Chapter 3

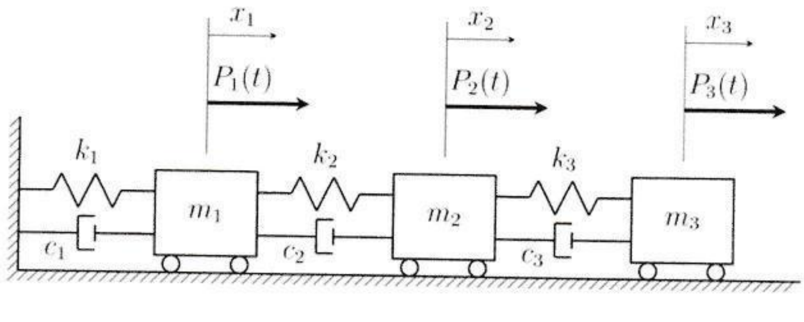
Multi-Degree-of-Freedom Systems

For an accurate description of the displacement configuration of a structure subjected to a dynamic loading, often displacements along more than one coordinate are necessary. Such a system is known as a *multi-degree-of-freedom system*.

We begin by discussing the formulation of the equations of motion for multi-degree-of-freedom (MDOF) systems.

3.1 Formulation of the Equation of Motion for an MDOF System

We begin with an example of a MDOF system with three degrees-of-freedom, as shown in Figure 3.1. Note that the displacements are time-varying, but are simply designated as x rather than $x(t)$ for succinctness.



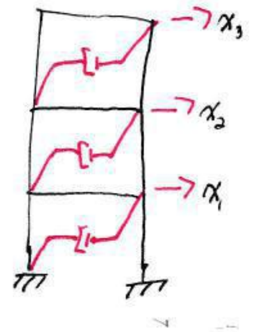
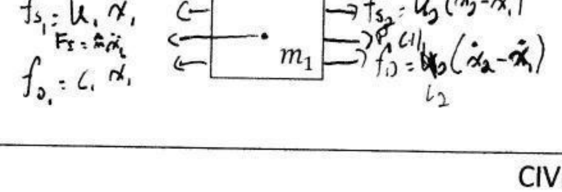


Figure 3.1 Three degree-of-freedom spring-mass-damper model

Example 3.1 Derive the equations of motion for the 3DOF system shown in Figure 3.1.

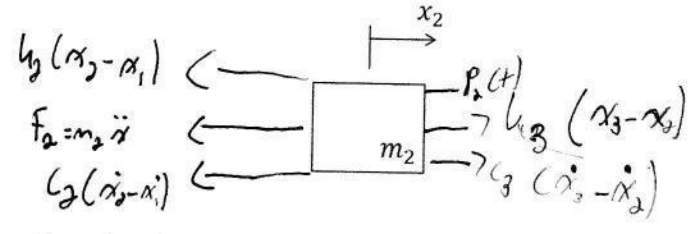
Solution: First, consider the free body diagram of m_1.



The equation of motion is

$$m_1\ddot{x}_1 + k_1 x_1 + c_1\dot{x}_1 + k_2(x_2-x_1) - c_2(\dot{x}_2-\dot{x}_1) = P_1(t) \tag{3.1}$$

Next, consider the free body diagram of m_2.

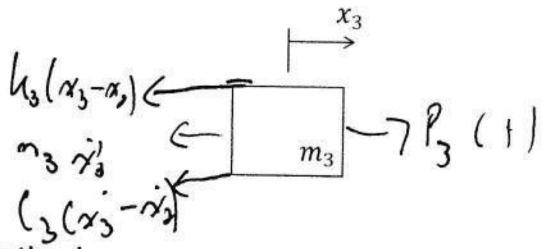


The equation of motion is

$$m_2\ddot{x}_2 + k_2(x_2-x_1) + c_2(\dot{x}_2-\dot{x}_1) - k_3(x_3-x_2) - c_3(\dot{x}_3-\dot{x}_2) = P_2 \tag{3.2}$$

$$m_2\ddot{x}_2 + [c_2+c_3]\dot{x}_2 + [k_2+k_3]x_2 - c_2\dot{x}_1 - k_2x_1 - k_3x_3 - c_3\dot{x}_3 = P_2$$

Finally, consider the free body diagram of m_3.



The equation of motion is

$$m_3\ddot{x}_3 + c_3\dot{x}_3 - c_3\dot{x}_2 + k_3x_3 - k_3x_2 = P_3 \tag{3.3}$$

Equations 3.1 through 3.3 can be written in matrix form as

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\\ \ddot{x}_3\end{bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} \tag{3.4}$$

$$= \begin{bmatrix}P_1(t)\\ P_2(t)\\ P_3(t)\end{bmatrix} \tag{3.5}$$

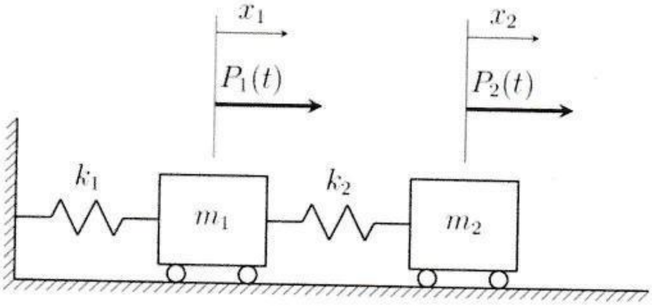
where $\boldsymbol{M}$ is the *mass matrix*, $\boldsymbol{C}$ is the *damping matrix*, and $\boldsymbol{K}$ is the *stiffness matrix*.

$$M\ddot{x} + C\dot{x} + kx = P(t)$$

We can also formulate MDOF equations of motion using an analytical mechanics approach. *Lagrange's Equation* is a powerful method but will not be covered in this course.

3.2 Solution for a 2-DOF System

We will now consider the solution for a 2-DOF system. We begin by finding the equation of motion for a 2-DOF system with no damping.



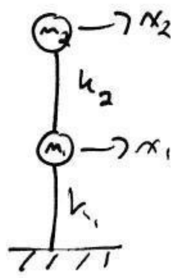
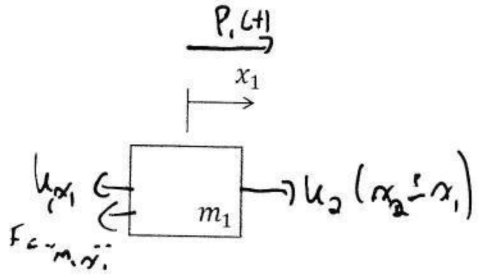


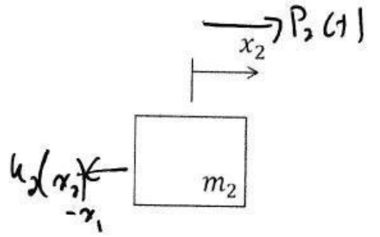
Figure 3.2 Two degree-of-freedom spring-mass model

Using Newtonian Mechanics, we can derive the equation of motion. For mass m_1, the free body diagram is:

using Bluebeam



Similarly, for the mass m_2, the free body diagram is



The equations of motion are

$$P_1(t) = k_1x_1 - k_2(x_2 - x_1) + m_1\ddot{x}_1 \quad , \; m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = P_1 \tag{3.6a}$$

$$P_2(t) = k_2x_2 + m_2\ddot{x}_2 \tag{3.6b}$$

which can be written in matrix form as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix} \tag{3.7}$$

$$M\ddot{x} + Kx = P$$

Several observations can be made from this result.

- Equation 3.7 is uncoupled in the mass terms, but coupled in the stiffness terms.
- Coupling is not an inherent property of the system; rather, it is a consequence of the coordinate system used.
- Both the mass and stiffness matrices are symmetric.

We will now turn to solving this system. Assuming the solution is of the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sin(\omega_n t + \alpha) \quad , \tag{3.8}$$

where X_1 and X_2 are the amplitudes of x_1 and x_2, respectively. As before, ω_n is the circular natural frequency, and α is the phase angle. Substituting Equation 3.8 into Equations 3.6 gives,

$$-m_1\omega_n^2 X_1 \sin(\omega_n t + \alpha) + (k_1 + k_2)X_1 \sin(\omega_n t + \alpha) - k_2 X_2 \sin(\omega_n t + \alpha) = P_1(t) \tag{3.9a}$$

$$-m_2\omega_n^2 X_2 \sin(\omega_n t + \alpha) - k_2 X_1 \sin(\omega_n t + \alpha) + k_2 X_2 \sin(\omega_n t + \alpha) = P_2(t) \tag{3.9b}$$

3.2.1 Free Vibration Response

For the free vibration response ($P_1(t) = P_2(t) = 0$), the solution becomes

$$[-m_1\omega_n^2 X_1 + (k_1 + k_2)X_1 - k_2 X_2] \sin(\omega_n t + \alpha) = 0 \tag{3.10a}$$

Homogeneous system

$$[-m_2\omega_n^2 X_2 - k_2 X_1 + k_2 X_2] \sin(\omega_n t + \alpha) = 0 \tag{3.10b}$$

For a nontrivial solution, X_1 and $X_2 \neq 0$ @ same time

∴ Non invertible, determinate = 0

$$\begin{bmatrix} (k_1+k_2) - m_1\omega_n^2 & -k_2 \\ -k_2 & k_2 - m_2\omega_n^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.11}$$

Therefore,

$$AD - BC = 0 \qquad \begin{vmatrix} A & B \\ C & D \end{vmatrix} \tag{3.12}$$

Expanding the determinant gives

$$\omega_n^4 - \left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right)\omega_n^2 + \frac{k_1 k_2}{m_1 m_2} = 0 \tag{3.13}$$

Equation 3.13 is known as the characteristic equation of the system. The roots are

$$\omega_n^2 = \frac{1}{2}\left[\left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right) \pm \sqrt{\left(\left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right)^2 - 4\frac{k_1 k_2}{m_1 m_2}\right)}\right] \tag{3.14}$$

The roots $\omega_{n,1}$ and $\omega_{n,2}$ are real and positive. The lower of $\omega_{n,1}$ and $\omega_{n,2}$ is known as the first or ***fundamental*** *circular natural frequency* of the system.

∴ has 2 undamped natural frequencies; can never exactly determine amplitudes

From Equation 3.14, there are two roots. Corresponding to each value of ω_n^2, we can determine a ratio of X_2/X_1 in Equation 3.11. However, it is not possible to determine X_1 and X_2 exactly, since Equation 3.11 is homogeneous. That is, if $\boldsymbol{X} = [X_1, X_2]$ is a solution, then $\alpha[X_1, X_2]$ is also a solution, where α is any scalar.

Example 3.2 For $k_1 = k$, $k_2 = 3k/4$, $m_1 = m$, and $m_2 = m/2$, determine the circular natural frequencies and the modes of vibration for the 2-DOF system.

<u>Solution</u>: From Equation 3.14, the circular natural frequencies are computed from

$$\begin{aligned}\omega_n^2 &= \frac{1}{2}\left[\left(\frac{k_1+k_2}{m_1}+\frac{k_2}{m_2}\right) \pm \sqrt{\left(\left(\frac{k_1+k_2}{m_1}+\frac{k_2}{m_2}\right)^2 - 4\frac{k_1 k_2}{m_1 m_2}\right)}\right] \\ &= \frac{1}{2}\frac{k}{m}\left[3.25 \pm \sqrt{(3.25)^2 - 6}\right] \\ &= \frac{1}{2}\frac{k}{m}\left(3.25 \pm \sqrt{4.563}\right)\end{aligned} \tag{3.15}$$

Therefore, the first and second circular natural frequencies are

$$\omega_{n,1} = 0.746\sqrt{k/m}\ \ rad/s \tag{3.16}$$

$$\omega_{n,2} = 1.64\sqrt{k/m}\ \ rad/s \tag{3.17}$$

Recall, $[(k_1+k_2) - m_1\omega_n^2]X_1 - k_2 X_2 = 0$

$$-k_2 X_1 + (k_2 - m_2\omega_n^2)X_2 = 0$$

(3.18a)

(3.18b)

Substituting $\omega_{n,1} = 0.746\sqrt{k/m}$ along with the other relevant quantities gives

$$\beta_1 = \frac{X_2}{X_1} = 1.591 \tag{3.19}$$

Similarly, substituting $\omega_{n,2} = 1.64\sqrt{k/m}$ into Equation 3.18

$$\beta_2 = \frac{X_2}{X_1} = -1.257 \tag{3.20}$$

We now have the mode shape vector for the first and second modes,

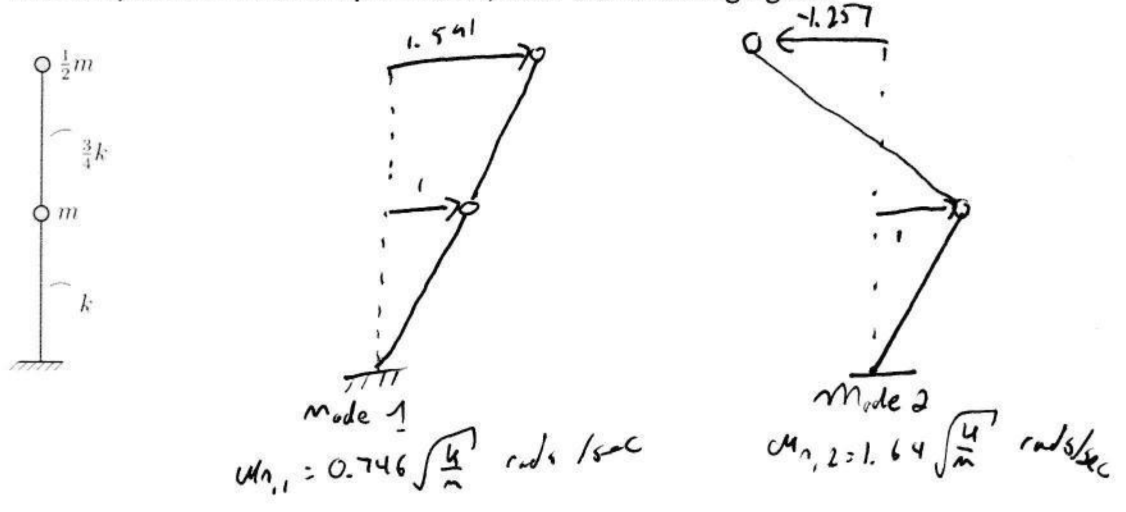
$\phi_1 = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.591 \end{bmatrix}$ (3.21a)

Corresponds to $\omega_{n,1}$

$\phi_2 = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}_2 = \begin{bmatrix} 1 \\ -1.257 \end{bmatrix}$ (3.21b)

Corresponds to $\omega_{n,2}$

We can plot the mode shape vectors, as in the following figure.



We can now write the solution for the 2-DOF system as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A_1 \phi_1 \sin(\omega_{n,1} t + \alpha_1) + A_2 \phi_2 \sin(\omega_{n,2} t + \alpha_2) \tag{3.22}$$

where A_1, A_2, α_1, and α_2 are constants which can be determined from initial conditions. If $X_1 = 1$, then Equation 3.22 can be rewritten as

$$x_1 = A_1 \sin(\omega_{n,1}t + \alpha_1) + A_2 \sin(\omega_{n,2}t + \alpha_2) \tag{3.23a}$$

$$x_2 = A_2\beta_1 \sin(\omega_{n,1}t + \alpha_1) + A_2\beta_2 \sin(\omega_{n,2}t + \alpha_2) \tag{3.23b}$$

or in matrix form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A_1 \begin{bmatrix} 1 \\ \beta_1 \end{bmatrix} \sin(\omega_{n,1} t + \alpha_1) + A_2 \begin{bmatrix} 1 \\ \beta_2 \end{bmatrix} \sin(\omega_{n,2} t + \alpha_2) \tag{3.24}$$

We can write Equation 3.24 in a slightly different form.

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = a_1 \begin{Bmatrix} 1 \\ \beta_1 \end{Bmatrix} \cos \omega_{n,1} t + b_2 \begin{Bmatrix} 1 \\ \beta_2 \end{Bmatrix} \sin \omega_{n,2} t + b_1 \begin{Bmatrix} 1 \\ \beta_1 \end{Bmatrix} \sin \omega_{n,1} t + a_2 \begin{Bmatrix} 1 \\ \beta_2 \end{Bmatrix} \cos \omega_{n,2} t \tag{3.25}$$

where $A_1 \sin \alpha_1 = a_1$, $A_1 \cos \alpha_1 = b_1$, $A_2 \sin \alpha_2 = a_2$, and $A_2 \cos \alpha_2 = b_2$ are constants. Equation 3.25 can be rewritten as

$$\boldsymbol{x} = \sum_{i=1}^{2} \left(a_i \boldsymbol{\phi}_i \cos \omega_{n,i} t + b_i \boldsymbol{\phi}_i \sin \omega_{n,i} t \right) = \sum_{i=1}^{2} \phi_i A_i \sin(\omega_{n,i} t + \alpha_i) \tag{3.26}$$

where

$$\boldsymbol{\phi}_i = \begin{Bmatrix} 1 \\ \beta_i \end{Bmatrix} \tag{3.27}$$

and the constants a_i and b_i can be determined from initial conditions.

- The natural modes and frequencies are intrinsic properties of the system and depend only on the mass and stiffness.
- The modes of a system are the simplest possible cases of undamped free vibration response – one exactly described by simple harmonic motion.
- When an undamped system is vibrating in one of its natural modes, all DOFs are oscillating in harmonic motion with the same natural frequency.
- As we can see from the results in Example 3.2, the general response of an MDOF system is given by a combination of the natural modes. This is a very important observation that will become clear later.

Before proceeding further, let's investigate some important properties of the mass matrix, $\boldsymbol{M}$, and the stiffness matrix, $\boldsymbol{K}$.

Note :

- m and k are symmetric : ∴ $M^T = M$, $k^T = k$
- for system w/o rigid body modes, m & k are positive, definite matrices

$$V = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{K}\boldsymbol{x} > 0 \qquad T = \frac{1}{2}\dot{\boldsymbol{x}}^T\boldsymbol{M}\dot{\boldsymbol{x}} > 0$$

(3.28a) (3.28b)

Kx^2 = potential energy. If system able to store potential energy.

Mx^2 = kinetic energy. If system able to store kinetic energy

3.3 Modes of the System

3.3.1 Background on Eigenvalues and Eigenvectors

- Eigenvalues and eigenvectors play a central role in structural dynamics problems. We discuss the standard eigen problem of the form:

$$Ax = \lambda x \quad , A \text{ is vector, } \lambda \text{ is scalar} \tag{3.29}$$

where, $\boldsymbol{x}$ is a vector of length n and $\boldsymbol{A}$ is a matrix of size $n \times n$. λ is some scalar.

- Let's consider an example[1]. Let:

$$\boldsymbol{A} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \qquad \boldsymbol{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The products:

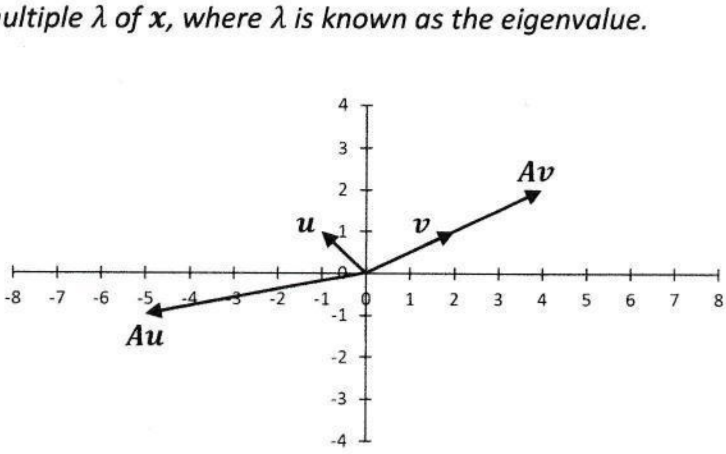
$$Au = \begin{bmatrix} -5 \\ -1 \end{bmatrix}, \quad Av = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

. v is eigenvector, u is not, w/ $\lambda = 2$

- Clearly, the product $\boldsymbol{Av}$ results in a scalar multiple (of 2) of the original vector (see the figure), $\boldsymbol{v}$, while the product $\boldsymbol{Au}$ does not. In this case $\boldsymbol{v}$ is an *eigenvector* of $\boldsymbol{A}$, and the corresponding *eigenvalue* is 2.

[1]Lay, D. (1994) *Linear Algebra and its Applications*, Addison-Wesley, USA.

Def.: An eigenvector of a matrix $\mathbf{A}$ *is a non-zero vector* $\mathbf{x}$ *such that the product* $\mathbf{Ax}$ *results in a scalar multiple* λ *of* $\mathbf{x}$*, where* λ *is known as the eigenvalue.*



- Now, let's examine some important properties of eigenvectors. In doing so, the eigen problem for the transpose of $\mathbf{A}$ can be written as:

$$A^T y = \bar{\lambda} y \tag{3.30}$$

Eq. 3.30 corresponds to:

$$\left[A^T - \bar{\lambda} I\right] y = 0, \quad y = [\ldots] \tag{3.31}$$

The characteristic equation to be solved for a non-trivial solution to exist for Eq. 3.31 is:

$$\det\left(A^T - \bar{\lambda} I\right) = 0 \tag{3.32}$$

But, we know from the properties of determinants that $|\mathbf{A}| = |\mathbf{A}^T|$. Hence, Eq. 3.32 is the same as: $\det(A^T) = \det(A)$

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \tag{3.33}$$

and we can conclude that, $\bar{\lambda} = \lambda$.

Now, we can write Eq. 3.30 as:

$$\mathbf{A}^T \mathbf{y} = \lambda \mathbf{y} \tag{3.34}$$

Taking the transpose on both sides of Eq. 3.34, we get:

$$y^T A = \lambda y^T \tag{3.35}$$

Now, let's write Eq. 3.35 and Eq. 3.29 for the i^{th} and j^{th} eigenvectors as follows:

$$y_i^T A = \lambda_i y_i^T \tag{3.36}$$

$$A x_j = \lambda_j x_j \tag{3.37}$$

Post-multiplying Eq. 3.36 by $\boldsymbol{x}_j$ and pre-multiplying Eq. 3.37 by $\boldsymbol{y}_i^T$,

$$y_i^T A x_j = \lambda_i y_i^T x_j \quad (3.38) \qquad\qquad y_i^T A x_j = \lambda_j y_i^T x_j \quad (3.39)$$

Subtracting them, we get:

$$(\lambda_i - \lambda_j) y_i^T x_j = 0 \tag{3.40}$$

$\neq 0$

This leads to the conclusion that:

$$y_i^T x_j = 0, \quad \lambda_i \neq \lambda_j \tag{3.41}$$

since y and x are vectors, this is equivalent to $y \cdot x = 0$

where, $\boldsymbol{y}$ are called the *left* eigenvectors and $\boldsymbol{x}$ are the *right* eigenvectors.

- The results show that the left and right eigenvectors corresponding to two distinct eigenvalues are orthogonal to each other. If $\boldsymbol{A}$ is symmetric, we do not need to differentiate between the left and right eigenvectors; they are the same.
- It can also be shown that the eigenvectors corresponding to distinct eigenvalues of $\boldsymbol{A}$ are linearly independent. $x_1, x_2, x_3 \ldots\ldots x_1 \neq a_1 x_2 + a_2 x_3 \ldots$ since distinct

  Hence, any arbitrary vector $\boldsymbol{u}$ can be written as a linear combination of N eigenvectors of $\boldsymbol{A}$. That is:

$$u_{n\times 1} = c_1 x_1 + c_2 x_2 + \cdots + c_N x_N \tag{3.42}$$

To calculate the coefficients, we simply pre-multiply the equation by $\boldsymbol{y}_i^T$ to get:

$$y_i^T u = c_i y_i^T x_i \tag{3.43}$$

This results in:

$$c_i = \frac{y_i^T u}{y_i^T x_i} \tag{3.44}$$

Of course, if $\boldsymbol{A}$ is symmetric, then $\boldsymbol{y}$ and $\boldsymbol{x}$ are the same.

- The solution to the eigen problem (i.e. determining the eigenvalues and eigenvectors) is usually accomplished numerically in practice.

3.3.2 Natural Frequencies and Modes

Now, let's revisit the idea of the natural frequencies and modes of an MDOF system. We will look at the similarities between the standard eigenproblem described earlier, and free vibration problem we encounter in structural dynamics. Consider an arbitrary N-DOF system without damping.

(3.45)

Benjamin Klasser

- We assume that the free vibration of the system in one of its natural modes takes the form:

$$M\ddot{x} + kx = 0$$

$N \times N \quad N \cdot 1 \quad N \times N \quad N \cdot 1$

$$x = \phi_i y_i(t)$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{bmatrix} A\sin(\omega_n t + \alpha) \cdot \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{bmatrix} \left(B\cos\omega_n t + C\cos\omega_n t\right) \tag{3.46}$$

where $\boldsymbol{\phi}_n$ is a vector containing the amplitudes of $\boldsymbol{x}$, and $y_i(t)$ is nothing more than undamped free vibration response in mode i.

- Substituting Eq. 3.46 into Eq. 3.45 yields

$$\left[-\omega_n^2 M \phi_n + K\phi_n\right] y_i(t) = 0 \tag{3.47}$$

- For a nontrivial solution,

$$K\phi_n = \omega_n^2 M \phi_n \tag{3.48}$$

- If we pre-multiply both sides of Eq. 3.45 by $\boldsymbol{M}^{-1}$, we get:

$$M^{-1}K\phi_n = \omega_n^2 \phi_n \tag{3.49}$$

We can easily see that Eq. 3.46 is the same as the standard eigenproblem in Eq. 3.29 with $\boldsymbol{M}^{-1}\boldsymbol{K}$ being analogous to $\boldsymbol{A}$ and ω_n^2 having the same interpretation as λ.

- The matrix eigenvalue problem in Eq. 3.48 in which ω_n^2 are the eigenvalues and the amplitude vector $\boldsymbol{\phi}_n$ are the corresponding eigenvectors can also be posed as

$$\left(-\omega_n^2 M + K\right)\phi_n = 0 \tag{3.50}$$

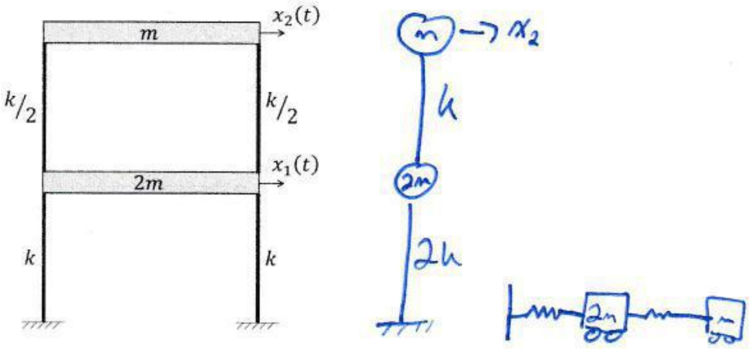
- Which has a nontrivial solution if

$$\det\left[-\omega_n^2 M + K\right] = 0 \tag{3.51}$$

- If $\boldsymbol{M}$ and $\boldsymbol{K}$ are real and positive definite, the eigenvalues ω_n^2 will be real and positive, and the eigenvectors will be real.
- For an N-DOF system, solving Eq. 3.51 will yield N real and positive roots for ω_n^2.
- For each ω_n, the corresponding eigenvector $\boldsymbol{\phi}_n$ can be obtained from Eq. 3.50. However, as we saw in Section 3.2, we cannot solve for exact values. It is only possible to know the relative values in $\boldsymbol{\phi}_n$.
- The eigenvectors $\boldsymbol{\phi}_n$ correspond to the response amplitude and are called the *mode shape vectors*. As we saw in Example 3.2, mode shape vectors give us a general sense of what the free vibration response looks like in the natural modes.

- The natural frequencies and associated mode shape vectors are numbered from the lowest frequency to highest from 1 to N. $\omega_{n,1}$ is called the *fundamental* circular natural frequency.

Example 3.3 Determine the natural periods and sketch the mode shapes for the two-storey frame shown below.



Solution: The equation of motion is

$$\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.52}$$

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{0}$$

Assume $x = \phi_n A \sin(\omega_n t + \alpha)$

To determine the natural frequencies and modes, we solve the eigenvalue problem:

$$(-\omega_n^2 \boldsymbol{M} + \boldsymbol{K})\boldsymbol{\phi}_n = \boldsymbol{0}$$

$$\begin{bmatrix} 3k - 2\omega_n^2 m & -k \\ -k & k - \omega_n^2 m \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}_n = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.53}$$

Which has nontrivial solutions if

$$\begin{vmatrix} 3k - 2\omega_n^2 m & -k \\ -k & k - \omega_n^2 m \end{vmatrix} = 0$$

$$(3k - 2\omega_n^2 m)(k - \omega_n^2 m) - k^2 = 0 \tag{3.54}$$

$$\omega_n^4 - \frac{5k}{2m}\omega_n^2 + \frac{k^2}{m^2} = 0$$

The 2-DOF system yields two roots,

$$\omega_{n,1}^2 = \frac{k}{2m}, \quad \omega_{n,1} = 0.707\sqrt{\frac{k}{m}} \text{ rads/sec} \tag{3.55a}$$

$$\omega_{n,1}^2 = \frac{2k}{m}, \quad \omega_{n,2} = 1.414\sqrt{\frac{k}{m}} \text{ rads/sec} \tag{3.55b}$$

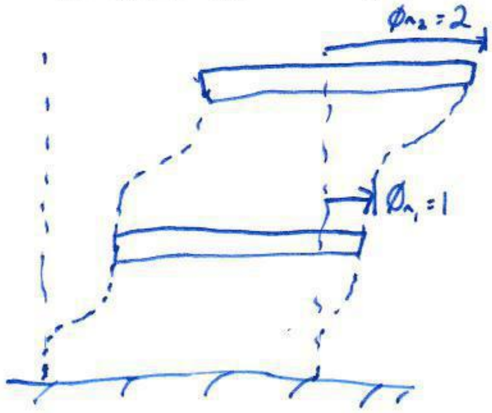
We can now solve for the mode shape vectors. For simplicity take $\phi_1 = 1$.

$$\begin{bmatrix} 3k - 2\omega_n^2 m & -k \\ -k & k - \omega_n^2 m \end{bmatrix} \begin{bmatrix} 1 \\ \phi_2 \end{bmatrix}_n = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.56}$$

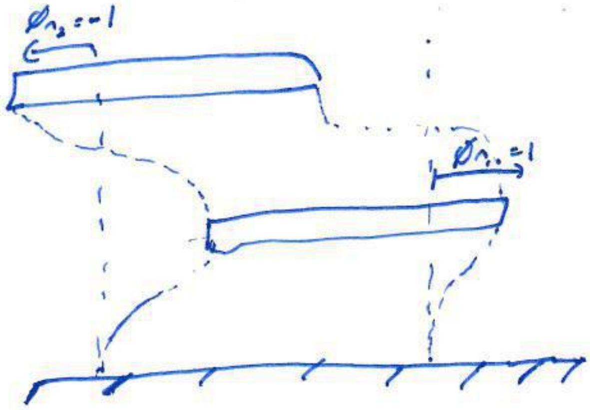
Subbing $\omega_{n,1}$ in, $3k - 2\omega_n^2 m - \phi_2 k = 0$, $\phi_2 = 2$, Mode shape 1 $= \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\omega_{n,2} \rightarrow 3k - 2\omega_{n,2}^2 m - \phi_2 k = 0$, $\phi_2 = -1$ $= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

MODE 1 ($\omega_{n,1} = 0.707\sqrt{k/m}$ rad/s)



MODE 2 ($\omega_{n,2} = 1.414\sqrt{k/m}$ rad/s)



3.3.3 Orthogonality Property

Recall the matrix eigenvalue problem

$$\boldsymbol{K\phi}_n = \omega_n^2 \boldsymbol{M\phi}_n \tag{3.57}$$

Equation 3.57 can be written for each frequency and mode as

Mode i $K\phi_{n,i} = \omega_{n,i}^2 M \phi_{n,i}$ (3.58a)

Mode j $K\phi_{n,j} = \omega_{n,j}^2 M \phi_{n,j}$ (3.58b)

Pre-multiplying Equation 3.58a by $\boldsymbol{\phi}_{n,j}^T$

$$\phi_{n,j}^T K \phi_{n,i} = \omega_{n,i}^2 \phi_{n,j}^T M \phi_{n,i} \tag{3.59}$$

Pre-multiplying Equation 3.58b by $\boldsymbol{\phi}_{n,i}^T$

$$\boldsymbol{\phi}_{n,i}^T \boldsymbol{K\phi}_{n,j} = \omega_{n,j}^2 \boldsymbol{\phi}_{n,i}^T \boldsymbol{M\phi}_{n,j} \tag{3.60}$$

Taking the transpose of both sides in Equation 3.59 gives

$$[LS]^T = [RS]^T, \quad [\phi_{n,i}^T K \phi_{n,j}]^T = \omega_{n,i}^2 [\phi_{n,i}^T M \phi_{n,j}]^T$$
$$\phi_{n,i}^T K \phi_{n,j} = \omega_{n,i}^2 \phi_{n,i}^T M \phi_{n,j} \tag{3.61}$$

Subtracting Equation 3.61 from Equation 3.60,

$$(\omega_{n,j}^2 - \omega_{n,i}^2)\,\phi_{n,i}^T M \phi_{n,j} = 0 \tag{3.62}$$

Since $\omega_{n,i}^2 \neq \omega_{n,j}^2 \neq 0$ (assuming $i \neq j$), it follows that

$$\phi_{n,i}^T M \phi_{n,j} = 0 \tag{3.63}$$

which is known as the mass orthogonality property of the modes. Substituting this result back into Equation 3.60, we find

$$\phi_{n,i}^T K \phi_{n,j} = 0 \quad \text{for } i \neq j \tag{3.64}$$

demonstrating the orthogonality of the stiffness matrix and the modes. Finally, pre-multiplying Equation 3.58a by $\boldsymbol{\phi}_{n,i}^T$ (or Equation 3.58b by $\boldsymbol{\phi}_{n,j}^T$, we get

$$\frac{\phi_{n,i}^T K \phi_{n,i}}{\phi_{n,i}^T M \phi_{n,i}} = \omega_{n,i}^2 \qquad \underline{K \phi_{n,i} = \omega_{n,i}^2 M \phi_{n,i}} \tag{3.65}$$

We now introduce the *modal matrix*, $\mathbf{\Phi}$.

$$\mathbf{\Phi} = \begin{array}{c} \begin{array}{cccc} \phi_{n,1} & \phi_{n,2} & & \phi_{n,N} \end{array} \\ \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N1} & \phi_{N2} & \cdots & \phi_{NN} \end{bmatrix} \end{array} = \begin{bmatrix} \phi_{n,1} & \phi_{n,2} & \cdots & \phi_{n,N} \end{bmatrix} \tag{3.66}$$

The orthogonality of natural modes implies the following:

$$\Phi^T M \Phi = \begin{bmatrix} \phi_{n,1}^T M \phi_{n,1} & 0 \\ 0 & \phi_{n,2}^T M \phi_{n,2} \end{bmatrix} \tag{3.67a}$$

$$\Phi^T K \Phi = \begin{bmatrix} \phi_{n,1}^T K \phi_{n,1} & 0 \\ 0 & \ddots \end{bmatrix} \tag{3.67b}$$

It follows that the following matrices are diagonal.

$$\phi_1^T M \phi_2 = 0 \tag{3.68a}$$

SWITCH

$$\phi_1^T K \phi_2 = 0 \tag{3.68b}$$

Let's return to the solution of a undamped 2-DOF system under free vibration, given in Equation 3.26.

$x_0 = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix}$

$$\boldsymbol{x} = \sum_{i=1}^{2} \boldsymbol{\phi}_i \left(a_i \cos \omega_{n,i} t + b_i \sin \omega_{n,i} t \right) \tag{3.67}$$

If $\boldsymbol{x}(0) = \boldsymbol{x}_0$ and $\dot{\boldsymbol{x}}(0) = \boldsymbol{0}$ are the displacement and velocity initial conditions, respectively, we can write the initial displacement of the system as

$\dot{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$x(0) = x_0 = a_1 \phi_1 + a_2 \phi_2 \tag{3.68}$$

Let us specify a particular configuration, or "shape", for the displacement initial condition. Furthermore, let us specify the initial displacement condition in the form of the first mode of vibration, $\boldsymbol{\phi}_1$. Therefore,

$\begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix} = \begin{bmatrix} \phi_1' \\ \phi_2' \end{bmatrix}$

$$\boldsymbol{x}_0 = \boldsymbol{\phi}_1 = a_1 \boldsymbol{\phi}_1 + a_2 \boldsymbol{\phi}_2 \tag{3.69}$$

Pre-multiplying each term in Equation 3.69 by $\boldsymbol{\phi}_1^T \boldsymbol{M}$, we get

$$\phi_1^T M \phi_1 = a_1 \phi_1^T M \phi_1 + a_2 \cancel{\phi_1^T M \phi_2} \to 0 \tag{3.70}$$

(if initial excitement only MODE2), will only be MODE 2

$= a_1 = 1$

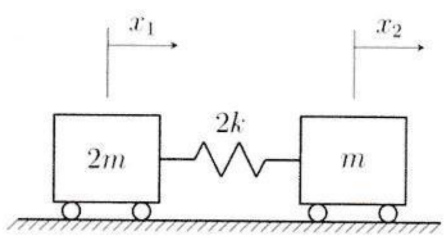
This implies $a_1 = 1$, which is the only non-zero coefficient. This means the structure continues to vibrate in the first mode.

- The coefficients a_1 and a_2 represent the coefficients in an expansion of the initial deflection configuration in terms of the natural modes.
- In general, all modes of the system are excited. However, the extent to which each mode participates in the total response depends on the initial conditions.

3.3.4 Rigid Body Modes

Let us investigate what happens when a system is not properly constrained by considering the following example.

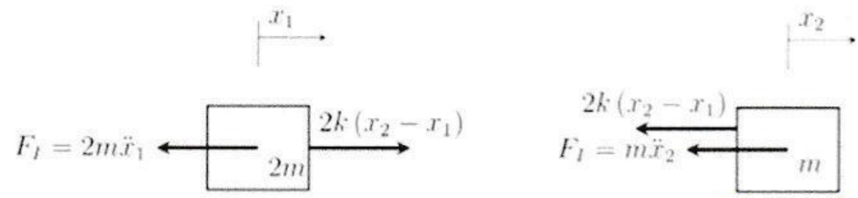
Example 3.4 Find the fundamental circular natural frequency and the corresponding natural mode for the system shown in Figure 3.3.



(Not stable

Figure 3.3 System exhibiting a rigid body natural mode

<u>Solution</u>: Using Newtonian Mechanics, we can derive the equations of motion. From the free body diagrams



$-m\ddot{x}_2 - 2k(x_2 - x_1)$

The equations of motion are

$-2m\ddot{x}_1 + 2k(x_2 - x_1)$

$$\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -2k \\ -2k & 2k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3.71a) \quad (3.71b)$$

Assuming the solution is of the form

lower uppercase

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sin(\omega_n t + \alpha) \quad (3.72)$$

For the non-trivial solution,

$$\begin{bmatrix} (2k - 2m\omega_n^2) & -2k \\ -2k & (k - m\omega_n^2) \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (3.73)$$

We can find the roots by setting the following determinant to zero:

$$\begin{vmatrix} 2k - 2m\omega_n^2 & -2k \\ -2k & k - m\omega_n^2 \end{vmatrix} = 0 \quad (3.74)$$

Expanding the determinant to find the characteristic equation and solving for ω_n^2, we find the following roots:

$$\omega_{n,1}^2 = 0 \quad (3.75a) \qquad\qquad \omega_{n,2}^2 = \frac{3k}{m} \quad (3.75b)$$

zero stiffness

The fact that $\omega_{n,1}^2 = 0$ indicates the presence of a rigid body mode. Substituting this result into Equation 3.73 and selecting $X_2 = 1$, we find the corresponding natural mode.

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad (3.76)$$

which is also indicative of a rigid body mode. For the second natural mode,

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_2 = \begin{Bmatrix} 1 \\ -2 \end{Bmatrix} \quad (3.77)$$

3.3.5 Mode Normalization / Scaling

We have already seen that the mode shape vectors (the eigenvectors) can only be determined to a scaling factor. That is, for a 2-DOF system, only the relative values can be obtained for $\{X_1 \;\; X_2\}^T$. We can set $X_1 = 1$ and find $\beta_1 = (X_2/X_1)_1$ to obtain $\boldsymbol{\phi}_1 = \{1 \;\; \beta_1\}^T$. Similarly, $\boldsymbol{\phi}_2 = \{1 \;\; \beta_2\}^T$, where $\beta_2 = (X_2/X_1)_2$. In other words, the elements of a modal matrix are only known in a relative sense; their absolute magnitudes are indeterminate. We will now investigate other ways to scale the natural modes.

3.3.6 Mass Normalization

As discussed above, we can only ever determine relative values for the mode shape vectors. Any vector proportional to $\boldsymbol{\phi}_i$ yields the same mode shape and satisfies the eigenvalue problem. As a result, it is typical to normalize the mode shape vectors in some way. It may be done such that the largest element in $\boldsymbol{\phi}_i$ is unity or such that the element corresponding to a certain DOF is unity. It is also common to normalize modes such that

$$M_{ii} = \phi_i^T M \phi_i = 1 \tag{3.78}$$

This is known as *mass normalization*. Note that due to mode orthogonality, $\boldsymbol{\phi}_i^T \boldsymbol{M} \boldsymbol{\phi}_j = 0$ for $i \neq j$. Equation 3.78 can also be written as

$$\Phi^T M \Phi = I \leftarrow \text{identity matrix} \tag{3.79}$$

where $\boldsymbol{\Phi} = [\boldsymbol{\phi}_1 \;\; \boldsymbol{\phi}_2 \;\; \cdots \;\; \boldsymbol{\phi}_N]$. Hence, from Equation 3.65,

$$\frac{\phi_{n,i}^T K \phi_{n,i}}{\phi_{n,i}^T M \phi_{n,i}}$$

$$\Phi^T K \Phi = \begin{bmatrix} \omega_{n,1}^2 & 0 & & 0 \\ 0 & \omega_{n,2}^2 & & 0 \\ 0 & 0 & \ddots & \omega_{n,N}^2 \end{bmatrix} = \Lambda \leftarrow \text{big lambda} \tag{3.80}$$

Let us consider an illustrative example.

Example 3.5 Determine the mass normalized modes, $\boldsymbol{\Phi} = [\boldsymbol{\phi}_1 \;\; \boldsymbol{\phi}_2]$ for $\boldsymbol{M} = \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix}$, assuming we have already calculated the natural modes $\mathbf{X} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$.

$\Phi \leftarrow$ phi

Solution: Begin by letting $\boldsymbol{\phi}_1 = \alpha \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$ and $\boldsymbol{\phi}_2 = \beta \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$, where α and β are to be determined. We would like to normalize the modes such that $\boldsymbol{\phi}_1^T \boldsymbol{M} \boldsymbol{\phi}_1 = 1$.

$$\begin{bmatrix} \alpha & 2\alpha \end{bmatrix} \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \alpha \\ 2\alpha \end{bmatrix} = 1 \tag{3.81}$$

$$2\alpha^2 m + 4\alpha^2 m = 1, \quad \alpha = \frac{1}{\sqrt{6m}}, \quad \phi_1 = \frac{1}{\sqrt{6m}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Solving for α, we find that

$$\alpha = \frac{1}{\sqrt{6m}} \tag{3.82}$$

Similarly, for $\boldsymbol{\phi}_2^T \boldsymbol{M} \boldsymbol{\phi}_2 = 1$, we have

$$\begin{bmatrix} \beta & -\beta \end{bmatrix} \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \beta \\ -\beta \end{bmatrix} = 1 , \tag{3.83}$$

Solving for β, we find that

$$\beta = \frac{1}{\sqrt{3m}} \rightarrow \phi_2 = \frac{1}{\sqrt{3m}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{3.84}$$

Therefore, the mass normalized modal matrix is

$$\boldsymbol{\Phi} = \begin{bmatrix} \frac{1}{\sqrt{6m}} & \frac{1}{\sqrt{3m}} \\ \frac{2}{\sqrt{6m}} & -\frac{1}{\sqrt{3m}} \end{bmatrix} \tag{3.85}$$

Verify $\Phi^T M \Phi = I$

$\Phi^T K \Phi = \Lambda$

3.3.7 Mode Superposition

Recall the solution for the undamped MDOF system under free vibration.

(3.86)

Let

$$\boldsymbol{\Phi} = [\boldsymbol{\phi}_1 \ \boldsymbol{\phi}_2 \ \cdots \ \boldsymbol{\phi}_N] = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N1} & \phi_{N2} & \cdots & \phi_{NN} \end{bmatrix} \tag{3.87}$$

be the modal matrix. Expanding Equation 3.86, we have

$$x_1 = a_1\phi_{11}\cos\omega_{n,1}t + b_1\phi_{11}\sin\omega_{n,1}t + a_2\phi_{12}\cos\omega_{n,2}t + b_2\phi_{12}\sin\omega_{n,2}t + \cdots \\ + a_N\phi_{1N}\cos\omega_{n,N}t + b_N\phi_{1N}\sin\omega_{n,N}t \tag{3.88a}$$

$$x_2 = a_1\phi_{21}\cos\omega_{n,1}t + b_1\phi_{21}\sin\omega_{n,1}t + a_2\phi_{22}\cos\omega_{n,2}t + b_2\phi_{22}\sin\omega_{n,2}t + \cdots \\ + a_N\phi_{2N}\cos\omega_{n,N}t + b_N\phi_{2N}\sin\omega_{n,N}t \tag{3.88b}$$

$$\vdots$$

3.3.7 Mode Superposition

Recall the solution for the undamped MDOF system under free vibration.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \sum_{i=1}^{N} \phi_i \left(a_i \cos \omega_{n,i} t + b_i \sin \omega_{n,i} t \right) = \sum_{i=1}^{N} \phi_i A_i \sin \left(\omega_{n,i} t + \alpha_i \right) \tag{3.86}$$

Let

$$\boldsymbol{\Phi} = [\phi_1 \ \phi_2 \ \cdots \ \phi_N] = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N1} & \phi_{N2} & \cdots & \phi_{NN} \end{bmatrix} \tag{3.87}$$

be the modal matrix. Expanding Equation 3.86, we have

$$\begin{aligned} x_1 = {} & a_1 \phi_{11} \cos \omega_{n,1} t + b_1 \phi_{11} \sin \omega_{n,1} t + a_2 \phi_{12} \cos \omega_{n,2} t + b_2 \phi_{12} \sin \omega_{n,2} t + \cdots \\ & + a_N \phi_{1N} \cos \omega_{n,N} t + b_N \phi_{1N} \sin \omega_{n,N} t \end{aligned} \tag{3.88a}$$

$$\begin{aligned} x_2 = {} & a_1 \phi_{21} \cos \omega_{n,1} t + b_1 \phi_{21} \sin \omega_{n,1} t + a_2 \phi_{22} \cos \omega_{n,2} t + b_2 \phi_{22} \sin \omega_{n,2} t + \cdots \\ & + a_N \phi_{2N} \cos \omega_{n,N} t + b_N \phi_{2N} \sin \omega_{n,N} t \end{aligned} \tag{3.88b}$$

$$x_N = a_1\phi_{N1}\cos\omega_{n,1}t + b_1\phi_{N1}\sin\omega_{n,1}t + a_2\phi_{N2}\cos\omega_{n,2}t + b_2\phi_{N2}\sin\omega_{n,2}t + \cdots + a_N\phi_{NN}\cos\omega_{n,N}t + b_N\phi_{NN}\sin\omega_{n,N}t \tag{3.88c}$$

In matrix form, Equations 3.88 can be written as

$$\boldsymbol{x} = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N1} & \phi_{N2} & \cdots & \phi_{NN} \end{bmatrix} \begin{Bmatrix} a_1\cos\omega_{n,1}t + b_1\sin\omega_{n,1}t \\ a_2\cos\omega_{n,2}t + b_2\sin\omega_{n,2}t \\ \vdots \\ a_N\cos\omega_{n,N}t + b_N\sin\omega_{n,N}t \end{Bmatrix} \tag{3.89}$$

$\boxed{x = \boldsymbol{\Phi y}}$ very powerful

The modal matrix, $\boldsymbol{\Phi}$, we have calculated can be very powerful to solve the equations of motion for MDOF system. Begin with the equations of motion for an undamped MDOF system under free vibration.

$$\boldsymbol{M\ddot{x}} + \boldsymbol{Kx} = \boldsymbol{0} \tag{3.90}$$

gone from x plane of displacement and acceleration to y plane of harmonic functions ∴ Coordinate Transformation.

Substituting the coordinate transformation found in Equation 3.89 into Equation 3.90

$$x = \Phi y \tag{3.91}$$

we get

$$M\Phi\ddot{y} + K\Phi y = 0 \tag{3.92}$$

Pre-multiplying throughout by $\boldsymbol{\Phi}^T$,

$$\Phi^T M \Phi \ddot{y} + \Phi^T K \Phi y = 0 \tag{3.93}$$

Using Equations 3.79 and 3.80, Equation 3.93 can be written as

$$I\ddot{y} + \Lambda y = 0 \qquad I = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} \omega_{n,1}^2 & & 0 \\ & \omega_{n,2}^2 & \\ 0 & & \ddots \; \omega_{n,N}^2 \end{bmatrix} \tag{3.94}$$

Since **I** and $\boldsymbol{\Lambda}$ are diagonal, we can write Equation 3.94 as N uncoupled SDOF equations using the *modal coordinate* y.

No longer vector Quantities, Now N SDOF systems

$$\ddot{y}_1 + \omega_{n,1}^2 y_1 = 0 \tag{3.95a}$$

$$\ddot{y}_2 + \omega_{n,2}^2 y_2 = 0 \tag{3.95b}$$

$$\vdots$$

$$\ddot{y}_N + \omega_{n,N}^2 y_N = 0 \tag{3.95c}$$

Recall from our study of undamped SDOF systems under free vibration, the solutions are

$$y(t) = y_{0,1}\cos\omega_{n,1}t + \frac{\dot{y}_{0,1}}{\omega_{n,1}}\sin\omega_{n,1}t \tag{3.96a}$$

$$y_2 = y_{0,2} \cos \omega_{n,2} t + \frac{\dot{y}_{0,2}}{\omega_{n,2}} \sin \omega_{n,2} t \tag{3.96b}$$

$$\vdots$$

$$y_N = y_{0,N} \cos \omega_{n,N} t + \frac{\dot{y}_{0,N}}{\omega_{n,N}} \sin \omega_{n,N} t \tag{3.96c}$$

where $\boldsymbol{y}_0 = \boldsymbol{y}(0) = [y_{0,1} \ y_{0,2} \ \cdots \ y_{0,N}]$ and $\dot{\boldsymbol{y}}_0 = \dot{\boldsymbol{y}}(0) = [\dot{y}_{0,1} \ \dot{y}_{0,2} \ \cdots \ \dot{y}_{0,N}]$ are the initial displacement and velocity, respectively.

3.3.8 Initial Conditions in Modal Coordinates

We have seen earlier that the equation of motion for a undamped MDOF system under free vibration

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{0} \tag{3.97}$$

can be written as

$$\mathbf{I}\ddot{\boldsymbol{y}} + \boldsymbol{\Lambda}\boldsymbol{y} = \boldsymbol{0} \tag{3.98}$$

The solutions to the equations of motion can be found by determining the solution to each of the uncoupled equations of motion as if they were the response of a SDOF system.

$$y_1 = y_{0,1} \cos \omega_{n,1} t + \frac{\dot{y}_{0,1}}{\omega_{n,1}} \sin \omega_{n,1} t \tag{3.99a}$$

$$y_2 = y_{0,2} \cos \omega_{n,2} t + \frac{\dot{y}_{0,2}}{\omega_{n,2}} \sin \omega_{n,2} t \tag{3.99b}$$

$$\vdots$$

$$y_N = y_{0,N} \cos \omega_{n,N} t + \frac{\dot{y}_{0,N}}{\omega_{n,N}} \sin \omega_{n,N} t \tag{3.99c}$$

where $\boldsymbol{y}_0 = [y_{0,1} \ y_{0,2} \ \cdots \ y_{0,N}]$ and $\dot{\boldsymbol{y}}_0 = [\dot{y}_{0,1} \ \dot{y}_{0,2} \ \cdots \ \dot{y}_{0,N}]$ are the displacement and velocity initial conditions, respectively, in modal coordinates. The solution to Equations 3.99 can by transformed back into the physical coordinates using the following coordinate transformation:

$$x = \Phi y \tag{3.100}$$

However, before we can solve Equations 3.99, we require $\boldsymbol{y}_0$ and $\dot{\boldsymbol{y}}_0$. It is impractical to specify the initial conditions in the modal domain. Therefore, we require a coordinate transformation to transform the physical initial conditions into the modal domain. The relationship between the physical and modal coordinate systems for the initial conditions is the same for the response. That is,

$$x_0 = \Phi y_0 \qquad \dot{x}_0 = \Phi \dot{y}_0 \tag{3.101}$$

$$\Phi^{-1} x_0 = \Phi^{-1} \Phi y_0$$
$$= I y_0$$

not used except for 2x2 and 3x3 → what we do is Eq 3.102.

Pre-multiplying both sides of the Equation 3.101 by $\boldsymbol{\Phi}^T \boldsymbol{M}$ gives

→Must be mass normalized

$$\Phi^T M x_0 = \Phi^T M \Phi y_0 = I y_0 = y_0 \tag{3.102}$$

Therefore, the displacement initial conditions in the modal coordinates are related to the physical coordinates by the following transformation:

$$y_0 = \Phi^T M x_0 \tag{3.103}$$

A similar relationship exists for the velocity initial condition

$$\dot{y}_0 = \Phi^T M \dot{x}_0 \tag{3.104}$$

Example 3.6 Find the solution for the 2-DOF system shown in Figure 3.4 for m_1 = 1 kg, m_2 = 4 kg, $k_1 = k_3$ = 10 N/m, and k_2 = 2 N/m. The initial conditions are $\boldsymbol{x}_0 = [1\ \ 1]^T$ and $\dot{\boldsymbol{x}}_0 = [0\ \ 0]^T$.

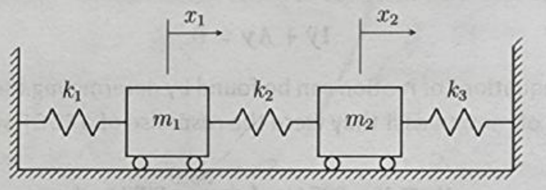
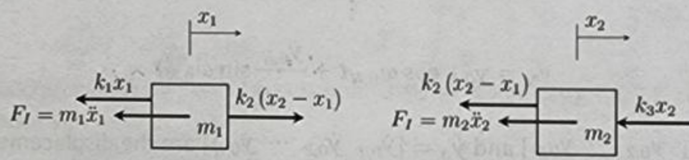


Figure 3.4 Two degree-of-freedom spring-mass model under free vibration

Solution: Begin by the equations of motion for the system. From the free-body diagrams:



the equations of motion are:

$$m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = 0 \tag{3.105a}$$

$$m_2\ddot{x}_2 - k_2x_1 + (k_2+k_3)x_2 = 0 \tag{3.105b}$$

which can also be written as

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{0} \tag{3.106a}$$

$$\boldsymbol{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad (3.106b) \qquad \boldsymbol{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \quad (3.106c)$$

Assuming the typical form of the solution for free vibration and expanding the following determinant,

$$\begin{vmatrix} (k_1+k_2)-m_1\omega_n^2 & -k_2 \\ -k_2 & (k_2+k_3)-m_2\omega_n^2 \end{vmatrix} = 0 \tag{3.107}$$

we get the characteristic equation of the system.

$$\omega_n^4 - \frac{m_1(k_2+k_3)+m_2(k_1+k_2)}{m_1 m_2}\omega_n^2 + \frac{(k_1+k_2)(k_2+k_3)-k_2^2}{m_1 m_2} = 0 \tag{3.108}$$

Solving for the roots ω_n^2, we find

$$\omega_{n,1}^2 = 2.89\ (\mathrm{rad/s})^2 \quad (3.109\mathrm{a}) \qquad \omega_{n,2}^2 = 12.11\ (\mathrm{rad/s})^2 \quad (3.109\mathrm{b})$$

Selecting $X_1 = 1$ for both modes of vibration, we find the following natural modes:

$$\boldsymbol{X} = \begin{bmatrix} 1 & 1 \\ 4.555 & -0.0549 \end{bmatrix} \tag{3.110}$$

$$\begin{array}{cc} \Phi_1 & \Phi_2 \end{array}$$

The mass normalized modes are

$$\boldsymbol{\Phi} = \begin{bmatrix} 0.109 & 0.994 \\ 0.497 & -0.055 \end{bmatrix} \tag{3.111}$$

The initial conditions in modal coordinates are

$$y_0 = \Phi^T M x_0$$

$$= \begin{bmatrix} 0.109 & 0.497 \\ 0.994 & -0.055 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{3.112a}$$

$$= \begin{bmatrix} 2.097 \\ 0.776 \end{bmatrix} \qquad \dot{y}_0 = \Phi^T M \dot{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.112b}$$

From Equation 3.99, the solution is $M\ddot{x} + Kx = 0 \rightarrow I\ddot{y} + \Lambda y = 0$

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{matrix} 2.097 \cos 1.7t \\ 0.776 \cos 3.48t \end{matrix} \tag{3.113}$$

Transforming back into physical coordinates, the response is $y_i(t) = y_{0,i}\cos\omega_{n,i}t + \frac{\dot{y}_{0,i}}{\omega_{n,i}}\sin\omega_{n,i}t$

$$x = \Phi y$$

$$= \begin{bmatrix} 0.109 & 0.994 \\ 0.497 & -0.055 \end{bmatrix}\begin{bmatrix} 2.097\cos 1.7t \\ 0.776\cos 3.48t \end{bmatrix} \tag{3.114}$$

Mode superposition

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0.229\cos 1.7t + 0.771\cos 3.48t \\ 1.042\cos 1.7t - 0.042\cos 3.48t \end{bmatrix}$$

3.4 Damped MDOF Systems

We will now consider the damped MDOF system under free vibration.

$$M\ddot{x} + C\dot{x} + Kx = 0 \tag{3.115}$$

Assume that we can use the same modal transformation used in the undamped case.

$$x = \Phi y \tag{3.116}$$

Using this transformation, we get

$$M\Phi\ddot{y} + C\Phi\dot{y} + K\Phi y = 0 \tag{3.117a}$$

$$\Phi^T M\Phi\ddot{y} + \Phi^T C\Phi\dot{y} + \Phi^T K\Phi y = 0 \tag{3.117b}$$

$$I\ddot{y} + \boxed{\Phi^T C\Phi}\dot{y} + \Lambda y = 0 \tag{3.117c}$$

which is only uncoupled when $\Phi^T C\Phi$ is diagonal. We will now consider under what conditions $\Phi^T C\Phi$ is diagonal.

3.4.1 Damping Matrix

Classical damping

We first introduce three cases known as proportional damping. When C is proportional, it is possible to uncouple the equations of motion using the undamped modes. Any other form of the damping matrix is known as non-proportional damping.

Mass proportional damping Let the damping matrix be related to the mass matrix as follows:

$$C = \alpha M \tag{3.118}$$

where α is a proportionality factor.

$$\Phi^T C\Phi = \Phi^T \alpha M\Phi = \alpha \Phi^T M\Phi = \alpha I \tag{3.119}$$

Therefore, $\Phi^T C\Phi$ is diagonal.

Stiffness proportional damping Let the damping matrix be related to the stiffness matrix as follows:

$$C = \beta K \tag{3.120}$$

where β is a proportionality factor.

$$\Phi^T C\Phi = \Phi^T \beta K\Phi = \beta \Phi^T K\Phi = \beta\Lambda \tag{3.121}$$

Therefore, $\Phi^T C\Phi$ is diagonal.

Rayleigh damping Let the damping matrix be related to the mass and stiffness matrices as follows:

$$C = \alpha M + \beta K \tag{3.122}$$

where α and β are proportionality factors.

$$\Phi^T C \Phi = \Phi^T(\alpha M + \beta K)\Phi = \alpha I + \beta \Lambda \tag{3.123}$$

Therefore, $\boldsymbol{\Phi}^T \boldsymbol{C} \boldsymbol{\Phi}$ is diagonal.

The damping matrix for practical structures cannot be simply calculated from the structural dimensions and member sizes and the corresponding individual damping of each member. Instead, the damping matrix must be estimated from the modal damping ratios, which account for the energy dissipation of the system. Modal damping ratios are typically based on the designer's experience with similar structures and excitations in similar stress ranges.

Recall from our discussion of SDOF systems, the equation of motion in the form

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{3.124}$$

can be also written as

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0 \tag{3.125a}$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \tag{3.125b}$$

From our previous discussion that mass proportional damping, where $\boldsymbol{C} = \alpha\boldsymbol{M}$, stiffness proportional damping, where $\boldsymbol{C} = \beta\boldsymbol{K}$, and Rayleigh damping, where $\boldsymbol{C} = \alpha\boldsymbol{M} + \beta\boldsymbol{K}$, are all forms of proportional damping that result in $\boldsymbol{\Phi}^T \boldsymbol{C} \boldsymbol{\Phi}$ being diagonal. If the system is to take the same form as the SDOF system in Equation 3.125b, the diagonal damping matrix must be of the form,

For every modal response, different damping ratio!

$$\Phi^T C \Phi = \begin{bmatrix} 2\zeta_1\omega_{n,1} & & 0 \\ & 2\zeta_2\omega_{n,2} & \\ 0 & & \ddots \; 2\zeta_N\omega_{n,N} \end{bmatrix} \tag{3.126}$$

where N is the number of degrees of freedom.

In practice, the modal damping ratios are generally specified for the various modes of vibration. With this information, we will now consider the development of the damping matrix, $\boldsymbol{C}$, by means of a 2-DOF damping system under free vibration. The equations of motion in the modal domain (in terms of $\boldsymbol{y} = \boldsymbol{\Phi}^T \boldsymbol{M}\boldsymbol{x}$) are as follows: For 2 DOF

$$\ddot{y}_1 + 2\zeta_1\omega_{n,1}\dot{y}_1 + \omega_{n,1}^2 y_1 = 0 \tag{3.127a}$$

$$\ddot{y}_2 + 2\zeta_2\omega_{n,2}\dot{y}_2 + \omega_{n,2}^2 y_2 = 0 \tag{3.127b}$$

We begin by assuming that ζ_1 or ζ_2 are given, or both. The damping matrix $\boldsymbol{C}$ is dependant on our assumption of either mass proportional, stiffness proportional, or Rayleigh damping. We will investigate each case separately. As long as reasonable damping, good enough

Mass proportional Assuming that $\boldsymbol{C}$ is of the form

$$\boldsymbol{C} = \alpha \boldsymbol{M} \tag{3.128}$$

Pre- and post-multiplying by $\boldsymbol{\Phi}^T$ and $\boldsymbol{\Phi}$, respectively, we get

$$\begin{gathered} \Phi^T C \Phi = \alpha \Phi^T M \Phi = \alpha I \\ \begin{bmatrix} 2\zeta_1 \omega_{n,1} & 0 \\ 0 & 2\zeta_2 \omega_{n,2} \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \end{gathered} \tag{3.129}$$

Therefore,

$$\alpha = 2\zeta_1 \omega_{n,1} \tag{3.130}$$

or

$$\alpha = 2\zeta_2 \omega_{n,2} \tag{3.131}$$

only one alpha.

This implies that α can be related to ζ_1 or ζ_2, but not both. Assuming that α is calculated using Equation 3.130, we can recalculate the damping ratio for the remaining mode as

$$\zeta_2 = \frac{\alpha}{2\omega_{n,2}} \tag{3.132}$$

Therefore, for a N-DOF system, α is calculated by selecting which damping ratio α is related to, and the remaining damping ratios are found in a similar fashion to Equation 3.132. For mass proportional damping, the damping ratio is inversely proportional to the natural frequency.

Stiffness proportional Assuming that $\boldsymbol{C}$ is of the form

$$\boldsymbol{C} = \beta \boldsymbol{K} \tag{3.133}$$

Pre- and post-multiplying by $\boldsymbol{\Phi}^T$ and $\boldsymbol{\Phi}$, respectively, we get

$$\begin{aligned} \boldsymbol{\Phi}^T \boldsymbol{C} \boldsymbol{\Phi} &= \beta \boldsymbol{\Phi}^T \boldsymbol{K} \boldsymbol{\Phi} \\ &= \beta \boldsymbol{\Lambda} \\ \begin{bmatrix} 2\zeta_1 \omega_{n,1} & 0 \\ 0 & 2\zeta_2 \omega_{n,2} \end{bmatrix} &= \begin{bmatrix} \beta \omega_{n,1}^2 & 0 \\ 0 & \beta \omega_{n,2}^2 \end{bmatrix} \end{aligned} \tag{3.134}$$

Therefore,

$$\beta = \frac{2\zeta_1}{\omega_{n,1}} \tag{3.135}$$

or

$$\beta = \frac{2\zeta_2}{\omega_{n,2}} \tag{3.136}$$

Similar to mass proportional damping, β can be related to ζ_1 or ζ_2, but not both. Assuming that β is calculated using Equation 3.135, we can recalculate the damping ratio for the remaining mode as

$$2\zeta_2\omega_{n,2} = \beta\omega_{n,2}^2 \quad \Rightarrow \quad \zeta_2 = \frac{\beta\omega_{n,2}}{2} \tag{3.137}$$

Therefore, for a N-DOF system, β is calculated by selecting which damping ratio β is related to, and the remaining damping ratios are found in a similar fashion to Equation 3.137. For stiffness proportional damping, the damping ratio is directly proportional to the natural frequency.

Rayleigh damping Assuming $\boldsymbol{C}$ is of the form

$$\boldsymbol{C} = \alpha\boldsymbol{M} + \beta\boldsymbol{K} \tag{3.138}$$

Pre- and post-multiplying by $\boldsymbol{\Phi}^T$ and $\boldsymbol{\Phi}$, respectively, we get

$$\begin{aligned} \boldsymbol{\Phi}^T\boldsymbol{C}\boldsymbol{\Phi} &= \alpha\boldsymbol{\Phi}^T\boldsymbol{M}\boldsymbol{\Phi} + \beta\boldsymbol{\Phi}^T\boldsymbol{K}\boldsymbol{\Phi} \\ &= \alpha\mathbf{I} + \beta\boldsymbol{\Lambda} \\ \begin{bmatrix} 2\zeta_1\omega_{n,1} & 0 \\ 0 & 2\zeta_2\omega_{n,2} \end{bmatrix} &= \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} + \begin{bmatrix} \beta\omega_{n,1}^2 & 0 \\ 0 & \beta\omega_{n,2}^2 \end{bmatrix} \end{aligned} \tag{3.139}$$

which can be written as

$$2\zeta_1\omega_{n,1} = \alpha + \beta\omega_{n,1}^2 \tag{3.140a}$$

$$2\zeta_2\omega_{n,2} = \alpha + \beta\omega_{n,2}^2 \tag{3.140b}$$

Subtracting Equation 3.140b from Equation 3.140a, we get

$$\beta = \frac{2[\zeta_1\omega_{n,1} - \zeta_2\omega_{n,2}]}{\omega_{n,1}^2 - \omega_{n,2}^2} \tag{3.141}$$

After β is determined, we can solve for α using

$$\alpha = 2\zeta_1\omega_{n,1} - \beta\omega_{n,1}^2 \tag{3.142}$$

or

$$\alpha = 2\zeta_2\omega_{n,2} - \beta\omega_{n,2}^2 \tag{3.143}$$

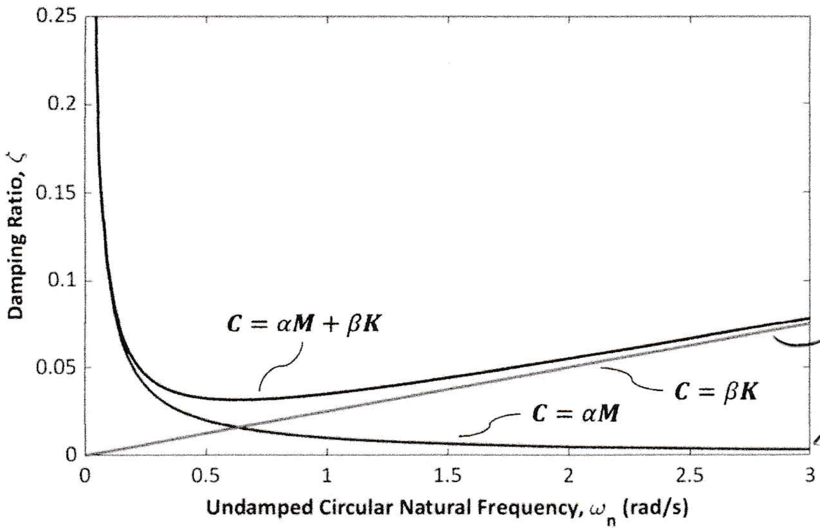
The relationship between frequency and damping ratio for mass and stiffness proportional damping, and Rayleigh damping is illustrated in Figure 3.5.

Easy to construct since forcing it to be diagonal in modal coordinates

Which are to use?

Depends

- Rayleigh the used



$\zeta = \frac{1}{2}\beta\omega_n$

$\zeta = \frac{\alpha}{2\omega_n}$

Figure 3.5 Frequency versus damping ratio for mass proportional, stiffness proportional, and Rayleigh damping

Example 3.7 Find the damping matrix for the 3-DOF system with the following natural frequencies:

$$\omega_{n,1} = 11.62 \text{ rad/s} \qquad \omega_{n,2} = 27.50 \text{ rad/s} \qquad \omega_{n,3} = 45.90 \text{ rad/s}$$

Assume Rayleigh damping and that $\boldsymbol{M}$ and $\boldsymbol{K}$ are known. Use 5% and 3% critical damping in the first two modes, respectively.

Solution: From Equation 3.141

$$\beta = \frac{2[\zeta_1\omega_{n,1} - \zeta_2\omega_{n,2}]}{\omega_{n,1}^2 - \omega_{n,2}^2}$$

$$= \frac{2[0.05 \cdot 11.62 - 0.03 \cdot 27.50]}{[11.62^2 - 27.5^2]}$$

$$= 0.000786$$

Using Equation 3.142 or 3.143, we find that

$$\alpha = 2\zeta_1\omega_{n,1} - \beta\omega_{n,1}^2$$

$$= 2 \cdot 0.05 \cdot 11.62 - 0.000786 \cdot 11.62^2$$

The damping matrix is then computed as

$$C = \alpha M + \beta K$$

$$\overbrace{\begin{bmatrix} 2\zeta_1\omega_{n,1} & 0 & 0 \\ 0 & 2\zeta_2\omega_{n,2} & 0 \\ 0 & 0 & 2\zeta_3\omega_{n,3} \end{bmatrix}}^{\Phi^T C \Phi} = \overbrace{\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}}^{\Phi^T M \Phi} + \overbrace{\begin{bmatrix} \beta\omega_{n,1}^2 & 0 & 0 \\ 0 & \beta\omega_{n,2}^2 & 0 \\ 0 & 0 & \beta\omega_{n,3}^2 \end{bmatrix}}^{\Phi^T K \Phi}$$

We can now compute the damping ratio for the third mode.

$$2\zeta_3 \omega_{n,3} = \alpha + \beta \omega_{n,3}^2$$

$$\zeta_3 = \frac{1.056 + (0.000786)(45.9)^2}{2 \cdot 45.9} = 0.0295$$

Therefore,

$$\alpha = 1.056 \qquad \beta = 0.000786 \qquad \zeta_3 = 0.0295 \approx 3\%$$

3.4.2 Mode Superposition for Damped Systems

Now that we have established a methodology to calculate the damping matrix, we now turn our attention to solving the equations for motion for a damped MDOF under free vibration. The equations of motion are

$$M\ddot{x} + C\dot{x} + Kx = 0 \tag{3.144}$$

Which can be expressed in modal coordinates using the modal transformation $x = \Phi y$ as

$$I\ddot{y} + \underbrace{\Phi^T C \Phi y}_{\text{diagonalized}} + \Lambda y = 0 \tag{3.145}$$

For the case of proportional damping, the equations of motion in modal coordinates are fully uncoupled and can be written as

$$\ddot{y}_1 + 2\zeta_1\omega_{n,1}\dot{y}_1 + \omega_{n,1}^2 y_1 = 0 \tag{3.146a}$$

if not mass normalized, there will be nonunit coeff here

$$\ddot{y}_2 + 2\zeta_2\omega_{n,2}\dot{y}_2 + \omega_{n,2}^2 y_2 = 0 \tag{3.146b}$$

$$\vdots$$

$$\ddot{y}_N + 2\zeta_N\omega_{n,N}\dot{y}_N + \omega_{n,N}^2 y_N = 0 \tag{3.146c}$$

From our study of free vibration of damped SDOF systems, the solution to Equations 3.146 is

$$y_1 = e^{-\zeta_1\omega_{n,1}t}\left(y_{0,1}\cos\omega_{D,1}t + \frac{\dot{y}_{0,1} + y_{0,1}\zeta_1\omega_{n,1}}{\omega_{D,1}}\sin\omega_{D,1}t\right) \tag{3.147a}$$

$$y_2 = e^{-\zeta_2\omega_{n,2}t}\left(y_{0,2}\cos\omega_{D,2}t + \frac{\dot{y}_{0,2} + y_{0,2}\zeta_2\omega_{n,2}}{\omega_{D,2}}\sin\omega_{D,2}t\right) \tag{3.147b}$$

$$\vdots$$

$$y_N = e^{-\zeta_N\omega_{n,N}t}\left(y_{0,N}\cos\omega_{D,N}t + \frac{\dot{y}_{0,N} + y_{0,N}\zeta_N\omega_{n,N}}{\omega_{D,N}}\sin\omega_{D,N}t\right) \tag{3.147c}$$

where

$$\omega_{D,i} = \omega_{n,i}\sqrt{1-\zeta_i^2} \qquad i = 1,2,\ldots,N \tag{3.148}$$

Example 3.8 Solve the equations of motion for a damped 2-DOF system with the following system matrices and initial conditions:

$$\boldsymbol{M} = \begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix}\text{ kg} \qquad \boldsymbol{K} = \begin{bmatrix}10 & -4\\ -4 & 10\end{bmatrix}\text{ N/m}$$

$$\boldsymbol{x}_0 = [1 \quad 1]^T\text{ m} \qquad \dot{\boldsymbol{x}}_0 = [0 \quad 0]^T\text{ m/s}$$

Assume Rayleigh damping with 5% critical damping in both modes.

Solution: The natural frequencies are found by solving the characteristic equation.

$$\omega_{n,1} = 1.616\text{ rad/s} \qquad \omega_{n,2} = 3.274\text{ rad/s}$$

The mass normalized modes are ← don't have to, but makes it easier

$$\boldsymbol{\Phi} = \begin{bmatrix}0.2983 & 0.9545\\ 0.5511 & -0.1722\end{bmatrix} \qquad \Phi^T M \Phi = I$$

For Rayleigh damping $\qquad \Phi^T K \Phi = \Lambda$

$$\alpha = 0.1082 \qquad \beta = 0.0204$$

Therefore, the damping matrix is

$$\boldsymbol{C} = \alpha\boldsymbol{M} + \beta\boldsymbol{K}$$

$$= \begin{bmatrix}0.3127 & -0.0818\\ -0.0818 & 0.5291\end{bmatrix}\text{ N}\cdot\text{s/m} \qquad \Phi^T C \Phi = \begin{bmatrix}2\zeta_1\omega_{n,1} & 0\\ 0 & 2\zeta_2\omega_{n,2}\end{bmatrix}$$

It can be easily verified that the matrix $\boxed{\boldsymbol{\Phi}^T\boldsymbol{C}\boldsymbol{\Phi}}$ is indeed diagonal.

The initial conditions in the modal domain are

$$y_0 = \Phi^T M x_0 \quad , \quad \dot{y}_0 = \Phi^T M \dot{x}_0 = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}y_{0,1}\\ y_{0,2}\end{bmatrix} = \begin{bmatrix}1.952\\ 0.438\end{bmatrix}$$

The modal response can be calculated as

$$y_1 = e^{-\zeta_1 \omega_{n,1} t}\left(y_{0,1}\cos\omega_{D,1}t + \frac{\dot{y}_{0,1} + y_{0,1}\zeta_1\omega_{n,1}}{\omega_{D,1}}\sin\omega_{D,1}t\right)$$

$$y_2 = e^{-\zeta_2 \omega_{n,2} t}\left(y_{0,2}\cos\omega_{D,2}t + \frac{\dot{y}_{0,2} + y_{0,2}\zeta_2\omega_{n,2}}{\omega_{D,2}}\sin\omega_{D,2}t\right)$$

Substituting in the relevant quantities, we get

$$\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{bmatrix} e^{-0.081}\,(1.952\cos 1.614t + 0.098\sin 1.614t) \\ e^{-0.164t}(0.483\cos 3.270t + 0.022\sin 3.270t) \end{bmatrix}$$

The displacement response in physical coordinates is found by

$$x = \Phi y \qquad \Phi$$

$$= \begin{bmatrix} 0.2983 & 0.9548 \\ 0.5511 & -0.1722 \end{bmatrix}\begin{bmatrix} \quad \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0.2983\,y_1(t) + 0.9548\,y_2(t) \\ 0.5511\,y_1(t) - 0.1722\,y_2(t) \end{bmatrix}$$

if pre and post multiply by eigenvectors, goes to 0. whole idea of orthogonality

3.4.3 Caughey Damping — More than 2 modes, $C = \alpha_1 M + \alpha_2 K + \alpha_3 \square$

We have seen that the expressions for Rayleigh damping allow us to determine the damping matrix by specifying damping in two modes. We now introduce Caughey damping, which allows us to determine the damping matrix by specifying damping in three modes.

Before writing the relevant equations for Caughey damping, we need to derive an additional orthogonality relation. Recall the characteristic equation

$$[\boldsymbol{K} - \omega_n^2 \boldsymbol{M}]\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = 0 \tag{3.149}$$

or

$$[K - \omega_n^2 M]\,\phi_i = 0, \quad K\phi_i = \omega_{n,i}^2 M \phi_i \tag{3.150}$$

Pre-multiplying both sides by $\boldsymbol{\phi}_i^T \boldsymbol{K}\boldsymbol{M}^{-1}$, we get (mode shapes mass normalized)

$$\begin{aligned} \phi_i^T K M^{-1} K \phi_i &= \omega_{n,i}^2\, \phi_i^T K M^{-1} M \phi_i \\ &= \omega_{n,i}^2\, \phi_i^T K I \phi_i \\ &= \omega_{n,i}^4 \end{aligned} \tag{3.151}$$

Therefore,

$$\phi_i^T K M^{-1} K \phi_i = \omega_{n,i}^4 \qquad \Phi^T K M^{-1} K \Phi = \Lambda^2 \tag{3.152a}$$

$$\phi_i^T K M M^{-1} K \phi_j = 0 \tag{3.152b}$$

← 2 different ϕs yield 2 → from orthogonality property

Assume the damping matrix $\boldsymbol{C}$ takes the following form:

$$C = \alpha_0 M + \alpha_1 K + \alpha_2 K M^{-1} K \tag{3.153}$$

Pre- and post-multiplying each term by $\boldsymbol{\Phi}^T$ and $\boldsymbol{\Phi}$, respectively, gives

$$\boldsymbol{\Phi}^T \boldsymbol{C} \boldsymbol{\Phi} = \alpha_0 \mathbf{I} + \alpha_1 \boldsymbol{\Lambda} + \alpha_2 \boldsymbol{\Lambda}^2 \tag{3.154}$$

which is diagonal, and therefore a form of proportional damping. Let us revisit Example 3.7 using Caughey damping.

Example 3.9 Find the damping matrix for the 3-DOF system with the following natural frequencies:

$$\omega_{n,1} = 11.62 \text{ rad/s} \qquad \omega_{n,2} = 27.50 \text{ rad/s} \qquad \omega_{n,3} = 45.90 \text{ rad/s}$$

Assume Caughey damping and that $\boldsymbol{M}$ and $\boldsymbol{K}$ are known. Use 5% critical damping for all three modes.

Solution: Using Equation 3.154, we get

$$2\zeta_1\omega_{n,1} = \alpha_0 + \alpha_1\omega_{n,1}^2 + \alpha_2\omega_{n,1}^4$$
$$2\zeta_2\omega_{n,2} = \alpha_0 + \alpha_1\omega_{n,2}^2 + \alpha_2\omega_{n,2}^4$$
$$2\zeta_3\omega_{n,3} = \alpha_0 + \alpha_1\omega_{n,3}^2 + \alpha_2\omega_{n,3}^4$$

Substituting in the known quantities we get

$$2(0.05)(11.62) = \alpha_0 + \alpha_1(11.62)^2 + \alpha_2(11.62)^4$$
$$2(0.05)(27.50) = \alpha_0 + \alpha_1(27.50)^2 + \alpha_2(27.50)^4$$
$$2(0.05)(45.90) = \alpha_0 + \alpha_1(45.90)^2 + \alpha_2(45.90)^4$$

Solving the three equations simultaneously gives

$$\alpha_0 = 0.755 \qquad \alpha_1 = 0.0031 \qquad \alpha_2 = -0.606 \times 10^{-6}$$

If $\boldsymbol{M}$ and $\boldsymbol{K}$ are known, the damping matrix can be calculated as

$$\boldsymbol{C} = \alpha_0\boldsymbol{M} + \alpha_1\boldsymbol{K} + \alpha_2\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}$$

3.4.4 Direct Method

The final method we will consider to develop the damping matrix is known as the direct method. The result is known as *modal damping*, and permits the damping matrix to be generated considering the damping ratios for all modes. If the damping ratio is specified for all modes, that is, $\zeta_1, \zeta_2, \ldots, \zeta_N$ are given, we can calculate the damping matrix as follows. Recall that

$$\boldsymbol{\Phi}^T\boldsymbol{C}\boldsymbol{\Phi} = \begin{bmatrix} 2\zeta_1\omega_{n,1} & 0 & \cdots & 0 \\ 0 & 2\zeta_2\omega_{n,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2\zeta_N\omega_{n,N} \end{bmatrix} = \widehat{\boldsymbol{C}} \tag{3.155}$$

Pre- and post-multiplying both sides by $(\boldsymbol{\Phi}^T)^{-1}$ and $\boldsymbol{\Phi}^{-1}$, respectively, gives

$$\begin{aligned} (\boldsymbol{\Phi}^T)^{-1}\boldsymbol{\Phi}^T\boldsymbol{C}\boldsymbol{\Phi}\boldsymbol{\Phi}^{-1} &= (\boldsymbol{\Phi}^T)^{-1}\widehat{\boldsymbol{C}}\boldsymbol{\Phi}^{-1} \\ \boldsymbol{I}\boldsymbol{C}\boldsymbol{I} &= (\boldsymbol{\Phi}^T)^{-1}\widehat{\boldsymbol{C}}\boldsymbol{\Phi}^{-1} \\ \boldsymbol{C} &= (\boldsymbol{\Phi}^T)^{-1}\widehat{\boldsymbol{C}}\boldsymbol{\Phi}^{-1} \end{aligned} \tag{3.156}$$

Using the orthogonality property, a convenient expression for $\boldsymbol{\Phi}^{-1}$ can be developed. Recall that

$$\Phi^T M \Phi = I \tag{3.157}$$

Pre-multiplying both sides by $(\boldsymbol{\Phi}^T)^{-1}$ gives

$$\begin{aligned} (\Phi^T)^{-1}\Phi^T M \Phi &= (\Phi^T)^{-1} I \\ M\Phi &= (\Phi^T)^{-1} \end{aligned} \tag{3.158}$$

Similarly, post-multiplying both sides by $\boldsymbol{\Phi}^{-1}$, we get:

$$\begin{aligned} \Phi^T M \Phi \Phi^{-1} &= I \Phi^{-1} \\ \Phi^T M &= \Phi^{-1} \end{aligned} \tag{3.159}$$

Substituting the expressions for $(\boldsymbol{\Phi}^T)^{-1}$ and $\boldsymbol{\Phi}^{-1}$, into Equation 3.156 results in

$$C = (M\Phi)\hat{C}(\Phi^T M) \tag{3.160}$$

Note: $\boldsymbol{\Phi}$ contains mass normalized modes. If the modes are not mass normalized, then the above expressions will change to:

$$\boldsymbol{C} = (\boldsymbol{M}\boldsymbol{\Phi}\boldsymbol{M}_r^{-1})\widehat{\boldsymbol{C}}(\boldsymbol{M}_r^{-1}\boldsymbol{\Phi}^T\boldsymbol{M}) \tag{3.161}$$

where,

$$\boldsymbol{M}_r = \boldsymbol{\Phi}^T\boldsymbol{M}\boldsymbol{\Phi} \tag{3.162}$$

3.5 Modal Analysis for MDOF Systems

- The treatment of general N-DOF systems is similar to the 2-DOF system we studied earlier.
- All the properties, such as orthogonality apply to N-DOF systems as well.
- The damping matrix, $\boldsymbol{C}$, requires special attention, as damping must be proportional in order for the equations of motion to be converted into N SDOF systems in the modal domain.
- Having already considered the response to free vibration, we now turn our attention the study of modal analysis for forced vibration.

3.5.1 Modal Analysis for Forced Vibration

Consider the equation of motion for a MDOF system subjected to forced vibration.

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{F}(t) \tag{3.163}$$

where $\boldsymbol{M}$ is the $N \times N$ mass matrix, $\boldsymbol{C}$ is the $N \times N$ damping matrix, $\boldsymbol{K}$ is the $N \times N$ stiffness matrix, $\boldsymbol{F}(t)$ is the $N \times 1$ force vector, and N is the number of degrees of freedom. Using the following transformation,

$$\boldsymbol{x} = \boldsymbol{\Phi}\boldsymbol{y} \tag{3.164}$$

where $\boldsymbol{\Phi}$ is the $N \times N$ matrix of undamped modes. Pre-multiplying by $\boldsymbol{\Phi}^T$, we get

$$\Phi^T M \Phi \ddot{y} + \Phi^T C \Phi \dot{y} + \Phi^T K \Phi y = \Phi^T F(t) \tag{3.165a}$$

$$I \ddot{y} + \Phi^T C \Phi \dot{y} + \Lambda y = \Phi^T F(t) \tag{3.165b}$$

where

$$\boldsymbol{\Phi}^T\boldsymbol{C}\boldsymbol{\Phi} = \begin{bmatrix} 2\zeta_1\omega_{n,1} & 0 & \cdots & 0 \\ 0 & 2\zeta_2\omega_{n,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2\zeta_N\omega_{n,N} \end{bmatrix} \tag{3.166}$$

The N uncoupled equations of motion can be written as

$$\begin{aligned} \ddot{y}_1 + 2\zeta_1\omega_{n,1}\dot{y}_1 + \omega_{n,1}^2 y_1 &= \boldsymbol{\phi}_1^T\boldsymbol{F}(t) \\ &= P_1(t) \end{aligned} \tag{3.167a}$$

Scalar

$$\begin{aligned} \ddot{y}_2 + 2\zeta_2\omega_{n,2}\dot{y}_2 + \omega_{n,2}^2 y_2 &= \boldsymbol{\phi}_2^T\boldsymbol{F}(t) \\ &= P_2(t) \end{aligned} \tag{3.167b}$$

$$\vdots$$

$$\begin{aligned} \ddot{y}_N + 2\zeta_N\omega_{n,N}\dot{y}_N + \omega_{n,N}^2 y_N &= \boldsymbol{\phi}_N^T\boldsymbol{F}(t) \\ &= P_N(t) \end{aligned} \tag{3.167c}$$

The solution to any of the above SDOF system in Equations 3.167 is

$$y_i(t) = \frac{1}{\omega_{D,i}} \int_0^t P_i(\tau) e^{-\zeta_i \omega_{n,i}(t-\tau)} \sin \omega_{D,i}(t-\tau)\, d\tau \quad , i = \tag{3.168}$$

If there are initial conditions present, the response due to the initial conditions should be added to Equation 3.168.

Once the modal responses are calculated using Equation 3.168, the responses in the original physical coordinates can be found by the transformation in Equation 3.164. We will consider the principle of modal superposition to the following forced excitation case.

Example 3.10 Determine the response $\boldsymbol{x}(t)$ to the step force input, $\boldsymbol{F}(t) = [0 \quad 0 \quad F_0]^T$, assuming zero initial conditions. The mass and stiffness matrix are

$$\boldsymbol{M} = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \boldsymbol{K} = k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

Assume Rayleigh damping where

$$\alpha = 0.2\sqrt{\frac{k}{m}} \qquad \beta = 0.01\sqrt{\frac{m}{k}}$$

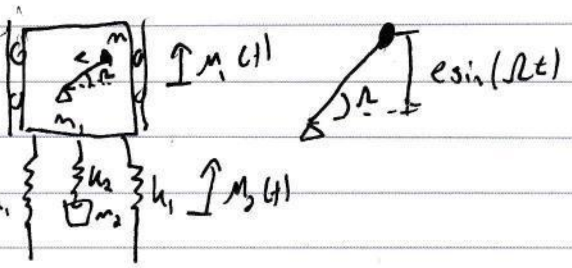
Solution: Begin by calculating the undamped natural frequencies and mass normalized modes.

$$\omega_{n,1} = 0.373\sqrt{\frac{k}{m}}\ \text{rad/s} \qquad \omega_{n,2} = 1.321\sqrt{\frac{k}{m}}\ \text{rad/s} \qquad \omega_{n,3} = 2.029\sqrt{\frac{k}{m}}\ \text{rad/s}$$

$$\begin{aligned} \boldsymbol{\Phi} &= [\boldsymbol{\phi}_1 \quad \boldsymbol{\phi}_2 \quad \boldsymbol{\phi}_3] \\ &= \frac{1}{\sqrt{m}}\begin{bmatrix} 0.269 & -0.878 & -0.395 \\ 0.501 & -0.223 & 0.836 \\ 0.582 & 0.299 & -0.269 \end{bmatrix} \end{aligned}$$

Using Equation 3.140a, we can compute ζ_1, ζ_2, and ζ_3.

(1)

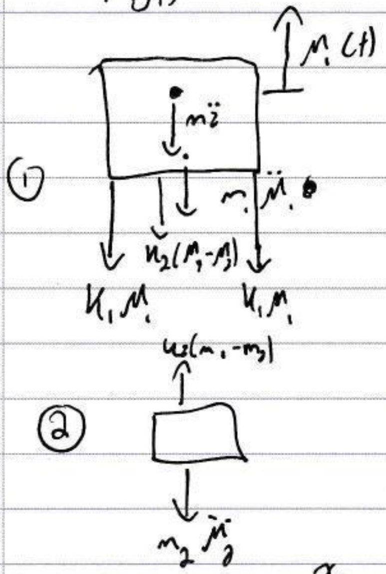


for rotating mass m_1 $\quad z(t) = x_1(t) + e\sin\Omega t$

$$\ddot{z}(t) = \ddot{x}_1(t) - e\Omega^2\sin(\Omega t)$$

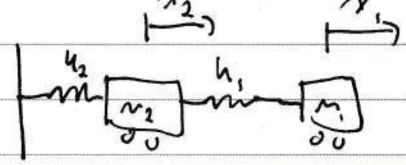
FBD

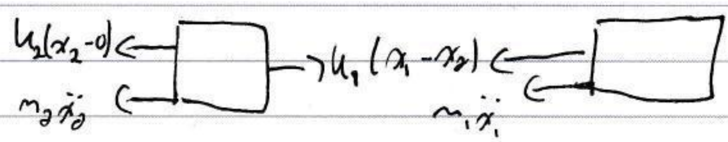
(1)



(2)

$\approx$





Date

① $\sum F_y = 0, \quad -m_1\ddot{x}_1 - 2k_1 x_1 - k_2(x_1 - x_2) - m\ddot{z} = 0$

$= -m_1\ddot{x}_1 - 2k_1 x_1 - k_2 x_1 + k_2 x_2 - m(\ddot{x}_1(t) - e\Omega^2 \sin\Omega t)$

$= (m_1 + m)\ddot{x}_1 + 2(k_1 + k_2)x_1 - k_2 x_2 = me\Omega^2 \sin\Omega t$

② $\sum F_y = 0$

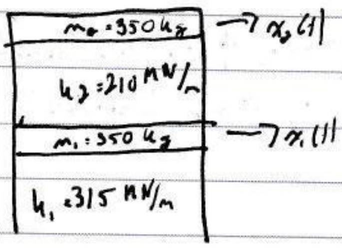
$m_2\ddot{x}_2 - k_2(x_1 - x_2) = 0, \quad m_2\ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0$

∴ for ①,

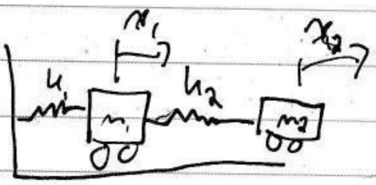
$$\begin{bmatrix} m_1 + m & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} me\Omega^2 \sin\Omega t \\ 0 \end{bmatrix}$$

Date

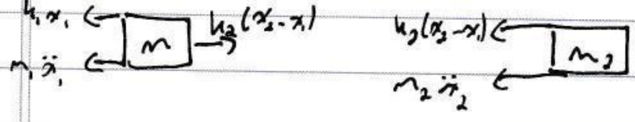
Question 2 - 2 storey shear building



=



a) Equation of motion



, $\Sigma F_x = 0$

$m_2 \ddot{x}_2 + k_2 (x_2 - x_1) = 0$, $m_2 \ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0$

$m_1 \ddot{x}_1 + k_1 x_1 - k_2 (x_2 - x_1) = 0$

$m_1 \ddot{x}_1 + (k_2 + k_1) x_1 - k_2 x_2 = 0$

$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{M} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}}_{K} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

b) Natural frequencies and mode shapes

$\omega_n^2 = k/m$ (SDOF)

$(-\omega_n^2 M + K) \phi_n = 0$

$\uparrow = 0$ $\therefore$ determinate

Date

$\det|K - \omega_n^2 M| = 0$

$$10^3 \begin{bmatrix} 350 & 0 \\ 0 & 350 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + 10^6 \begin{bmatrix} 525 & -210 \\ -210 & 210 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} 525000 - 350\omega_n^2 & -210000 \\ -210000 & 210000 - 350\omega_n^2 \end{vmatrix}$$

$\therefore \omega_n^2 = 300, 1800$

@ 300, $\therefore \omega_{n,1} = 17.321$ rads/sec, $T_{n,1} = 0.363$ (Fundamental Period)

@ 1800, $\omega_n = 42.426$ rads/sec, $T_{n,2} = 0.148$

For Mode 1, plug in 300

$$\begin{vmatrix} 525000 - 350(300) & -210000 \\ -210000 & 210000 - 350(300) \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0, \text{ letting } x_1 = 1$$

$\phi_{n,1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

For Mode 2, letting $\omega_n^2 = 1800$, $\phi_{n,2} = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$

Page

Date

c) Mass normalize the shape modes

$$(\alpha \phi_{n,:}^T) M (\alpha \phi_{n,:}) = 1$$

$$\alpha^2 \phi_{n,:}^T M \phi_{n,:} = 1$$

Mode 1, $\alpha_1^2 \phi_{n,1}^T M \phi_{n,1} = 1$

$$\alpha_1^2 [1 \;\; 2] \, 10^3 \begin{bmatrix} 350 & 0 \\ 0 & 350 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1$$

$$1750 \alpha_1^2 = 1, \quad \alpha_1 = 0.0239$$

Mode 2, $\alpha_2^2 [1 \;\; -0.5] \, 10^3 \begin{bmatrix} 350 & 0 \\ 0 & 350 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = 1$

$$437.5 \alpha_2^2 = 1, \quad \alpha_2 = 0.0478$$

$$\therefore \phi_n = \begin{array}{cc} \phi_{n,1} & \phi_{n,2} \end{array}$$
$$\phi_n = \begin{bmatrix} \alpha_1 (1) & \alpha_2 (1) \\ \alpha_1 (2) & \alpha_2 (-0.5) \end{bmatrix}$$

$$= \begin{bmatrix} 0.0239 & 0.0478 \\ 0.0478 & -0.0239 \end{bmatrix}$$