CIVE 205

Solid Mechanics II

COURSE NOTES

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Acknowledgement:

Most of these notes are adopted from the course notes “Introductory Structural Mechanics” by Dr. S.T. Ariaratnam to whom the instructor is indebted for his help. Thanks go to Dr. W. Brodland, Dr. R. Gracie, Dr. T. Hegazy, and Dr. K. Soudki for sharing their notes.

Table of Contents:

Text Book:

“Mechanics of Materials,” 7th Ed. by F.P. Beer, E. R. Johnston, Jr., J. T. DeWolf, D.F. Mazurek, McGraw-Hill.

Additional References:

“Statics and Mechanics of Materials,” 3rd Ed. by R.C. Hibbeler, Pearson Prentice Hall.

“Fundamentals of Structural Analysis,”4th Ed. by K.M. Leet, C. Uang, and A.M. Gilbert, McGraw Hill.

CivE 205– Solid Mechanics II

Part 1:

Beam Theory – A Review

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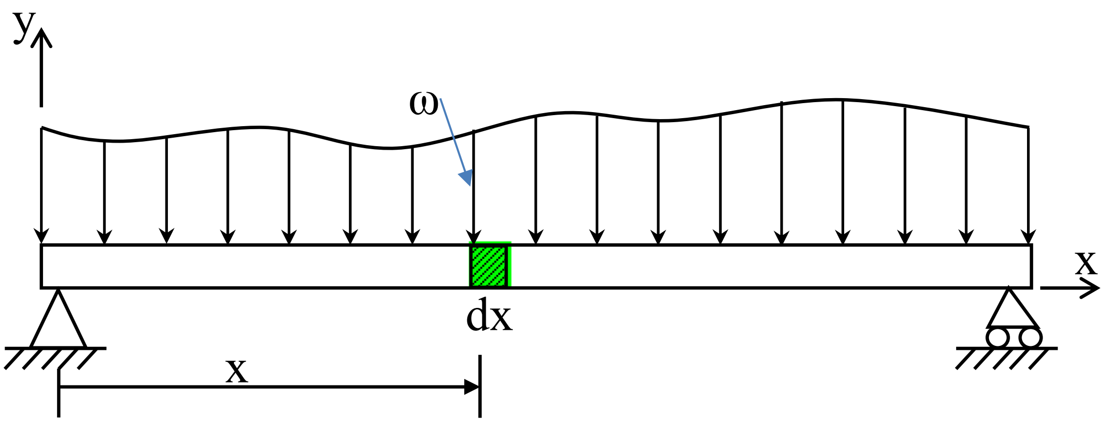
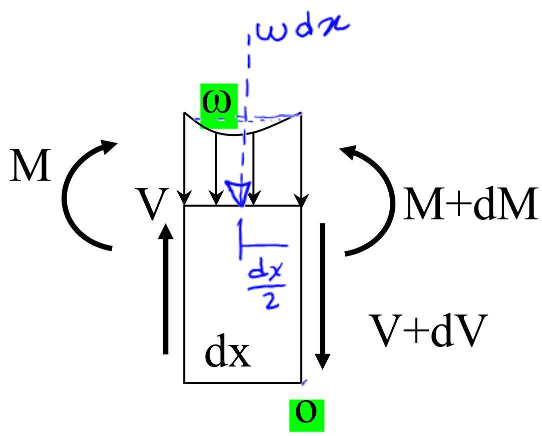
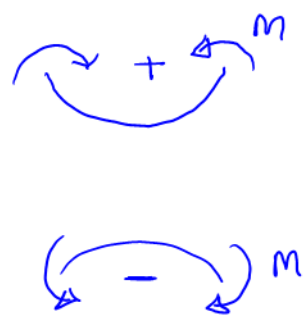
SHEARING FORCES AND BENDING MOMENTS IN BEAMS

For a beam subjected to a distributed load ω:







$\Sigma F_y = 0$

$$-\omega dx + \cancel{V} - (\cancel{V} + dV) = 0 \rightarrow \boxed{\frac{dV}{dx} = -\omega}$$

Pg 360
Sec 5-2
Text-book

Slope of the shear force diagram = - distributed load intensity at that point.

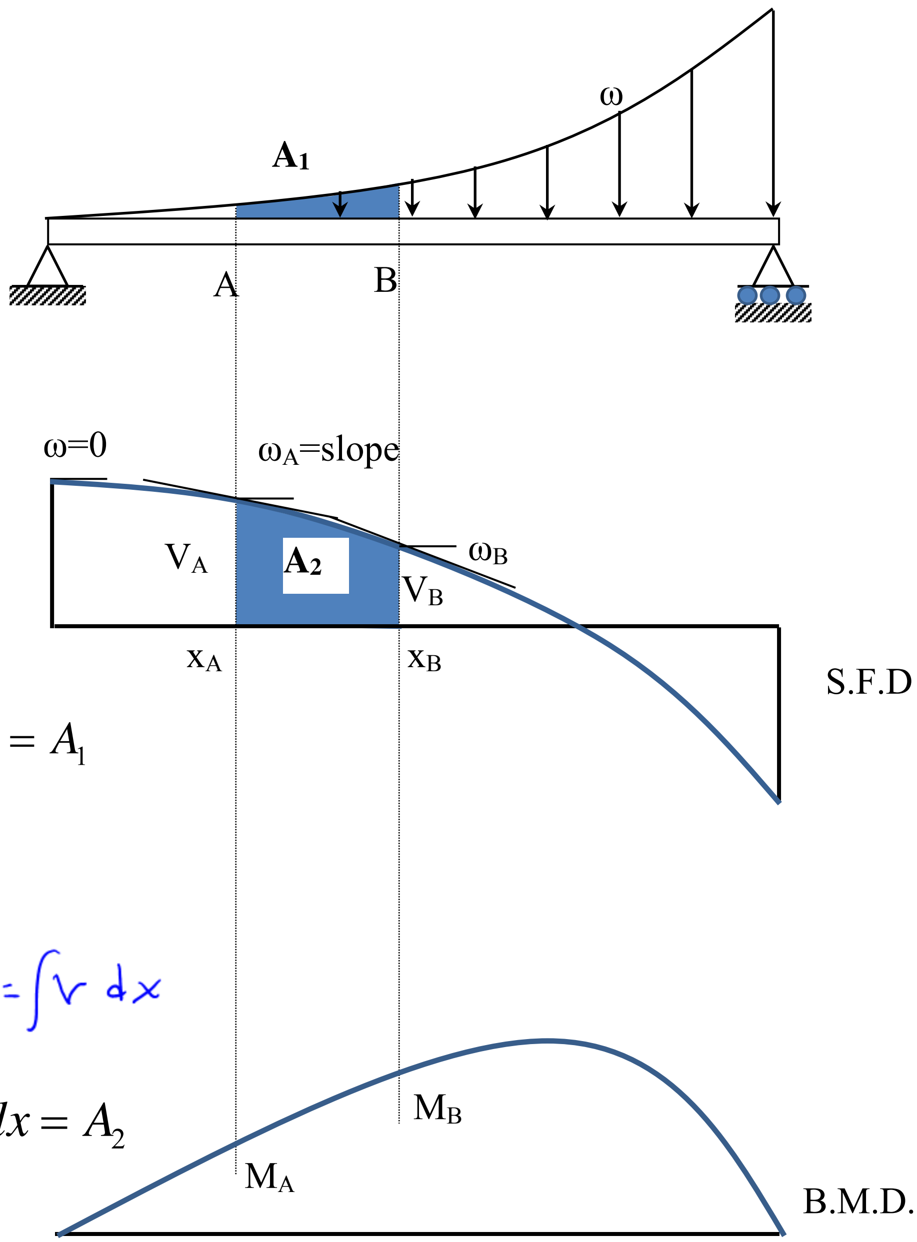
$\Sigma M_o = 0$ dx is very small $\Rightarrow dx^2 \simeq 0$

$$M-(M+dM) + Vdx - \omega \overbrace{\cancel{dx(dx/2)}}^{=0} = 0 \rightarrow \boxed{\frac{dM}{dx} = V}$$

Slope of the bending moment diagram =the shear force at that point.

Changes in V and M

Distributed Load



Shear Force

$$\frac{dV}{dx} = -w$$

$$dV = -w\,dx$$

$$\int_{V_A}^{V_B} dV = -\int w\,dx$$

$$V_B - V_A = -\int_{x_A}^{x_B} \omega.dx = A_1$$

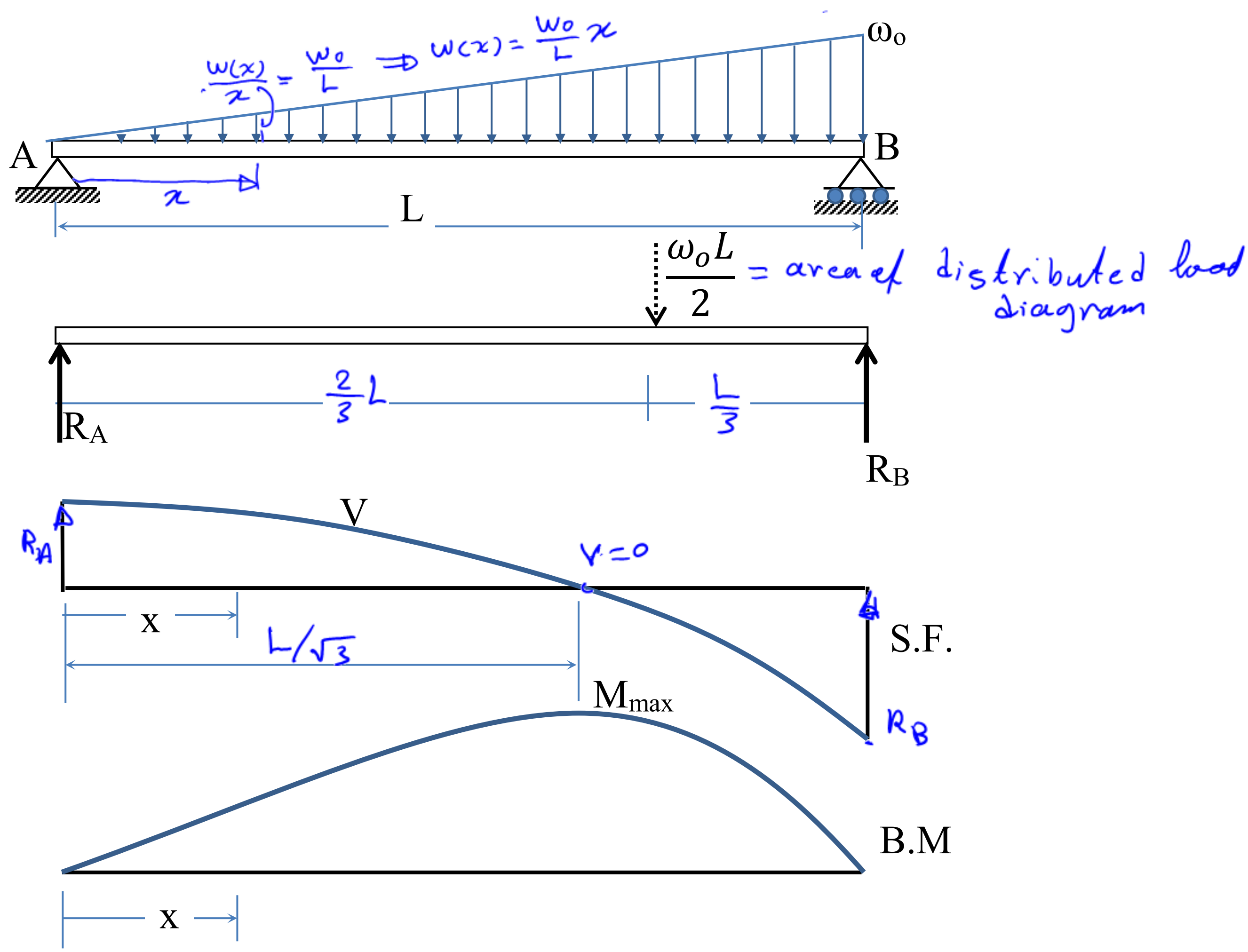
Bending Moment

$$\frac{dM}{dx} = V \Rightarrow \int dM = \int V\,dx$$

$$M_B - M_A = \int_{x_A}^{x_B} V.dx = A_2$$

These relationships are very helpful in sketching shear force and bending moment diagrams.

Example 1.1: Draw the shear force and bending moment diagrams?



Reactions:

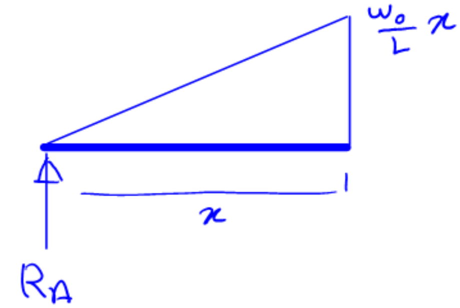
$$\sum M_A = \frac{\omega_o L}{2}\frac{2L}{3} - R_B L = 0$$

$$\boldsymbol{R_B = \frac{\omega_o L}{3}}$$

$$\sum F_y = R_A + R_B - \frac{\omega_o L}{2} = 0$$

$$\boldsymbol{R_A = \frac{\omega_o L}{6}}$$

a) Shearing Force:



$$V(x) = \int -w(x)\,dx = \int -\frac{w_o x}{L}\,dx$$
$$= -\frac{w_o x^2}{2L} + C_1$$

To find C_1, use B-C @ $x=0$ $V = R_A \Rightarrow R_A = \frac{-w_o(0)^2}{2L} + C_1$

$$\Rightarrow C_1 = R_A = \frac{w_o L}{6}$$

$$V(x) = \frac{w_o L}{6} - \frac{w_o x^2}{2L}$$

$$V(x) = 0 = \frac{w_o L}{6} - \frac{w_o x^2}{2L} \Rightarrow x = \frac{L}{\sqrt{3}}$$ (The location of Zero shear)

b) Bending Moment:

$$M(x) = \int V(x)\,dx = \int\left(\frac{w_o L}{6} - \frac{w_o x^2}{2L}\right)dx = \frac{w_o L x}{6} - \frac{w_o x^3}{6L} + C_2$$

Use B.C to find C_2: for simply supported beam, @ $x=0$, $M=0$ and $x=L$, $M=0$

@ $x=0$ $M=0$ $\Rightarrow C_2 = 0$

Therefore $M(x) = \frac{w_o L x}{6} - \frac{w_o x^3}{6L}$

M_{max} occurs when $\frac{dM}{dx} = 0$ But $\frac{dM}{dx} = V$ and $V=0$ @ $x = \frac{L}{\sqrt{3}}$

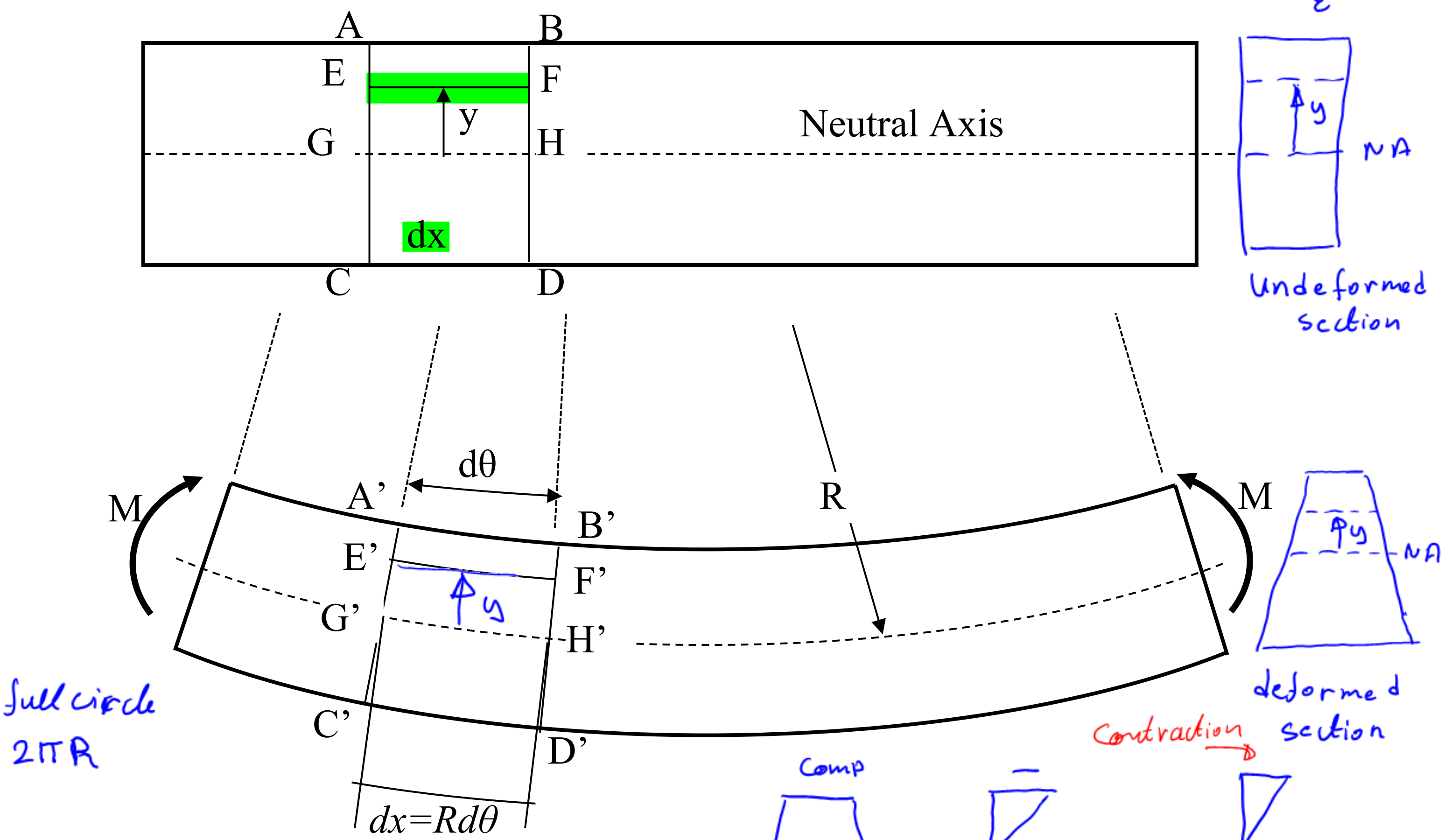
$$M_{max} = \frac{w_o}{6}\left(\frac{L^2}{\sqrt{3}} - \frac{L^2}{3\sqrt{3}}\right) = \frac{w_o L^2}{9\sqrt{3}}$$

P 238 and Pg 600 (Textbook)

Review of Beam Flexure:

Assumptions:

1) Linearly elastic material ($\sigma = E\varepsilon$)
2) Plane cross-sections remain plane



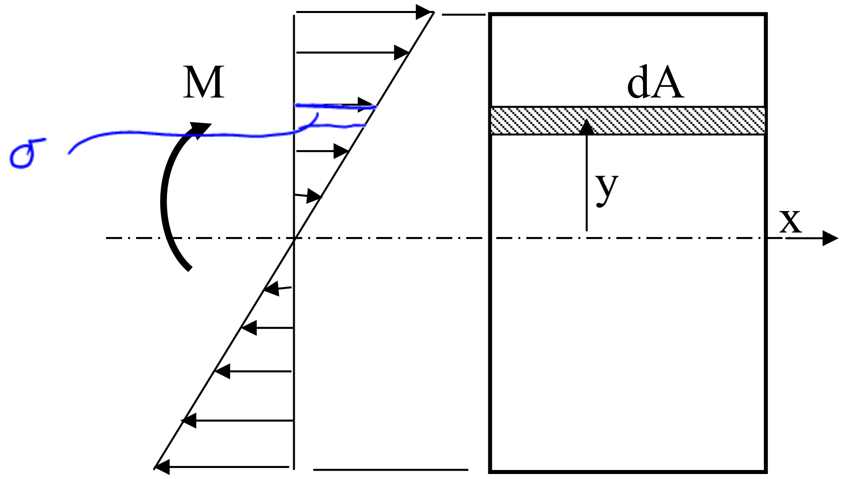
$$\varepsilon = \frac{\Delta l}{l} \quad E'F'$$

EF = dx

$$= \frac{(R-y)d\theta - R.d\theta}{\underbrace{R.d\theta}_{dx}} = -\frac{y}{R}$$

$$\sigma = E\underbrace{\varepsilon}_{= -\frac{y}{R}} = -E\frac{y}{R}$$

Stress (Hooke's Law)



Stress $= -E\frac{y}{R}$

For equilibrium $\Sigma F_x = 0$

$$\sum F_x = \int_{Area} \sigma . dA = -\frac{E}{R}\int_{Area} y.dA = 0$$

Therefore, neutral surface defined by $\int_{Area} y.dA = 0$ (1st moment of area)

$\Sigma M = 0$

$$M + \int_{Area} \sigma . y.dA = 0$$ Equilibrium

$$M = -(-\frac{E}{R})\int_{Area} y^2 .dA$$

Set $I = \int_{Area} y^2 dA$ 2nd moment of area (moment of Inertia)

Then $\boxed{M = \frac{EI}{R}}$ $\frac{M}{EI} = \frac{1}{R}$

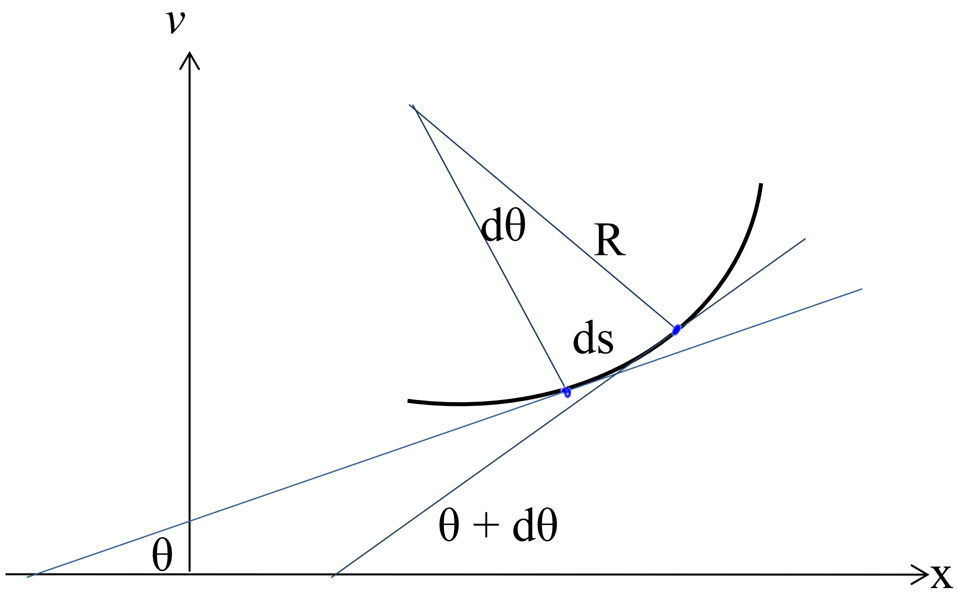
Where EI = flexural rigidity or bending stiffness

R = radius of curvature

y = the distance from the neutral surface

what is $\frac{1}{R}$ in terms of deformation? $\Rightarrow$

Elastic Curve

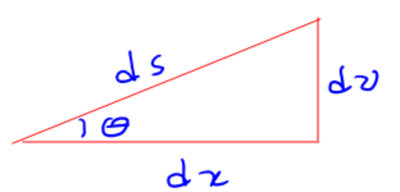


$$R.d\theta = ds$$

Rearrange

$$\boxed{\frac{d\theta}{ds} = \frac{1}{R}}$$ Curvature

This is exact and good for large deflection



For small deflections and small slopes $\dfrac{dv}{dx} << 1$, $ds \approx dx$

which is the case of beams in structures

$$\frac{dv}{dx} = \tan\theta \cong \theta$$

$$\frac{1}{R} \approx \frac{d\theta}{ds} \approx \frac{d\theta}{dx} = \frac{d^2v}{dx^2}$$

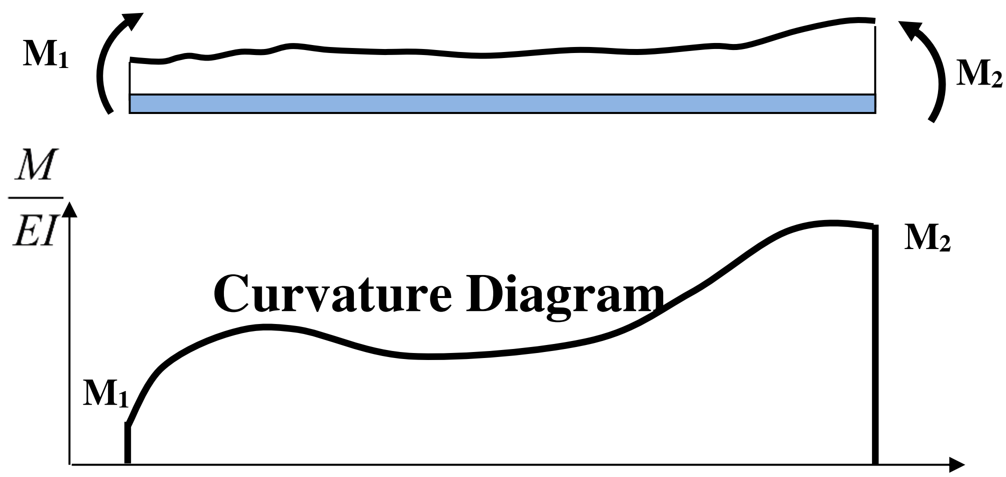
Therefore $\dfrac{1}{R} \cong \dfrac{d^2v}{dx^2}$

But $\dfrac{1}{R} = \dfrac{M}{EI}$ (Pg 1.7)

Note: From calculus, the exact expression for curvature is: $\boxed{\dfrac{1}{R} = \dfrac{d^2v/dx^2}{\left(1+(dv/dx)^2\right)^{3/2}}}$

Or $$\frac{M}{EI} = \frac{d^2 v}{dx^2}$$ the flexural differential equation

Therefore, the M/EI diagram represents the **curvature diagram** which has a significant role in beam deflections.



Beam Rotation and Deflection

By successive integration

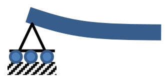
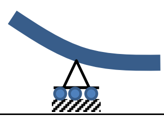
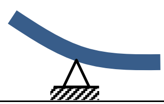
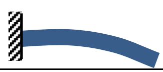
Curvature $$\frac{d\theta}{ds} = \frac{d^2 v}{dx^2} = \frac{M}{EI}$$

Rotation (Slope) $$\theta = \frac{dv}{dx} = \int \frac{M}{EI} dx + C_3$$ in radians

Deflection $$v = \int \theta dx + C_4 = \int \left(\int \frac{M}{EI} dx \right) dx + C_3 x + C_4$$

Where C_3 and C_4 are constants of integration which are determined by boundary conditions.

Boundary Conditions

End Support		**Boundary Conditions**
Roller		$M = 0$ $\Delta = 0$
Pin		$M = 0$ $\Delta = 0$
Roller		$\Delta = 0$ $M \neq 0$
Pin		$\Delta = 0$ $M \neq 0$
Fixed End		$\Delta = 0$ $\theta = 0$

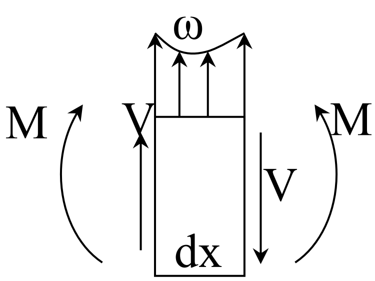
Summary of Beam Results

Category	**Statics**	**Deformation**
Distributed load *(ω)*	$\dfrac{dV}{dx}$	$EI\dfrac{d^4v}{dx^4}$
Shear Force *(V)*	$\int \omega dx + c_1$	$EI\dfrac{d^3v}{dx^3}$
Bending Moment *(M)*	$\int V dx + c_2$	$EI\dfrac{d^2v}{dx^2}$
Slope *(θ)*	$\int \dfrac{M}{EI} dx + c_3$	$\dfrac{dv}{dx}$
Deflection *(v)*		$\int \theta dx + c_4$

integrate

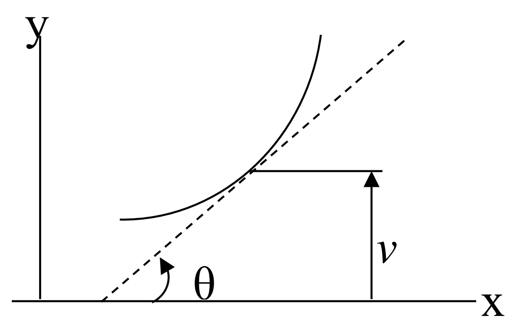
differentiate

Positive sign conventions used in the above:



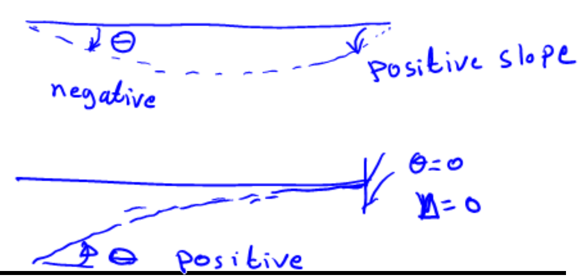
STATICS

+

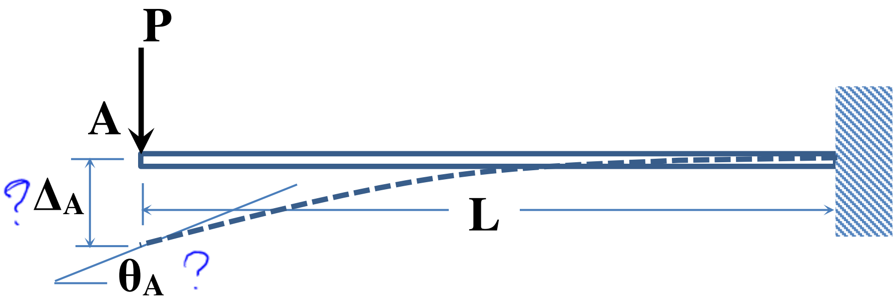


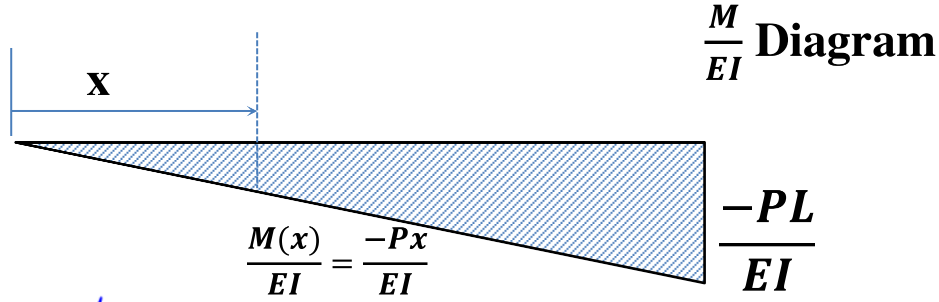
DEFORMATION

+



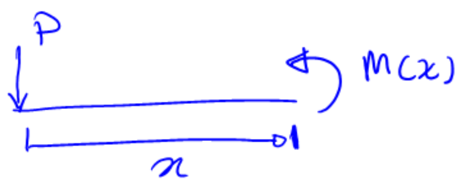
Example 1.2: Find deflection and rotation (slope) at point A.





Bending moment eqn:

$$M(x) = -Px$$



$$\frac{d^2v}{dx^2} = \frac{M}{EI} = \frac{-Px}{EI}$$

$$\theta = \frac{dv}{dx} = \int \frac{M}{EI}\,dx + C_3 = \frac{-P}{EI}\int x\,dx + C_3$$

$$\theta = -\frac{Px^2}{2EI} + C_3 \quad \text{-----} \triangleright \; (1)$$

Beam deflection

$$v = \int \frac{dv}{dx}\,dx + C_4 = \int \left(\frac{-Px^2}{2EI} + C_3\right)dx + C_4$$

$$= \frac{-Px^3}{6EI} + C_3 x + C_4 \quad \text{-----} \triangleright \; (2)$$

To find C_3 & C_4, use B.Cs

@ $x = L$, $\frac{dv}{dx} = \theta = 0$

Sub in ①

$$0 = \frac{-PL^2}{2EI} + C_3 \implies C_3 = \frac{PL^2}{2EI}$$

$\therefore$ slope equation: $\theta = \dfrac{-Px^2}{2EI} + \dfrac{PL^2}{2EI}$ ---→ ③

another B.C: @ $x = L$, $v = 0$

Sub in eq ②

$$0 = \frac{-PL^3}{6EI} + \frac{PL^3}{2EI} + C_4 \implies C_4 = \frac{-PL^3}{3EI}$$

$\therefore$ Deflection eq: $v = \dfrac{-Px^3}{6EI} + \dfrac{PL^2}{2EI}x - \dfrac{PL^3}{3EI}$ ---→ ④

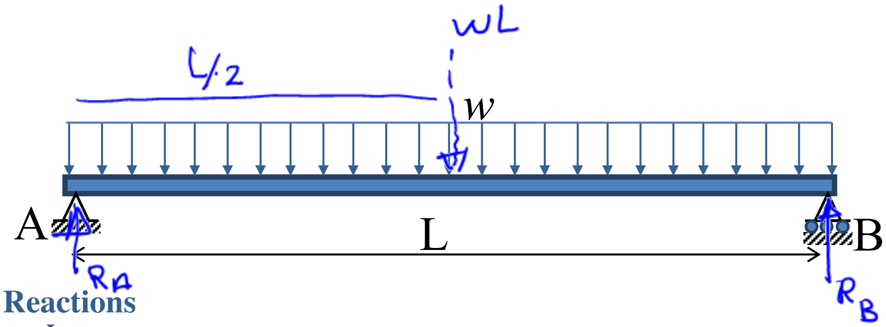
To find slope @ A: $x = 0$ in eq ③

$$\boxed{\theta_A = \frac{PL^2}{2EI}}$$

To find deflection @ A: $x = 0$ in eq ④

$$\boxed{\Delta_A = v_A = \frac{-PL^3}{3EI}} \downarrow$$

Example 1.3: Find 1) Slope at A , and 2) Deflection at mid span?



1) **Find Reactions**

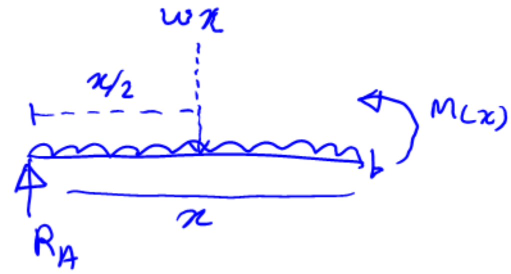
$$\sum M_A = wL\frac{L}{2} - R_B L = 0$$

$$\boldsymbol{R_B = \frac{wL}{2}}$$

$$\sum F_y = R_A + R_B - wL = 0$$

$$\boldsymbol{R_A = \frac{wL}{2}}$$

2) **Bending Moment**



$$M(x) = R_A x - wx\frac{x}{2}$$

$$M(x) = \frac{wL}{2}x - \frac{wx^2}{2}$$

3) **Deformation**

$$\frac{M(x)}{EI} = \frac{d\theta}{dx} = \frac{d^2v}{dx^2}$$

slope : $\frac{dv}{dx} = \int \frac{M(x)}{EI}dx = \frac{w}{2EI}\int (Lx - x^2)\,dx + C_3$

$$= \frac{w}{2EI}\left(\frac{Lx^2}{2} - \frac{x^3}{3}\right) + C_3 \quad \dashrightarrow \quad ①$$

Deflection:

$$v(x) = \int \frac{dv}{dx}\,dx + C_4$$

$$= \frac{w}{2EI}\int\left(\frac{Lx^2}{2} - \frac{x^3}{3}\right)dx + C_3 x + C_4$$

$$= \frac{w}{2EI}\left(\frac{Lx^3}{6} - \frac{x^4}{12}\right) + C_3 x + C_4 \qquad \longrightarrow (2)$$

To find C_3 & C_4, use B.Cs

At $x = 0$, $v = 0$ sub. in eq. (2) $\Rightarrow C_4 = 0$

At $x = L$, $v = 0$ sub in eq (2)

$$\Rightarrow 0 = \frac{w}{2EI}\left(\frac{L^4}{6} - \frac{L^4}{12}\right) + C_3 L \qquad \Rightarrow C_3 = \frac{-wL^3}{24EI}$$

Therefore!

Slope eq: $$\frac{dv}{dx} = \frac{w}{2EI}\left(\frac{Lx^2}{2} - \frac{x^3}{3}\right) - \frac{wL^3}{24EI} \qquad \longrightarrow (3)$$

deflection eq: $$v(x) = \frac{w}{12EI}\left(Lx^3 - \frac{x^4}{2}\right) - \frac{wL^3}{24EI}x \qquad \longrightarrow (4)$$

To find slope @ A, sub $x = 0$ in eq (3)

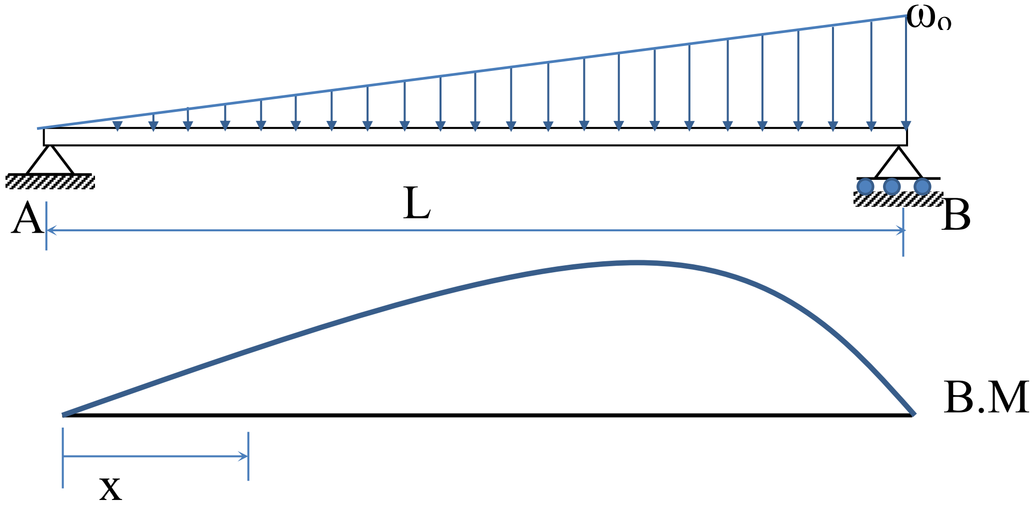
$$\boxed{\theta_A = \frac{-wL^3}{24EI}}$$

To find deflection @ midspan, sub. $x = \frac{L}{2}$ in eq (4)

$$v\left(\frac{L}{2}\right) = \frac{w}{12EI}\left[L\cdot\left(\frac{L}{2}\right)^3 - \frac{(L/2)^4}{2}\right] - \frac{wL^3}{24EI}\left(\frac{L}{2}\right)$$

$$\boxed{= -\frac{5wL^4}{384EI}} \downarrow$$

Example 1.4: For Example 1.1, find 1) Max deflection and its location, and 2) Slope at B?



From example (1.1) $M(x) = \frac{\omega_o L x}{6} - \frac{\omega_o x^3}{6L}$

$$\text{Slope} = \int \frac{M(x)}{EI} dx + C_3$$

$$= \frac{w_0}{6EI} \int \left(Lx - \frac{x^3}{L}\right) dx + C_3$$

$$= \frac{w_0}{6EI} \left(L\frac{x^2}{2} - \frac{x^4}{4L}\right) + C_3 \quad \text{-----} \rightarrow (1)$$

Deflection:

$$v(x) = \int \frac{dv}{dx} dx + C_3 x + C_4 = \frac{w_0}{6EI} \int \left(\frac{Lx^2}{2} - \frac{x^4}{4L}\right) dx + C_3 x + C_4$$

$$= \frac{w_0}{6EI} \left(\frac{Lx^3}{6} - \frac{x^5}{20L}\right) + C_3 x + C_4 \quad \text{-------} \rightarrow (2)$$

B.Cs: @ $x=0$ $\quad v(0)=0 \Rightarrow$ from eq ② $\quad C_4 = 0$

@ $x=L$ $\quad v(L) = 0 = \frac{w_o}{6EI}\left(\frac{L^4}{6} - \frac{L^4}{20}\right) + C_3 L$

$$\Rightarrow C_3 = -\frac{7}{360}\frac{w_o L^3}{EI}$$

Therefore

Slope eq: $\frac{dv}{dx} = \frac{w_o}{6EI}\left(\frac{Lx^2}{2} - \frac{x^4}{4L}\right) - \frac{7}{360}\frac{w_o L^3}{EI} \longrightarrow$ ③

Def eq $\quad v(x) = \frac{w_o}{6EI}\left(\frac{Lx^3}{6} - \frac{x^5}{20L}\right) - \frac{7}{360}\frac{w_o L^3}{EI}x \longrightarrow$ ④

max deflection occurs when $\frac{dv}{dx} = 0$

using eq③ $\quad 0 = \cancel{\frac{w_o}{6EI}}\left(\frac{Lx^2}{2} - \frac{x^4}{4L} - \frac{7L^3}{60}\right)$ $\quad$ (multiply by $-4L$)

$$x^4 - 2L^2x^2 + \frac{7}{15}L^4 = 0$$

quadratic eq $\quad ax^2 + bx^2 + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x^2 = \frac{-(2L^2) \pm \sqrt{(-2L^2)^2 - 4(1)\left(\frac{7}{15}L^4\right)}}{2}$$

$x^2 = L^2(1 \pm 0.73)$

$x = \underline{1.32L}$ $\quad$ or $\quad \boxed{x = 0.52L}$ ✓

Not valid
($> L$)

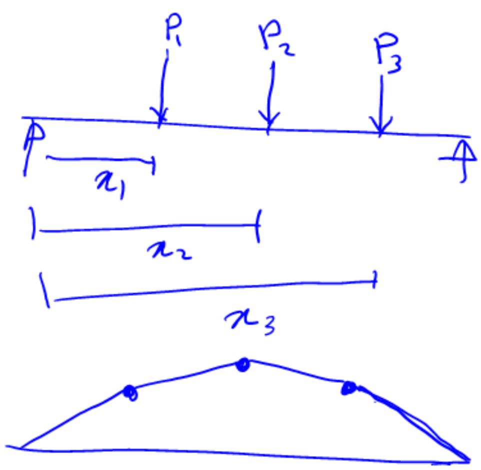
Sub $x = 0.52L$ in eq (4) to find v_{max}

$$\boxed{v_{max} = -0.0062 \frac{w_o L^4}{EI}}$$

Slope @ B

$$\theta_B = \frac{dv}{dx}(L) = \frac{w_o}{6EI}\left(\frac{L^3}{2} - \frac{L^3}{4}\right) - \frac{7}{360}\frac{w_o L^3}{EI} = \frac{8}{360}\frac{w_o L^3}{EI}$$

$$\text{or } \boxed{\theta_B = 0.022 \frac{w_o L^3}{EI}}$$



CivE 205– Solid Mechanics II

Part 2:

Moment Area Theorems

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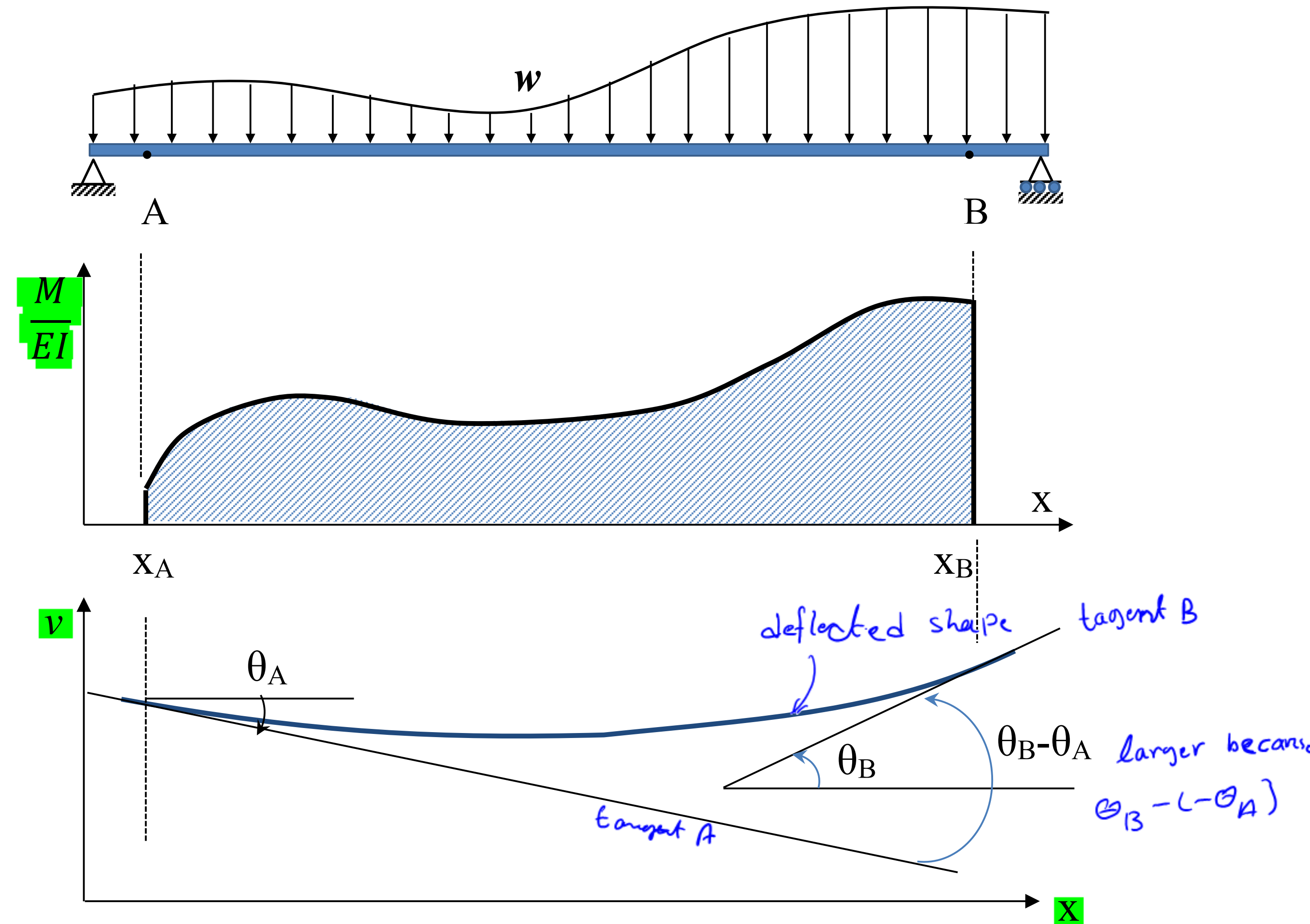
1st Moment area Theorem:

Chapter 9 Pg 649 text book



from (Pg 1-8)

$$\frac{d\theta}{dx} = \frac{d^2v}{dx^2} = \frac{M(x)}{EI} \Longrightarrow d\theta = \frac{M(x)}{EI}\,dx$$

integrate $\displaystyle\int_{\theta_A}^{\theta_B} d\theta = \int_{x_A}^{x_B} \frac{M(x)}{EI}\,dx \Longrightarrow$

$$\theta_B - \theta_A = \int_{x_A}^{x_B} \frac{M}{EI}\,dx$$

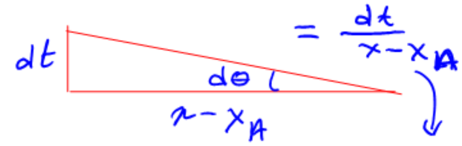
1st Moment area theorem "**<u>The change in slope from point A to point B on the elastic curve equals the area under the M/EI diagram between points A and B</u>**."

Note:

- Angle $\theta_B - \theta_A$ and the M/EI area have the same sign.
- Positive $\theta_B - \theta_A$ indicates counterclockwise rotation of tangent at B relative to the tangent at A.

2nd Moment Area Theorem:

$\tan d\theta \simeq d\theta = \frac{dt}{x-x_A}$

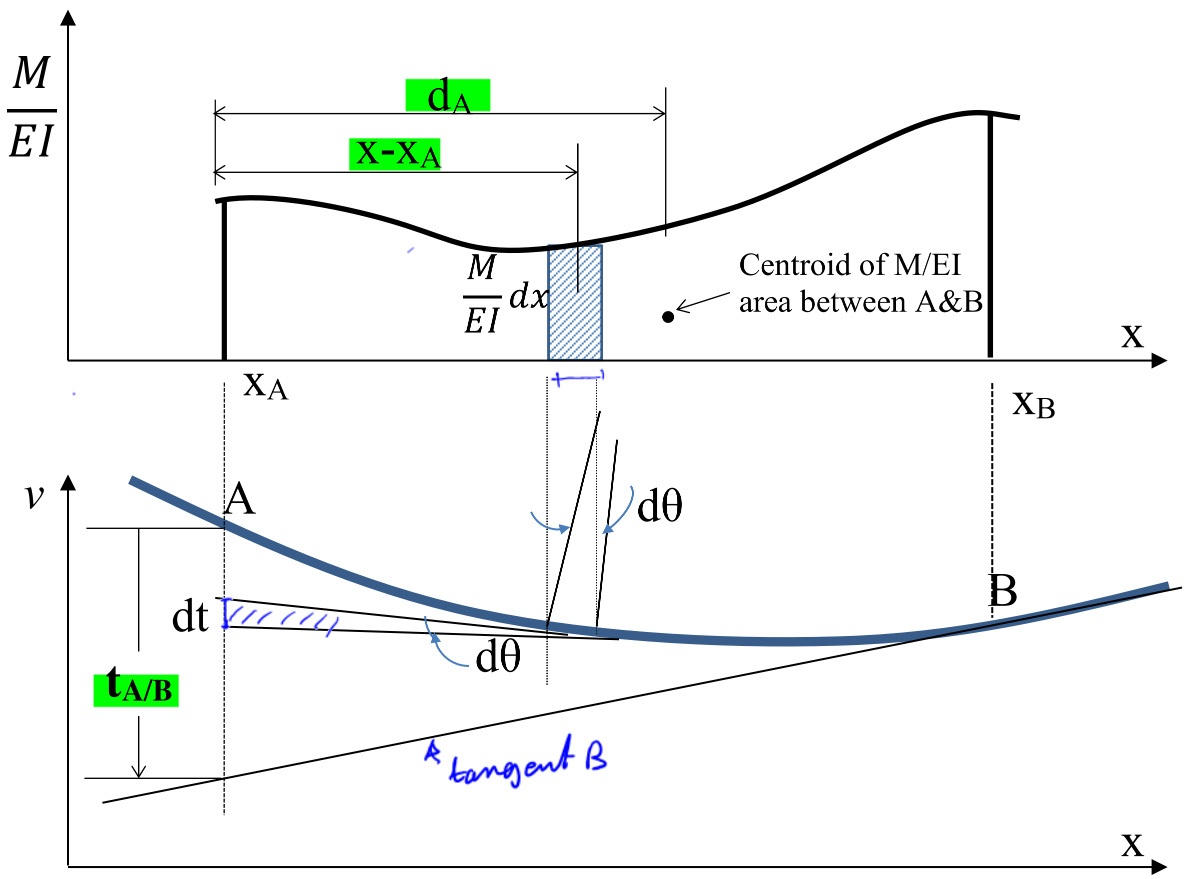


$dt = (x-x_A)d\theta$

For small deformations

$dt = (x - x_A)d\theta$

(dt) is the small vertical distance between the tangents drawn to the deflected curve at the ends of a small element of length dx.



$\int dt$

The total vertical distance is:

$$t_{A/B} = \int_{x_A}^{x_B} (x - x_A)d\theta$$

but $d\theta = \frac{M(x)}{EI}dx$

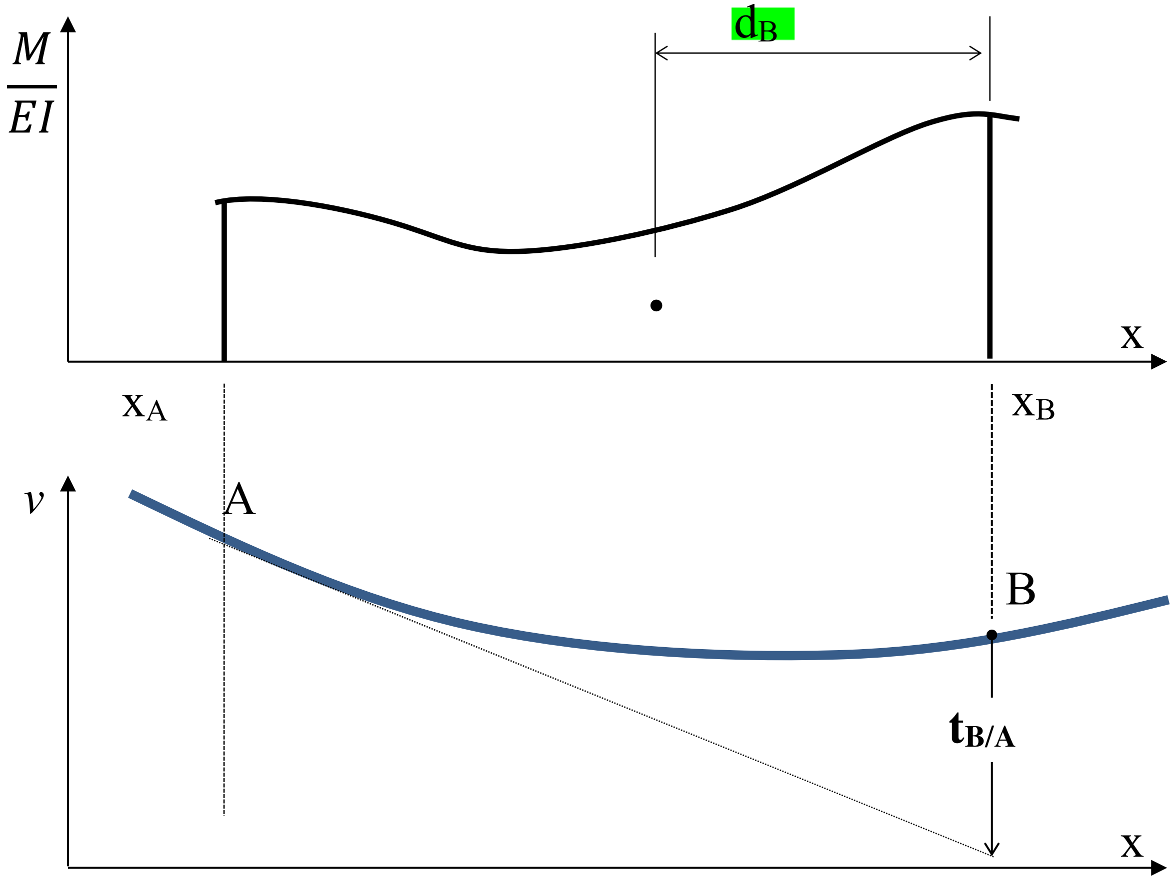
$$= \int_{x_A}^{x_B} \underbrace{(x - x_A)}_{\text{arm}} \overbrace{\frac{M(x)}{EI}dx}^{\text{area}}$$

This is the moment of the area under the M/EI diagram about the vertical line through A.

$$t_{A/B} = d_A \int_{x_A}^{x_B} \frac{M(x)}{EI} dx$$

Where d_A is the centroidal distance from A.

Similarly:

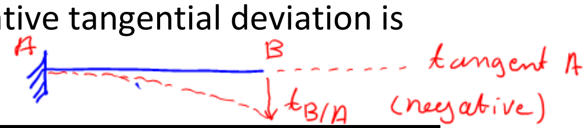


$$t_{B/A} = d_B \int_{x_A}^{x_B} \frac{M(x)}{EI} dx$$

" $t_{B/A}$

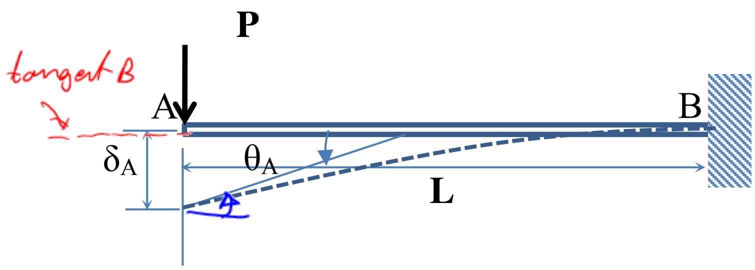
2nd Moment area theorem **"The tangential deviation of a point (B) from the tangent drawn at another point (A) equals the moment area of the M/EI diagram bounded by these points (A and B) about an axis through the first point (B)"**

- $t_{B/A} \neq t_{A/B}$ why? $d_A \neq d_B$
- The point with a positive tangential deviation is located above the corresponding tangent, and a point with a negative tangential deviation is located below that tangent.

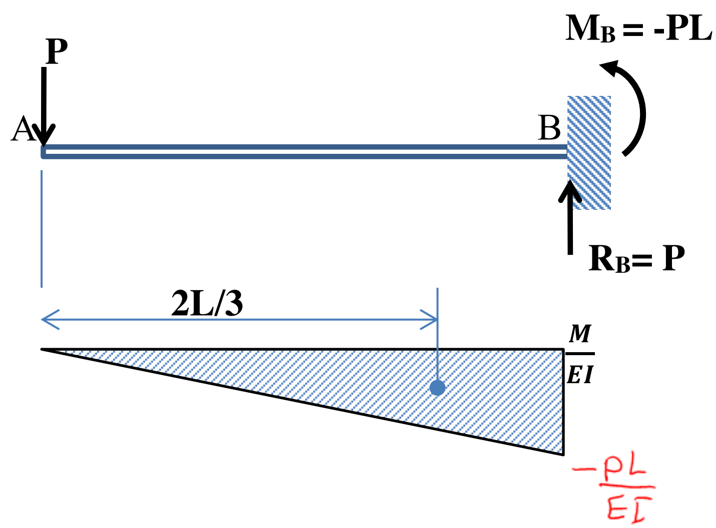


Example 2.1: Determine the rotation and deflection at A? (Assume Constant EI)

1) Sketch the Deflected Shape



2) Draw Bending Moment Diagram



3) Determine rotation at A:

– tangent @ B is horizontal (fixed support)

– change in slope from A to B

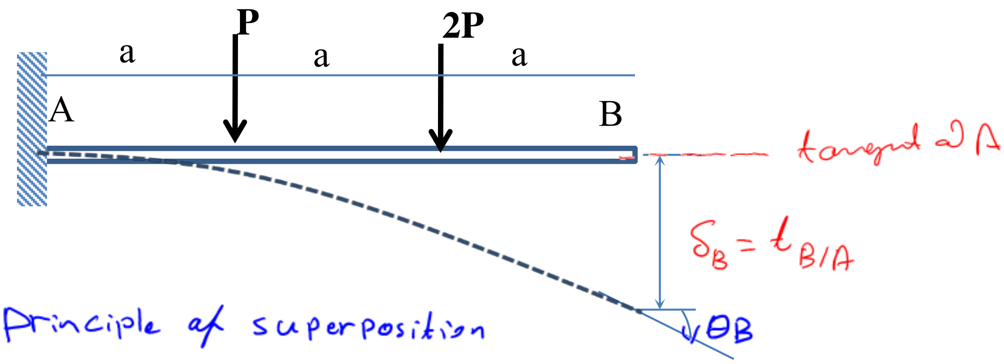
$$\theta_B - \theta_A = 0 - \theta_A = \cancel{-}\theta_A = \cancel{-}\frac{1}{2}\frac{PL}{EI}\cdot L$$

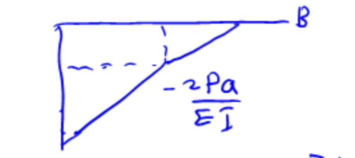
$$\theta_A = \frac{PL^2}{2EI}$$

4) Determine δ_A

$$\delta_A = t_{A/B} = \overbrace{\frac{1}{2}\left(\frac{-PL}{EI}\right)\times L}^{area} * \overbrace{\frac{2}{3}L}^{d_A} = -\frac{PL^3}{3EI}$$

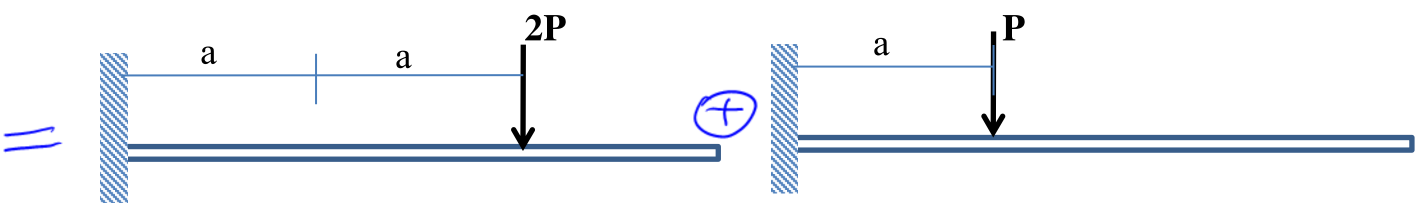
Example 2.2: Find the slope and deflection at B?

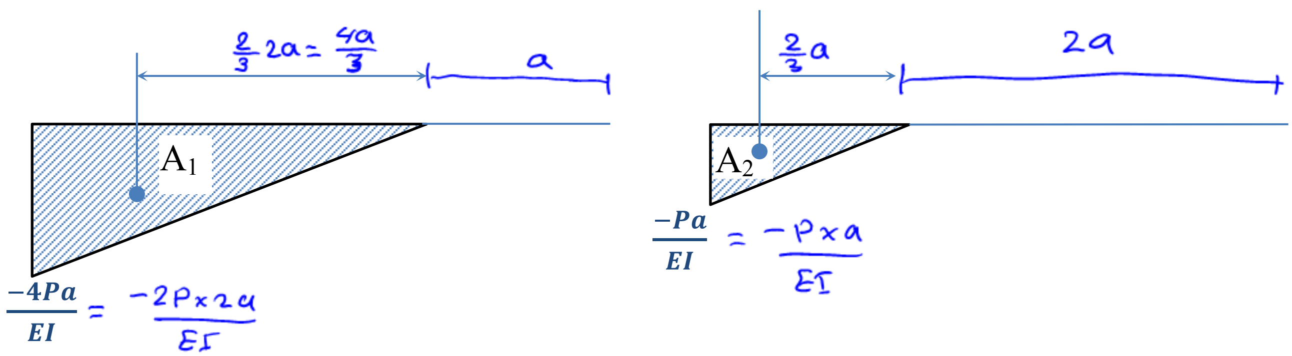




$\left[-2P\times 2a + (-P\cdot a)\right]/EI$
$= -\frac{5Pa}{EI}$

Principle of superposition





$$A_1 = \frac{1}{2}(2a)\left(-\frac{4Pa}{EI}\right) = \frac{-4Pa^2}{EI} \qquad A_2 = \frac{1}{2}a\left(\frac{-Pa}{EI}\right) = -\frac{Pa^2}{2EI}$$

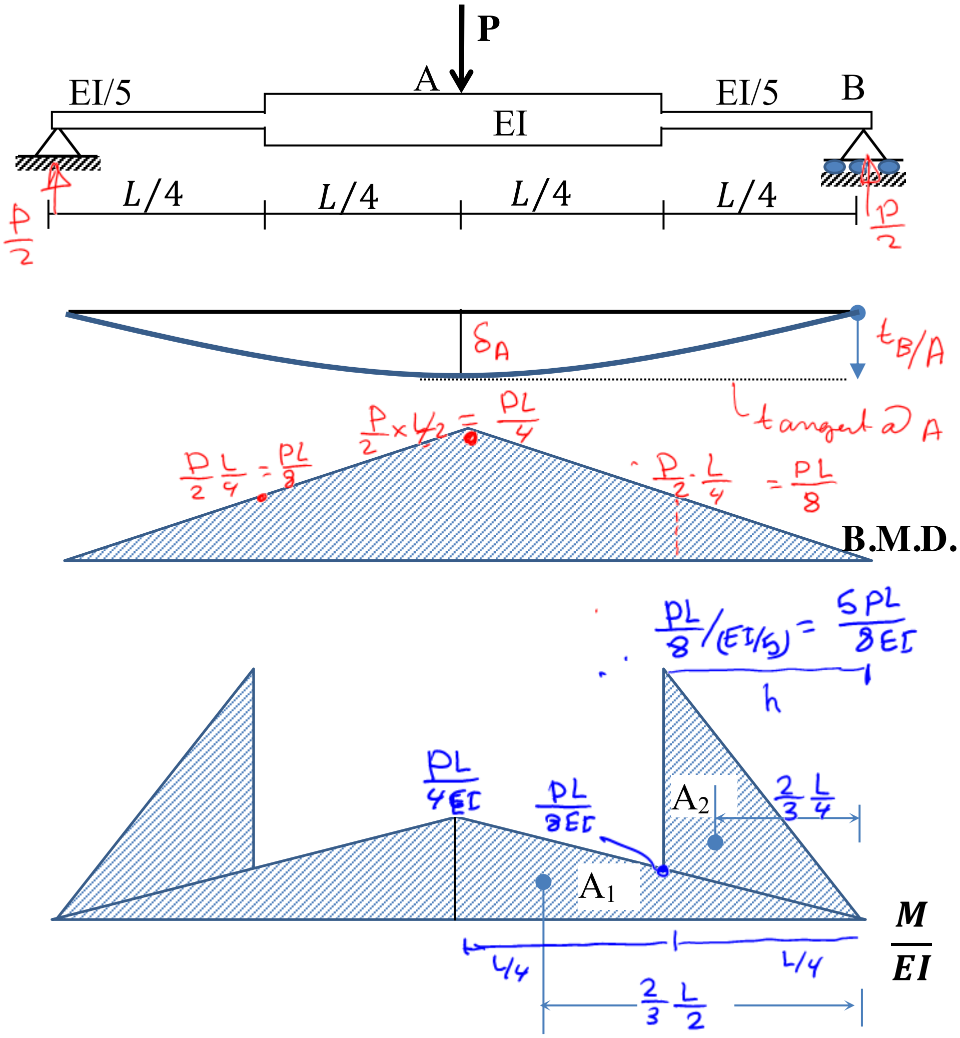
$$\theta_B - \theta_A = \theta_B - 0 = A_1 + A_2 = -\frac{4Pa^2}{EI} + \left(-\frac{Pa^2}{2EI}\right) = \boxed{-\frac{9Pa^2}{2EI}}$$

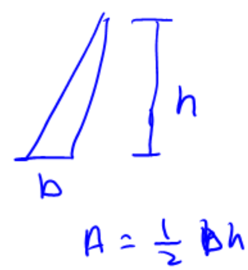
$$\delta_B = t_{B/A} = A_1\left(\frac{4a}{3} + a\right) + A_2\left(\frac{2a}{3} + 2a\right)$$

$$= -\frac{4Pa^2}{EI}\left(\frac{7a}{3}\right) + \left(-\frac{Pa^2}{2EI}\right)\left(\frac{8a}{3}\right)$$

$$\boxed{= -\frac{32Pa^3}{3EI} \downarrow}$$

Example 2.3: Find mid-span deflection





Using the symmetry of loading and structure:

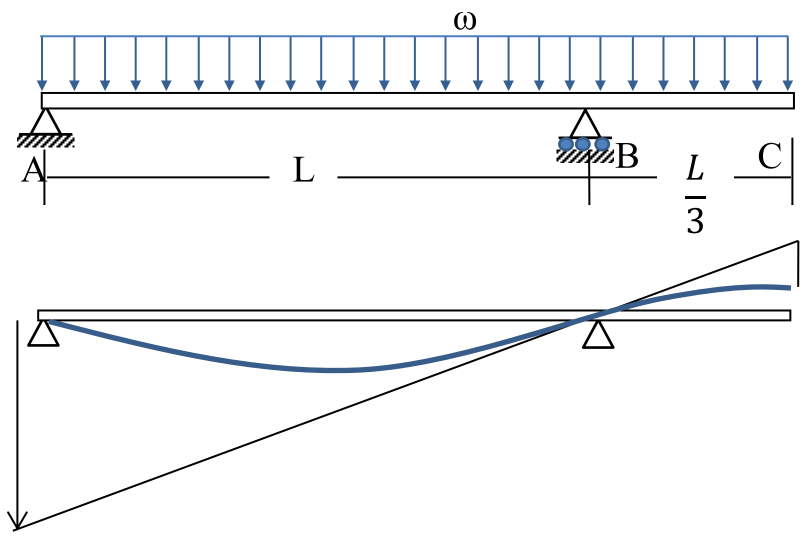
$$A_1 = \frac{1}{2}\frac{L}{2} * \frac{PL}{4EI} = \frac{PL^2}{16EI}$$

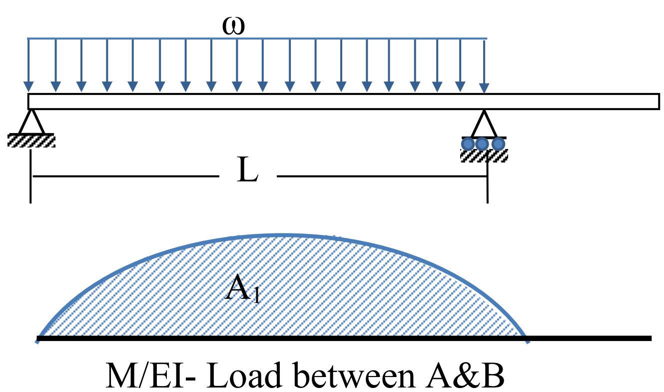
$$A_2 = \frac{1}{2}\frac{L}{4}\left(\frac{5}{8}-\frac{1}{8}\right)\frac{PL}{EI} = \frac{PL^2}{16EI}$$

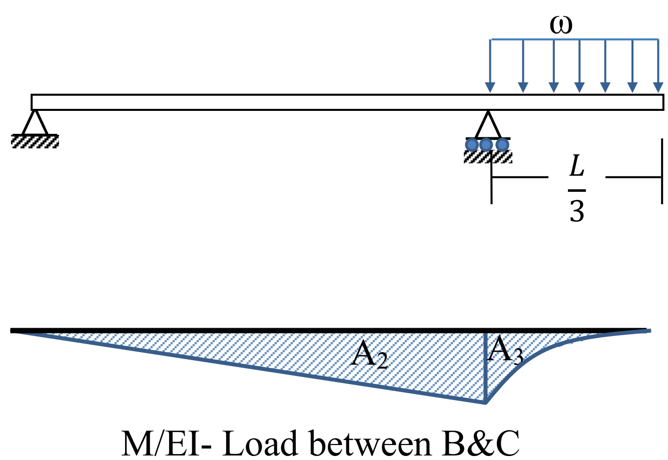
Because of the symmetry, the central deflection equals the tangential deflection of B from the horizontal tangent @ A (only sign to be changed)

$$\delta_A = t_{B/A} = -\left[\frac{2}{3}\frac{L}{4}\left(\frac{PL^2}{16EI}\right) + \left(\frac{2}{3}\frac{L}{2}\right) * \left(\frac{PL^2}{16EI}\right)\right] = \frac{-PL^3}{32EI} \downarrow$$

Example 2.4: Find the vertical deflection at C?

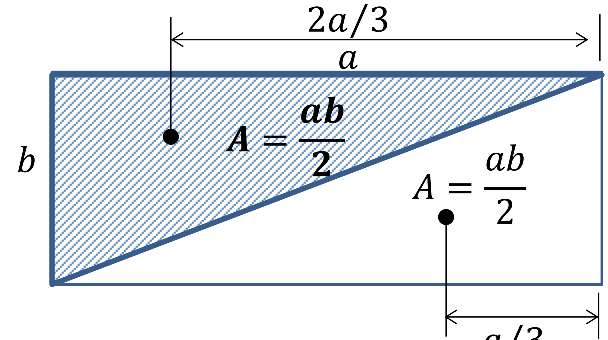




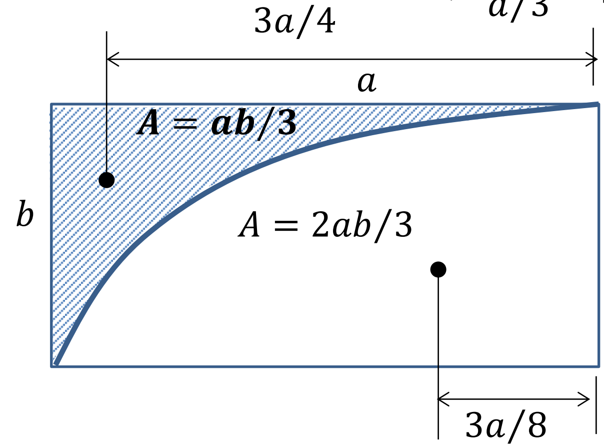


Areas and Centroids

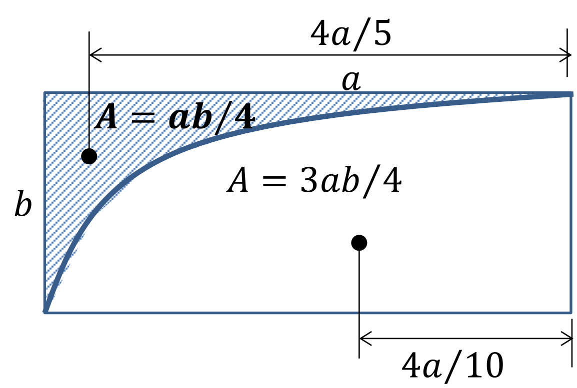
Linear $y = CX$



Parabolic $y = Cx^2$

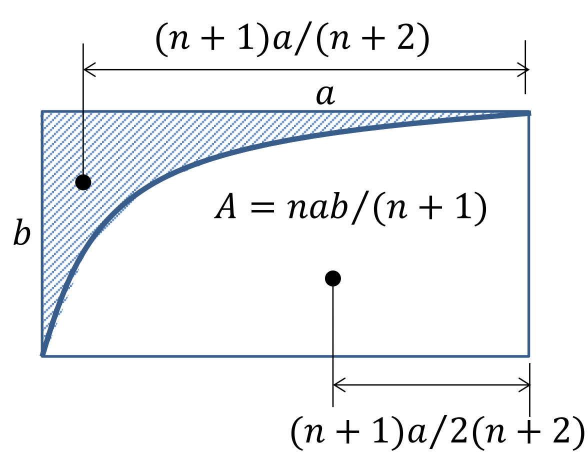


Cubic $y = Cx^3$



General
$y = Cx^n$

$A = ab/(n+1)$



CivE 205–Solids Mechanics II

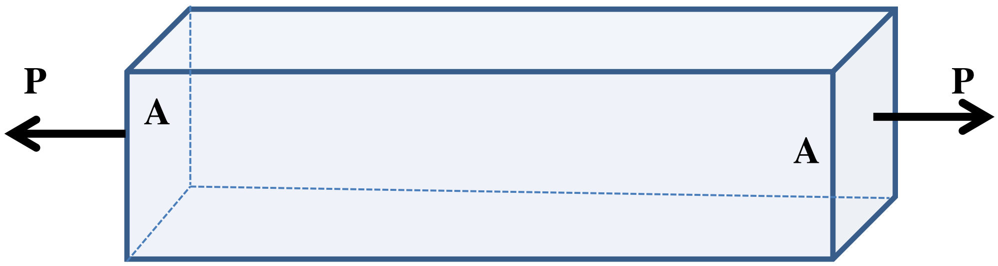
Part 3:

Analysis of Stress

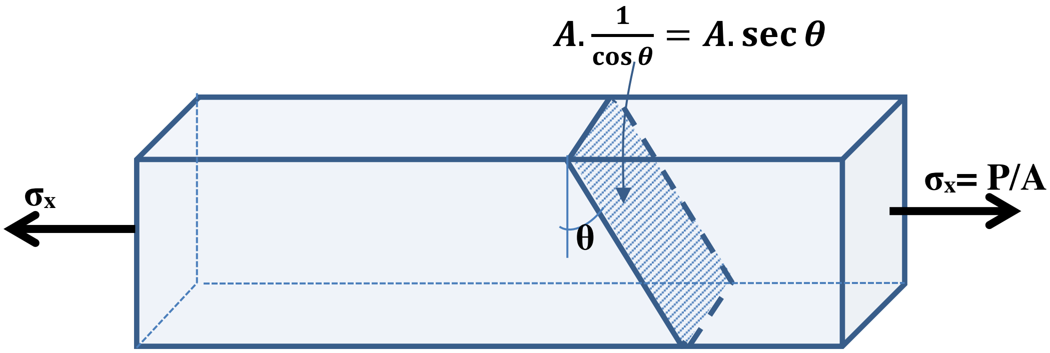
© A. Al-Mayah

AXIAL STRESS SYSTEM

A long uniform bar of cross section A is subjected to a tensile force P



Normal stress in axial direction (σ_x= P/A)



Normal Force =$P.cos\theta$

Shear Force=$P.sin\theta$

Hence $\sigma = \frac{P\cos\theta}{A\sec\theta} = \sigma_x cos^2\theta$ Normal Stress

$\tau = \frac{P\sin\theta}{A\sec\theta} = \sigma_x\ sin\theta\cos\theta = \frac{1}{2}\sigma_x\sin 2\theta$ Shear stress (using triangle identity)

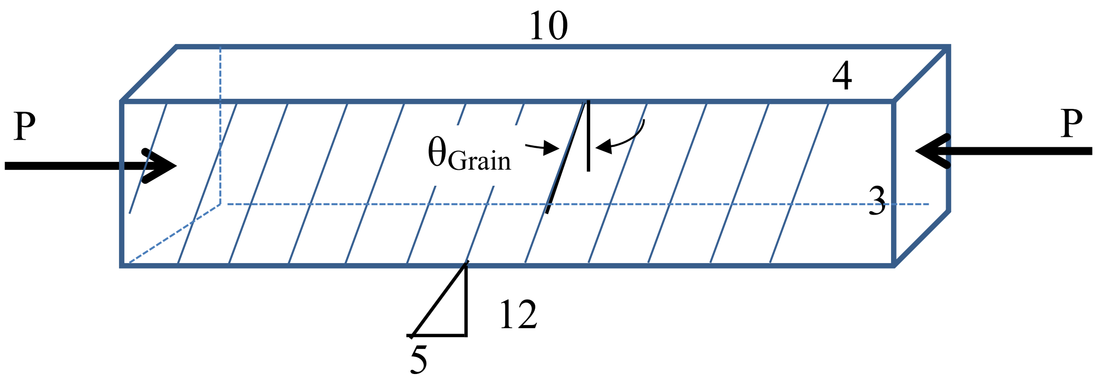
At θ=0 τ=0

At θ=45° τ=$\frac{1}{2}\sigma_x\sin 90$ $\boldsymbol{\tau = \frac{1}{2}\sigma_x = \tau_{max}}$

- **<u>In a tensile test, shearing stress is always present</u>**

- **Shearing stress is greatest on planes inclined at 45° to the longitudinal axis**
- **This has an important implication in practical strength problems**

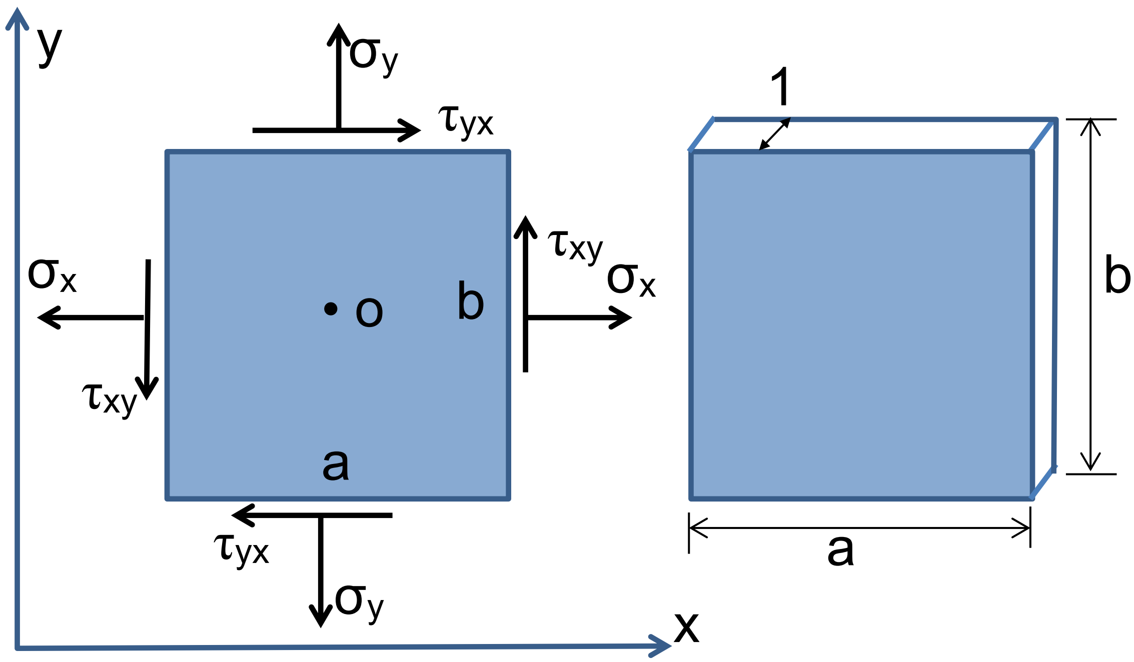
Example 3.1: Block of wood (10x4x3 in) loaded as shown. Maximum allowable shear stress along grain direction =150 psi. (Any other direction 200 psi). Maximum compression stress normal to grain direction is 600 psi. Find maximum allowable P.



TWO DIMENSIONAL STRESS SYSTEM

Plane stress: One in which the stresses at a point in the body act in one plane.

Consider a rectangular element under plane stress in the x-y plane



For rotational equilibrium,

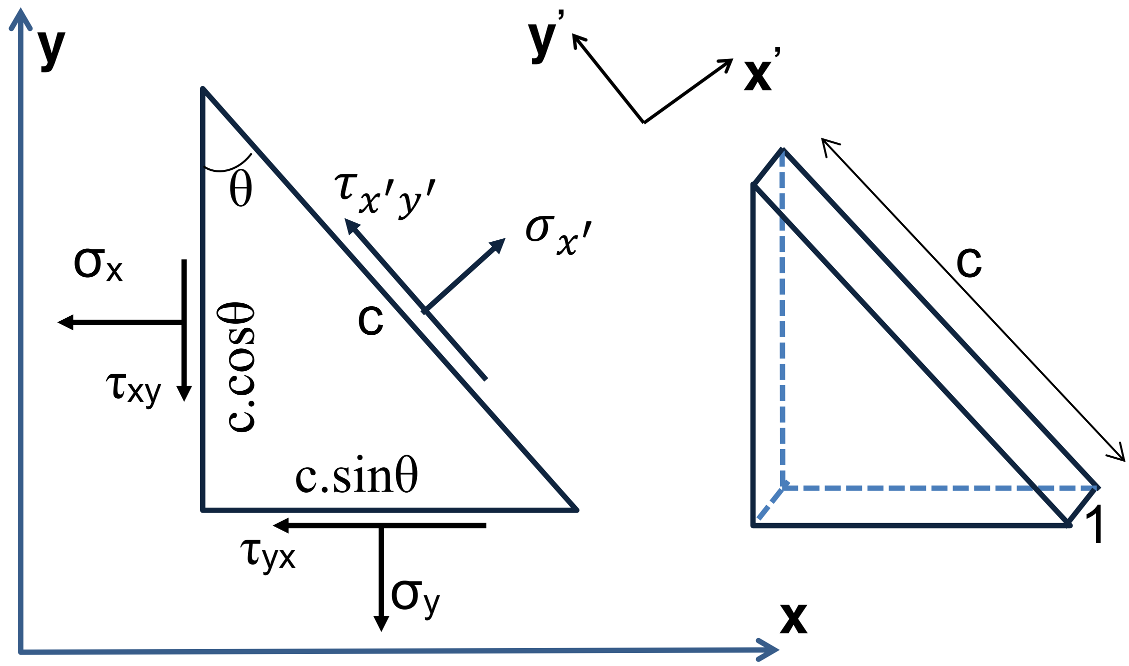
set $\sum M_o = 0$ $\qquad$ $\tau_{xy}(b \times 1).a = \tau_{yx}(a \times 1).b$

$\tau_{xy}(ab) = \tau_{yx}(ba)$ $\qquad$ $\boldsymbol{\tau_{xy} = \tau_{yx}}$

<u>Shearing Stresses on perpendicular planes are equal and complementary</u>

Stresses on an inclined plane (in plane stress):

Equilibrium Analysis: (Thickness= 1 Unit length)



$$\sum F_{x'} = 0$$

$$\sigma_{x'}(c) = \sigma_x(c\cos\theta)\cos\theta + \sigma_y(c\ \sin\theta)\sin\theta + \tau_{xy}(c\cos\theta)\sin\theta + \tau_{xy}(c\sin\theta)\cos\theta$$

Or

$$\sigma_{x'} = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta \qquad \text{-----------(1)}$$

$$\sum F_{y'} = 0$$

$$\tau_{x'y'}(c) = -\sigma_x(c\cos\theta)\sin\theta + \sigma_y(c\sin\theta)\cos\theta + \tau_{xy}(c\cos\theta)\cos\theta - \tau_{xy}(c\sin\theta)\sin\theta$$

Or $\tau_{x'y'} = -\left(\sigma_x - \sigma_y\right)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$ ----------------(2)

Using trigonometric identities: $\cos^2\theta = \frac{1+\cos 2\theta}{2}$, $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$, $\cos^2\theta - \sin^2\theta = \cos 2\theta$

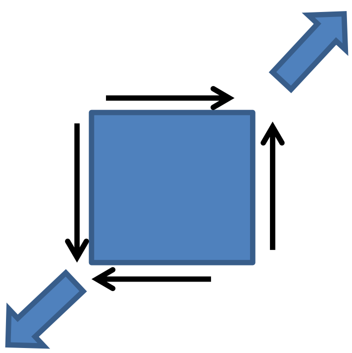
These **stress transformation equations** become:

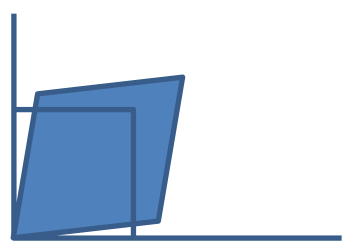
$$\boldsymbol{\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta}$$

$$\boldsymbol{\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta}$$

$$\boldsymbol{\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta} \qquad \text{By substituting } (90+\theta)$$

Note: Positive shear stress and corresponding deformation:





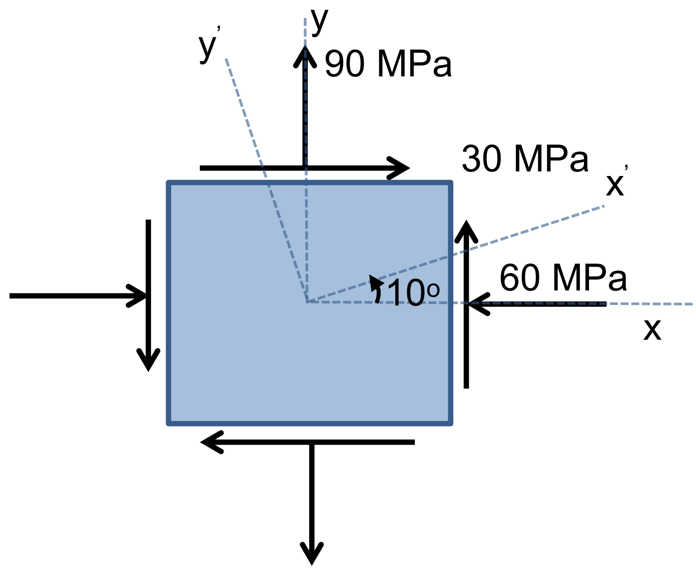
COMPUTER APPLICATIONS

For computer applications, equations (1 and 2) can be rewritten in the following form:

$$\begin{Bmatrix}\sigma_{x'}\\ \sigma_{y'}\\ \tau_{x'y'}\end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}$$

Where c = cos (θ), and s = sin(θ)

Examples 3.2: Given stress state, find stresses on an element at orientation 10° counterclockwise



Principal Stresses and Principal Planes:

Shear Stress is zero when

$$\frac{\sigma_x - \sigma_y}{2} \sin 2\theta = \tau_{xy} \cos 2\theta$$

Or $\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$ →(a)

Or $2\theta = \tan^{-1} \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$ or $\tan^{-1} \frac{2\tau_{xy}}{\sigma_x - \sigma_y} + 180°$

Hence

$$\boldsymbol{\theta = \frac{1}{2} \tan^{-1} \frac{2\tau_{xy}}{\sigma_x - \sigma_y}} \quad \text{or} \quad \boldsymbol{\frac{1}{2} \tan^{-1} \frac{2\tau_{xy}}{\sigma_x - \sigma_y} + 90°}$$

-There are two planes, which are separated by 90° on which shear stress is zero

-These are called **<u>PRINCIPAL PLANES</u>**

The normal stresses are maximum and minimum when:

$$\frac{d\sigma}{d\theta} = -(\sigma_x - \sigma_y) \sin 2\theta + 2\tau_{xy} \cos 2\theta = 0$$

i.e. when $\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$ (Identical to equation a).

- **Therefore, the <u>PRINCIPAL STRESSES</u> are also the <u>maximum and minimum normal stresses</u> in the material**

Principal stress values (Mohr's circle)

-To make use of a geometrical interpretation of the STRESS TRANSFORMATION equations, rewrite the equations:

$$\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\boldsymbol{\tau_{x'y'}} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

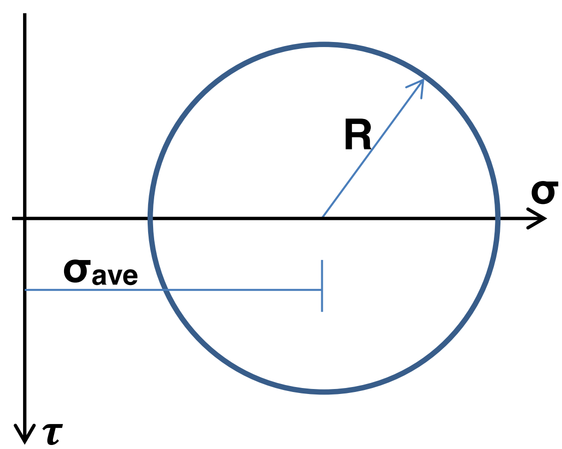
Square both equations, add and simplify result:

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \boldsymbol{\tau_{x'y'}}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2{}_{xy}$$

Set:

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2{}_{xy}}$$

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2}$$



Then

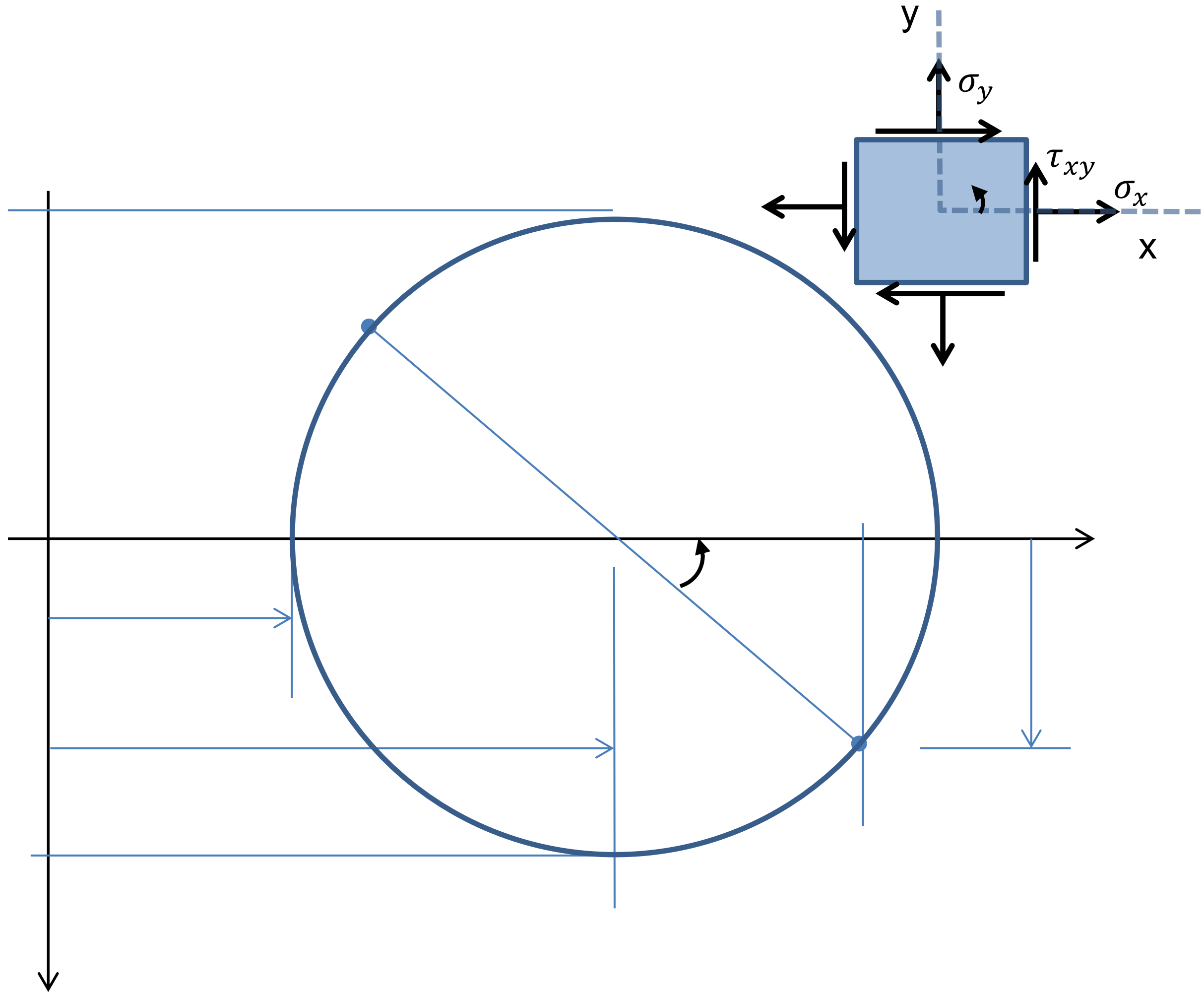
$$(\boldsymbol{\sigma_{x'}} - \boldsymbol{\sigma_{ave}})^2 + \boldsymbol{\tau_{x'y'}}^2 = \boldsymbol{R}^2$$

Equation of a circle

- Prof. Otto Mohr suggested this circle for use in stress analysis problems in 1895 (Germany)
- Mohr's circle can be drawn if stresses on any two perpendicular planes are known.
- Stresses acting on perpendicular planes at any other orientation can be found (i.e. at angle θ)
- The physical value of the angle is half of that measured in Mohr`s circle (example: the angle between σ_1, and σ_2 is 90° however, they are at 180° in the circle)

Mohr's circle drawing procedure:

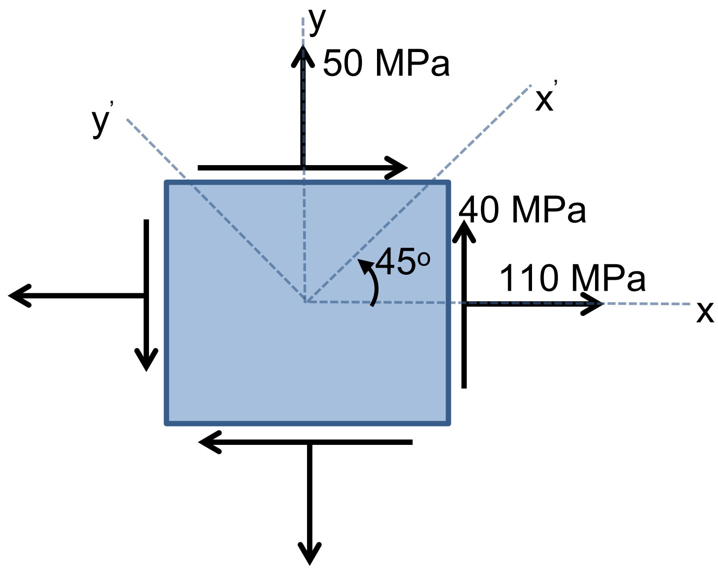
1. Draw a coordinate system: Normal stress on horizontal axis (+ve to the right), Shear stress on vertical axis (+ ve downwards).
2. Plot the center (C)of the circle using $\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2}$
3. Plot point A (σ_x, τ_{xy}) using positive sign convention
4. Draw the circle where line AC represents the radius of the circle.



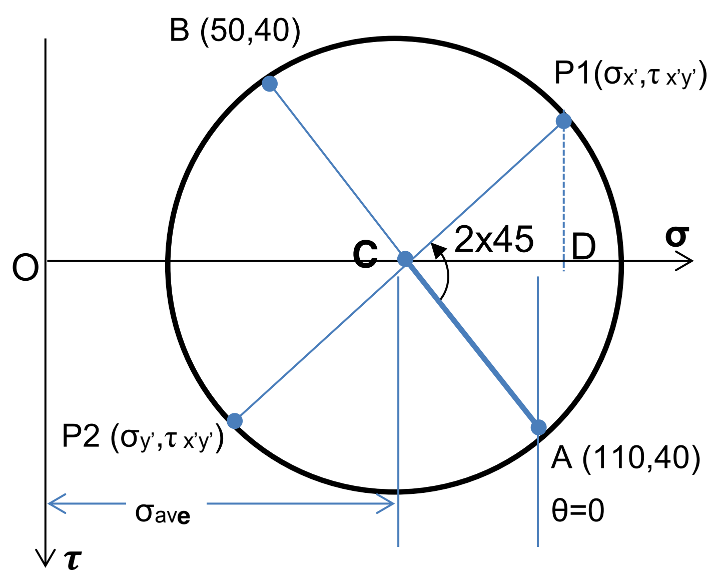
Note:

- **Θ is +ve counterclockwise**
- **σ is plotted in the horizontal directions**

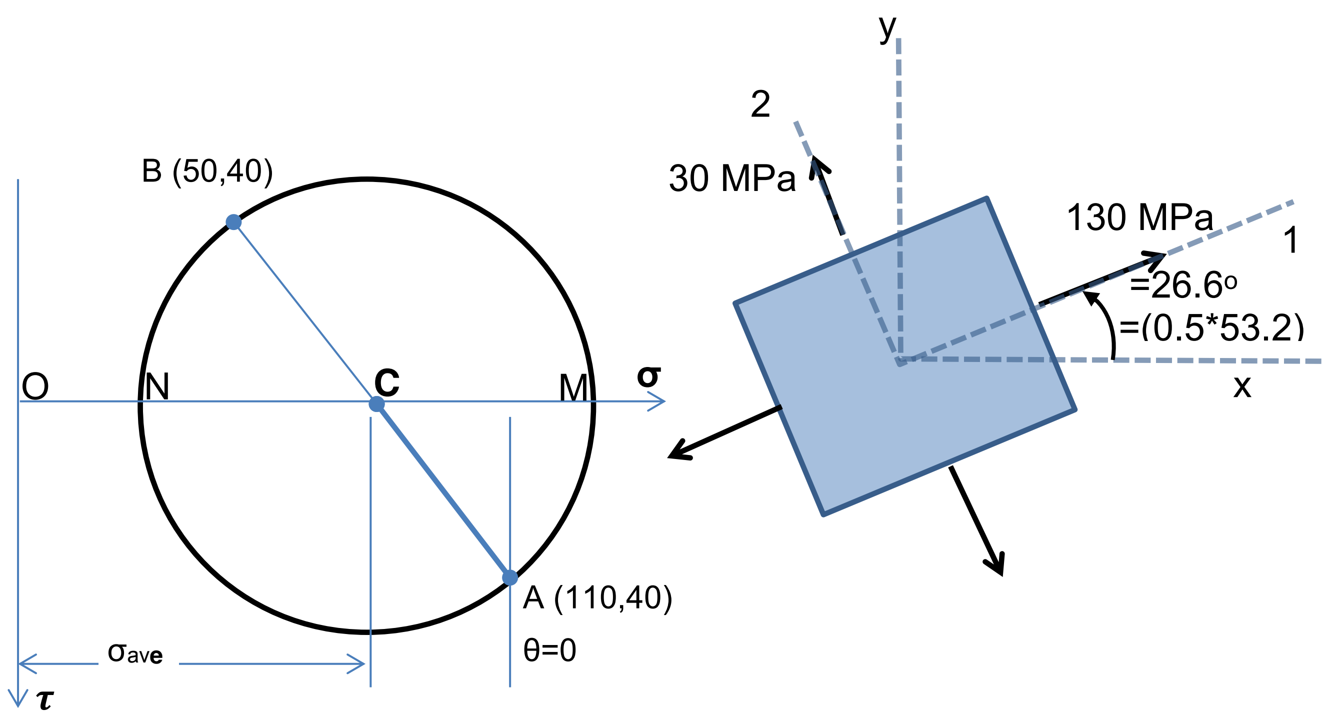
Example 3.3: Given stress state. Determine a) stresses on an element at orientation 45° counterclockwise, b) principal stresses and principal plane, c) maximum shear stress?



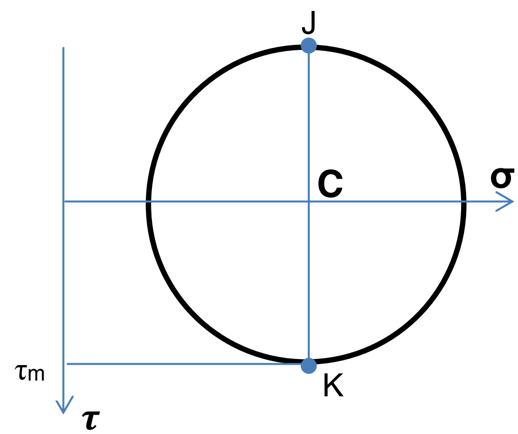
a)

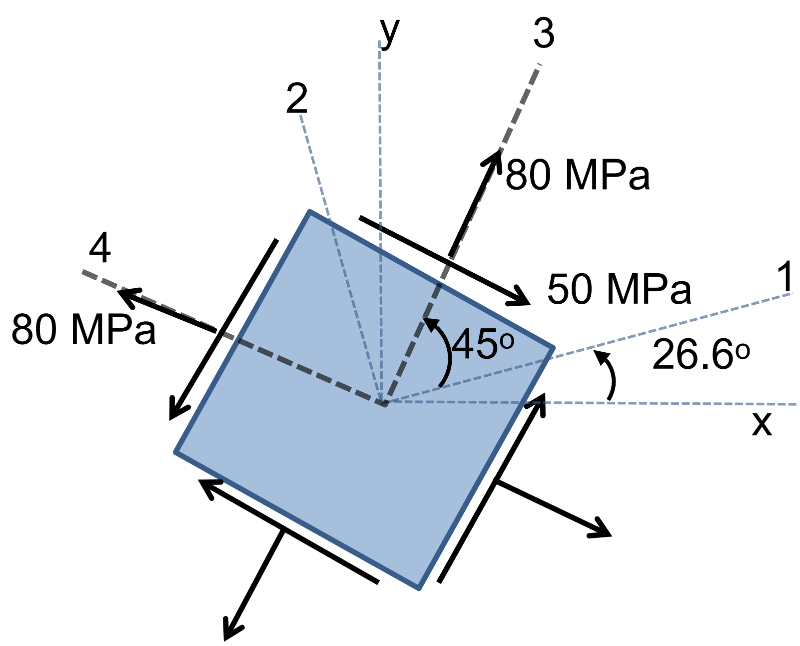


b) Principal Normal Stresses and principal plane:



c) Maximum Shearing Stress:





THREE DIMENSIONAL STRESS SYSTEM

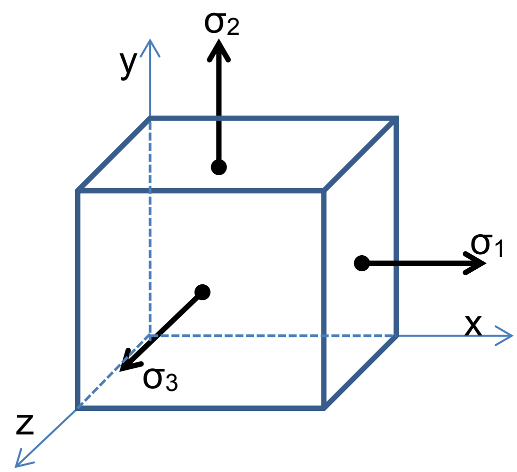
So far, we have considered the case of plane stress (i.e. σ_z , τ_{zx}, and τ_{zy}) and the transformations of stress associated with the rotation about the z axis.

More general solution can be reached by considering the case of three dimensional.

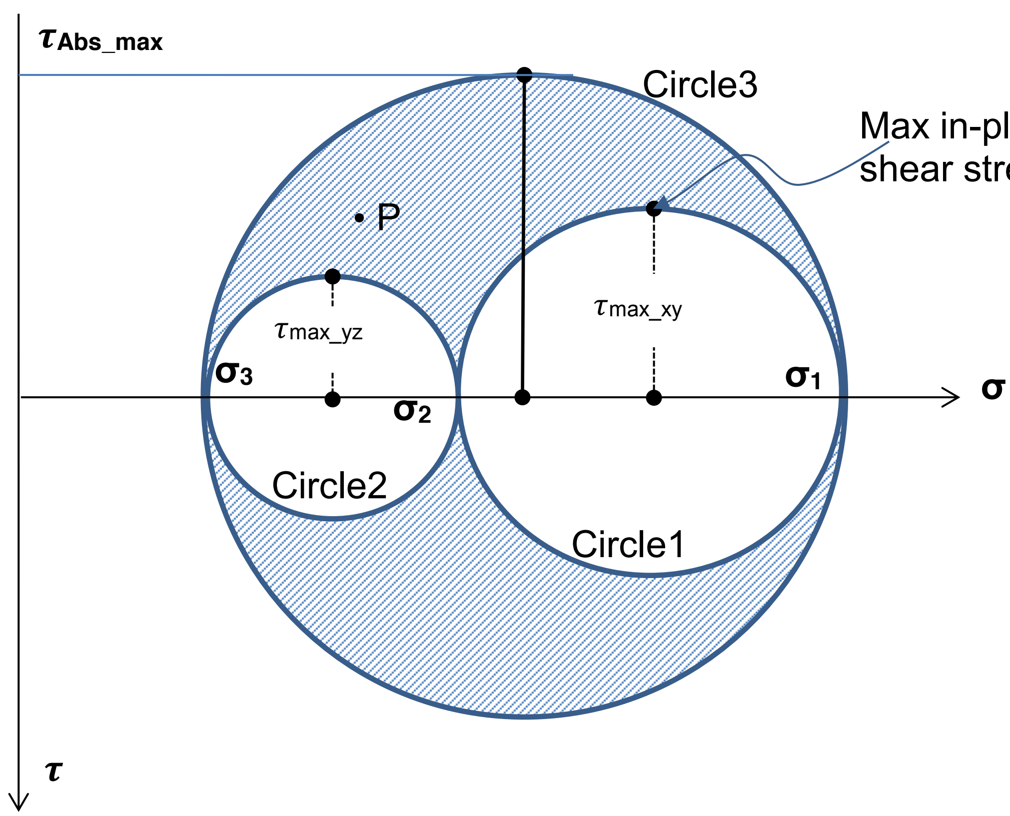
In any three dimensional stress systems, we can always reduce the problem to one which is cast in term of the three, mutually perpendicular stresses:

i.e. σ_1 , σ_2, and σ_3

No shearing stresses act on principal planes (σ_1 , σ_2, and σ_3 are shown as tensile stresses)

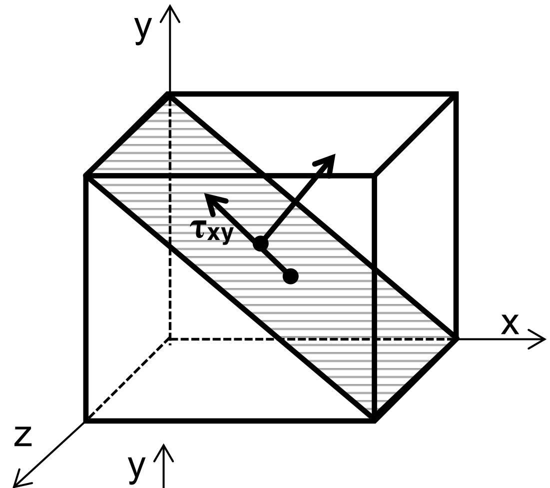


Convenient representation by Mohr's circles assuming $\sigma_1 > \sigma_2 > \sigma_3$



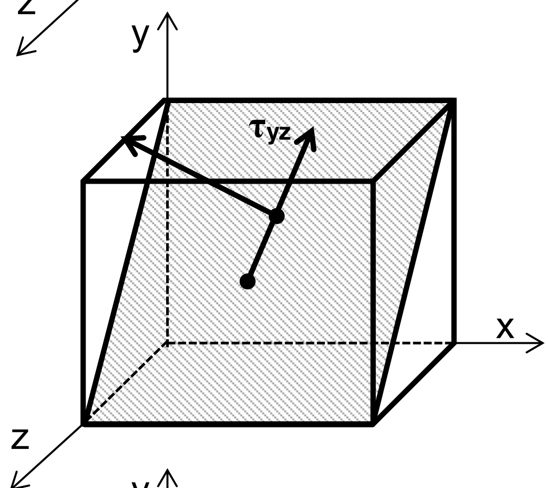
Circle 1 to describe stresses on plane (xy)

$$\tau_{\max\_xy} = \frac{1}{2}(\sigma_1 - \sigma_2)$$



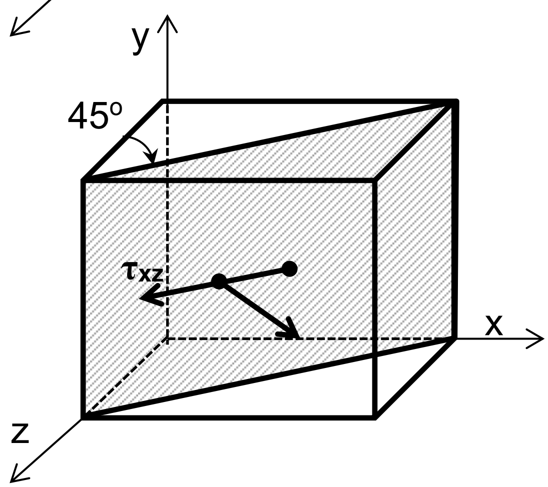
Circle 2 to describe stresses on plane (yz)

$$\tau_{\max\_yz} = \frac{1}{2}(\sigma_2 - \sigma_3)$$



Circle 3 to describe stresses on plane (xz)

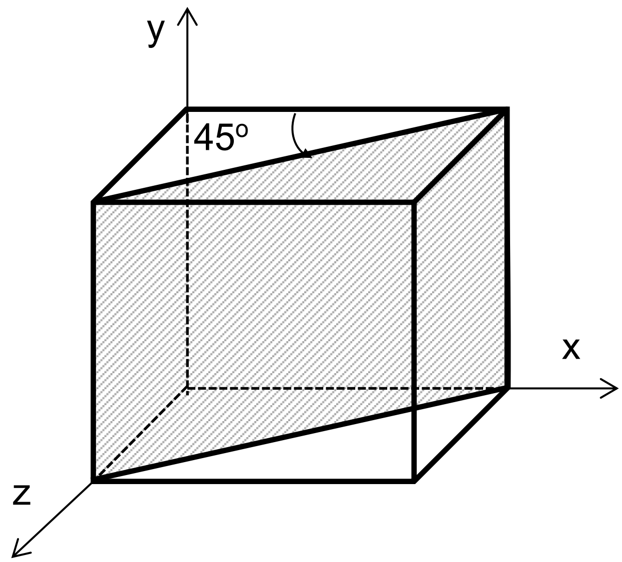
$$\tau_{\max\_xz} = \frac{1}{2}(\sigma_1 - \sigma_3)$$



For any plane inclined to three principal axes, the stresses σ and τ on that plane are defined by a point such as P in the shaded area of Mohr's diagram.

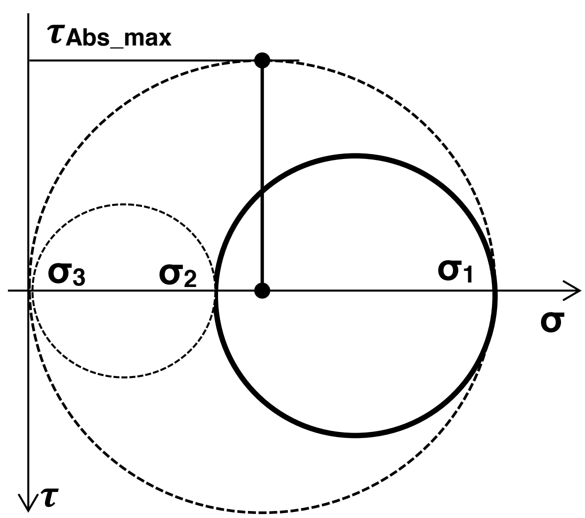
Discussion:

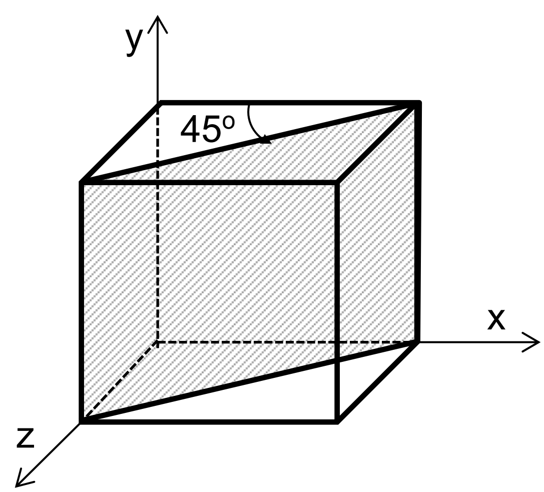
- In case $\sigma_1 > \sigma_2 > \sigma_3$: The absolute maximum shear stress is $\tau_{\text{Abs\_max}} = \frac{1}{2}(\sigma_1 - \sigma_3)$ and occurs at a plane inclined at 45° to x-y plane (Circle 3)



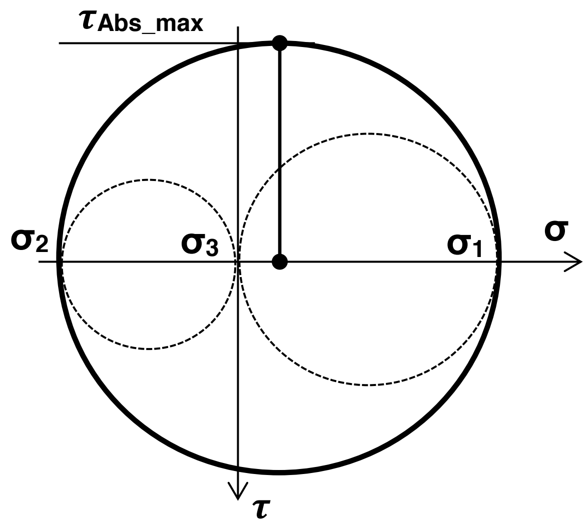
- If σ_3 =0 , there are two possible cases:

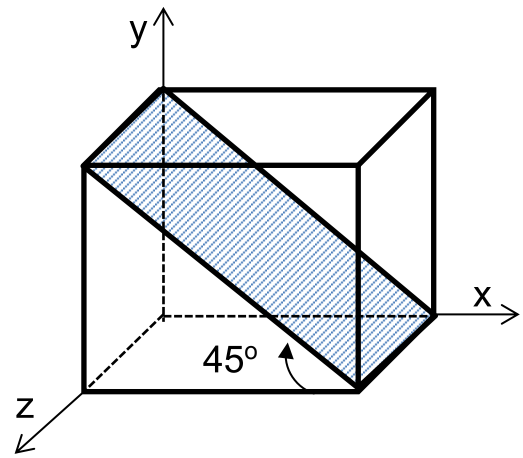
1) $\sigma_1 > \sigma_2$ (both tensile): $\tau_{\text{Abs\_max}} = \frac{\sigma_1}{2}$ (@45° to (x-y) plane





2) σ_1 is tensile, σ_2 is compressive: $\tau_{\text{Abs\_max}} = \frac{1}{2}(\sigma_1 - \sigma_2)$ (@45° to (x-z) plane

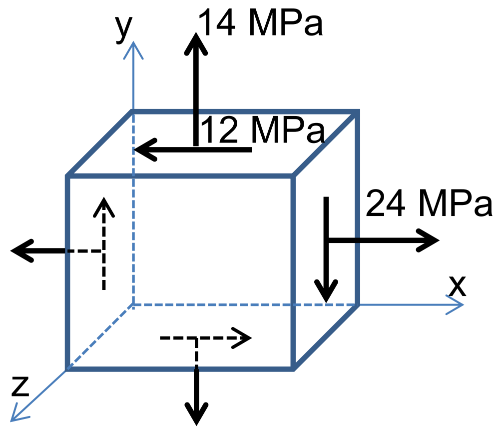


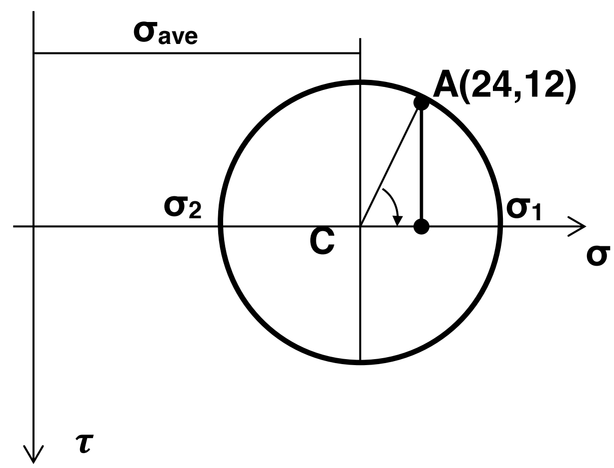


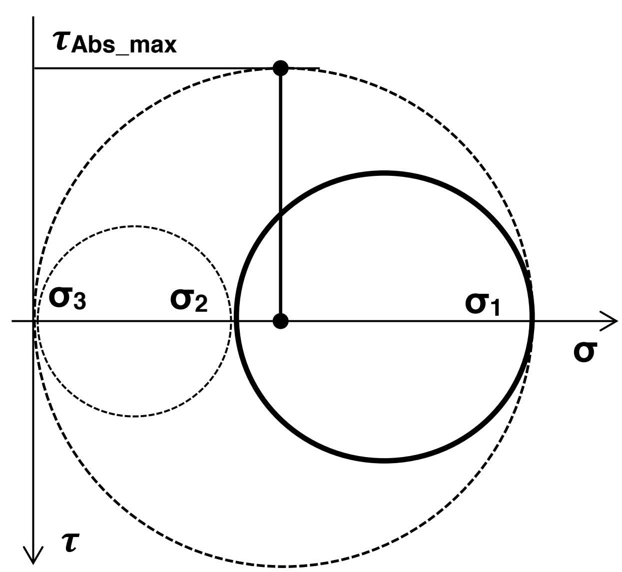
Note:

- When one of the principal stresses is zero (say σ_3) we have a 2D system of stresses; the absolute maximum shear stress is represented by the radius of the largest of the three circles. i.e. the max of $\frac{1}{2}(\sigma_1 - \sigma_2), \frac{\sigma_1}{2}$
- The absence of one stress in a direction perpendicular to a 2D stress system has a significant effect on the maximum shearing stress in the material and cannot be disregarded. This outcome is applied in the establishment of Tresca failure criterion (To be discussed in the upcoming Chapter).

Example 3.4: For the given stress system, determine 1) the principal stresses and principal plane, 2) the absolute maximum shear stress.







Part 4:

Analysis of Strain

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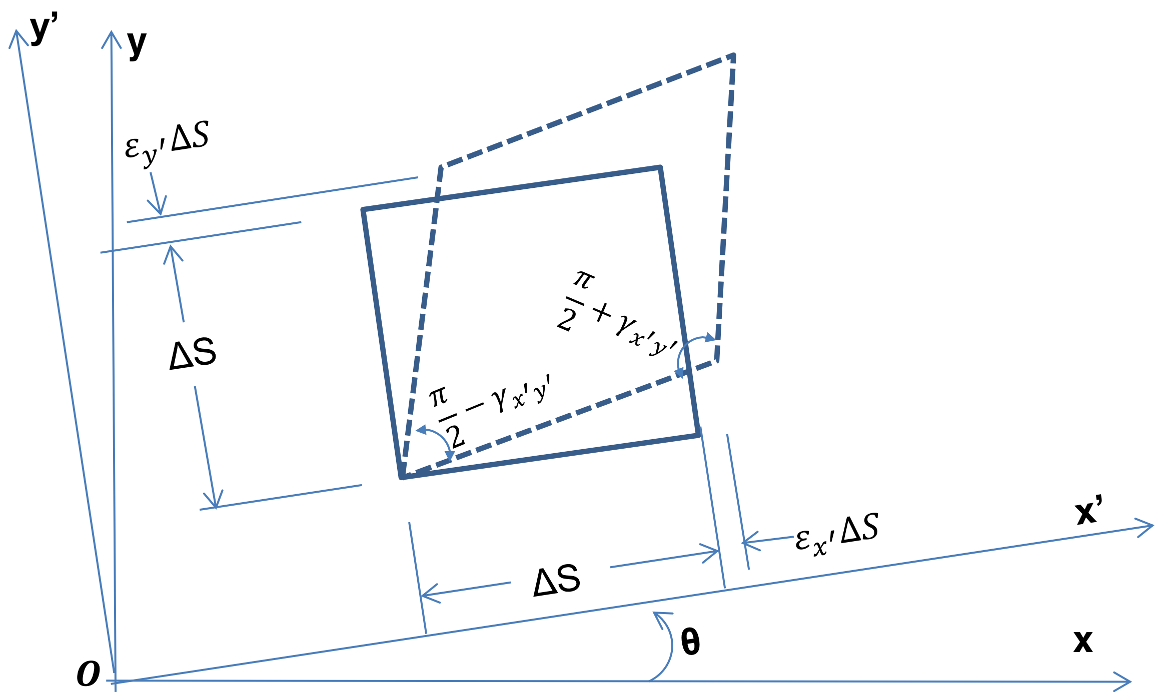
Spring 2020

TRANSFORMATION OF STRAIN

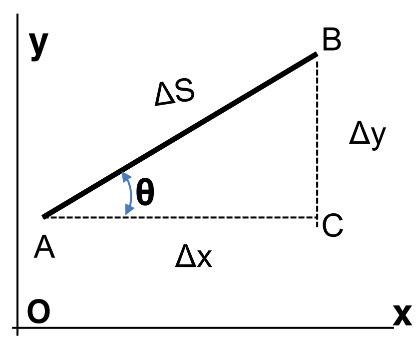
Often it is necessary to know strains associated with other orientations, i.e. (x',y') rotated by an angle θ. The strains for the new reference frame are: $\varepsilon_{x'}, \varepsilon_{y'}, \gamma_{x'y'}$



Take a segment of line ΔS that is subjected to $(\varepsilon_x, \varepsilon_y, \gamma_{xy})$

$$\Delta x = \Delta S \cos\theta$$

$$\Delta y = \Delta S \sin\theta$$



After Deformation:

$$\left[\Delta S\left(1+\varepsilon(\theta)\right)\right]^2 = [\Delta x(1+\varepsilon_x)]^2 + \left[\Delta y\left(1+\varepsilon_y\right)\right]^2 - 2[\Delta x(1+\varepsilon_x)]\left[\Delta y\left(1+\varepsilon_y\right)\right]\cos\left(\frac{\pi}{2}+\gamma_{xy}\right) \quad \text{(Law of cosines)}$$

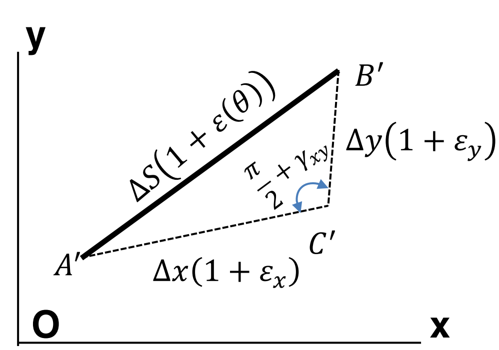
Simplify each term:

$\left(1+\varepsilon(\theta)\right)^2 = 1 + 2\varepsilon(\theta) + \left(\varepsilon(\theta)\right)^2 = 1 + 2\varepsilon(\theta)$ for small deformation

Similarly, $(1+\varepsilon_x)^2 = 1 + 2\varepsilon_x$ and $\left(1+\varepsilon_y\right)^2 = 1 + 2\varepsilon_y$

$$\left(\frac{\Delta x}{\Delta S}\right)^2 = \cos^2\theta \qquad , \qquad \left(\frac{\Delta y}{\Delta S}\right)^2 = \sin^2\theta$$

$\cos\left(\frac{\pi}{2}+\gamma_{xy}\right) = -\sin\gamma_{xy} = -\gamma_{xy}$ for small deformation



$$(1+\varepsilon_x)(1+\varepsilon_y)\cos\left(\frac{\pi}{2}+\gamma_{xy}\right) = -\gamma_{xy}(1+\varepsilon_x)(1+\varepsilon_y) = -\gamma_{xy}(1+\varepsilon_y+\varepsilon_x+\varepsilon_x\varepsilon_y)$$
$$\cong -\gamma_{xy}$$

$$1+2\varepsilon(\theta) = (1+2\varepsilon_x)\cos^2\theta + (1+2\varepsilon_y)\sin^2\theta + 2\gamma_{xy}\sin\theta\cos\theta$$

$$\varepsilon(\theta) = \varepsilon_x\cos^2\theta + \varepsilon_y\sin^2\theta + \gamma_{xy}\sin\theta\cos\theta$$

Take Line x' axis parallel to OB,

$$\varepsilon_{x'} = \varepsilon_x\cos^2\theta + \varepsilon_y\sin^2\theta + \gamma_{xy}\sin\theta\cos\theta$$

Similarly, $\varepsilon_{y'} = \varepsilon_x\sin^2\theta + \varepsilon_y\cos^2\theta - \gamma_{xy}\sin\theta\cos\theta$

$$\gamma_{x'y'} = -2(\varepsilon_x - \varepsilon_y)\sin\theta\cos\theta + \gamma_{xy}(\cos^2 - \sin^2\theta).$$

Alternatively, since

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}\ ,\quad \sin^2\theta = \frac{1-\cos 2\theta}{2}\ ,\quad \cos\theta\ \sin\theta = \frac{1}{2}\sin 2\theta$$

$$\boldsymbol{\varepsilon_{x'} = \frac{\varepsilon_x+\varepsilon_y}{2} + \frac{\varepsilon_x-\varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta}$$

$$\boldsymbol{\varepsilon_{y'} = \frac{\varepsilon_x+\varepsilon_y}{2} - \frac{\varepsilon_x-\varepsilon_y}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta}$$ by replacing θ with $\theta + 90$ in $\boldsymbol{\varepsilon_{x'}}$ equ

It can be shown that:

$$\boldsymbol{\frac{\gamma_{x'y'}}{2} = -\frac{\varepsilon_x-\varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta}$$

These are the **STRAIN TRANSFORMATION EQUATIONS**.

- Note the similarity to the stress transformation equations.
- If the strain components in one set of axes are specified then the strain components in any other set can be determined using these equations.
- These equations lend themselves to a convenient geometrical interpretation by means of **Mohr's Circle for Strains.**

COMPUTER APPLICATIONS

For computer applications, strain equations can be rewritten in the following form:

$$\begin{Bmatrix}\varepsilon_{x'}\\ \varepsilon_{x'}\\ \gamma_{x'y'}\end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & sc \\ s^2 & c^2 & -sc \\ -2sc & 2sc & c^2-s^2\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}$$

Where c = cos (θ), and s = sin(θ)

Mohr's Circle for Strains (plane Strain)

Proceed as in the plane stress case

Squaring and adding and simplifying equations we get:

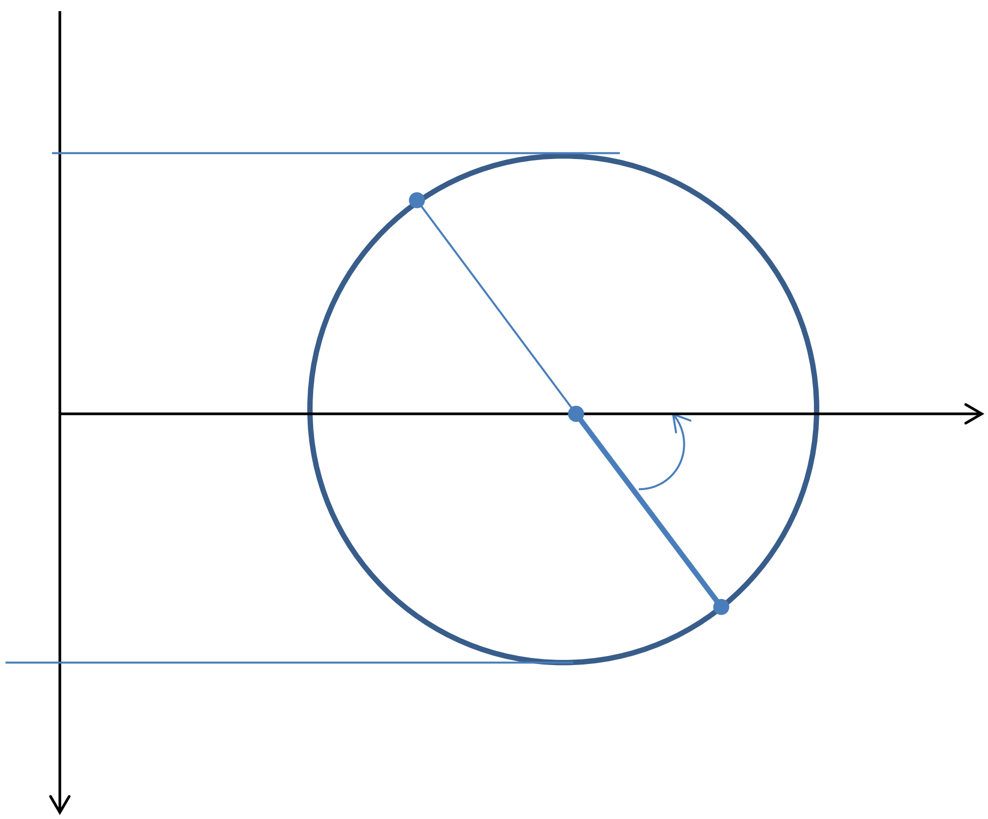
$$\left(\varepsilon_{x'} - \frac{\varepsilon_x+\varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{x'y'}}{2}\right)^2 = \left(\frac{\varepsilon_x-\varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2$$

$$R = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\varepsilon_{ave} = \frac{\varepsilon_x + \varepsilon_y}{2}$$

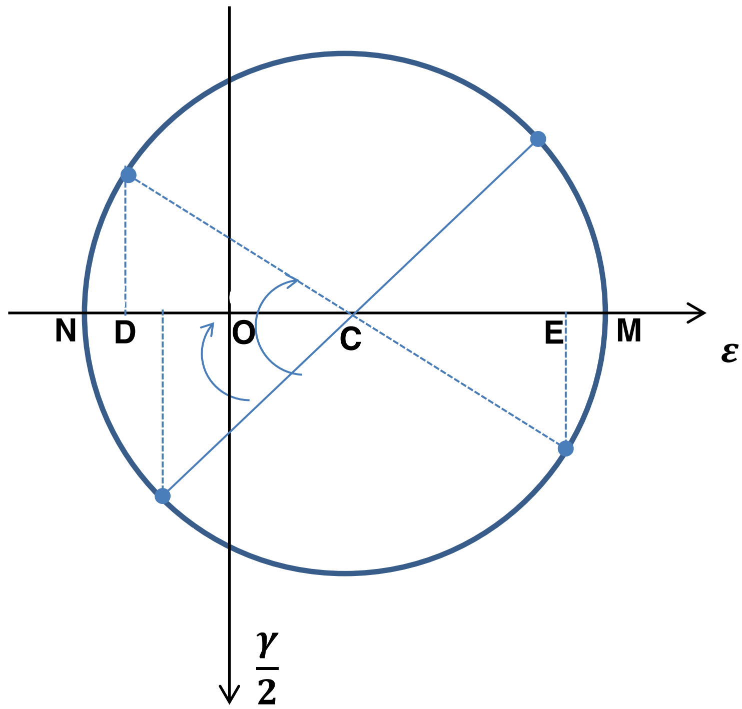
$R^2 = \left(\varepsilon_{x'} - \frac{\varepsilon_x+\varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{x'y'}}{2}\right)^2$ Equation of a circle of radius R

$$R^2 = (\varepsilon_{x'} - \varepsilon_{ave})^2 + \left(\frac{\gamma_{x'y'}}{2}\right)^2$$



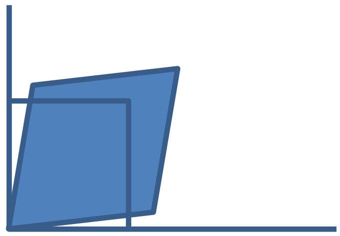
ε_1 and ε_2 are the normal strains and γ_{max} is maximum shear strain

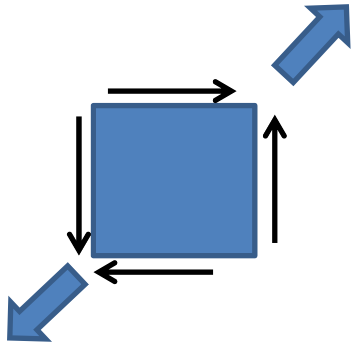
Example 4.1: Given ε_x= -200μ, ε_y= 1000μ, γ_{xy} = 900μ. 1) Find strains associated with axes x\`y\` inclined at 30° clockwise with respect to the x-y axes. 2) Find the principal strains and their directions (Principal axes). 3) Find max shear strain and the orientation of an element under max shear strain.



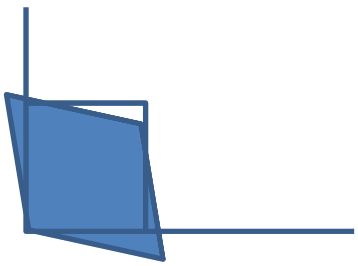
Notes

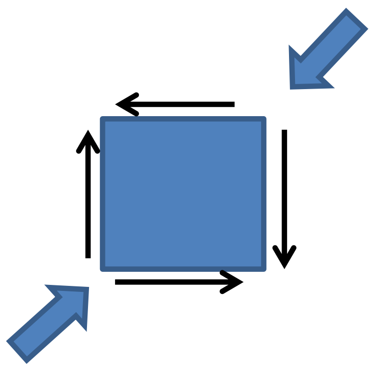
Positive shear strain and shear stress





Negative shear strain and stress



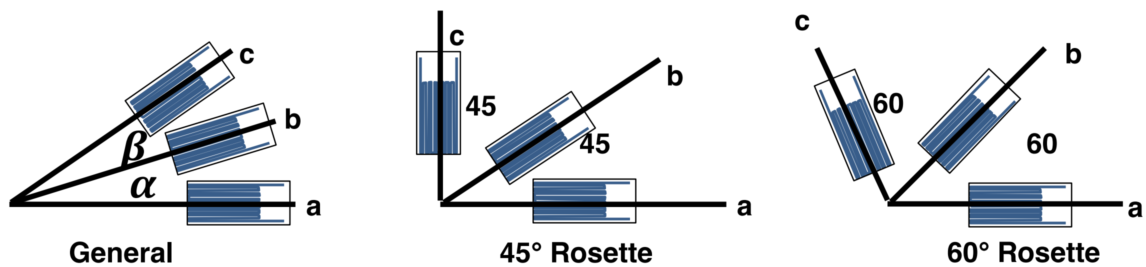


STRAIN GAGES

- Cemented to surface of material
- Electrical resistance changes when wires (number of folds) are stretched or compressed with the material under consideration
- Resistance changes are measured and interpreted in terms of deformation changes

Strain Gages Rosettes

- Three strain gages mounted in a way to measure normal strains at a point in 3 different directions



- Measured strains are analyzed using Mohr's circle and the strains transformation equation as follows (From Previous work):

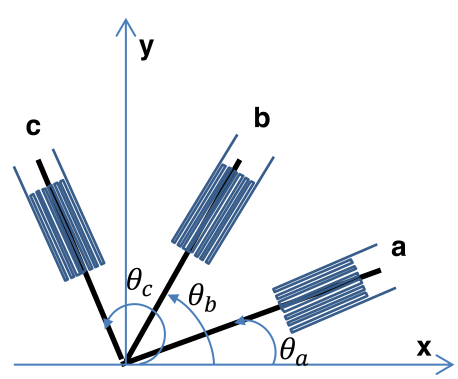
$$\varepsilon_a = \varepsilon_x \cos^2\theta_a + \varepsilon_y \sin^2\theta_a + \gamma_{xy}\sin\theta_a\cos\theta_a$$

$$\varepsilon_b = \varepsilon_x \cos^2\theta_b + \varepsilon_y \sin^2\theta_b + \gamma_{xy}\sin\theta_b\cos\theta_b$$

$$\varepsilon_c = \varepsilon_x \cos^2\theta_c + \varepsilon_y \sin^2\theta_c + \gamma_{xy}\sin\theta_c\cos\theta_c$$

ε_a, ε_b, and ε_c are known

θ_a, θ_b, and θ_c are known



Therefore, ε_x, ε_y, and γ_{xy} can be calculated which can be used in conjunction with Mohr's circle to find **Principal Strains.**

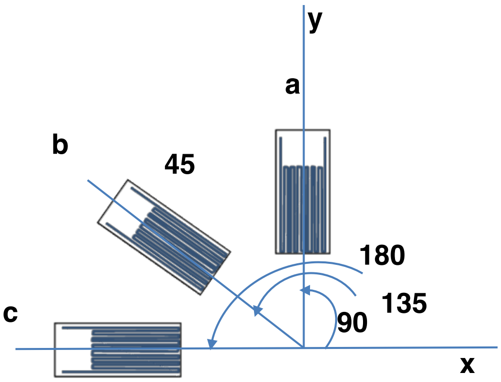
Examples 4.2: Calculate the maximum and minimum strains for 45° rosette with measured strains

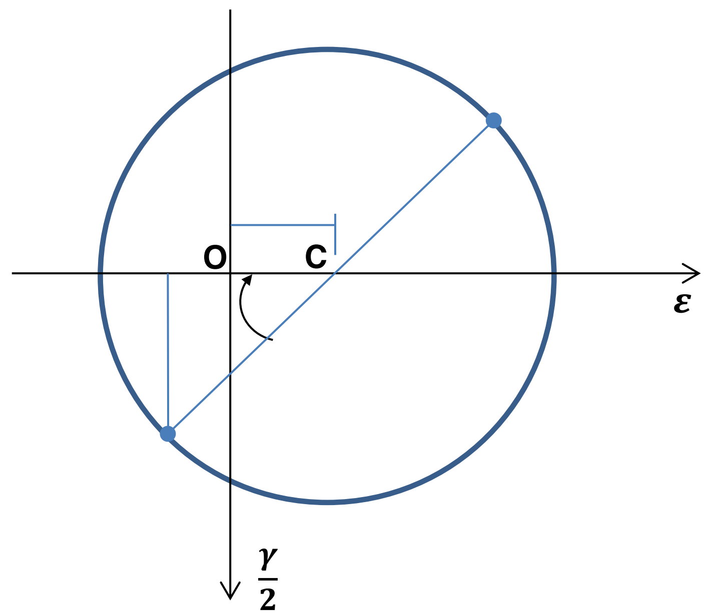
ε_a= 800 μ

ε_b= -600 μ

ε_c= -400 μ

Solution:





STRESS-STRAIN RELATIONSHIP

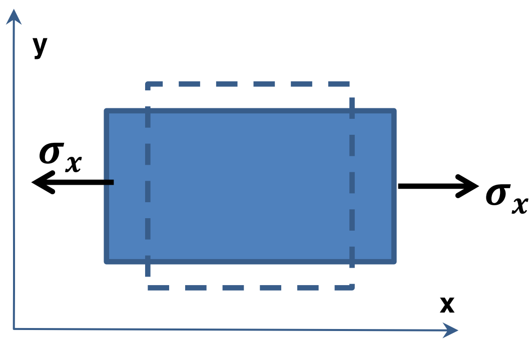
(Isotropic Materials) (2D stress state)

In a body subjected to mutually perpendicular normal stresses σ_x and σ_y, the strain in either direction depends on both stresses through the material constants **E (Young's Modulus, Modulus of Elasticity)** and **v (nu) (Poisson's ratio). The relationship between stress and strain is called Hooke's law (Robert Hooke 1676).**

E (Young's Modulus), named after a British Scientist Thomas Young 1807

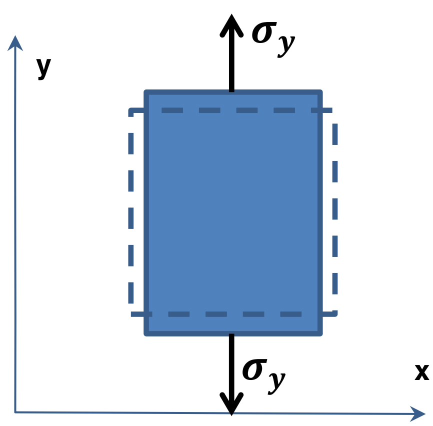
v (nu) (Poisson's ratio): named after French scientist Siméon D. Poisson (1781-1840).

1) Stress in x direction



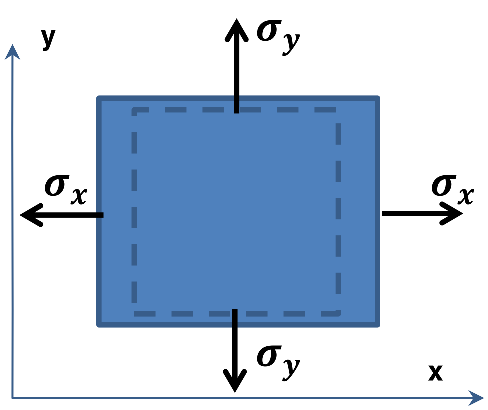
$$\epsilon_x = \frac{\sigma_x}{E}$$
$$\epsilon_y = -\frac{v\sigma_x}{E}$$

2) Stress in y direction



$$\epsilon_x = -\frac{v\sigma_y}{E}$$
$$\epsilon_y = \frac{\sigma_y}{E}$$

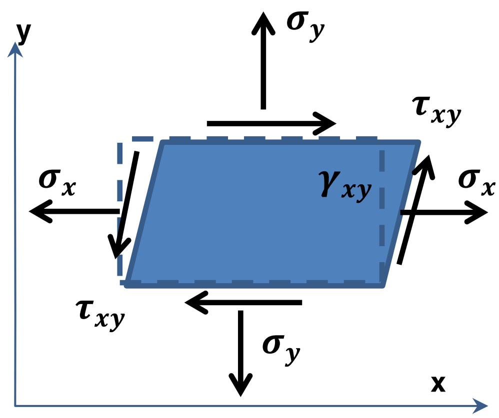
3) Stresses in x and y directions (adding 1+2 above)



Note: When shearing stresses are also present, it is assumed that τ_{xy} has no effect on strains in x and y directions, and σ_x and σ_y have no effect on γ_{xy}.

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

G is the shear modulus



3D General Hooke's Law:

$$E\varepsilon_x = \sigma_x - \nu(\sigma_y + \sigma_z) \qquad G\gamma_{xy} = \tau_{xy}$$

$$E\varepsilon_y = \sigma_y - \nu(\sigma_x + \sigma_z) \qquad G\gamma_{yz} = \tau_{yz}$$

$$E\varepsilon_z = \sigma_z - \nu(\sigma_x + \sigma_y) \qquad G\gamma_{zx} = \tau_{zx}$$

- Six linear equations relating six stress components to six strain components
- For two dimensional principal stresses (σ_1, σ_2), shear stress is zero on principal planes, therefore no shearing strains.
- Normal strains are ($E\varepsilon_1 = \sigma_1 - \nu\sigma_2$, $E\varepsilon_2 = \sigma_2 - \nu\sigma_1$) **Principal Strains**
- Principal strains occur in directions parallel to principal stresses. This is true for three dimensional stress states.

Computer Application:

The 3D general Hooke's law can be written as follows:

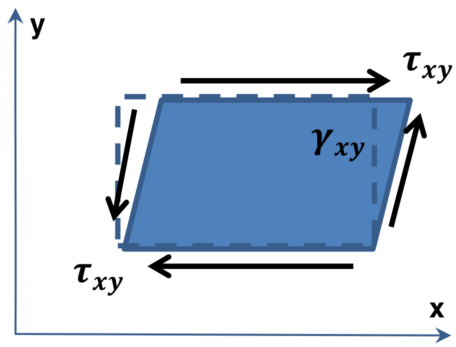
$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

Note: $\frac{E}{G} = 2(1+\nu)$

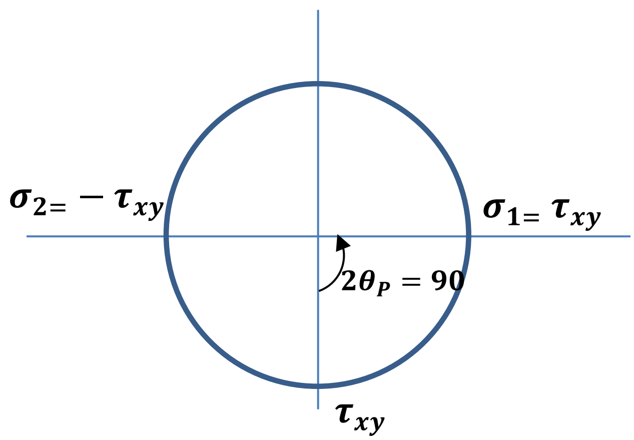
Relationship between E, v and G

Consider 2dimensional state of pure shear stress

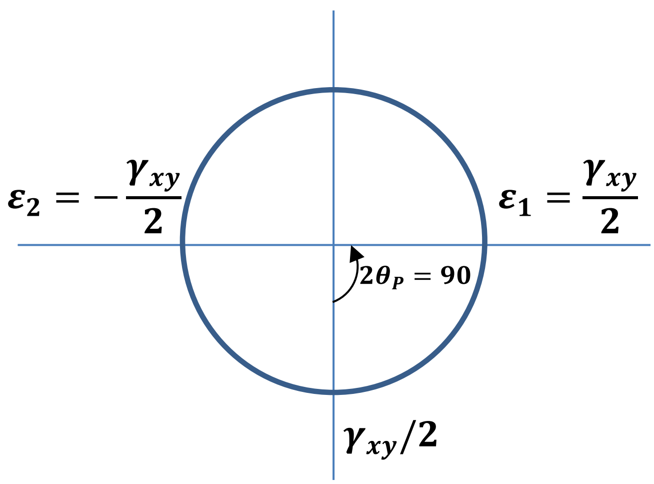
$G\gamma_{xy} = \tau_{xy}$



Mohr's circle for the stress



Mohr's circle for strain

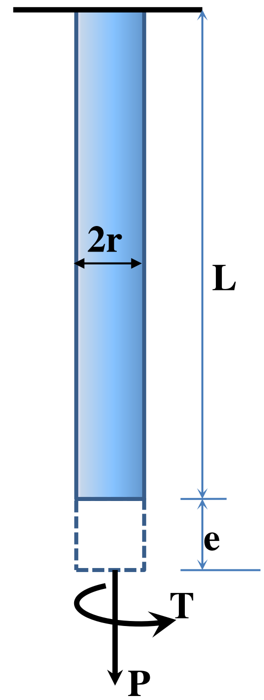


Hence $\frac{E}{2} = G(1+v)$

$$\boxed{G = \frac{E}{2(1+v)}}$$

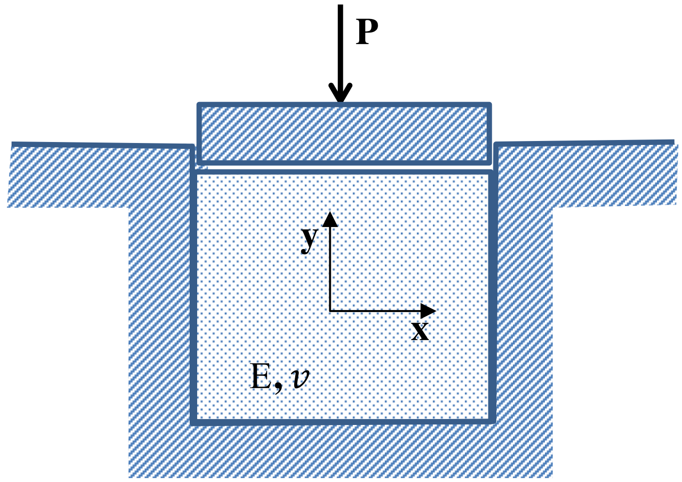
Hence G can be calculated from measured values of E and v. There are only 2 independent material constants in an isotropic material.

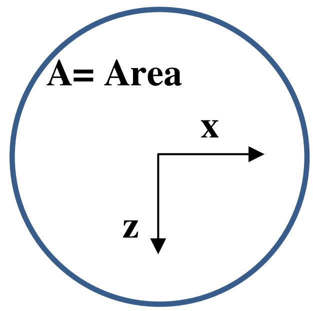
Example 4.3: A circular bar under the action of axial force and torque will elongate by an amount e and twist by an amount ϕ. Determine the Poisson's ratio of the bar?



Example 4.4: Find the pressure on side walls and (ε_y)

E, υ are constants (Isotropic material)





Part 5:

Strain Energy _An Introduction

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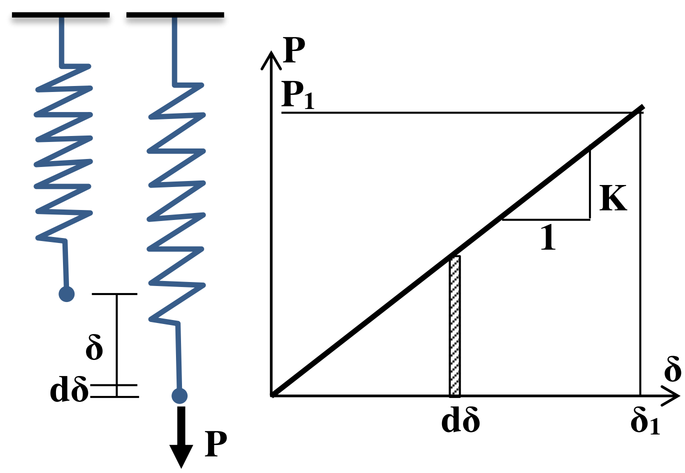
- When an elastic body is deformed through the action of external forces, recoverable energy is stored within the body in the form of strain energy.
- Since the energy is related to the deformation (strain) of the material, it is called strain energy.
- External Work=Internal Energy (Energy conservation)

dW =Work done by P through a small displacement $d\delta$



Total work $W =$

For linear spring, the strain energy

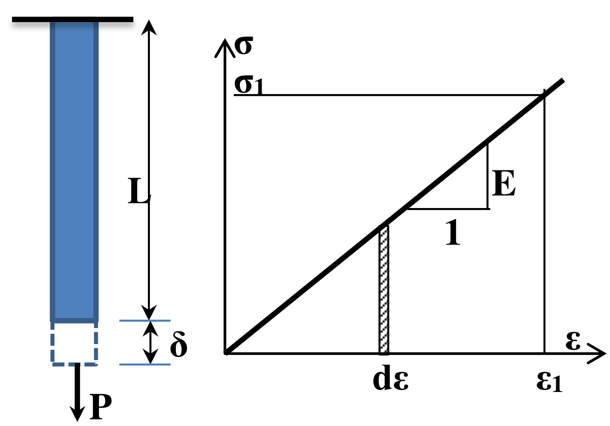
$U = \frac{1}{2} P_1 \delta_1$

Examples

<u>**Uniaxial state of stress**</u> <u>(Strain)</u>

(Note: Ignore the effect of gravity)

$\varepsilon = \frac{\delta}{L}$ $\sigma = \frac{P}{A}$



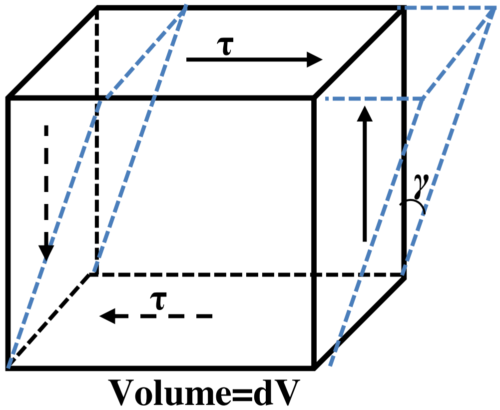
Total Wrok W=Total Strain Energy U

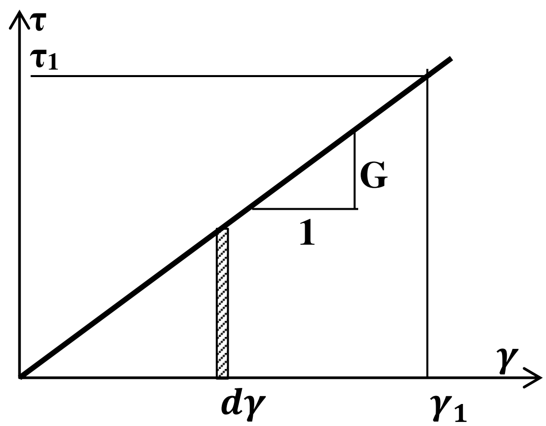
$W =$

AL=Total volume of Material

$$\boldsymbol{u = \frac{1}{2}\sigma_1 \varepsilon_1}$$

For a state of **Pure Shear Stress** (one plane)

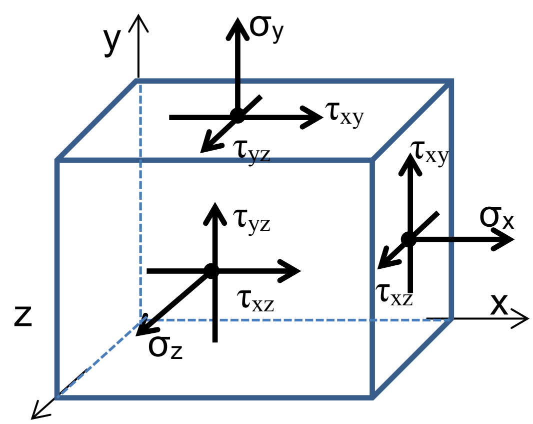




$$u = \frac{1}{2}\tau_1\gamma_1$$

3D state of stress: For an element of volume (dV) under loading, there are 6 stress components σ_x, σ_y , σ_z, τ_{xy} , τ_{yz}, and τ_{xz} and strain components ε_x, ε_y , ε_z, γ_{xy}, γ_{yz}, and γ_{xz}.

Strain Energy (Assuming material obyes Generalized Hooke's law):

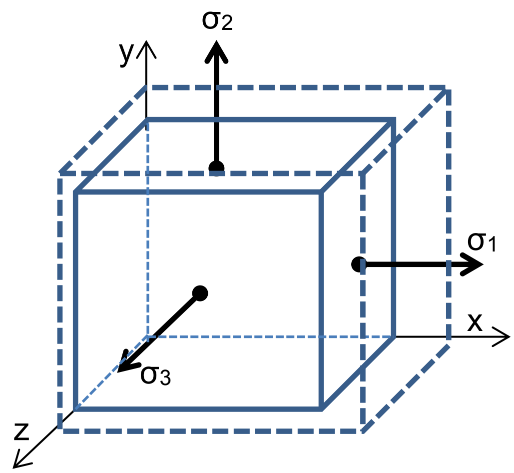


And the strain energy density

$$u = \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{xz}\gamma_{xz}\right)$$

i.e. the general case represents a linear addition (superposition) of the 6 separate effects.

Principal Stresses-Principal Strains



- Strain energy desnisty is a scalar quantity
- Independent of the orientation of xyz axes
- Easier to deal with strain energy using principal directions

For an element aligned with principal directions of stress and strains (1,2, and 3)

Stresses: σ_1, σ_2 , σ_3 (No shear stresses)

Strains : ε_1, ε_2 , ε_3 (No Shear strains)

The strain energy density

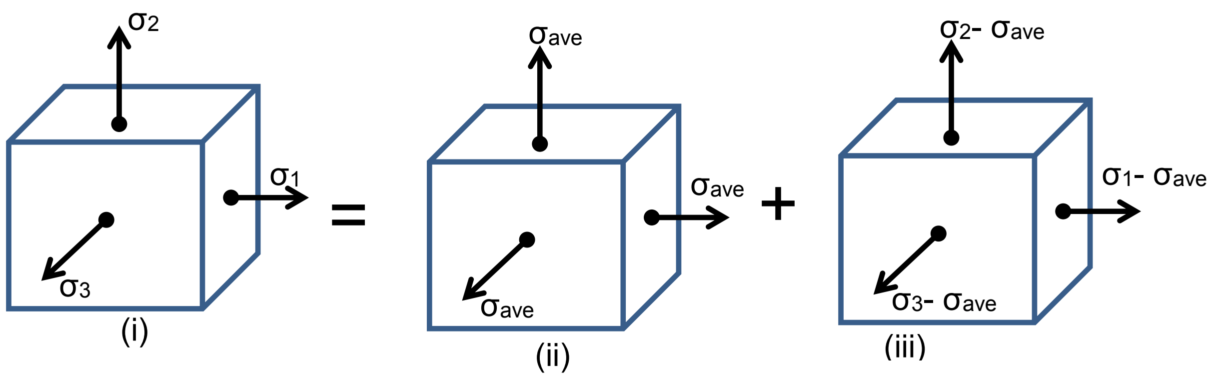
Substituting Hooke's law in the equation (No shear)

$$\boldsymbol{u = \frac{1}{2E}\left(\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) - 2v(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)\right)}$$

(strain energy density)

Two parts can be recognized: First part associated with volume change only, the second associated with distoration only

Consider σ_1, σ_2 , σ_3 to be the sum of two stress states:



Where σ_{ave}**=(σ_1+ σ_2+ σ_3)/3**=average normal stress

$$\boldsymbol{\varepsilon_i = \varepsilon_{ii} + \varepsilon_{iii}}$$

a) The average stress acting in state (ii) causes only changes in volume but no distortion at any orientation of axes:

By Hooke's law

$$\boldsymbol{E\varepsilon_1{}^{ii} = E\varepsilon_2{}^{ii} = E\varepsilon_3{}^{ii} = \sigma_{ave} - v(\sigma_{ave} + \sigma_{ave}) = \sigma_{ave}(1 - 2v)}$$

Strain energy density for state (ii);

$$u_v = \frac{1}{2}(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3) = \frac{1}{2}(\sigma_{ave}\varepsilon_1 + \sigma_{ave}\varepsilon_2 + \sigma_{ave}\varepsilon_3)$$

$$\boldsymbol{u_v = \frac{1-2v}{6E}(\sigma_1 + \sigma_2 + \sigma_3)^2}$$

$\boldsymbol{u_v}$= Volumetric strain energy density

---

*Bulk Modulus $\boldsymbol{K = \frac{E}{3(1-2v)}}$

Resistance against change in volume (i.e. $\boldsymbol{\sigma_{ave}}$=Ke where e is volume change/per unit volume which is called **Dilatation $\boldsymbol{e = \varepsilon_1{}^{ii} + \varepsilon_2{}^{ii} + \varepsilon_3{}^{ii}}$

$\boldsymbol{u_v = \frac{1}{2}\frac{\sigma_{ave}^2}{K}}$

b) Stress state (iii) results in **no volume change** i.e. the sum of its assocaited normal strains=0

Proof: (Hooke's law)

$$E\left(\varepsilon_1{}^{iii} + \varepsilon_2{}^{iii} + \varepsilon_3{}^{iii}\right) = (\sigma_1 - \sigma_{ave}) - v(\sigma_2 + \sigma_3 - 2\sigma_{ave})$$
$$+(\sigma_2 - \sigma_{ave}) - v(\sigma_1 + \sigma_3 - 2\sigma_{ave})$$
$$+(\sigma_3 - \sigma_{ave}) - v(\sigma_1 + \sigma_2 - 2\sigma_{ave})$$
$$= (\sigma_1 + \sigma_2 + \sigma_3)(1 - 2v) - 3\sigma_{ave}(1 - 2v) = 0$$

As 3 $\sigma_{ave} = \sigma_1 + \sigma_2 + \sigma_3$

Therefore stress state (iii) produces only Distortion

Its associated Energy distortion (u_d) is calculated as follows:

Total $u = u_v + u_d$ Therefore $\mathbf{u_d = u - u_v}$ Hence:

$$u_d = \frac{1}{6E}\left(3(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - 6v(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3) - (1 - 2v)(\sigma_1 + \sigma_2 + \sigma_3)^2\right)$$

$$\boldsymbol{u_d = \frac{1+v}{6E}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]}$$

u_d = Strain energy density for distortion

The strain energy of distortion depends only on differences between principal stresses.

This equation is important in the development of theries of failure to be discussed

Note: $\boldsymbol{\frac{1+v}{6E} = \frac{1}{12G}}$

CivE 205–Solid Mechanics II

Part 6:

Theories of Failure

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Spring 2020

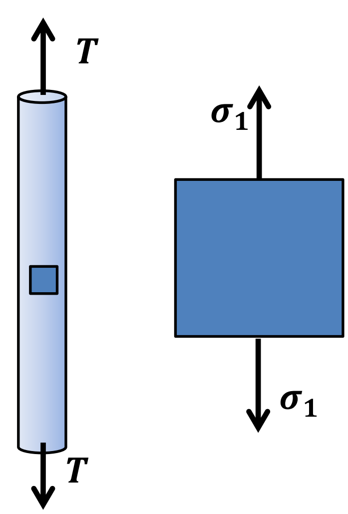
- Theories of failure are developed to establish criteria that define materials failure
- Types of failure:
    - Yielding of ductile materials
    - Fracture of brittle materials
    - Excessive deformation (elastic and inelastic)
    - Buckling:
        - Member or structure
        - Rapid and without warning
        - Usually catastrophic in nature
- Most of the information of strength of materials relies on simple:
    - Tensile tests for metals
    - Compression tests for brittle materials (concrete, and stone)
    - Limited information on combined stresses

In order to establish a relationship between the material behaviour under simple test and its strength under combined stresses, theories of failure are applied.

Theory 1: Maximum Shear Stress Theory (Tresca Criterion)

(H.E. Tresca 1814-1885)

- In good agreement with experiment in ductile materials
- Easy to apply
- Based on the observation that yielding of ductile materials initiated by slipping along shear planes that have maximum shear values.

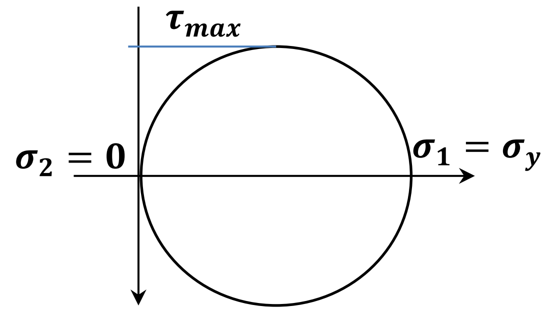


In a simple tensile test, failure occurs when the maximum shear stress τ_{max} reaches its corresponding value at yield point ($\boldsymbol{\sigma_1 = \sigma_y = \sigma_f}$).

For the following cases of loading:

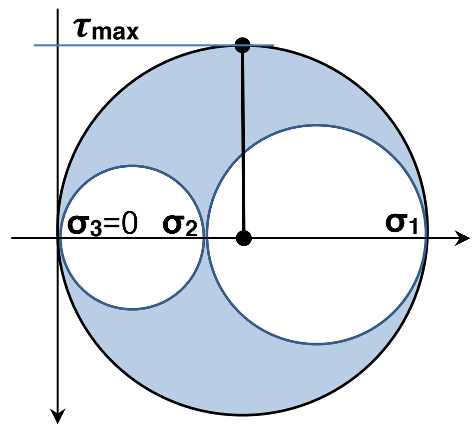
1) Under uniaxial tensile loading

$$\tau_{max} = \frac{\sigma_1}{2} = \frac{\sigma_y}{2}$$



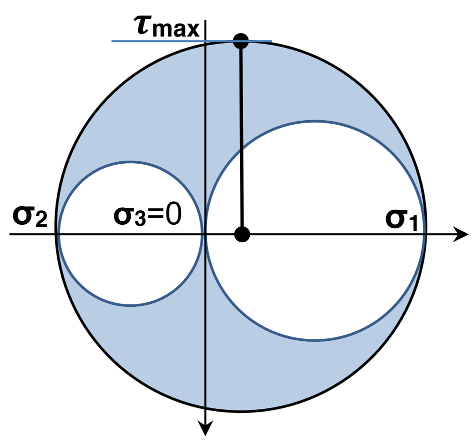
2) **For 3 dimensional stress states when** $\sigma_3=0$ and **two in-plane principal stresses** σ_1, and $\sigma_2 \neq 0$. Using Mohr's circle for 3D stress, <u>two stresses have the same sign</u>:

$$\tau_{max} = \frac{\sigma_1}{2} = \frac{\sigma_y}{2}$$



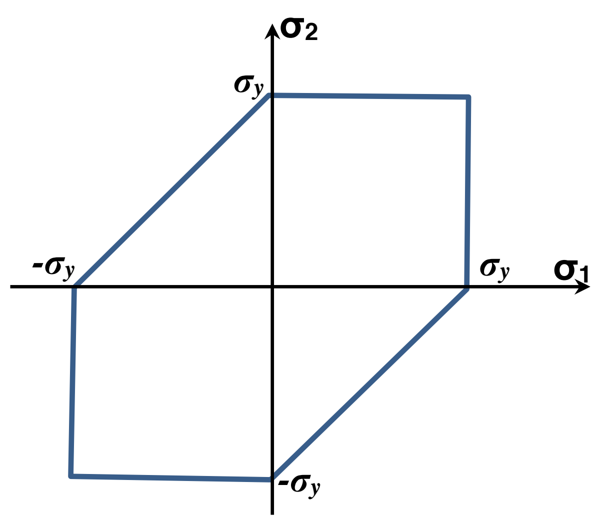
When <u>two stresses have opposite signs</u>:

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$$



<u>Tresca hexagon</u>

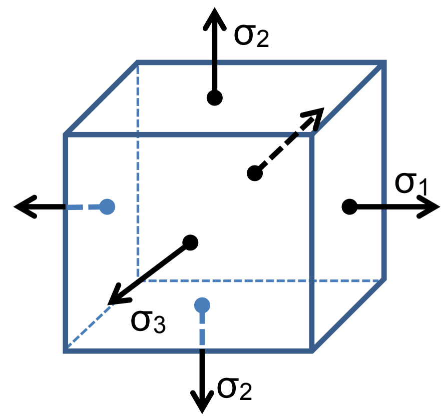
Failure occurs outside the hexagon as the material reaches its failure load.



Theory 2: Energy of Distortion Theory (von Mises Criterion)

Von Mises &H. Hencky (1883-1953) and independently by (M Huber 1904)

- Failure occurs when the strain energy of distortion at any point in a stressed body reaches the same value it has when yielding occurs in tensile test.
- It is very much in favour for ductile materials

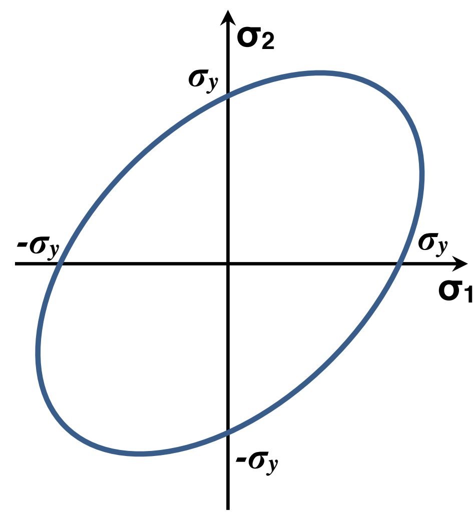


$$\boldsymbol{u_d = \frac{1+v}{6E}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]}$$

σ_1, σ_2, and σ_3 are principal stresses

In uniaxial stress:

$$\boldsymbol{u_d = \frac{1+v}{6E}[2\sigma_1^2] = \frac{1+v}{6E}[2\sigma_y^2]}$$



In plane stress (σ_3=0)

$\boldsymbol{\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2=\sigma_y^2}$ **(an ellipse equation)**

In three dimensional stress states

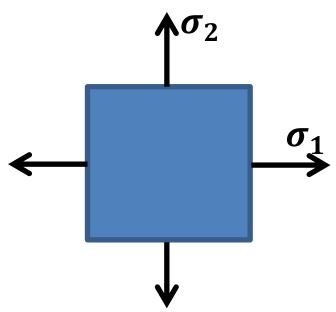
$$\boldsymbol{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2=2\sigma_y^2}$$

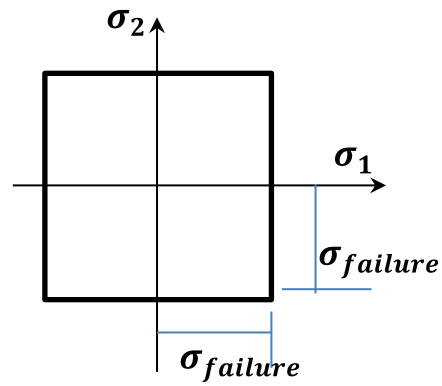
Yielding occurs when state of stress lies on the ellipse

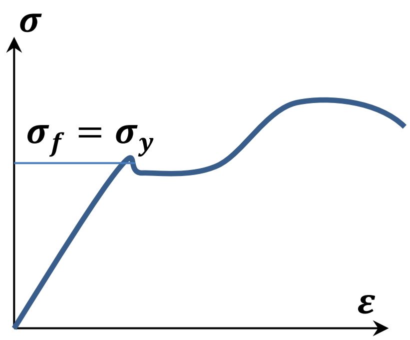
Theory 3: Maximum Normal Stress Theory (Rankine's theory)

Rankine, W.J.M (1820-1872), Coulomb (1736-1806)

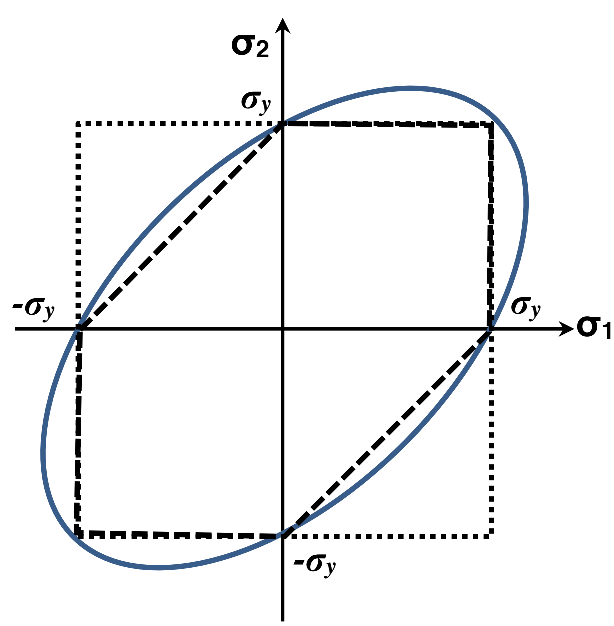
- It predicts failure when the maximum principal normal stress reaches the value of the axial failure stress in tension (or compression) test.
- Any combination of σ_1 and σ_2 in a biaxial stress which lies inside is safe
- This theory assumes that failure in tension and compression occurs at same stress
- It is in favour for brittle materials







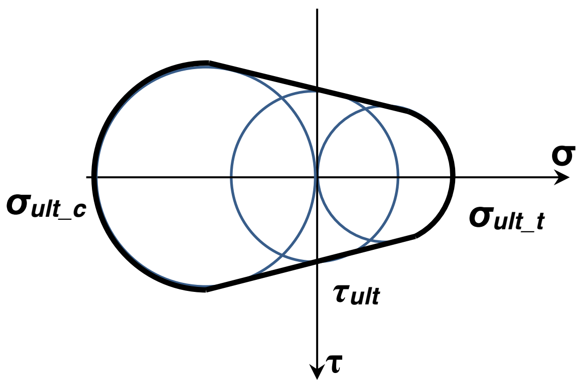
Combining the above three theories:

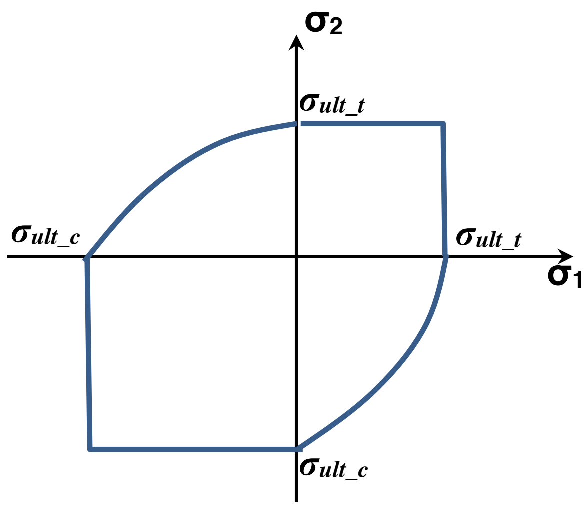


Other Criteria:

i. Mohr's Failure Criterion:

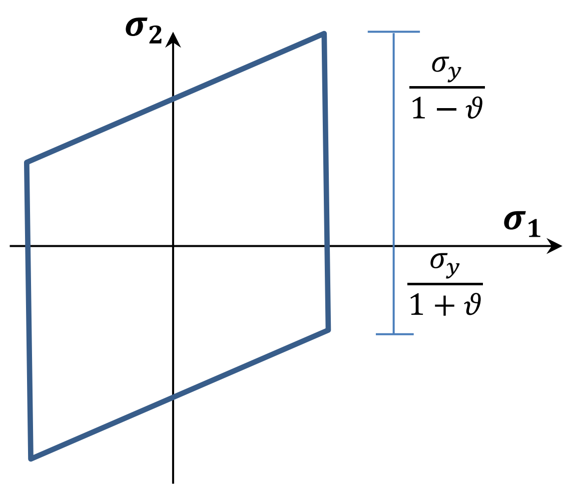
- Applicable to predict failure of brittle materials.
- Used when tension and compression material properties are different.
- Tensile, compressive and shear ultimate stresses must be found using uniaxial tensile test, uniaxial compressive test, and torsion, respectively.





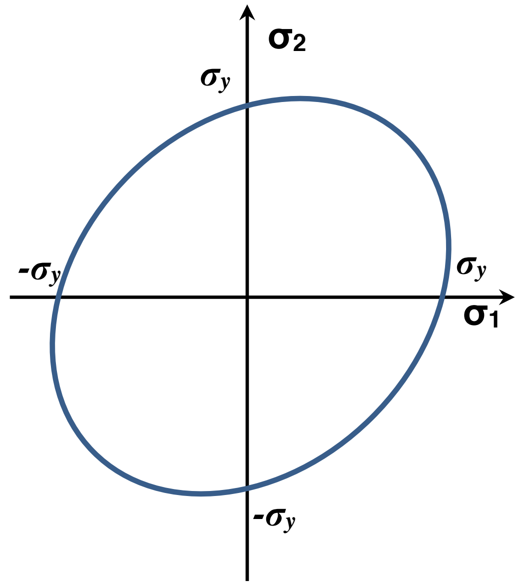
ii. Maximum axial strain theory (Saint- Venant) (rarely applied)

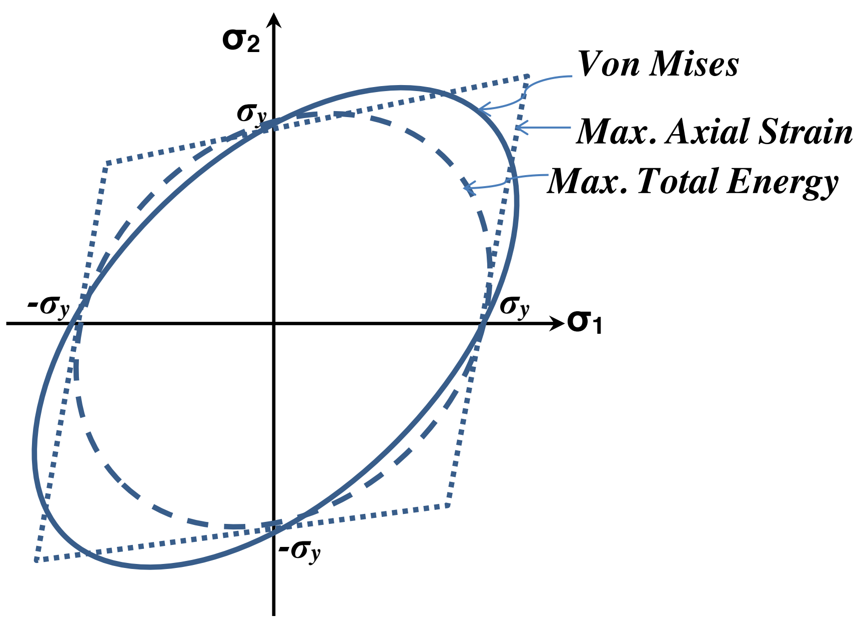
- It states that failure occurs when the largest principal (normal) strain reaches a limited value (Yield strain) in axial tensile test.
- It over-estimates the strength of ductile materials in tension. This is mainly related to the fact that as tension in one direction reduces strain in perpendicular direction, two equal tensions will cause failure at much higher values.



iii. Maximum total energy theory (rarely applied)

- It predicts that failure occurs when the total strain energy exceeds the total strain energy at yield in a simple tensile test.
- It does not agree with experiments for nearly hydrostatic pressure (tension)

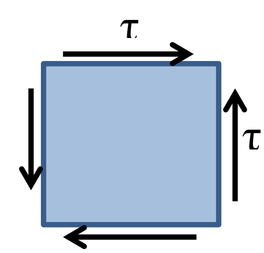


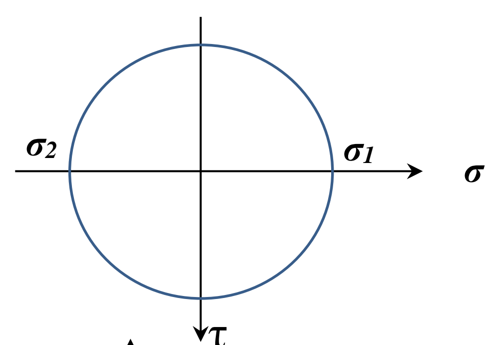


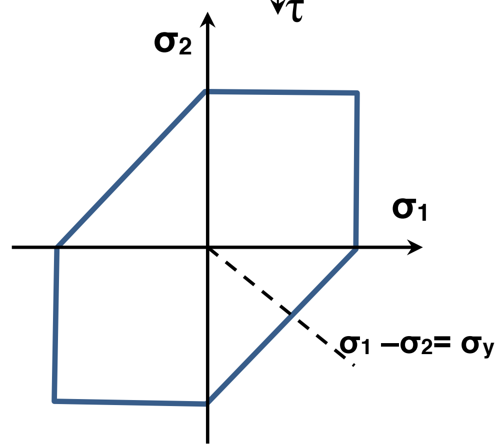
BRITTLE MATERIALS

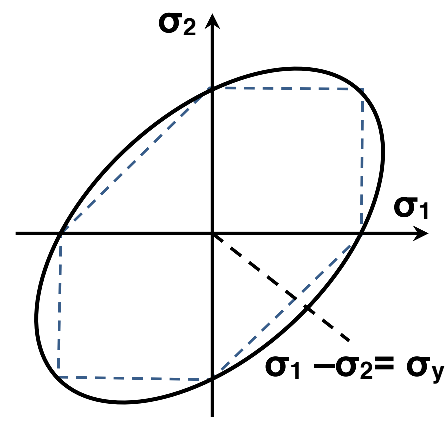
- Examples (concrete, chalk, cast iron etc.)
- Failure occurs at relatively low strains and there is little, or no, permanent yielding on the plane of maximum shearing stress.
- They are weak in tension in compare to compression (concrete: Compression strength= 30MPa, Tension=1.5MPa). Weakness in tension is attributed to the large number of cracks in brittle material that cause stress concentration.
- Cracking in concrete and other brittle materials occurs on planes inclined at 45^o to the directions of the applied shearing stresses.
- Maximum normal stress theory is useful for brittle materials.

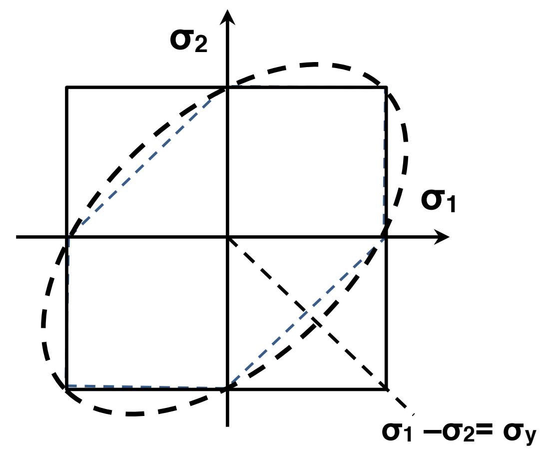
Example 6.1: Determine the maximum shear stress (τ_f) that can be carried by a material that has yield strength of σ_y and subjected to pure shear stress using Tresca, von Mises and Rankine's criteria?



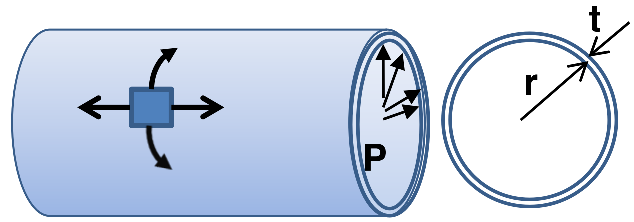


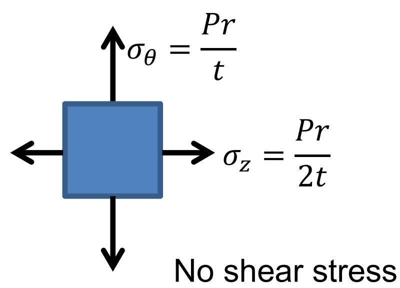


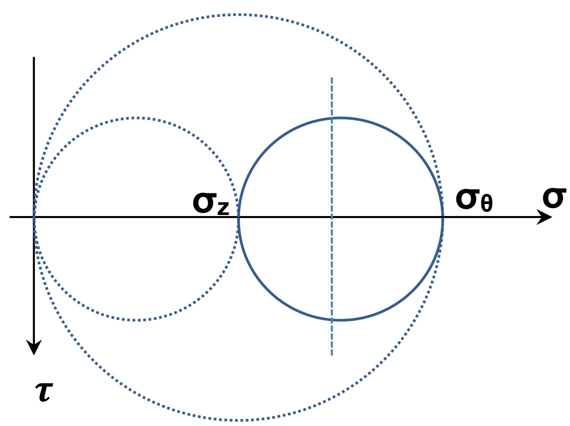

σ2
σ1
σ1 –σ2= σy


σ2
σ1
σ1 –σ2= σy

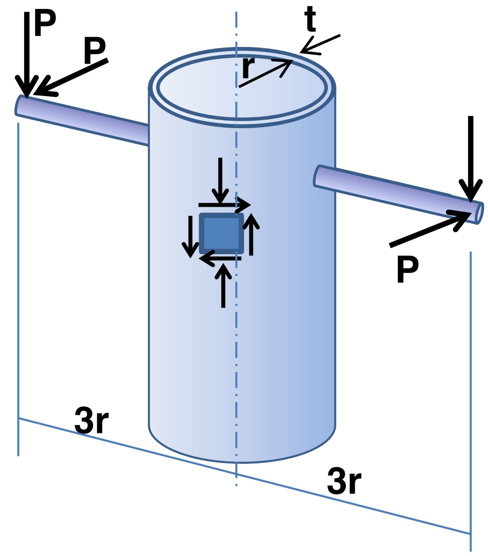
Example 6.2: Thin-walled cylindrical vessel of radius *r* and thickness *t* is subjected to internal pressure P. The yield strength of the vessel material is σ_y. Calculate the max. pressure (P) that can be applied using Tresca and von Mises Criteria?

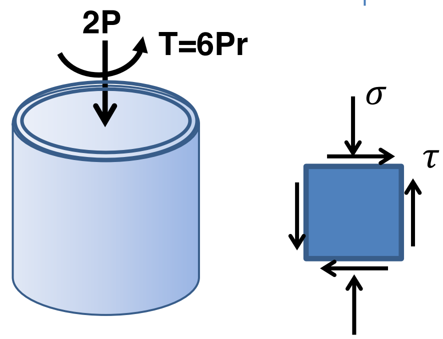


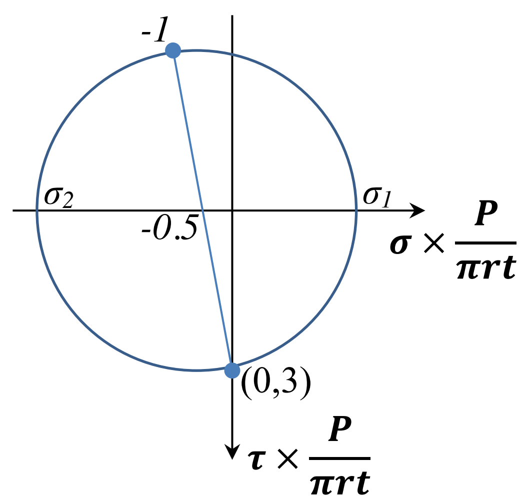




Example 6.3: Using Tresca yield criterion, find the required thickness of the torque wrench given P, r, σ_Y?







CivE 205–Solid Mechanics II

Part 7:

Buckling of Columns

Department of Civil & Environmental Engineering
University of Waterloo
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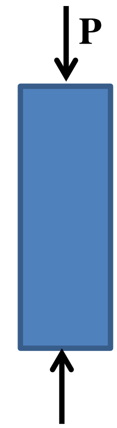
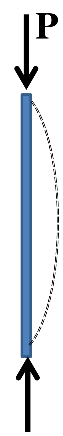
TYPES OF COLUMNS AND STABILITY



<u>Short, stocky columns</u>, when compressed will ultimately fail by crushing. The material experiences excessive stresses.

Stress at failure=P/A exceeds a limiting value



<u>Long and thin (slender) columns</u> may fail suddenly by change in configuration or shape at some critical value of the applied axial load.

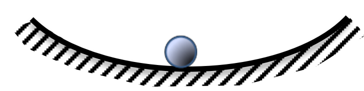
Totally elastic phenomenon

$P=P_{critical}$ (Instability; buckling)

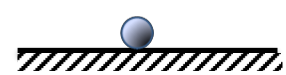
This problem of instability in slender columns was first analyzed by **<u>Leonard Euler</u>** (1707-1783)

<u>What is instability</u>?

Consider a ball resting on a smooth surface.

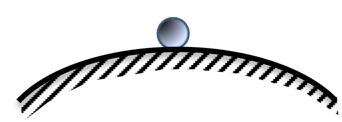


<u>Stable Equilibrium:</u> A small horizontal disturbance acting on the ball will not cause excessive lateral displacement.



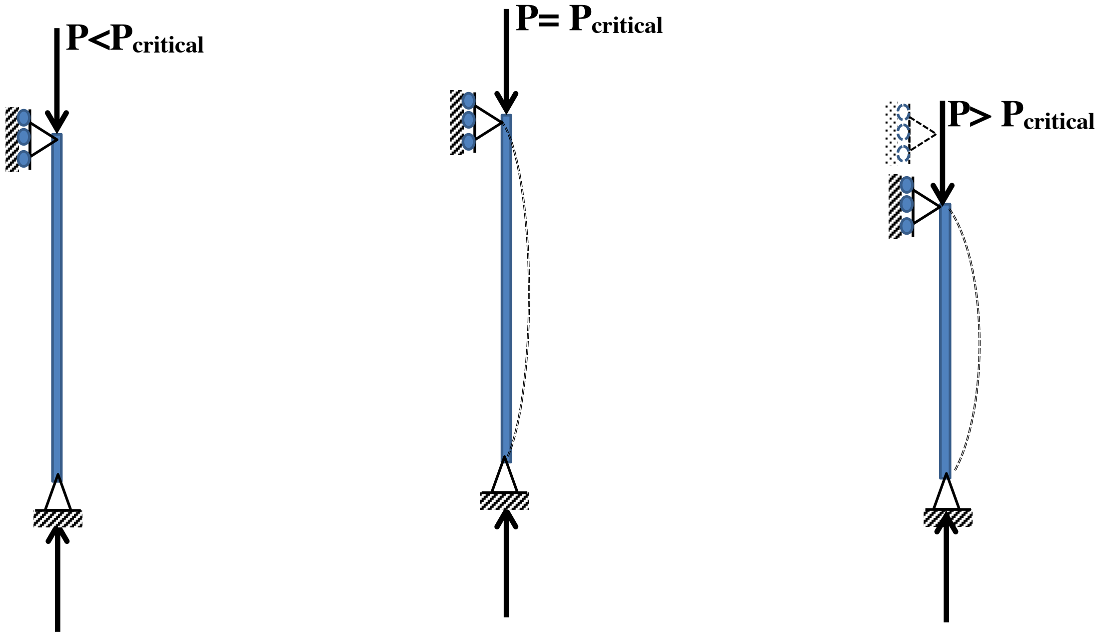
<u>Neutral Equilibrium:</u> an infinite number of adjacent equilibrium states exist.

<u>Unstable Equilibrium:</u> A slight lateral disturbance will cause large movement of the ball from its original equilibrium state.



Axially loaded columns behave a lot like the ball

The limit of stability is reached when the axial load reaches its critical value ($P_{critical}$)



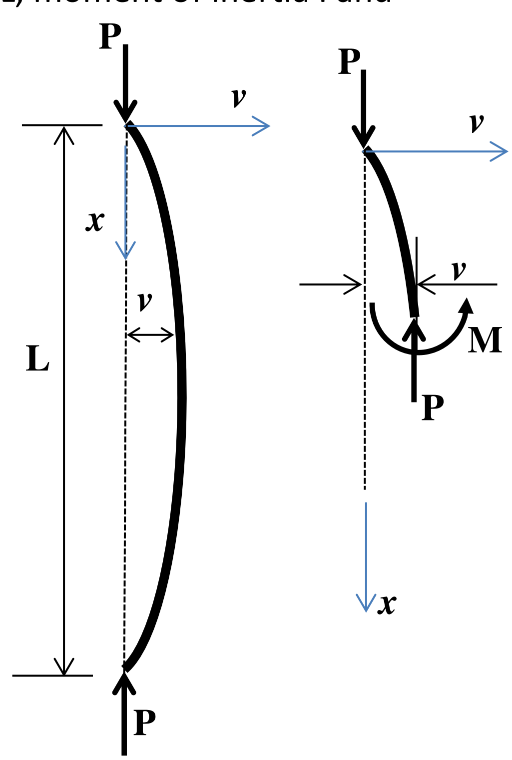
The straight configuration is in equilibrium and stable.

The straight and all adjacent bent configurations are in equilibrium but not stable.

The straight configuration is in equilibrium but unstable. Large lateral deflection will therefore develop.

ANALYSIS OF AN AXIALLY LOADED COLUMN

Consider a pinned-end prismatic column of length L, moment of inertia I and modulus of elasticity E, subject to a concentric axial load P.



For a small lateral deflection v:

$$\frac{M}{EI} = \frac{d^2v}{dx^2}$$

Moment equilibrium at the cut section requires:

The general solution for the differential equation of equilibrium is

$$v = A\cos\sqrt{\frac{P}{EI}}x + \mathrm{B}\sin\sqrt{\frac{P}{EI}}x$$

Where A and B are constants of integration.

For a pinned-end columns, boundary conditions:

At x=0, v=0 → v(0)=0=Acos 0+Bx0 → A=0

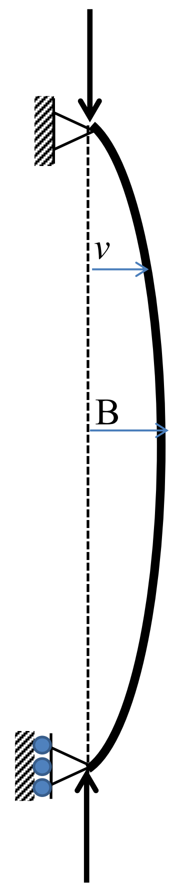
At x=L, v=0 →

v(L)=0 $= \mathrm{B}\sin\sqrt{\frac{P}{EI}}L$ Either B=0 then both A and B are zero and the column is straight (no deflection)

Or $\sin\sqrt{\frac{P}{EI}}L = 0$ B is indeterminate (i.e. B can take any value) and the column takes on the deflected form

$$v = \mathrm{B}\sin\sqrt{\frac{P}{EI}}x$$

<u>The deflected (buckled configuration):</u>



This buckled form is possible only when $\sin\sqrt{\frac{P}{EI}}L = 0$

Only the non-zero values of $\sqrt{\frac{P}{EI}}L$ need to be considered.

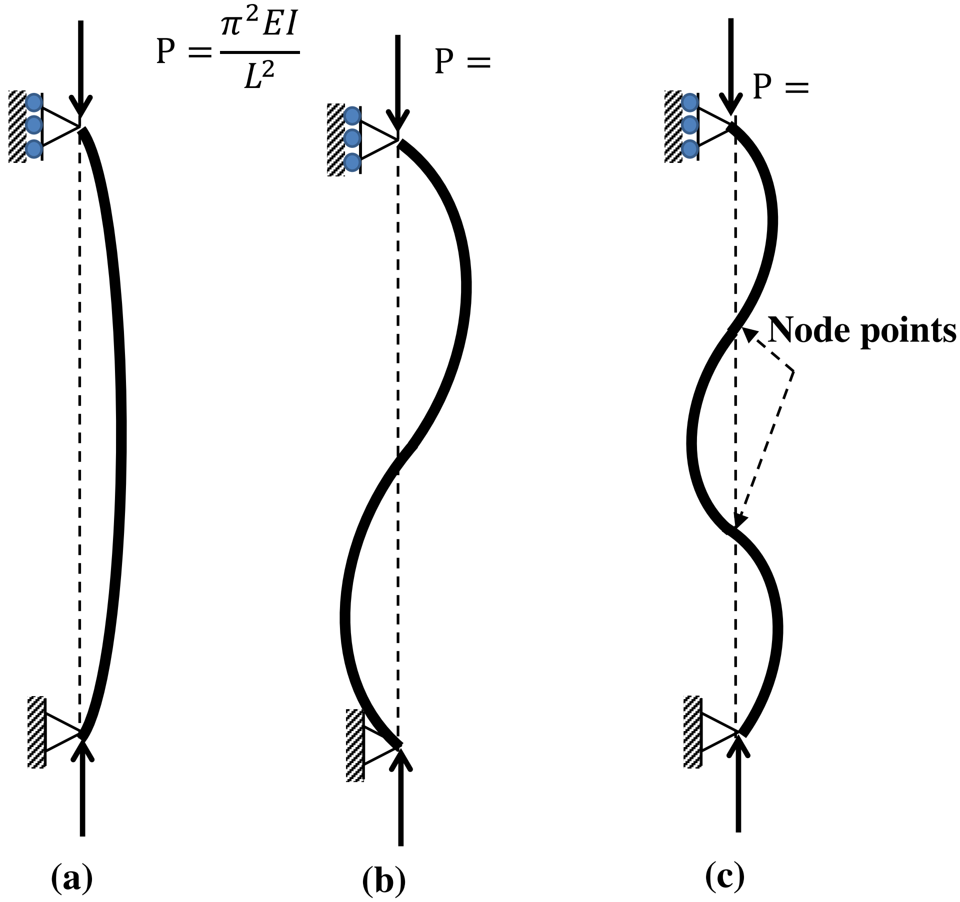
Therefore,

$\mathrm{P} = \frac{\pi^2 EI}{L^2} = \frac{4\pi^2 EI}{L^2}$,…. etc. are the possible critical values for P.

The lowest of these values is

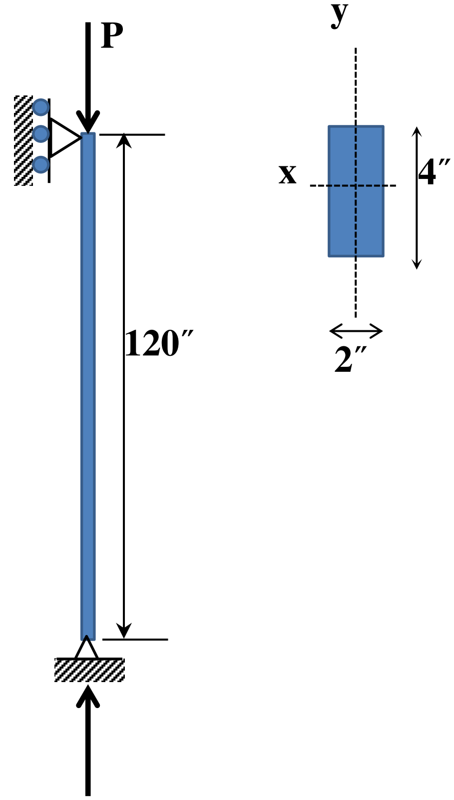
$$\boxed{\mathbf{P} = \mathbf{P}_{cr} = \frac{\pi^2 EI}{L^2}}$$ EULER FORMULA

Modes of buckling for a pin supported column



External restraints are applied at the node points to form buckling nodes in (b) and (c).

Example 7.1: Find the buckling load of a pin-supported column (2x4 in) made of pine wood ($E=1\times10^6$ lb/in^2). Check the adequacy of the column if the ultimate crushing stress of pine is 3000 lb/in^2



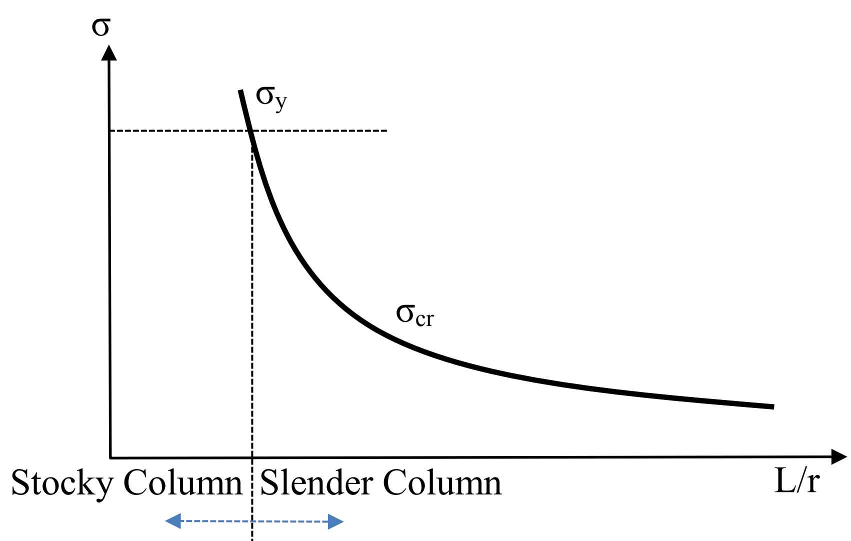
Critical Buckling Stress (σ_{cr}):

Notes: $P_{cr} = \frac{\pi^2 EI}{L^2}$ $\frac{P_{cr}}{A} = \sigma_{cr} = \frac{\pi^2 EI}{AL^2} = \frac{\pi^2 E}{\left(\frac{A}{I}\right)L^2}$

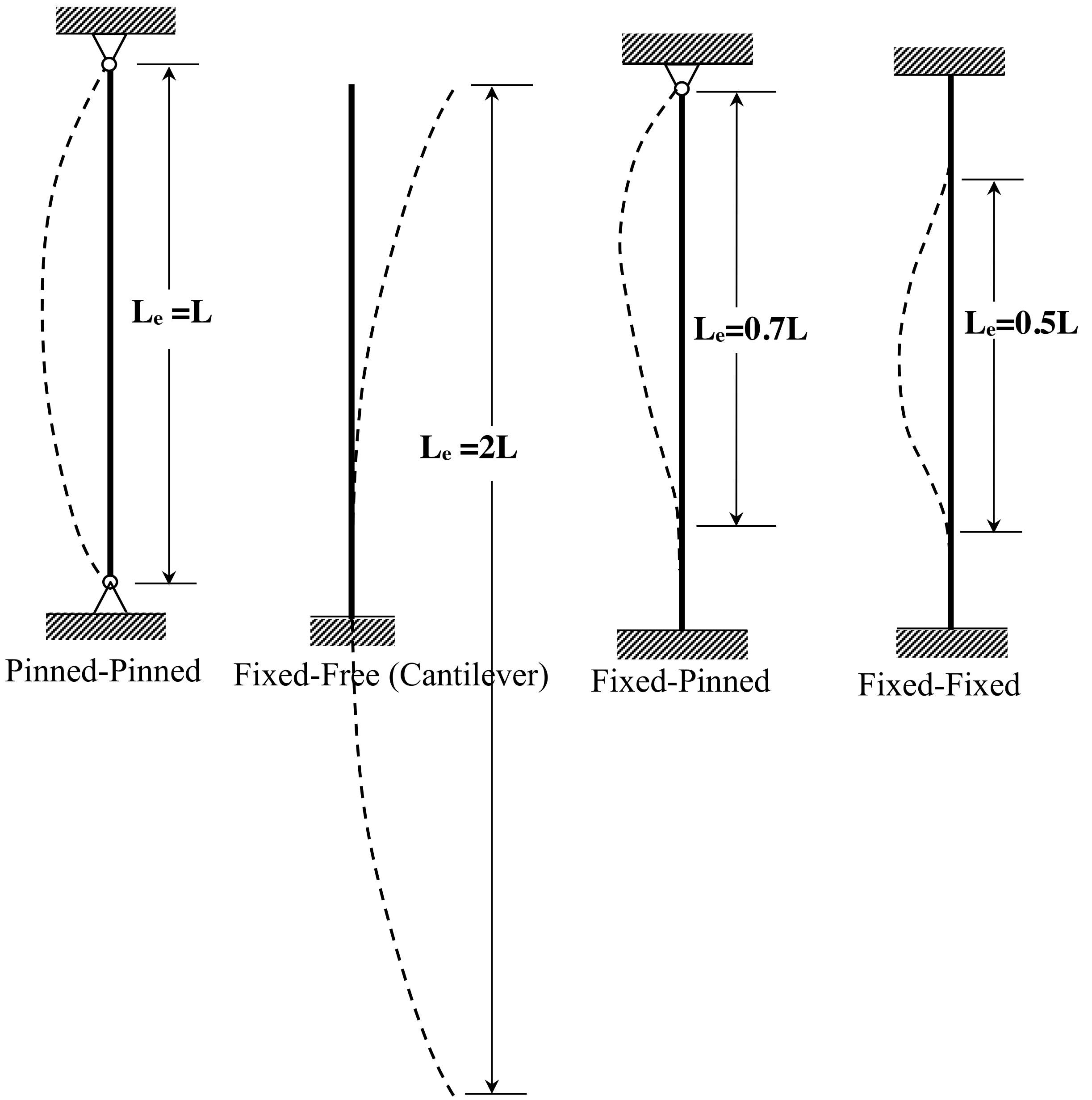
$\sqrt{\frac{I}{A}} = r$ radius of gyration

Therefore,

$$\boldsymbol{\sigma_{cr} = \frac{\pi^2 E}{(L/r)^2}}$$



Columns with other support conditions & effective length (L_e) concept



The critical load of different support conditions can always be expressed as:

$$P_{cr} = \frac{\pi^2 EI}{L_e^2}$$

where L_e is the effective length which is the length of a pinned-pinned column of the same EI value that would yield the same critical load (e.g. The effective length of the cantilever column is twice the actual length).

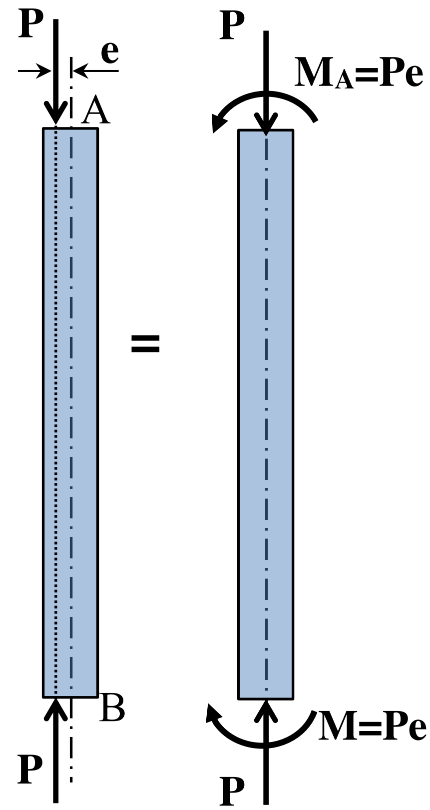
$\frac{L_e}{r}$ =Effective slenderness ratio of a column, then $\sigma_{cr} = \frac{\pi^2 E}{\left(L_e/r\right)^2}$ is critical stress

Example 7.2: Find the lightest W shape for 7.0 m long steel column to support an axial load of 450 kN with a factor of safety of 3. Assume the column is a) pinned-pinned, and b) fixed- fixed. Use ultimate strength of 250 MPa, and E=200 GPa (Design the column based on I only).

				Flange			Axis $X–X$			Axis $Y–Y$		
Designation	Theoretical mass (kg/m)	Area (mm^2)	Depth (mm)	Width (mm)	Thickness (mm)	Web Thickness (mm)	I (10^6 mm^4)	$S = \frac{I}{c}$ (10^3 mm^3)	$r = \sqrt{I/A}$ (mm)	I (10^6 mm^4)	$S = \frac{I}{c}$ (10^3 mm^3)	$r = \sqrt{I/A}$ (mm)
W250 ×167	167.4	21 300	289	265	31.8	19.2	300	2 080	119	98.8	746	68.1
×149	148.9	19 000	282	263	28.4	17.3	259	1 840	117	86.2	656	67.4
×131	131.1	16 700	275	261	25.1	15.4	221	1 610	115	74.5	571	66.8
×115	114.8	14 600	269	259	22.1	13.5	189	1 410	114	64.1	495	66.3
×101	101.2	12 900	264	257	19.6	11.9	164	1 240	113	55.5	432	65.6
×89	89.6	11 400	260	256	17.3	10.7	143	1 100	112	48.4	378	65.2
×80	80.1	10 200	256	255	15.6	9.4	126	982	111	43.1	338	65.0
×73	72.9	9 280	253	254	14.2	8.6	113	891	110	38.8	306	64.7
×67	67.1	8 550	257	204	15.7	8.9	104	806	110	22.2	218	51.0
×58	58.2	7 420	252	203	13.5	8.0	87.3	693	108	18.8	186	50.3
×49	49.0	6 250	247	202	11.0	7.4	70.6	572	106	15.1	150	49.2
×45	44.9	5 720	266	148	13.0	7.6	71.1	534	111	7.03	95.1	35.1
×39	38.7	4 920	262	147	11.2	6.6	60.1	459	111	5.94	80.8	34.7
×33	32.7	4 170	258	146	9.1	6.1	48.9	379	108	4.73	64.7	33.7
×28	28.5	3 630	260	102	10.0	6.4	40.0	307	105	1.78	34.8	22.1
×25	25.3	3 230	257	102	8.4	6.1	34.2	266	103	1.49	29.2	21.5
×22	22.4	2 850	254	102	6.9	5.8	28.9	227	101	1.23	24.0	20.8
×18	17.9	2 270	251	101	5.3	4.8	22.4	179	99.3	0.913	18.1	20.1
W200 ×100	99.5	12 700	229	210	23.7	14.5	113	989	94.3	36.6	349	53.7
×86	86.7	11 100	222	209	20.6	13.0	94.7	853	92.4	31.4	300	53.2
×71	71.5	9 110	216	206	17.4	10.2	76.6	709	91.7	25.4	246	52.8
×59	59.4	7 560	210	205	14.2	9.1	61.1	582	89.9	20.4	199	51.9
×52	52.3	6 660	206	204	12.6	7.9	52.7	512	89.0	17.8	175	51.7
×46	46.0	5 860	203	203	11.0	7.2	45.5	448	88.1	15.3	151	51.1
×42	41.7	5 310	205	166	11.8	7.2	40.9	399	87.7	9.00	108	41.2
×36	35.9	4 580	201	165	10.2	6.2	34.4	342	86.7	7.64	92.6	40.8

PIN-SUPPORTED COLUMN WITH ECCENTRIC LOADING

In practical situations, it is nearly impossible to load a column exactly at its centre of stiffness.



Suppose the axial load (P) has an equal eccentricity (e) at both ends.

$M = -P(v + e)$

$EI\frac{d^2v}{dx^2} = -P(v + e)$

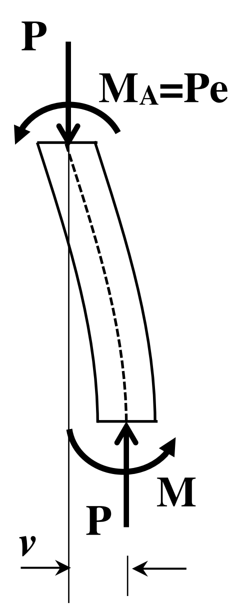
$\frac{d^2v}{dx^2} + \frac{P}{EI}v = -\frac{P}{EI}e$

Let $p^2 = \frac{P}{EI}$

$\frac{d^2v}{dx^2} + p^2v = -p^2e$

$v = A\cos px + \mathrm{B}\sin px - e$

To find A and B use boundary conditions:



Due to symmetry in loading, $v_{\max}$ occurs at mid-span (x=L/2):

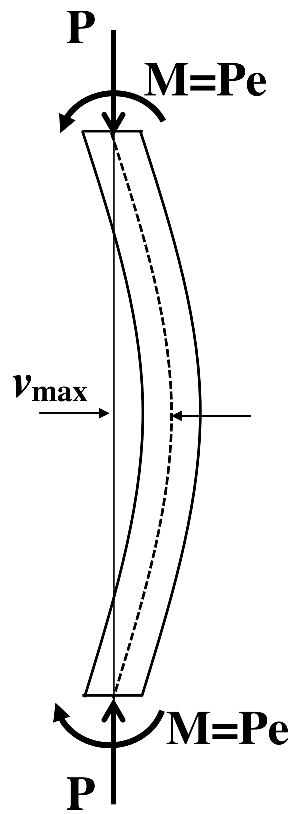
$$v_{max} = e\cos\frac{pL}{2} + e\tan\frac{pL}{2}\sin\frac{pL}{2} - e$$

$$v_{max} = e\left(\frac{\cos^2\frac{pL}{2}+\sin^2\frac{pL}{2}}{\cos\frac{pL}{2}} - 1\right)$$

$$\boldsymbol{v_{max} = e\left(\sec\sqrt{\frac{P}{EI}}\frac{L}{2} - 1\right)}$$

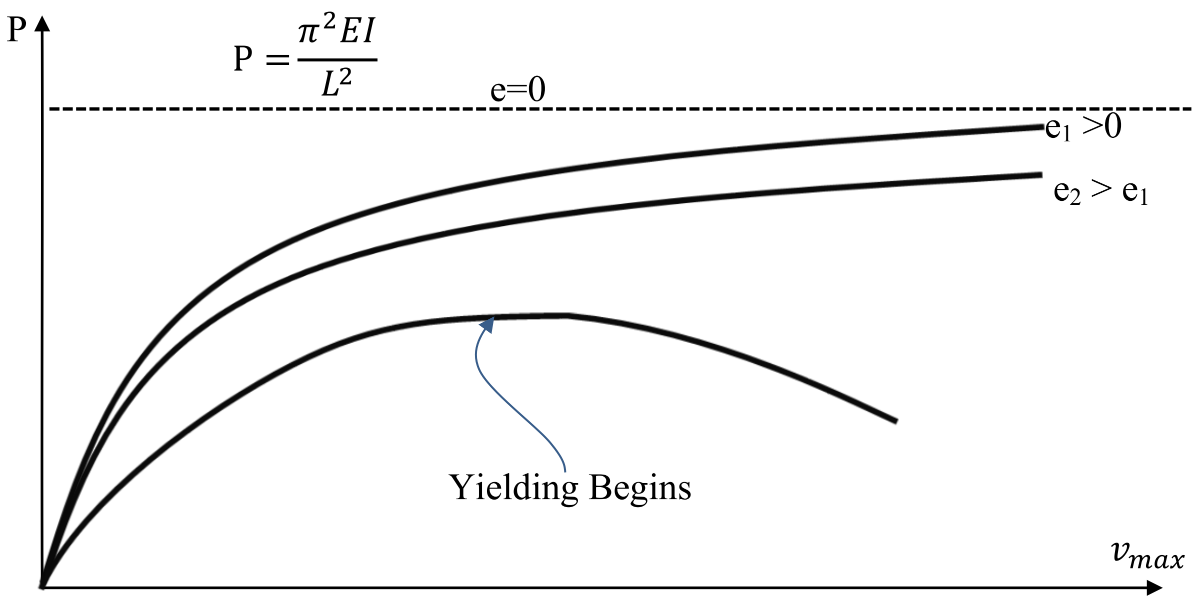
Or, it can be expressed in terms of P_{cr}

$$\boldsymbol{v_{max} = e\left(\sec\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}} - 1\right)}$$



- When $\frac{L}{2}.\sqrt{\frac{P}{EI}} = 0$, P=0 $\quad v_{max} = e[\sec(0) - 1] = 0$
- When $\frac{L}{2}.\sqrt{\frac{P}{EI}} = \frac{\pi}{2} \quad \rightarrow \quad \sec\left(\frac{L}{2}.\sqrt{\frac{P}{EI}}\right) \rightarrow \infty$ and

  $\mathrm{P} = \frac{\pi^2 EI}{L^2} = P_{cr}$ then $\quad v_{max} \rightarrow \infty$

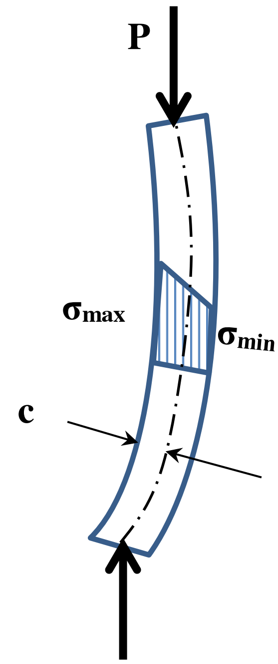


The deflections grow from the onset of loading and become very large as P approach P_{cr} (the Euler load for an equivalent pin-supported column which is concentrically loaded.

THE MAXIMUM STRESS IN ECCENTRIC LOADING

Occurs at mid-height where $M=M_{max}= P.(v_{max}+e)$

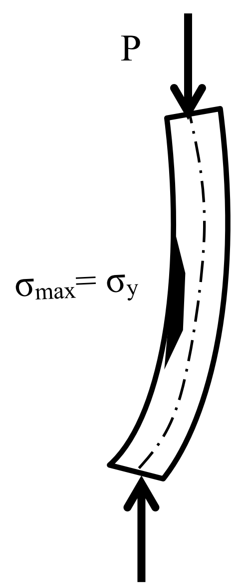
Stress at the extreme fibre on the concave side



$$\sigma_{max} = \frac{P}{A}\left(1 + \frac{ce}{r^2}\sec\left(\frac{L}{2}.\sqrt{\frac{P}{EI}}\right)\right)$$

Or, it can be expressed in terms of P_{cr}

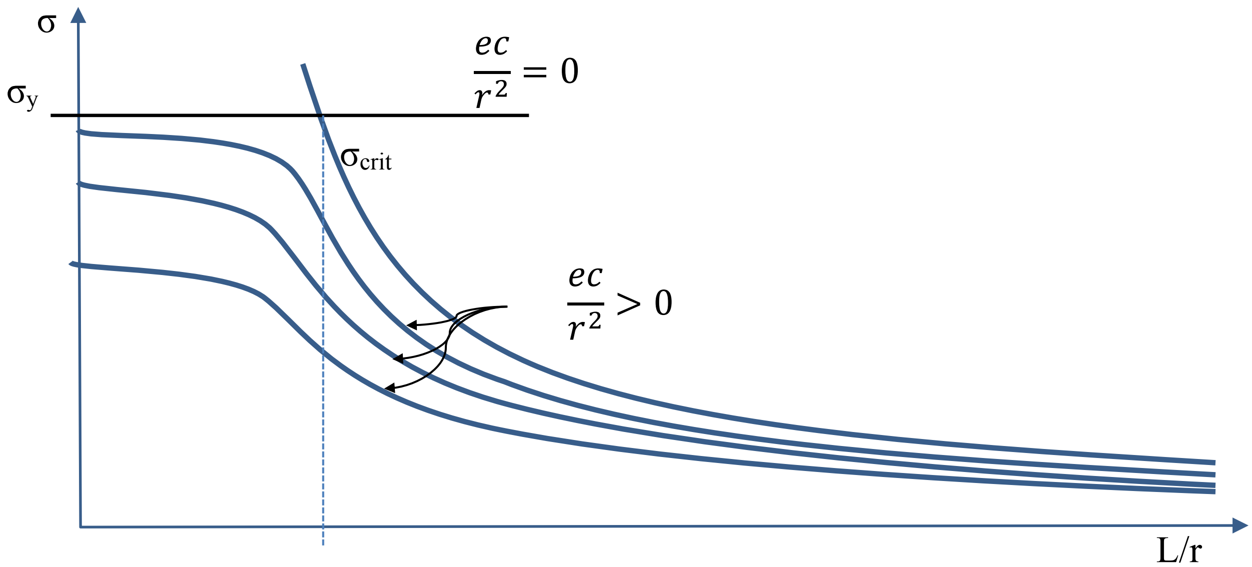
$$\sigma_{max} = \frac{P}{A}\left(1 + \frac{ce}{r^2}\sec\left(\frac{\pi}{2}.\sqrt{\frac{P}{P_{cr}}}\right)\right)$$



This is known in design as the **SECANT FORMULA**

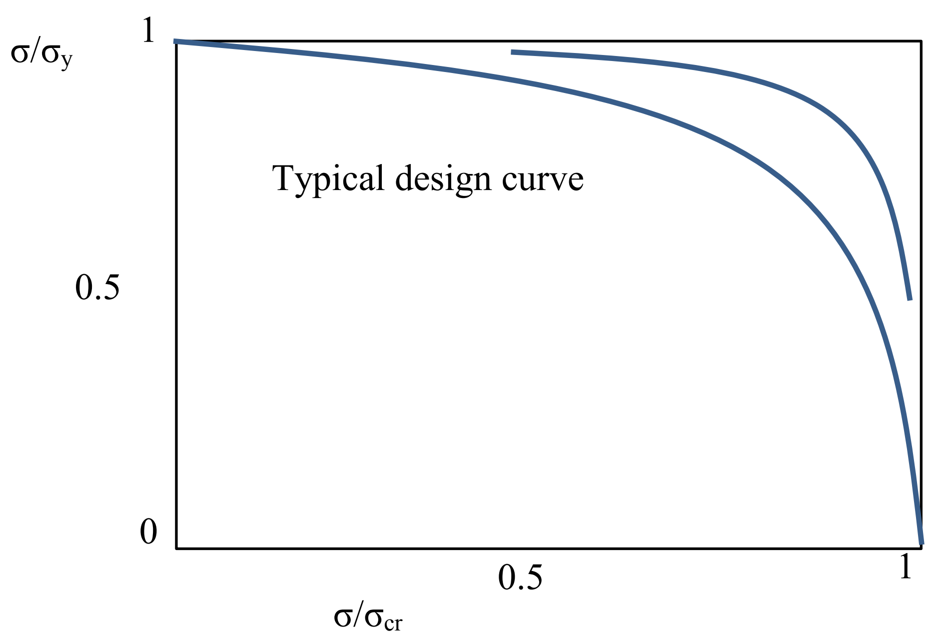
If $\sigma_{max}= \sigma_{yield}$, the column begins to fail by yielding in the extreme concave fibre. Further loading will cause a gradual spread of the yielding zone through the section and very little load can be carried beyond initial yield.

On the basis of the secant formula a stress vs. slenderness ratio plot can be developed for different values of $\left(\frac{ec}{r^2}\right)$.

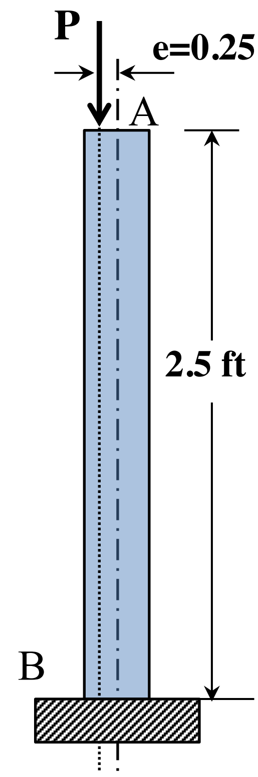


Material properties (σ_y) interact with elastic buckling behaviour to lower the failure load of the real columns to a fraction of the ideal curves (σ_y and σ_{cr}).

This type of plot may be redone on a (interaction plot) as shown below.



Example 7.3: A square column (1.75x1.75 in) is subjected to an axial load P at a distance of 0.25 in from the centre line. Determine a) the load P that causes a horizontal deflection of 0.5 in on the free end of the column, b) the corresponding maximum stress. ($E=10.1\times10^6$psi)



Part 8:

Energy Methods

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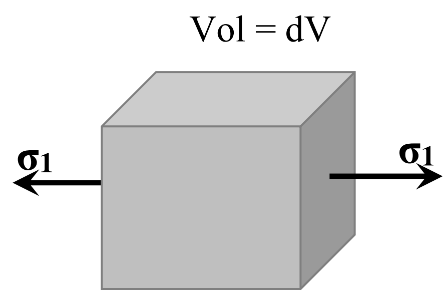
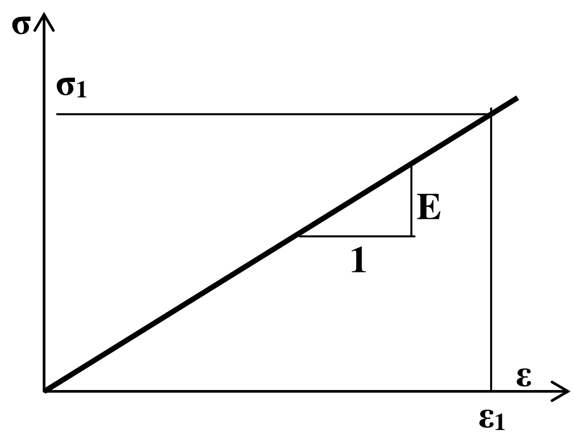
ELASTIC STRAIN ENERGY AND EXTERNAL WORK

- The words of ENERGY and WORK are practically synonyms
- When external forces are gradually applied on a system; the energy associated with the deformation of the system increases by the amount of "work done".
- For an elastic system, the "work done" on the system is stored in a form "elastic energy". The elastic energy is fully recoverable.



- For a uniaxial stress state in an elastic material obeying Hooke's law, the strain energy per unit volume (*u=strain energy density*) is:

$u = \frac{1}{2}\sigma_1\varepsilon_1$ = area under stress-strain curve.



- The strain energy (U) is the increase in energy associated with the deformation experienced by the member under external load:

$$U = \frac{1}{2}\int \frac{\sigma_1^2}{E} dV$$

Example in Mechanics:

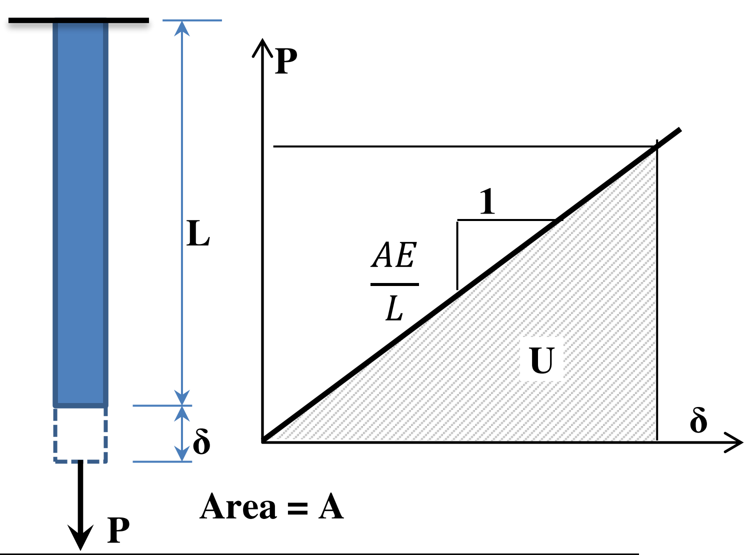
1) **Uniform bar under tension or compression**

$\sigma = E\varepsilon = \frac{P}{A}$

E=Modulus of Elasticity, set $dV=A.dx$

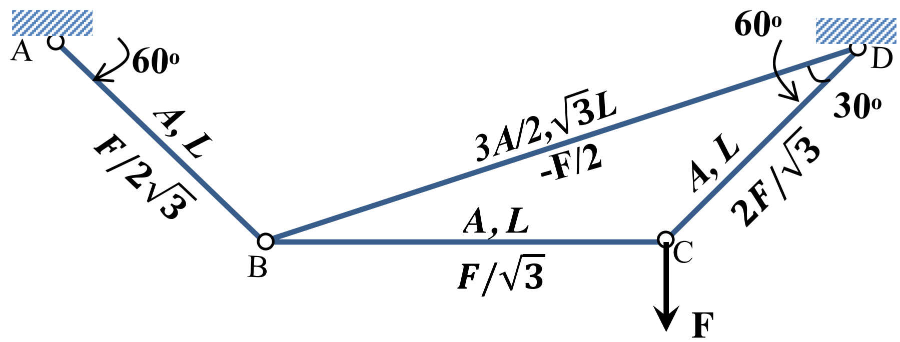
$$U = \frac{1}{2}\int \frac{\sigma^2}{E} dV = \frac{1}{2}\int_0^L \frac{P^2}{A^2\,E}(A.dx)$$

$$\boldsymbol{U = \frac{P^2 L}{2EA}}$$



Note: For trusses, the total strain energy is the summation of the strain energy of all members.

Example 8.1 Pin jointed frame: Areas and lengths are given. Forces are calculated by statics. E is constant for all members. Determine the strain energy in the frame.



Member	Area (A)	Length (L)	Force (P)	$U_i = \dfrac{P^2 L}{2EA}$
AB	A	L	$F/2\sqrt{3}$	
BC	A	L	$F/\sqrt{3}$	
CD	A	L	$2F/\sqrt{3}$	
BD	3A/2	$\sqrt{3}$L	-F/2	

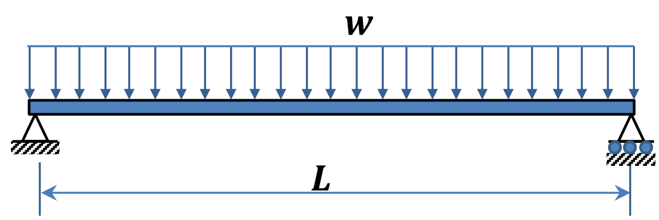
2) Strain Energy due to bending in a beam

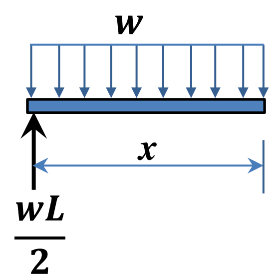
$\sigma = E\varepsilon = \frac{My}{I}$, $dU = \frac{1}{2}\sigma\varepsilon\, dV$ set $dV=dA.dx$

$$U = \frac{1}{2}\int \frac{\sigma^2}{E} dV = \frac{1}{2}\int \frac{\left(\frac{My}{I}\right)^2}{E} dV = \frac{1}{2}\int \frac{M^2}{EI^2}(y^2 dA)dx$$

$$\boldsymbol{U = \frac{1}{2}\int_0^L \frac{M^2}{EI} dx}$$

Example 8.2 A simply supported beam is under uniformly distributed load along its length (L). Determine the strain energy due to bending. (EI is Constant).





3) Strain energy due to shearing forces in a beam

$dU = \frac{1}{2}\tau\gamma\, dV$ $\gamma = \frac{\tau}{G}$ set $dV = dA.dx$ G=shear modulus

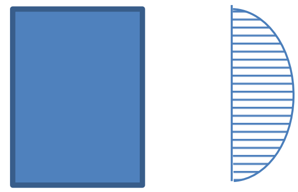
$$U = \frac{1}{2}\int \tau\gamma\; dV = \frac{1}{2}\int \tau\frac{\tau}{G}(dA)dx = \frac{1}{2}\int \frac{\tau^2 A}{G}\, dx$$

A= cross sectional area → $\tau_{avg} = \frac{V}{A}$

$$U = \frac{1}{2}\int_0^L \frac{V^2}{GA}dx$$

Note: This assumes that shear stress is uniformly distributed over the cross-section.

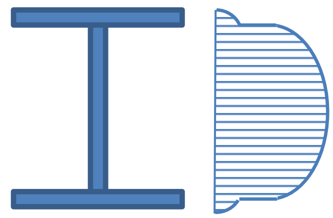
But shear stresses are not uniformly distributed: Examples:



K =1.2

K= is a form factor which adjusts the result to take account of uneven shear stress distribution (See the footnote). Therefore:

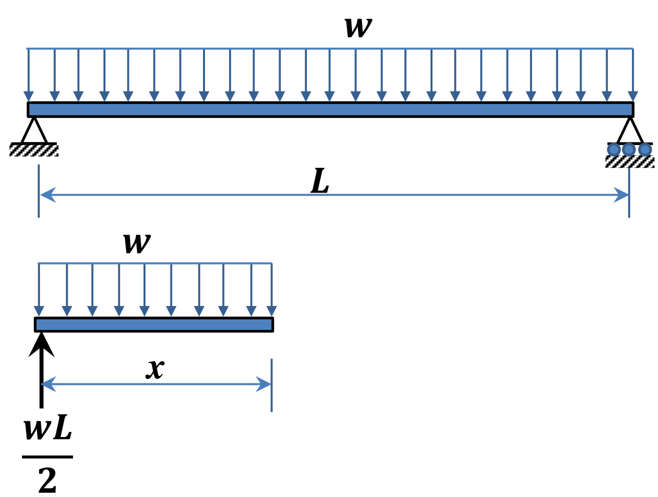
$$U = \frac{K}{2}\int_0^L \frac{V^2}{GA}dx$$



K≃1.2

Note: Since most beams are relatively slender, the strain energy due to bending is much greater that strain energy due to shear. Therefore, **the strain energy due to shear is usually ignored, except in short deep beams**.

Example 8.3: Determine the strain energy due to shear in Example 8.2?



Note: The exact value of K: $K = \frac{A}{I^2}\int_A \frac{Q^2}{t^2}dA$

4) Strain Energy in Circular shafts in torsion

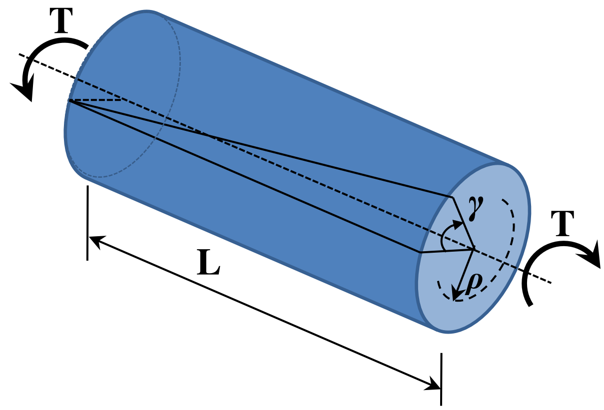
Integrating over the length of the shaft:

$$U = \frac{1}{2}\int \tau\gamma \; dV = \frac{1}{2}\int \tau\frac{\tau}{G} \; dV = \frac{1}{2}\int \frac{\tau^2}{G} \; dV$$

$\tau = \frac{T\rho}{J}$ J= polar moment of inertia= $\int \rho^2 dA$

$$U = \frac{1}{2}\int \frac{T^2}{GJ^2} \left(\int \rho^2 dA\right) dx$$

$$U = \frac{1}{2}\int_0^L \frac{T^2}{JG} dx$$



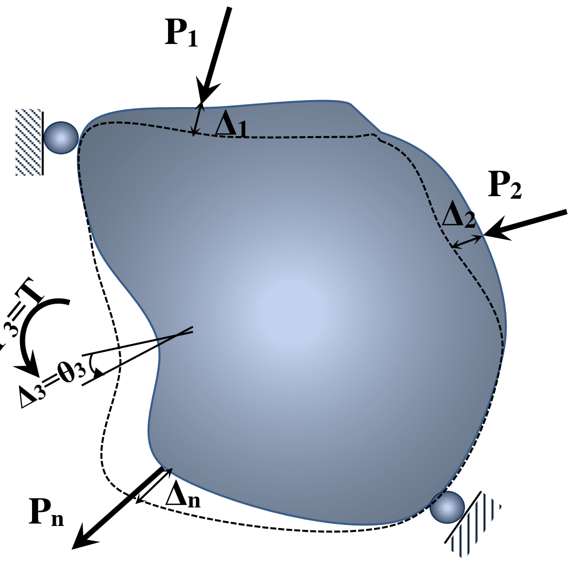
For uniform shaft with constant torque:

$$\boldsymbol{U = \frac{T^2 L}{2GJ}}$$

Conclusions:

System	Load	Stiffness	U
Uniaxial loading (e.g. Rod)	P	EA	$U = \frac{P^2 L}{2EA}$
Bending Moment (e.g. beam)	M	EI	$U = \frac{1}{2}\int_0^L \frac{M^2}{EI} dx$
Shear force (e.g. beam)	V	GA	$U = \frac{1}{2}\int_0^L \frac{V^2}{GA} dx$
Torque (e.g. Shaft)	T	GJ	$U = \frac{T^2 L}{2JG}$

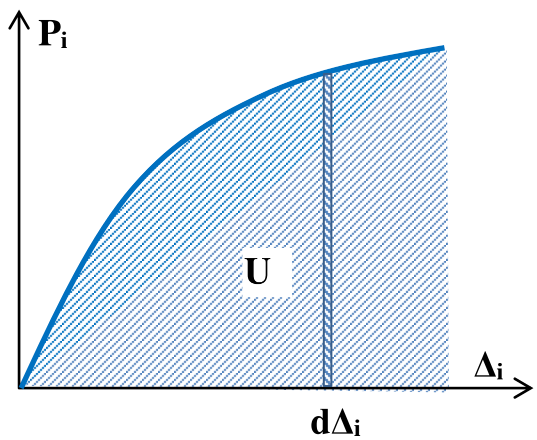
COMPLEMENTARY ENERGY



- For a general (non-linear) elastic system subjected to loads $P_1, P_2, \ldots P_n$.

W=Work done by loads $=\sum_{i=1}^{n}\int_0^{\Delta} P_i d\Delta_i$

- P_i can be a torque acting through a rotation (P_3=T).
- Δ_i is the deformation corresponding to the load and it can be a rotation as in the case of $\Delta_3 = \theta_3$.
- Work is the area under load-deflection curve.



Internal work=External work

(system is in equilibrium at all deflections)

$$U = \int_0^{\Delta_1} P_1 d\Delta_1 + \int_0^{\Delta_2} P_2 d\Delta_2 \ldots\ldots.. \int_0^{\Delta_n} P_n d\Delta_n$$

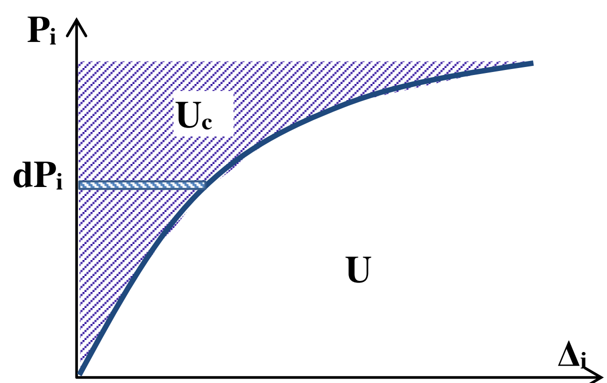
(Strain Energy)

Differentiate both sides with respect to (Δ_i) while keeping all other parameters constants (partial derivative).

$\boldsymbol{\frac{\partial U}{\partial \Delta_i} = P_i}$ Castigliano's 1st Theorem (Albert Castigliano 1847-1884)

This theory is not used very often.

<u>Complementary work (W_c) (Not real work):</u>



$$W_c = \sum_{i=1}^{n} \int_0^{P_i} \Delta_i dP_i$$

If the system remains in equilibrium at all times: $U_c = W_c$

Complementary strain energy (U_c) is the area above the curve:

$$U_c = \int_0^{P_1} \Delta_1 dP_1 + \int_0^{P_2} \Delta_2 dP_2 \ldots\ldots\ldots. + \int_0^{P_n} \Delta_n dP_n$$

Partial differentiation with respect to P_i :

$$\frac{\partial U_c}{\partial P_i} = \Delta_i$$ Castigliano's 2nd theorem (Deflection Theorem)

This theorem is very useful especially when the system is linear.

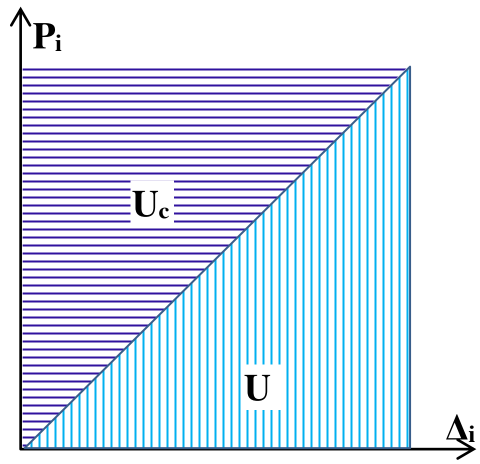
For Linear behaviour only:

Complementary strain energy=strain energy

i.e $U_c = U$

$$\frac{\partial U}{\partial P_i} = \Delta_i$$

This is form in which it can be used



System	U	Deflection or Rotation
Uniaxial loading (e.g. Truss)	$U = \sum_{i=1}^{n} \frac{P_i^2 L_i}{2EA_i}$	$\Delta_i = \frac{\partial U}{\partial F_i} = \sum_{i=1}^{n} \frac{P_i L_i}{EA_i} \frac{\partial P_i}{\partial F_i}$
Bending Moment (e.g. beam)	$U = \frac{1}{2} \int_0^L \frac{M^2}{EI} dx$	$\Delta_i = \frac{\partial U}{\partial F_i} = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial F_i} dx$
		$\theta_i = \frac{\partial U}{\partial M_i} = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial M_i} dx$

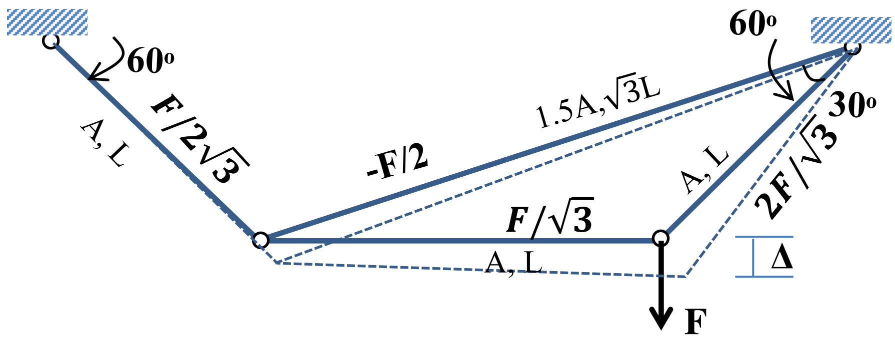
*F_i= the applied load at point (i) where the deflection (Δ_i) is required

**M_i= the applied moment at point (i) where the rotation (θ_i) is required

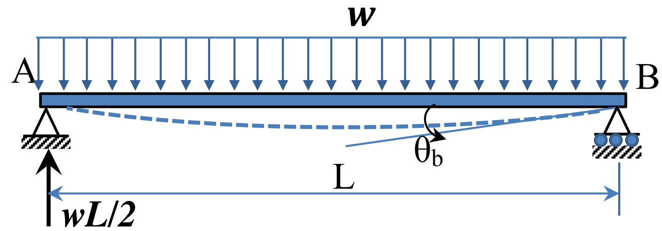
Notes:

- The deflection (Δ_i) of a structure at point (i) can be calculated by Castigliano's theorem only if a load P_i is applied at point (i). Otherwise, "dummy" load can be applied at (i) in the same direction of requested deflection.
- Dummy moment can be applied at (i) in order to calculate slope.
- In some cases, it is simpler to differentiate with respect to P_i before integration.

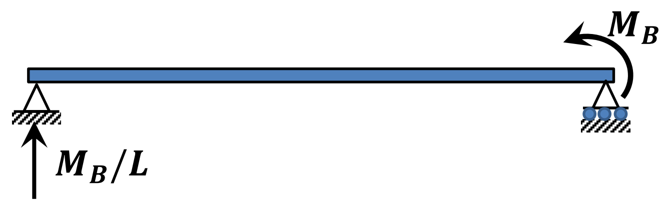
Example 8.4 Pin-jointed frame made of linear material. Determine the deflection (**Δ**) of the joint where the load P is applied.



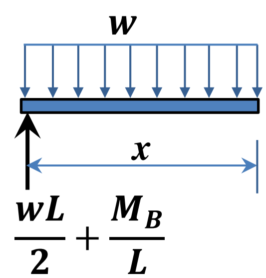
Example 8.5 Beam in bending is under uniform distributed load. Find rotation at (B)? (EI is constant).



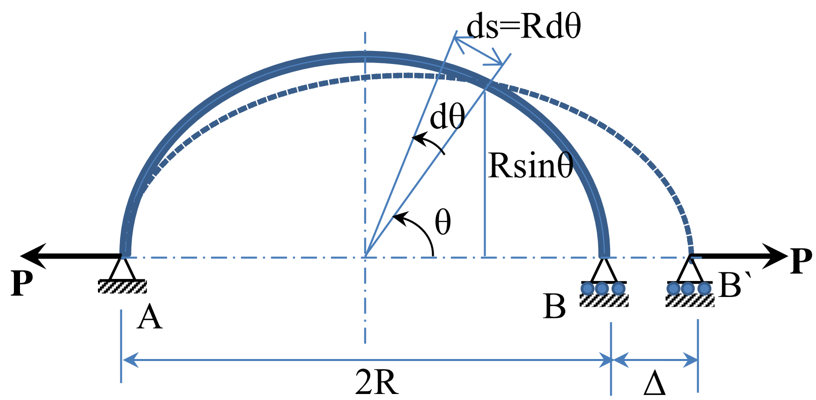
Apply temporary "dummy" moment (M_B) at B to be set to zero after Castigliano's theorem has been applied.

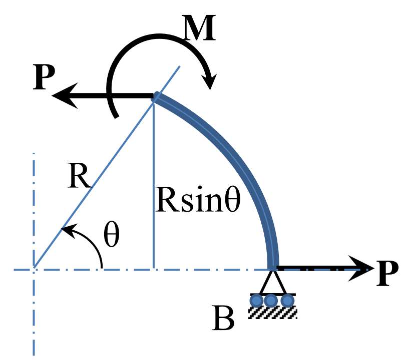


Combine BM (superposition is valid for +linear systems)

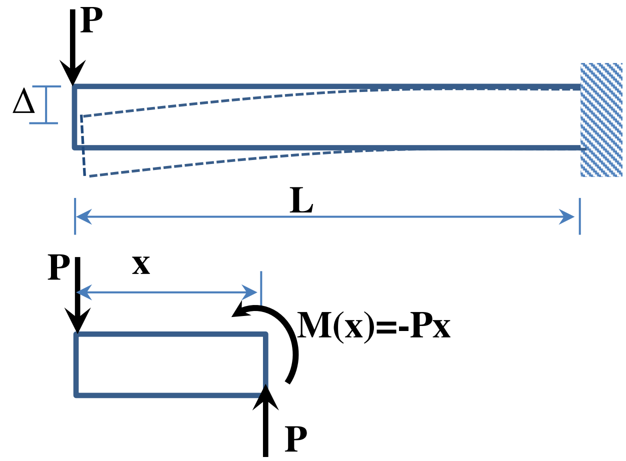


Example 8.6 A circular arch of constant cross section EI is subjected to a concentrated load at point B. Determine the deformation of B. Consider deformation due to bending only.





Example 8.7 A cantilever beam is subjected to a concentrated load at its free end. Determine the deflection of the free end due to bending. (EI is constant)



Part 9:

The Principle of Virtual Work

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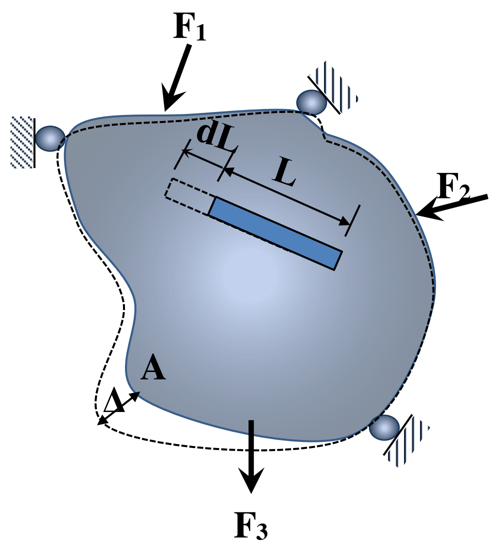
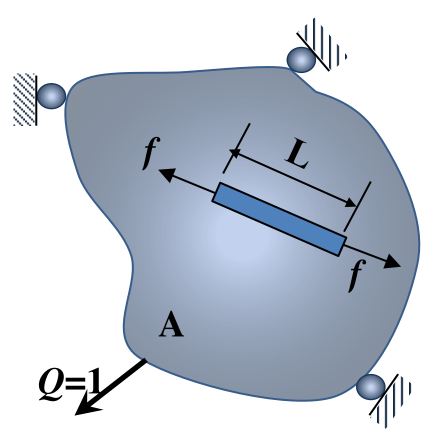
- Developed by John Bernoulli 1717, later formalized by J.C. Maxwell 1864, O. Mohr 1874.
- Based on the conservation of energy.
- Used in several mechanics applications.
- Applied to calculate displacement and slope at any points on a structure.
- Applied to find equilibrium conditions.

1. Consider a particle under the action of three forces, F1, F_2, and F_3.



2. Conservation of energy cannot be applied to calculate deformation at any points where load is not applied.
3. Therefore, to calculate the displacement (Δ) of point A, an imaginary (virtual) load is applied at A in the direction of the displacement.



4. This external virtual load results in internal virtual load *f* in an element of the body.
5. In order for the system to be in equilibrium: External virtual work must equal internal work i.e.

External Virtual work (EVW)=Internal Virtual work (IVW)

(**Virtual external force × real external displacement= virtual internal load × real internal displacement**)

$$1.\Delta = \sum f.dL$$

1=external virtual load (Q)
Δ=external displacement caused by actual load
f= internal virtual load
dL=internal real displacement caused by internal real load

Principle of Virtual Work:
"If a structure is in equilibrium under the action of a set of external forces and is subjected to a set of displacements compatible with the constraints of the structure, the total work done by the external and internal forces during the displacements must be zero."

Notes:

- The forces can be in a form of moment to calculate the deformation in a form of rotation.
- The set of forces need not be related to the set of displacements as cause-and-effect and either set can be real while the other is imaginary.

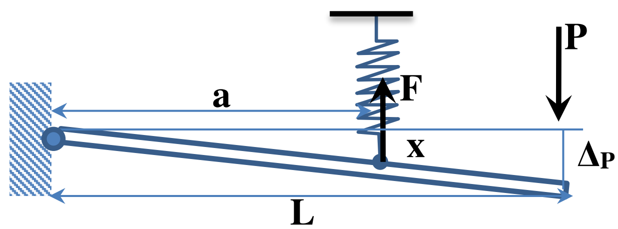
Beam Example:

Self-equilibrated forces are applied to the structure in its undeflected position. Then the real deflections due to the real load are allowed to take place. These real deflections are (x) and (**Δ)** at distance (a) and (c) from the hinge, respectively)

Q is virtual external force at a distance (c) from the hinge.

Apply: (**Virtual <u>external</u> force *Q* × real <u>external</u> displacement Δ = virtual <u>internal</u> load *f* × real <u>internal</u> displacement *x***)

QΔ = fx This is a compatibility condition



Force system + Real displacements

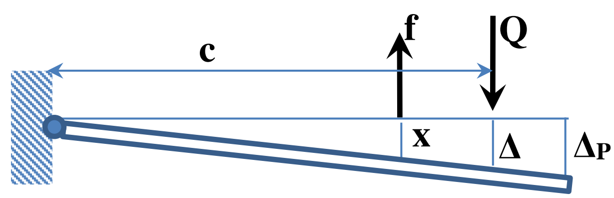
For a linear system

$F=Kx$ (where K=spring stiffness)

Or

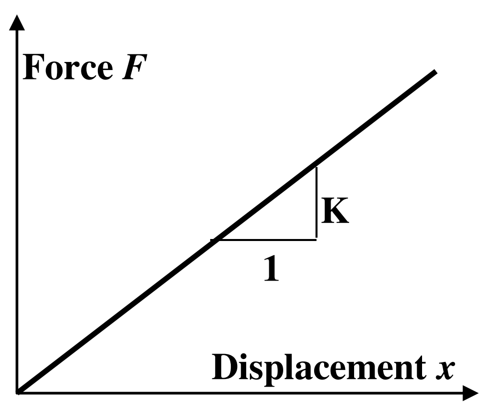
$x=F/K$ (deflection due to real load)

$$Q\Delta=\frac{f.F}{K}$$



Virtual forces f and Q satisfy ***(fa=Qc)***

If Q is a unit load, and f corresponds to this value of Q:



$$\Delta=\frac{\boldsymbol{f.F}}{\boldsymbol{K}}$$

f = the internal virtual spring force due to unit virtual applied load.

F = the internal real spring force due to real applied load.

Δ = the required real deflection due to real loads.

The more general form is

$$\Delta= f.x$$

Where x is a real displacement caused by:

a) External real forces
b) Temperature changes

c) Misfit of component parts
d) Support settlements etc.

This general form is valid also for nonlinear system.

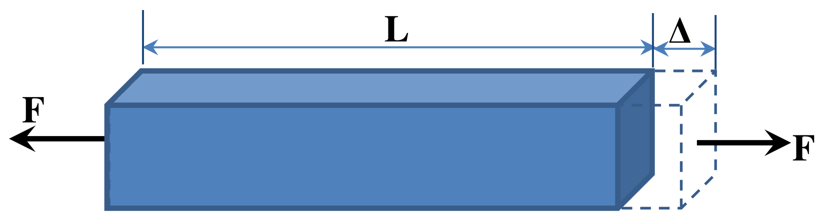
Note: If a *rotation* at a point on a linear structure is required, Δ is an angle and Q is a torque (of unit value) applied in the direction of Δ.

TENSILE OR COMPRESSION STRUCTURAL MEMBERS:

Since $\Delta = \frac{FL}{EA}$ (real Δ and F)

Let *f*= virtual member force in equilibrium with external virtual applied load:

$$\mathbf{1.\Delta = \frac{F.f.L}{AE}}$$



To determine the displacement of truss joints:

$$\mathbf{1.\Delta = \sum_{i=1}^{n} \frac{F_i.f_i.L_i}{A_i E_i}}$$

Where,

F_i= real internal force in truss member caused by applied real loads

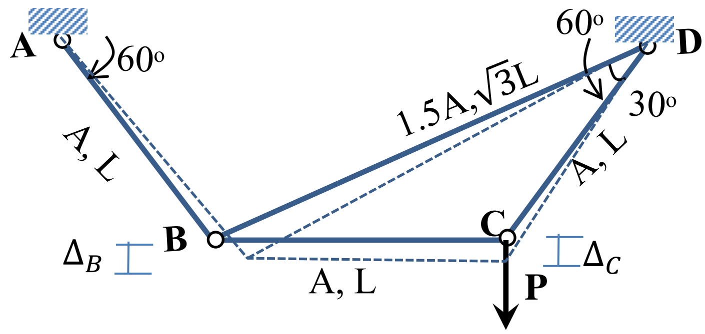
f_i= virtual internal force in truss member caused by external virtual load

L_i=length of a member

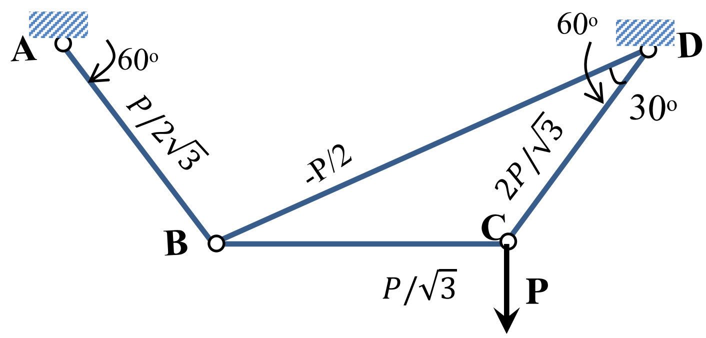
A_i= area of a member

E_i= modulus of elasticity of a member

Example 9.1 Deflection of Trusses_ load P is applied to 4-bars truss with given areas and lengths. Find vertical deflections at B and C.



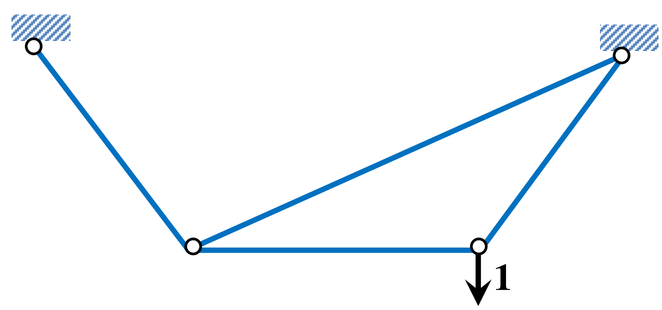
1) Calculate all internal members forces caused by the applied load P



2) To find Δ_C:

- Apply unit vertical force at C
- Calculate the internal virtual member forces (f_{Ci}) caused by the applied unit load at C.

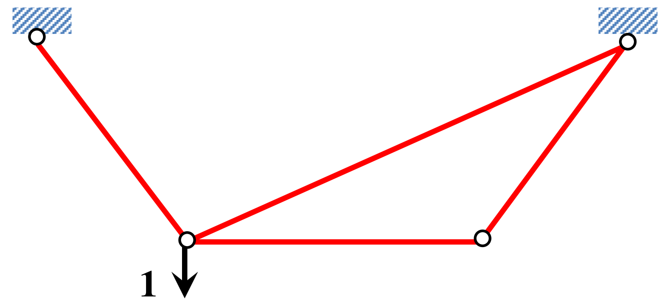
$$\Delta_c = \sum_{i=1}^{n} \frac{F_i . f_{Ci} . L_i}{A_i E_i}$$



3) To find Δ_B:

- Apply unit vertical force at B
- Calculate the internal virtual member forces (f_{Bi}) caused by the applied unit load at B.

$$\Delta_B = \sum_{i=1}^{n} \frac{F_i . f_{Bi} . L_i}{A_i E_i}$$



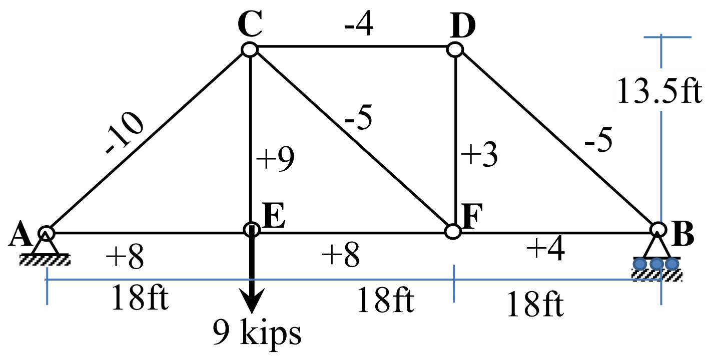
Member	F_i	f_{Ci}	f_{Bi}	L_i	A_i	$\frac{F_i . f_{Ci} . L_i}{A_i E_i}$	$\frac{F_i . f_{Bi} . L_i}{A_i E_i}$
AB	$\frac{1}{2\sqrt{3}}P$			L	A		
BC	$\frac{1}{\sqrt{3}}P$			L	A		
CD	$\frac{2}{\sqrt{3}}P$			L	A		
BD	$-\frac{1}{2}P$			$\sqrt{3}L$	1.5A		
					Sum		

Example 9.2 A concentrated load of 9 kips is applied at point E in the truss. The internal forces are calculated using statics. E=30x10^3 kips/in^2

Areas: Compression members =10 in^2

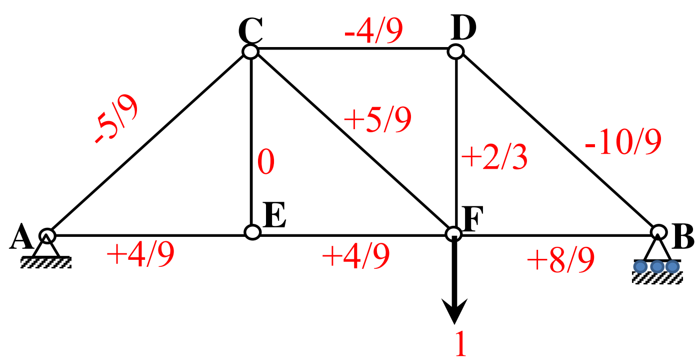
Tension members =5 in^2

Find the deflection at F.



Solution:

Apply unit load at F



Member	F_i (kips)	f_i	L_i (in)	A_i (in^2)	$\frac{F_i f_i L_i}{A_i}$
AC	-10		270	10	
CD	-4		216	10	
BD	-5		270	10	
CE	+9				
CF	-5		270	10	
DF	+3		162	5	
AE	+8		216	5	
EF	+8		216	5	
FB	+4		216	5	
				Sum	

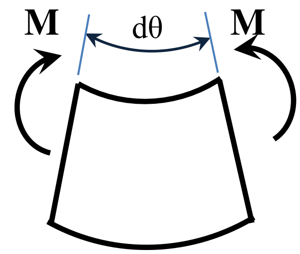
By virtual work

$$1.\Delta_F = \sum \frac{F_i f_i L_i}{A_i E} =$$

BENDING MOMENT OF BEAMS:

$$d\theta = \frac{M}{EI}dx$$

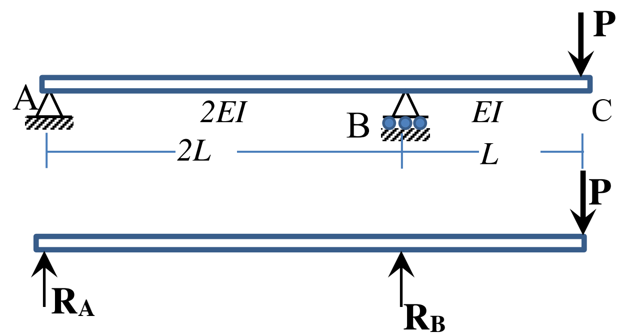
Let m= virtual internal moment due to virtual external load (I.V.W.)



$$d(I.V.W) = \frac{M.m}{EI}dx \quad \text{for an element}$$

$$\boldsymbol{Total(I.V.W) = \int_0^L \frac{M.m}{EI}dx}$$

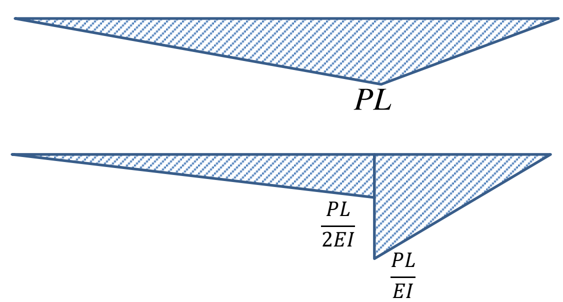
Example 9.3 Find the rotation at A due to the concentrated load P applied at C.



I.V.W=E.V.W

$$\int \frac{M.m}{EI}dx = 1.\theta_A$$

- Since, the rotation is required at point A, a virtual unit moment is required at A.
- Draw the bending moment for real loads.



$$\sum M(A) = 0$$

$$R_B(2L) = P(3L) \rightarrow R_B = \frac{3}{2}P$$

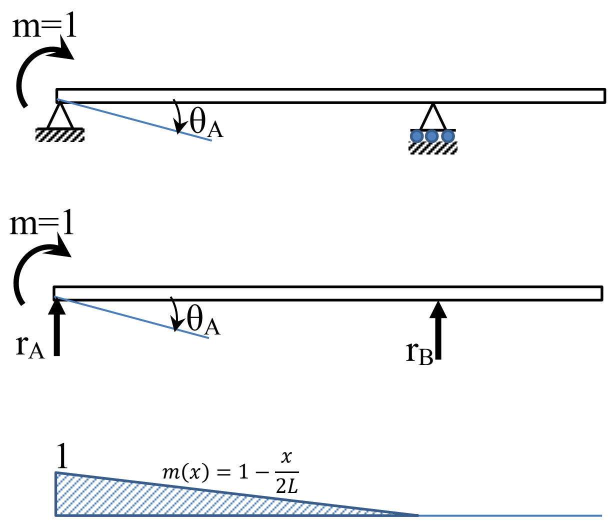
$$\sum F(y) = 0 \quad R_B + R_A = P$$

$$R_A = -\frac{P}{2}$$

- Draw the bending moment for virtual load to identify the regions of no virtual moment to simplify the virtual load calculations.

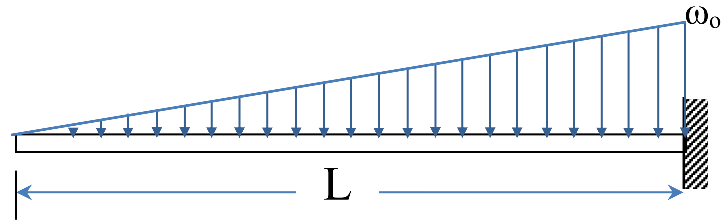
$\sum m(B) = 0 \quad r_A(2L) + 1 = 0 \;\rightarrow\; r_A = \frac{-1}{2L}$

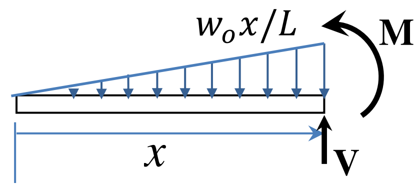
$$\sum F(y) = 0 \quad r_B + r_A = 0 \;\rightarrow\; r_b = \frac{1}{2L}$$

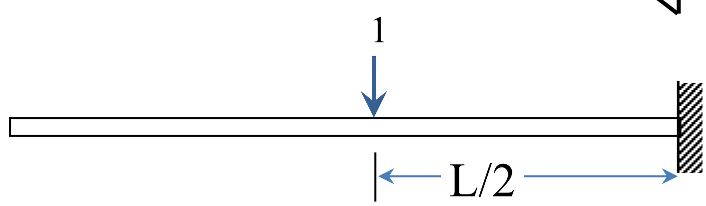


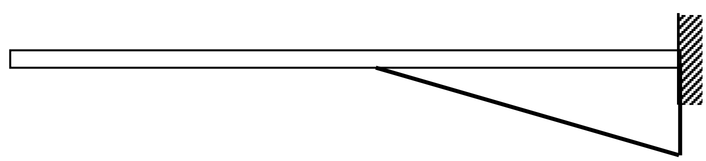
- In this example, the effect of virtual bending moment is limited to the segment AB.

Example9.4: Find the deflection at the mid span of the cantilever beam. EI is constant.









Torsion of Shafts

Let t= internal virtual torque due to external load

$$d(I.V.W) = \frac{T.t.dx}{GJ}$$

$$Total(I.V.W) = \int_0^L \frac{T.t}{GJ} dx$$

And for a uniform shaft:

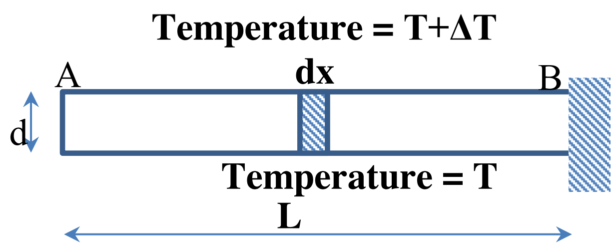
$$\boldsymbol{Total(I.V.W) = \frac{T.t.L}{GJ}}$$

System	Load	Strain Energy U	Internal Virtual Wok
Uniaxial loading (e.g. Rod)	$\boldsymbol{F}$	$\boldsymbol{U = \frac{F^2L}{2EA}}$	$\boldsymbol{\frac{F.f.L}{AE}}$
Bending Moment (e.g. beam)	$\boldsymbol{M}$	$\boldsymbol{U = \frac{1}{2}\int_0^L \frac{M^2}{EI} dx}$	$\boldsymbol{\int_0^L \frac{M.m}{EI} dx}$
Torque (e.g. Shaft)	$\boldsymbol{T}$	$\boldsymbol{U = \frac{T^2L}{2JG}}$	$\boldsymbol{\frac{T.t.L}{GJ}}$

OTHER APPLICATIONS OF THE VIRTUAL WORK METHOD TO FIND DEFLECTIONS

1) <u>Deflection due to temperature</u>:

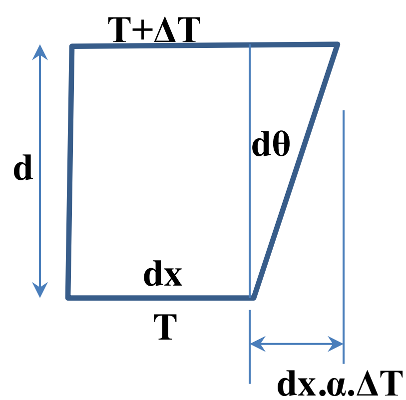
Example 9.5 Determine the deflection due to temperature for the cantilever beams. Temperature at the top of the beam is **T+ΔT, and the temperature inside is T** Coefficient of linear expansion = α

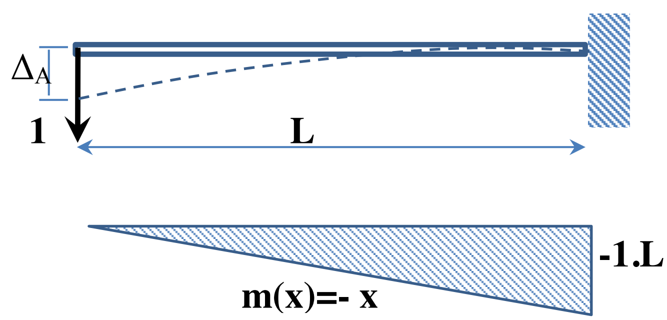


<u>Deformation due to temperature gradient</u>

Linear temperature change through depth of beam: (Real element deformation)

$$d\theta = \tan^{-1}\frac{\alpha.\Delta T.dx}{d} \cong \frac{\alpha.\Delta T.dx}{d}$$





(Internal Virtual Bending Moment)

2) Discrepancies in member lengths:

Example 9.6: Suppose some members of a truss were made too long or too short. What is the resulting joint displacement at A.

+ ve sign → longer than intended

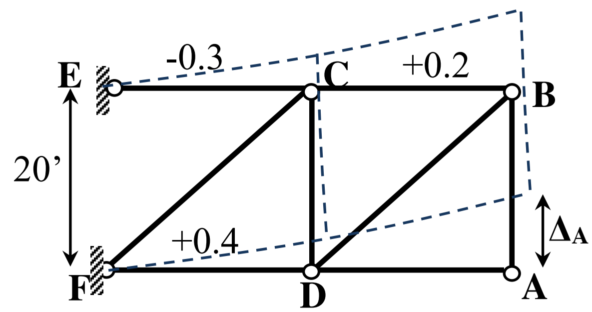
- ve sign → shorter than intended

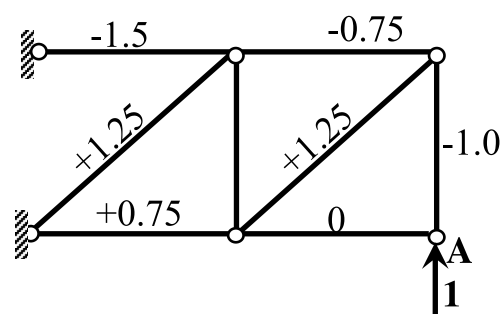
CB is *0.2 in* longer than specified

FD is *0.4 in* longer than specified

EC is *0.3 in* shorter than specified

Δ_A is the real displacements

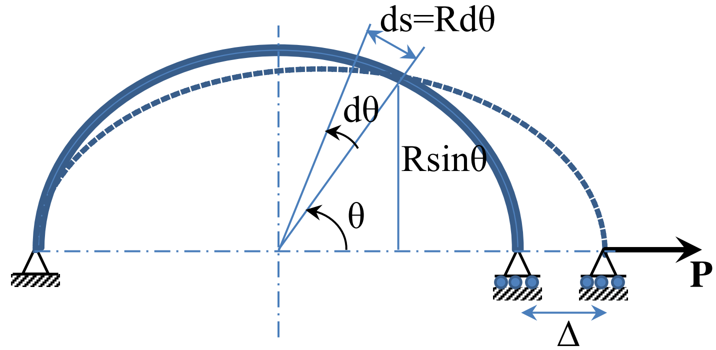




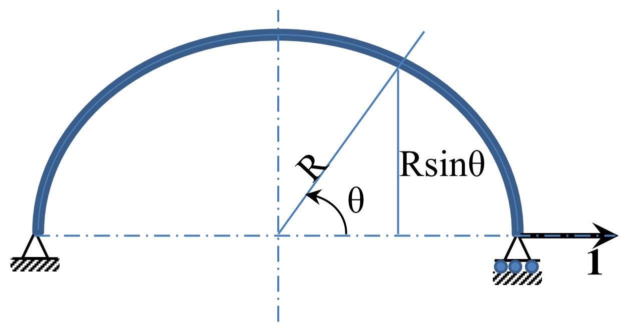
3) Arch Deformation:

Example 9.7 Consider bending deformations only (EI is constant)

$$M(\theta) = PR\sin(\theta)$$



$$m(\theta) = 1.R\sin(\theta)$$



METHOD OF VIRTUAL WORK APPLIED TO STATICALLY INDETERMINATE STRUCTURES:

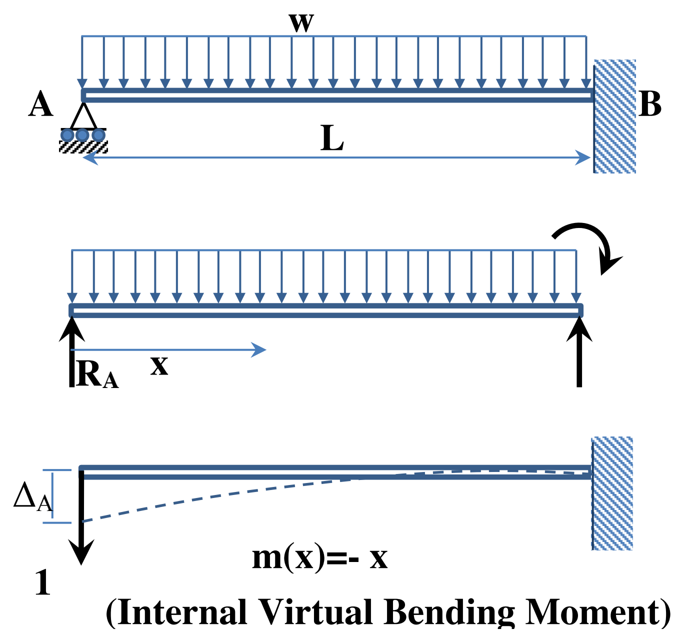
Redundancy can be either a support reaction or an internal force.

A statically determinate structure is formed by removing the redundancy:

- Take away the extra support; OR
- Cut the redundant member.

Then the primary structure (the statically determinate structure) is analysed bearing in mind the compatibility situation.

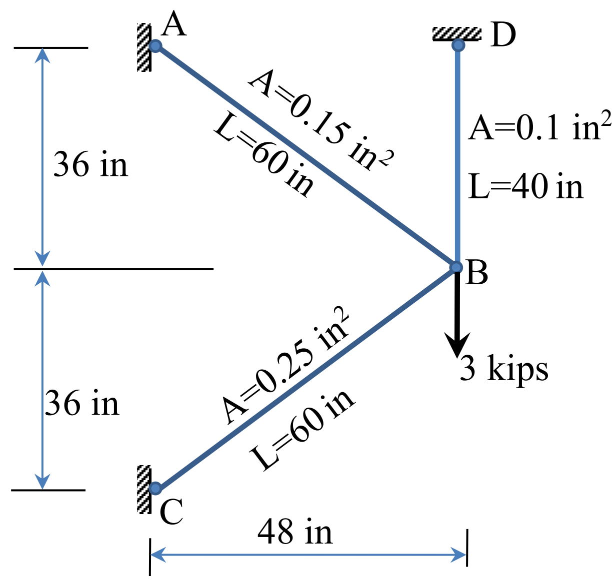
Example 9.8 Find the reaction at A (R_A)?



Example 9.9

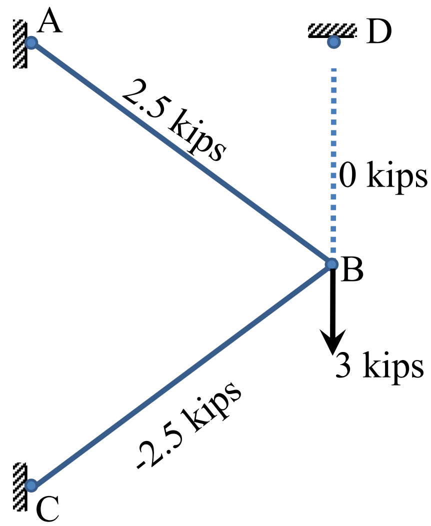
Find the forces in the pin-jointed steel structure.

$E = 30 \times 10^6\ psi$



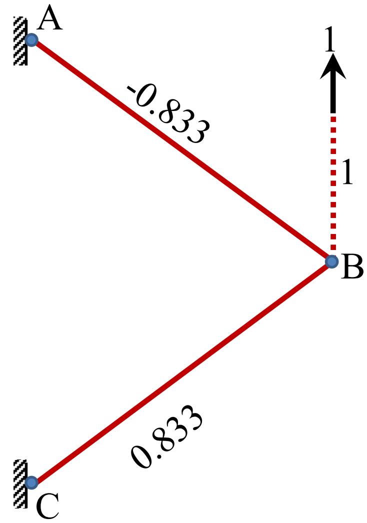
To make the truss a statically determinate structure, **<u>cut one of the members</u>** (say BD) and calculate the internal forces.

The statically determinate structure with actual forces (F_i) caused by actual applied force. Each force causes elongation $\frac{F_i L_i}{EA}$ in each member.



In the statically determinate structure, Δ_d must be found by applying virtual vertical unit force at D. This unit tension causes forces f_i.

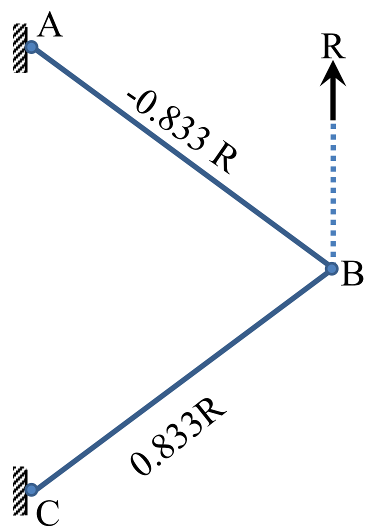
Member	F_i	f_i	L_i (in)	A_i (in^2)	$\frac{F_i f_i L_i}{A_i}$
AB	2.5	−0.833	60	0.15	
BC	-2.5	0.833	60	0.25	
BD	0	1	40	0.1	



$$\Delta_d = \sum \frac{F_i f_i L_i}{EA_i} =$$

But the actual displacement Δ at D is zero in the statically indeterminate structure (provided there is no initial lack of fit (i.e. BD bar is not too long or short). Therefore, a force must be applied at D to return it to its original location.

Assume the actual force in BD is R. This redundant force gives rise to internal forces R.f in the remaining members. The additional elongations in the members due to R are given by $\frac{(Rf_i)L_i}{A_iE}$



Apply virtual work method:

External virtual work=Internal virtual work

$$1.\Delta = \sum \frac{F_i f_i L_i}{A_i E} + R \sum \frac{f_i f_i L_i}{A_i E}$$

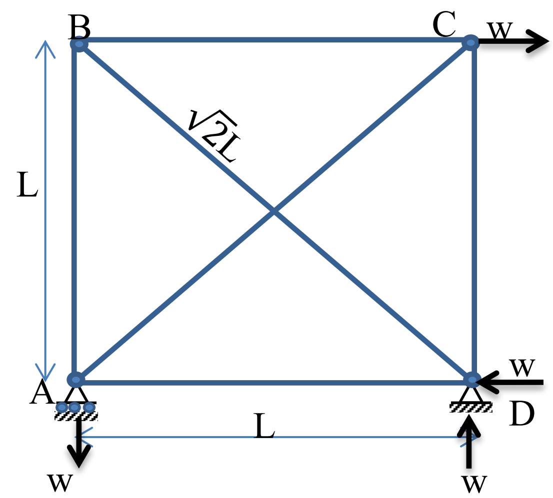
Then

$$R = -\sum \frac{FfL}{AE} / \sum \frac{f^2 L}{AE}$$

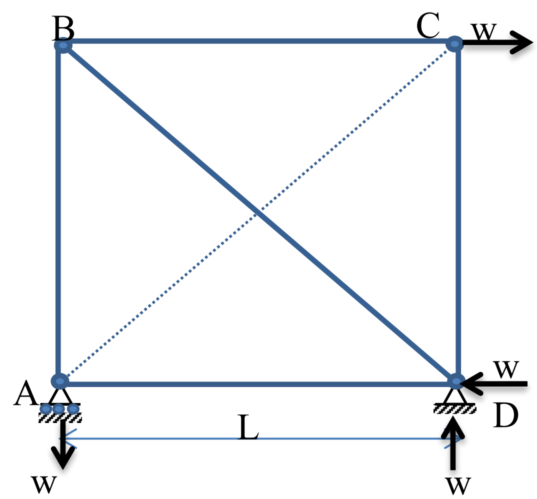
Member	f_i	L_i (in)	A_i (in^2)	$\frac{f_i f_i L_i}{A_i}$
AB	−0.833	60	0.15	
BC	0.833	60	0.25	
BD	1	40	0.1	

<u>**Example 9.10**: Plane trusses with a single redundant</u>

AE is constant for all members.

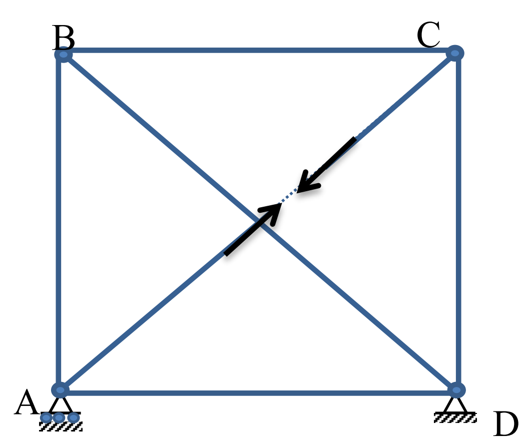


To make the truss as a statically determinate structure**, cut one of the members (say AC**).

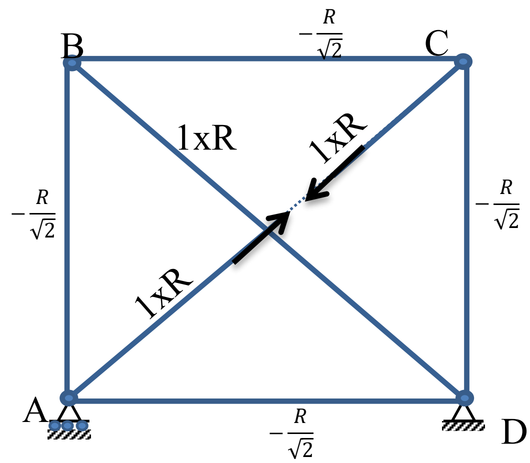


Calculate actual forces (F_i) caused by actual applied force in the statically determinate structure. Each force causes elongation $\frac{F_i L_i}{EA}$ in each member.

Apply a unit tension load in the cut member in the statically determinate structure. This unit tension causes forces f_i.



Assume the actual force in AC is R. This redundant force gives rise to internal forces R.f in the remaining members. The additional elongations in the members due to R are given by $\frac{RfL}{AE}$



Apply virtual work method:

External virtual work=Internal virtual work

$$1.\Delta = \sum \frac{FfL}{AE} + R\sum \frac{f.fL}{AE}$$

Δ overlaps at the cut in bar AC

But consistency of deformation requires that Δ=0 (provided there is no initial lack of fit (i.e. AC bar is not too long or short). Then

$R = \sum \frac{FfL}{AE} / \sum \frac{f^2L}{AE}$

Member	F	f	L (in)	$F.f.L$	$f^2.L$
AB			L		
BC			L		
CD			L		
DA			L		
AC			$\sqrt{2}$L		
BD			$\sqrt{2}$L		

$R =$

Based on the calculated R, all other forces can be calculated.

Part 10:

Frames

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- Frames are structural elements composed of beams, columns, and beam-columns.
- Unlike trusses where all members are hinged together, frames are generally connected by rigid joints. Therefore, frames are commonly known as "rigid frames".
- Rigid joints are moment-resisting joints that are constructed using riveting, or welding.
- Frames are subjected to bending moments, shear and axial forces.

STABILITY AND DETERMINACY:

The degree of stability and determinacy of frames are determined by comparing the number of unknowns with the number of available independent equations.

The number of unknowns:

Each member of a frame is subjected to an axial force, shear, and bending moment. For a frame with a number of reactions (***r***) and a number of members (***b***), the total number of unknowns is: $3b+r$

Available equations:

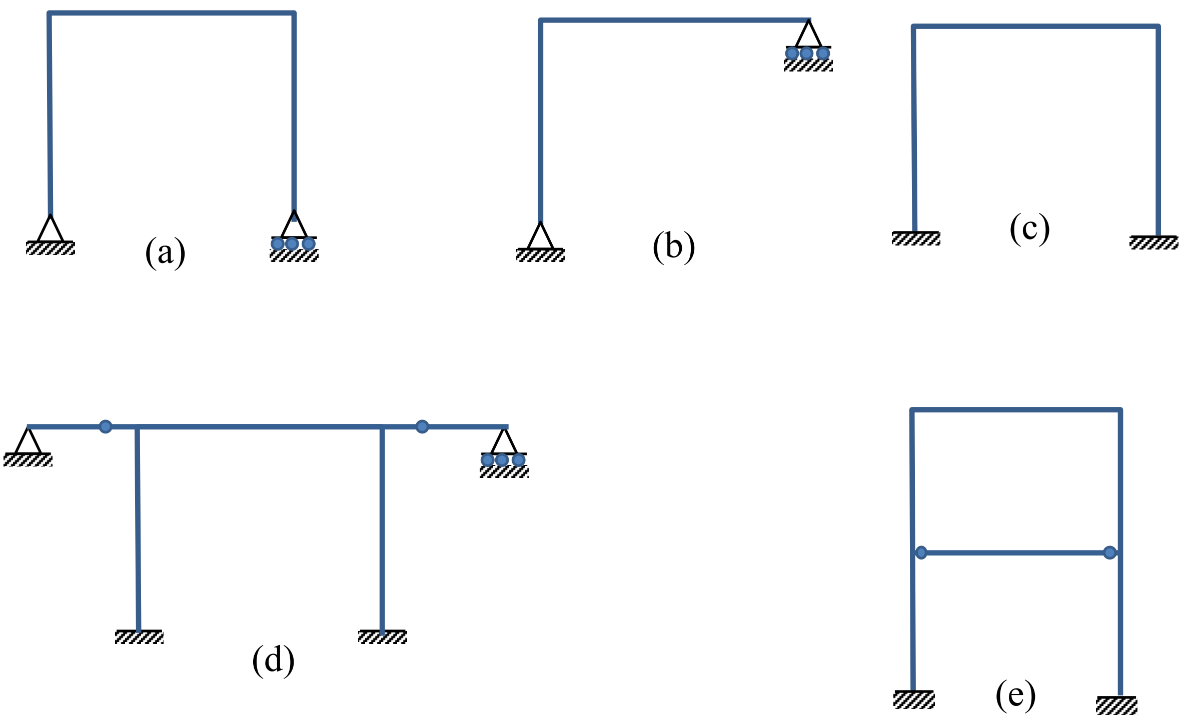
For a number of joints (n), there are three static equilibrium equations for each joint (total equations =3n). In some frames, hinges are introduced providing (s) special equations. Therefore, the number of available equations is: $3n+s$

If $3n+s > 3b+r$, **the frame is unstable**

If $3n+s = 3b+r$, **the frame is statically determinate**

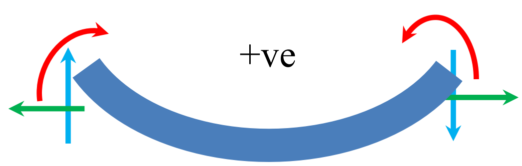
If $3n+s < 3b+r$, **the frame is statically indeterminate**

b = number of members
r =number of reactions
n = number of joints
s = number of special equations introduced by hinges



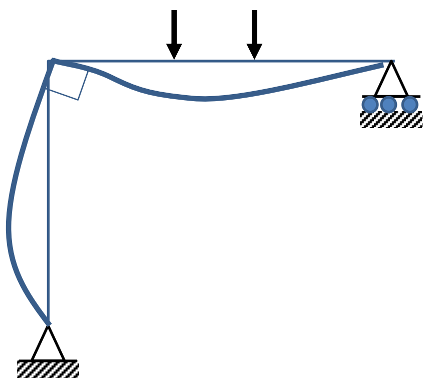
Frame	b	r	n	s	3b+r	3n+s	Determinacy
a	3	3	4	0	12	12	
b	2	3	3	0	9	9	
c	3	6	4	0	15	12	
d	5	9	6	2	24	20	
e	6	6	6	2	24	20	

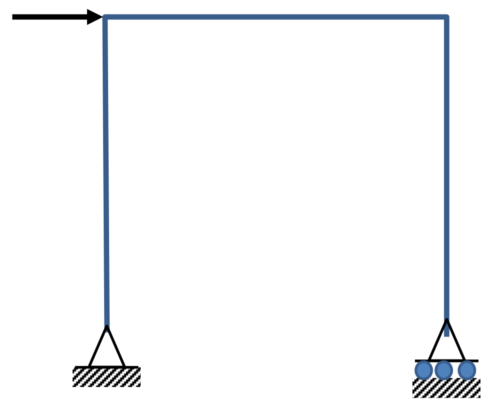
Sign conventions:

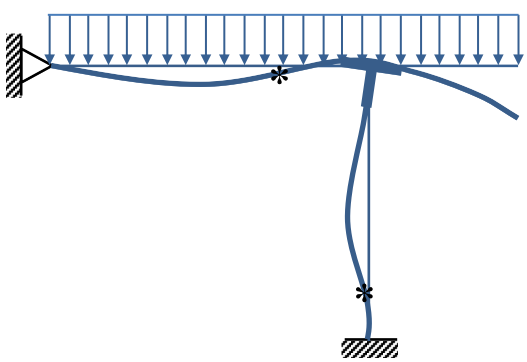


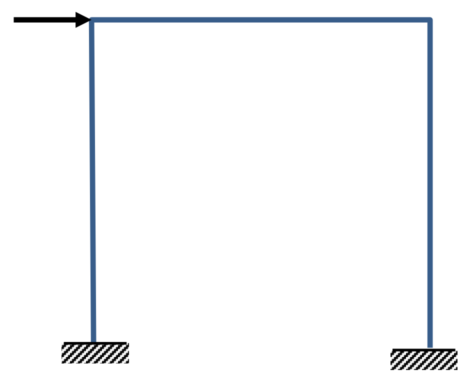
Deflected Shapes:

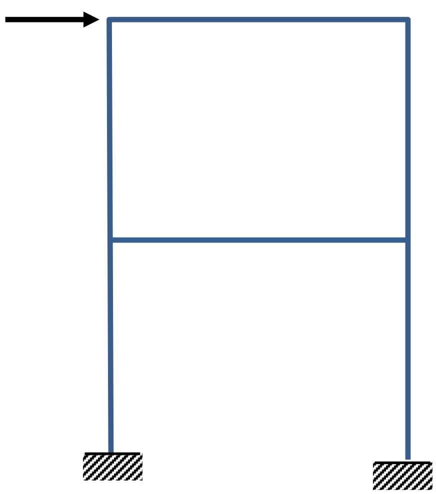
- Original angle of a rigid joint must be preserved
- Original length of each member is unchanged (little axial deformation in compare to flexural deformation)
- Deflected shape must satisfy boundary conditions
- Curvature must be consistent with the moment.

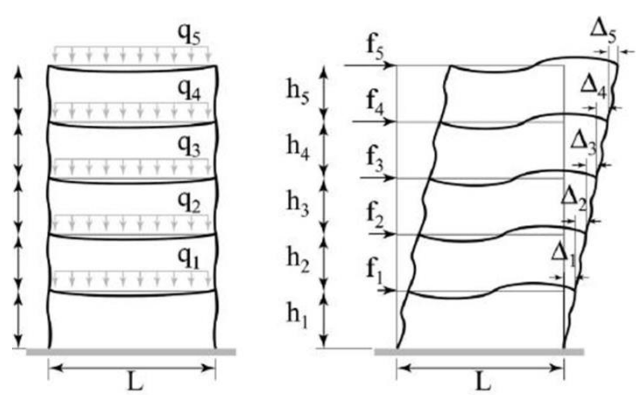






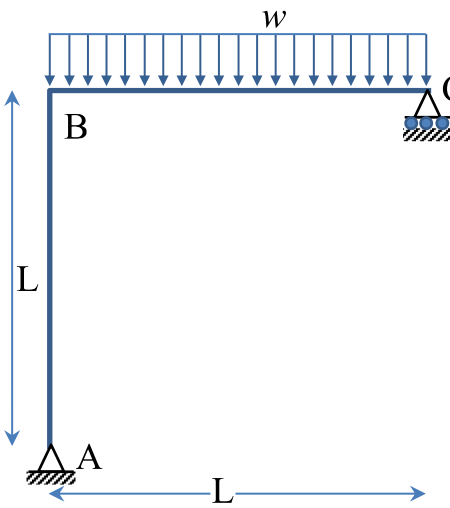


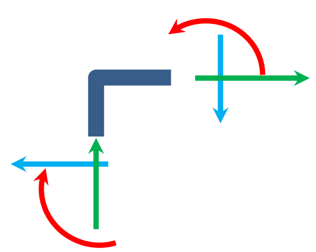


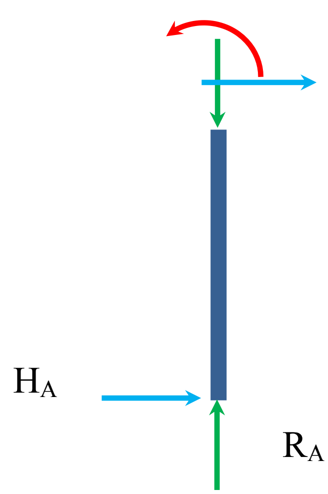


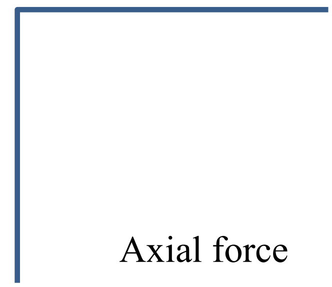
(a) Gravity loads (b) Lateral forces

Example 10.1: Draw the Axial Force, SF and BM diagrams

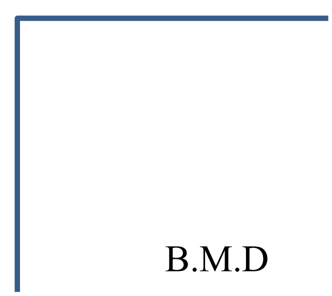




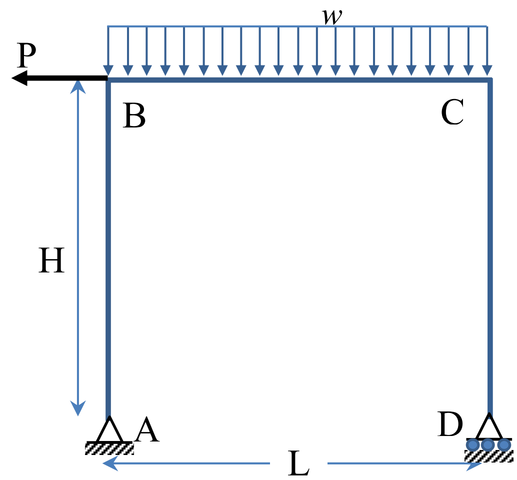


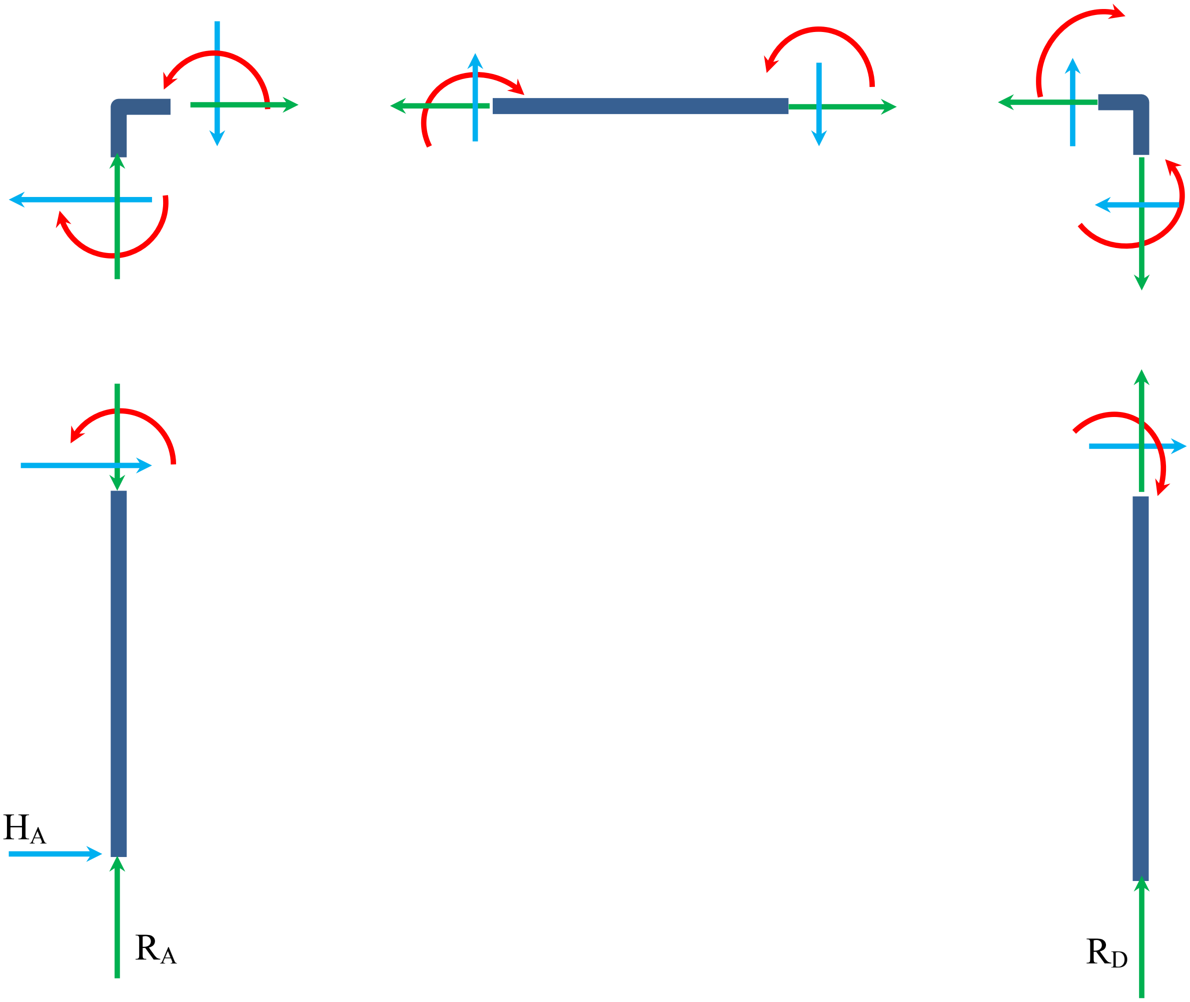

Axial force

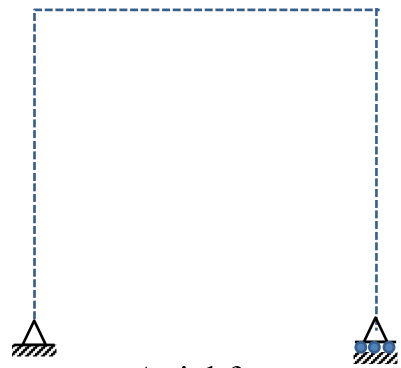

S.F.D


B.M.D

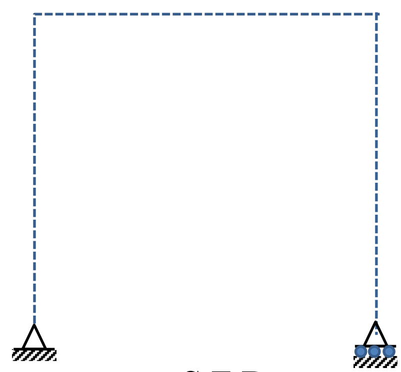
Example 10.2 Draw the Axial Force, SF and BM diagrams



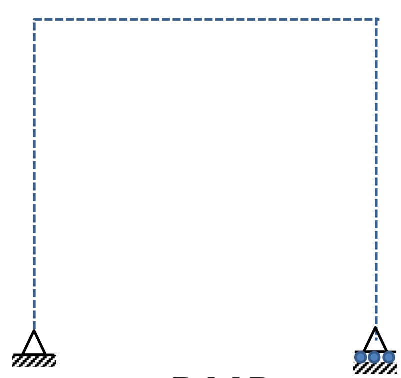




Axial forces

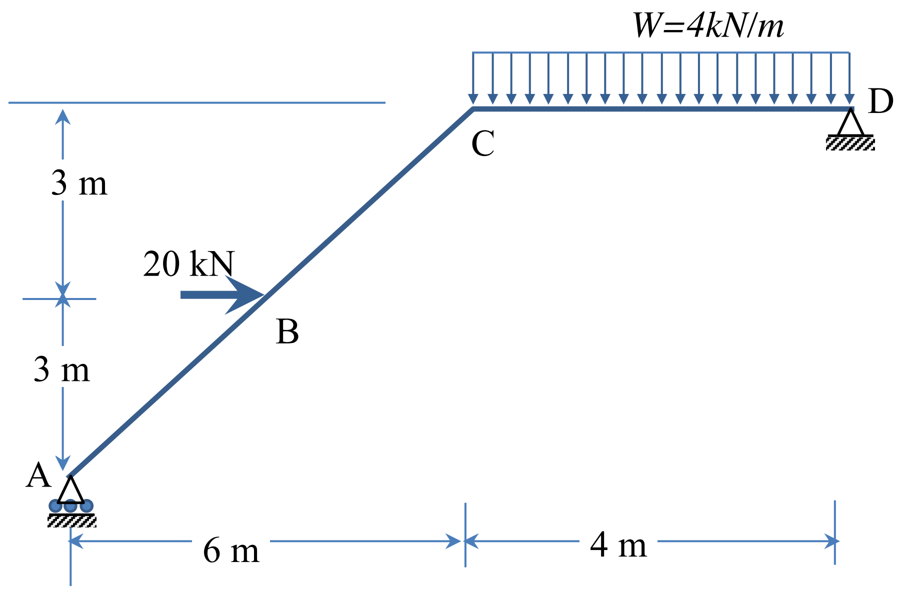


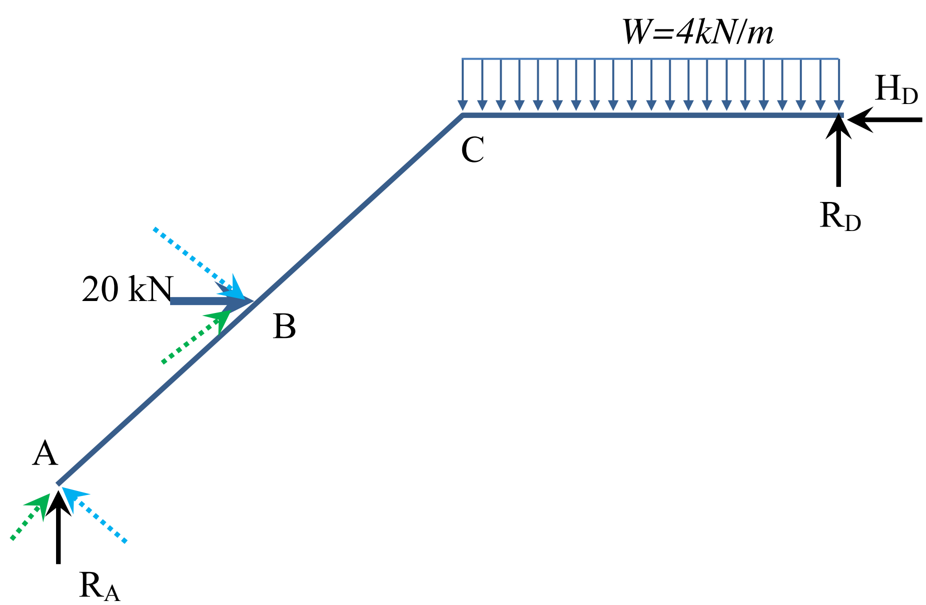
S.F.D

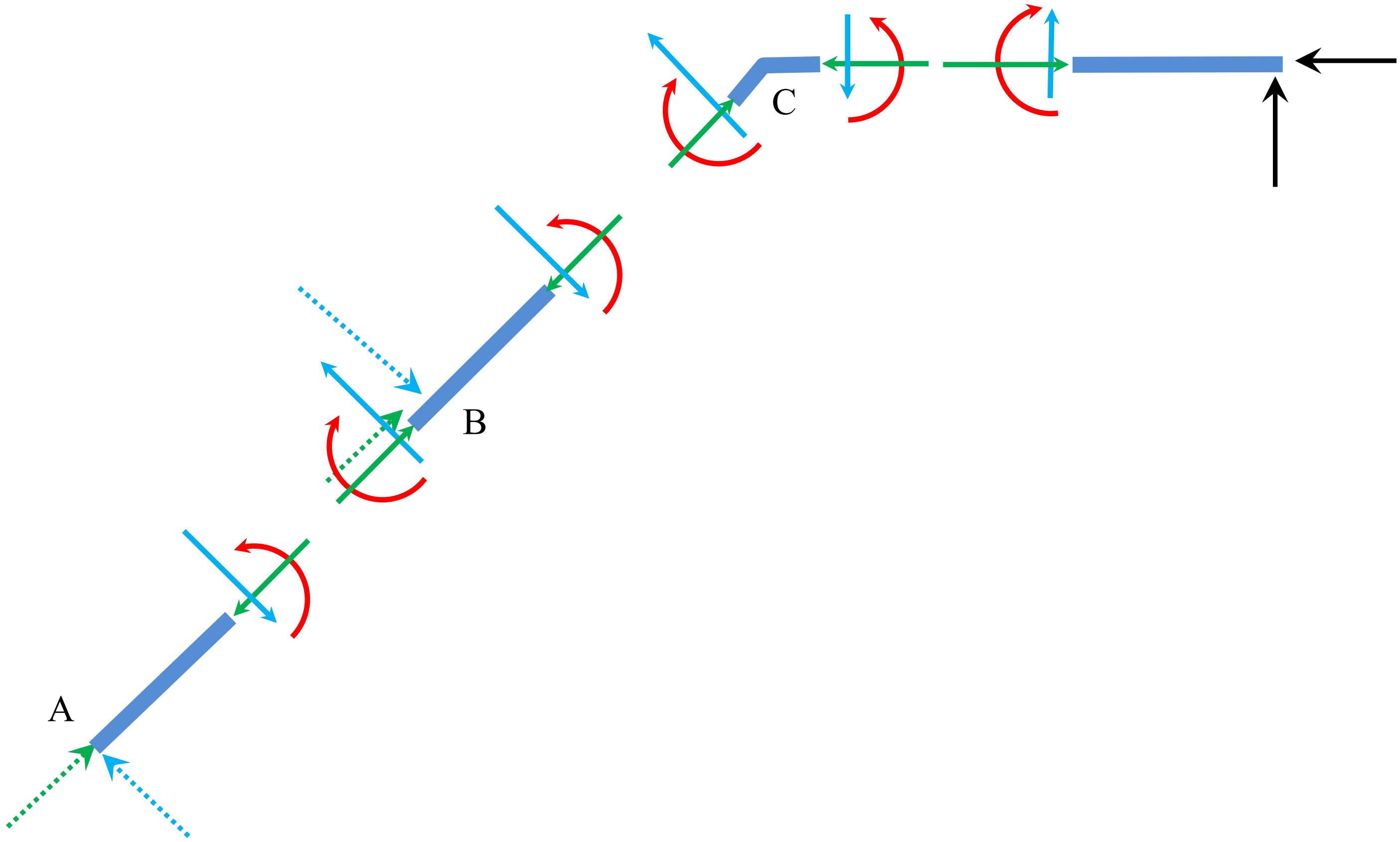


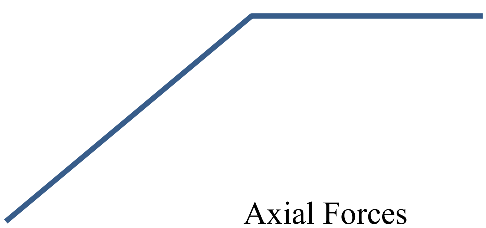
B.M.D

Example 10.3 Draw the Axial Force, SF and BM diagrams

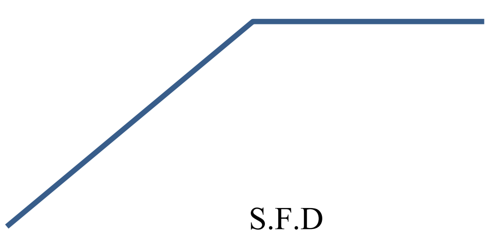




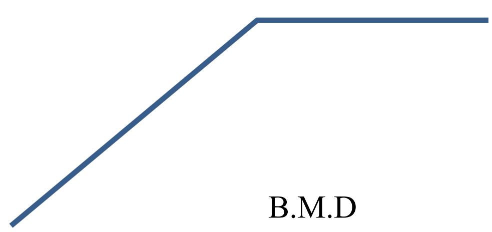




Axial Forces

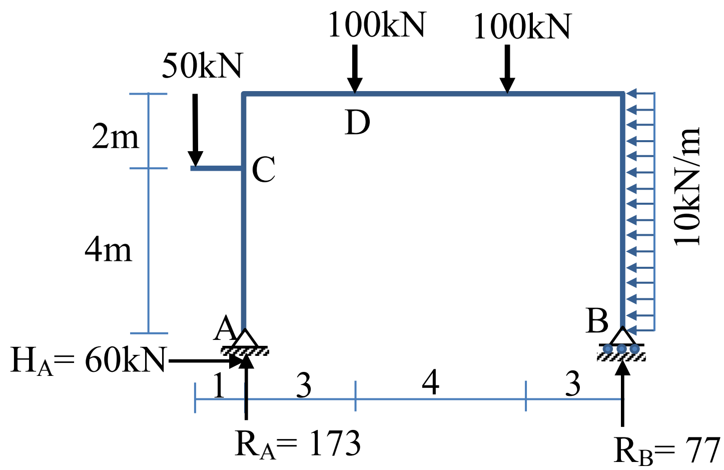


S.F.D



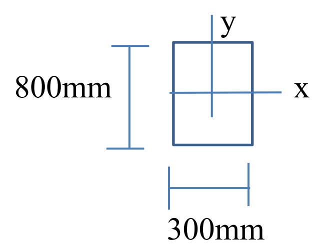
B.M.D

Example 10.4 Calculate normal stresses at point D and just below point C. The dimensions of the cross sections at C and D are (300 ×800 mm) and (1000 ×300 mm), respectively as shown.

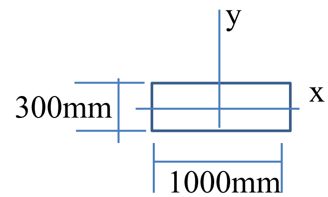


Reactions:

H_A=60 kN, R_A=173 kN, R_B=77 kN



Section D



Section C

CivE 205 – Solid Mechanics II

Part 11:

Arches

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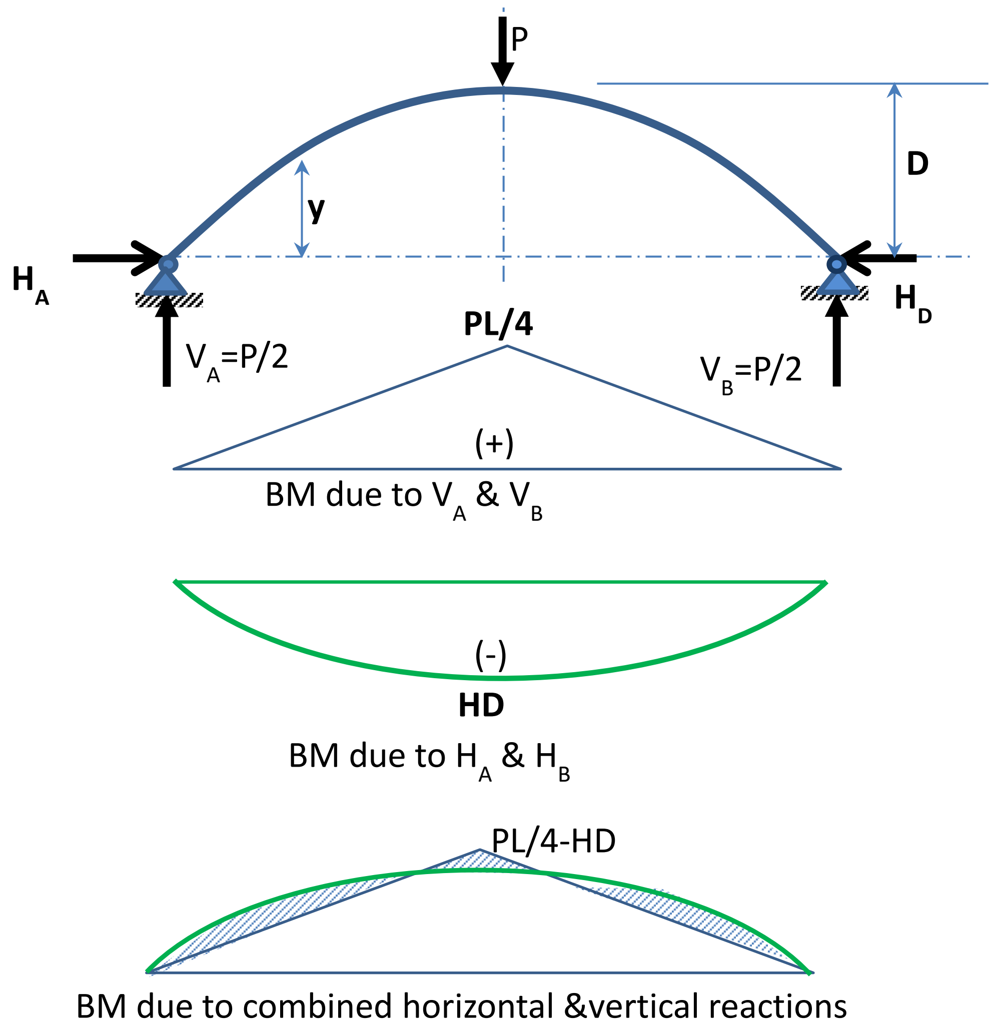
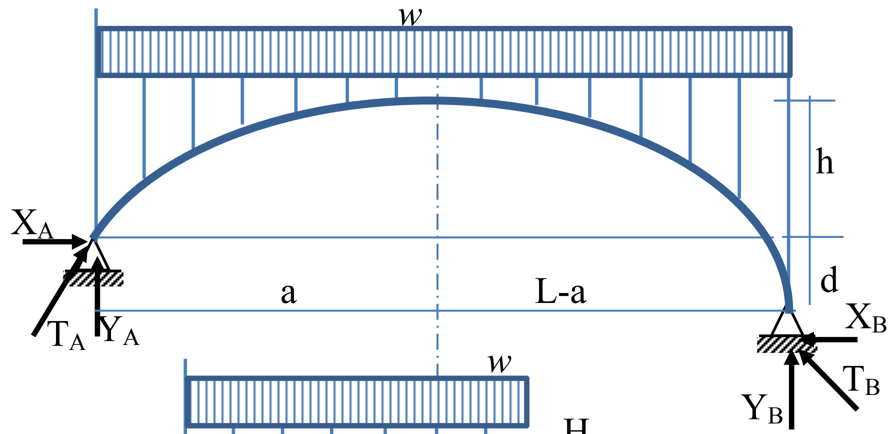
- An arch is a curved beam usually loaded vertically and supported at two ends.
- To prevent support movement, ends must be pinned or fixed. This introduces horizontal support reactions in addition to vertical reactions. Both cause bending moments, but of opposite sign.
- The bending moments are much reduced from simple beams due to "arch action".
- The horizontal thrust is of primary importance in arch.



What shape must the arch have to experience axial thrust only?

Consider an arch subjected to a uniformly distributed load,
Apex of arch(y)=h @x=a



At apex x=a, y=h

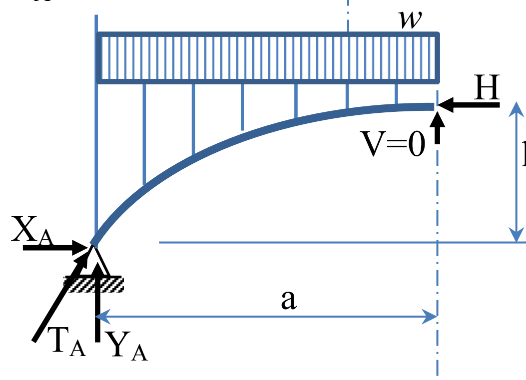
$\sum F_x = X_A - H = 0 \qquad X_A = H$

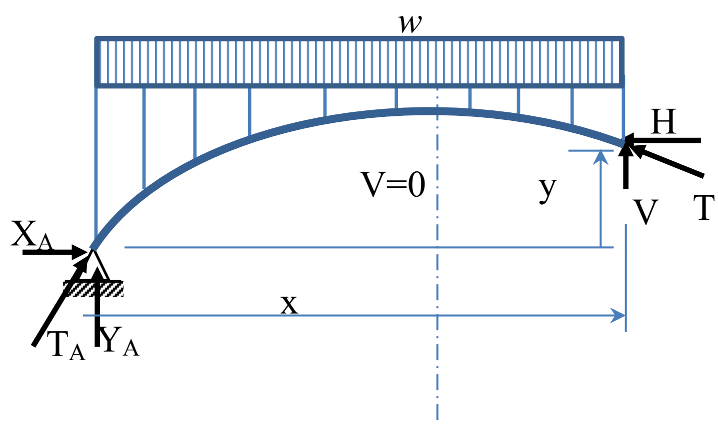
$\sum F_y = Y_A + V - wa = 0$

But V=0 → $\boldsymbol{Y_A = wa}$

$\sum M_A = H(h) - \frac{wa^2}{2} = 0$

$\boldsymbol{X_A = H = \frac{wa^2}{2h}}$



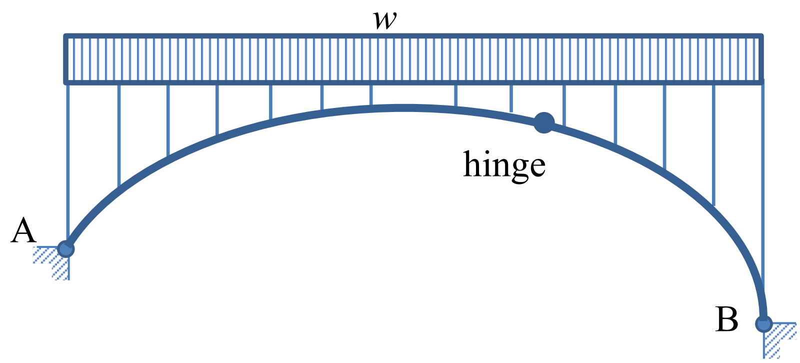


Since there is zero moment everywhere, we can introduce a hinge at any point in the arch without any effect. Note: the parabola is the thrust (or pressure line) for a uniformly distributed load.

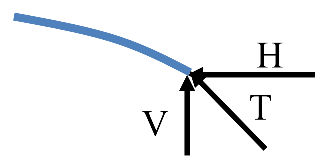
Three Hinged Arch:

Parabolic arch: Internal forces same as above.

No bending moment anywhere.



Thrust at any point in the arch:



$$T = \frac{wa^2}{2h}\sqrt{\frac{4h^2}{a^4}(x-a)^2+1}$$

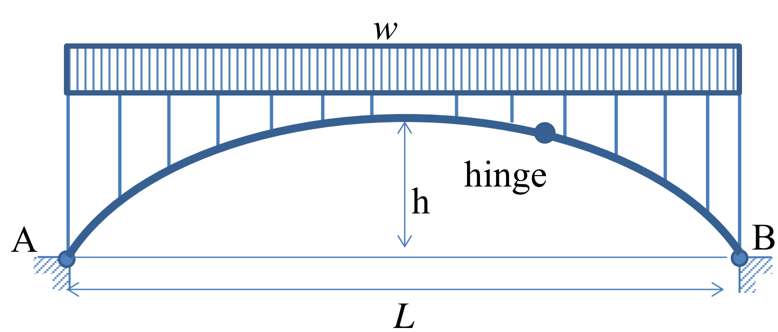
Since h is constant, the maximum thrust occurs at the point of maximum slope in the arch (i.e. at B)

$$T_{max} = \frac{wa^2}{2h}\sqrt{\frac{4h^2}{a^4}(L-a)^2+1}$$

<u>IF A and B are at the same elevation:</u>

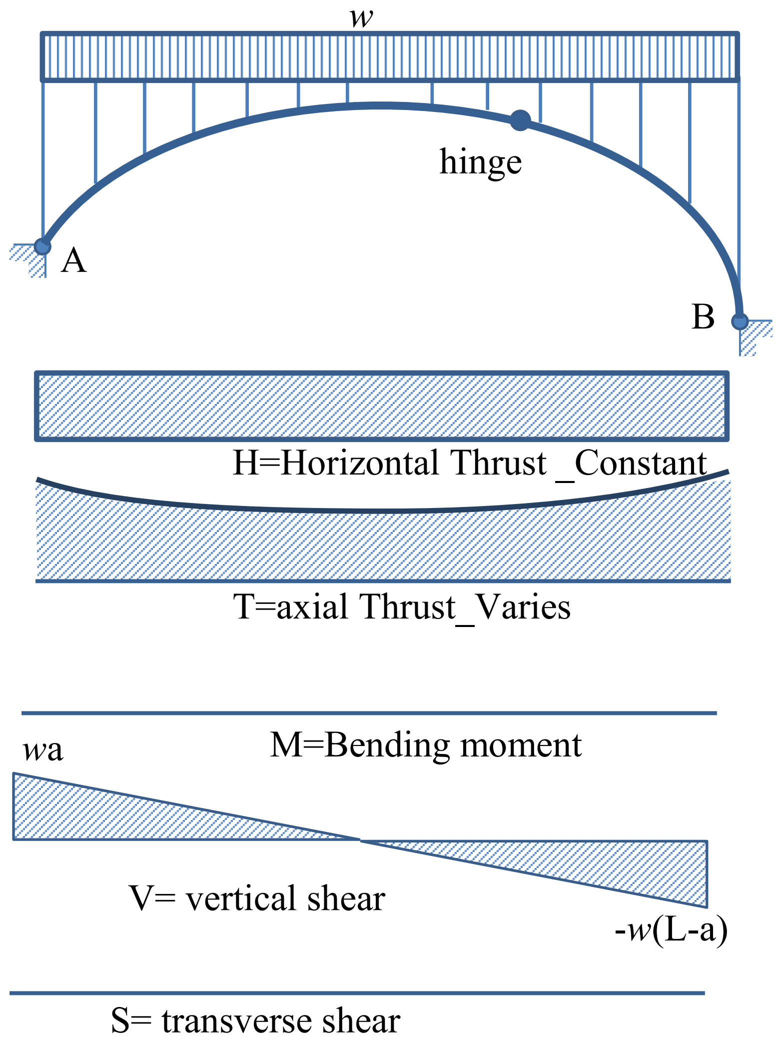
Then a=L/2 and

$$T_{max} = \frac{wL^2}{8h}\sqrt{16\frac{h^2}{L^2}+1}$$



<u>Force diagrams for hinged parabolic arch subjected to distributed loading:</u>

It appears that a parabolic arch is the right shape for **uniformly distributed load** covering the whole span.



Note: For partially loaded arch, or for concentrated loads, it is clear that bending moments will exist in general.

Part 12:

Suspension Cable- Suspension Bridge

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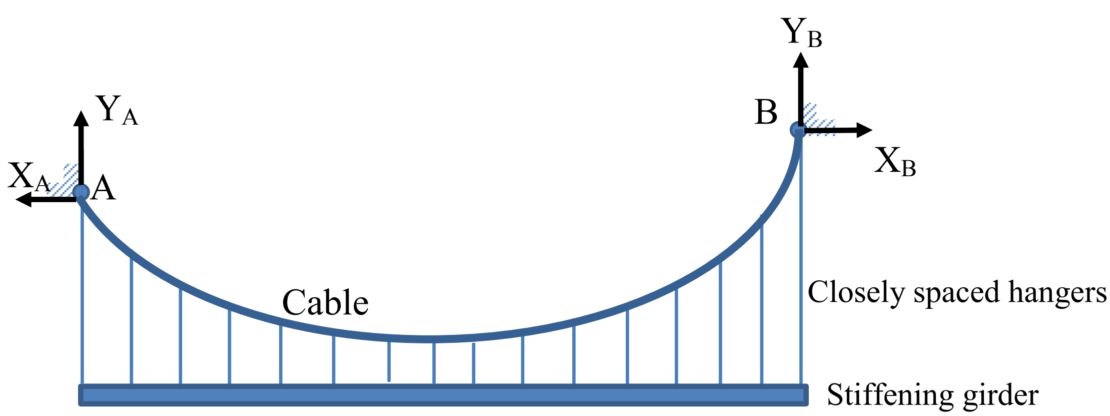
Simplified Theory (Assumptions):

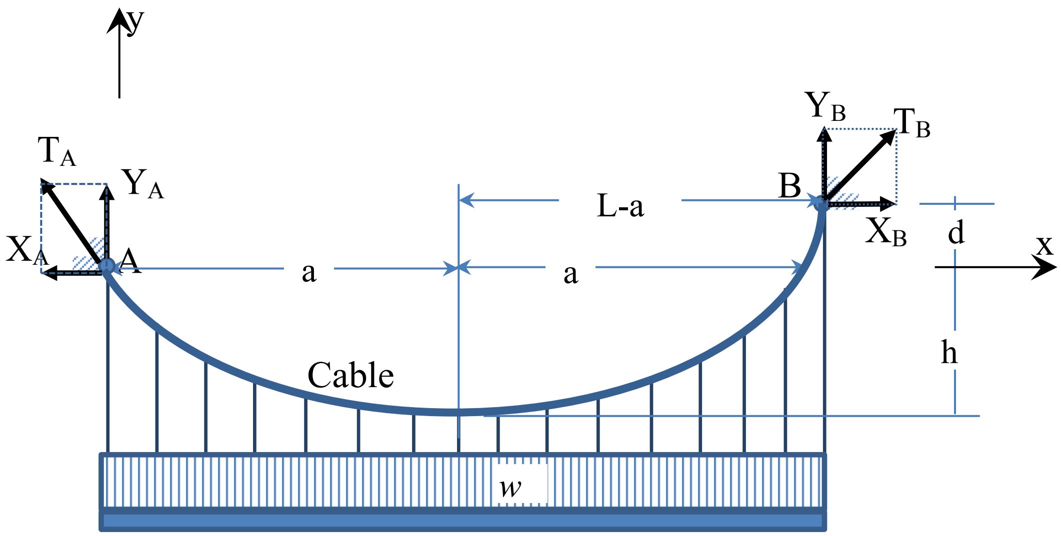
- Cable takes tensile (axial) forces only
- Cable does not change shape under load
- Cable has no weight
- Load on girder causes equal tension in all hangers
- Hanger loads on cable can be approximated as a uniformly distributed load.

Two main components:

- Cable
- Stiffening girder



Cable



Analysis for this suspension cable is same as that for the arch under uniformly distributed load except that the sense in which forces act changes (i.e. tensions rather than thrusts.)

The maximum tension in cable

$$T_{max} = \frac{wa^2}{2h}\sqrt{\frac{4h^2}{a^4}(L-a)^2 + 1}$$

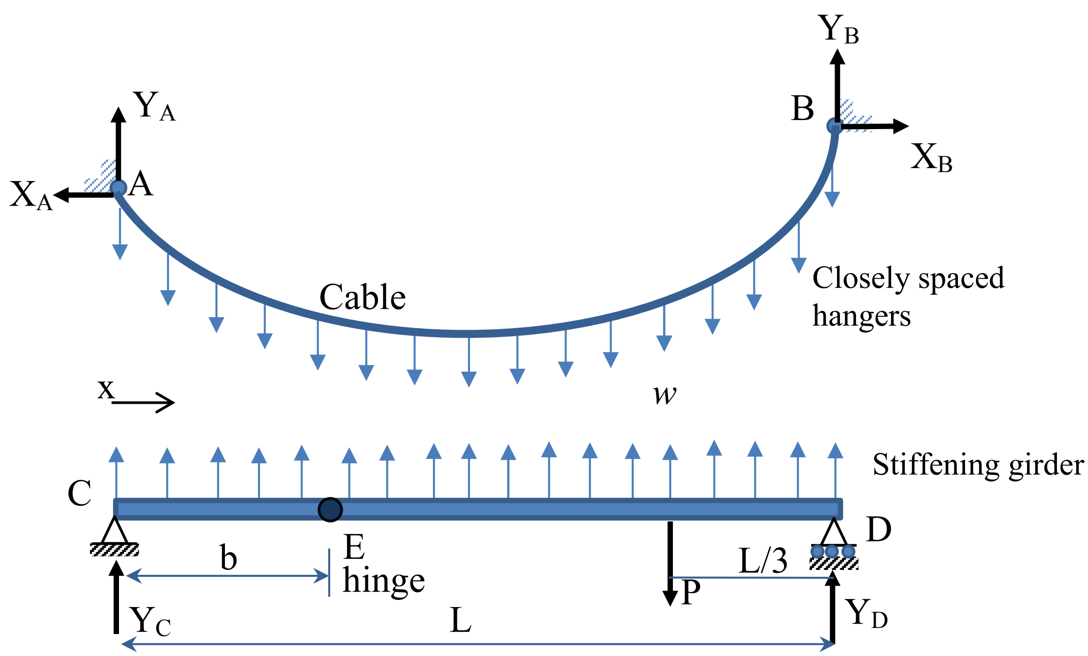
Stiffening Girder

- Let P be a concentrated load on the girder (say at L/3 from D)
- Assume that P gives uniformly distributed load of the intensity *w* on the cable.

3 Unknowns: Y_C, Y_D, and w

- Introduce a hinge at E in the girder to make it statically determinate.

Find the distributed load *w* in the girder due to the applied load of P.?



Three equilibrium equations:

$\sum F_y = 0$ $\sum M_o = 0$ $\sum M_E = 0$

Note: $\sum F_x = 0$ is identically satisfied

Equilibrium analysis of member CE gives $V_E = Y_C = -\frac{wb}{2}$

Equilibrium of member ED:

$$\sum F_y = -P + Y_D + \frac{wb}{2} + w(L-b) = 0$$

$$\boldsymbol{Y_D = P - w(L - \tfrac{b}{2})}$$

$$\sum M_D = \frac{wb}{2}(L-b) + w(L-b)\frac{(L-b)}{2} - P(\tfrac{L}{3}) = 0$$

$$\boldsymbol{w = \frac{2PL}{3L(L-b)}}$$

Once *w* is known, all the cable forces are known by the previous cable analysis.

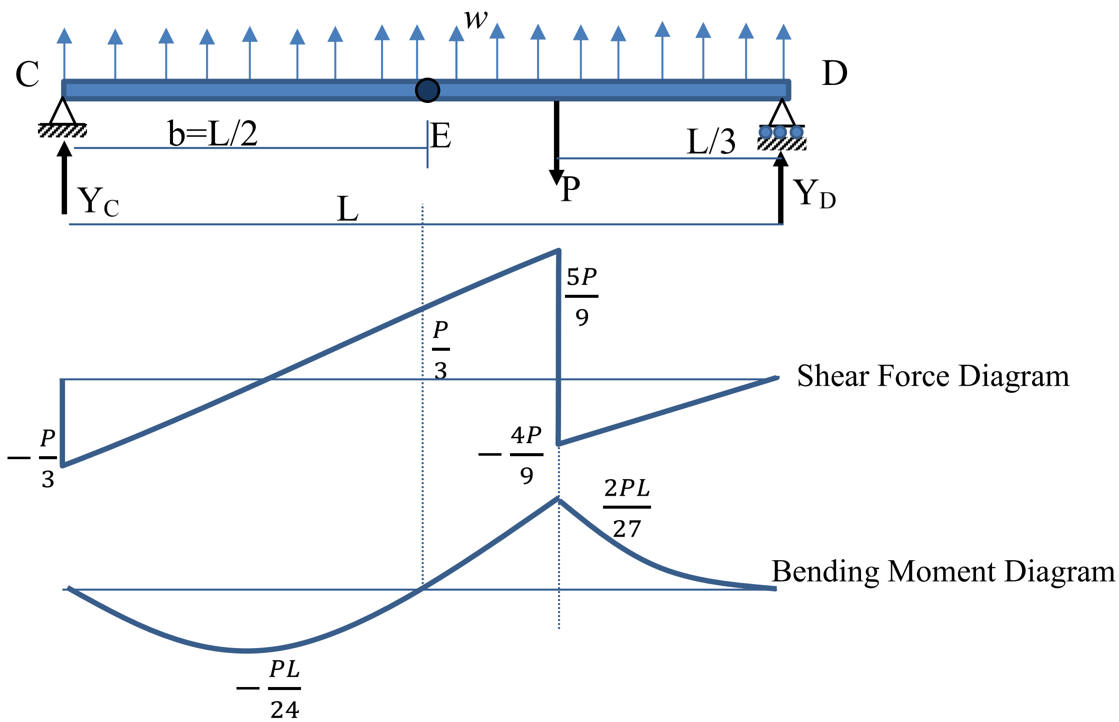
Example: for b=L/2 in the above example, and P at a distance L/3 from D

Then;

$$w = \frac{2PL}{3L(L-L/2)} = \frac{4P}{3L} \qquad Y_C = -\frac{wb}{2} = -\frac{4P}{3L}\frac{L}{4} = -\frac{P}{3}$$

$$V_E = \frac{P}{3} \qquad Y_D = P - \frac{4P}{3L}\left(L - \frac{L}{4}\right) = 0$$

Bending moment and shear force diagrams:



Suppose P is applied at (L/2) (Load just to right of hinge)

$w = \frac{2P}{L}$ $Y_C = -\frac{P}{2} = Y_D$ $V_E = -\frac{P}{2}$