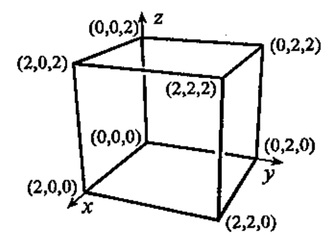
CHAPTER 11

EXERCISES 11.1

2. Length $= \sqrt{(1+3)^2 + (-2-2)^2 + (5-4)^2} = \sqrt{33}$

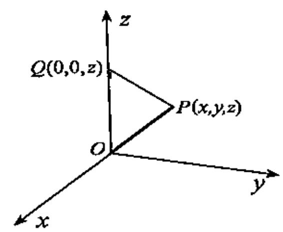
3. We find the squares of the lengths of the lines joining $P(2,0,4\sqrt{2})$, $Q(3,-1,5\sqrt{2})$, and $R(4,-2,4\sqrt{2})$: $\|PQ\|^2 = (1)^2 + (-1)^2 + (\sqrt{2})^2 = 4$, $\|PR\|^2 = (2)^2 + (-2)^2 = 8$, $\|QR\|^2 = (1)^2 + (-1)^2 + (-\sqrt{2})^2 = 4$. Since $\|PQ\| = \|QR\|$, the triangle is isosceles, and because $\|PR\|^2 = \|PQ\|^2 + \|QR\|^2$, the triangle is right angled.

4. The diagram to the right indicates the vertices of the cube.



5. If we draw a line from $P(x,y,z)$ perpendicular to the z-axis, the coordinates of Q are $(0,0,z)$. The length of the perpendicular is

$$\begin{aligned}\|PQ\| &= \sqrt{\|OP\|^2 - \|OQ\|^2}\\ &= \sqrt{x^2+y^2+z^2-z^2}\\ &= \sqrt{x^2+y^2}.\end{aligned}$$



Similar derivations give distances to the x- and y-axes.

6. (a) $\sqrt{2^2+3^2+(-4)^2} = \sqrt{29}$ (b) $\sqrt{3^2+(-4)^2} = 5$ (c) $\sqrt{2^2+(-4)^2} = 2\sqrt{5}$ (d) $\sqrt{2^2+3^2} = \sqrt{13}$

7. (a) $\sqrt{1^2+(-5)^2+(-6)^2} = \sqrt{62}$ (b) $\sqrt{(-5)^2+(-6)^2} = \sqrt{61}$ (c) $\sqrt{1^2+(-6)^2} = \sqrt{37}$
(d) $\sqrt{1^2+(-5)^2} = \sqrt{26}$

8. (a) $\sqrt{4^2+3^2} = 5$ (b) $\sqrt{3^2} = 3$ (c) $\sqrt{4^2} = 4$ (d) $\sqrt{4^2+3^2} = 5$

9. (a) $\sqrt{(-2)^2+1^2+(-3)^2} = \sqrt{14}$ (b) $\sqrt{1^2+(-3)^2} = \sqrt{10}$ (c) $\sqrt{(-2)^2+(-3)^2} = \sqrt{13}$
(d) $\sqrt{(-2)^2+1^2} = \sqrt{5}$

10. The lengths of the lines joining $P(1,3,5)$, $Q(-2,0,3)$, and $R(7,9,9)$ are

$$\|PQ\| = \sqrt{(-3)^2+(-3)^2+(-2)^2} = \sqrt{22}, \quad \|QR\| = \sqrt{(9)^2+(9)^2+(6)^2} = 3\sqrt{22},$$

$$\|PR\| = \sqrt{(6)^2+(6)^2+(4)^2} = 2\sqrt{22}.$$

Since $\|QR\| = \|PQ\| + \|PR\|$, the three points are collinear.

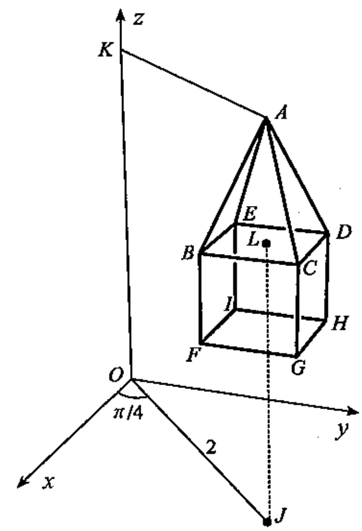
11. The coordinates of the point can be taken in the form $(x, 3x, 0)$. The fact that the point is equidistant from $(1,3,2)$ and $(2,4,5)$ is expressed as $(x-1)^2 + (3x-3)^2 + (-2)^2 = (x-2)^2 + (3x-4)^2 + (-5)^2$. The only solution of this equation is $x = 31/8$, and therefore the required point is $(31/8, 93/8, 0)$.

12. A point $P(x,y,z)$ is equidistant from $(-3,0,4)$ and $(2,1,5)$ if and only if $(x+3)^2 + (y-0)^2 + (z-4)^2 = (x-2)^2 + (y-1)^2 + (z-5)^2$ and this equation reduces to $10x + 2y + 2z = 5$. The equation should describe a plane.

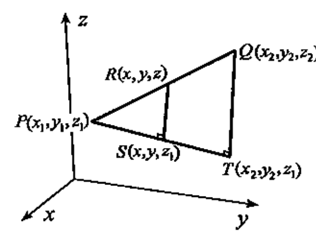
13. (a) If the third vertex is on the z-axis, its coordinates must be $P(0,0,z)$. Because this point is equidistant from $Q(\sqrt{3}-3, 2+2\sqrt{3}, 2\sqrt{3}-1)$ and $R(2\sqrt{3}, 4, \sqrt{3}-2)$, we can write that $(\sqrt{3}-3)^2 + (2+2\sqrt{3})^2 + (2\sqrt{3}-1-z)^2 = (2\sqrt{3})^2 + (4)^2 + (\sqrt{3}-2-z)^2$. The solution of this equation is $z = \sqrt{3}$. Since $\|PQ\| = \|QR\| = 4\sqrt{2}$, the triangle is equilateral.
(b) If the third vertex is on the x-axis, its coordinates must be $P(x,0,0)$. For P to be equidistant from Q and R, we can write that $(\sqrt{3}-3-x)^2 + (2+2\sqrt{3})^2 + (2\sqrt{3}-1)^2 = (2\sqrt{3}-x)^2 + (4)^2 + (\sqrt{3}-2)^2$. The solution of this equation is $x = -1$. Since $\|PQ\| \neq \|QR\|$, the triangle is isosceles but not equilateral.

14. Because $\|OJ\| = \|KA\|$, the x and y coordinates of A and J are the same, namely, $\sqrt{2}$ and $\sqrt{2}$. The coordinates of A are therefore $(\sqrt{2}, \sqrt{2}, 5)$. The length of BD is
$\sqrt{\|BC\|^2 + \|CD\|^2} = \sqrt{(1/2)^2 + (1/2)^2} = \sqrt{2}/2.$
The length of AL is
$\sqrt{\|AD\|^2 - \|LD\|^2} = \sqrt{(3/4)^2 - (\sqrt{2}/4)^2} = \sqrt{7}/4.$
Consequently, the coordinates of the remaining corners are

$B(\sqrt{2}+1/4, \sqrt{2}-1/4, 5-\sqrt{7}/4)$,
$C(\sqrt{2}+1/4, \sqrt{2}+1/4, 5-\sqrt{7}/4)$,
$D(\sqrt{2}-1/4, \sqrt{2}+1/4, 5-\sqrt{7}/4)$,
$E(\sqrt{2}-1/4, \sqrt{2}-1/4, 5-\sqrt{7}/4)$,
$F(\sqrt{2}+1/4, \sqrt{2}-1/4, 9/2-\sqrt{7}/4)$,
$G(\sqrt{2}+1/4, \sqrt{2}+1/4, 9/2-\sqrt{7}/4)$,
$H(\sqrt{2}-1/4, \sqrt{2}+1/4, 9/2-\sqrt{7}/4)$,
$I(\sqrt{2}-1/4, \sqrt{2}-1/4, 9/2-\sqrt{7}/4)$.



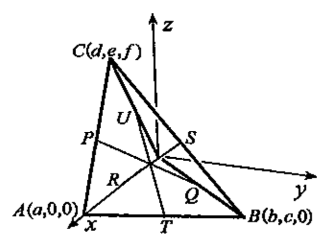
15. Because triangles PTQ and PSR are similar, ratios of corresponding sides are equal,
$$\frac{\|PQ\|}{\|PR\|} = \frac{\|QT\|}{\|RS\|} = \frac{z_2 - z_1}{z - z_1}.$$
Thus, $2 = (z_2 - z_1)/(z - z_1)$, from which $z = (z_1 + z_2)/2$. A similar proof gives the corresponding formulas for the x- and y-coordinates of R.



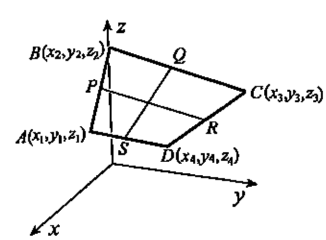
16. (a) According to Exercise 15, the required coordinates are $\left(\dfrac{1+3}{2}, \dfrac{-1+2}{2}, \dfrac{-3-4}{2}\right) = \left(2, \dfrac{1}{2}, -\dfrac{7}{2}\right)$.

(b) If the coordinates of R are (x, y, z), then, because Q is midway between P and R, $(3, 2, -4) = \left(\dfrac{x+1}{2}, \dfrac{y-1}{2}, \dfrac{z-3}{2}\right)$. When coordinates are equated, we obtain $x = 5$, $y = 5$, and $z = -5$.

17. The coordinates of the midpoints of the six sides of the tetrahedron have coordinates
$$P\left(\frac{a+d}{2}, \frac{e}{2}, \frac{f}{2}\right), \quad Q\left(\frac{b}{2}, \frac{c}{2}, 0\right), \quad R\left(\frac{a}{2}, 0, 0\right),$$
$$S\left(\frac{b+d}{2}, \frac{c+e}{2}, \frac{f}{2}\right), \quad T\left(\frac{a+b}{2}, \frac{c}{2}, 0\right), \quad U\left(\frac{d}{2}, \frac{e}{2}, \frac{f}{2}\right).$$
Midpoints of PQ, RS, and TU all have the same coordinates, $\left(\dfrac{a+b+d}{4}, \dfrac{e+c}{4}, \dfrac{f}{4}\right)$. This proves the required result.

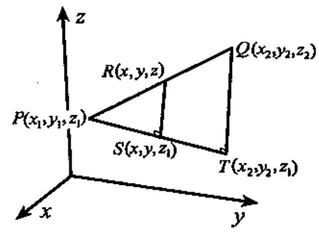


18. If coordinates of the vertices are as shown in the figure, then coordinates of the midpoints of the sides are
$$P\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right), \quad Q\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, \frac{z_2+z_3}{2}\right),$$
$$R\left(\frac{x_3+x_4}{2}, \frac{y_3+y_4}{2}, \frac{z_3+z_4}{2}\right), \quad S\left(\frac{x_4+x_1}{2}, \frac{y_4+y_1}{2}, \frac{z_4+z_1}{2}\right).$$
Midpoints of the line segments PR and QS both have coordinates
$$\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right),$$
and the line segments therefore intersect in this point.



19. Because triangles PTQ and PSR are similar, ratios of corresponding sides are equal,

$$\frac{\|PQ\|}{\|PR\|} = \frac{\|QT\|}{\|RS\|} = \frac{z_2 - z_1}{z - z_1}.$$



If we subtract 1 from each side of this equation,

$$\frac{\|PQ\|}{\|PR\|} - 1 = \frac{z_2 - z_1}{z - z_1} - 1$$

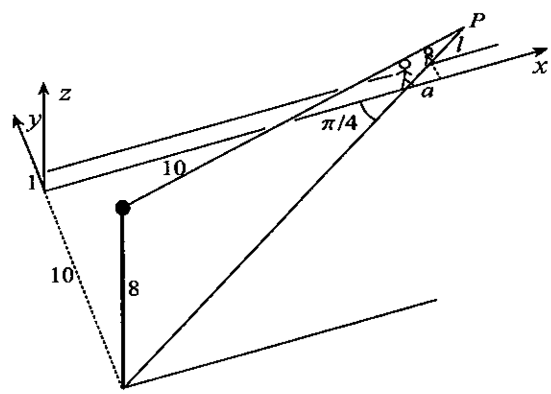
$$\Longrightarrow \quad \frac{\|PQ\| - \|PR\|}{\|PR\|} = \frac{z_2 - z}{z - z_1}$$

$$\Longrightarrow \quad \frac{r_2}{r_1} = \frac{z_2 - z}{z - z_1}.$$

Thus, $r_2 z - r_2 z_1 = r_1 z_2 - r_1 z \Longrightarrow z = \dfrac{r_1 z_2 + r_2 z_1}{r_1 + r_2}$. A similar proof gives the corresponding formulas for the x- and y-coordinates of R.

20. The x-coordinate of the tip of the shadow is $10 + a = 10 + 1 = 11$. The y-coordinate is 1. Let P be the point where the tip of the shadow would fall on the ground were the building not there. If z is the z-coordinate of the tip of the shadow on the wall, then from similar triangles, we may write

$$\frac{8}{2} = \frac{11\sqrt{2} + l}{\sqrt{2} + l} \quad \text{and} \quad \frac{8}{z} = \frac{11\sqrt{2} + l}{l}.$$



From the second of these, we obtain $l = \dfrac{11\sqrt{2}z}{8 - z}$, which substituted into the first gives

$$4\left(\sqrt{2} + \frac{11\sqrt{2}z}{8 - z}\right) = 11\sqrt{2} + \frac{11\sqrt{2}z}{8 - z}.$$

The solution of this equation is $z = 7/5$.

EXERCISES 11.2

See answers in text for even numbered exercises.

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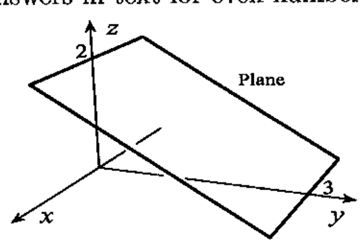

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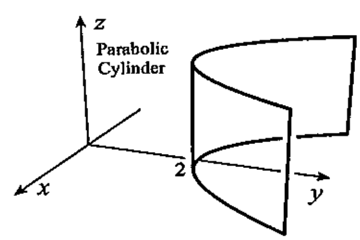

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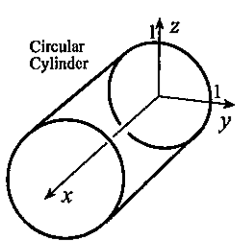

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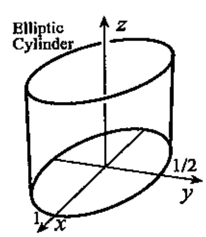

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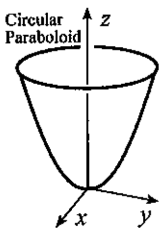

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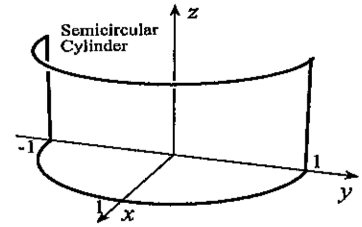

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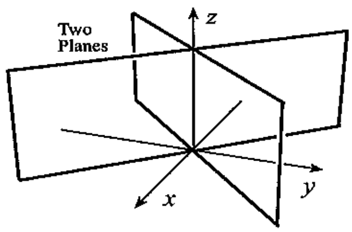

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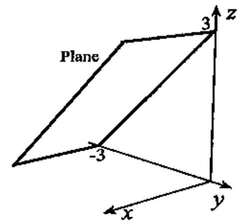

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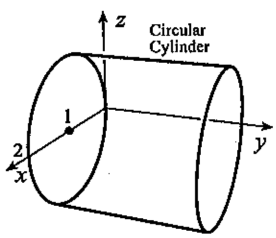

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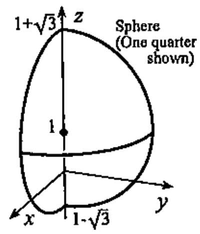

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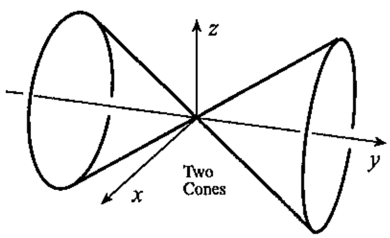

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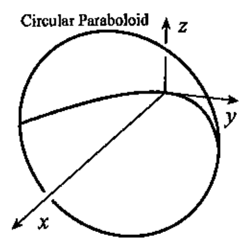

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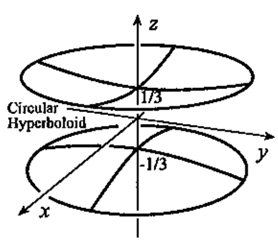

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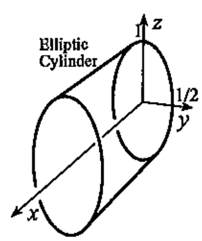

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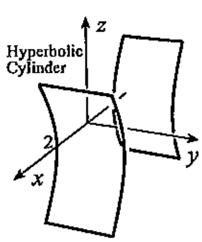

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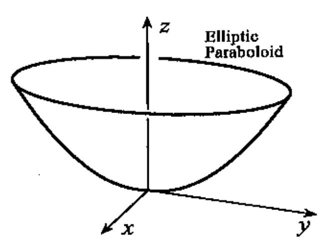

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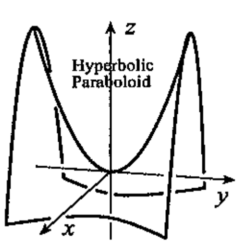

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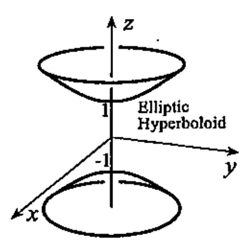

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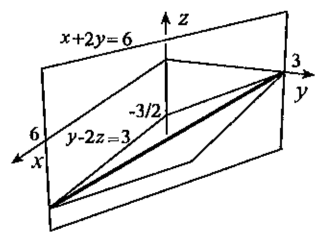

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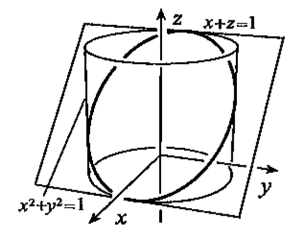

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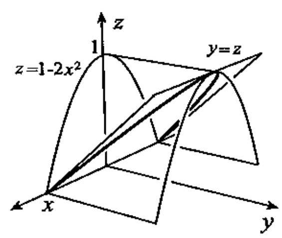

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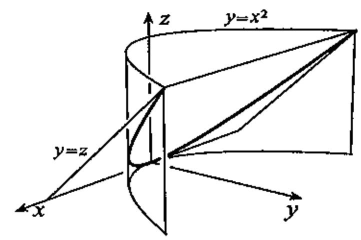

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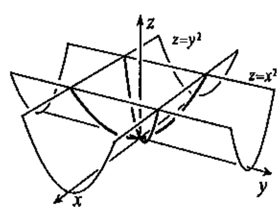
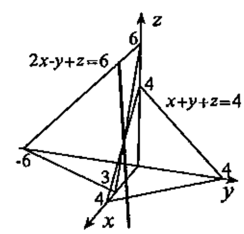
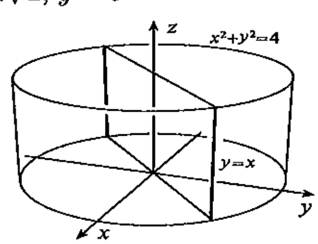


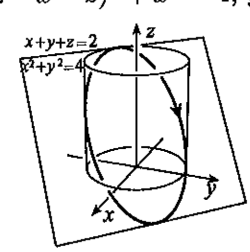
47. $x - 2y = 2,\ z = 0;$
$3y + z = 2,\ x = 0;$
$3x + 2z = 10,\ y = 0$



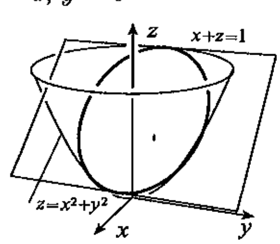
49. $x = \pm\sqrt{2},\ y = \pm\sqrt{2},\ z = 0;$
$y = \pm\sqrt{2},\ x = 0;$
$x = \pm\sqrt{2},\ y = 0$



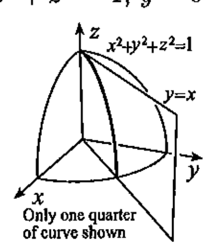
51. $x^2 + y^2 = 4,\ z = 0;$
$(2 - y - z)^2 + y^2 = 4,\ x = 0;$
$(2 - x - z)^2 + x^2 = 4,\ y = 0$



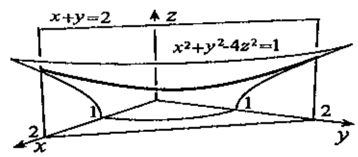
53. $(x + 1/2)^2 + y^2 = 5/4,\ z = 0;$
$y^2 + (z - 3/2)^2 = 5/4,\ x = 0;$
$x + z = 1,\ y = 0$



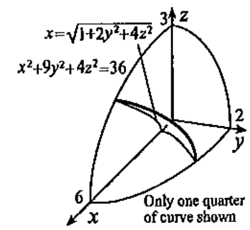
55. $y = x,\ z = 0,\ -1/\sqrt{2} \le x \le 1/\sqrt{2};$
$2y^2 + z^2 = 1,\ x = 0;$
$2x^2 + z^2 = 1,\ y = 0$



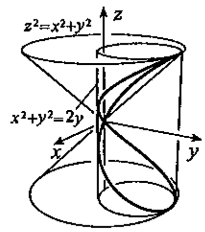
57. $4z^2 - 2(x-1)^2 = 1$, $y = 0$



59. $11y^2 + 8z^2 = 35$, $x = 0$



61. $z^2 = 2 \pm 2\sqrt{1-x^2}$, $y = 0$



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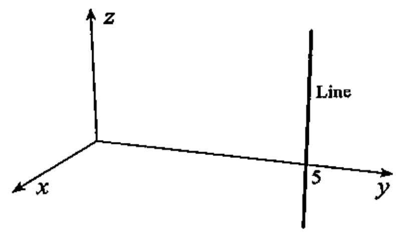

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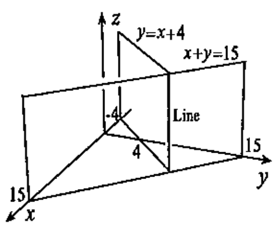

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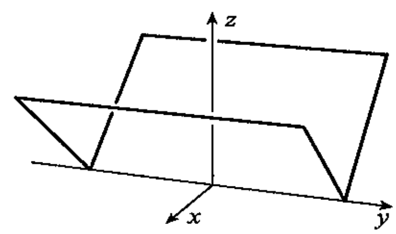

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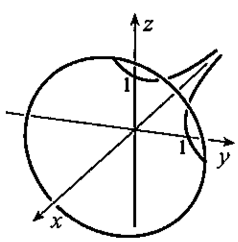
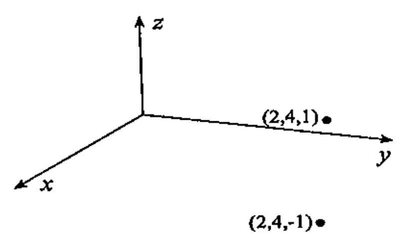


71.



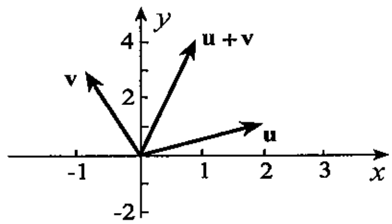
EXERCISES 11.3

1. $3\mathbf{u} - 2\mathbf{v} = 3(1,3,6) - 2(-2,0,4) = (7,9,10)$
2. $2\mathbf{w} + 3\mathbf{v} = 2(4,3,-2) + 3(-2,0,4) = (2,6,8)$
3. $\mathbf{w} - 3\mathbf{u} - 3\mathbf{v} = (4,3,-2) - 3(1,3,6) - 3(-2,0,4) = (7,-6,-32)$
4. $\hat{\mathbf{v}} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{(-2,0,4)}{\sqrt{20}} = \left(-\dfrac{1}{\sqrt{5}}, 0, \dfrac{2}{\sqrt{5}}\right)$
5. $2\hat{\mathbf{w}} - 3\mathbf{v} = \dfrac{2(4,3,-2)}{\sqrt{29}} - 3(-2,0,4) = \left(\dfrac{8}{\sqrt{29}} + 6, \dfrac{6}{\sqrt{29}}, \dfrac{-4}{\sqrt{29}} - 12\right)$
6. $|\mathbf{v}|\mathbf{v} - 2|\hat{\mathbf{v}}|\mathbf{w} = \sqrt{20}(-2,0,4) - 2(1)(4,3,-2) = (-4\sqrt{5} - 8, -6, 8\sqrt{5} + 4)$
7. $(15 - 2|\mathbf{w}|)(\mathbf{u} + \mathbf{v}) = (15 - 2\sqrt{29})[(1,3,6) + (-2,0,4)] = (-15 + 2\sqrt{29}, 45 - 6\sqrt{29}, 150 - 20\sqrt{29})$
8. $|3\mathbf{u}|\mathbf{v} - |-2\mathbf{v}|\mathbf{u} = 3\sqrt{46}(-2,0,4) - 2\sqrt{20}(1,3,6) = (-6\sqrt{46} - 4\sqrt{5}, -12\sqrt{5}, 12\sqrt{46} - 24\sqrt{5})$
9. $|2\mathbf{u} + 3\mathbf{v} - \mathbf{w}|\hat{\mathbf{w}} = |(2(1,3,6) + 3(-2,0,4) - (4,3,-2)|\hat{\mathbf{w}}$

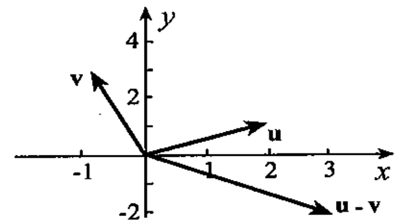
$$= |(-8,3,26)|\hat{\mathbf{w}} = \sqrt{(-8)^2 + 3^2 + 26^2}\frac{(4,3,-2)}{\sqrt{29}} = (4\sqrt{749/29}, 3\sqrt{749/29}, -2\sqrt{749/29})$$

10. $\dfrac{\mathbf{v}-\mathbf{w}}{|\mathbf{v}+\mathbf{w}|} = \dfrac{\mathbf{v}-\mathbf{w}}{|(-2,0,4)+(4,3,-2)|} = \dfrac{(-2,0,4)-(4,3,-2)}{|(2,3,2)|} = \dfrac{(-6,-3,6)}{\sqrt{17}} = \left(-\dfrac{6}{\sqrt{17}}, -\dfrac{3}{\sqrt{17}}, \dfrac{6}{\sqrt{17}}\right)$

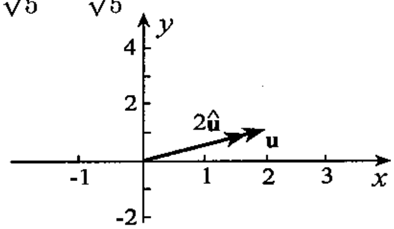
11. $\mathbf{u}+\mathbf{v} = \hat{\mathbf{i}} + 4\hat{\mathbf{j}}$



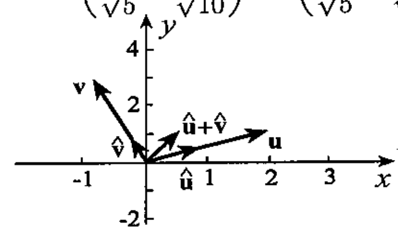
12. $\mathbf{u}-\mathbf{v} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}}$



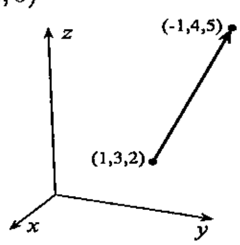
13. $2\hat{\mathbf{u}} = \dfrac{4}{\sqrt{5}}\hat{\mathbf{i}} + \dfrac{2}{\sqrt{5}}\hat{\mathbf{j}}$



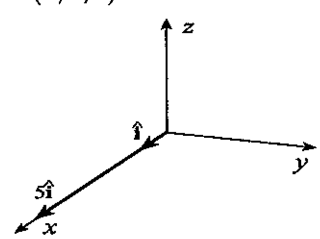
14. $\hat{\mathbf{v}} + \hat{\mathbf{u}} = \left(\dfrac{2}{\sqrt{5}} - \dfrac{1}{\sqrt{10}}\right)\hat{\mathbf{i}} + \left(\dfrac{1}{\sqrt{5}} + \dfrac{3}{\sqrt{10}}\right)\hat{\mathbf{j}}$



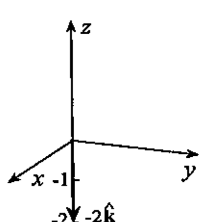
15. $(-2,1,3)$



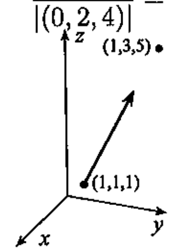
16. $5\hat{\mathbf{i}} = (5,0,0)$



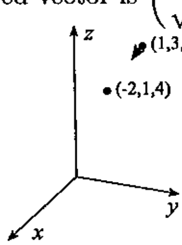
17. $-2\hat{\mathbf{k}} = (0,0,-2)$



18. The vector from $(1,1,1)$ to $(1,3,5)$ is $(1,3,5)-(1,1,1) = (0,2,4)$. The vector of length 3 in this direction is
$$\frac{3(0,2,4)}{|(0,2,4)|} = \left(0, \frac{3}{\sqrt{5}}, \frac{6}{\sqrt{5}}\right)$$

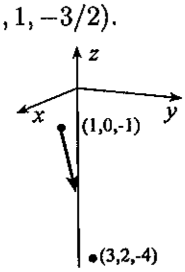


19. A vector in the correct direction is $(1,3,6)-(-2,1,4) = (3,2,2)$. The required vector is $\left(\dfrac{3}{\sqrt{17}}, \dfrac{2}{\sqrt{17}}, \dfrac{2}{\sqrt{17}}\right)$

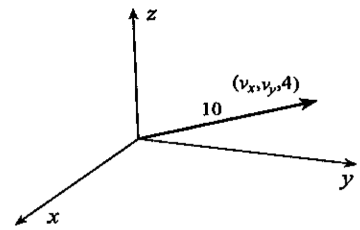


20. The vector from $(1,0,-1)$ to $(3,2,-4)$ is $(2,2,-3)$. A vector half as long is

$(1,1,-3/2)$.



21. If components of the vector are $(v_x, v_x, 4)$, the fact that its length is 10 requires $10 = \sqrt{2v_x{}^2 + 16} \implies v_x = \pm\sqrt{42}$. The required vector is $(\sqrt{42}, \sqrt{42}, 4)$.



22. Let (v_x, v_y, v_z) be the components of the vector, and draw perpendiculars from the tip P of the vector to the x- and y-axes. Since triangle OPQ is right-angled at Q, $\|OQ\|/\|OP\| = \cos(\pi/4)$. Consequently,

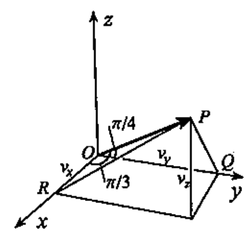
$$v_y = \|OQ\| = \|OP\|\cos(\pi/4) = \frac{5}{2}\left(\frac{1}{\sqrt{2}}\right) = \frac{5}{2\sqrt{2}}.$$

Similarly, $v_x = \dfrac{5}{2}\cos(\pi/3) = \dfrac{5}{2}\left(\dfrac{1}{2}\right) = \dfrac{5}{4}$.

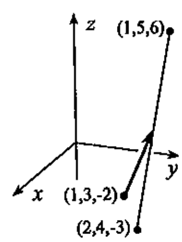
Finally, because the length of the vector is 5/2,

$$\frac{5}{2} = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{25/16 + 25/8 + v_z^2}.$$

Thus, $v_z{}^2 = \dfrac{25}{4} - \dfrac{25}{16} - \dfrac{25}{8} = \dfrac{25}{16} \implies v_z = 5/4$, and the vector has components $(5/4, 5\sqrt{2}/4, 5/4)$.



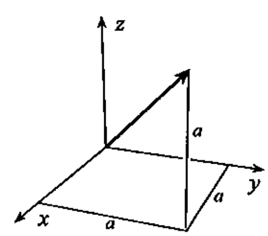
23. Since the midpoint of the line segment joining $(2,4,-3)$ and $(1,5,6)$ has coordinates $(3/2, 9/2, 3/2)$, the required vector is $(3/2, 9/2, 3/2) - (1,3,-2) = (1/2, 3/2, 7/2)$.



24. If the vector makes equal angles with the positive coordinate axes, its components must all be equal. If we take its components as (a, a, a), then the fact that its length is 2 requires

$$2 = \sqrt{a^2 + a^2 + a^2} = \sqrt{3}a \implies a = 2/\sqrt{3}.$$

The required vector is $(2/\sqrt{3}, 2/\sqrt{3}, 2/\sqrt{3})$.

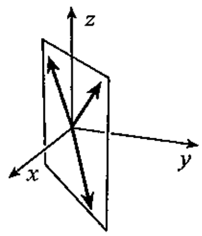


25. If coordinates of S are (x, y, z), the requirement $\mathbf{PQ} = \mathbf{RS}$ is expressed as $(-3, -1, 5) = (x-6, y-5, z+2)$. When we equate components, $x = 3$, $y = 4$, and $z = 3$.

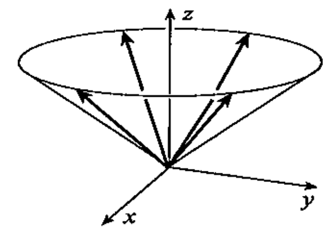
26. If we set $\mathbf{u} = (u_x, u_y, u_z)$, $\mathbf{v} = (v_x, v_y, v_z)$, and $\mathbf{w} = (w_x, w_y, w_z)$, then

$$\begin{aligned}
\mathbf{u} + \mathbf{v} &= (u_x + v_x, u_y + v_y, u_z + v_z) = (v_x + u_x, v_y + u_y, v_z + u_z) = \mathbf{v} + \mathbf{u};\\
(\mathbf{u} + \mathbf{v}) + \mathbf{w} &= (u_x + v_x, u_y + v_y, u_z + v_z) + (w_x, w_y, w_z)\\
&= (u_x + v_x + w_x, u_y + v_y + w_y, u_z + v_z + w_z)\\
&= (u_x, u_y, u_z) + (v_x + w_x, v_y + w_y, v_z + w_z) = \mathbf{u} + (\mathbf{v} + \mathbf{w});\\
\lambda(\mathbf{u} + \mathbf{v}) &= \lambda(u_x + v_x, u_y + v_y, u_z + v_z)\\
&= \big(\lambda(u_x + v_x), \lambda(u_y + v_y), \lambda(u_z + v_z)\big)\\
&= (\lambda u_x + \lambda v_x, \lambda u_y + \lambda v_y, \lambda u_z + \lambda v_z)\\
&= (\lambda u_x, \lambda u_y, \lambda u_z) + (\lambda v_x, \lambda v_y, \lambda v_z) = \lambda\mathbf{u} + \lambda\mathbf{v};\\
(\lambda + \mu)\mathbf{v} &= \big((\lambda + \mu)v_x, (\lambda + \mu)v_y, (\lambda + \mu)v_z\big)\\
&= (\lambda v_x + \mu v_x, \lambda v_y + \mu v_y, \lambda v_z + \mu v_z)\\
&= (\lambda v_x, \lambda v_y, \lambda v_z) + (\mu v_x, \mu v_y, \mu v_z) = \lambda\mathbf{v} + \mu\mathbf{v}.
\end{aligned}$$

27. All vectors lie in the plane $y = x$.



28. All vectors lie on the cone $z = \sqrt{x^2 + y^2}$.



29. When we equate components of $(5, -18, -32) = \mathbf{w} = \lambda\mathbf{u} + \mu\mathbf{v} = \lambda(3, 2, -4) + \mu(1, 6, 5) = (3\lambda + \mu, 2\lambda + 6\mu, -4\lambda + 5\mu)$, we obtain $3\lambda + \mu = 5$, $2\lambda + 6\mu = -18$, and $-4\lambda + 5\mu = -32$. The solution of these is $\lambda = 3$ and $\mu = -4$.

30. The slope of the curve $y = f(x) = x^2$ at $(2, 4)$ is $f'(2) = 4$. A vector along the tangent line is $(1, 4)$. A vector of length 3 along this line is $\mathbf{T} = \dfrac{3(1,4)}{\sqrt{1^2+4^2}} = \left(\dfrac{3}{\sqrt{17}}, \dfrac{12}{\sqrt{17}}\right)$.

31. The force due to the 3-coulomb charge at $(1, 1, 2)$ is

$$\mathbf{F}_1 = \frac{(2)(3)}{4\pi\epsilon_0(1+1+4)}\frac{(-1,-1,-2)}{\sqrt{6}} = \frac{(-1,-1,-2)}{4\sqrt{6}\pi\epsilon_0}\ \text{N}.$$

The force due to the 3-coulomb charge at $(2, -1, -2)$ is

$$\mathbf{F}_2 = \frac{(2)(3)}{4\pi\epsilon_0(4+1+4)}\frac{(-2,1,2)}{3} = \frac{(-2,1,2)}{18\pi\epsilon_0}\ \text{N}.$$

The resultant force due to both charges is

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \frac{(-1,-1,-2)}{4\sqrt{6}\pi\epsilon_0} + \frac{(-2,1,2)}{18\pi\epsilon_0} = \frac{1}{4\pi\epsilon_0}\left(\frac{-1}{\sqrt{6}} - \frac{4}{9}, \frac{-1}{\sqrt{6}} + \frac{2}{9}, \frac{-2}{\sqrt{6}} + \frac{4}{9}\right)\ \text{N}.$$

32. The force due to the mass at $(5, 1, 3)$ is

$$\mathbf{F}_1 = \frac{G(5)(10)}{3^2+(-1)^2+1^2}\frac{(3,-1,1)}{\sqrt{3^2+(-1)^2+1^2}} = \frac{50G}{11\sqrt{11}}(3,-1,1)\ \text{N}.$$

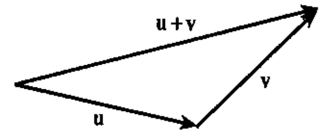
The force due to the mass at $(-1, 2, 1)$ is

$$\mathbf{F}_2 = \frac{G(5)(10)}{(-3)^2+0^2+(-1)^2}\frac{(-3,0,-1)}{\sqrt{(-3)^2+(-1)^2}} = \frac{5G}{\sqrt{10}}(-3,0,-1)\ \text{N}.$$

The resultant force due to both masses is

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \frac{50G(3,-1,1)}{11\sqrt{11}} + \frac{5G(-3,0,-1)}{\sqrt{10}} = 5G\left(\frac{30}{11\sqrt{11}} - \frac{3}{\sqrt{10}}, \frac{-10}{11\sqrt{11}}, \frac{10}{11\sqrt{11}} - \frac{1}{\sqrt{10}}\right)\ \text{N}.$$

33. Since the three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}+\mathbf{v}$ form the sides of a triangle, the triangle inequality simply agrees with the fact that the length of one side of a triangle is less than or equal to the sum of the lengths of the other two sides. To prove it algebraically, we set $\mathbf{u} = (u_x, u_y, u_z)$ and $\mathbf{v} = (v_x, v_y, v_z)$, then



$$|\mathbf{u}| + |\mathbf{v}| \geq |\mathbf{u}+\mathbf{v}| \iff (|\mathbf{u}| + |\mathbf{v}|)^2 \geq |\mathbf{u}+\mathbf{v}|^2$$

$$\iff \left(\sqrt{{u_x}^2 + {u_y}^2 + {u_z}^2} + \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2}\right)^2 \geq (u_x+v_x)^2 + (u_y+v_y)^2 + (u_z+v_z)^2$$

$$\iff \quad 2\sqrt{u_x{}^2+u_y{}^2+u_z{}^2}\sqrt{v_x{}^2+v_y{}^2+v_z{}^2} \ge 2(u_xv_x+u_yv_y+u_zv_z)$$

$$\iff \quad (u_x{}^2+u_y{}^2+u_z{}^2)(v_x{}^2+v_y{}^2+v_z{}^2) \ge (u_xv_x+u_yv_y+u_zv_z)^2$$

$$\iff \quad (u_xv_y-u_yv_x)^2+(u_yv_z-u_zv_y)^2+(u_xv_z-u_zv_x)^2 \ge 0$$

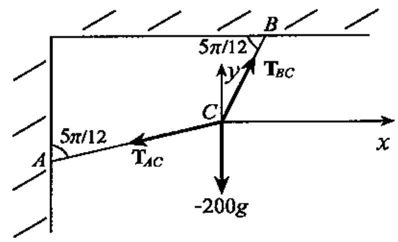
and this is clearly always true.

34. Let T_{AC} and T_{BC} be tensions in cables AC and BC. For equilibrium when both cables are taut, x- and y-components of all forces acting at C must sum to zero:

$0 = T_{BC}\cos 5\pi/12 - T_{AC}\sin 5\pi/12,$

$0 = T_{BC}\sin 5\pi/12 - T_{AC}\cos 5\pi/12 - 200g,$

where $g = 9.81$. When these are solved for T_{AC} and T_{BC}, the result, (to the nearest newton), is $T_{AC} = 586$ N and $T_{BC} = 2188$ N.

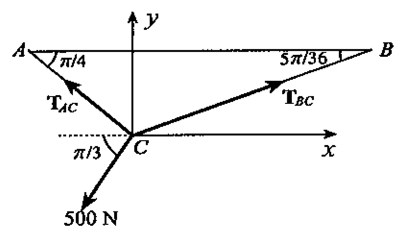


35. Let T_{AC} and T_{BC} be tensions in cables AC and BC. For equilibrium when both cables are taut, x- and y-components of all forces acting at C must sum to zero:

$0 = -T_{AC}\cos \pi/4 + T_{BC}\cos 5\pi/36 - 500\cos \pi/3,$

$0 = T_{AC}\sin \pi/4 + T_{BC}\sin 5\pi/36 - 500\sin \pi/3.$

When these are solved for T_{AC} and T_{BC}, the result, (to the nearest newton), is $T_{AC} = 305$ N and $T_{BC} = 514$ N.



36. For equilibrium when both cables are taut, x- and y-components of all forces acting at C must sum to zero:

$0 = 750\cos 5\pi/36 - 600\cos \pi/4 - |\mathbf{F}|\cos\theta,$

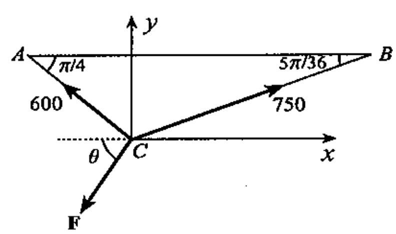
$0 = 750\sin 5\pi/36 + 600\sin \pi/4 - |\mathbf{F}|\sin\theta.$

When we write

$|\mathbf{F}|\sin\theta = 750\sin 5\pi/36 + 600/\sqrt{2},$

$|\mathbf{F}|\cos\theta = 750\cos 5\pi/36 - 600/\sqrt{2},$

and divide one equation by the other,



$$\tan\theta = \frac{750\sin 5\pi/36 + 600/\sqrt{2}}{750\cos 5\pi/36 - 600/\sqrt{2}} \quad \Longrightarrow \quad \theta = 1.24 \text{ radians.}$$

This in turn implies that $|\mathbf{F}| = 784$ N.

37. Assuming that there is no friction in the pulleys, the tension is the same at all points in the rope. For equilibrium, the sum of the x- and y-components of all forces acting on the pulley at O must be zero. If we assume that the two ropes from O to A are parallel, then

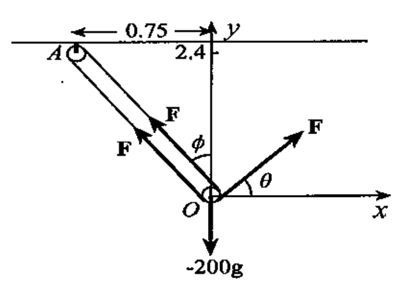
$0 = |\mathbf{F}|\cos\theta - 2|\mathbf{F}|\sin\phi,$

$0 = |\mathbf{F}|\sin\theta + 2|\mathbf{F}|\cos\phi - 200g,$

where $g = 9.81$. Since $\tan\phi = 0.75/2.4 = 5/16$, it follows that $\sin\phi = 5/\sqrt{281}$ and $\cos\phi = 16/\sqrt{281}$. From the first equation above,

$\cos\theta = 2\sin\phi = 10/\sqrt{281} \Longrightarrow \theta = \pm 0.9316$ radians.

When $\theta = 0.9316$, the second equation gives $|\mathbf{F}| = 200g/(\sin\theta + 2\cos\phi) = 724$ N. When $\theta = -0.9316$, we obtain $|\mathbf{F}| = 1773$ N.

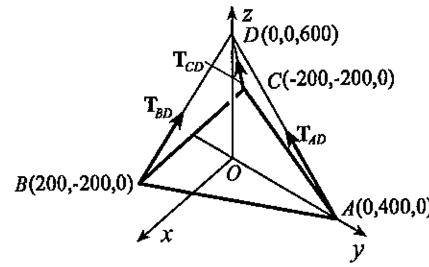


38. Let T_{AD}, T_{BD}, and T_{CD} denote tensions in the wires. The sum of all forces acting on the plate must be zero. There is the weight of the plate $\mathbf{W} = (0, 0, -16g)$, where $g = 9.81$, and tensions in the wires,

$$\mathbf{T}_{AD} = T_{AD}\left(\frac{\mathbf{AD}}{|\mathbf{AD}|}\right) = \frac{T_{AD}(0, -400, 600)}{\sqrt{400^2 + 600^2}} = \frac{T_{AD}(0, -2, 3)}{\sqrt{13}},$$

$$\mathbf{T}_{BD} = T_{BD}\left(\frac{\mathbf{BD}}{|\mathbf{BD}|}\right) = \frac{T_{BD}(-200, 200, 600)}{\sqrt{200^2 + 200^2 + 600^2}} = \frac{T_{BD}(-1, 1, 3)}{\sqrt{11}},$$

$$\mathbf{T}_{CD} = T_{CD}\left(\frac{\mathbf{CD}}{|\mathbf{CD}|}\right) = \frac{T_{CD}(200, 200, 600)}{\sqrt{200^2 + 200^2 + 600^2}} = \frac{T_{CD}(1, 1, 3)}{\sqrt{11}}.$$



Hence, $\mathbf{0} = (0, 0, -16g) + \dfrac{T_{AD}(0, -2, 3)}{\sqrt{13}} + \dfrac{T_{BD}(-1, 1, 3)}{\sqrt{11}} + \dfrac{T_{CD}(1, 1, 3)}{\sqrt{11}}$. When we equate components to zero,

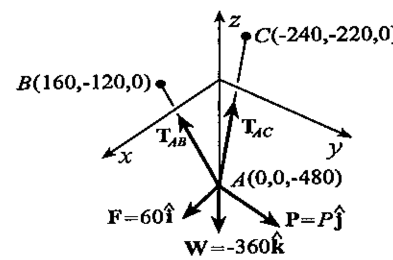
$$0 = -\frac{T_{BD}}{\sqrt{11}} + \frac{T_{CD}}{\sqrt{11}}, \quad 0 = -\frac{2T_{AD}}{\sqrt{13}} + \frac{T_{BD}}{\sqrt{11}} + \frac{T_{CD}}{\sqrt{11}}, \quad 0 = -16g + \frac{3T_{AD}}{\sqrt{13}} + \frac{3T_{BD}}{\sqrt{11}} + \frac{3T_{CD}}{\sqrt{11}}.$$

These can be solved for $T_{AD} = 16\sqrt{13}g/9$ N and $T_{BD} = T_{CD} = 16\sqrt{11}g/9$ N.

39. Let T_{AB} and T_{AC} denote tensions in the cables. The sum of all forces acting at A must be zero. There is $\mathbf{F} = 60\hat{\mathbf{i}}$, $\mathbf{P} = P\hat{\mathbf{j}}$, $\mathbf{W} = -360\hat{\mathbf{k}}$,

$$\mathbf{T}_{AB} = T_{AB}\left(\frac{\mathbf{AB}}{|\mathbf{AB}|}\right) = \frac{T_{AB}(160, -120, 480)}{\sqrt{160^2 + 120^2 + 480^2}} = \frac{T_{AB}(4, -3, 12)}{13},$$

$$\mathbf{T}_{AC} = T_{AC}\left(\frac{\mathbf{AC}}{|\mathbf{AC}|}\right) = \frac{T_{AC}(-240, -220, 480)}{\sqrt{240^2 + 220^2 + 480^2}} = \frac{T_{AC}(-12, -11, 24)}{29}.$$



Hence, $\mathbf{0} = 60\hat{\mathbf{i}} + P\hat{\mathbf{j}} - 360\hat{\mathbf{k}} + \dfrac{T_{AB}(4, -3, 12)}{13} + \dfrac{T_{AC}(-12, -11, 24)}{29}$. When we equate components to zero,

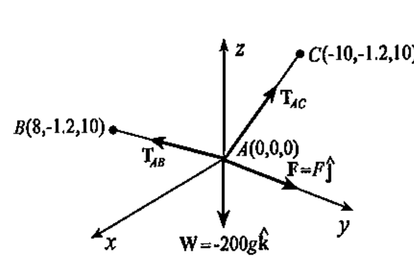
$$0 = 60 + \frac{4T_{AB}}{13} - \frac{12T_{AC}}{29}, \quad 0 = P - \frac{3T_{AB}}{13} - \frac{11T_{AC}}{29}, \quad 0 = -360 + \frac{12T_{AB}}{13} + \frac{24T_{AC}}{29}.$$

These can be solved for $T_{AB} = 156$ N, $T_{AC} = 261$ N, and $P = 135$ N.

40. Let T_{AB} and T_{AC} denote tensions in the cables. The sum of all forces acting at A must be zero. There is $\mathbf{F} = F\hat{\mathbf{j}}$, $\mathbf{W} = -200g\hat{\mathbf{k}}$ $(g = 9.81)$,

$$\mathbf{T}_{AB} = T_{AB}\left(\frac{\mathbf{AB}}{|\mathbf{AB}|}\right) = \frac{T_{AB}(8, -1.2, 10)}{\sqrt{8^2 + 1.2^2 + 10^2}} = \frac{T_{AB}(20, -3, 25)}{\sqrt{1034}},$$

$$\mathbf{T}_{AC} = T_{AC}\left(\frac{\mathbf{AC}}{|\mathbf{AC}|}\right) = \frac{T_{AC}(-10, -1.2, 10)}{\sqrt{10^2 + 1.2^2 + 10^2}} = \frac{T_{AC}(-25, -3, 25)}{\sqrt{1259}}.$$

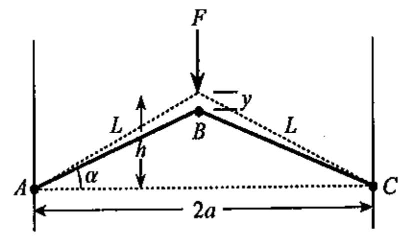


Hence, $\mathbf{0} = F\hat{\mathbf{j}} - 200g\hat{\mathbf{k}} + \dfrac{T_{AB}(20, -3, 25)}{\sqrt{1034}} + \dfrac{T_{AC}(-25, -3, 25)}{\sqrt{1259}}$. When we equate components to zero,

$$0 = \frac{20T_{AB}}{\sqrt{1034}} - \frac{25T_{AC}}{\sqrt{1259}}, \quad 0 = F - \frac{3T_{AB}}{\sqrt{1034}} - \frac{3T_{AC}}{\sqrt{1259}}, \quad 0 = -200g + \frac{25T_{AB}}{\sqrt{1034}} + \frac{25T_{AC}}{\sqrt{1259}}.$$

These can be solved for $T_{AB} = 40\sqrt{1034}g/9$ N, $T_{AC} = 32\sqrt{1259}g/9$ N, and $F = 24g$ N.

41. If we let P be the magnitude of the force in each bar when B is deflected downward by an amount y, then vertical components of forces acting at B give $2P\sin\alpha - F = 0 \Longrightarrow F = 2P\sin\alpha$. From equation 7.43, $P = (AE/L)[L - \sqrt{(h-y)^2 + a^2}]$, and therefore



$$F = \frac{2AE}{L}[L - \sqrt{(h-y)^2+a^2}]\frac{h-y}{\sqrt{(h-y)^2+a^2}} = \frac{2AE}{L}(h-y)\left[\frac{L}{\sqrt{(h-y)^2+a^2}} - 1\right]$$
$$= \frac{2AE}{L}(h-y)\left[\frac{L}{\sqrt{y^2-2hy+L^2}} - 1\right].$$

42. Since $0\mathbf{u} - 2\mathbf{v} + \mathbf{w} = \mathbf{0}$, the vectors are linearly dependent.

43. If we set $\mathbf{0} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = a(1,1,1) + b(2,1,3) + c(1,6,4)$
$= (a+2b+c, a+b+6c, a+3b+4c)$,
then $a+2b+c=0$, $a+b+6c=0$, $a+3b+4c=0$. Since the only solution of this system is $a=b=c=0$, the vectors are linearly independent.

44. If we set $\mathbf{0} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = a(-1,3,-5) + b(2,4,-1) + c(3,11,-7)$
$= (-a+2b+3c, 3a+4b+11c, -5a-b-7c)$,
then $-a+2b+3c=0$, $3a+4b+11c=0$, $-5a-b-7c=0$. This system of equations has an infinite number of solutions representable in the form $a=-c$, $b=-2c$, where c is arbitrary. The vectors are therefore linearly dependent.

45. If we set $\mathbf{0} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = a(4,2,6) + b(1,3,-2) + c(7,1,4)$
$= (4a+b+7c, 2a+3b+c, 6a-2b+4c)$,
then $4a+b+7c=0$, $2a+3b+c=0$, $6a-2b+4c=0$. Since the only solution of this system is $a=b=c=0$, the vectors are linearly independent.

46. If the coordinates of A, B, and C are (x_1,y_1,z_1), (x_2,y_2,z_2), and (x_3,y_3,z_3) respectively, then

$$D = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right) \quad \text{and} \quad E = \left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}, \frac{z_1+z_3}{2}\right).$$

Thus, $\mathbf{DE} = \left(\dfrac{x_3-x_2}{2}, \dfrac{y_3-y_2}{2}, \dfrac{z_3-z_2}{2}\right)$ and $\mathbf{BC} = (x_3-x_2, y_3-y_2, z_3-z_2)$. Clearly, $\mathbf{DE} = (1/2)\mathbf{BC}$, and the result is therefore verified.

47. The coordinates of the midpoint D of side BC are $\big((x_2+x_3)/2, (y_2+y_3)/2, (z_2+z_3)/2\big)$. With the result of Exercise 11.1-19, the point on AD two-thirds of the way from A to D has coordinates

$$\left(\frac{2}{3}\left(\frac{x_2+x_3}{2}\right)+\frac{x_1}{3}, \frac{2}{3}\left(\frac{y_2+y_3}{2}\right)+\frac{y_1}{3}, \frac{2}{3}\left(\frac{z_2+z_3}{2}\right)+\frac{z_1}{3}\right) = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right).$$

The points that divide medians from B and C in the same ratio have the same coordinates. It follows therefore that all three medians intersect in this point.

48. If we take x- and y-components of the vector equation $M(\overline{x},\overline{y}) = \sum_{i=1}^{n} m_i\mathbf{r}_i = \sum_{i=1}^{n} m_i(x_i,y_i)$, we obtain

$M\overline{x} = \sum_{i=1}^{n} m_i x_1$ and $M\overline{y} = \sum_{i=1}^{n} m_i y_i$. These are equations 7.31 and 7.32.

49. If $\mathbf{w} = w_x\hat{\mathbf{i}} + w_y\hat{\mathbf{j}}$ is any vector in the xy-plane, then for

$$\mathbf{w} = w_x\hat{\mathbf{i}} + w_y\hat{\mathbf{j}} = \lambda\mathbf{u} + \rho\mathbf{v} = (\lambda u_x + \rho v_x)\hat{\mathbf{i}} + (\lambda u_y + \rho v_y)\hat{\mathbf{j}}$$

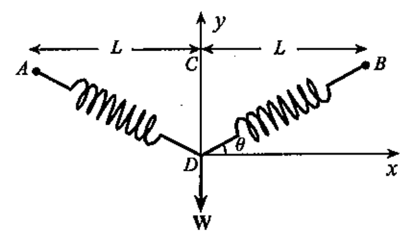
we must have $w_x = \lambda u_x + \rho v_x$, $w_y = \lambda u_y + \rho v_y$. The solution of these is

$$\lambda = \frac{w_x v_y - v_x w_y}{u_x v_y - u_y v_x}, \qquad \rho = \frac{u_x w_y - w_x u_y}{u_x v_y - u_y v_x}.$$

These are seen to exist provided $u_x v_y - u_y v_x \neq 0$.

50. (a) By summing vertical components of forces acting at D,

$$\begin{aligned} 0 &= -W + 2k(\sqrt{L^2+y^2} - L)\sin\theta \\ &= -W + 2k(\sqrt{L^2+y^2} - L)\frac{y}{\sqrt{L^2+y^2}}. \end{aligned}$$



Hence, $\quad W = 2ky\left(1 - \dfrac{L}{\sqrt{L^2+y^2}}\right).$

(b) When y is very much less than L, we use the binomial expansion to write,

$$W = 2ky\left[1 - \frac{1}{\sqrt{1+(y/L)^2}}\right] = 2ky\left\{1 - \left[1 - \frac{1}{2}\left(\frac{y}{L}\right)^2 + \cdots\right]\right\}.$$

If we neglect higher order terms, $\quad W \approx 2ky\left(\dfrac{y^2}{2L^2}\right) = \dfrac{ky^3}{L^2}.$

51. If we denote the position of the sleeve by P, a distance x from O, the magnitude of the force on the sleeve due to spring one is $k_1(\sqrt{x^2+l^2} - l)$. The direction of this force is along $\mathbf{PA}$, and a unit vector in this direction is

$$\widehat{\mathbf{PA}} = \frac{\mathbf{PA}}{|\mathbf{PA}|} = \frac{(0-x, l-0)}{\sqrt{x^2+l^2}} = \frac{(-x,l)}{\sqrt{x^2+l^2}}.$$

Consequently, the force due to spring one is

$$\mathbf{F}_1 = k_1(\sqrt{x^2+l^2} - l)\frac{(-x,l)}{\sqrt{x^2+l^2}} = k_1\left(1 - \frac{l}{\sqrt{x^2+l^2}}\right)(-x,l).$$

Similarly, the force due to spring two is

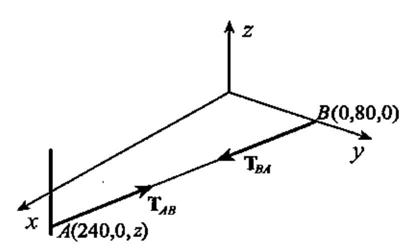
$$\mathbf{F}_2 = k_2(\sqrt{(L-x)^2+l^2} - l)\frac{(L-x,l)}{\sqrt{(L-x)^2+l^2}} = k_2\left(1 - \frac{l}{\sqrt{(L-x)^2+l^2}}\right)(L-x,l).$$

The resultant force on the sleeve is therefore

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = k_1\left(1 - \frac{l}{\sqrt{x^2+l^2}}\right)(-x,l) + k_2\left(1 - \frac{l}{\sqrt{(L-x)^2+l^2}}\right)(L-x,l).$$

52. If coordinates of A are denoted by $(240, 0, z)$, then, $440 = \sqrt{240^2+80^2+z^2} \Longrightarrow z = -360$. Let the tension in the cable be T. Because there is no friction between collar A and the rod, the reaction of the rod $\mathbf{R}_A = N_x\hat{\mathbf{i}} + N_y\hat{\mathbf{j}}$ has no z-component.
Similarly, the reaction of the rod on collar B has no y-component, $\mathbf{R}_B = M_x\hat{\mathbf{i}} + M_z\hat{\mathbf{k}}$. When we sum forces on the collars and equate them to zero,

$$\mathbf{0} = -450\hat{\mathbf{k}} + \mathbf{R}_A + \mathbf{T}_{AB}, \quad \mathbf{0} = P\hat{\mathbf{j}} + \mathbf{R}_B + \mathbf{T}_{BA},$$



where $\mathbf{T}_{AB}$ and $\mathbf{T}_{BA}$ are tensions in the wire. Since

$$\mathbf{T}_{AB} = T\left(\frac{\mathbf{AB}}{|\mathbf{AB}|}\right) = \frac{T(-240, 80, 360)}{\sqrt{240^2+80^2+360^2}} = \frac{T(-6,2,9)}{11},$$

we obtain

$$\mathbf{0} = -450\hat{\mathbf{k}} + (N_x\hat{\mathbf{i}} + N_y\hat{\mathbf{j}}) + \frac{T(-6,2,9)}{11}, \qquad \mathbf{0} = P\hat{\mathbf{j}} + (M_x\hat{\mathbf{i}} + M_z\hat{\mathbf{k}}) - \frac{T(-6,2,9)}{11}.$$

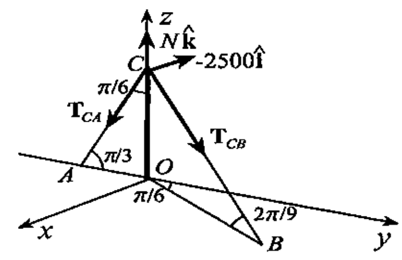
When we equate components,

$$0 = N_x - \frac{6T}{11}, \quad 0 = N_y + \frac{2T}{11}, \quad 0 = -450 + \frac{9T}{11}, \quad 0 = M_x + \frac{6T}{11}, \quad 0 = P - \frac{2T}{11}, \quad 0 = M_z - \frac{9T}{11}.$$

These give $T = 550$ N and $P = 100$ N.

53. Suppose $N\hat{\mathbf{k}}$ is the reaction of the pole at C. When we equate the sum of the forces acting at C to zero,

$$\begin{aligned}\mathbf{0} &= -2500\hat{\mathbf{i}} + N\hat{\mathbf{k}} + \mathbf{T}_{CA} + \mathbf{T}_{CB}\\ &= -2500\hat{\mathbf{i}} + N\hat{\mathbf{k}} + T_{CA}\left(-\frac{\hat{\mathbf{j}}}{2} - \frac{\sqrt{3}\hat{\mathbf{k}}}{2}\right)\\ &\quad + T_{CB}\left(\frac{1}{2}\cos 2\pi/9, \frac{\sqrt{3}}{2}\cos 2\pi/9, -\sin 2\pi/9\right).\end{aligned}$$



We now equate components,

$$0 = -2500 + \frac{T_{CB}}{2}\cos 2\pi/9, \quad 0 = -\frac{T_{CA}}{2} + \frac{\sqrt{3}T_{CB}}{2}\cos 2\pi/9, \quad 0 = N - \frac{\sqrt{3}T_{CA}}{2} - T_{CB}\sin 2\pi/9.$$

These can be solved for $T_{CA} = 8660$ N, $T_{CB} = 6527$ N, and $N = 11\,695$ N.

54. (a) Since $i = \phi$, the z-component of $\hat{\mathbf{v}}$ must be the negative of the z-components of $\hat{\mathbf{u}}$; that is, $v_z = -u_z$. As a result, $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ both have the same projections in the xy-plane, and hence they have the same x- and y-components. The components of $\hat{\mathbf{v}}$ are therefore $(u_x, u_y, -u_z)$.
(b) The z-components of $\hat{\mathbf{u}}$ and $\hat{\mathbf{w}}$ are $u_z = -\cos i$ and $w_z = -\cos\theta$, and therefore

$$w_z = -\sqrt{1-\sin^2\theta} = -\sqrt{1 - \frac{{n_1}^2}{{n_2}^2}\sin^2 i} = -\sqrt{1 - \frac{{n_1}^2}{{n_2}^2}(1-\cos^2 i)} = -\sqrt{1 - \frac{{n_1}^2}{{n_2}^2}(1-{u_z}^2)}.$$

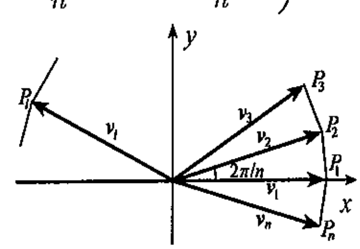
The projections of $\hat{\mathbf{u}}$ and $\hat{\mathbf{w}}$ in the xy-plane must be in the same direction so that we may write $(w_x, w_y, 0) = \lambda(u_x, u_y, 0)$. In addition, the lengths of these vectors must be

$$\sqrt{{w_x}^2 + {w_y}^2} = \sin\theta, \qquad \sqrt{{u_x}^2 + {u_y}^2} = \sin i.$$

Because $n_1 \sin i = n_2 \sin\theta$, it follows that ${n_1}^2({u_x}^2 + {u_y}^2) = {n_2}^2({w_x}^2 + {w_y}^2) = {n_2}^2(\lambda^2{u_x}^2 + \lambda^2{u_y}^2)$.
This equation implies that $\lambda = n_1/n_2$. Consequently, $\hat{\mathbf{w}} = \left(\frac{n_1}{n_2}u_x, \frac{n_1}{n_2}u_y, -\sqrt{1 - \frac{{n_1}^2}{{n_2}^2}(1-{u_z}^2)}\right)$.

55. Suppose that the length of each vector is a and the angle between any two successive vectors is $2\pi/n$. The i[th] vector $\mathbf{v}_i$ in the diagram has components $\mathbf{v}_i = a\left(\cos\frac{2\pi(i-1)}{n}, \sin\frac{2\pi(i-1)}{n}\right)$.
The sum of these vectors is

$$\begin{aligned}\sum_{i=1}^{n}\mathbf{v}_i &= a\left(\sum_{i=1}^{n}\cos\frac{2\pi(i-1)}{n}, \sum_{i=1}^{n}\sin\frac{2\pi(i-1)}{n}\right)\\ &= a\left(\sum_{i=0}^{n-1}\cos\frac{2\pi i}{n}, \sum_{i=0}^{n-1}\sin\frac{2\pi i}{n}\right).\end{aligned}$$



According to Exercises 6.3–12 and 13,

$$\sum_{i=1}^{m} \sin i\theta = \frac{\sin\dfrac{(m+1)\theta}{2}\sin\dfrac{m\theta}{2}}{\sin\dfrac{\theta}{2}} \qquad \text{and} \qquad \sum_{i=1}^{m} \cos i\theta = \frac{\cos\dfrac{(m+1)\theta}{2}\sin\dfrac{m\theta}{2}}{\sin\dfrac{\theta}{2}}.$$

With these,

$$\sum_{i=0}^{n-1} \sin\frac{2\pi i}{n} = 0 + \frac{\sin\left[\dfrac{n}{2}\left(\dfrac{2\pi}{n}\right)\right]\sin\left[\dfrac{n-1}{2}\left(\dfrac{2\pi}{n}\right)\right]}{\sin\left[\dfrac{1}{2}\left(\dfrac{2\pi}{n}\right)\right]} = 0,$$

and

$$\begin{aligned}\sum_{i=0}^{n-1} \cos\frac{2\pi i}{n} &= 1 + \frac{\cos\left[\dfrac{n}{2}\left(\dfrac{2\pi}{n}\right)\right]\sin\left[\dfrac{n-1}{2}\left(\dfrac{2\pi}{n}\right)\right]}{\sin\left[\dfrac{1}{2}\left(\dfrac{2\pi}{n}\right)\right]} = 1 - \frac{\sin\dfrac{(n-1)\pi}{n}}{\sin\dfrac{\pi}{n}}\\ &= 1 - \frac{\sin\pi\,\cos\dfrac{\pi}{n} - \cos\pi\,\sin\dfrac{\pi}{n}}{\sin\dfrac{\pi}{n}} = 0.\end{aligned}$$

This completes the proof.

EXERCISES 11.4

1. $\mathbf{u}\cdot\mathbf{v} = (2)(0) + (-3)(1) + (1)(-1) = -4$
2. $(\mathbf{v}\cdot\mathbf{w})\mathbf{u} = [(0)(6) + (1)(-2) + (-1)(3)](2,-3,1) = (-10,15,-5)$
3. $(2\mathbf{u} - 3\mathbf{v})\cdot\mathbf{w} = (4,-9,5)\cdot(6,-2,3) = (4)(6) + (-9)(-2) + (5)(3) = 57$
4. $2\hat{\mathbf{i}}\cdot\hat{\mathbf{u}} = (2,0,0)\cdot\dfrac{(2,-3,1)}{\sqrt{4+9+1}} = \dfrac{4}{\sqrt{14}}$
5. $|2\mathbf{u}|\mathbf{v}\cdot\mathbf{w} = 2\sqrt{4+9+1}[(0)(6) + (1)(-2) + (-1)(3)] = -10\sqrt{14}$
6. $(3\mathbf{u} - 4\mathbf{w})\cdot(2\hat{\mathbf{i}} + 3\mathbf{u} - 2\mathbf{v}) = (-18,-1,-9)\cdot(8,-11,5) = -144 + 11 - 45 = -178$
7. $\mathbf{w}\cdot\hat{\mathbf{w}} = (6,-2,3)\cdot\dfrac{(6,-2,3)}{7} = 7$
8. $\dfrac{(105\mathbf{u} + 240\mathbf{v})\cdot(105\mathbf{u} + 240\mathbf{v})}{|105\mathbf{u} + 240\mathbf{v}|^2} = \dfrac{|105\mathbf{u} + 240\mathbf{v}|^2}{|105\mathbf{u} + 240\mathbf{v}|^2} = 1$
9. $|\mathbf{u} - \mathbf{v} + \hat{\mathbf{k}}|(\hat{\mathbf{j}} + \mathbf{w})\cdot\hat{\mathbf{k}} = |(2,-4,3)|(6,-1,3)\cdot(0,0,1) = 3\sqrt{29}$
10. $\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{w} - (\mathbf{u} + \mathbf{w})\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{w} - \mathbf{u}\cdot\mathbf{v} - \mathbf{w}\cdot\mathbf{v} = 0$
11. $\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 2 & 0 \\ -2 & -3 & 5 \end{vmatrix} = 10\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$
12. $(-3\mathbf{u})\times(2\mathbf{v}) = -6(\mathbf{u}\times\mathbf{v}) = -6\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 1 & 4 \\ -1 & 2 & 0 \end{vmatrix} = -6[-8\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 7\hat{\mathbf{k}}] = 48\hat{\mathbf{i}} + 24\hat{\mathbf{j}} - 42\hat{\mathbf{k}}$
13. Using the result of Exercise 11, $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (3,1,4)\cdot(10,5,7) = 63$.
14. $\hat{\mathbf{u}}\times\hat{\mathbf{w}} = \dfrac{\mathbf{u}}{|\mathbf{u}|}\times\dfrac{\mathbf{w}}{|\mathbf{w}|} = \dfrac{1}{|\mathbf{u}||\mathbf{w}|}\mathbf{u}\times\mathbf{w} = \dfrac{1}{\sqrt{9+1+16}\sqrt{4+9+25}}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 1 & 4 \\ -2 & -3 & 5 \end{vmatrix}$

$$= \frac{1}{\sqrt{26}\sqrt{38}}(17\hat{\mathbf{i}} - 23\hat{\mathbf{j}} - 7\hat{\mathbf{k}}) = \frac{1}{2\sqrt{247}}(17\hat{\mathbf{i}} - 23\hat{\mathbf{j}} - 7\hat{\mathbf{k}})$$

15. $((3\mathbf{u})\times\mathbf{w})+(\mathbf{u}\times\mathbf{v}) = 3\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 1 & 4\\ -2 & -3 & 5\end{vmatrix} + \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 1 & 4\\ -1 & 2 & 0\end{vmatrix}$

$= 3(17,-23,-7)+(-8,-4,7) = (43,-73,-14)$

16. $\mathbf{u}\times(3\mathbf{v}-\mathbf{w}) = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 1 & 4\\ -1 & 9 & -5\end{vmatrix} = -41\hat{\mathbf{i}}+11\hat{\mathbf{j}}+28\hat{\mathbf{k}}$

17. Since $\mathbf{u}\times\mathbf{v} = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 1 & 4\\ -1 & 2 & 0\end{vmatrix} = (-8,-4,7)$, we find $\dfrac{\mathbf{w}\times\mathbf{u}}{|\mathbf{u}\times\mathbf{v}|} = \dfrac{1}{\sqrt{129}}\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ -2 & -3 & 5\\ 3 & 1 & 4\end{vmatrix} = \dfrac{(-17,23,7)}{\sqrt{129}}$.

18. $\mathbf{u}\times\mathbf{w} - \mathbf{u}\times\mathbf{v}+\mathbf{u}\times(2\mathbf{u}+\mathbf{v}) = \mathbf{u}\times(\mathbf{w}-\mathbf{v}+2\mathbf{u}+\mathbf{v}) = \mathbf{u}\times(\mathbf{w}+2\mathbf{u})$

$= \mathbf{u}\times\mathbf{w}$ (since $\mathbf{u}\times\mathbf{u}=\mathbf{0}$)

$= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 1 & 4\\ -2 & -3 & 5\end{vmatrix} = 17\hat{\mathbf{i}}-23\hat{\mathbf{j}}-7\hat{\mathbf{k}}$

19. $(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 1 & 4\\ -1 & 2 & 0\end{vmatrix}\times\mathbf{w} = (-8,-4,7)\times\mathbf{w} = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ -8 & -4 & 7\\ -2 & -3 & 5\end{vmatrix} = (1,26,16)$

20. $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = \mathbf{u}\times\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ -1 & 2 & 0\\ -2 & -3 & 5\end{vmatrix} = \mathbf{u}\times(10,5,7) = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 1 & 4\\ 10 & 5 & 7\end{vmatrix} = (-13,19,5)$

21. Since $(1,2)\cdot(3,5) = 13$, the vectors are not perpendicular.

22. Since $(2,4)\cdot(-8,4) = -16+16 = 0$, the vectors are perpendicular.

23. Since $(1,3,6)\cdot(-2,1,-4) = -23$, the vectors are not perpendicular.

24. Since $(2,3,-6)\cdot(-6,6,1) = -12+18-6 = 0$, the vectors are perpendicular.

25. If θ is the angle between the vectors, then

$$\theta = \mathrm{Cos}^{-1}\left[\frac{(3,4)\cdot(2,-5)}{|(3,4)|\,|(2,-5)|}\right] = \mathrm{Cos}^{-1}\left(\frac{-14}{5\sqrt{29}}\right) = 2.12 \text{ radians.}$$

26. If θ is the angle between the vectors, then

$$\theta = \mathrm{Cos}^{-1}\left[\frac{(1,6)\cdot(-4,7)}{|(1,6)|\,|(-4,7)|}\right] = \mathrm{Cos}^{-1}\left(\frac{38}{\sqrt{37}\sqrt{65}}\right) = 0.684 \text{ radians.}$$

27. If θ is the angle between the vectors, then

$$\theta = \mathrm{Cos}^{-1}\left[\frac{(4,2,3)\cdot(1,5,6)}{|(4,2,3)|\,|(1,5,6)|}\right] = \mathrm{Cos}^{-1}\left(\frac{32}{\sqrt{29}\sqrt{62}}\right) = 0.716 \text{ radians.}$$

28. If θ is the angle between the vectors, then

$$\theta = \mathrm{Cos}^{-1}\left[\frac{(3,1,-1)\cdot(-2,1,4)}{|(3,1,-1)|\,|(-2,1,4)|}\right] = \mathrm{Cos}^{-1}\left(\frac{-9}{\sqrt{11}\sqrt{21}}\right) = 2.20 \text{ radians.}$$

29. If θ is the angle between the vectors, then

$$\theta = \mathrm{Cos}^{-1}\left[\frac{(2,0,5)\cdot(0,3,0)}{|(2,0,5)|\,|(0,3,0)|}\right] = \mathrm{Cos}^{-1}(0) = \frac{\pi}{2} \text{ radians.}$$

30. Since $(-2,-6,4) = -2(1,3,-2)$, the vectors are in opposite directions, and $\theta = \pi$.

31. One such vector is $\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 3 & 5\\ -2 & 1 & 4\end{vmatrix} = (7,-14,7)$. All such vectors can be represented in the form $\lambda(1,-2,1)$.

32. One such vector is $\hat{\mathbf{j}} \times (1,-1,-9) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 1 & 0 \\ 1 & -1 & -9 \end{vmatrix} = (-9,0,-1)$. All such vectors can be represented by $\lambda(9,0,1)$.

33. Two of the sides of the triangle are represented by the vectors $(6,1,-1)$ and $(-5,2,1)$. A vector perpendicular to the triangle is therefore $(6,1,-1) \times (-5,2,1) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 6 & 1 & -1 \\ -5 & 2 & 1 \end{vmatrix} = (3,-1,17)$. All vectors perpendicular to the triangle are of the form $\lambda(3,-1,17)$.

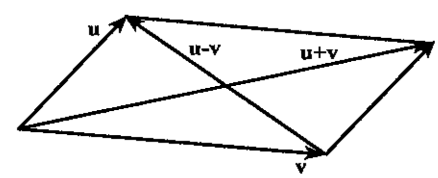
34. If we set $\mathbf{u} = (u_x, u_y, u_z)$, $\mathbf{v} = (v_x, v_y, v_z)$, and $\mathbf{w} = (w_x, w_y, w_z)$, then

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u_x v_x + u_y v_y + u_z v_z = v_x u_x + v_y u_y + v_z u_z = \mathbf{v} \cdot \mathbf{u}, \\ \mathbf{u} \cdot (\lambda \mathbf{v} + \rho \mathbf{w}) &= (u_x, u_y, u_z) \cdot (\lambda v_x + \rho w_x, \lambda v_y + \rho w_y, \lambda v_z + \rho w_z) \\ &= u_x(\lambda v_x + \rho w_x) + u_y(\lambda v_y + \rho w_y) + u_z(\lambda v_z + \rho w_z) \\ &= \lambda(u_x v_x + u_y v_y + u_z v_z) + \rho(u_x w_x + u_y w_y + u_z w_z) = \lambda(\mathbf{u} \cdot \mathbf{v}) + \rho(\mathbf{u} \cdot \mathbf{w}). \end{aligned}$$

35. With equation 11.23,

$$\begin{aligned} |\mathbf{u}+\mathbf{v}|^2 + |\mathbf{u}-\mathbf{v}|^2 &= (\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}+\mathbf{v}) + (\mathbf{u}-\mathbf{v}) \cdot (\mathbf{u}-\mathbf{v}) \\ &= \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} \\ &= 2\mathbf{u}\cdot\mathbf{u} + 2\mathbf{v}\cdot\mathbf{v} = 2|\mathbf{u}|^2 + 2|\mathbf{v}|^2. \end{aligned}$$

In the diagram, $|\mathbf{u}+\mathbf{v}|$ and $|\mathbf{u}-\mathbf{v}|$ are the lengths of the diagonals of the parallelogram with $\mathbf{u}$ and $\mathbf{v}$ as coterminal sides. The law states that the sum of the squares of the lengths of the diagonals is equal to the sum of the squares of the lengths of the four sides.



36. If α is the angle between $\mathbf{v}$ and $\hat{\mathbf{i}}$, then $\mathbf{v} \cdot \hat{\mathbf{i}} = |\mathbf{v}||\hat{\mathbf{i}}| \cos\alpha \implies \cos\alpha = \dfrac{\mathbf{v} \cdot \hat{\mathbf{i}}}{|\mathbf{v}|}$.

Since the angle between two vectors always lies between 0 and π, and such angles coincide with the principal values of the inverse cosine function, we may write that

$$\alpha = \mathrm{Cos}^{-1}\left(\frac{\mathbf{v} \cdot \hat{\mathbf{i}}}{|\mathbf{v}|}\right) = \mathrm{Cos}^{-1}\left(\frac{v_x}{|\mathbf{v}|}\right).$$

Similar discussions lead to the formulas for β and γ.

37. $\alpha = \mathrm{Cos}^{-1}\left(\dfrac{1}{\sqrt{1+4+9}}\right) = 1.30$; $\beta = \mathrm{Cos}^{-1}\left(\dfrac{2}{\sqrt{14}}\right) = 1.01$; $\gamma = \mathrm{Cos}^{-1}\left(\dfrac{-3}{\sqrt{14}}\right) = 2.50$.

38. $\alpha = \mathrm{Cos}^{-1} 0 = \pi/2$; $\beta = \mathrm{Cos}^{-1}\left(\dfrac{1}{\sqrt{1+9}}\right) = 1.25$; $\gamma = \mathrm{Cos}^{-1}\left(\dfrac{-3}{\sqrt{10}}\right) = 2.82$.

39. $\alpha = \mathrm{Cos}^{-1}\left(\dfrac{-1}{\sqrt{1+4+36}}\right) = 1.73$; $\beta = \mathrm{Cos}^{-1}\left(\dfrac{-2}{\sqrt{41}}\right) = 1.89$; $\gamma = \mathrm{Cos}^{-1}\left(\dfrac{6}{\sqrt{41}}\right) = 0.36$.

40. $\alpha = \mathrm{Cos}^{-1}\left(\dfrac{-2}{\sqrt{4+9+16}}\right) = 1.95$; $\beta = \mathrm{Cos}^{-1}\left(\dfrac{3}{\sqrt{29}}\right) = 0.980$; $\gamma = \mathrm{Cos}^{-1}\left(\dfrac{4}{\sqrt{29}}\right) = 0.734$.

41. If we set $\mathbf{u} = (u_x, u_y, u_z)$, $\mathbf{v} = (v_x, v_y, v_z)$, and $\mathbf{w} = (w_x, w_y, w_z)$, then

$$-(\mathbf{v} \times \mathbf{u}) = -\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ v_x & v_y & v_z \\ u_x & u_y & u_z \end{vmatrix} = -[(v_y u_z - v_z u_y)\hat{\mathbf{i}} + (v_z u_x - v_x u_z)\hat{\mathbf{j}} + (v_x u_y - v_y u_x)\hat{\mathbf{k}}$$

and this is the same as $\mathbf{u} \times \mathbf{v}$ (see equation 11.28). Property 11.31b is verified when we compare components of

$$\mathbf{u}\times(\lambda\mathbf{v}+\rho\mathbf{w})=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ u_x & u_y & u_z\\ \lambda v_x+\rho w_x & \lambda v_y+\rho w_y & \lambda v_z+\rho w_z\end{vmatrix}$$
$$=[u_y(\lambda v_z+\rho w_z)-u_z(\lambda v_y+\rho w_y)]\hat{\mathbf{i}}+[u_z(\lambda v_x+\rho w_x)-u_x(\lambda v_z+\rho w_z)]\hat{\mathbf{j}}$$
$$+[u_x(\lambda v_y+\rho w_y)-u_y(\lambda v_x+\rho w_x)]\hat{\mathbf{k}}$$

and

$$\lambda\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ u_x & u_y & u_z\\ v_x & v_y & v_z\end{vmatrix}+\rho\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ u_x & u_y & u_z\\ w_x & w_y & w_z\end{vmatrix}=\lambda[(u_yv_z-u_zv_y)\hat{\mathbf{i}}+(u_zv_x-u_xv_z)\hat{\mathbf{j}}+(u_xv_y-u_yv_x)\hat{\mathbf{k}}]$$
$$+\rho[(u_yw_z-u_zw_y)\hat{\mathbf{i}}+(u_zw_x-u_xw_z)\hat{\mathbf{j}}+(u_xw_y-u_yw_x)\hat{\mathbf{k}}]$$

42. One example serves to show this. See Exercises 19 and 20 for an example.

43. (a) Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$, and ϕ be the angle between $\mathbf{u}$ and $\mathbf{w}$. Because $\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot\mathbf{w}$ and $\mathbf{u}\times\mathbf{v}=\mathbf{u}\times\mathbf{w}$,

$$|\mathbf{u}||\mathbf{v}|\cos\theta=|\mathbf{u}||\mathbf{w}|\cos\phi \qquad \text{and} \qquad |\mathbf{u}||\mathbf{v}|\sin\theta=|\mathbf{u}||\mathbf{w}|\sin\phi,$$

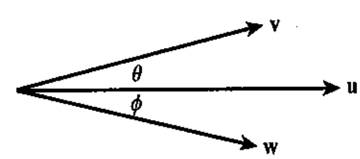
or,

$$|\mathbf{v}|\cos\theta=|\mathbf{w}|\cos\phi \qquad \text{and} \qquad |\mathbf{v}|\sin\theta=|\mathbf{w}|\sin\phi.$$

If these equations are squared and added,

$$|\mathbf{v}|^2\cos^2\theta+|\mathbf{v}|^2\sin^2\theta=|\mathbf{w}|^2\cos^2\phi+|\mathbf{w}|^2\sin^2\phi \quad\Longrightarrow\quad |\mathbf{v}|^2=|\mathbf{w}|^2.$$

Thus, $|\mathbf{v}|=|\mathbf{w}|$, and it follows that $\cos\theta=\cos\phi$ and $\sin\theta=\sin\phi$. These require $\theta=\phi$, and therefore the angle between $\mathbf{u}$ and $\mathbf{v}$ is the same as that between $\mathbf{u}$ and $\mathbf{w}$.
Finally, if all three vectors are placed at the same point (see figure), $\mathbf{v}$ and $\mathbf{w}$ cannot lie on opposite sides of $\mathbf{u}$ for the right-hand rule would then give $\mathbf{u}\times\mathbf{v}=-\mathbf{u}\times\mathbf{w}$.
Thus, $\mathbf{v}=\mathbf{w}$.



(b) This is a matter of logic. If $\mathbf{v}=\mathbf{w}$, then certainly $\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot\mathbf{w}$ and $\mathbf{u}\times\mathbf{v}=\mathbf{u}\times\mathbf{w}$. Consequently, if one of $\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot\mathbf{w}$ and $\mathbf{u}\times\mathbf{v}=\mathbf{u}\times\mathbf{w}$, is not satisfied, then $\mathbf{v}$ cannot be equal to $\mathbf{w}$.

44. (a) $\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}=(6,-1,0)\cdot\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 3 & 4\\ -2 & -1 & 4\end{vmatrix}=(6,-1,0)\cdot(16,-12,5)=108$

(b) This is verified when we compare for general $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$,

$$\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}=(u_x,u_y,u_z)\cdot\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ v_x & v_y & v_z\\ w_x & w_y & w_z\end{vmatrix}$$
$$=(u_x,u_y,u_z)\cdot[(v_yw_z-v_zw_y)\hat{\mathbf{i}}+(v_zw_x-v_xw_z)\hat{\mathbf{j}}+(v_xw_y-v_yw_x)\hat{\mathbf{k}}]$$
$$=u_x(v_yw_z-v_zw_y)+u_y(v_zw_x-v_xw_z)+u_z(v_xw_y-v_yw_x)$$

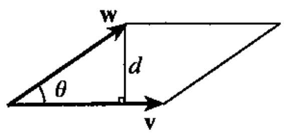
and

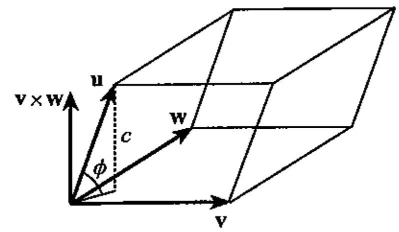
$$\mathbf{u}\times\mathbf{v}\cdot\mathbf{w}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ u_x & u_y & u_z\\ v_x & v_y & v_z\end{vmatrix}\cdot(w_x,w_y,w_z)$$
$$=[(u_yv_z-u_zv_y)\hat{\mathbf{i}}+(u_zv_x-u_xv_z)\hat{\mathbf{j}}+(u_xv_y-u_yv_x)\hat{\mathbf{k}}]\cdot(w_x,w_y,w_z)$$
$$=(u_yv_z-u_zv_y)w_x+(u_zv_x-u_xv_z)w_y+(u_xv_y-u_yv_x)w_z.$$

(c) The volume of the parallelepiped is the area of one of the parallelogram sides multiplied by the perpendicular distance to the opposite side. The area of the parallelogram with sides $\mathbf{v}$ and $\mathbf{w}$ in the left figure below is $|\mathbf{v}|d = |\mathbf{v}||\mathbf{w}|\sin\theta = |\mathbf{v}\times\mathbf{w}|$. The perpendicular distance between the parallel sides, one of which contains $\mathbf{v}$ and $\mathbf{w}$ (right figure), is $c = |\mathbf{u}|\sin\phi$. The volume of the parallelepiped is

$$(|\mathbf{u}|\sin\phi)|\mathbf{v}\times\mathbf{w}| = |\mathbf{u}||\mathbf{v}\times\mathbf{w}|\cos(\pi/2-\phi) = |\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}|.$$

The absolute values are included to take care of the possibility that the scalar product could be negative.





(d) This result follows from the fact that the three vectors are coplanar if and only if the volume of the parallelepiped with the vectors as coterminal sides is zero.

45. If $\mathbf{u} = (u_x, u_y, u_z)$, $\mathbf{v} = (v_x, v_y, v_z)$, $\mathbf{w} = (w_x, w_y, w_z)$, and $\mathbf{r} = (r_x, r_y, r_z)$, then

$$\begin{aligned}(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{w}\times\mathbf{r}) &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ u_x & u_y & u_z\\ v_x & v_y & v_z\end{vmatrix}\cdot\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ w_x & w_y & w_z\\ r_x & r_y & r_z\end{vmatrix}\\
&= [(u_yv_z - u_zv_y)\hat{\mathbf{i}} + (u_zv_x - u_xv_z)\hat{\mathbf{j}} + (u_xv_y - u_yv_x)\hat{\mathbf{k}}]\cdot\\
&\quad\ [(w_yr_z - w_zr_y)\hat{\mathbf{i}} + (w_zr_x - w_xr_z)\hat{\mathbf{j}} + (w_xr_y - w_yr_x)\hat{\mathbf{k}}]\\
&= (u_yv_z - u_zv_y)(w_yr_z - w_zr_y) + (u_zv_x - u_xv_z)(w_zr_x - w_xr_z)\\
&\quad + (u_xv_y - u_yv_x)(w_xr_y - w_yr_x),\end{aligned}$$

and

$$\begin{aligned}(\mathbf{u}\cdot\mathbf{w})(\mathbf{v}\cdot\mathbf{r}) - (\mathbf{u}\cdot\mathbf{r})(\mathbf{v}\cdot\mathbf{w}) &= (u_xw_x + u_yw_y + u_zw_z)(v_xr_x + v_yr_y + v_zr_z)\\
&\quad - (u_xr_x + u_yr_y + u_zr_z)(v_xw_x + v_yw_y + v_zw_z).\end{aligned}$$

When these two scalar expressions are expanded, they are identical.

46. If $\mathbf{u} = (u_x, u_y, u_z)$, $\mathbf{v} = (v_x, v_y, v_z)$, and $\mathbf{w} = (w_x, w_y, w_z)$, then

$$\begin{aligned}\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ u_x & u_y & u_z\\ v_yw_z - v_zw_y & v_zw_x - v_xw_z & v_xw_y - v_yw_x\end{vmatrix}\\
&= [u_y(v_xw_y - v_yw_x) - u_z(v_zw_x - v_xw_z)]\hat{\mathbf{i}} + [u_z(v_yw_z - v_zw_y) - u_x(v_xw_y - v_yw_x)]\hat{\mathbf{j}}\\
&\quad + [u_x(v_zw_x - v_xw_z) - u_y(v_yw_z - v_zw_y)]\hat{\mathbf{k}},\end{aligned}$$

and

$$\begin{aligned}(\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w} &= (u_xw_x + u_yw_y + u_zw_z)(v_x, v_y, v_z) - (u_xv_x + u_yv_y + u_zv_z)(w_x, w_y, w_z)\\
&= [v_x(u_xw_x + u_yw_y + u_zw_z) - w_x(u_xv_x + u_yv_y + u_zv_z)]\hat{\mathbf{i}}\\
&\quad + [v_y(u_xw_x + u_yw_y + u_zw_z) - w_y(u_xv_x + u_yv_y + u_zv_z)]\hat{\mathbf{j}}\\
&\quad + [v_z(u_xw_x + u_yw_y + u_zw_z) - w_z(u_xv_x + u_yv_y + u_zv_z)]\hat{\mathbf{k}}.\end{aligned}$$

Comparison of components shows that these vectors are identical.

47. If we cross $\mathbf{PQ}+\mathbf{QR}+\mathbf{RP}=\mathbf{0}$ with $\mathbf{PQ}$,

$$\mathbf{0}=(\mathbf{PQ}\times\mathbf{PQ})+(\mathbf{PQ}\times\mathbf{QR})+(\mathbf{PQ}\times\mathbf{RP}) \quad\Longrightarrow\quad \mathbf{PQ}\times\mathbf{QR}=-\mathbf{PQ}\times\mathbf{RP}.$$

When we take lengths (using 11.33),

$$|\mathbf{PQ}||\mathbf{QR}|\sin(\pi-B)=|\mathbf{PQ}||\mathbf{RP}|\sin(\pi-A) \quad\Longrightarrow\quad \frac{\sin B}{b}=\frac{\sin A}{a}.$$

Similarly, crossing $\mathbf{PQ}+\mathbf{QR}+\mathbf{RP}=\mathbf{0}$ with $\mathbf{QR}$ gives $\dfrac{\sin C}{c}=\dfrac{\sin A}{a}$.

48. If we let $\mathbf{w}=(|\mathbf{v}|\mathbf{u}+|\mathbf{u}|\mathbf{v})/\big||\mathbf{u}|\mathbf{v}+|\mathbf{v}|\mathbf{u}\big|$, then $\mathbf{w}$ is clearly a unit vector, since it is a vector divided by its own length. If we let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{u}\cdot\mathbf{v}=|\mathbf{u}||\mathbf{v}|\cos\theta$. If we let ϕ be the angle between $\mathbf{w}$ and $\mathbf{u}$, then

$$\mathbf{u}\cdot\mathbf{w}=|\mathbf{u}||\mathbf{w}|\cos\phi \quad\Longrightarrow\quad \cos\phi=\frac{|\mathbf{v}|(\mathbf{u}\cdot\mathbf{u})+|\mathbf{u}|(\mathbf{v}\cdot\mathbf{u})}{|\mathbf{u}|\big(\big||\mathbf{u}|\mathbf{v}+|\mathbf{v}|\mathbf{u}\big|\big)}=\frac{|\mathbf{v}||\mathbf{u}|^2+|\mathbf{u}|(\mathbf{v}\cdot\mathbf{u})}{|\mathbf{u}|\big(\big||\mathbf{u}|\mathbf{v}+|\mathbf{v}|\mathbf{u}\big|\big)}=\frac{|\mathbf{v}||\mathbf{u}|+(\mathbf{v}\cdot\mathbf{u})}{\big||\mathbf{u}|\mathbf{v}+|\mathbf{v}|\mathbf{u}\big|}.$$

Now,

$$\begin{aligned}(|\mathbf{u}||\mathbf{v}|+\mathbf{v}\cdot\mathbf{u})^2&=|\mathbf{u}|^2|\mathbf{v}|^2+2|\mathbf{u}||\mathbf{v}|\mathbf{v}\cdot\mathbf{u}+(\mathbf{v}\cdot\mathbf{u})^2\\&=|\mathbf{u}|^2|\mathbf{v}|^2+2|\mathbf{u}|^2|\mathbf{v}|^2\cos\theta+|\mathbf{u}|^2|\mathbf{v}|^2\cos^2\theta\\&=|\mathbf{u}|^2|\mathbf{v}|^2(1+\cos\theta)^2,\end{aligned}$$

and

$$\begin{aligned}\big||\mathbf{u}|\mathbf{v}+|\mathbf{v}|\mathbf{u}\big|^2&=(|\mathbf{u}|\mathbf{v}+|\mathbf{v}|\mathbf{u})\cdot(|\mathbf{u}|\mathbf{v}+|\mathbf{v}|\mathbf{u})\\&=|\mathbf{u}|^2|\mathbf{v}|^2+|\mathbf{v}|^2|\mathbf{u}|^2+2|\mathbf{u}||\mathbf{v}|(\mathbf{u}\cdot\mathbf{v})\\&=2|\mathbf{u}|^2|\mathbf{v}|^2(1+\cos\theta).\end{aligned}$$

It follows that

$$2\cos^2\phi-1=\frac{2[|\mathbf{u}|^2|\mathbf{v}|^2(1+\cos\theta)^2]}{2|\mathbf{u}|^2|\mathbf{v}|^2(1+\cos\theta)}-1=\cos\theta.$$

In other words $\phi=\theta/2$.

EXERCISES 11.5

1. The equation of the plane is $0=(4,3,-2)\cdot(x-1,y+1,z-3)=4x+3y-2z+5$.

2. A vector normal to the plane is $(4,2,3)-(2,1,5)=(2,1,-2)$. The equation of the plane is therefore $0=(2,1,-2)\cdot(x-2,y-1,z-5)=2x+y-2z+5$.

3. Since $(3,3,4)$ and $(0,1,1)$ are vectors that lie in the plane, a vector normal to the plane is

$$(3,3,4)\times(0,1,1)=\begin{vmatrix}\hat{\mathbf{i}}&\hat{\mathbf{j}}&\hat{\mathbf{k}}\\3&3&4\\0&1&1\end{vmatrix}=(-1,-3,3),\quad\text{as is}\quad(1,3,-3).$$

The equation of the plane is therefore $0=(1,3,-3)\cdot(x-1,y-3,z-2)=x+3y-3z-4$.

4. One vector in the plane is $(3,4,1)$. Since $(1,-5,-2)$ is a second point in the plane, a second vector in the plane is $(2,-4,3)-(1,-5,-2)=(1,1,5)$. A vector normal to the plane is

$$(3,4,1)\times(1,1,5)=\begin{vmatrix}\hat{\mathbf{i}}&\hat{\mathbf{j}}&\hat{\mathbf{k}}\\3&4&1\\1&1&5\end{vmatrix}=(19,-14,-1).$$

The equation of the plane is $0=(19,-14,-1)\cdot(x-2,y+4,z-3)=19x-14y-z-91$.

5. Two lines determine a plane only if they are parallel or they intersect. Since the lines are obviously not parallel, they must intersect. To confirm this, we substitute $x = t$ and $y = 2t$ into $x = 2y$ giving $t = 4t$ or $t = 0$. This gives the point $(0, 0, -1)$, which satisfies equations for both lines. Two vectors that lie in the plane are $(1, 1/2, 4)$ and $(1, 2, 6)$. A vector normal to the plane is

$$(2,1,8)\times(1,2,6) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 8 \\ 1 & 2 & 6 \end{vmatrix} = (-10,-4,3), \quad \text{as is} \quad (10,4,-3).$$

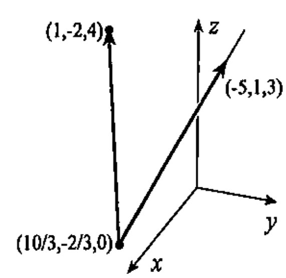
The equation of the plane is therefore $0 = (10, 4, -3) \cdot (x - 0, y - 0, z + 1) = 10x + 4y - 3z - 3$.

6. Two lines determine a plane only if they are parallel or they intersect. Since these lines are parallel (a vector along each is $(3, 4, 1)$), they determine a plane. Since $(1, 0, -2)$ and $(-1, 2, -5)$ are points on the lines, a second vector in the plane is $(1, 0, -2) - (-1, 2, -5) = (2, -2, 3)$. A vector normal to the plane is

$$(3,4,1)\times(2,-2,3) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 4 & 1 \\ 2 & -2 & 3 \end{vmatrix} = (14,-7,-14), \quad \text{as is} \quad (2,-1,-2).$$

The equation of the plane is $0 = (2, -1, -2) \cdot (x - 1, y, z + 2) = 2x - y - 2z - 6$.

7. The vectors $(1, -1, 2)$ and $(2, 1, 3)$ are normal to the planes $x - y + 2z = 4$ and $2x + y + 3z = 6$, respectively. A vector along the given line is



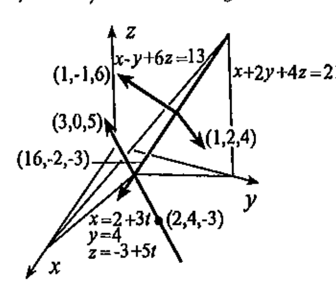
$$(1,-1,2)\times(2,1,3) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{vmatrix} = (-5,1,3).$$

Since $(10/3, -2/3, 0)$ (a point on the given line) and $(1, -2, 4)$ are points on the required plane, a second vector in this plane is $(10/3, -2/3, 0) - (1, -2, 4) = (7/3, 4/3, -4)$, as is $(7, 4, -12)$. A vector normal to the required plane is

$$(7,4,-12)\times(-5,1,3) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 7 & 4 & -12 \\ -5 & 1 & 3 \end{vmatrix} = (24,39,27), \quad \text{as is} \quad (8,13,9).$$

The equation of the plane is therefore $0 = (8, 13, 9) \cdot (x - 1, y + 2, z - 4) = 8x + 13y + 9z - 18$.

8. Two lines determine a plane only if they are parallel or they intersect. The vectors $(1, 2, 4)$ and $(1, -1, 6)$ are normal to the planes $x + 2y + 4z = 21$ and $x - y + 6z = 13$, respectively. A vector along the line determined by these planes is



$$(1,2,4)\times(1,-1,6) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 4 \\ 1 & -1 & 6 \end{vmatrix} = (16,-2,-3).$$

Since $(3, 0, 5)$ is a vector along the second line, the given lines are not parallel. To ensure that the lines intersect, we substitute $x = 2 + 3t$, $y = 4$, and $z = -3 + 5t$ into $x + 2y + 4z = 21$ giving $(2+3t)+8+4(-3+5t) = 21 \Longrightarrow t = 1$. This gives the point $(5, 4, 2)$, which also satisfies $x-y+6z = 13$. The lines therefore intersect at this point. A vector normal to the required plane is

$$(16,-2,-3)\times(3,0,5) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 16 & -2 & -3 \\ 3 & 0 & 5 \end{vmatrix} = (-10,-89,6).$$

The equation of the plane is therefore $0 = (10, 89, -6)(x - 2, y - 4, z + 3) = 10x + 89y - 6z - 394$.

9. Two lines determine a plane only if they are parallel or they intersect. Since the vectors $(3,4,0)$ and $(1,2,1)$ are normal to the planes $3x+4y=-6$ and $x+2y+z=2$, respectively, a vector along their line of intersection is

$$(3,4,0)\times(1,2,1)=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 4 & 0\\ 1 & 2 & 1\end{vmatrix}=(4,-3,2).$$

Similarly, a vector along the other line is

$$(0,2,3)\times(3,-2,-9)=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 0 & 2 & 3\\ 3 & -2 & -9\end{vmatrix}=(-12,9,-6),\quad \text{as is}\quad (4,-3,2).$$

Consequently, the two lines are parallel. Since $(-6,3,2)$ and $(-11,8,1)$ are points on the two lines, $(-6,3,2)-(-11,8,1)=(5,-5,1)$ is a vector that also lies in the required plane. A vector normal to this plane is

$$(4,-3,2)\times(5,-5,1)=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 4 & -3 & 2\\ 5 & -5 & 1\end{vmatrix}=(7,6,-5).$$

The equation of the required plane is therefore $0=(7,6,-5)\cdot(x+6,y-3,z-2)=7x+6y-5z+34$.

10. (a) Since the vectors $(1,1,-4)$ and $(2,3,5)$ are normal to the planes $x+y-4z=6$ and $2x+3y+5z=10$, respectively, a vector along their line of intersection is

$$(1,1,-4)\times(2,3,5)=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 1 & -4\\ 2 & 3 & 5\end{vmatrix}=(17,-13,1).$$

Since the plane is to be perpendicular to the xy-plane, $\hat{\mathbf{k}}$ must also lie in the plane, and a vector normal to the required plane is

$$(17,-13,1)\times\hat{\mathbf{k}}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 17 & -13 & 1\\ 0 & 0 & 1\end{vmatrix}=(-13,-17,0).$$

Since a point on the plane is $(8,-2,0)$, the equation of the plane is $0=(13,17,0)\cdot(x-8,y+2,z)=13x+17y-70$. Similar derivations give the other two equations.

11. The angle between the normals $(1,-2,4)$ and $(2,1,-1)$ to the planes is

$$\theta=\operatorname{Cos}^{-1}\left[\frac{(1,-2,4)\cdot(2,1,-1)}{|(1,-2,4)||(2,1,-1)|}\right]=\operatorname{Cos}^{-1}\left(\frac{-4}{\sqrt{21}\sqrt{6}}\right)=1.935\text{ radians}.$$

The acute angle between the planes is therefore 1.206 radians.

12. A vector equation is $\mathbf{r}=(x,y,z)=(1,-1,3)+t(2,4,-3)$. By equating components we obtain parametric equations $x=1+2t$, $y=-1+4t$, $z=3-3t$, and by solving each for t, symmetric equations are $\dfrac{x-1}{2}=\dfrac{y+1}{4}=\dfrac{z-3}{-3}$.

13. A vector equation is $\mathbf{r}=(x,y,z)=(-1,3,6)+t(2,-3,0)$. By equating components we obtain parametric equations $x=-1+2t$, $y=3-3t$, $z=6$. By solving the first and second for t, partial symmetric equations are $\dfrac{x+1}{2}=\dfrac{y-3}{-3}$, $z=6$.

14. Since a vector along the line is $(3,5,-5)$, a vector equation for the line is $\mathbf{r}=(x,y,z)=(2,-3,4)+t(3,5,-5)$. By equating components we obtain parametric equations $x=2+3t$, $y=-3+5t$, $z=4-5t$, and by solving each for t, symmetric equations are $\dfrac{x-2}{3}=\dfrac{y+3}{5}=\dfrac{z-4}{-5}$.

15. Since a vector along the line is $(0,6,6)$, a vector equation for the line is $\mathbf{r}=(x,y,z)=(-2,3,3)+t(0,1,1)$. By equating components we obtain parametric equations $x=-2$, $y=3+t$, $z=3+t$. By solving the second and third for t, partial symmetric equations are $y=z$, $x=-2$.

16. Since a vector along the line is $(0,0,1)$, a vector equation for the line is $\mathbf{r}=(x,y,z)=(1,3,4)+t(0,0,1)$. By equating components we obtain parametric equations $x=1$, $y=3$, $z=4+t$. Symmetric equations do not exist.

17. Since $(5,3,-2)$ is a vector along the line, a vector equation for the line is $\mathbf{r}=(x,y,z)=(1,-3,5)+t(5,3,-2)$. By equating components we obtain parametric equations $x=1+5t$, $y=-3+3t$, $z=5-2t$, and by solving each for t, symmetric equations are $\dfrac{x-1}{5}=\dfrac{y+3}{3}=\dfrac{z-5}{-2}$.

18. Since a vector along the line is $(1,0,-2)$, a vector equation for the line is $\mathbf{r}=(x,y,z)=(2,0,3)+u(1,0,-2)$. By equating components we obtain parametric equations $x=2+u$, $y=0$, $z=3-2u$, and by solving the first and last for u, partial symmetric equations are $x-2=\dfrac{z-3}{-2}$, $y=0$.

19. The given lines intersect at the point $(1,-4,2)$. Since a vector along the line is $(1,3,-2)-(2,-2,1)=(-1,5,-3)$, a vector equation for the line is $\mathbf{r}=(x,y,z)=(1,-4,2)+t(1,-5,3)$. By equating components we obtain parametric equations $x=1+t$, $y=-4-5t$, $z=2+3t$, and by solving each for t, symmetric equations are $x-1=\dfrac{y+4}{-5}=\dfrac{z-2}{3}$.

20. Parametric equations for the line are $x=t$, $y=2t-5$, $z=10-3(t)-4(2t-5)=30-11t$. By solving each for t, symmetric equations are $x=\dfrac{y+5}{2}=\dfrac{z-30}{-11}$. A vector equation is $\mathbf{r}=(x,y,z)=(0,-5,30)+t(1,2,-11)$.

21. Since the vectors $(1,1,0)$ and $(2,-1,1)$ are normal to the planes $x+y=3$ and $2x-y+z=-2$, respectively , a vector along the given line is $(1,1,0)\times(2,-1,1)=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 1 & 0\\ 2 & -1 & 1\end{vmatrix}=(1,-1,-3)$. A vector equation for the line is $\mathbf{r}=(x,y,z)=(-2,3,1)+t(1,-1,-3)$. By equating components we obtain parametric equations $x=-2+t$, $y=3-t$, $z=1-3t$, and by solving each for t, symmetric equations are $x+2=\dfrac{y-3}{-1}=\dfrac{z-1}{-3}$.

22. Parametric equations for the line are $x=3+2t$, $y=2+t$, $z=-1+4t$. When these values are substituted into $x-y+2z$, the result is $x-y+2z=(3+2t)-(2+t)+2(-1+4t)=-1+9t\not\equiv-1$. Consequently, the line does not lie in the plane.

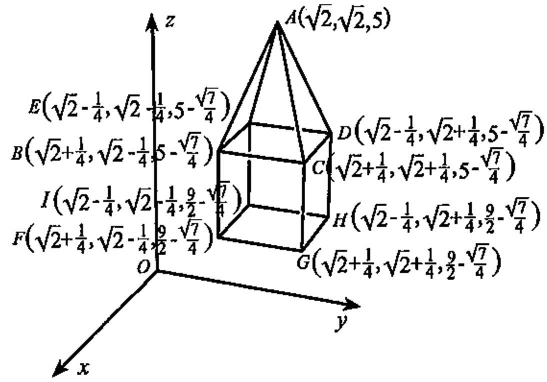
23. Since the slope of the line is $-A/B$, a vector along the line is $(B,-A)$. Because the scalar product of this vector with (A,B) is zero, (A,B) is perpendicular to the line.

24. The equation of every plane is of the form $Ax+By+Cz+D=0$. Because its intercept with the x-axis is a, it follows that $a=-D/A$ or $A=-D/a$. Similarly, $B=-D/b$ and $C=-D/c$. Thus, the equation of the plane is $-\dfrac{D}{a}x-\dfrac{D}{b}y-\dfrac{D}{c}z+D=0 \implies \dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$.

25. The bottom face has equation $z=0$. A vector normal to the left face is $(a,0,0)\times(d,e,f)=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ a & 0 & 0\\ d & e & f\end{vmatrix}$ $=(0,-af,ae)$, as is $(0,f,-e)$. The equation of this face is $0=(0,f,-e)\cdot(x,y,z)=fy-ez$. A vector normal to the right face is $(b,c,0)\times(d,e,f)=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ b & c & 0\\ d & e & f\end{vmatrix}=(cf,-bf,be-cd)$. The equation of this face is $0=(cf,-bf,be-cd)\cdot(x,y,z)=cfx-bfy+(be-cd)z$. A vector normal to the front face is $(b-a,c,0)\times(d-b,e-c,f)=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ b-a & c & 0\\ d-b & e-c & f\end{vmatrix}=(cf,f(a-b),be-cd-ae+ac)$. The equation of this face is $0=(cf,f(a-b),be-cd-ae+ac)\cdot(x-a,y,z)=cfx+f(a-b)y+(be-cd-ae+ac)z-acf$.

26. In Exercise 11.1–14, the coordinates of the corners of the birdhouse were determined as shown. The equation of face:
$FGHI$ is $z = 9/2 - \sqrt{7}/4$;
$BFIE$ is $y = \sqrt{2} - 1/4$;
$CGHD$ is $y = \sqrt{2} + 1/4$;
$BFCG$ is $x = \sqrt{2} + 1/4$;
$EIHD$ is $x = \sqrt{2} - 1/4$.
A vector normal to face ABC is

$$(4\mathbf{AB}) \times (4\mathbf{AC}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & -\sqrt{7} \\ 1 & 1 & -\sqrt{7} \end{vmatrix} = (2\sqrt{7}, 0, 2).$$



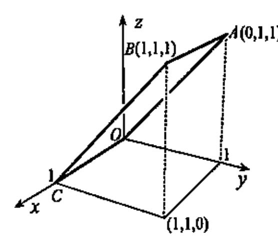
The equation of this face is therefore

$$0 = (\sqrt{7}, 0, 1) \cdot (x - \sqrt{2}, y - \sqrt{2}, z - 5)$$
$$= \sqrt{7}x + z - 5 - \sqrt{14}.$$

Similarly, equations for faces ACD, ADE, and AEB are $\sqrt{7}y + z - \sqrt{14} - 5 = 0$, $\sqrt{7}x - z = \sqrt{14} - 5$, and $\sqrt{7}y - z = \sqrt{14} - 5$.

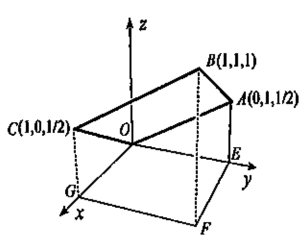
27. Since $\mathbf{P_1P_2}$ and $\mathbf{P_1P_3}$ are vectors that lie in the plane, $\mathbf{P_1P_2} \times \mathbf{P_1P_3}$ is a vector normal to the plane. Since $\mathbf{P_1P}$ is a vector in the plane for any point $P(x, y, z)$ in the plane, it follows that $\mathbf{P_1P} \cdot \mathbf{P_1P_2} \times \mathbf{P_1P_3} = 0$, and this must be the equation of the plane.

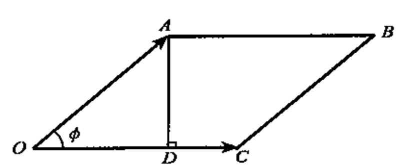
28. (a) Since the coordinates of A and B are $(0, 1, 1)$ and $(1, 1, 1)$, it follows that $\mathbf{AB} = (1, 0, 0)$ and $\mathbf{CB} = (0, 1, 1)$. Because $\mathbf{AB} = \mathbf{OC}$ and $\mathbf{CB} = \mathbf{OA}$, $OCBA$ is a rectangle with area $|\mathbf{OA}||\mathbf{OC}| = \sqrt{2}(1)$.



(b) The coordinates of the points A, B and C on the plane $x + y - 2z = 0$ directly above E, F, and D are $(0, 1, 1/2)$, $(1, 1, 1)$, and $(1, 0, 1/2)$, respectively (left figure below). Since $\mathbf{OA} = (0, 1, 1/2)$ is parallel to $\mathbf{CB} = (0, 1, 1/2)$, and $\mathbf{OC} = (1, 0, 1/2)$ is parallel to $\mathbf{AB} = (1, 0, 1/2)$, it follows that $OABC$ is a parallelogram with area (right figure below)

$$|\mathbf{OC}|(AD) = |\mathbf{OC}||\mathbf{OA}|\sin\phi = |\mathbf{OC} \times \mathbf{OA}| = \left\| \begin{matrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \end{matrix} \right\| = |(-1/2, -1/2, 1)| = \frac{\sqrt{6}}{2}.$$





(c) The coordinates of P, Q, R, and S, points in the plane $Ax + By + Cz + D = 0$ directly above O, G, F, and E are

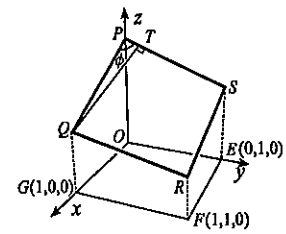
$$P(0, 0, -D/C),\ Q(1, 0, -(D + A)/C),\ R(1, 1, -(D + A + B)/C),\ S(0, 1, -(D + B)/C).$$

Consequently,

$$\mathbf{PQ} = (1, 0, -A/C),\ \mathbf{QR} = (0, 1, -B/C),\ \mathbf{PS} = (0, 1, -B/C),\ \mathbf{SR} = (1, 0, -A/C).$$

Because $\mathbf{PQ} = \mathbf{SR}$ and $\mathbf{PS} = \mathbf{QR}$, $PQRS$ is a parallelogram with area

$$|\mathbf{PS}|(QT) = |\mathbf{PS}||\mathbf{PQ}|\sin\phi = |\mathbf{PQ}\times\mathbf{PS}| = \left\|\begin{matrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 0 & -A/C\\ 0 & 1 & -B/C\end{matrix}\right\|$$
$$= |(A/C, B/C, 1)|$$
$$= \sqrt{A^2/C^2 + B^2/C^2 + 1}$$
$$= \frac{\sqrt{A^2+B^2+C^2}}{|C|}.$$



The acute angle between the xy-plane and the plane $Ax+By+Cz+D=0$ is defined as the acute angle between their normals (see Exercise 11). Normals to these planes are $\hat{\mathbf{k}}$ and (A,B,C). If θ is the angle between these vectors, then

$$\hat{\mathbf{k}}\cdot(A,B,C) = |\hat{\mathbf{k}}||(A,B,C)|\cos\theta = (1)\sqrt{A^2+B^2+C^2}\cos\theta.$$

Consequently, $\cos\theta = C/\sqrt{A^2+B^2+C^2}$. If $C>0$, then θ is acute and $\theta=\gamma$ where

$$\cos\gamma = \frac{C}{\sqrt{A^2+B^2+C^2}}.$$

If $C<0$, then θ is obtuse and $\gamma = \pi - \theta$ where

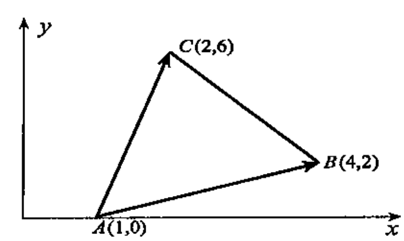
$$\cos(\pi-\gamma) = \frac{C}{\sqrt{A^2+B^2+C^2}} \quad\Longrightarrow\quad \cos\gamma = \frac{-C}{\sqrt{A^2+B^2+C^2}}.$$

Thus, $\cos\gamma = \dfrac{|C|}{\sqrt{A^2+B^2+C^2}}$ or $\sec\gamma = \dfrac{\sqrt{A^2+B^2+C^2}}{|C|}$.

EXERCISES 11.6

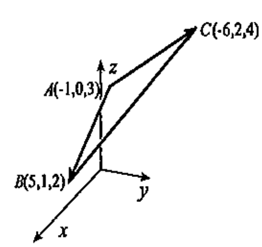
1. The area of the triangle is

$$\frac{1}{2}|\mathbf{AB}\times\mathbf{AC}| = \frac{1}{2}\left\|\begin{matrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 2 & 0\\ 1 & 6 & 0\end{matrix}\right\|$$
$$= \frac{1}{2}|(0,0,16)| = 8.$$



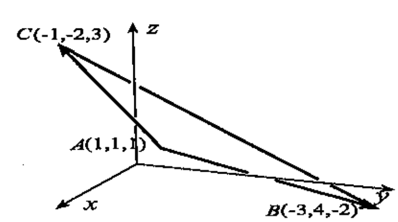
2. The area of the triangle is

$$\frac{1}{2}|\mathbf{AB}\times\mathbf{AC}| = \frac{1}{2}\left\|\begin{matrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 6 & 1 & -1\\ -5 & 2 & 1\end{matrix}\right\|$$
$$= \frac{1}{2}|(3,-1,17)| = \frac{\sqrt{299}}{2}.$$



3. The area of the triangle is

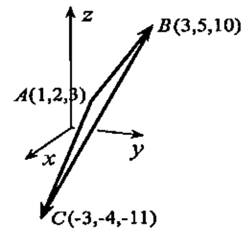
$$\frac{1}{2}|\mathbf{AB}\times\mathbf{AC}| = \frac{1}{2}\left\|\begin{matrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ -4 & 3 & -3\\ -2 & -3 & 2\end{matrix}\right\|$$
$$= \frac{1}{2}|(-3,14,18)| = \frac{\sqrt{529}}{2}.$$



4. The area of the triangle is

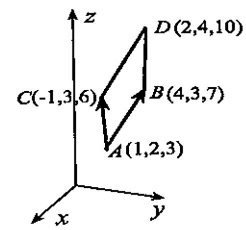
$$\frac{1}{2}|\mathbf{AB}\times\mathbf{AC}| = \frac{1}{2}\left\|\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 2 & 3 & 7\\ -4 & -6 & -14\end{vmatrix}\right\| = \frac{1}{2}|(0,0,0)| = 0.$$

The points must be collinear.



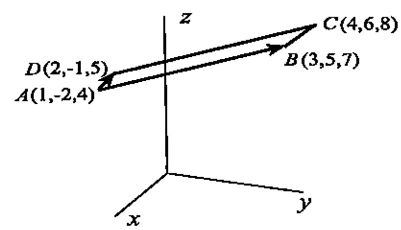
5. The area of the parallelogram is

$$|\mathbf{AB}\times\mathbf{AC}| = \left\|\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 1 & 4\\ -2 & 1 & 3\end{vmatrix}\right\| = |(-1,-17,5)| = 3\sqrt{35}.$$



6. The area of the parallelogram is

$$|\mathbf{AB}\times\mathbf{AD}| = \left\|\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 2 & 7 & 3\\ 1 & 1 & 1\end{vmatrix}\right\| = |(4,1,-5)| = \sqrt{42}.$$



7. The component is $\mathbf{v}\cdot\dfrac{(1,2,-3)}{\sqrt{14}} = -\dfrac{16}{\sqrt{14}}$.

8. Since $(4,-3,2)-(-1,2,3) = (5,-5,-1)$ is a vector in the required direction, the component is $\mathbf{v}\cdot\dfrac{(5,-5,-1)}{\sqrt{51}} = \dfrac{22}{\sqrt{51}}$.

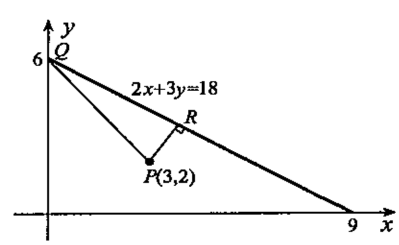
9. Since a vector normal to the plane is $(1,1,2)$, the component is $\mathbf{v}\cdot\dfrac{(1,1,2)}{\sqrt{6}} = \dfrac{3}{\sqrt{6}}$.

10. Since a vector along the line is $(-1,2,-3)$, the component is $\mathbf{v}\cdot\dfrac{(1,-2,3)}{\sqrt{14}} = \dfrac{18}{\sqrt{14}}$.

11. The required distance d is the component of $\mathbf{PQ}$ along $\mathbf{PR}$. Since the slope of the line is $-2/3$, a vector along it is $(-3,2)$, and a vector in the same direction as $\mathbf{PR}$ is $(2,3)$. Consequently,

$$d = \left|\mathbf{PQ}\cdot\widehat{\mathbf{PR}}\right| = \left|(-3,4)\cdot\frac{(2,3)}{\sqrt{4+9}}\right| = \frac{6}{\sqrt{13}}.$$

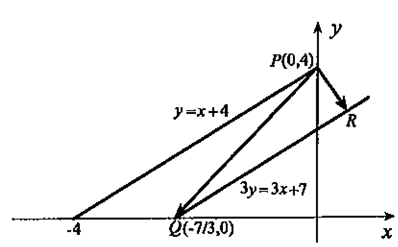
We could also have used formula 1.16.



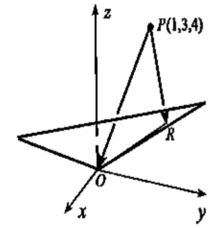
12. Since the lines are parallel, the required distance d is the component of $\mathbf{PQ}$ along $\mathbf{PR}$. Since the slope of the line is 1, a vector along it is $(1,1)$, and a vector in the same direction as $\mathbf{PR}$ is $(1,-1)$. Consequently,

$$d = \left|\mathbf{PQ}\cdot\widehat{\mathbf{PR}}\right| = \left|(-7/3,-4)\cdot\frac{(1,-1)}{\sqrt{1+1}}\right| = \frac{5}{3\sqrt{2}}.$$

We could also have used formula 1.16.

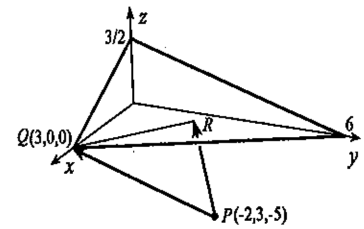


13. The required distance d is the component of **PO** along **PR**. Since a vector in the same direction as **PR** is $(1, 1, -2)$,



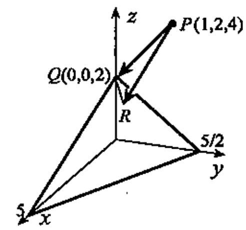
$$d = \left|\mathbf{PO} \cdot \widehat{\mathbf{PR}}\right|$$
$$= \left|(-1, -3, -4) \cdot \frac{(1, 1, -2)}{\sqrt{1 + 1 + 4}}\right| = \frac{4}{\sqrt{6}}.$$

14. The required distance d is the component of **PQ** along **PR**. Since a vector in the same direction as **PR** is $(2, 1, 4)$,



$$d = \left|\mathbf{PQ} \cdot \widehat{\mathbf{PR}}\right|$$
$$= \left|(5, -3, 5) \cdot \frac{(2, 1, 4)}{\sqrt{4 + 1 + 16}}\right| = \frac{27}{\sqrt{21}}.$$

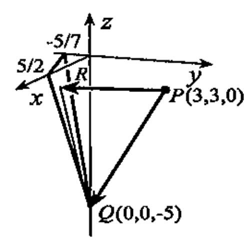
15. First we confirm that the line is parallel to the plane (else the distance is zero). Since a vector along the line is $(-1, 3, -2)$ and $(-1, 3, -2) \cdot (2, 4, 5) = 0$, the line is indeed parallel to the plane. Since a point on the line is $P(1, 2, 4)$, the required distance d is the component of **PQ** along **PR**. Since a vector in the same direction as **PR** is $(-2, -4, -5)$,



$$d = \left|\mathbf{PQ} \cdot \widehat{\mathbf{PR}}\right| = \left|(-1, -2, -2) \cdot \frac{(-2, -4, -5)}{\sqrt{4 + 16 + 25}}\right| = \frac{20}{3\sqrt{5}}.$$

16. First we confirm that the line is parallel to the plane (else the distance is zero). Since a vector along the line is

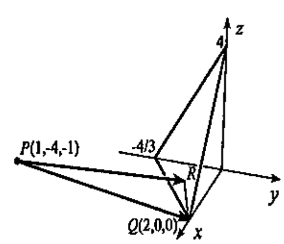
$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = (2, 1, -3),$$



and $(2, 1, -3) \cdot (2, -7, -1) = 0$, the line is indeed parallel to the plane. Since a point on the line is $P(3, 3, 0)$, the required distance d is the component of **PQ** along **PR**. Since a vector in the same direction as **PR** is $(2, -7, -1)$,

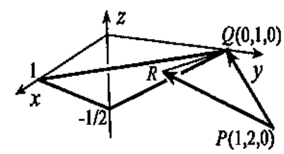
$$d = \left|\mathbf{PQ} \cdot \widehat{\mathbf{PR}}\right| = \left|(-3, -3, -5) \cdot \frac{(2, -7, -1)}{\sqrt{4 + 49 + 1}}\right| = \frac{20}{3\sqrt{6}}.$$

17. First we confirm that the line is parallel to the plane (else the distance is zero). Since a vector along the line is $(1, 1/3, -1)$ and $(1, 1/3, -1) \cdot (2, -3, 1) = 0$, the line is indeed parallel to the plane. Since a point on the line is $P(1, -4, -1)$, the required distance d is the component of **PQ** along **PR**. Since a vector in the same direction as **PR** is $(-2, 3, -1)$,



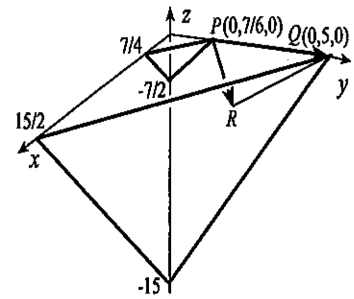
$$d = \left|\mathbf{PQ} \cdot \widehat{\mathbf{PR}}\right| = \left|(1, 4, 1) \cdot \frac{(-2, 3, -1)}{\sqrt{4 + 9 + 1}}\right| = \frac{9}{\sqrt{14}}.$$

18. First we confirm that the line is parallel to the plane (else the distance is zero). Since a vector along the line is $(-6,4,-1)$ and $(-6,4,-1)\cdot(1,1,-2)=0$, the line is indeed parallel to the plane. Since a point on the line is $P(1,2,0)$, the required distance is the component of **PQ** along **PR**. Since a vector in the same direction as **PR** is $(-1,-1,2)$,



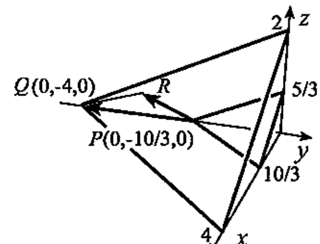
$$d=\left|\mathbf{PQ}\cdot\widehat{\mathbf{PR}}\right|=\left|(-1,-1,0)\cdot\frac{(-1,-1,2)}{\sqrt{6}}\right|=\frac{2}{\sqrt{6}}.$$

19. The required distance d is the component of **PQ** along **PR**. Since a vector in the same direction as **PR** is $(2,3,-1)$,



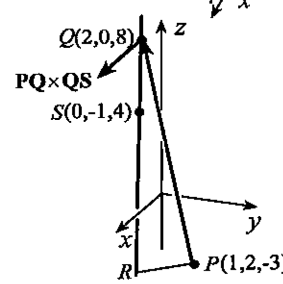
$$\begin{aligned}d&=\left|\mathbf{PQ}\cdot\widehat{\mathbf{PR}}\right|\\&=\left|(0,23/6,0)\cdot\frac{(2,3,-1)}{\sqrt{4+9+1}}\right|=\frac{23}{2\sqrt{14}}.\end{aligned}$$

20. The required distance d is the component of **PQ** along **PR**. Since a vector in the same direction as **PR** is $(1,-1,2)$,



$$\begin{aligned}d&=\left|\mathbf{PQ}\cdot\widehat{\mathbf{PR}}\right|\\&=\left|(0,-2/3,0)\cdot\frac{(1,-1,2)}{\sqrt{1+1+4}}\right|=\frac{2}{3\sqrt{6}}.\end{aligned}$$

21. The required distance is the component of **PQ** along **PR**. A vector perpendicular to **PQ** and **QS** is



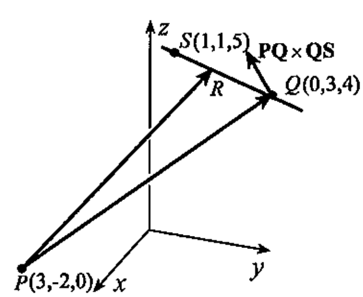
$$\mathbf{PQ}\times\mathbf{QS}=\begin{vmatrix}\hat{\mathbf{i}}&\hat{\mathbf{j}}&\hat{\mathbf{k}}\\1&-2&11\\-2&-1&-4\end{vmatrix}=(19,-18,-5).$$

A vector in direction **PR** is therefore

$$(\mathbf{PQ}\times\mathbf{QS})\times\mathbf{SQ}=\begin{vmatrix}\hat{\mathbf{i}}&\hat{\mathbf{j}}&\hat{\mathbf{k}}\\19&-18&-5\\2&1&4\end{vmatrix}=(-67,-86,55).$$

Thus, $$d=\left|\mathbf{PQ}\cdot\widehat{\mathbf{PR}}\right|=\left|(1,-2,11)\cdot\frac{(-67,-86,55)}{\sqrt{(-67)^2+(-86)^2+(55)^2}}\right|=\frac{710}{\sqrt{14\,910}}.$$

22. The required distance is the component of **PQ** along **PR**. A vector perpendicular to **PQ** and **QS** is



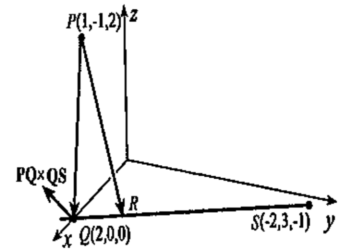
$$\mathbf{PQ}\times\mathbf{QS}=\begin{vmatrix}\hat{\mathbf{i}}&\hat{\mathbf{j}}&\hat{\mathbf{k}}\\-3&5&4\\1&-2&1\end{vmatrix}=(13,7,1).$$

A vector in direction **PR** is therefore

$$\begin{aligned}(\mathbf{PQ}\times\mathbf{QS})\times\mathbf{SQ}&=\begin{vmatrix}\hat{\mathbf{i}}&\hat{\mathbf{j}}&\hat{\mathbf{k}}\\13&7&1\\-1&2&-1\end{vmatrix}\\&=(-9,12,33)\quad\text{or}\quad(-3,4,11).\end{aligned}$$

Thus, $$d=\left|\mathbf{PQ}\cdot\widehat{\mathbf{PR}}\right|=\left|(-3,5,4)\cdot\frac{(-3,4,11)}{\sqrt{(-3)^2+(4)^2+(11)^2}}\right|=\frac{73}{\sqrt{146}}=\frac{\sqrt{146}}{2}.$$

23. The required distance is the component of **PQ** along **PR**. A vector perpendicular to **PQ** and **QS** is



$$\mathbf{PQ}\times\mathbf{QS} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & -2 \\ -4 & 3 & -1 \end{vmatrix} = (5,9,7).$$

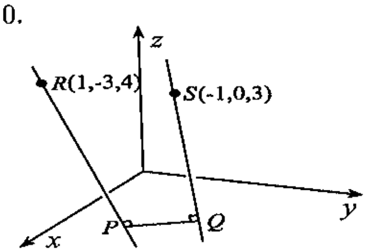
A vector in direction **PR** is therefore

$$(\mathbf{PQ}\times\mathbf{QS})\times\mathbf{SQ} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 5 & 9 & 7 \\ 4 & -3 & 1 \end{vmatrix} = (30,23,-51).$$

Thus, $$d = \left|\mathbf{PQ}\cdot\widehat{\mathbf{PR}}\right| = \left|(1,1,-2)\cdot\frac{(30,23,-51)}{\sqrt{(30)^2+(23)^2+(-51)^2}}\right| = \frac{155}{\sqrt{4030}}.$$

24. Since the point $(1,3,3)$ is on the line, the minimum distance is 0.

25. Following Example 11.24, the distance is the component of **RS** along **PQ**. A vector in direction **PQ** is

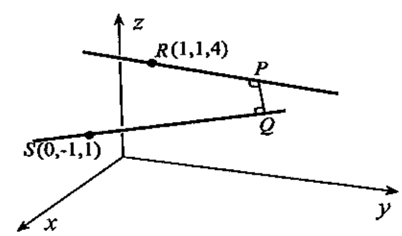


$$(2,3,-1)\times(1,2,-2) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 3 & -1 \\ 1 & 2 & -2 \end{vmatrix} = (-4,3,1).$$

Consequently,

$$d = \left|\mathbf{RS}\cdot\widehat{\mathbf{PQ}}\right| = \left|(-2,3,-1)\cdot\frac{(-4,3,1)}{\sqrt{26}}\right| = \frac{16}{\sqrt{26}}.$$

26. Following Example 11.24, the distance is the component of **RS** along **PQ**. A vector in direction **PQ** is

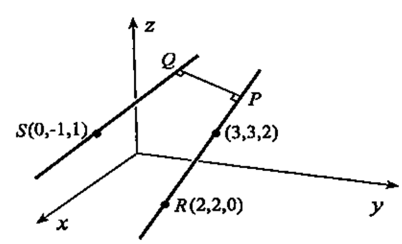


$$(1,3,2)\times(2,-1,2) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 3 & 2 \\ 2 & -1 & 2 \end{vmatrix} = (8,2,-7).$$

Consequently,

$$d = \left|\mathbf{RS}\cdot\widehat{\mathbf{PQ}}\right| = \left|(-1,-2,-3)\cdot\frac{(8,2,-7)}{\sqrt{117}}\right| = \frac{9}{\sqrt{117}}.$$

27. Following Example 11.24, the distance is the component of **RS** along **PQ**. A vector in direction **PQ** is

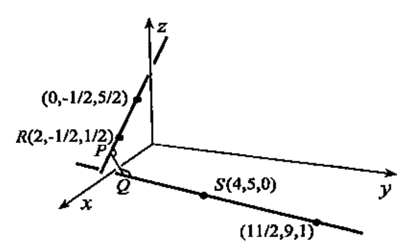


$$(1,1,2)\times(1,2,3) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = (-1,-1,1).$$

Consequently,

$$d = \left|\mathbf{RS}\cdot\widehat{\mathbf{PQ}}\right| = \left|(-2,-3,1)\cdot\frac{(-1,-1,1)}{\sqrt{3}}\right| = \frac{6}{\sqrt{3}}.$$

28. Following Example 11.24, the distance is the component of **RS** along **PQ**. A vector in direction **PQ** is



$$(-2,0,2)\times(3,8,2) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -2 & 0 & 2 \\ 3 & 8 & 2 \end{vmatrix} = (-16,10,-16).$$

Consequently,

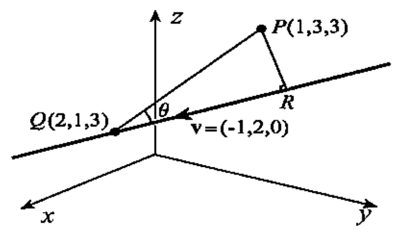
$$d = \left|\mathbf{RS}\cdot\widehat{\mathbf{PQ}}\right| = \left|(2,11/2,-1/2)\cdot\frac{(-8,5,-8)}{\sqrt{153}}\right| = \frac{31}{2\sqrt{153}}.$$

29. Since $\hat{\mathbf{v}} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$, and $\mathbf{v} = (1,2,3)$,

$$|\mathbf{PR}| = |\mathbf{PQ}|\sin\theta = |\mathbf{PQ}||\hat{\mathbf{v}}|\sin\theta = |\mathbf{PQ}\times\hat{\mathbf{v}}| = \frac{1}{|\mathbf{v}|}|\mathbf{PQ}\times\mathbf{v}|$$
$$= \frac{1}{\sqrt{14}}\left\|\begin{matrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 0 & -1 & -2\\ 1 & 2 & 3\end{matrix}\right\| = \frac{1}{\sqrt{14}}|(1,-2,1)| = \frac{\sqrt{6}}{\sqrt{14}} = \sqrt{\frac{3}{7}} = \frac{\sqrt{21}}{7}.$$

30. A point on the line is $Q(2,1,3)$ and a vector along the line is $\mathbf{v} = (-1,2,0)$. Using the technique of Exercise 29,

$$|\mathbf{PR}| = |\mathbf{PQ}|\sin\theta = |\mathbf{PQ}||\hat{\mathbf{v}}|\sin\theta$$
$$= |\mathbf{PQ}\times\hat{\mathbf{v}}| = \frac{1}{|\mathbf{v}|}|\mathbf{PQ}\times\mathbf{v}|$$
$$= \frac{1}{\sqrt{5}}\left\|\begin{matrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & -2 & 0\\ -1 & 2 & 0\end{matrix}\right\|$$
$$= \frac{1}{\sqrt{5}}|(0,0,0)| = 0.$$



Hence the point is on the line.

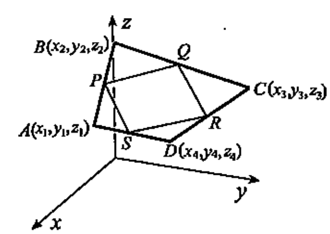
31. If $P(x^*,y^*,z^*)$ is any point in the first plane, then according to equation 11.41, the distance from P to the second plane is

$$\frac{|Ax^*+By^*+Cz^*+D_2|}{\sqrt{A^2+B^2+C^2}} = \frac{|-D_1+D_2|}{\sqrt{A^2+B^2+C^2}} = \frac{|D_1-D_2|}{\sqrt{A^2+B^2+C^2}},$$

(since $Ax^*+By^*+Cz^*+D_1 = 0$).

32. If coordinates of the vertices are as shown in the figure, then coordinates of the midpoints of the sides are

$$P\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right),\quad Q\left(\frac{x_2+x_3}{2},\frac{y_2+y_3}{2},\frac{z_2+z_3}{2}\right),$$
$$R\left(\frac{x_3+x_4}{2},\frac{y_3+y_4}{2},\frac{z_3+z_4}{2}\right),\quad S\left(\frac{x_4+x_1}{2},\frac{y_4+y_1}{2},\frac{z_4+z_1}{2}\right).$$



Since $\mathbf{PQ} = \left(\dfrac{x_3-x_1}{2},\dfrac{y_3-y_1}{2},\dfrac{z_3-z_1}{2}\right) = \mathbf{SR}$, and similarly, $\mathbf{PS} = \mathbf{QR}$, it follows that $PQRS$ is a parallelogram.

33. The lines form a triangle if each pair of lines intersects in a single point. For the first pair of lines, we set $4x-16 = 3y-24$ and $2x+10 = 3y+6$. The solution of these equations is $x = -2$, $y = 0$. Both of the original lines then give $z = 1$, and these lines therefore intersect in the point $A(-2,0,1)$.

To determine whether the first and third lines intersect in a point, we set $x = 1$ and $y = 5+t$ in $4x-16 = 3y-24$. The result is $t = -1$. This value of t in the third line gives the point $B(1,4,-3)$, and this point also satisfies the equations for the first line.

A similar procedure gives $C(1,2,3)$ as the point of intersection of the second and third lines.

According to equation 11.42, the area of triangle ABC is

$$\frac{1}{2}|\mathbf{AB}\times\mathbf{AC}| = \frac{1}{2}\left\|\begin{matrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 4 & -4\\ 3 & 2 & 2\end{matrix}\right\| = \frac{1}{2}|(16,-18,-6)| = \sqrt{154}.$$

34. Since $\hat{\mathbf{v}}\cdot\hat{\mathbf{w}} = 1/2 - 1/2 = 0$, $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$ are perpendicular. The components of $\mathbf{u}$ along $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$ are

$$\lambda = \mathbf{u}\cdot\hat{\mathbf{v}} = \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}} \qquad \text{and} \qquad \rho = \mathbf{u}\cdot\hat{\mathbf{w}} = \frac{2}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

35. Since $\hat{\mathbf{v}} \cdot \hat{\mathbf{w}} = 2/5 - 2/5 = 0$, $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$ are perpendicular. The components of $\mathbf{u}$ along $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$ are

$$\lambda = \mathbf{u} \cdot \hat{\mathbf{v}} = \frac{3+4}{\sqrt{5}} = \frac{7}{\sqrt{5}} \qquad \text{and} \qquad \rho = \mathbf{u} \cdot \hat{\mathbf{w}} = \frac{6-2}{\sqrt{5}} = \frac{4}{\sqrt{5}}.$$

36. Since $\hat{\mathbf{u}} \cdot \hat{\mathbf{v}} = (1/\sqrt{70})(-2+2+0) = 0$, $\hat{\mathbf{u}} \cdot \hat{\mathbf{w}} = [1/(5\sqrt{14})](6-6+0) = 0$, and $\hat{\mathbf{v}} \cdot \hat{\mathbf{w}} = [1/(14\sqrt{5})](-3-12+15) = 0$, the three unit vectors are mutually perpendicular. The components of $\mathbf{r}$ along $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, and $\hat{\mathbf{w}}$ are

$$\mathbf{r} \cdot \hat{\mathbf{u}} = \frac{1}{\sqrt{5}}(2+3) = \sqrt{5}; \; \mathbf{r} \cdot \hat{\mathbf{v}} = \frac{1}{\sqrt{14}}(-1+6-12) = -\frac{\sqrt{14}}{2}; \; \mathbf{r} \cdot \hat{\mathbf{w}} = \frac{1}{\sqrt{70}}(3-18-20) = -\frac{\sqrt{70}}{2}.$$

37. Since $\hat{\mathbf{u}} \cdot \hat{\mathbf{v}} = (1/\sqrt{18})(1+1-2) = 0$, $\hat{\mathbf{u}} \cdot \hat{\mathbf{w}} = [1/(\sqrt{6})](1-1) = 0$, and $\hat{\mathbf{v}} \cdot \hat{\mathbf{w}} = [1/\sqrt{12}](1-1) = 0$, the three unit vectors are mutually perpendicular. The components of $\mathbf{r}$ along $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, and $\hat{\mathbf{w}}$ are

$$\mathbf{r} \cdot \hat{\mathbf{u}} = \frac{1}{\sqrt{3}}(2-1) = \frac{1}{\sqrt{3}}; \; \mathbf{r} \cdot \hat{\mathbf{v}} = \frac{1}{\sqrt{6}}(2+2) = \frac{4}{\sqrt{6}}; \; \mathbf{r} \cdot \hat{\mathbf{w}} = \frac{1}{\sqrt{2}}(2) = \sqrt{2}.$$

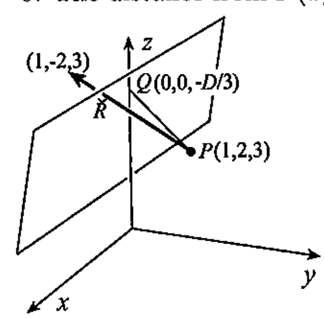
38. Since $\mathbf{v} \cdot \mathbf{w} = 6 - 6 = 0$, $\mathbf{v}$ and $\mathbf{w}$ are perpendicular. Because $\mathbf{u} \cdot \mathbf{v} = \lambda \mathbf{v} \cdot \mathbf{v} + \rho \mathbf{w} \cdot \mathbf{v} = \lambda |\mathbf{v}|^2$, it follows that $\lambda = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} = \dfrac{3-6}{1+9} = -\dfrac{3}{10}$. Similarly, $\rho = \dfrac{\mathbf{u} \cdot \mathbf{w}}{|\mathbf{w}|^2} = \dfrac{18+4}{36+4} = \dfrac{11}{20}$.

39. Since $\mathbf{u} \cdot \mathbf{v} = 1 - 1 = 0$, $\mathbf{v} \cdot \mathbf{w} = -1 + 2 - 1 = 0$ and $\mathbf{u} \cdot \mathbf{w} = -1 + 1 = 0$ the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are mutually perpendicular. Because $\mathbf{r} \cdot \mathbf{u} = \lambda \mathbf{u} \cdot \mathbf{u} + \rho \mathbf{v} \cdot \mathbf{u} + \mu \mathbf{w} \cdot \mathbf{u} = \lambda |\mathbf{u}|^2$, it follows that $\lambda = \dfrac{\mathbf{r} \cdot \mathbf{u}}{|\mathbf{u}|^2} = \dfrac{-2+4}{1+1} = 1$. Similarly, $\rho = \dfrac{\mathbf{r} \cdot \mathbf{v}}{|\mathbf{v}|^2} = \dfrac{-2-3-4}{1+1+1} = -3$, and $\mu = \dfrac{\mathbf{r} \cdot \mathbf{w}}{|\mathbf{w}|^2} = \dfrac{2-6+4}{1+4+1} = 0$.

40. The equation of the plane must be of the form $x - 2y + 3z + D = 0$. The distance from $P(1,2,3)$ to this plane is the projection of $\mathbf{PQ}$ along $\mathbf{PR}$, and hence

$$\begin{aligned} 2 &= |\mathbf{PQ} \cdot \widehat{\mathbf{PR}}| \\ &= \left| (-1, -2, -D/3 - 3) \cdot \frac{(1,-2,3)}{\sqrt{1+4+9}} \right| \\ &= \frac{1}{\sqrt{14}} |D + 6|. \end{aligned}$$

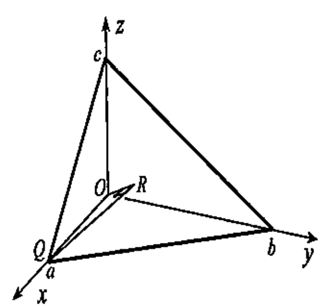
When this equation is solved, $D = -6 \pm 2\sqrt{14}$, and the two possible planes are $x - 2y + 3z = 6 \pm 2\sqrt{14}$.



41. The equation of the plane is $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$.
The distance p from the origin to the plane is the component of $\mathbf{OQ}$ along $\mathbf{OR}$,

$$\begin{aligned} p &= |\mathbf{OQ} \cdot \widehat{\mathbf{OR}}| \\ &= \left| (a, 0, 0) \cdot \frac{(1/a, 1/b, 1/c)}{\sqrt{1/a^2 + 1/b^2 + 1/c^2}} \right| \\ &= \frac{1}{\sqrt{1/a^2 + 1/b^2 + 1/c^2}}. \end{aligned}$$

When this equation is squared and inverted, the required result is obtained.



EXERCISES 11.7

1. $\mathbf{M} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -1 & 0 \\ 2 & 3 & -4 \end{vmatrix} = (4, 8, 8)$

2. $\mathbf{M} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 0 & 5 \\ 1 & 2 & 0 \end{vmatrix} = (-10, 5, -2)$

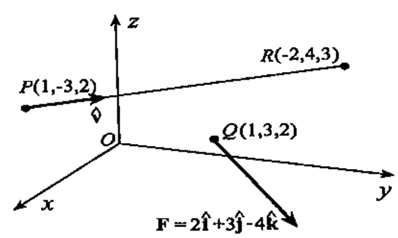
3. $\mathbf{M} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -3 & 0 \\ -1 & 0 & 3 \end{vmatrix} = (-9, -3, -3)$

4. $\mathbf{M} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & -1 & -1 \\ 3 & -1 & 4 \end{vmatrix} = (-5, 1, 4)$

5. $\mathbf{M} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -2 & 1 & 3 \\ 6 & 0 & 0 \end{vmatrix} = (0, 18, -6)$

6. (a) Since $\mathbf{v} = (-3, 7, 1)$ is a vector along the line through $P(1, -3, 2)$ and $R(-2, 4, 3)$, the moment about the line is

$$\begin{aligned} \text{Moment } &= \mathbf{PQ} \times \mathbf{F} \cdot (\pm\hat{\mathbf{v}}) \\ &= \pm(0, 6, 0) \times (2, 3, -4) \cdot \frac{(-3, 7, 1)}{\sqrt{59}} \\ &= \pm \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 6 & 0 \\ 2 & 3 & -4 \end{vmatrix} \cdot \frac{(-3, 7, 1)}{\sqrt{59}} \\ &= \pm(-24, 0, -12) \cdot \frac{(-3, 7, 1)}{\sqrt{59}} = \pm \frac{60}{\sqrt{59}}. \end{aligned}$$



(b) Since the moment of $\mathbf{F}$ about O is

$$\mathbf{M} = \mathbf{OQ} \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 3 & 2 \\ 2 & 3 & -4 \end{vmatrix} = (-18, 8, -3),$$

moments about the x-, y-, and z-axes are -18, 8, and -3, respectively.

(c) We use the point $S(2, 4, 1)$ on the line. If $\hat{\mathbf{v}}$ is a unit vector along the line, the moment of $\mathbf{F}$ about the line is

$$\begin{aligned} \text{Moment} = \mathbf{SQ} \times \mathbf{F} \cdot \hat{\mathbf{v}} &= (-1, -1, 1) \times (2, 3, -4) \cdot \frac{\pm(3, -2, 5)}{\sqrt{38}} \\ &= \pm \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & -1 & 1 \\ 2 & 3 & -4 \end{vmatrix} \cdot \frac{(3, -2, 5)}{\sqrt{38}} = \pm(1, -2, -1) \cdot \frac{(3, -2, 5)}{\sqrt{38}} = \pm \frac{2}{\sqrt{38}}. \end{aligned}$$

7. Since $P(3, -1, 0)$ is a point on the line and $\mathbf{v} = (2, 1, 4)$ is a vector along the line, the moment of $\mathbf{F}$ at $Q(-2, 3, 1)$ is

$$\begin{aligned} \text{Moment} = \pm\mathbf{PQ} \times \mathbf{F} \cdot \hat{\mathbf{v}} &= \pm(-5, 4, 1) \times (6, -5, 1) \cdot \frac{(2, 1, 4)}{\sqrt{21}} \\ &= \pm \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -5 & 4 & 1 \\ 6 & -5 & 1 \end{vmatrix} \cdot \frac{(2, 1, 4)}{\sqrt{21}} = \pm(9, 11, 1) \cdot \frac{(2, 1, 4)}{\sqrt{21}} = \pm \frac{33}{\sqrt{21}}. \end{aligned}$$

8. Since $P(0, 0, 1)$ is a point on the line and $\mathbf{v} = (1, 1, 1)$ is a vector along the line, the moment of $\mathbf{F}$ at $Q(6, -2, 1)$ is

$$\begin{aligned} \text{Moment} = \pm\mathbf{PQ} \times \mathbf{F} \cdot \hat{\mathbf{v}} &= \pm(6, -2, 0) \times (4, 0, -2) \cdot \frac{(1, 1, 1)}{\sqrt{3}} \\ &= \pm \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 6 & -2 & 0 \\ 4 & 0 & -2 \end{vmatrix} \cdot \frac{(1, 1, 1)}{\sqrt{3}} = \pm(4, 12, 8) \cdot \frac{(1, 1, 1)}{\sqrt{3}} = \pm 8\sqrt{3}. \end{aligned}$$

9. A point on the line is $P(-4,-2,4)$, and a vector along the line is

$$\mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = (-5,-2,3).$$

The moment of $\mathbf{F}$ at $Q(-1,-1,-2)$ is

$$\text{Moment} = \pm\mathbf{PQ}\times\mathbf{F}\cdot\hat{\mathbf{v}} = \pm(3,1,-6)\times(1,1,-1)\cdot\frac{(-5,-2,3)}{\sqrt{38}}$$
$$= \pm\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 1 & -6 \\ 1 & 1 & -1 \end{vmatrix}\cdot\frac{(-5,-2,3)}{\sqrt{38}} = \pm(5,-3,2)\cdot\frac{(-5,-2,3)}{\sqrt{38}} = \pm\frac{13}{\sqrt{38}}.$$

10. Moments about the axes are M_x, M_y, and M_z.

11. The moment of $\mathbf{F}$ about ℓ is $\pm(\mathbf{PQ}\times\mathbf{F}\cdot\hat{\mathbf{v}})$, where $\hat{\mathbf{v}}$ is a unit vector along ℓ. Since $\mathbf{PQ}\times\mathbf{F}$ is perpendicular to $\hat{\mathbf{v}}$, the scalar product is zero.

12. Since the vector $\mathbf{PQ}$ in equation 11.46 is equal to zero, the moment is zero.

13. The moment of $\mathbf{F}$ about ℓ is

$$\begin{aligned}
\mathrm{M} &= \mathbf{PQ}\times\mathbf{F}\cdot\hat{\mathbf{v}} = \hat{\mathbf{v}}\cdot\mathbf{PQ}\times\mathbf{F}\\
&= (v_x,v_y,v_z)\cdot(x_0-x_1,y_0-y_1,z_0-z_1)\times(F_x,F_y,F_z)\\
&= (v_x,v_y,v_z)\cdot\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x_0-x_1 & y_0-y_1 & z_0-z_1 \\ F_x & F_y & F_z \end{vmatrix}\\
&= (v_x,v_y,v_z)\cdot\big(F_z(y_0-y_1)-F_y(z_0-z_1),F_x(z_0-z_1)-F_z(x_0-x_1),F_y(x_0-x_1)-F_x(y_0-y_1)\big)\\
&= v_x[F_z(y_0-y_1)-F_y(z_0-z_1)]+v_y[F_x(z_0-z_1)-F_z(x_0-x_1)]+v_z[F_y(x_0-x_1)-F_x(y_0-y_1)].
\end{aligned}$$

The same result is obtained by expanding the determinant along the first row.

14. When the sleeve is at D, the magnitude of $\mathbf{F}$ is $|\mathbf{F}| = k[\sqrt{(1-x)^2+1/4}-l]$, and therefore $\mathbf{F} = k[\sqrt{(1-x)^2+1/4}-l]\dfrac{(1-x),1/2)}{\sqrt{(1-x)^2+1/4}}$. For a small displacement dx at position x, the amount of work done by $\mathbf{F}$ is approximately

$$\mathbf{F}\cdot(dx\,\hat{\mathbf{i}}) = k[\sqrt{(1-x)^2+1/4}-l]\frac{1-x}{\sqrt{(1-x)^2+1/4}}dx = k(1-x)\left[1-\frac{l}{\sqrt{(1-x)^2+1/4}}\right]dx.$$

The total work done between B and C is therefore

$$W = \int_0^1 k(1-x)\left[1-\frac{l}{\sqrt{(1-x)^2+1/4}}\right]dx = k\int_0^1\left[1-x-\frac{l(1-x)}{\sqrt{(1-x)^2+1/4}}\right]dx$$
$$= k\left\{x-\frac{x^2}{2}+l\sqrt{(1-x)^2+1/4}\right\}_0^1 = \frac{k}{2}[1+l(1-\sqrt{5})]\ \text{J}.$$

15. The work done by the resultant force of q_1 and q_2 is the sum of the individual amounts of work done by q_1 and q_2 separately. The force on q_3 due to q_1 when it is at position x on the x-axis is

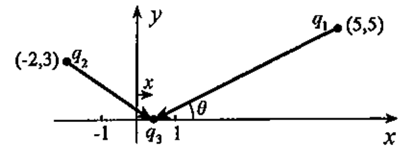
$$\mathbf{F}_1 = \frac{q_1q_3}{4\pi\epsilon_0[(x-5)^2+25]}\frac{(x-5,-5)}{\sqrt{(x-5)^2+25}} = \frac{q_1q_3(x-5,-5)}{4\pi\epsilon_0[(x-5)^2+25]^{3/2}}.$$

For a small displacement dx at position x, the amount of work done by $\mathbf{F}_1$ is

$$\mathbf{F}_1\cdot(dx\hat{\mathbf{i}}) = \frac{q_1q_3(x-5)}{4\pi\epsilon_0[(x-5)^2+25]^{3/2}}dx.$$

The work done by this force as q_3 moves from $x = 1$ to $x = -1$ is

$$W_1 = \int_1^{-1} \frac{q_1 q_3 (x-5)}{4\pi\epsilon_0 [(x-5)^2 + 25]^{3/2}} dx = \frac{q_1 q_3}{4\pi\epsilon_0} \left\{ \frac{-1}{\sqrt{(x-5)^2+25}} \right\}_1^{-1}$$
$$= \frac{q_1 q_3}{4\pi\epsilon_0} \left(\frac{1}{\sqrt{41}} - \frac{1}{\sqrt{61}} \right).$$

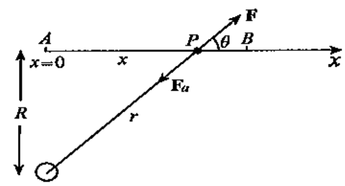


Similarly, the work done by q_2 is $W_2 = \dfrac{q_2 q_3}{4\pi\epsilon_0} \left(\dfrac{\sqrt{2}}{6} - \dfrac{1}{\sqrt{10}} \right)$.

The total work is $W_1 + W_2$.

16. At position P, the magnitude of the force $\mathbf{F}_a$ of attraction of the asteroid on the rocket is $|\mathbf{F}_a| = \dfrac{GMm}{x^2 + R^2}$, and therefore

$\mathbf{F}_a = \dfrac{GMm}{x^2+R^2} \dfrac{(-x, -R)}{\sqrt{x^2+R^2}}$. For a small displacement dx at P, the work done by an equal and opposite force $\mathbf{F}$ is approximately



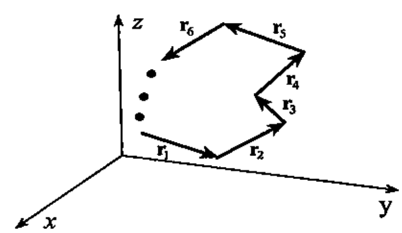
$$\mathbf{F} \cdot (dx\,\hat{\mathbf{i}}) = \frac{GMm}{x^2+R^2} \frac{x}{\sqrt{x^2+R^2}} dx$$
$$= \frac{GMmx}{(x^2+R^2)^{3/2}} dx.$$

The total work done between A and B is therefore

$$W = \int_0^R \frac{GMmx}{(x^2+R^2)^{3/2}} dx = GMm \left\{ -\frac{1}{\sqrt{x^2+R^2}} \right\}_0^R = \frac{GMm}{\sqrt{2}R}(\sqrt{2} - 1).$$

17. Suppose the sides of the polygon are denoted by $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ so that $\mathbf{r}_1 + \mathbf{r}_2 + \cdots + \mathbf{r}_n = \mathbf{0}$. The work done by a constant force $\mathbf{F}$ as an object moves around the polygon is

$$\mathbf{F} \cdot \mathbf{r}_1 + \mathbf{F} \cdot \mathbf{r}_2 + \cdots + \mathbf{F} \cdot \mathbf{r}_n$$
$$= \mathbf{F} \cdot (\mathbf{r}_1 + \cdots + \mathbf{r}_n) = \mathbf{F} \cdot \mathbf{0} = 0.$$

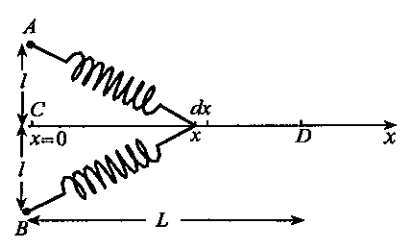


18. The work to stretch the two springs is three times the work to stretch the upper spring. The force necessary to hold the upper spring at position x on the x-axis against the upper spring is

$$\mathbf{F} = k(\sqrt{x^2+l^2} - l) \frac{(x, -l)}{\sqrt{x^2+l^2}}.$$

The work done by this force in moving the end of the spring a small amount dx along the x-axis is

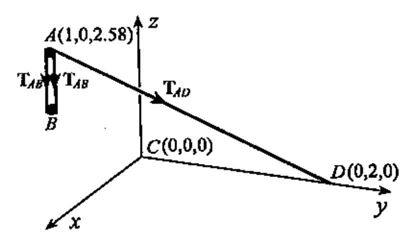
$$\mathbf{F} \cdot (dx\hat{\mathbf{i}}) = k(\sqrt{x^2+l^2} - l) \frac{x}{\sqrt{x^2+l^2}} dx$$
$$= k \left(x - \frac{lx}{\sqrt{x^2+l^2}} \right) dx.$$



The total work done in stretching both springs is therefore

$$W = 3 \int_0^L k \left(x - \frac{lx}{\sqrt{x^2+l^2}} \right) dx = 3k \left\{ \frac{x^2}{2} - l\sqrt{x^2+l^2} \right\}_0^L = 3k \left(\frac{L^2}{2} - l\sqrt{L^2+l^2} + l^2 \right).$$

19. Tensions in the ropes joining A to the boat B are the same, namely $\mathbf{T}_{AB} = -410\hat{\mathbf{k}}$. The tension in rope AD is



$$\mathbf{T}_{AD} = 410\left(\frac{\mathbf{AD}}{|\mathbf{AD}|}\right) = \frac{410(-1,2,-2.58)}{\sqrt{1^2+2^2+2.58^2}} = \frac{410(-1,2,-2.58)}{\sqrt{11.6564}}.$$

The moment of the sum of these tensions about C is

$$\mathbf{M} = \mathbf{CA}\times(2\mathbf{T}_{AB}+\mathbf{T}_{AD}) = (1,0,2.58)\times\left[2(-410\hat{\mathbf{k}}) + \frac{410(-1,2,-2.58)}{\sqrt{11.6564}}\right]$$

$$= \frac{410}{\sqrt{11.6564}}(1,0,2.58)\times(-1,2,-2\sqrt{11.6564}-2.58) = \frac{410}{\sqrt{11.6564}}\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 0 & 2.58\\ -1 & 2 & -2\sqrt{11.6564}-2.58\end{vmatrix}$$

$$= \frac{410}{\sqrt{11.6564}}(-5.16, 2\sqrt{11.6564}, 2) = (-619.7, 820, 240.2)\ \text{N}\cdot\text{m}.$$

20. Suppose we denote the components of $\mathbf{F}$ by $\mathbf{F} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}}$. Then $F_x^2+F_y^2+F_z^2 = 200^2$. Since the angle between $\mathbf{F}$ and $\hat{\mathbf{i}}$ is $\pi/3$ radians,

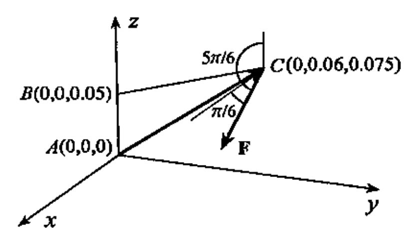
$F_x = \mathbf{F}\cdot\hat{\mathbf{i}} = |\mathbf{F}||\hat{\mathbf{i}}|\cos\pi/3 = 200(1/2) = 100.$

Similarly,

$F_z = \mathbf{F}\cdot\hat{\mathbf{k}} = |\mathbf{F}||\hat{\mathbf{k}}|\cos 5\pi/6 = 200(-\sqrt{3}/2) = -100\sqrt{3}.$

It follows that

$100^2 + F_y^2 + 3(100)^2 = 200^2 \Longrightarrow F_y = 0.$

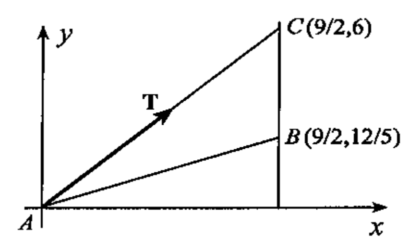


The moment of $\mathbf{F}$ about A is

$$\mathbf{M} = (0,0.06,0.075)\times(100,0,-100\sqrt{3}) = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 0 & 0.06 & 0.075\\ 100 & 0 & -100\sqrt{3}\end{vmatrix} = (-6\sqrt{3}, 15/2, -6)\ \text{N}\cdot\text{m}.$$

21. Since $\mathbf{BA} = (-9/2, -12/5)$, and

$$\mathbf{T} = 1500\left(\frac{\mathbf{AC}}{|\mathbf{AC}|}\right) = 1500\left[\frac{(9/2,6)}{\sqrt{81/4+36}}\right] = 300(3,4),$$

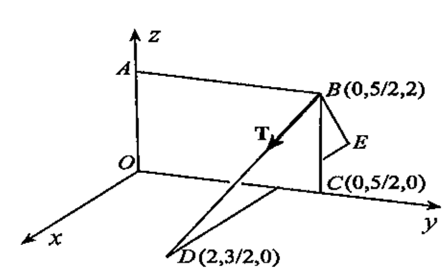


the moment of $\mathbf{T}$ about B is

$$\mathbf{M} = \mathbf{BA}\times\mathbf{T} = (-9/2,-12/5)\times 300(3,4) = 300\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ -9/2 & -12/5 & 0\\ 3 & 4 & 0\end{vmatrix} = (0,0,-3240).$$

The magnitude of this moment is 3240 N·m.

22. (a) Since $\mathbf{T} = 900\left(\dfrac{\mathbf{BD}}{|\mathbf{BD}|}\right) = \dfrac{900(2,-1,-2)}{3}$, the moment of $\mathbf{T}$ at B about O is



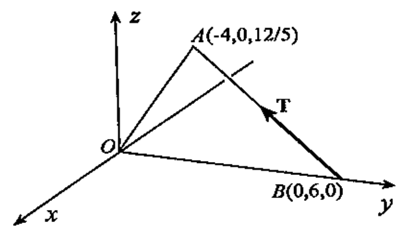
$$\mathbf{M} = \mathbf{OB}\times\mathbf{T} = (0,5/2,2)\times 300(2,-1,-2) = 300\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 0 & 5/2 & 2\\ 2 & -1 & -2\end{vmatrix} = 300(-3,4,-5)\ \text{N}\cdot\text{m}.$$

(b) The moment about the z-axis is -1500 N·m.

(c) Since the moment of $\mathbf{T}$ about B is zero, so also is the moment about any line through B (see Exercise 12).

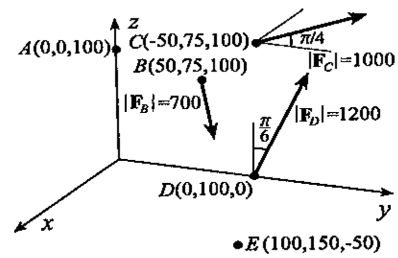
23. Since $\mathbf{T} = 1900\left(\dfrac{\mathbf{BA}}{|\mathbf{BA}|}\right) = \dfrac{1900(-4,-6,12/5)}{\sqrt{16+36+144/25}}$
$= 100(-10,-15,6),$
the moment of $\mathbf{M}$ at B about O is

$$\mathbf{M} = \mathbf{OB}\times\mathbf{T} = (0,6,0)\times 100(-10,-15,6)$$
$$= 100\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 0 & 6 & 0\\ -10 & -15 & 6\end{vmatrix} = 1200(3,0,5) \text{ N·m}.$$



24. Components of the forces acting at B, C, and D are

$$\mathbf{F}_B = 700\left(\frac{\mathbf{BE}}{|\mathbf{BE}|}\right) = \frac{700(50,75,-150)}{\sqrt{50^2+75^2+150^2}} = 100(2,3,-6),$$
$$\mathbf{F}_C = 1000\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right) = 500\sqrt{2}(-1,1,0),$$
$$\mathbf{F}_D = 1200\left(0,\frac{1}{2},\frac{\sqrt{3}}{2}\right) = 600(0,1,\sqrt{3}).$$



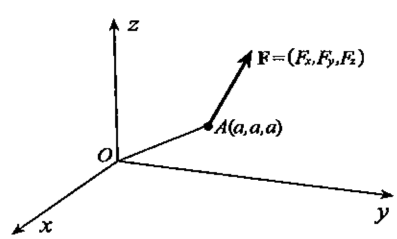
The sum of the moments of these forces about A is

$$\begin{aligned}\mathbf{M} &= \mathbf{AB}\times\mathbf{F}_B + \mathbf{AC}\times\mathbf{F}_C + \mathbf{AD}\times\mathbf{F}_D\\
&= (50,75,0)\times 100(2,3,-6) + (-50,75,0)\times 500\sqrt{2}(-1,1,0) + (0,100,-100)\times 600(0,1,\sqrt{3})\\
&= 100\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 50 & 75 & 0\\ 2 & 3 & -6\end{vmatrix} + 500\sqrt{2}\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ -50 & 75 & 0\\ -1 & 1 & 0\end{vmatrix} + 600\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 0 & 100 & -100\\ 0 & 1 & \sqrt{3}\end{vmatrix}\\
&= 100(-450,300,0) + 500\sqrt{2}(0,0,25) + 600(100\sqrt{3}+100,0,0)\\
&= 500(120\sqrt{3}+120-90,60,25\sqrt{2}) = (60\sqrt{3}+15,30,25/\sqrt{2}) \text{ N·m}.\end{aligned}$$

25. If $\mathbf{F} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}}$, then the moment of $\mathbf{F}$ about O is

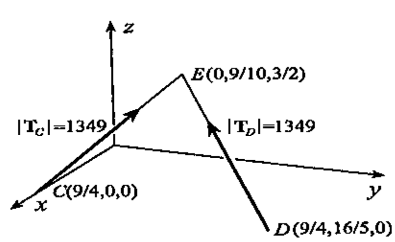
$$\mathbf{M} = \mathbf{OA}\times\mathbf{F} = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ a & a & a\\ F_x & F_y & F_z\end{vmatrix}$$
$$= a(F_z - F_y)\hat{\mathbf{i}} + a(F_x - F_z)\hat{\mathbf{j}} + a(F_y - F_x)\hat{\mathbf{k}}.$$

The sum of the moments of $\mathbf{F}$ about the coordinate axes is the sum of the components of this moment, and this is clearly seen to be zero.



26. Tensions in the cable exerted on C and D are

$$\mathbf{T}_C = 1349\left(\frac{\mathbf{CE}}{|\mathbf{CE}|}\right) = \frac{1349(-9/4,9/10,3/2)}{\sqrt{81/16+81/100+9/4}}$$
$$= 71(-15,6,10),$$
$$\mathbf{T}_D = 1349\left(\frac{\mathbf{DE}}{|\mathbf{DE}|}\right) = \frac{1349(-9/4,-23/10,3/2)}{\sqrt{81/16+529/100+9/4}}$$
$$= 19(-45,-46,30).$$



The moment of $\mathbf{T}_C$ about O is

$$\mathbf{M}_C = \mathbf{OC}\times\mathbf{T}_C = (9/4,0,0)\times 71(-15,6,10) = 71\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 9/4 & 0 & 0\\ -15 & 6 & 10\end{vmatrix} = 71(0,-45/2,27/2).$$

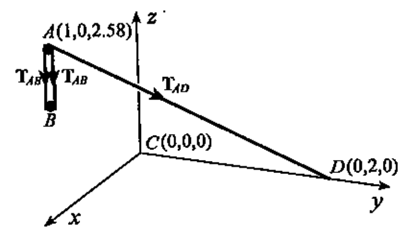
Moments of $\mathbf{T}_C$ about the coordinate axes are therefore 0, $-3195/2$ N·m, and $1917/2$ N·m. The moment of $\mathbf{T}_D$ about O is

$$\mathbf{M}_D = \mathbf{OD} \times \mathbf{T}_D = (9/4, 16/5, 0) \times 19(-45, -46, 30) = 19 \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 9/4 & 16/5 & 0 \\ -45 & -46 & 30 \end{vmatrix} = 19(96, -135/2, 81/2).$$

Moments of $\mathbf{T}_D$ about the coordinate axes are therefore 1824 N·m, $-2565/2$ N·m, and $1539/2$ N·m.

27. If T is the tension in the ropes , then $\mathbf{T}_{AB} = -T\hat{\mathbf{k}}$. The tension in rope AD is

$$\mathbf{T}_{AD} = T\left(\frac{\mathbf{AD}}{|\mathbf{AD}|}\right) = \frac{T(-1, 2, -2.58)}{\sqrt{1^2 + 2^2 + 2.58^2}} = \frac{T(-1, 2, -2.58)}{\sqrt{11.6564}}.$$



The moment of the sum of these tensions about C is

$$\mathbf{M} = \mathbf{CA} \times (2\mathbf{T}_{AB} + \mathbf{T}_{AD}) = (1, 0, 2.58) \times \left[2(-T\hat{\mathbf{k}}) + \frac{T(-1, 2, -2.58)}{\sqrt{11.6564}}\right]$$

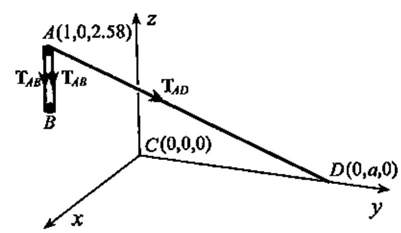
$$= \frac{T}{\sqrt{11.6564}}(1, 0, 2.58) \times (-1, 2, -2\sqrt{11.6564} - 2.58) = \frac{T}{\sqrt{11.6564}} \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & 2.58 \\ -1 & 2 & -2\sqrt{11.6564} - 2.58 \end{vmatrix}$$

$$= \frac{T}{\sqrt{11.6564}}(-5.16, 2\sqrt{11.6564}, 2) \text{ N·m}.$$

Since the absolute value of the x-component must be less than 375 N·m, it follows that $5.16T/\sqrt{11.6564} \leq 375 \Longrightarrow T \leq 248$ N.

28. Since tension in the ropes is 300 N, $\mathbf{T}_{AB} = -300\hat{\mathbf{k}}$. The tension in rope AD is

$$\mathbf{T}_{AD} = 300\left(\frac{\mathbf{AD}}{|\mathbf{AD}|}\right) = \frac{300(-1, a, -2.58)}{\sqrt{1^2 + a^2 + 2.58^2}} = \frac{300(-1, a, -2.58)}{\sqrt{a^2 + 7.6564}}.$$



The moment of the sum of these tensions about C is

$$\mathbf{M} = \mathbf{CA} \times (2\mathbf{T}_{AB} + \mathbf{T}_{AD}) = (1, 0, 2.58) \times \left[2(-300\hat{\mathbf{k}}) + \frac{300(-1, a, -2.58)}{\sqrt{a^2 + 7.6564}}\right]$$

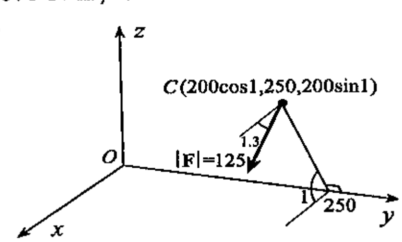
$$= \frac{300}{\sqrt{a^2 + 7.6564}}(1, 0, 2.58) \times (-1, a, -2\sqrt{a^2 + 7.6564} - 2.58)$$

$$= \frac{300}{\sqrt{a^2 + 7.6564}} \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & 2.58 \\ -1 & a & -2\sqrt{a^2 + 7.6564} - 2.58 \end{vmatrix} = \frac{300}{\sqrt{a^2 + 7.6564}}(-2.58a, 2\sqrt{a^2 + 7.6564}, a) \text{ N·m}.$$

Since the absolute value of the x-component must be less than 375 N·m, it follows that $300(2.58a)/\sqrt{a^2 + 7.6564} \leq 375$. This implies that $a \leq 1.532$ m.

29. Since components of $\mathbf{F}$ are $\mathbf{F} = 125(\cos 1.3, 0, -\sin 1.3)$, its moment about O is

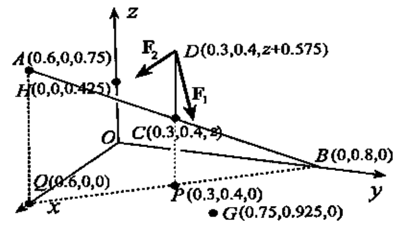
$$\mathbf{M} = \mathbf{OC} \times \mathbf{F}$$
$$= (200\cos 1, 250, 200\sin 1) \times 125(\cos 1.3, 0, -\sin 1.3)$$
$$= 125 \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 200\cos 1 & 250 & 200\sin 1 \\ \cos 1.3 & 0 & -\sin 1.3 \end{vmatrix}$$
$$= 125(-250\sin 1.3, 200\sin 1\cos 1.3 + 200\cos 1\sin 1.3, -250\cos 1.3).$$



Moments about the coordinate axes (in N·m) are components of this vector divided by 1000, namely, -30.1 N·m, 18.6 N·m, and -8.36 N·m.

30. If we denote the z-coordinate of C by z, then from similar triangles AQB and CPB,

$$\frac{\|PC\|}{\|AQ\|} = \frac{\|PB\|}{\|QB\|},$$



from which

$$\frac{z}{0.75} = \frac{\sqrt{0.3^2+0.4^2}}{\sqrt{0.6^2+0.8^2}} \quad \Longrightarrow \quad z = 0.375.$$

The z-coordinate of D is then $0.375 + 0.575 = 0.95$.
Components of $\mathbf{F}_1$ and $\mathbf{F}_2$ are

$$\mathbf{F}_1 = 1175\left(\frac{\mathbf{DG}}{|\mathbf{DG}|}\right) = \frac{1175(0.45, 0.525, -0.95)}{\sqrt{0.45^2+0.525^2+0.95^2}} = 1000(0.45, 0.525, -0.95),$$

$$\mathbf{F}_2 = 870\left(\frac{\mathbf{DH}}{|\mathbf{DH}|}\right) = \frac{870(-0.3, -0.4, -0.525)}{\sqrt{0.3^2+0.4^2+0.525^2}} = -1200(0.3, 0.4, 0.525).$$

Moments of these forces about C are

$$\mathbf{M}_1 = \mathbf{CD} \times \mathbf{F}_1 = (0,0,0.575) \times 1000(0.45, 0.525, -0.95) = 1000\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 0.575 \\ 0.45 & 0.525 & -0.95 \end{vmatrix}$$

$$= (-301.875, 258.75, 0),$$

$$\mathbf{M}_2 = \mathbf{CD} \times \mathbf{F}_2 = (0,0,0.575) \times (-1200)(0.3, 0.4, 0.525) = -1200\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 0.575 \\ 0.3 & 0.4 & 0.525 \end{vmatrix}$$

$$= -1200(-0.23, 0.1725, 0).$$

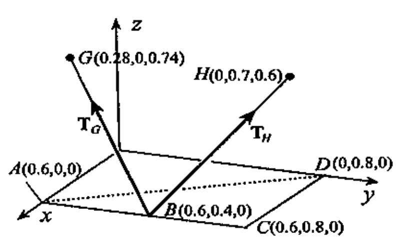
Moments of these forces about AB are

$$\mathbf{M}_1 \cdot \left(\frac{\mathbf{AB}}{|\mathbf{AB}|}\right) = (-301.875, 258.75, 0) \cdot \frac{(-0.6, 0.8, -0.75)}{\sqrt{0.6^2+0.8^2+0.75^2}} = 310.5 \text{ N·m},$$

$$\mathbf{M}_2 \cdot \left(\frac{\mathbf{AB}}{|\mathbf{AB}|}\right) = -1200(-0.23, 0.1725, 0) \cdot \frac{(-0.6, 0.8, -0.75)}{\sqrt{0.6^2+0.8^2+0.75^2}} = -264.96 \text{ N·m}.$$

31. Tensions in BH and BG are

$$\mathbf{T}_H = 1125\left(\frac{\mathbf{BH}}{|\mathbf{BH}|}\right) = \frac{1125(-0.6, 0.3, 0.6)}{\sqrt{0.6^2+0.3^2+0.6^2}}$$
$$= 1250(-0.6, 0.3, 0.6),$$



$$\mathbf{T}_G = 1125\left(\frac{\mathbf{BG}}{|\mathbf{BG}|}\right) = \frac{1125(-0.32, -0.4, 0.74)}{\sqrt{0.32^2+0.4^2+0.74^2}}$$
$$= 1250(-0.32, -0.4, 0.74).$$

Moments of these forces about A are

$$\mathbf{M}_H = \mathbf{AB} \times \mathbf{T}_H = 1250\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0.4 & 0 \\ -0.6 & 0.3 & 0.6 \end{vmatrix} = 1250(0.24, 0, 0.24),$$

$$\mathbf{M}_G = \mathbf{AB} \times \mathbf{T}_G = 1250\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0.4 & 0 \\ -0.32 & -0.4 & 0.74 \end{vmatrix} = 1250(0.296, 0, 0.128).$$

Moments of these forces about diagonal AD are

$$\mathbf{M}_H \cdot \left(\frac{\mathbf{AD}}{|\mathbf{AD}|}\right) = 1250(0.24, 0, 0.24) \cdot \frac{(-0.6, 0.8, 0)}{\sqrt{0.6^2+0.8^2}} = -180 \text{ N·m},$$

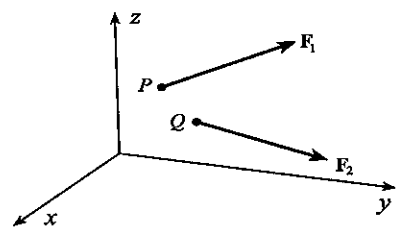
$$\mathbf{M}_G \cdot \left(\frac{\mathbf{AD}}{|\mathbf{AD}|}\right) = 1250(0.296, 0, 0.128) \cdot \frac{(-0.6, 0.8, 0)}{\sqrt{0.6^2+0.8^2}} = -222 \text{ N·m}.$$

32. The moment of $\mathbf{F}_1$ about the line of action of $\mathbf{F}_2$ is

$$\mathbf{QP} \times \mathbf{F}_1 \cdot \left(\frac{\mathbf{F}_2}{|\mathbf{F}_2|}\right),$$

and the moment of $\mathbf{F}_2$ about the line of action of $\mathbf{F}_1$ is

$$\mathbf{PQ} \times \mathbf{F}_2 \cdot \left(\frac{\mathbf{F}_1}{|\mathbf{F}_1|}\right).$$



If $\mathbf{PQ} = (a, b, c)$, $\mathbf{F}_1 = (F_x, F_y, F_z)$, and $\mathbf{F}_2 = (G_x, G_y, G_z)$, then

$$\mathbf{PQ} \times \mathbf{F}_2 \cdot \mathbf{F}_1 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a & b & c \\ G_x & G_y & G_z \end{vmatrix} \cdot (F_x, F_y, F_z) = F_x(bG_z - cG_y) + F_y(cG_x - aG_z) + F_z(aG_y - bG_x),$$

and

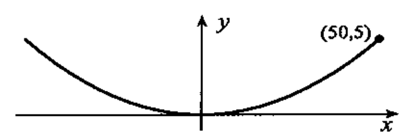
$$\mathbf{QP} \times \mathbf{F}_1 \cdot \mathbf{F}_2 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -a & -b & -c \\ F_x & F_y & F_z \end{vmatrix} \cdot (G_x, G_y, G_z) = G_x(-bF_z + cF_y) + G_y(-cF_x + aF_z) + G_z(-aF_y + bF_x).$$

Hence, $\mathbf{PQ} \times \mathbf{F}_2 \cdot \mathbf{F}_1 = \mathbf{QP} \times \mathbf{F}_1 \cdot \mathbf{F}_2$, and because $|\mathbf{F}_1| = |\mathbf{F}_2|$, the two moments are equal.

EXERCISES 11.8

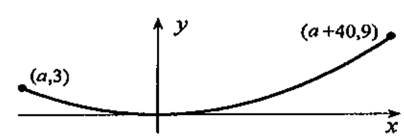
1. In the coordinate system shown, the equation of the cable is $y(x) = \dfrac{1000x^2}{2T_0}$, where T_0 is the tension in the cable at $x = 0$. Since $(50, 5)$ is a point on the cable,

$$5 = \frac{500(50)^2}{T_0} \quad \Longrightarrow \quad T_0 = 250\,000.$$



According to the first of equations 11.47, tension in the cable at any point is $T = T_0 \sec\theta = T_0\sqrt{1 + \tan^2\theta}$ $= T_0\sqrt{1 + (dy/dx)^2}$. This means that tension is a minimum when slope is a minimum, namely at $x = 0$. In other words, $T_0 = 250\,000$ N is the minimum tension in the cable.

2. In the coordinate system shown, the equation of the cable is $y(x) = \dfrac{1100x^2}{2T_0}$, where T_0 is the tension in the cable at $x = 0$. Since $(a, 3)$, and $(a + 40, 9)$ are points on the cable,

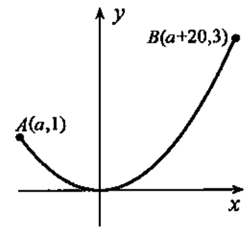


$$3 = \frac{550a^2}{T_0}, \qquad 9 = \frac{550(a + 40)^2}{T_0}.$$

These imply that $a = 20(1 - \sqrt{3})$ and $T_0 = 3.93 \times 10^4$. According to the first of equations 11.47, tension in the cable at any point is $T = T_0 \sec\theta = T_0\sqrt{1 + \tan^2\theta} = T_0\sqrt{1 + (dy/dx)^2}$. This means that tension is a minimum when slope is a minimum, namely at $x = 0$. In other words, $T_0 = 3.93 \times 10^4$ N is the minimum tension in the cable. Maximum tension is at $x = a + 40 = 60 - 20\sqrt{3}$ where slope is greatest; i.e., maximum tension is

$$T_0\sqrt{1 + [y'(60 - 20\sqrt{3})]^2} = T_0\sqrt{1 + \left[\frac{1100(60 - 20\sqrt{3})}{T_0}\right]^2} = 4.82 \times 10^4 \text{ N}.$$

3. In the coordinate system shown, the equation of the cable is $y(x) = \dfrac{wx^2}{2T_0}$, where T_0 is the tension in the cable at $x = 0$. Since $(a, 1)$, and $(a + 20, 3)$ are points on the cable,



$$1 = \frac{wa^2}{2T_0}, \qquad 3 = \frac{w(a + 20)^2}{2T_0}.$$

The first implies that $a = -\sqrt{2T_0/w}$, which substituted into the second gives

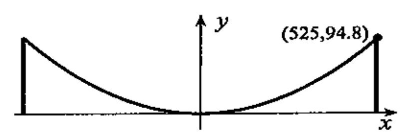
$$6T_0 = w(-\sqrt{2T_0/w} + 20)^2 \quad \Longrightarrow \quad T_0 + 10\sqrt{2w}\sqrt{T_0} - 100w = 0.$$

This is a quadratic in $\sqrt{T_0}$ with solutions

$$\sqrt{T_0} = \frac{-10\sqrt{2w} \pm \sqrt{200w + 400w}}{2} = 5\sqrt{w}(\sqrt{6} - \sqrt{2}) \quad \Longrightarrow \quad T_0 = 100(2 - \sqrt{3})w.$$

According to the first of equations 11.47, tension in the cable at any point is $T = T_0 \sec\theta = T_0\sqrt{1 + \tan^2\theta}$ $= T_0\sqrt{1 + (dy/dx)^2}$. This means that tension is a minimum when slope is a minimum, namely at $x = 0$. In other words, $T_0 = 100(2 - \sqrt{3})w$ is the minimum tension in the cable.

4. In the coordinate system shown, the equation of the cable is $y(x) = \dfrac{wx^2}{2T_0}$, where T_0 is the tension in the cable at $x = 0$. Since $(525, 94.8)$ is a point on the cable,



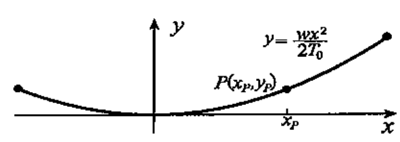
$$94.8 = \frac{142\,000(525)^2}{2T_0}.$$

This implies that $T_0 = 2.06 \times 10^8$. According to the first of equations 11.47, tension in the cable at any point is $T = T_0 \sec\theta = T_0\sqrt{1 + \tan^2\theta} = T_0\sqrt{1 + (dy/dx)^2}$. This means that tension is a minimum when slope is a minimum, namely at $x = 0$. In other words, $T_0 = 2.06 \times 10^8$ N is the minimum tension in the cable. Maximum tension is at $x = 525$ where slope is greatest; i.e., maximum tension is

$$T_0\sqrt{1 + [y'(525)]^2} = T_0\sqrt{1 + \left[\frac{w(525)}{T_0}\right]^2} = 2.19 \times 10^8 \text{ N}.$$

5. (a) With the coordinate system shown, the equation of the cable is $y = wx^2/(2T_0)$. The length of the cable from $x = 0$ to an arbitrary point $P(x_P, y_P)$ is

$$L(P) = \int_0^{x_P} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{x_P} \sqrt{1 + \frac{w^2x^2}{T_0^2}}\, dx.$$



If we set $x = (T_0/w)\tan\theta$, $dx = (T_0/w)\sec^2\theta\, d\theta$, and $\overline{\theta} = \mathrm{Tan}^{-1}(wx_P/T_0)$, then

$$\begin{aligned}
L(P) &= \int_0^{\overline{\theta}} \sec\theta\left(\frac{T_0}{w}\right)\sec^2\theta\, d\theta = \frac{T_0}{w}\int_0^{\overline{\theta}} \sec^3\theta\, d\theta \\
&= \frac{T_0}{2w}\{\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\}_0^{\overline{\theta}} \qquad \text{(see Example 8.9)} \\
&= \frac{T_0}{2w}(\sec\overline{\theta}\tan\overline{\theta} + \ln|\sec\overline{\theta} + \tan\overline{\theta}|) \\
&= \frac{T_0}{2w}\left[\frac{\sqrt{T_0^2 + w^2x_P^2}}{T_0}\left(\frac{wx_P}{T_0}\right) + \ln\left(\frac{\sqrt{T_0^2 + w^2x_P^2}}{T_0} + \frac{wx_P}{T_0}\right)\right]
\end{aligned}$$

$$= \frac{x_P}{2}\sqrt{1+\left(\frac{wx_P}{T_0}\right)^2} + \frac{T_0}{2w}\ln\left[\sqrt{1+\left(\frac{wx_P}{T_0}\right)^2} + \frac{wx_P}{T_0}\right].$$

(b) If we expand the root by the binomial expansion,

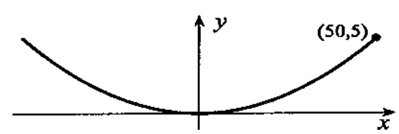
$$\begin{aligned}
L(P) &= \int_0^{x_P}\left[1+\frac{1}{2}\left(\frac{w^2x^2}{T_0^2}\right)+\frac{(1/2)(-1/2)}{2}\left(\frac{w^2x^2}{T_0^2}\right)^2+\cdots\right]dx \\
&= \left\{x+\frac{w^2x^3}{6T_0^2}-\frac{w^4x^5}{40T_0^4}+\cdots\right\}_0^{x_P}, \qquad \text{valid for } w^2x^2/T_0^2<1 \\
&= x_P+\frac{w^2x_P^3}{6T_0^2}-\frac{w^4x_P^5}{40T_0^4}+\cdots, \qquad \text{valid for } x_P<T_0/w.
\end{aligned}$$

Since $y_p = wx_P^2/(2T_0)$, this can be expressed in the form

$$L(P) = x_P\left(1+\frac{w^2x_P^2}{6T_0^2}-\frac{w^4x_P^4}{40T_0^4}+\cdots\right) = x_P\left[1+\frac{2}{3}\left(\frac{y_P}{x_P}\right)^2-\frac{2}{5}\left(\frac{y_P}{x_P}\right)^4+\cdots\right].$$

This is valid for $x_P < T_0/w = x_P^2/(2y_P) \Longrightarrow y_P/x_P < 1/2$.

6. In the coordinate system shown, the equation of the cable is $\quad y(x) = \dfrac{1000x^2}{2T_0}$, where T_0 is the tension in the cable at $x = 0$. Since $(50,5)$ is a point on the cable,



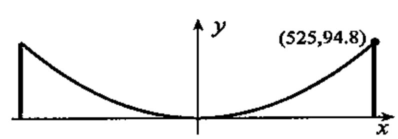
$$5 = \frac{500(50)^2}{T_0} \quad\Longrightarrow\quad T_0 = 250\,000.$$

Using the formula in Exercise 5(a), the length of the cable is

$$2\left\{\frac{50}{2}\sqrt{1+\left[\frac{1000(50)}{250\,000}\right]^2}+\frac{250\,000}{2(1000)}\ln\left[\sqrt{1+\left(\frac{1000(50)}{250\,000}\right)^2}+\frac{1000(50)}{250\,000}\right]\right\} = 100.663 \text{ m}.$$

With the two-term approximation in part (b), the length is $2(50)[1+(2/3)(5/50)^2] = 100.667$ m.

7. In the coordinate system shown, the equation of the cable is $y(x) = wx^2(2T_0)$, where T_0 is the tension in the cable at $x = 0$. Since $(525, 94.8)$ is a point on the cable,



$$94.8 = \frac{142\,000(525)^2}{2T_0} \quad\Longrightarrow\quad T_0 = 2.06\times 10^8.$$

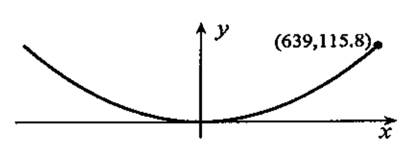
Using the formula in Exercise 5(a), the length of the cable is

$$2\left\{\frac{525}{2}\sqrt{1+\left[\frac{142\,000(525)}{2.06\times 10^8}\right]^2}+\frac{2.06\times 10^8}{2(142\,000)}\ln\left[\sqrt{1+\left(\frac{142\,000(525)}{2.06\times 10^8}\right)^2}+\frac{142\,000(525)}{2.06\times 10^8}\right]\right\} = 1072.5 \text{ m}.$$

With the two-term approximation in part (b), the length is $2(525)[1+(2/3)(94.8/525)^2] = 1072.8$ m.

8. When the sag is 115.8 m, the length is approximated by

$$2(639)\left[1+\frac{2}{3}\left(\frac{115.8}{639}\right)^2\right] = 1305.98.$$

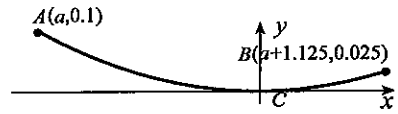


When the sag is 118.2, we find

$$2(639)\left[1+\frac{2}{3}\left(\frac{118.2}{639}\right)^2\right] = 1307.15.$$

The difference is 1.17 m.

9. (a) In the coordinate system shown, the equation of the curve for the paper is $y = wx^2/(2T_0)$, where T_0 is the tension at C. With points $A(a, 0.1)$, and $B(a + 1.125, 0.025)$ on the parabola, we must have



$$0.1 = \frac{0.3(9.81)a^2}{2T_0}, \qquad 0.025 = \frac{0.3(9.81)(a + 1.125)^2}{2T_0}.$$

These imply that

$$2T_0 = 3(9.81)a^2, \quad T_0 = 6(9.81)(a + 1.125)^2 \quad \Longrightarrow \quad \frac{3a^2}{2} = 6(a + 1.125)^2 \quad \Longrightarrow \quad 3a^2 + 9a + 5.0625 = 0.$$

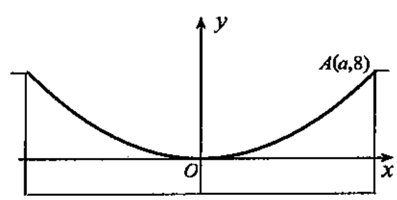
Solutions of this quadratic are $a = -0.75, -2.25$, only the first being acceptable. Hence, C is 75 cm to the right of A.

(b) According to the first of equations 11.47, tension in the cable at any point is $T = T_0 \sec\theta = T_0\sqrt{1 + \tan^2\theta} = T_0\sqrt{1 + (dy/dx)^2}$. This means that tension is a maximum when slope is a maximum, namely at A. With $T_0 = 3(9.81)(-0.75)^2/2$, maximum tension is

$$T_0\sqrt{1 + [y'(-0.75)]^2} = T_0\sqrt{1 + \left[\frac{w(-0.75)}{T_0}\right]^2} = 8.57 \text{ N}.$$

10. Using equation 11.51, the equation of the rope in the coordinate system shown is

$$y = \frac{T_0}{w}\left[\cosh\left(\frac{wx}{T_0}\right) - 1\right].$$



Since $A(a, 8)$ is on the curve,

$$8 = \frac{T_0}{w}\left[\cosh\left(\frac{wa}{T_0}\right) - 1\right].$$

According to Example 11.32, the length of one-half the rope is

$$20 = \frac{T_0}{w}\sinh\left(\frac{wa}{T_0}\right).$$

Finally, Example 11.33 indicates that maximum tension is at A, and therefore $350 = T_0 + 8w$. The first two equations give

$$1 = \cosh^2\left(\frac{wa}{T_0}\right) - \sinh^2\left(\frac{wa}{T_0}\right) = \left(\frac{8w}{T_0} + 1\right)^2 - \left(\frac{20w}{T_0}\right)^2 \quad \Longrightarrow \quad \frac{16w}{T_0^2}(T_0 - 21w) = 0.$$

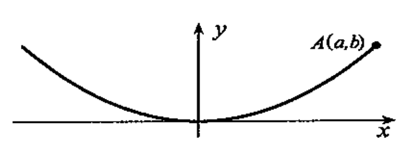
Thus, $T_0 = 21w$, and when this is substituted into $T_0 + 8w = 350$, we obtain $29w = 350 \Longrightarrow w = 350/29$. The mass of the rope is $40(350/29)/9.81 = 49.2$ kg. With $T_0 = 350 - 8(350/29) = 7350/29$,

$$20 = \frac{7350/29}{350/29}\sinh\left(\frac{350a/29}{7350/29}\right) = 21\,\sinh\left(\frac{a}{21}\right) \quad \Longrightarrow \quad a = 21\,\mathrm{Sinh}^{-1}\left(\frac{20}{21}\right) = 17.79.$$

Hence, the horizontal distance between the buildings is $2a = 35.6$ m.

11. Using equation 11.51, the equation of the tape in the coordinate system shown is

$$y = \frac{T_0}{w}\left[\cosh\left(\frac{wx}{T_0}\right) - 1\right],$$



where $w = 1.6(9.81)/50$. Since $A(a, b)$ is on the curve,

$$b = \frac{T_0}{w}\left[\cosh\left(\frac{wa}{T_0}\right) - 1\right].$$

According to Example 11.32, the length of one-half the tape is $25 = \dfrac{T_0}{w}\sinh\left(\dfrac{wa}{T_0}\right)$. Finally, Example

11.33 indicates that maximum tension is at A, and therefore $60 = T_0 + wb$. The first two equations give

$$1 = \cosh^2\left(\frac{wa}{T_0}\right) - \sinh^2\left(\frac{wa}{T_0}\right) = \left(\frac{wb}{T_0}+1\right)^2 - \left(\frac{25w}{T_0}\right)^2.$$

Substituting $wb = 60 - T_0$ gives

$$1 = \left(\frac{60-T_0}{T_0}+1\right)^2 - \left(\frac{25w}{T_0}\right)^2 = \left(\frac{60}{T_0}\right)^2 - \left[\frac{25(1.6)(9.81)}{50T_0}\right]^2 \quad \Longrightarrow \quad T_0 = \sqrt{60^2-(9.81)^2(0.8)^2}.$$

It now follows that

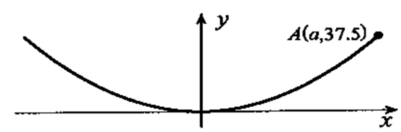
$$a = \frac{T_0}{w}\text{Sinh}^{-1}\left(\frac{25w}{T_0}\right) = \frac{\sqrt{60^2-(9.81)^2(0.8)^2}}{1.6(9.81)/50}\text{Sinh}^{-1}\left[\frac{25(1.6)(9.81)}{50\sqrt{60^2-(9.81)^2(0.8)^2}}\right] = 24.928.$$

The span of the tape is therefore $2a = 49.86$ m.

12. Using equation 11.51, the equation of the cable in the coordinate system shown is

$$y = \frac{T_0}{w}\left[\cosh\left(\frac{wx}{T_0}\right)-1\right],$$

where $w = 4(9.81)$. Since $A(a, 37.5)$ is on the curve,



$$37.5 = \frac{T_0}{w}\left[\cosh\left(\frac{wa}{T_0}\right)-1\right].$$

According to Example 11.32, the length of one-half the cable is $75 = \frac{T_0}{w}\sinh\left(\frac{wa}{T_0}\right)$. From these,

$$1 = \cosh^2\left(\frac{wa}{T_0}\right) - \sinh^2\left(\frac{wa}{T_0}\right) = \left(\frac{37.5w}{T_0}+1\right)^2 - \left(\frac{75w}{T_0}\right)^2 \quad \Longrightarrow \quad \frac{75w}{T_0^2}\left(T_0 - \frac{225w}{4}\right) = 0.$$

Thus, $T_0 = 225w/4$, and this in turn implies that

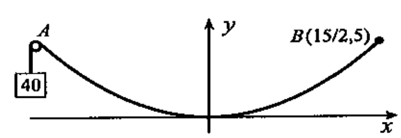
$$75 = \frac{225}{4}\sinh\left(\frac{4a}{225}\right) \quad \Longrightarrow \quad a = \frac{225}{4}\text{Sinh}^{-1}\left(\frac{4}{3}\right) = 61.797.$$

The span is therefore $2a = 123.6$ m. Maximum tension is at A where, according to Example 11.33, $T = 225(4)(9.81)/4 + 4(9.81)(37.5) = 3679$ N.

13. Using equation 11.51, the equation of the cable in the coordinate system shown is

$$y = \frac{T_0}{w}\left[\cosh\left(\frac{wx}{T_0}\right)-1\right].$$

Since $B(15/2, 5)$ is on the curve,



$$5 = \frac{T_0}{w}\left[\cosh\left(\frac{15w}{2T_0}\right)-1\right].$$

Since tension at A must be $40g$ N, it follows from Example 11.33 that $40g = T_0 + 5w$. These combine to give

$$5w = (40g - 5w)\left[\cosh\left(\frac{15w}{80g-10w}\right)-1\right].$$

When this is solved numerically, the result is $w = 34.672$. Mass per unit length of the cable is therefore $34.672/9.81 = 3.53$ kg/m. According to Example 11.32 length of the cable from A to B is

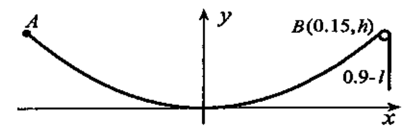
$$\frac{2T_0}{w}\sinh\left(\frac{15w}{2T_0}\right) = \frac{2[40g-5(34.672)]}{34.672}\sinh\left[\frac{15(34.672)}{2[40g-5(34.672)]}\right] = 18.78 \text{ m}.$$

14. Using equation 11.51, the equation of the cord in the coordinate system shown is

$$y = \frac{T_0}{w}\left[\cosh\left(\frac{wx}{T_0}\right) - 1\right].$$

Since $B(0.15, h)$ is on the curve,

$$h = \frac{T_0}{w}\left[\cosh\left(\frac{0.15w}{T_0}\right) - 1\right].$$



According to Example 11.32, the length of cord between A and B is

$$l = \frac{2T_0}{w}\sinh\left(\frac{0.15w}{T_0}\right).$$

Since the cord is in equilibrium, the maximum tension in the cord between A and B must be equal to the force of gravity on that part of the cord hanging to the right of B. Since maximum tension between A and B is at B, we obtain $T_0 + wh = (0.9 - l)w$. The first two equations give

$$1 = \cosh^2\left(\frac{0.15w}{T_0}\right) - \sinh^2\left(\frac{0.15w}{T_0}\right) = \left(\frac{wh}{T_0} + 1\right)^2 - \left(\frac{wl}{2T_0}\right)^2 \implies 4wh^2 + 8T_0h - wl^2 = 0.$$

When we substitute $l = 0.9 - (T_0 + wh)/w$,

$$4wh^2 + 8T_0h - w\left(0.9 - \frac{T_0 + wh}{w}\right)^2 = 0 \implies \frac{3wh^2}{T_0} + h\left(6 + \frac{1.8w}{T_0}\right) - \frac{w}{T_0}\left(0.9 - \frac{T_0}{w}\right)^2 = 0.$$

Solutions of this quadratic equation in h are

$$h = \frac{-(6 + 1.8w/T_0) \pm \sqrt{(6 + 1.8w/T_0)^2 + 12(w^2/T_0^2)(0.9 - T_0/w)^2}}{6w/T_0}$$
$$= \frac{-(6 + 1.8w/T_0) \pm \sqrt{48 + 12.96w^2/T_0^2}}{6w/T_0}.$$

When we choose the positive root and substitute into the equation $h = (T_0/w)[\cosh(0.15w/T_0) - 1]$,

$$-\frac{T_0}{w} - 0.3 + \frac{1}{6}\sqrt{12.96 + \frac{48T_0^2}{w^2}} = \frac{T_0}{w}\left[\cosh\left(\frac{0.15w}{T_0}\right) - 1\right],$$

which simplifies to

$$-1.8 + \sqrt{12.96 + \frac{48T_0^2}{w^2}} = \frac{6T_0}{w}\cosh\left(\frac{0.15w}{T_0}\right).$$

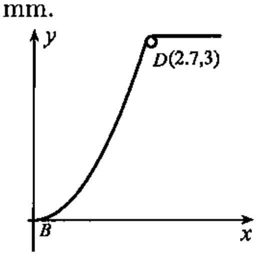
If we set $z = w/T_0$, solutions of $-1.8z + \sqrt{12.96z^2 + 48} - 6\cosh(0.15z) = 0$ are $z = 0.83939, 1.7331$. These give $h = 0.00946, 0.0196$. Hence, the smaller sag is 9.46 mm.

15. Using equation 11.51, the equation of the cable in the coordinate system shown is

$$y = \frac{T_0}{w}\left[\cosh\left(\frac{wx}{T_0}\right) - 1\right].$$

Since $D(2.7, 3)$ is on the curve,

$$3 = \frac{T_0}{w}\left[\cosh\left(\frac{2.7w}{T_0}\right) - 1\right].$$

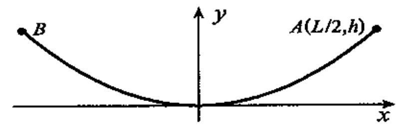


When w is set equal to $2.7(9.81)$, the equation can be solved numerically for $T_0 = 41.145$. Maximum tension occurs at D, and this tension is equal to the required force F. According to Example 11.33,

$$F = T = T_0 + wh = T_0 + 3w = 41.145 + 3(2.7)(9.81) = 120.6 \text{ N}.$$

16. Using equation 11.51, the equation of the cable in the coordinate system shown is

$$y = \frac{T_0}{w}\left[\cosh\left(\frac{wx}{T_0}\right) - 1\right].$$



If h is the sag, then

$$h = \frac{T_0}{w}\left[\cosh\left(\frac{wL}{2T_0}\right) - 1\right].$$

Maximum tension occurs at A and according to Example 11.33 is given by

$$T = T_0 + wh = T_0 + T_0\left[\cosh\left(\frac{wL}{2T_0}\right) - 1\right] = T_0\cosh\left(\frac{wL}{2T_0}\right).$$

Critical points of $T(T_0)$ are defined

$$0 = \frac{dT}{dT_0} = \cosh\left(\frac{wL}{2T_0}\right) - \frac{wL}{2T_0}\sinh\left(\frac{wL}{2T_0}\right) \quad\Longrightarrow\quad \frac{wL}{2T_0}\tanh\left(\frac{wL}{2T_0}\right) = 1.$$

This equation can be solved numerically for $wL/(2T_0) = 1.200$. It now follows that

$$h = \frac{L}{2.4}[\cosh(1.200) - 1] \quad\Longrightarrow\quad \frac{h}{L} = \frac{\cosh(1.200) - 1}{2.4} = 0.338.$$

EXERCISES 11.9

1. $t \geq 1$ **2.** All real t **3.** $-1 \leq t \leq 1$

4. $t > -4$ **5.** All real t **6.** $\dfrac{d\mathbf{u}}{dt} = \hat{\mathbf{i}} - 2t\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

7. $$\begin{aligned}\frac{d}{dt}[f(t)\mathbf{v}(t)] &= f'(t)\mathbf{v}(t) + f(t)\mathbf{v}'(t) = 2t(\hat{\mathbf{i}} - 2t\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}}) + (t^2+3)(-2\hat{\mathbf{j}} + 6t\hat{\mathbf{k}})\\ &= 2t\hat{\mathbf{i}} - 6(t^2+1)\hat{\mathbf{j}} + 6t(2t^2+3)\hat{\mathbf{k}}\end{aligned}$$

8. $$\begin{aligned}\frac{d}{dt}[g(t)\mathbf{u}(t)] &= g'(t)\mathbf{u}(t) + g(t)\mathbf{u}'(t) = (6t^2-3)(t\hat{\mathbf{i}} - t^2\hat{\mathbf{j}} + 2t\hat{\mathbf{k}}) + (2t^3-3t)(\hat{\mathbf{i}} - 2t\hat{\mathbf{j}} + 2\hat{\mathbf{k}})\\ &= 2t(4t^2-3)\hat{\mathbf{i}} + t^2(9-10t^2)\hat{\mathbf{j}} + 4t(4t^2-3)\hat{\mathbf{k}}\end{aligned}$$

9. $$\begin{aligned}\frac{d}{dt}(\mathbf{u}\times\mathbf{v}) &= \mathbf{u}\times\frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt}\times\mathbf{v} = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ t & -t^2 & 2t\\ 0 & -2 & 6t\end{vmatrix} + \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & -2t & 2\\ 1 & -2t & 3t^2\end{vmatrix}\\ &= [(-6t^3+4t)\hat{\mathbf{i}} - 6t^2\hat{\mathbf{j}} - 2t\hat{\mathbf{k}}] + [(-6t^3+4t)\hat{\mathbf{i}} + (2-3t^2)\hat{\mathbf{j}}]\\ &= 4t(2-3t^2)\hat{\mathbf{i}} + (2-9t^2)\hat{\mathbf{j}} - 2t\hat{\mathbf{k}}\end{aligned}$$

10. $$\begin{aligned}\frac{d}{dt}(\mathbf{u}\times t\mathbf{v}) &= \frac{d}{dt}\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ t & -t^2 & 2t\\ t & -2t^2 & 3t^3\end{vmatrix} = \frac{d}{dt}[(4t^3-3t^5)\hat{\mathbf{i}} + (2t^2-3t^4)\hat{\mathbf{j}} - t^3\hat{\mathbf{k}}]\\ &= (12t^2-15t^4)\hat{\mathbf{i}} + (4t-12t^3)\hat{\mathbf{j}} - 3t^2\hat{\mathbf{k}}\end{aligned}$$

11. $$\frac{d}{dt}(2\mathbf{u}\cdot\mathbf{v}) = 2\frac{d}{dt}(t + 2t^3 + 6t^3) = 2(1+24t^2)$$

12. $$\frac{d}{dt}(3\mathbf{u}+4\mathbf{v}) = 3\frac{d\mathbf{u}}{dt} + 4\frac{d\mathbf{v}}{dt} = 3(\hat{\mathbf{i}} - 2t\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) + 4(-2\hat{\mathbf{j}} + 6t\hat{\mathbf{k}}) = 3\hat{\mathbf{i}} - (6t+8)\hat{\mathbf{j}} + (6+24t)\hat{\mathbf{k}}$$

13. $$\int \mathbf{u}(t)\,dt = \int (t\hat{\mathbf{i}} - t^2\hat{\mathbf{j}} + 2t\hat{\mathbf{k}})\,dt = \frac{t^2}{2}\hat{\mathbf{i}} - \frac{t^3}{3}\hat{\mathbf{j}} + t^2\hat{\mathbf{k}} + \mathbf{C}$$

14. $$\begin{aligned}\frac{d}{dt}[f(t)\mathbf{u} + g(t)\mathbf{v}] &= f'(t)\mathbf{u} + f(t)\mathbf{u}'(t) + g'(t)\mathbf{v} + g(t)\mathbf{v}'(t)\\ &= 2t(t\hat{\mathbf{i}} - t^2\hat{\mathbf{j}} + 2t\hat{\mathbf{k}}) + (t^2+3)(\hat{\mathbf{i}} - 2t\hat{\mathbf{j}} + 2\hat{\mathbf{k}})\\ &\quad + (6t^2-3)(\hat{\mathbf{i}} - 2t\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}}) + (2t^3-3t)(-2\hat{\mathbf{j}} + 6t\hat{\mathbf{k}})\\ &= 9t^2\hat{\mathbf{i}} + (6t-20t^3)\hat{\mathbf{j}} + (6-21t^2+30t^4)\hat{\mathbf{k}}\end{aligned}$$

15. $\int 4\mathbf{v}(t)\,dt = 4\int(\hat{\mathbf{i}} - 2t\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}})\,dt = 4(t\hat{\mathbf{i}} - t^2\hat{\mathbf{j}} + t^3\hat{\mathbf{k}}) + \mathbf{C}$

16. $$\frac{d}{dt}[t(\mathbf{u}\times\mathbf{v})] = \frac{d}{dt}\left[t\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ t & -t^2 & 2t\\ 1 & -2t & 3t^2\end{vmatrix}\right] = \frac{d}{dt}[(4t^3 - 3t^5)\hat{\mathbf{i}} + (2t^2 - 3t^4)\hat{\mathbf{j}} - t^3\hat{\mathbf{k}}$$
$$= (12t^2 - 15t^4)\hat{\mathbf{i}} + (4t - 12t^3)\hat{\mathbf{j}} - 3t^2\hat{\mathbf{k}}$$

17. $\int f(t)\mathbf{u}(t)\,dt = \int(t^2+3)(t\hat{\mathbf{i}} - t^2\hat{\mathbf{j}} + 2t\hat{\mathbf{k}})\,dt = (t^4/4 + 3t^2/2)\hat{\mathbf{i}} - (t^5/5 + t^3)\hat{\mathbf{j}} + (t^4/2 + 3t^2)\hat{\mathbf{k}} + \mathbf{C}$

18. $$\int[3g(t)\mathbf{v}(t) + \mathbf{u}(t)]\,dt = \int[3(2t^3 - 3t)(\hat{\mathbf{i}} - 2t\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}}) + (t\hat{\mathbf{i}} - t^2\hat{\mathbf{j}} + 2t\hat{\mathbf{k}})]\,dt$$
$$= \int[(6t^3 - 8t)\hat{\mathbf{i}} + (-12t^4 + 17t^2)\hat{\mathbf{j}} + (18t^5 - 27t^3 + 2t)\hat{\mathbf{k}}]\,dt$$
$$= \left(\frac{3t^4}{2} - 4t^2\right)\hat{\mathbf{i}} + \left(-\frac{12t^5}{5} + \frac{17t^3}{3}\right)\hat{\mathbf{j}} + \left(3t^6 - \frac{27t^4}{4} + t^2\right)\hat{\mathbf{k}} + \mathbf{C}$$

19. $\int[f(t)\mathbf{u}\cdot\mathbf{v}]\,dt = \int(t^2+3)(t + 2t^3 + 6t^3)\,dt = \int(8t^5 + 25t^3 + 3t)\,dt = \dfrac{4t^6}{3} + \dfrac{25t^4}{4} + \dfrac{3t^2}{2} + C$

20. $$\mathbf{u}\times\frac{d\mathbf{v}}{dt} - f(t)\mathbf{u}\cdot\frac{d\mathbf{v}}{dt}\mathbf{v} = \mathbf{u}\times(-2\hat{\mathbf{j}} + 6t\hat{\mathbf{k}}) - f(t)\mathbf{u}\cdot(-2\hat{\mathbf{j}} + 6t\hat{\mathbf{k}})\mathbf{v}$$
$$= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ t & -t^2 & 2t\\ 0 & -2 & 6t\end{vmatrix} - (t^2+3)(t\hat{\mathbf{i}} - t^2\hat{\mathbf{j}} + 2t\hat{\mathbf{k}})\cdot(-2\hat{\mathbf{j}} + 6t\hat{\mathbf{k}})\mathbf{v}$$
$$= (-6t^3 + 4t)\hat{\mathbf{i}} - 6t^2\hat{\mathbf{j}} - 2t\hat{\mathbf{k}} - (t^2+3)(2t^2 + 12t^2)(\hat{\mathbf{i}} - 2t\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}})$$
$$= (-14t^4 - 6t^3 - 42t^2 + 4t)\hat{\mathbf{i}} + (28t^5 + 84t^3 - 6t^2)\hat{\mathbf{j}} + (-42t^6 - 126t^4 - 2t)\hat{\mathbf{k}}$$

21. $$\mathbf{u}\cdot\frac{d\mathbf{v}}{dt} - \mathbf{v}\cdot\int\mathbf{u}(t)\,dt = (t\hat{\mathbf{i}} - t^2\hat{\mathbf{j}} + 2t\hat{\mathbf{k}})\cdot(-2\hat{\mathbf{j}} + 6t\hat{\mathbf{k}})$$
$$-(\hat{\mathbf{i}} - 2t\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}})\cdot\left(\frac{t^2}{2}\hat{\mathbf{i}} - \frac{t^3}{3}\hat{\mathbf{j}} + t^2\hat{\mathbf{k}} + a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}\right)$$
$$= 2t^2 + 12t^2 - \frac{t^2}{2} - \frac{2t^4}{3} - 3t^4 - a + 2bt - 3ct^2 = -\frac{11t^4}{3} + C_1 + C_2t + C_3t^2$$

22. If $\mathbf{u} = (u_x, u_y, u_z)$ and $\mathbf{v} = (v_x, v_y, v_z)$, then
$$\frac{d}{dt}(\mathbf{u}+\mathbf{v}) = \frac{d}{dt}[(u_x+v_x)\hat{\mathbf{i}} + (u_y+v_y)\hat{\mathbf{j}} + (u_z+v_z)\hat{\mathbf{k}}] = \left[\frac{d}{dt}(u_x+v_x)\right]\hat{\mathbf{i}} + \left[\frac{d}{dt}(u_y+v_y)\right]\hat{\mathbf{j}} + \left[\frac{d}{dt}(u_z+v_z)\right]\hat{\mathbf{k}}$$
$$= \left(\frac{du_x}{dt}\hat{\mathbf{i}} + \frac{du_y}{dt}\hat{\mathbf{j}} + \frac{du_z}{dt}\hat{\mathbf{k}}\right) + \left(\frac{dv_x}{dt}\hat{\mathbf{i}} + \frac{dv_y}{dt}\hat{\mathbf{j}} + \frac{dv_z}{dt}\hat{\mathbf{k}}\right) = \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{v}}{dt}.$$

23. If we set $\mathbf{u} = u_x\hat{\mathbf{i}} + u_y\hat{\mathbf{j}} + u_z\hat{\mathbf{k}}$ and $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$, then
$$\frac{d}{dt}(f\mathbf{v}) = \frac{d}{dt}[f(t)v_x\hat{\mathbf{i}} + f(t)v_y\hat{\mathbf{j}} + f(t)v_z\hat{\mathbf{k}}]$$
$$= \left[f'(t)v_x + f(t)\frac{dv_x}{dt}\right]\hat{\mathbf{i}} + \left[f'(t)v_y + f(t)\frac{dv_y}{dt}\right]\hat{\mathbf{j}} + \left[f'(t)v_z + f(t)\frac{dv_z}{dt}\right]\hat{\mathbf{k}}$$
$$= f'(t)(v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}) + f(t)\left(\frac{dv_x}{dt}\hat{\mathbf{i}} + \frac{dv_y}{dt}\hat{\mathbf{j}} + \frac{dv_z}{dt}\hat{\mathbf{k}}\right) = f'(t)\mathbf{v}(t) + f(t)\frac{d\mathbf{v}}{dt},$$
$$\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v}) = \frac{d}{dt}(u_xv_x + u_yv_y + u_zv_z) = u_x\frac{dv_x}{dt} + v_x\frac{du_x}{dt} + u_y\frac{dv_y}{dt} + v_y\frac{du_y}{dt} + u_z\frac{dv_z}{dt} + v_z\frac{du_z}{dt}$$
$$= (u_x\hat{\mathbf{i}} + u_y\hat{\mathbf{j}} + u_z\hat{\mathbf{k}})\cdot\left(\frac{dv_x}{dt}\hat{\mathbf{i}} + \frac{dv_y}{dt}\hat{\mathbf{j}} + \frac{dv_z}{dt}\hat{\mathbf{k}}\right) + \left(\frac{du_x}{dt}\hat{\mathbf{i}} + \frac{du_y}{dt}\hat{\mathbf{j}} + \frac{du_z}{dt}\hat{\mathbf{k}}\right)\cdot(v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}})$$
$$= \mathbf{u}\cdot\frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt}\cdot\mathbf{v},$$

$$\begin{aligned}\frac{d}{dt}(\mathbf{u}\times\mathbf{v}) &= \frac{d}{dt}[(u_yv_z - u_zv_y)\hat{\mathbf{i}} + (u_zv_x - u_xv_z)\hat{\mathbf{j}} + (u_xv_y - u_yv_x)\hat{\mathbf{k}}]\\ &= \left(u_y\frac{dv_z}{dt} + v_z\frac{du_y}{dt} - u_z\frac{dv_y}{dt} - v_y\frac{du_z}{dt}\right)\hat{\mathbf{i}} + \left(u_z\frac{dv_x}{dt} + v_x\frac{du_z}{dt} - u_x\frac{dv_z}{dt} - v_z\frac{du_x}{dt}\right)\hat{\mathbf{j}}\\ &\quad + \left(u_x\frac{dv_y}{dt} + v_y\frac{du_x}{dt} - u_y\frac{dv_x}{dt} - v_x\frac{du_y}{dt}\right)\hat{\mathbf{k}}\\ &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ u_x & u_y & u_z\\ \dfrac{dv_x}{dt} & \dfrac{dv_y}{dt} & \dfrac{dv_z}{dt}\end{vmatrix} + \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \dfrac{du_x}{dt} & \dfrac{du_y}{dt} & \dfrac{du_z}{dt}\\ v_x & v_y & v_z\end{vmatrix} = \mathbf{u}\times\frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt}\times\mathbf{v}.\end{aligned}$$

24. Using equation 11.59b, $\dfrac{d}{dt}[\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})] = \mathbf{u}\cdot\dfrac{d}{dt}(\mathbf{v}\times\mathbf{w}) + \dfrac{d\mathbf{u}}{dt}\cdot\mathbf{v}\times\mathbf{w}$, and now using equation 11.59c,

$$\frac{d}{dt}[\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})] = \mathbf{u}\cdot\left(\mathbf{v}\times\frac{d\mathbf{w}}{dt} + \frac{d\mathbf{v}}{dt}\times\mathbf{w}\right) + \frac{d\mathbf{u}}{dt}\cdot\mathbf{v}\times\mathbf{w} = \mathbf{u}\cdot\mathbf{v}\times\frac{d\mathbf{w}}{dt} + \mathbf{u}\cdot\frac{d\mathbf{v}}{dt}\times\mathbf{w} + \frac{d\mathbf{u}}{dt}\cdot\mathbf{v}\times\mathbf{w}.$$

25. If $\mathbf{v}$ has constant length, then $\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2 =$ constant. Differentiation with 11.59b gives

$$0 = \frac{d\mathbf{v}}{dt}\cdot\mathbf{v} + \mathbf{v}\cdot\frac{d\mathbf{v}}{dt} = 2\left(\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\right).$$

But this implies that $\mathbf{v}$ and $d\mathbf{v}/dt$ are perpendicular.

26. If we set $\mathbf{v} = v_x(s)\hat{\mathbf{i}} + v_y(s)\hat{\mathbf{j}} + v_z(s)\hat{\mathbf{k}}$, then

$$\frac{d\mathbf{v}}{dt} = \frac{dv_x}{dt}\hat{\mathbf{i}} + \frac{dv_y}{dt}\hat{\mathbf{j}} + \frac{dv_z}{dt}\hat{\mathbf{k}} = \frac{dv_x}{ds}\frac{ds}{dt}\hat{\mathbf{i}} + \frac{dv_y}{ds}\frac{ds}{dt}\hat{\mathbf{j}} + \frac{dv_z}{ds}\frac{ds}{dt}\hat{\mathbf{k}} = \left(\frac{dv_x}{ds}\hat{\mathbf{i}} + \frac{dv_y}{ds}\hat{\mathbf{j}} + \frac{dv_z}{ds}\hat{\mathbf{k}}\right)\frac{ds}{dt} = \frac{d\mathbf{v}}{ds}\frac{ds}{dt}.$$

27. Suppose first of all that the limit satisfies the definition of the exercise. Then given any $\epsilon > 0$, there exists a $\delta > 0$ such that $|\mathbf{v}(t) - \mathbf{V}| < \epsilon$ whenever $0 < |t - t_0| < \delta$. If components of $\mathbf{v}$ and $\mathbf{V}$ are denoted by $\mathbf{v} = v_x(t)\hat{\mathbf{i}} + v_y(t)\hat{\mathbf{j}} + v_z(t)\hat{\mathbf{k}}$ and $\mathbf{V} = V_x\hat{\mathbf{i}} + V_y\hat{\mathbf{j}} + V_z\hat{\mathbf{k}}$, then for $0 < |t - t_0| < \delta$,

$$|\mathbf{v}(t) - \mathbf{V}| = \sqrt{[v_x(t) - V_x]^2 + [v_y(t) - V_y]^2 + [v_z(t) - V_z]^2} < \epsilon.$$

But this implies that $|v_x(t) - V_x| < \epsilon$, $|v_y(t) - V_y| < \epsilon$, and $|v_z(t) - V_z| < \epsilon$ for $0 < |t - t_0| < \delta$. But this means that

$$\lim_{t\to t_0} v_x(t) = V_x, \quad \lim_{t\to t_0} v_y(t) = V_y, \quad \lim_{t\to t_0} v_z(t) = V_z.$$

Thus,

$$\lim_{t\to t_0}\mathbf{v}(t) = \mathbf{V} = V_x\hat{\mathbf{i}} + V_y\hat{\mathbf{j}} + V_z\hat{\mathbf{k}} = \left[\lim_{t\to t_0} v_x(t)\right]\hat{\mathbf{i}} + \left[\lim_{t\to t_0} v_y(t)\right]\hat{\mathbf{j}} + \left[\lim_{t\to t_0} v_z(t)\right]\hat{\mathbf{k}};$$

that is, 11.54 is satisfied. Conversely, suppose 11.54 is satisfied and we denote the component limits by

$$V_x = \lim_{t\to t_0} v_x(t), \quad V_y = \lim_{t\to t_0} v_y(t), \quad V_z = \lim_{t\to t_0} v_z(t).$$

Given any $\epsilon > 0$, there exists a $\delta_1 > 0$ such that

$$|v_x(t) - V_x| < \frac{\epsilon}{\sqrt{3}} \qquad \text{whenever } 0 < |t - t_0| < \delta_1;$$

a $\delta_2 > 0$ such that

$$|v_y(t) - V_y| < \frac{\epsilon}{\sqrt{3}} \qquad \text{whenever } 0 < |t - t_0| < \delta_2;$$

and a $\delta_3 > 0$ such that

$$|v_z(t) - V_z| < \frac{\epsilon}{\sqrt{3}} \qquad \text{whenever } 0 < |t - t_0| < \delta_3.$$

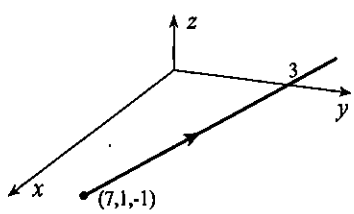
If we set $\mathbf{V} = V_x\hat{\mathbf{i}} + V_y\hat{\mathbf{j}} + V_z\hat{\mathbf{k}}$, and choose δ as the smallest of δ_1, δ_2, and δ_3, then whenever $0 < |t - t_0| < \delta$,

$$|\mathbf{v}(t) - \mathbf{V}| = \sqrt{[v_x(t) - V_x]^2 + [v_y(t) - V_y]^2 + [v_z(t) - V_z]^2} < \sqrt{\epsilon^2/3 + \epsilon^2/3 + \epsilon^2/3} = \epsilon;$$

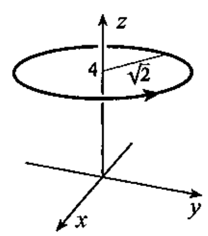
that is, the limit satisfies the definition of the exercise.

EXERCISES 11.10

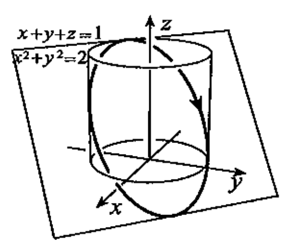
1. If we set $z = t$, then $y = 2t + 3$, and $x = 6 - 2(2t + 3) - 3t = -7t$. These are parametric equations for the curve. A vector representation is $\mathbf{r} = -7t\hat{\mathbf{i}} + (2t + 3)\hat{\mathbf{j}} + t\hat{\mathbf{k}}$.



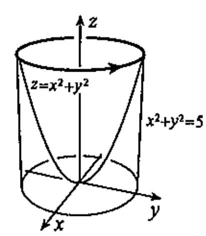
2. If we set $x = \sqrt{2}\cos t$, then $y = \pm\sqrt{2}\sin t$. For y to increase in the first octant, we choose $y = \sqrt{2}\sin t$. Parametric and vector equations are therefore $x = \sqrt{2}\cos t$, $y = \sqrt{2}\sin t$, $z = 4$, $\mathbf{r} = \sqrt{2}\cos t\,\hat{\mathbf{i}} + \sqrt{2}\sin t\,\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$, $0 \le t < 2\pi$.



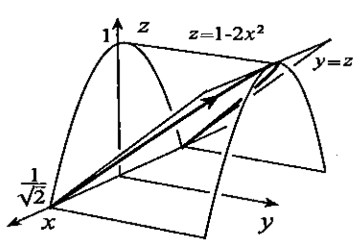
3. If we set $x = \sqrt{2}\cos t$, then $y = \pm\sqrt{2}\sin t$, and for correct direction we choose $y = -\sqrt{2}\sin t$. Parametric and vector equations are therefore $x = \sqrt{2}\cos t$, $y = -\sqrt{2}\sin t$, $z = 1 + \sqrt{2}(\sin t - \cos t)$, $\mathbf{r} = \sqrt{2}\cos t\,\hat{\mathbf{i}} - \sqrt{2}\sin t\,\hat{\mathbf{j}}$ $+(1 + \sqrt{2}\sin t - \sqrt{2}\cos t)\hat{\mathbf{k}}$, $0 \le t < 2\pi$



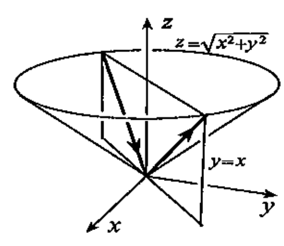
4. If we set $x = \sqrt{5}\cos t$, then $y = \pm\sqrt{5}\sin t$, and for correct direction we choose $y = \sqrt{5}\sin t$. Parametric and vector equations are therefore $x = \sqrt{5}\cos t$, $y = \sqrt{5}\sin t$, $z = 5$, $\mathbf{r} = \sqrt{5}\cos t\,\hat{\mathbf{i}} + \sqrt{5}\sin t\,\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$, $0 \le t < 2\pi$.



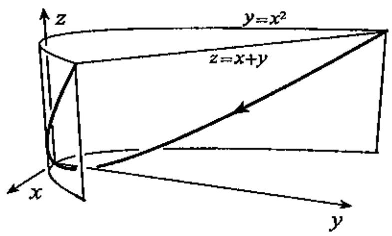
5. If we set $x = -t$ so that x decreases along the curve, then parametric and vector equations for the curve are $x = -t$, $y = 1 - 2t^2$, $z = 1 - 2t^2$, and $\mathbf{r} = -t\hat{\mathbf{i}} + (1 - 2t^2)(\hat{\mathbf{j}} + \hat{\mathbf{k}})$.



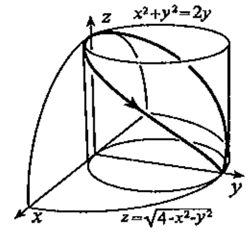
6. If we choose x as parameter by setting $x = t$, then $y = t$ and $z = \sqrt{t^2 + t^2} = \sqrt{2}|t|$. Parametric and vector equations are $x = t$, $y = t$, $z = \sqrt{2}|t|$, $\mathbf{r} = t(\hat{\mathbf{i}} + \hat{\mathbf{j}}) + \sqrt{2}|t|\hat{\mathbf{k}}$.



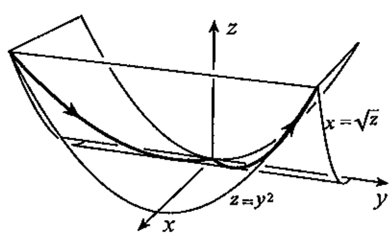
7. If we set $x = t$ so that x increases along the curve, then parametric and vector equations for the curve are $x = t$, $y = t^2$, $z = t + t^2$, and $\mathbf{r} = t\hat{\mathbf{i}} + t^2\hat{\mathbf{j}} + (t + t^2)\hat{\mathbf{k}}$.



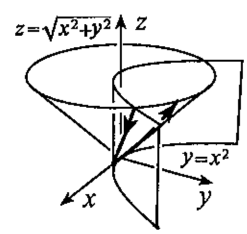
8. If we set $x = \cos t$, then $y = 1 \pm \sin t$, and for correct direction we choose $y = 1 + \sin t$. Parametric and vector equations are therefore
$x = \cos t$, $y = 1 + \sin t$,
$z = \sqrt{4 - \cos^2 t - (1 + \sin t)^2} = \sqrt{2 - 2\sin t}$,
$\mathbf{r} = \cos t\,\hat{\mathbf{i}} + (1 + \sin t)\hat{\mathbf{j}} + \sqrt{2 - 2\sin t}\,\hat{\mathbf{k}}$,
$0 \le t < 2\pi$.



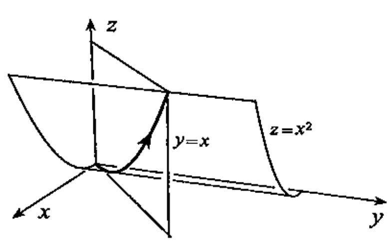
9. If we set $y = t$, then parametric and vector equations for the curve are
$x = |t|$, $y = t$, $z = t^2$, and
$\mathbf{r} = |t|\hat{\mathbf{i}} + t\hat{\mathbf{j}} + t^2\hat{\mathbf{k}}$.



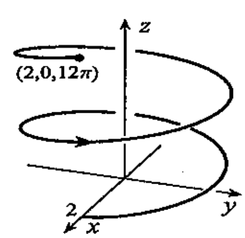
10. If we set $x = -t$ (so that x decreases as t increases), parametric and vector equations are
$x = -t$, $y = t^2$, $z = \sqrt{t^2 + t^4}$,
$\mathbf{r} = -t\hat{\mathbf{i}} + t^2\hat{\mathbf{j}} + \sqrt{t^2 + t^4}\,\hat{\mathbf{k}}$.



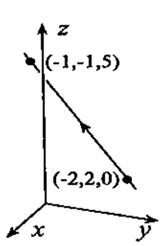
11. The curve is the first octant intersection of the plane $y = x$ and the parabolic cylinder $z = x^2$.



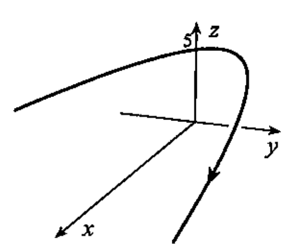
12. Since $x^2 + y^2 = 4$, the curve is two turns of a helix that rises a distance of 6π in each turn.



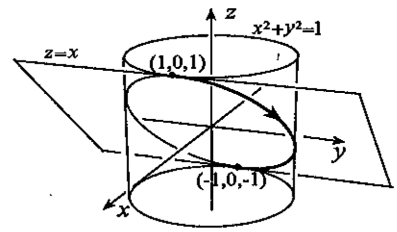
13. This is a straight line through the points $(-2, 2, 0)$ and $(-1, -1, 5)$.



14. Since $x = t^2 - t = y^2 - y$, the curve is a parabola in the plane $z = 5$.



15. The curve is half the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z = x$.



EXERCISES 11.11

1. Since $\mathbf{r} = \sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} + t\hat{\mathbf{k}}$, a tangent vector is $\mathbf{T} = \dfrac{d\mathbf{r}}{dt} = \cos t\,\hat{\mathbf{i}} - \sin t\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$. A unit tangent vector is $\hat{\mathbf{T}} = \dfrac{\cos t\,\hat{\mathbf{i}} - \sin t\,\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{2}}$.

2. Since $\mathbf{r} = t\hat{\mathbf{i}} + t^2\hat{\mathbf{j}} + t^3\hat{\mathbf{k}}$, a tangent vector is $\mathbf{T} = \dfrac{d\mathbf{r}}{dt} = \hat{\mathbf{i}} + 2t\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}}$. A unit tangent vector is $\hat{\mathbf{T}} = \dfrac{\hat{\mathbf{i}} + 2t\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}}}{\sqrt{1+4t^2+9t^4}}$.

3. Since $\mathbf{r} = (t-1)^2\hat{\mathbf{i}} + (t+1)^2\hat{\mathbf{j}} - t\hat{\mathbf{k}}$, a tangent vector is $\mathbf{T} = \dfrac{d\mathbf{r}}{dt} = 2(t-1)\hat{\mathbf{i}} + 2(t+1)\hat{\mathbf{j}} - \hat{\mathbf{k}}$. A unit tangent vector is $\hat{\mathbf{T}} = \dfrac{2(t-1)\hat{\mathbf{i}} + 2(t+1)\hat{\mathbf{j}} - \hat{\mathbf{k}}}{\sqrt{4(t-1)^2+4(t+1)^2+1}} = \dfrac{2(t-1)\hat{\mathbf{i}} + 2(t+1)\hat{\mathbf{j}} - \hat{\mathbf{k}}}{\sqrt{8t^2+9}}$.

4. Since x decreases along the curve, we set $x = -t$ for parametric equations, in which case $y = 5 + t$, $z = t^2 - 5 - t$. A vector equation for the curve is $\mathbf{r} = -t\hat{\mathbf{i}} + (5+t)\hat{\mathbf{j}} + (t^2 - t - 5)\hat{\mathbf{k}}$, $-5 \le t \le 0$. A tangent vector is $\mathbf{T} = \dfrac{d\mathbf{r}}{dt} = -\hat{\mathbf{i}} + \hat{\mathbf{j}} + (2t-1)\hat{\mathbf{k}}$, and a unit tangent vector is

$$\hat{\mathbf{T}} = \frac{-\hat{\mathbf{i}} + \hat{\mathbf{j}} + (2t-1)\hat{\mathbf{k}}}{\sqrt{1+1+(2t-1)^2}} = \frac{-\hat{\mathbf{i}} + \hat{\mathbf{j}} + (2t-1)\hat{\mathbf{k}}}{\sqrt{4t^2-4t+3}}.$$

5. If we set $x = 2\cos t$, then $y = 2\sin t$ and $z = 4 - 2\cos t - 2\sin t$. A vector representation for the curve is $\mathbf{r} = 2\cos t\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}} + (4 - 2\cos t - 2\sin t)\hat{\mathbf{k}}$. A tangent vector is $\mathbf{T} = \dfrac{d\mathbf{r}}{dt} = -2\sin t\,\hat{\mathbf{i}} + 2\cos t\,\hat{\mathbf{j}} + (2\sin t - 2\cos t)\hat{\mathbf{k}}$, and a unit tangent vector is

$$\hat{\mathbf{T}} = \frac{-2\sin t\,\hat{\mathbf{i}} + 2\cos t\,\hat{\mathbf{j}} + (2\sin t - 2\cos t)\hat{\mathbf{k}}}{\sqrt{4\sin^2 t + 4\cos^2 t + (2\sin t - 2\cos t)^2}} = \frac{-\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} + (\sin t - \cos t)\hat{\mathbf{k}}}{\sqrt{2 - \sin 2t}}.$$

6. Since $\mathbf{T} = \dfrac{d\mathbf{r}}{dt} = -4\sin t\,\hat{\mathbf{i}} + 6\cos t\,\hat{\mathbf{j}} + 2\cos t\,\hat{\mathbf{k}}$, the unit tangent vector at $(2\sqrt{2}, 3\sqrt{2}, \sqrt{2})$ is

$$\hat{\mathbf{T}}(\pi/4) = \frac{-2\sqrt{2}\hat{\mathbf{i}} + 3\sqrt{2}\hat{\mathbf{j}} + \sqrt{2}\hat{\mathbf{k}}}{\sqrt{8+18+2}} = \frac{-2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{14}}.$$

7. Since $\mathbf{T} = \dfrac{d\mathbf{r}}{dt} = -5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 4\hat{\mathbf{k}}$, the unit tangent vector at every point on the line is $\hat{\mathbf{T}} = \dfrac{-5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 4\hat{\mathbf{k}}}{\sqrt{42}}$.

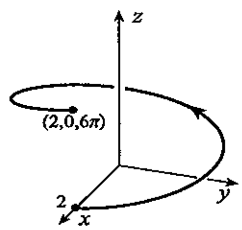
8. With $x = \sqrt{2}\cos t$, $y = -\sqrt{2}\sin t$, $z = \sqrt{2}$, $0 \le t < 2\pi$, a tangent vector at $(1, 1, \sqrt{2})$ is $\mathbf{T}(7\pi/4) = (-\sqrt{2}\sin t, -\sqrt{2}\cos t, 0)_{|t=7\pi/4} = (1, -1, 0)$. Hence, $\hat{\mathbf{T}}(7\pi/4) = (1, -1, 0)/\sqrt{2}$.

9. With $x = t^2 + 1$, $y = t$, $z = t^2 + 6$, a tangent vector is $\dfrac{d\mathbf{r}}{dt} = 2t\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2t\hat{\mathbf{k}}$. A unit tangent vector at $(5, 2, 10)$ is $\hat{\mathbf{T}}(2) = \dfrac{4\hat{\mathbf{i}} + \hat{\mathbf{j}} + 4\hat{\mathbf{k}}}{\sqrt{33}}$.

10. If we set $x = 2\cos t$, then $y = 1 \pm 2\sin t$. For z to decrease when y is negative, we choose $y = 1 - 2\sin t$, and then $\mathbf{r} = 2\cos t\,\hat{\mathbf{i}} + (1 - 2\sin t)\hat{\mathbf{j}} + 2\cos t\,\hat{\mathbf{k}}$, $0 \le t < 2\pi$. Since $\mathbf{T} = d\mathbf{r}/dt = -2\sin t\,\hat{\mathbf{i}} - 2\cos t\,\hat{\mathbf{j}} - 2\sin t\,\hat{\mathbf{k}}$, $\mathbf{T}(0) = -2\hat{\mathbf{j}}$. Consequently, $\hat{\mathbf{T}}(0) = -\hat{\mathbf{j}}$.

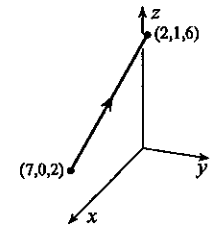
11. With equation 11.78,

$$L = \int_0^{2\pi} \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 9}\,dt = \sqrt{13}\Big\{t\Big\}_0^{2\pi} = 2\sqrt{13}\pi.$$



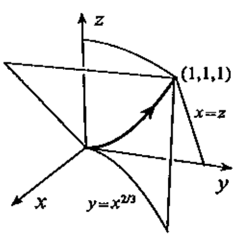
12. Since this is a straight line segment from $(7,0,2)$ to $(2,1,6)$, its length is

$$\sqrt{5^2 + (-1)^2 + (-4)^2} = \sqrt{42}.$$



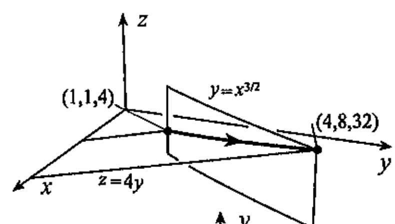
13. With equation 11.78,

$$L = \int_0^1 \sqrt{(3t^2)^2 + (2t)^2 + (3t^2)^2}\,dt = \int_0^1 t\sqrt{4 + 18t^2}\,dt. = \left\{\frac{(4+18t^2)^{3/2}}{54}\right\}_0^1 = \frac{11\sqrt{22} - 4}{27}.$$

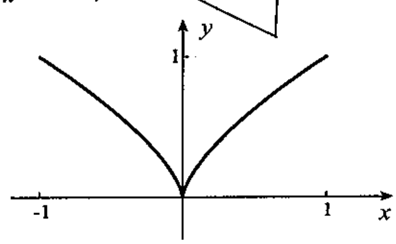


14. With equation 11.78,

$$L = \int_1^4 \sqrt{1^2 + \left(\frac{3\sqrt{t}}{2}\right)^2 + (6\sqrt{t})^2}\,dt = \int_1^4 \sqrt{1 + 153t/4}\,dt = \left\{\frac{8}{459}\left(1 + \frac{153t}{4}\right)^{3/2}\right\}_1^4 = \frac{616\sqrt{616} - 157\sqrt{157}}{459}.$$



15. There may be a corner at a point at which all derivatives vanish. For example, $d\mathbf{r}/dt = \mathbf{0}$ when $t = 0$ for the curve $x = t^3$, $y = t^2$, $z = 0$. The figure shows that the curve reverses direction at $(0,0)$.



16. Since a tangent vector is

$$\mathbf{T} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} = (-\sin t + \sin t + t\cos t)\hat{\mathbf{i}} + (\cos t - \cos t + t\sin t)\hat{\mathbf{j}} = t\cos t\,\hat{\mathbf{i}} + t\sin t\,\hat{\mathbf{j}},$$

a unit tangent vector is $\hat{\mathbf{T}} = \dfrac{t\cos t\,\hat{\mathbf{i}} + t\sin t\,\hat{\mathbf{j}}}{\sqrt{t^2\cos^2 t + t^2\sin^2 t}} = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}$.

17. This follows from the fact that $\mathbf{r}(t+h) - \mathbf{r}(t)$ in limit 11.65 always points in the direction in which t increases along the curve (see Figure 11.102).

18. (a) If we use equation 11.66, $\mathbf{T}(0) = (2t\hat{\mathbf{i}} + 3t^2\hat{\mathbf{j}} + 2t\hat{\mathbf{k}})_{|t=0} = \mathbf{0}$.

(b) A tangent vector at any point except $t = 0$ is $\mathbf{T} = 2t\hat{\mathbf{i}} + 3t^2\hat{\mathbf{j}} + 2t\hat{\mathbf{k}} = t(2\hat{\mathbf{i}} + 3t\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$. It follows then that $\mathbf{S}(t) = 2\hat{\mathbf{i}} + 3t\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ must also be tangent at any point (except possibly when $t = 0$). The only way $\mathbf{S}(t)$ can assign a tangent vector to the curve continuously is for $\mathbf{S}(0) = \lim_{t\to 0} \mathbf{S}(t) = 2\hat{\mathbf{i}} + 2\hat{\mathbf{k}}$.

EXERCISES 11.12

1. From $\hat{\mathbf{T}} = \dfrac{(\cos t, -\sin t, 1)}{\sqrt{2}}$, a vector in the direction of $\hat{\mathbf{N}}$ is $\mathbf{N} = \dfrac{d\hat{\mathbf{T}}}{dt} = \dfrac{1}{\sqrt{2}}(-\sin t, -\cos t, 0)$. Consequently, the principal normal is $\hat{\mathbf{N}} = (-\sin t, -\cos t, 0)$. The binormal is

$$\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}} = \frac{1}{\sqrt{2}}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos t & -\sin t & 1 \\ -\sin t & -\cos t & 0 \end{vmatrix} = \frac{1}{\sqrt{2}}(\cos t, -\sin t, -1).$$

2. From $\hat{\mathbf{T}} = \dfrac{(1, 2t, 3t^2)}{\sqrt{1+4t^2+9t^4}}$, a vector in the direction of $\hat{\mathbf{N}}$ is

$$\begin{aligned}\mathbf{N} = \frac{d\hat{\mathbf{T}}}{dt} &= \frac{-(8t+36t^3)}{2(1+4t^2+9t^4)^{3/2}}(1, 2t, 3t^2) + \frac{(0, 2, 6t)}{\sqrt{1+4t^2+9t^4}} \\ &= \frac{1}{(1+4t^2+9t^4)^{3/2}}\left[-(4t+18t^3)(1, 2t, 3t^2) + (1+4t^2+9t^4)(0, 2, 6t)\right] \\ &= \frac{1}{(1+4t^2+9t^4)^{3/2}}(-4t-18t^3, 2-18t^4, 6t+12t^3).\end{aligned}$$

Consequently, the principal normal is

$$\hat{\mathbf{N}} = \frac{(-2t-9t^3, 1-9t^4, 3t+6t^3)}{\sqrt{(2t+9t^3)^2+(1-9t^4)^2+(3t+6t^3)^2}} = \frac{(-2t-9t^3, 1-9t^4, 3t+6t^3)}{\sqrt{1+13t^2+54t^4+117t^6+81t^8}}.$$

The direction of the binormal is

$$\begin{aligned}\mathbf{B} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2t & 3t^2 \\ -2t-9t^3 & 1-9t^4 & 3t+6t^3 \end{vmatrix} \\ &= (6t^2+12t^4-3t^2+27t^6)\hat{\mathbf{i}} + (-6t^3-27t^5-3t-6t^3)\hat{\mathbf{j}} + (1-9t^4+4t^2+18t^4)\hat{\mathbf{k}} \\ &= (1+4t^2+9t^4)(3t^2\hat{\mathbf{i}} - 3t\hat{\mathbf{j}} + \hat{\mathbf{k}}).\end{aligned}$$

Thus, $\hat{\mathbf{B}} = \dfrac{(3t^2, -3t, 1)}{\sqrt{9t^4+9t^2+1}}$.

3. From $\hat{\mathbf{T}} = \dfrac{(2t-2, 2t+2, -1)}{\sqrt{(2t-2)^2+(2t+2)^2+1}} = \dfrac{(2t-2, 2t+2, -1)}{\sqrt{8t^2+9}}$, a vector in the direction of $\hat{\mathbf{N}}$ is

$$\begin{aligned}\mathbf{N} = \frac{d\hat{\mathbf{T}}}{dt} &= \frac{-8t}{(8t^2+9)^{3/2}}(2t-2, 2t+2, -1) + \frac{(2, 2, 0)}{\sqrt{8t^2+9}} \\ &= \frac{1}{(8t^2+9)^{3/2}}\left[-8t(2t-2, 2t+2, -1) + (8t^2+9)(2, 2, 0)\right] \\ &= \frac{1}{(8t^2+9)^{3/2}}(16t+18, -16t+18, 8t).\end{aligned}$$

Consequently, the principal normal is

$$\hat{\mathbf{N}} = \frac{(8t+9, -8t+9, 4t)}{\sqrt{(8t+9)^2+(-8t+9)^2+16t^2}} = \frac{(9+8t, 9-8t, 4t)}{3\sqrt{18+16t^2}}.$$

The direction of the binormal is

$$\mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2t-2 & 2t+2 & -1 \\ 9+8t & 9-8t & 4t \end{vmatrix} = (8t^2+9, -8t^2-9, -32t^2-36) = (8t^2+9)(1, -1, -4).$$

Thus, $\hat{\mathbf{B}} = (1, -1, -4)/\sqrt{18} = (1, -1, -4)/(3\sqrt{2})$.

4. With parametric equations $x = -t$, $y = 5 + t$, $z = t^2 - t - 5$, (see Exercise 11.11–4),

$$\hat{\mathbf{T}} = \frac{(-1,1,2t-1)}{\sqrt{1+1+(2t-1)^2}} = \frac{(-1,1,2t-1)}{\sqrt{4t^2-4t+3}}.$$

A vector in the direction of $\hat{\mathbf{N}}$ is

$$\begin{aligned}\mathbf{N} = \frac{d\hat{\mathbf{T}}}{dt} &= \frac{-(4t-2)}{(4t^2-4t+3)^{3/2}}(-1,1,2t-1) + \frac{(0,0,2)}{\sqrt{4t^2-4t+3}}\\ &= \frac{2}{(4t^2-4t+3)^{3/2}}\left[-(2t-1)(-1,1,2t-1) + (4t^2-4t+3)(0,0,1)\right]\\ &= \frac{2}{(4t^2-4t+3)^{3/2}}(2t-1,1-2t,2).\end{aligned}$$

Consequently, the principal normal is

$$\hat{\mathbf{N}} = \frac{(2t-1,1-2t,2)}{\sqrt{(2t-1)^2+(1-2t)^2+4}} = \frac{(2t-1,1-2t,2)}{\sqrt{8t^2-8t+6}}.$$

The direction of the binormal is

$$\begin{aligned}\mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 1 & 2t-1 \\ 2t-1 & 1-2t & 2 \end{vmatrix} &= (2+4t^2-4t+1)\hat{\mathbf{i}} + (4t^2-4t+1+2)\hat{\mathbf{j}} + (-1+2t-2t+1)\hat{\mathbf{k}}\\ &= (3-4t+4t^2)(\hat{\mathbf{i}}+\hat{\mathbf{j}}).\end{aligned}$$

Thus, $\hat{\mathbf{B}} = (\hat{\mathbf{i}}+\hat{\mathbf{j}})/\sqrt{2}$.

5. With parametric equations $x = 2\cos t$, $y = 2\sin t$, $z = 2\cos t$, $0 \le t \le \pi$,

$$\hat{\mathbf{T}} = \frac{(-2\sin t, 2\cos t, -2\sin t)}{\sqrt{4\sin^2 t + 4\cos^2 t + 4\sin^2 t}} = \frac{(-\sin t, \cos t, -\sin t)}{\sqrt{1+\sin^2 t}}.$$

A vector in the direction of $\hat{\mathbf{N}}$ is

$$\begin{aligned}\mathbf{N} = \frac{d\hat{\mathbf{T}}}{dt} &= \frac{-\sin t\cos t}{(1+\sin^2 t)^{3/2}}(-\sin t, \cos t, -\sin t) + \frac{(-\cos t, -\sin t, -\cos t)}{\sqrt{1+\sin^2 t}}\\ &= \frac{1}{(1+\sin^2 t)^{3/2}}\left[-\sin t\cos t(-\sin t,\cos t,-\sin t) + (1+\sin^2 t)(-\cos t,-\sin t,-\cos t)\right]\\ &= \frac{1}{(1+\sin^2 t)^{3/2}}(-\cos t, -2\sin t, -\cos t).\end{aligned}$$

Consequently, the principal normal is

$$\hat{\mathbf{N}} = \frac{(-\cos t, -2\sin t, -\cos t)}{\sqrt{\cos^2 t + 4\sin^2 t + \cos^2 t}} = -\frac{(\cos t, 2\sin t, \cos t)}{\sqrt{2+2\sin^2 t}}.$$

The direction of the binormal is

$$\mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -\sin t & \cos t & -\sin t \\ -\cos t & -2\sin t & -\cos t \end{vmatrix} = (-\cos^2 t - 2\sin^2 t, 0, 2\sin^2 t + \cos^2 t) = (\cos^2 t + 2\sin^2 t)(-1,0,1).$$

Thus, $\hat{\mathbf{B}} = (-1,0,1))/\sqrt{2}$.

6. From $\hat{\mathbf{T}} = \dfrac{(-4\sin t, 6\cos t, 2\cos t)}{\sqrt{16\sin^2 t + 36\cos^2 t + 4\cos^2 t}} = \dfrac{(-2\sin t, 3\cos t, \cos t)}{\sqrt{4+6\cos^2 t}}$, a vector in the direction of $\hat{\mathbf{N}}$ is

$$\mathbf{N} = \frac{d\hat{\mathbf{T}}}{dt} = \frac{6\cos t\,\sin t}{(4+6\cos^2 t)^{3/2}}(-2\sin t, 3\cos t, \cos t) + \frac{(-2\cos t, -3\sin t, -\sin t)}{\sqrt{4+6\cos^2 t}}.$$

At $(2\sqrt{2}, 3\sqrt{2}, \sqrt{2})$, we may take $t = \pi/4$, in which case

$$\mathbf{N}(\pi/4) = \frac{3}{7\sqrt{7}}(-\sqrt{2}, 3/\sqrt{2}, 1/\sqrt{2}) + \frac{(-\sqrt{2}, -3/\sqrt{2}, -1/\sqrt{2})}{\sqrt{7}} = -\frac{4}{7\sqrt{14}}(5,3,1).$$

Hence, the principal normal at $(2\sqrt{2}, 3\sqrt{2}, \sqrt{2})$ is $\hat{\mathbf{N}} = -\dfrac{(5,3,1)}{\sqrt{35}}$. Since a tangent vector at the point is $\mathbf{T}(\pi/4) = (-\sqrt{2}, 3/\sqrt{2}, 1/\sqrt{2}) = (-2,3,1)/\sqrt{2}$, the direction of the binormal at the point is

$$\mathbf{B}(\pi/4) = (-2,3,1)\times[-(5,3,1)] = -\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -2 & 3 & 1 \\ 5 & 3 & 1 \end{vmatrix} = -(0,7,-21).$$

Thus, $\hat{\mathbf{B}}(\pi/4) = (0,-1,3)/\sqrt{10}$.

7. From $\hat{\mathbf{T}} = \dfrac{(-5,1,12t^2)}{\sqrt{26+144t^4}}$, a vector in the direction of $\hat{\mathbf{N}}$ is

$$\mathbf{N} = \frac{d\hat{\mathbf{T}}}{dt} = \frac{-576t^3}{2(26+144t^4)^{3/2}}(-5,1,12t^2) + \frac{(0,0,24t)}{\sqrt{26+144t^4}}.$$

At $(7,0,2)$, $t=-1$, in which case

$$\mathbf{N}(-1) = \frac{576}{2(170)^{3/2}}(-5,1,12) + \frac{(0,0,-24)}{\sqrt{170}} = \frac{48(-30,6,-13)}{170^{3/2}}.$$

Hence, the principal normal at $(7,0,2)$ is $\hat{\mathbf{N}} = \dfrac{(-30,6,-13)}{\sqrt{1105}}$. Since a tangent vector at the point is $\mathbf{T}(-1) = (-5,1,12)$, the direction of the binormal at the point is

$$\mathbf{B}(-1) = (-5,1,12)\times[(-30,6,-13)] = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -5 & 1 & 12 \\ -30 & 6 & -13 \end{vmatrix} = (-85,-425,0).$$

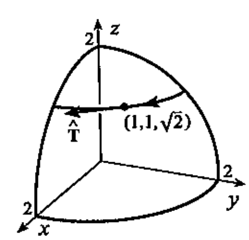
Thus, $\hat{\mathbf{B}}(-1) = -(1,5,0)/\sqrt{26}$.

8. This curve is the circle $x^2+y^2=2$ in the plane $z=\sqrt{2}$. The unit tangent and principal normal vectors at $(1,1,\sqrt{2})$ are

$$\hat{\mathbf{T}} = \frac{(1,-1,0)}{\sqrt{2}}, \qquad \hat{\mathbf{N}} = \frac{(-1,-1,0)}{\sqrt{2}}.$$

The binormal is

$$\hat{\mathbf{B}} = \hat{\mathbf{T}}\times\hat{\mathbf{N}} = \frac{1}{2}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 0 \\ -1 & -1 & 0 \end{vmatrix} = \frac{1}{2}(0,0,-2) = (0,0,-1).$$



9. With parametric equations $x = t^2+1$, $y=t$, $z=t^2+6$, a unit tangent vector to the curve is $\hat{\mathbf{T}} = \dfrac{(2t,1,2t)}{\sqrt{8t^2+1}}$. A vector in the direction of $\hat{\mathbf{N}}$ is

$$\mathbf{N} = \frac{d\hat{\mathbf{T}}}{dt} = \frac{-8t}{(1+8t^2)^{3/2}}(2t,1,2t) + \frac{(2,0,2)}{\sqrt{1+8t^2}}.$$

At $(5,2,10)$, $t=2$, in which case

$$\mathbf{N}(2) = \frac{-16}{33\sqrt{33}}(4,1,4) + \frac{(2,0,2)}{\sqrt{33}} = \frac{2(1,-8,1)}{33\sqrt{33}}.$$

Hence, the principal normal at $(5,2,10)$ is $\hat{\mathbf{N}} = \dfrac{(1,-8,1)}{\sqrt{66}}$. Since a tangent vector at the point is $\mathbf{T}(2) = (4,1,4)$, the direction of the binormal at the point is

$$\mathbf{B}(2) = (4,1,4) \times (1,-8,1) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 4 & 1 & 4 \\ 1 & -8 & 1 \end{vmatrix} = (33,0,-33).$$

Thus, $\hat{\mathbf{B}}(2) = (1,0,-1)/\sqrt{2}$.

10. With the parametric equations $x = 2\cos t$, $y = 1 - 2\sin t$, $z = 2\cos t$, (see Exercise 11.11–10),

$$\hat{\mathbf{T}} = \frac{(-2\sin t, -2\cos t, -2\sin t)}{\sqrt{4\sin^2 t + 4\cos^2 t + 4\sin^2 t}} = -\frac{(\sin t, \cos t, \sin t)}{\sqrt{1+\sin^2 t}}.$$

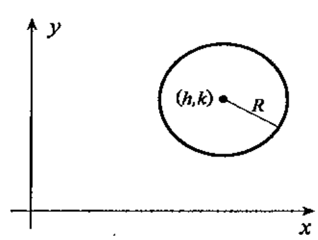
A vector in the direction of $\hat{\mathbf{N}}$ is

$$\mathbf{N} = \frac{d\hat{\mathbf{T}}}{dt} = \frac{\sin t \cos t}{(1+\sin^2 t)^{3/2}}(\sin t, \cos t, \sin t) - \frac{(\cos t, -\sin t, \cos t)}{\sqrt{1+\sin^2 t}}.$$

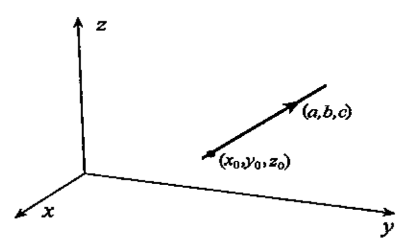
At $(2,1,2)$, $t=0$ and $\mathbf{N}(0) = -(1,0,1)$. Thus, the principal normal at $(2,1,2)$ is $\hat{\mathbf{N}}(0) = -(1,0,1)/\sqrt{2}$. Since $\hat{\mathbf{T}}(0) = -(0,1,0)$, the binormal at $(2,1,2)$ is

$$\hat{\mathbf{B}}(0) = -(0,1,0) \times \frac{-(1,0,1)}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \frac{1}{\sqrt{2}}(1,0,-1).$$

11. For a circle, the radius of curvature is the radius $\rho = R$ of the circle, and the curvature is $\kappa = 1/R$.



12. This is a straight line in space for which $\ddot{\mathbf{r}} = \mathbf{0}$, and therefore $\kappa = 0$. Its radius of curvature is undefined.

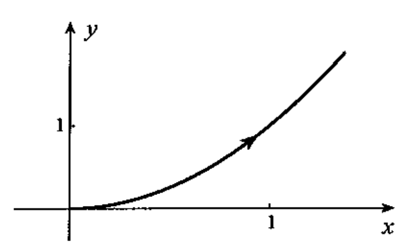


13. $$\kappa(t) = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \frac{|(1,2t,0)\times(0,2,0)|}{|(1,2t,0)|^3}$$

$$= \frac{1}{(1+4t^2)^{3/2}}\left\|\begin{matrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2t & 0 \\ 0 & 2 & 0 \end{matrix}\right\|$$

$$= \frac{|(0,0,2)|}{(1+4t^2)^{3/2}} = \frac{2}{(1+4t^2)^{3/2}}$$

and $\rho(t) = \dfrac{1}{\kappa} = \dfrac{(1+4t^2)^{3/2}}{2}$.



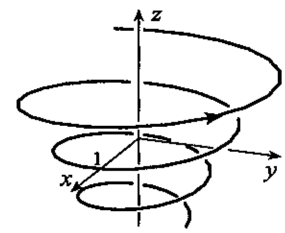
14. With $\dot{\mathbf{r}} = (e^t\cos t - e^t \sin t, e^t \sin t + e^t \cos t, 1)$ and $\ddot{\mathbf{r}} = (-2e^t \sin t, 2e^t \cos t, 0)$, we obtain

$$\dot{\mathbf{r}} \times \ddot{\mathbf{r}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ e^t(\cos t - \sin t) & e^t(\sin t + \cos t) & 1 \\ -2e^t \sin t & 2e^t \cos t & 0 \end{vmatrix} = (-2e^t\cos t, -2e^t \sin t, 2e^{2t}).$$

Consequently,

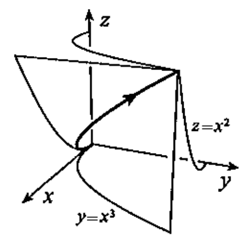
$$\kappa(t) = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \frac{\sqrt{4e^{2t}\cos^2 t + 4e^{2t}\sin^2 t + 4e^{4t}}}{[(e^t\cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2 + 1]^{3/2}}$$
$$= \frac{2e^t\sqrt{1+e^{2t}}}{(1+2e^{2t})^{3/2}},$$

and $\rho(t) = \dfrac{1}{\kappa} = \dfrac{(1+2e^{2t})^{3/2}}{2e^t\sqrt{1+e^{2t}}}$.



15. $$\kappa(t) = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \frac{|(1,3t^2,2t) \times (0,6t,2)|}{|(1,3t^2,2t)|^3}$$
$$= \frac{1}{(1+4t^2+9t^4)^{3/2}} \left\| \begin{matrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 3t^2 & 2t \\ 0 & 6t & 2 \end{matrix} \right\|$$
$$= \frac{|(-6t^2,-2,6t)|}{(1+4t^2+9t^4)^{3/2}} = \frac{2\sqrt{1+9t^2+9t^4}}{(1+4t^2+9t^4)^{3/2}}$$

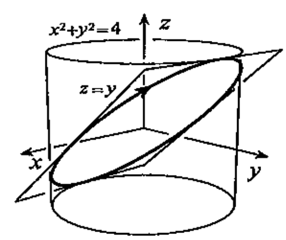
and $\rho(t) = \dfrac{1}{\kappa} = \dfrac{(1+4t^2+9t^4)^{3/2}}{2\sqrt{1+9t^2+9t^4}}$.



16. Since $\dot{\mathbf{r}} \times \ddot{\mathbf{r}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -2\sin t & 2\cos t & 2\cos t \\ -2\cos t & -2\sin t & -2\sin t \end{vmatrix} = (0,-4,4)$,

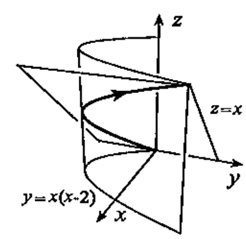
$$\kappa(t) = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \frac{\sqrt{16+16}}{(4\sin^2 t + 4\cos^2 t + 4\cos^2 t)^{3/2}}$$
$$= \frac{1}{\sqrt{2}(1+\cos^2 t)^{3/2}},$$

and $\rho(t) = \dfrac{1}{\kappa} = \sqrt{2}(1+\cos^2 t)^{3/2}$.



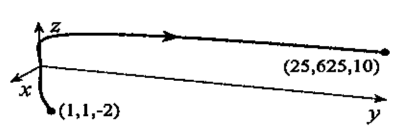
17. $$\kappa(t) = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \frac{|(1,2t,1) \times (0,2,0)|}{|(1,2t,1)|^3} = \frac{1}{(2+4t^2)^{3/2}} \left\| \begin{matrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2t & 1 \\ 0 & 2 & 0 \end{matrix} \right\|$$
$$= \frac{|(-2,0,2)|}{(2+4t^2)^{3/2}} = \frac{1}{(1+2t^2)^{3/2}}$$

and $\rho(t) = \dfrac{1}{\kappa} = (1+2t^2)^{3/2}$.



18. $$\kappa(t) = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \frac{|(2t,4t^3,2) \times (2,12t^2,0)|}{|(2t,4t^3,2)|^3} = \frac{1}{(4t^2+16t^6+4)^{3/2}} \left\| \begin{matrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2t & 4t^3 & 2 \\ 2 & 12t^2 & 0 \end{matrix} \right\|$$
$$= \frac{|(-24t^2,4,16t^3)|}{8(1+t^2+4t^6)^{3/2}} = \frac{4\sqrt{36t^4+1+16t^6}}{8(1+t^2+4t^6)^{3/2}}$$
$$= \frac{\sqrt{1+36t^4+16t^6}}{2(1+t^2+4t^6)^{3/2}},$$

and $\rho(t) = \dfrac{1}{\kappa} = \dfrac{2(1+t^2+4t^6)^{3/2}}{\sqrt{1+36t^4+16t^6}}$.



19. The curvature is a maximum at the points $(\pm a, 0)$ and a minumum at $(0, \pm b)$.

20. For a smooth curve $\mathbf{r} = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$,

$$\kappa(t) = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \frac{|(\dot{x},\dot{y},0) \times (\ddot{x},\ddot{y},0)|}{|(\dot{x},\dot{y},0)|^3} = \frac{1}{(\dot{x}^2+\dot{y}^2)^{3/2}} \left\| \begin{matrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dot{x} & \dot{y} & 0 \\ \ddot{x} & \ddot{y} & 0 \end{matrix} \right\|$$

$$= \frac{1}{(\dot{x}^2+\dot{y}^2)^{3/2}}|(0,0,\dot{x}\ddot{y}-\ddot{x}\dot{y})| = \frac{|\dot{x}\ddot{y}-\ddot{x}\dot{y}|}{(\dot{x}^2+\dot{y}^2)^{3/2}} = \frac{|\dot{y}\ddot{x}-\dot{x}\ddot{y}|}{(\dot{x}^2+\dot{y}^2)^{3/2}}.$$

21. According to equation 11.90, curvature is identically equal to zero when $0 = \kappa(s) = \left|\frac{d\hat{\mathbf{T}}}{ds}\right| = \left|\frac{d^2\mathbf{r}}{ds^2}\right|$. But for $d^2\mathbf{r}/ds^2 = \mathbf{0}$, we must have $\mathbf{r} = \mathbf{C}s + \mathbf{D}$, where $\mathbf{C}$ and $\mathbf{D}$ are constant vectors. This defines a straight line.

22. Suppose (x, y) is a point of inflection at which $f''(x) = 0$. Because

$$f''(x) = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3} \qquad \text{(see Exercise 9.1–38)},$$

it follows from Exercise 20 that at such a point of inflection, $\kappa = 0$. If (x, y) is a point of inflection at which $f''(x)$ does not exist, then neither does the curvature.

23. (a) $\hat{\mathbf{T}} = \frac{(1, 2t)}{\sqrt{1+4t^2}}$ According to Example 11.47,

$$\hat{\mathbf{N}} = \operatorname{sgn}\left[(2t)(0) - (1)(2)\right]\frac{(2t, -1)}{\sqrt{1+4t^2}} = \frac{(-2t, 1)}{\sqrt{1+4t^2}}.$$

The direction of the binormal is $\mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2t & 0 \\ -2t & 1 & 0 \end{vmatrix} = (0, 0, 1+4t^2)$. Thus, $\hat{\mathbf{B}} = (0, 0, 1)$.

(b) $F_T = \mathbf{F}\cdot\hat{\mathbf{T}} = (t^2, t^4)\cdot\frac{(1, 2t)}{\sqrt{1+4t^2}} = \frac{t^2+2t^5}{\sqrt{1+4t^2}}$; $F_N = \mathbf{F}\cdot\hat{\mathbf{N}} = (t^2, t^4)\cdot\frac{(-2t, 1)}{\sqrt{1+4t^2}} = \frac{t^4-2t^3}{\sqrt{1+4t^2}}$

(c) $\mathbf{F} = F_T\hat{\mathbf{T}} + F_N\hat{\mathbf{N}} = \frac{t^2+2t^5}{\sqrt{1+4t^2}}\hat{\mathbf{T}} + \frac{t^4-2t^3}{\sqrt{1+4t^2}}\hat{\mathbf{N}}$

24. (a) $\hat{\mathbf{T}} = \frac{(-2\sin t, 2\cos t)}{\sqrt{4\sin^2 t + 4\cos^2 t}} = (-\sin t, \cos t)$ According to Example 11.47,

$$\hat{\mathbf{N}} = \operatorname{sgn}\left[(2\cos t)(-2\cos t) - (-2\sin t)(-2\sin t)\right]\frac{(2\cos t, 2\sin t)}{\sqrt{4\sin^2 t + 4\cos^2 t}} = -(\cos t, \sin t).$$

The binormal is $\hat{\mathbf{B}} = \hat{\mathbf{T}}\times\hat{\mathbf{N}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix} = (0, 0, 1)$.

(b)
$$\begin{aligned} F_T &= \mathbf{F}\cdot\hat{\mathbf{T}} = (x^2, y^2)\cdot(-\sin t, \cos t) = -x^2\sin t + y^2\cos t \\ &= -(2\cos t)^2\sin t + (2\sin t)^2\cos t = 2\sin 2t(\sin t - \cos t) \\ F_N &= \mathbf{F}\cdot\hat{\mathbf{N}} = (x^2, y^2)\cdot(-\cos t, -\sin t) = -x^2\cos t - y^2\sin t \\ &= -(2\cos t)^2\cos t - (2\sin t)^2\sin t = -4(\cos^3 t + \sin^3 t) \end{aligned}$$

(c) $\mathbf{F} = F_T\hat{\mathbf{T}} + F_N\hat{\mathbf{N}} = 2\sin 2t(\sin t - \cos t)\hat{\mathbf{T}} - 4(\cos^3 t + \sin^3 t)\hat{\mathbf{N}}$

25.
$$\begin{aligned} F_T &= \mathbf{F}\cdot\hat{\mathbf{T}} = (t^2, 2t, -3)\cdot\frac{(1, 2t, 2t)}{\sqrt{1+8t^2}} = \frac{5t^2-6t}{\sqrt{1+8t^2}} \\ F_N &= \mathbf{F}\cdot\hat{\mathbf{N}} = (t^2, 2t, -3)\cdot\frac{(-4t, 1, 1)}{\sqrt{2+16t^2}} = \frac{-4t^3+2t-3}{\sqrt{2+16t^2}} \\ F_B &= \mathbf{F}\cdot\hat{\mathbf{B}} = (t^2, 2t, -3)\cdot\frac{(0, -1, 1)}{\sqrt{2}} = \frac{-2t-3}{\sqrt{2}} \end{aligned}$$

$$\mathbf{F} = F_T\hat{\mathbf{T}} + F_N\hat{\mathbf{N}} + F_B\hat{\mathbf{B}} = \frac{5t^2-6t}{\sqrt{1+8t^2}}\hat{\mathbf{T}} + \frac{-4t^3+2t-3}{\sqrt{2+16t^2}}\hat{\mathbf{N}} - \frac{2t+3}{\sqrt{2}}\hat{\mathbf{B}}$$

26. $\hat{\mathbf{T}} = \dfrac{(-\sin t, \cos t, 1)}{\sqrt{\sin^2 t + \cos^2 t + 1}} = \dfrac{(-\sin t, \cos t, 1)}{\sqrt{2}}$ A vector in the direction of $\hat{\mathbf{N}}$ is

$$\mathbf{N} = \frac{d\hat{\mathbf{T}}}{dt} = \frac{(-\cos t, -\sin t, 0)}{\sqrt{2}},$$

and therefore the principal normal is $\hat{\mathbf{N}} = -(\cos t, \sin t, 0)$.

$$\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}} = \frac{1}{\sqrt{2}} \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{1}{\sqrt{2}}(\sin t, -\cos t, 1).$$

$$\begin{aligned} F_T = \mathbf{F} \cdot \hat{\mathbf{T}} &= (x, xy^2, 1) \cdot \frac{(-\sin t, \cos t, 1)}{\sqrt{2}} = \frac{1}{\sqrt{2}}(-x \sin t + xy^2 \cos t + 1) \\ &= \frac{1}{\sqrt{2}}(-\cos t \sin t + \cos^2 t \sin^2 t + 1) \\ F_N = \mathbf{F} \cdot \hat{\mathbf{N}} &= (x, xy^2, 1) \cdot (-\cos t, -\sin t, 0) = -(x \cos t + xy^2 \sin t) \\ &= -(\cos^2 t + \cos t \sin^3 t) \\ F_B = \mathbf{F} \cdot \hat{\mathbf{B}} &= (x, xy^2, 1) \cdot \frac{(\sin t, -\cos t, 1)}{\sqrt{2}} = \frac{1}{\sqrt{2}}(x \sin t - xy^2 \cos t + 1) \\ &= \frac{1}{\sqrt{2}}(\cos t \sin t - \cos^2 t \sin^2 t + 1) \end{aligned}$$

$$\begin{aligned} \mathbf{F} = F_T\hat{\mathbf{T}} + F_N\hat{\mathbf{N}} + F_B\hat{\mathbf{B}} &= \frac{1}{\sqrt{2}}(1 - \cos t \sin t + \cos^2 t \sin^2 t)\hat{\mathbf{T}} \\ &- \cos t(\cos t + \sin^3 t)\hat{\mathbf{N}} + \frac{1}{\sqrt{2}}(1 + \cos t \sin t - \cos^2 t \sin^2 t)\hat{\mathbf{B}} \end{aligned}$$

27. (a) Yes
(b) Since the circle of curvature is in the plane of $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ (see Figure 11.116 and Definition 11.20), and the centre is along $\hat{\mathbf{N}}$, it follows that $\hat{\mathbf{T}}$, the unit tangent vector to the curve, must also be tangent to the circle of curvature.
(c) They have the same curvature if and only if they have the same radius of curvature. The radius of curvature of the curve is ρ. The radius of curvature of a circle is its radius, and the radius of the circle of curvature is ρ.

28. If ϕ is the angle shown, then $\hat{\mathbf{T}} = (\cos \phi, \sin \phi)$. Consequently,

$$\kappa = \left|\frac{d\hat{\mathbf{T}}}{ds}\right| = \left|\left(-\sin \phi \frac{d\phi}{ds}, \cos \phi \frac{d\phi}{ds}\right)\right| = \left|\frac{d\phi}{ds}\right|.$$

29. Since $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$, and $\hat{\mathbf{B}}$ form a set of three mutually perpendicular vectors at each point on a curve, there is no question that $\hat{\mathbf{N}}$ must be in the same direction as, or opposite to, $\hat{\mathbf{B}} \times \hat{\mathbf{T}}$; that is, $\hat{\mathbf{N}} = \lambda(\hat{\mathbf{B}} \times \hat{\mathbf{T}})$. The scalar product of this result with $\hat{\mathbf{N}}$ gives $1 = \hat{\mathbf{N}} \cdot \hat{\mathbf{N}} = \lambda(\hat{\mathbf{N}} \cdot \hat{\mathbf{B}} \times \hat{\mathbf{T}})$. But the scalar triple product $\hat{\mathbf{N}} \cdot \hat{\mathbf{B}} \times \hat{\mathbf{T}}$ is the volume of the rectangular parallelepiped formed by the vectors $\hat{\mathbf{N}}$, $\hat{\mathbf{B}}$, and $\hat{\mathbf{T}}$ (see Exercise 11.4–44). Since this volume is 1, we obtain $\lambda = 1$. Thus, $\hat{\mathbf{N}} = \hat{\mathbf{B}} \times \hat{\mathbf{T}}$, and

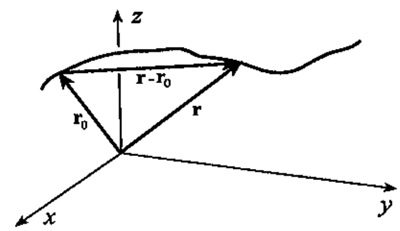
$$\frac{d\hat{\mathbf{N}}}{ds} = \frac{d\hat{\mathbf{B}}}{ds} \times \hat{\mathbf{T}} + \hat{\mathbf{B}} \times \frac{d\hat{\mathbf{T}}}{ds} = -\tau\hat{\mathbf{N}} \times \hat{\mathbf{T}} + \hat{\mathbf{B}} \times \kappa\hat{\mathbf{N}} = -\tau(\hat{\mathbf{N}} \times \hat{\mathbf{T}}) + \kappa(\hat{\mathbf{B}} \times \hat{\mathbf{N}}).$$

Now $\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}$ so that $-\hat{\mathbf{B}} = \hat{\mathbf{N}} \times \hat{\mathbf{T}}$. By a similar proof to the above we can show that $\hat{\mathbf{T}} = \hat{\mathbf{N}} \times \hat{\mathbf{B}}$. Hence, $\dfrac{d\hat{\mathbf{N}}}{ds} = -\tau(-\hat{\mathbf{B}}) + \kappa(-\hat{\mathbf{T}}) = \tau\hat{\mathbf{B}} - \kappa\hat{\mathbf{T}}$.

30. According to the second Frenet-Serret formula, torsion along a curve vanishes if and only if the binormal vector is constant. We therefore prove that a curve lies in a plane if and only if its binormal is constant. If a curve lies in a plane, then $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ will both lie in the plane. It follows that $\hat{\mathbf{B}}$ will be perpendicular to the plane, and will always therefore be in the same direction. Since its length is one, $\hat{\mathbf{B}}$ must be a constant vector.

Conversely, suppose now that $\hat{\mathbf{B}}$ is constant along a curve. Let $\mathbf{r}(s)$ denote the position vector for points along the curve where s is length along the curve measured from some initial point denoted by $\mathbf{r}_0 = \mathbf{r}(0)$. Consider

$$\frac{d}{ds}[\hat{\mathbf{B}} \cdot (\mathbf{r} - \mathbf{r}_0)] = \frac{d\hat{\mathbf{B}}}{ds} \cdot (\mathbf{r} - \mathbf{r}_0) + \hat{\mathbf{B}} \cdot \frac{d\mathbf{r}}{ds}$$
$$= 0 + \hat{\mathbf{B}} \cdot \hat{\mathbf{T}} = 0.$$



This implies that $\hat{\mathbf{B}} \cdot (\mathbf{r} - \mathbf{r}_0)$ is constant along the curve. Since it has value 0 at $\mathbf{r} = \mathbf{r}_0$, it has value 0 everywhere. This means that $\mathbf{r} - \mathbf{r}_0$ is perpendicular to $\hat{\mathbf{B}}$ for every point on the curve. In other words, the curve must lie in the plane containing $\mathbf{r}_0$ and perpendicular to $\hat{\mathbf{B}}$.

EXERCISES 11.13

1. $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = \dfrac{t}{\sqrt{t^2+1}}\hat{\mathbf{i}} + \dfrac{2t^2+1}{\sqrt{t^2+1}}\hat{\mathbf{j}} = \dfrac{t\hat{\mathbf{i}} + (2t^2+1)\hat{\mathbf{j}}}{\sqrt{t^2+1}}$; $\quad |\mathbf{v}| = \dfrac{1}{\sqrt{t^2+1}}\sqrt{t^2+(2t^2+1)^2} = \sqrt{\dfrac{4t^4+5t^2+1}{t^2+1}}$

$\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{1}{(t^2+1)^{3/2}}\hat{\mathbf{i}} + \dfrac{2t^3+3t}{(t^2+1)^{3/2}}\hat{\mathbf{j}} = \dfrac{\hat{\mathbf{i}} + (2t^3+3t)\hat{\mathbf{j}}}{(t^2+1)^{3/2}}$

2. $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = \left(1 - \dfrac{1}{t^2}\right)\hat{\mathbf{i}} + \left(1 + \dfrac{1}{t^2}\right)\hat{\mathbf{j}} = \dfrac{1}{t^2}[(t^2-1)\hat{\mathbf{i}} + (t^2+1)\hat{\mathbf{j}}]$

$|\mathbf{v}| = \dfrac{1}{t^2}\sqrt{(t^2-1)^2+(t^2+1)^2} = \dfrac{\sqrt{2t^4+2}}{t^2}$; $\quad \mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{2}{t^3}\hat{\mathbf{i}} - \dfrac{2}{t^3}\hat{\mathbf{j}} = \dfrac{2}{t^3}(\hat{\mathbf{i}} - \hat{\mathbf{j}})$

3. $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = \cos t\,\hat{\mathbf{i}} - 3\sin t\,\hat{\mathbf{j}} + \cos t\,\hat{\mathbf{k}}$; $\quad |\mathbf{v}| = \sqrt{\cos^2 t + 9\sin^2 t + \cos^2 t} = \sqrt{2 + 7\sin^2 t}$;

$\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = -\sin t\,\hat{\mathbf{i}} - 3\cos t\,\hat{\mathbf{j}} - \sin t\,\hat{\mathbf{k}}$

4. $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = 2t\hat{\mathbf{i}} + 2e^t(t+1)\hat{\mathbf{j}} - \dfrac{2}{t^3}\hat{\mathbf{k}}$; $\quad |\mathbf{v}| = \sqrt{4t^2 + 4e^{2t}(t+1)^2 + 4/t^6} = 2\sqrt{t^2 + e^{2t}(t+1)^2 + 1/t^6}$;

$\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = 2\hat{\mathbf{i}} + 2e^t(t+2)\hat{\mathbf{j}} + \dfrac{6}{t^4}\hat{\mathbf{k}}$

5. $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = -2te^{-t^2}\hat{\mathbf{i}} + (\ln t + 1)\hat{\mathbf{j}}$; $\quad |\mathbf{v}| = \sqrt{4t^2e^{-2t^2} + (\ln t + 1)^2}$; $\quad \mathbf{a} = \dfrac{d\mathbf{v}}{dt} = 2e^{-t^2}(2t^2-1)\hat{\mathbf{i}} + (1/t)\hat{\mathbf{j}}$

6. If $\mathbf{a} = 3t^2\hat{\mathbf{i}} + (t+1)\hat{\mathbf{j}} - 4t^3\hat{\mathbf{k}}$, then $\mathbf{v} = t^3\hat{\mathbf{i}} + \left(\dfrac{t^2}{2} + t\right)\hat{\mathbf{j}} - t^4\hat{\mathbf{k}} + \mathbf{C}$. Since $\mathbf{v}(0) = \mathbf{0}$, it follows that $\mathbf{C} = \mathbf{0}$, and $\mathbf{v} = t^3\hat{\mathbf{i}} + \left(\dfrac{t^2}{2} + t\right)\hat{\mathbf{j}} - t^4\hat{\mathbf{k}}$. Integration now gives $\mathbf{r} = \dfrac{t^4}{4}\hat{\mathbf{i}} + \left(\dfrac{t^3}{6} + \dfrac{t^2}{2}\right)\hat{\mathbf{j}} - \dfrac{t^5}{5}\hat{\mathbf{k}} + \mathbf{D}$. Since $\mathbf{r}(0) = (1,2,-1)$, we find $(1,2,-1) = \mathbf{D}$, and $\mathbf{r} = \left(\dfrac{t^4}{4} + 1\right)\hat{\mathbf{i}} + \left(\dfrac{t^3}{6} + \dfrac{t^2}{2} + 2\right)\hat{\mathbf{j}} - \left(\dfrac{t^5}{5} + 1\right)\hat{\mathbf{k}}$.

7. If $\mathbf{a} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}}/(t+1)^3$, then $\mathbf{v} = 3t\hat{\mathbf{i}} - \dfrac{\hat{\mathbf{j}}}{2(t+1)^2} + \mathbf{C}$. Since $\mathbf{v}(0) = \mathbf{0}$, it follows that $\mathbf{0} = -\hat{\mathbf{j}}/2 + \mathbf{C}$, and $\mathbf{v} = 3t\hat{\mathbf{i}} - \dfrac{\hat{\mathbf{j}}}{2(t+1)^2} + \dfrac{\hat{\mathbf{j}}}{2}$. Integration now gives $\mathbf{r} = \dfrac{3t^2}{2}\hat{\mathbf{i}} + \dfrac{\hat{\mathbf{j}}}{2(t+1)} + \dfrac{t\hat{\mathbf{j}}}{2} + \mathbf{D}$. Since $\mathbf{r}(0) = (1,2,-1)$, we find $(1,2,-1) = \hat{\mathbf{j}}/2 + \mathbf{D} \implies \mathbf{D} = (1,3/2,-1)$. Thus, $\mathbf{r} = \left(\dfrac{3t^2}{2} + 1\right)\hat{\mathbf{i}} + \left(\dfrac{1}{2t+2} + \dfrac{t}{2} + \dfrac{3}{2}\right)\hat{\mathbf{j}} - \hat{\mathbf{k}}$.

8. The velocity and acceleration are $\mathbf{v} = \hat{\mathbf{i}} + 2t\hat{\mathbf{j}}$ and $\mathbf{a} = 2\hat{\mathbf{j}}$. According to equation 11.112b, the tangential component of acceleration is

$$a_T = \frac{d}{dt}|\mathbf{v}| = \frac{d}{dt}\sqrt{1+4t^2} = \frac{4t}{\sqrt{1+4t^2}}.$$

According to equation 11.113, the normal component of acceleration is

$$a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{4 - \frac{16t^2}{1+4t^2}} = \sqrt{\frac{4}{1+4t^2}} = \frac{2}{\sqrt{1+4t^2}}.$$

9. The velocity and acceleration are $\mathbf{v} = -\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{a} = -\cos t\,\hat{\mathbf{i}} - \sin t\,\hat{\mathbf{j}}$. According to equation 11.112b, the tangential component of acceleration is

$$a_T = \frac{d}{dt}|\mathbf{v}| = \frac{d}{dt}\sqrt{\sin^2 t + \cos^2 t + 1} = 0.$$

According to equation 11.113, the normal component of acceleration is

$$a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{\cos^2 t + \sin^2 t} = 1.$$

10. From equation 11.112b, $a_N = |\mathbf{v}|\left|\frac{d\hat{\mathbf{T}}}{dt}\right| = |\mathbf{v}|\left|\frac{d\hat{\mathbf{T}}}{ds}\right|\left|\frac{ds}{dt}\right| = |\mathbf{v}|\kappa|\mathbf{v}| = \frac{|\mathbf{v}|^2}{\rho}.$

11. (a) $\text{KE} = \frac{1}{2}m|\mathbf{v}|^2 = \frac{1}{2}\left(\frac{2}{1000}\right)\left(\frac{4t^4+5t^2+1}{t^2+1}\right) = \frac{4t^4+5t^2+1}{1000(t^2+1)}$ J

(b) $\text{KE} = \frac{1}{2}m|\mathbf{v}|^2 = \frac{1}{2}\left(\frac{2}{1000}\right)\left(\frac{2t^4+2}{t^4}\right) = \frac{t^4+1}{500t^4}$ J

(c) $\text{KE} = \frac{1}{2}m|\mathbf{v}|^2 = \frac{1}{2}\left(\frac{2}{1000}\right)(2+7\sin^2 t) = \frac{2+7\sin^2 t}{1000}$ J

(d) $\text{KE} = \frac{1}{2}m|\mathbf{v}|^2 = \frac{1}{2}\left(\frac{2}{1000}\right)4\left[t^2 + e^{2t}(t+1)^2 + \frac{1}{t^6}\right] = \frac{1}{250}\left[t^2 + e^{2t}(t+1)^2 + \frac{1}{t^6}\right]$ J

(e) $\text{KE} = \frac{1}{2}m|\mathbf{v}|^2 = \frac{1}{2}\left(\frac{2}{1000}\right)\left[4t^2e^{-2t^2} + (\ln t+1)^2\right] = \frac{4t^2e^{-t^2} + (\ln t+1)^2}{1000}$ J

12. (a) If $d^2y/dt^2 = 2$, then $dy/dt = 2t + C$. Since $y'(0) = 0$, $C = 0$, and $dy/dt = 2t$. Thus, $y(t) = t^2 + D$. Since $y(0) = 0$, $D = 0$, and $y(t) = t^2$. Consequently, $4t^2 = x^2$, from which $x = 2t$. Thus, $dx/dt = 2$ and $d^2x/dt^2 = 0$.

(b) If $d^2x/dt^2 = 24t^2$, then $dx/dt = 8t^3 + C$. Since $x'(0) = 0$, $C = 0$ and $dx/dt = 8t^3$. Hence, $x(t) = 2t^4 + D$. Since $x(0) = 0$, $D = 0$ and $x(t) = 2t^4$. From $4y = x^2$, we obtain $y = x^2/4 = t^8$. Differentiation now gives $dy/dt = 8t^7$ and $d^2y/dt^2 = 56t^6$.

13. Velocity and displacement will be parallel if for some value of λ,

$$\mathbf{v} = \lambda\mathbf{r} \quad \Longrightarrow \quad \mathbf{v} = \hat{\mathbf{i}} + (3t^2 - 6t + 2)\hat{\mathbf{j}} = \lambda[t\hat{\mathbf{i}} + (t^3 - 3t^2 + 2t)\hat{\mathbf{j}}].$$

When we equate components, $1 = \lambda t$, $3t^2 - 6t + 2 = \lambda(t^3 - 3t^2 + 2t)$. Substituting $\lambda = 1/t$ into the second leads to the equation $2t^2 - 3t = 0$ with solutions $t = 0, 3/2$. Since $\mathbf{r}(0) = \mathbf{0}$, we cannot discuss parallelism at $t = 0$. The position of the particle at $t = 3/2$ is $(3/2, -3/8)$.

14. If $dx/dt = 5$, then $dy/dt = 3x^2\,dx/dt - 2dx/dt = 5(3x^2 - 2)$. The acceleration of the particle is therefore

$$\mathbf{a} = \frac{d^2x}{dt^2}\hat{\mathbf{i}} + \frac{d^2y}{dt^2}\hat{\mathbf{j}} = (0)\hat{\mathbf{i}} + 5\left(6x\frac{dx}{dt}\right)\hat{\mathbf{j}} = 30x(5)\hat{\mathbf{j}} = 150x\hat{\mathbf{j}}.$$

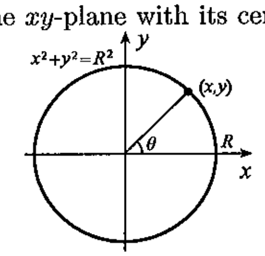
15. Antidifferentiation of the acceleration gives $\mathbf{v}(t) = -t^5\hat{\mathbf{i}} - \left(\frac{t^4}{2} + t\right)\hat{\mathbf{j}} + \mathbf{C}$. Since $\mathbf{v}(0) = \mathbf{0}$, it follows that $\mathbf{C} = \mathbf{0}$. The speed of the particle at $t = 2$ is $|\mathbf{v}(2)| = \sqrt{2^{10} + 100} = 2\sqrt{281}$.

16. Parametric equations for the particle's path are $x = h + R\cos\theta$, $y = k + R\sin\theta$, in which case $\mathbf{v} = \left(-R\sin\theta\frac{d\theta}{dt}\right)\hat{\mathbf{i}} + \left(R\cos\theta\frac{d\theta}{dt}\right)\hat{\mathbf{j}}$. Thus, $|\mathbf{v}| = \sqrt{R^2\sin^2\theta\left(\frac{d\theta}{dt}\right)^2 + R^2\cos^2\theta\left(\frac{d\theta}{dt}\right)^2} = R\frac{d\theta}{dt} = R\omega$.

17. If we choose $t \geq 0$ and $t = 0$ when the particle is at the point $(2, 0)$, its position can be described by $x = 2\cos(4\pi t)$, $y = 2\sin(4\pi t)$. It is at the point $(1, -\sqrt{3})$ when $1 = 2\cos(4\pi t)$, and $-\sqrt{3} = 2\sin(4\pi t)$. This happens for the first time at $t = 5/12$ s. The velocity of the particle at this time is

$$\mathbf{v}(5/12) = -8\pi\sin\left(\frac{5\pi}{3}\right)\hat{\mathbf{i}} + 8\pi\cos\left(\frac{5\pi}{3}\right)\hat{\mathbf{j}} = 4\pi(\sqrt{3}\hat{\mathbf{i}} + \hat{\mathbf{j}}) \text{ m/s.}$$

18. (a) We choose a coordinate system so that the circle lies in the xy-plane with its centre at the origin. Then, $x = R\cos\theta$, $y = R\sin\theta$, and



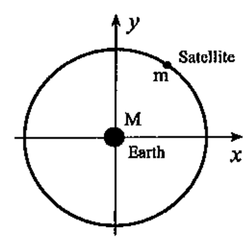
$$\mathbf{v} = \left(-R\sin\theta\frac{d\theta}{dt}, R\cos\theta\frac{d\theta}{dt}\right) = \omega R(-\sin\theta, \cos\theta),$$

where $\omega = d\theta/dt$. Since $|\mathbf{v}| = \omega R$ (see Exercise 16), and $|\mathbf{v}|$ is constant, so also is ω. Hence,

$$\mathbf{a} = \omega R\left(-\cos\theta\frac{d\theta}{dt}, -\sin\theta\frac{d\theta}{dt}\right) = -\omega^2 R(\cos\theta, \sin\theta),$$

and $|\mathbf{a}| = \omega^2 R = \left(\frac{|\mathbf{v}|^2}{R^2}\right) R = \frac{|\mathbf{v}|^2}{R}$.

(b) According to Newton's universal law of gravitation, the magnitude of the force of the earth on the satellite is



$$|\mathbf{F}| = \frac{GMm}{r^2}.$$

According to Newton's second law, $\mathbf{F} = m\mathbf{a}$, and therefore

$$m|\mathbf{a}| = \frac{GMm}{r^2} \quad \Longrightarrow \quad |\mathbf{a}| = \frac{GM}{r^2}.$$

But from part (a), $|\mathbf{a}| = |\mathbf{v}|^2/r$, and therefore $\frac{|\mathbf{v}|^2}{r} = \frac{GM}{r^2}$, from which

$$|\mathbf{v}| = \sqrt{\frac{GM}{r}} = \sqrt{\frac{6.67 \times 10^{-11}(4/3)\pi(6370 \times 10^3)^3(5.52 \times 10^3)}{6570 \times 10^3}} = 7.79 \times 10^3.$$

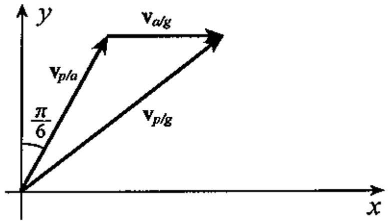
The speed of the satellite is therefore 7.79 km/s.

19. Differentiation of $\mathbf{OP}_1 + \mathbf{P}_1\mathbf{P}_2 = \mathbf{OP}_2$ gives

$$\frac{d\mathbf{OP}_1}{dt} + \frac{d\mathbf{P}_1\mathbf{P}_2}{dt} = \frac{d\mathbf{OP}_2}{dt} \quad \Longrightarrow \quad \mathbf{v}_{P_1/O} + \mathbf{v}_{P_2/P_1} = \mathbf{v}_{P_2/O}.$$

Since $\mathbf{v}_{P_2/O} = -\mathbf{v}_{O/P_2}$ and $\mathbf{v}_{P_2/P_1} = -\mathbf{v}_{P_1/P_2}$, the above equation can indeed be expressed in the form $\mathbf{v}_{P_1/O} + \mathbf{v}_{O/P_2} = \mathbf{v}_{P_1/P_2}$.

20. Let $\mathbf{v}_{\mathrm{p/a}}$ be the velocity of the plane with respect to the air, $\mathbf{v}_{\mathrm{a/g}}$ the velocity of the air with respect to the ground, and $\mathbf{v}_{\mathrm{p/g}}$ the velocity of the plane with respect to the ground. According to Exercise 19,



$$\mathbf{v}_{\mathrm{p/a}} + \mathbf{v}_{\mathrm{a/g}} = \mathbf{v}_{\mathrm{p/g}},$$

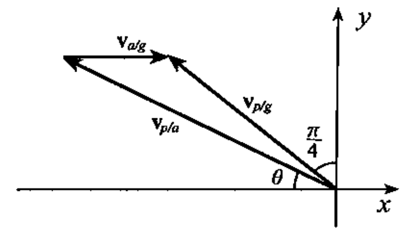
where $\mathbf{v}_{\mathrm{p/a}} = 650[\cos(\pi/3)\hat{\mathbf{i}} + \sin(\pi/3)\hat{\mathbf{j}}] = 325(\hat{\mathbf{i}} + \sqrt{3}\hat{\mathbf{j}})$, and $\mathbf{v}_{\mathrm{a/g}} = 40\hat{\mathbf{i}}$. Thus,

$$\mathbf{v}_{\mathrm{p/g}} = 325(\hat{\mathbf{i}} + \sqrt{3}\hat{\mathbf{j}}) + 40\hat{\mathbf{i}} = 365\hat{\mathbf{i}} + 325\sqrt{3}\hat{\mathbf{j}} \text{ km/hr},$$

and $|\mathbf{v}_{\mathrm{p/g}}| = \sqrt{365^2 + (325\sqrt{3})^2} = 670.9$ km/hr.

21. Let $\mathbf{v}_{\text{p/a}}$ be the velocity of the plane with respect to the air, $\mathbf{v}_{\text{a/g}}$ the velocity of the air with respect to the ground, and $\mathbf{v}_{\text{p/g}}$ the velocity of the plane with respect to the ground. According to Exercise 19, $\mathbf{v}_{\text{p/a}} + \mathbf{v}_{\text{a/g}} = \mathbf{v}_{\text{p/g}}$, where

$$\mathbf{v}_{\text{p/a}} = 600[-\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}], \qquad \mathbf{v}_{\text{a/g}} = 50\hat{\mathbf{i}},$$
$$\mathbf{v}_{\text{p/g}} = v[-\cos(\pi/4)\hat{\mathbf{i}} + \sin(\pi/4)\hat{\mathbf{j}}] = \frac{v}{\sqrt{2}}(-\hat{\mathbf{i}} + \hat{\mathbf{j}}),$$



where v is the speed of the plane with respect to the ground. When we substitute these into the above equation

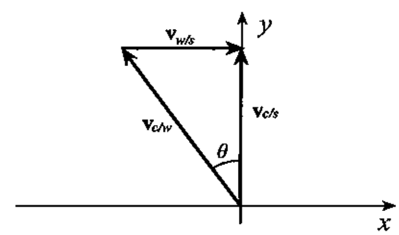
$$600(-\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) + 50\hat{\mathbf{i}} = \frac{v}{\sqrt{2}}(-\hat{\mathbf{i}} + \hat{\mathbf{j}}),$$

and equate components, $-600\cos\theta + 50 = -\dfrac{v}{\sqrt{2}}$, $600\sin\theta = \dfrac{v}{\sqrt{2}}$. Eliminating v leads to the equation $\cos\theta - \sin\theta = 1/12$, which we square

$$\cos^2\theta - 2\sin\theta\cos\theta + \sin^2\theta = \frac{1}{144} \quad\Longrightarrow\quad \sin 2\theta = \frac{143}{144}.$$

The appropriate angle satisfying this equation (between 0 and $\pi/4$) is 0.726 radians. The plane should therefore take the bearing of west 0.726 radians north. Ground speed of the plane is $600\sqrt{2}\sin(0.726)$, which when divided into 1000 results in a trip time of 1.8 hours.

22. Let us take the positive x-direction as the direction in which the river flows. Then the velocity of the water relative to the shore is $\mathbf{v}_{\text{w/s}} = 3\hat{\mathbf{i}}$. If the canoe points in direction θ as shown, $\mathbf{v}_{\text{c/w}} = -4\sin\theta\,\hat{\mathbf{i}} + 4\cos\theta\,\hat{\mathbf{j}}$. If v is the speed of the canoe with respect to the shore, then $\mathbf{v}_{\text{c/s}} = v\hat{\mathbf{j}}$. Since $\mathbf{v}_{\text{c/s}} = \mathbf{v}_{\text{c/w}} + \mathbf{v}_{\text{w/s}}$, we obtain $v\,\hat{\mathbf{j}} = -4\sin\theta\,\hat{\mathbf{i}} + 4\cos\theta\,\hat{\mathbf{j}} + 3\hat{\mathbf{i}}$. When



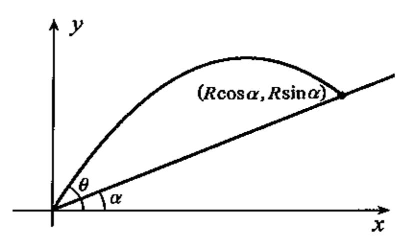
we equate components, $-4\sin\theta + 3 = 0$ and $4\cos\theta = v$. Consequently, $\theta = \text{Sin}^{-1}(3/4)$ radians, and $v = 4\sqrt{1 - 9/16} = \sqrt{7}$ km/hr. To cross the river takes $(200/1000)/v = 1/(5\sqrt{7})$ hr or $12/\sqrt{7}$ min.

23. (a) The acceleration of the cannon ball is $\mathbf{a} = -g\hat{\mathbf{j}}$ so that $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{C}$. If we take as $t = 0$ the time when the cannon ball leaves the cannon, then when the cannon ball is fired at angle θ,

$$\mathbf{v}(0) = S(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}),$$

from which

$$S(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) = \mathbf{C}.$$



Integration of $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{C}$ gives $\mathbf{r} = -gt^2\hat{\mathbf{j}}/2 + \mathbf{C}t + \mathbf{D}$. Since $\mathbf{r}(0) = \mathbf{0}$, it follows that $\mathbf{D} = \mathbf{0}$, and

$$\mathbf{r} = -\frac{1}{2}gt^2\hat{\mathbf{j}} + St(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) = (St\cos\theta)\hat{\mathbf{i}} + \left(-\frac{1}{2}gt^2 + St\sin\theta\right)\hat{\mathbf{j}}.$$

The cannon ball strikes the ground at position $(R\cos\alpha, R\sin\alpha)$. When we equate $R\cos\alpha$ and $R\sin\alpha$ to the x- and y-components of the displacement vector,

$$R\cos\alpha = St\cos\theta, \qquad R\sin\alpha = -\frac{1}{2}gt^2 + St\sin\theta.$$

Substituting $t = R\cos\alpha/(S\cos\theta)$ into the second gives

$$R\sin\alpha = -\frac{1}{2}g\left(\frac{R\cos\alpha}{S\cos\theta}\right)^2 + S\left(\frac{R\cos\alpha}{S\cos\theta}\right)\sin\theta.$$

When this is solved for R, the required expression is obtained.
(b) The function $R(\theta)$ must be maximized on the interval $\alpha \leq \theta \leq \pi/2$. For critical points of the function, we solve

$$0 = R'(\theta) = \frac{2S^2}{g\cos^2\alpha}[-\sin\theta\sin(\theta-\alpha) + \cos\theta\cos(\theta-\alpha)] = \frac{2S^2}{g\cos^2\alpha}\cos(2\theta-\alpha).$$

The only solution of this equation in the specified interval is $\theta = \pi/4 + \alpha/2$. Since $\theta = \alpha$ and $\theta = \pi/2$ leads to $R = 0$, maximum range is obtained for $\theta = \pi/4 + \alpha/2$.

24. If its constant acceleration is $\mathbf{a}$, then $\mathbf{v} = \mathbf{a}t + \mathbf{C}$. If the particle starts at $t = 0$, then $\mathbf{v}(0) = \mathbf{0}$, so that $\mathbf{C} = \mathbf{0}$, and $\mathbf{v} = \mathbf{a}t$. Integration now gives $\mathbf{r} = \mathbf{a}t^2/2 + \mathbf{D}$. Since $\mathbf{r}(0) = (1,2,3)$, $(1,2,3) = \mathbf{D}$, and $\mathbf{r} = \mathbf{a}t^2/2 + (1,2,3)$. For the particle to be at $(4,5,7)$ when $t = 2$,

$$(4,5,7) = \frac{1}{2}\mathbf{a}(2)^2 + (1,2,3) \quad\Longrightarrow\quad \mathbf{a} = \frac{1}{2}[(4,5,7)-(1,2,3)] = \left(\frac{3}{2},\frac{3}{2},2\right).$$

25. Since $\hat{\mathbf{T}} = \dfrac{(2t,4t,2t+5)}{\sqrt{4t^2+16t^2+(2t+5)^2}} = \dfrac{(2t,4t,2t+5)}{\sqrt{24t^2+20t+25}}$,

$$\begin{aligned}\frac{d\hat{\mathbf{T}}}{dt} &= \frac{-(24t+10)}{(24t^2+20t+25)^{3/2}}(2t,4t,2t+5) + \frac{(2,4,2)}{\sqrt{24t^2+20t+5}}\\ &= \frac{1}{(24t^2+20t+25)^{3/2}}[-(24t+10)(2t,4t,2t+5) + (24t^2+20t+25)(2,4,2)]\\ &= \frac{10(2t+5,4t+10,-10t)}{(24t^2+20t+25)^{3/2}}.\end{aligned}$$

According to 11.112b,

$$\begin{aligned}a_N &= |(2t,4t,2t+5)|\left|\frac{10(2t+5,4t+10,-10t)}{(24t^2+20t+25)^{3/2}}\right|\\ &= \sqrt{24t^2+20t+25}\left[\frac{10\sqrt{(2t+5)^2+(4t+10)^2+100t^2}}{(24t^2+20t+25)^{3/2}}\right]\\ &= \frac{10\sqrt{5}}{\sqrt{24t^2+20t+25}}.\end{aligned}$$

Since $\mathbf{v} = (2t,4t,2t+5)$ and $\mathbf{a} = (2,4,2)$,

$$a_T = \frac{d}{dt}|\mathbf{v}| = \frac{d}{dt}\sqrt{24t^2+20t+25} = \frac{24t+10}{\sqrt{24t^2+20t+25}}.$$

Equation 11.113 gives $a_N = \sqrt{24 - \dfrac{(24t+10)^2}{24t^2+20t+25}} = \dfrac{10\sqrt{5}}{\sqrt{24t^2+20t+25}}$.

26. Since $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = \dfrac{-t}{\sqrt{1-t^2}}\hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{a} = \left[\dfrac{-1}{\sqrt{1-t^2}} - \dfrac{t^2}{(1-t^2)^{3/2}}\right]\hat{\mathbf{i}} = \dfrac{-1}{(1-t^2)^{3/2}}\hat{\mathbf{i}}$, acceleration and velocity are perpendicular if, and when, $0 = \mathbf{v}\cdot\mathbf{a} = \left[\dfrac{-t}{\sqrt{1-t^2}}\right]\left[\dfrac{-1}{(1-t^2)^{3/2}}\right]$. This only occurs at $t = 0$ when the particle is at position $(3,0)$.

27. (a)(i) When $\mathbf{a}$ is constant, integration of $d\mathbf{v}/dt = \mathbf{a}$ gives $\mathbf{v} = \mathbf{a}t + \mathbf{C}$. If we take $t = 0$ as initial time, then $\mathbf{v}(0) = \mathbf{0}$, and this implies that $\mathbf{C} = \mathbf{0}$. Integration of $d\mathbf{r}/dt = \mathbf{a}t$ gives $\mathbf{r} = (1/2)\mathbf{a}t^2 + \mathbf{D}$. When we equate components of this equation,

$$x = \frac{1}{2}a_x t^2 + D_x, \quad y = \frac{1}{2}a_y t^2 + D_y, \quad z = \frac{1}{2}a_z t^2 + D_z.$$

Solving each for $t^2/2$ and equating results gives $\dfrac{x-D_x}{a_x}=\dfrac{y-D_y}{a_y}=\dfrac{z-D_z}{a_z}$. These are symmetric equations for a line.
(ii) If the acceleration is constant, then (as in part (a)), $\mathbf{v}=\mathbf{a}t+\mathbf{C}$. If $\mathbf{v}(0)=\mathbf{V}$, then $\mathbf{v}=\mathbf{a}t+\mathbf{V}$. Integration of this gives $\mathbf{r}=(1/2)\mathbf{a}t^2+\mathbf{V}t+\mathbf{D}$. When we equate components of this equation,

$$x=\frac{1}{2}a_xt^2+V_xt+D_x,\quad y=\frac{1}{2}a_yt^2+V_yt+D_y,\quad z=\frac{1}{2}a_zt^2+V_zt+D_z.$$

Since $\mathbf{V}$ and $\mathbf{a}$ are parallel, we can say that $\mathbf{a}=\lambda\mathbf{V}$; that is, $a_x=\lambda V_x$, $a_y=\lambda V_y$, and $a_z=\lambda V_z$. Thus,

$$x=\frac{1}{2}\lambda V_xt^2+V_xt+D_x,\quad y=\frac{1}{2}\lambda V_yt^2+V_yt+D_y,\quad z=\frac{1}{2}\lambda V_zt^2+V_zt+D_z.$$

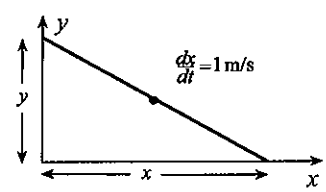
Solving each of these for $\lambda t^2/2+t$ and equating results gives $\dfrac{x-D_x}{V_x}=\dfrac{y-D_y}{V_y}=\dfrac{z-D_z}{V_z}$. These are symmetric equations for a line.
(b) No. A stone thrown at an angle to the ground is subjected to the constant acceleration due to gravity, but it follows a parabolic path.

28. Because the acceleration $\mathbf{a}$ is constant, $\mathbf{v}=\mathbf{a}t+\mathbf{C}$. Since $\mathbf{v}(t_0)=\mathbf{v}_0$, it follows that $\mathbf{v}_0=\mathbf{a}t_0+\mathbf{C}$, and $\mathbf{v}=\mathbf{a}t+\mathbf{v}_0-\mathbf{a}t_0=\mathbf{a}(t-t_0)+\mathbf{v}_0$. Integration now gives $\mathbf{r}=\mathbf{a}(t-t_0)^2/2+\mathbf{v}_0t+\mathbf{D}$. Since $\mathbf{r}(t_0)=\mathbf{r}_0$, we obtain $\mathbf{r}_0=\mathbf{v}_0t_0+\mathbf{D}$, and therefore

$$\mathbf{r}=\frac{1}{2}\mathbf{a}(t-t_0)^2+\mathbf{v}_0t+\mathbf{r}_0-\mathbf{v}_0t_0=\frac{1}{2}\mathbf{a}(t-t_0)^2+\mathbf{v}_0(t-t_0)+\mathbf{r}_0.$$

29. (a) Coordinates of the middle point of the ladder are $(x/2, y/2)$. Differentiation of $x^2+y^2=64$ with respect to time t gives

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0 \Longrightarrow \frac{dy}{dt}=-\frac{x}{y}\frac{dx}{dt}.$$

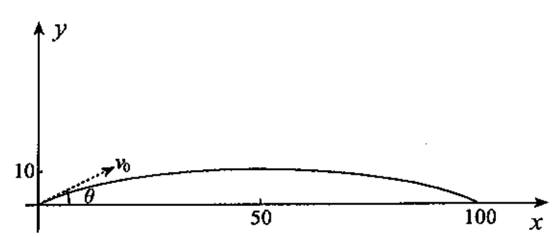


When $x=3$ and $y=\sqrt{55}$, we find that $\dfrac{dy}{dt}=-\dfrac{3}{\sqrt{55}}$.
The velocity of the midpoint of the ladder is therefore $\dfrac{1}{2}\left(\hat{\mathbf{i}}-\dfrac{3\hat{\mathbf{j}}}{\sqrt{55}}\right)$ m/s. Since dx/dt is constant, the x-component of the acceleration is always 0. Furthermore,

$$\frac{d^2y}{dt^2}=-\left[\frac{y\left(\frac{dx}{dt}\right)-x\left(\frac{dy}{dt}\right)}{y^2}\right]\frac{dx}{dt}.$$

When $x=3$, $\dfrac{d^2y}{dt^2}=-\left[\dfrac{\sqrt{55}(1)-3(-3/\sqrt{55})}{55}\right](1)=-\dfrac{64}{55^{3/2}}$. Thus, the acceleration of the midpoint of the ladder is $-32\hat{\mathbf{j}}/55^{3/2}$ m/s^2.
(b) The limit of dy/dt as y approaches 0 is "infinity"; that is, the midpoint of the ladder strikes the floor infinitely fast.

30. The acceleration of the arrow is $\mathbf{a}=-g\hat{\mathbf{j}}$, so that $\mathbf{v}=-gt\hat{\mathbf{j}}+\mathbf{C}$. If we take $t=0$ to be the time when the arrow leaves the bow, then when the bow is held at angle θ, and the initial speed of the arrow is v_0, $\mathbf{v}(0)=v_0(\cos\theta\,\hat{\mathbf{i}}+\sin\theta\,\hat{\mathbf{j}})$. This gives $v_0(\cos\theta\,\hat{\mathbf{i}}+\sin\theta\,\hat{\mathbf{j}})=\mathbf{C}$. Integration of $\mathbf{v}=-gt\hat{\mathbf{j}}+\mathbf{C}$ gives $\mathbf{r}=-gt^2\hat{\mathbf{j}}/2+\mathbf{C}t+\mathbf{D}$.



If the arrow starts from the origin, then $\mathbf{r}(0)=\mathbf{0}$, from which $\mathbf{D}=\mathbf{0}$, and therefore

$$\mathbf{r} = -\frac{1}{2}gt^2\hat{\mathbf{j}} + v_0t(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) = (v_0t\cos\theta)\hat{\mathbf{i}} + \left(-\frac{1}{2}gt^2 + v_0t\sin\theta\right)\hat{\mathbf{j}}.$$

If T is the time for the arrow to reach maximum height, we can say that $\mathbf{r}(T) = 50\hat{\mathbf{i}}+10\hat{\mathbf{j}}$ and $\mathbf{r}(2T) = 100\hat{\mathbf{i}}$. These imply that

$$50\hat{\mathbf{i}} + 10\hat{\mathbf{j}} = (v_0T\cos\theta)\hat{\mathbf{i}} + \left(-\frac{1}{2}gT^2 + v_0T\sin\theta\right)\hat{\mathbf{j}}, \quad 100\hat{\mathbf{i}} = [v_0(2T)\cos\theta]\hat{\mathbf{i}} + \left[-\frac{1}{2}g(2T)^2 + v_0(2T)\sin\theta\right]\hat{\mathbf{j}}.$$

When we equate components, we obtain four equations, three of which are independent,

$$50 = v_0T\cos\theta, \quad 10 = -\frac{1}{2}gT^2 + v_0T\sin\theta, \quad 0 = -2gT^2 + 2v_0T\sin\theta.$$

We eliminate T and solve for v_0 and θ. The result is $v_0 = 37.7$ m/s and $\theta = 0.381$ radians.

31. If we let T be the tension in the cord and apply Newton's second law to each of the masses, we obtain

$$M\frac{dv}{dt} = T - \frac{\mu vA}{h}, \qquad m\frac{dv}{dt} = mg - T.$$

Elimination of T gives

$$M\frac{dv}{dt} = mg - m\frac{dv}{dt} - \frac{\mu vA}{h} \quad \Longrightarrow \quad (m+M)\frac{dv}{dt} = mg - \frac{\mu vA}{h} \quad \Longrightarrow \quad \frac{dv}{mg - \mu Av/h} = \frac{1}{m+M}dt.$$

This is a separated differential equation with solutions defined implicitly by

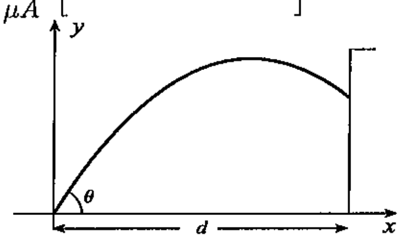
$$-\frac{h}{\mu A}\ln\left|mg - \frac{\mu Av}{h}\right| = \frac{t}{m+M} + C \qquad \Longrightarrow \qquad \ln\left|mg - \frac{\mu Av}{h}\right| = -\frac{\mu At}{h(m+M)} - \frac{\mu AC}{h}.$$

Exponentiation gives

$$mg - \frac{\mu Av}{h} = Ee^{-\mu At/(hm+hM)}, \qquad \text{where } E = \pm e^{-\mu AC/h}.$$

Since $v(0) = 0$, it follows that $mg = E$, and therefore

$$mg - \frac{\mu Av}{h} = mge^{-\mu At/(hm+hM)} \qquad \Longrightarrow \qquad v = \frac{hmg}{\mu A}\left[1 - e^{-\mu At/(hm+hM)}\right].$$



32. The acceleration of any droplet of water is $\mathbf{a} = -g\hat{\mathbf{j}}$, so that $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{C}$. If we take as $t = 0$ the time when the droplet leaves the nozzle, then when the hose is held at angle θ,

$$\mathbf{v}(0) = S(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}),$$

from which $S(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) = \mathbf{C}$. Integration of $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{C}$ gives $\mathbf{r} = -gt^2\hat{\mathbf{j}}/2 + \mathbf{C}t + \mathbf{D}$. Since $\mathbf{r}(0) = \mathbf{0}$, it follows that $\mathbf{D} = \mathbf{0}$, and

$$\mathbf{r} = -\frac{1}{2}gt^2\hat{\mathbf{j}} + St(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) = (St\cos\theta)\hat{\mathbf{i}} + \left(-\frac{1}{2}gt^2 + St\sin\theta\right)\hat{\mathbf{j}}.$$

The droplet strikes the building where $x = d$, in which case $d = St\cos\theta$. This equation implies that $t = d/(S\cos\theta)$, and this is the time that it strikes the building if it is fired at angle θ. The height it reaches on the building when it is fired at angle θ is therefore

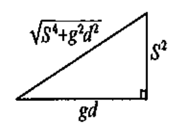
$$y(\theta) = -\frac{1}{2}g\left(\frac{d}{S\cos\theta}\right)^2 + S\sin\theta\left(\frac{d}{S\cos\theta}\right) = -\frac{gd^2}{2S^2\cos^2\theta} + d\tan\theta.$$

The problem then is to maximize $y(\theta)$ considering those values of θ which guarantee that the droplet does strike the building. There is a smallest value, say α, below which the droplet does not reach the wall, and a largest value, say β, beyond which the droplet also fails to reach the wall (and these values depend on d and S). Thus, we should maximize $y(\theta)$ for $\alpha \le \theta \le \beta$ where $y(\alpha) = y(\beta) = 0$. For critical values of $y(\theta)$, we solve

$$0 = \frac{dy}{d\theta} = \frac{gd^2}{S^2 \cos^3\theta}(-\sin\theta) + d\sec^2\theta = \frac{d(-gd\sin\theta + S^2\cos\theta)}{S^2\cos^3\theta}.$$

Thus, $\tan\theta = S^2/(gd)$. We accept from this equation only the acute angle, and this must be the value which maximizes $y(\theta)$, so that maximum $y(\theta)$ is

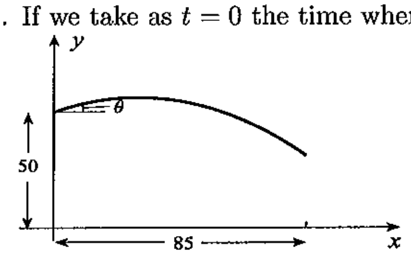
$$\frac{-gd^2}{2S^2}\left(\frac{S^4 + g^2d^2}{g^2d^2}\right) + \frac{dS^2}{gd} = \frac{S^4 - g^2d^2}{2gS^2}.$$



Notice that this result also implies that the water reaches the wall if, and only if,

$$S^4 - g^2d^2 > 0 \quad \text{or} \quad S^2 > gd.$$

33. The acceleration of the stone is $\mathbf{a} = -g\hat{\mathbf{j}}$, so that $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{C}$. If we take as $t = 0$ the time when the stone is thrown, then when it is thrown at angle θ,



$$\mathbf{v}(0) = 25(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}),$$

from which

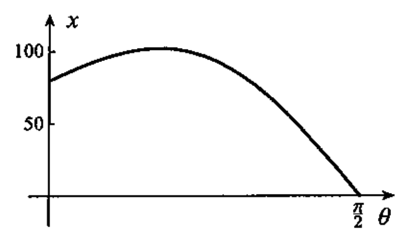
$$25(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) = \mathbf{C}.$$

Integration of $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{C}$ gives $\mathbf{r} = -gt^2\hat{\mathbf{j}}/2 + \mathbf{C}t + \mathbf{D}$. Since $\mathbf{r}(0) = 50\hat{\mathbf{j}}$, it follows that $\mathbf{D} = 50\hat{\mathbf{j}}$, and

$$\mathbf{r} = -\frac{1}{2}gt^2\hat{\mathbf{j}} + 25t(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) + 50\hat{\mathbf{j}} = (25t\cos\theta)\hat{\mathbf{i}} + \left(-\frac{1}{2}gt^2 + 25t\sin\theta + 50\right)\hat{\mathbf{j}}.$$

The stone hits water level when $y = 0$ in which case

$$0 = -\frac{1}{2}gt^2 + 25t\sin\theta + 50.$$



from which

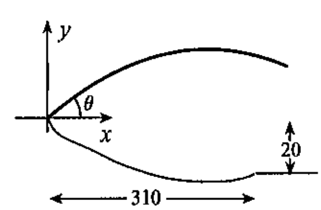
$$t = \frac{-25\sin\theta \pm \sqrt{625\sin^2\theta + 100g}}{-g}.$$

Using the positive solution, the x-coordinate of the stone at water level is

$$x = 25\cos\theta\left(\frac{25\sin\theta + \sqrt{625\sin^2\theta + 100g}}{g}\right).$$

A plot of this function shows that there are indeed angles for which $x \ge 85$.

34. The acceleration of the ball after it leaves the tee is $\mathbf{a} = -g\hat{\mathbf{j}}$ so that $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{C}$. If we take $t = 0$ at the instant the ball is struck, and it begins at angle θ, then



$$\mathbf{v}(0) = S(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}})$$

where $S > 0$ is its initial speed off the tee. This implies that $\mathbf{C} = \mathbf{v}(0)$. Integration gives

$$\mathbf{r} = -\frac{1}{2}gt^2\hat{\mathbf{j}} + \mathbf{C}t + \mathbf{D}.$$

Since $\mathbf{r}(0) = \mathbf{0}$, it follows that $\mathbf{D} = \mathbf{0}$, and

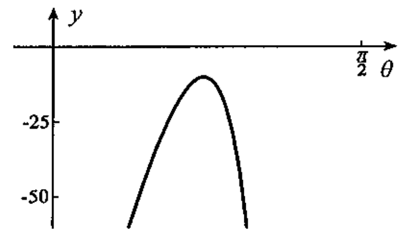
$$\mathbf{r} = -\frac{1}{2}gt^2\hat{\mathbf{j}} + St(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) = (St\cos\theta)\hat{\mathbf{i}} + \left(-\frac{1}{2}gt^2 + St\sin\theta\right)\hat{\mathbf{j}}.$$

According to Exercise 23, maximum range along a level fairway is attained for $\theta = \pi/4$, and in this case

$$R = \frac{2S^2(1/\sqrt{2})(1/\sqrt{2})}{g} = \frac{S^2}{g}.$$

Since maximum range is 300 m, $300 = S^2/g$, or, $S = \sqrt{300g} = 54.25$ m/s. In other words, maximum speed of the ball off the tee is 54.25 m/s. The ball covers a horizontal displacement of 310 m when $310 = (54.25)t\cos\theta$, or $t = (310/54.25)\sec\theta$. The y-displacement at this instant is

$$y = -\frac{1}{2}g\left(\frac{310}{54.25}\sec\theta\right)^2 + 54.25\left(\frac{310}{54.25}\sec\theta\right)\sin\theta$$
$$= -\frac{(310)(155)g}{(54.25)^2}\sec^2\theta + 310\tan\theta.$$



A plot of this function shows that there are angles for which $y = -20$.

35. (a) The acceleration of the projectile is $\mathbf{a} = -g\hat{\mathbf{j}}$, so that $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{C}$. If we take $t = 0$ when the projectile begins its trajectory, then $\mathbf{v}(0) = v(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}})$, and this implies that $\mathbf{C} = \mathbf{v}(0)$. Integration of $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{C}$ gives $\mathbf{r} = -gt^2\hat{\mathbf{j}}/2 + \mathbf{C}t + \mathbf{D}$. Since $\mathbf{r}(0) = h\hat{\mathbf{j}}$, it follows that $\mathbf{D} = h\hat{\mathbf{j}}$, and

$$\mathbf{r} = -\frac{1}{2}gt^2\hat{\mathbf{j}} + vt(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) + h\hat{\mathbf{j}} = vt\cos\theta\,\hat{\mathbf{i}} + \left(-\frac{1}{2}gt^2 + vt\sin\theta + h\right)\hat{\mathbf{j}}.$$

The projectile hits the ground when $y = 0$, in which case

$$0 = -\frac{1}{2}gt^2 + vt\sin\theta + h \quad\Longrightarrow\quad t = \frac{-v\sin\theta \pm \sqrt{v^2\sin^2\theta + 2gh}}{-g}.$$

Using the positive solution, the range of the projectile is

$$R = v\cos\theta\left(\frac{v\sin\theta + \sqrt{v^2\sin^2\theta + 2gh}}{g}\right) = \frac{v^2\cos\theta}{g}\left(\sin\theta + \sqrt{\sin^2\theta + \frac{2gh}{v^2}}\right).$$

(b) For critical points of $R(\theta)$, we solve

$$0 = \frac{dR}{d\theta} = \frac{v^2}{g}\left(-\sin^2\theta + \cos^2\theta - \sin\theta\sqrt{\sin^2\theta + \frac{2gh}{v^2}} + \frac{\sin\theta\cos^2\theta}{\sqrt{\sin^2\theta + 2gh/v^2}}\right).$$

From this equation,

$$(\sin^2\theta - \cos^2\theta)\sqrt{\sin^2\theta + \frac{2gh}{v^2}} = \sin\theta\cos^2\theta - \sin\theta\left(\sin^2\theta + \frac{2gh}{v^2}\right) = \sin\theta(\cos^2\theta - \sin^2\theta) - \frac{2gh}{v^2}\sin\theta,$$

from which $-\cos 2\theta\sqrt{\sin^2\theta + \dfrac{2gh}{v^2}} = \sin\theta\cos 2\theta - \dfrac{2gh}{v^2}\sin\theta$. Squaring gives

$$\cos^2 2\theta\left(\sin^2\theta + \frac{2gh}{v^2}\right) = \sin^2\theta\cos^2 2\theta - \frac{4gh}{v^2}\sin^2\theta\cos 2\theta + \frac{4g^2h^2}{v^4}\sin^2\theta,$$

from which

$$\cos^2 2\theta = \frac{2gh}{v^2}\sin^2\theta - 2\sin^2\theta\cos 2\theta = \left(\frac{2gh}{v^2} - 2\cos 2\theta\right)\left(\frac{1-\cos 2\theta}{2}\right) = \frac{gh}{v^2} - \cos 2\theta - \frac{gh}{v^2}\cos 2\theta + \cos^2 2\theta.$$

Thus, $\quad \cos 2\theta = \dfrac{gh/v^2}{1 + gh/v^2} = \dfrac{gh}{v^2 + gh}$. The only solution of this equation in the interval $0 < \theta < \pi/2$ is $\theta = \dfrac{1}{2}\mathrm{Cos}^{-1}\left(\dfrac{gh}{v^2 + gh}\right)$. It is geometrically clear that there is an angle between $\theta = 0$ and $\theta = \pi/2$ that maximizes R, and since only one critical point has been obtained, it must maximize R.

(c) For $v = 13.7$ and $h = 2.25$, $\theta = \dfrac{1}{2}\mathrm{Cos}^{-1}\left[\dfrac{9.81(2.25)}{13.7^2 + 9.81(2.25)}\right] = 0.733$ radians.

(d) The height of the projectile $y = -gt^2/2 + vt\sin\theta + h$ is a maximum when $0 = -gt + v\sin\theta \Longrightarrow t = (v/g)\sin\theta$; that is, maximum height is $-\frac{g}{2}\left(\frac{v}{g}\sin\theta\right)^2 + v\sin\theta\left(\frac{v}{g}\sin\theta\right) + h = h + \frac{v^2}{2g}\sin^2\theta$. From the formula for the range R in part (a), $\frac{gR}{v^2\cos\theta} - \sin\theta = \sqrt{\sin^2\theta + \frac{2gh}{v^2}}$. When this is squared,

$$\frac{g^2R^2}{v^4\cos^2\theta} - \frac{2gR\tan\theta}{v^2} + \sin^2\theta = \sin^2\theta + \frac{2gh}{v^2} \quad \Longrightarrow \quad \frac{v^2}{2g} = \frac{R^2}{4\cos^2\theta(h + R\tan\theta)}.$$

Maximum height is therefore $h + \frac{R^2\sin^2\theta}{4\cos^2\theta(h + R\tan\theta)} = h + \frac{R^2\tan^2\theta}{4(h + R\tan\theta)}$.

36. The acceleration of the projectile is $\mathbf{a} = -g\hat{\mathbf{j}}$, integration of which gives $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{C}$. If $\mathbf{v}_0 = v_0\cos(\alpha+\beta)\,\hat{\mathbf{i}} + v_0\sin(\alpha+\beta)\,\hat{\mathbf{j}}$ is the initial velocity of the projectile at time $t = 0$ when it leaves the cannon, then $\mathbf{v}_0 = \mathbf{C}$, and $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{v}_0$. Integration gives $\mathbf{r} = -gt^2\hat{\mathbf{j}}/2 + \mathbf{v}_0 t + \mathbf{D}$. Since $\mathbf{r}(0) = \mathbf{0}$, it follows that $\mathbf{D} = \mathbf{0}$, and $\mathbf{r}(t) = -gt^2\hat{\mathbf{j}}/2 + \mathbf{v}_0 t$. In component form,

$$x = v_0\cos(\alpha+\beta)\,t, \qquad y = -\frac{1}{2}gt^2 + v_0\sin(\alpha+\beta)\,t.$$

The projectile strikes the inclined plane at a point satisfying $y = x\tan\alpha$, and this implies that

$$-\frac{1}{2}gt^2 + v_0\sin(\alpha+\beta)\,t = v_0\cos(\alpha+\beta)\tan\alpha\,t \quad \Longrightarrow \quad t = \frac{2v_0}{g}[\sin(\alpha+\beta) - \cos(\alpha+\beta)\tan\alpha].$$

Since the projectile strikes the ground horizontally, the y-component of velocity at the point of impact must be zero,

$$0 = -gt + v_0\sin(\alpha+\beta) \quad \Longrightarrow \quad t = \frac{v_0}{g}\sin(\alpha+\beta).$$

When we equate these two expressions for t,

$$\frac{2v_0}{g}[\sin(\alpha+\beta) - \cos(\alpha+\beta)\tan\alpha] = \frac{v_0}{g}\sin(\alpha+\beta) \quad \Longrightarrow \quad \sin(\alpha+\beta) = 2\cos(\alpha+\beta)\tan\alpha.$$

Multiplication by $\cos\alpha$ gives

$$(\sin\alpha\cos\beta + \cos\alpha\sin\beta)\cos\alpha = 2(\cos\alpha\cos\beta - \sin\alpha\sin\beta)\sin\alpha,$$

from which $\sin\alpha\cos\alpha\cos\beta = (\cos^2\alpha + 2\sin^2\alpha)\sin\beta$. This can be solved for

$$\tan\beta = \frac{\sin\alpha\cos\alpha}{\cos^2\alpha + 2\sin^2\alpha} = \frac{(1/2)\sin 2\alpha}{(1+\cos 2\alpha)/2 + (1-\cos 2\alpha)} = \frac{\sin 2\alpha}{3 - \cos 2\alpha},$$

and therefore $\beta = \mathrm{Tan}^{-1}\left(\frac{\sin 2\alpha}{3 - \cos 2\alpha}\right)$.

37. From equation 11.108,

$$\boldsymbol{\tau} = \mathbf{r}\times\mathbf{F} = \mathbf{r}\times\left[\frac{d}{dt}(m\mathbf{v})\right] = \frac{d}{dt}[\mathbf{r}\times(m\mathbf{v})] - \frac{d\mathbf{r}}{dt}\times(m\mathbf{v}) = \frac{d\mathbf{H}}{dt} - \mathbf{v}\times m\mathbf{v} = \frac{d\mathbf{H}}{dt}$$

since $\mathbf{v}\times m\mathbf{v} = \mathbf{0}$.

38. (a) Since length around the circumference of the tire is given by $R\theta$ and the time rate of change of this quantity is the speed of the centre of the tire, it follows that $S = R(d\theta/dt)$. Antidifferentiation gives $\theta = St/R + C$. Since $\theta = 0$ when $t = 0$, it follows that $C = 0$ and $\theta = St/R$.
(b) Since $x = R(\theta - \sin\theta)$ and $y = R(1 - \cos\theta)$,

$$\mathbf{v} = R\left(\frac{d\theta}{dt} - \cos\theta\frac{d\theta}{dt}\right)\hat{\mathbf{i}} + \left(R\sin\theta\frac{d\theta}{dt}\right)\hat{\mathbf{j}} = S[(1-\cos\theta)\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}]$$

$$|\mathbf{v}| = S\sqrt{(1-\cos\theta)^2 + \sin^2\theta} = S\sqrt{2 - 2\cos\theta}$$

$$\mathbf{a} = S\left(\sin\theta\frac{d\theta}{dt}\hat{\mathbf{i}} + \cos\theta\frac{d\theta}{dt}\hat{\mathbf{j}}\right) = \frac{S^2}{R}(\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}).$$

(c) $$a_T = \frac{d}{dt}|\mathbf{v}| = \frac{S\sin\theta}{\sqrt{2-2\cos\theta}}\frac{d\theta}{dt} = \frac{S^2\sin\theta}{R\sqrt{2-2\cos\theta}}$$

$$a_N = \sqrt{|\mathbf{a}|^2 - a_T{}^2} = \sqrt{\frac{S^4}{R^2} - \frac{S^4\sin^2\theta}{R^2(2-2\cos\theta)}} = \frac{S^2}{R}\sqrt{(1-\cos\theta)/2}$$

39. (a) Since length around the circumference of the tire is given by $R\theta$ and the time rate of change of this quantity is the speed of the centre of the tire, it follows that $S = R(d\theta/dt)$. Antidifferentiation gives $\theta = St/R + C$. Since $\theta = 0$ when $t = 0$, it follows that $C = 0$ and $\theta = St/R$. Since $x = R\theta - b\sin\theta$ and $y = R - b\cos\theta$,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \left(R\frac{d\theta}{dt} - b\cos\theta\frac{d\theta}{dt}\right)\hat{\mathbf{i}} + \left(b\sin\theta\frac{d\theta}{dt}\right)\hat{\mathbf{j}} = \frac{S}{R}\left[(R - b\cos\theta)\hat{\mathbf{i}} + b\sin\theta\hat{\mathbf{j}}\right]$$

$$|\mathbf{v}| = \frac{S}{R}\sqrt{(R-b\cos\theta)^2 + b^2\sin^2\theta} = \frac{S}{R}\sqrt{R^2 - 2Rb\cos\theta + b^2}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{S}{R}\left[\left(b\sin\theta\frac{d\theta}{dt}\right)\hat{\mathbf{i}} + \left(b\cos\theta\frac{d\theta}{dt}\right)\hat{\mathbf{j}}\right] = \frac{bS^2}{R^2}(\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}).$$

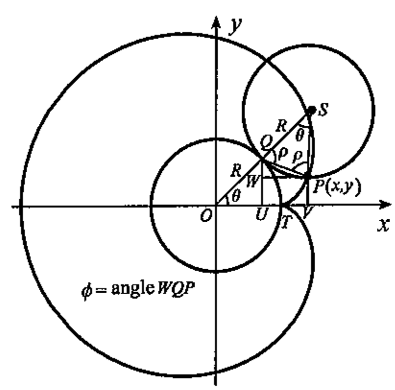
(b) $$a_T = \frac{d}{dt}|\mathbf{v}| = \frac{S}{R}\frac{2Rb\sin\theta(d\theta/dt)}{2\sqrt{R^2-2Rb\cos\theta+b^2}} = \frac{bS^2}{R}\frac{\sin\theta}{\sqrt{R^2-2Rb\cos\theta+b^2}}$$

$$a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \left(\frac{b^2S^4}{R^4} - \frac{b^2S^4}{R^2}\frac{\sin^2\theta}{R^2-2Rb\cos\theta+b^2}\right)^{1/2} = \frac{bS^2}{R^2}\left(1 - \frac{R^2\sin^2\theta}{R^2-2Rb\cos\theta+b^2}\right)^{1/2}$$

$$= \frac{bS^2}{R^2}\left[\frac{R^2-2Rb\cos\theta+b^2-R^2(1-\cos^2\theta)}{R^2-2Rb\cos\theta+b^2}\right]^{1/2}$$

$$= \frac{bS^2}{R^2}\left[\frac{(-b+R\cos\theta)^2}{R^2-2Rb\cos\theta+b^2}\right]^{1/2} = \frac{bS^2}{R^2}\frac{|-b+R\cos\theta|}{\sqrt{R^2-2Rb\cos\theta+b^2}}$$

40. (a) $$x = R + \|TV\| = R + \|UV\| - \|UT\|$$
$$= R + \|PQ\|\sin\phi - (R - \|OU\|)$$
$$= \|PQ\|\sin\phi + R\cos\theta.$$



When $(\pi/2 - \theta) + \phi + \rho = \pi$, and $\theta + 2\rho = \pi$, are solved for ρ, and results are equated, we obtain $\phi = 3\theta/2$. Hence, with $\|PQ\| = 2R\sin(\theta/2)$,

$$x = 2R\sin(\theta/2)\sin(3\theta/2) + R\cos\theta$$
$$= R(-\cos 2\theta + \cos\theta + \cos\theta)$$
$$= R(2\cos\theta - \cos 2\theta).$$

Furthermore,

$$y = \|UQ\| - \|QW\| = R\sin\theta - \|PQ\|\cos\phi$$
$$= R\sin\theta - 2R\sin(\theta/2)\cos(3\theta/2)$$
$$= R(\sin\theta + \sin\theta - \sin 2\theta)$$
$$= R(2\sin\theta - \sin 2\theta).$$

(b) When the point of contact has moved so that it makes angle θ with the positive x-axis, length along the stationary circle from $(R, 0)$ to the point of contact is $R\theta$. Since this length changes at a rate of $Rd\theta/dt = S$, it follows that $\theta = St/R$.

(c) $$\mathbf{v} = \frac{d\mathbf{r}}{dt} = R(-2\sin\theta + 2\sin 2\theta)\frac{d\theta}{dt}\hat{\mathbf{i}} + R(2\cos\theta - 2\cos 2\theta)\frac{d\theta}{dt}\hat{\mathbf{j}}$$
$$= 2S(\sin 2\theta - \sin\theta)\hat{\mathbf{i}} + 2S(\cos\theta - \cos 2\theta)\hat{\mathbf{j}};$$

$$\begin{aligned}|\mathbf{v}| &= 2S\sqrt{(\sin 2\theta - \sin\theta)^2 + (\cos\theta - \cos 2\theta)^2} = 2S\sqrt{2 - 2\sin 2\theta\sin\theta - 2\cos 2\theta\cos\theta}\\ &= 2\sqrt{2}S\sqrt{1-\cos\theta};\end{aligned}$$

$$\begin{aligned}\mathbf{a} &= 2S(2\cos 2\theta - \cos\theta)\frac{d\theta}{dt}\hat{\mathbf{i}} + 2S(-\sin\theta + 2\sin 2\theta)\frac{d\theta}{dt}\hat{\mathbf{j}}\\ &= \frac{2S^2}{R}[(2\cos 2\theta - \cos\theta)\hat{\mathbf{i}} + (2\sin 2\theta - \sin\theta)\hat{\mathbf{j}}].\end{aligned}$$

(d) $$a_T = \frac{d}{dt}|\mathbf{v}| = \frac{\sqrt{2}S\sin\theta}{\sqrt{1-\cos\theta}}\frac{d\theta}{dt} = \frac{\sqrt{2}S^2\sin\theta}{R\sqrt{1-\cos\theta}};$$

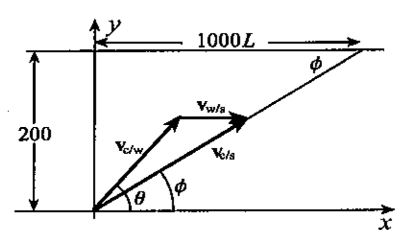
$$\begin{aligned}a_N &= \sqrt{|\mathbf{a}|^2 - a_T{}^2} = \sqrt{\frac{4S^4}{R^2}\left[(2\cos 2\theta - \cos\theta)^2 + (2\sin 2\theta - \sin\theta)^2\right] - \frac{2S^4\sin^2\theta}{R^2(1-\cos\theta)}}\\ &= \frac{S^2}{R}\sqrt{4[5 - 4\cos 2\theta\cos\theta - 4\sin 2\theta\sin\theta] - \frac{2\sin^2\theta}{1-\cos\theta}}\\ &= \frac{S^2}{R}\sqrt{20 - 16\cos\theta - \frac{2\sin^2\theta}{1-\cos\theta}} = \frac{S^2}{R}\sqrt{\frac{20 - 36\cos\theta + 16\cos^2\theta - 2(1-\cos^2\theta)}{1-\cos\theta}}\\ &= \frac{S^2}{R}\sqrt{\frac{18\cos^2\theta - 36\cos\theta + 18}{1-\cos\theta}} = \frac{3\sqrt{2}S^2}{R}\sqrt{1-\cos\theta}\end{aligned}$$

41. If $\mathbf{r}$ and $\frac{d\mathbf{r}}{dt}$ are always perpendicular, then $\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} = 0$. It follows that

$$\frac{d}{dt}(\mathbf{r}\cdot\mathbf{r}) = \mathbf{r}\cdot\frac{d\mathbf{r}}{dt} + \frac{d\mathbf{r}}{dt}\cdot\mathbf{r} = 2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} = 0,$$

and this implies that $|\mathbf{r}|^2 = \mathbf{r}\cdot\mathbf{r} =$ constant. But this means that the tip of $\mathbf{r}$ lies on a sphere.

42. Let us take the positive x-direction as the direction in which the river flows. Then the velocity of the water with respect to the shore is $\mathbf{v}_{w/s} = 3\hat{\mathbf{i}}$. If the canoe points in direction θ shown, $\mathbf{v}_{c/w} = 2\cos\theta\,\hat{\mathbf{i}} + 2\sin\theta\,\hat{\mathbf{j}}$. If v is the speed of the canoe with respect to the shore, then $\mathbf{v}_{c/s} = v\cos\phi\,\hat{\mathbf{i}} + v\sin\phi\,\hat{\mathbf{j}}$. Since $\mathbf{v}_{c/s} = \mathbf{v}_{c/w} + \mathbf{v}_{w/s}$, we obtain



$$v\cos\phi\,\hat{\mathbf{i}} + v\sin\phi\,\hat{\mathbf{j}} = 2\cos\theta\,\hat{\mathbf{i}} + 2\sin\theta\,\hat{\mathbf{j}} + 3\hat{\mathbf{i}}.$$

When we equate components, $v\cos\phi = 2\cos\theta + 3 = 0$ and $v\sin\phi = 2\sin\theta$. These imply that

$$4 = 4\cos^2\theta + 4\sin^2\theta = (v\cos\phi - 3)^2 + (v\sin\phi)^2 = v^2 - 6v\cos\phi + 9.$$

Thus, $v^2 - 6v\cos\phi + 5 = 0$, the solutions of which are $v = 3\cos\phi \pm \sqrt{9\cos^2\phi - 5}$. Clearly v exists only if $9\cos^2\phi - 5 \geq 0 \implies \cos\phi \geq \sqrt{5}/3$. But from the figure, $\cos\phi = 5L/\sqrt{1+25L^2}$, and therefore

$$\frac{5L}{\sqrt{1+25L^2}} \geq \frac{\sqrt{5}}{3} \quad\implies\quad L \geq \frac{\sqrt{5}}{10}.$$

When $L = \sqrt{5}/10$, there is one solution for v, namely, $v = 3\cos\phi$, in which case

$$2\sin\theta = 3\cos\phi\sin\phi = 3\left(\frac{\sqrt{5}}{3}\right)\left(\frac{2}{3}\right) \quad\implies\quad \theta = \mathrm{Sin}^{-1}\left(\frac{\sqrt{5}}{3}\right) = 0.841 \text{ radians}.$$

When $L > \sqrt{5}/10$, there are two solutions for v. The larger one $v = 3\cos\phi + \sqrt{9\cos^2\phi - 5}$ gives the shorter travel time. For this choice,

$$2\sin\theta = \sin\phi(3\cos\phi + \sqrt{9\cos^2\phi - 5}) = \frac{1}{\sqrt{1+25L^2}}\left(\frac{15L}{\sqrt{1+25L^2}} + \sqrt{\frac{225L^2}{1+25L^2} - 5}\right)$$
$$= \frac{15L + \sqrt{100L^2 - 5}}{1+25L^2}.$$

Therefore, $\theta = \mathrm{Sin}^{-1}\left(\dfrac{15L + \sqrt{100L^2-5}}{2+50L^2}\right)$.

43. If $\mathbf{F}_i$ is the force on m_i, then $\mathbf{F}_i = m_i\mathbf{a}_i$ for $i = 1, \ldots, n$, and $\mathbf{F} = \sum_{i=1}^{n}\mathbf{F}_i = \sum_{i=1}^{n} m_i\mathbf{a}_i$. The centre of mass $\overline{\mathbf{r}}$ of the system is given by $\overline{\mathbf{r}} = \dfrac{1}{M}\sum_{i=1}^{n} m_i\mathbf{r}_i$. Therefore,

$$\overline{\mathbf{a}} = \frac{d^2\overline{\mathbf{r}}}{dt^2} = \frac{1}{M}\sum_{i=1}^{n} m_i\frac{d^2\mathbf{r}_i}{dt^2} = \frac{1}{M}\sum_{i=1}^{n} m_i\mathbf{a}_i,$$

and $\mathbf{F} = M\overline{\mathbf{a}}$.

44. If we express the acceleration $\mathbf{a}$ of the particle in form 11.112a, write the force in the form $\mathbf{F} = \lambda(t)\mathbf{T}$, and substitute these into 11.109

$$\lambda(t)\mathbf{T} = m(a_T\hat{\mathbf{T}} + a_N\hat{\mathbf{N}}) \quad \Longrightarrow \quad a_N = 0.$$

But then 11.112b implies that $|\mathbf{v}|\left|\dfrac{d\hat{\mathbf{T}}}{dt}\right| = 0$, from which $\dfrac{d\hat{\mathbf{T}}}{dt} = 0$, and $\hat{\mathbf{T}}$ is a constant vector. But this means that the trajectory is a straight line.

45. (a) Using formula 9.16, $A(t) = \displaystyle\int_{\theta_0}^{\theta(t)} \frac{1}{2}r^2\,d\theta$. If we differentiate this with respect to t using equation 6.19,

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}.$$

(b) If $\hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$ is a unit vector pointing to the planet at any given time, then differentiation of $1 = \hat{\mathbf{r}}\cdot\hat{\mathbf{r}}$ with respect to time gives

$$0 = \hat{\mathbf{r}}\cdot\frac{d\hat{\mathbf{r}}}{dt} + \frac{d\hat{\mathbf{r}}}{dt}\cdot\hat{\mathbf{r}} = 2\left(\hat{\mathbf{r}}\cdot\frac{d\hat{\mathbf{r}}}{dt}\right).$$

This implies that $\hat{\mathbf{r}}$ and $d\hat{\mathbf{r}}/dt$ are perpendicular at any given time. If we write $\mathbf{r} = r\,\hat{\mathbf{r}}$, where r is therefore the length of $\mathbf{r}$, then $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = r\dfrac{d\hat{\mathbf{r}}}{dt} + \dfrac{dr}{dt}\hat{\mathbf{r}}$. When we take magnitudes of 11.115 and substitute this expression for $\mathbf{v}$,

$$|\mathbf{C}| = |\mathbf{r}\times\mathbf{v}| = \left|r\,\hat{\mathbf{r}}\times\left(r\frac{d\hat{\mathbf{r}}}{dt} + \frac{dr}{dt}\hat{\mathbf{r}}\right)\right| = \left|r^2\hat{\mathbf{r}}\times\frac{d\hat{\mathbf{r}}}{dt} + r\frac{dr}{dt}\hat{\mathbf{r}}\times\hat{\mathbf{r}}\right| = r^2\left|\frac{d\hat{\mathbf{r}}}{dt}\right|,$$

since $\hat{\mathbf{r}}\times\hat{\mathbf{r}} = \mathbf{0}$ and $\hat{\mathbf{r}}$ and $d\hat{\mathbf{r}}/dt$ are perpendicular. Since $d\hat{\mathbf{r}}/dt = (-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}})d\theta/dt$,

$$|\mathbf{C}| = r^2\left|(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}})\frac{d\theta}{dt}\right| \quad \Longrightarrow \quad \frac{d\theta}{dt} = \frac{|\mathbf{C}|}{r^2}.$$

(c) When we combine the results in (a) and (b),

$$\frac{dA}{dt} = \frac{1}{2}r^2\left(\frac{|\mathbf{C}|}{r^2}\right) = \frac{|\mathbf{C}|}{2}.$$

This implies that $\mathbf{A}(t) = \dfrac{|\mathbf{C}|}{2}t + k$, where k is a constant. Since $A(t)$ is a linear function, its changes in time intervals of the same length are the same, thus verifying Kepler's second law.

46. (a) Using the formula in part (c) of Exercise 45, the area swept out by the plane in one revolution is

$$A(t+P) - A(t) = \left[\frac{|\mathbf{C}|}{2}(t+P) + k\right] - \left[\frac{|\mathbf{C}|}{2}t + k\right] = \frac{P|\mathbf{C}|}{2}.$$

But the area of the ellipse is πab, so that $\dfrac{P|\mathbf{C}|}{2} = \pi ab \Longrightarrow P = \dfrac{2\pi ab}{|\mathbf{C}|}$.

(b) If we change 11.116 to polar coordinates by setting $x = r\cos\theta$ and $y = r\sin\theta$,

$$\sqrt{x^2+y^2} = \frac{\epsilon d}{1 + \dfrac{\epsilon x}{\sqrt{x^2+y^2}}} \quad \Longrightarrow \quad \sqrt{x^2+y^2} = \epsilon(d-x).$$

Squaring gives

$$x^2 + y^2 = \epsilon^2(d-x)^2 \quad \Longrightarrow \quad x^2 + \frac{2d\epsilon^2 x}{1-\epsilon^2} + \frac{y^2}{1-\epsilon^2} = \frac{\epsilon^2 d^2}{1-\epsilon^2}.$$

Completing the square on the x-terms results in

$$\left(x + \frac{d\epsilon^2}{1-\epsilon^2}\right)^2 + \frac{y^2}{1-\epsilon^2} = \frac{\epsilon^2 d^2}{1-\epsilon^2} + \frac{d^2\epsilon^4}{(1-\epsilon^2)^2} = \left(\frac{\epsilon d}{1-\epsilon^2}\right)^2.$$

This allows us to identify $a = \epsilon d/(1-\epsilon^2)$ and $b = \epsilon d/\sqrt{1-\epsilon^2}$. With $\epsilon = |\mathbf{b}|/(GM)$ and $d = |\mathbf{C}|^2/|\mathbf{b}|$,

$$a = \frac{|\mathbf{C}|^2/(GM)}{1 - |\mathbf{b}|^2/(G^2M^2)} = \frac{|\mathbf{C}|^2 GM}{G^2M^2 - |\mathbf{b}|^2}, \quad b = \frac{|\mathbf{C}|^2/(GM)}{\sqrt{1 - |\mathbf{b}|^2/(G^2M^2)}} = \frac{|\mathbf{C}|^2}{\sqrt{G^2M^2 - |\mathbf{b}|^2}}.$$

It follows therefore that $\dfrac{b^2}{a} = \dfrac{|\mathbf{C}|^4}{G^2M^2 - |\mathbf{b}|^2}\dfrac{G^2M^2 - |\mathbf{b}|^2}{|\mathbf{C}|^2 GM} = \dfrac{|\mathbf{C}|^2}{GM}$, and hence,

$$P^2 = \frac{4\pi^2 a^2 b^2}{|\mathbf{C}|^2} = 4\pi^2 a^2\left(\frac{a}{GM}\right) = \frac{4\pi^2 a^3}{GM}.$$

47. The first point in Consulting Project 18 at which use is made of the initial speed of the ice is in the equation $-9.81\sin\theta = (a/2)(d\theta/dt)^2 + C$. With $d\theta/dt = v_0/a$ when $\theta = \pi/2$, we find that $C = -9.81 - v_0^2/(2a)$. Hence,

$$-9.81\sin\theta = \frac{a}{2}\left(\frac{d\theta}{dt}\right)^2 - 9.81 - \frac{v_0^2}{2a} \quad \Longrightarrow \quad \left(\frac{d\theta}{dt}\right)^2 = \frac{19.62}{a}(1-\sin\theta) + \frac{v_0^2}{a^2}.$$

If we now substitute this into the equation $-N + 9.81m\sin\theta = ma\left(\dfrac{d\theta}{dt}\right)^2$, we obtain

$$-N + 9.81m\sin\theta = 19.62m(1-\sin\theta) + \frac{mv_0^2}{a} \quad \Longrightarrow \quad N = 9.81m(3\sin\theta - 2) - \frac{mv_0^2}{a}.$$

We find that $N = 0$ when

$$9.81m(3\sin\theta - 2) - \frac{mv_0^2}{a} = 0 \quad \Longrightarrow \quad \theta = \mathrm{Sin}^{-1}\left(\frac{2}{3} + \frac{v_0^2}{3(9.81)a}\right).$$

This will be valid provided $\dfrac{2}{3} + \dfrac{v_0^2}{3(9.81)a} \le 1 \Longrightarrow v_0 \le \sqrt{9.81a}$.

REVIEW EXERCISES

1. $2\mathbf{u} - 3\mathbf{w} + \mathbf{r} = 2(1,3,-2) - 3(0,2,1) + (2,0,-1) = (4,0,-8)$

2. $\mathbf{u}\cdot\mathbf{v}\times\mathbf{w} = \mathbf{u}\cdot\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 4 & -1 \\ 0 & 2 & 1 \end{vmatrix} = (1,3,-2)\cdot(6,-2,4) = 6 - 6 - 8 = -8$

3. $(3\mathbf{u}\times 4\mathbf{v}) - \mathbf{w} = 12\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 3 & -2 \\ 2 & 4 & -1 \end{vmatrix} - \mathbf{w} = 12(5,-3,-2) - (0,2,1) = (60,-38,-25)$

4. $3\mathbf{u}\times(4\mathbf{v}-\mathbf{w}) = 3\mathbf{u}\times(8,14,-5) = 3\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 3 & -2 \\ 8 & 14 & -5 \end{vmatrix} = 3(13,-11,-10) = (39,-33,-30)$

5. $|\mathbf{u}|\mathbf{v} - |\mathbf{v}|\mathbf{r} = \sqrt{14}(2,4,-1) - \sqrt{21}(2,0,-1) = (2\sqrt{14} - 2\sqrt{21}, 4\sqrt{14}, -\sqrt{14} + \sqrt{21})$

6. $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{r}-\mathbf{w}) = (3,7,-3)\cdot(2,-2,-2) = 6 - 14 + 6 = -2$

7. $(\mathbf{u}+\mathbf{v})\times(\mathbf{r}-\mathbf{w}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 7 & -3 \\ 2 & -2 & -2 \end{vmatrix} = (-20,0,-20)$

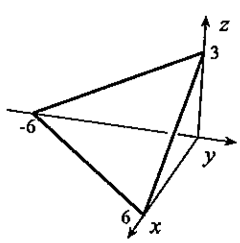
8. $$(\mathbf{u}\times\mathbf{v})\times(\mathbf{r}\times\mathbf{w}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 3 & -2 \\ 2 & 4 & -1 \end{vmatrix} \times \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 0 & -1 \\ 0 & 2 & 1 \end{vmatrix} = (5,-3,-2)\times(2,-2,4)$$
$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 5 & -3 & -2 \\ 2 & -2 & 4 \end{vmatrix} = (-16,-24,-4)$$

9. $(\mathbf{u}\cdot\mathbf{v})\mathbf{r} - 3(\mathbf{v}\cdot\mathbf{w})\mathbf{u} = 16(2,0,-1) - 3(7)(1,3,-2) = (11,-63,26)$

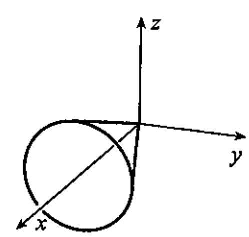
10. $\dfrac{2\mathbf{r}}{\mathbf{v}\cdot\mathbf{w}} + 3(\mathbf{v}+\mathbf{u}) = \dfrac{2}{8-1}(2,0,-1) + 3(3,7,-3) = \left(\dfrac{67}{7}, 21, -\dfrac{65}{7}\right)$

For Exercises 12, 14, 16, 18, 20, 22, 24, and 26, see answers in text.

11.



13.



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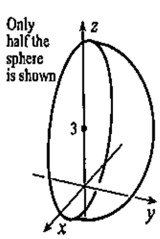

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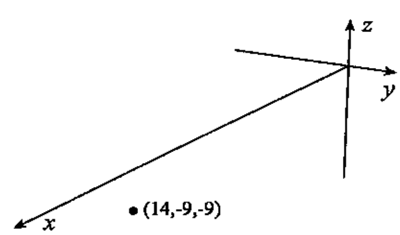

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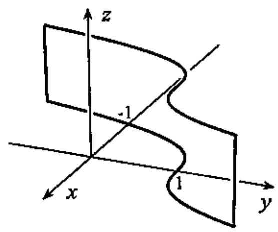

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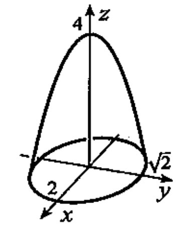

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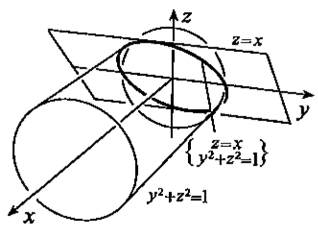

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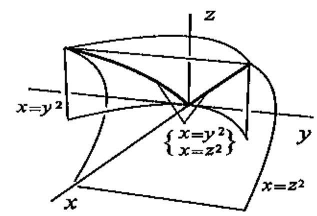


27. Parametric equations for the line are $x = -2 + 3t$, $y = 3 - 5t$, $z = 4t$.

28. Since a vector along the line is $(5, -2, 1)$, parametric equations for the line are $x = 6+5t$, $y = 6-2t$, $z = 2 + t$.

29. Since a vector along the line is $\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 0 \\ 2 & 3 & 6 \end{vmatrix} = (-6, -6, 5)$ as is $(6, 6, -5)$, parametric equations are $x = 6t$, $y = 6t$, $z = -5t$.

30. Since $(1, 3, 2)$ is a point on the required line, parametric equations must be of the form

$$x = 1 + au,\ \ y = 3 + bu,\ \ z = 2 + cu.$$

Because the line must be perpendicular to the given line,

$$0 = (a, b, c) \cdot (1, -2, 1) = a - 2b + c.$$

Finally, since the lines must intersect, we set

$$t + 2 = 1 + au,\ \ 3 - 2t = 3 + bu,\ \ 4 + t = 2 + cu.$$

When the first two of these are solved for t and u,

$$u = \frac{2}{b + 2a} \qquad \text{and} \qquad t = \frac{-b}{b + 2a}.$$

Substitution into the third gives $4a + b - 2c = 0$. When this is combined with $a - 2b + c = 0$, the result is $b = 2a$ and $c = 3a$. Consequently, parametric equations for the required line are $x = 1 + au$, $y = 3 + 2au$, $z = 2 + 3au$, or, $x = 1 + v$, $y = 3 + 2v$, $z = 2 + 3v$ (where we have set $v = au$).

31. Since $(2, -1, 0) - (1, 3, 2) = (1, -4, -2)$ and $(6, 1, 3) - (1, 3, 2) = (5, -2, 1)$ are vectors in the plane, a normal to the plane is $\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -4 & -2 \\ 5 & -2 & 1 \end{vmatrix} = (-8, -11, 18)$ as is $(8, 11, -18)$. The equation of the plane is $0 = 8(x - 1) + 11(y - 3) - 18(z - 2) = 8x + 11y - 18z - 5$.

32. Since parametric equations for the line are $x = 4 - t$, $y = t$, $z = t$, a vector along the line (and therefore normal to the plane is $(-1, 1, 1)$. The equation of the plane is

$$0 = (-1, 1, 1) \cdot (x - 1, y - 2, z + 1) = -x + y + z.$$

33. Since two points on the line are $(2,0,1)$ and $(6,-3,-6)$, two vectors in the plane are $(2,2,2)-(2,0,1) = (0,2,1)$ and $(2,2,2)-(6,-3,-6) = (-4,5,8)$. A normal to the plane is $\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 2 & 1 \\ -4 & 5 & 8 \end{vmatrix} = (11,-4,8)$. The equation of the plane is $0 = 11(x-2) - 4(y-2) + 8(z-2) = 11x - 4y + 8z - 30$.

34. Since the lines are not parallel, they determine a plane only if they intersect. To confirm this, we set $3t = 1 + 2t$ and $3t = 4 - t$. These both give $t = 1$ leading to the point of intersection $(3,3,3)$. A vector normal to the plane is

$$(3,2,-1) \times (1,1,1) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 2 & -1 \\ 1 & 1 & 1 \end{vmatrix} = (3,-4,1).$$

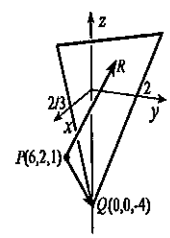
The equation of the plane is

$$0 = (3,-4,1) \cdot (x-3, y-3, z-3) = 3x - 4y + z.$$

35. $\sqrt{25+1+9} = \sqrt{35}$

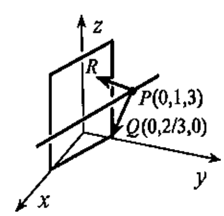
36. The required distance d is the projection of $\mathbf{PQ}$ on the direction $\mathbf{PR}$ normal to the plane:

$$\begin{aligned} d &= \left|\mathbf{PQ} \cdot \widehat{\mathbf{PR}}\right| \\ &= \left|(-6,-2,-5) \cdot \frac{(-6,-2,1)}{\sqrt{36+4+1}}\right| \\ &= 35/\sqrt{41}. \end{aligned}$$



37. The distance is zero unless the line and plane are parallel. A vector along the line is $(1,-1,1) \times (2,1,1) = (-2,1,3)$. Since $(-2,1,3) \cdot (1,-1,0) = -3 \neq 0$, the line plane are not parallel; they intersect. The minimum distance is therefore 0.

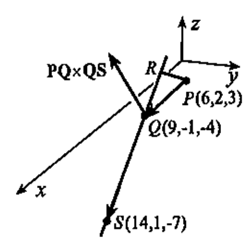
38. The distance is zero unless the line and plane are parallel. A vector along the line is $(1,-1,1) \times (2,1,1) = (-2,1,3)$. Since $(-2,1,3) \cdot (3,6,0) = 0$, the line and plane are parallel. The required distance d is the projection of $\mathbf{PQ}$ on the direction $\mathbf{PR}$ normal to the plane:



$$d = \left|\mathbf{PQ} \cdot \widehat{\mathbf{PR}}\right| = \left|(0,-1/3,-3) \cdot \frac{(-3,-6,0)}{\sqrt{9+36}}\right| = \frac{2}{3\sqrt{5}}.$$

39. The required distance is the component of $\mathbf{PQ}$ along $\mathbf{PR}$. A vector perpendicular to $\mathbf{PQ}$ and $\mathbf{QS}$ is

$$\begin{aligned} \mathbf{PQ} \times \mathbf{QS} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & -3 & -7 \\ 5 & 2 & -3 \end{vmatrix} \\ &= (23,-26,21). \end{aligned}$$



A vector in direction $\mathbf{PR}$ is therefore

$$(\mathbf{PQ} \times \mathbf{QS}) \times \mathbf{SQ} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 23 & -26 & 21 \\ -5 & -2 & 3 \end{vmatrix} = (-36,-174,-176).$$

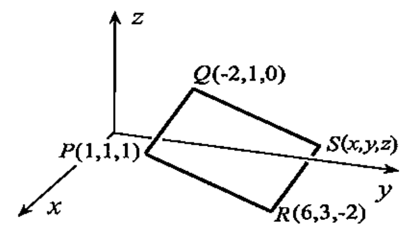
Thus,

$$d = \left|\mathbf{PQ} \cdot \widehat{\mathbf{PR}}\right| = \left|(3,-3,-7) \cdot \frac{(-18,-87,-88)}{\sqrt{(-18)^2 + (-87)^2 + (-88)^2}}\right| = \frac{823}{\sqrt{15637}}.$$

40. According to equation 11.42,

$$\begin{aligned}\text{Area} &= \frac{1}{2}|[(-2,1,0)-(1,1,1)]\times[(6,3,-2)-(1,1,1)]|\\ &= \frac{1}{2}|(-3,0,-1)\times(5,2,-3)| = \frac{1}{2}\left\|\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ -3 & 0 & -1\\ 5 & 2 & -3\end{vmatrix}\right\| = \frac{1}{2}|(2,-14,-6)| = \sqrt{59}.\end{aligned}$$

41. One such parallelogram is shown in the figure. The coordinates of S are



$$\begin{aligned}(x,y,z) &= \mathbf{OR}+\mathbf{RS} = \mathbf{OR}+\mathbf{PQ}\\ &= (6,3,-2)+(-3,0,-1)\\ &= (3,3,-3).\end{aligned}$$

The area of the parallelogram is

$$\text{area} = |\mathbf{PQ}\times\mathbf{PR}| = \left\|\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ -3 & 0 & -1\\ 5 & 2 & -3\end{vmatrix}\right\| = |(2,-14,-6)| = 2\sqrt{59}.$$

Similar procedures give two additional vertices $(-7,-1,3)$ and $(9,3,-1)$ with equal areas.

42. From $\mathbf{T} = \dfrac{d\mathbf{r}}{dt} = (2\cos t, -2\sin t, 1)$, we obtain the unit tangent vector

$$\hat{\mathbf{T}} = \frac{(2\cos t, -2\sin t, 1)}{\sqrt{4\cos^2 t + 4\sin^2 t + 1}} = \frac{(2\cos t, -2\sin t, 1)}{\sqrt{5}}.$$

A vector in the direction of $\hat{\mathbf{N}}$ is

$$\mathbf{N} = \frac{d\hat{\mathbf{T}}}{dt} = \frac{1}{\sqrt{5}}(-2\sin t, -2\cos t, 0) = -\frac{2}{\sqrt{5}}(\sin t, \cos t, 0),$$

and therefore $\hat{\mathbf{N}} = -(\sin t, \cos t, 0)$. The binormal is

$$\hat{\mathbf{B}} = \hat{\mathbf{T}}\times\hat{\mathbf{N}} = -\frac{1}{\sqrt{5}}\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 2\cos t & -2\sin t & 1\\ \sin t & \cos t & 0\end{vmatrix} = \frac{1}{\sqrt{5}}(\cos t, -\sin t, -2).$$

43. From $\hat{\mathbf{T}} = \dfrac{(3t^2, 4t, 1)}{\sqrt{9t^4+16t^2+1}}$, a vector in the direction of $\hat{\mathbf{N}}$ is

$$\begin{aligned}\mathbf{N} = \frac{d\hat{\mathbf{T}}}{dt} &= \frac{-(18t^3+16t)}{(9t^4+16t^2+1)^{3/2}}(3t^2,4t,1) + \frac{(6t,4,0)}{\sqrt{9t^4+16t^2+1}}\\ &= \frac{1}{(9t^4+16t^2+1)^{3/2}}\left[-(18t^3+16t)(3t^2,4t,1)+(9t^4+16t^2+1)(6t,4,0)\right]\\ &= \frac{1}{(9t^4+16t^2+1)^{3/2}}(48t^3+6t, -36t^4+4, -18t^3-16t).\end{aligned}$$

Consequently, the principal normal is

$$\hat{\mathbf{N}} = \frac{(24t^3+3t, 2-18t^4, -9t^3-8t)}{\sqrt{(24t^3+3t)^2+(2-18t^4)^2+(-9t^3-8t)^2}} = \frac{(24t^3+3t, 2-18t^4, -9t^3-8t)}{\sqrt{324t^8+657t^6+216t^4+73t^2+4}}.$$

The direction of the binormal is

$$\begin{aligned}\mathbf{B} &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3t^2 & 4t & 1\\ 24t^3+3t & 2-18t^4 & -9t^3-8t\end{vmatrix}\\ &= (-18t^4-32t^2-2)\hat{\mathbf{i}} + (27t^5+48t^3+3t)\hat{\mathbf{j}} + (-54t^6-96t^4-6t^2)\hat{\mathbf{k}}\\ &= (9t^4+16t^2+1)(-2\hat{\mathbf{i}}+3t\hat{\mathbf{j}}-6t^2\hat{\mathbf{k}}).\end{aligned}$$

Thus, $\hat{\mathbf{B}} = \dfrac{(-2, 3t, -6t^2)}{\sqrt{4 + 9t^2 + 36t^4}}$.

44. $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = (1, 2t, 2t) \qquad |\mathbf{v}| = \sqrt{1 + 4t^2 + 4t^2} = \sqrt{1 + 8t^2} \qquad \mathbf{a} = \dfrac{d\mathbf{v}}{dt} = (0, 2, 2)$

The normal component of velocity is always zero, and the tangential component of velocity is speed $|\mathbf{v}| = \sqrt{1 + 8t^2}$.

$$a_T = \frac{d}{dt}|\mathbf{v}| = \frac{8t}{\sqrt{1 + 8t^2}} \qquad a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \left(8 - \frac{64t^2}{1 + 8t^2}\right)^{1/2} = \frac{2\sqrt{2}}{\sqrt{1 + 8t^2}}$$

45. $\mathrm{W} = \displaystyle\int_1^4 \mathbf{F} \cdot dx\hat{\mathbf{i}} = \int_1^4 \frac{2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}}{x^2} \cdot dx\hat{\mathbf{i}} = \int_1^4 \frac{2}{x^2}dx = \left\{-\frac{2}{x}\right\}_1^4 = \frac{3}{2}$

46. (a) The acceleration of the ball after it leaves the table is $\mathbf{a} = -g\hat{\mathbf{j}}$ where $g = 9.81$, so that $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{C}$. If we take $t = 0$ at the instant the ball leaves the table, $\mathbf{v}(0) = \hat{\mathbf{i}}/2$. This implies that $\mathbf{C} = \hat{\mathbf{i}}/2$, and

$$\mathbf{r} = -\frac{1}{2}gt^2\hat{\mathbf{j}} + \frac{t}{2}\hat{\mathbf{i}} + \mathbf{D}.$$

Since $\mathbf{r}(0) = \mathbf{0}$, it follows that $\mathbf{D} = \mathbf{0}$, and $\mathbf{r} = -\frac{1}{2}gt^2\hat{\mathbf{j}} + \frac{t}{2}\hat{\mathbf{i}}$. The ball strikes the floor when the y-component of $\mathbf{r}$ is -1,

$$-1 = -\frac{1}{2}gt^2, \qquad \text{or,} \qquad t = \sqrt{\frac{2}{g}} = \sqrt{\frac{2}{9.81}}.$$

At this instant, $\mathbf{v} = -g\sqrt{\dfrac{2}{9.81}}\hat{\mathbf{j}} + \dfrac{1}{2}\hat{\mathbf{i}} = \dfrac{1}{2}\hat{\mathbf{i}} - \sqrt{19.62}\hat{\mathbf{j}}$, and therefore its speed when it strikes the floor is

$$|\mathbf{v}| = \sqrt{1/4 + 19.62} = \sqrt{19.87} \text{ m/s}.$$

(b) Its displacement vector when it strikes the floor is

$$\mathbf{r} = -\frac{1}{2}g\left(\frac{2}{g}\right)\hat{\mathbf{j}} + \frac{1}{2}\sqrt{\frac{2}{g}}\hat{\mathbf{i}} = 0.226\hat{\mathbf{i}} - \hat{\mathbf{j}}.$$

(c) After the rebound, the acceleration of the ball is once again $\mathbf{a} = -g\hat{\mathbf{j}}$, so that $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{C}$. If we redefine $t = 0$ at the time of the first bounce, then

$$\mathbf{C} = \mathbf{v}(0) = \frac{4}{5}\left(\frac{1}{2}\hat{\mathbf{i}} + \sqrt{19.62}\hat{\mathbf{j}}\right).$$

Integration now gives

$$\mathbf{r} = -\frac{1}{2}gt^2\hat{\mathbf{j}} + \mathbf{C}t + \mathbf{D}.$$

If we redefine $(x, y) = (0, 0)$ at the position of the first bounce, then $\mathbf{D} = \mathbf{0}$. The ball hits the floor for the second time when the y-component of $\mathbf{r}$ is zero,

$$0 = -\frac{1}{2}gt^2 + \frac{4}{5}\sqrt{19.62}t \qquad \text{or} \qquad t = \frac{16}{5\sqrt{19.62}}.$$

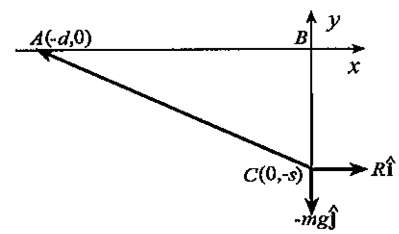
At this instant, the x-component of its displacement is

$$r_x = \frac{2}{5}\left(\frac{16}{5\sqrt{19.62}}\right) = 0.289.$$

Thus, the second bounce takes place 0.515 m from the point on the floor directly below the point it left the table.

47. With the coordinate system shown, the force of gravity on the sleeve is $-mg\hat{\mathbf{j}}$. The spring force is

$$k(\sqrt{d^2+s^2}-d)\frac{-d\hat{\mathbf{i}}+s\hat{\mathbf{j}}}{\sqrt{d^2+s^2}}.$$



The reaction of the rod on the sleeve is $R\hat{\mathbf{i}}$, there being no friction. At equilibrium,

$$k(\sqrt{d^2+s^2}-d)\frac{-d\hat{\mathbf{i}}+s\hat{\mathbf{j}}}{\sqrt{d^2+s^2}}-mg\hat{\mathbf{j}}+R\hat{\mathbf{i}}=\mathbf{0}.$$

The horizontal component of this equation determines R, and the vertical component gives

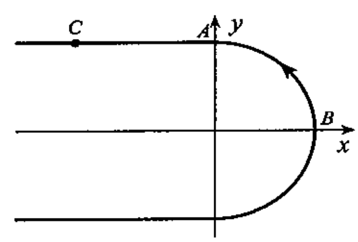
$$\frac{ks(\sqrt{d^2+s^2}-d)}{\sqrt{d^2+s^2}}-mg=0.$$

Thus, $(ks-mg)\sqrt{d^2+s^2}=kds$, and this is the required equation.

48. In Example 11.26, it was shown that $|\mathbf{F}|=k[\sqrt{(1-x)^2+1/4}-1/2]$. A vector in the direction of $\mathbf{F}$ is $(1,1/2)-(x,0)=(1-x,1/2)$, and therefore

$$\begin{aligned}\mathbf{F}&=k[\sqrt{(1-x)^2+1/4}-1/2]\frac{(1-x,1/2)}{\sqrt{(1-x)^2+1/4}}\\&=k(1-x)\left[1-\frac{1}{\sqrt{4(1-x)^2+1}}\right]\hat{\mathbf{i}}+\frac{k}{2}\left[1-\frac{1}{\sqrt{4(1-x)^2+1}}\right]\hat{\mathbf{j}}\ \text{N}.\end{aligned}$$

49. Suppose s is the constant speed of the train, and we take $t=0$ when the train passes through B. Along BA, $x=R\cos\omega t$, $y=R\sin\omega t$, where $\omega=s/R$. The train passes through A at $t=\pi R/(2s)$. The acceleration of the train along BA is



$$\mathbf{a}=\frac{d^2\mathbf{r}}{dt^2}=(-R\omega^2\cos\omega t)\hat{\mathbf{i}}+(-R\omega^2\sin\omega t)\hat{\mathbf{j}}.$$

Consequently, $\displaystyle\lim_{t\to\pi R/(2s)^-}\mathbf{a}(t)=-R\omega^2\hat{\mathbf{j}}$.

Along AC, $x=-s[t-\pi R/(2s)]$, $y=R$, so that acceleration on this part of the track is $\mathbf{a}=\mathbf{0}$. Hence $\mathbf{a}$ is discontinuous at A.