CHAPTER 2

EXERCISES 2.1

1. $\lim_{x\to 7} \frac{x^2-5}{x+2} = \frac{49-5}{7+2} = \frac{44}{9}$

2. $\lim_{x\to -2} \frac{x^3+8}{x+5} = \frac{0}{3} = 0$

3. $\lim_{x\to -5} \frac{x^2+3x+2}{x^2+25} = \frac{25-15+2}{25+25} = \frac{6}{25}$

4. $\lim_{x\to 0} \frac{x^2+3x}{3x^2-2x} = \lim_{x\to 0} \frac{x(x+3)}{x(3x-2)} = \lim_{x\to 0} \frac{x+3}{3x-2} = \frac{3}{-2} = -\frac{3}{2}$

5. $\lim_{x\to 3^+} \frac{2x-3}{x^2-5} = \frac{6-3}{9-5} = \frac{3}{4}$

6. $\lim_{x\to 2^-} \frac{2x-4}{3x+2} = \frac{0}{8} = 0$

7. $\lim_{x\to 0^-} \frac{x^4+5x^3}{3x^4-x^3} = \lim_{x\to 0^-} \frac{x^3(x+5)}{x^3(3x-1)} = \lim_{x\to 0^-} \frac{x+5}{3x-1} = \frac{5}{-1} = -5$

8. $\lim_{x\to 2^+} \frac{x^2+2x+4}{x-3} = \frac{4+4+4}{-1} = -12$

9. $\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2} (x+2) = 4$

10. $\lim_{x\to 3^+} \frac{x^2-9}{x-3} = \lim_{x\to 3^+} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3^+} (x+3) = 6$

11. $\lim_{x\to 5^-} \frac{x^2-25}{x-5} = \lim_{x\to 5^-} \frac{(x+5)(x-5)}{x-5} = \lim_{x\to 5^-} (x+5) = 10$

12. $\lim_{x\to 3} \frac{x^2-2x-3}{3-x} = \lim_{x\to 3} \frac{(x-3)(x+1)}{3-x} = \lim_{x\to 3} [-(x+1)] = -4$

13. $\lim_{x\to 2} \frac{x^2-4x+4}{x-2} = \lim_{x\to 2} \frac{(x-2)^2}{x-2} = \lim_{x\to 2} (x-2) = 0$

14. $\lim_{x\to 2} \frac{x^3-6x^2+12x-8}{x^2-4x+4} = \lim_{x\to 2} \frac{(x-2)^3}{(x-2)^2} = \lim_{x\to 2} (x-2) = 0$

15. $\lim_{x\to 1} \frac{x^3-6x^2+11x-6}{x^2-3x+2} = \lim_{x\to 1} \frac{(x-1)(x-2)(x-3)}{(x-1)(x-2)} = \lim_{x\to 1} (x-3) = -2$

16. $\lim_{x\to 2} \frac{x^3-6x^2+11x-6}{x^2-3x+2} = \lim_{x\to 2} \frac{(x-1)(x-2)(x-3)}{(x-1)(x-2)} = \lim_{x\to 2} (x-3) = -1$

17. $\lim_{x\to 3^+} \frac{x^3-6x^2+11x-6}{x^2-3x+2} = \frac{27-54+33-6}{9-9+2} = 0$

18. $\lim_{x\to 3^-} \frac{x^3-6x^2+11x-6}{x^2-3x+2} = \frac{27-54+33-6}{9-9+2} = 0$

19. $\lim_{x\to 0} \frac{x^3-6x^2+11x-6}{x^2-3x+2} = \frac{-6}{2} = -3$

20. $\lim_{x\to -1} \frac{12x+5}{x^2-2x+1} = \frac{-7}{4} = -\frac{7}{4}$

21. $\lim_{x\to 1} \sqrt{\frac{2-x}{2+x}} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$

22. $\lim_{x\to 5} \frac{\sqrt{1-x^2}}{3x+2}$ does not exist

23. $\lim_{x\to 0} \frac{\tan x}{\sin x} = \lim_{x\to 0} \frac{\sin x}{\sin x \cos x} = \lim_{x\to 0} \frac{1}{\cos x} = 1$

24. $\lim_{x\to \pi/4} \frac{\sin x}{\tan x} = \frac{1/\sqrt{2}}{1} = \frac{1}{\sqrt{2}}$

25. $\lim_{x\to 0} \frac{\sin 4x}{\sin 2x} = \lim_{x\to 0} \frac{2\sin 2x \cos 2x}{\sin 2x} = \lim_{x\to 0} (2\cos 2x) = 2$

26. $\lim_{x\to 0^+} \frac{\sin 6x}{\sin 3x} = \lim_{x\to 0^+} \frac{2\sin 3x \cos 3x}{\sin 3x} = \lim_{x\to 0^+} (2\cos 3x) = 2$

27. $\lim_{x\to 0^+} \frac{\sin 2x}{\tan x} = \lim_{x\to 0^+} \frac{2\sin x \cos x}{\sin x / \cos x} = \lim_{x\to 0^+} (2\cos^2 x) = 2$

28. $\lim_{x\to 2} \frac{x-2}{\sqrt{x}-\sqrt{2}} = \lim_{x\to 2} \frac{(\sqrt{x}+\sqrt{2})(\sqrt{x}-\sqrt{2})}{\sqrt{x}-\sqrt{2}} = \lim_{x\to 2} (\sqrt{x}+\sqrt{2}) = 2\sqrt{2}$

29. $\lim_{x\to 0} \frac{\sqrt{1-x}-\sqrt{1+x}}{x} = \lim_{x\to 0}\left(\frac{\sqrt{1-x}-\sqrt{1+x}}{x}\frac{\sqrt{1-x}+\sqrt{1+x}}{\sqrt{1-x}+\sqrt{1+x}}\right)$

$= \lim_{x\to 0}\frac{(1-x)-(1+x)}{x(\sqrt{1-x}+\sqrt{1+x})} = \lim_{x\to 0}\frac{-2}{\sqrt{1-x}+\sqrt{1+x}} = \frac{-2}{1+1} = -1$

30. $\lim_{x\to 5^+}\frac{|x^2-25|}{x^2-25} = \lim_{x\to 5^+}\frac{x^2-25}{x^2-25} = \lim_{x\to 5^+}(1) = 1$ **31.** $\lim_{x\to 5^-}\frac{|x^2-25|}{x^2-25} = \lim_{x\to 5^-}\frac{-(x^2-25)}{x^2-25} = -1$

32. Since $\lim_{x\to 5^+}\frac{|x^2-25|}{x^2-25} = 1$ (Exercise 30) and $\lim_{x\to 5^-}\frac{|x^2-25|}{x^2-25} = \lim_{x\to 5^-}\frac{-(x^2-25)}{x^2-25} = \lim_{x\to 5^-}(-1) = -1$, it follows that the given limit does not exist.

33. $\lim_{x\to 0^+}\frac{\sqrt{x+2}-\sqrt{2}}{\sqrt{x}} = \lim_{x\to 0^+}\left(\frac{\sqrt{x+2}-\sqrt{2}}{\sqrt{x}}\frac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}\right) = \lim_{x\to 0^+}\frac{x+2-2}{\sqrt{x}(\sqrt{x+2}+\sqrt{2})}$

$= \lim_{x\to 0^+}\frac{\sqrt{x}}{\sqrt{x+2}+\sqrt{2}} = 0$

34. $\lim_{x\to 0}\frac{1-\sqrt{x^2+1}}{2x^2} = \lim_{x\to 0}\left[\frac{1-\sqrt{x^2+1}}{2x^2}\frac{1+\sqrt{x^2+1}}{1+\sqrt{x^2+1}}\right] = \lim_{x\to 0}\frac{1-(x^2+1)}{2x^2\left(1+\sqrt{x^2+1}\right)}$

$= \lim_{x\to 0}\frac{-1}{2\left(1+\sqrt{x^2+1}\right)} = \frac{-1}{2(2)} = -\frac{1}{4}$

35. $\lim_{x\to -2}\frac{x+2}{\sqrt{-x}-\sqrt{2}} = \lim_{x\to -2}\left(\frac{x+2}{\sqrt{-x}-\sqrt{2}}\frac{\sqrt{-x}+\sqrt{2}}{\sqrt{-x}+\sqrt{2}}\right) = \lim_{x\to -2}\frac{(x+2)(\sqrt{-x}+\sqrt{2})}{-x-2}$

$= \lim_{x\to -2}[-(\sqrt{-x}+\sqrt{2})] = -2\sqrt{2}$

36. $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim_{x\to 0}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{x}\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$

$= \lim_{x\to 0}\frac{(1+x)-(1-x)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)} = \lim_{x\to 0}\frac{2}{\sqrt{1+x}+\sqrt{1-x}} = \frac{2}{1+1} = 1$

37. $\lim_{x\to -2^+}\frac{\sqrt{x+3}-\sqrt{-x-1}}{\sqrt{x+2}} = \lim_{x\to -2^+}\left(\frac{\sqrt{x+3}-\sqrt{-x-1}}{\sqrt{x+2}}\frac{\sqrt{x+3}+\sqrt{-x-1}}{\sqrt{x+3}+\sqrt{-x-1}}\right)$

$= \lim_{x\to -2^+}\frac{(x+3)-(-x-1)}{\sqrt{x+2}(\sqrt{x+3}+\sqrt{-x-1})} = \lim_{x\to -2^+}\frac{2(x+2)}{\sqrt{x+2}(\sqrt{x+3}+\sqrt{-x-1})}$

$= \lim_{x\to -2^+}\frac{2\sqrt{x+2}}{\sqrt{x+3}+\sqrt{-x-1}} = 0$

38. $\lim_{x\to 0}\frac{x}{\sqrt{x+4}-2} = \lim_{x\to 0}\left[\frac{x}{\sqrt{x+4}-2}\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}\right] = \lim_{x\to 0}\frac{x\left(\sqrt{x+4}+2\right)}{(x+4)-4} = \lim_{x\to 0}\left(\sqrt{x+4}+2\right) = 4$

39. $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{2+x}-\sqrt{2-x}} = \lim_{x\to 0}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{2+x}-\sqrt{2-x}}\frac{\sqrt{2+x}+\sqrt{2-x}}{\sqrt{2+x}+\sqrt{2-x}}\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

$= \lim_{x\to 0}\frac{(1+x-1+x)(\sqrt{2+x}+\sqrt{2-x})}{(2+x-2+x)(\sqrt{1+x}+\sqrt{1-x})} = \lim_{x\to 0}\frac{\sqrt{2+x}+\sqrt{2-x}}{\sqrt{1+x}+\sqrt{1-x}} = \sqrt{2}$

40. $\lim_{x\to 0}\frac{\sqrt{x+1}-\sqrt{2x+1}}{\sqrt{3x+4}-\sqrt{2x+4}} = \lim_{x\to 0}\left(\frac{\sqrt{x+1}-\sqrt{2x+1}}{\sqrt{3x+4}-\sqrt{2x+4}}\frac{\sqrt{x+1}+\sqrt{2x+1}}{\sqrt{x+1}+\sqrt{2x+1}}\frac{\sqrt{3x+4}+\sqrt{2x+4}}{\sqrt{3x+4}+\sqrt{2x+4}}\right)$

$= \lim_{x\to 0}\frac{(x+1-2x-1)(\sqrt{3x+4}+\sqrt{2x+4})}{(3x+4-2x-4)(\sqrt{x+1}+\sqrt{2x+1})}$

$= \lim_{x\to 0}\left(-\frac{\sqrt{3x+4}+\sqrt{2x+4}}{\sqrt{x+1}+\sqrt{2x+1}}\right) = -2$

41. $\lim_{x\to 1}\frac{\sqrt{x+3}-2}{x-1} = \lim_{x\to 1}\left(\frac{\sqrt{x+3}-2}{x-1}\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}\right) = \lim_{x\to 1}\frac{x+3-4}{(x-1)(\sqrt{x+3}+2)} = \lim_{x\to 1}\frac{1}{\sqrt{x+3}+2} = \frac{1}{4}$

42. $\displaystyle\lim_{x\to a}\frac{x^2-a^2}{x-a}=\lim_{x\to a}\frac{(x+a)(x-a)}{x-a}=\lim_{x\to a}(x+a)=2a$

43. $\displaystyle\lim_{x\to a}\frac{x^3-a^3}{x-a}=\lim_{x\to a}\frac{(x-a)(x^2+ax+a^2)}{x-a}=\lim_{x\to a}(x^2+ax+a^2)=3a^2$

44. $\displaystyle\lim_{x\to -a}\frac{x+a}{x^2+ax-x-a}=\lim_{x\to -a}\frac{x+a}{(x+a)(x-1)}=\lim_{x\to -a}\frac{1}{x-1}=-\frac{1}{a+1}$

45. $\displaystyle\lim_{x\to 0}\frac{\sin 2ax}{\sin ax}=\lim_{x\to 0}\frac{2\sin ax\cos ax}{\sin ax}=\lim_{x\to 0}(2\cos ax)=2$

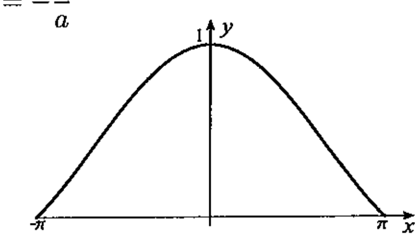
46. $\displaystyle\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a}=\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{(\sqrt{x}+\sqrt{a})(\sqrt{x}-\sqrt{a})}=\lim_{x\to a}\frac{1}{\sqrt{x}+\sqrt{a}}=\frac{1}{2\sqrt{a}}$

47. $$\lim_{x\to 0^+}\frac{\sqrt{x+a}-\sqrt{a}}{\sqrt{x}}=\lim_{x\to 0^+}\left[\frac{\sqrt{x+a}-\sqrt{a}}{\sqrt{x}}\,\frac{\sqrt{x+a}+\sqrt{a}}{\sqrt{x+a}+\sqrt{a}}\right]$$
$$=\lim_{x\to 0^+}\frac{x+a-a}{\sqrt{x}\left(\sqrt{x+a}+\sqrt{a}\right)}=\lim_{x\to 0^+}\frac{\sqrt{x}}{\sqrt{x+a}+\sqrt{a}}=0$$

48. $$\lim_{x\to 0}\frac{\sqrt{a+x}-\sqrt{a-x}}{x}=\lim_{x\to 0}\left[\frac{\sqrt{a+x}-\sqrt{a-x}}{x}\,\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}\right]$$
$$=\lim_{x\to 0}\frac{(a+x)-(a-x)}{x\left(\sqrt{a+x}+\sqrt{a-x}\right)}=\lim_{x\to 0}\frac{2}{\sqrt{a+x}+\sqrt{a-x}}=\frac{2}{\sqrt{a}+\sqrt{a}}=\frac{1}{\sqrt{a}}$$

49. $$\lim_{x\to 0}\frac{\sqrt{x^2+a^2}-\sqrt{2x^2+a^2}}{\sqrt{3x^2+4}-\sqrt{2x^2+4}}=\lim_{x\to 0}\left[\frac{\sqrt{x^2+a^2}-\sqrt{2x^2+a^2}}{\sqrt{3x^2+4}-\sqrt{2x^2+4}}\,\frac{\sqrt{x^2+a^2}+\sqrt{2x^2+a^2}}{\sqrt{x^2+a^2}+\sqrt{2x^2+a^2}}\,\frac{\sqrt{3x^2+4}+\sqrt{2x^2+4}}{\sqrt{3x^2+4}+\sqrt{2x^2+4}}\right]$$
$$=\lim_{x\to 0}\left[\frac{(x^2+a^2-2x^2-a^2)\left(\sqrt{3x^2+4}+\sqrt{2x^2+4}\right)}{(3x^2+4-2x^2-4)\left(\sqrt{x^2+a^2}+\sqrt{2x^2+a^2}\right)}\right]$$
$$=\lim_{x\to 0}\left[-\frac{\sqrt{3x^2+4}+\sqrt{2x^2+4}}{\sqrt{x^2+a^2}+\sqrt{2x^2+a^2}}\right]=-\frac{2}{a}$$

50. Although the plot does not show it, there should be a hole in the graph at $x=0$. The plot suggests that the limit is 1.



51. Since $-1\le\sin(1/x)\le 1$ for all x, it follows that $-|x|\le|x|\sin(1/x)\le|x|$. Since $-|x|$ is always nonpositive and $|x|$ is always nonnegative, we can say that $-|x|\le x\sin(1/x)\le|x|$. Because $\lim_{x\to 0}(-|x|)=\lim_{x\to 0}|x|=0$, the squeeze theorem implies that $\lim_{x\to 0}x\sin(1/x)=0$ also.

52. Since $-1\le\cos(3/x)\le 1$ for all x, it follows that $-x^4\le x^4\cos(3/x)\le x^4$. Because $\lim_{x\to 0}(-x^4)=\lim_{x\to 0}x^4=0$, the squeeze theorem implies that $\lim_{x\to 0}x^4\cos(3/x)=0$ also.

53. Left and right limits exist, but the "full" limit does not exist.

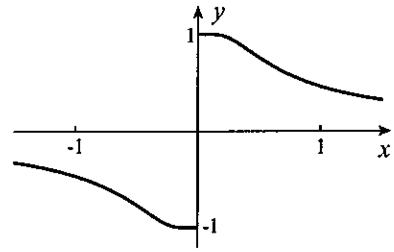
54. This statement is false. For example, $x^2<2x^2$ for all $x\neq 0$, but $\lim_{x\to 0}x^2=\lim_{x\to 0}2x^2=0$.

55. If we use the stated result, then,

$$\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}=\lim_{h\to 0}\frac{1}{h}\{(x+h-x)[(x+h)^{n-1}+(x+h)^{n-2}x+\cdots+(x+h)x^{n-2}+x^{n-1}]\}$$
$$=\lim_{h\to 0}[(x+h)^{n-1}+(x+h)^{n-2}x+\cdots+(x+h)x^{n-2}+x^{n-1}]=nx^{n-1}.$$

56. $$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=\lim_{h\to 0}\left[\frac{\sqrt{x+h}-\sqrt{x}}{h}\,\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right]=\lim_{h\to 0}\frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$
$$=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

57. A graph of the function is shown to the right. It indicates that the left-hand limit is -1, the right-hand limit is 1, and because these limits are different, the limit as $x \to 0$ does not exist. The function has no value at $x = 0$.

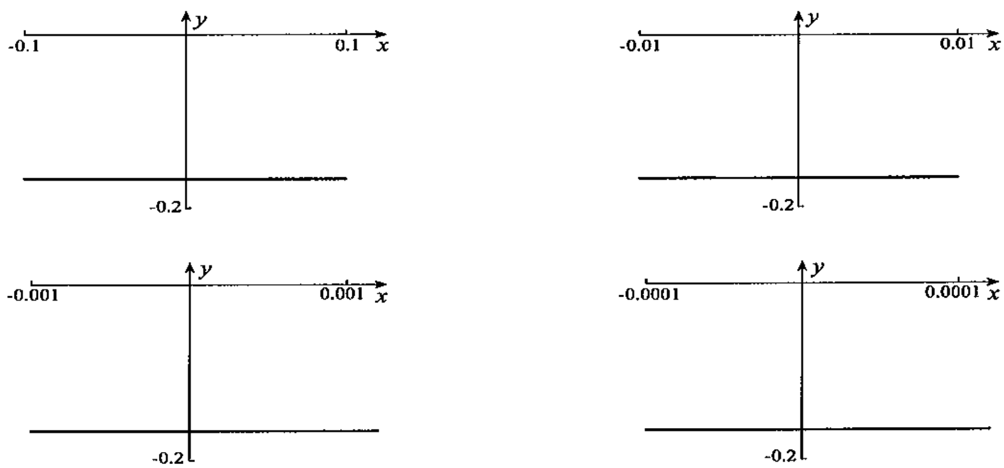


58. (a) Our calculator gave

x	0.1	0.01	0.001	0.0001	0.00001	0.000001	0.0000001
$(\sin x - x)/x^3$	-0.16658	-0.16667	-0.16667	-0.1667	-0.17	0.0	0.0

It would appear that the limit is 0.

(b) Plots of the function on the suggested intervals are shown below.



They suggest that the limit is approximately -0.17.

59. The only way for this limit to exist is for $\lim_{x\to a} g(x) = L$.

60. If we set $z = -x$, then $\displaystyle\lim_{x\to -a} f(x) = \lim_{z\to a} f(-z) = \lim_{z\to a} f(z) = L$.

61. If we set $z = -x$, then $\displaystyle\lim_{x\to -a^-} f(x) = \lim_{z\to a^+} f(-z) = \lim_{z\to a^+} f(z) = L$.

62. This limit cannot be determined.

63. If we set $z = -x$, then $\displaystyle\lim_{x\to -a} f(x) = \lim_{z\to a} f(-z) = \lim_{z\to a} [-f(z)] = -\lim_{z\to a} f(z) = -L$.

64. If we set $z = -x$, then $\displaystyle\lim_{x\to -a^-} f(x) = \lim_{z\to a^+} f(-z) = \lim_{z\to a^+} [-f(z)] = -\lim_{z\to a^+} f(z) = -L$.

65. There is not enough information to find $\lim_{x\to -a^+} f(x)$.

66. The limit is 0 if $F = 0$; it does not exist if $F \neq 0$.

67. (a) Since $\displaystyle\lim_{x\to 0^+} e^{-1/x} = 0$, it follows that $\displaystyle\lim_{x\to 0^+} \frac{a + ce^{-1/x}}{b + de^{-1/x}} = \frac{a}{b}$.

(b) If we multiply numerator and denominator by $e^{1/x}$ and use the fact that $\displaystyle\lim_{x\to 0^-} e^{1/x} = 0$, we obtain

$$\lim_{x\to 0^-} \frac{a + ce^{-1/x}}{b + de^{-1/x}} = \lim_{x\to 0^-} \frac{ae^{1/x} + c}{be^{1/x} + d} = \frac{c}{d}.$$

(c) The limit does not exist since left and right limits are not the same, unless $a/b = c/d$, in which case the limit is a/b.

EXERCISES 2.2

1. $\lim_{x\to 2^+} \frac{1}{x-2} = \infty$

2. $\lim_{x\to 2^-} \frac{1}{x-2} = -\infty$

3. $\lim_{x\to 2} \frac{1}{x-2}$ does not exist since $\lim_{x\to 2^-} \frac{1}{x-2} = -\infty$ and $\lim_{x\to 2^+} \frac{1}{x-2} = \infty$

4. $\lim_{x\to 2^+} \frac{1}{(x-2)^2} = \infty$

5. $\lim_{x\to 2^-} \frac{1}{(x-2)^2} = \infty$

6. $\lim_{x\to 2} \frac{1}{(x-2)^2} = \infty$

7. $\lim_{x\to 1} \frac{5x}{(x-1)^3}$ does not exist since $\lim_{x\to 1^-} \frac{5x}{(x-1)^3} = -\infty$ and $\lim_{x\to 1^+} \frac{5x}{(x-1)^3} = \infty$

8. $\lim_{x\to 1/2} \frac{6x^2+7x-5}{2x-1} = \lim_{x\to 1/2} \frac{(3x+5)(2x-1)}{2x-1} = \lim_{x\to 1/2} (3x+5) = \frac{13}{2}$

9. $\lim_{x\to 1} \frac{2x+3}{x^2-2x+1} = \lim_{x\to 1} \frac{2x+3}{(x-1)^2} = \infty$

10. $\lim_{x\to 2} \frac{x-2}{x^2-4x+4} = \lim_{x\to 2} \frac{x-2}{(x-2)^2} = \lim_{x\to 2} \frac{1}{x-2}$

Since $\lim_{x\to 2^+} \frac{1}{x-2} = \infty$ and $\lim_{x\to 2^-} \frac{1}{x-2} = -\infty$, the given limit does not exist.

11. $\lim_{x\to 0} \csc x$ does not exist since $\lim_{x\to 0^-} \csc x = -\infty$ and $\lim_{x\to 0^+} \csc x = \infty$.

12. $\lim_{x\to \pi/4} \sec(x-\pi/4) = 1$

13. This limit does not exist since $\lim_{x\to 3\pi/4^-} \sec(x-\pi/4) = \infty$ and $\lim_{x\to 3\pi/4^+} \sec(x-\pi/4) = -\infty$.

14. $\lim_{x\to 0^+} \cot x = \infty$

15. $\lim_{x\to \pi/2^+} \tan x = -\infty$

16. $\lim_{x\to \pi/2^-} \tan x = \infty$

17. $\lim_{x\to 1} \frac{x^2-2x+1}{x^3-3x^2+3x-1} = \lim_{x\to 1} \frac{(x-1)^2}{(x-1)^3} = \lim_{x\to 1} \frac{1}{x-1}$ which does not exist since $\lim_{x\to 1^-} \frac{1}{x-1} = -\infty$ and $\lim_{x\to 1^+} \frac{1}{x-1} = \infty$

18. $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x^2} = \lim_{x\to 0} \left[\frac{\sqrt{1+x}-1}{x^2} \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}\right] = \lim_{x\to 0} \frac{(1+x)-1}{x^2(\sqrt{1+x}+1)} = \lim_{x\to 0} \frac{1}{x(\sqrt{1+x}+1)}$

Since $\lim_{x\to 0^+} \frac{1}{x(\sqrt{x+1}+1)} = \infty$ and $\lim_{x\to 0^-} \frac{1}{x(\sqrt{x+1}+1)} = -\infty$, the given limit does not exist.

19. $\lim_{x\to 0} \frac{2x}{1-\sqrt{x^2+1}} = \lim_{x\to 0} \left(\frac{2x}{1-\sqrt{x^2+1}} \frac{1+\sqrt{x^2+1}}{1+\sqrt{x^2+1}}\right) = \lim_{x\to 0} \frac{2x(1+\sqrt{x^2+1})}{1-(x^2+1)} = \lim_{x\to 0} \frac{-2(1+\sqrt{x^2+1})}{x}$

This limit does not exist since $\lim_{x\to 0^-} \frac{-2(1+\sqrt{x^2+1})}{x} = \infty$ and $\lim_{x\to 0^+} \frac{-2(1+\sqrt{x^2+1})}{x} = -\infty$.

20. Since $\lim_{x\to 4^+} \frac{|4-x|}{x^2-8x+16} = \lim_{x\to 4^+} \frac{x-4}{(x-4)^2} = \lim_{x\to 4^+} \frac{1}{x-4} = \infty$

and $\lim_{x\to 4^-} \frac{|4-x|}{x^2-8x+16} = \lim_{x\to 4^-} \frac{4-x}{(x-4)^2} = \lim_{x\to 4^-} \frac{-1}{x-4} = \infty$, the given limit does not exist.

21. $\lim_{x\to 0^+} \ln(4x) = -\infty$

22. $\lim_{x\to 1} \frac{1}{\ln|x-1|} = 0$

23. $\lim_{x\to 0} e^{1/x}$ does not exist since $\lim_{x\to 0^+} e^{1/x} = \infty$.

24. $\lim_{x\to 0} e^{1/|x|} = \infty$

25. $\lim_{x\to a^+} \frac{x-a}{x^2-2ax+a^2} = \lim_{x\to a^+} \frac{x-a}{(x-a)^2} = \lim_{x\to a^+} \frac{1}{x-a} = \infty$

26. $\lim_{x\to a} \frac{|x-a|}{x^2-2ax+a^2} = \lim_{x\to a} \frac{|x-a|}{(x-a)^2} = \lim_{x\to a} \frac{1}{|x-a|} = \infty$

27. $\lim_{x\to 0^-} \frac{\sqrt{a+x}-\sqrt{a}}{x^2} = \lim_{x\to 0^-} \left[\frac{\sqrt{a+x}-\sqrt{a}}{x^2} \frac{\sqrt{a+x}+\sqrt{a}}{\sqrt{a+x}+\sqrt{a}}\right]$

$$= \lim_{x\to 0^-} \frac{a+x-a}{x^2\left(\sqrt{a+x}+\sqrt{a}\right)} = \lim_{x\to 0^-} \frac{1}{x\left(\sqrt{a+x}+\sqrt{a}\right)} = -\infty$$

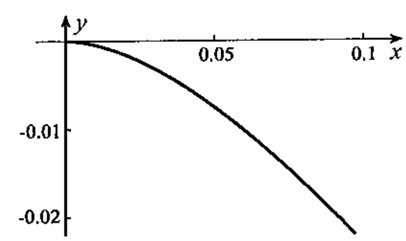
28. Since $\lim_{x\to -a^-} e^{1/(|x|-a)} = \infty$ and $\lim_{x\to -a^+} e^{1/(|x|-a)} = 0$, the limit does not exist.

29. (a) The table suggests that the limit is 0.

x	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
x^2	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
$\ln x$	-2.30	-4.61	-6.91	-9.21	-11.5
$x^2 \ln x$	-2.30×10^{-2}	-4.61×10^{-4}	-6.91×10^{-6}	-9.21×10^{-8}	-1.15×10^{-9}

x	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}
x^2	10^{-12}	10^{-14}	10^{-16}	10^{-18}	10^{-20}
$\ln x$	-13.8	-16.1	-18.4	-20.7	-23.0
$x^2 \ln x$	-1.38×10^{-11}	-1.61×10^{-13}	-1.84×10^{-15}	-2.07×10^{-17}	-2.30×10^{-19}

(b) The graph of $x^2 \ln x$ to the right also suggests that the limit is 0.



30.

x	1.0	0.1	0.05	0.01	0.005
x^{10}	1	10^{-10}	9.77×10^{-14}	10^{-20}	9.77×10^{-24}
$e^{1/x}$	2.72	2.20×10^{4}	4.85×10^{8}	2.69×10^{43}	7.23×10^{86}
$x^{10}e^{1/x}$	2.72	2.20×10^{-6}	4.74×10^{-5}	2.69×10^{23}	7.06×10^{63}

Thus, $\lim_{x\to 0^+} x^{10}e^{1/x} = \infty$.

EXERCISES 2.3

1. $\lim_{x\to\infty} \frac{x+1}{2x-1} = \lim_{x\to\infty} \frac{1+\frac{1}{x}}{2-\frac{1}{x}} = \frac{1}{2}$

2. $\lim_{x\to\infty} \frac{1-x}{3+2x} = \lim_{x\to\infty} \frac{\frac{1}{x}-1}{\frac{3}{x}+2} = -\frac{1}{2}$

3. $\lim_{x\to\infty} \frac{x^2+1}{2x^3+5} = \lim_{x\to\infty} \frac{1+\frac{1}{x^2}}{2x+\frac{5}{x^2}} = 0$

4. $\lim_{x\to\infty} \frac{1-4x^3}{3+2x-x^2} = \lim_{x\to\infty} \frac{\frac{1}{x^2}-4x}{\frac{3}{x^2}+\frac{2}{x}-1} = \infty$

5. $\lim_{x\to-\infty} \frac{2+x-x^2}{3+4x^2} = \lim_{x\to-\infty} \frac{\frac{2}{x^2}+\frac{1}{x}-1}{\frac{3}{x^2}+4} = -\frac{1}{4}$

6. $\lim_{x\to-\infty} \frac{x^3-2x^2}{3x^3+4x^2} = \lim_{x\to-\infty} \frac{1-\frac{2}{x}}{3+\frac{4}{x}} = \frac{1}{3}$

7. $\lim_{x\to-\infty} \frac{x^3-2x^2+x+1}{x^4+3x} = \lim_{x\to-\infty} \frac{1-\frac{2}{x}+\frac{1}{x^2}+\frac{1}{x^3}}{x+\frac{3}{x^2}} = 0$

8. $\lim_{x\to-\infty} \frac{x^3-2x^2+x+1}{x^2-x+1} = \lim_{x\to-\infty} \frac{x-2+\frac{1}{x}+\frac{1}{x^2}}{1-\frac{1}{x}+\frac{1}{x^2}} = -\infty$

9. $\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{2x+1} = \lim_{x\to\infty} \frac{\sqrt{1+\frac{1}{x^2}}}{2+\frac{1}{x}} = \frac{1}{2}$

10. $\lim_{x\to\infty} \frac{3x-1}{\sqrt{5+4x^2}} = \lim_{x\to\infty} \frac{3-\frac{1}{x}}{\sqrt{\frac{5}{x^2}+4}} = \frac{3}{2}$

11. This limit does not exist because the function is not defined for $x < -1/\sqrt{2}$.

12. $\lim_{x\to-\infty} \frac{\sqrt{1-2x}}{x+2} = \lim_{x\to-\infty} \frac{\frac{\sqrt{1-2x}}{\sqrt{-x}}}{\frac{x+2}{\sqrt{-x}}} = \lim_{x\to-\infty} \frac{\sqrt{-\frac{1}{x}+2}}{-\sqrt{-x}+\frac{2}{\sqrt{-x}}} = 0$

13. $\lim_{x\to\infty} \sqrt{\frac{2+x}{x-2}} = \lim_{x\to\infty} \sqrt{\frac{\frac{2}{x}+1}{1-\frac{2}{x}}} = 1$

14. $\lim_{x\to\infty} \frac{\sqrt{3+x}}{\sqrt{x}} = \lim_{x\to\infty} \sqrt{\frac{3}{x}+1} = 1$

15. $\lim_{x\to\infty} (x^2-x^3) = \lim_{x\to\infty} x^2(1-x) = -\infty$

16. $\lim_{x\to\infty} \left(x+\frac{1}{x}\right) = \infty$

17. $\lim_{x\to\infty} \frac{x}{\sqrt{x+5}} = \lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{1+\frac{5}{x}}} = \infty$

18. $\lim_{x\to-\infty} \frac{x^2}{\sqrt{3-x}} = \lim_{x\to-\infty} \frac{\frac{x^2}{\sqrt{-x}}}{\frac{\sqrt{3-x}}{\sqrt{-x}}} = \lim_{x\to-\infty} \frac{-x\sqrt{-x}}{\sqrt{1-\frac{3}{x}}} = \infty$

19. $\lim_{x\to-\infty} \frac{x}{\sqrt[3]{4+x^3}} = \lim_{x\to-\infty} \frac{1}{\sqrt[3]{\frac{4}{x^3}+1}} = 1$

20. $\lim_{x\to\infty} \frac{3x}{\sqrt[3]{2+4x^3}} = \lim_{x\to\infty} \frac{3}{\sqrt[3]{\frac{2}{x^3}+4}} = \frac{3}{\sqrt[3]{4}}$

21. $\lim_{x\to\infty} \frac{1}{2x}\cos x = 0$

22. $\lim_{x\to-\infty} \frac{1}{2x}\cos x = 0$

23. $\lim_{x\to\infty} \frac{\sin 4x}{x^2} = 0$

24. $\lim_{x\to\infty} \frac{\sin^2 x}{x} = 0$

25. $\lim_{x\to-\infty} \tan x$ does not exist

26. $\lim_{x\to\infty} \frac{1}{x}\tan x$ does not exist

27. $\lim_{x\to\infty} (\sqrt{x^2+1}-x) = \lim_{x\to\infty} \left[(\sqrt{x^2+1}-x)\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\right] = \lim_{x\to\infty} \frac{x^2+1-x^2}{\sqrt{x^2+1}+x} = 0$

28. $\lim_{x\to\infty} \left(\sqrt{x^2+4}-x\right) = \lim_{x\to\infty} \left[\left(\sqrt{x^2+4}-x\right)\frac{\sqrt{x^2+4}+x}{\sqrt{x^2+4}+x}\right] = \lim_{x\to\infty} \left[\frac{(x^2+4)-x^2}{\sqrt{x^2+4}+x}\right] = 0$

29. $\lim_{x\to\infty} (\sqrt{2x^2+1}-x) = \lim_{x\to\infty} \left[(\sqrt{2x^2+1}-x)\frac{\sqrt{2x^2+1}+x}{\sqrt{2x^2+1}+x}\right] = \lim_{x\to\infty} \frac{2x^2+1-x^2}{\sqrt{2x^2+1}+x}$

$$= \lim_{x\to\infty} \frac{x^2+1}{\sqrt{2x^2+1}+x} = \lim_{x\to\infty} \frac{x+\frac{1}{x}}{\sqrt{2+\frac{1}{x^2}}+1} = \infty$$

30. $\lim_{x\to-\infty} \left(\sqrt{2x^2+1}-x\right) = \infty$

31. $\lim_{x\to\infty} \frac{\sqrt{3x^2+2}}{x+4} = \lim_{x\to\infty} \frac{\sqrt{3+\frac{2}{x^2}}}{1+\frac{4}{x}} = \sqrt{3}$

32. $\lim_{x\to\infty} \frac{\sqrt{4x^2+7}}{2x+3} = \lim_{x\to\infty} \frac{\sqrt{4+\frac{7}{x^2}}}{2+\frac{3}{x}} = 1$

33. $\lim_{x\to-\infty} \frac{\sqrt{3x^2+2}}{x+4} = \lim_{x\to-\infty} \frac{\frac{1}{x}\sqrt{3x^2+2}}{1+\frac{4}{x}} = \lim_{x\to-\infty} \frac{-\sqrt{3+\frac{2}{x^2}}}{1+\frac{4}{x}} = -\sqrt{3}$

34. $\lim_{x\to-\infty} \frac{\sqrt{4x^2+7}}{2x+3} = \lim_{x\to-\infty} \frac{\frac{\sqrt{4x^2+7}}{x}}{\frac{2x+3}{x}} = \lim_{x\to-\infty} \frac{-\sqrt{4+\frac{7}{x^2}}}{2+\frac{3}{x}} = -1$

35. $\lim_{x\to\infty}(\sqrt{x^2+4}-\sqrt{x^2-1}) = \lim_{x\to\infty}\left[(\sqrt{x^2+4}-\sqrt{x^2-1})\frac{\sqrt{x^2+4}+\sqrt{x^2-1}}{\sqrt{x^2+4}+\sqrt{x^2-1}}\right]$

$$= \lim_{x\to\infty}\frac{(x^2+4)-(x^2-1)}{\sqrt{x^2+4}+\sqrt{x^2-1}} = \lim_{x\to\infty}\frac{5}{\sqrt{x^2+4}+\sqrt{x^2-1}} = 0$$

36. $\lim_{x\to\infty}(\sqrt[3]{1+x}-\sqrt[3]{x}) = \lim_{x\to\infty}\left\{\left[(1+x)^{1/3}-x^{1/3}\right]\frac{(1+x)^{2/3}+(1+x)^{1/3}x^{1/3}+x^{2/3}}{(1+x)^{2/3}+(1+x)^{1/3}x^{1/3}+x^{2/3}}\right\}$

$$= \lim_{x\to\infty}\left[\frac{(1+x)-x}{(1+x)^{2/3}+(1+x)^{1/3}x^{1/3}+x^{2/3}}\right] = 0$$

37. $\lim_{x\to\infty}(\sqrt{x^2+x}-x) = \lim_{x\to\infty}\left[(\sqrt{x^2+x}-x)\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}\right] = \lim_{x\to\infty}\frac{(x^2+x)-x^2}{\sqrt{x^2+x}+x}$

$$= \lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x}}+1} = \frac{1}{2}$$

38. $\lim_{x\to-\infty}\left(\sqrt{x^2+x}-x\right) = \infty$

39. $\lim_{x\to\infty}\frac{x^2+ax-2}{ax^2+5} = \lim_{x\to\infty}\frac{1+\frac{a}{x}-\frac{2}{x^2}}{a+\frac{5}{x^2}} = \frac{1}{a}$

40. $\lim_{x\to\infty}\frac{x}{\sqrt{ax^2+3x+2}} = \lim_{x\to\infty}\frac{1}{\sqrt{a+\frac{3}{x}+\frac{2}{x^2}}} = \frac{1}{\sqrt{a}}$

41. $\lim_{x\to\infty}(\sqrt{x^2+ax}-x) = \lim_{x\to\infty}\left[(\sqrt{x^2+ax}-x)\frac{\sqrt{x^2+ax}+x}{\sqrt{x^2+ax}+x}\right] = \lim_{x\to\infty}\frac{(x^2+ax)-x^2}{\sqrt{x^2+ax}+x}$

$$= \lim_{x\to\infty}\frac{a}{\sqrt{1+\frac{a}{x}}+1} = \frac{a}{2}$$

42. $\lim_{x\to-\infty}\frac{\sqrt{ax^2+7}}{x-3a} = \lim_{x\to-\infty}\frac{\frac{\sqrt{ax^2+7}}{x}}{\frac{x-3a}{x}} = \lim_{x\to-\infty}\frac{-\sqrt{a+\frac{7}{x^2}}}{1-\frac{3a}{x}} = -\sqrt{a}$

43. A vertical asymptote is $x=-3/4$. Since $\lim_{x\to\pm\infty}\frac{2-x}{3+4x} = -\frac{1}{4}$, the horizontal asymptote is $y=-1/4$. With $f(x)$ expressed in the form $f(x) = -\frac{1}{4}+\frac{11/4}{4x+3}$, we can say that the for large negative x, $f(x) < -1/4$, and for large positive x, $f(x) > -1/4$. Hence, the graph approaches the horizontal asymptote from below as $x\to-\infty$ and from above as $x\to\infty$.

44. A vertical asymptote is $x=5/2$. Since $\lim_{x\to\pm\infty}\frac{x+3}{2x-5} = \frac{1}{2}$, the horizontal asymptote is $y=1/2$. With $f(x)$ expressed in the form $f(x) = \frac{1}{2}+\frac{11/2}{2x-5}$, we can say that the for large negative x, $f(x) < 1/2$, and for large positive x, $f(x) > 1/2$. Hence, the graph approaches the horizontal asymptote from below as $x\to-\infty$ and from above as $x\to\infty$.

45. Since $\lim_{x\to\infty}\frac{3x-1}{\sqrt{5+2x^2}} = \lim_{x\to\infty}\frac{3-1/x}{\sqrt{5/x^2+2}} = \frac{3}{\sqrt{2}}$, $y=3/\sqrt{2}$ is a horizontal asymptote as $x\to\infty$. Since $\lim_{x\to-\infty}\frac{3x-1}{\sqrt{5+2x^2}} = \lim_{x\to-\infty}\frac{3-1/x}{-\sqrt{5/x^2+2}} = -\frac{3}{\sqrt{2}}$, $y=-3/\sqrt{2}$ is a horizontal asymptote as $x\to-\infty$. To determine whether the graph approaches $y=3/\sqrt{2}$ from above or below as $x\to\infty$, we write

$$f(x) = \frac{3x-1}{\sqrt{5+2x^2}} = \sqrt{\frac{(3x-1)^2}{5+2x^2}} = \sqrt{\frac{9x^2-6x+1}{2x^2+5}} = \sqrt{\frac{9}{2}-\frac{6x+43/2}{2x^2+5}}.$$

This shows that $f(x) < 3/\sqrt{2}$ for large x, and the graph therefore approaches the asymptote from below. Similarly, for large negative values of x, we express $f(x)$ in the form $f(x) = -\sqrt{\frac{9}{2}-\frac{6x+43/2}{2x^2+5}}$, and this shows that the graph of $f(x)$ approaches $y=-3/\sqrt{2}$ from below as $x\to-\infty$.

46. A vertical asymptote is $x = -3/2$. Since $\lim_{x\to\infty} \dfrac{\sqrt{5x^2+7}}{2x+3} = \lim_{x\to\infty} \dfrac{\sqrt{5+7/x^2}}{2+3/x} = \dfrac{\sqrt{5}}{2}$, $y = \sqrt{5}/2$ is a horizontal asymptote as $x \to \infty$. Since $\lim_{x\to-\infty} \dfrac{\sqrt{5x^2+7}}{2x+3} = \lim_{x\to-\infty} \dfrac{-\sqrt{5+7/x^2}}{2+3/x} = -\dfrac{\sqrt{5}}{2}$, $y = -\sqrt{5}/2$ is a horizontal asymptote as $x \to -\infty$. To determine whether the graph approaches $y = \sqrt{5}/2$ from above or below as $x \to \infty$, we write

$$f(x) = \frac{\sqrt{5x^2+7}}{2x+3} = \sqrt{\frac{5x^2+7}{(2x+3)^2}} = \sqrt{\frac{5x^2+7}{4x^2+12x+9}} = \sqrt{\frac{5}{4} - \frac{15x+17/4}{4x^2+12x+9}}.$$

This shows that $f(x) < \sqrt{5}/2$ for large x, and the graph therefore approaches the asymptote from below. Similarly, for large negative values of x, we express $f(x)$ in the form $f(x) = -\sqrt{\dfrac{5}{4} - \dfrac{15x+17/4}{4x^2+12x+9}}$, and this shows that the graph of $f(x)$ approaches $y = -\sqrt{5}/2$ from below as $x \to -\infty$.

47. Since $3 + 2x - x^2 = (3-x)(1+x)$, horizontal asymptotes are $x = 3$ and $x = -1$. With $f(x)$ expressed in the form $f(x) = \dfrac{1-4x^3}{3+2x-x^2} = 4x + 8 + \dfrac{28x+23}{x^2-2x-3}$, we see that $y = 4x+8$ is an oblique asymptote that is approached from above as $x \to \infty$, and from below as $x \to -\infty$.

48. Since $x^2 - 3x + 1 = 0$ for $x = (3 \pm \sqrt{9-4})/2 = (3 \pm \sqrt{5})/2$, vertical asymptotes occur at these values of x. With $f(x)$ expressed in the form $f(x) = \dfrac{3x^3+2x-1}{1-3x+x^2} = 3x + 9 + \dfrac{26x-10}{x^2-3x+1}$, we see that $y = 3x+9$ is an oblique asymptote that is approached from above as $x \to \infty$, and from below as $x \to -\infty$.

49. $\displaystyle \lim_{x\to-\infty} \frac{\sqrt{ax^2+bx+c}}{dx+e} = \lim_{x\to-\infty} \frac{\frac{1}{x}\sqrt{ax^2+bx+c}}{d+\frac{e}{x}} = \lim_{x\to-\infty} \frac{-\sqrt{a+\frac{b}{x}+\frac{c}{x^2}}}{d+\frac{e}{x}} = -\frac{\sqrt{a}}{d}$

50. Clearly a and d must both be positive else neither square root is defined for large x. If we rationalize,

$$\begin{aligned}
&\lim_{x\to\infty} \left(\sqrt{ax^2+bx+c} - \sqrt{dx^2+ex+f}\right) \\
&= \lim_{x\to\infty} \left[\left(\sqrt{ax^2+bx+c} - \sqrt{dx^2+ex+f}\right)\frac{\sqrt{ax^2+bx+c}+\sqrt{dx^2+ex+f}}{\sqrt{ax^2+bx+c}+\sqrt{dx^2+ex+f}}\right] \\
&= \lim_{x\to\infty} \left[\frac{(ax^2+bx+c)-(dx^2+ex+f)}{\sqrt{ax^2+bx+c}+\sqrt{dx^2+ex+f}}\right].
\end{aligned}$$

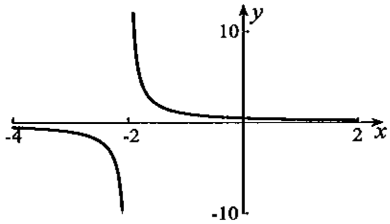
For this limit to exist, we must have $a = d$. When this is the case, the limit becomes

$$\lim_{x\to\infty} \frac{(b-e)x+(c-f)}{\sqrt{ax^2+bx+c}+\sqrt{ax^2+ex+f}} = \lim_{x\to\infty} \frac{(b-e)+\frac{c-f}{x}}{\sqrt{a+\frac{b}{x}+\frac{c}{x^2}}+\sqrt{a+\frac{e}{x}+\frac{f}{x^2}}} = \frac{b-e}{2\sqrt{a}}.$$

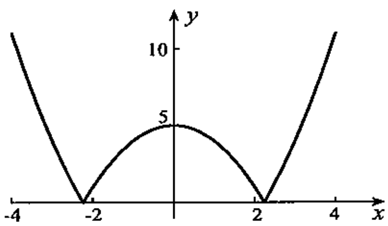
EXERCISES 2.4

1. The function has an infinite discontinuity at $x = -2$.

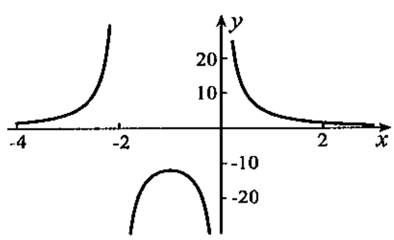
2. For $x \neq -4$, $f(x) = \dfrac{(4-x)(4+x)}{x+4} = 4 - x$. The graph of the function is therefore the straight line $y = 4 - x$ with the point at $x = -4$ deleted. The computer does not show the hole at the removable discontinuity $x = -4$.



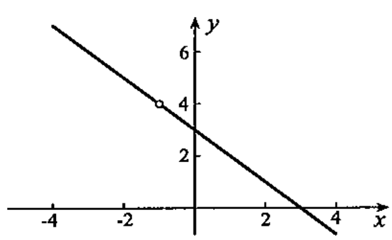
3. The function has no discontinuities.



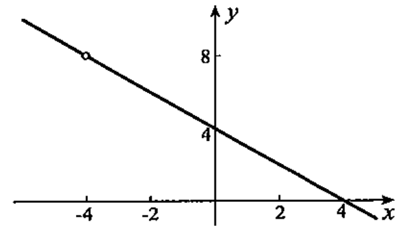
5. The function has infinite discontinuities at $x = 0, -2$.



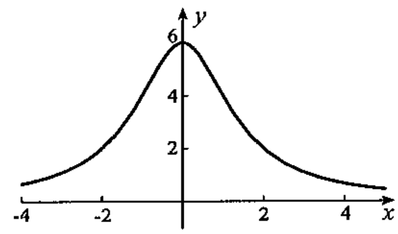
7. For $x \neq -1$, $f(x) = \dfrac{(3-x)(1+x)}{x+1} = 3 - x$.
The graph of the function is therefore the straight line $y = 3 - x$ with the point at $x = -1$ deleted. The computer does not show the hole at the removable discontinuity $x = -1$.



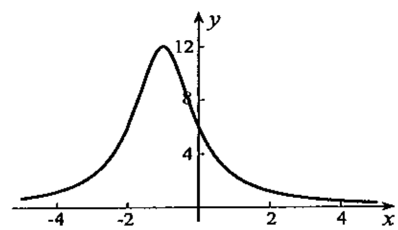
9. For $x \neq 2$, $f(x) = \dfrac{(x-2)(x^2+5)}{x-2} = x^2 + 5$.
The graph of the function is therefore the parabola $y = x^2 + 5$ with the point at $x = 2$ deleted. The computer does not show the hole at the removable discontinuity $x = 2$.



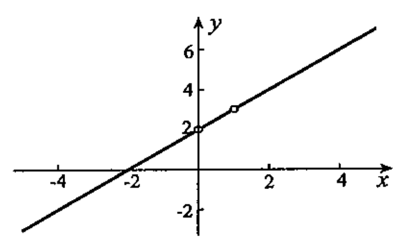
4. The function has no discontinuities.



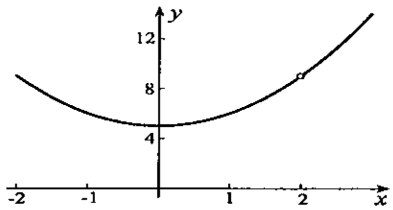
6. The function has no discontinuities.

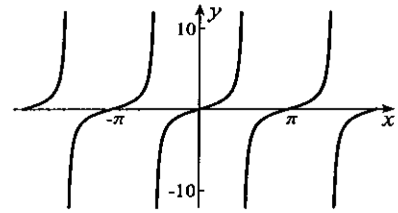


8. For $x \neq 0, 1$, $f(x) = \dfrac{x(x+2)(x-1)}{x(x-1)} = x + 2$.
The graph of the function is therefore the straight line $y = x + 2$ with the points at $x = 0, 1$ deleted. The computer does not show holes at the removable discontinuities $x = 0, 1$.



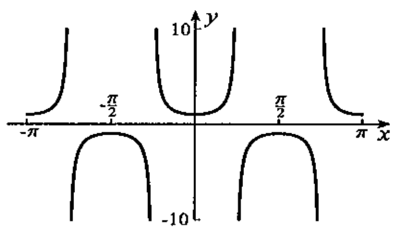
10. The tangent function has infinite discontinuities at $x = (2n+1)\pi/2$, where n is an integer.

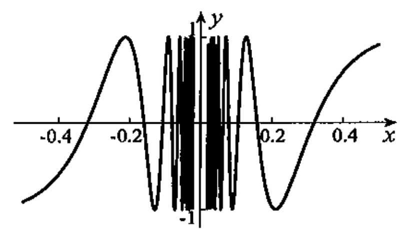




11. The function has infinite discontinuities when $2x = \dfrac{(2n+1)\pi}{2} \Longrightarrow x = \dfrac{(2n+1)\pi}{4}$, where n is an integer.

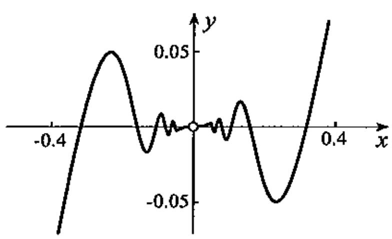
12. The function is discontinuous at $x = 0$. The discontinuity is not removable, jump, or infinite.

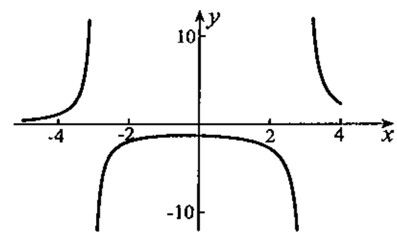




13. The function has a removable discontinuity at $x = 0$. The computer does not show the hole at $x = 0$.

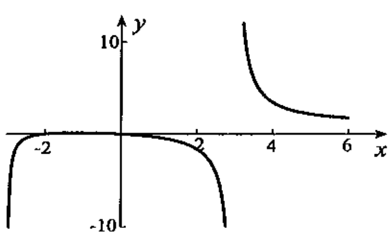
14. The function has infinite discontinuities at $x = \pm 3$.

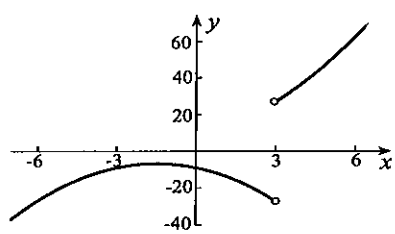




15. The function has infinite discontinuities at $x = \pm 3$.

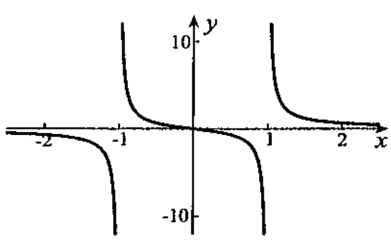
16. The function has a jump discontinuity at $x = 3$. The computer does not show the empty circles at $x = 3$.

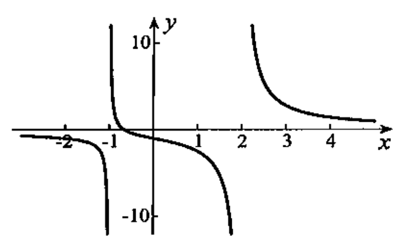




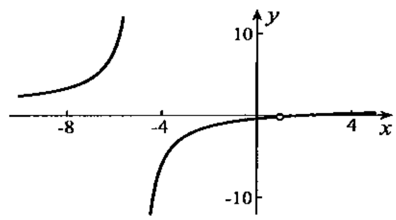
17. The function has infinite discontinuities at $x = \pm 1$.

18. The function has infinite discontinuities at $x = -1, 2$.

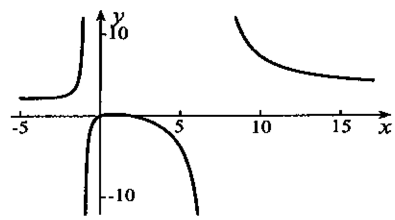




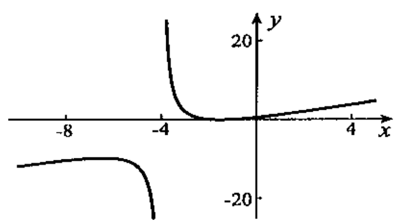
19. For $x \neq 1$, $f(x) = \dfrac{(x-1)(x-2)}{(x-1)(x+5)} = \dfrac{x-2}{x+5}$. The graph of the function is therefore the curve $y = (x-2)/(x+5)$ with the point at $x = 1$ deleted. The computer does not show the hole at the removable discontinuity $x = 1$. There is also an infinite discontinuity at $x = -5$.



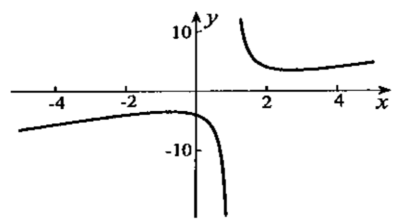
20. The function has infinite discontinuities at $x = -1, 7$.



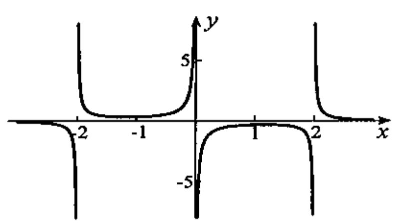
21. The function has an infinite discontinuity at $x = -4$.



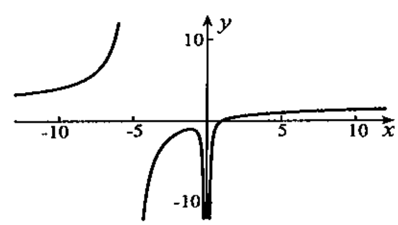
22. The function has an infinite discontinuity at $x = 1$.



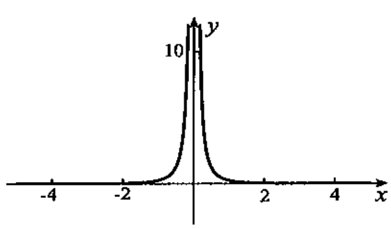
23. The function has infinite discontinuities at $x = 0, \pm 2$.



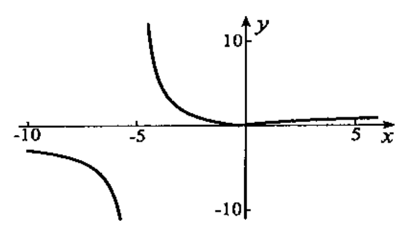
24. The function has infinite discontinuities at $x = 0, -5$.



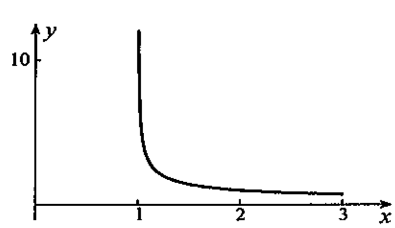
25. The function has an infinite discontinuity at $x = 0$.



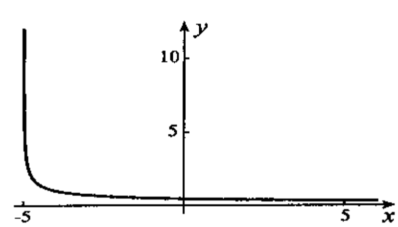
26. The function has an infinite discontinuity at $x = -5$.



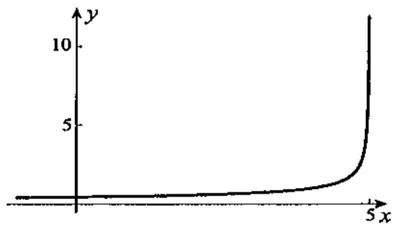
27. The function is continuous for $x > 1$.



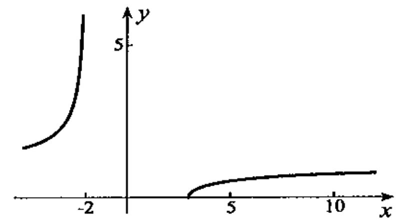
28. The function is continuous for $x > -5$.



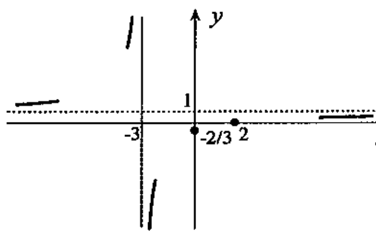
29. The function is continuous for $x < 5$.

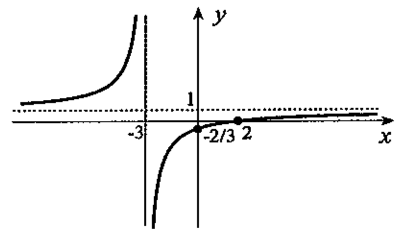


30. The function is continuous for $x < -2$ and $x \geq 3$.

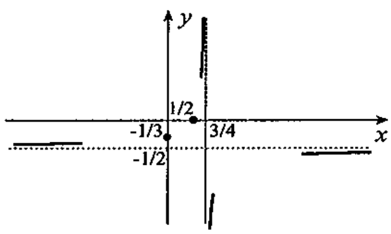


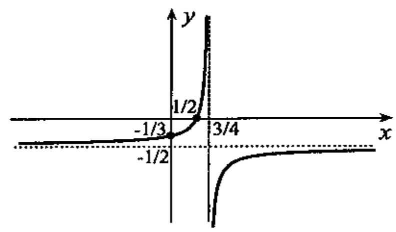
31. Right- and left-limits as $x \to -3$ and $x \to \pm\infty$ lead to the vertical and horizontal asymptotes in the left drawing below. With x- and y-intercepts, we finish the graph as shown to the right.



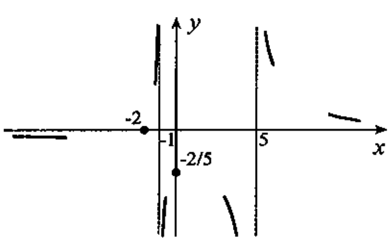


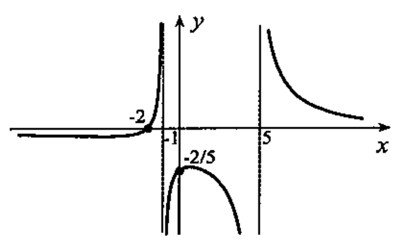
32. Right- and left-limits as $x \to 3/4$ and $x \to \pm\infty$ lead to the vertical and horizontal asymptotes in the left drawing below. With x- and y-intercepts, we finish the graph as shown to the right.



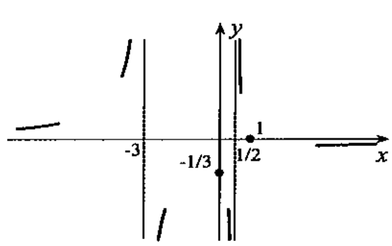


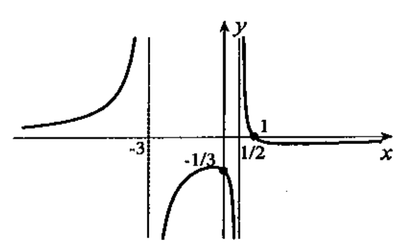
33. First we factor the denominator $f(x) = \dfrac{x+2}{(x-5)(x+1)}$. Right- and left-limits as $x \to -1$ and $x \to 5$, and limits as $x \to \pm\infty$ lead to the vertical and horizontal asymptotes in the left drawing below. With x- and y-intercepts, we finish the graph as shown to the right.



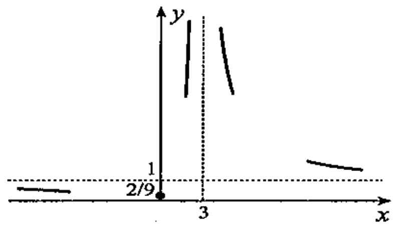


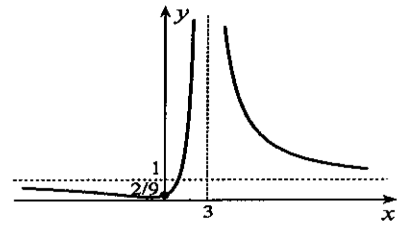
34. First we factor the denominator $f(x) = \dfrac{1-x}{(2x-1)(x+3)}$. Right- and left-limits as $x \to -3$ and $x \to 1/2$, and limits as $x \to \pm\infty$ lead to the vertical and horizontal asymptotes in the left drawing below. With x- and y-intercepts, we finish the graph as shown to the right.



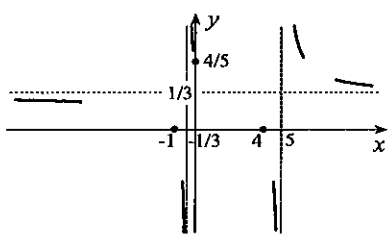


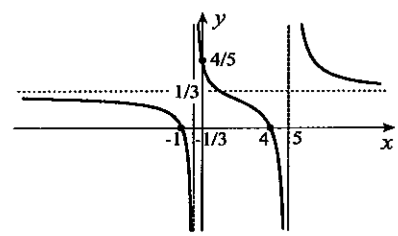
35. First we factor the denominator $f(x) = \dfrac{x^2+x+2}{(x-3)^2}$. Right- and left-limits as $x \to 3$ lead to the vertical asymptote in the left drawing below. To take limits as $x \to \pm\infty$, we use long division to write $f(x)$ in the form $f(x) = 1 + \dfrac{7x-7}{x^2-6x+9}$. Limits as $x \to \pm\infty$ lead to the horizontal asymptote in the same drawing. With a y-intercept equal to 2/9 and no x-intercept, we finish the graph as shown to the right.



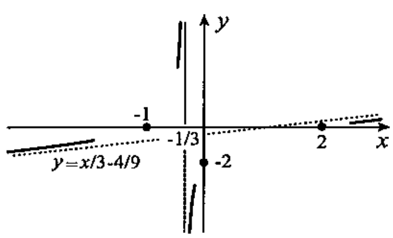


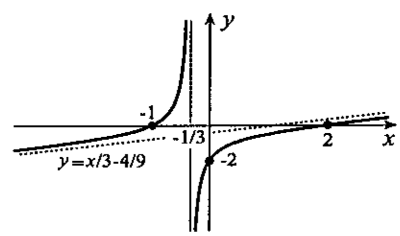
36. First we factor numerator and denominator $f(x) = \dfrac{(x-4)(x+1)}{(3x+1)(x-5)}$. Right- and left-limits as $x \to -1/3$ and $x \to 5$ lead to the vertical asymptotes in the left drawing below. To take limits as $x \to \pm\infty$, we use long division to write $f(x)$ in the form $f(x) = \dfrac{1}{3} + \dfrac{5x/3 - 7/3}{3x^2 - 14x - 5}$. Limits as $x \to \pm\infty$ lead to the horizontal asymptote in the same drawing. With x- and y-intercepts, we finish the graph as shown to the right.



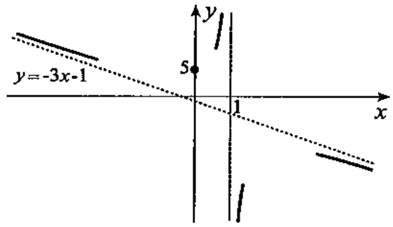


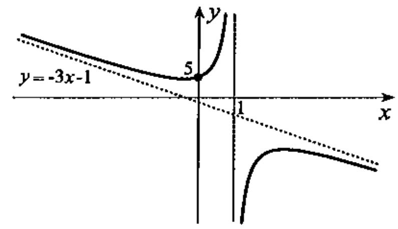
37. First we factor the numerator $f(x) = \dfrac{(x-2)(x+1)}{3x+1}$. Right- and left-limits as $x \to -1/3$ lead to the vertical asymptote in the left drawing below. The graph has an oblique asymptote that we can identify with long division, $f(x) = \dfrac{x}{3} - \dfrac{4}{9} - \dfrac{14/9}{3x+1}$. The line $y = x/3 - 4/9$ is the oblique asymptote. With x-intercepts at -1 and 2, and y-intercept equal to -2, we finish the graph as shown to the right.



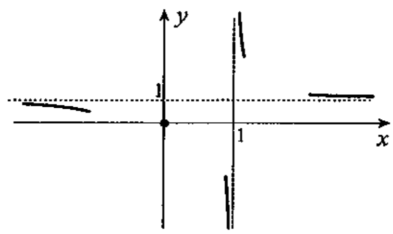


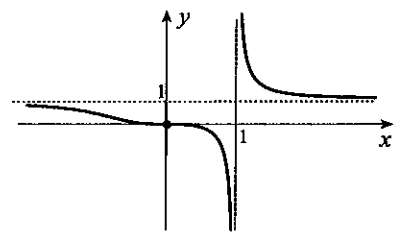
38. Right- and left-limits as $x \to 1$ lead to the vertical asymptote in the left drawing below. The graph has an oblique asymptote that we can identify with long division, $f(x) = -3x - 1 + \dfrac{6}{1-x}$. The line $y = -3x - 1$ is the oblique asymptote. With no x-intercepts, and y-intercept equal to 5, we finish the graph as shown to the right.



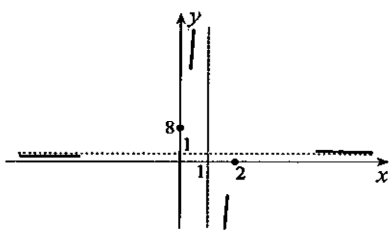


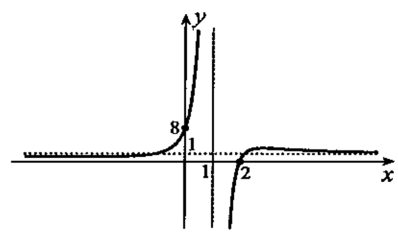
39. First we factor the denominator $f(x) = \dfrac{x^3}{(x-1)(x^2+x+1)}$. Right- and left-limits as $x \to 1$, and limits as $x \to \pm\infty$ lead to the vertical and horizontal asymptotes in the left drawing below. With x- and y-intercepts both at the origin, we finish the graph as shown to the right.



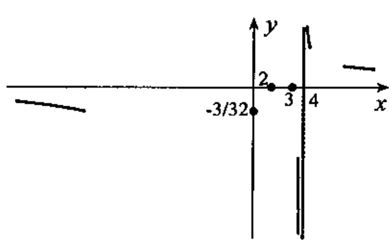


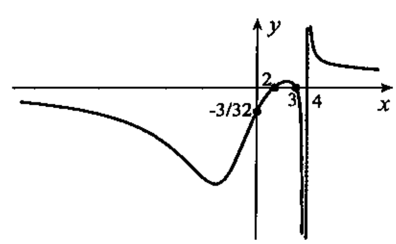
40. First we factor numerator and denominator $f(x) = \dfrac{(x-2)(x^2+x+4)}{(x-1)^3}$. Right- and left-limits as $x \to 1$ lead to the vertical asymptote in the left drawing below. To take limits as $x \to \pm\infty$, we use long division to write $f(x)$ in the form $f(x) = 1 + \dfrac{2x^2-x-7}{x^3-3x^2+3x-1}$. Limits as $x \to \pm\infty$ lead to the horizontal asymptote in the same drawing. With x- and y-intercepts, we finish the graph as shown to the right.



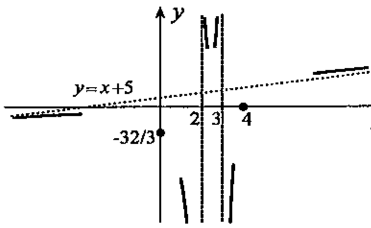


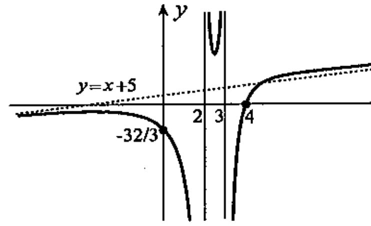
41. First we factor numerator and denominator $f(x) = \dfrac{(x-2)(x-3)}{(x-4)(x^2+4x+16)}$. Right- and left-limits as $x \to 4$, and limits as $x \to \pm\infty$ lead to the vertical and horizontal asymptotes in the left drawing below. With x- and y-intercepts, we finish the graph as shown to the right.





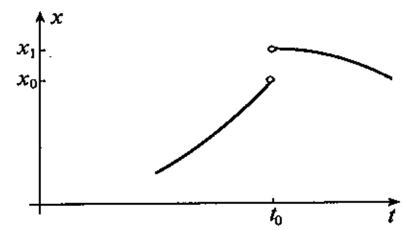
42. First we factor the numerator and denominator $f(x) = \dfrac{(x-4)(x^2+4x+16)}{(x-2)(x-3)}$. Right- and left-limits as $x \to 2$ and $x \to 3$ lead to the vertical asymptotes in the left drawing below. The graph as an oblique asymptote that we can identify with long division, $f(x) = x + 5 + \dfrac{19x-94}{x^2-5x+6}$. The line $y = x+5$ is the oblique asymptote. With x-intercept 4, and y-intercept equal to $-32/3$, we finish the graph as shown to the right.



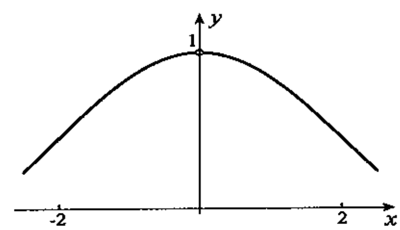


43. Yes, since $f(0) = 0 = \lim_{x\to 0} f(x)$ (see Exercise 51 in Exercises 2.1).

44. No. If it were to have a discontinuity at $t = t_0$ as in the figure to the right, the particle would disappear at position x_0 and reappear instantaneously at position x_1.

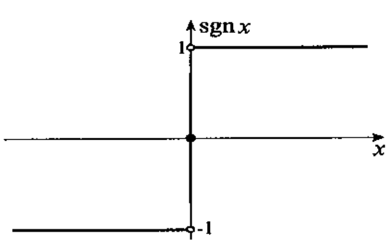


45. The graph of the function to the right was generated on a computer but the hole was added manually. It indicates that $\lim_{x\to 0} x^{-1}\sin x = 1$, and therefore the discontinuity is removable.

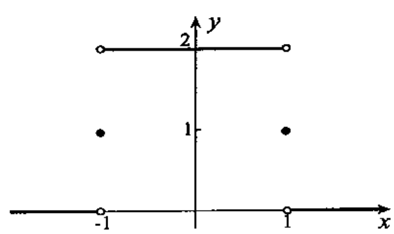


46. If we set $h = x - a$ in the left side of 2.4a, then $f(a) = \lim\limits_{x\to a} f(x) = \lim\limits_{h+a\to a} f(a+h) = \lim\limits_{h\to 0} f(a+h)$.

47. (a) The graph indicates that the function is discontinous at $x = 0$.

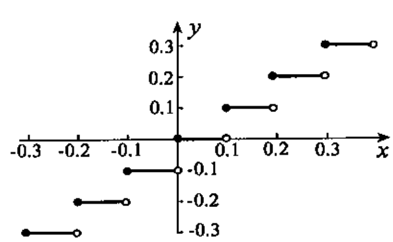


(b) The function is discontinuous at $x = \pm 1$.



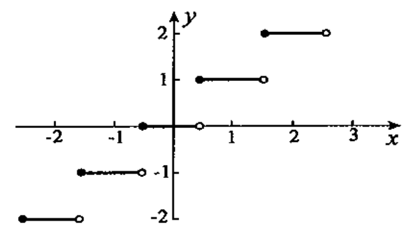
48. (a) The function is discontinuous at $x = n/10$, where n is an integer.
(b) Let a positive number x be denoted by $z.abc\cdots$ where z is the integer part, and a, b, and c are the first three decimals. Then

$$f(z.abc\cdots) = (1/10)\lfloor za.bc\cdots\rfloor = \tfrac{1}{10}(za) = z.a.$$



49. The function $\lfloor 100x+1\rfloor/100$. For example, if $x = -2.357$, then $\lfloor 100(-2.357)+1\rfloor/100 = \lfloor -234.7\rfloor/100 = -235/100 = -2.35$.

50. (a) The function is discontinuous at $x = n + 1/2$, where n is an integer.
(b) Let a positive number x be denoted by $z.a\cdots$ where z is the integer part, and a is the first decimal. Suppose that a is equal to 0, 1, 2, 3, or 4.
The integer part of $x + 1/2$ is z and the first decimal in the number $x + 1/2$ is 5, 6, 7, 8, or 9, and therefore $f(x) = \lfloor x + 1/2 \rfloor = z$. On the other hand, suppose that a is equal to 5, 6, 7, 8, or 9. Then the integer part of $x + 1/2$ is $z + 1$, and its first decimal is 0, 1, 2, 3, or 4. Hence, $f(x) = \lfloor x + 1/2 \rfloor = z + 1$.



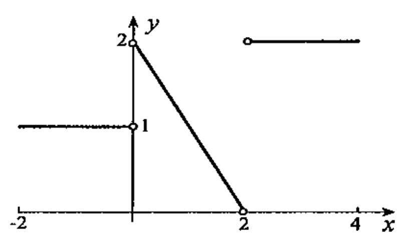
51. (a) $\lfloor 10x + 1/2 \rfloor / 10$ (b) $\lfloor 100x + 1/2 \rfloor / 100$ (c) $\lfloor 10^n x + 1/2 \rfloor / 10^n$

52. There are no points at which the function is continuous since $\lim_{x\to a} f(x)$ does not exist for any a.

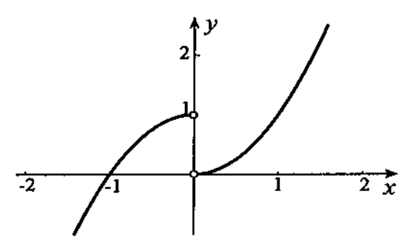
53. The function is continuous only at $x = 0$.

EXERCISES 2.5

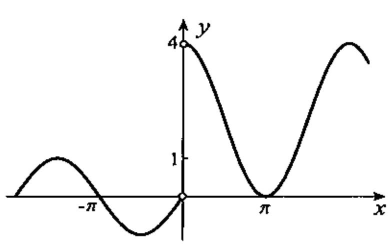
1. $$\begin{aligned} f(x) &= [1 - h(x)] + (2 - x)[h(x) - h(x-2)] \\ &\quad + 2h(x-2) \\ &= 1 + (1 - x)h(x) + x\,h(x-2) \end{aligned}$$



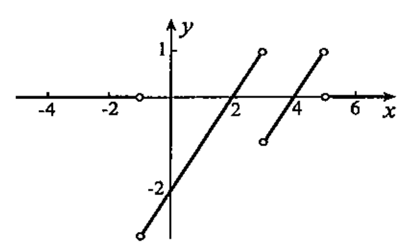
2. $$\begin{aligned} f(x) &= (1 - x^2)[1 - h(x)] + x^2 h(x) \\ &= 1 - x^2 + (2x^2 - 1)h(x) \end{aligned}$$



3. $$\begin{aligned} f(x) &= \sin x[1 - h(x)] + (2 + 2\cos x)h(x) \\ &= \sin x + (2 + 2\cos x - \sin x)h(x) \end{aligned}$$

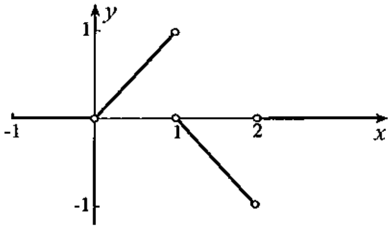


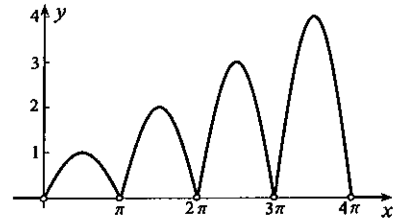
4. $$\begin{aligned} f(x) &= (x - 2)[h(x+1) - h(x-3)] \\ &\quad + (x - 4)[h(x-3) - h(x-5)] \\ &= (x - 2)h(x+1) - 2h(x-3) \\ &\quad + (4 - x)h(x-5) \end{aligned}$$



5. $$\begin{aligned} f(x) &= x[h(x) - h(x-1)] \\ &\quad + (1 - x)[h(x-1) - h(x-2)] \\ &= x\,h(x) + (1 - 2x)h(x-1) \\ &\quad + (x - 1)h(x-2) \end{aligned}$$

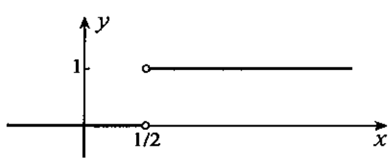
6. $$\begin{aligned} f(x) &= \sin x[h(x) - h(x-\pi)] \\ &\quad + 2\sin(x-\pi)[h(x-\pi) - h(x-2\pi)] \\ &\quad + 3\sin(x-2\pi)[h(x-2\pi) - h(x-3\pi)] \\ &\quad + 4\sin(x-3\pi)[h(x-3\pi) - h(x-4\pi)] \\ &= \sin x\,[h(x) - h(x-\pi)] \\ &\quad - 2\sin x\,[h(x-\pi) - h(x-2\pi)] \\ &\quad + 3\sin x\,[h(x-2\pi) - h(x-3\pi)] \\ &\quad - 4\sin x\,[h(x-3\pi) - h(x-4\pi)] \\ &= \sin x\,[h(x) - 3h(x-\pi) + 5h(x-2\pi) \\ &\quad - 7h(x-3\pi) + 4h(x-4\pi)] \end{aligned}$$





7. $-F[h(t) - h(t - T)]$
8. $10 \sin 4t[h(t-1) - h(t-1-\pi)]$
9. $-50\delta(t-4)$
10. $[100 + 2(t-10)][h(t-10) - h(t-60)] = (80 + 2t)[h(t-10) - h(t-60)]$
11. $60[\delta(t) + \delta(t-10) + \delta(t-20) + \delta(t-30) + \delta(t-40) + \delta(t-50) + \delta(t-60)]$
12. $-(2mg/L)[h(x) - h(x - L/2)]$
13. $-F\delta(x - L/3)$
14. $F_1\delta(x - x_1) - F_2\delta(x - x_2)$
15. $-[3mg/(2L)][h(x - L/3) - h(x - L)]$
16. $h(x-a) - h(x-b) + h(x-c)$
17. $h(t) - h(t-1) + h(t-2) - h(t-3) + h(t-4) - \cdots$
18. Yes, except at $x = a$ where $h(x-a)h(x-b)$ is undefined whereas $h(x-b) = 0$.

19.



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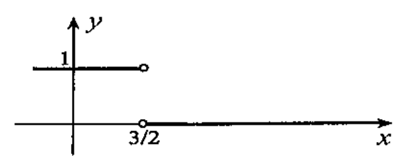

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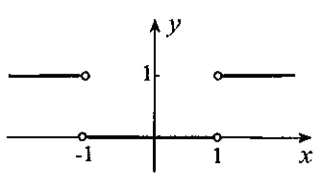

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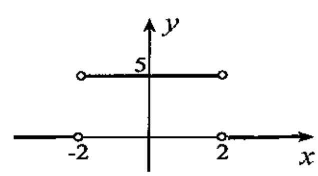

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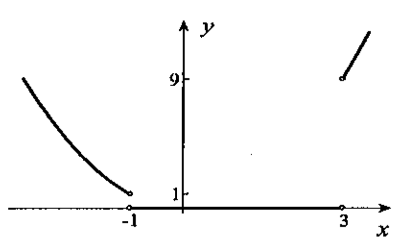

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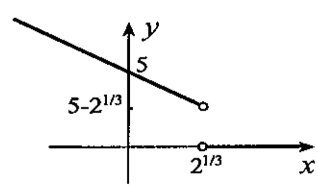


25. The function is

$$[h(t) - h(t-1)] + 2[h(t-1) - h(t-2)] + 3[h(t-2) - h(t-3)] + 4h(t-3)$$
$$= h(t) + h(t-1) + h(t-2) + h(t-3).$$

EXERCISES 2.6

1. Suppose $\epsilon > 0$ is given. We must show that we can choose x sufficiently close to 1 so that $|(x+5)-6| < \epsilon$. To do this, we rewrite the inequality with the x's in the combination $x-1$, $|x-1| < \epsilon$. Thus, if we choose $0 < |x-1| < \epsilon$, then $|(x+5)-6| < \epsilon$; that is, we can make $x+5$ within ϵ of 6 by choosing x within ϵ of 1.

2. Suppose $\epsilon > 0$ is given. We must show that we can choose x sufficiently close to 2 so that $|(2x-3)-1| < \epsilon$. To do this, we rewrite the inequality with the x's in the combination $x-2$,

$$|(2x-3)-1| = |2(x-2)| = 2|x-2|.$$

We must now choose x so that $2|x-2| < \epsilon$. But this will be true if $|x-2| < \epsilon/2$. In other words, if we choose x to satisfy $0 < |x-2| < \epsilon/2$, then

$$|(2x-3)-1| = 2|x-2| < 2\left(\frac{\epsilon}{2}\right) = \epsilon.$$

We have shown that we can make $2x-3$ within ϵ of 1 by choosing x within $\epsilon/2$ of 2.

3. Suppose $\epsilon > 0$ is given. We must show that we can choose x sufficiently close to 0 so that $|(x^2+3)-3| < \epsilon$. To do this, we rewrite the inequality in the form $|x|^2 < \epsilon$. But this will be true if $|x| < \sqrt{\epsilon}$. In other words, if we choose x to satisfy $0 < |x| < \sqrt{\epsilon}$, then

$$|(x^2+3)-3| = |x|^2 < (\sqrt{\epsilon})^2 = \epsilon.$$

We have shown that we can make x^2+3 within ϵ of 3 by choosing x within $\sqrt{\epsilon}$ of 0.

4. Suppose $\epsilon > 0$ is given. We must show that we can choose x sufficiently close to 1 so that $|(x^2+4)-5| < \epsilon$. To do this, we rewrite the inequality with the x's in the combination $x-1$,

$$|(x^2+4)-5| = |(x-1)^2 + 2(x-1)|.$$

We must now choose x so that

$$|(x-1)^2 + 2(x-1)| < \epsilon.$$

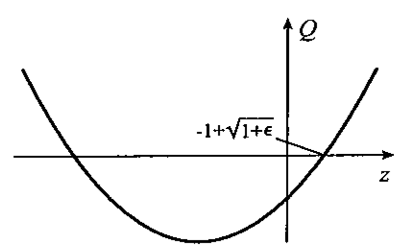
Since $|(x-1)^2 + 2(x-1)| \le |x-1|^2 + 2|x-1|$, the above inequality is satisfied if x is chosen so that

$$|x-1|^2 + 2|x-1| < \epsilon.$$

Suppose we set $z = |x-1|$, and consider the parabola $Q(z) = z^2 + 2z - \epsilon$ in the figure. It crosses the z-axis when

$$z = \frac{-2 \pm \sqrt{4+4\epsilon}}{2} = -1 \pm \sqrt{1+\epsilon}.$$

The graph shows that $Q(z) < 0$ whenever $0 < z < -1 + \sqrt{1+\epsilon}$. In other words, if $0 < |x-1| < \sqrt{1+\epsilon} - 1$, then $|x-1|^2 + 2|x-1| < \epsilon$, and therefore $|(x-1)^2 + 2(x-1)| < \epsilon$.



5. Suppose $\epsilon > 0$ is given. We must show that we can choose x sufficiently close to -2 so that $|(3-x^2)+1| < \epsilon$. To do this, we rewrite the inequality with the x's in the combination $x+2$,

$$|(3-x^2)+1| = |x^2-4| = |(x+2)^2 - 4(x+2)|.$$

We must now choose x so that

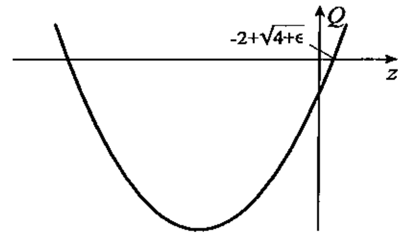
$$|(x+2)^2 - 4(x+2)| < \epsilon.$$

Since $|(x+2)^2 - 4(x+2)| \le |x+2|^2 + 4|x+2|$, the above inequality is satisfied if x is chosen so that

$$|x+2|^2 + 4|x+2| < \epsilon.$$

Suppose we set $z = |x+2|$, and consider the parabola $Q(z) = z^2 + 4z - \epsilon$ in the figure. It crosses the z-axis when

$$z = \frac{-4 \pm \sqrt{16+4\epsilon}}{2} = -2 \pm \sqrt{4+\epsilon}.$$



The graph shows that $Q(z) < 0$ whenever $0 < z < -2 + \sqrt{4+\epsilon}$. In other words, if $0 < |x+2| < \sqrt{4+\epsilon} - 2$, then $|x+2|^2 + 4|x+2| < \epsilon$, and therefore $|(x+2)^2 - 4(x+2)| < \epsilon$.

6. Suppose $\epsilon > 0$ is given. We must show that we can choose x sufficiently close to 3 so that $|(x^2 - 7x) + 12| < \epsilon$. To do this, we rewrite the inequality with the x's in the combination $x - 3$,

$$|(x^2 - 7x) + 12| = |x^2 - 7x + 12| = |(x-3)^2 - (x-3)|.$$

We must now choose x so that

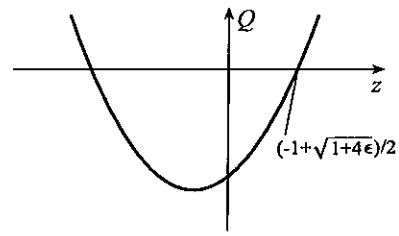
$$|(x-3)^2 - (x-3)| < \epsilon.$$

Since $|(x-3)^2 - (x-3)| \leq |x-3|^2 + |x-3|$, the above inequality is satisfied if x is chosen so that

$$|x-3|^2 + |x-3| < \epsilon.$$

Suppose we set $z = |x-3|$, and consider the parabola $Q(z) = z^2 + z - \epsilon$ in the figure. It crosses the z-axis when

$$z = \frac{-1 \pm \sqrt{1+4\epsilon}}{2}.$$



The graph shows that $Q(z) < 0$ whenever $0 < z < \left(-1 + \sqrt{1+4\epsilon}\right)/2$. In other words, if $0 < |x-3| < \left(\sqrt{1+4\epsilon} - 1\right)/2$, then $|x-3|^2 + |x-3| < \epsilon$, and therefore $|(x-3)^2 - (x-3)| < \epsilon$.

7. Suppose $\epsilon > 0$ is given. We must show that we can choose x sufficiently close to -1 so that $|(x^2 - 3x + 4) - 8| < \epsilon$. To do this, we rewrite the inequality with the x's in the combination $x + 1$,

$$|(x^2 - 3x + 4) - 8| = |x^2 - 3x - 4| = |(x+1)^2 - 5(x+1)|.$$

We must now choose x so that

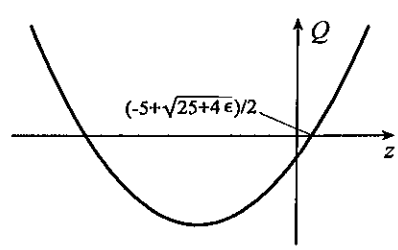
$$|(x+1)^2 - 5(x+1)| < \epsilon.$$

Since $|(x+1)^2 - 5(x+1)| \leq |x+1|^2 + 5|x+1|$, the above inequality is satisfied if x is chosen so that

$$|x+1|^2 + 5|x+1| < \epsilon.$$

Suppose we set $z = |x+1|$, and consider the parabola $Q(z) = z^2 + 5z - \epsilon$ in the figure. It crosses the z-axis when

$$z = \frac{-5 \pm \sqrt{25+4\epsilon}}{2}.$$



The graph shows that $Q(z) < 0$ whenever $0 < z < (-5 + \sqrt{25+4\epsilon})/2$. In other words, if $0 < |x+1| < (\sqrt{25+4\epsilon} - 5)/2$, then $|x+1|^2 + 5|x+1| < \epsilon$, and therefore $|(x+1)^2 + 5(x+1)| < \epsilon$.

8. Suppose $\epsilon > 0$ is given. We must show that we can choose x sufficiently close to 1 so that $|(x^2 + 3x + 5) - 9| < \epsilon$. To do this, we rewrite the inequality with the x's in the combination $x - 1$,

$$|(x^2 + 3x + 5) - 9| = |x^2 + 3x - 4| = |(x-1)^2 + 5(x-1)|.$$

We must now choose x so that

$$|(x-1)^2 + 5(x-1)| < \epsilon.$$

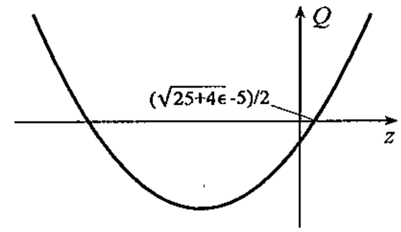
Since $|(x-1)^2 + 5(x-1)| \le |x-1|^2 + 5|x-1|$, the above inequality is satisfied if x is chosen so that

$$|x-1|^2 + 5|x-1| < \epsilon.$$

Suppose we set $z = |x-1|$ and consider the parabola $Q(z) = z^2 + 5z - \epsilon$ in the figure. It crosses the z-axis when

$$z = \frac{-5 \pm \sqrt{25+4\epsilon}}{2}.$$

The graph shows that $Q(z) < 0$ whenever $0 < z < \left(\sqrt{25+4\epsilon} - 5\right)/2$. In other words, if $0 < |x-1| < \left(\sqrt{25+4\epsilon} - 5\right)/2$, then $|x-1|^2 + 5|x-1| < \epsilon$, and therefore $|(x-1)^2 + 5(x-1)| < \epsilon$.



9. Suppose $\epsilon > 0$ is given. We must show that we can choose x sufficiently close to 2 so that $|(x+2)/(x-1) - 4| < \epsilon$. To do this, we rewrite the inequality with the x's in the combination $x-2$,

$$\left|\frac{x+2}{x-1} - 4\right| = \left|\frac{-3x+6}{x-1}\right| = \frac{3|x-2|}{|(x-2)+1|}.$$

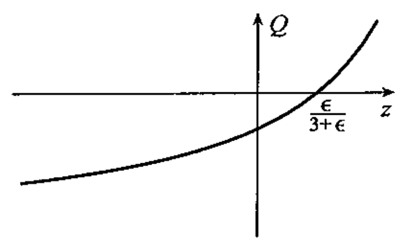
We must now choose x so that

$$\frac{3|x-2|}{|(x-2)+1|} < \epsilon.$$

Since $\dfrac{3|x-2|}{|(x-2)+1|} \le \dfrac{3|x-2|}{1-|x-2|}$, provided $|x-2| < 1$, the above inequality is satisfied if x is chosen so that

$$\frac{3|x-2|}{1-|x-2|} < \epsilon.$$

Suppose we set $z = |x-2|$ and consider the curve $Q(z) = 3z/(1-z) - \epsilon$ in the figure. It crosses the z-axis when $z = \epsilon/(3+\epsilon)$. The graph shows that $Q(z) < 0$ whenever $0 < z < \epsilon/(3+\epsilon)$. In other words, if $0 < |x-2| < \epsilon/(3+\epsilon)$, then $3|x-2|/(1-|x-2|) < \epsilon$, and therefore $3|(x-2)/|(x-2)+1| < \epsilon$.



10. $\lim_{x\to a^+} f(x) = L$ if given any $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x)-L| < \epsilon$ whenever $a < x < a+\delta$.

11. $\lim_{x\to a^-} f(x) = L$ if given any $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x)-L| < \epsilon$ whenever $a-\delta < x < a$.

12. $\lim_{x\to\infty} f(x) = L$ if given any $\epsilon > 0$, there exists an $X > 0$ such that $|f(x) - L| < \epsilon$ whenever $x > X$.

13. $\lim_{x\to-\infty} f(x) = L$ if given any $\epsilon > 0$, there exists an $X < 0$ such that $|f(x) - L| < \epsilon$ whenever $x < X$.

14. $\lim_{x\to a} f(x) = \infty$ if given any $M > 0$, there exists a $\delta > 0$ such that $f(x) > M$ whenever $0 < |x-a| < \delta$.

15. $\lim_{x\to a} f(x) = -\infty$ if given any $M < 0$, there exists a $\delta > 0$ such that $f(x) < M$ whenever $0 < |x-a| < \delta$.

16. $\lim_{x\to\infty} f(x) = \infty$ if given any $M > 0$, there exists an $X > 0$ such that $f(x) > M$ whenever $x > X$.

17. $\lim_{x\to\infty} f(x) = -\infty$ if given any $M < 0$, there exists an $X > 0$ such that $f(x) < M$ whenever $x > X$.

18. $\lim_{x\to-\infty} f(x) = \infty$ if given any $M > 0$, there exists an $X < 0$ such that $f(x) > M$ whenever $x < X$.

19. $\lim_{x\to-\infty} f(x) = -\infty$ if given any $M < 0$, there exists an $X < 0$ such that $f(x) < M$ whenever $x < X$.

20. Suppose to the contrary that $f(x)$ has two limits L_1 and L_2 as x approaches a where $L_2 > L_1$ and $L_2 - L_1 = \epsilon$. Since $\lim_{x\to a} f(x) = L_1$, there exists a δ_1 such that $|f(x) - L_1| < \epsilon/3$ when $0 < |x-a| < \delta_1$; that is, when x is in the interval $0 < |x-a| < \delta_1$, the function is within $\epsilon/3$ of L_1. On the other hand, since $\lim_{x\to a} f(x) = L_2$, there exists a δ_2 such that $|f(x) - L_2| < \epsilon/3$ when $0 < |x-a| < \delta_2$; that is, the function is within $\epsilon/3$ of L_2 in the interval $0 < |x-a| < \delta_2$. But this is impossible because L_1 and L_2 are ϵ apart. Consequently, $f(x)$ can have at most one limit as x approaches a.

21. We use the definition in Exercise 14: $\lim_{x\to 1} 1/(x-1)^2 = \infty$ if given any $M > 0$, there exists a $\delta > 0$ such that $1/(x-1)^2 > M$ whenever $0 < |x-1| < \delta$. The inequality $1/(x-1)^2 > M \Longleftrightarrow (x-1)^2 < 1/M \Longleftrightarrow |x-1| < 1/\sqrt{M}$. Consequently, if we choose $\delta = 1/\sqrt{M}$, then whenever $0 < |x-1| < \delta = 1/\sqrt{M}$, we must have $1/(x-1)^2 > M$.

22. We use the definition in Exercise 15: $\lim_{x\to -2} [-1/(x+2)^2] = -\infty$ if given any number $M < 0$, there exists a $\delta > 0$ such that $-1/(x+2)^2 < M$ whenever $0 < |x+2| < \delta$. The inequality $-1/(x+2)^2 < M \Longleftrightarrow (x+2)^2 < -1/M \Longleftrightarrow |x+2| < 1/\sqrt{-M}$. Consequently, if we choose $\delta = 1/\sqrt{-M}$, then whenever $0 < |x+2| < \delta = 1/\sqrt{-M}$, we must have $-1/(x+2)^2 < M$.

23. We use the definition in Exercise 16: $\lim_{x\to\infty} (x+5) = \infty$ if given any $M > 0$, there exists an $X > 0$ such that $x+5 > M$ whenever $x > X$. The inequality $x+5 > M \Longleftrightarrow x > M-5$. Consequently, if we choose $X = M-5$, then whenever $x > X$, we must have $x+5 > M$.

24. We use the definition in Exercise 17: $\lim_{x\to\infty} (5-x^2) = -\infty$ if given any number $M < 0$, there exists an $X > 0$ such that $5-x^2 < M$ whenever $x > X$. The inequality $5-x^2 < M \Longleftrightarrow x^2 > 5-M$, which for positive x implies that $x > \sqrt{5-M}$. Consequently, if we choose $X = \sqrt{5-M}$, then whenever $x > X$, we must have $5-x^2 < M$.

25. We use the definition in Exercise 12: $\lim_{x\to\infty} (x+2)/(x-1) = 1$ if given any $\epsilon > 0$, there exists an $X > 0$ such that $|(x+2)/(x-1) - 1| < \epsilon$ whenever $x > X$. The inequality $|(x+2)/(x-1) - 1| < \epsilon \Longleftrightarrow 3/|x-1| < \epsilon \Longleftrightarrow |x-1| > 3/\epsilon$. This inequality is satisfied if $x > 1 + 3/\epsilon$. In other words, if we choose $X = 1 + 3/\epsilon$, then whenever $x > X$, $|(x+2)/(x-1) - 1| < \epsilon$.

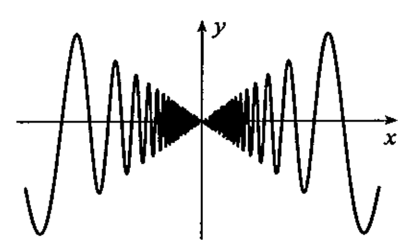
26. We use the definition in Exercise 13: $\lim_{x\to -\infty} (x+2)/(x-1) = 1$ if given any number $\epsilon > 0$, there exists an $X < 0$ such that $|(x+2)/(x-1) - 1| < \epsilon$ whenever $x < X$. The inequality $|(x+2)/(x-1)-1| < \epsilon \Longleftrightarrow 3/|x-1| < \epsilon \Longleftrightarrow |x-1| > 3/\epsilon$. This inequality is satisfied if $x < 1 - 3/\epsilon$. In other words, if we choose $X = 1 - 3/\epsilon$, then whenever $x < X$, $|(x+2)/(x-1) - 1| < \epsilon$.

27. We use the definition in Exercise 18: $\lim_{x\to -\infty} (5-x) = \infty$ if given any $M > 0$, there exists an $X < 0$ such that $5-x > M$ whenever $x < X$. The inequality $5-x > M \Longleftrightarrow x < 5-M$. Consequently, if we choose $X = 5-M$, then whenever $x < X$, we must have $5-x > M$.

28. We use the definition in Exercise 19: $\lim_{x\to -\infty} (3+x-x^2) = -\infty$ if given any number $M < 0$, there exists an $X < 0$ such that $3+x-x^2 < M$ whenever $x < X$. The inequality $3+x-x^2 < M \Longleftrightarrow M > -(x-1/2)^2 + 13/4 \Longleftrightarrow (x-1/2)^2 > 13/4 - M \Longleftrightarrow |x-1/2| > \sqrt{13/4-M}$. This is satisfied for negative x if $x - 1/2 < -\sqrt{13/4-M}$, or, $x < 1/2 - \sqrt{13/4-M}$. Thus, if we choose $X = 1/2 - \sqrt{13/4-M}$, then whenever $x < X$, we must have $3+x-x^2 < M$.

29. Let $\epsilon = L$. Because $\lim_{x\to a} f(x) = L$, there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x-a| < \delta$. Thus, in the interval $I: 0 < |x-a| < \delta$, we have $-\epsilon < f(x) - L < \epsilon \Longleftrightarrow L - \epsilon < f(x) < L + \epsilon$. But with $\epsilon = L$, this implies that in I, $0 < f(x) < 2L$.

30. No. A graph of the function $g(x) = x\sin(1/x)$ is shown to the right. It has limit 0 as $x \to 0$. If values at $x = 1/(n\pi)$ are redefined as 1, then all values of $f(x)$ no longer approach 0 as x approaches 0. Every interval around $x = 0$ contains an infinity of points at which the value of $f(x)$ is equal to 1.



31. Since $\lim_{x\to a} f(x) = F$, there exists a $\delta_1 > 0$ such that

$$|f(x) - F| < \epsilon/2 \qquad \text{whenever } 0 < |x - a| < \delta_1.$$

Since $\lim_{x\to a} g(x) = G$, there exists a $\delta_2 > 0$ such that

$$|g(x) - G| < \epsilon/2 \qquad \text{whenever } 0 < |x - a| < \delta_2.$$

It follows that whenever $0 < |x - a| < \delta$, where δ is the smaller of δ_1 and δ_2,

$$\big|[f(x) + g(x)] - (F + G)\big| = \big|[f(x) - F] + [g(x) - G]\big| \le |f(x) - F| + |g(x) - G| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

32. Since $\lim_{x\to a} f(x) = F$, there exists a $\delta_1 > 0$ such that

$$|f(x) - F| < \epsilon/2 \qquad \text{whenever } 0 < |x - a| < \delta_1.$$

Since $\lim_{x\to a} g(x) = G$, there exists a $\delta_2 > 0$ such that

$$|g(x) - G| < \epsilon/2 \qquad \text{whenever } 0 < |x - a| < \delta_2.$$

It follows that whenever $0 < |x - a| < \delta$, where δ is the smaller of δ_1 and δ_2,

$$\big|[f(x) - g(x)] - (F - G)\big| = \big|[f(x) - F] - [g(x) - G]\big| \le |f(x) - F| + |g(x) - G| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

33. Suppose c is any constant, and $\epsilon > 0$ is any given number. Since $\lim_{x\to a} f(x) = F$, there exists a $\delta > 0$ such that $|f(x) - F| < \epsilon/|c|$ whenever $0 < |x - a| < \delta$. Hence, whenever $0 < |x - a| < \delta$, we can say that

$$|cf(x) - cL| = |c||f(x) - L| < |c|\left(\frac{\epsilon}{|c|}\right) = \epsilon.$$

34. (a)
$$\begin{aligned} |f(x)g(x) - FG| &= \big|[f(x)g(x) - f(x)G] + [f(x)G - FG]\big| \\ &\le |f(x)g(x) - f(x)G| + |f(x)G - FG| = |f(x)||g(x) - G| + |G||f(x) - F| \end{aligned}$$

(b) If $\lim_{x\to a} f(x) = F$, it follows that $\lim_{x\to a} |f(x)| = |F|$. There must exist a δ_1 such that whenever $0 < |x - a| < \delta_1$,

$$\big||f(x)| - |F|\big| < 1, \qquad \text{or,} \qquad -1 < |f(x)| - |F| < 1.$$

But then for such x, $|f(x)| < |F| + 1$.
Since $\lim_{x\to a} g(x) = G$, given any $\epsilon > 0$, there exists δ_2 such that when $0 < |x - a| < \delta_2$,

$$|g(x) - G| < \frac{\epsilon}{2(|F| + 1)}.$$

Since $\lim_{x\to a} f(x) = F$, there exists δ_3 such that when $0 < |x - a| < \delta_3$,

$$|f(x) - F| < \frac{\epsilon}{2|G| + 1}.$$

(c) If we set δ equal to the minimum of δ_1, δ_2, and δ_3, then for $0 < |x - a| < \delta$,

$$\begin{aligned} |f(x)g(x) - FG| &\le |f(x)||g(x) - G| + |G||f(x) - F| && \text{(from part (a))} \\ &< |f(x)||g(x) - G| + (|G| + 1/2)|f(x) - F| \\ &< (|F| + 1)\frac{\epsilon}{2(|F| + 1)} + \frac{1}{2}(2|G| + 1)\frac{\epsilon}{2|G| + 1} && \text{(from part (b))} \\ &= \epsilon. \end{aligned}$$

35. (a)
$$\begin{aligned} \left|\frac{f(x)}{g(x)} - \frac{F}{G}\right| &= \left|\frac{f(x)G - g(x)F}{g(x)G}\right| = \left|\frac{[f(x)G - FG] + [FG - g(x)F]}{g(x)G}\right| \\ &\le \frac{|f(x)G - FG| + |FG - g(x)F|}{|G||g(x)|} = \frac{|f(x) - F|}{|g(x)|} + \frac{|F||g(x) - G|}{|G||g(x)|}. \end{aligned}$$

(b) Because $\lim_{x\to a} g(x) = G$, it follows that $\lim_{x\to a} |g(x)| = |G|$. Hence with $\epsilon = |G|/2$, there exists a $\delta_1 > 0$ such that whenever $0 < |x-a| < \delta_1$,

$$\big||g(x)| - |G|\big| < \frac{|G|}{2} \quad\Longleftrightarrow\quad -\frac{|G|}{2} < |g(x)| - |G| < \frac{|G|}{2} \quad\Longleftrightarrow\quad \frac{|G|}{2} < |g(x)| < \frac{3|G|}{2}.$$

Because $\lim_{x\to a} f(x) = F$, we can say that for any $\epsilon > 0$, there exists a $\delta_2 > 0$ such that whenever $0 < |x-a| < \delta_2$,

$$|f(x) - F| < \frac{\epsilon|G|}{4}.$$

Because $\lim_{x\to a} g(x) = G$, we can say that for any $\epsilon > 0$, there exists a $\delta_3 > 0$ such that whenever $0 < |x-a| < \delta_3$,

$$|g(x) - G| < \frac{\epsilon|G|^2}{4(|F|+1)}.$$

(c) It now follows that for $0 < |x-a| < \delta$ where δ is the smallest of δ_1, δ_2, and δ_3,

$$\left|\frac{f(x)}{g(x)} - \frac{F}{G}\right| < \frac{\epsilon|G|}{4}\frac{2}{|G|} + \frac{|F|+1}{|G|}\frac{2}{|G|}\frac{\epsilon|G|^2}{4(|F|+1)} = \epsilon.$$

This completes the proof.

REVIEW EXERCISES

1. $\displaystyle\lim_{x\to 1}\frac{x^2-2x}{x+5} = \frac{1-2}{1+5} = -\frac{1}{6}$

2. $\displaystyle\lim_{x\to -1}\frac{x^2-1}{x+1} = \lim_{x\to -1}\frac{(x+1)(x-1)}{x+1} = \lim_{x\to -1}(x-1) = -2$

3. $\displaystyle\lim_{x\to -2}\frac{x^2+4x+4}{x+3} = 0$

4. $\displaystyle\lim_{x\to\infty}\frac{x+5}{x-3} = \lim_{x\to\infty}\frac{1+\frac{5}{x}}{1-\frac{3}{x}} = 1$

5. $\displaystyle\lim_{x\to -\infty}\frac{x^2+3x+2}{2x^2-5} = \lim_{x\to -\infty}\frac{1+\frac{3}{x}+\frac{2}{x^2}}{2-\frac{5}{x^2}} = \frac{1}{2}$

6. $\displaystyle\lim_{x\to -\infty}\frac{5-x^3}{3+4x^3} = \lim_{x\to -\infty}\frac{\frac{5}{x^3}-1}{\frac{3}{x^3}+4} = -\frac{1}{4}$

7. $\displaystyle\lim_{x\to\infty}\frac{3x^3+2x-5}{x^2+5x} = \lim_{x\to\infty}\frac{3x+\frac{2}{x}-\frac{5}{x^2}}{1+\frac{5}{x}} = \infty$

8. $\displaystyle\lim_{x\to\infty}\frac{4-3x+x^2}{3+5x^3} = \lim_{x\to\infty}\frac{\frac{4}{x^2}-\frac{3}{x}+1}{\frac{3}{x^2}+5x} = 0$

9. $\displaystyle\lim_{x\to 2^+}\frac{x^2-2x}{x^2+2x} = 0$

10. $\displaystyle\lim_{x\to 2^-}\frac{x^2-4x+4}{x-2} = \lim_{x\to 2^-}\frac{(x-2)^2}{x-2} = \lim_{x\to 2^-}(x-2) = 0$

11. $\displaystyle\lim_{x\to 0}\frac{x^2+2x}{3x-2x^2} = \lim_{x\to 0}\frac{x+2}{3-2x} = \frac{2}{3}$

12. $\displaystyle\lim_{x\to 1}\frac{x^2+5x}{(x-1)^2} = \infty$

13. $\displaystyle\lim_{x\to 1}\frac{\sqrt{x}-1}{x} = 0$

14. $\displaystyle\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1}\frac{\sqrt{x}-1}{(\sqrt{x}-1)(\sqrt{x}+1)} = \lim_{x\to 1}\frac{1}{\sqrt{x}+1} = \frac{1}{2}$

15. $\displaystyle\lim_{x\to 1/2}\frac{(2-4x)^3}{x(2x-1)^2} = \lim_{x\to 1/2}\frac{8(1-2x)}{x} = 0$

16. $\displaystyle\lim_{x\to\infty}\frac{\cos 5x}{x} = 0$

17. This limit does not exist.

18. $\displaystyle\lim_{x\to -\infty}\frac{\sqrt{3x^2+4}}{2x+5} = \lim_{x\to -\infty}\frac{\frac{\sqrt{3x^2+4}}{x}}{\frac{2x+5}{x}} = \lim_{x\to -\infty}\frac{-\sqrt{3+\frac{4}{x^2}}}{2+\frac{5}{x}} = -\frac{\sqrt{3}}{2}$

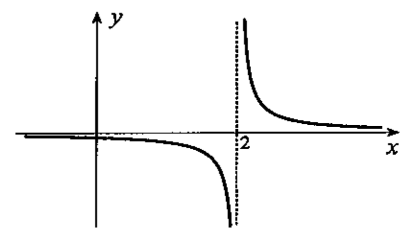
19. $\displaystyle\lim_{x\to\infty}\frac{\sqrt{3x^2+4}}{2x+5}=\lim_{x\to\infty}\frac{\sqrt{3+\frac{4}{x^2}}}{2+\frac{5}{x}}=\frac{\sqrt{3}}{2}$

20. $$\lim_{x\to\infty}\left(\sqrt{2x+1}-\sqrt{3x-1}\right)=\lim_{x\to\infty}\left[\left(\sqrt{2x+1}-\sqrt{3x-1}\right)\frac{\sqrt{2x+1}+\sqrt{3x-1}}{\sqrt{2x+1}+\sqrt{3x-1}}\right]$$

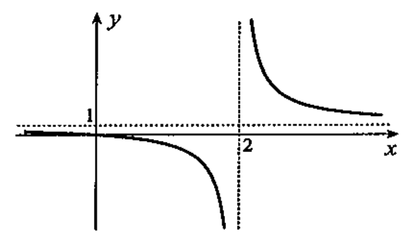
$$=\lim_{x\to\infty}\frac{(2x+1)-(3x-1)}{\sqrt{2x+1}+\sqrt{3x-1}}=\lim_{x\to\infty}\frac{2-x}{\sqrt{2x+1}+\sqrt{3x-1}}$$

$$=\lim_{x\to\infty}\frac{\frac{2-x}{\sqrt{x}}}{\frac{\sqrt{2x+1}+\sqrt{3x-1}}{\sqrt{x}}}=\lim_{x\to\infty}\frac{\frac{2}{\sqrt{x}}-\sqrt{x}}{\sqrt{2+\frac{1}{x}}+\sqrt{3-\frac{1}{x}}}=-\infty$$

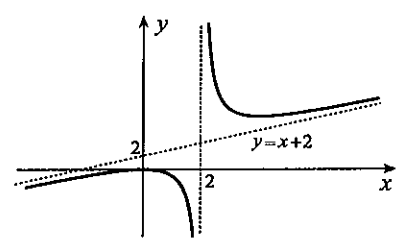
21. The limits $\displaystyle\lim_{x\to 2^+}f(x)=\infty$, $\displaystyle\lim_{x\to 2^-}f(x)=-\infty$, $\displaystyle\lim_{x\to\infty}f(x)=0^+$, and $\displaystyle\lim_{x\to-\infty}f(x)=0^-$ lead to the graph to the right. The discontinuity at $x=2$ is infinite.



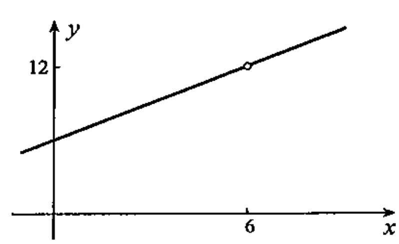
22. The limits $\displaystyle\lim_{x\to 2^+}f(x)=\infty$, $\displaystyle\lim_{x\to 2^-}f(x)=-\infty$, $\displaystyle\lim_{x\to\infty}f(x)=1^+$, and $\displaystyle\lim_{x\to-\infty}f(x)=1^-$ lead to the graph to the right. The discontinuity at $x=2$ is infinite.



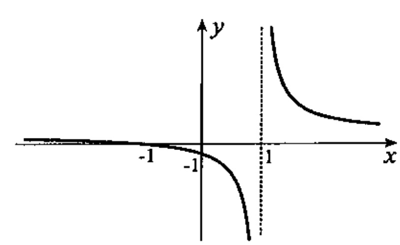
23. We calculate $\displaystyle\lim_{x\to 2^+}f(x)=\infty$ and $\displaystyle\lim_{x\to 2^-}f(x)=-\infty$. From $f(x)=x+2+\dfrac{4}{x-2}$, we obtain the oblique asymptote $y=x+2$, approached from above as $x\to\infty$ and from below as $x\to-\infty$. These give the graph to the right. The discontinuity at $x=2$ is infinite.



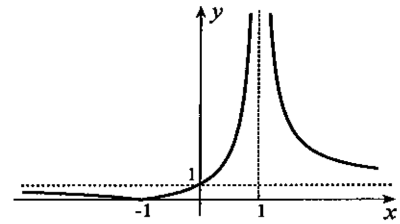
24. For $x\neq 6$, $f(x)=x+6$. Consequently, the graph is a straight line with the point at $x=6$ removed. The discontinuity at $x=6$ is removable.



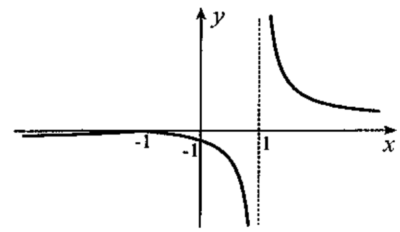
25. The limits $\displaystyle\lim_{x\to 1^+}f(x)=\infty$, $\displaystyle\lim_{x\to 1^-}f(x)=-\infty$, $\displaystyle\lim_{x\to\infty}f(x)=1^+$, and $\displaystyle\lim_{x\to-\infty}f(x)=1^-$ lead to the graph to the right. The discontinuity at $x=1$ is infinite.



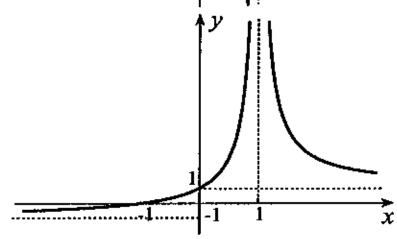
26. The limits $\lim_{x\to 1^+} f(x) = \infty$, $\lim_{x\to 1^-} f(x) = \infty$, $\lim_{x\to\infty} f(x) = 1^+$, and $\lim_{x\to -\infty} f(x) = 1^-$ lead to the graph to the right. The discontinuity at $x = 1$ is infinite.



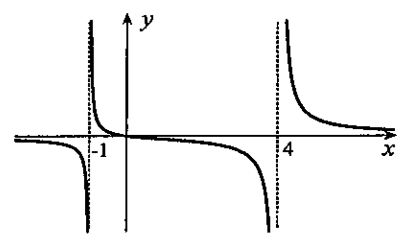
27. The limits $\lim_{x\to 1^+} f(x) = \infty$, $\lim_{x\to 1^-} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = 1^+$, and $\lim_{x\to -\infty} f(x) = -1^+$ lead to the graph to the right. The discontinuity at $x = 1$ is infinite.



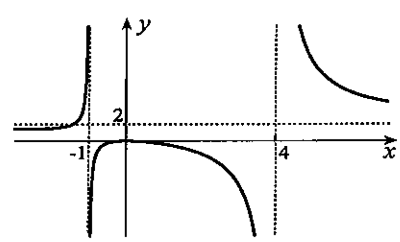
28. The limits $\lim_{x\to 1^+} f(x) = \infty$, $\lim_{x\to 1^-} f(x) = \infty$, $\lim_{x\to\infty} f(x) = 1^+$, and $\lim_{x\to -\infty} f(x) = -1^+$ lead to the graph to the right. The discontinuity at $x = 1$ is infinite.



29. With $f(x) = \dfrac{2x}{(x-4)(x+1)}$, we calculate $\lim_{x\to 4^+} f(x) = \infty$, $\lim_{x\to 4^-} f(x) = -\infty$, $\lim_{x\to -1^+} f(x) = \infty$, and $\lim_{x\to -1^-} f(x) = -\infty$. Furthermore, $\lim_{x\to\infty} f(x) = 0^+$ and $\lim_{x\to -\infty} f(x) = 0^-$. The graph is shown to the right. The discontinuities at $x = -1$, 4 are infinite.

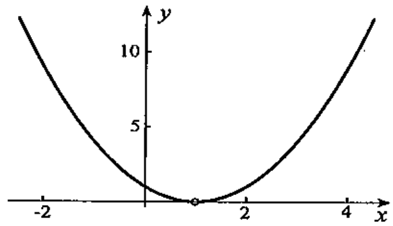


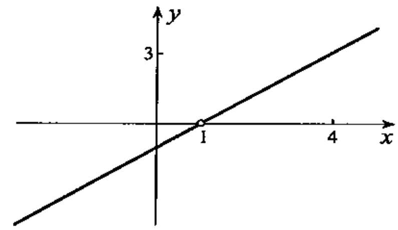
30. With $f(x) = \dfrac{2x^2}{(x-4)(x+1)}$, we calculate $\lim_{x\to 4^+} f(x) = \infty$, $\lim_{x\to 4^-} f(x) = -\infty$, $\lim_{x\to -1^+} f(x) = -\infty$, and $\lim_{x\to -1^-} f(x) = \infty$. Furthermore, with $f(x) = 2 + \dfrac{6x+8}{x^2-3x-4}$, we find $\lim_{x\to\infty} f(x) = 2^+$ and $\lim_{x\to -\infty} f(x) = 2^-$. The graph is shown to the right. The discontinuities at $x = -1$, 4 are infinite.



31. For $x \neq 1$, $f(x) = \dfrac{(x-1)^3}{x-1} = (x-1)^2$. The graph of the function is therefore the parabola $y = (x-1)^2$ with the point at $x = 1$ deleted. The function has a removable discontinuity at $x = 1$.

32. For $x \neq 1$, $f(x) = \dfrac{(x-1)^3}{(x-1)^2} = x - 1$. The graph of the function is therefore the straight line $y = x - 1$ with the point at $x = 1$ deleted. The function has a removable discontinuity at $x = 1$.





33. The function can be expressed in the form

$$f(x) = x^2[1 - h(x)] + x[h(x) - h(x-4)] + (5-2x)h(x-4) = x^2 + (x - x^2)h(x) + (5-3x)h(x-4).$$

34. The function can be expressed in the form

$$\begin{aligned} f(x) &= (3 + x^3)[1 - h(x+1)] + (x^2+2)[h(x+1) - h(x-2)] + 4h(x-2) \\ &= 3 + x^3 - (x^3 - x^2 + 1)h(x+1) + (2 - x^2)h(x-2). \end{aligned}$$

35. The function is discontinuous at $x = \pm\sqrt{n}$, where $n > 0$ is an integer.