28. Since $\displaystyle\int_{C'} (x-y)(dx+dy) = 0$, we may write

$$\begin{aligned}
\int_C (x-y)(dx+dy) &= \oint_{C+C'} (x-y)(dx+dy) \\
&= -\iint_R (1+1)\,dA \\
&= -2(\text{Area of } R) = -2(2\pi) = -4\pi.
\end{aligned}$$



29. If C' is the y-axis from $(0,1)$ to $(0,-1)$, then

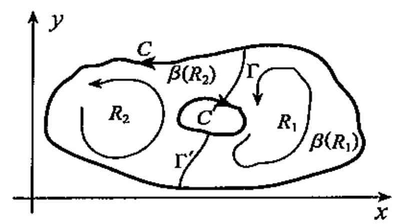
$$\begin{aligned}
&\int_C (e^y - y\sin x)\,dx + (\cos x + xe^y)\,dy \\
&\quad = \oint_{C'+C} (e^y - y\sin x)\,dx + (\cos x + xe^y)\,dy - \int_{C'} (e^y - y\sin x)\,dx + (\cos x + xe^y)\,dy \\
&\quad = \iint_R (-\sin x + e^y - e^y + \sin x)\,dA \\
&\qquad + \int_{-C'} (e^y - y\sin x)\,dx + (\cos x + xe^y)\,dy \\
&\quad = \int_{-1}^{1} dy = 2.
\end{aligned}$$



30. (a) If we draw curves Γ and Γ' as shown, then P and Q have continuous first partial derivatives in a domain that contains R_1 and its boundary, and also in a domain that contains R_2 and its boundary. If we apply Green's theorem to these regions, denoting their boundaries by $\beta(R_1)$ and $\beta(R_2)$,

$$\oint_{\beta(R_1)} P\,dx + Q\,dy = \iint_{R_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA,$$

$$\oint_{\beta(R_2)} P\,dx + Q\,dy = \iint_{R_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$



When these results are added,

$$\oint_{\beta(R_1)} P\,dx + Q\,dy + \oint_{\beta(R_2)} P\,dx + Q\,dy = \iint_{R_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint_{R_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Now $R_1 + R_2 = R$. Furthermore, tracing $\beta(R_1)$ and $\beta(R_2)$ in the directions indicated is equivalent to tracing C and C' in the directions indicated, plus Γ and Γ' each traversed once in one direction and then in the reverse direction. Consequently,

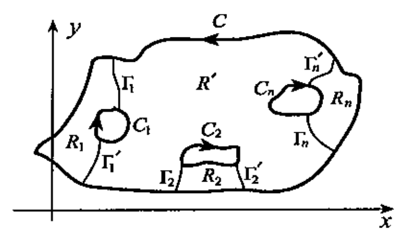
$$\oint_C P\,dx + Q\,dy + \oint_{C'} P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

(b) In this case we draw curves Γ_i and Γ_i' from C_i to C as shown. This divides R into regions R_i and R' to which we apply Green's theorem,

$$\oint_{\beta(R_i)} P\,dx + Q\,dy = \iint_{R_i} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

for $i = 1, \ldots, n$, and

$$\oint_{\beta(R')} P\,dx + Q\,dy = \iint_{R'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$



Since $R_1 + R_2 + \cdots + R_n + R' = R$, and traversing the $\beta(R_i)$ and $\beta(R')$ is equivalent to traversing C and the C_i in the directions shown, when we add these equations we obtain

$$\oint_C P\,dx + Q\,dy + \sum_{i=1}^{n} \oint_{C_i} P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

(c) If $\partial Q/\partial x = \partial P/\partial y$ in R, then in part (a), $\displaystyle\oint_C P\,dx + Q\,dy = \oint_{-C'} P\,dx + Q\,dy$,

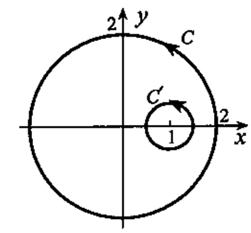
and in part (b), $\displaystyle\oint_C P\,dx + Q\,dy = \sum_{i=1}^{n} \oint_{-C_i} P\,dx + Q\,dy.$

31. The circle C': $(x-1)^2+y^2=1/4$ is interior to C, and everywhere except at $(1,0)$,

$$\frac{\partial}{\partial x}\left[\frac{-(x-1)}{(x-1)^2+y^2}\right]=\frac{\partial}{\partial y}\left[\frac{y}{(x-1)^2+y^2}\right].$$

Consequently, from Exercise 30(c), (and using the parametric equations $C': x=1+(1/2)\cos t,\ y=(1/2)\sin t,\ -\pi<t\le\pi$),

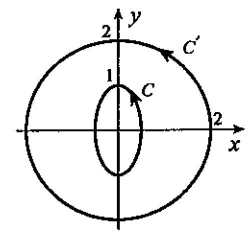
$$\begin{aligned}\oint_C \frac{y\,dx-(x-1)\,dy}{(x-1)^2+y^2} &= \oint_{C'}\frac{y\,dx-(x-1)\,dy}{(x-1)^2+y^2}\\ &=\int_{-\pi}^{\pi}\frac{(1/2)\sin t(-(1/2)\sin t\,dt)-(1/2)\cos t(1/2)\cos t\,dt}{1/4}\\ &=\int_{-\pi}^{\pi}-dt=-2\pi.\end{aligned}$$



32. The circle C': $x^2+y^2=4$ encloses C, and everywhere except at $(0,0)$,

$$\frac{\partial}{\partial x}\left[\frac{x^3}{(x^2+y^2)^2}\right]=\frac{\partial}{\partial y}\left[\frac{-x^2y}{(x^2+y^2)^2}\right].$$

Hence, by Exercise 30(c), (and using the parametric equations $C': x=2\cos t,\ y=2\sin t,\ -\pi<t\le\pi$),



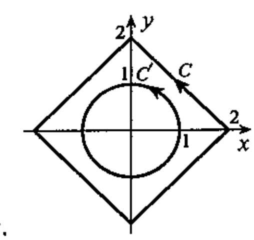
$$\begin{aligned}\oint_C\frac{-x^2y\,dx+x^3\,dy}{(x^2+y^2)^2}&=\oint_{C'}\frac{-x^2y\,dx+x^3\,dy}{(x^2+y^2)^2}\\ &=\int_{-\pi}^{\pi}\frac{-4\cos^2 t\,2\sin t(-2\sin t\,dt)+(2\cos t)^3(2\cos t\,dt)}{16}\\ &=\int_{-\pi}^{\pi}(\cos^2 t\,\sin^2 t+\cos^4 t)\,dt=\int_{-\pi}^{\pi}\cos^2 t\,dt\\ &=\int_{-\pi}^{\pi}\left(\frac{1+\cos 2t}{2}\right)dt=\frac{1}{2}\left\{t+\frac{\sin 2t}{2}\right\}_{-\pi}^{\pi}=\pi.\end{aligned}$$

33. The circle C': $x^2+y^2=1$ is interior to C, and everywhere except at $(0,0)$,

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)=\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right).$$

Hence, by Exercise 30(c), (and using the parametric equations $C': x=\cos t,\ y=\sin t,\ -\pi<t\le\pi$),

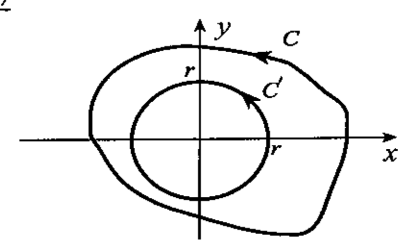
$$\begin{aligned}\oint_C\frac{-y\,dx+x\,dy}{x^2+y^2}&=\oint_{C'}\frac{-y\,dx+x\,dy}{x^2+y^2}\\ &=\int_{-\pi}^{\pi}-\sin t(-\sin t\,dt)+\cos t(\cos t\,dt)=\int_{-\pi}^{\pi}dt=2\pi.\end{aligned}$$



34. Suppose C is a curve enclosing the origin in the counterclockwise sense. It is always possible to find a circle C' of radius $r>0$ centred at the origin which is interior to C. Since $\partial Q/\partial x=\partial P/\partial y$ in a domain containing C and C' and the area between them, it follows by Exercise 30(c) that

$$\begin{aligned}\oint_C\frac{-y\,dx+x\,dy}{x^2+y^2}&=\oint_{C'}\frac{-y\,dx+x\,dy}{x^2+y^2}\\ &=\int_{-\pi}^{\pi}\frac{(-r\sin t)(-r\sin t\,dt)+r\cos t(r\cos t\,dt)}{r^2}\\ &=\int_{-\pi}^{\pi}dt=2\pi,\end{aligned}$$

where we have used the parametric equations $C': x=r\cos t,\ y=r\sin t,\ -\pi<t\le\pi$. If C encloses the origin in the opposite direction, then the value of the line integral is -2π.



35. (a) Since $\nabla[(1/2)\ln(x^2+y^2)] = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}})/(x^2+y^2)$, the line integral is independent of path in any domain not containing $(0,0)$.
(b) Since the line integral is independent of path in the domain consisting of the xy-plane with $(0,0)$ removed, the value of the line integral is zero.

36. According to Exercise 13,

$$\oint_C \frac{\partial P}{\partial n} ds = \oint_C \nabla P \cdot \hat{\mathbf{n}}\, ds = \iint_R \nabla \cdot \nabla P\, dA = \iint_R \left[\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial y}\right)\right] dA = \iint_R \nabla^2 P\, dA.$$

If P satisfies Laplace's equation in R, then $\nabla^2 P = 0$, and $\displaystyle\oint_C \frac{\partial P}{\partial n} ds = 0$.

37. Using Green's theorem in the form of Exercise 13, we may write that

$$\oint_C (P\nabla Q) \cdot \hat{\mathbf{n}}\, ds = \iint_R \nabla \cdot (P\nabla Q)\, dA.$$

Using identity 14.11 on the right side gives

$$\oint_C P\frac{\partial Q}{\partial n} ds = \iint_R (\nabla P \cdot \nabla Q + P\nabla \cdot \nabla Q)\, dA = \iint_R \nabla P \cdot \nabla Q\, dA + \iint_R P\,\nabla^2 Q\, dA.$$

38. If we reverse the roles of P and Q in Exercise 37,

$$\oint_C \frac{Q\partial P}{\partial n} ds = \iint_R Q\nabla^2 P\, dA + \iint_R \nabla Q \cdot \nabla P\, dA.$$

When we subtract this result from that in Exercise 37, we obtain

$$\oint_C \left(P\frac{\partial Q}{\partial n} - Q\frac{\partial P}{\partial n}\right) ds = \iint_R (P\nabla^2 Q - Q\nabla^2 P)\, dA.$$

39. According to Exercise 19(a) in Section 14.4, if C is the unit circle $x^2+y^2=1$ directed counterclockwise, the value of the line integral is 2π. Notice that

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2}, \qquad \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2},$$

and this is valid everywhere except at $(0,0)$. According to Exercise 30(c), the value of the line integral around every other curve (besides $x^2+y^2=1$) that encloses the origin in the same direction is also equal to 2π. For a curve encircling the origin n times, the value is $\pm 2\pi n$, the $\pm$ depending on the direction of the curve. If a curve does not encircle the origin, then Green's theorem can be involked to yield a value of zero. The only possible values are therefore $2\pi n$ (n an integer).

EXERCISES 14.7

1.
$$\begin{aligned}\iint_S (x^2y+z)\, dS &= \iint_{S_{xy}} (x^2y+z)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA\\
&= \iint_{S_{xy}} (x^2y+6-2x-3y)\sqrt{1+(-2)^2+(-3)^2}\, dA\\
&= \sqrt{14}\int_0^3\int_0^{2-2x/3} (x^2y+6-2x-3y)\, dy\, dx = \sqrt{14}\int_0^3 \left\{\frac{x^2y^2}{2}+6y-2xy-\frac{3y^2}{2}\right\}_0^{2-2x/3} dx\\
&= \frac{2\sqrt{14}}{9}\int_0^3 (27-18x+12x^2-6x^3+x^4)\, dx = \frac{2\sqrt{14}}{9}\left\{27x-9x^2+4x^3-\frac{3x^4}{2}+\frac{x^5}{5}\right\}_0^3 = \frac{39\sqrt{14}}{5}\end{aligned}$$

2. $\displaystyle\iint_S (x^2+y^2)z\,dS$

$$= \iint_{S_{xy}} (x^2+y^2)(x+y)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

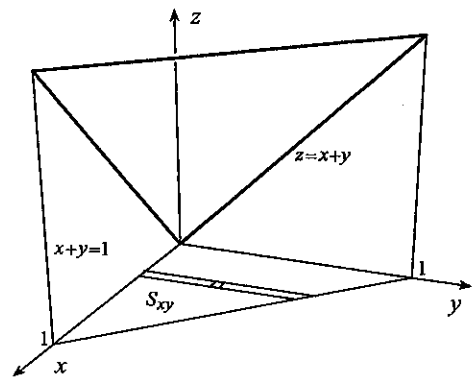
$$= \iint_{S_{xy}} (x^2+y^2)(x+y)\sqrt{1+(1)^2+(1)^2}\,dA$$

$$= \sqrt{3}\int_0^1\int_0^{1-x}(x^3+xy^2+x^2y+y^3)\,dy\,dx$$

$$= \sqrt{3}\int_0^1 \left\{x^3y+\frac{xy^3}{3}+\frac{x^2y^2}{2}+\frac{y^4}{4}\right\}_0^{1-x} dx$$

$$= \frac{\sqrt{3}}{12}\int_0^1 [4x-6x^2+12x^3-10x^4+3(1-x)^4]\,dx$$

$$= \frac{\sqrt{3}}{12}\left\{2x^2-2x^3+3x^4-2x^5-\frac{3}{5}(1-x)^5\right\}_0^1 = \frac{2\sqrt{3}}{15}$$



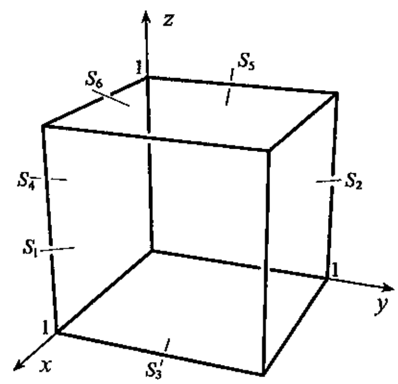
3. Integrals over S_3, S_4, and S_5 in the coordinate planes vanish.

$$\iint_{S_1} xyz\,dS = \iint_{S_{yz}} yz\,dA$$

$$= \int_0^1\int_0^1 yz\,dz\,dy = \int_0^1 \left\{\frac{yz^2}{2}\right\}_0^1 dy$$

$$= \frac{1}{2}\int_0^1 y\,dy = \frac{1}{2}\left\{\frac{y^2}{2}\right\}_0^1 = \frac{1}{4}$$

Integrals over S_2 and S_6 are the same.
Hence, the integral over S is $3(1/4) = 3/4$.



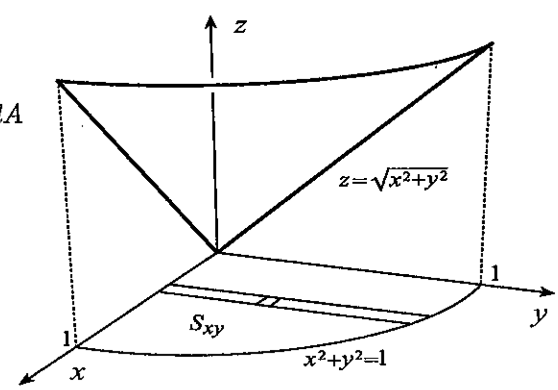
4. $$\iint_S xy\,dS = \iint_{S_{xy}} xy\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$= \iint_{S_{xy}} xy\sqrt{1+\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2+\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}\,dA$$

$$= \sqrt{2}\iint_{S_{xy}} xy\,dA = \sqrt{2}\int_0^1\int_0^{\sqrt{1-x^2}} xy\,dy\,dx$$

$$= \sqrt{2}\int_0^1 \left\{\frac{xy^2}{2}\right\}_0^{\sqrt{1-x^2}} dx = \frac{1}{\sqrt{2}}\int_0^1 (x-x^3)\,dx$$

$$= \frac{1}{\sqrt{2}}\left\{\frac{x^2}{2}-\frac{x^4}{4}\right\}_0^1 = \frac{1}{4\sqrt{2}}$$

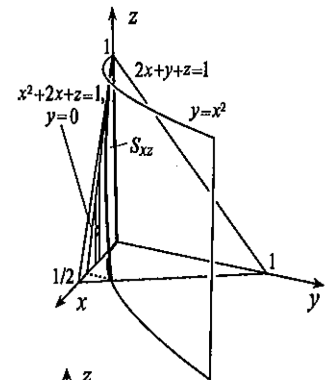


5. $$\iint_S \frac{1}{\sqrt{z-y+1}}dS = \iint_{S_{xy}} \frac{1}{\sqrt{z-y+1}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$= \iint_{S_{xy}} \frac{1}{\sqrt{x^2/2+y-y+1}}\sqrt{1+(x)^2+(1)^2}\,dA = \sqrt{2}\iint_{S_{xy}} dA = \sqrt{2}(1) = \sqrt{2}$$

6. $\displaystyle\iint_S \sqrt{4y+1}\,dS = \iint_{S_{xz}} \sqrt{4x^2+1}\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dA = \iint_{S_{xz}} \sqrt{4x^2+1}\sqrt{1+(2x)^2}\,dA$

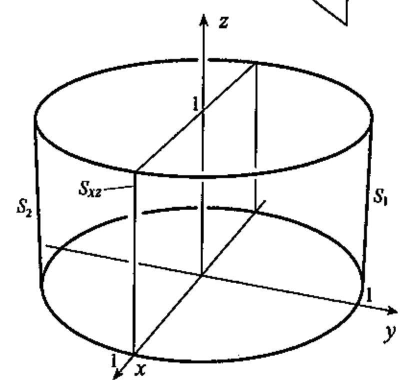
$$= \iint_{S_{xz}} (1+4x^2)\,dA = \int_0^{\sqrt{2}-1}\int_0^{1-2x-x^2}(1+4x^2)\,dz\,dx$$
$$= \int_0^{\sqrt{2}-1}(1+4x^2)(1-2x-x^2)\,dx$$
$$= \int_0^{\sqrt{2}-1}(1-2x+3x^2-8x^3-4x^4)\,dx$$
$$= \left\{x-x^2+x^3-2x^4-\frac{4x^5}{5}\right\}_0^{\sqrt{2}-1} = \frac{44\sqrt{2}-61}{5}$$



7. $\displaystyle\iint_S x^2z\,dS = \iint_{S_1} x^2z\,dS + \iint_{S_2} x^2z\,dS$ On S_1 and S_2,

$$dS = \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dA$$
$$= \sqrt{1+\left(\frac{\mp x}{\sqrt{1-x^2}}\right)^2}\,dA = \frac{1}{\sqrt{1-x^2}}dA.$$

Hence,



$$\iint_S x^2z\,dS = \iint_{S_{xz}} \frac{x^2z}{\sqrt{1-x^2}}dA + \iint_{S_{xz}} \frac{x^2z}{\sqrt{1-x^2}}dA$$
$$= 4\int_0^1\int_0^1 \frac{x^2z}{\sqrt{1-x^2}}dz\,dx = 4\int_0^1\left\{\frac{x^2z^2}{2\sqrt{1-x^2}}\right\}_0^1 dx = 2\int_0^1 \frac{x^2}{\sqrt{1-x^2}}dx.$$

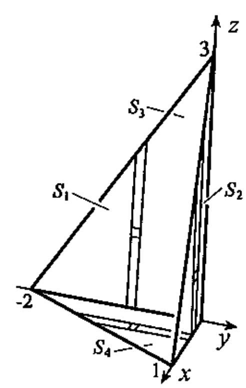
If we set $x=\sin\theta$ and $dx=\cos\theta\,d\theta$,

$$\iint_S x^2z\,dS = 2\int_0^{\pi/2}\frac{\sin^2\theta}{\cos\theta}(\cos\theta\,d\theta) = 2\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right)d\theta = \left\{\theta-\frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = \frac{\pi}{2}.$$

8. $\displaystyle\iint_S (x+y)\,dS = \iint_{S_1}(x+y)\,dS + \iint_{S_2}(x+y)\,dS$

$$+ \iint_{S_3}(x+y)\,dS + \iint_{S_4}(x+y)\,dS$$
$$= \iint_{S_{1xy}}(x+y)\sqrt{1+(-3)^2+(3/2)^2}\,dA$$
$$+ \iint_{S_{2xz}} x\,dA + \iint_{S_{3yz}} y\,dA + \iint_{S_{4xy}}(x+y)\,dA.$$

Since $S_{1xy}=S_{4xy}$,



$$\iint_S (x+y)\,dS = \frac{9}{2}\iint_{S_{1xy}}(x+y)\,dA + \iint_{S_{2xz}} x\,dA + \iint_{S_{3yz}} y\,dA$$
$$= \frac{9}{2}\int_0^1\int_{2x-2}^0 (x+y)\,dy\,dx + \int_0^1\int_0^{3-3x} x\,dz\,dx + \int_{-2}^0\int_0^{3+3y/2} y\,dz\,dy$$

$$= \frac{9}{2}\int_0^1 \left\{xy + \frac{y^2}{2}\right\}_{2x-2}^{0} dx + \int_0^1 \left\{xz\right\}_0^{3-3x} dx + \int_{-2}^{0}\left\{yz\right\}_0^{3+3y/2} dy$$

$$= \frac{9}{4}\int_0^1 [-4x^2 + 4x - 4(x-1)^2]\,dx + 3\int_0^1 (x - x^2)\,dx + \frac{3}{2}\int_{-2}^{0}(2y + y^2)\,dy$$

$$= \frac{9}{4}\left\{\frac{-4x^3}{3} + 2x^2 - \frac{4(x-1)^3}{3}\right\}_0^1 + 3\left\{\frac{x^2}{2} - \frac{x^3}{3}\right\}_0^1 + \frac{3}{2}\left\{y^2 + \frac{y^3}{3}\right\}_{-2}^{0} = -3.$$

9. For projection in the xy-plane, $dS = \sqrt{1 + (-2x)^2 + (-8y)^2}\,dA = \sqrt{1 + 4x^2 + 64y^2}\,dA$. Thus,

$$\iint_S f(x,y,z)\,dS = \int_0^1 \int_0^{\sqrt{4-4y^2}} f(x, y, 4 - x^2 - 4y^2)\sqrt{1 + 4x^2 + 64y^2}\,dx\,dy$$

For projection in the xz-plane,

$$dS = \sqrt{1 + \left(\frac{-x}{2\sqrt{4 - x^2 - z}}\right)^2 + \left(\frac{-1}{4\sqrt{4 - x^2 - z}}\right)^2}\,dA = \frac{1}{4}\sqrt{\frac{65 - 12x^2 - 16z}{4 - x^2 - z}}\,dA.$$

Thus,

$$\iint_S f(x,y,z)\,dS = \frac{1}{4}\int_0^2 \int_0^{4-x^2} f(x, \sqrt{4 - x^2 - z}/2, z)\sqrt{\frac{65 - 12x^2 - 16z}{4 - x^2 - z}}\,dz\,dx.$$

For projection in the yz-plane,

$$dS = \sqrt{1 + \left(\frac{-4y}{\sqrt{4 - 4y^2 - z}}\right)^2 + \left(\frac{-1}{2\sqrt{4 - 4y^2 - z}}\right)^2}\,dA = \frac{1}{2}\sqrt{\frac{17 + 48y^2 - 4z}{4 - 4y^2 - z}}\,dA.$$

Thus,

$$\iint_S f(x,y,z)\,dS = \frac{1}{2}\int_0^1 \int_0^{4-4y^2} f(\sqrt{4 - 4y^2 - z}, y, z)\sqrt{\frac{17 + 48y^2 - 4z}{4 - 4y^2 - z}}\,dz\,dy.$$

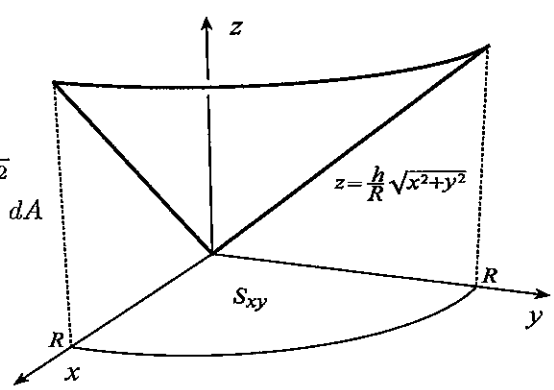
10. If S is that portion of the cone in the first octant,

$$\text{Area} = 4\iint_S dS = 4\iint_{S_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$= 4\iint_{S_{xy}} \sqrt{1 + \left(\frac{hx}{R\sqrt{x^2+y^2}}\right)^2 + \left(\frac{hy}{R\sqrt{x^2+y^2}}\right)^2}\,dA$$

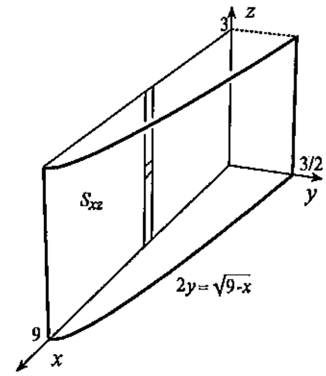
$$= \frac{4\sqrt{R^2+h^2}}{R}\iint_{S_{xy}} dA = \frac{4\sqrt{R^2+h^2}}{R}(\text{Area of } S_{xy})$$

$$= \frac{4\sqrt{R^2+h^2}}{R}\left(\frac{1}{4}\pi R^2\right) = \pi R\sqrt{R^2+h^2}.$$



11. Since xyz^3 is an odd function of y and the surface is symmetric about the xz-plane, the integral has value zero.

12. $$\iint_S xyz\,dS = \iint_{S_{xz}} xyz\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dA$$
$$= \iint_{S_{xz}} xyz\sqrt{1+\left(\frac{-1}{4\sqrt{9-x}}\right)^2}\,dA$$
$$= \iint_{S_{xz}} xyz\sqrt{\frac{145-16x}{16(9-x)}}\,dA$$
$$= \frac{1}{4}\int_0^9\int_0^3 xz\frac{\sqrt{9-x}}{2}\sqrt{\frac{145-16x}{9-x}}\,dz\,dx$$
$$= \frac{1}{8}\int_0^9\left\{x\sqrt{145-16x}\,\frac{z^2}{2}\right\}_0^3 dx = \frac{9}{16}\int_0^9 x\sqrt{145-16x}\,dx$$



If we now set $u = 145 - 16x$, then $du = -16\,dx$, and

$$\iint_S xyz\,dS = \frac{9}{16}\int_{145}^1\left(\frac{145-u}{16}\right)\sqrt{u}\left(\frac{du}{-16}\right) = \frac{-9}{4096}\left\{\frac{290u^{3/2}}{3}-\frac{2u^{5/2}}{5}\right\}_{145}^1 = \frac{3(145^{5/2}-361)}{5120}.$$

13. If S_{xy} is the projection of the surface in the xy-plane,

$$dS = \sqrt{1+\left(\frac{-x}{\sqrt{a^2-x^2-y^2}}\right)^2+\left(\frac{-y}{\sqrt{a^2-x^2-y^2}}\right)^2}\,dA = \frac{a}{\sqrt{a^2-x^2-y^2}}dA.$$

Thus,

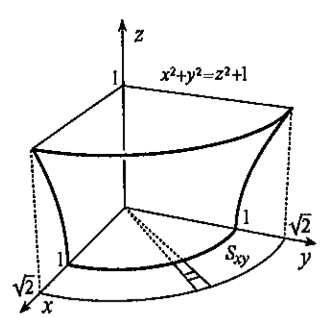
$$\iint_S \frac{1}{\sqrt{2az-z^2}}dS = \iint_{S_{xy}}\frac{1}{\sqrt{x^2+y^2}}\frac{a}{\sqrt{a^2-x^2-y^2}}dA = a\int_0^{\pi/2}\int_0^{a\sin\theta}\frac{1}{r\sqrt{a^2-r^2}}r\,dr\,d\theta$$
$$= a\int_0^{\pi/2}\int_0^{a\sin\theta}\frac{1}{\sqrt{a^2-r^2}}dr\,d\theta.$$

If we set $r = a\sin\phi$ and $dr = a\cos\phi\,d\phi$,

$$\iint_S \frac{1}{\sqrt{2az-z^2}}dS = a\int_0^{\pi/2}\int_0^{\theta}\frac{1}{a\cos\phi}(a\cos\phi\,d\phi) = a\int_0^{\pi/2}\left\{\phi\right\}_0^{\theta}d\theta = a\int_0^{\pi/2}\theta\,d\theta = a\left\{\frac{\theta^2}{2}\right\}_0^{\pi/2} = \frac{a\pi^2}{8}.$$

14. We quadruple the integral over that part of the surface in the first octant.

$$\iint_S z\,dS = 4\iint_{S_{xy}} z\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$
$$= 4\iint_{S_{xy}} z\sqrt{1+\left(\frac{x}{z}\right)^2+\left(\frac{y}{z}\right)^2}\,dA$$
$$= 4\iint_{S_{xy}} z\sqrt{\frac{x^2+y^2+z^2}{z^2}}\,dA$$
$$= 4\iint_{S_{xy}}\sqrt{2x^2+2y^2-1}\,dA$$
$$= 4\int_0^{\pi/2}\int_1^{\sqrt{2}}\sqrt{2r^2-1}\,r\,dr\,d\theta$$
$$= 4\int_0^{\pi/2}\left\{\frac{(2r^2-1)^{3/2}}{6}\right\}_1^{\sqrt{2}}d\theta = \frac{2(3\sqrt{3}-1)}{3}\left\{\theta\right\}_0^{\pi/2} = \frac{\pi(3\sqrt{3}-1)}{3}$$



15. We quadruple the integral over that part of the surface in the first octant.

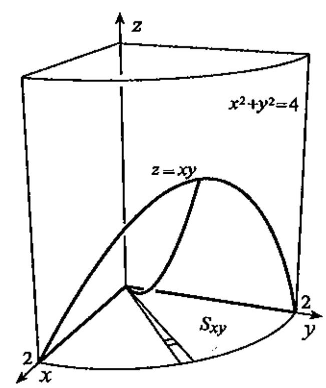
$$\iint_S x^2y^2\,dS = 4\iint_{S_{xy}} x^2y^2\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA = 4\iint_{S_{xy}} x^2y^2\sqrt{1+(2x)^2+(2y)^2}\,dA$$
$$= 4\int_0^{\pi/2}\int_0^1 r^4\cos^2\theta\sin^2\theta\sqrt{1+4r^2}\,r\,dr\,d\theta$$

If we set $u = 1+4r^2$ and $du = 8r\,dr$,

$$\iint_S x^2y^2\,dS = 4\int_0^{\pi/2}\int_1^5 \cos^2\theta\sin^2\theta\left(\frac{u-1}{4}\right)^2\sqrt{u}\left(\frac{du}{8}\right)d\theta$$
$$= \frac{1}{32}\int_0^{\pi/2}\int_1^5 \cos^2\theta\sin^2\theta(u^{5/2}-2u^{3/2}+\sqrt{u})\,du\,d\theta$$
$$= \frac{1}{32}\int_0^{\pi/2}\left\{\cos^2\theta\sin^2\theta\left(\frac{2u^{7/2}}{7}-\frac{4u^{5/2}}{5}+\frac{2u^{3/2}}{3}\right)\right\}_1^5 d\theta = \frac{125\sqrt{5}-1}{210}\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^2 d\theta$$
$$= \frac{125\sqrt{5}-1}{840}\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right)d\theta = \frac{125\sqrt{5}-1}{1680}\left\{\theta-\frac{1}{4}\sin 4\theta\right\}_0^{\pi/2} = \frac{(125\sqrt{5}-1)\pi}{3360}.$$

16. We quadruple the integral over that part of the surface in the first octant.

$$\iint_S x^2\,dS = 4\iint_{S_{xy}} x^2\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$
$$= 4\iint_{S_{xy}} x^2\sqrt{1+y^2+x^2}\,dA$$
$$= 4\int_0^2\int_0^{\pi/2} r^2\cos^2\theta\sqrt{1+r^2}\,r\,d\theta\,dr$$
$$= 4\int_0^2\int_0^{\pi/2} r^3\sqrt{1+r^2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta\,dr$$
$$= 2\int_0^2\left\{r^3\sqrt{1+r^2}\left(\theta+\frac{\sin 2\theta}{2}\right)\right\}_0^{\pi/2}dr = \pi\int_0^2 r^3\sqrt{1+r^2}\,dr$$



If we set $u = 1+r^2$, then $du = 2r\,dr$, and

$$\iint_S x^2\,dS = \pi\int_1^5 (u-1)\sqrt{u}\left(\frac{du}{2}\right) = \frac{\pi}{2}\int_1^5 (u^{3/2}-\sqrt{u})\,du = \frac{\pi}{2}\left\{\frac{2u^{5/2}}{5}-\frac{2u^{3/2}}{3}\right\}_1^5 = \frac{(50\sqrt{5}+2)\pi}{15}.$$

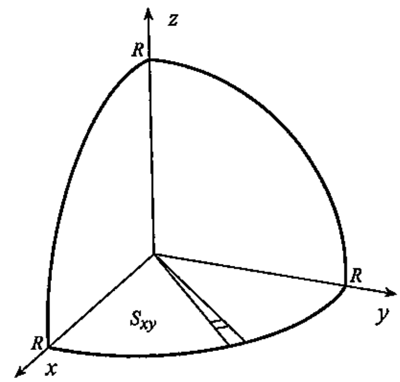
17. $$\iint_S z(y+x^2)\,dS = \iint_{S_{xz}} z(y+x^2)\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dA = \iint_{S_{xz}} z(1-x^2+x^2)\sqrt{1+(-2x)^2}\,dA$$
$$= 2\int_0^1\int_0^2 z\sqrt{1+4x^2}\,dz\,dx = 2\int_0^1\left\{\frac{z^2}{2}\sqrt{1+4x^2}\right\}_0^2 dx = 4\int_0^1\sqrt{1+4x^2}\,dx$$

If we set $x = (1/2)\tan\theta$ and $dx = (1/2)\sec^2\theta\,d\theta$, and use Example 8.9,

$$\iint_S z(y+x^2)\,dS = 4\int_0^{\mathrm{Tan}^{-1}2}\sec\theta(1/2)\sec^2\theta\,d\theta = 2\int_0^{\mathrm{Tan}^{-1}2}\sec^3\theta\,d\theta$$
$$= \left\{\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\right\}_0^{\mathrm{Tan}^{-1}2} = 2\sqrt{5}+\ln(\sqrt{5}+2).$$

18. The surface integral over S is eight times that over that part of the the upper hemisphere $z = \sqrt{R^2 - x^2 - y^2}$ in the first octant. Since

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA = \sqrt{1 + \frac{x^2}{R^2 - x^2 - y^2} + \frac{y^2}{R^2 - x^2 - y^2}}\,dA = \frac{R}{\sqrt{R^2 - x^2 - y^2}}dA,$$



it follows that

$$\iint_S dS = 8\iint_{S_{xy}} \frac{R}{\sqrt{R^2 - x^2 - y^2}}\,dA = 8R\int_0^{\pi/2}\int_0^R \frac{1}{\sqrt{R^2 - r^2}}\,r\,dr\,d\theta$$
$$= 8R\int_0^{\pi/2}\left\{-\sqrt{R^2 - r^2}\right\}_0^R d\theta = 8R^2\left\{\theta\right\}_0^{\pi/2} = 4\pi R^2.$$

Alternatively, using area element 14.56,

$$\iint_S dS = 8\int_0^{\pi/2}\int_0^{\pi/2} R^2 \sin\phi\,d\phi\,d\theta = 8R^2\int_0^{\pi/2}\left\{-\cos\phi\right\}_0^{\pi/2} d\theta = 8R^2\left\{\theta\right\}_0^{\pi/2} = 4\pi R^2.$$

19. If S_{xy} is the projection of the first octant part of the sphere in the xy-plane,

$$dS = \sqrt{1 + \left(\frac{-x}{\sqrt{1 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{1 - x^2 - y^2}}\right)^2}\,dA = \frac{1}{\sqrt{1 - x^2 - y^2}}dA.$$

Thus,

$$\iint_S x^2z^2\,dS = 8\iint_{S_{xz}} x^2(1 - x^2 - y^2)\frac{1}{\sqrt{1 - x^2 - y^2}}dA = 8\int_0^{\pi/2}\int_0^1 r^2\cos^2\theta\sqrt{1 - r^2}\,r\,dr\,d\theta.$$

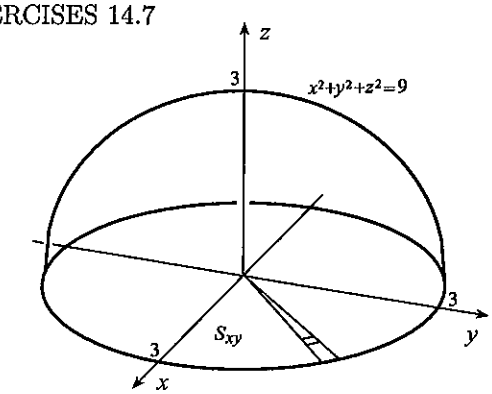
If we set $u = 1 - r^2$ and $du = -2r\,dr$,

$$\iint_S x^2z^2\,dS = 8\int_0^{\pi/2}\int_1^0 (1 - u)\sqrt{u}\cos^2\theta\left(\frac{du}{-2}\right)d\theta = 4\int_0^{\pi/2}\int_0^1 (\sqrt{u} - u^{3/2})\cos^2\theta\,du\,d\theta$$
$$= 4\int_0^{\pi/2}\left\{\left(\frac{2u^{3/2}}{3} - \frac{2u^{5/2}}{5}\right)\cos^2\theta\right\}_0^1 d\theta = \frac{16}{15}\int_0^{\pi/2}\left(\frac{1 + \cos 2\theta}{2}\right)d\theta$$
$$= \frac{8}{15}\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = \frac{4\pi}{15}.$$

Alternatively, using area element 14.56 with $R = 1$,

$$\iint_S x^2z^2\,dS = 8\int_0^{\pi/2}\int_0^{\pi/2}(\sin^2\phi\cos^2\theta)\cos^2\phi\sin\phi\,d\phi\,d\theta$$
$$= 8\int_0^{\pi/2}\int_0^{\pi/2}\cos^2\theta(1 - \cos^2\phi)\cos^2\phi\sin\phi\,d\phi\,d\theta$$
$$= 8\int_0^{\pi/2}\left\{\cos^2\theta\left(-\frac{1}{3}\cos^3\phi + \frac{1}{5}\cos^5\phi\right)\right\}_0^{\pi/2} d\theta = \frac{16}{15}\int_0^{\pi/2}\left(\frac{1 + \cos 2\theta}{2}\right)d\theta$$
$$= \frac{8}{15}\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = \frac{4\pi}{15}.$$

20. The hemisphere projects one-to-one onto the circle $S_{xy}: x^2+y^2 \le 9,\ z=0$. Since



$$\begin{aligned} dS &= \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA \\ &= \sqrt{1+\frac{x^2}{9-x^2-y^2}+\frac{y^2}{9-x^2-y^2}}\,dA \\ &= \frac{3}{\sqrt{9-x^2-y^2}}\,dA, \end{aligned}$$

it follows that

$$\begin{aligned} \iint_S (x^2-y^2)\,dS &= \iint_{S_{xy}} \frac{3(x^2-y^2)}{\sqrt{9-x^2-y^2}}\,dA = 3\int_0^3\int_{-\pi}^{\pi} \frac{(r^2\cos^2\theta - r^2\sin^2\theta)}{\sqrt{9-r^2}}\,r\,d\theta\,dr \\ &= 3\int_0^3\int_{-\pi}^{\pi} \frac{r^3}{\sqrt{9-r^2}}\cos 2\theta\,d\theta\,dr = 3\int_0^3 \left\{\frac{r^3}{\sqrt{9-r^2}}\frac{\sin 2\theta}{2}\right\}_{-\pi}^{\pi} dr = 0. \end{aligned}$$

Alternatively, using $dS = R^2\sin\phi\,d\phi\,d\theta$, with $R=3$,

$$\begin{aligned} \iint_S (x^2-y^2)\,dS &= \int_{-\pi}^{\pi}\int_0^{\pi/2} (9\sin^2\phi\,\cos^2\theta - 9\sin^2\phi\,\sin^2\theta)\,9\sin\phi\,d\phi\,d\theta \\ &= 81\int_{-\pi}^{\pi}\int_0^{\pi/2} \sin^3\phi\,\cos 2\theta\,d\phi\,d\theta = 81\int_{-\pi}^{\pi}\int_0^{\pi/2} \sin\phi(1-\cos^2\phi)\cos 2\theta\,d\phi\,d\theta \\ &= 81\int_{-\pi}^{\pi} \left\{\left(-\cos\phi + \frac{\cos^3\phi}{3}\right)\cos 2\theta\right\}_0^{\pi/2} d\theta = 54\left\{\frac{\sin 2\theta}{2}\right\}_{-\pi}^{\pi} = 0. \end{aligned}$$

21. If S_{xy} is the projection of the sphere in the xy-plane,

$$dS = \sqrt{1+\left(\frac{-x}{\sqrt{R^2-x^2-y^2}}\right)^2+\left(\frac{-y}{\sqrt{R^2-x^2-y^2}}\right)^2}\,dA = \frac{R}{\sqrt{R^2-x^2-y^2}}\,dA.$$

Thus, $\displaystyle\iint_S (x^2+y^2)\,dS = 2\iint_{S_{xy}} (x^2+y^2)\frac{R}{\sqrt{R^2-x^2-y^2}}\,dA = 8R\int_0^{\pi/2}\int_0^R \frac{r^2}{\sqrt{R^2-r^2}}\,r\,dr\,d\theta.$

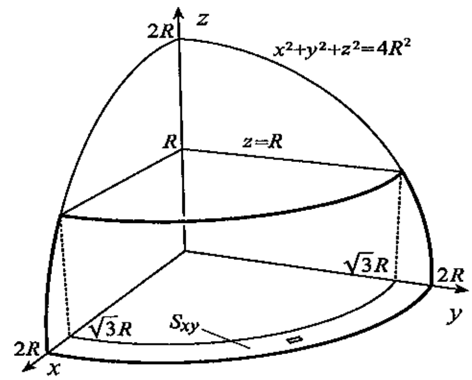
If we set $u = R^2 - r^2$ and $du = -2r\,dr$,

$$\begin{aligned} \iint_S (x^2+y^2)\,dS &= 8R\int_0^{\pi/2}\int_{R^2}^{0} \frac{R^2-u}{\sqrt{u}}\left(\frac{du}{-2}\right)d\theta = 4R\int_0^{\pi/2}\int_0^{R^2}\left(\frac{R^2}{\sqrt{u}} - \sqrt{u}\right)du\,d\theta \\ &= 4R\int_0^{\pi/2}\left\{2R^2\sqrt{u} - \frac{2u^{3/2}}{3}\right\}_0^{R^2} d\theta = \frac{16R^4}{3}\Big\{\theta\Big\}_0^{\pi/2} = \frac{8\pi R^4}{3}. \end{aligned}$$

Alternatively, if we use area element 14.56,

$$\begin{aligned} \iint_S (x^2+y^2)\,dS &= 8\int_0^{\pi/2}\int_0^{\pi/2} R^2\sin^2\phi R^2\sin\phi\,d\phi\,d\theta = 8R^4\int_0^{\pi/2}\int_0^{\pi/2}(1-\cos^2\phi)\sin\phi\,d\phi\,d\theta \\ &= 8R^4\int_0^{\pi/2}\left\{-\cos\phi + \frac{1}{3}\cos^3\phi\right\}_0^{\pi/2} d\theta = \frac{16R^4}{3}\Big\{\theta\Big\}_0^{\pi/2} = \frac{8\pi R^4}{3}. \end{aligned}$$

22. The first octant part of the surface projects one-to-one onto the area $S_{xy}: 3R^2 \le x^2+y^2 \le 4R^2$ in the first quadrant of the xy-plane. Since



$$\begin{aligned} dS &= \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA \\ &= \sqrt{1+\frac{x^2}{4R^2-x^2-y^2}+\frac{y^2}{4R^2-x^2-y^2}}\,dA \\ &= \frac{2R}{\sqrt{4R^2-x^2-y^2}}\,dA, \end{aligned}$$

it follows that

$$\begin{aligned} \iint_S \frac{1}{x^2+y^2}\,dS &= 4\iint_{S_{xy}} \frac{1}{x^2+y^2}\frac{2R}{\sqrt{4R^2-x^2-y^2}}\,dA \\ &= 8R\int_{\sqrt{3}R}^{2R}\int_0^{\pi/2}\frac{1}{r^2\sqrt{4R^2-r^2}}\,r\,d\theta\,dr = 4\pi R\int_{\sqrt{3}R}^{2R}\frac{1}{r\sqrt{4R^2-r^2}}\,dr. \end{aligned}$$

If we set $r=2R\sin\phi$ and $dr = 2R\cos\phi\,d\phi$, then

$$\begin{aligned} \iint_S \frac{1}{x^2+y^2}\,dS &= 4\pi R\int_{\pi/3}^{\pi/2}\frac{1}{2R\sin\phi\,2R\cos\phi}2R\cos\phi\,d\phi = 2\pi\int_{\pi/3}^{\pi/2}\csc\phi\,d\phi \\ &= 2\pi\Big\{\ln|\csc\phi-\cot\phi|\Big\}_{\pi/3}^{\pi/2} = \pi\ln 3. \end{aligned}$$

Alternatively, using area element 14.56,

$$\begin{aligned} \iint_S \frac{1}{x^2+y^2}\,dS &= 4\int_0^{\pi/2}\int_{\pi/3}^{\pi/2}\frac{1}{4R^2\sin^2\phi\,\cos^2\theta+4R^2\sin^2\phi\,\sin^2\theta}4R^2\sin\phi\,d\phi\,d\theta \\ &= 4\int_0^{\pi/2}\int_{\pi/3}^{\pi/2}\csc\phi\,d\phi\,d\theta = 4\int_0^{\pi/2}\Big\{\ln|\csc\phi-\cot\phi|\Big\}_{\pi/3}^{\pi/2}d\theta = \pi\ln 3. \end{aligned}$$

23. The thickness of material as a function of ϕ is $\rho(\phi) = \dfrac{0.004\phi}{\pi}+0.001 = \dfrac{4\phi+\pi}{1000\pi}$. Using area element 14.56 with $R=1$,

$$V = \iint_S \rho\,dS = \int_{-\pi}^{\pi}\int_0^{\pi}\frac{4\phi+\pi}{1000\pi}\sin\phi\,d\theta\,d\phi = \frac{1}{1000\pi}\int_0^{\pi}\Big\{(4\phi+\pi)\sin\phi\,\theta\Big\}_{-\pi}^{\pi}d\phi = \frac{1}{500}\int_0^{\pi}(4\phi+\pi)\sin\phi\,d\phi.$$

If we set $u=4\phi+\pi$, $dv=\sin\phi\,d\phi$, $du = 4\,d\phi$, and $v=-\cos\phi$,

$$V = \frac{1}{500}\left[\Big\{-(4\phi+\pi)\cos\phi\Big\}_0^{\pi} - \int_0^{\pi}-4\cos\phi\,d\phi\right] = \frac{1}{500}\left(6\pi+4\Big\{\sin\phi\Big\}_0^{\pi}\right) = \frac{3\pi}{250}\ \mathrm{m}^3.$$

24. If S projects one-to-one onto S_{xy}, then an element of area dS on S is related to its projection dA in the xy-plane by $dS = \sqrt{1+\left(\dfrac{\partial z}{\partial x}\right)^2+\left(\dfrac{\partial z}{\partial y}\right)^2}\,dA$, where

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(z)}} = -\frac{F_x}{F_z} \quad\text{and}\quad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial(F)}{\partial(y)}}{\dfrac{\partial(F)}{\partial(z)}} = -\frac{F_y}{F_z}.$$

Thus, $dS = \sqrt{1+\left(-\dfrac{F_x}{F_z}\right)^2+\left(-\dfrac{F_y}{F_z}\right)^2}\,dA = \dfrac{\sqrt{(F_x)^2+(F_y)^2+(F_z)^2}}{|F_z|}\,dA = \dfrac{|\nabla F|}{|F_z|}\,dA$, and

$$\iint_S f(x,y,z)\,dS = \iint_{S_{xy}} f[x,y,g(x,y)]\frac{|\nabla F|}{|F_z|}\,dA,$$

where $z = g(x,y)$ on S.

25. (a) If S_{xy} is the projection of the first octant part of $z^2 = x^2 + y^2$ in the xy-plane,

$$dS = \sqrt{1 + \left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}\,dA = \sqrt{2}\,dA.$$

Thus, $A = 4\iint_{S_{xy}} \sqrt{2}\,dA = 4\sqrt{2}\left(\frac{\pi}{2}\right) = 2\sqrt{2}\pi.$

(b) The projection in the xz-plane of the curve of intersection of the cone and the cylinder in the first octant has equation $z = \sqrt{2x}$. Since $dS = \sqrt{1 + \left(\frac{1-x}{\sqrt{2x-x^2}}\right)^2}\,dA = \frac{1}{\sqrt{2x-x^2}}dA,$

$$\begin{aligned}A &= 4\iint_{S_{xz}} \frac{1}{\sqrt{2x-x^2}}dA = 4\int_0^2\int_0^{\sqrt{2x}} \frac{1}{\sqrt{2x-x^2}}dz\,dx = 4\int_0^2 \frac{\sqrt{2x}}{\sqrt{2x-x^2}}dx\\ &= 4\sqrt{2}\int_0^2 \frac{1}{\sqrt{2-x}}dx = 4\sqrt{2}\Big\{-2\sqrt{2-x}\Big\}_0^2 = 16.\end{aligned}$$

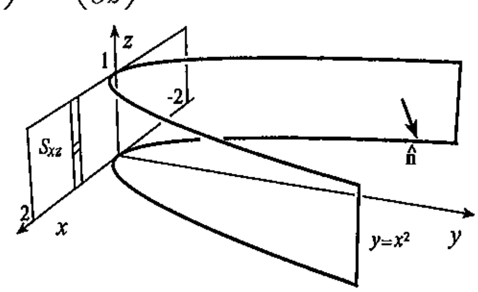
EXERCISES 14.8

1. Since $\hat{\mathbf{n}} = (1,1,1)/\sqrt{3}$,

$$\begin{aligned}\iint_S (x\hat{\mathbf{i}} + y\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS &= \iint_{S_{xy}} \frac{x+y}{\sqrt{3}}\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA = \frac{1}{\sqrt{3}}\iint_{S_{xy}} (x+y)\sqrt{1+(-1)^2+(-1)^2}\,dA\\ &= \int_0^3\int_0^{3-x}(x+y)\,dy\,dx = \int_0^3\left\{\frac{1}{2}(x+y)^2\right\}_0^{3-x}dx = \frac{1}{2}\int_0^3 (9-x^2)\,dx = \frac{1}{2}\left\{9x - \frac{x^3}{3}\right\}_0^3 = 9.\end{aligned}$$

2. Since $\hat{\mathbf{n}} = \dfrac{\nabla(y-x^2)}{|\nabla(y-x^2)|} = \dfrac{(-2x,1,0)}{\sqrt{4x^2+1}}$,

$$\begin{aligned}\iint_S (yz^2\hat{\mathbf{i}} + ye^x\hat{\mathbf{j}} + x\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS &= \iint_{S_{xz}} \frac{-2xyz^2 + ye^x}{\sqrt{4x^2+1}}\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\,dA\\ &= \iint_{S_{xz}} \frac{-2x^3z^2 + x^2e^x}{\sqrt{4x^2+1}}\sqrt{1+(2x)^2}\,dA\\ &= \int_{-2}^2\int_0^1 (x^2e^x - 2x^3z^2)\,dz\,dx = \int_{-2}^2\left\{x^2ze^x - \frac{2x^3z^3}{3}\right\}_0^1 dx\\ &= \int_{-2}^2\left(x^2e^x - \frac{2x^3}{3}\right)dx = \left\{x^2e^x - 2xe^x + 2e^x - \frac{x^4}{6}\right\}_{-2}^2\\ &= 2e^2 - 10e^{-2}.\end{aligned}$$

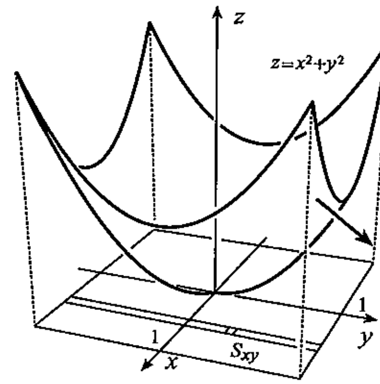


3. Since $\hat{\mathbf{n}} = \dfrac{\nabla(x^2+y^2+z^2-1)}{|\nabla(x^2+y^2+z^2-1)|} = \dfrac{(x,y,z)}{\sqrt{x^2+y^2+z^2}} = (x,y,z)$,

$$\iint_S (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = \iint_S (x^2+y^2+z^2)\,dS = \iint_S dS = \frac{1}{2}(4\pi) = 2\pi.$$

4. Since $\hat{\mathbf{n}} = \dfrac{\nabla(x^2+y^2-z)}{|\nabla(x^2+y^2-z)|} = \dfrac{(2x, 2y, -1)}{\sqrt{1+4x^2+4y^2}}$,

$$\begin{aligned}
&\iint_S (yz\hat{\mathbf{i}} + zx\hat{\mathbf{j}} + xy\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS \\
&= \iint_{S_{xy}} \frac{2xyz + 2xyz - xy}{\sqrt{1+4x^2+4y^2}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \\
&= \iint_{S_{xy}} \frac{4xy(x^2+y^2) - xy}{\sqrt{1+4x^2+4y^2}} \sqrt{1 + (2x)^2 + (2y)^2}\, dA \\
&= \int_{-1}^{1}\int_{-1}^{1} (4x^3y + 4xy^3 - xy)\, dy\, dx \\
&= \int_{-1}^{1} \left\{2x^3y^2 + xy^4 - \frac{xy^2}{2}\right\}_{-1}^{1} dx = 0.
\end{aligned}$$



5. Let S_1 be the hemisphere and S_2 be that part of the xy-plane bounded by $x^2+y^2=4$. On S_1, $\hat{\mathbf{n}} = \dfrac{\nabla(x^2+y^2+z^2-1)}{|\nabla(x^2+y^2+z^2-1)|} = \dfrac{(x,y,z)}{\sqrt{x^2+y^2+z^2}} = \dfrac{(x,y,z)}{2}$, and on S_2, $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$. Thus,

$$\oiint_S (z\hat{\mathbf{i}} - x\hat{\mathbf{j}} + y\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \frac{1}{2}\iint_{S_1} (xz - xy + yz)\, dS + \iint_{S_2} -y\, dS.$$

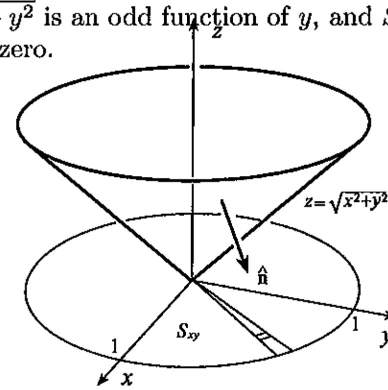
The integral over S_2 is zero since y is an odd function of y and S_2 is symmetric about the xz-plane. If S_{xy} is the projection of S_1 in the xy-plane,

$$\begin{aligned}
\oiint_S (z\hat{\mathbf{i}} - x\hat{\mathbf{j}} + y\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS &= \frac{1}{2}\iint_{S_{xy}} (xz - xy + yz)\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \\
&= \frac{1}{2}\iint_{S_{xy}} (xz - xy + yz)\sqrt{1 + \left(\frac{-x}{\sqrt{4-x^2-y^2}}\right)^2 + \left(\frac{-y}{\sqrt{4-x^2-y^2}}\right)^2}\, dA \\
&= \frac{1}{2}\iint_{S_{xy}} [(x+y)\sqrt{4-x^2-y^2} - xy]\frac{2}{\sqrt{4-x^2-y^2}} dA.
\end{aligned}$$

Since $x\sqrt{4-x^2-y^2} - xy$ is an odd function of x, and $y\sqrt{4-x^2-y^2}$ is an odd function of y, and S_{xy} is symmetric about the x- and y-axes, this integral is also equal to zero.

6. Since $\hat{\mathbf{n}} = \dfrac{\nabla(x^2+y^2-z^2)}{|\nabla(x^2+y^2-z^2)|} = \dfrac{(2x, 2y, -2z)}{\sqrt{4x^2+4y^2+4z^2}} = \dfrac{(x, y, -z)}{\sqrt{2}z}$,

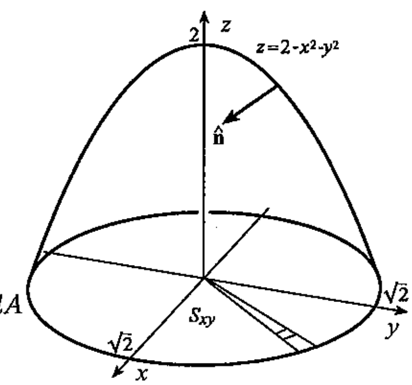
$$\begin{aligned}
\iint_S (x\hat{\mathbf{i}} + y\hat{\mathbf{j}}) \cdot \hat{\mathbf{n}}\, dS &= \iint_{S_{xy}} \frac{x^2+y^2}{\sqrt{2}z} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \\
&= \frac{1}{\sqrt{2}}\iint_{S_{xy}} \frac{x^2+y^2}{z}\sqrt{1 + \left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}\, dA \\
&= \frac{1}{\sqrt{2}}\iint_{S_{xy}} \frac{x^2+y^2}{\sqrt{x^2+y^2}}\sqrt{2}\, dA = 4\int_0^{\pi/2}\int_0^1 r\, r\, dr\, d\theta = 4\int_0^{\pi/2}\left\{\frac{r^3}{3}\right\}_0^1 d\theta = \frac{4}{3}\Big\{\theta\Big\}_0^{\pi/2} = \frac{2\pi}{3}.
\end{aligned}$$



7. Since $\hat{\mathbf{n}} = \dfrac{\nabla(x^2+y^2-9)}{|\nabla(x^2+y^2-9)|} = \dfrac{(2x,2y,0)}{\sqrt{4x^2+4y^2}} = \dfrac{(x,y,0)}{3}$,

$$\begin{aligned}\iint_S (xyz\hat{\mathbf{i}} - x\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS &= \frac{1}{3}\iint_{S_{xz}} (x^2yz - xy)\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dA\\ &= \frac{1}{3}\iint_{S_{xz}} (x^2z - x)y\sqrt{1+\left(\frac{-x}{\sqrt{9-x^2}}\right)^2}\,dA\\ &= \frac{1}{3}\iint_{S_{xz}} (x^2z - x)\sqrt{9-x^2}\frac{3}{\sqrt{9-x^2}}\,dA\\ &= \int_{-3/\sqrt{2}}^{3/\sqrt{2}}\int_0^2 (x^2z - x)\,dz\,dx = \int_{-3/\sqrt{2}}^{3/\sqrt{2}}\left\{\frac{x^2z^2}{2} - xz\right\}_0^2 dx\\ &= \int_{-3/\sqrt{2}}^{3/\sqrt{2}} (2x^2 - 2x)\,dx = \left\{\frac{2x^3}{3} - x^2\right\}_{-3/\sqrt{2}}^{3/\sqrt{2}} = 9\sqrt{2}.\end{aligned}$$

8. Since $\hat{\mathbf{n}} = \dfrac{\nabla(2-x^2-y^2-z)}{|\nabla(2-x^2-y^2-z)|} = \dfrac{(-2x,-2y,-1)}{\sqrt{1+4x^2+4y^2}}$,

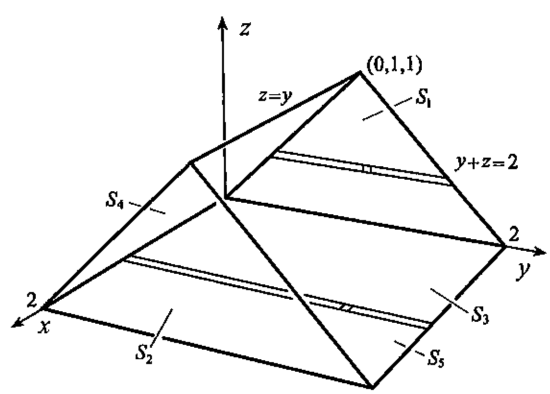


$$\begin{aligned}&\iint_S (x^2y\hat{\mathbf{i}} + xy\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS\\ &= \iint_{S_{xy}} \frac{-2x^3y - 2xy^2 - z}{\sqrt{1+4x^2+4y^2}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA\\ &= \iint_{S_{xy}} \frac{-(2x^3y + 2xy^2 + 2 - x^2 - y^2)}{\sqrt{1+4x^2+4y^2}}\sqrt{1+(-2x)^2+(-2y)^2}\,dA\\ &= -\iint_{S_{xy}} [2x^3y + 2xy^2 + 2 - (x^2+y^2)]\,dA.\end{aligned}$$

Because the first two terms are odd functions of x and S_{xy} is symmetric about the y-axis, their double integral vanishes, and

$$\iint_S (x^2y\hat{\mathbf{i}} + xy\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = -\int_{-\pi}^{\pi}\int_0^{\sqrt{2}} (2-r^2)\,r\,dr\,d\theta = -\int_{-\pi}^{\pi}\left\{r^2 - \frac{r^4}{4}\right\}_0^{\sqrt{2}} d\theta = -\Big\{\theta\Big\}_{-\pi}^{\pi} = -2\pi.$$

9. On S_1, $\hat{\mathbf{n}} = -\hat{\mathbf{i}}$, and therefore



$$\begin{aligned}&\iint_{S_1} (yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS\\ &= \iint_{S_1} -yz\,dA = \int_0^1\int_z^{2-z} -yz\,dy\,dz\\ &= -\int_0^1\left\{\frac{y^2z}{2}\right\}_z^{2-z} dz = -\frac{1}{2}\int_0^1 (4z - 4z^2)\,dz\\ &= -\frac{1}{2}\left\{2z^2 - \frac{4z^3}{3}\right\}_0^1 = -\frac{1}{3}.\end{aligned}$$

On S_2, $\hat{\mathbf{n}} = \hat{\mathbf{i}}$, and therefore

$$\iint_{S_2} (yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = \iint_{S_2} yz\,dS = \iint_{S_1} yz\,dA = \frac{1}{3}.$$

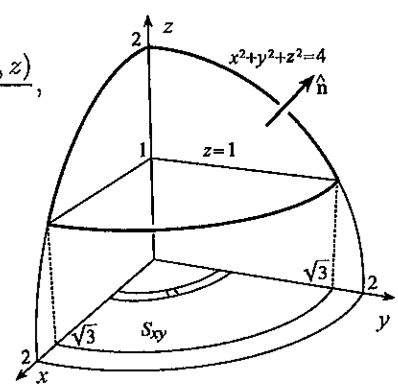
On S_3, $\mathbf{n} = (0,1,1)/\sqrt{2}$, and therefore

$$\iint_{S_3}(yz\hat{\mathbf{i}}+xz\hat{\mathbf{j}}+xy\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS=\iint_{S_3}\left(\frac{xz+xy}{\sqrt{2}}\right)dS=\frac{1}{\sqrt{2}}\iint_{S_{xy}}x(2)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$
$$=\sqrt{2}\iint_{S_{xy}}x\sqrt{1+(-1)^2}\,dA=2\int_0^2\int_1^2 x\,dy\,dx$$
$$=2\int_0^2\left\{xy\right\}_1^2dx=2\int_0^2 x\,dx=2\left\{\frac{x^2}{2}\right\}_0^2=4.$$

On S_4, $\mathbf{n}=(0,-1,1)/\sqrt{2}$, and therefore
$$\iint_{S_4}(yz\hat{\mathbf{i}}+xz\hat{\mathbf{j}}+xy\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS=\iint_{S_4}\left(\frac{-xz+xy}{\sqrt{2}}\right)dS=\frac{1}{\sqrt{2}}\iint_S x(0)\,dS=0.$$

On S_5, $\mathbf{n}=-\hat{\mathbf{k}}$, and therefore
$$\iint_{S_5}(yz\hat{\mathbf{i}}+xz\hat{\mathbf{j}}+xy\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS=\iint_{S_5}-xy\,dS=\int_0^2\int_0^2-xy\,dy\,dx=-\int_0^2\left\{\frac{xy^2}{2}\right\}_0^2dx=-\frac{1}{2}\int_0^2 4x\,dx$$
$$=-2\left\{\frac{x^2}{2}\right\}_0^2=-4.$$

Thus, $\displaystyle\oiint_S(yz\hat{\mathbf{i}}+xz\hat{\mathbf{j}}+xy\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS=-\frac{1}{3}+\frac{1}{3}+4+0-4=0.$



10. Since $\hat{\mathbf{n}}=\dfrac{\nabla(x^2+y^2+z^2-4)}{|\nabla(x^2+y^2+z^2-4)|}=\dfrac{(2x,2y,2z)}{\sqrt{4x^2+4y^2+4z^2}}=\dfrac{(x,y,z)}{2}$,

$$\iint_S(x\hat{\mathbf{i}}+y\hat{\mathbf{j}})\cdot\hat{\mathbf{n}}\,dS$$
$$=\iint_{S_{xy}}\frac{x^2+y^2}{2}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$
$$=\frac{1}{2}\iint_{S_{xy}}(x^2+y^2)\sqrt{1+\left(\frac{-x}{z}\right)^2+\left(\frac{-y}{z}\right)^2}\,dA$$
$$=\frac{1}{2}\iint_{S_{xy}}(x^2+y^2)\left(\frac{2}{z}\right)dA$$
$$=\iint_{S_{xy}}\frac{x^2+y^2}{\sqrt{4-x^2-y^2}}\,dA=\int_0^{\sqrt{3}}\int_{-\pi}^{\pi}\frac{r^2}{\sqrt{4-r^2}}\,r\,d\theta\,dr=2\pi\int_0^{\sqrt{3}}\frac{r^3}{\sqrt{4-r^2}}\,dr.$$

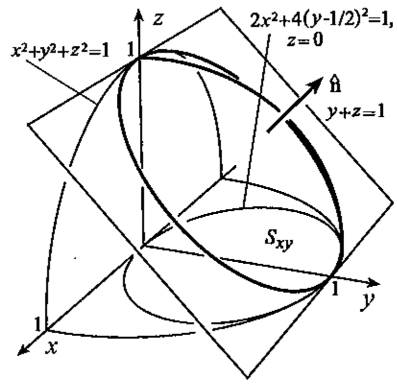
If we set $u=4-r^2$ and $du=-2r\,dr$, then
$$\iint_S(x\hat{\mathbf{i}}+y\hat{\mathbf{j}})\cdot\hat{\mathbf{n}}\,dS=2\pi\int_4^1\frac{(4-u)}{\sqrt{u}}\left(-\frac{du}{2}\right)=\pi\int_1^4\left(\frac{4}{\sqrt{u}}-\sqrt{u}\right)du=\pi\left\{8\sqrt{u}-\frac{2u^{3/2}}{3}\right\}_1^4=\frac{10\pi}{3}.$$

11. Since $\hat{\mathbf{n}}=\dfrac{\nabla(x^2+y^2+z^2-a^2)}{|\nabla(x^2+y^2+z^2-a^2)|}=\dfrac{(2x,2y,2z)}{\sqrt{4x^2+4y^2+4z^2}}=\dfrac{(x,y,z)}{a}$,
$$\oiint_S(x^2\hat{\mathbf{i}}+y^2\hat{\mathbf{j}}+z^2\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS=\oiint_S(x^3+y^3+z^3)\,dS.$$

Since each of x^3, y^3, and z^3 is an odd function, and S is symmetric about the coordinate planes, the surface integral must be equal to zero.

12. Since $\hat{\mathbf{n}} = \dfrac{\nabla(x^2+y^2+z^2-1)}{|\nabla(x^2+y^2+z^2-1)|}$

$$= \frac{(2x, 2y, 2z)}{\sqrt{4x^2+4y^2+4z^2}} = (x, y, z),$$



$$\iint_S (y\hat{\mathbf{i}} - x\hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS$$

$$= \iint_{S_{xy}} (yx - xy + z)\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

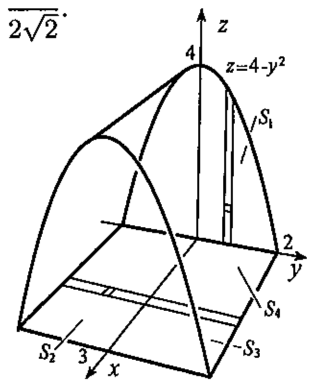
$$= \iint_{S_{xy}} z\sqrt{1 + \left(\frac{-x}{z}\right)^2 + \left(\frac{-y}{z}\right)^2}\, dA$$

$$= \iint_{S_{xy}} dA = \text{Area of } S_{xy}.$$

Since S_{xy} is the region inside the ellipse $2x^2 + 4(y - 1/2)^2 = 1$,

$$\iint_S (y\hat{\mathbf{i}} - x\hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \pi\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{\pi}{2\sqrt{2}}.$$

13. On S_1, $\hat{\mathbf{n}} = -\hat{\mathbf{i}}$, and therefore



$$\iint_{S_1} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iint_{S_1} (x - z^2)\, dS = \iint_{S_1} -z^2\, dA$$

$$= -2\int_0^2 \int_0^{4-y^2} z^2\, dz\, dy = -2\int_0^2 \left\{\frac{z^3}{3}\right\}_0^{4-y^2} dy$$

$$= -\frac{2}{3}\int_0^2 (64 - 48y^2 + 12y^4 - y^6)\, dy$$

$$= -\frac{2}{3}\left\{64y - 16y^3 + \frac{12y^5}{5} - \frac{y^7}{7}\right\}_0^2 = -\frac{4096}{105}.$$

On S_2, $\hat{\mathbf{n}} = \hat{\mathbf{i}}$, and therefore

$$\iint_{S_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iint_{S_2} (z^2 - x)\, dS = \iint_{S_2} z^2\, dS - \iint_{S_2} x\, dS = -\iint_{S_1} z^2\, dS - \iint_{S_1} 3\, dS$$

$$= \frac{4096}{105} - 6\int_0^2 \int_0^{4-y^2} dz\, dy = \frac{4096}{105} - 6\int_0^2 (4 - y^2)\, dy = \frac{4096}{105} - 6\left\{4y - \frac{y^3}{3}\right\}_0^2 = \frac{4096}{105} - 32.$$

On S_3, $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$, and therefore $\displaystyle\iint_{S_3} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iint_{S_3} -3z\, dS = 0.$

On S_4, $\hat{\mathbf{n}} = \dfrac{\nabla(y^2 + z - 4)}{|\nabla(y^2 + z - 4)|} = \dfrac{(0, 2y, 1)}{\sqrt{4y^2+1}}$, and therefore

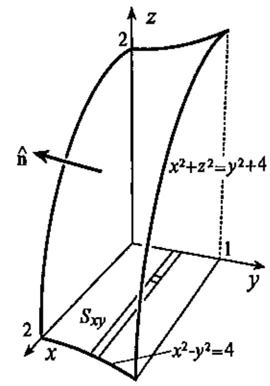
$$\iint_{S_4} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iint_{S_4} \frac{(-2xy^2 + 3z)}{\sqrt{1+4y^2}} dS = \iint_{S_{xy}} \frac{(-2xy^2 + 12 - 3y^2)}{\sqrt{1+4y^2}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$= \int_0^3 \int_{-2}^2 (12 - 2xy^2 - 3y^2)\, dy\, dx = \int_0^3 \left\{12y - \frac{2xy^3}{3} - y^3\right\}_{-2}^2 dx$$

$$= \int_0^3 \left(24 - \frac{16x}{3} - 8 + 24 - \frac{16x}{3} - 8\right) dx = \frac{32}{3}\int_0^3 (3 - x)\, dx = \frac{32}{3}\left\{3x - \frac{x^2}{2}\right\}_0^3 = 48.$$

Thus, $\displaystyle\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = -\frac{4096}{105} + \frac{4096}{105} - 32 + 48 = 16.$

14. The surface projects one-to-one onto the area S_{xy} in the xy-plane shown. Since



$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$
$$= \sqrt{1 + \frac{x^2}{4 + y^2 - x^2} + \frac{y^2}{4 + y^2 - x^2}}\, dA$$
$$= \sqrt{\frac{4 + 2y^2}{4 + y^2 - x^2}}\, dA,$$

and $\hat{\mathbf{n}} = \dfrac{\nabla(x^2 + z^2 - y^2 - 4)}{|\nabla(x^2 + z^2 - y^2 - 4)|} = \dfrac{(2x, -2y, 2z)}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \dfrac{(x, -y, z)}{\sqrt{x^2 + y^2 + 4 + y^2 - x^2}} = \dfrac{(x, -y, z)}{\sqrt{4 + 2y^2}}$,

it follows that

$$\iint_S (x^2\hat{\mathbf{i}} + xy\hat{\mathbf{j}} + xz\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \iint_{S_{xy}} \frac{x^3 - xy^2 + xz^2}{\sqrt{4 + 2y^2}} \sqrt{\frac{4 + 2y^2}{4 + y^2 - x^2}}\, dA$$
$$= \iint_{S_{xy}} \frac{4x}{\sqrt{4 + y^2 - x^2}}\, dA = 4\int_0^1 \int_0^{\sqrt{4+y^2}} \frac{x}{\sqrt{4 + y^2 - x^2}}\, dx\, dy$$
$$= 4\int_0^1 \left\{-\sqrt{4 + y^2 - x^2}\right\}_0^{\sqrt{4+y^2}} dy = 4\int_0^1 \sqrt{4 + y^2}\, dy.$$

If we set $y = 2\tan\theta$ and $dy = 2\sec^2\theta\, d\theta$, then

$$\iint_S (x^2\hat{\mathbf{i}} + xy\hat{\mathbf{j}} + xz\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = 4\int_0^{\overline{\theta}} 2\sec\theta\, 2\sec^2\theta\, d\theta \quad (\overline{\theta} = \mathrm{Tan}^{-1}(1/2))$$
$$= 8\Big\{\sec\theta\, \tan\theta + \ln|\sec\theta + \tan\theta|\Big\}_0^{\overline{\theta}} \quad \text{(see Example 8.9)}$$
$$= 2\sqrt{5} + 8\ln[(\sqrt{5} + 1)/2].$$

15. If S_{yz} is the projection of the surface in the yz-plane, then $dS = \sqrt{1 + z^2 + y^2}\, dA$ and $\hat{\mathbf{n}} = \dfrac{\nabla(x - yz)}{|\nabla(x - yz)|} = \dfrac{(1, -z, -y)}{\sqrt{1 + y^2 + z^2}}$. Thus,

$$\iint_S (x^2\hat{\mathbf{i}} + yz\hat{\mathbf{j}} - x\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \iint_S \frac{(x^2 - yz^2 + xy)}{\sqrt{1 + y^2 + z^2}} dS = \iint_{S_{yz}} \frac{(y^2z^2 - yz^2 + y^2z)}{\sqrt{1 + y^2 + z^2}} \sqrt{1 + y^2 + z^2}\, dA$$
$$= \int_0^1 \int_0^{\sqrt{1-y^2}} (y^2z^2 - yz^2 + y^2z)\, dz\, dy = \int_0^1 \left\{\frac{y^2z^3}{3} - \frac{yz^3}{3} + \frac{y^2z^2}{2}\right\}_0^{\sqrt{1-y^2}} dy$$
$$= \frac{1}{6}\int_0^1 [2y^2(1 - y^2)^{3/2} - 2y(1 - y^2)^{3/2} + 3y^2(1 - y^2)]\, dy.$$

If we set $y = \sin\theta$ and $dy = \cos\theta\, d\theta$ in the first term,

$$\iint_S (x^2\hat{\mathbf{i}} + yz\hat{\mathbf{j}} - x\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \frac{1}{6}\int_0^{\pi/2} 2\sin^2\theta\cos^3\theta\cos\theta\, d\theta + \frac{1}{6}\left\{\frac{2}{5}(1 - y^2)^{5/2} + y^3 - \frac{3y^5}{5}\right\}_0^1$$
$$= \frac{1}{3}\int_0^{\pi/2} \left(\frac{\sin 2\theta}{2}\right)^2 \left(\frac{1 + \cos 2\theta}{2}\right) d\theta = \frac{1}{24}\int_0^{\pi/2} \left(\frac{1 - \cos 4\theta}{2} + \sin^2 2\theta\cos 2\theta\right) d\theta$$
$$= \frac{1}{24}\left\{\frac{\theta}{2} - \frac{1}{8}\sin 4\theta + \frac{1}{6}\sin^3 2\theta\right\}_0^{\pi/2} = \frac{\pi}{96}.$$

16. We divide S into two parts

$S_1 : \ x^2 + y^2/4 + z^2 = 1, \ z \ge 0,$

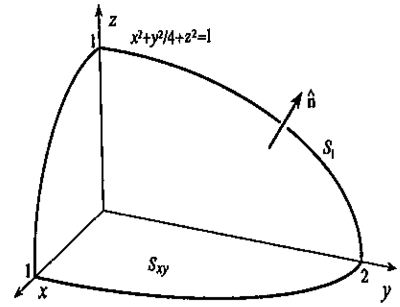
$S_2 : \ x^2 + y^2/4 + z^2 = 1, \ z \le 0,$

both of which project onto the ellipse

$$S_{xy} : \ x^2 + \frac{y^2}{4} \le 1, \ z = 0.$$

We have shown one-quarter of S_1 and S_{xy} in the figure. On S_1, $\partial z/\partial x = -x/z$ and $\partial z/\partial y = -y/(4z)$, so that

$$dS = \sqrt{1 + \left(\frac{x^2}{z^2}\right) + \left(\frac{y^2}{16z^2}\right)}\, dA = \frac{\sqrt{16z^2 + 16x^2 + y^2}}{4z}\, dA.$$



$$\text{Since } \hat{\mathbf{n}} = \frac{\nabla(x^2 + y^2/4 + z^2 - 1)}{|\nabla(x^2 + y^2/4 + z^2 - 1)|} = \frac{(2x, y/2, 2z)}{\sqrt{4x^2 + y^2/4 + 4z^2}} = \frac{(4x, y, 4z)}{\sqrt{16x^2 + y^2 + 16z^2}},$$

$$\begin{aligned}\iint_{S_1} (yx\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} + yz\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS &= \iint_{S_{xy}} \frac{(4x^2y + y^3 + 4yz^2)}{\sqrt{16x^2 + y^2 + 16z^2}} \frac{\sqrt{16z^2 + 16x^2 + y^2}}{4z}\, dA \\ &= \iint_{S_{xy}} \frac{y}{z}\, dA = \iint_{S_{xy}} \frac{y}{\sqrt{1 - x^2 - y^2/4}}\, dA.\end{aligned}$$

On S_2, $dS = \dfrac{\sqrt{16z^2 + 16x^2 + y^2}}{-4z}\, dA$ and $\hat{\mathbf{n}} = \dfrac{(4x, y, 4z)}{\sqrt{16x^2 + y^2 + 16z^2}}$, and therefore

$$\iint_{S_2} (yx\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} + yz\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \iint_{S_{xy}} \frac{y}{-z}\, dA = \iint_{S_{xy}} \frac{y}{\sqrt{1 - x^2 - y^2/4}}\, dA.$$

Hence, $\displaystyle\iint_S (yx\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} + yz\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = 2 \iint_{S_{xy}} \frac{y}{\sqrt{1 - x^2 - y^2/4}}\, dA = 0$, because the integrand is an odd function of y and S_{xy} is symmetric about the x-axis.

17. If the surface projects one-to-one onto area S_{xy} in the xy-plane, we can take its equation in the form $z = f(x, y)$. Then

$$\hat{\mathbf{n}} = \pm \frac{\left(-\dfrac{\partial z}{\partial x}, -\dfrac{\partial z}{\partial y}, 1\right)}{\sqrt{1 + \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2}} \quad \text{and} \quad dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA.$$

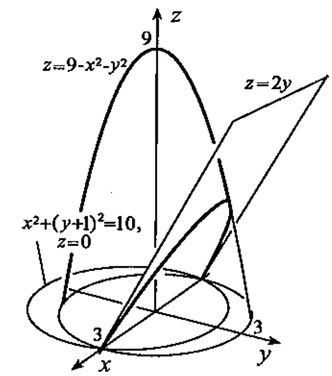
Thus, $\displaystyle\iint_S (P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \pm \iint_{S_{xy}} \left(-P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R\right) dA.$

Corresponding formulas when S projects onto S_{yz} and S_{xz} are

$$\pm \iint_{S_{yz}} \left(P - Q\frac{\partial x}{\partial y} - R\frac{\partial x}{\partial z}\right) dA \quad \text{and} \quad \pm \iint_{S_{xz}} \left(Q - P\frac{\partial y}{\partial x} - R\frac{\partial y}{\partial z}\right) dA.$$

18. (a) Since $\hat{\mathbf{n}} = \dfrac{\nabla(x^2+y^2+z-9)}{|\nabla(x^2+y^2+z-9)|} = \dfrac{(2x,2y,1)}{\sqrt{1+4x^2+4y^2}}$,

$$\iint_S (y\hat{\mathbf{i}} - x\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = \iint_{S_{xy}} \frac{(2yx-2xy+z)}{\sqrt{1+4x^2+4y^2}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$
$$= \iint_{S_{xy}} \frac{z}{\sqrt{1+4x^2+4y^2}}\sqrt{1+(-2x)^2+(-2y)^2}\,dA = \iint_{S_{xy}} (9-x^2-y^2)\,dA.$$



If we set up polar coordinates with the pole at $(0,-1)$ and polar axis parallel to the positive x-axis, then $x = r\cos\theta$ and $y = -1 + r\sin\theta$, and

$$\iint_S (y\hat{\mathbf{i}} - x\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = \int_{-\pi}^{\pi}\int_0^{\sqrt{10}} [9 - r^2\cos^2\theta - (-1+r\sin\theta)^2]\,r\,dr\,d\theta$$
$$= \int_{-\pi}^{\pi}\int_0^{\sqrt{10}} (8+2r\sin\theta - r^2)\,r\,dr\,d\theta = \int_{-\pi}^{\pi}\left\{4r^2 + \frac{2r^3\sin\theta}{3} - \frac{r^4}{4}\right\}_0^{\sqrt{10}} d\theta$$
$$= \int_{-\pi}^{\pi}\left(15 + \frac{20\sqrt{10}}{3}\sin\theta\right)d\theta = \left\{15\theta - \frac{20\sqrt{10}}{3}\cos\theta\right\}_{-\pi}^{\pi} = 30\pi.$$

(b) With $\hat{\mathbf{n}} = \dfrac{\nabla(z-2y)}{|\nabla(z-2y)|} = \dfrac{(0,-2,1)}{\sqrt{5}}$,

$$\iint_S (y\hat{\mathbf{i}} - x\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = \iint_{S_{xy}} \frac{(2x+z)}{\sqrt{5}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$
$$= \frac{1}{\sqrt{5}}\iint_{S_{xy}} (2x+2y)\sqrt{1+(2)^2}dA = 2\int_{-\pi}^{\pi}\int_0^{\sqrt{10}} (r\cos\theta - 1 + r\sin\theta)\,r\,dr\,d\theta$$
$$= 2\int_{-\pi}^{\pi}\left\{\frac{r^3}{3}(\cos\theta+\sin\theta) - \frac{r^2}{2}\right\}_0^{\sqrt{10}} d\theta = 2\int_{-\pi}^{\pi}\left[\frac{10\sqrt{10}}{3}(\cos\theta+\sin\theta) - 5\right]d\theta$$
$$= 2\left\{\frac{10\sqrt{10}}{3}(\sin\theta - \cos\theta) - 5\theta\right\}_{-\pi}^{\pi} = -20\pi.$$

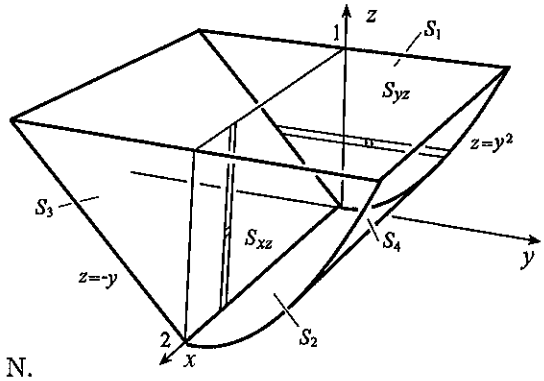
19. Normals to the surface are $\hat{\mathbf{n}} = \pm\nabla G/|\nabla G|$, and $dS = \sqrt{1+\left(\dfrac{\partial z}{\partial x}\right)^2+\left(\dfrac{\partial z}{\partial y}\right)^2}\,dA$. Since

$$\frac{\partial z}{\partial x} = -\frac{G_x}{G_z} \quad\text{and}\quad \frac{\partial z}{\partial y} = -\frac{G_y}{G_z},$$

it follows that $dS = \sqrt{1+\left(-\dfrac{G_x}{G_z}\right)^2+\left(-\dfrac{G_y}{G_z}\right)^2}\,dA = \dfrac{|\nabla G|}{|G_z|}dA.$

Thus, $\displaystyle\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \pm\iint_{S_{xy}} \mathbf{F}\cdot\left(\frac{\nabla G}{|\nabla G|}\right)\frac{|\nabla G|}{|G_z|}dA = \pm\iint_{S_{xy}} \frac{\mathbf{F}\cdot\nabla G}{|G_z|}dA.$

20. (a) The force on the end $S_1 : x = 0$ is in the negative x-direction and has magnitude



$$F_1 = \int_0^1 \int_{-z}^{\sqrt{z}} 9810(1-z)\,dy\,dz$$
$$= 9810 \int_0^1 \left\{ y(1-z) \right\}_{-z}^{\sqrt{z}} dz$$
$$= 9810 \int_0^1 (\sqrt{z} - z^{3/2} + z - z^2)\,dz$$
$$= 9810 \left\{ \frac{2z^{3/2}}{3} - \frac{2z^{5/2}}{5} + \frac{z^2}{2} - \frac{z^3}{3} \right\}_0^1 = 4251 \text{ N}.$$

Thus, $\mathbf{F}_1 = -4251\hat{\mathbf{i}}$ N. The force on $S_2 : x = 2$ is $\mathbf{F}_2 = 4251\hat{\mathbf{i}}$ N. Forces on all parts of $S_3 : y = -z$ are in the same direction, namely $-\hat{\mathbf{j}} - \hat{\mathbf{k}}$. The magnitude of the force is

$$F_3 = \iint_{S_3} P\,dS = \iint_{S_3} 9810(1-z)\,dS = 9810 \iint_{S_{xz}} (1-z)\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\,dA$$
$$= 9810 \iint_{S_{xz}} (1-z)\sqrt{1+1}\,dA = 9810\sqrt{2} \int_0^2 \int_0^1 (1-z)\,dz\,dx$$
$$= 9810\sqrt{2} \int_0^2 \left\{ z - \frac{z^2}{2} \right\}_0^1 dx = 4905\sqrt{2}\left\{x\right\}_0^2 = 9810\sqrt{2} \text{ N}.$$

Thus, $\mathbf{F}_3 = 9810\sqrt{2}\left(\dfrac{-\hat{\mathbf{j}} - \hat{\mathbf{k}}}{\sqrt{2}}\right) = -9810(\hat{\mathbf{j}} + \hat{\mathbf{k}})$ N. The force on an element dS on $S_4 : z = y^2$ points in the direction normal to S_4. At a point (x, y, z), the unit downward normal is $\hat{\mathbf{n}} = (0, 2y, -1)/\sqrt{1 + 4y^2}$. The magnitude of the force on dS is $P\,dS = 9810(1-z)\,dS$. The force on dS at (x, y, z) is therefore $\dfrac{9810(1-z)dS(0, 2y, -1)}{\sqrt{1+4y^2}}$. The y-component of the total force on S_4 is

$$F_{4_y} = \iint_{S_4} \frac{9810(1-z)(2y)}{\sqrt{1+4y^2}} dS = 19\,620 \iint_{S_{xz}} \frac{y(1-z)}{\sqrt{1+4y^2}} \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\,dA$$
$$= 19\,620 \iint_{S_{xz}} \frac{y(1-z)}{\sqrt{1+4y^2}} \sqrt{1 + \left(\frac{1}{2\sqrt{z}}\right)^2}\,dA = 19\,620 \int_0^2 \int_0^1 \left(\frac{1-z}{2}\right) dz\,dx$$
$$= 9810 \int_0^2 \left\{ z - \frac{z^2}{2} \right\}_0^1 dx = 4905\left\{x\right\}_0^2 = 9810 \text{ N}.$$

The z-component of the total force on S_4 is

$$F_{4_z} = \iint_{S_4} \frac{9810(1-z)(-1)}{\sqrt{1+4y^2}} dS = 9810 \iint_{S_{xz}} \frac{(z-1)}{\sqrt{1+4y^2}} \sqrt{1 + \left(\frac{1}{2\sqrt{z}}\right)^2}\,dA$$
$$= 9810 \int_0^2 \int_0^1 \left(\frac{z-1}{2\sqrt{z}}\right) dz\,dx = 4905 \int_0^2 \left\{ \frac{2z^{3/2}}{3} - 2\sqrt{z} \right\}_0^1 dx = -6540\left\{x\right\}_0^2 = -13080 \text{ N}.$$

Thus, $\mathbf{F}_4 = 9810\hat{\mathbf{j}} - 13080\hat{\mathbf{k}}$ N.

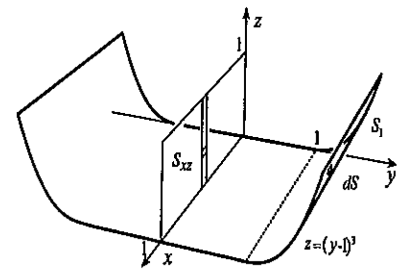
(b) The sum of the four forces is $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 = -22\,890\,\hat{\mathbf{k}}$ N. The magnitude of the weight of the water in the trough is

$$\int_0^1 \int_{-z}^{\sqrt{z}} \int_0^2 9810\,dx\,dy\,dz = 9810 \int_0^1 \int_{-z}^{\sqrt{z}} \left\{x\right\}_0^2 dy\,dx = 19\,620 \int_0^1 \left\{y\right\}_{-z}^{\sqrt{z}} dz$$
$$= 19\,620 \int_0^1 (\sqrt{z} + z)\,dz = 19\,620 \left\{ \frac{2z^{3/2}}{3} + \frac{z^2}{2} \right\}_0^1 = 22\,890 \text{ N}.$$

21. (a) The force on the bottom of the channel is equal to the weight of the water above it, namely, $\mathbf{F}_1 = -9810(2)(1)(1)\hat{\mathbf{k}} = -19\,620\hat{\mathbf{k}}$ N.

(b) The force on an elemental dS on $S_1 : z = (y-1)^3$ points in the direction normal to S_1. At a point (x, y, z), the unit downward normal to S_1 is $\hat{\mathbf{n}} = (0, 3(y-1)^2, -1)/\sqrt{1+9(y-1)^4}$. The magnitude of the force on dS is $P\,dS = 9810(1-z)\,dS$. The force on dS at (x, y, z) is therefore

$$\frac{9810(1-z)dS(0, 3(y-1)^2, -1)}{\sqrt{1+9(y-1)^4}}.$$



The y-component of the total force on S_1 is

$$\begin{aligned}
F_{2_y} &= \iint_{S_1} \frac{9810(1-z)(3)(y-1)^2}{\sqrt{1+9(y-1)^4}}\,dS = 29\,430 \iint_{S_{xz}} \frac{(1-z)(y-1)^2}{\sqrt{1+9(y-1)^4}} \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dA \\
&= 29\,430 \iint_{S_{xz}} \frac{(1-z)(y-1)^2}{\sqrt{1+9(y-1)^4}} \sqrt{1+\frac{1}{9(y-1)^4}}\,dA = 9810 \iint_{S_{xz}} (1-z)\,dA \\
&= 9810 \int_0^1\int_0^1 (1-z)\,dz\,dx = 9810 \int_0^1 \left\{z - \frac{z^2}{2}\right\}_0^1 dx = 4905\left\{x\right\}_0^1 = 4905 \text{ N.}
\end{aligned}$$

The z-component of the total force on S_1 is

$$\begin{aligned}
F_{2_z} &= \iint_{S_1} \frac{-9810(1-z)}{\sqrt{1+9(y-1)^4}}\,dS = -9810 \iint_{S_{xz}} \frac{1-z}{\sqrt{1+9(y-1)^4}} \sqrt{1+\frac{1}{9(y-1)^4}}\,dA \\
&= -9810 \iint_{S_{xz}} \frac{1-z}{3(y-1)^2}\,dA = -3270 \iint_{S_{xz}} \frac{1-z}{z^{2/3}}\,dA = -3270 \int_0^1\int_0^1 \left(\frac{1}{z^{2/3}} - z^{1/3}\right) dz\,dx \\
&= -3270 \int_0^1 \left\{3z^{1/3} - \frac{3z^{4/3}}{4}\right\}_0^1 dx = -3270\left(\frac{9}{4}\right)\left\{x\right\}_0^1 = -\frac{14\,715}{2} \text{ N.}
\end{aligned}$$

Thus, $\mathbf{F}_2 = 4905\hat{\mathbf{j}} - (14\,715/2)\hat{\mathbf{k}}$ N.

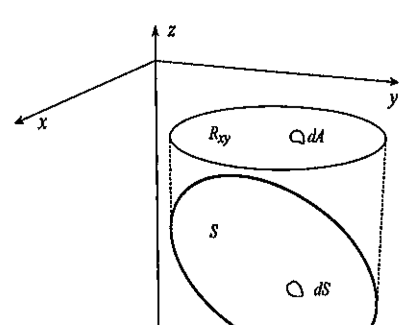
(c) By symmetry, the force on the left wall is $\mathbf{F}_3 = -4905\hat{\mathbf{j}} - (14\,715/2)\hat{\mathbf{k}}$ N.

The sum of all three forces is $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = -34\,335\hat{\mathbf{k}}$ N. The magnitude of the weight of the water in 1 metre of length is

$$\begin{aligned}
19\,620 + 2\int_0^1\int_1^2\int_{(y-1)^3}^1 9810\,dz\,dy\,dx &= 19\,620 + 19\,620 \int_0^1\int_1^2 [1-(y-1)^3]\,dy\,dx \\
&= 19\,620 + 19\,620 \int_0^1 \left\{y - \frac{(y-1)^4}{4}\right\}_1^2 dx = 19\,620 + 19\,620\left(\frac{3}{4}\right)\left\{x\right\}_0^1 = 34\,335 \text{ N.}
\end{aligned}$$

22. The magnitude of the fluid force on S is

$$\begin{aligned}
\iint_S P\,dS &= \iint_{R_{xy}} -\rho g z \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA \\
&= -\rho g \iint_{R_{xy}} z\sqrt{1+\left(\frac{-A}{C}\right)^2+\left(\frac{-B}{C}\right)^2}\,dA \\
&= -\frac{\rho g\sqrt{A^2+B^2+C^2}}{|C|} \iint_{R_{xy}} \left(\frac{-D-Ax-By}{C}\right) dA \\
&= \frac{\rho g\sqrt{A^2+B^2+C^2}}{C|C|} \iint_{R_{xy}} (D+Ax+By)\,dA.
\end{aligned}$$



A unit normal vector to S is $\pm(A,B,C)/\sqrt{A^2+B^2+C^2}$, where the plus or minus is chosen depending on the sign of C and whether we consider the top or bottom of S. The force on S is

$$\left[\frac{\rho g\sqrt{A^2+B^2+C^2}}{C|C|}\iint_{R_{xy}}(D+Ax+By)\,dA\right]\frac{\pm(A,B,C)}{\sqrt{A^2+B^2+C^2}}.$$

The magnitude of the z-component of this force is $F_z=\dfrac{\rho g}{C}\displaystyle\iint_{R_{xy}}(D+Ax+By)\,dA$. On the other hand, the weight of the column of fluid above S is

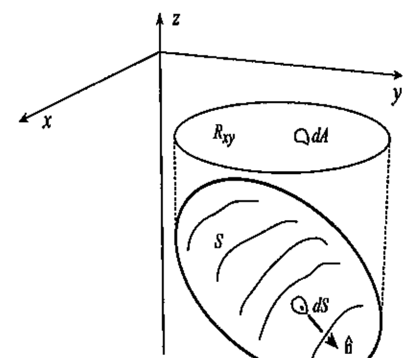
$$W=\iint_{R_{xy}}\int_{(-D-Ax-By)/C}^{0}\rho g\,dz\,dA=\iint_{R_{xy}}\rho g\left(\frac{D+Ax+By}{C}\right)dA=\frac{\rho g}{C}\iint_{R_{xy}}(D+Ax+By)\,dA.$$

Consequently, the magnitude of F_z is equal to W.

23. The force due to fluid pressure on a small area dS on (the top of) S is $(P\,dS)\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is the unit downward pointing normal to S at dS. The z-component of this force is $(P\,dS)\hat{\mathbf{n}}\cdot\hat{\mathbf{k}}$, and the total z-component of the fluid force on S is

$$F_z=\iint_S P\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\,dS=\iint_S -\rho g z\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\,dS.$$

If we take the equation for S in the form $z=f(x,y)$, then $\hat{\mathbf{n}}=\dfrac{(f_x,f_y,-1)}{\sqrt{1+(f_x)^2+(f_y)^2}}$, and $dS=\sqrt{1+(f_x)^2+(f_y)^2}\,dA$. Hence,



$$F_z=\iint_{R_{xy}}-\rho g f(x,y)\left[\frac{-1}{\sqrt{1+(f_x)^2+(f_y)^2}}\right]\sqrt{1+(f_x)^2+(f_y)^2}\,dA=\rho g\iint_{R_{xy}}f(x,y)\,dA.$$

This is a negative quantity since $f(x,y)<0$ for all (x,y) in R_{xy}. On the other hand, the weight of the column of fluid above S is

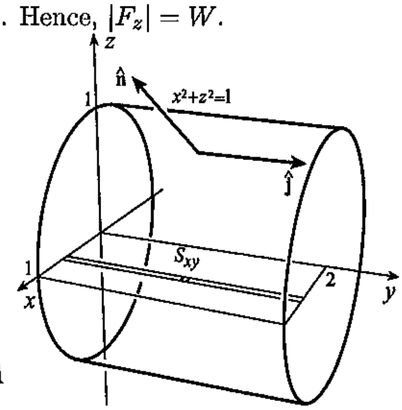
$$W=\iint_{R_{xy}}\int_{f(x,y)}^{0}\rho g\,dz\,dA=\rho g\iint_{R_{xy}}-f(x,y)\,dA=-\rho g\iint_{R_{xy}}f(x,y)\,dA.$$

This is a positive quantity since $f(x,y)<0$ for all (x,y) in R_{xy}. Hence, $|F_z|=W$.

24. The total amount of blood per second is

$$\iint_S\mathbf{F}\cdot\hat{\mathbf{n}}\,dS=\iint_S e^{-y}\,dS.$$

The same amount of diffusion occurs in each of the four octants. If S_{xy} is the projection of the first octant part of S in the xy-plane, then



$$\begin{aligned}\iint_S\mathbf{F}\cdot\hat{\mathbf{n}}\,dS&=4\iint_{S_{xy}}e^{-y}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA\\&=4\iint_{S_{xy}}e^{-y}\sqrt{1+\left(-\frac{x}{z}\right)^2}\,dA=4\iint_{S_{xy}}\frac{e^{-y}}{\sqrt{1-x^2}}\,dA\\&=4\int_0^1\int_0^2\frac{e^{-y}}{\sqrt{1-x^2}}\,dy\,dx=4\int_0^1\left\{\frac{-e^{-y}}{\sqrt{1-x^2}}\right\}_0^2dx=4(1-e^{-2})\int_0^1\frac{1}{\sqrt{1-x^2}}\,dx\\&=4(1-e^{-2})\left\{\mathrm{Sin}^{-1}x\right\}_0^1=2\pi(1-e^{-2}).\end{aligned}$$

25. (a) The normal to S is $\hat{\mathbf{n}} = \dfrac{(x,y,z)}{\sqrt{x^2+y^2+z^2}} = \dfrac{(x,y,z)}{a}$, and when S is projected into the xz-plane,

$$dS = \sqrt{1 + \left(\frac{-x}{\sqrt{a^2-x^2-z^2}}\right)^2 + \left(\frac{-z}{\sqrt{a^2-x^2-z^2}}\right)^2}\,dA = \frac{a}{\sqrt{a^2-x^2-z^2}}dA.$$

Thus, absorption in time dt is

$$\begin{aligned}\iint_S \mathbf{I}\cdot\hat{\mathbf{n}}\,dS\,dt &= dt\iint_S \frac{ye^{-t}}{ay^2}dS = \frac{e^{-t}\,dt}{a}\iint_{S_{xz}} \frac{a}{a^2-x^2-z^2}dA \\ &= 4e^{-t}\,dt\int_0^{\pi/2}\int_0^{\sqrt{3}a/2}\frac{1}{a^2-r^2}r\,dr\,d\theta = 4e^{-t}\,dt\int_0^{\pi/2}\left\{-\frac{1}{2}\ln|a^2-r^2|\right\}_0^{\sqrt{3}a/2}d\theta \\ &= 4\ln 2e^{-t}\,dt\Big\{\theta\Big\}_0^{\pi/2} = 2\pi\ln 2e^{-t}\,dt.\end{aligned}$$

(b) Absorption from $t=0$ to $t=5$ is $\displaystyle\int_0^5 2\pi\ln 2e^{-t}\,dt = 2\pi\ln 2\Big\{e^{-t}\Big\}_0^5 = 2\pi\ln 2(1-e^{-5})$.

EXERCISES 14.9

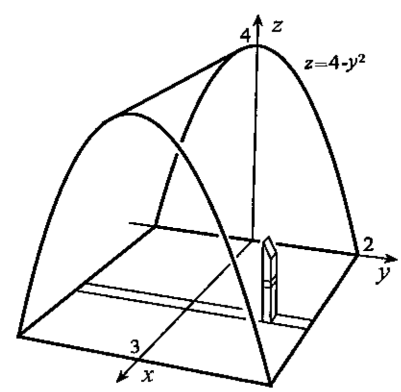
1. By the divergence theorem, $\displaystyle\oiint_S (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} - 2z\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = \iiint_V (1+1-2)\,dV = 0.$

2. By the divergence theorem, $\displaystyle\oiint_S (x^2\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} + z^2\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = \iiint_V (2x+2y+2z)\,dV$. Since the triple integral is twice the sum of the first moments of the sphere about the coordinate planes, it must have value zero.

3. By the divergence theorem, $\displaystyle\oiint_S (yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = \iiint_V (0+0+0)\,dV = 0.$

4. By the divergence theorem,

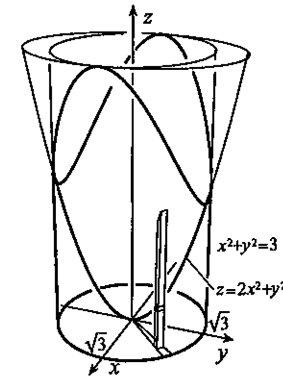
$$\begin{aligned}&\oiint_S [(z^2-x)\hat{\mathbf{i}} - xy\hat{\mathbf{j}} + 3z\hat{\mathbf{k}}]\cdot\hat{\mathbf{n}}\,dS = \iiint_V (-1-x+3)\,dV \\ &= \int_0^3\int_{-2}^2\int_0^{4-y^2}(2-x)\,dz\,dy\,dx \\ &= \int_0^3\int_{-2}^2 (2-x)(4-y^2)\,dy\,dx \\ &= \int_0^3\left\{(2-x)\left(4y-\frac{y^3}{3}\right)\right\}_{-2}^2 dx \\ &= \frac{32}{3}\left\{2x-\frac{x^2}{2}\right\}_0^3 = 16.\end{aligned}$$



5. By the divergence theorem,

$$\begin{aligned}\oiint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS &= -\iiint_V (xy+yz+xz)\,dV = -\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^1 (xy+yz+xz)\,dz\,dy\,dx \\ &= -\int_0^1\int_0^{\sqrt{1-x^2}}\left\{xyz+\frac{yz^2}{2}+\frac{xz^2}{2}\right\}_0^1 dy\,dx = -\frac{1}{2}\int_0^1\int_0^{\sqrt{1-x^2}}(2xy+y+x)\,dy\,dx \\ &= -\frac{1}{2}\int_0^1\left\{xy^2+\frac{y^2}{2}+xy\right\}_0^{\sqrt{1-x^2}}dx = -\frac{1}{4}\int_0^1 [2x(1-x^2)+(1-x^2)+2x\sqrt{1-x^2}]\,dx \\ &= -\frac{1}{4}\left\{x+x^2-\frac{x^3}{3}-\frac{x^4}{2}-\frac{2}{3}(1-x^2)^{3/2}\right\}_0^1 = -\frac{11}{24}.\end{aligned}$$

6. By the divergence theorem,



$$\oiint_S (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \iiint_V (1+1+2)\, dV$$

$$= 4\int_{-\pi}^{\pi}\int_0^{\sqrt{3}}\int_0^{2r^2\cos^2\theta + r^2\sin^2\theta} r\, dz\, dr\, d\theta$$

$$= 4\int_{-\pi}^{\pi}\int_0^{\sqrt{3}} r(2r^2\cos^2\theta + r^2\sin^2\theta)\, dr\, d\theta$$

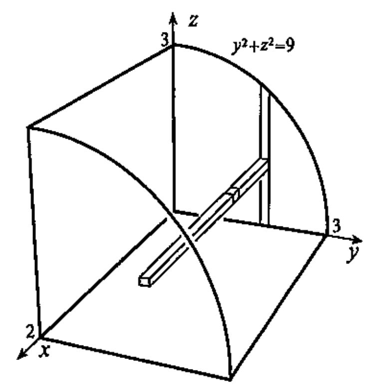
$$= 4\int_{-\pi}^{\pi}\left\{\frac{r^4}{2}\cos^2\theta + \frac{r^4}{4}\sin^2\theta\right\}_0^{\sqrt{3}} d\theta = 9\int_{-\pi}^{\pi}(2\cos^2\theta + \sin^2\theta)\, d\theta$$

$$= 9\int_{-\pi}^{\pi}\left(1 + \frac{1+\cos 2\theta}{2}\right) d\theta = 9\left\{\frac{3\theta}{2} + \frac{\sin 2\theta}{4}\right\}_{-\pi}^{\pi} = 27\pi.$$

7. By the divergence theorem, $\oiint_S (z\hat{\mathbf{i}} - x\hat{\mathbf{j}} + y\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \iiint_V (0+0+0)\, dV = 0.$

8. By the divergence theorem,

$$\oiint_S (2x^2y\hat{\mathbf{i}} - y^2\hat{\mathbf{j}} + 4xz^2\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \iiint_V (4xy - 2y + 8xz)\, dV$$



$$= 2\int_0^3\int_0^{\sqrt{9-y^2}}\int_0^2 (2xy - y + 4xz)\, dx\, dz\, dy$$

$$= 2\int_0^3\int_0^{\sqrt{9-y^2}}\left\{x^2y - xy + 2x^2z\right\}_0^2 dz\, dy$$

$$= 2\int_0^3\int_0^{\sqrt{9-y^2}} (2y + 8z)\, dz\, dy$$

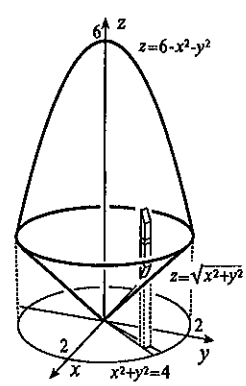
$$= 4\int_0^3\left\{yz + 2z^2\right\}_0^{\sqrt{9-y^2}} dy = 4\int_0^3 (y\sqrt{9-y^2} + 18 - 2y^2)\, dy$$

$$= 4\left\{-\frac{1}{3}(9-y^2)^{3/2} + 18y - \frac{2y^3}{3}\right\}_0^3 = 180.$$

9. By the divergence theorem, $\oiint_S (yx\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} + yz\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \iiint_V (y + 2y + y)\, dV = 4\iiint_V y\, dV$. Since the integrand is an odd function of y, and V is symmetric about the xz-plane, the value of the triple integral is zero.

10. By the divergence theorem,

$$\oiint_S (x^3\hat{\mathbf{i}} + y^3\hat{\mathbf{j}} - z^3\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \iiint_V (3x^2 + 3y^2 - 3z^2)\, dV$$



$$= 3\int_{-\pi}^{\pi}\int_0^2\int_r^{6-r^2} (r^2 - z^2)\, r\, dz\, dr\, d\theta$$

$$= 3\int_{-\pi}^{\pi}\int_0^2\left\{r^3z - \frac{rz^3}{3}\right\}_r^{6-r^2} dr\, d\theta$$

$$= \int_{-\pi}^{\pi}\int_0^2 (r^7 - 21r^5 - 2r^4 + 126r^3 - 216r)\, dr\, d\theta$$

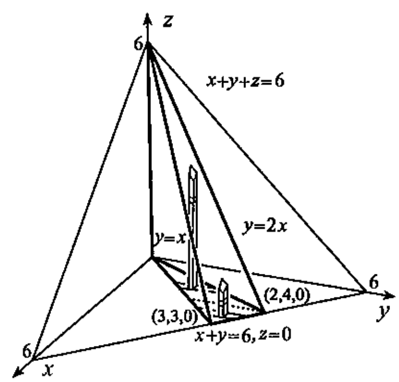
$$= \int_{-\pi}^{\pi}\left\{\frac{r^8}{8} - \frac{7r^6}{2} - \frac{2r^5}{5} + \frac{63r^4}{2} - 108r^2\right\}_0^2 d\theta$$

$$= \frac{-664}{5}\left\{\theta\right\}_{-\pi}^{\pi} = -\frac{1328\pi}{5}.$$

11. By the divergence theorem, $\oiint_S (y\hat{\mathbf{i}} - xy\hat{\mathbf{j}} + zy^2\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = -\iiint_V (-x+y^2)\,dV$. Since x is an odd function of x and V is symmetric about the yz-plane, this term contributes nothing to the integral. If we introduce polar coordinates $x = r\cos\theta$ and $z = r\sin\theta$ in the xz-plane, then

$$\begin{aligned}\oiint_S (y\hat{\mathbf{i}} - xy\hat{\mathbf{j}} + zy^2\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS &= -4\int_0^{\pi/2}\int_0^{2\sqrt{3}}\int_{\sqrt{4+r^2}}^4 y^2 r\,dy\,dr\,d\theta\\ &= -4\int_0^{\pi/2}\int_0^{2\sqrt{3}}\left\{\frac{ry^3}{3}\right\}_{\sqrt{4+r^2}}^4 dr\,d\theta\\ &= -\frac{4}{3}\int_0^{\pi/2}\int_0^{2\sqrt{3}}[64r - r(4+r^2)^{3/2}]\,dr\,d\theta\\ &= -\frac{4}{3}\int_0^{\pi/2}\left\{32r^2 - \frac{1}{5}(4+r^2)^{5/2}\right\}_0^{2\sqrt{3}} d\theta = -\frac{3712}{15}\left\{\theta\right\}_0^{\pi/2} = -\frac{1856\pi}{15}.\end{aligned}$$

12. By the divergence theorem,



$$\begin{aligned}&\oiint_S (xy\hat{\mathbf{i}} + z^2\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = \iiint_V (y+2z)\,dV\\ &= \int_0^2\int_x^{2x}\int_0^{6-x-y}(y+2z)\,dz\,dy\,dx\\ &\quad+\int_2^3\int_x^{6-x}\int_0^{6-x-y}(y+2z)\,dz\,dy\,dx\\ &= \int_0^2\int_x^{2x}\left\{yz+z^2\right\}_0^{6-x-y}dy\,dx\\ &\quad+\int_2^3\int_x^{6-x}\left\{yz+z^2\right\}_0^{6-x-y}dy\,dx\\ &= \int_0^2\int_x^{2x}[y(6-x-y)+(6-x-y)^2]\,dy\,dx + \int_2^3\int_x^{6-x}[y(6-x-y)+(6-x-y)^2]\,dy\,dx\\ &= \int_0^2\left\{3y^2 - \frac{xy^2}{2} - \frac{y^3}{3} - \frac{(6-x-y)^3}{3}\right\}_x^{2x}dx + \int_2^3\left\{3y^2 - \frac{xy^2}{2} - \frac{y^3}{3} - \frac{(6-x-y)^3}{3}\right\}_x^{6-x}dx\\ &= \int_0^2\left[9x^2 - \frac{23x^3}{6} - \frac{(6-3x)^3}{3} + \frac{(6-2x)^3}{3}\right]dx\\ &\quad+\int_2^3\left[3(6-x)^2 - \frac{(6-x)^3}{3} + \frac{(6-2x)^3}{3} - 18x + 3x^2 + \frac{x^3}{3}\right]dx\\ &= \left\{3x^3 - \frac{23x^4}{24} + \frac{(6-3x)^4}{36} - \frac{(6-2x)^4}{24}\right\}_0^2\\ &\quad+\left\{-(6-x)^3 + \frac{(6-x)^4}{12} - \frac{(6-2x)^4}{24} - 9x^2 + x^3 + \frac{x^4}{12}\right\}_2^3 = \frac{57}{2}.\end{aligned}$$

13. If we let S' be that part of the xy-plane bounded by $x^2 + 4y^2 = 36$, then

$$\oiint_{S+S'}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = \iiint_V (1+1+1)\,dV = 3\iiint_V dV.$$

Therefore, $\oiint_S (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = 3\iiint_V dV - \iint_{S'}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS$.

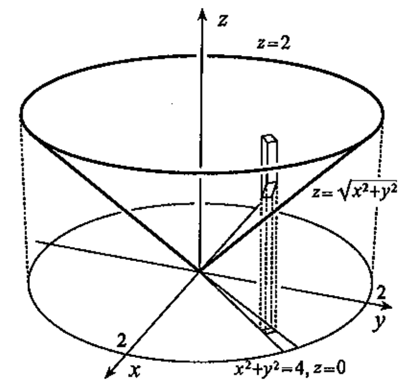
The volume of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ is $4\pi abc/3$ (see Exercise 27 in Section 13.9). Since $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$ on S', $\iint_S (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = 3\left(\frac{2\pi}{3}\right)(6)(3)(2) - \iint_{S'} -z\,dS = 72\pi$.

14. If we create a closed surface by including with S the surface $S' : z = 2,\ x^2 + y^2 \le 4$, then

$$\iint_S (xy\hat{\mathbf{i}} - yz\hat{\mathbf{j}} + x^2z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS + \iint_{S'} (xy\hat{\mathbf{i}} - yz\hat{\mathbf{j}} + x^2z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = -\iiint_V (y - z + x^2)\, dV$$

provided $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$ on S'. Now,

$$\iiint_V (y - z + x^2)\, dV = \int_{-\pi}^{\pi}\int_0^2\int_r^2 (-z + r^2\cos^2\theta)\, r\, dz\, dr\, d\theta$$
$$= \int_{-\pi}^{\pi}\int_0^2 r\left\{-\frac{z^2}{2} + zr^2\cos^2\theta\right\}_r^2 dr\, d\theta$$
$$= \int_{-\pi}^{\pi}\int_0^2 \left(-2r + 2r^3\cos^2\theta + \frac{r^3}{2} - r^4\cos^2\theta\right) dr\, d\theta$$
$$= \int_{-\pi}^{\pi} \left\{-r^2 + \frac{r^4\cos^2\theta}{2} + \frac{r^4}{8} - \frac{r^5\cos^2\theta}{5}\right\}_0^2 d\theta$$
$$= \int_{-\pi}^{\pi} \left(-4 + 8\cos^2\theta + 2 - \frac{32}{5}\cos^2\theta\right) d\theta$$
$$= \int_{-\pi}^{\pi} \left[-2 + \frac{4(1 + \cos 2\theta)}{5}\right] d\theta = \left\{-\frac{6\theta}{5} + \frac{2\sin 2\theta}{5}\right\}_{-\pi}^{\pi} = -\frac{12\pi}{5}.$$



We now calculate that

$$\iint_{S'} (xy\hat{\mathbf{i}} - yz\hat{\mathbf{j}} + x^2z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \iint_{S'} -x^2z\, dS = -2\iint_{S'} x^2\, dS = -2\iint_{S'_{xy}} x^2\, dA$$
$$= -2\int_{-\pi}^{\pi}\int_0^2 r^2\cos^2\theta\, r\, dr\, d\theta = -2\int_{-\pi}^{\pi}\left\{\frac{r^4}{4}\cos^2\theta\right\}_0^2 d\theta = -8\int_{-\pi}^{\pi}\cos^2\theta\, d\theta$$
$$= -8\int_{-\pi}^{\pi}\left(\frac{1 + \cos 2\theta}{2}\right) d\theta = -4\left\{\theta + \frac{\sin 2\theta}{2}\right\}_{-\pi}^{\pi} = -8\pi.$$

Thus, $\displaystyle\iint_S (xy\hat{\mathbf{i}} - yz\hat{\mathbf{j}} + x^2z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \frac{12\pi}{5} + 8\pi = \frac{52\pi}{5}.$

15. If S' is that part of the plane $z = 2y$ cut out by $z = 4 - x^2 - y^2$, then

$$\oiint_{S+S'} (y^2e^z\hat{\mathbf{i}} - xy\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \iiint_V (-x + 1)\, dV.$$

The integral of x over V is zero because V is symmetric about the yz-plane. Therefore,

$$\iint_S (y^2e^z\hat{\mathbf{i}} - xy\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \iiint_V dV - \iint_{S'} (y^2e^z\hat{\mathbf{i}} - xy\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS.$$

On S', $\hat{\mathbf{n}} = (0, 2, -1)/\sqrt{5}$, and if S' projects onto S_{xy} in the xy-plane, then $dS = \sqrt{1 + (2)^2}\, dA = \sqrt{5}\, dA$. Thus,

$$\iint_S (y^2e^z\hat{\mathbf{i}} - xy\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = 2\int_0^{\sqrt{5}}\int_{-1-\sqrt{5-x^2}}^{-1+\sqrt{5-x^2}}\int_{2y}^{4-x^2-y^2} dz\, dy\, dx - \iint_{S'} \frac{(-2xy - z)}{\sqrt{5}} dS$$
$$= 2\int_0^{\sqrt{5}}\int_{-1-\sqrt{5-x^2}}^{-1+\sqrt{5-x^2}} (4 - x^2 - y^2 - 2y)\, dy\, dx + \iint_{S_{xy}} \frac{(2xy + 2y)}{\sqrt{5}}\sqrt{5}\, dA$$
$$= 2\int_0^{\sqrt{5}}\int_{-1-\sqrt{5-x^2}}^{-1+\sqrt{5-x^2}} (4 - x^2 - y^2 - 2y)\, dy\, dx + 4\int_0^{\sqrt{5}}\int_{-1-\sqrt{5-x^2}}^{-1+\sqrt{5-x^2}} y\, dy\, dx$$
$$= 2\int_0^{\sqrt{5}}\int_{-1-\sqrt{5-x^2}}^{-1+\sqrt{5-x^2}} (4 - x^2 - y^2)\, dy\, dx$$

$$= 2\int_0^{\sqrt{5}} \left\{ (4-x^2)y - \frac{y^3}{3} \right\}_{-1-\sqrt{5-x^2}}^{-1+\sqrt{5-x^2}} dx = \frac{8}{3}\int_0^{\sqrt{5}} (2-x^2)\sqrt{5-x^2}\, dx.$$

If we set $x = \sqrt{5}\sin\theta$, and $dx = \sqrt{5}\cos\theta\, d\theta$, then

$$\begin{aligned}\iint_S (y^2 e^z \hat{\mathbf{i}} - xy\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS &= \frac{8}{3}\int_0^{\pi/2} (2 - 5\sin^2\theta)\sqrt{5}\cos\theta\sqrt{5}\cos\theta\, d\theta \\ &= \frac{40}{3}\int_0^{\pi/2} \left[1 + \cos 2\theta - \frac{5}{4}\left(\frac{1-\cos 4\theta}{2}\right)\right] d\theta \\ &= \frac{40}{3}\left\{\frac{3\theta}{8} + \frac{1}{2}\sin 2\theta + \frac{5}{32}\sin 4\theta\right\}_0^{\pi/2} = \frac{5\pi}{2}.\end{aligned}$$

16. By the divergence theorem, $\frac{1}{3}\oint\!\!\!\oint_S \mathbf{r}\cdot\hat{\mathbf{n}}\, dS = \frac{1}{3}\iiint_V \nabla\cdot\mathbf{r}\, dV = \frac{1}{3}\iiint_V (1+1+1)\, dV = \iiint_V dV = V.$

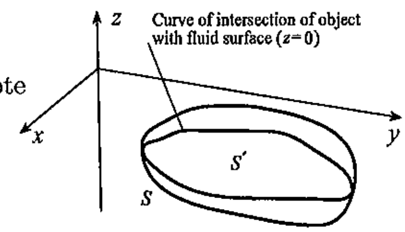
17. If we set $\mathbf{F} = \hat{\mathbf{n}}$ in the divergence theorem, then $\iiint_V \nabla\cdot\hat{\mathbf{n}}\, dV = \oint\!\!\!\oint_S \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}\, dS = \oint\!\!\!\oint_S dS = \text{area}(S).$

18. Using the discussion in Example 14.26, we can state that the total buoyant force must be

$$\iint_S (9.81\rho z\hat{\mathbf{k}}) \cdot (-\hat{\mathbf{n}})\, dS$$

where S is the submerged portion of the surface. Suppose we remove that part of the object above the fluid surface and denote by S' the remaining part of the solid in the fluid surface. Since

$$\iint_{S'} (9.81\rho z\hat{\mathbf{k}}) \cdot (-\hat{\mathbf{n}})\, dS = 0,$$



we may write that

$$\iint_S (9.81\rho z\hat{\mathbf{k}}) \cdot (-\hat{\mathbf{n}})\, dS + \iint_{S'} (9.81\rho z\hat{\mathbf{k}}) \cdot (-\hat{\mathbf{n}})\, dS = \iiint_V \nabla\cdot(9.81\rho z\hat{\mathbf{k}})\, dV,$$

or, $$\begin{aligned}\iint_S (9.81\rho z\hat{\mathbf{k}}) \cdot (-\hat{\mathbf{n}})\, dS = \iiint_V 9.81\rho\, dV &= 9.81\rho(\text{Volume of object below fluid surface}) \\ &= \text{Weight of fluid displaced by object.}\end{aligned}$$

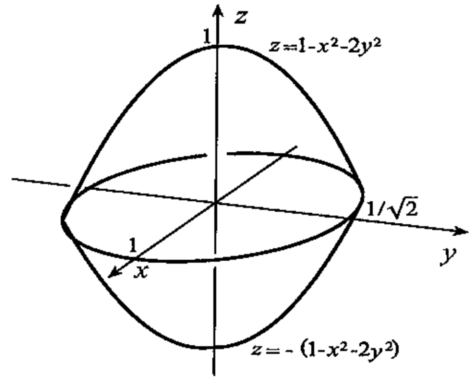
19. By the divergence theorem,

$$\begin{aligned}&\oint\!\!\!\oint_S [(x+y)\hat{\mathbf{i}} + y^3\hat{\mathbf{j}} + x^2 z\hat{\mathbf{k}}] \cdot \hat{\mathbf{n}}\, dS \\ &= \iiint_V (1 + 3y^2 + x^2)\, dV = 8\int_0^{\pi/2}\int_0^1\int_{\sqrt{2}z}^{\sqrt{z^2+1}} (1 + 3r^2\sin^2\theta + r^2\cos^2\theta)\, r\, dr\, dz\, d\theta \\ &= 8\int_0^{\pi/2}\int_0^1 \left\{\frac{r^2}{2} + \frac{3r^4}{4}\sin^2\theta + \frac{r^4}{4}\cos^2\theta\right\}_{\sqrt{2}z}^{\sqrt{z^2+1}} dr\, d\theta \\ &= 2\int_0^{\pi/2}\int_0^1 [2(z^2+1) + 3(z^2+1)^2\sin^2\theta + (z^2+1)^2\cos^2\theta - 4z^2 - 12z^4\sin^2\theta - 4z^4\cos^2\theta]\, dz\, d\theta \\ &= 2\int_0^{\pi/2}\int_0^1 [2 - 2z^2 + (-9z^4 + 6z^2 + 3)\sin^2\theta + (-3z^4 + 2z^2 + 1)\cos^2\theta]\, dz\, d\theta \\ &= 2\int_0^{\pi/2} \left\{2z - \frac{2z^3}{3} + \left(-\frac{9z^5}{5} + 2z^3 + 3z\right)\sin^2\theta + \left(-\frac{3z^5}{5} + \frac{2z^3}{3} + z\right)\cos^2\theta\right\}_0^1 d\theta \\ &= 2\int_0^{\pi/2} \left[\frac{4}{3} + \frac{8}{5}(1 - \cos 2\theta) + \frac{8}{15}(1 + \cos 2\theta)\right] d\theta \\ &= 2\left\{\frac{4\theta}{3} + \frac{8}{5}\left(\theta - \frac{1}{2}\sin 2\theta\right) + \frac{8}{15}\left(\theta + \frac{1}{2}\sin 2\theta\right)\right\}_0^{\pi/2} = \frac{52\pi}{15}.\end{aligned}$$

20. By the divergence theorem,

$$\oiint_S [(x+y)^2\hat{\mathbf{i}} + x^2y\hat{\mathbf{j}} - x^2z\hat{\mathbf{k}}] \cdot \hat{\mathbf{n}}\, dS$$
$$= -\iiint_V [2(x+y) + x^2 - x^2]\, dV$$
$$= -2\iiint_V (x+y)\, dV.$$

Since the triple integral is the sum of the first moments of V about the yz- and xz-coordinate planes, its value must be zero.



21. If S' is that part of the xy-plane bounded by $x^2+y^2=1$, then

$$\oiint_{S+S'} [(y^3+x^2y)\hat{\mathbf{i}} + (x^3-xy^2)\hat{\mathbf{j}} + z\hat{\mathbf{k}}] \cdot \hat{\mathbf{n}}\, dS = \iiint_V (2xy - 2xy + 1)\, dV = \iiint_V dV.$$

Since $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$ on S',

$$\iint_S [(y^3+x^2y)\hat{\mathbf{i}} + (x^3-xy^2)\hat{\mathbf{j}} + z\hat{\mathbf{k}}] \cdot \hat{\mathbf{n}}\, dS = \iiint_V dV - \oiint_{S'} [(y^3+x^2y)\hat{\mathbf{i}} + (x^3-xy^2)\hat{\mathbf{j}} + z\hat{\mathbf{k}}] \cdot (-\hat{\mathbf{k}})\, dS$$
$$= \frac{2\pi}{3} - \iint_{S'} -z\, dS = \frac{2\pi}{3}.$$

22. By the divergence theorem, $\oiint_S \mathbf{B}\cdot\hat{\mathbf{n}}\, dS = \iiint_V \nabla\cdot\mathbf{B}\, dV = \iiint_V \nabla\cdot(\nabla\times\mathbf{A})\, dV$. But according to equation 14.15, $\nabla\cdot(\nabla\times\mathbf{A}) = 0$, and hence the required result follows immediately.

23. The divergence theorem is the three-dimensional analogue of Green's theorem when it is written in the form in Exercise 13 of Section 14.6.

24. By the divergence theorem, $\oiint_S \nabla P\cdot\hat{\mathbf{n}}\, dS = \iiint_V \nabla\cdot\nabla P\, dV = \iiint_V \nabla^2 P\, dV$. If $\nabla^2 P = 0$ in V, then $\oiint_S \nabla P\cdot\hat{\mathbf{n}}\, dS = 0$.

25. If we set $\mathbf{F} = P\,\nabla Q$ in the divergence theorem, $\oiint_S (P\,\nabla Q)\cdot\hat{\mathbf{n}}\, dS = \iiint_V \nabla\cdot(P\,\nabla Q)\, dV$. Using identity 14.11 on the right gives

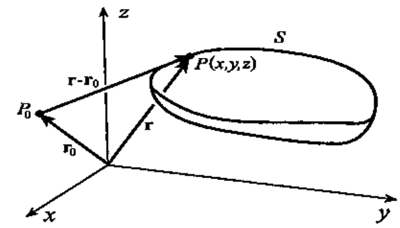
$$\oiint_S (P\,\nabla Q)\cdot\hat{\mathbf{n}}\, dS = \iiint_V (\nabla P\cdot\nabla Q + P\nabla\cdot\nabla Q)\, dV = \iiint_V (\nabla P\cdot\nabla Q + P\nabla^2 Q)\, dV.$$

26. If we reverse the roles of P and Q in Exercise 25, $\oiint_S Q\nabla P\cdot\hat{\mathbf{n}}\, dS = \iiint_V (Q\nabla^2 P + \nabla Q\cdot\nabla P)\, dV$. When we subtract this result from that in Exercise 25, we obtain

$$\oiint_S (P\nabla Q - Q\nabla P)\cdot\hat{\mathbf{n}}\, dS = \iiint_V (P\nabla^2 Q - Q\nabla^2 P)\, dV.$$

27. Exercises 24–26 in this section are the three-dimensional analogues of Exercises 36–38 in Section 14.6.

28. If P_0 is outside V, then $\mathbf{r}-\mathbf{r}_0 \neq \mathbf{0}$ on or inside S, and by the divergence theorem,
$\displaystyle\oiint_S \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\cdot\hat{\mathbf{n}}\,dS = \iiint_V \nabla\cdot\left(\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\right)dV.$
Now,



$$\nabla\cdot\left(\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\right) = \nabla\cdot\left(\frac{(x-x_0)\hat{\mathbf{i}}+(y-y_0)\hat{\mathbf{j}}+(z-z_0)\hat{\mathbf{k}}}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}}\right),$$

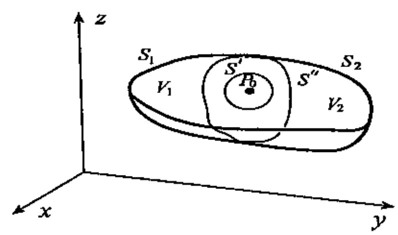
and it is a straightforward calculation to show that

$$\frac{\partial}{\partial x}\left(\frac{x-x_0}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}}\right) = \frac{(y-y_0)^2+(z-z_0)^2-2(x-x_0)^2}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{5/2}}.$$

With similar calculations for the partial derivatives of the y- and z-components with respect to y and z, we obtain

$$\begin{aligned}\nabla\cdot\left(\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\right) &= \frac{1}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{5/2}}\{[(y-y_0)^2+(z-z_0)^2-2(x-x_0)^2] \\ &\quad + [(x-x_0)^2+(z-z_0)^2-2(y-y_0)^2] + [(x-x_0)^2+(y-y_0)^2-2(z-z_0)^2]\} \\ &= 0.\end{aligned}$$

Thus, when P_0 is outside S, $\displaystyle\oiint_S \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\cdot\hat{\mathbf{n}}\,dS = 0.$
Now suppose that P_0 is inside S. We construct a sphere S' of radius R and centre P_0 which is entirely within V. We now join S and S' by a surface S'' so that S'' divides V into two parts V_1 and V_2, with bounding surfaces S_1 and S_2. Since P_0 is outside both V_1 and V_2, we have



$$\oiint_{S_1} \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\cdot\hat{\mathbf{n}}\,dS = 0, \qquad \oiint_{S_2} \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\cdot\hat{\mathbf{n}}\,dS = 0.$$

When these results are added together,

$$0 = \oiint_{S_1} \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\cdot\hat{\mathbf{n}}\,dS + \oiint_{S_2} \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\cdot\hat{\mathbf{n}}\,dS.$$

But in evaluating integrals over S_1 and S_2, integration is performed over S'' twice, once with $\hat{\mathbf{n}}$ in one direction, and once with $\hat{\mathbf{n}}$ in the opposite direction. As a result, the contributions over S'' vanish, leaving only the surface integrals over S and S', where $\hat{\mathbf{n}}$ is the outer normal on S and the inner normal on S',

$$0 = \oiint_S \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\cdot\hat{\mathbf{n}}\,dS + \oiint_{S'} \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\cdot\hat{\mathbf{n}}\,dS.$$

On S' : $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = R^2$, $\quad \hat{\mathbf{n}} = -\dfrac{1}{R}(x-x_0, y-y_0, z-z_0) = -\dfrac{\mathbf{r}-\mathbf{r}_0}{R}$, and $|\mathbf{r}-\mathbf{r}_0| = R$, so that

$$\oiint_{S'} \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\cdot\hat{\mathbf{n}}\,dS = \oiint_{S'} \frac{(-R\hat{\mathbf{n}})}{R^3}\cdot\hat{\mathbf{n}}\,dS = -\frac{1}{R^2}\oiint_{S'} dS = -\frac{1}{R^2}(\text{Area of } S') = -\frac{1}{R^2}(4\pi R^2) = -4\pi.$$

Consequently, when P_0 is enclosed by S, $\displaystyle\oiint_S \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\cdot\hat{\mathbf{n}}\,dS = -\oiint_{S'} \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\cdot\hat{\mathbf{n}}\,dS = 4\pi.$

EXERCISES 14.10

1. According to Stokes's theorem, $\oint_C x^2y\,dx + y^2z\,dy + z^2x\,dz = \iint_S \nabla \times (x^2y, y^2z, z^2x) \cdot \hat{\mathbf{n}}\,dS$ where S is any surface with C as boundary. Now,

$$\nabla \times (x^2y, y^2z, z^2x) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & y^2z & z^2x \end{vmatrix} = (-y^2, -z^2, -x^2).$$

If we choose S as that part of the plane $z = 4$ inside C, then $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$, and

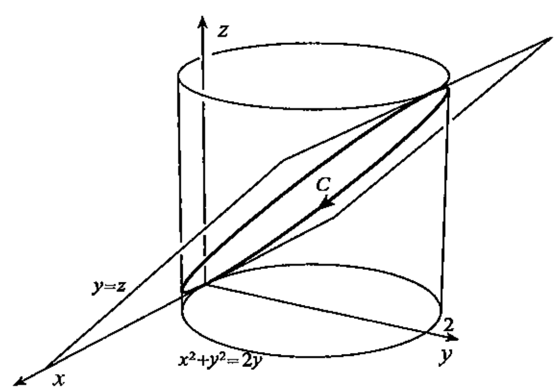
$$\begin{aligned}\oint_C x^2y\,dx + y^2z\,dy + z^2x\,dz &= \iint_S (-y^2, -z^2 - x^2) \cdot (-\hat{\mathbf{k}})\,dS = \iint_S x^2\,dS \\ &= \iint_{S_{xy}} x^2\,dA = 4\int_0^{\pi/2}\int_0^2 r^2\cos^2\theta\, r\,dr\,d\theta = 4\int_0^{\pi/2}\left\{\frac{r^4}{4}\cos^2\theta\right\}_0^2 d\theta \\ &= 16\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta = 8\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = 4\pi.\end{aligned}$$

2. According to Stokes's theorem,

$$\oint_C y^2\,dx + xy\,dy + xz\,dz = \iint_S \nabla \times (y^2, xy, xz) \cdot \hat{\mathbf{n}}\,dS$$

where S is any surface with C as boundary. Now,

$$\nabla \times (y^2, xy, xz) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & xy & xz \end{vmatrix} = (0, -z, -y).$$



If we choose S as that part of $z = y$ bounded by C, then $\hat{\mathbf{n}} = (0, -1, 1)/\sqrt{2}$, and

$$\oint_C y^2\,dx + xy\,dy + xz\,dz = \iint_S (0, -z, -y) \cdot \frac{(0,-1,1)}{\sqrt{2}}dS = \frac{1}{\sqrt{2}}\iint_S (z - y)\,dS = 0.$$

3. According to Stokes's theorem, $\oint_C (xyz + 2yz)\,dx + xz\,dy + 2xy\,dz = \iint_S \nabla \times (xyz + 2yz, xz, 2xy) \cdot \hat{\mathbf{n}}\,dS$ where S is any surface with C as boundary. Now,

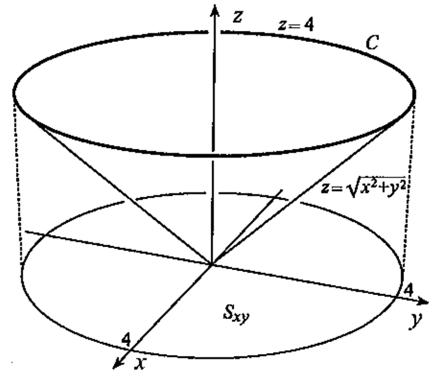
$$\nabla \times (xyz + 2yz, xz, 2xy) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz + 2yz & xz & 2xy \end{vmatrix} = (x, xy, -z - xz).$$

If we choose S as that part of the plane $z = 1$ inside C, then $\hat{\mathbf{n}} = \hat{\mathbf{k}}$, and

$$\begin{aligned}\oint_C (xyz + 2yz)\,dx + xz\,dy + 2xy\,dz &= \iint_S (x, xy, -z - xz) \cdot \hat{\mathbf{k}}\,dS = \iint_S -(z + xz)\,dS \\ &= -\iint_S z\,dS - \iint_S xz\,dS = -\iint_S dS - 0 = -3\pi.\end{aligned}$$

4. By Stokes's theorem, $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}\,dS$ where S is any surface with C as boundary and $\mathbf{F} = (2xy+y)\hat{\mathbf{i}} + (x^2+xy-3y)\hat{\mathbf{j}} + 2xz\hat{\mathbf{k}}$. Now

$$\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy+y & x^2+xy-3y & 2xz \end{vmatrix} = -2z\hat{\mathbf{j}} + (y-1)\hat{\mathbf{k}}.$$



If we choose S as that part of the plane $z=4$ inside C, then $\hat{\mathbf{n}} = \pm\hat{\mathbf{k}}$, depending on the direction along C, and $\nabla\times\mathbf{F}\cdot\hat{\mathbf{n}} = \pm(y-1)$. Hence

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \pm\iint_S (y-1)\,dS = \pm\left[\iint_{S_{xy}} y\,dA - \iint_{S_{xy}} dA\right] = \pm(0-16\pi) = \pm 16\pi.$$

5. According to Stokes's theorem, $\oint_C x^2\,dx + y^2\,dy + (x^2+y^2)\,dz = \iint_S \nabla\times(x^2,y^2,x^2+y^2)\cdot\hat{\mathbf{n}}\,dS$ where S is any surface with C as boundary. Now,

$$\nabla\times(x^2,y^2,x^2+y^2) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & y^2 & x^2+y^2 \end{vmatrix} = (2y,-2x,0).$$

If we choose S as that part of the plane $x+y+z=1$ bounded by C, then $\hat{\mathbf{n}} = (-1,-1,-1)/\sqrt{3}$, and

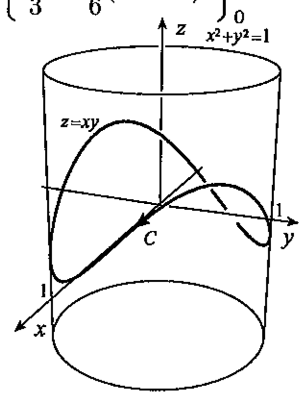
$$\begin{aligned}\oint_C x^2\,dx + y^2\,dy + (x^2+y^2)\,dz &= \iint_S \frac{-2y+2x}{\sqrt{3}}dS = \frac{2}{\sqrt{3}}\iint_{S_{xy}} (x-y)\sqrt{1+(-1)^2+(-1)^2}\,dA \\ &= 2\int_0^1\int_0^{1-x}(x-y)\,dy\,dx = 2\int_0^1 \left\{-\frac{1}{2}(x-y)^2\right\}_0^{1-x} dx \\ &= \int_0^1 [x^2-(2x-1)^2]\,dx = \left\{\frac{x^3}{3} - \frac{1}{6}(2x-1)^3\right\}_0^1 = 0.\end{aligned}$$

6. By Stokes's theorem,

$$\oint_C y\,dx + x\,dy + (x^2+y^2+z^2)\,dz = \iint_S \nabla\times(y,x,x^2+y^2+z^2)\cdot\hat{\mathbf{n}}\,dS$$

where S is any surface with C as boundary. Now

$$\nabla\times(y,x,x^2+y^2+z^2) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & x & x^2+y^2+z^2 \end{vmatrix} = (2y,-2x,0).$$



If we choose S as that part of $z=xy$ inside C, then $\hat{\mathbf{n}} = (y,x,-1)/\sqrt{1+x^2+y^2}$. Thus,

$$\begin{aligned}\oint_C y\,dx + x\,dy + (x^2+y^2+z^2)\,dz &= \iint_S \frac{2y^2-2x^2}{\sqrt{1+x^2+y^2}}\,dS \\ &= 2\iint_{S_{xy}} \frac{y^2-x^2}{\sqrt{1+x^2+y^2}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA \\ &= 2\iint_{S_{xy}} \frac{y^2-x^2}{\sqrt{1+y^2+x^2}}\sqrt{1+y^2+x^2}\,dA = 2\iint_{S_{xy}} y^2\,dA - 2\iint_{S_{xy}} x^2\,dA = 0\end{aligned}$$

since these integrals are equal.

7. According to Stokes's theorem, $\oint_C zy^2\,dx + xy\,dy + (y^2+z^2)\,dz = \iint_S \nabla \times (zy^2, xy, y^2+z^2)\cdot \hat{\mathbf{n}}\,dS$ where S is any surface with C as boundary. Now,

$$\nabla \times (zy^2, xy, y^2+z^2) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ zy^2 & xy & y^2+z^2 \end{vmatrix} = (2y, y^2, y-2yz).$$

If we choose S as that part of the plane $y=3$ inside C, then $\hat{\mathbf{n}} = -\hat{\mathbf{j}}$, and

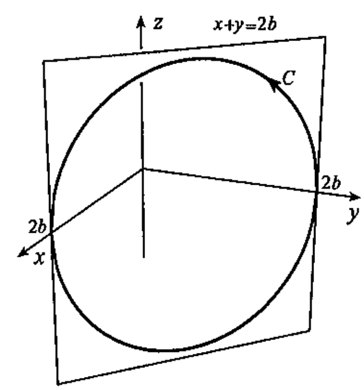
$$\begin{aligned}\oint_C zy^2\,dx + xy\,dy + (y^2+z^2)\,dz &= \iint_S (2y, y^2, y-2yz)\cdot(-\hat{\mathbf{j}})\,dS \\ &= \iint_S -y^2\,dS = -9\iint_S dS = -9(9\pi) = -81\pi.\end{aligned}$$

8. By Stokes's theorem, $\oint_C y\,dx + z\,dy + x\,dz = \iint_S \nabla\times(y,z,x)\cdot\hat{\mathbf{n}}\,dS$ where S is any surface with C as boundary. Now

$$\begin{aligned}\nabla\times(y,z,x) &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} \\ &= (-1,-1,-1).\end{aligned}$$

If we choose S as that part of $x+y=2b$ inside C, then $\hat{\mathbf{n}} = (1,1,0)/\sqrt{2}$, and,

$$\begin{aligned}&\oint_C y\,dx + z\,dy + x\,dz \\ &= \iint_S \frac{-2}{\sqrt{2}}\,dS = -\sqrt{2}\iint_S dS \\ &= -\sqrt{2}(\text{Area of } S) = -2\sqrt{2}\pi b^2.\end{aligned}$$



9. According to Stokes's theorem, $\oint_C y^2\,dx + (x+y)\,dy + yz\,dz = \iint_S \nabla\times(y^2, x+y, yz)\cdot\hat{\mathbf{n}}\,dS$ where S is any surface with C as boundary. Now,

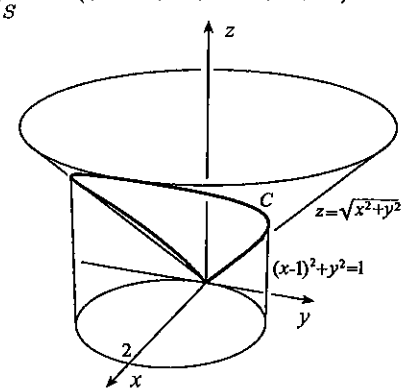
$$\nabla\times(y^2, x+y, yz) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & x+y & yz \end{vmatrix} = (z, 0, 1-2y).$$

If we choose S as that part of the plane $x+y+z=2$ bounded by C, then $\hat{\mathbf{n}} = (1,1,1)/\sqrt{3}$, and

$$\begin{aligned}\oint_C y^2\,dx + (x+y)\,dy + yz\,dz &= \iint_S (z,0,1-2y)\cdot\frac{(1,1,1)}{\sqrt{3}}dS = \frac{1}{\sqrt{3}}\iint_S (z+1-2y)\,dS \\ &= \frac{1}{\sqrt{3}}\iint_{S_{xy}} (2-x-y+1-2y)\sqrt{1+(-1)^2+(-1)^2}\,dA \\ &= \iint_{S_{xy}} 3\,dA - \iint_{S_{xy}} x\,dA - 3\iint_{S_{xy}} y\,dA = 3(2\pi) - 0 - 0 = 6\pi.\end{aligned}$$

10. By Stokes's theorem, $\oint_C (x+y)^2\,dx + (x+y)^2\,dy + yz^3\,dz = \iint_S \nabla\times\big((x+y)^2,(x+y)^2,yz^3\big)\cdot\hat{\mathbf{n}}\,dS$ where S is any surface with C as boundary. Now

$$\nabla\times\big((x+y)^2,(x+y)^2,yz^3\big) = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ (x+y)^2 & (x+y)^2 & yz^3\end{vmatrix} = (z^3,0,0).$$



If we choose S as that part of $z=\sqrt{x^2+y^2}$ inside C, then

$$\hat{\mathbf{n}} = \frac{\pm\nabla(x^2+y^2-z^2)}{|\nabla(x^2+y^2-z^2)|} = \frac{\pm(2x,2y,-2z)}{\sqrt{4x^2+4y^2+4z^2}} = \frac{\pm(x,y,-z)}{\sqrt{2}z},$$

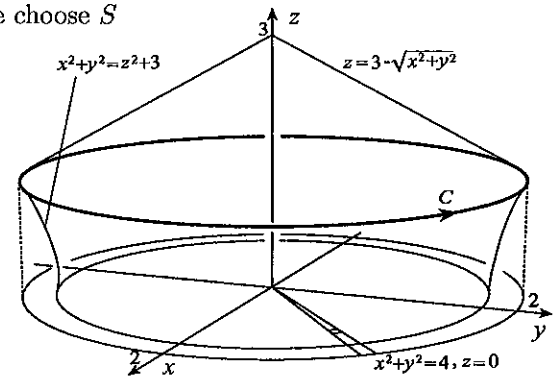
the sign depending on the direction along C. Hence,

$$\begin{aligned}
\oint_C (x+y)^2\,dx + (x+y)^2\,dy + yz^3\,dz &= \pm\iint_S \frac{xz^3}{\sqrt{2}z}\,dS = \frac{\pm 1}{\sqrt{2}}\iint_S xz^2\,dS\\
&= \frac{\pm 1}{\sqrt{2}}\iint_{S_{xy}} x(x^2+y^2)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA\\
&= \frac{\pm 1}{\sqrt{2}}\iint_{S_{xy}} x(x^2+y^2)\sqrt{1+\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2+\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}\,dA\\
&= \pm\iint_{S_{xy}} x(x^2+y^2)\,dA = \pm 2\int_0^{\pi/2}\int_0^{2\cos\theta} r^4\cos\theta\,dr\,d\theta = \pm 2\int_0^{\pi/2}\left\{\frac{r^5}{5}\cos\theta\right\}_0^{2\cos\theta} d\theta\\
&= \frac{\pm 64}{5}\int_0^{\pi/2}\cos^6\theta\,d\theta = \frac{\pm 64}{5}\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^3 d\theta\\
&= \frac{\pm 8}{5}\int_0^{\pi/2}\left[1+3\cos 2\theta+\frac{3}{2}(1+\cos 4\theta)+\cos 2\theta(1-\sin^2 2\theta)\right] d\theta\\
&= \frac{\pm 8}{5}\left\{\frac{5\theta}{2}+2\sin 2\theta+\frac{3\sin 4\theta}{8}-\frac{\sin^3 2\theta}{6}\right\}_0^{\pi/2} = \pm 2\pi.
\end{aligned}$$

11. According to Stokes's theorem, $\oint_C xy\,dx - zx\,dy + yz\,dz = \iint_S \nabla\times(xy,-zx,yz)\cdot\hat{\mathbf{n}}\,dS$ where S is any surface with C as boundary. Now, $\nabla\times(xy,-zx,yz) = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ xy & -zx & yz\end{vmatrix} = (z+x,0,-z-x)$. If we choose S as that part of $z=x+y$ inside C, then $\hat{\mathbf{n}} = (-1,-1,1)/\sqrt{3}$, and

$$\begin{aligned}
\oint_C xy\,dx - zx\,dy + yz\,dz &= \iint_S (z+x,0,-z-x)\cdot\frac{(-1,-1,1)}{\sqrt{3}}dS = -\frac{2}{\sqrt{3}}\iint_S (z+x)\,dS\\
&= -\frac{2}{\sqrt{3}}\iint_{S_{xy}} (x+y+x)\sqrt{1+(1)^2+(1)^2}\,dA = -2\int_0^1\int_0^{1-x}(2x+y)\,dy\,dx\\
&= -2\int_0^1\left\{\frac{1}{2}(2x+y)^2\right\}_0^{1-x} dx = \int_0^1 [4x^2-(x+1)^2]\,dx\\
&= \left\{\frac{4x^3}{3}-\frac{1}{3}(x+1)^3\right\}_0^1 = -1.
\end{aligned}$$

12. The curve of intersection lies in the plane $z=1$. If we choose S as that part of the plane interior to C, then



$$\oint_C y^3\,dx - x^3\,dy + xyz\,dz$$
$$= \iint_S \nabla\times(y^3,-x^3,xyz)\cdot\hat{\mathbf{n}}\,dS.$$

Now, $\nabla\times(y^3,-x^3,xyz) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^3 & -x^3 & xyz \end{vmatrix}$
$$= (xz, -yz, -3x^2-3y^2).$$

Since $\hat{\mathbf{n}}=\hat{\mathbf{k}}$ on S,

$$\oint_C y^3\,dx - x^3\,dy + xyz\,dz = \iint_S (-3x^2-3y^2)\,dS = -3\iint_{S_{xy}}(x^2+y^2)\,dA$$
$$= -3\int_{-\pi}^{\pi}\int_0^2 r^2\,r\,dr\,d\theta = -3\int_{-\pi}^{\pi}\left\{\frac{r^4}{4}\right\}_0^2 d\theta = -12\Big\{\theta\Big\}_{-\pi}^{\pi} = -24\pi.$$

13. If S is that part of the plane $y=x$ inside C, then according to Stokes's theorem,

$$I = \oint_C z(x+y)^2\,dx + (y-x)^2\,dy + z^2\,dz = \iint_S \nabla\times\big(z(x+y)^2,(y-x)^2,z^2\big)\cdot\hat{\mathbf{n}}\,dS.$$

Now, $\nabla\times\big(z(x+y)^2,(y-x)^2,z^2\big) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z(x+y)^2 & (y-x)^2 & z^2 \end{vmatrix} = \big(0,(x+y)^2,2(x-y-xz-yz)\big).$

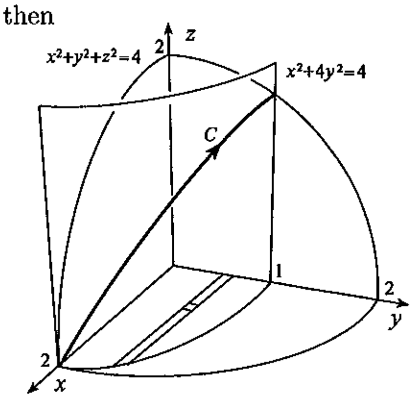
Since $\hat{\mathbf{n}} = (-1,1,0)/\sqrt{2}$,

$$I = \iint_S \big(0,(x+y)^2,2(x-y-xz-yz)\big)\cdot\frac{(-1,1,0)}{\sqrt{2}}dS = \frac{1}{\sqrt{2}}\iint_S (x+y)^2\,dS$$
$$= \frac{1}{\sqrt{2}}\iint_{S_{xz}}(x+x)^2\sqrt{1+(1)^2}\,dA = 4\iint_{S_{xz}} x^2\,dA.$$

If we set up polar coordinates $x=r\cos\theta$ and $z=r\sin\theta$ in the xz-plane,

$$I = 16\int_0^{\pi/2}\int_0^a r^2\cos^2\theta\,r\,dr\,d\theta = 16\int_0^{\pi/2}\left\{\frac{r^4}{4}\cos^2\theta\right\}_0^a d\theta$$
$$= 4a^4\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta = 2a^4\left\{\theta+\frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = \pi a^4.$$

14. If we choose S as the upper part of the sphere bounded by C, then



$$\oint_C -2y^3x^2\,dx + x^3y^2\,dy + z\,dz$$
$$= \iint_S \nabla\times(-2y^3x^2,x^3y^2,z)\cdot\hat{\mathbf{n}}\,dS.$$

Now,

$$\nabla\times(-2y^3x^2,x^3y^2,z) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -2y^3x^2 & x^3y^2 & z \end{vmatrix}$$
$$= (0,0,9x^2y^2),$$

and $\hat{\mathbf{n}} = \dfrac{(x,y,z)}{\sqrt{x^2+y^2+z^2}} = \dfrac{(x,y,z)}{2}$. Thus,

$$\oint_C -2y^3x^2\,dx + x^3y^2\,dy + z\,dz = \iint_S \frac{z}{2}(9x^2y^2)\,dS = \frac{9}{2}\iint_S x^2y^2\sqrt{4-x^2-y^2}\,dS$$
$$= \frac{9}{2}\iint_{S_{xy}} x^2y^2\sqrt{4-x^2-y^2}\sqrt{1+\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}}\,dA$$
$$= 9\iint_{S_{xy}} x^2y^2\,dA = 36\int_0^1\int_0^{\sqrt{4-4y^2}} x^2y^2\,dx\,dy$$
$$= 36\int_0^1 \left\{\frac{x^3y^2}{3}\right\}_0^{\sqrt{4-4y^2}} dy = 96\int_0^1 y^2(1-y^2)^{3/2}\,dy.$$

If we set $y = \sin\theta$ and $dy = \cos\theta\,d\theta$, then

$$\oint_C -2y^3x^2\,dx + x^3y^2\,dy + z\,dz = 96\int_0^{\pi/2}\sin^2\theta\,\cos^3\theta\,\cos\theta\,d\theta = 96\int_0^{\pi/2}\sin^2\theta\,\cos^4\theta\,d\theta$$
$$= 96\int_0^{\pi/2}\frac{\sin^2 2\theta}{4}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$
$$= 12\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}+\sin^2 2\theta\,\cos 2\theta\right)d\theta$$
$$= 12\left\{\frac{\theta}{2}-\frac{\sin 4\theta}{8}+\frac{\sin^3 2\theta}{6}\right\}_0^{\pi/2} = 3\pi.$$

15. (a) With parametric equations $x = \cos t$, $y = \sin t$, $z = \sqrt{3}$, $-\pi \le t \le \pi$,

$$I = \oint_C 2x^2y\,dx - yz\,dy + xz\,dz = \int_{-\pi}^{\pi} 2\cos^2 t\sin t(-\sin t\,dt) - \sqrt{3}\sin t(\cos t\,dt)$$
$$= -\int_{-\pi}^{\pi}\left(\frac{1}{2}\sin^2 2t + \sqrt{3}\sin t\cos t\right)dt = -\frac{1}{2}\int_{-\pi}^{\pi}\left[\left(\frac{1-\cos 4t}{2}\right)+2\sqrt{3}\sin t\cos t\right]dt$$
$$= -\frac{1}{2}\left\{\frac{t}{2}-\frac{1}{8}\sin 4t+\sqrt{3}\sin^2 t\right\}_{-\pi}^{\pi} = -\frac{\pi}{2}.$$

(b) Since $\nabla\times(2x^2y, -yz, xz) = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 2x^2y & -yz & xz\end{vmatrix} = (y, -z, -2x^2)$,

and $\hat{\mathbf{n}} = \dfrac{\nabla(x^2+y^2+z^2-4)}{|\nabla(x^2+y^2+z^2-4)|} = \dfrac{(2x, 2y, 2z)}{\sqrt{4x^2+4y^2+4z^2}} = \dfrac{(x,y,z)}{2}$,

$$I = \iint_S (y, -z, -2x^2)\cdot\frac{(x,y,z)}{2}dS = \frac{1}{2}\iint_S (xy - yz - 2x^2z)\,dS$$
$$= \frac{1}{2}\iint_S (xy - y\sqrt{4-x^2-y^2} - 2x^2\sqrt{4-x^2-y^2})\,dS$$
$$= \frac{1}{2}\iint_{S_{xy}} [xy - (y+2x^2)\sqrt{4-x^2-y^2}]\sqrt{1+\left(\frac{-x}{\sqrt{4-x^2-y^2}}\right)^2+\left(\frac{-y}{\sqrt{4-x^2-y^2}}\right)^2}\,dA$$
$$= \frac{1}{2}\iint_{S_{xy}} [xy - (y+2x^2)\sqrt{4-x^2-y^2}]\frac{2}{\sqrt{4-x^2-y^2}}\,dA$$
$$= \iint_{S_{xy}}\left(\frac{xy}{\sqrt{4-x^2-y^2}} - y - 2x^2\right)dA.$$

Integrals of the first two terms give zero, and therefore

$$I = -2\iint_{S_{xy}} x^2\,dA = -8\int_0^{\pi/2}\int_0^1 r^2\cos^2\theta\, r\,dr\,d\theta = -8\int_0^{\pi/2}\left\{\frac{r^4}{4}\cos^2\theta\right\}_0^1 d\theta$$
$$= -2\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta = -\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = -\frac{\pi}{2}.$$

(c) Since $\hat{\mathbf{n}} = \dfrac{\nabla(z-\sqrt{3}(x^2+y^2))}{|\nabla(z-\sqrt{3}(x^2+y^2))|} = \dfrac{(-2\sqrt{3}x, -2\sqrt{3}y, 1)}{\sqrt{1+12x^2+12y^2}}$,

$$I = \iint_S (y,-z,-2x^2)\cdot\frac{(-2\sqrt{3}x,-2\sqrt{3}y,1)}{\sqrt{1+12x^2+12y^2}}dS = \iint_S \frac{-2\sqrt{3}xy + 2\sqrt{3}yz - 2x^2}{\sqrt{1+12x^2+12y^2}}dS$$
$$= \iint_{S_xy} \frac{2\sqrt{3}y(\sqrt{3}x^2+\sqrt{3}y^2) - 2\sqrt{3}xy - 2x^2}{\sqrt{1+12x^2+12y^2}}\sqrt{1+(2\sqrt{3}x)^2+(2\sqrt{3}y)^2}\,dA$$
$$= \iint_{S_{xy}} [2\sqrt{3}y(\sqrt{3}x^2+\sqrt{3}y^2) - 2\sqrt{3}xy - 2x^2]\,dA.$$

Integrals of the first two terms give zero, leaving $I = \iint_{S_{xy}} -2x^2\,dA$, the same integral as in part (b).

(d) Since $\hat{\mathbf{n}} = \hat{\mathbf{k}}$, $I = \iint_S (y,-z,-2x^2)\cdot\hat{\mathbf{k}}\,dS = \iint_S -2x^2\,dS = \iint_{S_{xy}} -2x^2\,dA = -\frac{\pi}{2}$, the same integral as in part (b).

16. Both surfaces have the curve $C: x^2+y^2=1,\ z=0$ as boundary. Consequently, by Stokes's theorem, both surface integrals are equal to $\oint_C \mathbf{F}\cdot d\mathbf{r}$ and are therefore equal to each other.

REVIEW EXERCISES

1. $\nabla f = (2xy^3 - y)\hat{\mathbf{i}} + (3x^2y^2 - x)\hat{\mathbf{j}} + \hat{\mathbf{k}}$

2. $\nabla\cdot\mathbf{F} = 3x^2y + x^2/y^2$

3. $$\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin(xy) & \cos(xy) & xy \end{vmatrix} = x\hat{\mathbf{i}} - y\hat{\mathbf{j}} + [-y\sin(xy) - x\cos(xy)]\hat{\mathbf{k}}$$

4. $$\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y+z & x+y+z & x+y+z \end{vmatrix} = (1-1)\hat{\mathbf{i}} + (1-1)\hat{\mathbf{j}} + (1-1)\hat{\mathbf{k}} = \mathbf{0}$$

5. $$\nabla f = \left(\frac{2x}{x^2+y^2+z^2}\right)\hat{\mathbf{i}} + \left(\frac{2y}{x^2+y^2+z^2}\right)\hat{\mathbf{j}} + \left(\frac{2z}{x^2+y^2+z^2}\right)\hat{\mathbf{k}}$$

6. $\nabla\cdot\mathbf{F} = ye^x + ze^y + xe^z$

7. $$\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & xyz & 0 \end{vmatrix} = -xy\hat{\mathbf{i}} + yz\hat{\mathbf{k}}$$

8. $$\nabla f = \frac{1}{\sqrt{1-(x+y)^2}}(\hat{\mathbf{i}} + \hat{\mathbf{j}})$$

9. $\nabla\cdot\mathbf{F} = -2y + 2y^2z$

10. $$\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \mathrm{Cot}^{-1}(xyz) & 0 & 0 \end{vmatrix} = \frac{-xy}{1+x^2y^2z^2}\hat{\mathbf{j}} + \frac{xz}{1+x^2y^2z^2}\hat{\mathbf{k}}$$

11. $$\int_C y\,ds = \int_{-1}^2 x^3\sqrt{1+(3x^2)^2}\,dx = \int_{-1}^2 x^3\sqrt{1+9x^4}\,dx = \left\{\frac{(1+9x^4)^{3/2}}{54}\right\}_{-1}^2 = \frac{145\sqrt{145} - 10\sqrt{10}}{54}$$

12. $$\iint_S (x^2+yz)\,dS = \iint_{S_{xy}} [x^2+y(2-x-y)]\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

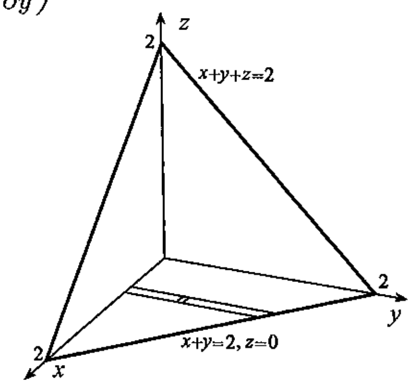
$$= \iint_{S_{xy}} (x^2+2y-xy-y^2)\sqrt{1+(-1)^2+(-1)^2}\,dA$$

$$= \sqrt{3}\int_0^2\int_0^{2-x}(x^2+2y-xy-y^2)\,dy\,dx$$

$$= \sqrt{3}\int_0^2\left\{x^2y+y^2-\frac{xy^2}{2}-\frac{y^3}{3}\right\}_0^{2-x}dx$$

$$= \frac{\sqrt{3}}{6}\int_0^2[24x^2-9x^3-12x+6(2-x)^2-2(2-x)^3]\,dx$$

$$= \frac{\sqrt{3}}{6}\left\{8x^3-\frac{9x^4}{4}-6x^2-2(2-x)^3+\frac{(2-x)^4}{2}\right\}_0^2 = 2\sqrt{3}$$



13. Since $\hat{\mathbf{n}} = \dfrac{(2x,2y,-1)}{\sqrt{1+4x^2+4y^2}}$,

$$\iint_S (x\hat{\mathbf{i}}+y\hat{\mathbf{j}})\cdot\hat{\mathbf{n}}\,dS = \iint_{S_{xy}} \frac{2x^2+2y^2}{\sqrt{1+4x^2+4y^2}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$= \iint_{S_{xy}} \frac{2x^2+2y^2}{\sqrt{1+4x^2+4y^2}}\sqrt{1+(2x)^2+(2y)^2}\,dA = 8\int_0^1\int_0^1(x^2+y^2)\,dy\,dx$$

$$= 8\int_0^1\left\{x^2y+\frac{y^3}{3}\right\}_0^1 dx = \frac{8}{3}\int_0^1(3x^2+1)\,dx = \frac{8}{3}\left\{x^3+x\right\}_0^1 = \frac{16}{3}$$

14. Since $\hat{\mathbf{n}} = \dfrac{\nabla(x^2+y^2-z)}{|\nabla(x^2+y^2-z)|} = \dfrac{(2x,2y,-1)}{\sqrt{1+4x^2+4y^2}}$,

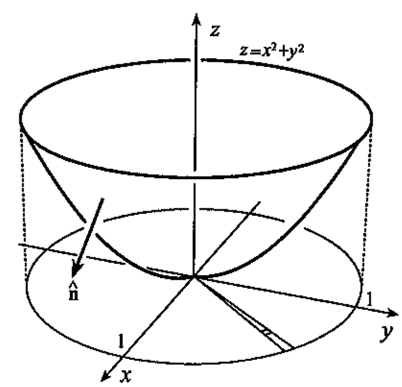
$$\iint_S (x\hat{\mathbf{i}}+y\hat{\mathbf{j}})\cdot\hat{\mathbf{n}}\,dS$$

$$= \iint_{S_{xy}} \frac{2x^2+2y^2}{\sqrt{1+4x^2+4y^2}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$= \iint_{S_{xy}} \frac{2(x^2+y^2)}{\sqrt{1+4x^2+4y^2}}\sqrt{1+(2x)^2+(2y)^2}\,dA$$

$$= 2\iint_{S_{xy}}(x^2+y^2)\,dA = 2\int_{-\pi}^{\pi}\int_0^1 r^2\,r\,dr\,d\theta$$

$$= 2\int_{-\pi}^{\pi}\left\{\frac{r^4}{4}\right\}_0^1 d\theta = \frac{1}{2}\left\{\theta\right\}_{-\pi}^{\pi} = \pi.$$



15. Since $\nabla(x^2/2+y^2/2-z^3/3) = x\hat{\mathbf{i}}+y\hat{\mathbf{j}}-z^2\hat{\mathbf{k}}$, the line integral is independent of path in space. Because C is a closed curve, the line integral has value zero.

16. With parametric equations $C:\ x=-t,\ y=\sqrt{1+t^2},\ z=\sqrt{1-2t^2},\ -1/\sqrt{2}\le t\le 1/\sqrt{2}$,

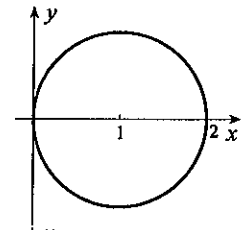
$$\int_C xy\,dx+xz\,dz = \int_{-1\sqrt{2}}^{1/\sqrt{2}}\left[-t\sqrt{1+t^2}(-dt)-t\sqrt{1-2t^2}\left(\frac{-2t}{\sqrt{1-2t^2}}\right)dt\right]$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}}(t\sqrt{1+t^2}+2t^2)\,dt = \left\{\frac{1}{3}(1+t^2)^{3/2}+\frac{2t^3}{3}\right\}_{-1/\sqrt{2}}^{1/\sqrt{2}} = \frac{\sqrt{2}}{3}.$$

17. By Green's theorem,

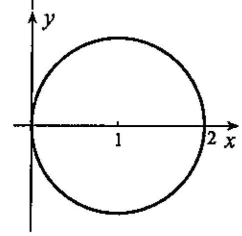
$$\oint_C 2xy^3\,dx + (3x^2y^2 + 2xy)\,dy$$
$$= \iint_R (6xy^2 + 2y - 6xy^2)\,dA = 2\iint_R y\,dA.$$

This integral has value zero since y is an odd function of y and R is symmetric about the x-axis.



18. By Green's theorem,

$$\oint_C 2xy^3\,dx + (3x^2y^2 + x^2)\,dy$$
$$= \iint_R (6xy^2 + 2x - 6xy^2)\,dA$$
$$= 2\iint_R x\,dA = 2\overline{x}(\text{Area of } R) = 2(1)\pi(1)^2 = 2\pi.$$

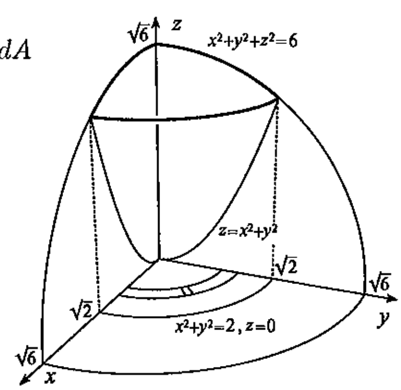


19. By the divergence theorem,

$$\oiint_S (x^2\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} + z^2\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = \iiint_V (2x + 2y + 2z)\,dV = 2\int_0^1\int_z^{2-z}\int_0^1 (x + y + z)\,dx\,dy\,dz$$
$$= 2\int_0^1\int_z^{2-z}\left\{\frac{x^2}{2} + xy + xz\right\}_0^1 dy\,dz = \int_0^1\int_z^{2-z}(1 + 2y + 2z)\,dy\,dz$$
$$= \int_0^1\left\{\frac{1}{4}(1 + 2y + 2z)^2\right\}_z^{2-z} dz = \frac{1}{4}\int_0^1 [25 - (1 + 4z)^2]\,dz$$
$$= \frac{1}{4}\left\{25z - \frac{1}{12}(1 + 4z)^3\right\}_0^1 = \frac{11}{3}$$

20. We quadruple the integral over that part of the surface in the first octant.

$$\iint_S (x^2 + y^2)\,dS = 4\iint_{S_{xy}} (x^2 + y^2)\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA$$
$$= 4\iint_{S_{xy}} (x^2 + y^2)\sqrt{1 + \left(-\frac{x}{z}\right)^2 + \left(-\frac{y}{z}\right)^2}\,dA$$
$$= 4\iint_{S_{xy}} (x^2 + y^2)\frac{\sqrt{6}}{z}\,dA$$
$$= 4\sqrt{6}\iint_{S_{xy}} \frac{x^2 + y^2}{\sqrt{6 - x^2 - y^2}}\,dA$$
$$= 4\sqrt{6}\int_0^{\sqrt{2}}\int_0^{\pi/2} \frac{r^2}{\sqrt{6 - r^2}}\,r\,d\theta\,dr$$
$$= 2\sqrt{6}\pi\int_0^{\sqrt{2}} \frac{r^3}{\sqrt{6 - r^2}}\,dr$$



If we set $u = 6 - r^2$ and $du = -2r\,dr$, then

$$\iint_S (x^2 + y^2)\,dS = 2\sqrt{6}\pi\int_6^4 \frac{6 - u}{\sqrt{u}}\left(-\frac{du}{2}\right) = \sqrt{6}\pi\int_6^4\left(-\frac{6}{\sqrt{u}} + \sqrt{u}\right)du$$
$$= \sqrt{6}\pi\left\{-12\sqrt{u} + \frac{2u^{3/2}}{3}\right\}_6^4 = \frac{8\pi(18 - 7\sqrt{6})}{3}.$$

21. Since $\hat{\mathbf{n}} = \dfrac{\nabla(x^2+y^2+z^2-6)}{|\nabla(x^2+y^2+z^2-6)|} = \dfrac{(2x,2y,2z)}{\sqrt{4x^2+4y^2+4z^2}} = \dfrac{(x,y,z)}{\sqrt{6}}$,

$$\iint_S (x^2+y^2)\hat{\mathbf{i}}\cdot\hat{\mathbf{n}}\,dS = \iint_S \frac{x(x^2+y^2)}{\sqrt{6}}dS = \frac{1}{\sqrt{6}}\iint_{S_{xy}} x(x^2+y^2)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$
$$= \frac{1}{\sqrt{6}}\iint_{S_{xy}} x(x^2+y^2)\sqrt{1+\left(-\frac{x}{\sqrt{6-x^2-y^2}}\right)^2+\left(-\frac{y}{\sqrt{6-x^2-y^2}}\right)^2}\,dA$$
$$= \frac{1}{\sqrt{6}}\iint_{S_{xy}} x(x^2+y^2)\frac{\sqrt{6}}{\sqrt{6-x^2-y^2}}dA = \iint_{S_{xy}} \frac{x(x^2+y^2)}{\sqrt{6-x^2-y^2}}dA = 0.$$

The integrand is an odd function of x and S_{xy} is symmetric about the y-axis.

22. By Stokes's theorem,

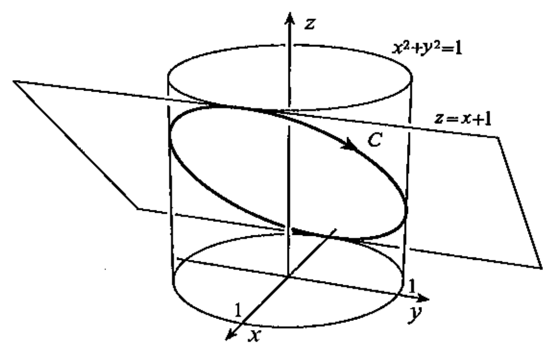
$$\oint_C (x^2\hat{\mathbf{i}}+y\hat{\mathbf{j}}-xz\hat{\mathbf{k}})\cdot d\mathbf{r} = \iint_S \nabla\times(x^2,y,-xz)\cdot\hat{\mathbf{n}}\,dS$$

where S is any surface with C as boundary. Now

$$\nabla\times(x^2,y,-xz) = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ x^2 & y & -xz\end{vmatrix} = (0,z,0).$$

If we choose S as that part of $z=x+1$ inside C, then $\hat{\mathbf{n}} = (-1,0,1)/\sqrt{2}$, and

$$\oint_C (x^2\hat{\mathbf{i}}+y\hat{\mathbf{j}}-xz\hat{\mathbf{k}})\cdot d\mathbf{r} = \iint_S 0\,dS = 0.$$



23. If S is that part of $z=x+1$ inside C, then by Stokes's theorem,

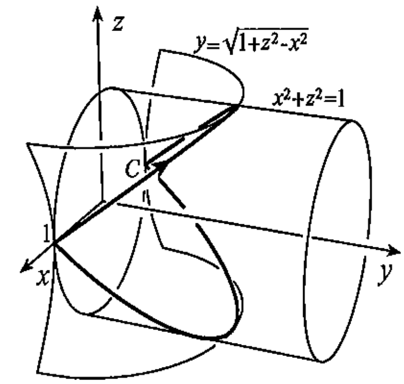
$$\oint_C (xy\hat{\mathbf{i}}+z\hat{\mathbf{j}}-x^2\hat{\mathbf{k}})\cdot d\mathbf{r} = \iint_S \nabla\times(xy,z,-x^2)\cdot\hat{\mathbf{n}}\,dS.$$

Since $\nabla\times(xy,z,-x^2) = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ xy & z & -x^2\end{vmatrix} = (-1,2x,-x)$, and $\hat{\mathbf{n}} = (-1,0,1)/\sqrt{2}$,

$$\oint_C (xy\hat{\mathbf{i}}+z\hat{\mathbf{j}}-x^2\hat{\mathbf{k}})\cdot d\mathbf{r} = \iint_S (-1,2x,-x)\cdot\frac{(-1,0,1)}{\sqrt{2}}dS = \frac{1}{\sqrt{2}}\iint_S (1-x)\,dS$$
$$= \frac{1}{\sqrt{2}}\iint_S (1-x)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA = \frac{1}{\sqrt{2}}\iint_{S_{xy}} (1-x)\sqrt{1+(1)^2}\,dA$$
$$= \iint_{S_{xy}} dA = \pi$$

24. By Stokes's theorem, $\oint_C y\,dx + 2x\,dy - 3z^2\,dz = \iint_S \nabla \times (y, 2x, -3z^2) \cdot \hat{\mathbf{n}}\,dS$ where S is any surface with C as boundary. Now,

$$\nabla \times (y, 2x, -3z^2) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & 2x & -3z^2 \end{vmatrix} = (0, 0, 1).$$



If we choose S as that part of $y = \sqrt{1 + z^2 - x^2}$ inside C, then

$$\hat{\mathbf{n}} = \frac{\nabla(z^2 - x^2 - y^2 + 1)}{|\nabla(z^2 - x^2 - y^2 + 1)|} = \frac{(-2x, -2y, 2z)}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{(-x, -y, z)}{\sqrt{x^2 + y^2 + z^2}}.$$

Hence,

$$\oint_C y\,dx + 2x\,dy - 3z^2\,dz = \iint_S \frac{z}{\sqrt{x^2 + y^2 + z^2}}\,dS$$

$$= \iint_{S_{xz}} \frac{z}{\sqrt{x^2 + y^2 + z^2}} \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\,dA$$

$$= \iint_{S_{xz}} \frac{z}{\sqrt{x^2 + y^2 + z^2}} \sqrt{1 + \left(-\frac{x}{y}\right)^2 + \left(\frac{z}{y}\right)^2}\,dA$$

$$= \iint_{S_{xz}} \frac{z}{y}\,dA = \iint_{S_{xz}} \frac{z}{\sqrt{1 + z^2 - x^2}}\,dA = 0,$$

since the integrand is an odd function of z and S_{xz} is symmetric about the x-axis.

25. If S is that part of the plane $y = z$ inside C, then by Stokes's theorem,

$$I = \oint_C (xy + 4x^3y^2)\,dx + (z + 2x^4y)\,dy + (z^5 + x^2z^2)\,dz = \iint_S \nabla \times (xy + 4x^3y^2, z + 2x^4y, z^5 + x^2z^2) \cdot \hat{\mathbf{n}}\,dS.$$

Since $\nabla \times (xy + 4x^3y, z + 2x^4y, z^5 + x^2z^2) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy + 4x^3y^2 & z + 2x^4y & z^5 + x^2z^2 \end{vmatrix} = (-1, -2xz^2, -x),$

and $\hat{\mathbf{n}} = (0, -1, 1)/\sqrt{2}$,

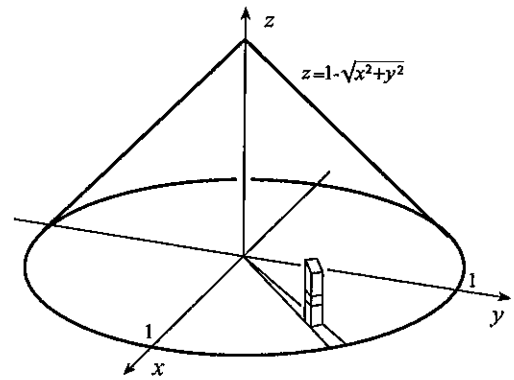
$$I = \iint_S (-1, -2xz^2, -x) \cdot \frac{(0, -1, 1)}{\sqrt{2}}\,dS = \frac{1}{\sqrt{2}} \iint_S (2xz^2 - x)\,dS$$

$$= \frac{1}{\sqrt{2}} \iint_{S_{xy}} (2xz^2 - x)\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA = \frac{1}{\sqrt{2}} \iint_{S_{xy}} (2xy^2 - x)\sqrt{1 + (1)^2}\,dA$$

$$= \iint_{S_{xy}} (2xy^2 - x)\,dA = 0$$

(since $2xy^2 - x$ is an odd function of x and S_{xy} is symmetric about the y-axis).

26. If S' is that part of the xy-plane bounded by $x^2+y^2=1$, $z=0$, then
$$\iint_{S'}(x^2yz\hat{\mathbf{i}}-x^2yz\hat{\mathbf{j}}-xyz^2\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS=0.$$
By the divergence theorem,
$$\oiint_{S+S'}(x^2yz\hat{\mathbf{i}}-x^2yz\hat{\mathbf{j}}-xyz^2\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS=\iiint_V(2xyz-x^2z-2xyz)\,dV$$



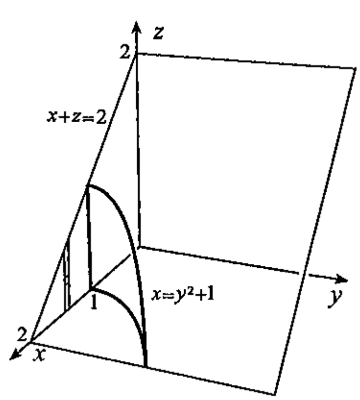
Thus,

$$\begin{aligned}\iint_S(x^2yz\hat{\mathbf{i}}-x^2yz\hat{\mathbf{j}}-xyz^2\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS&=-\iiint_V x^2z\,dV-\iint_{S'}(x^2yz\hat{\mathbf{i}}-x^2yz\hat{\mathbf{j}}-xyz^2\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS\\&=-\int_{-\pi}^{\pi}\int_0^1\int_0^{1-r}zr^2\cos^2\theta\,r\,dz\,dr\,d\theta=-\int_{-\pi}^{\pi}\int_0^1\left\{\frac{z^2r^3\cos^2\theta}{2}\right\}_0^{1-r}dr\,d\theta\\&=-\frac{1}{2}\int_{-\pi}^{\pi}\int_0^1(r^3-2r^4+r^5)\cos^2\theta\,dr\,d\theta\\&=-\frac{1}{2}\int_{-\pi}^{\pi}\left\{\frac{r^4}{4}-\frac{2r^5}{5}+\frac{r^6}{6}\right\}_0^1\cos^2\theta\,d\theta\\&=\frac{-1}{120}\int_{-\pi}^{\pi}\left(\frac{1+\cos2\theta}{2}\right)d\theta=\frac{-1}{240}\left\{\theta+\frac{\sin2\theta}{2}\right\}_{-\pi}^{\pi}=\frac{-\pi}{120}.\end{aligned}$$

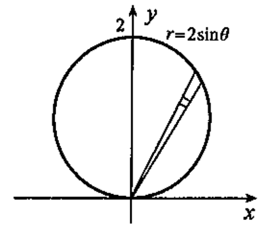
27. $$\begin{aligned}\iint_S dS&=\iint_{S_{xy}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA=\iint_{S_{xy}}\sqrt{1+(2x)^2+(-2y)^2}\,dA\\&=4\int_0^{\pi/2}\int_0^2\sqrt{1+4r^2}\,r\,dr\,d\theta=4\int_0^{\pi/2}\left\{\frac{1}{12}(1+4r^2)^{3/2}\right\}_0^2d\theta\\&=\frac{1}{3}(17\sqrt{17}-1)\Big\{\theta\Big\}_0^{\pi/2}=\frac{(17\sqrt{17}-1)\pi}{6}\end{aligned}$$

28. $$\begin{aligned}\iint_S y\,dS&=\iint_{S_{xz}}y\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dA\\&=\iint_{S_{xz}}y\sqrt{1+[1/(2y)]^2}\,dA=\frac{1}{2}\iint_{S_{xz}}\sqrt{4y^2+1}\,dA\\&=\frac{1}{2}\iint_{S_{xz}}\sqrt{4(x-1)+1}\,dA=\frac{1}{2}\iint_{S_{xz}}\sqrt{4x-3}\,dA\\&=\frac{1}{2}\int_1^2\int_0^{2-x}\sqrt{4x-3}\,dz\,dx=\frac{1}{2}\int_1^2(2-x)\sqrt{4x-3}\,dx\end{aligned}$$

If we set $u=4x-3$ and $du=4\,dx$ in the second term,
$$\begin{aligned}\iint_S y\,dS&=\left\{\frac{1}{6}(4x-3)^{3/2}\right\}_1^2-\frac{1}{2}\int_1^5\left(\frac{u+3}{4}\right)\sqrt{u}\left(\frac{du}{4}\right)\\&=\frac{1}{6}(5\sqrt{5}-1)-\frac{1}{32}\left\{\frac{2u^{5/2}}{5}+2u^{3/2}\right\}_1^5=\frac{25\sqrt{5}-11}{120}.\end{aligned}$$

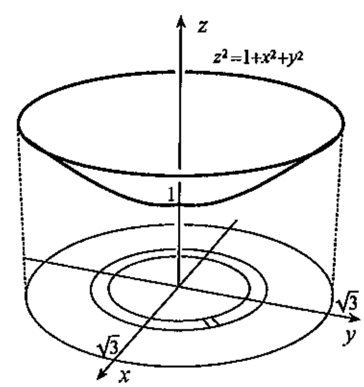


29. By Green's theorem,



$$\oint_C (ye^{xy} + xy^2e^{xy})\,dx + (xe^{xy} + x^2ye^{xy} + x^3y)\,dy$$
$$= \iint_R (e^{xy} + xye^{xy} + 2xye^{xy} + x^2y^2e^{xy} + 3x^2y$$
$$-e^{xy} - xye^{xy} - 2xye^{xy} - x^2y^2e^{xy})\,dA$$
$$= 6\int_0^{\pi/2}\int_0^{2\sin\theta} (r^2\cos^2\theta)(r\sin\theta)r\,dr\,d\theta$$
$$= 6\int_0^{\pi/2} \left\{\frac{r^5}{5}\cos^2\theta\sin\theta\right\}_0^{2\sin\theta} d\theta$$
$$= \frac{192}{5}\int_0^{\pi/2}\cos^2\theta\sin^6\theta\,d\theta = \frac{192}{5}\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^2\left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta$$
$$= \frac{12}{5}\int_0^{\pi/2}\sin^2 2\theta(1 - 2\cos 2\theta + \cos^2 2\theta)\,d\theta = \frac{12}{5}\int_0^{\pi/2}\left[\frac{1-\cos 4\theta}{2} - 2\sin^2 2\theta\cos 2\theta + \left(\frac{\sin 4\theta}{2}\right)^2\right] d\theta$$
$$= \frac{12}{5}\int_0^{\pi/2}\left[\frac{1-\cos 4\theta}{2} - 2\sin^2 2\theta\cos 2\theta + \frac{1}{4}\left(\frac{1-\cos 8\theta}{2}\right)\right] d\theta$$
$$= \frac{12}{5}\left\{\frac{5\theta}{8} - \frac{1}{8}\sin 4\theta - \frac{1}{3}\sin^3 2\theta - \frac{1}{64}\sin 8\theta\right\}_0^{\pi/2} = \frac{3\pi}{4}$$

30. Since $\hat{\mathbf{n}} = \dfrac{\nabla(x^2+y^2-z^2+1)}{|\nabla(x^2+y^2-z^2+1)|} = \dfrac{(2x,2y,-2z)}{\sqrt{4x^2+4y^2+4z^2}} = \dfrac{(x,y,-z)}{\sqrt{x^2+y^2+z^2}}$,



$$\iint_S (x\hat{\mathbf{i}} + y\hat{\mathbf{j}})\cdot\hat{\mathbf{n}}\,dS = \iint_{S_{xy}} \frac{x^2+y^2}{\sqrt{x^2+y^2+z^2}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$
$$= \iint_{S_{xy}} \frac{x^2+y^2}{\sqrt{x^2+y^2+z^2}}\sqrt{1+\left(\frac{x}{z}\right)^2+\left(\frac{y}{z}\right)^2}\,dA$$
$$= \iint_{S_{xy}} \frac{x^2+y^2}{z}\,dA$$
$$= \iint_{S_{xy}} \frac{x^2+y^2}{\sqrt{1+x^2+y^2}}\,dA$$
$$= \int_0^{\sqrt{3}}\int_{-\pi}^{\pi}\frac{r^2}{\sqrt{1+r^2}}r\,d\theta\,dr = 2\pi\int_0^{\sqrt{3}}\frac{r^3}{\sqrt{1+r^2}}dr.$$

If we set $u = 1 + r^2$ and $du = 2r\,dr$, then

$$\iint_S (x\hat{\mathbf{i}} + y\hat{\mathbf{j}})\cdot\hat{\mathbf{n}}\,dS = 2\pi\int_1^4 \frac{u-1}{\sqrt{u}}\left(\frac{du}{2}\right) = \pi\int_1^4\left(\sqrt{u} - \frac{1}{\sqrt{u}}\right)du = \pi\left\{\frac{2u^{3/2}}{3} - 2\sqrt{u}\right\}_1^4 = \frac{8\pi}{3}.$$

31. (a) With $dS = \sqrt{1+\left(\dfrac{-x}{\sqrt{1-x^2-y^2}}\right)^2+\left(\dfrac{-y}{\sqrt{1-x^2-y^2}}\right)^2}\,dA = \dfrac{1}{\sqrt{1-x^2-y^2}}dA$,

$$\iint_S x^2y^2z^2\,dS = \iint_{S_{xy}} x^2y^2(1-x^2-y^2)\frac{1}{\sqrt{1-x^2-y^2}}dA$$
$$= 4\int_0^{\sqrt{2\sqrt{2}-2}}\int_0^{\sqrt{1-x^2-x^4/4}} x^2y^2\sqrt{1-x^2-y^2}\,dy\,dx$$

(b) With $dS = \sqrt{1 + \left(\frac{-y}{\sqrt{1-y^2-z^2}}\right)^2 + \left(\frac{-z}{\sqrt{1-y^2-z^2}}\right)^2}\, dA = \frac{1}{\sqrt{1-y^2-z^2}} dA,$

$$\iint_S x^2y^2z^2\, dS = \iint_{S_{yz}} y^2z^2(1-y^2-z^2)\frac{1}{\sqrt{1-y^2-z^2}} dA = 4\int_0^1 \int_{-1+\sqrt{2-y^2}}^{\sqrt{1-y^2}} y^2z^2\sqrt{1-y^2-z^2}\, dz\, dy.$$

(c) With $dS = \sqrt{1 + \left(\frac{-x}{\sqrt{1-x^2-z^2}}\right)^2 + \left(\frac{-z}{\sqrt{1-x^2-z^2}}\right)^2}\, dA = \frac{1}{\sqrt{1-x^2-z^2}} dA,$

$$\iint_S x^2y^2z^2\, dS = 4\iint_{S_{xz}} x^2z^2(1-x^2-z^2)\frac{1}{\sqrt{1-x^2-z^2}} dA = 4\int_0^{\sqrt{2\sqrt{2}-2}} \int_{x^2/2}^{\sqrt{1-x^2}} x^2z^2\sqrt{1-x^2-z^2}\, dz\, dx.$$

32. $\nabla(|\mathbf{r}|^n) = \nabla[(x^2+y^2+z^2)^{n/2}] = \frac{n}{2}(x^2+y^2+z^2)^{n/2-1}(2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + 2z\hat{\mathbf{k}})$
$= n(x^2+y^2+z^2)^{(n-2)/2}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) = n|\mathbf{r}|^{n-2}\mathbf{r}$

33. If $\mathbf{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$, then

$$\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}},$$

and
$$\nabla\times(\nabla\times\mathbf{F}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} & \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} & \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \end{vmatrix}$$
$$= \left(\frac{\partial^2 Q}{\partial y\partial x} - \frac{\partial^2 P}{\partial y^2} - \frac{\partial^2 P}{\partial z^2} + \frac{\partial^2 R}{\partial z\partial x}\right)\hat{\mathbf{i}} + \left(\frac{\partial^2 R}{\partial z\partial y} - \frac{\partial^2 Q}{\partial z^2} - \frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 P}{\partial x\partial y}\right)\hat{\mathbf{j}}$$
$$+ \left(\frac{\partial^2 P}{\partial x\partial z} - \frac{\partial^2 R}{\partial x^2} - \frac{\partial^2 R}{\partial y^2} + \frac{\partial^2 Q}{\partial y\partial z}\right)\hat{\mathbf{k}}.$$

On the other hand, $\nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F} = \nabla\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) - \nabla^2(P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}})$
$$= \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 Q}{\partial x\partial y} + \frac{\partial^2 R}{\partial z\partial x} - \frac{\partial^2 P}{\partial x^2} - \frac{\partial^2 P}{\partial y^2} - \frac{\partial^2 P}{\partial z^2}\right)\hat{\mathbf{i}}$$
$$+ \left(\frac{\partial^2 P}{\partial y\partial x} + \frac{\partial^2 Q}{\partial y^2} + \frac{\partial^2 R}{\partial y\partial z} - \frac{\partial^2 Q}{\partial x^2} - \frac{\partial^2 Q}{\partial y^2} - \frac{\partial^2 Q}{\partial z^2}\right)\hat{\mathbf{j}}$$
$$+ \left(\frac{\partial^2 P}{\partial z\partial x} + \frac{\partial^2 Q}{\partial z\partial y} + \frac{\partial^2 R}{\partial z^2} - \frac{\partial^2 R}{\partial x^2} - \frac{\partial^2 R}{\partial y^2} - \frac{\partial^2 R}{\partial z^2}\right)\hat{\mathbf{k}}$$
$$= \left(\frac{\partial^2 Q}{\partial y\partial x} - \frac{\partial^2 P}{\partial y^2} - \frac{\partial^2 P}{\partial z^2} + \frac{\partial^2 R}{\partial z\partial x}\right)\hat{\mathbf{i}}$$
$$+ \left(\frac{\partial^2 R}{\partial z\partial y} - \frac{\partial^2 Q}{\partial z^2} - \frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 P}{\partial x\partial y}\right)\hat{\mathbf{j}}$$
$$+ \left(\frac{\partial^2 P}{\partial x\partial z} - \frac{\partial^2 R}{\partial x^2} - \frac{\partial^2 R}{\partial y^2} + \frac{\partial^2 Q}{\partial y\partial z}\right)\hat{\mathbf{k}}.$$