Differential Equations Assignment #3 Benjamin Klassen June 9 2020

① $\frac{dy}{dx} = \frac{4x^3y^2}{x^4y+2}$, $\frac{dx}{dy} = \frac{x^4y+2}{4x^3y^2}$

$$\frac{dx}{dy} = \frac{x}{4y} + \frac{1}{2x^3y^2}$$

$$\frac{dx}{dy} - \frac{1}{4y}x = \frac{1}{2y^2}x^{-3}$$, Case 1 $y \neq 0$

Case 2, $y = 0$

$$\frac{dx}{dy}x^3 - \frac{1}{4y}x^4 = \frac{1}{2y^2}$$, let $u = x^4$, $\frac{du}{dy} = \frac{du}{dx} \cdot \frac{dx}{dy}$

$\frac{du}{dx} = 4x^3$

$$\frac{1}{4}\frac{du}{dy} - \frac{1}{4y}u = \frac{1}{2y^2}$$

$$\frac{du}{dy} - \frac{1}{y}u = \frac{2}{y^2}$$

$$u = e^{\int \frac{1}{y}dy} \cdot \left[2\int \frac{1}{y^2} e^{-\int \frac{1}{y}dy} dy + C\right]$$

$$u = y \cdot \left[2\int \frac{1}{y^2} e^{\ln|y|^{-1}} + C\right]$$

$$= y \cdot \left[2\int \frac{1}{y^3} dy + C\right]$$

$$= y \cdot \left[2 \cdot \frac{y^{-2}}{-2} + C\right]$$

$$u = \frac{-1}{y} + Cy$$

$$x^4 = \frac{-1}{y} + Cy$$, $y = 0$

(2) $6y^2dx - x(2x^3+y)dy = 0$

$6y^2dx = x(2x^3+y)dy$

$\frac{dx}{dy} = \frac{x(2x^3+y)}{6y^2}$

$= \frac{2x^4}{6y^2} + \frac{x}{6y}$

$\frac{dx}{dy} - \frac{1}{6y}x = \frac{1}{3y^2}x^4$

$\frac{dx}{dy}x^{-4} - \frac{1}{6y}x^{-3} = \frac{1}{3y^2}$, let $u = x^{-3}$, $\frac{du}{dy} = \frac{du}{dx}\cdot\frac{dx}{dy}$

$\frac{du}{dx} = -3x^{-4}$ $\frac{dx}{dy} = -\frac{1}{3}x^4\frac{du}{dy}$

$-\frac{1}{3}\frac{du}{dy} - \frac{1}{6y}u = \frac{1}{3y^2}$

$\frac{du}{dy} + \frac{1}{2y}u = -\frac{1}{y^2}$, $y \neq 0$ (Case 1)

$u = e^{-\frac{1}{2}\int\frac{1}{y}dy}\cdot\left[-\int\frac{1}{y^2}e^{\frac{1}{2}\int\frac{1}{y}dy}dy + C\right]$

$= e^{\ln|y^{-1/2}|}\cdot\left[-\int\frac{1}{y^2}\cdot\frac{1}{y^{1/2}}\right.$

$= y^{-1/2}\cdot\left[-\int\frac{1}{y^{3/2}}dy + C\right]$

$u = y^{-1/2}\left[2y^{-1/2} + C\right]$

$x^{-3} = \frac{2}{y} + \frac{C}{\sqrt{y}}$

$x^{-3}y = 2 + Cy^{1/2}$

$y = 2x^3 + Cx^3y^{1/2}$

$(y-2x^3)^2 = Cx^6y$, $y = 0$

③ $xy'' = y'(\ln y' - \ln x)$

$x\frac{d^2y}{dx^2} = \frac{dy}{dx}\left(\ln\frac{dy}{dx} - \ln x\right)$, dependent-variable y absent, 2nd-order, linear

$x\frac{du}{dx} = u(\ln u - \ln x)$,

$\frac{du}{dx} = \frac{u}{x}\left(\ln\frac{u}{x}\right)$, $v = \frac{u}{x}$.

$v + x\frac{dv}{dx} = v(\ln v)$

$v\ln v - v = x\frac{dv}{dx}$

$\int\frac{1}{x}dx = \int\frac{1}{v\ln v - v}dv$, case 1, $v\ln v - v = 0$, is impossible since $\ln v \neq 0$

$\ln x + C = \int\frac{1}{v(\ln v - 1)}dv$ $\quad \frac{d(\ln v - 1)}{dv} = \frac{1}{v}$, $d(\ln v - 1) = \frac{1}{v}dv$

$= \int\frac{1}{(\ln v - 1)}d(\ln v - 1)$

$\ln|\ln v - 1| = \ln x + C$, $C = \ln C$

$\ln|\ln v - 1| = \ln|xC|$

$\ln v - 1 = xC$

$\ln|\frac{u}{x}| = xC + 1$

$\frac{u}{x} = e^{C_1x+1}$

$\frac{dy}{dx} = xe^{Cx+1}$ $\quad \frac{de^{Cx+1}}{dx} = C_1e^{C_1x+1}$

$d(e^{Cx+1}) = Ce^{C_1x+1}dx$

$\int dy = \int xe^{Cx+1}dx =$

$y = \frac{1}{C}\int x\,d(e^{Cx+1})$, $= \frac{1}{C}\left[xe^{C_1x+1} - \int e^{Cx+1}dx\right]$ $\longrightarrow$

$$y = \frac{1}{c}\left[x e^{c_1 x+1} - \frac{1}{c} e^{c_1 x+1}\right.$$

$$= \frac{e^{c_1 x+1}}{c_1}\left(x - \frac{1}{c_1}\right) + C_2$$

④ $3yy'y''-(y')^3+1=0$, x - absent

$3yu^2\frac{du}{dy}-u^3+1=0$

$3yu^2\frac{du}{dy}=u^3-1$

Case 1, $u^3-1=0$

$\int\frac{u^2}{u^3-1}du=\int\frac{1}{3y}dy$

$\frac{1}{3}\int\frac{1}{u^3-1}d(u^3-1)=\frac{1}{3}\ln|y|+C$

$\frac{1}{3}\ln|u^3-1|=\frac{1}{3}\ln|y|+C$

$\ln|u^3-1|=\ln|yC|$

$u^3-1=yC$

$\frac{dy}{dx}=(yC+1)^{1/3}$

$\int\frac{1}{(yC_1+1)^{1/3}}dy=\int dx$

$-1/3$

$u=yC_1+1$

$\frac{du}{dy}=C_1$

$\frac{1}{C_1}\int\frac{1}{u^{1/3}}du=\int dx$, $yC_1+1\neq0$, $y\neq\frac{-1}{C_1}$

$\frac{3}{2C_1}(yC_1+1)^{2/3}=x+C_2$ $\therefore 3(C_1y+1)^{2/3}-2C_1x=C_2$

(5) $x^3 y'' - x^2 y' = 3 - x^2$, y-absent, linear

$x^3 \frac{du}{dx} - x^2 u = 3 - x^2$

$\frac{du}{dx} - \frac{1}{x} u = \frac{3 - x^2}{x^3}$

$u = e^{\int \frac{1}{x}} \cdot \left[\int \frac{3 - x^2}{x^3} e^{-\int 1/x \, dx} dx + C \right]$

$u = x \cdot \left[\int \frac{3 - x^2}{x^4} dx + C \right]$

$u = x \cdot \left[\int \frac{3}{x^4} dx - \int \frac{1}{x^2} dx + C \right]$

x^{-4}

$u = x \cdot \left[-1x^{-3} + x^{-1} + C \right]$

$\frac{dy}{dx} = 1 - x^{-2} + Cx$

$y = x + x^{-1} + \frac{C_1 x^2}{2} + C_2$ new C_1

$y = \frac{1}{x} + x + C_1 x^2 + C_2$

⑥ $yy'' = (y')^2(1 - y'\sin y - yy'\cos y)$, x-absent, 2nd-order

$u = \frac{dy}{dx}$ $\quad \frac{dy^2}{dx^2} = u \cdot \frac{du}{dy}$

$yu \cdot \frac{du}{dy} = u^2(1 - u\sin y - yu\cos y)$

$\frac{du}{dy} = \frac{u}{y}(1 - u\sin y - yu\cos y)$

$\frac{du}{dy} = \frac{u}{y} - \frac{u^2}{y}\sin y - u^2\cos y$

$\frac{du}{dy} - \frac{1}{y}u = -u^2\left(\frac{\sin y}{y} + \cos y\right)$, Bernoulli

$\frac{1}{u^2}\frac{du}{dy} - \frac{1}{yu} = -\left(\frac{\sin y}{y} + \cos y\right)$, let $1/u = v$, $\frac{dv}{dy} = \frac{dv}{du} \cdot \frac{du}{dy}$

$-\frac{dv}{dy} - \frac{v}{y} = -\left(\frac{\sin y}{y} + \cos y\right)$ $\quad \frac{dv}{du} = \frac{-1}{u^2}$

$\frac{dv}{dy} + \frac{v}{y} = \frac{\sin y}{y} + \cos y$

$v = e^{-\int \frac{1}{y}dy} \cdot \left[\int \left[\frac{\sin y}{y} + \cos y\right] e^{\int \frac{1}{y}dy} dy + C\right]$

$v = \frac{1}{y}\left[\int \sin y + y\cos y + C\right]$

$v = \frac{-\cos y}{y} + \frac{C_1}{y} + \left[\int y\cos y\right]\frac{1}{y}$ $\quad \int y\, d(\sin y) = y\sin y - \int \sin y\, dy$

$= \left[y\sin y + \cos y\right]\frac{1}{y}$

$= \frac{-\cos y}{y} + \frac{C_1}{y} + \sin y + \frac{\cos y}{y}$

$\frac{dx}{dy} = \frac{C_1}{y} + \sin y$

$\int dx = \int \frac{C_1}{y} + \sin y\, dy$, $x + C_2 = C_1\ln|y| - \cos y$

$x = C_1\ln|y| - \cos y + C_2$