CHAPTER 10

EXERCISES 10.1

1. This sequence has limit 0.

2. This sequence diverges.

3. This sequence has limit 3.

4. This sequence has limit 0.

5. This sequence diverges.

6. This sequence has limit 0.

7. This sequence diverges.

8. This sequence has limit $\lim_{n\to\infty} \dfrac{n}{n^2+n+2} = \lim_{n\to\infty} \dfrac{1}{n+1+2/n} = 0.$

9. This sequence has limit 0.

10. This sequence has limit $\pi/2$.

11. This sequence has limit 0.

12. This sequence diverges.

13. This sequence has limit 2 (since all terms are equal to 2).

14. This sequence has limit 0.

15. This sequence has limit $\lim_{n\to\infty} \dfrac{n+1}{2n+3} = \lim_{n\to\infty} \dfrac{1+1/n}{2+3/n} = \dfrac{1}{2}.$

16. This sequence has limit $\lim_{n\to\infty} \dfrac{2n+3}{n^2-5} = \lim_{n\to\infty} \dfrac{2+3/n}{n-5/n} = 0.$

17. This sequence has limit $\lim_{n\to\infty} \dfrac{n^2+5n-4}{n^2+2n-2} = \lim_{n\to\infty} \dfrac{1+5/n-4/n^2}{1+2/n-2/n^2} = 1.$

18. This sequence has limit 0.

19. This sequence has limit 0.

20. This sequence has limit $\lim_{n\to\infty} \dfrac{1}{1+1/n}\mathrm{Tan}^{-1} n = \dfrac{\pi}{2}.$

21. The general term is $\dfrac{2^n-1}{2^n}.$

22. The general term is $\dfrac{3n+1}{n^2}.$

23. The general term is $(-1)^{n+1}\dfrac{\ln(n+1)}{\sqrt{n+1}}.$

24. The general term is $\dfrac{1+(-1)^{n+1}}{2}.$

25. The general term is $\sqrt{2}\sin\dfrac{(2n-1)\pi}{4}.$

26. The limit of the sequence $\{\ln n/\sqrt{n}\}$ as $n\to\infty$ is equal to the limit of the function $\ln x/\sqrt{x}$ as $x\to\infty$, provided the limit of the function exists. When we use L'Hôpital's rule on the limit of the function,

$$\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = \lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/(2\sqrt{x})} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0.$$

27. The limit of the sequence $\{(n^3+1)/e^n\}$ as $n\to\infty$ is equal to the limit of the function $(x^3+1)/e^x$ as $x\to\infty$, provided the limit of the function exists. When we use L'Hôpital's rule on the limit of the function,

$$\lim_{n\to\infty} \frac{n^3+1}{e^n} = \lim_{x\to\infty} \frac{x^3+1}{e^x} = \lim_{x\to\infty} \frac{3x^2}{e^x} = \lim_{x\to\infty} \frac{6x}{e^x} = \lim_{x\to\infty} \frac{6}{e^x} = 0.$$

28. The limit of the sequence $\{n\sin(4/n)\}$ as $n\to\infty$ is equal to the limit of the function $x\sin(4/x)$ as $x\to\infty$, provided the limit of the function exists. When we use L'Hôpital's rule,

$$\lim_{n\to\infty} n\sin\left(\frac{4}{n}\right) = \lim_{x\to\infty} x\sin\left(\frac{4}{x}\right) = \lim_{x\to\infty} \frac{\sin(4/x)}{1/x} = \lim_{x\to\infty} \frac{-(4/x^2)\cos(4/x)}{-1/x^2} = 4.$$

29. The limit of the sequence $\{[(n+5)/(n+3)]^n\}$ as $n \to \infty$ is equal to the limit of the function $[(x+5)/(x+3)]^x$ as $x \to \infty$, provided the limit of the function exists. We set L equal to the limit of the function, take logarithms, and use L'Hôpital's rule,

$$\ln L = \ln\left[\lim_{x\to\infty}\left(\frac{x+5}{x+3}\right)^x\right] = \lim_{x\to\infty} x\ln\left(\frac{x+5}{x+3}\right) = \lim_{x\to\infty}\frac{\ln\left(\dfrac{x+5}{x+3}\right)}{1/x}$$
$$= \lim_{x\to\infty}\frac{\dfrac{x+3}{x+5}\left[\dfrac{(x+3)-(x+5)}{(x+3)^2}\right]}{-1/x^2} = \lim_{x\to\infty}\frac{2x^2}{(x+3)(x+5)} = 2.$$

Thus, $L = e^2$, and this is also the limit of the sequence.

30. Certainly the sequence diverges; terms get arbitrarily large for large n. On the other hand, as n increases, the difference between terms approaches $\lim_{n\to\infty}[\ln n - \ln(n+1)] = \lim_{n\to\infty}\ln\left(\frac{n}{n+1}\right) = 0.$

31. (a) The first ten terms are 2,3,5,7,11,13,17,19,23,29. (b) No one has developed a formula for all primes.

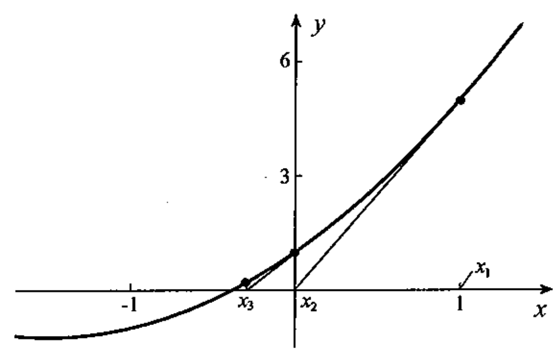
32. The figure indicates that with initial approximation $x_1 = 1$, the sequence defined by Newton's iterative procedure has a limit near $-1/2$. Iteration of

$$x_1 = 1, \quad x_{n+1} = x_n - \frac{x_n^2 + 3x_n + 1}{2x_n + 3}$$

leads to

$x_2 = 0,\qquad x_3 = -1/3,$
$x_4 = -0.381,\qquad x_5 = -0.381\,966,$
$x_6 = -0.381\,966\,01,\qquad x_7 = -0.381\,966\,01.$

Since $f(-0.381\,965\,95) = 1.4\times10^{-7}$ and $f(-0.381\,966\,05) = -8.7\times10^{-8}$, we can say that to seven decimals $x = -0.381\,966\,0$.



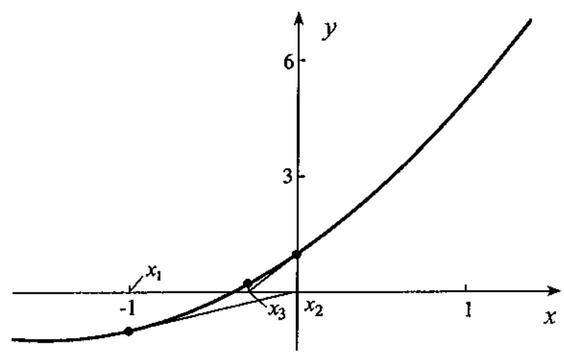
33. The figure indicates that with initial approximation $x_1 = -1$, the sequence defined by Newton's iterative procedure has a limit near $-1/2$. Iteration of

$$x_1 = -1, \quad x_{n+1} = x_n - \frac{x_n^2 + 3x_n + 1}{2x_n + 3}$$

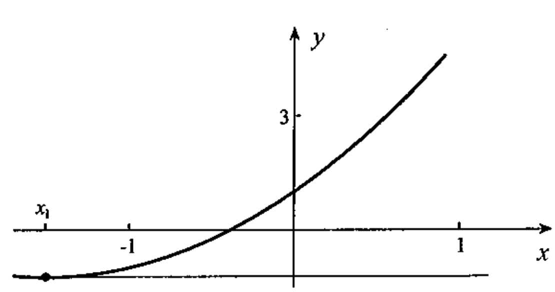
leads to

$x_2 = 0,\qquad x_3 = -1/3,$
$x_4 = -0.381,\qquad x_5 = -0.381\,966,$
$x_6 = -0.381\,966\,01,\qquad x_7 = -0.381\,966\,01.$

Since $f(-0.381\,965\,95) = 1.4\times10^{-7}$ and $f(-0.381\,966\,05) = -8.7\times10^{-8}$, we can say that to seven decimals $x = -0.381\,966\,0$.



34. The figure indicates that with initial approximation $x_1 = -1.5$, the sequence defined by Newton's iterative procedure does not have a limit. This is because $x_1 = -1.5$ is a critical point of the function.



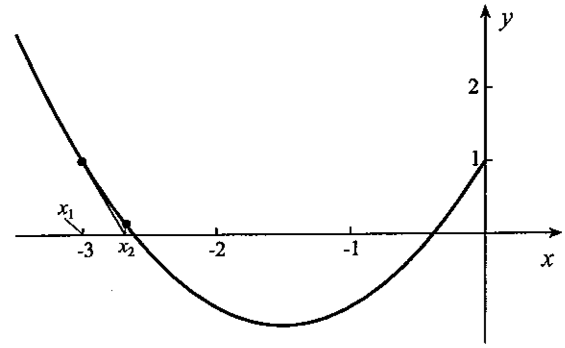
35. The figure indicates that with initial approximation $x_1 = -3$, the sequence defined by Newton's iterative procedure has a limit near -3. Iteration of

$$x_1 = -3, \quad x_{n+1} = x_n - \frac{x_n^2 + 3x_n + 1}{2x_n + 3}$$

leads to

$$x_2 = -2.667, \qquad x_3 = -2.6191,$$
$$x_4 = -2.618\,034\,5, \qquad x_5 = -2.618\,033\,99,$$
$$x_6 = -2.618\,033\,99.$$

Since $f(-2.618\,034\,05) = 1.4 \times 10^{-7}$ and $f(-2.618\,033\,95) = -8.7 \times 10^{-8}$, we can say that to seven decimals $x = -2.618\,034\,0$.



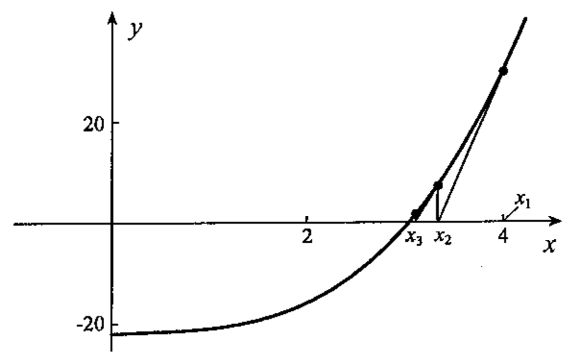
36. The figure indicates that with initial approximation $x_1 = 4$, the sequence defined by Newton's iterative procedure has a limit near 3. Iteration of

$$x_1 = 4, \quad x_{n+1} = x_n - \frac{x_n^3 - x_n^2 + x_n - 22}{3x_n^2 - 2x_n + 1}$$

leads to

$$x_2 = 3.268, \qquad x_3 = 3.060\,9,$$
$$x_4 = 3.044\,8, \qquad x_5 = 3.044\,723\,15,$$
$$x_6 = 3.044\,723\,15.$$

Since $f(3.044\,723\,05) = -2.2 \times 10^{-6}$ and $f(3.044\,723\,15) = 3.5 \times 10^{-8}$, we can say that to seven decimals $x = 3.044\,723\,1$.



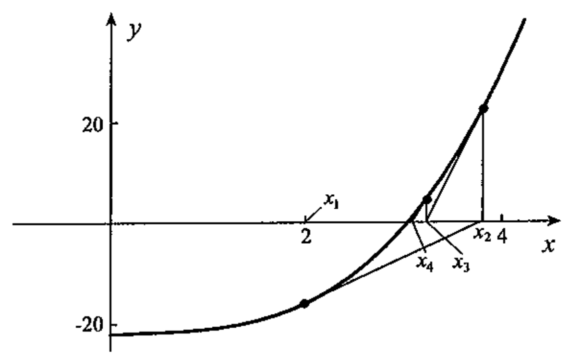
37. The figure indicates that with initial approximation $x_1 = 2$, the sequence defined by Newton's iterative procedure has a limit near 3. Iteration of

$$x_1 = 2, \quad x_{n+1} = x_n - \frac{x_n^3 - x_n^2 + x_n - 22}{3x_n^2 - 2x_n + 1}$$

leads to

$$x_2 = 3.778, \qquad x_3 = 3.187,$$
$$x_4 = 3.0515, \qquad x_5 = 3.044\,740,$$
$$x_6 = 3.044\,723\,15, \qquad x_7 = 3.044\,723\,15.$$

Since $f(3.044\,723\,05) = -2.2 \times 10^{-6}$ and $f(3.044\,723\,15) = 3.5 \times 10^{-8}$, we can say that to seven decimals $x = 3.044\,723\,1$.



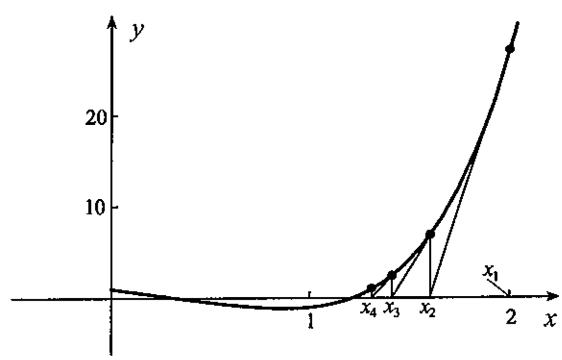
38. The figure indicates that with initial aproximation $x_1 = 2$, the sequence defined by Newton's iterative procedure has a limit near 1. Iteration of

$$x_1 = 2, \quad x_{n+1} = x_n - \frac{x_n^5 - 3x_n + 1}{5x_n^4 - 3}$$

gives

$$x_2 = 1.649, \qquad x_3 = 1.406,$$
$$x_4 = 1.268, \qquad x_5 = 1.220,$$
$$x_6 = 1.215, \qquad x_7 = 1.214\,65,$$
$$x_8 = 1.214\,648\,04, \qquad x_9 = 1.214\,648\,04.$$

Since $f(1.214\,647\,95) = -7.3 \times 10^{-7}$ and $f(1.214\,648\,05) = 5.8 \times 10^{-8}$, we can say that to seven decimals $x = 1.214\,648\,0$.



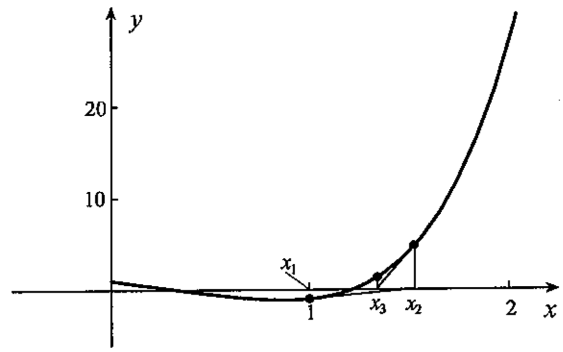
39. The figure indicates that the sequence defined by Newton's iterative procedure has a limit. Iteration of

$$x_1 = 1, \quad x_{n+1} = x_n - \frac{x_n^5 - 3x_n + 1}{5x_n^4 - 3}$$

gives

$x_2 = 1.5,$ $\quad x_3 = 1.317,$
$x_4 = 1.233,$ $\quad x_5 = 1.2154,$
$x_6 = 1.214\,649,$ $\quad x_7 = 1.214\,648\,04,$
$x_8 = 1.214\,648\,04.$

Since $f(1.214\,647\,95) = -7.3 \times 10^{-7}$ and $f(1.214\,648\,05) = 5.8 \times 10^{-8}$, we can say that to seven decimals $x = 1.214\,648\,0$.



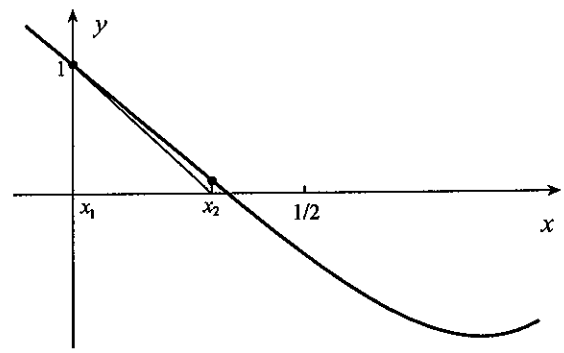
40. The figure indicates that with initial approximation $x_1 = 0$, the sequence defined by Newton's iterative procedure has a limit near 0.3. Iteration of

$$x_1 = 0, \quad x_{n+1} = x_n - \frac{x_n^5 - 3x_n + 1}{5x_n^4 - 3}$$

gives

$x_2 = 1/3,$ $\quad x_3 = 0.334\,7,$
$x_4 = 0.334\,734\,14,$ $\quad x_5 = 0.334\,734\,14.$

Since $f(0.334\,734\,05) = 2.7 \times 10^{-7}$ and $f(0.334\,734\,15) = -2.4 \times 10^{-8}$, we can say that to seven decimals $x = 0.334\,734\,1$.



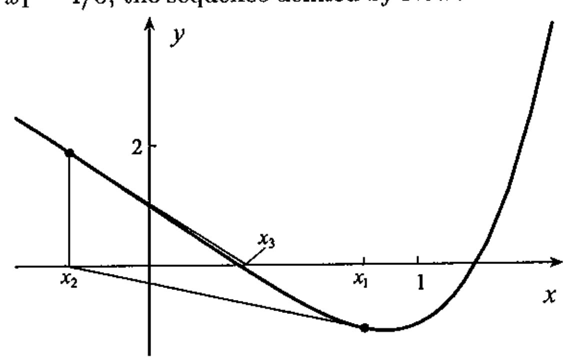
41. The figure indicates that with initial approximation $x_1 = 4/5$, the sequence defined by Newton's iterative procedure has a limit near 0.3. Iteration of

$$x_1 = 4/5, \quad x_{n+1} = x_n - \frac{x_n^5 - 3x_n + 1}{5x_n^4 - 3}$$

gives

$x_2 = -0.326,$ $\quad x_3 = 0.345,$
$x_4 = 0.334\,72,$ $\quad x_5 = 0.334\,734\,14,$
$x_6 = 0.334\,734\,14.$

Since $f(0.334\,734\,05) = 2.7 \times 10^{-7}$ and $f(0.334\,734\,15) = -2.4 \times 10^{-8}$, we can say that to seven decimals $x = 0.334\,734\,1$.



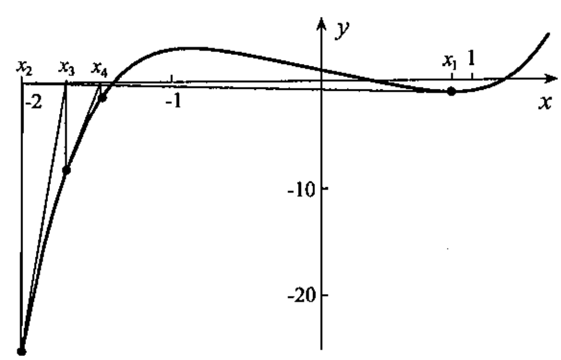
42. The figure indicates that with initial approximation $x_1 = 0.85$, the sequence defined by Newton's iterative procedure has a limit near -1.5. Iteration of

$$x_1 = 0.85, \quad x_{n+1} = x_n - \frac{x_n^5 - 3x_n + 1}{5x_n^4 - 3}$$

gives

$x_2 = -1.987,$ $\quad x_3 = -1.667,$
$x_4 = -1.474,$ $\quad x_5 = -1.399,$
$x_6 = -1.389,$ $\quad x_7 = -1.388\,792\,06,$
$x_8 = -1.388\,791\,98,$ $\quad x_9 = -1.388\,791\,98.$

Since $f(-1.388\,791\,95) = 5.4 \times 10^{-7}$ and $f(-1.388\,792\,05) = -1.0 \times 10^{-6}$, we can say that to seven decimals $x = -1.388\,792\,0$.



43. The figure indicates that with initial approximation $x_1 = -2$, the sequence defined by Newton's iterative procedure has a limit near -1.5. Iteration of

$$x_1 = -2, \quad x_{n+1} = x_n - \frac{x_n^5 - 3x_n + 1}{5x_n^4 - 3}$$

gives

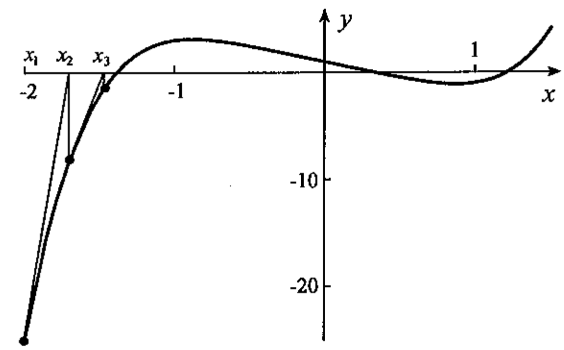
$x_2 = -1.675, \qquad x_3 = -1.478,$

$x_4 = -1.4004, \qquad x_5 = -1.3890,$

$x_6 = -1.388\,792, \qquad x_7 = -1.388\,791\,98,$

$x_8 = -1.388\,791\,98.$

Since $f(-1.388\,791\,95) = 5.4 \times 10^{-7}$ and $f(-1.388\,792\,05) = -1.0 \times 10^{-6}$, we can say that to seven decimals $x = -1.388\,792\,0$.



44. Iteration of $x_1 = 2, \quad x_{n+1} = 2 + \dfrac{1}{x_n}$ gives

$x_2 = 2.5, \quad x_3 = 2.4, \quad x_4 = 2.416\,67, \quad x_5 = 2.413\,79, \quad x_6 = 2.414\,29,$

$x_7 = 2.414\,20, \quad x_8 = 2.414\,22, \quad x_9 = 2.414\,21, \quad x_{10} = 2.414\,21.$

Since $f(2.414\,15) = -1.8 \times 10^{-4}$ and $f(2.414\,25) = 1.0 \times 10^{-4}$, it follows that to 4 decimals, $x = 2.4142$.

45. Iteration of $x_1 = -1, \quad x_{n+1} = -\dfrac{1}{6}(x_n^3 + 3)$ gives

$x_2 = -1/3, \quad x_3 = -0.4938, \quad x_4 = -0.4799, \quad x_5 = -0.4816, \quad x_6 = -0.481\,38, \quad x_7 = -0.481\,41.$

Since $f(-0.481\,35) = 3.7 \times 10^{-4}$ and $f(-0.481\,45) = -3.0 \times 10^{-4}$, it follows that to 4 decimals, $x = -0.4814$.

46. Iteration of $x_1 = 0, \quad x_{n+1} = \dfrac{1}{120}(x_n^4 + 20)$ gives

$$x_2 = 1/6, \quad x_3 = 0.166\,67, \quad x_4 = 0.166\,67.$$

Since $f(0.166\,65) = 2.8 \times 10^{-3}$ and $f(0.166\,75) = -9.2 \times 10^{-3}$, it follows that to 4 decimals, $x = 0.1667$.

47. Iteration of $x_1 = 3, \quad x_{n+1} = \dfrac{2x_n^2 + 3x_n - 1}{x_n^2}$ gives

$x_2 = 2.889, \quad x_3 = 2.9186, \quad x_4 = 2.910\,49, \quad x_5 = 2.912\,70,$

$x_6 = 2.912\,10, \quad x_7 = 2.912\,26, \quad x_8 = 2.912\,22.$

Since $f(2.912\,15) = -8.5 \times 10^{-4}$ and $f(2.912\,25) = 2.2 \times 10^{-4}$, the root is $x = 2.9122$ to 4 decimal places.

48. Iteration of $x_1 = 0, \quad x_{n+1} = \dfrac{1}{2}(1 + x_n^2)^{1/3}$ gives

$x_2 = 1/2, \quad x_3 = 0.5386, \quad x_4 = 0.5443, \quad x_5 = 0.545\,17,$

$x_6 = 0.545\,31, \quad x_7 = 0.545\,33, \quad x_8 = 0.545\,33.$

Since $f(0.545\,25) = -4.9 \times 10^{-4}$ and $f(0.545\,35) = 1.2 \times 10^{-4}$, it follows that to 4 decimals, $x = 0.5453$.

49. With $x_1 = 3.5$, and $x_{n+1} = \dfrac{6x_n^2 - 11x_n + 7}{x_n^2}$, iteration gives

$x_2 = 3.4286, \quad x_3 = 3.3872, \quad x_4 = 3.3626, \quad x_5 = 3.3478, \quad x_6 = 3.3388,$

$x_7 = 3.333\,34, \quad x_8 = 3.330\,00, \quad x_9 = 3.327\,96, \quad x_{10} = 3.326\,71, \quad x_{11} = 3.325\,94,$

$x_{12} = 3.325\,47, \quad x_{13} = 3.325\,18, \quad x_{14} = 3.325\,00, \quad x_{15} = 3.324\,89, \quad x_{16} = 3.324\,82,$

$x_{17} = 3.324\,78, \quad x_{18} = 3.324\,76, \quad x_{19} = 3.324\,74.$

With $f(x) = x^3 - 6x^2 + 11x - 7$, we calculate that $f(3.324\,65) = -2.9 \times 10^{-4}$ and $f(3.324\,75) = 1.4 \times 10^{-4}$. The root is therefore $x = 3.3247$ to 4 decimals.

50. With $x_1 = 0$, and $x_{n+1} = \dfrac{x_n^4 - 3x_n^2 + 1}{3}$, iteration gives

$x_2 = 1/3,\quad x_3 = 0.226,\quad x_4 = 0.283,\quad x_5 = 0.255,\quad x_6 = 0.270,$
$x_7 = 0.262,\quad x_8 = 0.266,\quad x_9 = 0.2642,\quad x_{10} = 0.2652,\quad x_{11} = 0.2647,$
$x_{12} = 0.2649,\quad x_{13} = 0.264\,80,\quad x_{14} = 0.264\,85,\quad x_{15} = 0.264\,83,\quad x_{16} = 0.264\,84.$

With $f(x) = x^4 - 3x^2 - 3x + 1$, we calculate that $f(0.264\,75) = 3.9 \times 10^{-4}$ and $f(0.264\,85) = -6.6 \times 10^{-5}$. The root is therefore $x = 0.2648$ to 4 decimals.

51. With $x_1 = 0.5$ and $x_{n+1} = \dfrac{50 + 50x_n^2 - 4x_n^3 - x_n^4}{100}$, iteration gives

$x_2 = 0.6194,\quad x_3 = 0.6809,\quad x_4 = 0.7170,\quad x_5 = 0.7397,\quad x_6 = 0.7544,$
$x_7 = 0.7641,\quad x_8 = 0.7707,\quad x_9 = 0.7751,\quad x_{10} = 0.7782,\quad x_{11} = 0.7803,$
$x_{12} = 0.7817,\quad x_{13} = 0.7827,\quad x_{14} = 0.7834,\quad x_{15} = 0.7839,\quad x_{16} = 0.7842,$
$x_{17} = 0.7844,\quad x_{18} = 0.7846,\quad x_{19} = 0.7847,\quad x_{20} = 0.7848,\quad x_{21} = 0.784\,83,$
$x_{22} = 0.784\,85,\quad x_{23} = 0.784\,86.$

With $f(x) = x^4 + 4x^3 - 50x^2 + 100x - 50$, we calculate that $f(0.784\,85) = -1.2 \times 10^{-3}$ and $f(0.784\,95) = 1.9 \times 10^{-3}$. Thus to 4 decimals, $x = 0.7849$.

52. With $x_1 = 0.75$, and $x_{n+1} = \sqrt{1 - \sin^2 x_n} = \sqrt{\cos^2 x_n} = \cos x_n$, iteration gives

$x_2 = 0.732,\quad x_3 = 0.744,\quad x_4 = 0.736,\quad x_5 = 0.741,\quad x_6 = 0.738,$
$x_7 = 0.740,\quad x_8 = 0.7385,\quad x_9 = 0.7395.$

With $f(x) = \sin^2 x - 1 + x^2$, we calculate that $f(0.739\,05) = -8.7 \times 10^{-5}$ and $f(0.739\,15) = 1.6 \times 10^{-4}$. The root is therefore $x = 0.7391$ to 4 decimals.

53. By cross-multiplying, $(1 + x^4)\sec x = 2$, and therefore the equation can be rearranged into the form $x = (2\cos x - 1)^{1/4}$. With $x_1 = 0.5$ and $x_{n+1} = (2\cos x_n - 1)^{1/4}$, iteration gives

$x_2 = 0.932,\quad x_3 = 0.662,\quad x_4 = 0.872,\quad x_5 = 0.732,\quad x_6 = 0.836,$
$x_7 = 0.764,\quad x_8 = 0.816,\quad x_9 = 0.780,\quad x_{10} = 0.806,\quad x_{11} = 0.788,$
$x_{12} = 0.800,\quad x_{13} = 0.792,\quad x_{14} = 0.798,\quad x_{15} = 0.793,\quad x_{16} = 0.797,$
$x_{17} = 0.7941,\quad x_{18} = 0.7962,\quad x_{19} = 0.7947,\quad x_{20} = 0.7958,\quad x_{21} = 0.7950,$
$x_{22} = 0.7956,\quad x_{23} = 0.7951,\quad x_{24} = 0.7955,\quad x_{25} = 0.7952,\quad x_{26} = 0.7954,$
$x_{27} = 0.7953.$

With $f(x) = \sec x - 2(1+x^4)^{-1}$, we calculate that $f(0.795\,25) = -2.6 \times 10^{-4}$ and $f(0.795\,35) = 9.2 \times 10^{-5}$. To 4 decimals then, $x = 0.7953$.

54. With $x_1 = 0.5$, and $x_{n+1} = \dfrac{e^{x_n} + e^{-x_n}}{10}$, iteration gives

$$x_2 = 0.226,\quad x_3 = 0.205,\quad x_4 = 0.2042,\quad x_5 = 0.204\,18,\quad x_6 = 0.204\,18.$$

With $f(x) = e^x + e^{-x} - 10x$, we calculate that $f(0.204\,15) = 3.2 \times 10^{-4}$ and $f(0.204\,25) = -6.4 \times 10^{-4}$. The root is therefore $x = 0.2042$ to 4 decimals.

55. (a) Iteration of $x_1 = 1$, $x_{n+1} = x_n - \dfrac{x_n^4 - 15x_n + 2}{4x_n^3 - 15}$ gives

$$x_2 = -0.09,\quad x_3 = 0.1333,\quad x_4 = 0.133\,354\,4,\quad x_5 = 0.133\,354\,4.$$

With $f(x) = x^4 - 15x + 2$, we calculate that $f(0.133\,353\,5) = 1.4 \times 10^{-5}$ and $f(0.133\,354\,5) = -1.2 \times 10^{-6}$. The root is therefore $x = 0.133\,354$ to 6 decimals.
(b) Iteration gives

$$x_2 = 0.2,\quad x_3 = 0.133\,44,\quad x_4 = 0.133\,354\,47,\quad x_5 = 0.133\,354\,42.$$

This leads to the same root as in part (a).
(c) Iteration of the sequence in part (a) beginning with $x_1 = 2.5$ gives

$$x_2 = 2.425, \quad x_3 = 2.4201, \quad x_3 = 2.420\,061\,9, \quad x_4 = 2.420\,061\,9.$$

Since $f(2.420\,061\,5) = -1.5 \times 10^{-5}$ and $f(2.420\,062\,5) = 2.7 \times 10^{-5}$, the root is $x = 2.420\,062$.
(d) Iteration beginning with $x_1 = 2$ gives

$$x_2 = 1.2, \quad x_3 = 0.271\,6, \quad x_4 = 0.133\,696, \quad x_5 = 0.133\,355.$$

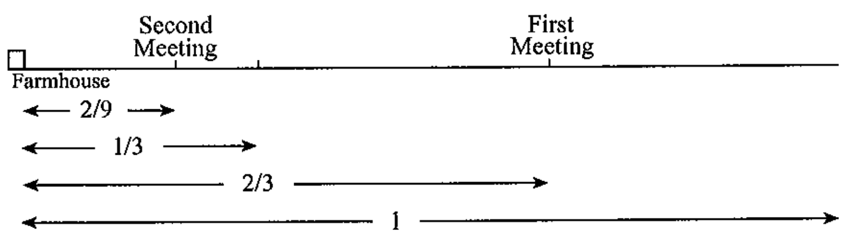
The sequence is converging to the root in part (a). Beginning with $x_1 = 3$, we obtain $x_2 = 5.5$ and $x_3 = 61.1$. The sequence is diverging.

56. (a) $d_1 = 2(0.99)(20) = 40(0.99)$ m
$d_2 = 2(0.99)[(0.99)(20)] = 40(0.99)^2$ m
$d_3 = 2(0.99)[(20)(0.99)^2] = 40(0.99)^3$ m
The pattern emerging is $d_n = 40(0.99)^n$ m.

(b) When an object falls from rest under gravity, the distance that it falls as a function of time t is given by $d = 4.905t^2$. Consequently, the time to fall from peak height between n^{th} and $(n+1)^{\text{th}}$ bounces is given by $d_n/2 = 4.905t^2$. When this equation is solved for t, the result is $t = \sqrt{d_n/9.81}$, and therefore

$$t_n = 2\sqrt{d_n/9.81} = 2\sqrt{40(0.99)^n/9.81} = \frac{4}{\sqrt{0.981}}(0.99)^{n/2} \text{ s}.$$

57. The dog reaches the farmer for the first time $2/3$ km from the farmhouse. When the dog returns to the farmhouse (travelling $2/3$ km), the farmer moves to a distance $1/3$ km from the farmhouse. The dog then runs $(2/3)(1/3) = 2/9$ km in reaching the farmer for the second time. Thus, $d_1 = 2/3 + 2/9 = 8/9$ km. When the dog returns to the farmhouse for the second time, the farmer moves to a distance $1/9$ km from the farmhouse. The dog then runs $(2/3)(1/9) = 2/27$ km in reaching the farmer for the third time. Thus, $d_2 = 2/9 + 2/27 = 8/27$ km. The pattern emerging is $d_n = 8/3^{n+1}$ km.



58. Since each of the 12 straight line segments in the middle figure has length $P/9$,

$$P_1 = \frac{12P}{9} = \frac{4P}{3}.$$

Since each of the 48 straight line segments in the right figure has length $P/27$,

$$P_2 = \frac{48P}{27} = \frac{4^2P}{3^2}.$$

The next perimeter is $P_3 = 4(48)\dfrac{P}{81} = \dfrac{4^3P}{3^3}$. The pattern emerging is $P_n = \dfrac{4^nP}{3^n}$. The limit of P_n as $n \to \infty$ does not exist.

59. (a) Since $y(3) = 11.8$ and $y(4) = -3.0$, the solution is between 3 and 4. To find it more accurately we use

$$t_1 = 3.8, \qquad t_{n+1} = t_n - \frac{1181(1 - e^{-t_n/10}) - 98.1t_n}{118.1e^{-t_n/10} - 98.1}.$$

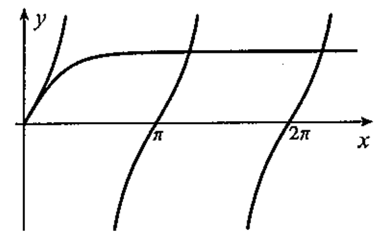
Iteration gives $t_2 = 3.833\,4$ and $t_3 = 3.833\,2$. Since $y(3.825) = 0.14$ and $y(3.835) = -0.03$, it follows that to 2 decimals $t = 3.83$ s.

(b) If air resistance is ignored, the acceleration of the stone is $a = dv/dt = -9.81$. Antidifferentiation gives $v(t) = -9.81t + C$. Since $v(0) = 20$, it follows that $C = 20$, and $v(t) = dy/dt = -9.81t + 20$. Antidifferentiation now gives $y(t) = -4.905t^2 + 20t + D$. Since $y(0) = 0$, we find that $D = 0$, and the height of the stone is $y(t) = -4.905t^2 + 20t$. When we set $0 = y = -4.905t^2 + 20t$, the positive solution is 4.08 s.

60. The figure shows graphs of $y = \tan x$ and $y = (e^x - e^{-x})/(e^x + e^{-x}) = \tanh x$ for $x \geq 0$. They intersect at $x = 0$ and values near 4 and 7. We use Newton's iterative procedure

$$x_{n+1} = x_n - \frac{\tan x_n - \tanh x_n}{\sec^2 x_n - \operatorname{sech}^2 x_n}$$

with $x_1 = 4$ to locate the smaller root.



Iteration gives $x_2 = 3.932\,25$, $x_3 = 3.926\,63$, $x_4 = 3.926\,60$, $x_5 = 3.926\,60$. When we divide this by 20π, the result is 0.0625. A similar procedure gives the next natural frequency 0.1125.

61. Since the area of an equilateral triangle with sides of length l is $\sqrt{3}l^2/4$, the area of the first triangle in Exercise 58 is $\dfrac{\sqrt{3}}{4}\left(\dfrac{P}{3}\right)^2 = \dfrac{\sqrt{3}P^2}{36}$. The middle figure adds three triangles each of area $\sqrt{3}(P/9)^2/4$ to the area in the first figure, and therefore

$$A_1 = \frac{\sqrt{3}P^2}{36} + \frac{3\sqrt{3}}{4}\left(\frac{P^2}{81}\right) = \frac{\sqrt{3}P^2}{36} + \frac{\sqrt{3}P^2}{3 \cdot 36}.$$

The right figure adds twelve triangles each of area $\sqrt{3}(P/27)^2/4$ to the middle figure, and therefore

$$A_2 = A_1 + \frac{12\sqrt{3}}{4}\left(\frac{P}{27}\right)^2 = \frac{\sqrt{3}P^2}{36} + \frac{\sqrt{3}P^2}{3 \cdot 36} + \frac{4\sqrt{3}P^2}{3^3 \cdot 36}.$$

The next figure in the sequence would add 48 triangles each of area $\sqrt{3}(P/81)^2/4$, and therefore

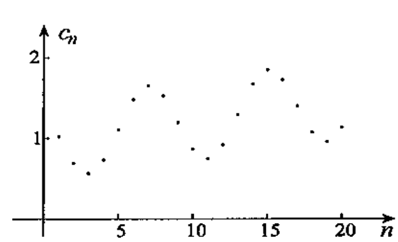
$$A_3 = A_2 + \frac{48\sqrt{3}}{4}\left(\frac{P}{81}\right)^2 = \frac{\sqrt{3}P^2}{36} + \frac{\sqrt{3}P^2}{3 \cdot 36} + \frac{4\sqrt{3}P^2}{3^3 \cdot 36} + \frac{4^2\sqrt{3}P^2}{3^5 \cdot 36}.$$

The pattern emerging is $\quad A_n = \dfrac{\sqrt{3}P^2}{36}\left(1 + \dfrac{1}{3} + \dfrac{4}{3^3} + \dfrac{4^2}{3^5} + \cdots + \dfrac{4^{n-1}}{3^{2n-1}}\right).$

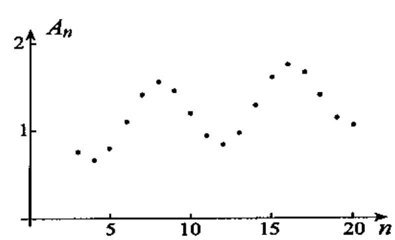
62. The next two terms are 1113213211, 31131211131221. Reason as follows: The second term is 11 because there is one 1 in the first term; the third term is 21 because the second term has two 1's; the fourth term is 1211 because the third term has one 2 followed by one 1; the fifth term is 111221 because the fourth term is one 1, followed by one 2, followed by two 1's; etc.

63. Plots of the sequences are shown below.

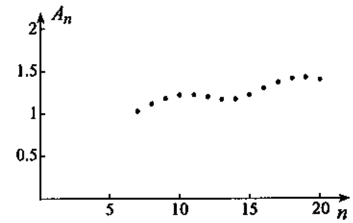
(a)



(b)



64. The plot of the seven-point averager is shown to the right.



65. An explicit formula for this FIR is $F_n = \dfrac{n}{n+1} + 2\left(\dfrac{n-1}{n}\right) - \left(\dfrac{n-2}{n-1}\right)$. When we substitute $n = 3, \ldots, 12$, we obtain the first 10 terms,

$$\frac{19}{12}, \frac{49}{30}, \frac{101}{60}, \frac{181}{105}, \frac{295}{168}, \frac{449}{252}, \frac{649}{360}, \frac{901}{495}, \frac{1211}{660}, \frac{1585}{858}.$$

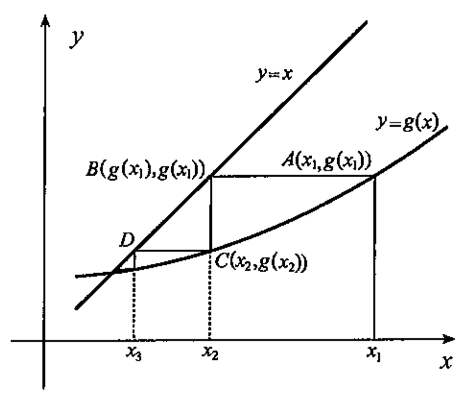
66. An explicit formula for this FIR is

$$F_n = \frac{1}{n^2}\sin\left(\frac{n}{3}\right) - \frac{2}{(n-1)^2}\sin\left(\frac{n-1}{3}\right) + \frac{3}{(n-2)^2}\sin\left(\frac{n-2}{3}\right) - \frac{4}{(n-3)^2}\sin\left(\frac{n-3}{3}\right).$$

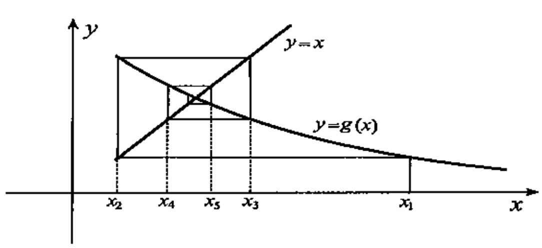
When we substitute $n = 4, \ldots, 13$, we obtain the first 10 terms,

$$-0.9712, -0.4196, -0.2461, -0.1593, -0.1059, -0.0693, -0.0430, -0.0237, -0.0096, 0.0002.$$

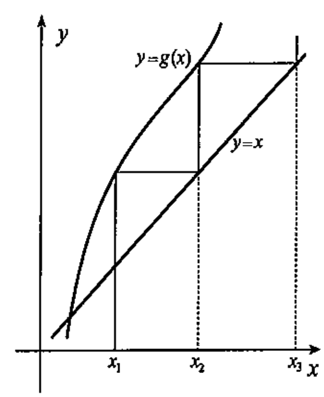
67. (a) The height of the curve $y = g(x)$ at the point A with x-coordinate x_1 is $y = g(x_1)$. If we proceed horizontally to the line $y = x$, the coordinates of the point B on the line are $\big(g(x_1), g(x_1)\big)$. But the second term in the sequence established by the method of successive substitutions is $x_2 = g(x_1)$. Hence the x-coordinate of B is x_2. The height of the curve $y = g(x)$ at C is $y = g(x_2)$. The point D has coordinates $\big(g(x_2), g(x_2)\big)$, and hence, the x-coordinate of D is $x_3 = g(x_2)$. Continuation leads to the interpretation of the $\{x_n\}$ as shown in the figure.



(b)

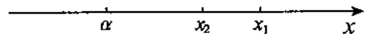


(c)



(d) It appears that the slope of $y = g(x)$ near the required root dictates whether the sequence converges. For slopes near zero (figures in (a) and (b)), the sequence converges, but for large slopes (figure in (c)), the sequence diverges.

(e)



If we apply the mean value theorem (Theorem 3.19) to $g(x)$ on the interval between α and x_1,

$$g(x_1) = g(\alpha) + g'(c)(x_1 - \alpha)$$

where c is between α and x_1. Since $\alpha = g(\alpha)$, $x_2 = g(x_1)$, and $|g'(c)| \le a$, we may write that

$$x_2 = \alpha + g'(c)(x_1 - \alpha) \quad \Longrightarrow \quad |x_2 - \alpha| = |g'(c)||x_1 - \alpha| \le a|x_1 - \alpha|.$$

What this means is that x_2 is closer to α than x_1. If we repeat this procedure for $x_3 = g(x_2)$ on the interval between α and x_2, we obtain

$$|x_3 - \alpha| \le a|x_2 - \alpha| \le a^2|x_1 - \alpha|.$$

Continuation of this process gives $|x_n - \alpha| \le a^{n-1}|x_1 - \alpha|$. It now follows that

$$\lim_{n\to\infty} |x_n - \alpha| \le \lim_{n\to\infty} a^{n-1}|x_1 - \alpha| = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} x_n = \alpha.$$

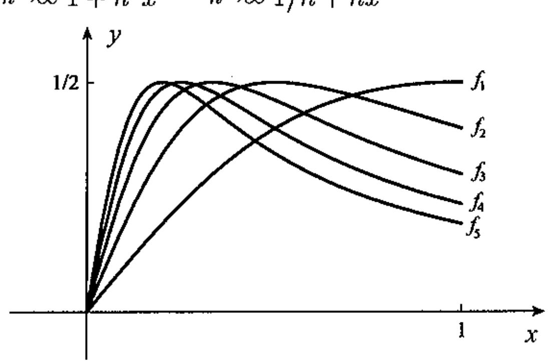
68. Newton's iterative procedure defines the sequence $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$. If we define $F(x) = x - f(x)/f'(x)$, then $x_{n+1} = F(x_n)$. According to part (e) of Exercise 67, a sequence of this type converges to a root $x = \alpha$ of $x = F(x)$ if on the interval $|x - \alpha| \le |x_1 - \alpha|$ we have $|F'(x)| \le a < 1$. Since $F'(x) = 1 - [(f')^2 - ff'']/(f')^2$, we will have convergence if $1 > a \ge \left|1 - \dfrac{(f')^2 - ff''}{(f')^2}\right| = \left|\dfrac{ff''}{(f')^2}\right|$. Thus, Newton's sequence converges to α if on $|x-\alpha| \le |x_1-\alpha|$, $|ff''/(f')^2| \le a < 1$. In other words, if it is possible to choose x_1 close enough to α to guarantee $|ff''/(f')^2| \le a < 1$, on the interval $|x - \alpha| \le |x_1 - \alpha|$, then Newton's sequence converges to α. To show that this is always possible, we let M be the maximum value of $|f''|$ on the open interval containing α in which $f''(x)$ is known to exist. Because $f'(\alpha) \ne 0$, there exists an open interval I containing α in which $f'(x) \ne 0$ (by continuity of $f'(x)$). Let m be the minimum value of $|f'(x)|$ on I. Since $f(x)$ is continuous at $x = \alpha$, where $f(\alpha) = 0$, there exists an open interval $|x - \alpha| < \delta$ contained in I which $|f(x)| < am^2/M$ for any a such that $0 < a < 1$. Consequently, for $|x - \alpha| < \delta$,

$$\left|\frac{ff''}{(f')^2}\right| < \frac{am^2}{M}\frac{M}{m^2} = a < 1.$$

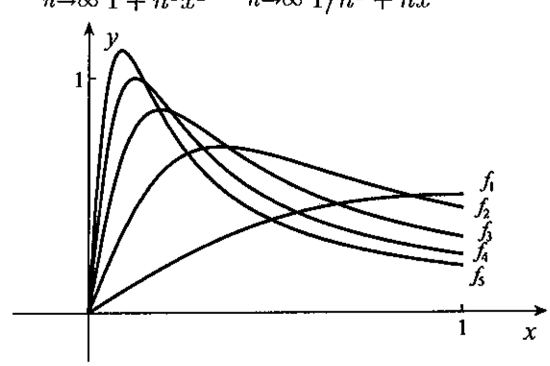
Thus, if $|x_1 - \alpha| = \delta$, we may say that for all x in $|x - a| < |x_1 - \alpha|$, $|ff''/(f')^2| < a < 1$, and Newton's iterative sequence converges to α.

EXERCISES 10.2

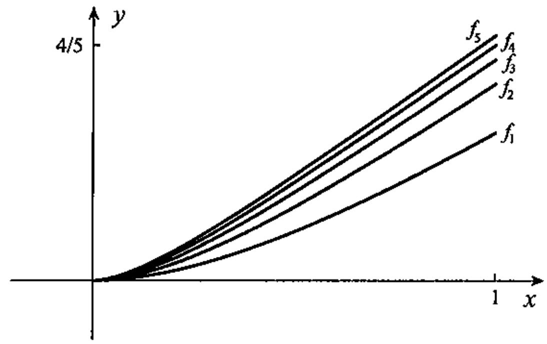
1. The limit function is $f(x) = 0$, since for each x in $0 \le x \le 1$,
$$\lim_{n\to\infty} \frac{nx}{1 + n^2x^2} = \lim_{n\to\infty} \frac{x}{1/n + nx^2} = 0.$$



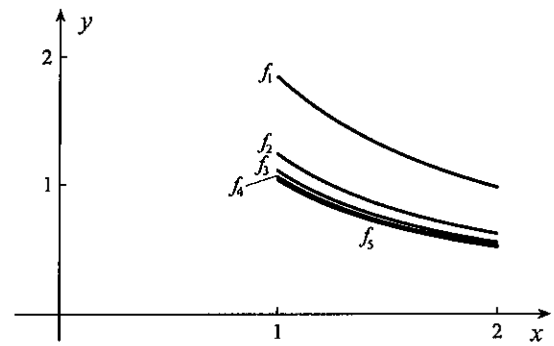
2. The limit function is $f(x) = 0$, since for each x in $0 \le x \le 1$,
$$\lim_{n\to\infty} \frac{n^2x}{1 + n^3x^2} = \lim_{n\to\infty} \frac{x}{1/n^2 + nx^2} = 0.$$



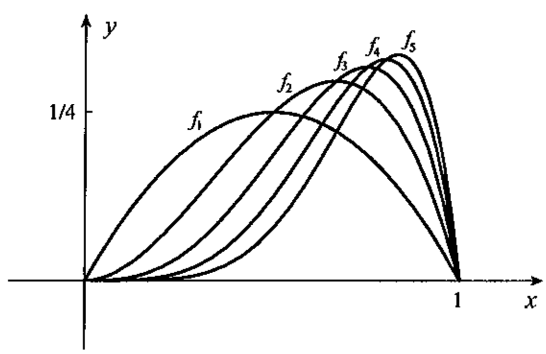
3. The limit function is $f(x) = x$, since for each x in $0 \le x \le 1$,
$$\lim_{n\to\infty} \frac{nx^2}{1+nx} = \lim_{n\to\infty} \frac{x^2}{1/n + x} = x.$$



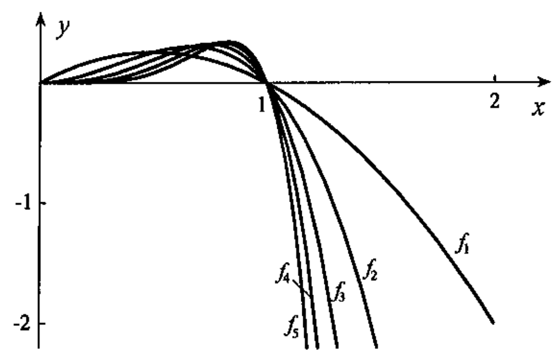
4. The limit function is $f(x) = 1/x$.



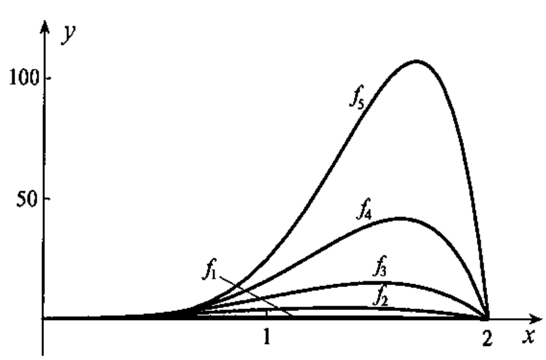
5. Since $f_n(0) = f_n(1) = 0$, the limit function $f(x)$ has values $f(0) = f(1) = 0$. For fixed x in $0 < x < 1$,
$$f(x) = \lim_{n\to\infty} nx^n(1-x) = \lim_{n\to\infty} \frac{n(1-x)}{x^{-n}}$$
$$= \lim_{n\to\infty} \frac{1-x}{-x^{-n}\ln x} = \lim_{n\to\infty} \frac{x^n(x-1)}{\ln x} = 0.$$



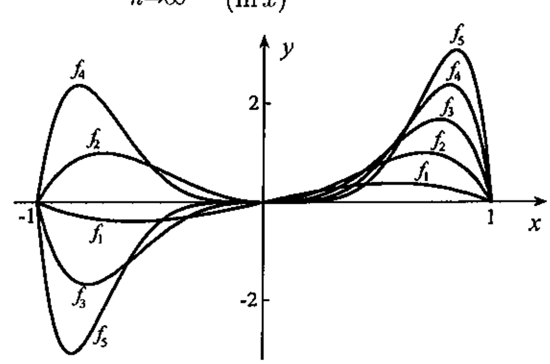
6. There is no limit function.



7. There is no limit function.

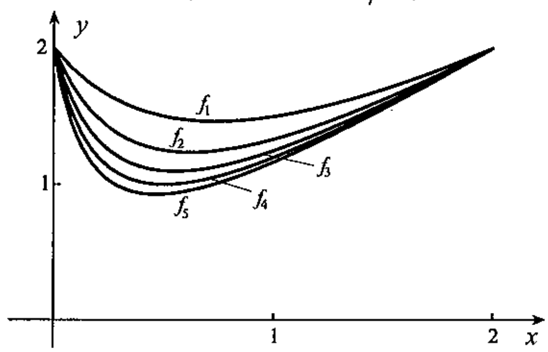


8. Since $f_n(0) = f_n(1) = 0$, the limit function $f(x)$ has values $f(0) = f(1) = 0$. For fixed x in $0 < x < 1$,
$$f(x) = \lim_{n\to\infty} n^2x^n(1-x^2) = \lim_{n\to\infty} \frac{n^2(1-x^2)}{x^{-n}}$$
$$= \lim_{n\to\infty} \frac{2n(1-x^2)}{-x^{-n}\ln x} = \lim_{n\to\infty} \frac{2(1-x^2)}{x^{-n}(\ln x)^2}$$
$$= \lim_{n\to\infty} \frac{2(1-x^2)x^n}{(\ln x)^2} = 0.$$

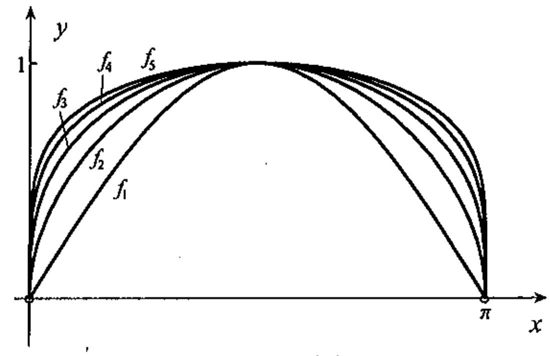


9. The limit function $f(x)$ has value 2 at $x = 0$, and for all other vaues of x,

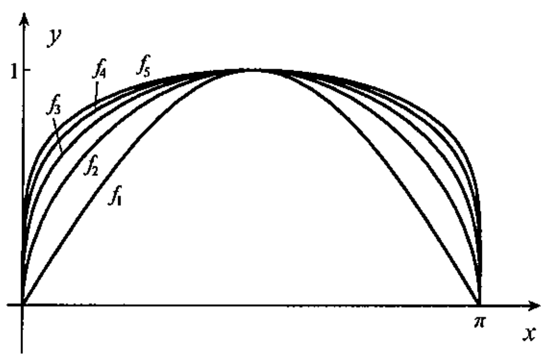
$$f(x) = \lim_{n\to\infty} \frac{2 + nx^2}{1 + nx} = \lim_{n\to\infty} \frac{2/n + x^2}{1/n + x} = x.$$



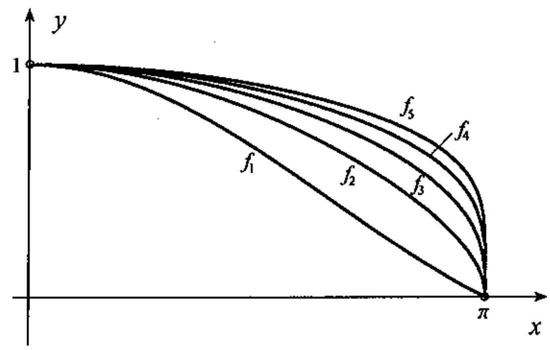
10. The limit function is $f(x) = 1$.



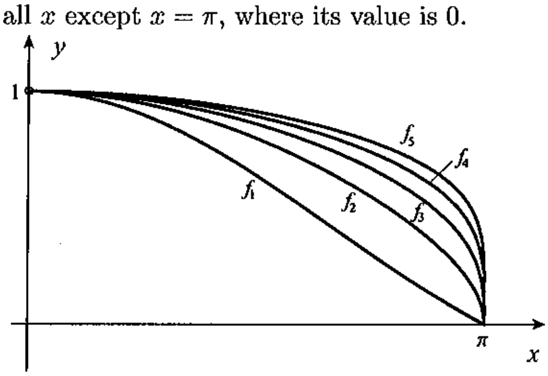
11. The limit function $f(x)$ has value 1 for all x except $x = 0, \pi$, where its value is 0.



12. The limit function is $f(x) = 1$.

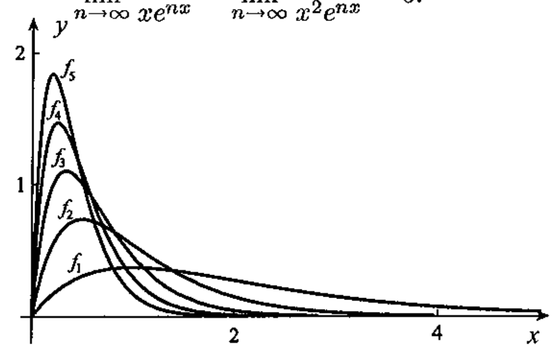


13. The limit function $f(x)$ has value 1 for all x except $x = \pi$, where its value is 0.



14. The limit function is $f(x) = \lim_{n\to\infty} \dfrac{n^2 x}{e^{nx}}$

$$= \lim_{n\to\infty} \frac{2nx}{xe^{nx}} = \lim_{n\to\infty} \frac{2x}{x^2 e^{nx}} = 0.$$

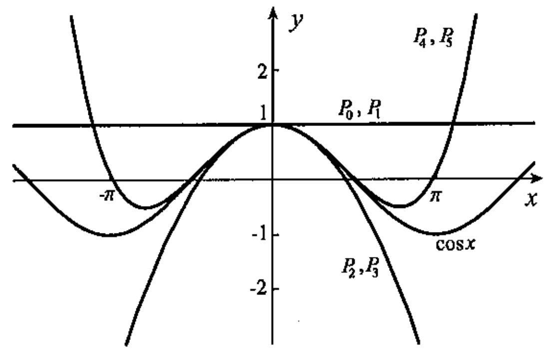


15. The sequence $\{x^n\}$ converges to 0 for $-1 < x < 1$, to 1 for $x = 1$, and diverges for all other values of x. Hence, the sequence $\{(1 - x^n)/(1 - x)\}$ converges to $1/(1 - x)$ for $-1 < x < 1$ and diverges for all other values of x.

EXERCISES 10.3

1. Since $f(0) = 1$, $f'(0) = -\sin 0 = 0$, $f''(0) = -\cos 0 = -1$, $f'''(0) = \sin 0 = 0$, $f''''(0) = \cos 0 = 1$, etc., Taylor's remainder formula gives

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \text{term in } x^n + R_n,$$



where $R_n = \frac{d^{n+1}}{dx^{n+1}}(\cos x)_{|x=z_n} \frac{x^{n+1}}{(n+1)!}$. The n^{th} derivative of $\cos x$ is $\pm \sin x$ or $\pm \cos x$, so that

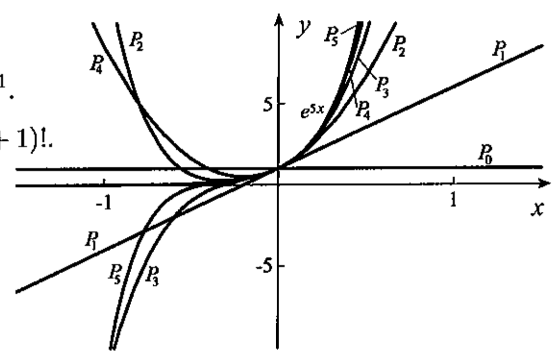
$$\left| \frac{d^{n+1}}{dx^{n+1}} \cos x_{|x=z_n} \right| \le 1.$$

Hence, $|R_n| \le |x|^{n+1}/(n+1)!$. But according to Example 10.5, $\lim_{n\to\infty} |x|^n/n! = 0$ for any x whatsoever. It follows that $\lim_{n\to\infty} R_n = 0$, and the Maclaurin series for $\cos x$ therefore converges to $\cos x$ for all x. We may write

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots, \quad -\infty < x < \infty.$$

2. Since $f^{(n)}(x) = 5^n e^{5x}$, Taylor's remainder formula for e^{5x} and $c = 0$ gives

$$e^{5x} = 1 + 5x + \frac{5^2}{2!}x^2 + \frac{5^3}{3!}x^3 + \cdots + \frac{5^n}{n!}x^n + R_n,$$



where $R_n = \frac{d^{n+1}}{dx^{n+1}}(e^{5x})_{|x=z_n} \frac{x^{n+1}}{(n+1)!} = \frac{5^{n+1}e^{5z_n}}{(n+1)!}x^{n+1}$.
If $x < 0$, then $x < z_n < 0$, and $|R_n| < 5^{n+1}|x|^{n+1}/(n+1)!$. According to Example 10.5, $\lim_{n\to\infty} |x|^n/n! = 0$ for any x whatsoever, and therefore $\lim_{n\to\infty} 5^{n+1}|x|^{n+1}/(n+1)! = 0$ also. Thus, if $x < 0$, $\lim_{n\to\infty} R_n = 0$. If $x > 0$,

then $0 < z_n < x$, and

$$|R_n| < \frac{5^{n+1}e^{5x}}{(n+1)!}|x|^{n+1} = e^{5x}\left(\frac{5^{n+1}|x|^{n+1}}{(n+1)!}\right).$$

But we have just indicated that $\lim_{n\to\infty} 5^{n+1}|x|^{n+1}/(n+1)! = 0$, and therefore $\lim_{n\to\infty} R_n = 0$ for $x > 0$ also. Thus, for any x whatsoever, the sequence $\{R_n\}$ has limit 0, and the Maclaurin series for e^{5x} converges to e^{5x},

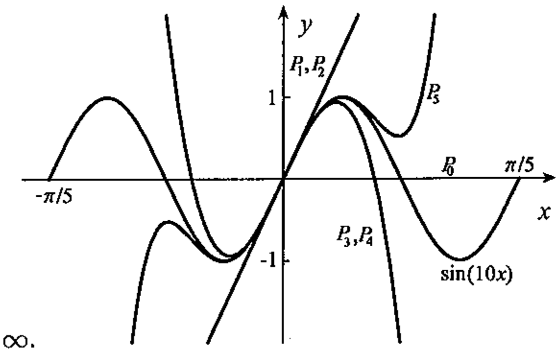
$$e^{5x} = \sum_{n=0}^{\infty} \frac{5^n}{n!}x^n, \quad -\infty < x < \infty.$$

3. Since $f(0) = \sin(0) = 0$, $f'(0) = 10\cos 0 = 10$, $f''(0) = -10^2 \sin 0 = 0$, $f'''(0) = -10^3 \cos 0 = -10^3$, $f''''(0) = 10^4 \sin 0 = 0$, etc., Taylor's remainder formula gives

$$\sin(10x) = 10x - \frac{10^3x^3}{3!} + \frac{10^5x^5}{5!} + \cdots + \text{term in } x^n + R_n,$$

where $R_n = \frac{d^{n+1}}{dx^{n+1}}[\sin(10x)]_{|x=z_n} \frac{x^{n+1}}{(n+1)!}$. The n^{th} derivative of $\sin(10x)$ is $\pm 10^n \sin(10x)$ or $\pm 10^n \cos(10x)$, so that

$$\left| \frac{d^{n+1}}{dx^{n+1}}[\sin(10x)]_{|x=z_n} \right| \leq 10^{n+1}.$$



Hence, $|R_n| \leq 10^{n+1}|x|^{n+1}/(n+1)!$. According to Example 10.5, $\lim_{n\to\infty} |x|^n/n! = 0$ for any x whatsoever, and therefore $\lim_{n\to\infty} 10^{n+1}|x|^{n+1}/(n+1)! = 0$ also. It follows that $\lim_{n\to\infty} R_n = 0$, and the Maclaurin series for $\sin(10x)$ therefore converges to $\sin(10x)$ for all x.

We may write

$$\sin(10x) = 10x - \frac{10^3x^3}{3!} + \frac{10^5x^5}{5!} + \cdots, \quad -\infty < x < \infty.$$

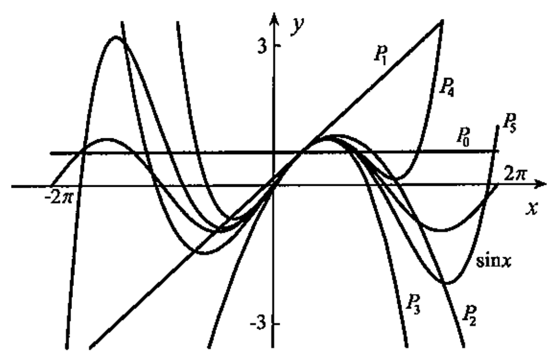
4. Since $f(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$, $f'(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$, $f''(\pi/4) = -\sin(\pi/4) = -1/\sqrt{2}$, $f'''(\pi/4) = -\cos(\pi/4) = -1/\sqrt{2}$, $f''''(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$, etc., Taylor's remainder formula gives

$$\sin x = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(x - \pi/4) - \frac{1}{2!\sqrt{2}}(x - \pi/4)^2 - \frac{1}{3!\sqrt{2}}(x - \pi/4)^3 + \ldots + \text{term in } (x - \pi/4)^n + R_n,$$

where $R_n = \frac{d^{n+1}}{dx^{n+1}}(\sin x)_{|x=z_n} \frac{(x - \pi/4)^{n+1}}{(n+1)!}$.

The n^{th} derivative of $\sin x$ is $\pm \sin x$ or $\pm \cos x$, so that

$$\left| \frac{d^{n+1}}{dx^{n+1}}(\sin x)_{|x=z_n} \right| \leq 1.$$



Hence, $|R_n| \leq |x - \pi/4|^{n+1}/(n+1)!$. According to Example 10.5, $\lim_{n\to\infty} |x|^n/n! = 0$ for any x whatsoever, and therefore $\lim_{n\to\infty} |x - \pi/4|^{n+1}/(n+1)! = 0$ also. It follows that $\lim_{n\to\infty} R_n = 0$, and the Taylor series for $\sin x$ about $\pi/4$ therefore converges to $\sin x$ for all x.

We may write

$$\sin x = \frac{1}{\sqrt{2}} \left[1 + (x - \pi/4) - \frac{1}{2!}(x - \pi/4)^2 - \frac{1}{3!}(x - \pi/4)^3 + \cdots \right], \quad -\infty < x < \infty.$$

5. Since $f^{(n)}(x) = 2^n e^{2x}$, Taylor's remainder formula for e^{2x} and $c = 1$ gives

$$e^{2x} = e^2 + 2e^2(x-1) + \frac{2^2e^2}{2!}(x-1)^2 + \frac{2^3e^2}{3!}(x-1)^3 + \cdots + \frac{2^ne^2}{n!}(x-1)^n + R_n,$$

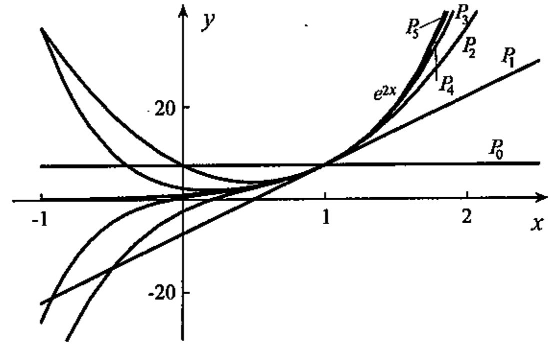
where $R_n = \frac{d^{n+1}}{dx^{n+1}}(e^{2x})_{|x=z_n} \frac{(x-1)^{n+1}}{(n+1)!} = \frac{2^{n+1}e^{2z_n}}{(n+1)!}(x-1)^{n+1}$. If $x < 1$, then $x < z_n < 1$, and $|R_n| < 2^{n+1}e^2|x-1|^{n+1}/(n+1)!$. According to Example 10.5, $\lim_{n\to\infty} |x|^n/n! = 0$ for any x whatsoever, and therefore $\lim_{n\to\infty} 2^{n+1}e^2|x-1|^{n+1}/(n+1)! = 0$ also.

Thus, if $x < 1$, $\lim_{n\to\infty} R_n = 0$. If $x > 1$, then $1 < z_n < x$, and

$$|R_n| < \frac{2^{n+1}e^{2x}}{(n+1)!}|x-1|^{n+1} = e^{2x}\left[\frac{2^{n+1}|x-1|^{n+1}}{(n+1)!}\right].$$

But we have just indicated that $\lim_{n\to\infty} 2^{n+1}|x-1|^{n+1}/(n+1)! = 0$, and therefore $\lim_{n\to\infty} R_n = 0$ for $x > 1$ also. Thus, for any x whatsoever, the sequence $\{R_n\}$ has limit 0, and the Taylor series for e^{2x} converges to e^{2x},

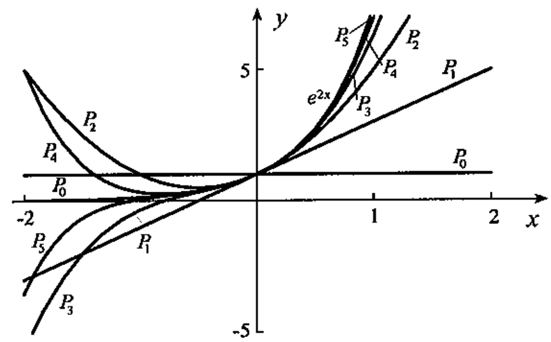
$$e^{2x} = \sum_{n=0}^{\infty} \frac{2^n e^2}{n!}(x-1)^n, \quad -\infty < x < \infty.$$



6. Since $f^{(n)}(0) = 2^n$, the Maclaurin series for e^{2x} is

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}x^n = 1 + 2x + \frac{2^2x^2}{2!} + \cdots.$$

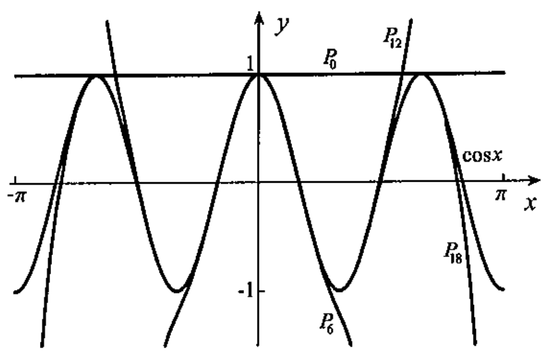
Plots of the polynomials suggest that the series converges to e^{2x} for all x.



7. Since $f(0) = 1$, $f'(0) = 0$, $f''(0) = -3^2$, $f'''(0) = 0$, $f''''(0) = 3^4$, etc., the Maclaurin series for $\cos 3x$ is

$$1 - \frac{3^2x^2}{2!} + \frac{3^4x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n}}{(2n)!}x^{2n}.$$

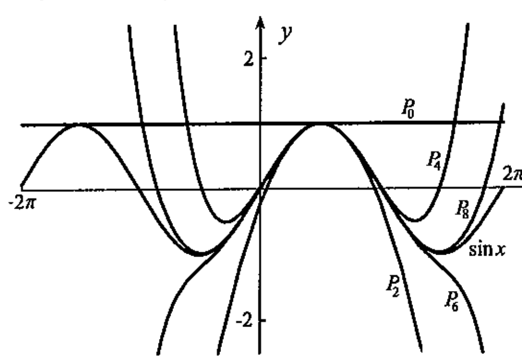
Plots of the polynomials suggest that the series converges to $\cos 3x$ for all x.



8. Since $f(\pi/2) = 1$, $f'(\pi/2) = 0$, $f''(\pi/2) = -1$, $f'''(\pi/2) = 0$, and $f''''(\pi/2) = 1$, the Taylor series for $\sin x$ about $x = \pi/2$ is

$$1 - \frac{(x-\pi/2)^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x-\pi/2)^{2n}.$$

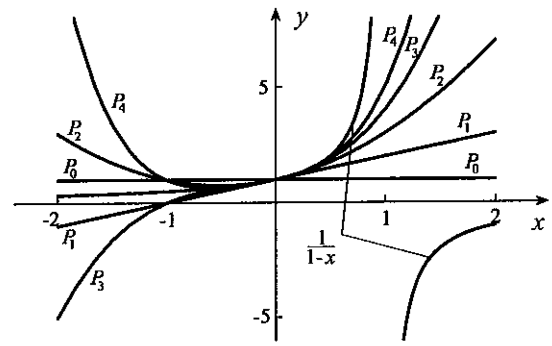
Plots of the polynomials suggest that the series converges to $\sin x$ for all x.



9. Since $f^{(n)}(0) = n!$, the Maclaurin series for $1/(1-x)$ is

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots .$$

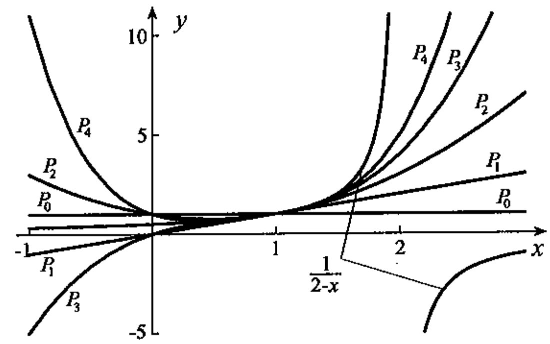
Plots of the polynomials suggest that the series converges to $1/(1-x)$ for $-1 < x < 1$.



10. Since $f^{(n)}(1) = n!$, the Taylor series for $1/(2-x)$ about $x = 1$ is

$$\sum_{n=0}^{\infty} (x-1)^n = 1 + (x-1) + (x-1)^2 + \cdots .$$

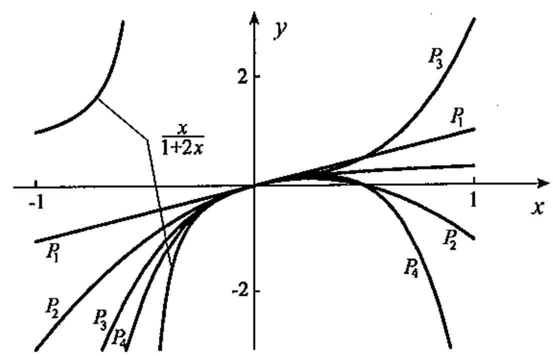
Plots of the polynomials suggest that the series converges to $1/(2-x)$ only for $0 < x < 2$.



11. By writing $f(x)$ in the form $1/2 - (1/2)/(1+2x)$ and taking derivatives, we quickly discover that $f^{(n)}(0) = (-1)^{n+1}2^{n-1}n!$ for $n \geq 1$. The Maclaurin series for $f(x)$ is therefore

$$\sum_{n=1}^{\infty} (-1)^{n+1}2^{n-1}x^n = x - 2x^2 + 4x^3 - 8x^4 + \cdots .$$

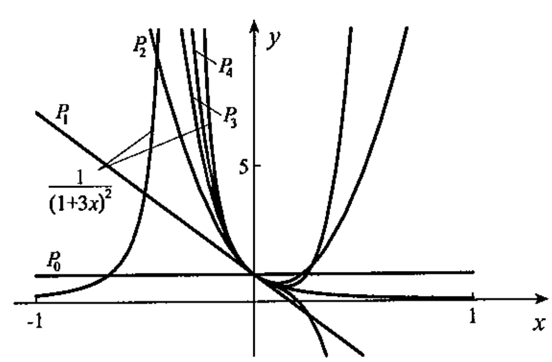
Plots of the polynomials suggest that the series converges to $x/(1+2x)$ only for $-1/2 < x < 1/2$.



12. Since $f^{(n)}(0) = (-1)^n 3^n (n+1)!$, the Maclaurin series for $1/(1+3x)^2$ is

$$\sum_{n=0}^{\infty} (-1)^n 3^n (n+1) x^n = 1 - 6x + 3^2(3)x^2 + \cdots .$$

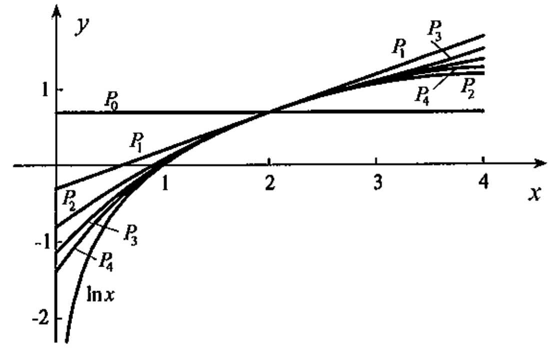
Plots of the polynomials suggest that the series converges to $1/(1+3x)^2$ only for $-1/3 < x < 1/3$.



13. Since $f^{(n)}(2) = (-1)^{n+1}(n-1)!/2^n$ for $n \geq 1$, the Taylor series for $\ln x$ about $x = 2$ is

$$\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\,2^n}(x-2)^n$$
$$= \ln 2 + \frac{(x-2)}{2} - \frac{(x-2)^2}{2 \cdot 2^2} + \cdots .$$

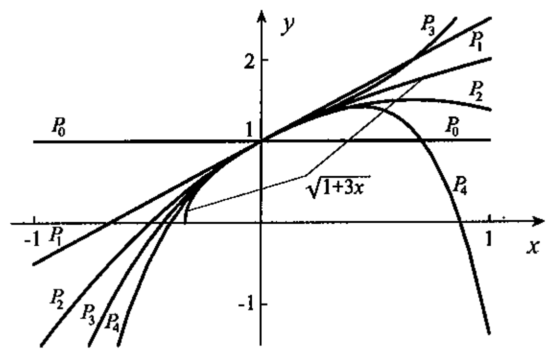
Plots of the polynomials suggest that the series converges to $\ln x$ only for $0 < x < 4$.



14. Calculating derivatives of the function leads to the formula $f^{(n)}(0) = \dfrac{(-1)^{n+1}3^n[1 \cdot 3 \cdot 5 \cdots (2n-3)]}{2^n}$ for $n \geq 2$, together with $f(0) = 1$ and $f'(0) = 3/2$. The Maclaurin series for $\sqrt{1+3x}$ is therefore

$$1 + \frac{3x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}3^n[1 \cdot 3 \cdot 5 \cdots (2n-3)]}{2^n\, n!}x^n.$$

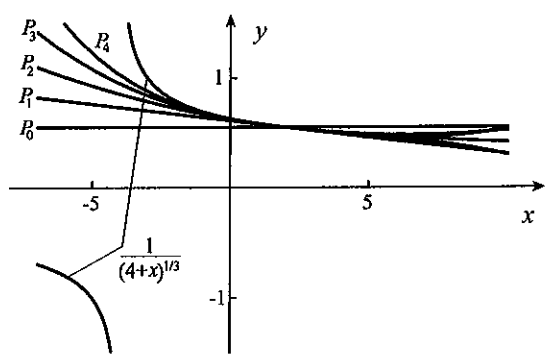
Plots of the polynomials suggest that the series converges to $\sqrt{1+3x}$ only for $-1/3 \leq x \leq 1/3$.



15. Calculating derivatives of the function leads to the formula $f^{(n)}(2) = \dfrac{(-1)^n[1 \cdot 4 \cdot 7 \cdots (3n-2)]}{2^n 3^{2n} 6^{1/3}}$ for $n \geq 1$. The Taylor series for $1/(4+x)^{1/3}$ about $x = 2$ is therefore

$$\frac{1}{6^{1/3}} + \sum_{n=1}^{\infty} \frac{(-1)^n[1 \cdot 4 \cdot 7 \cdots (3n-2)]}{2^n 3^{2n} 6^{1/3} n!}(x-2)^n.$$

Plots of the polynomials suggest that the series converges to $1/(4+x)^{1/3}$ only for $-4 < x < 8$.



16. If I' is the open interval in which $f'(x)$ and $f''(x)$ are continuous, and we apply Taylor's remainder formula to $f(x)$ at x_0 in I', we obtain

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(z_1)}{2!}(x-x_0)^2 = f(x_0) + \frac{f''(z_1)}{2}(x-x_0)^2,$$

where z_1 is between x_0 and x. Suppose that $f''(x_0) > 0$. Because $f''(x)$ is continuous at x_0, there exists an open interval I containing x_0 in which $f''(x) > 0$. For any x in this interval, it follows that $f''(z_1) > 0$ also. As a result, for any x in I, $f(x) > f(x_0)$, and $f(x)$ must have a relative minimum at x_0. A similar discussion shows that when $f''(x_0) < 0$, the function has a relative maximum at x_0. If $f''(x_0) = 0$, no conclusion can be reached.

17. If I' is the open interval in which $f(x)$ has derivatives of all orders, and we apply Taylor's remainder formula to $f(x)$ at x_0 in I', we obtain

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(z_n)}{(n+1)!}(x-x_0)^{n+1}$$
$$= f(x_0) + \frac{f^{(n+1)}(z_n)}{(n+1)!}(x-x_0)^{n+1}$$

where z_n is between x_0 and x.

(i) Consider first the case that n is even, and suppose that $f^{(n+1)}(x_0) > 0$. (A similar proof follows in the case that $f^{(n+1)}(x_0) < 0$.) Because $f^{(n+1)}(x)$ is continuous at x_0, there exists an open interval I containing x_0 in which $f^{(n+1)}(x) > 0$. For any x in this interval, it follows that $f^{(n+1)}(z_n) > 0$ also. As a result, when $x < x_0$, $f(x) < f(x_0)$, and when $x > x_0$, $f(x) > f(x_0)$. This implies that x_0 must yield a horizontal point of inflection.

(ii) Consider now when n is odd and $f^{(n+1)}(x_0) > 0$. In this case, for any x in I, $f(x) > f(x_0)$ and $f(x)$ must have a relative minimum at x_0.

(iii) When n is odd and $f^{(n+1)}(x_0) < 0$, $f(x) < f(x_0)$ in I, and $f(x)$ has a relative maximum at x_0.

18. (a) This follows from $\displaystyle\int_c^x f'(t)\,dt = \Big\{f(t)\Big\}_c^x = f(x) - f(c)$.

(b) If we set $u = f'(t)$, $du = f''(t)\,dt$, $dv = dt$, and $v = t - x$, then

$$f(x) = f(c) + \Big\{(t-x)f'(t)\Big\}_c^x - \int_c^x (t-x)f''(t)\,dt = f(c) + f'(c)(x-c) + \int_c^x (x-t)f''(t)\,dt.$$

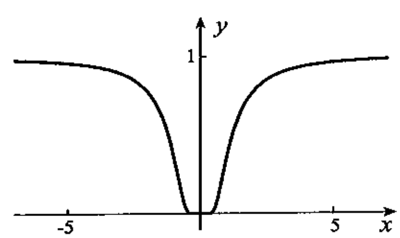
(c) If we now set $u = f''(t)$, $du = f'''(t)\,dt$, $dv = (x-t)\,dt$, and $v = -(1/2)(x-t)^2$,

$$\begin{aligned} f(x) &= f(c) + f'(c)(x-c) + \left\{-\frac{(x-t)^2 f''(t)}{2}\right\}_c^x - \int_c^x -\frac{1}{2}(x-t)^2 f'''(t)\,dt \\ &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{1}{2}\int_c^x (x-t)^2 f'''(t)\,dt. \end{aligned}$$

(d) One more integration by parts should convince us that the formula is correct. If we set $u = f'''(t)$, $du = f''''(t)\,dt$, $dv = (x-t)^2\,dt$, and $v = -(1/3)(x-t)^3$,

$$\begin{aligned} f(x) &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{1}{2}\left\{-\frac{(x-t)^3 f'''(t)}{3}\right\}_c^x - \frac{1}{2}\int_c^x -\frac{1}{3}(x-t)^3 f''''(t)\,dt \\ &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \frac{1}{3!}\int_c^x (x-t)^3 f''''(t)\,dt. \end{aligned}$$

19. (a) Limits as $x \to 0^+$ and $x \to \infty$ together with symmetry about the y-axis give the graph to the right.



(b) If we can show that

$$\lim_{x\to 0^+} \frac{e^{-1/x^2}}{x^n} = 0,$$

then the limit from the left must also be zero.

Suppose we set $L = \displaystyle\lim_{x\to 0^+} \frac{e^{-1/x^2}}{x^n}$, and take logarithms,

$$\ln L = -\lim_{x\to 0^+}\left(\frac{1}{x^2} + n\ln x\right) = -\lim_{x\to 0^+}\left(\frac{1 + nx^2\ln x}{x^2}\right).$$

Since $\displaystyle\lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x^2} = \lim_{x\to 0^+}\frac{1/x}{-2/x^3} = \lim_{x\to 0^+}(-x^2/2) = 0$, it follows that $\ln L \to -\infty$ as $x \to 0^+$. Therefore, $L = 0$.

(c) $f'(0) = \displaystyle\lim_{h\to 0}\frac{f(h) - f(0)}{h} = \lim_{h\to 0}\frac{e^{-1/h^2}}{h} = 0$, by part (b). Suppose that k is some integer for which $f^{(k)}(0) = 0$. Then

$$f^{(k+1)}(0) = \lim_{h\to 0}\frac{f^{(k)}(h) - f^k(0)}{h} = \lim_{h\to 0}\frac{f^{(k)}(h)}{h}.$$

Now, any number of differentiations of $f(x) = e^{-1/x^2}$ gives rise to terms of the form $Ae^{-1/x^2}/x^n$, where n is a positive integer, and A is a constant. It follows that $f^{(k)}(h)/h$ must consist of terms of the form $Ae^{-1/h^2}/h^n$ which have limit zero as $h \to 0$. Thus, $f^{(k+1)}(0) = 0$, and by mathematical induction, $f^{(n)}(0) = 0$ for all $n \geq 1$.

(d) The Maclaurin series for $f(x)$ is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots = 0 + 0 + 0 + \cdots .$$

(e) This series converges to $f(x)$ only at $x = 0$.

EXERCISES 10.4

1. Since the radius of convergence is $R = \lim_{n\to\infty} \left|\frac{1/n}{1/(n+1)}\right| = 1$, the open interval of convergence is $-1 < x < 1$.

2. Since the radius of convergence is $R = \lim_{n\to\infty} \left|\frac{n^2}{(n+1)^2}\right| = 1$, the open interval of convergence is $-1 < x < 1$.

3. Since the radius of convergence is $R = \lim_{n\to\infty} \left|\frac{1/(n+1)^3}{1/(n+2)^3}\right| = 1$, the open interval of convergence is $-1 < x < 1$.

4. Since the radius of convergence is $R = \lim_{n\to\infty} \left|\frac{n^2 3^n}{(n+1)^2 3^{n+1}}\right| = \frac{1}{3}$, the open interval of convergence is $-1/3 < x < 1/3$.

5. Since the radius of convergence is $R = \lim_{n\to\infty} \left|\frac{1/2^n}{1/2^{n+1}}\right| = 2$, the open interval of convergence is $-1 < x < 3$.

6. Since the radius of convergence is $R = \lim_{n\to\infty} \left|\frac{(-1)^n n^3}{(-1)^{n+1}(n+1)^3}\right| = 1$, the open interval of convergence is $-4 < x < -2$.

7. Since the radius of convergence is $R = \lim_{n\to\infty} \left|\frac{1/\sqrt{n}}{1/\sqrt{n+1}}\right| = 1$, the open interval of convergence is $-3 < x < -1$.

8. Since the radius of convergence is $R = \lim_{n\to\infty} \left|\frac{2^n\left(\frac{n-1}{n+2}\right)^2}{2^{n+1}\left(\frac{n}{n+3}\right)^2}\right| = \frac{1}{2}$, the open interval of convergence is $7/2 < x < 9/2$.

9. If we set $y = x^2$, then $\sum_{n=1}^{\infty} \frac{1}{n^2}x^{2n} = \sum_{n=1}^{\infty} \frac{1}{n^2}y^n$. Since $R_y = \lim_{n\to\infty} \left|\frac{1/n^2}{1/(n+1)^2}\right| = 1$, it follows that $R_x = \sqrt{R_y} = 1$. The open interval of convergence is therefore $-1 < x < 1$.

10. If we set $y = x^3$, then $\sum_{n=0}^{\infty} (-1)^n x^{3n} = \sum_{n=0}^{\infty} (-1)^n y^n$. Since $R_y = \lim_{n\to\infty} \left|\frac{(-1)^n}{(-1)^{n+1}}\right| = 1$, it follows that $R_x = {R_y}^{1/3} = 1$. The open interval of convergence is therefore $-1 < x < 1$.

11. Since the radius of convergence is $R = \lim_{n\to\infty} \left|\frac{2^n(n-1)/(n+1)}{2^{n+1}n/(n+2)}\right| = 1/2$, the open interval of convergence is $-1/2 < x < 1/2$.

12. If we set $y = x^3$, then $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}x^{3n+1} = y^{1/3}\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}y^n$. Since $R_y = \lim_{n\to\infty} \left|\frac{1/\sqrt{n+1}}{1/\sqrt{n+2}}\right| = 1$, it follows that $R_x = R_y^{1/3} = 1$. The open interval of convergence is therefore $-1 < x < 1$.

13. If we set $y = x^2$, then $\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} x^{2n+1} = \pm\sqrt{y} \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} y^n$. Since $R_y = \lim_{n\to\infty} \left| \frac{(-1)^n/3^n}{(-1)^{n+1}/3^{n+1}} \right| = 3$, it follows that $R_x = \sqrt{R_y} = \sqrt{3}$. The open interval of convergence is therefore $-\sqrt{3} < x < \sqrt{3}$.

14. Since the radius of convergence is $R = \lim_{n\to\infty} \left| \frac{(-e)^n/n^2}{(-e)^{n+1}/(n+1)^2} \right| = \frac{1}{e}$, the open interval of convergence is $-1/e < x < 1/e$.

15. Since the radius of convergence is $R = \lim_{n\to\infty} \left| \frac{n^2/3^{2n}}{(n+1)^2/3^{2n+2}} \right| = 9$, the open interval of convergence is $-9 < x < 9$.

16. Since the radius of convergence is $R = \lim_{n\to\infty} \left| \frac{n^n}{(n+1)^{n+1}} \right| = \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^n \left(\frac{1}{n+1} \right) = \frac{1}{e}(0) = 0$, the series converges only for $x = 0$.

17. Since the radius of convergence is $R = \lim_{n\to\infty} \left| \frac{1/n^2}{1/(n+1)^2} \right| = 1$, the open interval of convergence is $-11 < x < -9$.

18. Since the radius of convergence is $R = \lim_{n\to\infty} \left| \frac{n^3 3^n}{(n+1)^3 3^{n+1}} \right| = \frac{1}{3}$, the open interval of convergence is $-1/3 < x < 1/3$.

19. If we set $y = x^2$, then $\sum_{n=1}^{\infty} \frac{3^n}{(n+1)^2} x^{2n} = \sum_{n=1}^{\infty} \frac{3^n}{(n+1)^2} y^n$. Since $R_y = \lim_{n\to\infty} \left| \frac{3^n/(n+1)^2}{3^{n+1}/(n+2)^2} \right| = 1/3$, it follows that $R_x = \sqrt{R_y} = 1/\sqrt{3}$. The open interval of convergence is therefore $-1/\sqrt{3} < x < 1/\sqrt{3}$.

20. If we set $y = x^3$, the series becomes $\sum_{n=0}^{\infty} y^n/5^n$. Since $R_y = \lim_{n\to\infty} \left| \frac{1/5^n}{1/5^{n+1}} \right| = 5$, it follows that $R_x = {R_y}^{1/3} = 5^{1/3}$. The open interval of convergence is therefore $-5^{1/3} < x < 5^{1/3}$.

21. Using L'Hôpital's rule, $R = \lim_{n\to\infty} \left| \frac{1/\ln n}{1/\ln(n+1)} \right| = \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} = \lim_{n\to\infty} \frac{1/(n+1)}{1/n} = 1$. The open interval of convergence is therefore $-1 < x < 1$.

22. Using L'Hôpital's rule, $R = \lim_{n\to\infty} \left| \dfrac{\dfrac{1}{n^2 \ln n}}{\dfrac{1}{(n+1)^2 \ln(n+1)}} \right| = \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} = \lim_{n\to\infty} \frac{1/(n+1)}{1/n} = 1$. The open interval of convergence is therefore $-1 < x < 1$.

23. Since $R = \lim_{n\to\infty} \left| \frac{(n!)^3/(3n)!}{[(n+1)!]^3/(3n+3)!} \right| = \lim_{n\to\infty} \frac{(n!)^3(3n+3)(3n+2)(3n+1)(3n)!}{(3n)!(n+1)^3(n!)^3} = 27$, the open interval of convergence is $-27 < x < 27$.

24. Since $R = \lim_{n\to\infty} \left| \frac{2\cdot 4\cdot 6\cdots(2n)}{3\cdot 5\cdot 7\cdots(2n+1)} \frac{3\cdot 5\cdots(2n+3)}{2\cdot 4\cdots(2n+2)} \right| = \lim_{n\to\infty} \frac{2n+3}{2n+2} = 1$, the open interval of convergence is $-1 < x < 1$.

25. Since $R = \lim_{n\to\infty} \left| \dfrac{\dfrac{[1\cdot 3\cdots(2n+1)]^2}{2^{2n}(2n)!}}{\dfrac{[1\cdot 3\cdots(2n+3)]^2}{2^{2n+2}(2n+2)!}} \right| = \lim_{n\to\infty} \frac{4(2n+2)(2n+1)}{(2n+3)^2} = 4$, the open interval of convergence is $-4 < x < 4$.

26. $\sum_{n=0}^{\infty} \frac{1}{4^n} x^{3n} = \sum_{n=0}^{\infty} \left(\frac{x^3}{4} \right)^n = \frac{1}{1 - x^3/4} = \frac{4}{4 - x^3}$ provided $\left| \frac{x^3}{4} \right| < 1 \implies |x| < 4^{1/3}$

27. $\sum_{n=1}^{\infty} (-e)^n x^n = \sum_{n=1}^{\infty} (-ex)^n = \frac{-ex}{1 + ex}$ provided $|-ex| < 1 \implies |x| < 1/e$

28. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3^{2n}}(x-1)^n=\sum_{n=1}^{\infty}\left(\frac{x-1}{9}\right)^n=\frac{\frac{x-1}{9}}{1-\frac{x-1}{9}}=\frac{x-1}{10-x}$ provided $\left|\frac{x-1}{9}\right|<1\implies|x-1|<9$

29. $\displaystyle\sum_{n=2}^{\infty}(x+5)^{2n}=\frac{(x+5)^4}{1-(x+5)^2}$ provided $|(x+5)^2|<1\implies|x+5|<1$

30. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{4n}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(x^2)^{2n}=\cos(x^2)$ valid for all x

31. $\displaystyle\sum_{n=0}^{\infty}\frac{5^n}{n!}x^n=\sum_{n=0}^{\infty}\frac{1}{n!}(5x)^n=e^{5x}$ valid for all x

32. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{2n+1}(2n+1)!}x^{2n+2}=x\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{x}{3}\right)^{2n+1}=x\sin(x/3)$ valid for all x

33. $\displaystyle\sum_{n=0}^{\infty}\frac{(-3)^n}{n!}(x+1)^n=\sum_{n=0}^{\infty}\frac{1}{n!}[-3(x+1)]^n=e^{-3(x+1)}$ valid for all x

34. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}x^n=-1+\sum_{n=0}^{\infty}\frac{1}{n!}(-x)^n=-1+e^{-x}$ valid for all x

35. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)!}(x+1)^{2n+3}=-(x+1)^2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(x+1)^{2n+1}=-(x+1)^2\sin(x+1)$ valid for all x

36. $\displaystyle\sum_{n=0}^{\infty}\frac{2^n}{n!}(x-1/2)^n=\sum_{n=0}^{\infty}\frac{1}{n!}(2x-1)^n=e^{2x-1}$ valid for all x

37. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(2n)!}x^{4n+4}=x^4\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{x^2}{2}\right)^{2n}=x^4\cos(x^2/2)$ valid for all x

38. (a) $\displaystyle J_0(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(n!)^2}x^{2n}=1-\frac{x^2}{2^2}+\frac{x^4}{2^4(2!)^2}-\frac{x^6}{2^6(3!)^2}+\frac{x^8}{2^8(4!)^2}-\cdots$

$$J_1(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+1}(n!)(n+1)!}x^{2n+1}=\frac{x}{2}-\frac{x^3}{2^3 2!}+\frac{x^5}{2^5 2!3!}-\frac{x^7}{2^7 3!4!}+\frac{x^9}{2^9 4!5!}-\cdots$$

$$J_m(x)=\frac{x^m}{2^m m!}-\frac{x^{m+2}}{2^{m+2}(m+1)!}+\frac{x^{m+4}}{2^{m+4}2!(m+2)!}-\frac{x^{m+6}}{2^{m+6}3!(m+3)!}+\frac{x^{m+8}}{2^{m+8}4!(m+4)!}-\cdots\cdots$$

(b) $\displaystyle R=\lim_{n\to\infty}\left|\frac{(-1)^n}{2^{2n+m}n!(n+m)!}\frac{2^{2n+m+2}(n+1)!(n+m+1)!}{(-1)^{n+1}}\right|=\lim_{n\to\infty}2^2(n+1)(n+m+1)=\infty$
The interval of convergence is therefore $-\infty<x<\infty$.

39. (a) $\displaystyle 1+\sum_{n=1}^{\infty}\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\,\gamma(\gamma+1)\cdots(\gamma+n-1)}x^n$

(b) $\displaystyle R=\lim_{n\to\infty}\left|\frac{\dfrac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\,\gamma(\gamma+1)\cdots(\gamma+n-1)}}{\dfrac{\alpha(\alpha+1)\cdots(\alpha+n)\beta(\beta+1)\cdots(\beta+n)}{(n+1)!\,\gamma(\gamma+1)\cdots(\gamma+n)}}\right|=\lim_{n\to\infty}\frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)}=1$

EXERCISES 10.5

1. $\displaystyle\frac{1}{3x+2}=\frac{1}{2(1+3x/2)}=\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{3x}{2}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n3^n}{2^{n+1}}x^n,\quad |-3x/2|<1\implies|x|<2/3$

2. $f(x) = \dfrac{1}{4+x^2} = \dfrac{1}{4}\left(\dfrac{1}{1+x^2/4}\right) = \dfrac{1}{4}\displaystyle\sum_{n=0}^{\infty}\left(-\dfrac{x^2}{4}\right)^n = \sum_{n=0}^{\infty}\dfrac{(-1)^n}{4^{n+1}}x^{2n}, \quad \left|-\dfrac{x^2}{4}\right| < 1 \Longrightarrow |x| < 2$

3. Since $\cos x = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}, \quad -\infty < x < \infty,$ it follows that

$$\cos(x^2) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(x^2)^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{4n}, \quad -\infty < x < \infty.$$

4. Since $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}x^n, \quad -\infty < x < \infty,$ it follows that

$$e^{5x} = \sum_{n=0}^{\infty}\frac{1}{n!}(5x)^n = \sum_{n=0}^{\infty}\frac{5^n}{n!}x^n, \quad -\infty < x < \infty.$$

5. Since $f(x) = e^x = e^3\,e^{x-3}$, and the Maclaurin series $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}x^n$ converges for all x, it follows that

$$e^x = e^3\sum_{n=0}^{\infty}\frac{1}{n!}(x-3)^n = \sum_{n=0}^{\infty}\frac{e^3}{n!}(x-3)^n, \quad -\infty < x < \infty.$$

6. Since $f(x) = e^{1-2x} = e\,e^{-2x}$, and the Maclaurin series $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}x^n$ converges for all x, it follows that

$$e^{1-2x} = e\sum_{n=0}^{\infty}\frac{1}{n!}(-2x)^n = \sum_{n=0}^{\infty}\frac{e(-1)^n2^n}{n!}x^n, \quad -\infty < x < \infty.$$

7. Since $f(x) = e^{1-2x} = e^{3-2(x+1)} = e^3\,e^{-2(x+1)}$, and the Maclaurin series $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}x^n$ converges for all x, it follows that

$$e^{1-2x} = e^3\sum_{n=0}^{\infty}\frac{1}{n!}[-2(x+1)]^n = \sum_{n=0}^{\infty}\frac{e^3(-1)^n2^n}{n!}(x+1)^n, \quad -\infty < x < \infty.$$

8.
$$\begin{aligned}\cosh x &= \frac{1}{2}(e^x + e^{-x}) = \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{1}{n!}x^n + \sum_{n=0}^{\infty}\frac{1}{n!}(-x)^n\right] = \frac{1}{2}\sum_{n=0}^{\infty}\frac{[1+(-1)^n]}{n!}x^n \\ &= \frac{1}{2}\left(2 + \frac{2}{2!}x^2 + \frac{2}{4!}x^4 + \cdots\right) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots = \sum_{n=0}^{\infty}\frac{1}{(2n)!}x^{2n}, \quad -\infty < x < \infty\end{aligned}$$

9.
$$\begin{aligned}\sinh x &= \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{1}{n!}x^n - \sum_{n=0}^{\infty}\frac{1}{n!}(-x)^n\right] = \frac{1}{2}\sum_{n=0}^{\infty}\frac{[1-(-1)^n]}{n!}x^n \\ &= \frac{1}{2}\left(2x + \frac{2}{3!}x^3 + \frac{2}{5!}x^5 + \cdots\right) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}x^{2n+1}, \quad -\infty < x < \infty\end{aligned}$$

10. This function is its own Maclaurin series.

11. Since $f(-2) = 33$, $f'(-2) = -46$, $f''(-2) = 54$, $f'''(-2) = -48$, $f''''(-2) = 24$, and $f^{(n)}(-2) = 0$ for $n \geq 5$, formula 10.17 gives

$$\begin{aligned}f(x) &= 33 - 46(x+2) + \frac{54}{2!}(x+2)^2 - \frac{48}{3!}(x+2)^3 + \frac{24}{4!}(x+2)^4 \\ &= 33 - 46(x+2) + 27(x+2)^2 - 8(x+2)^3 + (x+2)^4.\end{aligned}$$

12. $\dfrac{1}{x+3} = \dfrac{1}{5+(x-2)} = \dfrac{1}{5\left(1+\dfrac{x-2}{5}\right)} = \dfrac{1}{5}\sum_{n=0}^{\infty}\left(-\dfrac{x-2}{5}\right)^n = \sum_{n=0}^{\infty}\dfrac{(-1)^n}{5^{n+1}}(x-2)^n, \quad \left|-\dfrac{x-2}{5}\right| < 1 \implies$
$-3 < x < 7$

13. Long division gives

$$\frac{x}{2x+5} = \frac{1}{2} - \frac{5/2}{2x+5} = \frac{1}{2} - \frac{5/2}{2(x-1)+7} = \frac{1}{2} - \frac{5}{14\left[1+\dfrac{2(x-1)}{7}\right]} = \frac{1}{2} - \frac{5}{14}\sum_{n=0}^{\infty}\left[-\frac{2}{7}(x-1)\right]^n$$

$$= \frac{1}{7} + \sum_{n=1}^{\infty}\frac{5(-1)^{n+1}2^{n-1}}{7^{n+1}}(x-1)^n, \quad \left|-\frac{2(x-1)}{7}\right| < 1 \implies -\frac{5}{2} < x < \frac{9}{2}$$

14. Long division gives

$$\frac{x^2}{3-4x} = -\frac{x}{4} - \frac{3}{16} + \frac{9/16}{3-4x} = -\frac{1}{4}(x-2) - \frac{11}{16} + \frac{9/16}{-5-4(x-2)} = -\frac{11}{16} - \frac{1}{4}(x-2) - \frac{9/80}{1+\dfrac{4(x-2)}{5}}$$

$$= -\frac{11}{16} - \frac{1}{4}(x-2) - \frac{9}{80}\sum_{n=0}^{\infty}\left[-\frac{4}{5}(x-2)\right]^n$$

$$= -\frac{11}{16} - \frac{1}{4}(x-2) - \frac{9}{80}\left[1 - \frac{4}{5}(x-2) + \sum_{n=2}^{\infty}\frac{(-1)^n4^n}{5^n}(x-2)^n\right]$$

$$= -\frac{4}{5} - \frac{4}{25}(x-2) + \sum_{n=2}^{\infty}\frac{9(-1)^{n+1}4^{n-2}}{5^{n+1}}(x-2)^n, \quad \left|-\frac{4(x-2)}{5}\right| < 1 \implies \frac{3}{4} < x < \frac{13}{4}$$

15. With the binomial expansion 10.33b,

$$\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} = 1 - \frac{x}{2} + \frac{(-1/2)(-3/2)}{2!}x^2 + \frac{(-1/2)(-3/2)(-5/2)}{3!}x^3 + \cdots, \quad -1 < x \le 1$$

$$= 1 - \frac{x}{2} + \frac{3}{2^2 2!}x^2 - \frac{3\cdot 5}{2^3 3!}x^3 + \cdots = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n[1\cdot 3\cdot 5\cdots(2n-1)]}{2^n n!}x^n$$

$$= 1 + \sum_{n=1}^{\infty}\frac{(-1)^n[1\cdot 2\cdot 3\cdot 4\cdots(2n)]}{2^n n![2\cdot 4\cdot 6\cdots(2n)]}x^n = \sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{2^{2n}(n!)^2}x^n$$

16. Term-by-term integration of $\dfrac{1}{1+2x} = \sum_{n=0}^{\infty}(-2x)^n = \sum_{n=0}^{\infty}(-1)^n2^nx^n$ gives

$$\frac{1}{2}\ln|1+2x| = \sum_{n=0}^{\infty}\frac{(-1)^n2^n}{n+1}x^{n+1} + C.$$

Setting $x=0$ gives $C=0$, and therefore $\ln|1+2x| = \sum_{n=0}^{\infty}\dfrac{(-1)^n2^{n+1}}{n+1}x^{n+1}$. Since the radius of convergence of the geometric series is $1/2$, this is also the radius of convergence for the series of the logarithm function. The open interval of convergence is therefore $-1/2 < x < 1/2$, and the absolute values may be dropped,

$$\ln(1+2x) = \sum_{n=0}^{\infty}\frac{(-1)^n2^{n+1}}{n+1}x^{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^n}{n}x^n.$$

17. With the binomial expansion 10.33b,

$$
\begin{aligned}
(1+3x)^{3/2} &= 1+\left(\frac{3}{2}\right)(3x)+\frac{(3/2)(1/2)}{2!}(3x)^2+\frac{(3/2)(1/2)(-1/2)}{3!}(3x)^3+\cdots, \quad -1 \le 3x \le 1\\
&= 1+\frac{9}{2}x+\frac{3^3}{2^2 2!}x^2-\frac{3^4}{2^3 3!}x^3+\frac{3^5(1)(3)}{2^4 4!}x^4-\frac{3^6(1)(3)(5)}{2^5 5!}x^5+\cdots\\
&= 1+\frac{9}{2}x+\frac{27}{8}x^2-\frac{27}{16}x^3+\sum_{n=4}^{\infty}\frac{(-1)^n[1\cdot 3\cdot 5\cdots(2n-5)]3^{n+1}}{2^n n!}x^n\\
&= 1+\frac{9}{2}x+\frac{27}{8}x^2-\frac{27}{16}x^3+\sum_{n=4}^{\infty}\frac{(-1)^n[1\cdot 2\cdot 3\cdot 4\cdots(2n-5)(2n-4)]3^{n+1}}{[2\cdot 4\cdots(2n-4)]2^n n!}x^n\\
&= 1+\frac{9}{2}x+\frac{27}{8}x^2-\frac{27}{16}x^3+\sum_{n=4}^{\infty}\frac{(-1)^n(2n-4)!\,3^{n+1}}{2^{2n-2}n!\,(n-2)!}x^n\\
&= 1+\frac{9}{2}x+\sum_{n=2}^{\infty}\frac{(-1)^n(2n-4)!\,3^{n+1}}{2^{2n-2}n!\,(n-2)!}x^n, \quad -\frac{1}{3}\le x\le\frac{1}{3}.
\end{aligned}
$$

18. Termwise integration of

$$\frac{1}{x}=\frac{1}{2+(x-2)}=\frac{1}{2[1+(x-2)/2]}=\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x-2}{2}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(x-2)^n$$

gives $\ln|x|=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)2^{n+1}}(x-2)^{n+1}+C$. Setting $x=2$ gives $C=\ln 2$, and therefore $\ln|x|=\ln 2+\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)2^{n+1}}(x-2)^{n+1}$. Since the radius of convergence of the geometric series is 2, this is also the radius of convergence for the series of the logarithm function. The open interval of convergence is therefore $0<x<4$, and the absolute values may be dropped,

$$\ln x=\ln 2+\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)2^{n+1}}(x-2)^{n+1}=\ln 2+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\,2^n}(x-2)^n.$$

19. Termwise integration of

$$\frac{1}{x+3}=\frac{1}{2+(x+1)}=\frac{1}{2[1+(x+1)/2]}=\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x+1}{2}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(x+1)^n$$

gives $\ln|x+3|=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)2^{n+1}}(x+1)^{n+1}+C$. Setting $x=-1$ gives $C=\ln 2$, and therefore $\ln|x+3|=\ln 2+\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)2^{n+1}}(x+1)^{n+1}$. Since the radius of convergence of the geometric series is 2, this is also the radius of convergence for the series of the logarithm function. The open interval of convergence is therefore $-3<x<1$, and the absolute values may be dropped,

$$\ln(x+3)=\ln 2+\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)2^{n+1}}(x+1)^{n+1}=\ln 2+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\,2^n}(x+1)^n.$$

20. $\frac{1}{x}=\frac{1}{4+(x-4)}=\frac{1}{4[1+(x-4)/4]}=\frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{x-4}{4}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}(x-4)^n$, provided $\left|-\frac{x-4}{4}\right|<1 \Longrightarrow 0<x<8$

21. With the binomial expansion 10.33b,

$$\frac{1}{(x+2)^3} = \frac{1}{8(1+x/2)^3} = \frac{1}{8}\left(1+\frac{x}{2}\right)^{-3} = \frac{1}{8}\left[1+(-3)\left(\frac{x}{2}\right)+\frac{(-3)(-4)}{2!}\left(\frac{x}{2}\right)^2+\frac{(-3)(-4)(-5)}{3!}\left(\frac{x}{2}\right)^3+\cdots\right]$$

$$= \frac{1}{8}\left[1-\frac{3x}{2}+\frac{3\cdot 4}{2^3}x^2-\frac{4\cdot 5}{2^4}x^3+\frac{5\cdot 6}{2^5}x^4+\cdots\right]$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(n+2)}{2^{n+4}}x^n, \quad \text{valid for } -1<\frac{x}{2}<1 \Longrightarrow -2<x<2.$$

22. With the binomial expansion 10.33b,

$$\frac{1}{(2-x)^2} = \frac{1}{[-1-(x-3)]^2} = \frac{1}{[1+(x-3)]^2} = [1+(x-3)]^{-2}$$

$$= 1-2(x-3)+\frac{(-2)(-3)}{2!}(x-3)^2+\frac{(-2)(-3)(-4)}{3!}(x-3)^3+\cdots$$

$$= \sum_{n=0}^{\infty}(-1)^n(n+1)(x-3)^n, \quad \text{provided } -1<x-3<1 \Longrightarrow 2<x<4.$$

23. With the binomial expansion 10.33b,

$$\frac{1}{(x+3)^2} = \frac{1}{[4+(x-1)]^2} = \frac{1}{16[1+(x-1)/4]^2} = \frac{1}{16}\left(1+\frac{x-1}{4}\right)^{-2}$$

$$= \frac{1}{16}\left[1-2\left(\frac{x-1}{4}\right)+\frac{(-2)(-3)}{2!}\left(\frac{x-1}{4}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{x-1}{4}\right)^3+\cdots\right]$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{4^{n+2}}(x-1)^n, \quad \text{provided } -1<\frac{x-1}{4}<1 \Longrightarrow -3<x<5.$$

24. $$\frac{1}{x^2+8x+15} = \frac{1}{(x+3)(x+5)} = \frac{1/2}{x+3}+\frac{-1/2}{x+5} = \frac{1/6}{1+x/3}-\frac{1/10}{1+x/5}$$

$$= \frac{1}{6}\sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n-\frac{1}{10}\sum_{n=0}^{\infty}\left(-\frac{x}{5}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2(3^{n+1})}x^n+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2(5^{n+1})}x^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2}\left(\frac{1}{3^{n+1}}-\frac{1}{5^{n+1}}\right)x^n, \quad \text{valid for } -3<x<3.$$

25. Term-by-term integration of $\dfrac{1}{1+x^2} = 1-x^2+x^4-x^6+\cdots$ gives

$$\mathrm{Tan}^{-1}x = \left(x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots\right)+C.$$

Substitution of $x=0$ gives $C=0$, and therefore $\mathrm{Tan}^{-1}x = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$. The open interval of convergence is $-1<x<1$.

26. With the binomial expansion 10.33b,

$$\sqrt{x+3} = \sqrt{3}\sqrt{1+x/3} = \sqrt{3}\left[1+\left(\frac{1}{2}\right)\left(\frac{x}{3}\right)+\frac{(1/2)(-1/2)}{2!}\left(\frac{x}{3}\right)^2+\frac{(1/2)(-1/2)(-3/2)}{3!}\left(\frac{x}{3}\right)^3+\cdots\right]$$

$$= \sqrt{3}\left[1+\frac{x}{6}-\frac{1}{2^2 3^2 2!}x^2+\frac{(1)(3)}{2^3 3^3 3!}x^3-\frac{(1)(3)(5)}{2^4 3^4 4!}x^4+\cdots\right]$$

$$= \sqrt{3}\left[1+\frac{x}{6}-\frac{x^2}{72}+\sum_{n=3}^{\infty}\frac{(-1)^{n+1}[1\cdot 3\cdot 5\cdots(2n-3)]}{2^n 3^n n!}x^n\right]$$

$$= \sqrt{3}\left[1 + \frac{x}{6} - \frac{x^2}{72} + \sum_{n=3}^{\infty} \frac{(-1)^{n+1}[1 \cdot 2 \cdot 3 \cdot 4 \cdots (2n-3)(2n-2)]}{[2 \cdot 4 \cdot 6 \cdots (2n-2)]6^n n!} x^n\right]$$

$$= \sqrt{3}\left[1 + \frac{x}{6} - \frac{x^2}{72} + \sum_{n=3}^{\infty} \frac{2(-1)^{n+1}(2n-2)!}{12^n n!\,(n-1)!} x^n\right]$$

$$= \sqrt{3} + \sum_{n=1}^{\infty} \frac{2\sqrt{3}(-1)^{n+1}(2n-2)!}{12^n n!\,(n-1)!} x^n, \quad \text{valid for } -1 \le \frac{x}{3} \le 1 \Longrightarrow |x| \le 3.$$

27. With the binomial expansion 10.33b,

$$\sqrt{x+3} = \sqrt{5 + (x-2)} = \sqrt{5}\sqrt{1 + (x-2)/5}$$

$$= \sqrt{5}\left[1 + \frac{1}{2}\left(\frac{x-2}{5}\right) + \frac{(1/2)(-1/2)}{2!}\left(\frac{x-2}{5}\right)^2 + \frac{(1/2)(-1/2)(-3/2)}{3!}\left(\frac{x-2}{5}\right)^3 + \cdots\right]$$

$$= \sqrt{5}\left[1 + \frac{1}{10}(x-2) - \frac{1}{10^2 2!}(x-2)^2 + \frac{1 \cdot 3}{10^3 3!}(x-2)^3 + \cdots\right]$$

$$= \sqrt{5} + \frac{\sqrt{5}}{10}(x-2) + \sum_{n=2}^{\infty} \frac{\sqrt{5}(-1)^{n+1}[1 \cdot 3 \cdot 5 \cdots (2n-3)]}{10^n n!}(x-2)^n$$

$$= \sqrt{5} + \frac{\sqrt{5}}{10}(x-2) + \sum_{n=2}^{\infty} \frac{\sqrt{5}(-1)^{n+1}[1 \cdot 2 \cdot 3 \cdots (2n-2)]}{[2 \cdot 4 \cdot 6 \cdots (2n-2)]10^n n!}(x-2)^n$$

$$= \sqrt{5} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-2)!}{5^{n-1/2}2^{2n-1}n!\,(n-1)!}(x-2)^n, \quad \text{valid for } -1 \le \frac{x-2}{5} \le 1 \Longrightarrow -3 \le x \le 7$$

28. With the binomial expansion 10.33b,

$$(1-2x)^{1/3} = [-1 - 2(x-1)]^{1/3} = -[1 + 2(x-1)]^{1/3}$$

$$= -\left\{1 + \frac{2(x-1)}{3} + \frac{(1/3)(-2/3)}{2!}[2(x-1)]^2 + \frac{(1/3)(-2/3)(-5/3)}{3!}[2(x-1)])^3 + \cdots\right\}$$

$$= -1 - \frac{2}{3}(x-1) + \frac{2^2 2}{3^2 2!}(x-1)^2 - \frac{2^3(2 \cdot 5)}{3^3 3!}(x-1)^3 + \cdots$$

$$= -1 - \frac{2}{3}(x-1) + \sum_{n=2}^{\infty} \frac{(-1)^n 2^n [2 \cdot 5 \cdot 8 \cdots (3n-4)]}{3^n n!}(x-1)^n,$$

$$= -1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^n[(-1) \cdot 2 \cdot 5 \cdot 8 \cdots (3n-4)]}{3^n n!}(x-1)^n,$$

valid for $-1 \le 2(x-1) \le 1 \Longrightarrow 1/2 \le x \le 3/2$.

29. With the binomial expansion 10.33b,

$$\frac{x^2}{(1+x^2)^2} = x^2(1+x^2)^{-2} = x^2\left[1 + (-2)(x^2) + \frac{(-2)(-3)}{2!}(x^2)^2 + \frac{(-2)(-3)(-4)}{3!}(x^2)^3 + \cdots\right]$$

$$= x^2 - 2x^4 + 3x^6 - 4x^8 + \cdots$$

$$= \sum_{n=1}^{\infty} n(-1)^{n+1}x^{2n}, \quad \text{valid for } -1 < x^2 < 1 \Longrightarrow -1 < x < 1$$

30. With the binomial expansion 10.33b,

$$x(1-x)^{1/3} = x\left[1 + \left(\frac{1}{3}\right)(-x) + \frac{(1/3)(-2/3)}{2!}(-x)^2 + \frac{(1/3)(-2/3)(-5/3)}{3!}(-x)^3 + \cdots\right]$$

$$= x - \frac{x^2}{3} - \frac{2}{3^2 2!}x^3 - \frac{(2)(5)}{3^3 3!}x^4 - \frac{(2)(5)(8)}{3^4 4!}x^5 + \cdots$$

$$= x - \frac{x^2}{3} - \sum_{n=3}^{\infty} \frac{2\cdot 5\cdot 8\cdots(3n-7)}{3^{n-1}(n-1)!}x^n$$

$$= x + \sum_{n=2}^{\infty} \frac{(-1)\cdot 2\cdot 5\cdot 8\cdots(3n-7)}{3^{n-1}(n-1)!}x^n, \quad \text{valid for } -1 \le x \le 1.$$

31. We extend the calculations in Example 10.24 to obtain another nonzero term. When we equate coefficients of x^6, we obtain $0 = a_6 - a_4/2! + a_2/4! - a_0/6!$, and this implies that $a_6 = 0$. Coefficients of x^7 give $-1/7! = a_7 - a_5/2! + a_3/4! - a_1/6! \Longrightarrow a_7 = 17/315$. Consequently,

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots,$$

and if we replace x by $2x$, $\tan 2x = 2x + \dfrac{8x^3}{3} + \dfrac{64x^5}{15} + \dfrac{2176x^7}{315} + \cdots$.

32. If we set $\sec x = \dfrac{1}{\cos x} = a_0 + a_1x + a_2x^2 + \cdots$, then

$$1 = \left(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right).$$

We now multiply the power series on the right and equate coefficients:

1: $1 = a_0$

x: $0 = a_1$

x^2: $0 = -a_0/2! + a_2 \Longrightarrow a_2 = 1/2$

x^3: $0 = -a_1/2! + a_3 \Longrightarrow a_3 = 0$

x^4: $0 = a_0/4! - a_2/2! + a_4 \Longrightarrow a_4 = 5/24$

x^5: $0 = a_1/4! - a_3/2! + a_5 \Longrightarrow a_5 = 0$

x^6: $0 = -a_0/6! + a_2/4! - a_4/2! + a_6 \Longrightarrow a_6 = 61/720$

Thus, $\sec x = 1 + \dfrac{1}{2}x^2 + \dfrac{5}{24}x^4 + \dfrac{61}{720}x^6 + \cdots$. Long division could also be used.

33.
$$e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$
$$= x + x^2 + \left(\frac{1}{2!} - \frac{1}{3!}\right)x^3 + \left(\frac{1}{3!} - \frac{1}{3!}\right)x^4 + \left(\frac{1}{4!} - \frac{1}{2!3!} + \frac{1}{5!}\right)x^5 + \cdots$$
$$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \cdots$$

34.
$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) = \frac{1}{2}\left[1 + \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(2x)^{2n}\right] = \frac{1}{2}\left[1 + 1 + \sum_{n=1}^{\infty}\frac{(-1)^n 2^{2n}}{(2n)!}x^{2n}\right]$$
$$= 1 + \sum_{n=1}^{\infty}\frac{(-1)^n 2^{2n-1}}{(2n)!}x^{2n}, \quad -\infty < x < \infty$$

35.
$$\frac{1}{x^6 - 3x^3 - 4} = \frac{1}{(x^3-4)(x^3+1)} = \frac{-1/5}{1+x^3} + \frac{1/5}{x^3-4} = \frac{-1/5}{1+x^3} - \frac{1/20}{1 - x^3/4}$$
$$= -\frac{1}{5}\sum_{n=0}^{\infty}(-x^3)^n - \frac{1}{20}\sum_{n=0}^{\infty}\left(\frac{x^3}{4}\right)^n$$
$$= \sum_{n=0}^{\infty} -\frac{1}{5}\left[(-1)^n + \frac{1}{4^{n+1}}\right]x^{3n}, \quad \text{valid for } -1 < x < 1.$$

36. The Maclaurin series for $\mathrm{Sin}^{-1}(x^2)$ can be obtained by replacing x by x^2 in the series for $\mathrm{Sin}^{-1}x$ in Example 10.26:

$$\mathrm{Sin}^{-1}(x^2) = \sum_{n=0}^{\infty} \frac{(2n)!}{(2n+1)2^{2n}(n!)^2}(x^2)^{2n+1} = \sum_{n=0}^{\infty} \frac{(2n)!}{(2n+1)2^{2n}(n!)^2}x^{4n+2}, \quad |x| < 1.$$

37. $$\frac{2x^2+4}{x^2+4x+3} = 2 - \frac{8x+2}{(x+3)(x+1)} = 2 - \frac{11}{x+3} + \frac{3}{x+1} = 2 - \frac{11/3}{1+x/3} + \frac{3}{1+x}$$
$$= 2 - \frac{11}{3}\sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n + 3\sum_{n=0}^{\infty}(-x)^n = \left(2 - \frac{11}{3} + 3\right) + \sum_{n=1}^{\infty}\left[-\frac{11}{3}\left(-\frac{1}{3}\right)^n + 3(-1)^n\right]x^n$$
$$= \frac{4}{3} + \sum_{n=1}^{\infty}(-1)^n\left(3 - \frac{11}{3^{n+1}}\right)x^n, \quad \text{valid for } -1 < x < 1.$$

38. If we integrate the series $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$, $|x| < 1$, we obtain $-\ln|1-x| = \displaystyle\sum_{n=0}^{\infty}\frac{1}{n+1}x^{n+1} + C$. Substitution of $x = 0$ gives $C = 0$, and therefore $\ln|1-x| = \displaystyle\sum_{n=0}^{\infty}\frac{-1}{n+1}x^{n+1} = \sum_{n=1}^{\infty} -\frac{1}{n}x^n$. The open interval of convergence is $-1 < x < 1$ so that absolute values may be dropped. If we replace x by $x/\sqrt{2}$ and $-x/\sqrt{2}$, we find

$$f(x) = \ln(1 + x/\sqrt{2}) - \ln(1 - x/\sqrt{2}) = \sum_{n=1}^{\infty} -\frac{1}{n}\left(-\frac{x}{\sqrt{2}}\right)^n - \sum_{n=1}^{\infty} -\frac{1}{n}\left(\frac{x}{\sqrt{2}}\right)^n$$
$$= \sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n2^{n/2}} + \frac{1}{n2^{n/2}}\right]x^n = \sum_{n=1}^{\infty}\left[\frac{1+(-1)^{n+1}}{n2^{n/2}}\right]x^n.$$

When n is even the coefficient of x^n is zero, and therefore

$$f(x) = \sum_{n=0}^{\infty}\frac{2}{(2n+1)2^{(2n+1)/2}}x^{2n+1} = \sum_{n=0}^{\infty}\frac{\sqrt{2}}{(2n+1)2^n}x^{2n+1}.$$

Since the added series both have open interval of convergence $-\sqrt{2} < x < \sqrt{2}$, this is the open interval of convergence for the combined series.

39. If $\sum_{n=0}^{\infty} a_n(x-c)^n = \sum_{n=0}^{\infty} b_n(x-c)^n$, then $\sum_{n=0}^{\infty}(a_n - b_n)(x-c)^n = 0$. The right side of this equation is the Maclaurin series for the function identically equal to zero, and as such, its coefficients must all be zero; that is, $a_n - b_n = 0$ for all n.

40. The right side of this equation is the Maclaurin series for the function identically equal to zero, and as such, its coefficients must all be zero; that is, $a_n = 0$ for all n.

41. $\displaystyle\sum_{n=0}^{\infty} P_n(t) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{t}{30}\right)^n e^{-t/30} = e^{-t/30}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{t}{30}\right)^n = e^{-t/30}(e^{t/30}) = 1$ The sum represents the probability that either nobody, or just one person, or two people, or three people, etc., drink from the fountain. Since one of these situations must occur, the probability is one.

42. (a) $\displaystyle\sum_{n=1}^{\infty} np(1-p)^{n-1} = p\sum_{n=1}^{\infty} n(1-p)^{n-1}$ If we differentiate the series $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$, $|x| < 1$, term-by-term, we obtain $\dfrac{1}{(1-x)^2} = \displaystyle\sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1}$, $|x| < 1$. We now substitute $x = 1-p$ into this result, $\dfrac{1}{[1-(1-p)]^2} = \displaystyle\sum_{n=1}^{\infty} n(1-p)^{n-1}$. Multiplication by p gives $\dfrac{1}{p} = \displaystyle\sum_{n=1}^{\infty} np(1-p)^{n-1}$.

(b) The probability of throwing a six is $p = 1/6$, and therefore $\displaystyle\sum_{n=1}^{\infty} np(1-p)^{n-1} = \frac{1}{1/6} = 6.$

43. $$\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt = \frac{2}{\sqrt{\pi}}\int_0^x\left[\sum_{n=0}^{\infty}\frac{1}{n!}(-t^2)^n\right]dt = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^x t^{2n}\,dt$$
$$= \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left\{\frac{t^{2n+1}}{2n+1}\right\}_0^x = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)n!}x^{2n+1}$$

44. Integrating the Maclaurin series for $\cos(\pi t^2/2)$ (see Example 10.21) term-by-term gives
$$C(x) = \int_0^x\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{\pi t^2}{2}\right)^{2n}\right]dt = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\frac{\pi^{2n}}{2^{2n}}\int_0^x t^{4n}\,dt$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\frac{\pi^{2n}}{2^{2n}}\left\{\frac{t^{4n+1}}{4n+1}\right\}_0^x = \sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{(4n+1)2^{2n}(2n)!}x^{4n+1},$$
valid for $-\infty < x < \infty$. A similar procedure leads to the Maclaurin series for $S(x)$.

45. With the binomial expansion 10.33b,
$$\frac{x}{(4+3x)^2} = \frac{x}{16}\left(1+\frac{3x}{4}\right)^{-2} = \frac{x}{16}\left[1-2\left(\frac{3x}{4}\right)+\frac{(-2)(-3)}{2!}\left(\frac{3x}{4}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{3x}{4}\right)^3+\cdots\right]$$
$$= \frac{x}{16}\sum_{n=0}^{\infty}\frac{(-1)^n3^n(n+1)}{4^n}x^n = \sum_{n=0}^{\infty}\frac{(-1)^n3^n(n+1)}{4^{n+2}}x^{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}3^{n-1}n}{4^{n+1}}x^n.$$
But the coefficient of x^n in the Maclaurin series is $f^{(n)}(0)/n!$, and therefore
$$\frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n+1}3^{n-1}n}{4^{n+1}} \quad\Longrightarrow\quad f^{(n)}(0) = \frac{(-1)^{n+1}3^{n-1}n\,n!}{4^{n+1}}.$$

46. The Maclaurin series for $f(x) = xe^{-2x}$ is
$$xe^{-2x} = x\sum_{n=0}^{\infty}\frac{1}{n!}(-2x)^n = \sum_{n=0}^{\infty}\frac{(-1)^n2^n}{n!}x^{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^{n-1}}{(n-1)!}x^n.$$
But the coefficient of x^n in the Maclaurin series is $f^{(n)}(0)/n!$, and therefore
$$\frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n+1}2^{n-1}}{(n-1)!} \quad\Longrightarrow\quad f^{(n)}(0) = \frac{(-1)^{n+1}2^{n-1}n!}{(n-1)!} = n(-1)^{n+1}2^{n-1}.$$

47. The Taylor series for $f(x) = 1/(3+x)$ about $x = 2$ is
$$\frac{1}{3+x} = \frac{1}{5+(x-2)} = \frac{1}{5[1+(x-2)/5]} = \frac{1}{5}\sum_{n=0}^{\infty}\left(-\frac{x-2}{5}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{5^{n+1}}(x-2)^n.$$
But the coefficient of $(x-2)^n$ in the Taylor series is $f^{(n)}(2)/n!$, and therefore
$$\frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{5^{n+1}} \quad\Longrightarrow\quad f^{(n)}(2) = \frac{(-1)^n n!}{5^{n+1}}.$$

48. The Taylor series for $f(x) = xe^{-x}$ about $x = 2$ is
$$xe^{-x} = [(x-2)+2]e^{-(x-2)-2} = e^{-2}[2+(x-2)]\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(x-2)^n$$
$$= e^{-2}\left[\sum_{n=0}^{\infty}\frac{2(-1)^n}{n!}(x-2)^n + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(x-2)^{n+1}\right]$$
$$= e^{-2}\left[\sum_{n=0}^{\infty}\frac{2(-1)^n}{n!}(x-2)^n + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(n-1)!}(x-2)^n\right]$$
$$= e^{-2}\left[2+\sum_{n=1}^{\infty}\frac{(-1)^n(2-n)}{n!}(x-2)^n\right].$$

But the coefficient of $(x-2)^n$ in the Taylor series is $f^{(n)}(2)/n!$, and therefore

$$\frac{f^{(n)}(2)}{n!} = \frac{(-1)^n(2-n)e^{-2}}{n!} \quad \Longrightarrow \quad f^{(n)}(2) = \frac{(-1)^n(2-n)n!}{e^2 n!} = \frac{(n-2)(-1)^{n+1}}{e^2}.$$

49. Since the Maclaurin series for $x^2 \sin 2x$

$$x^2 \sin 2x = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(2x)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{2n+3}$$

contains only odd powers of x, the even derivatives of $x^2 \sin 2x$ must all be zero.

50. Since the Maclaurin series for e^{-x^2}, namely, $\quad e^{-x^2} = \sum_{n=0}^{\infty} \frac{1}{n!}(-x^2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} \quad$ contains only even powers of x, the odd derivatives of e^{-x^2} must all be zero.

51. Using the definition of $J_m(x)$ as the Maclaurin series in Exercise 38 of Section 10.4, we may write

$$\begin{aligned}
2m\, J_m(x) - x\, J_{m-1}(x) &= 2m \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+m} n!(n+m)!} x^{2n+m} - x \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+m-1} n!(n+m-1)!} x^{2n+m-1} \\
&= \sum_{n=0}^{\infty} \frac{m(-1)^n}{2^{2n+m-1} n!(n+m)!} x^{2n+m} - \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+m-1} n!(n+m-1)!} x^{2n+m} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n(m-n-m)}{2^{2n+m-1} n!\,(n+m)!} x^{2n+m} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{2n+m-1}(n-1)!\,(n+m)!} x^{2n+m} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+m+1} n!\,(n+m+1)!} x^{2n+m+2} = x \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+m+1} n!\,(n+m+1)!} x^{2n+m+1} \\
&= x\, J_{m+1}(x).
\end{aligned}$$

52. Using the definition of $J_m(x)$ as the Maclaurin series in Exercise 38 of Section 10.4, we may write

$$J_{m-1}(x) - J_{m+1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+m-1} n!(n+m-1)!} x^{2n+m-1} - \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+m+1} n!(n+m+1)!} x^{2n+m+1}.$$

We lower n by 1 in the second summation, and separate out the first term in the first summation,

$$\begin{aligned}
J_{m-1}(x) - J_{m+1}(x) &= \frac{1}{2^{m-1}(m-1)!} x^{m-1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n+m-1} n!(n+m-1)!} x^{2n+m-1} \\
&\quad + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n+m-1}(n-1)!(n+m)!} x^{2n+m-1} \\
&= \frac{1}{2^{m-1}(m-1)!} x^{m-1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n+m-1}(n-1)!(n+m-1)!} \left(\frac{1}{n} + \frac{1}{n+m} \right) x^{2n+m-1} \\
&= \frac{1}{2^{m-1}(m-1)!} x^{m-1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n+m-1}(n-1)!(n+m-1)!} \left[\frac{2n+m}{n(n+m)} \right] x^{2n+m-1} \\
&= \frac{m}{2^{m-1} m!} x^{m-1} + \sum_{n=1}^{\infty} \frac{(2n+m)(-1)^n}{2^{2n+m-1} n!(n+m)!} x^{2n+m-1} \\
&= \sum_{n=0}^{\infty} \frac{(2n+m)(-1)^n}{2^{2n+m-1} n!(n+m)!} x^{2n+m-1}.
\end{aligned}$$

Term-by-term differentiation of the series for $J_m(x)$ gives $J'_m(x) = \sum_{n=0}^{\infty} \frac{(2n+m)(-1)^n}{2^{2n+m} n!(n+m)!} x^{2n+m-1}$.

Hence, $J_{m-1}(x) - J_{m+1}(x) = 2J'_m(x)$.

53. With the binomial expansion 10.33b,

$$\begin{aligned}\frac{1}{\sqrt{1-2\mu x+x^2}} &= 1-\frac{1}{2}(x^2-2\mu x)+\frac{(-1/2)(-3/2)}{2!}(x^2-2\mu x)^2+\frac{(-1/2)(-3/2)(-5/2)}{3!}(x^2-2\mu x)^3+\cdots\\ &= 1+\frac{1}{2}(2\mu x-x^2)+\frac{3}{8}(4\mu^2x^2-4\mu x^3+x^4)+\frac{5}{16}(8\mu^3x^3-12\mu^2x^4+6\mu x^5-x^6)+\cdots\\ &= 1+(\mu)x+\left(-\frac{1}{2}+\frac{3\mu^2}{2}\right)x^2+\left(-\frac{3\mu}{2}+\frac{5\mu^3}{2}\right)x^3+\cdots.\end{aligned}$$

Thus, $P_0(\mu)=1$, $P_1(\mu)=\mu$, $P_2(\mu)=(3\mu^2-1)/2$, and $P_3(\mu)=(5\mu^3-3\mu)/2$.

54. (a) If we substitute the Maclaurin series for e^x into $x=(e^x-1)\left(1+B_1x+\dfrac{B_2}{2!}x^2+\cdots\right)$,

$$\begin{aligned}x &= \left[\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)-1\right]\left(1+B_1x+\frac{B_2x^2}{2!}+\cdots\right)\\ &= \left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}+\cdots\right)\left(1+B_1x+\frac{B_2x^2}{2!}+\frac{B_3x^3}{3!}+\frac{B_4x^4}{4!}+\frac{B_5x^5}{5!}+\cdots\right).\end{aligned}$$

When we multiply the series on the right and equate coefficients of powers of x left and right:

x: $1=1$

x^2: $0=\dfrac{1}{2!}+B_1 \implies B_1=-\dfrac{1}{2}$

x^3: $0=\dfrac{1}{3!}+\dfrac{B_1}{2!}+\dfrac{B_2}{2!} \implies B_2=\dfrac{1}{6}$

x^4: $0=\dfrac{1}{4!}+\dfrac{B_1}{3!}+\dfrac{B_2}{(2!)^2}+\dfrac{B_3}{3!} \implies B_3=0$

x^5: $0=\dfrac{1}{5!}+\dfrac{B_1}{4!}+\dfrac{B_2}{2!3!}+\dfrac{B_3}{2!3!}+\dfrac{B_4}{4!} \implies B_4=-\dfrac{1}{30}$

x^6: $0=\dfrac{1}{6!}+\dfrac{B_1}{5!}+\dfrac{B_2}{2!4!}+\dfrac{B_3}{(3!)^2}+\dfrac{B_4}{2!4!}+\dfrac{B_5}{5!} \implies B_5=0$

(b) Suppose we set
$$\begin{aligned}f(x) &= \frac{x}{e^x-1}-1-B_1x=\frac{x}{e^x-1}-1+\frac{x}{2}=\frac{2x-2(e^x-1)+x(e^x-1)}{2(e^x-1)}\\ &= \frac{xe^x-2e^x+x+2}{2(e^x-1)}=\frac{B_2}{2!}x^2+\frac{B_3}{3!}x^3+\cdots.\end{aligned}$$

Since

$$\begin{aligned}f(-x) &= \frac{-x}{e^{-x}-1}-1-\frac{x}{2}=\frac{xe^x}{e^x-1}-1-\frac{x}{2}\\ &= \frac{2xe^x-2(e^x-1)-x(e^x-1)}{2(e^x-1)}=\frac{xe^x-2e^x+x+2}{2(e^x-1)}=f(x),\end{aligned}$$

$f(x)$ is an even function. But the Maclaurin series for $f(x)$ can represent an even function only if all odd powers are absent. In other words, $0=B_3=B_5=\cdots$.

55. $e^{x(t-1/t)/2}=e^{xt/2}\,e^{-x/(2t)}=\left[\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{xt}{2}\right)^n\right]\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{x}{2t}\right)^n\right]=\left[\sum_{n=0}^{\infty}\frac{(x/2)^n}{n!}t^n\right]\left[\sum_{n=0}^{\infty}\frac{(-x/2)^n}{n!}\left(\frac{1}{t}\right)^n\right].$

When these series are multiplied together, the coefficient of t^n is

$$\begin{aligned}\frac{(x/2)^n}{n!}+\frac{(x/2)^{n+1}}{(n+1)!}\frac{(-x/2)}{1!}+\frac{(x/2)^{n+2}}{(n+2)!}\frac{(-x/2)^2}{2!}+\cdots &= \sum_{m=0}^{\infty}\frac{(x/2)^{n+m}}{(n+m)!}\frac{(-x/2)^m}{m!}\\ &= \sum_{m=0}^{\infty}\frac{(-1)^m}{2^{2m+n}m!\,(n+m)!}x^{2m+n}=J_n(x).\end{aligned}$$

EXERCISES 10.6

1. The radius of convergence of the series is $R = \lim_{n\to\infty}\left|\dfrac{n}{n+1}\right| = 1$. If we set $S(x) = \sum_{n=1}^{\infty} nx^{n-1}$, then term-by-term integration gives

$$\int S(x)\,dx + C = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x},$$

since the series is geometric. Differentiation now gives $S(x) = \dfrac{(1-x)(1)-x(-1)}{(1-x)^2} = \dfrac{1}{(1-x)^2}$.

2. The radius of convergence of the series is $R = \lim_{n\to\infty}\left|\dfrac{n(n-1)}{(n+1)n}\right| = 1$. If we set $S(x) = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$, then term-by-term integration gives $\int S(x)\,dx + C = \sum_{n=2}^{\infty} nx^{n-1}$. A second integration leads to

$$\int\left[\int S(x)\,dx + C\right]dx + D = \sum_{n=2}^{\infty} x^n = \frac{x^2}{1-x},$$

since the series is geometric. Differentiation now gives

$$\int S(x)\,dx + C = \frac{(1-x)(2x)-x^2(-1)}{(1-x)^2} = \frac{2x-x^2}{(1-x)^2}.$$

A second differentiation provides $S(x)$,

$$S(x) = \frac{(1-x)^2(2-2x)-(2x-x^2)2(1-x)(-1)}{(1-x)^4} = \frac{2}{(1-x)^3}.$$

3. The radius of convergence of the series is $R = \lim_{n\to\infty}\left|\dfrac{n+1}{n+2}\right| = 1$. If we set $S(x) = \sum_{n=1}^{\infty}(n+1)x^{n-1}$, then $x\,S(x) = \sum_{n=1}^{\infty}(n+1)x^n$. Term-by-term integration gives

$$\int x\,S(x)\,dx + C = \sum_{n=1}^{\infty} x^{n+1} = \frac{x^2}{1-x},$$

since the series is geometric. Differentiation now gives

$$x\,S(x) = \frac{(1-x)(2x)-x^2(-1)}{(1-x)^2} = \frac{2x-x^2}{(1-x)^2} \quad\Longrightarrow\quad S(x) = \frac{2-x}{(1-x)^2}.$$

4. The radius of convergence of the series is $R = \lim_{n\to\infty}\left|\dfrac{n^2}{(n+1)^2}\right| = 1$. If we set $S(x) = \sum_{n=1}^{\infty} n^2x^{n-1}$, then term-by-term integration gives $\int S(x)\,dx + C = \sum_{n=1}^{\infty} nx^n$. When $x \neq 0$, we can divide by x, $\dfrac{1}{x}\int S(x)\,dx + \dfrac{C}{x} = \sum_{n=1}^{\infty} nx^{n-1}$. Integration now gives,

$$\int\left[\frac{1}{x}\int S(x)\,dx\right]dx + C\ln|x| + D = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x}.$$

If we now differentiate, $\dfrac{1}{x}\displaystyle\int S(x)\,dx + \dfrac{C}{x} = \dfrac{(1-x)(1)-x(-1)}{(1-x)^2} = \dfrac{1}{(1-x)^2}$. Multiplication by x and a further differentiation gives

$$S(x) = \frac{d}{dx}\left[\frac{x}{(1-x)^2}\right] = \frac{(1-x)^2(1)-x(2)(1-x)(-1)}{(1-x)^4} = \frac{x+1}{(1-x)^3}.$$

Since the sum of the series at $x=0$ is 1, and this is $S(0)$, the formula $S(x) = (x+1)/(1-x)^3$ can be used for all x in $|x|<1$.

5. If we divide the series into two parts, $\displaystyle\sum_{n=1}^{\infty}(n^2+2n)x^n = \sum_{n=1}^{\infty}n^2x^n + 2\sum_{n=1}^{\infty}nx^n$, the first series is x times that in Exercise 4, and the second is x times that in Exercise 1. Hence,

$$\sum_{n=1}^{\infty}(n^2+2n)x^n = \frac{x(x+1)}{(1-x)^3} + \frac{2x}{(1-x)^2} = \frac{3x-x^2}{(1-x)^3}.$$

6. The radius of convergence of the series is $R = \displaystyle\lim_{n\to\infty}\left|\frac{1/(n+1)}{1/(n+2)}\right| = 1$. If we set $S(x) = \displaystyle\sum_{n=0}^{\infty}\frac{1}{n+1}x^n$, then $x\,S(x) = \displaystyle\sum_{n=0}^{\infty}\frac{1}{n+1}x^{n+1}$. Term-by-term differentiation gives $\dfrac{d}{dx}[x\,S(x)] = \displaystyle\sum_{n=0}^{\infty}x^n = \frac{1}{1-x}$, since the series is geometric. We now integrate,

$$x\,S(x) = \int\frac{1}{1-x}dx = -\ln(1-x) + C.$$

Substitution of $x=0$ gives $C=0$, and therefore $S(x) = -\dfrac{1}{x}\ln(1-x)$. This is valid for $-1<x<1$, but not at $x=0$. It is interesting to note, however, that the limit of $S(x)$ as x approaches zero is 1 and this is the sum of the series at $x=0$.

7. If we set $y=x^2$, the series becomes $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} = \pm\sqrt{y}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}y^n$. The radius of convergence of this series is $R_y = \displaystyle\lim_{n\to\infty}\left|\frac{(-1)^n/(2n+1)}{(-1)^{n+1}/(2n+3)}\right| = 1$. The radius of convergence of the original series is therefore $R_x = 1$. If we set $S(x) = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$, then term-by-term differentiation gives $S'(x) = \displaystyle\sum_{n=0}^{\infty}(-1)^nx^{2n} = \frac{1}{1+x^2}$, since the series is geometric. Integration now gives $S(x) = \mathrm{Tan}^{-1}x + C$. Since $S(0)=0$, it follows that $C=0$, and $S(x) = \mathrm{Tan}^{-1}x$.

8. If we set $y=x^2$, the series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}x^{2n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}y^n$. The radius of convergence of this series is $R_y = \displaystyle\lim_{n\to\infty}\left|\frac{(-1)^n/n}{(-1)^{n+1}/(n+1)}\right| = 1$. The radius of convergence of the original series is therefore $R_x = 1$. If we set $S(x) = \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}x^{2n}$, then term-by-term differentiation gives

$$S'(x) = \sum_{n=1}^{\infty}2(-1)^nx^{2n-1} = \frac{-2x}{1+x^2},$$

since the series is geometric. Integration now leads to $S(x) = -\ln(1+x^2) + C$. Since $S(0)=0$, it follows that $C=0$, and $S(x) = -\ln(1+x^2)$.

9. If we set $y = x^2$, the series becomes $\displaystyle\sum_{n=2}^{\infty} n3^n x^{2n} = \sum_{n=2}^{\infty} n3^n y^n$. The radius of convergence of this series is $R_y = \displaystyle\lim_{n\to\infty}\left|\frac{n3^n}{(n+1)3^{n+1}}\right| = 1/3$. The radius of convergence of the original series is therefore $R_x = 1/\sqrt{3}$. If we set $S(x) = \displaystyle\sum_{n=2}^{\infty} n3^n x^{2n}$, then $\dfrac{S(x)}{x} = \displaystyle\sum_{n=2}^{\infty} n3^n x^{2n-1}$, provided $x \neq 0$. Term-by-term integration of this equation gives

$$\int \frac{S(x)}{x}dx = \sum_{n=2}^{\infty} \frac{3^n}{2}x^{2n} = \frac{9x^4/2}{1-3x^2},$$

since the series is geometric. Differentiation now gives

$$\frac{S(x)}{x} = \frac{9}{2}\left[\frac{(1-3x^2)(4x^3) - x^4(-6x)}{(1-3x^2)^2}\right] = \frac{9(4x^3-6x^5)}{2(1-3x^2)^2} \quad\Longrightarrow\quad S(x) = \frac{9x^4(2-3x^2)}{(1-3x^2)^2}.$$

Since the sum of the series at $x = 0$ is 0, and this is $S(0)$, the formula for $S(x)$ can be used for all x in $|x| < 1/\sqrt{3}$.

10. The radius of convergence of the series is $R = \displaystyle\lim_{n\to\infty}\left|\frac{(n+1)/(n+2)}{(n+2)/(n+3)}\right| = 1$. If we set $S(x) = \displaystyle\sum_{n=0}^{\infty}\left(\frac{n+1}{n+2}\right)x^n$, and integrate, $\displaystyle\int S(x)\,dx = \sum_{n=0}^{\infty}\frac{1}{n+2}x^{n+1} + C$. Multiplication by x gives $x\displaystyle\int S(x)\,dx = \sum_{n=0}^{\infty}\frac{1}{n+2}x^{n+2} + Cx$. Differentiation now gives

$$\frac{d}{dx}\left[x\int S(x)\,dx\right] = \sum_{n=0}^{\infty} x^{n+1} + C = \frac{x}{1-x} + C,$$

since the series is geometric. Integration now yields

$$x\int S(x)\,dx = \int \frac{x}{1-x}dx + Cx + D = -x - \ln|1-x| + Cx + D.$$

If we set $x = 0$ in this equation we find that $D = 0$. When we drop absolute values and divide by x,

$$\int S(x)\,dx = -1 - \frac{1}{x}\ln(1-x) + C, \quad x \neq 0.$$

When we differentiate this equation, we obtain $S(x) = \dfrac{1}{x^2}\ln(1-x) + \dfrac{1}{x(1-x)}$. This formula can only be used for values of x in the interval $-1 < x < 1$, but not $x = 0$. The sum at $x = 0$ is $1/2$.

11. The radius of convergence of the series is $R = \displaystyle\lim_{n\to\infty}\left|\frac{(n+1)/n!}{(n+2)/(n+1)!}\right| = \infty$. If we set $S(x) = \displaystyle\sum_{n=1}^{\infty}\left(\frac{n+1}{n!}\right)x^n$, and integrate,

$$\int S(x)\,dx = \sum_{n=1}^{\infty}\frac{x^{n+1}}{n!} + C = x\sum_{n=1}^{\infty}\frac{x^n}{n!} + C = x(e^x - 1) + C.$$

Differentiation now gives

$$S(x) = (e^x - 1) + x(e^x) = (x+1)e^x - 1.$$

12. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{(2n+1)!}x^{2n+1} = x\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} = x\cos x$

13. If we set $y = x^2$, the series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(n+2)}{(2n)!}x^{2n} = \pm\sqrt{y}\sum_{n=0}^{\infty}\frac{(-1)^n(n+2)}{(2n)!}y^n$. The radius of convergence of this series is $R_y = \displaystyle\lim_{n\to\infty}\left|\frac{(-1)^n(n+2)/(2n)!}{(-1)^{n+1}(n+3)/(2n+2)!}\right| = \infty$. The radius of convergence of the original series is therefore $R_x = \infty$ also. If we set $S(x) = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(n+2)}{(2n)!}x^{2n}$, and multiply by x^3, $x^3S(x) = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(n+2)}{(2n)!}x^{2n+3}$. Integration now gives

$$\int x^3S(x)\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n(n+2)}{(2n)!(2n+4)}x^{2n+4} + C = \frac{x^4}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} + C = \frac{x^4}{2}\cos x + C.$$

We now differentiate to get

$$x^3S(x) = 2x^3\cos x - \frac{x^4}{2}\sin x \qquad\Longrightarrow\qquad S(x) = 2\cos x - \frac{x}{2}\sin x.$$

14. If we set $y = x^2$, the series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(2n+3)2^n}{n!}x^{2n} = \sum_{n=1}^{\infty}\frac{(2n+3)2^n}{n!}y^n$. The radius of convergence of this series is $R_y = \displaystyle\lim_{n\to\infty}\left|\frac{(2n+3)2^n/n!}{(2n+5)2^{n+1}/(n+1)!}\right| = \infty$. The radius of convergence of the original series is therefore $R_x = \infty$ also. If we set $S(x) = \displaystyle\sum_{n=1}^{\infty}\frac{(2n+3)2^n}{n!}x^{2n}$, and multiply by x^2, $x^2S(x) = \displaystyle\sum_{n=1}^{\infty}\frac{(2n+3)2^n}{n!}x^{2n+2}$. Integration now gives

$$\int x^2S(x)\,dx = \sum_{n=1}^{\infty}\frac{2^n}{n!}x^{2n+3} + C = x^3\sum_{n=1}^{\infty}\frac{1}{n!}(2x^2)^n + C = x^3(e^{2x^2}-1) + C.$$

We now differentiate to get

$$x^2S(x) = 3x^2(e^{2x^2}-1) + x^3(4xe^{2x^2}) \qquad\Longrightarrow\qquad S(x) = (4x^2+3)e^{2x^2} - 3.$$

15. If we set $y = x^2$, the series becomes $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n-1)}{(2n)!}x^{2n+1} = \pm\sqrt{y}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n-1)}{(2n)!}y^n$. The radius of convergence of this series is $R_y = \displaystyle\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(2n-1)/(2n)!}{(-1)^{n+2}(2n+1)/(2n+2)!}\right| = \infty$. The radius of convergence of the original series is therefore $R_x = \infty$ also. If we set $S(x) = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n-1)}{(2n)!}x^{2n+1}$, and divide by x^3, $\displaystyle\frac{S(x)}{x^3} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n-1)}{(2n)!}x^{2n-2}$, $x \neq 0$. Integration now gives

$$\int\frac{S(x)}{x^3}\,dx = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n)!}x^{2n-1} + C = -\frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} + C = -\frac{1}{x}\cos x + C.$$

We now differentiate to get

$$\frac{S(x)}{x^3} = \frac{1}{x^2}\cos x + \frac{1}{x}\sin x \qquad\Longrightarrow\qquad S(x) = x\cos x + x^2\sin x.$$

This gives the sum of the series at $x = 0$ also.

EXERCISES 10.7

1. Taylor's remainder formula for e^x and $c = 0$ gives $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + R_3$, where $R_3 = \dfrac{d^4}{dx^4}e^x{}_{|x=z_3}\dfrac{x^4}{4!} = e^{z_3}\dfrac{x^4}{24}$, and $0 < z_3 < x$. Since $x \le 0.01$, we can say that

$$R_3 < e^x\frac{x^4}{24} \le e^{0.01}\frac{(0.01)^4}{24} = 4.2 \times 10^{-10}.$$

2. Taylor's remainder formula for e^x and $c = 0$ gives $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + R_3$, where $R_3 = \dfrac{d^4}{dx^4}e^x{}_{|x=z_3}\dfrac{x^4}{4!} = e^{z_3}\dfrac{x^4}{24}$, and $0 < z_3 < x$. Since $x < 0.01$, we can say that

$$R_3 < e^x\frac{x^4}{24} < e^{0.01}\frac{(0.01)^4}{24} = 4.2 \times 10^{-10}.$$

3. Taylor's remainder formula for e^x and $c = 0$ gives $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + R_3$, where $R_3 = \dfrac{d^4}{dx^4}e^x{}_{|x=z_3}\dfrac{x^4}{4!} = e^{z_3}\dfrac{x^4}{24}$, and $x < z_3 < 0$. Since $-0.01 \le x < 0$, we can say that

$$|R_3| < e^0\frac{|x|^4}{24} \le \frac{|-0.01|^4}{24} = 4.2 \times 10^{-10}.$$

4. According to Exercise 2, a maximum error on $0 \le x \le 0.01$ is 4.2×10^{-10}. For $-0.01 \le x < 0$, $R_3 = e^{z_3}\dfrac{x^4}{24}$ where $x < z_3 < 0$. Since $x \ge -0.01$, it follows that

$$|R_3| < e^0\frac{|x|^4}{24} \le \frac{|-0.01|^4}{24} < 4.2 \times 10^{-10}.$$

5. Taylor's remainder formula for $\sin x$ and $c = 0$ gives $\sin x = x - \dfrac{x^3}{3!} + R_4$, where $R_4 = \dfrac{d^5}{dx^5}\sin x_{|x=z_4}\dfrac{x^5}{5!} = (\cos z_4)\dfrac{x^5}{120}$, and $0 < z_4 < x$. Since $0 \le x \le 1$, we can say that

$$R_4 < (1)\frac{x^5}{120} \le \frac{(1)^5}{120} = \frac{1}{120}.$$

6. Taylor's remainder formula for $\cos x$ and $c = 0$ gives $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + R_5$, where $R_5 = \dfrac{d^6}{dx^6}\cos x_{|x=z_5}\dfrac{x^6}{6!} = -(\cos z_5)\dfrac{x^6}{6!}$, and z_5 is between 0 and x. Since $|x| \le 0.1$, we can say that

$$|R_5| < (1)\frac{|x|^6}{6!} \le \frac{(0.1)^6}{6!} < 1.4 \times 10^{-9}.$$

7. The first four derivatives of $f(x) = \ln(1-x)$ are $f'(x) = -1/(1-x)$, $f''(x) = -1/(1-x)^2$, $f'''(x) = -2/(1-x)^3$, and $f''''(x) = -6/(1-x)^4$. Taylor's remainder formula for $\ln(1-x)$ and $c = 0$ gives $\ln(1-x) = -x - x^2/2 - x^3/3 + R_3(x)$, where $R_3(x) = f''''(z_3)\dfrac{x^4}{4!} = \dfrac{-x^4}{4(1-z_3)^4}$, and $0 < z_3 < x$. Since $0 \le x \le 0.01$, we can say that

$$|R_3| < \frac{x^4}{4(1-x)^4} \le \frac{(0.01)^4}{4(1-0.01)^4} < 2.7 \times 10^{-9}.$$

8. The first four derivatives of $f(x) = 1/(1-x)^3$ are $f'(x) = 3/(1-x)^4$, $f''(x) = 12/(1-x)^5$, $f'''(x) = 60/(1-x)^6$, and $f''''(x) = 360/(1-x)^7$. Taylor's remainder formula for $1/(1-x)^3$ and $c = 0$ gives $\dfrac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + R_3(x)$, where $R_3(x) = f''''(z_3)\dfrac{x^4}{4!} = \dfrac{15x^4}{(1-z_3)^7}$, and z_3 is between 0 and x. Since $|x| < 0.2$, we can say that

$$|R_3| < \frac{15|x|^4}{(1-0.2)^7} < \frac{15(0.2)^4}{(1-0.2)^7} < 0.115.$$

9. Taylor's remainder formula for $\sin 3x$ and $c = 0$ gives $\sin 3x = 3x - \dfrac{9x^3}{2} + \dfrac{81x^5}{40} + R_6$, where $R_6 = \dfrac{d^7}{dx^7}\sin 3x_{|x=z_6}\dfrac{x^7}{7!} = -3^7(\cos 3z_6)\dfrac{x^7}{7!}$, and z_6 is between 0 and x. Since $|x| < \pi/100$, we can say that

$$|R_6| < 3^7(1)\frac{|x|^7}{7!} < 3^7\frac{(\pi/100)^7}{7!} < 1.4 \times 10^{-11}.$$

10. The first five derivatives of $f(x) = \ln x$ are $f'(x) = 1/x$, $f''(x) = -1/x^2$, $f'''(x) = 2/x^3$, $f''''(x) = -6/x^4$, and $f'''''(x) = 24/x^5$. Taylor's remainder formula with $c = 1$ gives

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + R_4,$$

where $R_4 = f^{(5)}(z_4)\dfrac{(x-1)^5}{5!} = \dfrac{24}{z_4^5}\dfrac{(x-1)^5}{5!} = \dfrac{(x-1)^5}{5z_4^5}$ and z_4 is between 1 and x. Since $1/2 \le x \le 3/2$, we can say that

$$|R_4| < \frac{|x-1|^5}{5(1/2)^5} \le \frac{(1/2)^5}{5(1/2)^5} = 0.2.$$

11. Taylor's remainder formula for $\sin x$ gives

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{d^n}{dx^n}(\sin x)_{|x=0}\frac{x^n}{n!} + R_n(0,x)$$

where $R_n(0,x) = \dfrac{d^{n+1}(\sin x)}{dx^{n+1}}_{|x=z_n}\dfrac{x^{n+1}}{(n+1)!}$ and z_n is between 0 and x. Therefore

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots + \frac{1}{x}R_n(0,x).$$

When we take definite integrals,

$$\begin{aligned}\int_0^1 \frac{\sin x}{x}dx &= \int_0^1 \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots + \frac{1}{x}R_n(0,x)\right] dx\\ &= \left\{x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \cdots\right\}_0^1 + \int_0^1 \frac{1}{x}R_n(0,x)\,dx,\\ &= 1 - \frac{1}{3\cdot 3!} + \frac{1}{5\cdot 5!} - \cdots + \int_0^1 \frac{1}{x}R_n(0,x)\,dx.\end{aligned}$$

Now, $\left|\displaystyle\int_0^1 \frac{1}{x}R_n(0,x)\,dx\right| \le \displaystyle\int_0^1 \left|\frac{1}{x}\frac{d^{n+1}(\sin x)}{dx^{n+1}}_{|x=z_n}\frac{x^{n+1}}{(n+1)!}\right| dx$. Since $\left|\dfrac{d^{n+1}(\sin x)}{dx^{n+1}}_{|x=z_n}\right| \le 1$, it follows that

$$\left|\int_0^1 \frac{1}{x}R_n(0,x)\,dx\right| \le \int_0^1 \frac{x^n}{(n+1)!}dx = \left\{\frac{x^{n+1}}{(n+1)(n+1)!}\right\}_0^1 = \frac{1}{(n+1)(n+1)!}.$$

When $n = 6$, this is less than 0.000 029. Hence, if we approximate the integral with the first three terms, namely, $1 - \dfrac{1}{3\cdot 3!} + \dfrac{1}{5\cdot 5!} = 0.946\,111$, then we can say that

$$0.946\,111 - 0.000\,029 < \int_0^1 \frac{\sin x}{x} dx < 0.946\,111 + 0.000\,029;$$

that is, $0.946\,082 < \int_0^1 \frac{\sin x}{x} dx < 0.946\,140$. To three decimals, then, the value of the integral is 0.946.

12. If we set $u = x^2$ and $du = 2x\,dx$, then $\int_0^{1/2} \cos(x^2)\,dx = \frac{1}{2}\int_0^{1/4} \frac{\cos u}{\sqrt{u}} du$. Taylor's remainder formula for $\cos u$ gives

$$\cos u = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \cdots + \frac{d^n(\cos u)}{du^n}_{|u=0} \frac{u^n}{n!} + R_n(0,u),$$

where $R_n(0,u) = \frac{d^{n+1}(\cos u)}{du^{n+1}}_{|u=z_n} \frac{u^{n+1}}{(n+1)!}$. Consequently,

$$\begin{aligned}\int_0^{1/2} \cos(x^2)\,dx &= \frac{1}{2}\int_0^{1/4} \frac{1}{\sqrt{u}}\left[1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \cdots + R_n(0,u)\right] du \\ &= \frac{1}{2}\int_0^{1/4}\left[\frac{1}{\sqrt{u}} - \frac{u^{3/2}}{2!} + \frac{u^{7/2}}{4!} - \cdots + \frac{1}{\sqrt{u}}R_n(0,u)\right] du \\ &= \frac{1}{2}\left\{2\sqrt{u} - \frac{2u^{5/2}}{5\cdot 2!} + \frac{2u^{9/2}}{9\cdot 4!} - \cdots\right\}_0^{1/4} + \frac{1}{2}\int_0^{1/4} \frac{1}{\sqrt{u}} R_n(0,u)\,du \\ &= \frac{1}{2} - \frac{1}{5\cdot 2^5\cdot 2!} + \frac{1}{9\cdot 2^9 \cdot 4!} - \cdots + \frac{1}{2}\int_0^{1/4}\frac{1}{\sqrt{u}}R_n(0,u)\,du.\end{aligned}$$

Now,

$$\begin{aligned}\left|\frac{1}{2}\int_0^{1/4}\frac{1}{\sqrt{u}}R_n(0,u)\,du\right| &\le \frac{1}{2}\int_0^{1/4}\frac{1}{\sqrt{u}}|R_n(0,u)|\,du \le \frac{1}{2}\int_0^{1/4}\frac{1}{\sqrt{u}}\frac{u^{n+1}}{(n+1)!}du \\ &= \frac{1}{2}\int_0^{1/4}\frac{u^{n+1/2}}{(n+1)!}du = \frac{1}{2(n+1)!}\left\{\frac{u^{n+3/2}}{n+3/2}\right\}_0^{1/4} = \frac{1}{(2n+3)(n+1)!4^{n+3/2}}.\end{aligned}$$

When $n = 2$, this is less than 1.9×10^{-4}. Hence, if we approximate the integral with the first two terms, namely, $\frac{1}{2} - \frac{1}{5\cdot 2^5 \cdot 2!} = \frac{159}{320}$, then we can say that

$$\frac{159}{320} - 0.000\,19 < \int_0^{1/2}\cos(x^2)\,dx < \frac{159}{320} + 0.000\,19,$$

that is, $0.496\,685 < \int_0^{1/2}\cos(x^2)\,dx < 0.497\,065$. To three decimals, the value of the integral is 0.497.

13. Taylor's remainder formula for $\sin x$ gives

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{d^n}{dx^n}(\sin x)_{|x=0}\frac{x^n}{n!} + R_n(0,x)$$

where $R_n(0,x) = \frac{d^{n+1}(\sin x)}{dx^{n+1}}_{|x=z_n}\frac{x^{n+1}}{(n+1)!}$ and z_n is between 0 and x. Therefore

$$x^{11}\sin x = x^{12} - \frac{x^{14}}{3!} + \frac{x^{16}}{5!} - \cdots + x^{11}R_n(0,x).$$

When we take definite integrals,

$$\begin{aligned}\int_{-1}^{1} x^{11}\sin x\,dx &= \int_{-1}^{1}\left[x^{12}-\frac{x^{14}}{3!}+\frac{x^{16}}{5!}-\cdots+x^{11}R_n(0,x)\right]dx\\ &= \left\{\frac{x^{13}}{13}-\frac{x^{15}}{15\cdot 3!}+\frac{x^{17}}{17\cdot 5!}-\cdots\right\}_{-1}^{1}+\int_{-1}^{1}x^{11}R_n(0,x)\,dx,\\ &= \frac{2}{13}-\frac{2}{15\cdot 3!}+\frac{2}{17\cdot 5!}-\cdots+\int_{-1}^{1}x^{11}R_n(0,x)\,dx.\end{aligned}$$

Now,

$$\begin{aligned}\left|\int_{-1}^{1}x^{11}R_n(0,x)\,dx\right| &\le \int_{-1}^{1}\left|x^{11}\frac{d^{n+1}(\sin x)}{dx^{n+1}}_{|x=z_n}\frac{x^{n+1}}{(n+1)!}\right|dx \le \int_{-1}^{1}\frac{|x^{n+12}|}{(n+1)!}dx\\ &= \frac{2}{(n+1)!}\int_{0}^{1}x^{n+12}\,dx = \frac{2}{(n+1)!}\left\{\frac{x^{n+13}}{n+13}\right\}_0^1 = \frac{2}{(n+13)(n+1)!}.\end{aligned}$$

When $n=6$, this is less than 2.1×10^{-5}. Hence, if we approximate the integral with the first three terms, namely, $\frac{2}{13}-\frac{2}{15\cdot 3!}+\frac{2}{17\cdot 5!}=0.132\,604$, then we can say that

$$0.132\,604-0.000\,021<\int_{-1}^{1}x^{11}\sin x\,dx<0.132\,604+0.000\,021,$$

that is, $0.132\,583<\int_{-1}^{1}x^{11}\sin x\,dx<0.132\,625$. To three decimals, the value of the integral is 0.133.

14. If we set $w=x^2$ and $dw=2x\,dx$, then $\int_0^{0.3}e^{-x^2}\,dx=\frac{1}{2}\int_0^{0.09}\frac{e^{-w}}{\sqrt{w}}dw$. Taylor's remainder formula applied to e^{-w} gives

$$e^{-w}=1-w+\frac{w^2}{2!}-\frac{w^3}{3!}+\cdots+\frac{(-1)^n w^n}{n!}+R_n(0,w)$$

where $R_n(0,w)=\frac{d^{n+1}}{dw^{n+1}}(e^{-w})_{|w=w_n}\frac{w^{n+1}}{(n+1)!}=\frac{(-1)^{n+1}e^{-w_n}w^{n+1}}{(n+1)!}$. Consequently,

$$\begin{aligned}\int_0^{0.3}e^{-x^2}dx &= \frac{1}{2}\int_0^{0.09}\frac{1}{\sqrt{w}}\left[1-w+\frac{w^2}{2!}-\frac{w^3}{3!}+\cdots+\frac{(-1)^n w^n}{n!}+R_n(0,w)\right]dw\\ &= \frac{1}{2}\int_0^{0.09}\left[\frac{1}{\sqrt{w}}-\sqrt{w}+\frac{w^{3/2}}{2!}-\frac{w^{5/2}}{3!}+\cdots+\frac{(-1)^n w^{n-1/2}}{n!}+\frac{1}{\sqrt{w}}R_n(0,w)\right]dw\\ &= \frac{1}{2}\left\{2\sqrt{w}-\frac{2w^{3/2}}{3}+\frac{2w^{5/2}}{5\cdot 2!}-\frac{2w^{7/2}}{7\cdot 3!}+\cdots+\frac{2(-1)^n w^{n+1/2}}{(2n+1)n!}\right\}_0^{0.09}+\frac{1}{2}\int_0^{0.09}\frac{R_n(0,w)}{\sqrt{w}}dw\\ &= \sqrt{0.09}-\frac{(0.09)^{3/2}}{3}+\frac{(0.09)^{5/2}}{5\cdot 2!}-\frac{(0.09)^{7/2}}{7\cdot 3!}+\cdots+\frac{(-1)^n(0.09)^{n+1/2}}{(2n+1)n!}+\frac{1}{2}\int_0^{0.09}\frac{R_n(0,w)}{\sqrt{w}}dw.\end{aligned}$$

Now,

$$\frac{1}{2}\left|\int_0^{0.09}\frac{R_n(0,w)}{\sqrt{w}}dw\right|\le\frac{1}{2}\int_0^{0.09}\frac{1}{\sqrt{w}}\left|\frac{(-1)^{n+1}e^{-w_n}w^{n+1}}{(n+1)!}\right|dw=\frac{1}{2(n+1)!}\int_0^{0.09}e^{-w_n}w^{n+1/2}dw.$$

Since $0<w_n<w<0.09$, we can say $e^{-w_n}\le 1$. Thus,

$$\frac{1}{2}\left|\int_0^{0.09}\frac{R_n(0,w)}{\sqrt{w}}dw\right|\le\frac{1}{2(n+1)!}\left\{\frac{2w^{n+3/2}}{2n+3}\right\}_0^{0.09}=\frac{(0.09)^{n+3/2}}{(2n+3)(n+1)!}.$$

When $n = 2$, this is less than 3.0×10^{-6}. Hence, if we approximate the integral with the first three terms, namely, $\sqrt{0.09} - \dfrac{(0.09)^{3/2}}{3} + \dfrac{(0.09)^{5/2}}{5 \cdot 2!} = 0.291\,243$, then we can say that

$$0.291\,243 - 0.000\,003 < \int_0^{0.3} e^{-x^2}\,dx < 0.291\,243 + 0.000\,003,$$

that is, $0.291\,240 < \displaystyle\int_0^{0.3} e^{-x^2}\,dx < 0.291\,246$. To three decimals, the value of the integral is 0.291.

15. Using the result of Example 10.24, $\displaystyle\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0}\left(1 + \frac{x^2}{3} + \frac{2x^4}{15} + \cdots\right) = 1.$

16. $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{1}{x^2}\left[1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)\right] = \lim_{x\to 0}\left(\frac{1}{2!} - \frac{x^2}{4!} + \cdots\right) = \frac{1}{2}$

17.
$$\lim_{x\to 0} \frac{(1-\cos x)^2}{3x^4} = \lim_{x\to 0} \frac{1}{3x^4}\left[1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)\right]^2 = \lim_{x\to 0} \frac{1}{3x^4}\left(\frac{x^2}{2!} - \frac{x^4}{4!} + \cdots\right)^2$$
$$= \lim_{x\to 0} \frac{1}{3x^4}\left(\frac{x^4}{4} - \frac{x^6}{24} + \cdots\right) = \lim_{x\to 0}\left(\frac{1}{12} - \frac{x^2}{72} + \cdots\right) = \frac{1}{12}.$$

18. $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0} \frac{1}{x}\left[\left(1 + \frac{x}{2} - \frac{(1/2)(-1/2)}{2!}x^2 + \cdots\right) - 1\right] = \lim_{x\to 0}\left[\frac{1}{2} + \frac{x}{8} + \cdots\right] = \frac{1}{2}$

19. $\displaystyle\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim_{x\to\infty} x\left(\frac{1}{x} - \frac{1}{3!x^3} + \frac{1}{5!x^5} - \cdots\right) = \lim_{x\to\infty}\left(1 - \frac{1}{3!x^2} + \frac{1}{5!x^4} - \cdots\right) = 1$

20.
$$\frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{1 + e^{-2x}}{1 - e^{-2x}} = \frac{1 + \left[1 - 2x + \dfrac{(-2x)^2}{2!} + \dfrac{(-2x)^3}{3!} + \cdots\right]}{1 - \left[1 - 2x + \dfrac{(-2x)^2}{2!} + \dfrac{(-2x)^3}{3!} + \cdots\right]} = \frac{2 - 2x + 2x^2 - \dfrac{4x^3}{3} + \cdots}{2x - 2x^2 + \dfrac{4x^3}{3} + \cdots}$$

Long division gives $\dfrac{e^x + e^{-x}}{e^x - e^{-x}} = \dfrac{1}{x} + \dfrac{x}{3} + \cdots$.

Thus, $\displaystyle\lim_{x\to 0}\left(\frac{e^x + e^{-x}}{e^x - e^{-x}} - \frac{1}{x}\right) = \lim_{x\to 0}\left[\left(\frac{1}{x} + \frac{x}{3} + \cdots\right) - \frac{1}{x}\right] = 0.$

21. Taylor's remainder formula for $\sin(x/3)$ gives

$$\sin(x/3) = \frac{x}{3} - \frac{x^3}{3^3 \cdot 3!} + \frac{x^5}{3^5 \cdot 5!} - \frac{x^7}{3^7 \cdot 7!} + \cdots + \frac{d^n}{dx^n}[\sin(x/3)]_{|x=0}\frac{x^n}{n!} + R_n(0,x)$$

where $R_n(0,x) = \dfrac{d^{n+1}[\sin(x/3)]}{dx^{n+1}}_{|x=z_n}\dfrac{x^{n+1}}{(n+1)!}$ and z_n is between 0 and x. Since the $(n+1)^{\text{th}}$ derivative of $\sin(x/3)$ is $\pm 3^{-n-1}\sin(x/3)$ or $\pm 3^{-n-1}\cos(x/3)$, and $|x| \le 4$, it follows that

$$|R_n(0,x)| \le \frac{|x|^{n+1}}{3^{n+1}(n+1)!} \le \frac{4^{n+1}}{3^{n+1}(n+1)!}.$$

The smallest integer for which this is less than 10^{-3} is $n = 7$. Thus, the series should be truncated after $x^7/(3^7 \cdot 7!)$.

22. We set $u = x^3$ and consider the function $f(u) = 1/\sqrt{1+u}$ on the interval $0 < u < 1/8$. Since the n^{th} derivative of $f(u)$ is $f^{(n)}(u) = \dfrac{(-1)^n[1 \cdot 3 \cdot 5 \cdots (2n-1)]}{2^n(1+u)^{n+1/2}}$, Taylor's remainder formula gives

$$f(u) = 1 - \frac{u}{2} + \frac{3u^2}{8} - \frac{5u^3}{16} + \cdots + \frac{(-1)^n[1 \cdot 3 \cdot 5 \cdots (2n-1)]}{2^n\,n!}u^n + R_n(0,u),$$

where $R_n(0,u) = \dfrac{f^{(n+1)}(z_n)}{(n+1)!}u^{n+1}$, and $0 < z_n < u$. Since $0 < u < 1/8$, we can say that

$$|R_n(0,u)| = \frac{1\cdot 3\cdot 5\cdots(2n+1)}{2^{n+1}|1+z_n|^{n+3/2}(n+1)!}|u|^{n+1} < \frac{1\cdot 3\cdot 5\cdots(2n+1)}{2^{n+1}|1+0|^{n+3/2}(n+1)!}|u|^{n+1}$$
$$< \frac{1\cdot 3\cdot 5\cdots(2n+1)}{2^{n+1}(n+1)!}\left(\frac{1}{8}\right)^{n+1} = \frac{1\cdot 3\cdot 5\cdots(2n+1)}{2^{4n+4}(n+1)!}.$$

The smallest integer for which this is less than 10^{-4} is $n=3$. Thus, we should approximate $1/\sqrt{1+u}$ with $1-u/2+3u^2/8-5u^3/16$, or approximate $1/\sqrt{1+x^3}$ with

$$1-\frac{x^3}{2}+\frac{3x^6}{8}-\frac{5x^9}{16}.$$

23. Since the n^{th} derivative of $f(x)=\ln(1-x)$ is $f^{(n)}(x)=-(n-1)!/(1-x)^n$, Taylor's remainder formula gives

$$f(x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\cdots-\frac{x^n}{n}+R_n(0,x),$$

where $R_n(0,x)=\dfrac{f^{(n+1)}(z_n)}{(n+1)!}x^{n+1}=\dfrac{-n!\,x^{n+1}}{(n+1)!(1-z_n)^{n+1}}=\dfrac{-x^{n+1}}{(n+1)(1-z_n)^{n+1}}$, and z_n is between 0 and x. Since $|x|<1/3$, we can say that

$$|R_n(0,x)| = \frac{|x|^{n+1}}{(n+1)|1-z_n|^{n+1}} < \frac{|x|^{n+1}}{(n+1)|1-1/3|^{n+1}} < \frac{(1/3)^{n+1}}{(n+1)(2/3)^{n+1}} = \frac{1}{(n+1)2^{n+1}}.$$

The smallest integer for which this is less than 10^{-2} is $n=4$. Thus, we should approximate $\ln(1-x)$ with $-x-x^2/2-x^3/3-x^4/4$.

24. Taylor's remainder formula for $\cos^2 x=(1+\cos 2x)/2$ gives

$$\cos^2 x = \frac{1}{2}(1+\cos 2x) = \frac{1}{2}\left[1+\left(1-\frac{2^2x^2}{2!}+\frac{2^4x^4}{4!}-\cdots+\frac{f^{(n)}(0)}{n!}x^n+R_n(0,x)\right)\right]$$
$$= 1-x^2+\frac{x^4}{3}-\cdots+\frac{f^{(n)}(0)}{2n!}x^n+\frac{1}{2}R_n(0,x),$$

where $R_n(0,x)=\dfrac{f^{(n+1)}(z_n)}{(n+1)!}x^{n+1}$ and z_n is between 0 and x. Since the $(n+1)^{\text{th}}$ derivative of $f(x)$ is $\pm 2^{n+1}\sin 2x$ or $\pm 2^{n+1}\cos 2x$, and $|x|\le 0.1$, it follows that

$$\frac{1}{2}|R_n(0,x)| \le \frac{2^{n+1}|x|^{n+1}}{2(n+1)!} < \frac{2^n}{(n+1)!10^{n+1}}.$$

The smallest integer for which this is less than 10^{-3} is $n=2$. Thus, the function should be approximated by $1-x^2$.

25. If we substitute $y=f(x)=\sum_{n=0}^{\infty}a_nx^n$ into the differential equation,

$$0 = -4+\sum_{n=0}^{\infty}3a_nx^n+\sum_{n=0}^{\infty}na_nx^{n-1} = -4+\sum_{n=0}^{\infty}3a_nx^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$
$$= (-4+3a_0+a_1)+\sum_{n=1}^{\infty}[3a_n+(n+1)a_{n+1}]x^n.$$

When we equate coefficients to zero:

$$-4+3a_0+a_1=0 \quad\text{and}\quad 3a_n+(n+1)a_{n+1}=0,\quad n\ge 1.$$

The first implies that $a_1=4-3a_0$ and the second gives the recursive formula $a_{n+1}=\dfrac{-3a_n}{(n+1)}$, $n\ge 1$. Iteration leads to

$$a_2 = -\frac{3a_1}{2} = \frac{-3(4-3a_0)}{2}, \quad a_3 = -\frac{3a_2}{3} = \frac{3^2(4-3a_0)}{3!}, \quad a_4 = -\frac{3a_3}{4} = -\frac{3^3(4-3a_0)}{4!}, \quad \cdots .$$

Thus,

$$\begin{aligned} y = f(x) &= a_0 + (4-3a_0)x - \frac{3(4-3a_0)}{2!}x^2 + \frac{3^2(4-3a_0)}{3!}x^3 + \cdots \\ &= a_0 + \frac{(4-3a_0)}{3}\left[3x - \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \cdots\right] \\ &= a_0 + \frac{(4-3a_0)}{3}(1-e^{-3x}) = \frac{4}{3} + \frac{(3a_0-4)}{3}e^{-3x}. \end{aligned}$$

26. If we substitute $y = f(x) = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation,

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty}(n+1)n a_{n+1}x^{n-1} + \sum_{n=1}^{\infty} n a_n x^{n-1} \\ &= \sum_{n=1}^{\infty}[(n+1)n a_{n+1} + n a_n]x^{n-1}. \end{aligned}$$

When we equate coefficients to zero:

$$(n+1)n a_{n+1} + n a_n = 0 \quad \Longrightarrow \quad a_{n+1} = -\frac{a_n}{n+1}, \quad n \geq 1.$$

This recursive definition implies that

$$a_2 = -\frac{a_1}{2}, \quad a_3 = -\frac{a_2}{3} = \frac{a_1}{3!}, \quad a_4 = -\frac{a_3}{4} = -\frac{a_1}{4!}, \quad \cdots .$$

Thus, $$\begin{aligned} y = f(x) &= a_0 + a_1\left(x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots\right) = a_0 + a_1\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}x^n \\ &= a_0 - a_1(e^{-x}-1) = (a_0+a_1) - a_1 e^{-x}. \end{aligned}$$

27. If we substitute $y = f(x) = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation,

$$0 = -3x - \sum_{n=0}^{\infty} 4a_n x^n + \sum_{n=0}^{\infty} n a_n x^n = -4a_0 + (-3 - 4a_1 + a_1)x + \sum_{n=2}^{\infty}(n-4)a_n x^n.$$

When we equate coefficients to zero:

$$a_0 = 0, \qquad -3 - 3a_1 = 0, \qquad (n-4)a_n = 0, \quad n \geq 2.$$

These imply that $a_1 = -1$, a_4 is undetermined, and all other coefficients vanish. Thus, $y = f(x) = -x + a_4 x^4$.

28. If we substitute $y = f(x) = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation,

$$\begin{aligned} 0 &= 4x\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + 2\sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} 4n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} 2n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=2}^{\infty} 4n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} 2n a_n x^{n-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n-1} \\ &= (2a_1 + a_0) + \sum_{n=2}^{\infty}[(4n^2 - 4n + 2n)a_n + a_{n-1}]x^{n-1}. \end{aligned}$$

We now equate coefficients of powers of x to zero. From the coefficient of x^0 we obtain $2a_1 + a_0 = 0$ which implies that $a_1 = -a_0/2$. From the remaining coefficients, we obtain

$$(4n^2 - 2n)a_n + a_{n-1} = 0 \quad \Longrightarrow \quad a_n = \frac{-a_{n-1}}{2n(2n-1)}, \quad n \geq 2.$$

When we iterate this recursive definition:

$$a_2 = \frac{-a_1}{4 \cdot 3} = \frac{a_0}{4!}, \quad a_3 = \frac{-a_2}{6 \cdot 5} = -\frac{a_0}{6!}, \quad \cdots$$

The solution is therefore $\quad y = f(x) = a_0 \left(1 - \frac{x}{2} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots\right) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n.$

29. If we substitute $y = f(x) = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation,

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + a_n]x^n. \end{aligned}$$

When we equate coefficients of powers of x to zero, we obtain the recursive formula

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}, \quad n \geq 0.$$

Iteration gives

$$a_2 = -\frac{a_0}{2!}, \quad a_4 = \frac{a_0}{4!}, \quad a_6 = -\frac{a_0}{6!}, \quad \ldots, \quad \text{and} \quad a_3 = -\frac{a_1}{3!}, \quad a_5 = \frac{a_1}{5!}, \quad a_7 = -\frac{a_1}{7!}, \quad \ldots .$$

The solution is therefore

$$y = f(x) = a_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + a_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) = a_0 \cos x + a_1 \sin x.$$

30. If we substitute $y = f(x) = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation,

$$\begin{aligned} 0 &= x\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} a_{n-1}x^{n-1} = a_0 + \sum_{n=2}^{\infty} [n(n-1)a_n + a_{n-1}]x^{n-1}. \end{aligned}$$

We now equate coefficients of powers of x to zero. From the coefficient of x^0 we obtain $a_0 = 0$. From the remaining coefficients, we obtain

$$n(n-1)a_n + a_{n-1} = 0 \quad \Longrightarrow \quad a_n = \frac{-a_{n-1}}{n(n-1)}, \quad n \geq 2.$$

When we iterate this recursive definition:

$$a_2 = \frac{-a_1}{2 \cdot 1}, \quad a_3 = \frac{-a_2}{3 \cdot 2} = \frac{a_1}{3!\,2!}, \quad a_4 = \frac{-a_3}{4 \cdot 3} = \frac{-a_1}{4!\,3!}, \quad \ldots .$$

The solution is therefore

$$y = f(x) = a_1 \left(x - \frac{x^2}{2 \cdot 1} + \frac{x^3}{3! \cdot 2!} - \frac{x^4}{4! \cdot 3!} + \cdots\right) = a_1 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!(n-1)!} x^n.$$

31. According to Exercise 23, Taylor's remainder formula for $\ln(1-x)$ is

$$\ln(1-x) = -x - \frac{x^2}{2} - \cdots + R_n(0,x), \quad \text{where } R_n(0,x) = \frac{-x^{n+1}}{(n+1)(1-z_n)^{n+1}},$$

and z_n is between 0 and x. The maximum error when only the first term is used is $R_1(0,x) = \dfrac{-x^2}{2(1-z_1)^2}$. If we set $x = 0.000\,000\,000\,1$, then $z_1 < 0.000\,000\,000\,1$, and we can say that

$$|R_1(0, 0.000\,000\,000\,1)| < \frac{(0.000\,000\,000\,1)^2}{2(0.999\,999\,999\,9)^2} < 3.4 \times 10^{-21}.$$

Hence, $\ln(0.999\,999\,999\,9) = -10^{-10}$, and this is definitely accurate to more than 15 decimal places.

32.
$$\begin{aligned}
K &= c^2(m-m_0) = c^2 m_0\left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right) \\
&= c^2 m_0\left\{\left[1 - \frac{1}{2}\left(-\frac{v^2}{c^2}\right) + \frac{(-1/2)(-3/2)}{2!}\left(-\frac{v^2}{c^2}\right)^2 + \frac{(-1/2)(-3/2)(-5/2)}{3!}\left(-\frac{v^2}{c^2}\right)^3 + \cdots\right] - 1\right\} \\
&= c^2 m_0\left\{\frac{v^2}{2c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right\} = \frac{1}{2}m_0 v^2 + m_0 c^2\left(\frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right)
\end{aligned}$$

33. Using the binomial expansion,

$$\frac{P_0}{P} = 1 + \left(\frac{k}{k-1}\right)\left(\frac{k-1}{2}\right)M^2 + \cdots = 1 + \frac{kM^2}{2} + \cdots.$$

34. When we expand P_s/P_0 with the binomial expansion,

$$\begin{aligned}
\frac{P_s}{P_0} &= 1 + \left(\frac{k}{k-1}\right)\left(\frac{k-1}{2}\right)M_0^2 + \frac{1}{2}\left(\frac{k}{k-1}\right)\left(\frac{k}{k-1} - 1\right)\left(\frac{k-1}{2}\right)^2 M_0^4 \\
&\quad + \frac{1}{3!}\left(\frac{k}{k-1}\right)\left(\frac{k}{k-1} - 1\right)\left(\frac{k}{k-1} - 2\right)\left(\frac{k-1}{2}\right)^3 M_0^6 + \cdots \\
&= 1 + \frac{k}{2}M_0^2 + \frac{k}{8}M_0^4 + \frac{k(2-k)}{48}M_0^6 + \cdots \\
&= 1 + \frac{1}{2}M_0^2\left(\frac{\rho_0 c_0^2}{P_0}\right) + \frac{1}{8}M_0^4\left(\frac{\rho_0 c_0^2}{P_0}\right) + \frac{1}{48}M_0^6(2-k)\left(\frac{\rho_0 c_0^2}{P_0}\right) + \cdots.
\end{aligned}$$

Multiplication by P_0, and replacement of M_0^2 by V_0^2/c_0^2 in the last three terms gives

$$\begin{aligned}
P_s &= P_0 + \frac{1}{2}\rho_0 c_0^2\left(\frac{V_0^2}{c_0^2}\right) + \frac{1}{8}\rho_0 c_0^2\left(\frac{V_0^2}{c_0^2}\right)M_0^2 + \frac{1}{48}(2-k)\rho_0 c_0^2\left(\frac{V_0^2}{c_0^2}\right)M_0^4 + \cdots \\
&= P_0 + \frac{1}{2}\rho_0 V_0^2 + \frac{1}{8}\rho_0 V_0^2 M_0^2 + \frac{1}{48}(2-k)\rho_0 V_0^2 M_0^4 + \cdots \\
&= P_0 + \frac{1}{2}\rho_0 V_0^2\left[1 + \frac{M_0^2}{4} + \left(\frac{2-k}{24}\right)M_0^4 + \cdots\right].
\end{aligned}$$

35. (a) Using formula 9.3, the length of the ellipse is four times that in the first quadrant,

$$L = 4\int_0^{\pi/2}\sqrt{(-a\sin t)^2 + (b\cos t)^2}\,dt = 4b\int_0^{\pi/2}\sqrt{\frac{a^2}{b^2}\sin^2 t + (1-\sin^2 t)}\,dt = 4b\int_0^{\pi/2}\sqrt{1-k^2\sin^2 t}\,dt,$$

where $k^2 = 1 - a^2/b^2$.
(b) If we expand the integrand with the binomial expansion 10.33b, and integrate term-by-term,

$$\begin{aligned}
L &= 4b\int_0^{\pi/2}\left[1+\frac{1}{2}(-k^2\sin^2 t)+\frac{(1/2)(-1/2)}{2}(-k^2\sin^2 t)^2+\cdots\right]dt\\
&= 4b\int_0^{\pi/2}\left[1-\frac{k^2}{2}\left(\frac{1-\cos 2t}{2}\right)-\frac{k^4}{8}\left(\frac{1-\cos 2t}{2}\right)^2+\cdots\right]dt\\
&= 4b\int_0^{\pi/2}\left[1-\frac{k^2}{4}(1-\cos 2t)-\frac{k^4}{32}\left(1-2\cos 2t+\frac{1+\cos 4t}{2}\right)+\cdots\right]dt\\
&= 4b\left\{t-\frac{k^2}{4}\left(t-\frac{\sin 2t}{2}\right)-\frac{k^4}{32}\left(\frac{3t}{2}-\sin 2t+\frac{\sin 4t}{8}\right)+\cdots\right\}_0^{\pi/2}\\
&= 4b\left[\frac{\pi}{2}-\frac{k^2}{4}\left(\frac{\pi}{2}\right)-\frac{k^4}{32}\left(\frac{3\pi}{4}\right)+\cdots\right]\\
&= 2\pi b\left(1-\frac{k^2}{4}-\frac{3k^4}{64}+\cdots\right).
\end{aligned}$$

36. (a) If we substitute $e^{-\beta^2/(4\alpha x)}=\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{\beta^2}{4\alpha x}\right)^n$, we obtain

$$W(\alpha,\beta)=\int_1^\infty\frac{1}{x}e^{-\alpha x}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{\beta^2}{4\alpha x}\right)^n dx=\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{\beta^2}{4\alpha}\right)^n\int_1^\infty\frac{e^{-\alpha x}}{x^{n+1}}dx=\sum_{n=0}^{\infty}\frac{(-1)^n\beta^{2n}}{4^n\alpha^n n!}E_{n+1}(\alpha).$$

(b) We use integration by parts with $u=e^{-\alpha x}$ and $dv=\dfrac{1}{x^{n+1}}dx$,

$$\begin{aligned}
E_{n+1}(\alpha)&=\int_1^\infty\frac{e^{-\alpha x}}{x^{n+1}}dx=\left\{-\frac{e^{-\alpha x}}{nx^n}\right\}_1^\infty-\int_1^\infty-\frac{1}{nx^n}(-\alpha)e^{-\alpha x}\,dx=\frac{e^{-\alpha}}{n}-\frac{\alpha}{n}\int_1^\infty\frac{e^{-\alpha x}}{x^n}dx\\
&=\frac{1}{n}[e^{-\alpha}-\alpha E_n(\alpha)].
\end{aligned}$$

37. If we substitute the Maclaurin series for $e^{ch/(\lambda kT)}$,

$$\Psi(\lambda)=\frac{8\pi ch\lambda^{-5}}{\left(1+\dfrac{ch}{\lambda kT}+\dfrac{c^2h^2}{2\lambda^2k^2T^2}+\cdots\right)-1}=\frac{8\pi ch}{\lambda^5\left(\dfrac{ch}{\lambda kT}+\dfrac{c^2h^2}{2\lambda^2k^2T^2}+\cdots\right)}=\frac{8\pi ch}{\dfrac{ch}{kT}\lambda^4+\dfrac{c^2h^2}{2k^2T^2}\lambda^3+\cdots}.$$

If we long divide the denominator into the numerator, the result is

$$\Psi(\lambda)=\frac{8\pi kT}{\lambda^4}+\text{ terms in }\lambda^{-5},\ \lambda^{-6},\text{ etc..}$$

Thus, for large λ, $\Psi(\lambda)$ can be approximated by $8\pi kT/\lambda^4$.

38. (a) We write $E=\dfrac{q}{4\pi\epsilon_0x^2\left(1-\dfrac{d}{2x}\right)^2}-\dfrac{q}{4\pi\epsilon_0x^2\left(1+\dfrac{d}{2x}\right)^2}=\dfrac{q}{4\pi\epsilon_0x^2}\left[\left(1-\dfrac{d}{2x}\right)^{-2}-\left(1+\dfrac{d}{2x}\right)^{-2}\right].$

(b) If we expand each term with the binomial expansion 10.33b,

$$E=\frac{q}{4\pi\epsilon_0x^2}\left\{\left[1-2\left(-\frac{d}{2x}\right)+\cdots\right]-\left[1-2\left(\frac{d}{2x}\right)+\cdots\right]\right\}.$$

When d is very much less than x, we omit higher order terms in d/x, and write

$$E\approx\frac{q}{4\pi\epsilon_0x^2}\left(1+\frac{d}{x}-1+\frac{d}{x}\right)=\frac{qd}{2\pi\epsilon_0x^3}.$$

39. The cross-sectional area of the liquid is the area of the sector less the area of the triangle above it,

$$A = \frac{1}{2}R^2\theta - 2\left(\frac{1}{2}\right)\left(R\sin\frac{\theta}{2}\right)\left(R\cos\frac{\theta}{2}\right) = \frac{R^2}{2}(\theta - \sin\theta).$$

Since $d = 2R\sin\dfrac{\theta}{2}$ and $h = R - R\cos\dfrac{\theta}{2}$,

$$hd = 2R\sin\frac{\theta}{2}\left(R - R\cos\frac{\theta}{2}\right) = R^2\left(2\sin\frac{\theta}{2} - \sin\theta\right).$$

The required ratio is

$$\frac{A}{hd} = \frac{\dfrac{R^2}{2}(\theta - \sin\theta)}{R^2\left(2\sin\dfrac{\theta}{2} - \sin\theta\right)} = \frac{\theta - \sin\theta}{2\left(2\sin\dfrac{\theta}{2} - \sin\theta\right)}.$$

If we expand the sine functions in their Maclaurin series

$$\begin{aligned}\frac{A}{hd} &= \frac{\theta - \left(\theta - \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} - \cdots\right)}{2\left[2\left(\dfrac{\theta}{2} - \dfrac{(\theta/2)^3}{3!} + \dfrac{(\theta/2)^5}{5!} - \cdots\right) - \left(\theta - \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} - \cdots\right)\right]}\\ &= \frac{\dfrac{\theta^3}{3!} - \dfrac{\theta^5}{5!} + \cdots}{\dfrac{\theta^3}{4} - \dfrac{\theta^5}{64} + \cdots} \qquad \text{(and by long division)}\\ &= \frac{2}{3} + \frac{\theta^2}{120} + \cdots .\end{aligned}$$

For small θ, we can use the approximation $\quad \dfrac{A}{hd} \approx \dfrac{2}{3} + \dfrac{\theta^2}{120}$.

EXERCISES 10.8

1. True If a sequence satisfies 10.35a, then it satisfies 10.35b; that is, every increasing sequence is nondecreasing.

2. False The sequence $\{n\}$ is increasing but has no upper bound.

3. True The first term of an increasing sequence is a lower bound.

4. False The sequence $\{-n\}$ is decreasing but has no lower bound.

5. False The sequence $\{n\}$ is increasing with lower bound 1, but it does not have a limit.

6. True An increasing sequence has a lower bound. If it also has an upper bound, then it has a limit according to Theorem 10.7.

7. False The sequence $\{(-1)^n\}$ does not converge, but its terms are all ± 1.

8. True For a sequence to be increasing and decreasing, its terms would have to satisfy $c_{n+1} > c_n$ and $c_{n+1} < c_n$ for all n. This is impossible.

9. True The sequence $\{1\}$ is an example.

10. True This is part of the corollary to Theorem 10.7.

11. False The sequence $\{(-1)^n/n\}$ is bounded and has limit 0, but it is not monotonic.

12. False The sequence $\{(-1)^n/n\}$ is bounded, not monotonic, and it has limit 0.

13. True Suppose that $\lim_{n\to\infty} c_n = L$. Then there exists an integer N such that for all $n > N$, all terms of the sequence satisfy $|c_n - L| < 1$. This implies that for $n > N$, terms satisfy $c_n < L + 1$. If we set U equal to the biggest of $L+1$ and the terms $c_1, c_2, \ldots, c_N$, then we can say that all terms of the sequence are less than U. Thus, U is an upper bound for the sequence. A similar proof shows that the sequence has a lower bound.

14. False The sequence $\{(-1)^n/n\}$ is not montonic, but it has limit 0.

15. False L could be equal to U. For example all terms of the sequence $\{(n-1)/n\}$ are less than 1, and the limit of the sequence is 1.

16. False The sequence $\{(-1)^n\}$ has no limit, but the sequence $\{[(-1)^n]^2\} = \{1\}$ has limit 1.

17. False The oscillating sequence $\{(-1)^n\}$ does not converge.

18. True Such a sequence displays the up-down-up-down nature required of an oscillating sequence.

19. True Terms of such a sequence must approach 0.

20. False Each term of the sequence $\{-3^n\}$ is less than half the previous term, but the sequence does not have a limit.

21. True If L is the limit of the sequence and U is the smallest of the upper bounds, then L cannot be smaller than U, otherwise there would be smaller upper bounds than U. On the other hand, L cannot be larger than U because U would not then be an upper bound. Hence, L must be equal to U.

22. True. The only other possibility is that the sequence $\{c_n\}$ has a limit L which is not equal to a. Suppose $L > a$ and we set $\epsilon = L - a$. Then there exists an integer N such that for $n > N$, $|c_n - L| < L - a$; that is $-(L-a) < c_n - L < L - a$. Thus, for $n > N$,

$$L - (L-a) < c_n < L + (L-a) \quad \Longrightarrow \quad a < c_n < 2L - a.$$

But this contradicts the fact that $c_n = a$ for an infinity of values of n. Hence L cannot be greater than a. A similar proof shows that L cannot be less than a.

23. False According to Theorem 10.8, absolute values of differences must also approach 0. For example, absolute values of the differences of the terms of the oscillating sequence $\{[1 + (-1)^n]/2 + (-1/2)^n\}$ decrease, but the sequence does not have a limit. The easiest way to see this is to plot about ten terms of the sequence.

24. False The oscillating sequence $\{[1 + (-1)^{n+1}]/(2n)\}$ has limit zero, but terms are all nonnegative.

25. The first four terms of the sequence are $c_1 = 1$, $c_2 = 13/10 = 1.3$, $c_3 = 1.4197$, $c_4 = 1.486\,147$. The sequence appears to be increasing; that is $c_{n+1} > c_n$. This is certainly true for $n = 1$ as $c_2 > c_1$. Suppose $c_{k+1} > c_k$ for some integer k. Then, ${c_{k+1}}^3 > {c_k}^3$, and ${c_{k+1}}^3 + 12 > {c_k}^3 + 12$. Thus, $({c_{k+1}}^3 + 12)/10 > ({c_k}^3 + 12)/10$, and this means that $c_{k+2} > c_{k+1}$. By mathematical induction then, $c_{n+1} > c_n$ for all $n \ge 1$. The first term $c_1 = 1$ must be a lower bound. We suspect that $U = 1.8$ is an upper bound; that is, $c_n \le 1.8$. This is true for $n = 1$. Suppose $c_k \le 1.8$ for some integer k. Then $c_{k+1} = ({c_k}^3 + 12)/10 \le (1.8^3 + 12)/10 < 1.8$. Hence, by mathematical induction, $c_n \le 1.8$ for all n.

Theorem 10.7 now implies that the sequence has a limit, call it L. By taking limits on both sides of the recursive definition of the sequence we obtain

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{1}{10}({c_n}^3 + 12).$$

It follows that L must satisfy $L = (L^3 + 12)/10 \Longrightarrow L^3 - 10L + 12 = 0$. Roots of this equation are 2 and $-1 \pm \sqrt{7}$. Since 1.8 is an upper bound for the sequence, it follows that $L = -1 + \sqrt{7}$.

26. The first four terms of the sequence are $c_1 = 0$, $c_2 = 5/12$, $c_3 = 0.419\,178$, $c_4 = 0.419\,240$. The sequence appears to be increasing; that is $c_{n+1} > c_n$. This is certainly true for $n = 1$ as $c_2 > c_1$. Suppose $c_{k+1} > c_k$ for some integer k. Then, ${c_{k+1}}^4 > {c_k}^4$, and ${c_{k+1}}^4 + 5 > {c_k}^4 + 5$. Thus, $({c_{k+1}}^4 + 5)/12 > ({c_k}^4 + 5)/12$, and this means that $c_{k+2} > c_{k+1}$. By mathematical induction then, $c_{n+1} > c_n$ for all $n \ge 1$. The first term $c_1 = 0$ must be a lower bound. We suspect that $U = 1$ is an upper bound; that is, $c_n \le 1$. This is true for $n = 1$. Suppose $c_k \le 1$ for some integer k. Then $c_{k+1} = ({c_k}^4 + 5)/12 \le (1 + 5)/12 = 1/2 < 1$. Hence, by mathematical induction, $c_n \le 1$ for all n.

Theorem 10.7 now implies that the sequence has a limit, call it L. By taking limits on both sides of the recursive definition of the sequence we obtain

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{1}{12}({c_n}^4 + 5).$$

It follows that L must satisfy $L = (L^4 + 5)/12 \Longrightarrow 0 = L^4 - 12L + 5 = f(L)$. Since $f(0.419\,240\,5) = 6.5 \times 10^{-6}$ and $f(0.419\,241\,5) = -5.2 \times 10^{-6}$, the limit of the sequence (to six decimals) is 0.419 241.

27. It is clear that all terms of the sequence are positive and therefore there is no difficulty with taking square roots. The first four terms of the sequence are $c_1 = 3$, $c_2 = \sqrt{8} = 2.828$, $c_3 = 2.7979$, $c_4 = 2.7925$. The sequence appears to be decreasing; that is $c_{n+1} < c_n$. This is certainly true for $n = 1$ as $c_2 < c_1$. Suppose $c_{k+1} < c_k$ for some integer k. Then, $5 + c_{k+1} < 5 + c_k$, and $\sqrt{5 + c_{k+1}} < \sqrt{5 + c_k}$. Thus, $c_{k+2} < c_{k+1}$, and by mathematical induction, $c_{n+1} < c_n$ for all $n \geq 1$. The first term $c_1 = 3$ must be an upper bound. Clearly $V = 0$ is a lower bound.

Theorem 10.7 now implies that the sequence has a limit, call it L. By taking limits on both sides of the recursive definition of the sequence we obtain

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \sqrt{5 + c_n}.$$

It follows that L must satisfy $L = \sqrt{5 + L} \Longrightarrow L^2 - L - 5 = 0$. Of the two solutions $(1 \pm \sqrt{21})/2$ of this equation, only $(1 + \sqrt{21})/2$ lies between the bounds. Hence, $L = (1 + \sqrt{21})/2$.

28. It is clear that all terms of the sequence are positive and therefore there is no difficulty with taking square roots. The first four terms of the sequence are $c_1 = 1$, $c_2 = \sqrt{6} = 2.449$, $c_3 = 2.7294$, $c_4 = 2.7802$. The sequence appears to be increasing; that is $c_{n+1} > c_n$. This is certainly true for $n = 1$ as $c_2 > c_1$. Suppose $c_{k+1} > c_k$ for some integer k. Then, $5 + c_{k+1} > 5 + c_k$, and $\sqrt{5 + c_{k+1}} > \sqrt{5 + c_k}$. Thus, $c_{k+2} > c_{k+1}$, and by mathematical induction, $c_{n+1} > c_n$ for all $n \geq 1$. The first term $c_1 = 1$ must be a lower bound. We suspect that $U = 5$ is an upper bound; that is, $c_n \leq 5$. This is true for $n = 1$. Suppose $c_k \leq 5$ for some integer k. Then $c_{k+1} = \sqrt{5 + c_k} \leq \sqrt{5 + 5} = \sqrt{10} < 5$. Hence, by mathematical induction, $c_n \leq 5$ for all n.

Theorem 10.7 now implies that the sequence has a limit, call it L. By taking limits on both sides of the recursive definition of the sequence we obtain

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \sqrt{5 + c_n}.$$

It follows that L must satisfy $L = \sqrt{5 + L} \Longrightarrow L^2 - L - 5 = 0$. Of the two solutions $(1 \pm \sqrt{21})/2$ of this equation, only $(1 + \sqrt{21})/2$ lies between the bounds. Hence, $L = (1 + \sqrt{21})/2$.

29. It is clear that all terms of the sequence are positive and therefore there is no difficulty with taking square roots. The first four terms of the sequence are $c_1 = 5$, $c_2 = 1 + \sqrt{11} = 4.317$, $c_3 = 4.212$, $c_4 = 4.196$. The sequence appears to be decreasing; that is $c_{n+1} < c_n$. This is certainly true for $n = 1$ as $c_2 < c_1$. Suppose $c_{k+1} < c_k$ for some integer k. Then, $6 + c_{k+1} < 6 + c_k$, and $\sqrt{6 + c_{k+1}} < \sqrt{6 + c_k}$. Thus, $1 + \sqrt{6 + c_{k+1}} < 1 + \sqrt{6 + c_k}$, and this means that $c_{k+2} < c_{k+1}$. By mathematical induction then, $c_{n+1} < c_n$ for all $n \geq 1$. The first term $c_1 = 5$ must be an upper bound, and $V = 0$ is a lower bound.

Theorem 10.7 now implies that the sequence has a limit, call it L. By taking limits on both sides of the recursive definition of the sequence we obtain

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \left[1 + \sqrt{6 + c_n}\right].$$

It follows that L must satisfy $L = 1 + \sqrt{6 + L} \Longrightarrow (L - 1)^2 = 6 + L$. This equation simplifies to $L^2 - 3L - 5 = 0$, and of the two solutions $(3 \pm \sqrt{29})/2$ of this equation, only $(3 + \sqrt{29})/2$ lies between the bounds. Hence, $L = (3 + \sqrt{29})/2$.

30. It is clear that all terms of the sequence are positive and therefore there is no difficulty with taking square roots. The first four terms of the sequence are $c_1 = 3$, $c_2 = 4$, $c_3 = 4.1623$, $c_4 = 4.1878$. The sequence appears to be increasing; that is $c_{n+1} > c_n$. This is certainly true for $n = 1$ as $c_2 > c_1$. Suppose $c_{k+1} > c_k$ for some integer k. Then, $6 + c_{k+1} > 6 + c_k$, and $\sqrt{6 + c_{k+1}} > \sqrt{6 + c_k}$. Thus, $1 + \sqrt{6 + c_{k+1}} > 1 + \sqrt{6 + c_k}$, and this means that $c_{k+2} > c_{k+1}$. By mathematical induction then,

$c_{n+1} > c_n$ for all $n \geq 1$. The first term $c_1 = 3$ must be a lower bound. We suspect that $U = 10$ is an upper bound; that is, $c_n \leq 10$. This is true for $n = 1$. Suppose $c_k \leq 10$ for some integer k. Then $c_{k+1} = 1 + \sqrt{6 + c_k} \leq 1 + \sqrt{6 + 10} = 5 < 10$. Hence, by mathematical induction, $c_n \leq 10$ for all n.

Theorem 10.7 now implies that the sequence has a limit, call it L. By taking limits on both sides of the recursive definition of the sequence we obtain

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \left[1 + \sqrt{6 + c_n}\right].$$

It follows that L must satisfy $L = 1 + \sqrt{6 + L} \Longrightarrow (L - 1)^2 = 6 + L$. This equation simplifies to $L^2 - 3L - 5 = 0$, and of the two solutions $(3 \pm \sqrt{29})/2$ of this equation, only $(3 + \sqrt{29})/2$ lies between the bounds. Hence, $L = (3 + \sqrt{29})/2$.

31. The first four terms of the sequence are $c_1 = 1$, $c_2 = 2$, $c_3 = 2.268$, $c_4 = 2.347$. Previous exercises have indicated that when it is anticipated that a sequence is monotonic and bounded, it is advantageous to first verify monotony. If a sequence is known to be increasing (or nondecreasing), then a lower bound must be the first term of the sequence. On the other hand, if a sequence is known to be decreasing (or nonincreasing), its first term is an upper bound. Unfortunately, it is not always possible to verify monotony without knowledge of bounds. This is such an example. Try to prove that this sequence is decreasing before reading the rest of this solution, and discover why the proof fails. In addition, to guarantee that all terms of the sequence are well-defined, we must know that no one of them can be greater than 5.

We begin by proving that upper and lower bounds are 4 and 0; that is, $0 \leq c_n \leq 4$. This is certainly true for $n = 1$. Suppose $0 \leq c_k \leq 4$ for some integer k. Then $0 \geq -c_k \geq -4$, from which $5 \geq 5 - c_k \geq 1$. It follows that $\sqrt{5} \geq \sqrt{5 - c_k} \geq 1$, and $-\sqrt{5} \leq -\sqrt{5 - c_k} \leq -1$. Thus, $4 - \sqrt{5} \leq 4 - \sqrt{5 - c_k} \leq 3$. This means that $0 < 4 - \sqrt{5} \leq c_{k+1} \leq 3 < 4$, and therefore by mathematical induction, $0 \leq c_n \leq 4$ for all n.

Now we verify that the sequence is increasing, $c_{n+1} > c_n$. This is certainly true for $n = 1$ as $c_2 > c_1$. Suppose $c_{k+1} > c_k$ for some integer k. Then, $5 - c_{k+1} < 5 - c_k$. Because all terms of the sequence are between 0 and 4, both sides of this inequality are positive, and we can take square roots, $\sqrt{5 - c_{k+1}} < \sqrt{5 - c_k}$. Thus, $4 - \sqrt{5 - c_{k+1}} > 4 - \sqrt{5 - c_k}$, and this means that $c_{k+2} > c_{k+1}$. By mathematical induction, $c_{n+1} > c_n$ for all $n \geq 1$.

Theorem 10.7 now implies that the sequence has a limit, call it L. By taking limits on both sides of the recursive definition of the sequence we obtain

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \left[4 - \sqrt{5 - c_n}\right].$$

It follows that L must satisfy $L = 4 - \sqrt{5 - L} \Longrightarrow (L - 4)^2 = 5 - L$. This equation simplifies to $L^2 - 7L + 11 = 0$, and of the two solutions $(7 \pm \sqrt{5})/2$, only $(7 - \sqrt{5})/2$ lies between the bounds. Hence, $L = (7 - \sqrt{5})/2$.

32. The first four terms of the sequence are $c_1 = 4$, $c_2 = 3$, $c_3 = 2.5858$, $c_4 = 2.4462$. Previous exercises have indicated that when it is anticipated that a sequence is monotonic and bounded, it is advantageous to first verify monotony. If a sequence is known to be increasing (or nondecreasing), then a lower bound must be the first term of the sequence. On the other hand, if a sequence is known to be decreasing (or nonincreasing), its first term is an upper bound. Unfortunately, it is not always possible to verify monotony without knowledge of bounds. This is such an example. Try to prove that this sequence is decreasing before reading the rest of this solution, and discover why the proof fails. In addition, to guarantee that all terms of the sequence are well-defined, we must know that no one of them can be greater than 5.

We begin by proving that upper and lower bounds are 4 and 0; that is, $0 \leq c_n \leq 4$. This is certainly true for $n = 1$. Suppose $0 \leq c_k \leq 4$ for some integer k. Then $0 \geq -c_k \geq -4$, from which $5 \geq 5 - c_k \geq 1$. It follows that $\sqrt{5} \geq \sqrt{5 - c_k} \geq 1$, and $-\sqrt{5} \leq -\sqrt{5 - c_k} \leq -1$. Thus, $4 - \sqrt{5} \leq 4 - \sqrt{5 - c_k} \leq 3$. This means that $0 < 4 - \sqrt{5} \leq c_{k+1} \leq 3 < 4$, and therefore by mathematical induction, $0 \leq c_n \leq 4$ for all n.

Now we verify that the sequence is decreasing, $c_{n+1} < c_n$. This is certainly true for $n = 1$ as $c_2 < c_1$. Suppose $c_{k+1} < c_k$ for some integer k. Then, $5 - c_{k+1} > 5 - c_k$. Because all terms of the sequence are between 0 and 4, both sides of this inequality are positive, and we can take square roots,

$\sqrt{5 - c_{k+1}} > \sqrt{5 - c_k}$. Thus, $4 - \sqrt{5 - c_{k+1}} < 4 - \sqrt{5 - c_k}$, and this means that $c_{k+2} < c_{k+1}$. By mathematical induction, $c_{n+1} < c_n$ for all $n \geq 1$.

Theorem 10.7 now implies that the sequence has a limit, call it L. By taking limits on both sides of the recursive definition of the sequence we obtain

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \left[4 - \sqrt{5 - c_n}\right].$$

It follows that L must satisfy $L = 4 - \sqrt{5 - L} \Longrightarrow (L - 4)^2 = 5 - L$. This equation simplifies to $L^2 - 7L + 11 = 0$, and of the two solutions $(7 \pm \sqrt{5})/2$, only $(7 - \sqrt{5})/2$ lies between the bounds. Hence, $L = (7 - \sqrt{5})/2$.

33. The first four terms of the sequence are $c_1 = 2$, $c_2 = 1$, $c_3 = 1/2$, $c_4 = 2/5$. An initial attempt at proving that this sequence is decreasing fails. We require information about its bounds. Consider proving that upper and lower bounds are 2 and 0; that is, $0 \leq c_n \leq 2$. This is certainly true for $n = 1$. Suppose $0 \leq c_k \leq 2$ for some integer k. Then, $0 \geq -c_k \geq -2$, from which $3 \geq 3 - c_k \geq 1$. Inversion gives $1/3 \leq 1/(3 - c_k) \leq 1$, but this implies that $0 < 1/3 \leq c_{k+1} \leq 1 < 2$. By mathematical induction then, $0 \leq c_n \leq 2$ for all n. Now we verify that the sequence is decreasing, that its terms satisfy $c_{n+1} < c_n$. This is true for $n = 1$. Suppose $c_{k+1} < c_k$ for some integer k. Then $-c_{k+1} > -c_k$, and, $3 - c_{k+1} > 3 - c_k$. Because all terms of the sequence are between 0 and 2, both expressions in this inequality are positive. We may therefore invert and write $1/(3 - c_{k+1}) < 1/(3 - c_k)$; i.e., $c_{k+2} < c_{k+1}$. By mathematical induction then, $c_{n+1} < c_n$ for all n.

Because the sequence is monotonic and bounded, Theorem 10.7 guarantees that it has a limit L. By writing

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{1}{3 - c_n},$$

we obtain the equation $L = 1/(3 - L)$. Of the two solutions $(3 \pm \sqrt{5})/2$ of this equation, only $(3 - \sqrt{5})/2$ is between the bounds. Hence, $L = (3 - \sqrt{5})/2$.

34. The first four terms of the sequence are $c_1 = 1$, $c_2 = 1/2$, $c_3 = 1/3$, $c_4 = 3/10$. An initial attempt at proving that this sequence is decreasing fails. We require information about its bounds. Consider proving that upper and lower bounds are 1 and 0; that is, $0 \leq c_n \leq 1$. This is certainly true for $n = 1$. Suppose $0 \leq c_k \leq 1$ for some integer k. Then, $0 \geq -2c_k \geq -2$, from which $4 \geq 4 - 2c_k \geq 2$. Inversion gives $1/4 \leq 1/(4 - 2c_k) \leq 1/2$, but this implies that $0 < 1/4 \leq c_{k+1} \leq 1/2 < 1$. By mathematical induction then, $0 \leq c_n \leq 1$ for all n. Now we verify that the sequence is decreasing, that its terms satisfy $c_{n+1} < c_n$. This is true for $n = 1$. Suppose $c_{k+1} < c_k$ for some integer k. Then $-2c_{k+1} > -2c_k$, and, $4 - 2c_{k+1} > 4 - 2c_k$. Because all terms of the sequence are between 0 and 1, both expressions in this inequality are positive. We may therefore invert and write $1/(4 - 2c_{k+1}) < 1/(4 - 2c_k)$; i.e., $c_{k+2} < c_{k+1}$. By mathematical induction then, $c_{n+1} < c_n$ for all n.

Because the sequence is monotonic and bounded, Theorem 10.7 guarantees that it has a limit L. By writing

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{1}{4 - 2c_n},$$

we obtain the equation $L = 1/(4 - 2L)$. Of the two solutions $1 \pm 1/\sqrt{2}$ of this equation, only $1 - 1/\sqrt{2}$ is between the bounds. Hence, $L = 1 - 1/\sqrt{2}$.

35. The first four terms of the sequence are $c_1 = 1$, $c_2 = 7/8$, $c_3 = 0.7089$, $c_4 = 0.5843$. An initial attempt at proving that this sequence is decreasing fails. We require information about its bounds. We prove first therefore that $0 \leq c_n \leq 1$. This is true for $n = 1$. Suppose $0 \leq c_k \leq 1$ for some integer k. Then, $0 \geq -8{c_k}^2 \geq -8$, and $16 \geq 16 - 8{c_k}^2 \geq 8$. Inversion gives $1/16 \leq 1/(16 - 8{c_k}^2) \leq 1/8$. But then $7/16 \leq 7/(16 - 8{c_k}^2) \leq 7/8$, or, $0 < 7/16 \leq c_{k+1} \leq 7/8 < 1$. By mathematical induction then, $0 \leq c_n \leq 1$, and we have upper and lower bounds. Now we verify that the sequence is decreasing, that its terms satisfy $c_{n+1} < c_n$. This is true for $n = 1$. Suppose $c_{k+1} < c_k$ for some integer k. Then $-8{c_{k+1}}^2 > -8{c_k}^2$, and, $16 - 8{c_{k+1}}^2 > 16 - 8{c_k}^2$. Because all terms of the sequence are between 0 and 1, both expressions in this inequality are positive. We may therefore invert and write $1/(16 - 8{c_{k+1}}^2) < 1/(16 - 8{c_k}^2)$. In other words, $7/(16 - 8{c_{k+1}}^2) < 7/(16 - 8{c_k}^2)$, or, $c_{k+2} < c_{k+1}$. By mathematical induction then, $c_{n+1} < c_n$ for all n.

Because the sequence is monotonic and bounded, Theorem 10.7 guarantees that it has a limit L. By writing

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{7}{16 - 8{c_n}^2},$$

we obtain the equation $L = 7/(16 - 8L^2) \Longrightarrow 0 = 8L^3 - 16L + 7 = (2L - 1)(4L^2 + 2L - 7)$. Of the three solutions to this equation, only $1/2$ is between the bounds. Hence, $L = 1/2$.

36. The first four terms of the sequence are $c_1 = 0$, $c_2 = 7/16$, $c_3 = 0.443$, $c_4 = 0.485$. An initial attempt at proving that this sequence is increasing fails. We require information about its bounds. We prove first therefore that $0 \le c_n \le 1$. This is true for $n = 1$. Suppose $0 \le c_k \le 1$ for some integer k. Then, $0 \ge -8{c_k}^2 \ge -8$, and $16 \ge 16 - 8{c_k}^2 \ge 8$. Inversion gives $1/16 \le 1/(16 - 8{c_k}^2) \le 1/8$. But then $7/16 \le 7/(16 - 8{c_k}^2) \le 7/8$, or, $0 < 7/16 \le c_{k+1} \le 7/8 < 1$. By mathematical induction then, $0 \le c_n \le 1$, and we have upper and lower bounds. Now we verify that the sequence is increasing, that its terms satisfy $c_{n+1} > c_n$. This is true for $n = 1$. Suppose $c_{k+1} > c_k$ for some integer k. Then $-8{c_{k+1}}^2 < -8{c_k}^2$, and, $16 - 8{c_{k+1}}^2 < 16 - 8{c_k}^2$. Because all terms of the sequence are between 0 and 1, both expressions in this inequality are positive. We may therefore invert and write $1/(16 - 8{c_{k+1}}^2) > 1/(16 - 8{c_k}^2)$. In other words, $7/(16 - 8{c_{k+1}}^2) > 7/(16 - 8{c_k}^2)$, or, $c_{k+2} > c_{k+1}$. By mathematical induction then, $c_{n+1} > c_n$ for all n.

Because the sequence is monotonic and bounded, Theorem 10.7 guarantees that it has a limit L. By writing

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{7}{16 - 8{c_n}^2},$$

we obtain the equation $L = 7/(16 - 8L^2) \Longrightarrow 0 = 8L^3 - 16L + 7 = (2L - 1)(4L^2 + 2L - 7)$. Of the three solutions to this equation, only $1/2$ is between the bounds. Hence, $L = 1/2$.

37. It is advantageous to express the recursive definition in the form $c_{n+1} = \dfrac{4}{1 + \dfrac{4}{c_n}}$.

The first four terms of the sequence are $c_1 = 1$, $c_2 = 4/5$, $c_3 = 2/3$, $c_4 = 4/7$. The sequence appears to be decreasing, $c_{n+1} < c_n$. This is true for $n = 1$. Suppose $c_{k+1} < c_k$ for some integer k. Then,

$$\frac{1}{c_{k+1}} > \frac{1}{c_k} \quad \Longrightarrow \quad \frac{4}{c_{k+1}} > \frac{4}{c_k} \quad \Longrightarrow \quad 1 + \frac{4}{c_{k+1}} > 1 + \frac{4}{c_k}.$$

Thus,

$$\frac{1}{1 + \dfrac{4}{c_{k+1}}} < \frac{1}{1 + \dfrac{4}{c_k}}; \qquad \text{that is,} \qquad c_{k+2} < c_{k+1}.$$

By mathematical induction , $c_{n+1} < c_n$ for all n. It follows that $U = 1$ is an upper bound, and because no term can be negative, $V = 0$. By Theorem 10.7, the sequence has a limit L that we can obtain by solving the equation $L = 4L/(4 + L)$. The only solution of this equation is $L = 0$.

38. It is advantageous to express the recursive definition in the form $c_{n+1} = \dfrac{3}{1 + \dfrac{2}{c_n}}$.

The first four terms of the sequence are $c_1 = 4$, $c_2 = 2$, $c_3 = 3/2$, $c_4 = 9/7$. The sequence appears to be decreasing, $c_{n+1} < c_n$. This is true for $n = 1$. Suppose $c_{k+1} < c_k$ for some integer k. Then,

$$\frac{2}{c_{k+1}} > \frac{2}{c_k} \quad \Longrightarrow \quad 1 + \frac{2}{c_{k+1}} > 1 + \frac{2}{c_k}.$$

Thus,

$$\frac{1}{1+\dfrac{2}{c_{k+1}}} < \frac{1}{1+\dfrac{2}{c_k}} \implies \frac{3}{1+\dfrac{2}{c_{k+1}}} < \frac{3}{1+\dfrac{2}{c_k}}; \qquad \text{that is,} \qquad c_{k+2} < c_{k+1}.$$

By mathematical induction , $c_{n+1} < c_n$ for all n. It follows that $U = 1$ is an upper bound, and because no term can be negative, $V = 0$. By Theorem 10.7, the sequence has a limit L that we can obtain by solving the equation $L = 3L/(2+L)$. Of the two solutions 0 and 1 of this equation, we must choose $L = 0$.

39. It is advantageous to express the recursive definition in the form $c_{n+1} = \dfrac{2}{\dfrac{1}{c_n} + \dfrac{3}{c_n^2}}$.

The first four terms of the sequence are $c_1 = 2$, $c_2 = 8/5$, $c_3 = 1.1130$, $c_4 = 0.6024$. The sequence appears to be decreasing, $c_{n+1} < c_n$. This is true for $n = 1$. Suppose $c_{k+1} < c_k$ for some integer k. Then, $\dfrac{1}{c_{k+1}} > \dfrac{1}{c_k}$ and $\dfrac{3}{{c_{k+1}}^2} > \dfrac{3}{{c_k}^2}$. Addition of these gives $\dfrac{1}{c_{k+1}} + \dfrac{3}{{c_{k+1}}^2} > \dfrac{1}{c_k} + \dfrac{3}{{c_k}^2}$. Hence,

$$\frac{1}{\dfrac{1}{c_{k+1}} + \dfrac{3}{{c_{k+1}}^2}} < \frac{1}{\dfrac{1}{c_k} + \dfrac{3}{{c_k}^2}} \implies \frac{2}{\dfrac{1}{c_{k+1}} + \dfrac{3}{{c_{k+1}}^2}} < \frac{2}{\dfrac{1}{c_k} + \dfrac{3}{{c_k}^2}}.$$

But this states that $c_{k+2} < c_{k+1}$, and therefore by mathematical induction , $c_{n+1} < c_n$ for all n. It follows that $c_1 = 2$ is an upper bound, and because no term can be negative, $V = 0$ is a lower bound. By Theorem 10.7, the sequence has a limit L that we can obtain by solving the equation $L = 2L^2/(3+L)$. This equation can be expressed in the form $L^2 - 3L = 0$, and since $L = 3$ is greater that $U = 2$, the limit must be $L = 0$.

40. It is advantageous to express the recursive definition in the form $c_{n+1} = -1 + \dfrac{6}{4 - c_n}$.

The first four terms of the sequence are $c_1 = 3/2$, $c_2 = 7/5$, $c_3 = 17/13$, $c_4 = 43/35$. It would be prudent to first verify that the sequence is decreasing in which case an upper bound would be immediate. Unfortunately, information on bounds is required to complete the proof. We therefore begin by proving that $1 \le c_n \le 2$. This is certainly true for $n = 1$. Suppose $1 \le c_k \le 2$ for some integer k. Then, $-1 \ge -c_k \ge -2$, from which $3 \ge 4 - c_k \ge 2$. Thus, $1/3 \le 1/(4-c_k) \le 1/2$, and $2 \le 6/(4-c_k) \le 3$. Hence, $1 \le -1 + 6/(4-c_k) \le 2$; that is, $1 \le c_{k+1} \le 2$. By mathematical induction then, $1 \le c_n \le 2$ for all n. Now we verify that $c_{n+1} < c_n$. This is true for $n = 1$. Suppose $c_{k+1} < c_k$ for some integer k. Then $-c_{k+1} > -c_k$, and, $4 - c_{k+1} > 4 - c_k$. Since both sides are positive, we may invert,

$$\frac{1}{4 - c_{k+1}} < \frac{1}{4 - c_k} \implies -1 + \frac{6}{4 - c_{k+1}} < -1 + \frac{6}{4 - c_k}.$$

Thus, $c_{k+2} < c_{k+1}$, and by mathematical induction, $c_{n+1} < c_n$ for all n. Because the sequence is monotonic and bounded, Theorem 10.7 guarantees that it has a limit L that must satisfy the equation $L = (L+2)/(4-L)$. This equation reduces to $L^2 - 3L + 2 = 0$, and of the two solutions $L = 1$ and $L = 2$ only $L = 1$ could be the limit.

41. It is advantageous to express the recursive definition in the form $c_{n+1} = \dfrac{1}{2} + \dfrac{1}{2(5 - 2c_n)}$.

The first four terms of the sequence are $c_1 = 0$, $c_2 = 3/5$, $c_3 = 12/19 = 0.6316$, $c_4 = 0.6338$. It would be prudent to first verify that the sequence is increasing in which case a lower bound would be immediate. Unfortunately, information on bounds is required to complete the proof. We therefore begin by proving that $0 \le c_n \le 1$. This is certainly true for $n = 1$. Suppose $0 \le c_k \le 1$ for some integer k. Then, $0 \ge -2c_k \ge -2$, from which $5 \ge 5 - 2c_k \ge 3$. Thus, $2(5) \ge 2(5 - 2c_k) \ge 2(3)$, and we can say that $\dfrac{1}{10} \le \dfrac{1}{2(5-2c_k)} \le \dfrac{1}{6}$. It follows that $\dfrac{1}{2} + \dfrac{1}{10} \le \dfrac{1}{2} + \dfrac{1}{2(5-2c_k)} \le \dfrac{1}{2} + \dfrac{1}{6}$; that is, $0 < \dfrac{3}{5} \le c_{k+1} \le \dfrac{2}{3} < 1$. By mathematical induction then, $0 \le c_n \le 1$ for all n. Now we verify that $c_{n+1} > c_n$. This is true for $n = 1$. Suppose $c_{k+1} > c_k$ for some integer k. Then $-2c_{k+1} < -2c_k$, and, $5 - 2c_{k+1} < 5 - 2c_k$. Thus, $2(5 - 2c_{k+1}) < 2(5 - 2c_k)$. Since both sides are positive, we may invert,

$$\frac{1}{2(5-2c_{k+1})} > \frac{1}{2(5-2c_k)} \implies \frac{1}{2} + \frac{1}{2(5-2c_{k+1})} > \frac{1}{2} + \frac{1}{2(5-2c_k)}.$$

Thus, $c_{k+2} > c_{k+1}$, and by mathematical induction, $c_{n+1} > c_n$ for all n. Because the sequence is monotonic and bounded, Theorem 10.7 guarantees that it has a limit L that must satisfy the equation $L = (3-L)/(5-2L)$. This equation reduces to $2L^2 - 6L + 3 = 0$, and of the two solutions $(3 \pm \sqrt{3})/2$, only $(3-\sqrt{3})/2$ lies between the bounds of the sequence. Hence, $L = (3-\sqrt{3})/2$.

42. First we verify that 0 and 1 are bounds for the sequence; that is, $0 \le c_n \le 1$. This is true for $n = 1$. Suppose that $0 \le c_k \le 1$. Then, $0 \ge -c_k \ge -1$ and $0 \ge -{c_k}^2 \ge -1$. When we add these, $0 \ge -c_k - {c_k}^2 \ge -2$, and if we add 4, $4 \ge 4 - c_k - {c_k}^2 \ge 2$. Inverting this gives $1/4 \le 1/(4 - c_k - {c_k}^2) \le 1/2$. Hence, $0 < 1/4 \le c_{k+1} \le 1/2 < 1$. Consequently, by mathematical induction, $0 \le c_n \le 1$ for all $n \ge 1$.

We now verify that the sequence is decreasing by showing that $c_{n+1} < c_n$. This is true for $n = 1$ since $c_2 = 1/2 < c_1$. Suppose that $c_{k+1} < c_k$. Then $-c_{k+1} > -c_k$ and $-{c_{k+1}}^2 > -{c_k}^2$. When we add these, $-c_{k+1} - {c_{k+1}}^2 > -c_k - {c_k}^2$, and if we add 4, $4 - c_{k+1} - {c_{k+1}}^2 > 4 - c_k - {c_k}^2$. Because both sides of this inequality are positive, we may invert and reverse the sign, $1/(4 - c_{k+1} - {c_{k+1}}^2) < 1/(4 - c_k - {c_k}^2)$; that is, $c_{k+2} < c_{k+1}$. By mathematical induction, then, $c_{n+1} < c_n$ for all $n \ge 1$.

Because the sequence is monotonic and bounded, Theorem 10.7 guarantees that it has a limit that must satisfy the equation $L = 1/(4 - L - L^2)$. This equation reduces to $f(L) = L^3 + L^2 - 4L + 1 = 0$. The first nine terms of the sequence are

$x_1 = 1$	$x_2 = 1/2$	$x_3 = 4/13$	$x_4 = 0.278$	$x_5 = 0.274$
$x_6 = 0.273\,903$	$x_7 = 0.273\,892$	$x_8 = 0.273\,891$	$x_9 = 0.273\,891.$	

Since $f(0.273\,885) = 1.8 \times 10^{-5}$ and $f(0.273\,895) = -1.4 \times 10^{-5}$, the limit is 0.273 89 (to 5 decimals).

43. (a) If L is the limit of the sequence $\{c_n\}_1^\infty$, then L is also the limit of $\{c_{n-1}\}_2^\infty$. By Theorem 10.10, the sequence $\{c_n - c_{n-1}\}_2^\infty$ has limit $L - L = 0$.
(b) The sequence diverges because $\lim_{n\to\infty} \ln n = \infty$, yet

$$\lim_{n\to\infty}(c_{n+1} - c_n) = \lim_{n\to\infty}[\ln(n+1) - \ln n] = \lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right) = 0.$$

44. The first four terms of the sequence are $c_1 = 2$, $c_2 = 6/5 = 1.2$, $c_3 = 1.36$, and $c_4 = 1.328$. They are oscillating. To show that the entire sequence oscillates, we calculate

$$c_{n+1} - c_n = \frac{8 - c_n}{5} - \frac{8 - c_{n-1}}{5} = -\frac{1}{5}(c_n - c_{n-1}).$$

This shows that the differences $c_{n+1} - c_n$ alternate in sign, and therefore the sequence oscillates. Because $|c_{n+1} - c_n| = |c_n - c_{n-1}|/5$, absolute values $|c_{n+1} - c_n|$ decrease and approach 0. According to Theorem 10.8 the sequence has a limit L that can be obtained by taking limits on both sides of the recursive definition,

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty}\frac{8 - c_n}{5} \implies L = \frac{8 - L}{5} \implies L = \frac{4}{3}.$$

45. The first four terms of the sequence are $c_1 = 20$, $c_2 = 243/20 = 12.15$, $c_3 = 12.247$, and $c_4 = 12.245$. They are oscillating. To show that the entire sequence oscillates, we calculate

$$c_{n+1} - c_n = \left(12 + \frac{3}{c_n}\right) - \left(12 + \frac{3}{c_{n-1}}\right) = -\frac{3(c_n - c_{n-1})}{c_n c_{n-1}}.$$

Since all terms of the sequence are positive, the denominator of this expression is positive. It follows that $c_{n+1} - c_n$ has the opposite sign of $c_n - c_{n-1}$, and the sequence oscillates. To verify properties 2 and 3 of Theorem 10.8, we take absolute values in the above equation,

$$|c_{n+1} - c_n| = \frac{3|c_n - c_{n-1}|}{c_n c_{n-1}}.$$

Since all terms of the sequence are greater than 12 (the recursive definition makes this clear), it follows that

$$|c_{n+1} - c_n| < \frac{3|c_n - c_{n-1}|}{(12)(12)} = \frac{|c_n - c_{n-1}|}{48}.$$

This shows that the $|c_{n+1} - c_n|$ decrease and have limit 0. By Theorem 10.8 the sequence has a limit L that can be obtained by taking limits on both sides of the recursive definition,

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty}\left(12 + \frac{3}{c_n}\right) \quad\Longrightarrow\quad L = 12 + \frac{3}{L}.$$

Of the two solutions $6 \pm \sqrt{39}$ of this equation, only $L = 6 + \sqrt{39}$ is positive.

46. The first four terms of the sequence are $c_1 = 1$, $c_2 = 1/3$, $c_3 = 3/7$, and $c_4 = 7/17$. They are oscillating. To show that the entire sequence oscillates, we calculate

$$c_{n+1} - c_n = \frac{1}{2 + c_n} - \frac{1}{2 + c_{n-1}} = -\frac{(c_n - c_{n-1})}{(2 + c_n)(2 + c_{n-1})}.$$

Since all terms of the sequence are positive, the denominator of this expression is positive. It follows that $c_{n+1} - c_n$ has the opposite sign of $c_n - c_{n-1}$, and the sequence oscillates. To verify properties 2 and 3 of Theorem 10.8, we take absolute values in the above equation,

$$|c_{n+1} - c_n| = \frac{|c_n - c_{n-1}|}{(2 + c_n)(2 + c_{n-1})}.$$

Since all terms of the sequence are positive, it follows that

$$|c_{n+1} - c_n| < \frac{|c_n - c_{n-1}|}{(2)(2)} = \frac{1}{4}|c_n - c_{n-1}|.$$

This shows that the $|c_{n+1} - c_n|$ decrease and have limit 0. By Theorem 10.8 the sequence has a limit L that can be obtained by taking limits on both sides of the recursive definition,

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{1}{2 + c_n} \quad\Longrightarrow\quad L = \frac{1}{2 + L}.$$

Of the two solutions $-1 \pm \sqrt{2}$ of this equation, only $L = -1 + \sqrt{2}$ is positive.

47. The first four terms of the sequence are $c_1 = 10$, $c_2 = 1/23 = 0.044$, $c_3 = 23/71 = 0.324$, and $c_4 = 71/259 = 0.274$. They are oscillating. To show that the entire sequence oscillates, we calculate

$$c_{n+1} - c_n = \frac{1}{3 + 2c_n} - \frac{1}{3 + 2c_{n-1}} = -\frac{2(c_n - c_{n-1})}{(3 + 2c_n)(3 + 2c_{n-1})}.$$

Since all terms of the sequence are positive, the denominator of this expression is positive. It follows that $c_{n+1} - c_n$ has the opposite sign of $c_n - c_{n-1}$, and the sequence oscillates. To verify properties 2 and 3 of Theorem 10.8, we take absolute values in the above equation,

$$|c_{n+1} - c_n| = \frac{2|c_n - c_{n-1}|}{(3 + 2c_n)(3 + 2c_{n-1})} < \frac{2|c_n - c_{n-1}|}{(3)(3)} = \frac{2}{9}|c_n - c_{n-1}|.$$

This shows that the $|c_{n+1} - c_n|$ decrease and have limit 0. By Theorem 10.8 the sequence has a limit L that can be obtained by taking limits on both sides of the recursive definition,

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{1}{3 + 2c_n} \quad\Longrightarrow\quad L = \frac{1}{3 + 2L}.$$

Of the two solutions $(-3 \pm \sqrt{17})/4$ of this equation, only $L = (-3 + \sqrt{17})/4$ is positive.

48. The inequality $1 \le c_n \le 2$ is valid for $n = 1$. Suppose $1 \le c_k \le 2$. Then, $2 \le 1 + c_k \le 3$, and inverting, $1/2 \ge 1/(1 + c_k) \ge 1/3$. Multiplication by 3 gives $3/2 \ge 3/(1 + c_k) \ge 1$, and this states that $1 \le c_{k+1} \le 3/2 < 2$. Hence, by mathematical induction, $1 \le c_n \le 2$ for all $n \ge 1$.

The first four terms of the sequence are $c_1 = 1$, $c_2 = 3/2$, $c_3 = 6/5$, and $c_4 = 15/11$. They are oscillating. To show that the entire sequence oscillates, we calculate

$$c_{n+1} - c_n = \frac{3}{1+c_n} - \frac{3}{1+c_{n-1}} = -\frac{3(c_n - c_{n-1})}{(1+c_n)(1+c_{n-1})}.$$

Since all terms of the sequence are positive, the denominator of this expression is positive. It follows that $c_{n+1} - c_n$ has the opposite sign of $c_n - c_{n-1}$, and the sequence oscillates. To verify properties 2 and 3 of Theorem 10.8, we take absolute values in the above equation,

$$|c_{n+1} - c_n| = \frac{3|c_n - c_{n-1}|}{(1+c_n)(1+c_{n-1})} < \frac{3|c_n - c_{n-1}|}{(1+1)(1+1)} = \frac{3}{4}|c_n - c_{n-1}|.$$

This shows that the $|c_{n+1} - c_n|$ decrease and have limit 0. By Theorem 10.8 the sequence has a limit L that can be obtained by taking limits on both sides of the recursive definition,

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{3}{1+c_n} \quad \Longrightarrow \quad L = \frac{3}{1+L}.$$

Of the two solutions $(-1 \pm \sqrt{13})/2$ of this equation, only $L = (-1+\sqrt{13})/2$ lies between 1 and 2.

49. The first six terms of the sequence are $c_1 = 0$, $c_2 = 3$, $c_3 = 3/4$, $c_4 = 12/7$, $c_5 = 21/19$, and $c_6 = 57/40$. They are oscillating. The inequality $1 \le c_n \le 2$ is valid for $n = 4$. Suppose $1 \le c_k \le 2$. Then, $2 \le 1 + c_k \le 3$, and inverting, $1/2 \ge 1/(1+c_k) \ge 1/3$. Multiplication by 3 gives $3/2 \ge 3/(1+c_k) \ge 1$, and this states that $1 \le c_{k+1} \le 3/2 < 2$. Hence, by mathematical induction, $1 \le c_n \le 2$ for all $n \ge 4$.

To show that the entire sequence oscillates, we calculate

$$c_{n+1} - c_n = \frac{3}{1+c_n} - \frac{3}{1+c_{n-1}} = -\frac{3(c_n - c_{n-1})}{(1+c_n)(1+c_{n-1})}.$$

Since all terms of the sequence are positive, the denominator of this expression is positive. It follows that $c_{n+1} - c_n$ has the opposite sign of $c_n - c_{n-1}$, and the sequence oscillates. To verify properties 2 and 3 of Theorem 10.8, we take absolute values in the above equation,

$$|c_{n+1} - c_n| = \frac{3|c_n - c_{n-1}|}{(1+c_n)(1+c_{n-1})} < \frac{3|c_n - c_{n-1}|}{(1+1)(1+1)} = \frac{3}{4}|c_n - c_{n-1}|, \quad n \ge 5.$$

This shows that the $|c_{n+1} - c_n|$ decrease and have limit 0. By Theorem 10.8 the sequence has a limit L that can be obtained by taking limits on both sides of the recursive definition,

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{3}{1+c_n} \quad \Longrightarrow \quad L = \frac{3}{1+L}.$$

Of the two solutions $(-1 \pm \sqrt{13})/2$ of this equation, only $L = (-1+\sqrt{13})/2$ lies between 1 and 2.

50. The inequality $1/2 \le c_n \le 1$ is valid for $n = 1$. Suppose $1/2 \le c_k \le 1$. Then, $5/2 \le 5c_k \le 5 \Longrightarrow 9/2 \le 2 + 5c_k \le 7$, and inverting, $2/9 \ge 1/(2+5c_k) \ge 1/7$. Multiplication by 4 gives $8/9 \ge 4/(2+5c_k) \ge 4/7$, and this states that $1/2 < 4/7 \le c_{k+1} \le 8/9 < 1$. Hence, by mathematical induction, $1/2 \le c_n \le 1$ for all $n \ge 1$.

The first four terms of the sequence are $c_1 = 1$, $c_2 = 4/7$, $c_3 = 14/17$, and $c_4 = 17/26$. They are oscillating. To show that the entire sequence oscillates, we calculate

$$c_{n+1} - c_n = \frac{4}{2+5c_n} - \frac{4}{2+5c_{n-1}} = -\frac{20(c_n - c_{n-1})}{(2+5c_n)(2+5c_{n-1})}.$$

Since all terms of the sequence are positive, the denominator of this expression is positive. It follows that $c_{n+1} - c_n$ has the opposite sign of $c_n - c_{n-1}$, and the sequence oscillates. To verify properties 2 and 3 of Theorem 10.8, we take absolute values in the above equation,

$$|c_{n+1} - c_n| = \frac{20|c_n - c_{n-1}|}{(2+5c_n)(2+5c_{n-1})} < \frac{20|c_n - c_{n-1}|}{(2+5/2)(2+5/2)} = \frac{80}{81}|c_n - c_{n-1}|.$$

This shows that the $|c_{n+1} - c_n|$ decrease and have limit 0. By Theorem 10.8 the sequence has a limit L that can be obtained by taking limits on both sides of the recursive definition,

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{4}{2+5c_n} \implies L = \frac{4}{2+5L}.$$

Of the two solutions $(-1 \pm \sqrt{21})/5$ of this equation, only $L = (-1+\sqrt{21})/5$ is positive.

51. The first four terms of the sequence are $c_1 = 1$, $c_2 = 5$, $c_3 = 4.58$, and $c_4 = 4.63$. They are oscillating.

The inequality $4 \le c_n \le 5$ is valid for $n = 2$. Suppose $4 \le c_k \le 5$. Then, $-4 \ge -c_k \ge -5$, and adding 26, $22 \ge 26 - c_k \ge 21$. When we take square roots, $\sqrt{22} \ge \sqrt{26-c_k} \ge \sqrt{21}$, and this states that $4 < \sqrt{21} \le c_{k+1} \le \sqrt{22} < 5$. Hence, by mathematical induction, $4 \le c_n \le 5$ for all $n \ge 2$. This shows that all terms of the sequence are defined. Consider now

$$(c_{n+1})^2 - (c_n)^2 = (26 - c_n) - (26 - c_{n-1}) = -(c_n - c_{n-1}).$$

When we factor the left side into $(c_{n+1} + c_n)(c_{n+1} - c_n)$, the above equation can be rewritten in the form

$$c_{n+1} - c_n = \frac{-(c_n - c_{n-1})}{c_{n+1} + c_n}.$$

Since all terms of the sequence are positive, the denominator of this expression is positive. It follows that $c_{n+1} - c_n$ has the opposite sign of $c_n - c_{n-1}$, and the sequence oscillates. To verify properties 2 and 3 of Theorem 10.8, we take absolute values in the above equation,

$$|c_{n+1} - c_n| = \frac{|c_n - c_{n-1}|}{c_{n+1} + c_n} \le \frac{|c_n - c_{n-1}|}{4+4} = \frac{1}{8}|c_n - c_{n-1}|, \quad n \ge 2.$$

This shows that the $|c_{n+1} - c_n|$ decrease and have limit 0. By Theorem 10.8 the sequence has a limit L that can be obtained by taking limits on both sides of the recursive definition,

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \sqrt{26 - c_n} \implies L = \sqrt{26 - L}.$$

Of the two solutions $(-1 \pm \sqrt{105})/2$ of this equation, only $L = (-1+\sqrt{105})/2$ lies between 4 and 5.

52. The first four terms of the sequence are $c_1 = 4$, $c_2 = 2.83$, $c_3 = 3.39$, and $c_4 = 3.13$. They are oscillating.

First we prove that $2 \le c_n \le 4$ which is valid for $n = 1$. Suppose $2 \le c_k \le 4$. Then, $-2 \ge -c_k \ge -4$, and multiplying by 3, $-6 \ge -3c_k \ge -12$. Adding 20 gives $14 \ge 20 - 3c_k \ge 8$. When we take square roots, $\sqrt{14} \ge \sqrt{20 - 3c_k} \ge \sqrt{8}$, and this states that $2 < \sqrt{8} \le c_{k+1} \le \sqrt{14} < 4$. Hence, by mathematical induction, $2 \le c_n \le 4$ for all $n \ge 1$. This shows that all terms of the sequence are defined. Consider now

$$(c_{n+1})^2 - (c_n)^2 = (20 - 3c_n) - (20 - 3c_{n-1}) = -3(c_n - c_{n-1}).$$

When we factor the left side into $(c_{n+1} + c_n)(c_{n+1} - c_n)$, the above equation can be rewritten in the form

$$c_{n+1} - c_n = \frac{-3(c_n - c_{n-1})}{c_{n+1} + c_n}.$$

Since all terms of the sequence are positive, the denominator of this expression is positive. It follows that $c_{n+1} - c_n$ has the opposite sign of $c_n - c_{n-1}$, and the sequence oscillates. To verify properties 2 and 3 of Theorem 10.8, we take absolute values in the above equation,

$$|c_{n+1} - c_n| = \frac{3|c_n - c_{n-1}|}{c_{n+1} + c_n} \le \frac{3|c_n - c_{n-1}|}{2+2} = \frac{3}{4}|c_n - c_{n-1}|.$$

This shows that the $|c_{n+1} - c_n|$ decrease and have limit 0. By Theorem 10.8 the sequence has a limit L that can be obtained by taking limits on both sides of the recursive definition,

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \sqrt{20 - 3c_n} \implies L = \sqrt{20 - 3L}.$$

Of the two solutions $(-3 \pm \sqrt{89})/2$ of this equation, only $L = (-3 + \sqrt{89})/2$ is positive.

53. The first four terms of the sequence are $c_1 = 2$, $c_2 = 2/7$, $c_3 = 2$, and $c_4 = 2/7$. In other words, the sequence oscillates back and forth between 2 and 2/7, never converging.

54. The third term is $c_3 = \dfrac{c_2}{4c_2 - 1} = \dfrac{\dfrac{c_1}{4c_1 - 1}}{4\left(\dfrac{c_1}{4c_1 - 1}\right) - 1} = \left(\dfrac{c_1}{4c_1 - 1}\right)\left(\dfrac{4c_1 - 1}{4c_1 - 4c_1 + 1}\right) = c_1.$ In other words, terms of the sequence oscillate back and forth between c_1 and c_2, never converging unless $c_1 = c_2$. This occurs only when $c_1 = 0$ or $c_1 = 1/2$.

55. The first four terms of the sequence are $c_1 = 1$, $c_2 = 5$, $c_3 = 25/9$, and $c_4 = 125/41$. They are oscillating. The inequality $2 \le c_n \le 4$ is valid for $n = 3$. Suppose $2 \le c_k \le 4$. Then, $1/2 \ge 1/c_k \ge 1/4$, and $-1/2 \le -1/c_k \le -1/4$. Adding 2 gives

$$\frac{3}{2} \le 2 - \frac{1}{c_k} \le \frac{7}{4} \quad \Longrightarrow \quad \frac{2}{3} \ge \frac{1}{2 - \dfrac{1}{c_k}} \ge \frac{4}{7}.$$

Multiplication be 5 yields

$$\frac{10}{3} \ge \frac{5}{2 - \dfrac{1}{c_k}} \ge \frac{20}{7} \quad \Longrightarrow \quad 2 < \frac{20}{7} \le c_{k+1} \le \frac{10}{3} < 4.$$

Hence, by mathematical induction, $2 \le c_n \le 4$ for all $n \ge 3$. To show that the entire sequence oscillates, we consider

$$c_{n+1} - c_n = \frac{5c_n}{2c_n - 1} - \frac{5c_{n-1}}{2c_{n-1} - 1} = \frac{-5(c_n - c_{n-1})}{(2c_n - 1)(2c_{n-1} - 1)}.$$

Since $c_1 = 1$, $c_2 = 5$, and $2 \le c_n \le 4$ for $n \ge 3$, the denominator of this expression is positive. It follows that $c_{n+1} - c_n$ has the opposite sign of $c_n - c_{n-1}$, and the sequence oscillates. To verify properties 2 and 3 of Theorem 10.8, we take absolute values in the above equation,

$$|c_{n+1} - c_n| = \frac{5|c_n - c_{n-1}|}{(2c_n - 1)(2c_{n-1} - 1)} \le \frac{5|c_n - c_{n-1}|}{(3)(3)} = \frac{5}{9}|c_n - c_{n-1}|, \quad n \ge 4.$$

This shows that the $|c_{n+1} - c_n|$ decrease and have limit 0. By Theorem 10.8 the sequence has a limit L that can be obtained by taking limits on both sides of the recursive definition,

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{5c_n}{2c_n - 1} \quad \Longrightarrow \quad L = \frac{5L}{2L - 1}.$$

Of the two solutions 0 and 3 of this equation, only $L = 3$ could be the limit.

56. The first four terms of the sequence are $c_1 = 3$, $c_2 = 1.913$, $c_3 = 2.007$, and $c_4 = 1.999$. They are oscillating. First we prove that $1 \le c_n \le 3$ which is valid for $n = 1$. Suppose $1 \le c_k \le 3$. Then, $-1 \ge -c_k \ge -3$, and adding 10 gives $9 \ge 10 - c_k \ge 7$. When we take cube roots, $\sqrt[3]{9} \ge \sqrt[3]{10 - c_k} \ge \sqrt[3]{7}$, and this states that $1 < \sqrt[3]{7} \le c_{k+1} \le \sqrt[3]{9} < 3$. Hence, by mathematical induction, $1 \le c_n \le 3$ for all $n \ge 1$. Consider now

$$(c_{n+1})^3 - (c_n)^3 = (10 - c_n) - (10 - c_{n-1}) = -(c_n - c_{n-1}).$$

When we factor the left side into $(c_{n+1} - c_n)({c_{n+1}}^2 + c_{n+1}c_n + {c_n}^2)$, the above equation can be rewritten in the form

$$c_{n+1} - c_n = \frac{-(c_n - c_{n-1})}{{c_{n+1}}^2 + c_{n+1}c_n + {c_n}^2}.$$

Since all terms of the sequence are positive, the denominator of this expression is positive. It follows that $c_{n+1} - c_n$ has the opposite sign of $c_n - c_{n-1}$, and the sequence oscillates. To verify properties 2 and 3 of Theorem 10.8, we take absolute values in the above equation,

$$|c_{n+1} - c_n| = \frac{|c_n - c_{n-1}|}{c_{n+1}{}^2 + c_{n+1}c_n + c_n{}^2} \le \frac{|c_n - c_{n-1}|}{1+1+1} = \frac{1}{3}|c_n - c_{n-1}|.$$

This shows that the $|c_{n+1} - c_n|$ decrease and have limit 0. By Theorem 10.8 the sequence has a limit L that can be obtained by taking limits on both sides of the recursive definition,

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \sqrt[3]{10 - c_n} \quad \Longrightarrow \quad L = \sqrt[3]{10 - L} \quad \Longrightarrow \quad L = 2.$$

57. The second and third terms of the sequence are

$$c_2 = \frac{a}{ab-1}, \qquad c_3 = \frac{c_2}{bc_2 - 1} = \frac{\dfrac{a}{ab-1}}{\dfrac{ab}{ab-1} - 1} = \left(\frac{a}{ab-1}\right)\left(\frac{ab-1}{ab-ab+1}\right) = a.$$

In other words, terms of the sequence oscillate back and forth between c_1 and c_2, never converging unless $c_1 = c_2$. This occurs for $a = 0$ which is not permitted, and for $ab = 2$.

58. (i) Suppose $\epsilon > 0$ is any given number. Since $\lim_{n\to\infty} c_n = C$, there exists an N such that $|c_n - C| < \epsilon/|k|$ for all $n > N$. For such n,

$$|kc_n - kC| = |k||c_n - C| < |k|\left(\frac{\epsilon}{|k|}\right) = \epsilon.$$

This proves that $\lim_{n\to\infty} kc_n = kC$.
(ii) Suppose $\epsilon > 0$ is any given number. Since $\lim_{n\to\infty} c_n = C$, there exists an N_1 such that $|c_n - C| < \epsilon/2$ for all $n > N_1$. Similarly, since $\lim_{n\to\infty} d_n = D$, there exists an N_2 such that $|d_n - D| < \epsilon/2$ for all $n > N_2$. For all n greater than the larger of N_1 and N_2,

$$|(c_n + d_n) - (C + D)| = |(c_n - C) + (d_n - D)| \le |c_n - C| + |d_n - D| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This proves that $\lim_{n\to\infty} (c_n + d_n) = C + D$. The proof that $\lim_{n\to\infty} (c_n - d_n) = C - D$ is similar.

59. The first four terms of the sequence are $c_1 = 1$, $c_2 = 2/3$, $c_3 = 5/7$, $c_4 = 12/17$. All terms of the sequence are clearly positive, and therefore a lower bound is $V = 0$. Since

$$1 - c_n = 1 - \frac{1 + c_{n-1}}{1 + 2c_{n-1}} = \frac{c_{n-1}}{1 + 2c_{n-1}} > 0,$$

it follows that $c_n < 1$ for all n, and therefore 1 is an upper bound.

60. The first four terms of the sequence are $c_1 = -30$, $c_2 = -20$, $c_3 = -15$, $c_4 = -55/6$. The sequence appears to be increasing, $c_{n+1} > c_n$. This is true for $n = 1, 2$. Suppose that $c_k > c_{k-1}$ and $c_{k+1} > c_k$. Then, $c_k/3 > c_{k-1}/3$ and $c_{k+1}/2 > c_k/2$. Addition of these gives

$$\frac{c_{k+1}}{2} + \frac{c_k}{3} > \frac{c_k}{2} + \frac{c_{k-1}}{3} \quad \Longrightarrow \quad 5 + \frac{c_{k+1}}{2} + \frac{c_k}{3} > 5 + \frac{c_k}{2} + \frac{c_{k-1}}{3}.$$

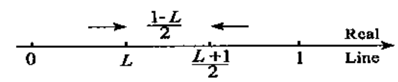
This states that $c_{k+2} > c_{k+1}$, and therefore, by mathematical induction, the sequence is increasing. The first term is therefore a lower bound. We now prove that 30 is an upper bound, $c_n \le 30$. Certainly c_1 and c_2 are both less than 30. Suppose that $c_{k-1} \le 30$ and $c_k \le 30$. Then

$$c_{k+1} = 5 + \frac{c_k}{2} + \frac{c_{k-1}}{3} \le 5 + \frac{30}{2} + \frac{30}{3} = 30.$$

Hence, by mathematical induction, $c_n \le 30$ for all $n \ge 1$. According to Theorem 10.7, the sequence has a limit that can be obtained by taking limits on both sides of the recursive definition,

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \left(5 + \frac{c_n}{2} + \frac{c_{n-1}}{3}\right) \quad \Longrightarrow \quad L = 5 + \frac{L}{2} + \frac{L}{3} \quad \Longrightarrow \quad L = 30.$$

61. (a) If $\lim_{n\to\infty} c_n = L$, then according to Definition 10.4, given $\epsilon = (1-L)/2$, there exists an integer N such that for all $n > N$,

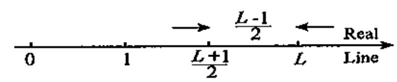


$$|c_n - L| < \epsilon = \frac{1-L}{2}.$$

This is equivalent to $\displaystyle -\frac{1-L}{2} < c_n - L < \frac{1-L}{2} \implies L - \frac{1-L}{2} < c_n < \frac{1-L}{2} + L.$

The right inequality implies that $c_n < (L+1)/2$.

(b) If $\lim_{n\to\infty} c_n = L$, then given $\epsilon = (L-1)/2$, there exists an integer N such that for all $n > N$,



$$|c_n - L| < \epsilon = \frac{L-1}{2}.$$

This is equivalent to

$$-\frac{L-1}{2} < c_n - L < \frac{L-1}{2} \implies L - \frac{L-1}{2} < c_n < \frac{L-1}{2} + L.$$

The left inequality implies that $c_n > (L+1)/2$.

62. If $\lim_{n\to\infty} c_n = L$, then given $\epsilon = 1$, there exists an integer N such that for all $n > N$, $|c_n - L| < \epsilon = 1$. This is equivalent to $-1 < c_n - L < 1 \implies L - 1 < c_n < L + 1$.

63. (a) The first ten terms of the sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55.

(b) The sequence is clearly increasing. It is bounded below by $V = c_1 = 1$, but it has no upper bound or limit.

(c) This is true for $n = 2$ since ${c_2}^2 - c_1c_3 = (1)^2 - (1)(2) = -1 = (-1)^{2+1}$. Suppose ${c_k}^2 - c_{k-1}c_{k+1} = (-1)^{k+1}$ for some integer k. Then,

$$\begin{aligned}
{c_{k+1}}^2 - c_kc_{k+2} &= {c_{k+1}}^2 - c_k(c_k + c_{k+1}) \qquad (\text{since } c_{k+2} = c_k + c_{k+1})\\
&= {c_{k+1}}^2 - {c_k}^2 - c_kc_{k+1}\\
&= -{c_k}^2 + c_{k+1}(c_{k+1} - c_k)\\
&= -{c_k}^2 + c_{k+1}(c_{k-1}) \qquad (\text{since } c_{k+1} = c_k + c_{k-1})\\
&= -[{c_k}^2 - c_{k-1}c_{k+1}]\\
&= -(-1)^{k+1} \qquad (\text{by assumption})\\
&= (-1)^{(k+1)+1}.
\end{aligned}$$

The result is therefore valid for $k+1$, and by mathematical induction it is true for all $n \ge 2$.

(d) When $n = 1$, the formula gives $\displaystyle \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right] = 1$, and this is c_1. Thus, the formula is correct for $n = 1$. When $n = 2$, the formula gives

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right] = \frac{1}{4\sqrt{5}}[(1+2\sqrt{5}+5) - (1-2\sqrt{5}+5)] = 1.$$

Thus, the formula is also true for $n = 2$. Suppose it is valid for integers $k-1$ and k; that is,

$$c_{k-1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right] \quad \text{and} \quad c_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\right].$$

Then,

$$c_{k+1} = c_{k-1} + c_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\right]$$
$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(1+\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(1+\frac{1-\sqrt{5}}{2}\right)\right]$$
$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{3-\sqrt{5}}{2}\right)\right].$$

Since $\left(\dfrac{1+\sqrt{5}}{2}\right)^2 = \dfrac{1}{4}(1+2\sqrt{5}+5) = \dfrac{3+\sqrt{5}}{2}$, and similarly, $\left(\dfrac{1-\sqrt{5}}{2}\right)^2 = \dfrac{3-\sqrt{5}}{2}$, we may write that

$$c_{k+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{1-\sqrt{5}}{2}\right)^2\right]$$
$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right].$$

Consequently the formula is valid for $k+1$, and by mathematical induction it is valid for all $n \geq 1$.
(e) The first four terms of the sequence are

$$b_1 = \frac{c_2}{c_1} = 1, \quad b_2 = \frac{c_3}{c_2} = 2, \quad b_3 = \frac{c_4}{c_3} = \frac{3}{2}, \quad b_4 = \frac{c_5}{c_4} = \frac{5}{3}.$$

The sequence is not monotonic. To investigate whether $\{b_n\}$ has a limit we use the explicit formula for c_n in part (d),

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}}.$$

We now divide both numerator and denominator by $[(1+\sqrt{5})/2]^n$,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\frac{1+\sqrt{5}}{2} - \left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n}{1 - \left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n} = \frac{1+\sqrt{5}}{2}$$

since $|(1-\sqrt{5})/(1+\sqrt{5})| < 1$.

64. We concentrate only on female rabbits. After the first month, we have the original adult female and a newborn female. Hence $R_1 = 1$. After the second month,we have the original adult female, a one-month old female, and a newborn female. Hence, $R_2 = 1$. If R_n is the number of adult females after n months, then R_{n+1} is the sum of R_n and the number of one-month old females after n months. But the number of one-month old females after n months is the number of adult females R_{n-1} after $n-1$ months. In other words, a recursive definition for the sequence is $R_1 = 1$, $R_2 = 1$, $R_{n+1} = R_n + R_{n-1}$. This is the Fibonacci sequence.

65. If we assume that $c_{k+1} > c_k$ for some integer k, then $2bc_{k+1} > 2bc_k$, and $a + 2bc_{k+1} > a + 2bc_k$. Taking square roots gives $\sqrt{a+2bc_{k+1}} > \sqrt{a+2bc_k}$; that is, $c_{k+2} > c_{k+1}$. Thus, the sequence is increasing if and only if

$$c_2 > c_1 \iff \sqrt{a+2bc_1} > d \iff a + 2bd > d^2 \iff d^2 - 2bd - a < 0.$$

Since $d^2 - 2bd - a = 0$ when $d = (2b \pm \sqrt{4b^2+4a})/2 = b \pm \sqrt{a+b^2}$, it follows that the sequence is increasing if and only if $d < b + \sqrt{a+b^2}$. When $d = b + \sqrt{a+b^2}$, all terms of the sequence are equal to d.

66. (a) $|c_nd_n - CD| = |(c_nd_n - Dc_n) + (Dc_n - CD)| \le |c_n||d_n - D| + |D||c_n - C|$

(b) If $\lim_{n\to\infty} c_n = C$, then $\lim_{n\to\infty} |c_n| = |C|$. According to Exercise 62, there exists an N_1 such that for $n > N_1$, $|c_n| < |C| + 1$.

Because $\lim_{n\to\infty} d_n = D$, given any $\epsilon > 0$, there exists an N_2 such that $|d_n - D| < \dfrac{\epsilon}{2(|C|+1)}$ for $n > N_2$. Similarly, there exists an N_3 such that for $n > N_3$, $|c_n - C| < \dfrac{\epsilon}{2|D|+1}$.

(c) When n is greater than the largest of N_1, N_2, and N_3,

$$\begin{aligned}|c_nd_n - CD| &\le |c_n||d_n - D| + |D||c_n - C| \\ &< (|C|+1)\left[\frac{\epsilon}{2(|C|+1)}\right] + |D|\left[\frac{\epsilon}{2|D|+1}\right] \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \qquad \left(\text{since } \frac{|D|}{2|D|+1} < \frac{1}{2}\right) \\ &= \epsilon.\end{aligned}$$

This proves that $\lim_{n\to\infty} c_nd_n = CD$.

67. (a)
$$\begin{aligned}\left|\frac{c_n}{d_n} - \frac{C}{D}\right| &= \left|\frac{Dc_n - Cd_n}{Dd_n}\right| = \left|\frac{D(c_n - C) + C(D - d_n)}{Dd_n}\right| \le \left|\frac{D(c_n - C)}{Dd_n}\right| + \left|\frac{C(D - d_n)}{Dd_n}\right| \\ &= \frac{|c_n - C|}{|d_n|} + \frac{|C||d_n - D|}{|D||d_n|}\end{aligned}$$

(b) If $\lim_{n\to\infty} d_n = D$, then $\lim_{n\to\infty} |d_n| = |D|$. Certainly there exists an N_1 such that for $n > N_1$, $|d_n| > |D|/2$, else the terms would not have limit $|D|$.

Because $\lim_{n\to\infty} c_n = C$, given any $\epsilon > 0$, there exists an N_2 such that for $n > N_2$, $|c_n - C| < \epsilon|D|/4$. Similarly, there exists an N_3 such that for $n > N_3$, $|d_n - D| < \epsilon|D|^2/(4|C|+1)$.

(c) When n is greater than the largest of N_1, N_2, and N_3,

$$\begin{aligned}\left|\frac{c_n}{d_n} - \frac{C}{D}\right| &\le \frac{|c_n - C|}{|d_n|} + \frac{|C||d_n - D|}{|D||d_n|} < \frac{\epsilon|D|/4}{|D|/2} + \frac{|C|\epsilon|D|^2/(4|C|+1)}{|D||D|/2} \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} \qquad \left(\text{since } \frac{2|C|}{4|C|+1} < \frac{1}{2}\right) \\ &= \epsilon.\end{aligned}$$

68. The first ten terms of the sequence with corresponding values of n written above them are

$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
1	2	$\frac{3}{2}$	$\frac{7}{4}$	$\frac{13}{8}$	$\frac{27}{16}$	$\frac{53}{32}$	$\frac{107}{64}$	$\frac{213}{128}$	$\frac{427}{256}$

The denominators are clearly powers of 2, and for $n \ge 2$ they are 2^{n-2}. Inspection of the numerators indicates that the n^{th} numerator is twice the $(n-1)^{\text{th}}$ numerator with 1 added if n is even and 1 subtracted if n is odd. Let us denote these numerators by d_n for $n \ge 2$. Then,

$$\begin{aligned}d_3 &= 3 = 2^2 - 1, \\ d_4 &= 7 = 2d_3 + 1 = 2(2^2 - 1) + 1 = 2^3 - 2 + 1, \\ d_5 &= 13 = 2d_4 - 1 = 2(2^3 - 2 + 1) - 1 = 2^4 - 2^2 + 2 - 1, \\ d_6 &= 27 = 2d_5 + 1 = 2(2^4 - 2^2 + 2 - 1) + 1 = 2^5 - 2^3 + 2^2 - 2 + 1, \\ d_7 &= 53 = 2d_6 - 1 = 2(2^5 - 2^3 + 2^2 - 2 + 1) - 1 = 2^6 - 2^4 + 2^3 - 2^2 + 2 - 1.\end{aligned}$$

The pattern emerging is that

$$d_n = 2^{n-1} - 2^{n-3} + 2^{n-4} - \cdots + (-1)^n.$$

If we multiply this by 2,

$$2d_n = 2^n - 2^{n-2} + 2^{n-3} - \cdots + 2(-1)^n,$$

and then add it to d_n,

$$3d_n = 2^n + 2^{n-1} - 2^{n-2} + (-1)^n.$$

Thus,

$$d_n = \frac{1}{3}\left[2^n + 2^{n-1} - 2^{n-2} + (-1)^n\right] = \frac{1}{3}\left[5 \cdot 2^{n-2} + (-1)^n\right].$$

Finally then, for $n \geq 2$,

$$c_n = \frac{d_n}{2^{n-2}} = \frac{1}{3 \cdot 2^{n-2}}\left[5 \cdot 2^{n-2} + (-1)^n\right] = \frac{5}{3} + \frac{(-1)^n}{3 \cdot 2^{n-2}}.$$

This formula also gives $c_1 = 1$.

EXERCISES 10.9

1. Since $\displaystyle\lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}$, the series diverges by the n^{th} term test.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{5^{n+1}} = \frac{1}{5}\sum_{n=1}^{\infty}\left(\frac{2}{5}\right)^n$, a geometric series with sum $\displaystyle\frac{1}{5}\left(\frac{2/5}{1-2/5}\right) = \frac{2}{15}$.

3. Since $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{2}\right) = 0 - 1 + 0 + 1 + 0 - 1 + 0 + 1 + \cdots$, terms do not approach zero, and the series diverges by the n^{th} term test.

4. Since $\displaystyle\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{1}{e}$ (see expression 1.68), the series diverges by the n^{th} term test.

5. This is a geometric series with common ratio 49/9, and therefore the series diverges.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{7^{n+3}}{3^{2n-2}} = \frac{7^3}{3^{-2}}\sum_{n=1}^{\infty}\left(\frac{7}{9}\right)^n$ is a geometric series with sum $\displaystyle 7^3(3)^2\left(\frac{7/9}{1-7/9}\right) = \frac{21\,609}{2}$.

7. Since $\displaystyle\lim_{n\to\infty}\sqrt{\frac{n^2-1}{n^2+1}} = 1$, the series diverges by the n^{th} term test.

8. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\pi}{2^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{2^n} = \sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n$ is a geometric series with sum $\displaystyle\frac{-1/2}{1+1/2} = -\frac{1}{3}$.

9. Since terms of the series become arbitrarily large as n increases, the series diverges by the n^{th} term test.

10. Since $\displaystyle\lim_{n\to\infty} \text{Tan}^{-1} n = \frac{\pi}{2}$, the series diverges by the n^{th} term test.

11. $\displaystyle 0.666\,666\ldots = 0.6 + 0.06 + 0.006 + \cdots = \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \cdots = \frac{6/10}{1-1/10} = \frac{2}{3}$

12.
$$\begin{aligned} 0.131\,313\,131\ldots &= 0.13 + 0.001\,3 + 0.000\,013 + \cdots = \frac{13}{100} + \frac{13}{10\,000} + \frac{13}{1\,000\,000} + \cdots \\ &= \frac{13/100}{1-1/100} = \frac{13}{99} \end{aligned}$$

13. $1.347\,346\,346\,346\ldots = 1.347 + 0.000\,346 + 0.000\,000\,346 + \cdots = \dfrac{1347}{1000} + \dfrac{346}{10^6} + \dfrac{346}{10^9} + \cdots$

$$= \frac{1347}{1000} + \frac{346/10^6}{1 - 1/10^3} = \frac{1\,345\,999}{999\,000}$$

14. $43.020\,502\,050\,205\ldots = 43 + 0.020\,5 + 0.000\,002\,05 + \cdots = 43 + \dfrac{205}{10^4} + \dfrac{205}{10^8} + \cdots$

$$= 43 + \frac{205/10^4}{1 - 1/10^4} = \frac{430\,162}{9999}$$

15. If $\sum c_n$ and $\sum d_n$ converge, then $\sum (c_n + d_n)$ converges.
Proof: Let $\{C_n\}$ and $\{D_n\}$ be the sequences of partial sums for $\sum c_n$ and $\sum d_n$ with limits C and D. The sequence of partial sums for $\sum (c_n + d_n)$, is $\{C_n + D_n\}$. According to part (ii) of Theorem 10.10, this sequence has limit $C + D$. Consequently, $\sum (c_n + d_n)$ converges to $C + D$.

16. If $\sum c_n$ converges and $\sum d_n$ diverges, then $\sum (c_n + d_n)$ diverges.
Proof: Assume to the contrary that $\sum (c_n + d_n)$ converges. Let $\{C_n\}$ and $\{D_n\}$ be the sequences of partial sums for $\sum c_n$ and $\sum d_n$. It follows that $\lim_{n\to\infty} C_n$ exists, call it C, but $\lim_{n\to\infty} D_n$ does not exist. $\{C_n + D_n\}$ is the sequence of partial sums for $\sum (c_n + d_n)$, and by assumption, it has a limit, call it E. But then according to part (ii) of Theorem 10.10, the sequence $\{(C_n + D_n) - C_n\} = \{D_n\}$ must have limit $E - C$, a contradiction. Consequently, our assumption that $\sum (c_n + d_n)$ converges must be incorrect.

17. If $\sum c_n$ and $\sum d_n$ diverge, then $\sum (c_n + d_n)$ may converge or diverge.
Proof: We give an example of each situation. The series $\sum n$ and $\sum (-n)$ both diverge, but their sum $\sum (n - n) = \sum 0$ has sum 0. On the other hand, the sum of $\sum n$ and $\sum n$ is $\sum 2n$ which diverges.

18. Since $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{4^n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{4^n}$ are both geometric series with sums

$$\sum_{n=1}^{\infty} \frac{2^n}{4^n} = \frac{1/2}{1 - 1/2} = 1 \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{3^n}{4^n} = \frac{3/4}{1 - 3/4} = 3,$$

then, by Exercise 15, $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n} = 1 + 3 = 4.$

19. Since $\displaystyle\sum_{n=1}^{\infty} (3/2)^n$ is a divergent geometric series, and $\displaystyle\sum_{n=1}^{\infty} (1/2)^n$ is a convergent geometric series, it follows from Exercise 16, that the given series diverges. (It also diverges by the n^{th} term test.)

20. Since $\displaystyle\lim_{n\to\infty} \frac{n^2 + 2^{2n}}{4^n} = \lim_{n\to\infty} \left(\frac{n^2}{4^n} + 1\right) = 1$, the series diverges by the n^{th} term test.

21. Since $\displaystyle\lim_{n\to\infty} \frac{2^n + 4^n - 8^n}{2^{3n}} = \lim_{n\to\infty} \left(\frac{1}{2^{2n}} + \frac{1}{2^n} - 1\right) = -1$, the series diverges by the n^{th} term test.

22. Since $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$, the n^{th} partial sum of the series is

$$S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$= 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

Since $\displaystyle\lim_{n\to\infty} S_n = 1$, it follows that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$

23. The total distance travelled is $20 + \displaystyle\sum_{n=1}^{\infty} 40(0.99)^n$. The series is geometric with sum $20 + \dfrac{40(0.99)}{1 - 0.99} = 3980$ m.

24. The total time taken to come to rest is

$$\sqrt{\frac{40}{9.81}} + t_1 + t_2 + t_3 + \cdots = \sqrt{\frac{40}{9.81}} + \sum_{n=1}^{\infty} t_n = \sqrt{\frac{40}{9.81}} + \sum_{n=1}^{\infty} \frac{4}{\sqrt{0.981}}(0.99)^{n/2}$$
$$= \sqrt{\frac{40}{9.81}} + \frac{4\sqrt{0.99}/\sqrt{0.981}}{1-\sqrt{0.99}} = 804 \text{ s.}$$

25. The total distance run by the dog is $\displaystyle \frac{2}{3} + \sum_{n=1}^{\infty} \frac{8}{3^{n+1}} = \frac{2}{3} + \frac{8/9}{1-1/3} = 2$ km.

We could also have reasoned this without series. Since the dog runs twice as fast as the farmer, and the farmer walks 1 km, the dog must run 2 km.

26. According to Exercise 10.1–61,

$$A_n = \frac{\sqrt{3}P^2}{36}\left(1 + \frac{1}{3} + \frac{4}{3^3} + \frac{4^2}{3^5} + \cdots + \frac{4^{n-1}}{3^{2n-1}}\right) \quad \text{(a finite geometric series after first term)}$$
$$= \frac{\sqrt{3}P^2}{36}\left\{1 + \frac{(1/3)[1-(4/9)^n]}{1-4/9}\right\} \quad \text{(using 10.39a)}$$
$$= \frac{\sqrt{3}P^2}{180}\left[8 - 3\left(\frac{4}{9}\right)^n\right].$$

Thus, $\displaystyle \lim_{n\to\infty} A_n = \frac{\sqrt{3}P^2}{180}(8) = \frac{2\sqrt{3}P^2}{45}$.

27. The inequality is certainly true for $x \geq 0$ and any n. To discuss the case when $x < 0$, we sum the geometric series

$$1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}.$$

When $x < 0$ and n is even, then $1 - x^{n+1} > 0$ and $1 - x > 0$. Hence, $(1-x^{n+1})/(1-x) > 0$. When n is odd, and $-1 \leq x < 0$, then $1 - x^{n+1} \geq 0$ and $1 - x > 0$. Hence, $(1-x^{n+1})/(1-x) > 0$. Finally, when n is odd, and $x < -1$, then $1 - x^{n+1} < 0$ and $1 - x > 0$. Hence, $(1-x^{n+1})/(1-x) < 0$. Consequently, the inequality is valid for all x when n is even, and for $x \geq -1$ when n is odd.

28. (a) If we subtract $S_n = 1 + r + r^2 + \cdots + r^{n-1}$ from $T_n = 1 + 2r + 3r^2 + \cdots + nr^{n-1}$, we obtain

$$T_n - S_n = r + 2r^2 + 3r^3 + \cdots + (n-1)r^{n-1} = r[1 + 2r + 3r^2 + \cdots + (n-1)r^{n-2}] = r(T_n - nr^{n-1}).$$

When we solve this for T_n and substitute for S_n,

$$T_n = \frac{S_n - nr^n}{1-r} = \frac{\dfrac{1-r^n}{1-r} - nr^n}{1-r} = \frac{1 - r^n - nr^n + nr^{n+1}}{(1-r)^2} = \frac{1-(n+1)r^n + nr^{n+1}}{(1-r)^2}.$$

If we now take limits as $n \to \infty$, we obtain

$$\sum_{n=1}^{\infty} nr^{n-1} = \lim_{n\to\infty} \frac{1-(n+1)r^n + nr^{n+1}}{(1-r)^2} = \frac{1}{(1-r)^2}, \quad \text{provided } |r| < 1.$$

(b) If we set $\displaystyle S(r) = \sum_{n=1}^{\infty} nr^{n-1}$, and integrate with respect to r,

$$\int S(r)\,dr + C = \sum_{n=1}^{\infty} r^n = \frac{r}{1-r}.$$

Differentiation now gives $\displaystyle S(r) = \frac{(1-r)(1) - r(-1)}{(1-r)^2} = \frac{1}{(1-r)^2}$.

29. $\dfrac{1}{2}+\dfrac{2}{2^2}+\dfrac{3}{2^3}+\dfrac{4}{2^4}+\cdots=\dfrac{1}{2}\left(1+\dfrac{2}{2}+\dfrac{3}{2^2}+\dfrac{4}{2^3}+\cdots\right)=\dfrac{1}{2}\left[\dfrac{1}{(1-1/2)^2}\right]=2$

30. $\dfrac{2}{5}+\dfrac{4}{25}+\dfrac{6}{125}+\dfrac{8}{625}+\cdots=\dfrac{2}{5}\left(1+\dfrac{2}{5}+\dfrac{3}{5^2}+\dfrac{4}{5^3}+\cdots\right)=\dfrac{2/5}{(1-1/5)^2}=\dfrac{5}{8}$

31. $\dfrac{2}{3}+\dfrac{3}{27}+\dfrac{4}{243}+\dfrac{5}{2187}+\cdots=3\left(1+\dfrac{2}{9}+\dfrac{3}{81}+\dfrac{4}{729}+\cdots\right)-3=3\left[\dfrac{1}{(1-1/9)^2}\right]-3=\dfrac{51}{64}$

32. $\dfrac{12}{5}+\dfrac{48}{25}+\dfrac{192}{125}+\dfrac{768}{625}+\cdots=\dfrac{12}{5}\left(1+\dfrac{4}{5}+\dfrac{16}{25}+\dfrac{64}{125}+\cdots\right)=\dfrac{12/5}{1-4/5}=12$

33. The probability that the first person wins on the first toss is $1/2$. The probability that the first person wins on the second toss is the product of the following three probabilities:

probability that first person throws a tail on the first toss $=1/2$;
probability that second person throws a tail on first toss $=1/2$;
probability that first person throws a head on second toss $=1/2$.

The resultant probabilty is $(1/2)(1/2)(1/2)=1/2^3$. The probability that the first person wins on the third toss is the product of the following five probabilities:

probability that first person throws a tail on the first toss $=1/2$;
probability that second person throws a tail on first toss $=1/2$;
probability that first person throws a tail on second toss $=1/2$.
probability that second person throws a tail on the second toss $=1/2$;
probability that first person throws a head on third toss $=1/2$;

The resultant probabilty is $1/2^5$.
Continuation of this process leads to the following probability that the first person to toss wins

$$\frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^5}+\frac{1}{2^7}+\cdots=\frac{1/2}{1-1/4}=\frac{2}{3}.$$

34. The probability that the first person wins on the first toss is $1/6$. The probability that the first person wins on the second toss is the product of the following three probabilities:

probability that first person does not throw a six on the first toss $=5/6$;
probability that second person does not throw a six on first toss $=5/6$;
probability that first person throws a six on second toss $=1/6$.

The resultant probabilty is $(5/6)(5/6)(1/6)=5^2/6^3$. The probability that the first person wins on the third toss is the product of the following five probabilities:

probability that first person does not throw a six on the first toss $=5/6$;
probability that second person does not throw a six on first toss $=5/6$;
probability that first person does not throw a six on second toss $=5/6$.
probability that second person does not throw a six on the second toss $=5/6$;
probability that first person throws a six on third toss $=1/6$;

The resultant probabilty is $5^4/6^5$.
Continuation of this process leads to the following probability that the first person to toss wins

$$\frac{1}{6}+\frac{5^2}{6^3}+\frac{5^4}{6^5}+\frac{5^6}{6^7}+\cdots=\frac{1/6}{1-25/36}=\frac{6}{11}.$$

35. Since the radius of convergence of the series is $R=\lim_{n\to\infty}\left|\dfrac{1/2^n}{1/2^{n+1}}\right|=2$, the open interval of convergence is $-2<x<2$. At $x=2$, the power series reduces to $\sum_{n=0}^{\infty}1$ which diverges by the n^{th} term test. At $x=-2$, it reduces to $\sum_{n=0}^{\infty}(-1)^n$ which also diverges by the n^{th} term test. The interval of convergence for the series is therefore $-2<x<2$.

36. Since the radius of convergence of the series is $R=\lim_{n\to\infty}\left|\dfrac{n^2 3^n}{(n+1)^2 3^{n+1}}\right|=1/3$, the open interval of convergence is $-1/3<x<1/3$. At $x=1/3$, the power series reduces to $\sum_{n=1}^{\infty}n^2$ which diverges by the n^{th} term test. At $x=-1/3$, it reduces to $\sum_{n=1}^{\infty}(-1)^n n^2$ which also diverges by the n^{th} term test. The interval of convergence for the series is therefore $-1/3<x<1/3$.

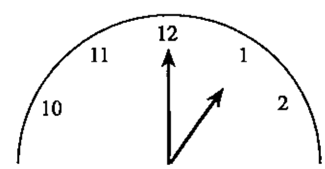
37. Since the radius of convergence of the series is $R = \lim_{n\to\infty} \left| \dfrac{2^n \left(\dfrac{n-1}{n+1}\right)^2}{2^{n+1}\left(\dfrac{n}{n+2}\right)^2} \right| = 1/2$, the open interval of convergence is $7/2 < x < 9/2$. At $x = 9/2$, the power series reduces to $\sum_{n=2}^{\infty} (n-1)^2/(n+1)^2$ which diverges by the n^{th} term test. At $x = 7/2$, it reduces to $\sum_{n=2}^{\infty} (-1)^n (n-1)^2/(n+1)^2$ which also diverges by the n^{th} term test. The interval of convergence for the series is therefore $7/2 < x < 9/2$.

38. If we set $y = x^3$, the series becomes $\sum_{n=0}^{\infty} (-1)^n x^{3n} = \sum_{n=0}^{\infty} (-1)^n y^n$. Since the radius of convergence of this series is $R_y = \lim_{n\to\infty} \left| \dfrac{(-1)^n}{(-1)^{n+1}} \right| = 1$, the radius of convergence of the given series is $R_x = 1$. The open interval of convergence is $-1 < x < 1$. At $x = 1$, the power series reduces to $\sum_{n=0}^{\infty} (-1)^n$ which diverges by the n^{th} term test. At $x = -1$, it reduces to $\sum_{n=0}^{\infty} 1$ which also diverges by the n^{th} term test. The interval of convergence for the series is therefore $-1 < x < 1$.

39. While Achilles makes up the head start L, the tortoise moves a further distance L/c. While Achilles makes up this distance, the tortoise moves a further distance $(L/c)/c = L/c^2$. Continuation of this process gives the following distance traveled by Achilles in catching the tortoise

$$L + \frac{L}{c} + \frac{L}{c^2} + \frac{L}{c^3} + \cdots = \frac{L}{1 - 1/c} = \frac{cL}{c-1}.$$

40. (a) The minute hand moves 12 times as fast as the hour hand. While the minute hand moves through the angle $\pi/6$ radians from 12 at 1:00 to 1 at 1:05, the hour hand moves a further $(\pi/6)/12$ radians. While the minute hand moves through this angle, the hour hand moves through a further angle $[(\pi/6)/12]/12 = (\pi/6)/12^2$. Continuation of this process leads to the following angle traveled by the minute hand in catching the hour hand

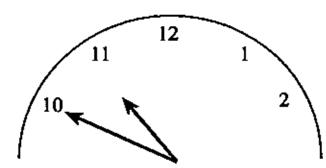


$$\frac{\pi}{6} + \frac{\pi/6}{12} + \frac{\pi/6}{12^2} + \frac{\pi/6}{12^3} + \cdots = \frac{\pi/6}{1 - 1/12} = \frac{2\pi}{11}.$$

This angle represents $\dfrac{2\pi}{11}\left(\dfrac{60}{2\pi}\right) = \dfrac{60}{11}$ minutes after 1:00.

(b) If we take time $t = 0$ at 1:00, the angle θ through which the minute hand moves in time t (in minutes) is $\theta = 2\pi t/60$. The angle ϕ that the hour hand makes with the vertical is $\phi = 2\pi t/720 + \pi/6$. These angles will be the same when $\dfrac{2\pi t}{60} = \dfrac{2\pi t}{720} + \dfrac{\pi}{6}$, the solution of which is $60/11$ minutes.

41. (a) The minute hand moves 12 times as fast as the hour hand. While the minute hand moves through the angle $5\pi/3$ radians from 12 at 10:00 to 10 at 10:50, the hour hand moves a further $(5\pi/3)/12$ radians. While the minute hand moves through this angle, the hour hand moves through a further angle $[(5\pi/3)/12]/12 = (5\pi/3)/12^2$. Continuation of this process leads to the following angle traveled by the minute hand in catching the hour hand



$$\frac{5\pi}{3} + \frac{5\pi/3}{12} + \frac{5\pi/3}{12^2} + \frac{5\pi/3}{12^3} + \cdots = \frac{5\pi/3}{1 - 1/12} = \frac{20\pi}{11}.$$

This angle represents $\dfrac{20\pi}{11}\left(\dfrac{60}{2\pi}\right) = \dfrac{600}{11}$ minutes after 10:00.

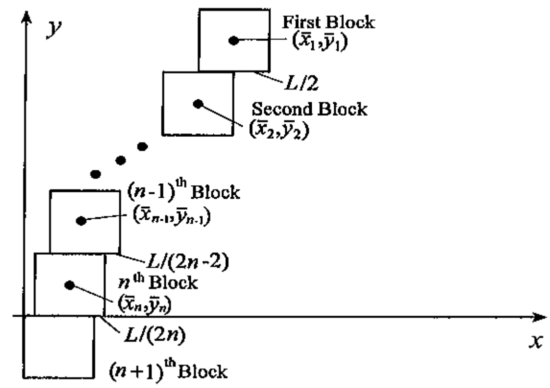
(b) If we take time $t = 0$ at 10:00, the angle θ through which the minute hand moves in time t (in minutes) is $\theta = 2\pi t/60$. The angle ϕ that the hour hand makes with the vertical is $\phi = 2\pi t/720 + 5\pi/3$. These angles will be the same when $\dfrac{2\pi t}{60} = \dfrac{2\pi t}{720} + \dfrac{5\pi}{3}$, the solution of which is $600/11$ minutes.

42. Suppose the length of each block is L. Taking the density of the blocks as unity, the mass of the top n blocks is nL^3. The first moment of the n^{th} block about the y-axis is

$$L^3\overline{x}_n = L^3\left(\frac{L}{2}+\frac{L}{2n}\right) = \frac{L^4}{2}\left(1+\frac{1}{n}\right).$$

The first moment of the $(n-1)^{\text{th}}$ block about the y-axis is

$$L^3\overline{x}_{n-1} = L^3\left[\frac{L}{2}+\frac{L}{2n}+\frac{L}{2(n-1)}\right] = \frac{L^4}{2}\left(1+\frac{1}{n}+\frac{1}{n-1}\right).$$



Continuing in this way, the moment of the first block about the y-axis is

$$L^3\overline{x}_1 = L^3\left[\frac{L}{2}+\frac{L}{2n}+\frac{L}{2(n-1)}+\cdots+\frac{L}{2}\right] = \frac{L^4}{2}\left(1+\frac{1}{n}+\frac{1}{n-1}+\cdots+\frac{1}{2}+1\right).$$

The x-coordinate of the centre of mass of the top n blocks is therefore

$$\begin{aligned}\overline{x} &= \frac{1}{nL^3}\left[\frac{L^4}{2}\left(1+\frac{1}{n}\right)+\frac{L^4}{2}\left(1+\frac{1}{n}+\frac{1}{n-1}\right)+\cdots+\frac{L^4}{2}\left(1+\frac{1}{n}+\frac{1}{n-1}+\cdots+\frac{1}{2}+1\right)\right]\\ &= \frac{L}{2n}\left[n(1)+n\left(\frac{1}{n}\right)+(n-1)\left(\frac{1}{n-1}\right)+\cdots+2\left(\frac{1}{2}\right)+1(1)\right] = \frac{L}{2n}(2n) = L.\end{aligned}$$

Thus, the centre of mass of the top n blocks is over the edge of the $(n+1)^{\text{th}}$ block. They will not tip, but they are in a state of precarious equilibrium.

The right edge of the top block sticks out the following distance over the right edge of the $(n+1)^{\text{th}}$ block

$$\frac{L}{2}+\frac{L}{4}+\frac{L}{6}+\cdots+\frac{L}{2n} = \frac{L}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right).$$

This is $L/2$ times the n^{th} partial sum of the harmonic series which we know becomes arbitrarily large as n increases. Hence, the top n blocks can be made to protrude arbitrarily far over the $(n+1)^{\text{th}}$ block.

43. Let $\{S_n\}$ be the sequence of partial sums of the given series. It converges to the sum of the series, call it S. If terms of the series are grouped together, then the sequence of partial sums of the new series, call it $\{T_n\}$, is a subsequence of $\{S_n\}$. But every subseqeuence of a convergent series must converge to the same limit as the sequence. Thus, $\{T_n\}$ converges to S also, and the grouped series has sum S.

44. To verify this, we first write the Laplace transform as an infinite series of integrals

$$F(s) = \int_0^\infty e^{-st}f(t)\,dt = \sum_{n=0}^{\infty}\int_{np}^{(n+1)p} e^{-st}f(t)\,dt.$$

If we change variables of integration in the n^{th} term with $u = t - np$, then

$$F(s) = \sum_{n=0}^{\infty}\int_0^p e^{-s(u+np)}f(u+np)\,du = \sum_{n=0}^{\infty} e^{-nps}\int_0^p e^{-su}f(u)\,du = \left(\int_0^p e^{-su}f(u)\,du\right)\left(\sum_{n=0}^{\infty} e^{-nps}\right).$$

Since the series is geometric with common ratio e^{-ps},

$$F(s) = \int_0^p e^{-su}f(u)\,du\left[\frac{1}{1-e^{-ps}}\right] = \frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt.$$

45. (a) When V is the voltage across the capacitor, and resistor R_2 (they are in parallel), the currents through these devices are $i_C = CdV/dt$ and $i_{R_2} = V/R_2$. The current through R_1 must be the sum of these, $i_{R_1} = V/R_2 + CdV/dt$. The voltage across R_1 is therefore $R_1(V/R_2 + CdV/dt)$, and it follows that for $V_{\text{in}} = \overline{V}$,

$$\overline{V} = V + R_1\left(\frac{V}{R_2} + C\frac{dV}{dt}\right) \quad\Longrightarrow\quad \frac{dV}{dt} + \tau V = \alpha\overline{V},$$

where $\tau = (R_1 + R_2)/(R_1R_2C)$ and $\alpha = 1/(R_1C)$.

(b) If we multiply the differential equation by $e^{\tau t}$, the left side becomes the derivative of a product,

$$e^{\tau t}\frac{dV}{dt} + \tau e^{\tau t}V = \alpha\overline{V}e^{\tau t} \Longrightarrow \frac{d}{dt}(Ve^{\tau t}) = \alpha\overline{V}e^{\tau t} \Longrightarrow Ve^{\tau t} = \frac{\alpha\overline{V}}{\tau}e^{\tau t} + D \Longrightarrow V = \frac{\alpha\overline{V}}{\tau} + De^{-\tau t}.$$

Using the condition that $\lim_{t\to 2(n-1)T^+} V(t) = V_{n-1}$, we obtain

$$V_{n-1} = \frac{\alpha\overline{V}}{\tau} + De^{-2\tau(n-1)T} \quad\Longrightarrow\quad D = \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{2\tau(n-1)T}.$$

Hence, for $2(n-1)T < t < (2n-1)T$,

$$V(t) = \frac{\alpha\overline{V}}{\tau} + \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{2\tau(n-1)T}e^{-\tau t} = \frac{\alpha\overline{V}}{\tau} + \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{-\tau[t-2(n-1)T]}.$$

At $t = (2n-1)T$,

$$V\big((2n-1)T\big) = \frac{\alpha\overline{V}}{\tau} + \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{-\tau[(2n-1)T-2(n-1)T]} = \frac{\alpha\overline{V}}{\tau} + \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{-\tau T}.$$

(c) When $V_{\text{in}} = 0$, the rectifier prevents the charge that has been stored in the capacitor from flowing back through R_1; it simply discharges itself through R_2. Consequently, $dV/dt + \sigma V = 0$ where $\sigma = 1/(R_2C)$.

(d) We separate the differential equation:

$$\frac{dV}{V} = -\sigma\,dt \qquad\Longrightarrow\qquad \ln|V| = -\sigma t + D \qquad\Longrightarrow\qquad V(t) = Ee^{-\sigma t}.$$

If we now use the fact that $\lim_{t\to(2n-1)T^+} V(t) = \frac{\alpha\overline{V}}{\tau} + \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{-\tau T}$, we obtain

$$\frac{\alpha\overline{V}}{\tau} + \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{-\tau T} = Ee^{-\sigma(2n-1)T} \quad\Longrightarrow\quad E = \frac{\alpha\overline{V}}{\tau}e^{\sigma(2n-1)T} + \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{-[\tau-\sigma(2n-1)]T}.$$

Hence, for $(2n-1)T < t < 2nT$, we have $\quad V(t) = \left[\frac{\alpha\overline{V}}{\tau} + \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{-\tau T}\right]e^{-\sigma[t-(2n-1)T]}.$

(e) When the function in (d) is evaluated at $t = 2nT$, its value is V_n; that is,

$$V_n = \left[\frac{\alpha\overline{V}}{\tau} + \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{-\tau T}\right]e^{-\sigma T} = pV_{n-1} + q,$$

where $p = e^{-T(\tau+\sigma)}$ and $q = (\alpha\overline{V}/\tau)(1 - e^{-\tau T})e^{-\sigma T}$. If we iterate this recursive definition,

$$V_1 = pV_0 + q, \quad V_2 = pV_1 + q = p^2V_0 + q(p+1), \quad V_3 = pV_2 + q = p^3V_0 + q(p^2 + p + 1).$$

The pattern emerging is $V_n = p^nV_0 + q(1 + p + p^2 + \cdots + p^{n-1}) = p^nV_0 + \dfrac{q(1-p^n)}{1-p}$.

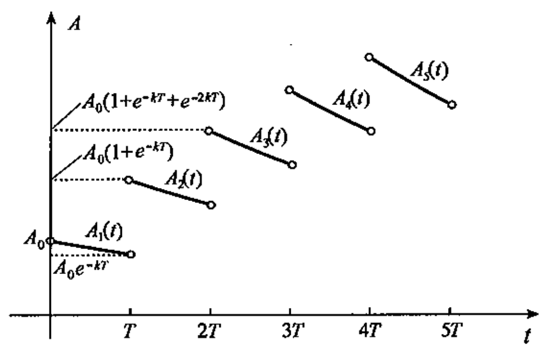
Since $V_0 = 0$, if the voltage across the capacitor is zero at time $t = 0$, we have

$$V_n = \frac{q(1-p^n)}{1-p} = \frac{\alpha\overline{V}}{\tau}(1 - e^{-\tau T})e^{-\sigma T}\left[\frac{1 - e^{-nT(\tau+\sigma)}}{1 - e^{-T(\tau+\sigma)}}\right].$$

46. (a) After time t, the amount of the first injection remaining is A_0e^{-kt}; the amount of the second injection remaining is $A_0e^{-k(t-T)}$; the amount of the third injection remaining is $A_0e^{-k(t-2T)}$; etc. At time t between the n^{th} and $(n+1)^{\text{th}}$ injection, the total amount remaining is

$$\begin{aligned}A_n(t) &= A_0e^{-kt} + A_0e^{-k(t-T)} + \cdots + A_0e^{-k[t-(n-1)T]}\\ &= A_0e^{-kt}\left[1 + e^{kT} + e^{2kT} + \cdots + e^{(n-1)kT}\right]\\ &= A_0e^{-kt}\left[\frac{1-(e^{kT})^n}{1-e^{kT}}\right] \qquad \text{(using 10.39a)}\\ &= A_0e^{-kt}\left[\frac{1-e^{knT}}{1-e^{kT}}\right] \qquad (n-1)T < t < nT.\end{aligned}$$

(b)



(c) $\displaystyle\lim_{n\to\infty} A_n[(n-1)T] = \lim_{n\to\infty} A_0e^{-k(n-1)T}\left[\frac{1-e^{knT}}{1-e^{kT}}\right]$

$$= \frac{A_0e^{kT}}{1-e^{kT}}\lim_{n\to\infty}(e^{-knT}-1) = \frac{-A_0e^{kT}}{1-e^{kT}} = \frac{A_0}{1-e^{-kT}}$$

EXERCISES 10.10

1. Since $l = \displaystyle\lim_{n\to\infty}\frac{\frac{1}{2n+1}}{\frac{1}{2n}} = 1$, and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, so also does the given series (by the limit comparison test).

2. Since $l = \displaystyle\lim_{n\to\infty}\frac{\frac{1}{4n-3}}{\frac{1}{4n}} = 1$, and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{4n} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, so also does the given series (by the limit comparison test).

3. Since $l = \displaystyle\lim_{n\to\infty}\frac{\frac{1}{2n^2+4}}{\frac{1}{2n^2}} = 1$, and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n^2} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, so also does the given series (by the limit comparison test).

4. Since $l = \displaystyle\lim_{n\to\infty}\frac{\frac{1}{5n^2-3n-1}}{\frac{1}{5n^2}} = 1$, and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{5n^2} = \frac{1}{5}\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, so also does the given series (by the limit comparison test).

5. Since $l = \lim_{n\to\infty} \dfrac{\dfrac{1}{n^3-1}}{\dfrac{1}{n^3}} = 1$, and $\sum_{n=2}^{\infty} \dfrac{1}{n^3}$ converges, so also does the given series (by the limit comparison test).

6. Since $l = \lim_{n\to\infty} \dfrac{\dfrac{n^2}{n^4-6n^2+5}}{\dfrac{1}{n^2}} = 1$, and $\sum_{n=4}^{\infty} \dfrac{1}{n^2}$ converges, so also does the given series (by the limit comparison test).

7. Since $l = \lim_{n\to\infty} \dfrac{\dfrac{1}{(2n-1)(2n+1)}}{\dfrac{1}{4n^2}} = 1$, and $\sum_{n=1}^{\infty} \dfrac{1}{4n^2} = \dfrac{1}{4}\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ converges, so also does the given series (by the limit comparison test).

8. Since $l = \lim_{n\to\infty} \dfrac{\dfrac{n-5}{n^2+3n-2}}{\dfrac{1}{n}} = 1$, and $\sum_{n=6}^{\infty} \dfrac{1}{n}$ diverges, so also does the series $\sum_{n=6}^{\infty} \dfrac{n-5}{n^2+3n-2}$ (by the limit comparison test). The given series therefore diverges also.

9. Since $\dfrac{1}{\ln n} > \dfrac{1}{n}$, and the harmonic series diverges, so also does the given series (by the comparison test).

10. The function $f(x) = x^2e^{-2x}$ is positive and continuous. Since $f'(x) = 2xe^{-2x} - 2x^2e^{-2x} = 2xe^{-2x}(1-x)$, the function is decreasing for $x \geq 1$. Integrating by parts, and understanding that limits must be taken for the infinite limit,

$$\begin{aligned}\int_1^{\infty} x^2e^{-2x}\,dx &= \left\{\frac{x^2e^{-2x}}{-2}\right\}_1^{\infty} - \int_1^{\infty} 2x\left(\frac{e^{-2x}}{-2}\right)dx = \frac{e^{-2}}{2} + \left\{\frac{xe^{-2x}}{-2}\right\}_1^{\infty} - \int_1^{\infty}\frac{e^{-2x}}{-2}dx \\ &= \frac{e^{-2}}{2} + \frac{e^{-2}}{2} - \left\{\frac{e^{-2x}}{4}\right\}_1^{\infty} = \frac{5}{4e^2}.\end{aligned}$$

Since the integral converges, so also does the series $\sum_{n=1}^{\infty} n^2e^{-2n}$ (by the integral test).

11. Since $l = \lim_{n\to\infty} \dfrac{\dfrac{\sqrt{n^2+2n-3}}{n^2+5}}{\dfrac{1}{n}} = 1$, and $\sum_{n=2}^{\infty} \dfrac{1}{n}$ diverges, so also does the given series (by the limit comparison test).

12. Since $l = \lim_{n\to\infty} \dfrac{\dfrac{\sqrt{n+5}}{n^3+3}}{\dfrac{1}{n^{5/2}}} = 1$, and $\sum_{n=1}^{\infty} \dfrac{1}{n^{5/2}}$ converges, so also does the given series (by the limit comparison test).

13. Since $l = \lim_{n\to\infty} \dfrac{\sqrt{\dfrac{n^2+2n+3}{2n^4-n}}}{\dfrac{1}{\sqrt{2}n}} = 1$, and $\sum_{n=1}^{\infty} \dfrac{1}{\sqrt{2}n} = \dfrac{1}{\sqrt{2}}\sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges, so also does the given series (by the limit comparison test).

14. Since $\dfrac{1}{n^2\ln n} < \dfrac{1}{n^2}$ for $n \geq 3$, and $\sum_{n=3}^{\infty} \dfrac{1}{n^2}$ converges, so also does $\sum_{n=3}^{\infty} \dfrac{1}{n^2\ln n}$ (by the comparison test). The original series therefore converges also.

15. Since $\frac{1}{2^n}\sin\left(\frac{\pi}{n}\right) < \frac{1}{2^n}$, and $\sum_{n=1}^{\infty}\frac{1}{2^n}$ is a convergent geometric series, the original series converges (by the comparison test).

16. Since $l = \lim_{n\to\infty}\dfrac{\dfrac{\sqrt{n^2+1}}{n^3}\mathrm{Tan}^{-1}n}{\dfrac{1}{n^2}\left(\dfrac{\pi}{2}\right)} = 1$, and $\sum_{n=1}^{\infty}\frac{\pi}{2n^2} = \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, so also does the given series (by the limit comparison test).

17. Since $l = \lim_{n\to\infty}\dfrac{\dfrac{2^n+n}{3^n+1}}{(2/3)^n} = 1$, and $\sum_{n=1}^{\infty}(2/3)^n$ is a convergent geometric series, the given series converges (by the limit comparison test).

18. Since $\frac{1+\ln^2 n}{n\ln^2 n} = \frac{1}{n\ln^2 n} + \frac{1}{n} > \frac{1}{n}$, and $\sum_{n=2}^{\infty}\frac{1}{n}$ diverges, so also does the given series (by the comparison test).

19. Since $l = \lim_{n\to\infty}\dfrac{\dfrac{1+1/n}{e^n}}{\dfrac{1}{e^n}} = 1$, and $\sum_{n=1}^{\infty}\frac{1}{e^n}$ is a convergent geometric series, the given series converges (by the limit comparison test).

20. Since $\frac{\ln(n+1)}{n+1} > \frac{1}{n+1}$ for $n \ge 2$, and $\sum_{n=2}^{\infty}\frac{1}{n+1}$ diverges (harmonic series with two terms missing), so also does $\sum_{n=2}^{\infty}\frac{\ln(n+1)}{n+1}$ (by the comparison test). It follows that the given series diverges also.

21. The function $f(x) = xe^{-x^2}$ is positive and continuous. Since $f'(x) = e^{-x^2} - 2x^2e^{-x^2} = (1-2x^2)e^{-x^2}$, the function is decreasing for $x \ge 1$. Since

$$\int_1^{\infty} xe^{-x^2}\,dx = \left\{\frac{e^{-x^2}}{-2}\right\}_1^{\infty} = \frac{1}{2e},$$

the improper integral converges. So also therefore does the series (by the integral test).

22. The function $1/[x(\ln x)^{1/3}]$ is positive, continuous, and decreasing for $x \ge 2$. Since

$$\int_2^{\infty}\frac{1}{x(\ln x)^{1/3}}dx = \left\{\frac{3}{2}(\ln x)^{2/3}\right\}_2^{\infty} = \infty,$$

the series $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^{1/3}}$ diverges (by the integral test).

23. If we set $y = x^2$, the series becomes $\sum_{n=1}^{\infty}\frac{1}{n}x^{2n} = \sum_{n=1}^{\infty}\frac{1}{n}y^n$. Since the radius of convergence of this series is $R_y = \lim_{n\to\infty}\left|\frac{1/n}{1/(n+1)}\right| = 1$, the radius of convergence of the original series is $R_x = \sqrt{R_y} = 1$. The open interval of convergence is $-1 < x < 1$. At the end points $x = \pm 1$, the series reduces to the harmonic series which diverges. The interval of convergence is therefore $-1 < x < 1$.

24. If we set $y = x^2$, the series becomes $\sum_{n=1}^{\infty}\frac{1}{n^2}x^{2n} = \sum_{n=1}^{\infty}\frac{1}{n^2}y^n$. Since the radius of convergence of this series is $R_y = \lim_{n\to\infty}\left|\frac{1/n^2}{1/(n+1)^2}\right| = 1$, the radius of convergence of the original series is $R_x = \sqrt{R_y} = 1$.

The open interval of convergence is $-1 < x < 1$. At the end points $x = \pm 1$, the series reduces to the convergent series $\sum_{n=1}^{\infty} 1/n^2$. The interval of convergence is therefore $-1 \le x \le 1$.

25. If we set $y = (x-1)^4$, the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n2^n}(x-1)^{4n} = \sum_{n=1}^{\infty} \frac{1}{n2^n} y^n$. Since the radius of convergence of this series is $R_y = \displaystyle\lim_{n\to\infty} \left| \frac{1/(n2^n)}{1/[(n+1)2^{n+1}]} \right| = 2$, the radius of convergence of the original series is $R_x = {R_y}^{1/4} = 2^{1/4}$. The open interval of convergence is $1 - 2^{1/4} < x < 1 + 2^{1/4}$. At the end points $x = 1 \pm 2^{1/4}$, the series reduces to the harmonic series which diverges. The interval of convergence is therefore $1 - 2^{1/4} < x < 1 + 2^{1/4}$.

26. If we set $y = (x+1)^2$, the series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{n3^n}{(n+1)^3}(x+2)^{2n} = \sum_{n=0}^{\infty} \frac{n3^n}{(n+1)^3} y^n$. Since the radius of convergence of this series is $R_y = \displaystyle\lim_{n\to\infty} \left| \frac{n3^n/(n+1)^3}{(n+1)3^{n+1}/(n+2)^3} \right| = 1/3$, the radius of convergence of the original series is $R_x = \sqrt{R_y} = 1/\sqrt{3}$. The open interval of convergence is $-2 - 1/\sqrt{3} < x < -2 + 1/\sqrt{3}$. At the end points $x = -2 \pm 1/\sqrt{3}$, the series reduces to $\sum_{n=0}^{\infty} n/(n+1)^3$. Since $l = \displaystyle\lim_{n\to\infty} \frac{n/(n+1)^3}{1/n^2} = 1$, and $\sum_{n=1}^{\infty} 1/n^2$ converges, so also does $\sum_{n=0}^{\infty} n/(n+1)^3$ (by the limit comparison test). The interval of convergence is therefore $-2 - 1/\sqrt{3} \le x \le -2 + 1/\sqrt{3}$.

27. When $p = 1$, the series is $\sum_{n=2}^{\infty} 1/(n \ln n)$. The function $f(x) = 1/(x \ln x)$ is positive, continuous and decreasing function for $x \ge 2$. Since

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \Big\{ \ln|\ln x| \Big\}_2^{\infty} = \infty,$$

the series $\sum_{n=2}^{\infty} 1/(n \ln n)$ diverges (by the integral test). When $p < 1$, then $1/(n^p \ln n) > 1/(n \ln n)$, and therefore the given series diverges for $p < 1$ also. When $p > 1$, then $1/(n^p \ln n) < 1/n^p$ for $n \ge 3$. Since $\sum_{n=3}^{\infty} 1/n^p$ converges for $p > 1$, it follows that the given series converges for $p > 1$.

28. When $p = 1$, the function $1/(x \ln x)$ is positive, continuous, and decreasing for $x \ge 2$. Since

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \Big\{ \ln|\ln x| \Big\}_2^{\infty} = \infty,$$

it follows by the integral test that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

When $p < 1$, $\quad \dfrac{1}{n(\ln n)^p} > \dfrac{1}{n \ln n}$, and therefore $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ diverges when $p < 1$ (by the comparison test).

For $p > 1$, the function $1/[x(\ln x)^p]$ is positive, continuous, and decreasing for $x \ge 2$. Since

$$\int_2^{\infty} \frac{1}{x(\ln x)^p} dx = \left\{ \frac{1}{(1-p)(\ln x)^{p-1}} \right\}_2^{\infty} = \frac{1}{(p-1)(\ln 2)^{p-1}},$$

the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges (by the integral test).

29. When $p = 1$, the series is $\sum_{n=2}^{\infty} 1/\ln n$. Since $1/\ln n > 1/n$, and $\sum_{n=2}^{\infty} 1/n$ diverges, so also does $\sum_{n=2}^{\infty} 1/\ln n$ (by the comparison test). Because $1/(\ln n)^p > 1/\ln n$ when $p < 1$, it follows that the given series diverges for all $p \le 1$. When $p > 1$, we set $f(x) = 1/(\ln x)^p$, a positive, continuous, decreasing function for $x \ge 2$. If we set $u = 1/(\ln x)^p$, $du = -p/[x(\ln x)^{p+1}]\,dx$, $dv = dx$, and $v = x$, then integration by parts gives

$$\int_2^{\infty} \frac{1}{(\ln x)^p} dx = \left\{ \frac{x}{(\ln x)^p} \right\}_2^{\infty} + p \int_2^{\infty} \frac{1}{(\ln x)^{p+1}} dx$$

$$= \lim_{x\to\infty}\left[\frac{x}{(\ln x)^p}\right] - \frac{2}{(\ln 2)^p} + p\int_2^\infty \frac{1}{(\ln x)^{p+1}}dx.$$

To evaluate $L = \lim_{x\to\infty} \dfrac{x}{(\ln x)^p}$, we take logarithms, $\ln L = \lim_{x\to\infty} [\ln x - p\ln(\ln x)]$. Consider instead

$$\lim_{x\to\infty} \frac{\ln x}{p\ln(\ln x)} = \lim_{x\to\infty} \frac{1/x}{\frac{p}{x\ln x}} = \lim_{x\to\infty} \frac{\ln x}{p} = \infty.$$

This implies that $\ln L = \lim_{x\to\infty} [\ln x - p\ln(\ln x)] = \infty$ also. Thus, $L = \lim_{x\to\infty} \dfrac{x}{(\ln x)^p} = \infty$. Since the $\int_2^\infty \dfrac{1}{(\ln x)^{p+1}}dx$ must be positive or diverge to "infinity", it follows that $\int_2^\infty \dfrac{1}{(\ln x)^p}dx = \infty$. The series $\sum_{n=2}^\infty 1/(\ln n)^p$ therefore diverges for $p > 1$ also. The series therefore diverges for all p.

EXERCISES 10.11

1. Since $L = \lim_{n\to\infty} \dfrac{\frac{e^{n+1}}{(n+1)^4}}{\frac{e^n}{n^4}} = e$, the series diverges (by the limit ratio test).

2. Since $L = \lim_{n\to\infty} \dfrac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty} \dfrac{1}{n+1} = 0$, the series converges (by the limit ratio test).

3. Since $L = \lim_{n\to\infty} \dfrac{\frac{(n+1)^3}{2^{n+1}}}{\frac{n^3}{2^n}} = \dfrac{1}{2}$, the series converges (by the limit ratio test).

4. Since $R = \lim_{n\to\infty} \left(\dfrac{1}{n^n}\right)^{1/n} = \lim_{n\to\infty} \dfrac{1}{n} = 0$, the series converges (by the limit root test).

5. Since $L = \lim_{n\to\infty} \dfrac{\frac{n(n-1)}{(n+1)^2 2^{n+1}}}{\frac{(n-1)(n-2)}{n^2 2^n}} = \dfrac{1}{2}$, the series converges (by the limit ratio test).

6. Since $L = \lim_{n\to\infty} \dfrac{\frac{(2n+2)!}{[(n+1)!]^2}}{\frac{(2n)!}{(n!)^2}} = \lim_{n\to\infty} \dfrac{(2n+2)(2n+1)}{(n+1)^2} = 4 > 1$, the series diverges (by the limit ratio test).

7. Since $l = \lim_{n\to\infty} \dfrac{\frac{\sqrt{n+1}}{n^{n+1/2}}}{\frac{1}{n^n}} = \lim_{n\to\infty} \dfrac{\sqrt{n+1}}{\sqrt{n}} = 1$, the given series converges if and only if the series $\sum\limits_{n=1}^{\infty} 1/n^n$ converges. Because $R = \lim_{n\to\infty} (1/n^n)^{1/n} = \lim_{n\to\infty} (1/n) = 0$. the series $\sum\limits_{n=1}^{\infty} 1/n^n$ converges (by the limit root test). Thus, the original series converges also.

8. Since $L = \lim_{n\to\infty} \dfrac{\frac{3^{-(n+1)} + 2^{-(n+1)}}{4^{-(n+1)} + 5^{-(n+1)}}}{\frac{3^{-n} + 2^{-n}}{4^{-n} + 5^{-n}}} = \lim_{n\to\infty} \left\{\dfrac{2^{-(n+1)}[1 + (3/2)^{-(n+1)}]}{4^{-(n+1)}[1 + (5/4)^{-(n+1)}]} \dfrac{4^{-n}[1 + (5/4)^{-n}]}{2^{-n}[1 + (3/2)^{-n}]}\right\} = 2$, the series diverges (by the limit ratio test).

9. Since $L = \lim_{n\to\infty} \dfrac{\dfrac{e^{-n-1}}{\sqrt{n+1+\pi}}}{\dfrac{e^{-n}}{\sqrt{n+\pi}}} = \dfrac{1}{e}$, the series converges (by the limit ratio test).

10. Since $L = \lim_{n\to\infty} \dfrac{\dfrac{2\cdot 4\cdots(2n+2)}{4\cdot 7\cdots(3n+4)}}{\dfrac{2\cdot 4\cdots(2n)}{4\cdot 7\cdots(3n+1)}} = \lim_{n\to\infty} \dfrac{2n+2}{3n+4} = \dfrac{2}{3}$, the series converges (by the limit ratio test).

11. Since $L = \lim_{n\to\infty} \dfrac{\dfrac{(n+1)^n}{3^n\, n!}}{\dfrac{n^{n-1}}{3^{n-1}(n-1)!}} = \lim_{n\to\infty} \dfrac{1}{3}\left(\dfrac{n+1}{n}\right)^n = \dfrac{e}{3} < 1$, the series converges (by the limit ratio test).

12. Since $L = \lim_{n\to\infty} \dfrac{(n+1)(3/4)^{n+1}}{n(3/4)^n} = \dfrac{3}{4}$, the series converges (by the limit ratio test).

13. Since $L = \lim_{n\to\infty} \dfrac{\dfrac{1+1/(n+1)}{e^{n+1}}}{\dfrac{1+1/n}{e^n}} = \dfrac{1}{e}$, the series converges (by the limit ratio test).

14. Since $\dfrac{2\cdot 4\cdots(2n)}{3\cdot 5\cdots(2n+1)}\left(\dfrac{1}{n^2}\right) = \left(\dfrac{2}{3}\right)\left(\dfrac{4}{5}\right)\cdots\left(\dfrac{2n}{2n+1}\right)\left(\dfrac{1}{n^2}\right) < \dfrac{1}{n^2}$, and $\sum_{n=1}^{\infty}\dfrac{1}{n^2}$ converges, so also does the original series (by the comparison test).

15. Since $l = \lim_{n\to\infty} \dfrac{\dfrac{n^n}{(n+1)^{n+1}}}{\dfrac{1}{en}} = e\lim_{n\to\infty}\left(\dfrac{n}{n+1}\right)\left(\dfrac{n}{n+1}\right)^n = 1$, and $\sum_{n=1}^{\infty} 1/(en) = (1/e)\sum_{n=1}^{\infty}(1/n)$ diverges, so also does the original series (by the limit comparison test).

16. Since $l = \lim_{n\to\infty} \dfrac{\dfrac{(n+1)^n}{n^{n+1}}}{\dfrac{e}{n}} = \lim_{n\to\infty}\dfrac{1}{e}\left(\dfrac{n+1}{n}\right)^n = 1$, and $\sum_{n=1}^{\infty}\dfrac{e}{n} = e\sum_{n=1}^{\infty}\dfrac{1}{n}$ diverges, so also does the original series (by the limit comparison test).

17. Since $L = \lim_{n\to\infty} \dfrac{\dfrac{(n+1)^4+3}{5^{(n+1)/2}}}{\dfrac{n^4+3}{5^{n/2}}} = \dfrac{1}{\sqrt{5}}$, the series converges (by the limit ratio test).

18. Since $L = \lim_{n\to\infty} \dfrac{\dfrac{2^{n+1}+(n+1)^2 3^{n+1}}{4^{n+1}}}{\dfrac{2^n+n^2 3^n}{4^n}} = \lim_{n\to\infty}\dfrac{1}{4}\left[\dfrac{2^{n+1}+(n+1)^2 3^{n+1}}{2^n+n^2 3^n}\right] = \lim_{n\to\infty}\dfrac{1}{4}\left[\dfrac{\dfrac{2^{n+1}}{3^{n+1}}+(n+1)^2}{\dfrac{2^n}{3^{n+1}}+\dfrac{n^2}{3}}\right]$ $= 3/4$, the series converges (by the limit ratio test).

19. Since $L = \lim_{n\to\infty} \dfrac{\dfrac{(n+1)^2 2^{n+1}-(n+1)}{(n+1)^3+1}}{\dfrac{n^2 2^n-n}{n^3+1}} = 2$, the series diverges (by the limit ratio test).

20. Since $L = \lim_{n\to\infty} \dfrac{\dfrac{(2n+2)!5^{2n+2}}{(3n+3)!}}{\dfrac{(2n)!5^{2n}}{(3n)!}} = \lim_{n\to\infty}\dfrac{25(2n+2)(2n+1)}{(3n+3)(3n+2)(3n+1)} = 0$, the series converges (by the limit ratio test).

21. The series diverges for $a = 0, 1$ by the n^{th} term test. When $a = 2$, the series converges by the limit ratio test since $L = \lim_{n\to\infty} \dfrac{\dfrac{[(n+1)!]^2}{(2n+2)!}}{\dfrac{(n!)^2}{(2n)!}} = \lim_{n\to\infty} \dfrac{(n+1)^2}{(2n+2)(2n+1)} = \dfrac{1}{4}$. When $a > 2$, terms of the series are less than those when $a = 2$, and the series therefore converges in these cases also.

EXERCISES 10.12

1. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1}$. Since $l = \lim_{n\to\infty} \dfrac{\dfrac{n}{n^3+1}}{\dfrac{1}{n^2}} = 1$, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so also does $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ (by the limit comparison test). The original series therefore converges absolutely.

2. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$. Since $l = \lim_{n\to\infty} \dfrac{\dfrac{n}{n^2+1}}{\dfrac{1}{n}} = 1$, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges so also does $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ (by the limit comparison test). The original series does not converge absolutely. Because the sequence $\{n/(n^2+1)\}$ is decreasing and has limit zero, the original series converges conditionally (by the alternating series test).

3. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the original series converges absolutely.

4. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3^n}$. Since $L = \lim_{n\to\infty} \dfrac{\dfrac{(n+1)^3}{3^{n+1}}}{\dfrac{n^3}{3^n}} = \dfrac{1}{3}$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ converges (by the limit ratio test). The original series therefore converges absolutely.

5. Since the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, the original series does not converge absolutely. Because the sequence $\{1/\sqrt{n}\}$ is decreasing and has limit zero, the original series converges conditionally (by the alternating series test).

6. Since $\lim_{n\to\infty} (-1)^n \dfrac{3^n}{n^3}$ does not exist, the series diverges (by the n^{th} term test).

7. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+n+1}$. Since $l = \lim_{n\to\infty} \dfrac{\dfrac{n}{n^2+n+1}}{\dfrac{1}{n}} = 1$, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges so also does $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+n+1}$ (by the limit comparison test). The original series does not converge absolutely. Because the sequence $\{n/(n^2+n+1)\}$ is decreasing and has limit zero, the original series converges conditionally (by the alternating series test).

8. Consider the series $\displaystyle\sum_{n=1}^{\infty} \left|\frac{n\sin(n\pi/4)}{2^n}\right|$. Since $\left|\dfrac{n\sin(n\pi/4)}{2^n}\right| \le \dfrac{n}{2^n}$, the series of absolute values converges if $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges. Because $L = \lim_{n\to\infty} \dfrac{\dfrac{n+1}{2^{n+1}}}{\dfrac{n}{2^n}} = \dfrac{1}{2}$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges (by the limit ratio test). Consequently, the original series converges absolutely.

9. Since $\lim_{n\to\infty} \left(\dfrac{n}{n+1}\right) = 1$, it follows that $\lim_{n\to\infty} (-1)^{n+1}\left(\dfrac{n}{n+1}\right)$ does not exist, and the series diverges (by the n^{th} term test).

10. Consider the series $\sum_{n=1}^{\infty} \frac{\sqrt{3n-2}}{n}$. Since $l = \lim_{n\to\infty} \frac{\frac{\sqrt{3n-2}}{n}}{\frac{\sqrt{3}}{\sqrt{n}}} = 1$, and $\sum_{n=1}^{\infty} \frac{\sqrt{3}}{\sqrt{n}}$ diverges so also does the series $\sum_{n=1}^{\infty} \frac{\sqrt{3n-2}}{n}$ (by the limit comparison test). The original series does not converge absolutely. Because the sequence $\{\sqrt{3n-2}/n\}$ is nonincreasing and has limit zero, the original series converges conditionally (by the alternating series test).

11. Since $\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e}$, it follows that $\lim_{n\to\infty} (-1)^{n+1} \left(\frac{n}{n+1}\right)^n$ does not exist, and the series diverges (by the n^{th} term test).

12. Consider the series $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+3}}{n^2+5}$. Since $l = \lim_{n\to\infty} \frac{\frac{\sqrt{n^2+3}}{n^2+5}}{\frac{1}{n}} = 1$, and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so also does the series $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+3}}{n^2+5}$ (by the limit comparison test). The original series does not therefore converge absolutely. Because the sequence $\{\sqrt{n^2+3}/(n^2+5)\}$ is decreasing and has limit zero, the original series converges conditionally (by the alternating series test).

13. Consider the series $\sum_{n=2}^{\infty} \frac{\ln n}{n}$. Since $(\ln n)/n > 1/n$ for $n \geq 3$, and $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges, so also does $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ (by the comparison test). The original series does not converge absolutely. Because the sequence $\{(\ln n)/n\}$ is decreasing and has limit zero, the original series converges conditionally (by the alternating series test).

14. Consider the series $\sum_{n=1}^{\infty} \left|\frac{\cos(n\pi/10)\text{Cot}^{-1}n}{n^3+5n}\right|$. Since $\left|\frac{\cos(n\pi/10)\text{Cot}^{-1}n}{n^3+5n}\right| < \frac{\pi/4}{n^3}$, and the series $\sum_{n=1}^{\infty} \frac{\pi/4}{n^3} = \frac{\pi}{4}\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, so also does the series of absolute values (by the comparison test). The original series therefore converges absolutely.

15. Since the radius of convergence of the series is $\lim_{n\to\infty} \left|\frac{1/n}{1/(n+1)}\right| = 1$, the open interval of convergence is $-1 < x < 1$. At $x = 1$, the power series reduces to the harmonic series which diverges. At $x = -1$, it reduces to the negative of the alternating harmonic series which converges conditionally. The interval of convergence is therefore $-1 \leq x < 1$.

16. With radius of convergence $\lim_{n\to\infty} \left|\frac{1/(n+1)^2}{1/(n+2)^2}\right| = 1$, the open interval of convergence is $-1 < x < 1$. At $x = 1$, the power series reduces to a p-series $\sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges. At $x = -1$, it reduces to $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2}$ which converges absolutely. The interval of convergence is therefore $-1 \leq x \leq 1$.

17. Since the radius of convergence of the series is $\lim_{n\to\infty} \left|\frac{\frac{1}{n2^n}}{\frac{1}{(n+1)2^{n+1}}}\right| = 2$, the open interval of convergence is $-1 < x < 3$. At $x = 3$, the power series reduces to the harmonic series which diverges. At $x = -1$, it reduces to the negative of the alternating harmonic series which converges conditionally. The interval of convergence is therefore $-1 \leq x < 3$.

18. With radius of convergence $\lim_{n\to\infty}\left|\dfrac{1/\sqrt{n}}{1/\sqrt{n+1}}\right| = 1$, the open interval of convergence is $-3 < x < -1$. At $x = -1$, the power series reduces to a p-series $\sum_{n=1}^{\infty}\dfrac{1}{\sqrt{n}}$ which diverges. At $x = -3$, it reduces to $\sum_{n=1}^{\infty}\dfrac{(-1)^n}{\sqrt{n}}$ which converges conditionally. The interval of convergence is therefore $-3 \le x < -1$.

19. Since the radius of convergence of the series is $\lim_{n\to\infty}\left|\dfrac{\dfrac{(n-1)2^n}{n^2+1}}{\dfrac{n2^{n+1}}{(n+1)^2+1}}\right| = \dfrac{1}{2}$, the open interval of convergence is $-1/2 < x < 1/2$. At $x = 1/2$, the power series reduces to $\sum_{n=0}^{\infty}\dfrac{n-1}{n^2+1}$. Since $l = \lim_{n\to\infty}\dfrac{\dfrac{n-1}{n^2+1}}{\dfrac{1}{n}} = 1$, and $\sum_{n=1}^{\infty}\dfrac{1}{n}$ diverges, the series diverges (by the limit comparison test). At $x = -1/2$, the power series reduces to $\sum_{n=0}^{\infty}\dfrac{(n-1)(-1)^n}{n^2+1}$. This series does not converge absolutely. Since the sequence $\{(n-1)/(n^2+1)\}$ is decreasing for $n \ge 2$ and has limit zero, it follows that the series $\sum_{n=0}^{\infty}\dfrac{(n-1)(-1)^n}{n^2+1}$ converges conditionally. The interval of convergence is therefore $-1/2 \le x < 1/2$.

20. If we set $y = x^3$, the series becomes $\sum_{n=0}^{\infty}\dfrac{1}{\sqrt{n+1}}x^{3n+1} = y^{1/3}\sum_{n=0}^{\infty}\dfrac{1}{\sqrt{n+1}}y^n$. Since the radius of convergence of this series is $R_y = \lim_{n\to\infty}\left|\dfrac{1/\sqrt{n+1}}{1/\sqrt{n+2}}\right| = 1$, it follows that $R_x = \sqrt{R_y} = 1$ also. The open interval of convergence is therefore $-1 < x < 1$. At $x = 1$, the power series reduces to a p-series $\sum_{n=0}^{\infty}\dfrac{1}{\sqrt{n+1}} = \sum_{n=1}^{\infty}\dfrac{1}{\sqrt{n}}$ which diverges. At $x = -1$, it reduces to $\sum_{n=0}^{\infty}\dfrac{(-1)^{3n+1}}{\sqrt{n+1}} = \sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{\sqrt{n}}$. This series does not converge absolutely but it does converge conditionally. Hence, the interval of convergence is $-1 \le x < 1$.

21. Since the radius of convergence of the series is $\lim_{n\to\infty}\left|\dfrac{1/\ln n}{1/\ln(n+1)}\right| = \lim_{n\to\infty}\dfrac{\ln(n+1)}{\ln n} = \lim_{n\to\infty}\dfrac{1/(n+1)}{1/n} = 1$, the open interval of convergence is $-1 < x < 1$. At $x = 1$, the power series reduces to $\sum_{n=2}^{\infty}\dfrac{1}{\ln n}$. This series diverges by the comparison test since $1/\ln n > 1/n$. At $x = -1$, the power series reduces to $\sum_{n=2}^{\infty}\dfrac{(-1)^n}{\ln n}$ which converges conditionally. The interval of convergence is therefore $-1 \le x < 1$.

22. With radius of convergence $\lim_{n\to\infty}\left|\dfrac{\dfrac{1}{n^2\ln n}}{\dfrac{1}{(n+1)^2\ln(n+1)}}\right| = \lim_{n\to\infty}\dfrac{\ln(n+1)}{\ln n} = \lim_{n\to\infty}\dfrac{1/(n+1)}{1/n} = 1$, the open interval of convergence of the series is $1 < x < 3$. At $x = 3$, the power series reduces to $\sum_{n=2}^{\infty}\dfrac{1}{n^2\ln n}$. This series converges by the comparison test since $1/(n^2\ln n) < 1/n^2$ for $n \ge 3$. At $x = 1$, the power series reduces to $\sum_{n=2}^{\infty}\dfrac{(-1)^n}{n^2\ln n}$ which converges absolutely. The interval of convergence is therefore $1 \le x \le 3$.

23. Consider the series $\displaystyle\sum_{n=1}^{\infty}\frac{|\sin(nx)|}{n^2}$. Since $|\sin(nx)/n^2| < 1/n^2$, the series converges (by the comparison test). Hence, the original series converges absolutely for all x.

24. If $\sum_{n=1}^{\infty} c_n$ converges absolutely, then $\sum_{n=1}^{\infty}|c_n|$ converges, and then $\lim_{n\to\infty}|c_n| = 0$. It follows that for n greater than or equal to some integer N, $0 < |c_n| < 1$. For such n, we have $|c_n|^p < |c_n|$. Consequently, $\sum_{n=N}^{\infty}|c_n^p| = \sum_{n=N}^{\infty}|c_n|^p$ converges (by the comparison test). Hence, $\sum_{n=1}^{\infty}|c_n|^p$ converges also.

25. Consider the series $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{(n+1)^{n+1}}$. Since $l = \displaystyle\lim_{n\to\infty}\frac{\dfrac{n^n}{(n+1)^{n+1}}}{\dfrac{1}{en}} = e\lim_{n\to\infty}\left(\frac{n}{n+1}\right)\left(\frac{n}{n+1}\right)^n = 1$, and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{en} = \frac{1}{e}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, so also does the series $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{(n+1)^{n+1}}$ (by the limit comparison test). Since the sequence $\{n^n/(n+1)^{n+1}\}$ is decreasing and has limit 0 (it looks like $1/(en)$ for large n), the original series converges conditionally.

EXERCISES 10.13

1. We obtain this result by setting $x = 2$ in the Maclaurin series for e^x (see Example 10.10).

2. We obtain this result by setting $x = 1$ in the Maclaurin series for $\sin x$ (see Example 10.9).

3. We obtain this result by setting $x = 3$ in the Maclaurin series for $\cos x$ (see Example 10.21).

4. If we set $x = -1$ in the Maclaurin series for e^x (Example 10.10),

$$e^{-1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!} = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!} \qquad \Longrightarrow \qquad \frac{1}{e} - 1 = \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}.$$

5. This is a geometric series with sum

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{2^{2n}} = \frac{-1/4}{1+1/4} = -\frac{1}{5}.$$

6. If we set $x = 2$ in the Maclaurin series for $\cos x$ (see example 10.21),

$$\cos 2 = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}2^{2n} = 1 - \frac{2^2}{2!} + \sum_{n=2}^{\infty}\frac{(-1)^n}{(2n)!}2^{2n}.$$

Consequently,

$$\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{(2n)!}2^{2n+3} = -8(\cos 2 - 1 + 2) = -8(1 + \cos 2).$$

7. If we set $x = 1/3$ in the Taylor series for $\ln x$ about $x = 1$ (see Example 10.22),

$$\ln(1/3) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(-\frac{2}{3}\right)^n = \sum_{n=1}^{\infty}\frac{-2^n}{n3^n} \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty}\frac{2^n}{n3^n} = -\ln(1/3) = \ln 3.$$

8. If we set $x = 1/2$ in the Taylor series for $\ln x$ about $x = 1$ (see Example 10.22),

$$\ln(1/2) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty}\frac{-1}{n2^n} \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty}\frac{1}{n2^n} = -\ln(1/2) = \ln 2.$$

9. If we set $x = 1/3$ in the Maclaurin series for $\sin x$ (see example 10.9),

$$\sin(1/3) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{3}\right)^{2n+1} = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n}{3^{2n+1}(2n+1)!}.$$

Consequently,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{3^{2n}(2n+1)!} = 3\left[\sin\left(\frac{1}{3}\right) - \frac{1}{3}\right] = 3\sin\left(\frac{1}{3}\right) - 1.$$

10. The series $\sum_{n=1}^{\infty} nx^n$ converges for $-1 < x < 1$. If we set $S(x) = \sum_{n=1}^{\infty} nx^n$, then $x^{-1}S(x) = \sum_{n=1}^{\infty} nx^{n-1}$. Term-by-term integration gives $\int \frac{S(x)}{x}dx + C = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$, since the series is geometric. We now differentiate with respect to x, getting $\frac{S(x)}{x} = \frac{(1-x)(1) - x(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}$.
Hence, $S(x) = \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$, and if we set $x = 1/2$, we obtain $\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1-1/2)^2} = 2$.

11. If we integrate the geometric series $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$, which converges for $-1 < x < 1$, term-by term,

$$\mathrm{Tan}^{-1}x + C = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots .$$

Substitution of $x = 0$ yields $C = 0$, and therefore $\mathrm{Tan}^{-1}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1}$. The open interval of convergence of this series is $-1 < x < 1$. At $x = \pm 1$, the Taylor series reduces to $\pm\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$, which converge conditionally. According to Theorem 10.20, we may therefore write that

$$\mathrm{Tan}^{-1}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1}, \quad -1 \le x \le 1.$$

When we set $x = 1$,

$$\mathrm{Tan}^{-1}(1) = \frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} \quad \Longrightarrow \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4} - 1.$$

12. Term-by-term differentiation of $\frac{1}{1+x^2} = 1 - x^2 + x^4 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n}$, $|x| < 1$ gives

$$\frac{-2x}{(1+x^2)^2} = \sum_{n=0}^{\infty} 2n(-1)^n x^{2n-1} \quad \Longrightarrow \quad \frac{x}{(1+x^2)^2} = \sum_{n=1}^{\infty} n(-1)^{n-1}x^{2n-1}.$$

If we set $x = 1/3$, then $\frac{1/3}{(1+1/9)^2} = \sum_{n=1}^{\infty} n(-1)^{n-1}\frac{1}{3^{2n-1}}$. Hence

$$\sum_{n=1}^{\infty} \frac{n(-1)^n}{3^{2n}} = -\frac{1}{3}\left[\frac{1/3}{(1+1/9)^2}\right] = -\frac{9}{100}.$$

13. The sum of the first ten terms is $\sum_{n=2}^{11} \frac{n^2}{(n^3+1)^4} = 0.000\,625\,322$. According to expression 10.48, the error in approximating the sum of this series with this value is less than

$$\int_{11}^{\infty} \frac{x^2}{(x^3+1)^4}\,dx = \left\{\frac{1}{-9(x^3+1)^3}\right\}_{11}^{\infty} = \frac{1}{9(12)^3}.$$

14. The sum of the first five terms is $\sum_{n=1}^{5} \frac{n}{e^{3n}} = 0.055\,140\,9$. According to expression 10.48, the error in approximating the sum of this series with this value is less than

$$\int_{5}^{\infty} xe^{-3x}\,dx = \left\{-\frac{x}{3}e^{-3x} - \frac{1}{9}e^{-3x}\right\}_{5}^{\infty} = \frac{5}{3}e^{-15} + \frac{1}{9}e^{-15} = \frac{16}{9}e^{-15} < 6 \times 10^{-7}.$$

15. According to expression 10.48, the error in approximating the sum of this series after the 100th term is less than

$$\int_{100}^{\infty} \frac{1}{x^2+1}\,dx = \left\{\mathrm{Tan}^{-1}x\right\}_{100}^{\infty} = \frac{\pi}{2} - \mathrm{Tan}^{-1}(100) < 0.01.$$

16. According to expression 10.48, the error in approximating the sum of this series after the 20th term is less than

$$\int_{20}^{\infty} \frac{1}{x^2}\sin\left(\frac{1}{x}\right)dx = \left\{\cos\left(\frac{1}{x}\right)\right\}_{20}^{\infty} = 1 - \cos\left(\frac{1}{20}\right) < 0.001\,25.$$

17. The sum of the first 3 three terms is $-0.012\,710$. Since the series is alternating and absolute values of the terms are decreasing and have limit zero, the maximum error is the absolute value of the 4th term, $1/(5^3 \cdot 3^5) < 3.3 \times 10^{-5}$.

18. The sum of the first 20 terms is $-0.947\,030$. Since the series is alternating and absolute values of the terms are decreasing and have limit zero, the maximum error is the absolute value of the 21st term, $1/21^4 < 5.2 \times 10^{-6}$.

19. The sum of the first 5 terms is 1.291 26. The error in using this to approximate the sum of the series is

$$\sum_{n=6}^{\infty} \frac{1}{n^n} = \frac{1}{6^6} + \frac{1}{7^7} + \frac{1}{8^8} + \cdots < \frac{1}{6^6} + \frac{1}{6^7} + \frac{1}{6^8} + \cdots = \frac{1/6^6}{1-1/6} = \frac{1}{5(6^6)}.$$

20. The sum of the first 10 terms is 0.693 065. The error in using this to approximate the sum of the series is

$$\begin{aligned}\sum_{n=11}^{\infty} \frac{1}{n2^n} &= \frac{1}{11\cdot 2^{11}} + \frac{1}{12\cdot 2^{12}} + \frac{1}{13\cdot 2^{13}} + \cdots \\ &< \frac{1}{11\cdot 2^{11}} + \frac{1}{11\cdot 2^{12}} + \frac{1}{11\cdot 2^{13}} + \cdots = \frac{1/(11\cdot 2^{11})}{1-1/2} = \frac{1}{11\cdot 2^{10}} < 9 \times 10^{-5}.\end{aligned}$$

21. The sum of the first fifteen terms is 0.434 732. The error in using this approximation for the sum of the series is

$$\begin{aligned}\sum_{n=16}^{\infty} \frac{1}{2^n}\sin\left(\frac{\pi}{n}\right) &= \frac{1}{2^{16}}\sin\left(\frac{\pi}{16}\right) + \frac{1}{2^{17}}\sin\left(\frac{\pi}{17}\right) + \frac{1}{2^{18}}\sin\left(\frac{\pi}{18}\right) + \cdots \\ &< \frac{1}{2^{16}}\sin\left(\frac{\pi}{16}\right) + \frac{1}{2^{17}}\sin\left(\frac{\pi}{16}\right) + \frac{1}{2^{18}}\sin\left(\frac{\pi}{16}\right) + \cdots \\ &= \frac{(1/2^{16})\sin(\pi/16)}{1-1/2} = \frac{1}{2^{15}}\sin\left(\frac{\pi}{16}\right) < 6 \times 10^{-6}.\end{aligned}$$

22. The sum of the first 20 terms is 1.067 49. The error in using this to approximate the sum of the series is

$$\sum_{n=22}^{\infty} \frac{2^n - 1}{3^n + n} = \frac{2^{22} - 1}{3^{22} + 22} + \frac{2^{23} - 1}{3^{23} + 23} + \cdots < \frac{2^{22}}{3^{22}} + \frac{2^{23}}{3^{23}} + \cdots = \frac{(2/3)^{22}}{1 - 2/3} < 4.01 \times 10^{-4}.$$

23. The sum of the first 20 terms is 1.351 66. The error in using this to approximate the sum of the series is

$$\sum_{n=22}^{\infty} \frac{2^n + 1}{3^n + n} = \frac{2^{22} + 1}{3^{22} + 22} + \frac{2^{23} + 1}{3^{23} + 23} + \frac{2^{24} + 1}{3^{24} + 24} + \cdots < \frac{2^{22} + 1}{3^{22}} + \frac{2^{23} + 1}{3^{23}} + \frac{2^{24} + 1}{3^{24}} + \cdots$$

$$= \left[\left(\frac{2}{3}\right)^{22} + \left(\frac{2}{3}\right)^{23} + \left(\frac{2}{3}\right)^{24} + \cdots\right] + \left(\frac{1}{3^{22}} + \frac{1}{3^{23}} + \frac{1}{3^{24}} + \cdots\right)$$

$$= \frac{(2/3)^{22}}{1 - 2/3} + \frac{1/3^{22}}{1 - 1/3} < 4.01 \times 10^{-4}.$$

24. The sum of the first 100 terms is $-0.688\,172\,2$. Since the series is alternating and absolute values of the terms are decreasing with limit zero, the maximum error is the absolute value of the 101^{st} term, $1/101$.

25. When this alternating series (with absolute values of terms decreasing and approaching zero) is truncated after N terms, the maximum error is the absolute value of the next term, $1/(N+1)^2$. It is less than 10^{-4} if $1/(N+1)^2 < 10^{-4}$. This occurs for $N \geq 100$.

26. When this series is truncated after N terms, the error is

$$\sum_{n=N+1}^{\infty} \frac{1}{n^2 4^n} = \frac{1}{(N+1)^2 4^{N+1}} + \frac{1}{(N+2)^2 4^{N+2}} + \cdots < \frac{1}{(N+1)^2 4^{N+1}} + \frac{1}{(N+1)^2 4^{N+2}} + \cdots$$

$$= \frac{1}{(N+1)^2 4^{N+1}} \left(\frac{1}{1 - 1/4}\right) = \frac{1}{3(N+1)^2 4^N}.$$

This quantity is less than 10^{-4} if $3(N+1)^2 4^N > 10^4$. This occurs for $N \geq 4$.

27. When this series is truncated after N terms, the error is

$$\sum_{n=N+1}^{\infty} \frac{2^n}{n!} = \frac{2^{N+1}}{(N+1)!} + \frac{2^{N+2}}{(N+2)!} + \frac{2^{N+3}}{(N+3)!} + \cdots$$

$$< \frac{2^{N+1}}{(N+1)!} + \frac{2^{N+2}}{(N+1)!(N+1)} + \frac{2^{N+3}}{(N+1)!(N+1)^2} + \cdots = \frac{2^{N+1}/(N+1)!}{1 - 2/(N+1)}.$$

This is less than 10^{-4} when $N \geq 10$.

28. (a) Using 10.48, the error is less than $\displaystyle\int_{10}^{\infty} e^{-x} \sin^2 x\, dx$. By writing the integrand in the form $e^{-x}(1 - \cos 2x)/2$, and using integration by parts, we obtain the following antiderivative,

$$\int_{10}^{\infty} e^{-x} \sin^2 x\, dx = \left\{-\frac{1}{5} e^{-x}(2 + \sin^2 x + 2 \sin x \cos x)\right\}_{10}^{\infty}$$

$$= \frac{e^{-10}}{5}(2 + \sin^2 10 + 2 \sin 10 \cos 10) < 2.92 \times 10^{-5}.$$

(b) The error in using ten terms to approximate the sum is

$$\sum_{n=11}^{\infty} e^{-n} \sin^2 n = e^{-11} \sin^2 11 + e^{-12} \sin^2 12 + e^{-13} \sin^2 13 + \cdots$$

$$< e^{-11} + e^{-12} + e^{-13} + \cdots = \frac{e^{-11}}{1 - 1/e} < 2.65 \times 10^{-5}.$$

The error in part (b) is better.

29. $$\int_0^1 \frac{\sin x}{x}\,dx = \int_0^1 \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) dx = \int_0^1 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right) dx$$
$$= \left\{x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \cdots\right\}_0^1 = 1 - \frac{1}{3\cdot 3!} + \frac{1}{5\cdot 5!} - \frac{1}{7\cdot 7!} + \cdots .$$

This is a convergent alternating series. To find a three-decimal approximation, we calculate partial sums until two successive sums agree to three decimals:

$$S_1 = 1, \qquad S_2 = S_1 - \frac{1}{3\cdot 3!} = 0.944\,44,$$
$$S_3 = S_2 + \frac{1}{5\cdot 5!} = 0.946\,11, \qquad S_4 = S_3 - \frac{1}{7\cdot 7!} = 0.946\,08.$$

Consequently, to three decimals the value of the integral is 0.946.

30. $$\int_0^{1/2} \cos(x^2)\,dx = \int_0^{1/2}\left[1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \cdots\right] dx = \int_0^{1/2}\left[1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots\right] dx$$
$$= \left\{x - \frac{x^5}{5\cdot 2!} + \frac{x^9}{9\cdot 4!} - \frac{x^{13}}{13\cdot 6!} + \cdots\right\}_0^{1/2} = \frac{1}{2} - \frac{1}{5\cdot 2^5\cdot 2!} + \frac{1}{9\cdot 2^9\cdot 4!} - \frac{1}{13\cdot 2^{13}\cdot 6!} + \cdots .$$

This is a convergent alternating series. To find a three-decimal approximation, we calculate partial sums until two successive sums agree to three decimals:

$$S_1 = \frac{1}{2}, \qquad S_2 = S_1 - \frac{1}{5\cdot 2^5\cdot 2!} = 0.496\,78, \qquad S_3 = S_2 + \frac{1}{9\cdot 2^9\cdot 4!} = 0.496\,88.$$

Consequently, to three decimals the value of the integral is 0.497.

31. $$\int_0^{2/3} \frac{1}{x^4+1}\,dx = \int_0^{2/3} (1 - x^4 + x^8 - x^{12} + \cdots)\,dx = \left\{x - \frac{x^5}{5} + \frac{x^9}{9} - \frac{x^{13}}{13} + \cdots\right\}_0^{2/3}$$
$$= \frac{2}{3} - \frac{(2/3)^5}{5} + \frac{(2/3)^9}{9} - \frac{(2/3)^{13}}{13} + \cdots .$$

This is a convergent alternating series. To find a three-decimal approximation, we calculate partial sums until two successive sums agree to three decimals:

$$S_1 = \frac{2}{3}, \qquad S_2 = S_1 - \frac{(2/3)^5}{5} = 0.640\,32,$$
$$S_3 = S_2 + \frac{(2/3)^9}{9} = 0.643\,22, \qquad S_4 = S_3 - \frac{(2/3)^{13}}{13} = 0.642\,82.$$

Consequently, to three decimals the value of the integral is 0.643.

32. $$\int_{-1}^1 x^{11}\sin x\,dx = 2\int_0^1 x^{11}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) dx = 2\int_0^1 \left(x^{12} - \frac{x^{14}}{3!} + \frac{x^{16}}{5!} - \cdots\right) dx$$
$$= 2\left\{\frac{x^{13}}{13} - \frac{x^{15}}{15\cdot 3!} + \frac{x^{17}}{17\cdot 5!} - \cdots\right\}_0^1 = 2\left(\frac{1}{13} - \frac{1}{15\cdot 3!} + \frac{1}{17\cdot 5!} - \frac{1}{19\cdot 7!} + \cdots\right).$$

This is a convergent alternating series. To find a three-decimal approximation, we calculate partial sums until two successive sums agree to three decimals:

$$S_1 = \frac{2}{13}, \qquad S_2 = S_1 - \frac{2}{15\cdot 3!} = 0.131\,62,$$
$$S_3 = S_2 + \frac{2}{17\cdot 5!} = 0.132\,60, \qquad S_4 = S_3 - \frac{2}{19\cdot 7!} = 0.132\,58.$$

Consequently, to three decimals the value of the integral is 0.133.

33. $$\int_0^{1/2} \frac{1}{\sqrt{1+x^3}}\,dx = \int_0^{1/2}\left[1 - \frac{x^3}{2} + \frac{(-1/2)(-3/2)}{2!}(x^3)^2 + \frac{(-1/2)(-3/2)(-5/2)}{3!}(x^3)^3 + \cdots\right] dx$$
$$= \int_0^{1/2}\left(1 - \frac{x^3}{2} + \frac{3x^6}{2^2\cdot 2!} - \frac{3\cdot 5x^9}{2^3\cdot 3!} + \cdots\right) dx$$
$$= \left\{x - \frac{x^4}{4\cdot 2} + \frac{3x^7}{7\cdot 2^2\cdot 2!} - \frac{3\cdot 5x^{10}}{10\cdot 2^3\cdot 3!} + \cdots\right\}_0^{1/2}$$

$$= \frac{1}{2} - \frac{1}{4 \cdot 2 \cdot 2^4} + \frac{3}{7 \cdot 2^2 \cdot 2! \cdot 2^7} - \frac{3 \cdot 5}{10 \cdot 2^3 \cdot 3! \cdot 2^{10}} + \cdots .$$

This is a convergent alternating series. To find a three-decimal approximation, we calculate partial sums until two successive sums agree to three decimals:

$$S_1 = \frac{1}{2}, \qquad S_2 = S_1 - \frac{1}{4 \cdot 2 \cdot 2^4} = 0.492\,19,$$

$$S_3 = S_2 + \frac{3}{7 \cdot 2^2 \cdot 2! \cdot 2^7} = 0.492\,61, \qquad S_4 = S_3 - \frac{3 \cdot 5}{10 \cdot 2^3 \cdot 3! \cdot 2^{10}} = 0.492\,58.$$

Consequently, to three decimals the value of the integral is 0.493.

34. $$\int_0^{0.3} e^{-x^2}\,dx = \int_0^{0.3}\left[1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots\right] dx = \int_0^{0.3}\left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right] dx$$

$$= \left\{x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots\right\}_0^{0.3} = 0.3 - \frac{(0.3)^3}{3} + \frac{(0.3)^5}{5 \cdot 2!} - \frac{(0.3)^7}{7 \cdot 3!} + \cdots .$$

This is a convergent alternating series. To find a three-decimal approximation, we calculate partial sums until two successive sums agree to three decimals:

$$S_1 = 0.3, \qquad S_2 = S_1 - \frac{(0.3)^3}{3} = 0.291\,00, \qquad S_3 = S_2 + \frac{(0.3)^5}{5 \cdot 2!} = 0.291\,24.$$

Consequently, to three decimals the value of the integral is 0.291.

35. Using the series from Example 10.23,

$$\int_{-0.1}^{0} \frac{1}{x-1}\ln(1-x)\,dx = \int_{-0.1}^{0}\left[x + \left(1 + \frac{1}{2}\right)x^2 + \left(1 + \frac{1}{2} + \frac{1}{3}\right)x^3 + \cdots\right] dx$$

$$= \left\{\frac{x^2}{2} + \left(1 + \frac{1}{2}\right)\frac{x^3}{3} + \left(1 + \frac{1}{2} + \frac{1}{3}\right)\frac{x^4}{4} + \cdots\right\}_{-0.1}^{0}$$

$$= -\frac{1}{2 \cdot 10^2} + \left(1 + \frac{1}{2}\right)\frac{1}{3 \cdot 10^3} - \left(1 + \frac{1}{2} + \frac{1}{3}\right)\frac{1}{4 \cdot 10^4} + \cdots .$$

This is a convergent alternating series. To find a three-decimal approximation, we calculate partial sums until two successive sums agree to three decimals:

$$S_1 = -0.005\,00, \qquad S_2 = S_1 + \left(1 + \frac{1}{2}\right)\frac{1}{3 \cdot 10^3} = -0.004\,50.$$

Consequently, to three decimals the value of the integral is -0.005.

36. $$\int_0^{1/2} \frac{1}{x^6 - 3x^3 - 4}dx = \int_0^{1/2} \frac{1}{(x^3-4)(x^3+1)}dx = \int_0^{1/2}\left(\frac{1/5}{x^3-4} + \frac{-1/5}{x^3+1}\right) dx$$

$$= \frac{1}{5}\int_0^{1/2}\left[\frac{-1}{4(1 - x^3/4)} - \frac{1}{1+x^3}\right] dx$$

$$= -\frac{1}{5}\int_0^{1/2}\left[\frac{1}{4}\left(1 + \frac{x^3}{4} + \frac{x^6}{4^2} + \frac{x^9}{4^3} + \cdots\right) + \left(1 - x^3 + x^6 - x^9 + \cdots\right)\right] dx$$

$$= -\frac{1}{5}\int_0^{1/2}\left[\frac{5}{4} - \left(1 - \frac{1}{4^2}\right)x^3 + \left(1 + \frac{1}{4^3}\right)x^6 - \cdots\right] dx$$

$$= -\frac{1}{5}\left\{\frac{5x}{4} - \frac{1}{4}\left(1 - \frac{1}{4^2}\right)x^4 + \frac{1}{7}\left(1 + \frac{1}{4^3}\right)x^7 - \cdots\right\}_0^{1/2}$$

$$= \frac{1}{5}\left[-\frac{5}{4}\left(\frac{1}{2}\right) + \frac{1}{4}\left(1 - \frac{1}{4^2}\right)\left(\frac{1}{2}\right)^4 - \frac{1}{7}\left(1 + \frac{1}{4^3}\right)\left(\frac{1}{2}\right)^7 + \cdots\right].$$

This is a convergent alternating series. To find a three-decimal approximation, we calculate partial sums until two successive sums agree to three decimals:

$$S_1 = -\frac{1}{8}, \quad S_2 = S_1 + \frac{1}{20 \cdot 2^4}\left(1 - \frac{1}{4^2}\right) = -0.122\,07, \quad S_3 = S_2 - \frac{1}{35 \cdot 2^7}\left(1 + \frac{1}{4^3}\right) = -0.122\,30.$$

Consequently, to three decimals the value of the integral is -0.122.

37. $$\int_0^{2\pi} \frac{1-\cos\theta}{\theta} d\theta = \int_0^{2\pi} \frac{1}{\theta}\left[1 - \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right)\right] d\theta = \int_0^{2\pi} \left[\frac{\theta}{2!} - \frac{\theta^3}{4!} + \frac{\theta^5}{6!} - \cdots\right] d\theta$$
$$= \left\{\frac{\theta^2}{2 \cdot 2!} - \frac{\theta^4}{4 \cdot 4!} + \frac{\theta^6}{6 \cdot 6!} - \cdots\right\}_0^{2\pi} = \frac{(2\pi)^2}{2 \cdot 2!} - \frac{(2\pi)^4}{4 \cdot 4!} + \frac{(2\pi)^6}{6 \cdot 6!} - \cdots$$

This is a convergent alternating series. To find a two-decimal approximation, we calculate partial sums until two successive sums agree to two decimals:

$$S_1 = \frac{(2\pi)^2}{2 \cdot 2!} = 9.8696, \qquad S_2 = S_1 - \frac{(2\pi)^4}{4 \cdot 4!} = -6.3652, \qquad S_3 = S_2 + \frac{(2\pi)^6}{6 \cdot 6!} = 7.7876,$$
$$S_4 = S_3 - \frac{(2\pi)^8}{8 \cdot 8!} = 0.3470, \qquad S_5 = S_4 + \frac{(2\pi)^{10}}{10 \cdot 10!} = 2.9896, \qquad S_6 = S_5 - \frac{(2\pi)^{12}}{12 \cdot 12!} = 2.3310,$$
$$S_7 = S_6 + \frac{(2\pi)^{14}}{14 \cdot 14!} = 2.4534, \qquad S_8 = S_7 - \frac{(2\pi)^{16}}{16 \cdot 16!} = 2.4358, \qquad S_9 = S_8 + \frac{(2\pi)^{18}}{18 \cdot 18!} = 2.4378.$$

Consequently, to two decimals the value of the integral is 2.44.

38. If we replace the integrand by its Maclaurin series, we have

$$\frac{\sqrt{\pi}}{2}\mathrm{erf}(1) = \int_0^1 \left[1 - t^2 + \frac{(-t^2)^2}{2!} + \frac{(-t^2)^3}{3!} + \cdots\right] dt = \left\{t - \frac{t^3}{3} + \frac{t^5}{5 \cdot 2!} - \frac{t^7}{7 \cdot 3!} + \cdots\right\}_0^1$$
$$= 1 - \frac{1}{3} + \frac{1}{5 \cdot 2!} - \frac{1}{7 \cdot 3!} + \frac{1}{9 \cdot 4!} - \cdots.$$

This is a convergent alternating series. To find a three-decimal approximation, we calculate partial sums until two successive sums agree to three decimals:

$$S_1 = 2/\sqrt{\pi} = 1.128\,38, \qquad S_2 = S_1 - (2/\sqrt{\pi})/3 = 0.7552\,25,$$
$$S_3 = S_2 + (2/\sqrt{\pi})/(5 \cdot 2!) = 0.865\,09, \qquad S_4 = S_3 - (2/\sqrt{\pi})/(7 \cdot 3!) = 0.838\,23,$$
$$S_5 = S_4 + (2/\sqrt{\pi})/(9 \cdot 4!) = 0.843\,45, \qquad S_6 = S_5 - (2/\sqrt{\pi})/(11 \cdot 5!) = 0.843\,31.$$

Consequently, to three decimals the value of the integral is 0.843.

39. (a) $S = 3.125\,100\,1 - 0.000\,090\,18\left(\dfrac{1}{1 - 1/10}\right) = 3.124\,999\,9.$

(b) $S_1 = 3.125\,100\,1$

$S_2 = 3.125\,100\,1 - 0.000\,090\,18 = 3.125\,009\,92$

$S_3 = 3.125\,100\,1 - 0.000\,090\,18(1 + 1/10) = 3.125\,000\,902$

$S_4 = 3.125\,100\,1 - 0.000\,090\,18(1 + 1/10 + 1/100) = 3.125\,000\,000\,2$

(c) $E_1 = 3.125\,100\,1 - 3.124\,999\,9 = 0.000\,100\,2$

$E_2 = 3.125\,009\,92 - 3.124\,999\,9 = 0.000\,010\,02$

$E_3 = 3.125\,000\,902 - 3.124\,999\,9 = 0.000\,001\,002$

$E_4 = 3.125\,000\,000\,2 - 3.124\,999\,9 = 0.000\,000\,100\,2$

Clearly the approximations get better as n increases in S_n.

(d) To three decimals, S_1, S_2, S_3, and S_4 predict an approximation of 3.125. To three decimals, S is also rounded to 3.125. On the other hand, to four decimals, S_1 predicts 3.1251, and S_2, S_3, and S_4 predict 3.1250. Rounded to four decimals, S is 3.1250.

REVIEW EXERCISES

1. The first five terms are $-1/10$, $-1/6$, $-3/28$, $-1/40$, and $1/18$. The sequence is not therefore monotonic. Since all further terms are positive, a lower bound is $V = -1$. Because

$$1 - \frac{n^2 - 5n + 3}{n^2 + 5n + 4} = \frac{10n + 1}{n^2 + 5n + 4} > 0,$$

$U = 1$ is an upper bound for the sequence. The limit of the sequence is 1.

2. The first three terms are $c_1 = 1$, $c_2 = 1/\sqrt{2}$, and $c_3 = \sqrt{3/8}$. The sequence appears to be decreasing; that is, $c_{n+1} < c_n$. This is true for $n = 1$. Suppose k is some integer for which $c_{k+1} < c_k$. Then

$${c_{k+1}}^2 + 1 < {c_k}^2 + 1 \qquad \Longrightarrow \qquad \frac{1}{2}\sqrt{{c_{k+1}}^2 + 1} < \frac{1}{2}\sqrt{{c_k}^2 + 1}.$$

In other words, $c_{k+2} < c_{k+1}$, and therefore by mathematical induction, the sequence is decreasing. It follows that $U = c_1 = 1$ is an upper bound, and clearly $V = 0$ is a lower bound. By Theorem 10.7 we conclude that the sequence has a limit L, and to find L we set

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{1}{2}\sqrt{{c_n}^2 + 1}.$$

This equation implies that $L = (1/2)\sqrt{L^2 + 1}$, the solution of which is $L = 1/\sqrt{3}$.

3. Since $\mathrm{Tan}^{-1}(1/n)$ decreases as n increases, and n^2+1 increases, it follows that the sequence is decreasing. The first term $\pi/8$ is an upper bound and 0 is clearly a lower bound. The limit of the sequence is 0.

4. First we note that because the first term of the sequence is 7, and all other terms are at least 15, there is no difficulty with the square root. The first three terms are $c_1 = 7$, $c_2 = 15+\sqrt{5}$, and $c_3 = 15+\sqrt{13 + \sqrt{5}}$. These terms are increasing, $c_3 > c_2 > c_1$. Suppose k is some integer for which $c_{k+1} > c_k$. Then

$$\sqrt{c_{k+1} - 2} > \sqrt{c_k - 2} \qquad \Longrightarrow \qquad 15 + \sqrt{c_{k+1} - 2} > 15 + \sqrt{c_k - 2}.$$

But this means that $c_{k+2} > c_{k+1}$, and therefore by mathematical induction, the sequence is increasing. A lower bound must be $V = c_1 = 7$. The first three terms are less than 100. Suppose k is some integer for which $c_k < 100$. Then $c_{k+1} = 15 + \sqrt{c_k - 2} < 15 + \sqrt{100 - 2} < 100$, and by mathematical induction, an upper bound is $U = 100$. By Theorem 10.7, the sequence has a limit L, and to find L we set

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} (15 + \sqrt{c_n - 2}).$$

This equation implies that $L = 15 + \sqrt{L - 2}$, the solution of which is $L = (31 + \sqrt{53})/2$.

5. The first four terms of the sequence are $c_1 = 6$, $c_2 = 19/3 = 6.333$, $c_3 = 6.316$, and $c_4 = 6.317$. They are oscillating. To show that the entire sequence oscillates, we calculate

$$c_{n+1} - c_n = \left(6 + \frac{2}{c_n}\right) - \left(6 + \frac{2}{c_{n-1}}\right) = -\frac{2(c_n - c_{n-1})}{c_n c_{n-1}}.$$

Since all terms of the sequence are positive, the denominator of this expression is positive. It follows that $c_{n+1} - c_n$ has the opposite sign of $c_n - c_{n-1}$, and the sequence oscillates. To verify properties 2 and 3 of Theorem 10.8, we take absolute values in the above equation,

$$|c_{n+1} - c_n| = \frac{2|c_n - c_{n-1}|}{c_n c_{n-1}}.$$

Since all terms of the sequence are greater than 6 (the recursive definition makes this clear), it follows that

$$|c_{n+1} - c_n| < \frac{2|c_n - c_{n-1}|}{(6)(6)} = \frac{|c_n - c_{n-1}|}{18}.$$

This shows that the $|c_{n+1} - c_n|$ decrease and have limit 0. By Theorem 10.8 the sequence has a limit L that can be obtained by taking limits on both sides of the recursive definition,

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} 6 + \frac{2}{c_n} \quad \Longrightarrow \quad L = 6 + \frac{2}{L}.$$

Of the two solutions $3 \pm \sqrt{11}$ of this equation, only $L = 3 + \sqrt{11}$ is positive.

6. The first four terms of the sequence are $c_1 = 6$, $c_2 = 1/29 = 0.034$, $c_3 = 0.195$, and $c_4 = 0.173$. They are oscillating. To show that the entire sequence oscillates, we calculate

$$c_{n+1} - c_n = \frac{1}{5 + 4c_n} - \frac{1}{5 + 4c_{n-1}} = -\frac{4(c_n - c_{n-1})}{(5 + 4c_n)(5 + 4c_{n-1})}.$$

Since all terms of the sequence are positive, the denominator of this expression is positive. It follows that $c_{n+1} - c_n$ has the opposite sign of $c_n - c_{n-1}$, and the sequence oscillates. To verify properties 2 and 3 of Theorem 10.8, we take absolute values in the above equation,

$$|c_{n+1} - c_n| = \frac{4|c_n - c_{n-1}|}{(5 + 4c_n)(5 + 4c_{n-1})} < \frac{4|c_n - c_{n-1}|}{(5)(5)} = \frac{4}{25}|c_n - c_{n-1}|.$$

This shows that the $|c_{n+1} - c_n|$ decrease and have limit 0. By Theorem 10.8 the sequence has a limit L that can be obtained by taking limits on both sides of the recursive definition,

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{1}{5 + 4c_n} \quad \Longrightarrow \quad L = \frac{1}{5 + 4L}.$$

Of the two solutions $(-5 \pm \sqrt{41})/8$ of this equation, only $L = (\sqrt{41} - 5)/8$ is positive.

7. If we rewrite the equation in the form $f(x) = x^3 + 7x^2 + 6x - 25 = 0$, then, with an initial approximation $x_1 = 1.5$, Newton's iterative procedure defines further approximations by

$$x_{n+1} = x_n - \frac{x_n^3 + 7x_n^2 + 6x_n - 25}{3x_n^2 + 14x_n + 6}.$$

Iteration gives $x_2 = 1.407$, $x_3 = 1.404\,32$, $x_4 = 1.404\,31$, and $x_5 = 1.404\,31$. Since $f(1.404\,305) = -2.7 \times 10^{-4}$ and $f(1.404\,315) = 4.5 \times 10^{-5}$, the root to five decimal places is 1.404 31. The method of successive approximations defines the sequence $x_1 = 1.5$, $x_{n+1} = \left(\dfrac{x_n + 5}{x_n + 4}\right)^2$. Iteration gives $x_2 = 1.40$, $x_3 = 1.405$, $x_4 = 1.404\,3$, $x_5 = 1.404\,31$, and $x_6 = 1.404\,31$. If we define $g(x) = x - (x + 5)^2/(x + 4)^2$, then $g(1.404305) = -9.3 \times 10^{-6}$ and $g(1.404\,315) = 1.5 \times 10^{-6}$. The root is 1.404 31 accurate to five decimal places.

8. When $|k| < 1$, the sequence converges to 0, and when $k = \pm 1$, it converges to 1. It diverges for all other values.

9. The first four terms of the sequence are $c_1 = 1$, $c_2 = \sqrt{2}$, $c_3 = \sqrt{3}$, and $c_4 = \sqrt{4}$. Just the way these were calculated makes it clear that an explicit formula for the terms is $c_n = \sqrt{n}$. This could be proved by mathematical induction.

10. Since $f'(x) = \dfrac{x(1/x) - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2}$, this function is decreasing ($f'(x) \le 0$) for $x \ge e$. It follows that the sequence $\{\ln(n)/n\}$ is decreasing for $n \ge 3$.

11. Since $l = \displaystyle\lim_{n\to\infty} \frac{\dfrac{n^2 - 3n + 2}{n^3 + 4n}}{\dfrac{1}{n}} = 1$, and $\displaystyle\sum_{n=1}^{\infty} 1/n$ diverges, so also does the original series (by the limit comparison test).

12. Since $l = \displaystyle\lim_{n\to\infty} \frac{\dfrac{n^2 + 5n + 3}{n^4 - 2n + 5}}{\dfrac{1}{n^2}} = 1$, and $\displaystyle\sum_{n=1}^{\infty} 1/n^2$ converges, so also does the original series (by the limit comparison test).

13. Since $L = \lim_{n\to\infty} \dfrac{\dfrac{5^{2n+2}}{(n+1)!}}{\dfrac{5^{2n}}{n!}} = \lim_{n\to\infty} \dfrac{25}{n+1} = 0$, the series converges (by the limit ratio test).

14. Since $L = \lim_{n\to\infty} \dfrac{\dfrac{(n+1)^2+3}{(n+1)3^{n+1}}}{\dfrac{n^2+3}{n3^n}} = \dfrac{1}{3}$, the series converges (by the limit ratio test).

15. Since $\dfrac{(\ln n)^2}{\sqrt{n}} > \dfrac{1}{\sqrt{n}}$ for $n \geq 3$, and $\sum_{n=3}^{\infty} 1/\sqrt{n}$ diverges, so also does the original series (by the comparison test).

16. Consider the series of absolute values $\sum_{n=1}^{\infty} \left(\dfrac{n+1}{n^2}\right)$. Since $l = \lim_{n\to\infty} \dfrac{(n+1)/n^2}{1/n} = 1$, and $\sum_{n=1}^{\infty} 1/n$ diverges, the original series does not converge absolutely. Because the sequence $\{(n+1)/n^2\}$ is decreasing and has limit 0, the original series converges conditionally (by the alternating series test).

17. Consider the series of absolute values $\sum_{n=1}^{\infty} \dfrac{n+1}{n^3}$. Since it is the term-by-term addition of the convergent $p = 2$ and $p = 3$ series, it converges also. The original series therefore converges absolutely.

18. Since $\lim_{n\to\infty} \mathrm{Cos}^{-1}(1/n) = \pi/2$, the series diverges (by the n^{th} term test).

19. Since $l = \lim_{n\to\infty} \dfrac{\dfrac{1}{n}\mathrm{Cos}^{-1}\left(\dfrac{1}{n}\right)}{\dfrac{1}{n}} = \dfrac{\pi}{2}$, and $\sum_{n=1}^{\infty} 1/n$ diverges, so also does the original series (by the limit comparison test).

20. Since $l = \lim_{n\to\infty} \dfrac{\dfrac{1}{n^2}\mathrm{Cos}^{-1}(1/n)}{\dfrac{1}{n^2}} = \dfrac{\pi}{2}$, and $\sum_{n=1}^{\infty} 1/n^2$ converges, so also does the original series (by the limit comparison test).

21. Since the series simplifies to $\sum_{n=1}^{\infty} 2^n$, it diverges (by the n^{th} term test).

22. Since $L = \lim_{n\to\infty} \dfrac{\dfrac{3\cdot 6\cdot 9\cdots(3n+3)}{(2n+2)!}}{\dfrac{3\cdot 6\cdot 9\cdots(3n)}{(2n)!}} = \lim_{n\to\infty} \dfrac{3n+3}{(2n+2)(2n+1)} = 0$, the series converges (by the limit ratio test).

23. Since $\lim_{n\to\infty} \sqrt{\dfrac{n^2+1}{n^2+5}} = 1$, the series diverges (by the n^{th} term test).

24. Because $\lim_{n\to\infty} (-1)^{n+1}\left(1 + \dfrac{1}{n}\right)^3$ does not exist, the series diverges (by the n^{th} term test).

25. Consider the series of absolute values $\sum_{n=1}^{\infty} \dfrac{1}{n^2}|\sin n|$. Since $(1/n^2)|\sin n| \leq 1/n^2$ and $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ converges, so also does the series of absolute values (by the comparison test). The original series therefore converges absolutely.

26. Since this is a geometric series with common ratio $10/125 = 2/25$, the series converges.

27. Consider the series of absolute values $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n}$. Since $\dfrac{\ln n}{n} > \dfrac{1}{n}$ for $n \geq 3$, and $\displaystyle\sum_{n=3}^{\infty} 1/n$ diverges, so also does the series of absolute values. The original series does not therefore converge absolutely. Since the sequence $\{(\ln n)/n\}$ is decreasing (for $n \geq 2$) with limit 0, the original series converges conditionally (by the alternating series test).

28. This is a geometric series with common ratio $1/e^{\pi}$, and therefore it converges.

29. Since the series is the sum of two convergent geometric series $\displaystyle\sum_{n=1}^{\infty}(2/3)^n$ and $\displaystyle\sum_{n=1}^{\infty}(1/6)^n$, it must converge.

30. Consider the series of absolute values $\displaystyle\sum_{n=1}^{\infty}\left|\frac{1}{\sqrt{n}}\cos(n\pi)\right| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$. Since this series diverges, the original series does not converge absolutely. Because the sequence $\{1/\sqrt{n}\}$ is decreasing with limit 0, the series converges conditionally (by the alternating series test).

31. Since the radius of convergence of the series is $R = \displaystyle\lim_{n\to\infty}\left|\frac{\dfrac{n+1}{n^2+1}}{\dfrac{n+2}{(n+1)^2+1}}\right| = 1$, the open interval of convergence is $-1 < x < 1$. At $x = 1$, the power series reduces to $\displaystyle\sum_{n=1}^{\infty}\frac{n+1}{n^2+1}$. Since $l = \displaystyle\lim_{n\to\infty}\frac{(n+1)/(n^2+1)}{1/n}$ $= 1$, and the harmonic series diverges, so also does the power series at $x = 1$. At $x = -2$, the power series reduces to $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(n+1)}{n^2+1}$. This series does not converge absolutely, but because the sequence $\{(n+1)/(n^2+1)\}$ is decreasing, with limit 0, the series converges conditionally. The interval of convergence is therefore $-1 \leq x < 1$.

32. Since the radius of convergence of the series is $R = \displaystyle\lim_{n\to\infty}\left|\frac{\dfrac{1}{n^2 2^n}}{\dfrac{1}{(n+1)^2 2^{n+1}}}\right| = 2$, the open interval of convergence is $-2 < x < 2$. At $x = 2$, the power series reduces to $\sum_{n=1}^{\infty} 1/n^2$ which converges. At $x = -2$, it becomes $\sum_{n=1}^{\infty}(-1)^n/n^2$ which converges absolutely. The interval of convergence is therefore $-2 \leq x \leq 2$.

33. Since the radius of convergence of the series is $R = \displaystyle\lim_{n\to\infty}\left|\frac{(n+1)^3}{(n+2)^3}\right| = 1$, the open interval of convergence is $-1 < x < 1$. At $x = 1$, the power series reduces to $\sum_{n=1}^{\infty}(n+1)^3$ which diverges. At $x = -1$, it becomes $\sum_{n=1}^{\infty}(-1)^n(n+1)^3$ which also diverges. The interval of convergence is therefore $-1 < x < 1$.

34. Since $R = \displaystyle\lim_{n\to\infty}\frac{1}{\sqrt[n]{1/n^n}} = \infty$, the series converges for all x.

35. Since this is a geometric series with common ratio $(x-2)/4$ it converges for $|(x-2)/4| < 1 \implies -2 < x < 6$.

36. Since the radius of convergence of the series is $R = \displaystyle\lim_{n\to\infty}\left|\frac{\sqrt{\dfrac{n+1}{n-1}}}{\sqrt{\dfrac{n+2}{n}}}\right| = 1$, the open interval of convergence is $-4 < x < -2$. At $x = -2$, the power series reduces to $\sum_{n=2}^{\infty}\sqrt{(n+1)/(n-1)}$ which diverges (by the n^{th} term test). At $x = -4$, it becomes $\sum_{n=2}^{\infty}(-1)^n\sqrt{(n+1)/(n-1)}$ which also diverges. The interval of convergence is therefore $-4 < x < -2$.

37. If we set $y = x^2$, the series becomes $\sum_{n=1}^{\infty} n3^n x^{2n} = \sum_{n=1}^{\infty} n3^n y^n$. Since the radius of convergence of this series is $R_y = \lim_{n\to\infty}\left|\frac{n3^n}{(n+1)3^{n+1}}\right| = \frac{1}{3}$, the radius of convergence of the x-series is $R_x = \sqrt{R_y} = 1/\sqrt{3}$. The open interval of convergence is $-1/\sqrt{3} < x < 1/\sqrt{3}$. At $x = \pm 1/\sqrt{3}$, the power series becomes $\sum_{n=1}^{\infty} n$ which diverges. The interval of convergence is therefore $-1/\sqrt{3} < x < 1/\sqrt{3}$.

38. If we set $y = x^3$, then $\sum_{n=1}^{\infty}\frac{2^n}{n}x^{3n} = \sum_{n=1}^{\infty}\frac{2^n}{n}y^n$. Since $R_y = \lim_{n\to\infty}\left|\frac{\frac{2^n}{n}}{\frac{2^{n+1}}{n+1}}\right| = \frac{1}{2}$, the radius of convergence of the original series is $R_x = {R_y}^{1/3} = 2^{-1/3}$. The open interval of convergence is $-2^{-1/3} < x < 2^{-1/3}$. At $x = 2^{-1/3}$, the power series reduces to the divergent harmonic series. At $x = -2^{-1/3}$, it becomes the alternating harmonic series which converges conditionally. The interval of convergence is therefore $-2^{-1/3} \le x < 2^{-1/3}$.

39. Using the binomial expansion 10.33b,

$$\begin{aligned}
\sqrt{1+x^2} = (1+x^2)^{1/2} &= 1 + \frac{x^2}{2} + \frac{(1/2)(-1/2)}{2!}(x^2)^2 + \frac{(1/2)(-1/2)(-3/2)}{3!}(x^2)^3 + \cdots \\
&= 1 + \frac{x^2}{2} - \frac{1}{2^2\cdot 2!}x^4 + \frac{3}{2^3\cdot 3!}x^6 - \frac{3\cdot 5}{2^4\cdot 4!}x^8 + \cdots \\
&= 1 + \frac{x^2}{2} + \sum_{n=2}^{\infty}\frac{(-1)^{n+1}[1\cdot 3\cdot 5\cdots(2n-3)]}{2^n n!}x^{2n}, \\
&= 1 + \frac{x^2}{2} + \sum_{n=2}^{\infty}\frac{(-1)^{n+1}[1\cdot 2\cdot 3\cdots(2n-3)(2n-2)]}{2^n n![2\cdot 4\cdot 6\cdots(2n-2)]}x^{2n}, \\
&= 1 + \frac{x^2}{2} + \sum_{n=2}^{\infty}\frac{(-1)^{n+1}(2n-2)!}{2^{2n-1}n!(n-1)!}x^{2n} \\
&= 1 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2n-2)!}{2^{2n-1}n!(n-1)!}x^{2n} \quad \text{valid for } -1 \le x \le 1.
\end{aligned}$$

40. $f(x) = e^5 e^x = e^5 \sum_{n=0}^{\infty}\frac{1}{n!}x^n = \sum_{n=0}^{\infty}\frac{e^5}{n!}x^n, \quad -\infty < x < \infty$

41. Since $\cos(x+\pi/4) = (1/\sqrt{2})(\cos x - \sin x)$, we may subtract the Maclaurin series for $\cos x$ and $\sin x$ and multiply by $1/\sqrt{2}$,

$$\cos(x+\pi/4) = \sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{2}(2n)!}x^{2n} + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{\sqrt{2}(2n+1)!}x^{2n+1}, \quad -\infty < x < \infty.$$

42. When we integrate $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$, $|x| < 1$, we obtain

$$\ln|1+x| = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1} + C.$$

Substitution of $x = 0$ gives $C = 0$, and therefore $\ln|1+x| = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$. At $x = 1$, the series reduces to $\sum_{n=1}^{\infty}(-1)^{n-1}/n$ which converges conditionally. The interval of convergence is therefore $-1 < x \le 1$. We now replace x by $2x$ and at the same time multiply by x,

$$x\ln(1+2x) = x\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(2x)^n = \sum_{n=1}^{\infty}\frac{2^n(-1)^{n-1}}{n}x^{n+1} = \sum_{n=2}^{\infty}\frac{2^{n-1}(-1)^n}{n-1}x^n,$$

valid for $-1/2 < x \le 1/2$.

43. $\sin x = \sin[(x-\pi/4)+\pi/4] = \dfrac{1}{\sqrt{2}}\sin(x-\pi/4) + \dfrac{1}{\sqrt{2}}\cos(x-\pi/4)$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{2}(2n+1)!}\left(x-\frac{\pi}{4}\right)^{2n+1} + \sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{2}(2n)!}\left(x-\frac{\pi}{4}\right)^{2n}, \quad -\infty < x < \infty$$

44. Partial fractions give $f(x) = \dfrac{x}{x^2+4x+3} = \dfrac{3/2}{x+3} - \dfrac{1/2}{x+1}$. Since

$$\frac{1}{x+1} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n, \quad |x| < 1,$$

and

$$\frac{1}{x+3} = \frac{1}{3(1+x/3)} = \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}x^n, \quad |x| < 3,$$

it follows that

$$f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2\cdot 3^n}x^n + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2}x^n = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2}\left(1-\frac{1}{3^n}\right)x^n, \quad |x| < 1.$$

45. $e^x = e^{3+(x-3)} = e^3\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}(x-3)^n = \sum_{n=0}^{\infty}\frac{e^3}{n!}(x-3)^n, \quad -\infty < x < \infty$

46. According to Exercise 42, $\ln(1+x) = \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$, $-1 < x \le 1$. With this,

$$f(x) = x\ln(1+x) + \ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^{n+1} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$$

$$= \sum_{n=2}^{\infty}\frac{(-1)^n}{n-1}x^n + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n = x + \sum_{n=2}^{\infty}(-1)^{n-1}\left(\frac{-1}{n-1}+\frac{1}{n}\right)x^n$$

$$= x + \sum_{n=2}^{\infty}\frac{(-1)^n}{n(n-1)}x^n, \quad -1 < x \le 1.$$

47. $x^3e^{x^2} = x^3\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}(x^2)^n = \sum_{n=0}^{\infty}\frac{1}{n!}x^{2n+3}, \quad -\infty < x < \infty$

48. $e^{-x^2} = \displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}(-x^2)^n = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots$

When this alternating series is truncated after the term in x^{2n}, the absolute value of the maximum possible error for given x is $|x|^{2n+2}/(n+1)!$. For $0 \le x \le 2$, the error is a maximum when $x = 2$, namely $2^{2n+2}/(n+1)!$. This error is less than 10^{-5} if $(n+1)!/2^{2n+2} > 10^5$. The smallest integer n for which this holds is $n = 18$. The terms that should be used are therefore $e^{-x^2} \approx \displaystyle\sum_{n=0}^{18}\frac{(-1)^n}{n!}x^{2n}$.

49. If we substitute $y = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation,

$$0 = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} -4a_n x^n = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} -4a_n x^n$$
$$= \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - 4a_n]x^n.$$

When we equate coefficients to zero, we obtain the recursive formula

$$a_{n+2} = \frac{4a_n}{(n+2)(n+1)}, \quad n \geq 0.$$

Iteration gives

$$a_2 = \frac{4a_0}{2 \cdot 1} = \frac{4a_0}{2!}, \quad a_4 = \frac{4a_2}{4 \cdot 3} = \frac{4^2 a_0}{4!}, \quad a^6 = \frac{4a_4}{6 \cdot 5} = \frac{4^3 a_0}{6!}, \cdots$$

$$a_3 = \frac{4a_1}{3 \cdot 2} = \frac{4a_1}{3!}, \quad a_5 = \frac{4a_3}{5 \cdot 4} = \frac{4^2 a_1}{5!}, \quad a^7 = \frac{4a_5}{7 \cdot 6} = \frac{4^3 a_1}{7!}, \cdots$$

The solution is therefore

$$y = a_0 \left(1 + \frac{4x^2}{2!} + \frac{4^2 x^4}{4!} + \frac{4^3 x^6}{6!} + \cdots\right) + a_1 \left(x + \frac{4x^3}{3!} + \frac{4^2 x^5}{5!} + \frac{4^3 x^7}{7!} + \cdots\right)$$
$$= a_0 \sum_{n=0}^{\infty} \frac{4^n}{(2n)!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{4^n}{(2n+1)!} x^{2n+1}.$$

Both series converge for $-\infty < x < \infty$.

50.
$$\sqrt{1 + \sin x} = \sqrt{[\cos^2(x/2) + \sin^2(x/2)] + 2\sin(x/2)\cos(x/2)}$$
$$= \sqrt{[\cos(x/2) + \sin(x/2)]^2} = \cos(x/2) + \sin(x/2)$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{x}{2}\right)^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{x}{2}\right)^{2n+1}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}(2n+1)!} x^{2n+1}$$

When x is not confined to $-\pi/2 \leq x \leq \pi/2$, we must write $\sqrt{1 + \sin x} = |\cos(x/2) + \sin(x/2)|$.

51. The numbers obtained are

0.5403, 0.8576, 0.6543 0.7935, 0.7014, 0.7640, 0.7221, 0.7504, 0.7314, 0.7442, 0.7356.

They seem to be converging to some limit. We are actually calculating the numbers in the recursively defined sequence $x_{n+1} = \cos x_n$. This implies that the numbers are converging to the root of the equation $x = \cos x$.