DE Assignment 7

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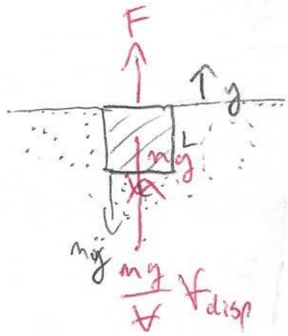
5.3



$$F - mg - m\ddot{y} + \frac{mg}{L^3} L^2(L-y) = 0$$

$$m\ddot{y} - \frac{mg}{L}(L-y) = F - mg$$

$$m\ddot{y} - mg + \frac{mgy}{L} = F - mg$$

$$\ddot{y} + \frac{g}{L}y = \frac{F}{m}$$

$$\left(D^2 + \frac{g}{L}\right)y = F/m$$

y_c, $\lambda^2 = -\frac{g}{L}$

$$\lambda = \pm i\sqrt{\frac{g}{L}}$$

$$\therefore y_c = C_1 \sin\left(\sqrt{\frac{g}{L}}\,t\right) + C_2 \cos\left(\sqrt{\frac{g}{L}}\,t\right)$$

y_p, $\frac{1}{D^2 + \frac{g}{L}} \frac{F}{m}$

$$\frac{\frac{L}{g}}{\frac{L}{g}D^2 + 1}\frac{F}{m} = \frac{L}{g}\left(1 - \frac{L}{g}D^2\right)F/m = \frac{LF}{gm}$$

$$\therefore y = C_1 \sin\left(\sqrt{\frac{g}{L}}\, t\right) + C_2 \cos\left(\sqrt{\frac{g}{L}}\, t\right) + \frac{FL}{mg}, \quad @\ t=0,\ y=0$$

$$0 = C_2 + \frac{FL}{mg}$$

$$C_2 = -\frac{FL}{mg}, \quad @\text{ time } t=0,\ \dot{y}=0$$

$$\dot{y} = C_1\sqrt{\frac{g}{L}} \cos\left(\sqrt{\frac{g}{L}}\, t\right) = 0, \quad C_1\sqrt{\frac{g}{L}} = 0, \quad \therefore C_1 = 0$$

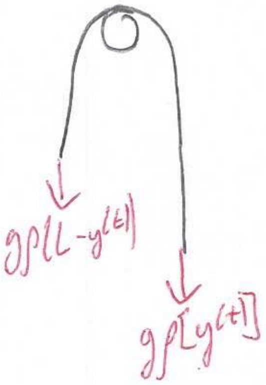
$$\therefore y(t) = \frac{FL}{mg}\left(1 - \cos\left(\sqrt{\frac{g}{L}}\, t\right)\right), \quad @\ t=T,\ y=L$$

$$L = \frac{FL}{mg}\left(1 - \cos\left(\sqrt{\frac{g}{L}}\, T\right)\right)$$

$$1 - \frac{mg}{F} = \cos\left(\sqrt{\frac{g}{L}}\, T\right)$$

$$\sqrt{\frac{L}{g}} \cos^{-1}\left(1 - \frac{mg}{F}\right) = T$$

(5.7)



$$g\rho[y(t)] - g\rho[L-y(t)] = m\ddot{y}$$

$$g\rho(2y(t)-L) = m\ddot{y}$$

$$m\ddot{y} - 2yg\rho = -g\rho L$$

$$\ddot{y}L - 2yg = -gL$$

$$(D^2L - 2g)y = -gL$$

y_c, $D^2L - 2g = 0$

$$\lambda = \pm\sqrt{\frac{2g}{L}}$$

$$y_c = C_1 e^{\sqrt{\frac{2g}{L}}t} + C_2 e^{-\sqrt{\frac{2g}{L}}t}$$

Since terms of constants and e^x, $y_c = A\sinh\left(\sqrt{\frac{2g}{L}}t\right) + B\cosh\left(\sqrt{\frac{2g}{L}}t\right)$

$$y_p, = \frac{1}{D^2L-2g}(-gL)$$

$$= \frac{-1/2}{1-\frac{D^2L}{2g}} - gL$$

$$= \frac{1}{2g}\left(1+\frac{D^2L}{2g}\right)gL$$

$$= L/2$$

$\therefore$ $y = A\sinh\left(\sqrt{\frac{2g}{L}}t\right) + B\cosh\left(\sqrt{\frac{2g}{L}}t\right) + L/2$, when $t=0$, $y=l$

$$l = B + L/2$$

$$B = l - L/2$$

$\therefore$ $\dot{y} = A\cosh\left(\sqrt{\frac{2g}{L}}t\right)\sqrt{\frac{2g}{L}} + B\sinh\left(\sqrt{\frac{2g}{L}}t\right)\sqrt{\frac{2g}{L}}$, when $t=0$, $\dot{y}=0$

$$0 = A = 0$$

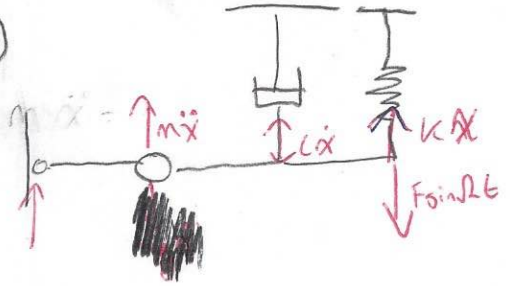
$$\therefore \quad y = (l - L/2)\cosh\left(\sqrt{\frac{2g}{L}}\,t\right) + L/2, \quad \text{when } t = T,\ y = L$$

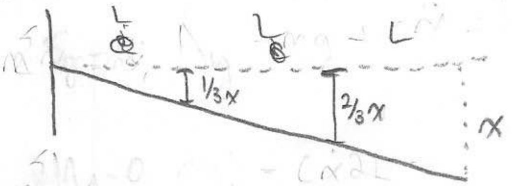
$$\frac{L - \frac{L}{2}}{l - L/2} = \cosh\left(\sqrt{\frac{2g}{L}}\,t\right)$$

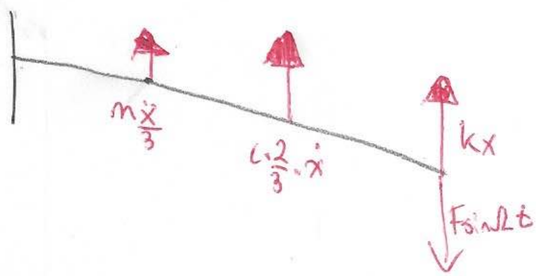
$$\cosh^{-1}\left(\frac{L/2}{l - L/2}\right) = \sqrt{\frac{2g}{L}}\,T$$

$$T = \cosh^{-1}\left(\frac{L/2}{l - L/2}\right)\sqrt{\frac{L}{2g}}$$

5.10







(1) $\Sigma M_A = 0,\ \frac{m}{3}\ddot{x}L + \frac{4c}{3}\dot{x}L + kx\cdot 3L - F\sin\Omega t(3L) = 0$

$$m\ddot{x}L + 4c\dot{x}L + kx9 = 9F\sin(\Omega t)$$

(2) $\ddot{x} + 4\frac{c}{m}\dot{x} + \frac{k}{m}\cdot 9x = \frac{9F}{m}\sin(\Omega t)$

$$\therefore \frac{9k}{m} = \omega_0^2,\ \omega_0 = 3\sqrt{\frac{k}{m}}$$

$$\omega_d = \omega_0\sqrt{1-\zeta^2}$$

$$2\zeta\omega_0 = 4\frac{c}{m}$$

$$2\zeta 3\sqrt{\frac{k}{m}} = 4\frac{c}{m}$$

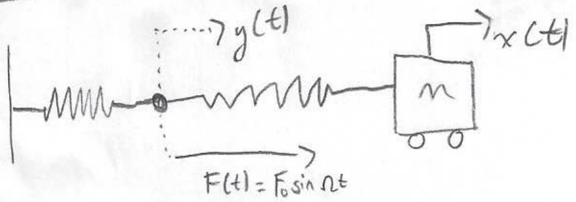
$$\zeta = \frac{2}{3}\frac{c\sqrt{m}}{\sqrt{k}\,m}$$

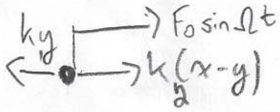
$$= \frac{2}{3}\cdot\frac{c}{\sqrt{km}}$$

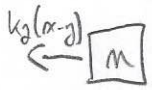
$$\omega_d = \omega_0\sqrt{1-\frac{4}{9}\frac{c^2}{km}}$$

$$= \omega_0\sqrt{1-\frac{4c^2}{9km}}$$

5.15

1. 

① 

② 

From ①, $F + k_2(x-y) - k_1(y) = \overset{0}{\cancel{m\ddot{y}}}$ From ②,

$F + k_2 x - y(k_2 + k_1) = 0$

$y = \dfrac{F + k_2 x}{k_2 + k_1}$

From ②, $-k_2(x-y) = m\ddot{x}$

Subbing ① into ②

$m\ddot{x} = -k_2\left(x - \dfrac{F + k_2 x}{k_2 + k_1}\right)$

$m\ddot{x} = -k_2 x\left(1 - \dfrac{k_2}{k^2 + k_1}\right) + \dfrac{k_2 F}{k_1 + k_2}$

$\therefore \ddot{x} + \left[\dfrac{k_2 - \dfrac{k_2^2}{k_2 + k_1}}{m}\right] x = \dfrac{\frac{k_2}{k_1 + k_2}}{m} F$. From original, $\dfrac{k_2}{m} - \dfrac{k_2}{m(k_2 + k_1)} = w_0^2$

$\dfrac{k_2(k_2 + k_1)}{m(k_2 + k_1)} - \dfrac{k_2}{m(k_2 + k_1)} = w_0^2$

$\dfrac{k_2^2 + k_1 k_2 - k^2}{m(k_2 + k_1)} = w_0^2$

$w_0 = \sqrt{\dfrac{k_1 k_2}{m(k_2 + k_1)}}$

Also, $RS = f \sin \Omega t$

$$\therefore f = \frac{k_2}{m(k_1+k_2)} F_0$$

2. x_c, $(D^2 + w_0^2)x = 0$

$$\lambda^2 = -w_0^2$$

$$\lambda = 0 \pm i w_0$$

$$x_c = A \cos w_0 t + B \sin w_0 t$$

x_p, $(D^2 + w_0^2)x = f \sin \Omega t$

$$x_p = \frac{1}{D^2 + w_0^2} f \sin \Omega t$$

$$= f \frac{1}{D^2 + w_0^2} \sin \Omega t$$

$$= f \frac{1}{-\Omega^2 + w_0^2} \sin \Omega t$$

$\therefore x(t) = A \cos w_0 t + B \sin w_0 t + f \cdot \frac{1}{w_0^2 - \Omega^2} \sin \Omega t$, when $t = 0$, $x(t) = 0$, $\dot{x}(t) = 0$

$x(0) = A$, $\therefore A = 0$

$\dot{x}(0) = -w_0 A \sin w_0 t + w_0 B \cos w_0 t + f \Omega \cdot \frac{1}{w_0^2 - \Omega^2} \cos \Omega t$

$0 = B + \frac{f\Omega}{w_0^2 - \Omega^2}$, $B = -\frac{f\Omega}{w_0^2 - \Omega^2} / w_0$

$$\therefore x(t) = \frac{f}{w_0^2 - \Omega^2}\left(\sin \Omega t - \frac{\Omega}{w_0} \sin w_0 t\right)$$

$$= k\mu \frac{(k+K - m\Omega^2 U^2) - cD}{[(k+K - m\Omega^2 U^2) + cD][(k+K - m\Omega^2 U^2) - cD]} \sin \Omega U t$$

$$= k\mu \frac{(k+K - m\Omega^2 U^2) - cD}{(k+K - m\Omega^2 U^2)^2 - c^2 D^2} \sin \Omega U t$$

$$= k\mu \frac{\overset{A}{(k+K - m\Omega^2 U^2)} \sin \Omega U t - \overset{B}{c\Omega U} \cos \Omega U t}{(k+K - m\Omega^2 U^2)^2 + c^2 \Omega U^2}$$

$$= k\mu \frac{1}{A^2+B^2} \sqrt{A^2+B^2} \left(\underbrace{\frac{A}{\sqrt{A^2+B^2}}}_{\cos\theta} \sin \Omega U t - \underbrace{\frac{B}{\sqrt{A^2+B^2}}}_{\sin\theta} \cos \Omega U t \right)$$

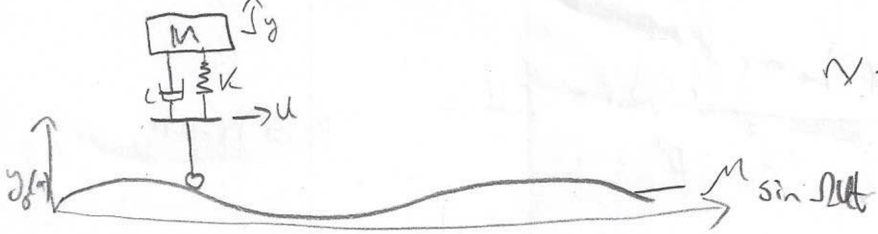
amplitude of steady state response

$$= \boxed{k\mu \frac{1}{\sqrt{(k+K - m\Omega^2 U^2 + c\Omega U)^2}}} \sin(\Omega U t - \theta)$$

5.16

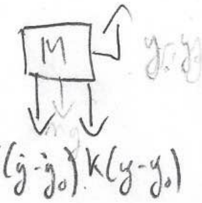
$y(t)$

M



$x = ut$, $\therefore \sin x = \sin ut$

(1)



$u = \frac{\omega_0}{\Omega}$

$ma = \sum F$, $m\ddot{y} = -c(\dot{y}-\dot{y}_0) - k(y-y_0)$

$m(\ddot{z} + \ddot{y}_0) = -c\dot{z} - kz$

$m\ddot{z} + m\ddot{y}_0 = -c\dot{z} - kz$

$\ddot{z} - \Omega^2 u \sin \Omega t = -\frac{c}{m}\dot{z} - \frac{k}{m}z$

$\ddot{z} + \frac{c}{m}\dot{z} + \frac{k}{m}z = \mu u^2 \Omega^2 \sin \Omega u t$

$\therefore c/m = 2\zeta\omega_0$, $\omega_0 = \sqrt{\frac{k}{m}}$

$(D^2 + 2\zeta\omega_0 D + \omega_0^2)z = \mu u^2 \Omega^2 \sin(\Omega u t)$

$$z_p = \frac{1}{D^2 - 2\zeta\omega_0 D + \omega_0^2} \mu u^2 \Omega^2 \sin(\Omega u t)$$

$$z_p = \mu u^2 \Omega^2 \cdot \frac{1}{-\Omega^2 u^2 - 2\zeta\omega_0 D + \omega_0^2} \sin(\Omega u t)$$

$$z_p = \mu u^2 \Omega^2 \cdot \frac{[(\omega_0^2 - \Omega^2 u^2) + 2\zeta\omega_0 D]}{[(\omega_0^2 - \Omega^2 u^2) - 2\zeta\omega_0 D][(\omega_0^2 - \Omega^2 u^2) + 2\zeta\omega_0 \Omega]} \sin \Omega u t$$

$$z_p = MU^2\Omega^2 \cdot \frac{\left[(w_0^2 - U^2\Omega^2) + 2\zeta w_0 D\right]}{(w_0^2-\Omega^2U^2)^2 - 4\zeta^2 w_0^2 D^2} \sin \Omega U t$$

$(a-b)(a+b)$

$$z_p = MU^2\Omega^2 \cdot \frac{\overset{A}{(w_0^2 - U^2\Omega^2)}\sin \Omega U t + \overset{B}{2\zeta w_0 \Omega U}\cos(\Omega U t)}{\left(\sqrt{(w_0^2-\Omega^2U^2)^2 + (2\zeta w_0 \Omega U)^2}\right)^2} \leftarrow \sqrt{A^2+B^2}$$

$$z_p = \frac{MU^2\Omega^2 \left[\cos(\Omega U t - \varphi)\right]}{\sqrt{(w_0^2-\Omega^2U^2)^2+(2\zeta w_0 \Omega U)^2}}$$, maximized when cos term = 1

$$\therefore |z_p(t)|_{max} = \frac{MU^2\Omega^2}{\sqrt{(w_0^2-\Omega^2U^2)^2+(2\zeta w_0 \Omega U)^2}}$$

3. Since $\zeta = 0$, $\ddot{z} + w_0^2 z = M\Omega^2U^2 \sin(w_0)$

Thus, from sin term, $w_0 = \Omega U$

$$U = \frac{w_0}{\Omega} = \frac{1}{\Omega}\sqrt{\frac{K}{M}}$$

(5.18)

$$\ddot{y}(t) + 2\zeta\omega_0\dot{y}(t) + \omega_0^2 y = -\ddot{x}_0(t)$$

$$\left(D^2 + 2\zeta\omega_0 D + \omega_0^2\right) y = -a\sin\Omega t$$

$x_0 = a\sin\Omega t$

$\dot{x}_0 = -a\Omega\cos\Omega t$

$\ddot{x}_0 = a\Omega^2\sin(\Omega t)$

$$y_p = \frac{1}{D^2 + 2\zeta\omega_0 D + \omega_0^2}\left(-a\sin(\Omega t)\right)$$

$$y_p = -a\frac{1}{-\Omega^2 + \omega_0^2 + 2\zeta\omega_0 D}\sin\Omega t$$

$$y_p = a\Omega^2\frac{(\omega_0^2-\Omega^2) - 2\zeta\omega_0 D}{\left[(\omega_0^2-\Omega^2) + 2\zeta\omega_0 D\right]\left[(\omega_0^2-\Omega^2) - 2\zeta\omega_0 D\right]}\sin\Omega t$$

$$y_p = a\Omega^2\frac{(\omega_0^2-\Omega^2) - 2\zeta\omega_0 D}{(\omega_0^2-\Omega^2)^2 - 2^2\zeta^2\omega_0^2 D^2}\sin\Omega t$$

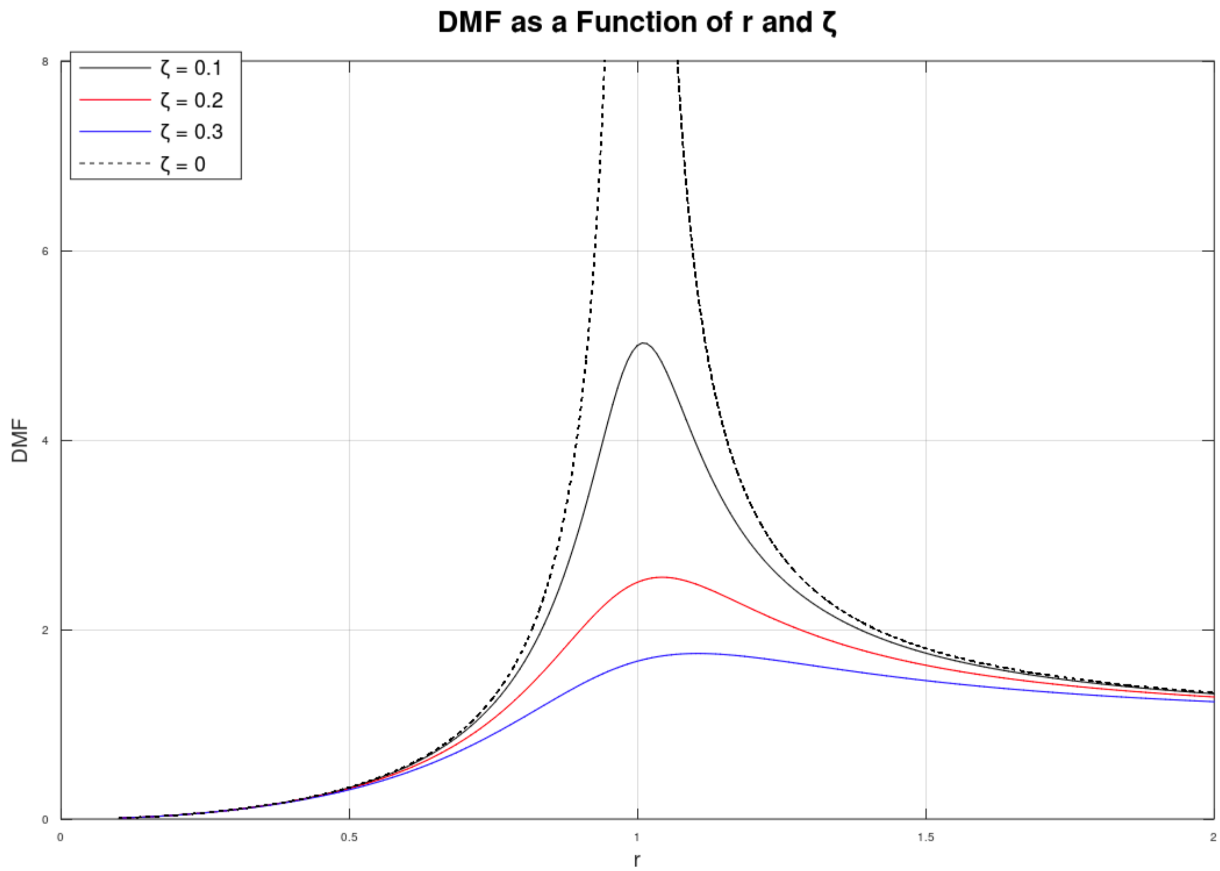
$$= a\Omega^2\frac{\overset{\cos\gamma}{(\omega_0^2-\Omega^2)}\sin\Omega t - \overset{\sin\gamma}{2\zeta\omega_0\Omega}\cos\Omega t}{\sqrt{(\omega_0^2-\Omega^2)^2 + 2^2\zeta^2\omega_0^2\Omega^2}\sqrt{(\omega_0^2-\Omega^2)^2 + 2^2\zeta^2\omega_0^2\Omega^2}}$$

$$= \frac{a\Omega^2\sin(\Omega t-\gamma)}{\sqrt{(\omega_0^2-\Omega^2)^2 + 2^2\zeta^2\omega_0^2\Omega^2}}$$, when maximized, $\sin(\Omega t-\gamma) = 1$

$$\therefore y_p = \frac{a\Omega^2}{\sqrt{(\omega_0^2-\Omega^2)^2 + 2^2\zeta^2\omega_0^2\Omega^2}}$$, $x_{0max} = a$

$$\therefore DMF = \frac{\Omega^2}{\sqrt{(\omega_0^2-\Omega^2)^2 + 2^2\zeta^2\omega_0^2\Omega^2}}\frac{\frac{1}{\omega_0^2}}{\frac{1}{\omega_0^2}} = \frac{r^2}{\sqrt{(\omega_0^2-\Omega^2)^2 + (2\zeta\omega_0\Omega)^2}\cdot\sqrt{\frac{1}{\omega_0^4}}}$$

$$= \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}, \quad r = \frac{\Omega}{\omega_0}$$


DMF as a Function of r and ζ
ζ = 0.1
ζ = 0.2
ζ = 0.3
ζ = 0
DMF
r
0
2
4
6
8
0
0.5
1
1.5
2