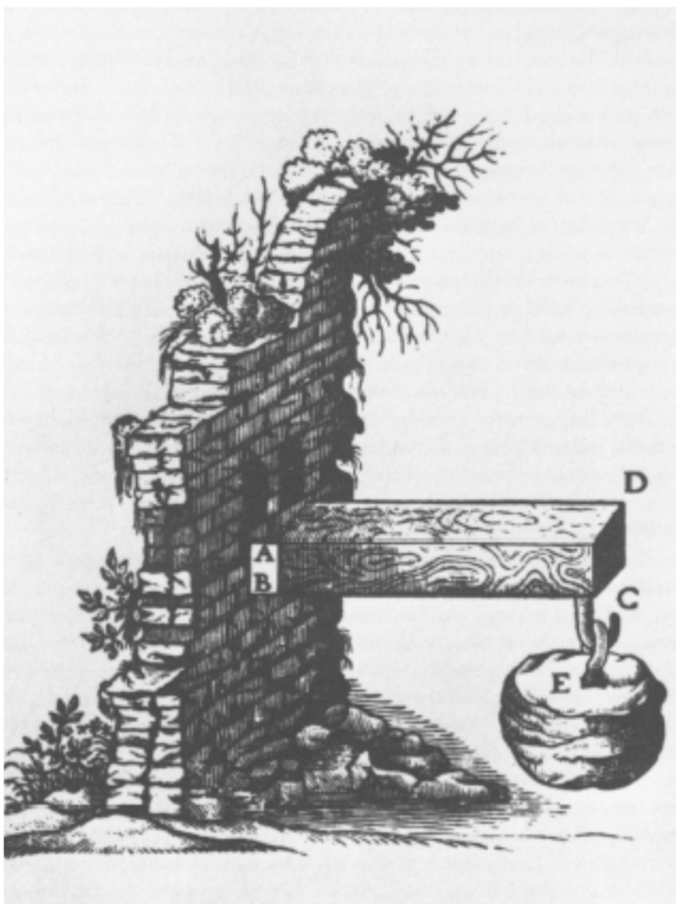
Department of Civil & Environmental Engineering

CIVE 303

Structural Analysis 1

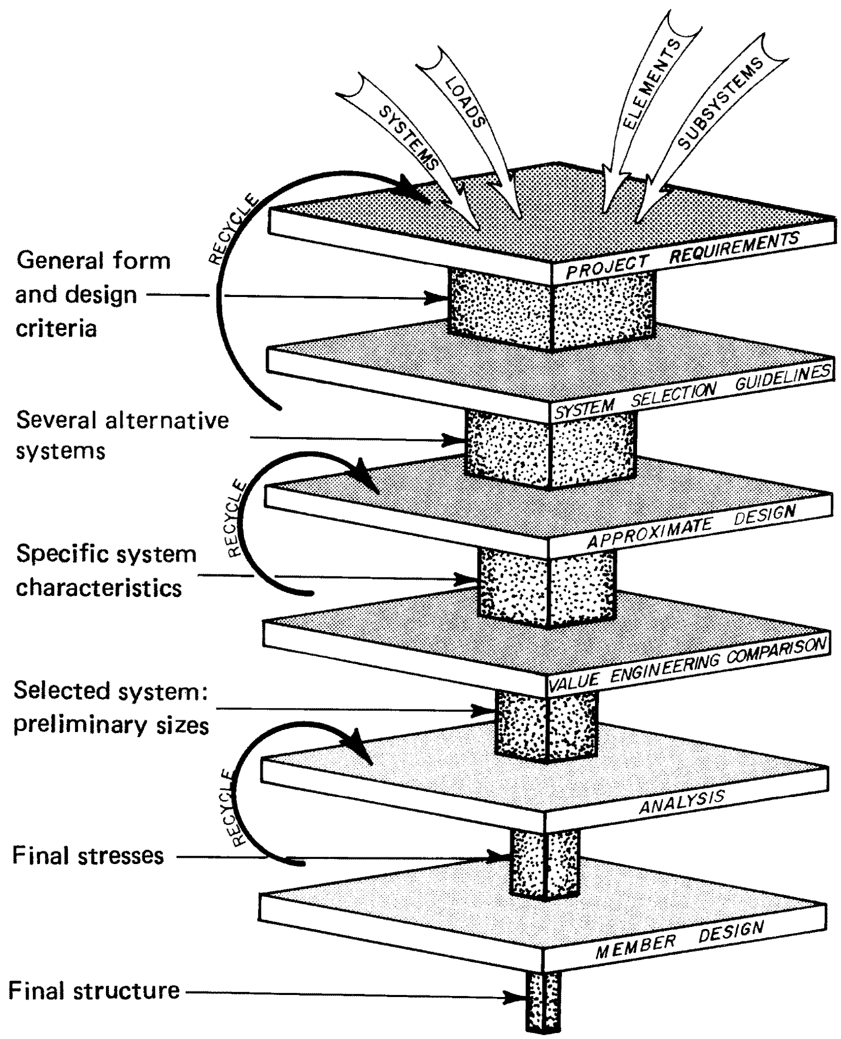


Galileo's Beam Theory
Image: M. Salvadori, *Why Buildings Stand Up*

Instructor: Cory Zurell

Introduction

Structural Analysis is a fundamental step in the process of structural design.



Coleman, R. A. *Structural Systems Design*, Prentice Hall, 1983.

A framework of codes and standards governs the design of building and bridge structures. These documents are generally based on Limit States Design principles.

Limit States Design is comprised of two fundamental criteria:

Ultimate Limit State

- Concerned with the safety of the structure
- Strength and stability
  - Includes fatigue (which is sometimes treated as a separate limit state – such as in the bridge code)
- Basic relationship is: factored resistance > factored load effects

$$\phi R \geq \sum \alpha_i S_i$$

- Structural analysis is used to determine member design forces based on code-specified conditions and load effects
- Member design then establishes the resistance necessary to resist the load effects

Serviceability Limit State

- Concerned with the useability of the structure
- Deflections, vibrations
- Subjective criteria
- Structural analysis is used to determine displacements/deflections, natural frequencies and mode shapes based on code-specified conditions and load effects

Structural Analysis (why we're here)

Given:

- Applied loads
- Imposed deformations
- Material properties (yield strength, rupture strength, E, G, etc.)
- Geometry (span, length, height, cross-section, etc.)
- Boundary conditions (support conditions, joints, etc.)

Need to know:

- Internal forces/stresses
- External reactions
- Deflections/deformations/rotations

But structural analysis is not exact.

> *"Engineering is the art of modelling materials we do not wholly understand, into shapes we cannot precisely analyse, so as to withstand forces we cannot properly assess, in such a way that the public has no reason to suspect the extent of our ignorance."*
>
> Dr. A. R. Dykes to the British Institution of Structural Engineers, 1976

Assumptions and Conventions

Structural engineering relies upon managing intrinsic uncertainty. Many assumptions and simplifications are employed in order to make analysis practical and efficient, and to result in designs that are conservative in relation to the degree of uncertainty.

Basic Assumptions

In the context of this course, the following assumptions are appropriate:

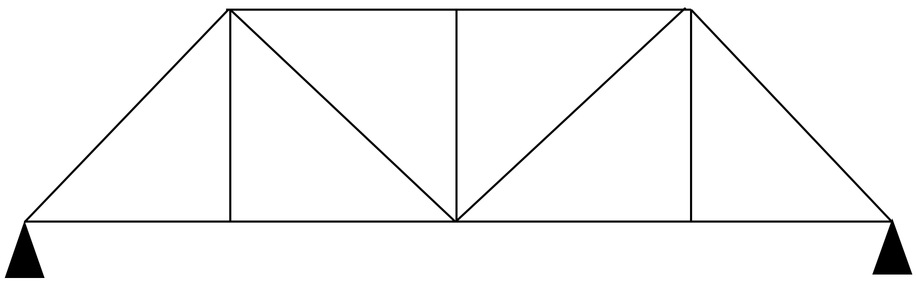
- Linear elastic, homogeneous, isotropic materials are assumed
    - Relationship between stress and strain is known
    - Compatibility of strain – plane sections remain plane

- Static Analysis
    - Loads are static (i.e. no or "low" variance with time) - nothing moves

- Small deformations/small rotations
    - Relative to the member size/span
    - Working from original geometry
    - No second-order effects

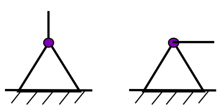
Idealized Framework

A wireframe idealization will be assumed for the purposes of analysis.

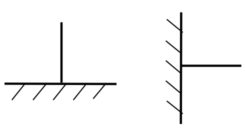


Idealized Supports/Boundary Conditions

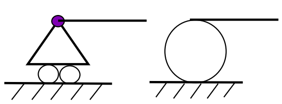
Supports are assumed in such a way to simplify analysis while ensuring conservative results.



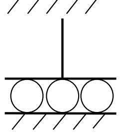
Pinned, $M=0$, $R_x/R_y \neq 0$



Fixed, $M / R_x / R_y \neq 0$



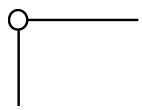
Roller, $M=0$, $R_x=0$, $R_y \neq 0$



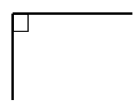
Fixed Roller, $M \neq 0$, $R_x=0$, $R_y \neq 0$

Internal Joints

True pinned or fixed joints are generally impractical to construct however they are generally assumed in analysis.



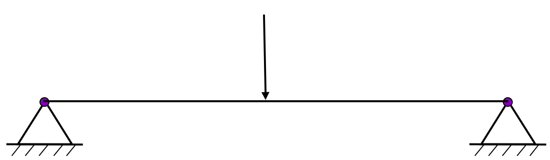
Pinned, M=0



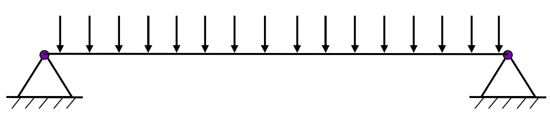
Fixed, M≠0

Idealized Loads

Apart from the self-weight of a structure, loads are seldom uniform or static however the majority of structural design generally assumes uniform, static loading.



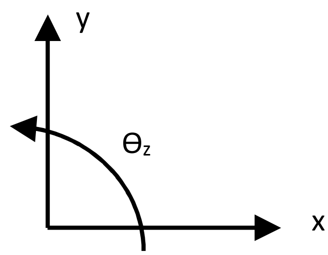
Point Loads



Distributed Loads

Conventions

This course focuses on two dimensional structures. For any given point on a two-dimensional structure, there are three degrees of freedom:



This results in three equations of equilibrium:

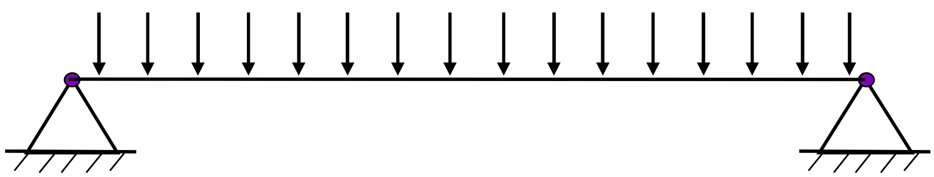
$$\sum F_x = 0$$
$$\sum F_y = 0$$
$$\sum M = 0$$

Internal Forces and Sign Conventions

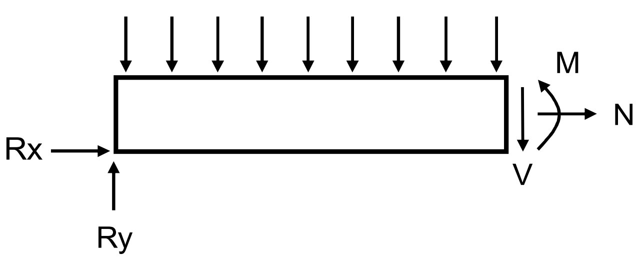
Axial Force

Shear Force

Bending Moment



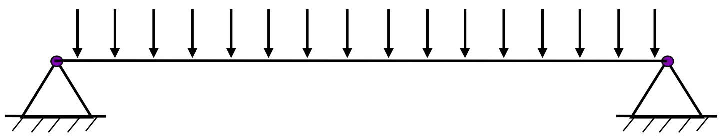
Overall



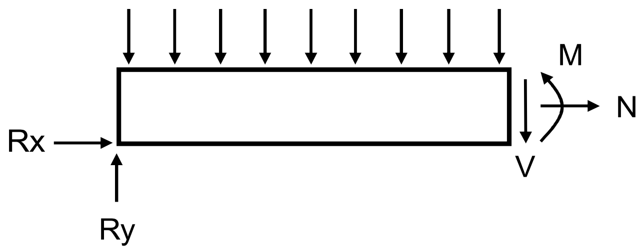
Review: Axial Force, Shear Force and Bending Moment Diagrams

There are various methods for constructing these diagrams:

1. Construct by cutting sections (the digital approach)



2. Construct by developing the equations (the analog approach)



$V(x) = R_y - wx$
$M(x) = R_y x - wx^2/2$

But, at discontinuities in loading/support conditions, one must determine new equations.

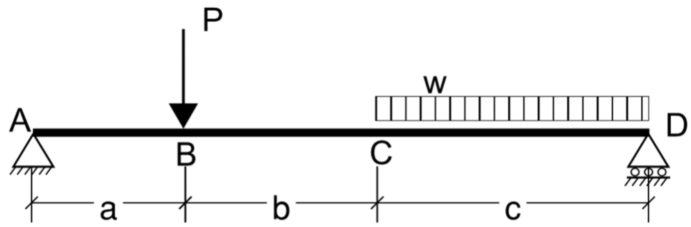
3. Construct using the known relationships between loads, shear, moment, etc. (the shortcut)

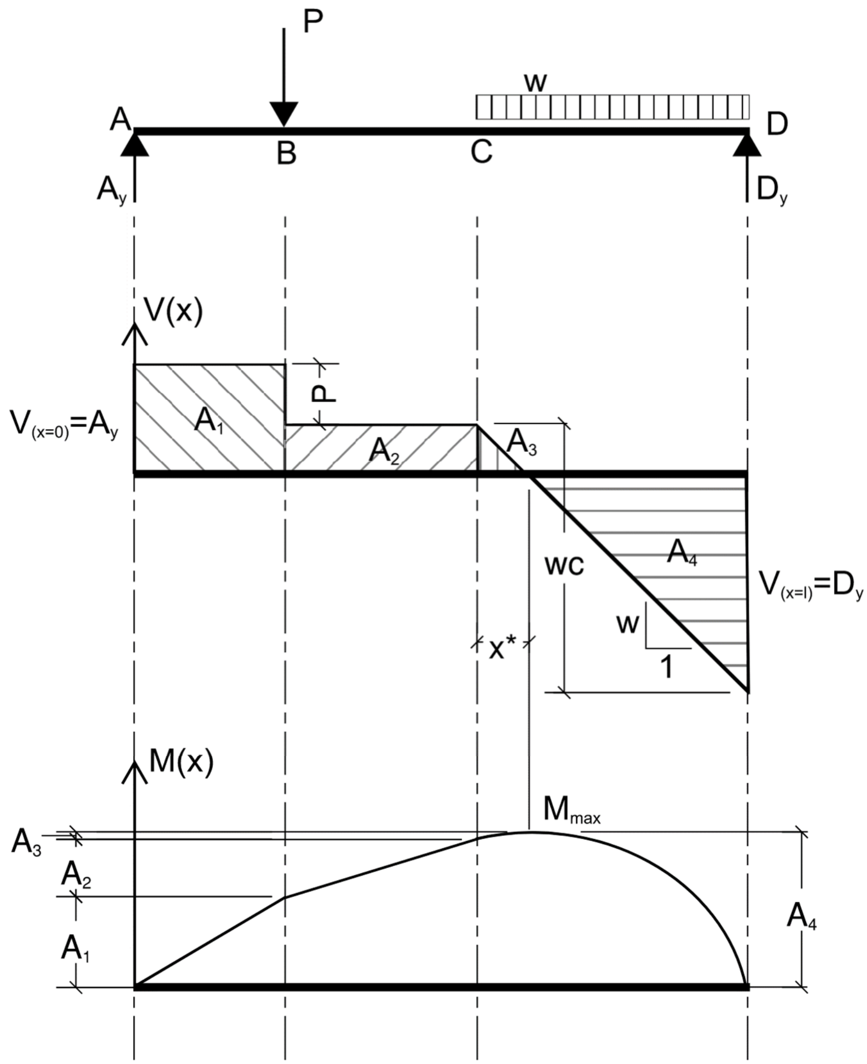
Recall some relationships between load, shear and moment:

- The variation in shear between concentrated loads is constant; the variation of moment is linear
- The variation of shear over length of uniformly distributed load is linear; the variation of moment is parabolic
- The variation of shear over a length of linearly varying load is parabolic; the variation of moment is cubic

In general:

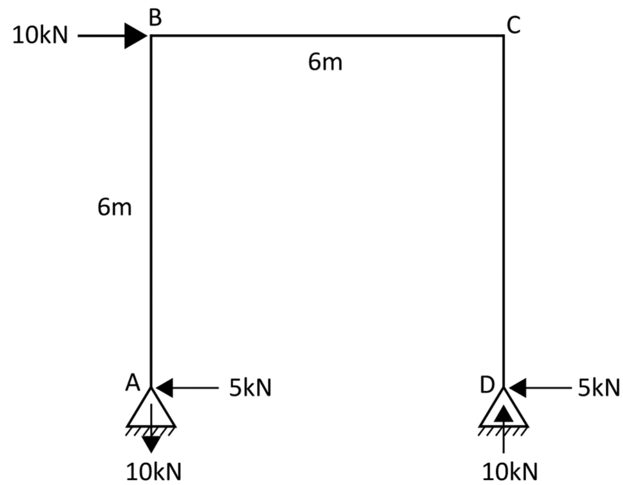
Load distribution of degree, n, results in a shear distribution of degree, n+1, and a moment distribution of degree, n+2.

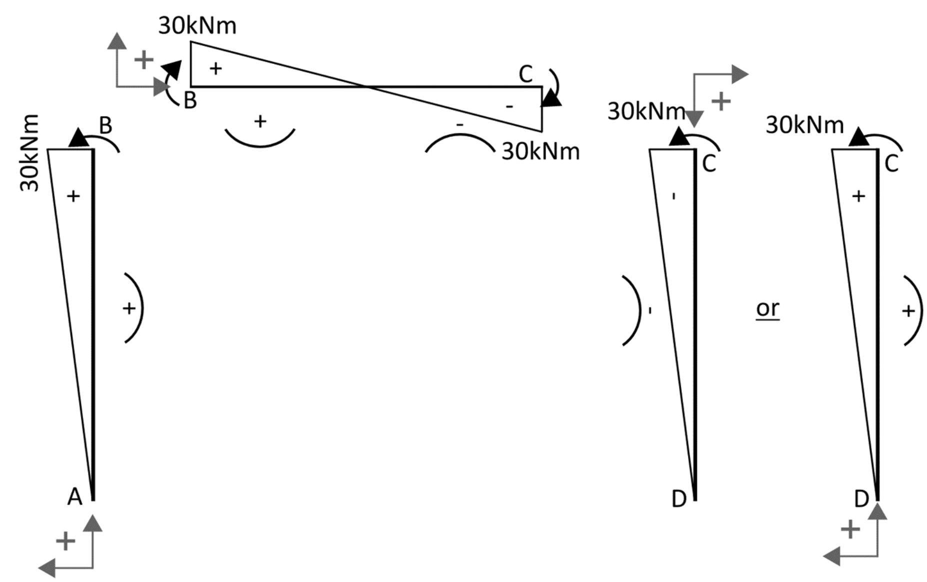




Moment Diagrams

Moment diagrams on frames can get confusing in that positive and negative depends upon one's perspective.

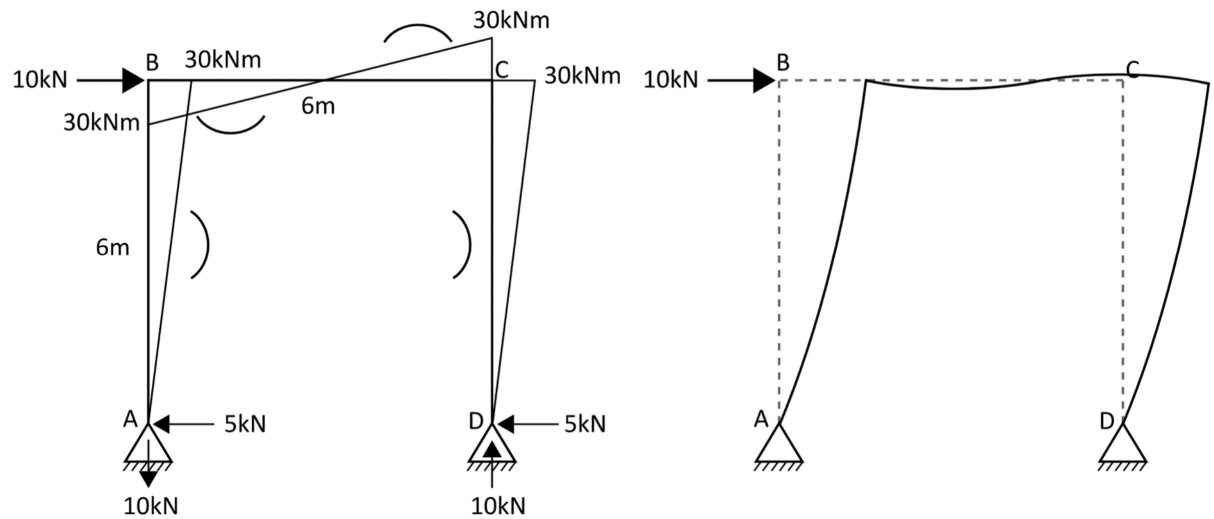




Rather than positive and negative, consider plotting the moment on the compression side or tension side of the bending member. Either way will produce the required results as long as you are consistent.

The convention of plotting moments on the tension side of the member is extremely common in the structural design industry. It provides a somewhat intuitive reflection of the anticipated deflections and, in concrete design, reflects the side of the member requiring reinforcing. That being said, computer analysis programs will typically display moments plotted on the compression side (i.e. positive is "up").

Plotted on the tension side:



Deflections

Deflections and rotations relate to the serviceability of a structure. While assessment of deflections is somewhat subjective, industry standards have been established over time to ensure structures perform adequately so as not to adversely impact occupants. Such impacts include:

- Cracking of finishes
- Vibration of floors
- Slope of occupied spaces
- Lateral drift of structures
- Etc.

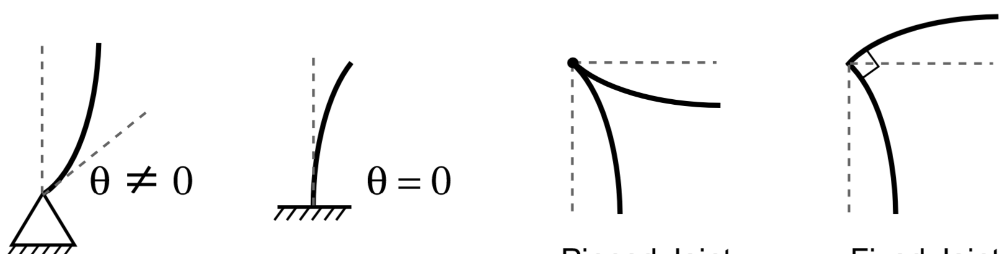
Sketching Deflected Shapes

Sketching is a valuable engineering skill. Being able to quickly represent a structure's deformations under load using an intuitive understanding of structural behaviour provides another quick means of checking one's work—to make sure the analysis results make sense.

Bending moment diagrams add to the information one has with which to make such quick deflection sketches. Moment diagrams convey sections of positive or negative curvature, leading to deflected shapes. Subject to the constraints of the interior joints, exterior supports and applied loads, a qualitative deflected shape can be accurately developed with little or no analysis. An approximate analysis can be performed based on the assumed deflected shape.

Positive Bending
Concave up

Negative Bending
Concave down



Pinned Base

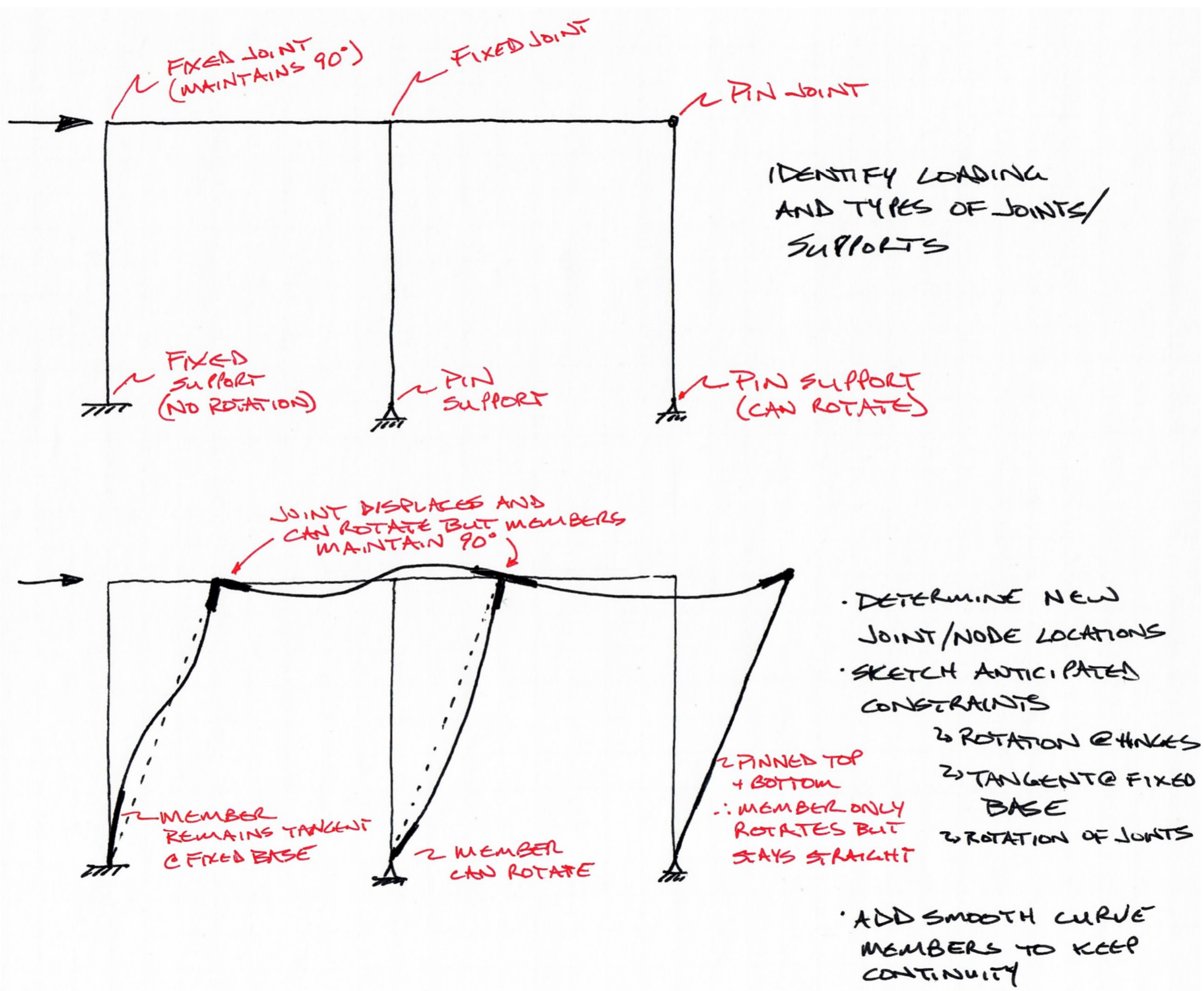
Fixed Base

Pinned Joint Members may rotate w.r.t. one another

Fixed Joint Members do not rotate w.r.t. one another

Drawing a deflected shape:

- Deflections are exaggerated
- Members generally stay the same length
- Fixed joints stay fixed – do not "open" or "close"
- Start with the joints – plot the new locations due to the applied loads – think about rotation of the joints
- Extend members between joints, providing flexure to accommodate joint rotations
- Maintain appropriate support conditions
- Consider inflection points



Deflection Analysis

Deflections can be determined analytically using various means:

- Double Integration
- Moment Area
- Conjugate Beam
- Castigliano's Theorem
- Virtual Work

Review of Virtual Work for Deflection Analysis

The principle of virtual work is based on conservation of energy.

$$U = W$$

(Internal Strain Energy stored in the structure) = (work done by external loads)

The method allows one to compute a deflection, or a component of a deflection at any point in a structure. It also allows the effects of temperature or support settlements to be evaluated.

Deflections of beams and frames are a result of flexural, shear and axial deformation of members. In most beam and frame systems, bending deflections are significantly larger than shear deformations and axial deformations, thus the latter two are <u>typically</u> neglected.

For deflections:

$$1 \times \Delta = \int_0^L m \left(\frac{M}{EI}\right) dx$$

Virtual Ext. Force **X** Real Ext. Displacement = Virtual Int. Load **X** Real Int. Displacement

Where:

1 = a virtual unit load applied at the location and in the direction for which one wants to determine the deflection, Δ

Δ = deflection at the location of interest caused by the real loads applied

m = internal bending moment caused by the virtual load, 1

M = internal bending moment caused by the real loads applied

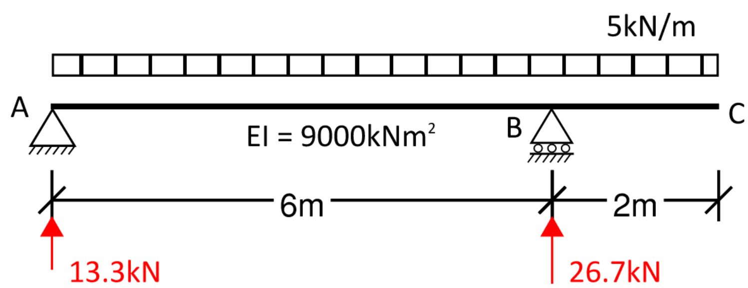
E = Young's modulus of the material

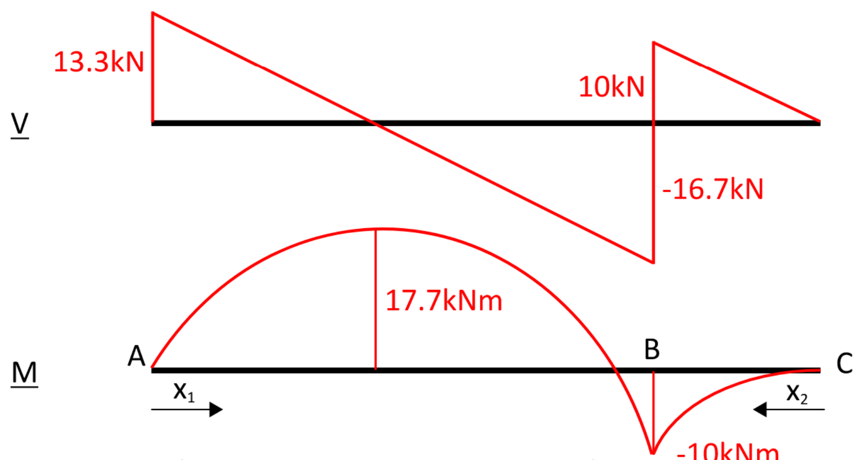
I = moment of inertia of the section

Evaluation of these integrals can be done in a number of ways:

- Integration of M(x) and m(x) functions
- "Simple" method
- Using table of product integrals – see A.2 in textbook

Review Example: Find the deflection at the tip of the cantilever.



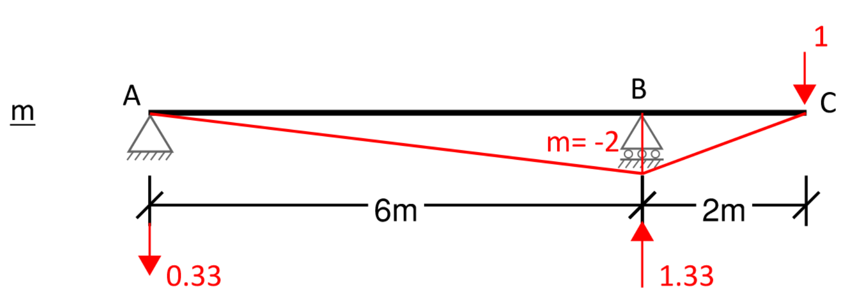


(Moment plotted on compression side)

From A to B: $$M(x_1) = 13.3x_1 - \frac{5}{2}{x_1}^2$$

From C to B: $$M(x_2) = -\frac{5}{2}{x_2}^2$$

Virtual Load



From A to B: $$M(x_1) = -0.33x_1$$

From <u>C to B</u>: $M(x_2) = -x_2$

Using Virtual Work:

$$1 \times \Delta_C = \int_A^B m(x_1)\frac{M(x_1)}{EI}dx_1 + \int_C^B m(x_2)\frac{M(x_2)}{EI}dx_2$$

$$1 \times \Delta_C = \frac{1}{EI}\int_0^6 \left(-\frac{x_1}{3}\right)(13.3x_1 - \frac{5}{2}{x_1}^2)dx_1 + \frac{1}{EI}\int_0^2 -x_2(-\frac{5}{2}{x_2}^2)\,dx_2$$

$$1 \times \Delta_C = \frac{1}{EI}\left(-\frac{4.44{x_1}^3}{3} + \frac{0.833{x_1}^3}{4}\right)_0^6 + \frac{1}{EI}\left(\frac{2.5{x_2}^4}{4}\right)_0^2$$

$$= \frac{1}{EI}(-320 + 270) + \frac{1}{EI}(10)$$

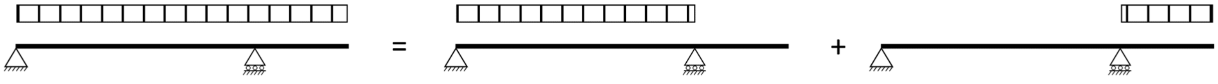
$$\Delta_C = -\left(\frac{40}{EI}\right) = -0.0044m = -4.4mm$$

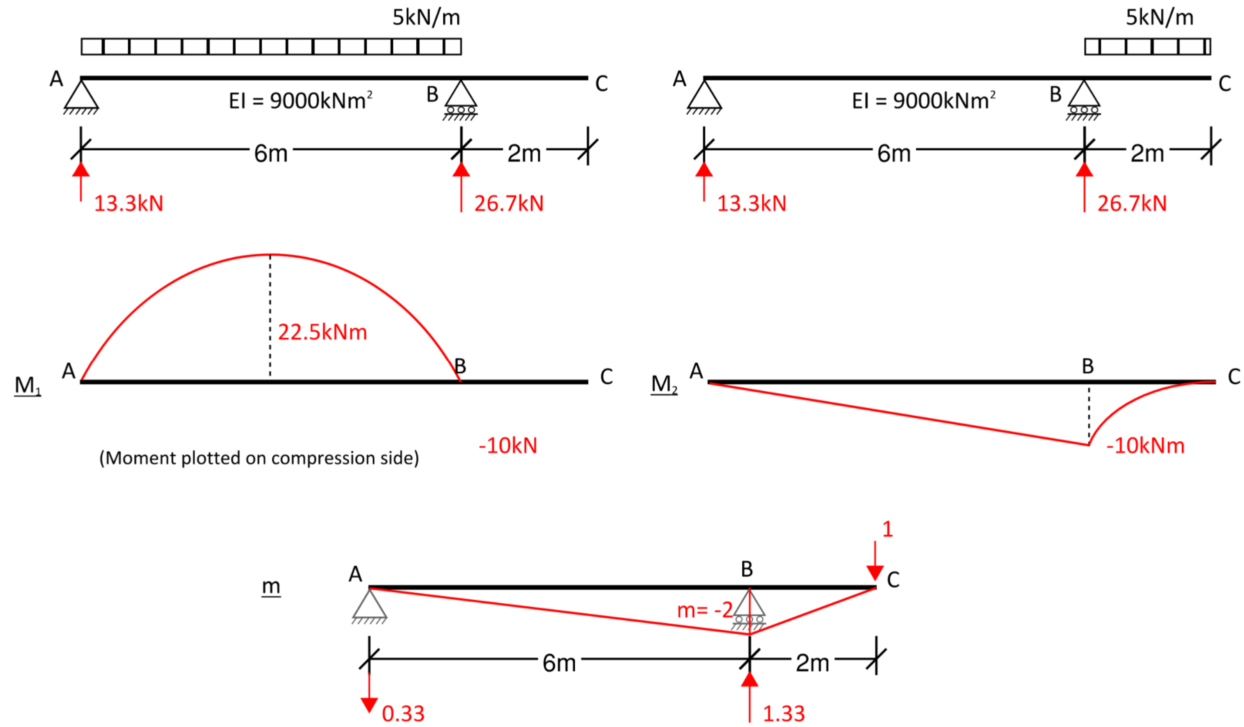
Negative sign indicates the direction of displacement is opposite to the direction of the applied load – i.e. the deflection is up.

<u>OR</u>

Using the "simple" method:

- Product integral is equal to area under moment diagram, M(x), times the value of m(x) corresponding to the centroid of the area under M(x)
- BUT – M(x) and m(x) are not continuous
  - Need to break up into continuous segments
  - Make it easier to use superposition





$$1 \times \Delta_C = \int m \frac{M}{EI} dx$$

$$= \frac{1}{EI}\left[\frac{2}{3}(6)(22.5)\left(-\frac{2}{2}\right) + \frac{1}{2}(6)(-10)\left(\frac{2}{3}(-2)\right) + \frac{1}{3}(2)(-10)\left(\frac{3}{4}(-2)\right)\right]$$

$$= \frac{1}{EI}[-90 + 40 + 10]$$

$$\Delta_C = -\frac{40}{EI} = -4.4mm$$

(The sign is relative to the direction of the virtual load, i.e. 4.4mm up)

<u>OR</u>

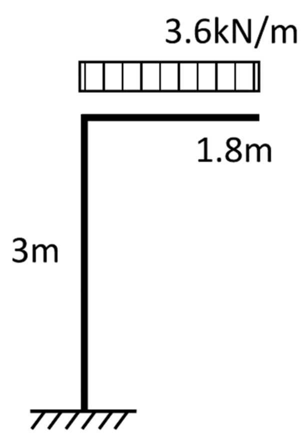
From beam tables:

$$\Delta = \frac{wa}{24EI}(4a^2l - l^3 + 3a^3) = -4.4mm$$

(Positive is assumed down, therefore tip deflection is up.)

So why use Virtual Work?

Consider a frame for a bus stop shelter. How much does the tip deflect horizontally and vertically under the indicated load?



For rotations:

$$1 \times \theta = \int_0^L m\left(\frac{M}{EI}\right)dx$$

Where:

1 = a virtual unit moment applied at the location and in the direction for which one wants to determine the rotation, θ

θ = deflection at the location of interest caused by the real loads applied

m = internal bending moment caused by the virtual moment, 1

M = internal bending moment caused by the real loads applied

E = Young's modulus of the material

I = moment of inertia of the section

Deflections of trusses are a result of axial shortening and lengthening of individual members rather than flexure of members.

$$1 \times \Delta = \sum_{i=1}^{n} \frac{F_i f_i L_i}{A_i E_i}$$

Virtual Ext. Force **X** Real Ext. Displacement = Virtual Int. Load **X** Real Int. Displacement

Where:

1 = a virtual unit load applied at the location and in the direction for which one wants to determine the deflection, Δ

Δ = deflection at the location of interest caused by the real loads applied

F_i = internal axial force in member, i, caused by the real applied loads

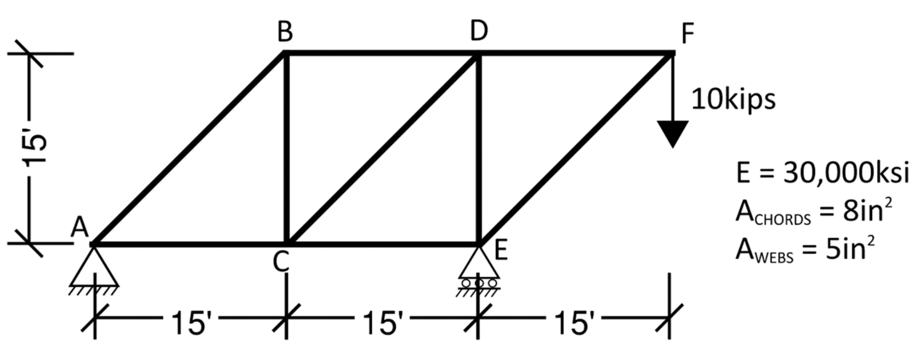
f_i = internal axial force in member, i, caused by the external virtual load, 1

L = length the member, i

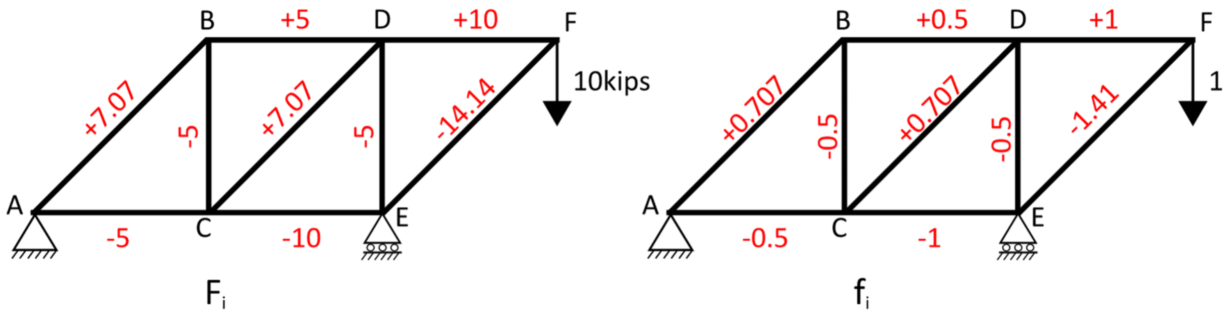
E = Young's modulus of the material

A_i = cross-sectional area of member, i

Review Example: Determine the vertical deflection of Node F



Analyse the truss under the applied loads as well as under an assumed virtual load applied to the node for which the deflection is desired.



A tabular format is frequently used to keep track of the information for each member.

Member	F_i	f_i	L_i (in)	A_i (in^2)	$F_if_iL_i/A_i$
AB	$+5\sqrt{2}$	$+0.5\sqrt{2}$	254.6	5	254.6
BC	-5	-0.5	180	5	90
CD	$+5\sqrt{2}$	$+0.5\sqrt{2}$	254.6	5	254.6
DE	-5	-0.5	180	5	90
EF	$-10\sqrt{2}$	$-\sqrt{2}$	254.6	5	1018.4
AC	-5	-0.5	180	8	56.25
CE	-10	-1	180	8	225
BD	+5	+0.5	180	8	56.25
DF	+10	+1	180	8	225
				$\sum =$	2270

$$1 \times \Delta_F = \sum_{i=1}^{n} \frac{F_i f_i L_i}{A_i E}$$

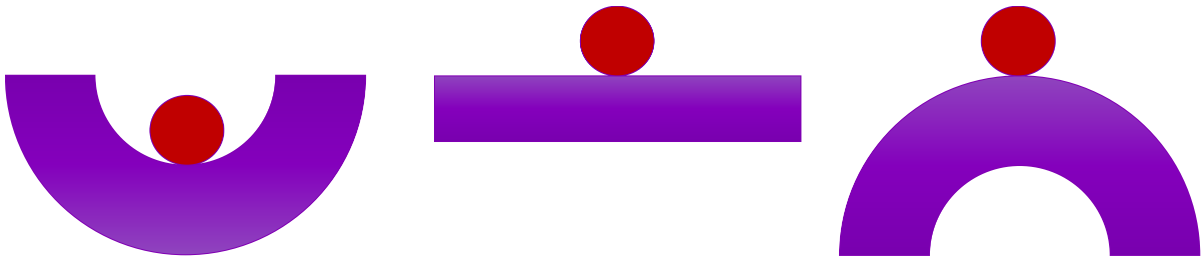
$$= 2270/30000$$

$$\Delta_F = 0.076'' \ (\sim 2\text{mm})$$

(result is positive, therefore the deflection is in the same direction as the applied virtual load)

Stability and Determinacy

Recall Equilibrium



Stable Equilibrium Neutral Equalibrium Unstable Equilibrium

Stability

In order to be in static equilibrium, a structure must be stable. Stability can be a condition external to the structure (a condition of the support conditions), or internal to the structure (a condition of the formulation of the structure itself).

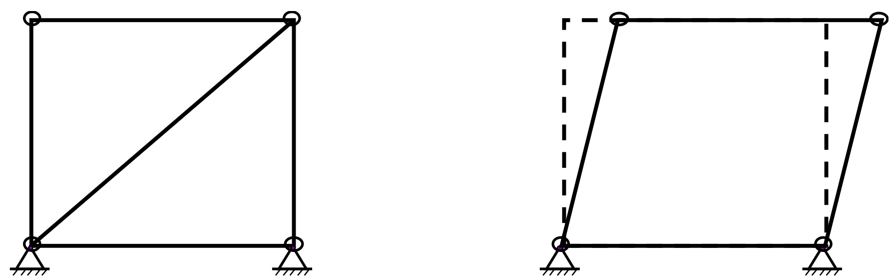
External Stability

Three conditions are required to ensure external stability:

- Three reactions
- Non-concurrent forces
- Non-parallel forces

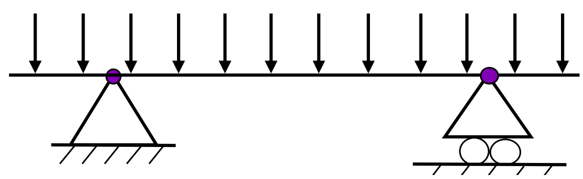
Internal Stability

A structure must have enough internal constraints so as not to form a mechanism.



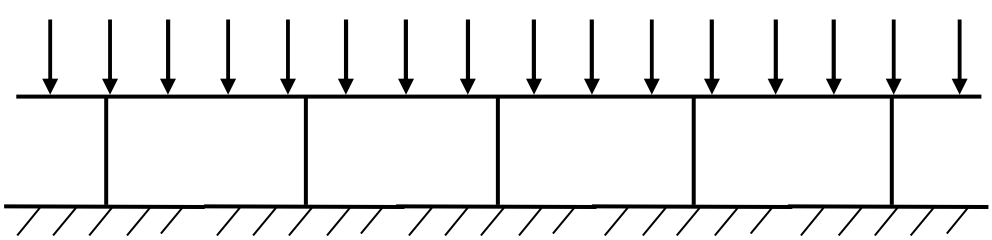
Statically Determinate Structures

In previous courses you have considered only statically determinate structures. That is, internal forces and external reactions may be determined by statics.



Statically Indeterminate Structures

In indeterminate structures, internal forces and external reactions cannot be determine by statics. That is, you still have 3 equations, but more than three unknowns. This is also known as redundancy.



Degrees of indeterminacy

Let R represent the number of external restraints.

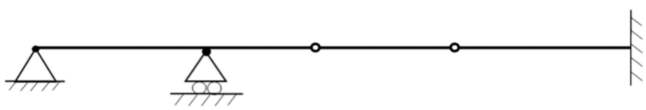
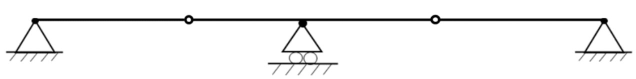
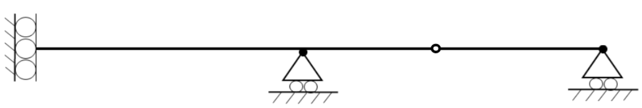
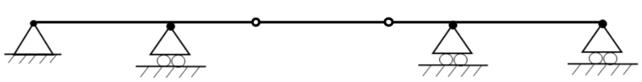
Let C represent the number of internal conditions.

The degree of indeterminacy is given by: $R - (3 + C)$

$R < (3 + C)$ The system is unstable

$R = (3 + C)$ The system is determinate provided it is stable

$R > (3 + C)$ The system is indeterminate to the degree $R-(3+C)$, provided it is stable

Beam system	R	C		
	6	2	R>(3+C) 6 > 5	Indeterminate to 1 degree, stable
	5	2	R=(3+C) 5 = 5	unstable
	4	1	R=(3+C) 4 = 4	unstable
	5	2	R=(3+C) 5 = 5	Determinate and stable
	7	2	R>(3+C) 7 > 5	Indeterminate to 2 degree, stable

For a truss:

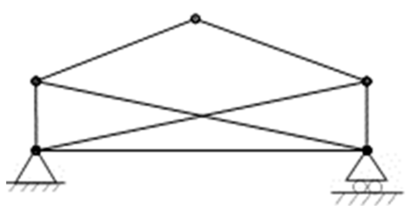
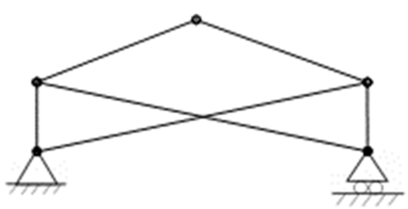
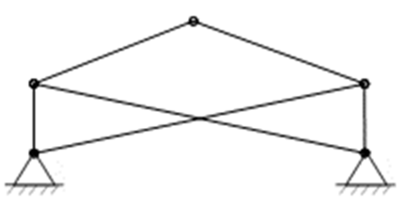
Let b represent the number of bars or members in the truss

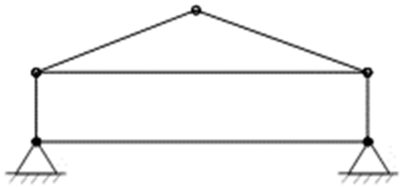
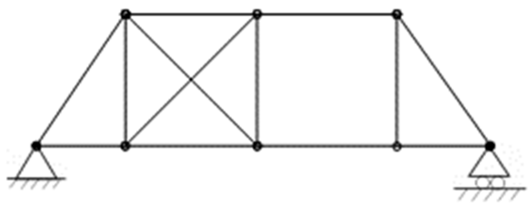
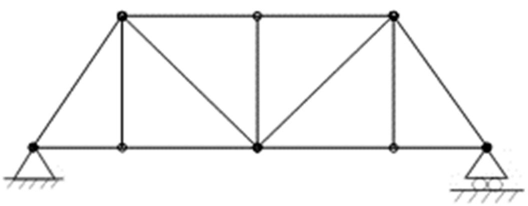
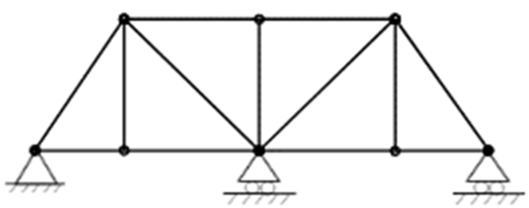
Let r represent the number of external reactions supporting the truss

Let j represent the number of joints in the truss (i.e. where bars intersect)

The degree of indeterminacy is given by: $b + r - 2j$

> $b + r < 2j$ The truss is statically unstable
>
> $b + r = 2j$ The truss is statically determinate (provided it is also stable)
>
> $b + r > 2j$ The truss is statically indeterminate (provided it is also stable)

Truss	b	r	j		
	7	3	5	10 = 10	Stable and determinate
	6	3	5	9 < 10	unstable
	6	4	5	10 = 10	Stable and determinate

	6	4	5	10 = 10	unstable
	13	3	8	16 = 16	unstable
	13	3	8	16 = 16	Determinate and stable
	13	4	8	17 > 16	Indeterminate to 1 degree, stable

Lastly, for a frame

Let b represent the number of members framing between joints

Let r represent the number of external reactions supporting the frame

Let j represent the number of joints in the frame

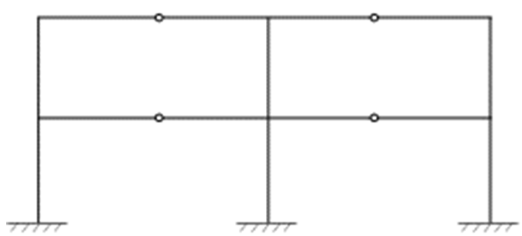
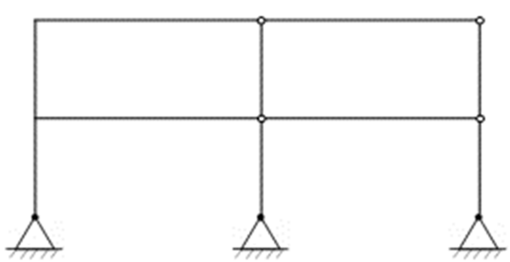
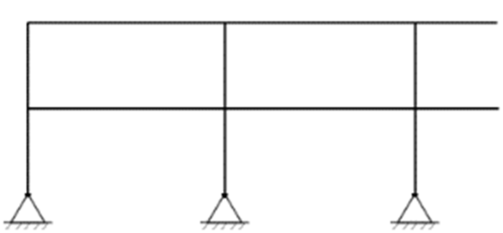
Let c represent the number of internal constraints in the frame

The degree of indeterminacy is given by: $(3b + r) - (3j + c)$

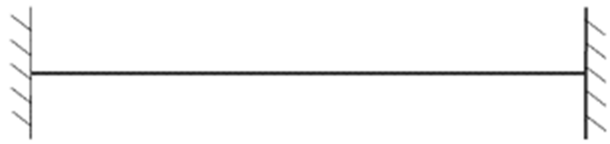
$3b + r < 3j + c$ The frame is statically unstable

$3b + r = 3j + c$ The frame is statically determinate (provided it is also stable)

$3b + r > 3j + c$ The frame is statically indeterminate (provided it is also stable)

Frame	b	r	j	c		
	10	9	9	0	39 > 27	Indeterminate to the twelfth degree
	10	9	9	4	39 > 31	Indeterminate to the eighth degree
	10	6	9	8	36 > 35	Indeterminate to one degree
	10	6	9	0	36 > 27	Indeterminate to the 9^{th} degree (ignore cantilevers)

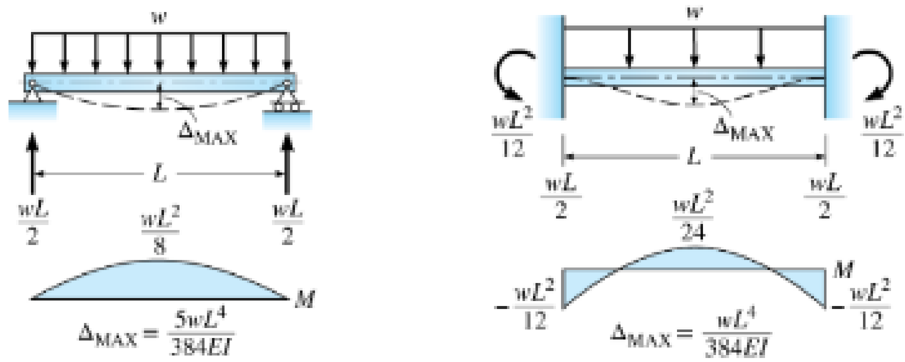
An alternate method of determining the degree of indeterminacy is to introduce releases into the system until it becomes strictly stable.



Indeterminate Structures

Why?

Indeterminate structures are generally constructed to achieve efficiencies or to simplify construction.



Advantages:

- Smaller moments and deflections, therefore smaller members can be used to achieve the same level of service
- Depending upon the material, they are easier to construct

Disadvantages:

- Fabrication/construction can be more costly depending upon the material used and connections required
- Unintentional stresses can be imposed due to restraints caused by the redundancies

Analysis of Indeterminate Structures

Indeterminate structures cannot be solved simply using static equilibrium. They have more internal or external constraints than are required for a strictly stable structure—that is, there are more unknowns then there are equations to solve.

Three conditions must simultaneously be satisfied in order to perform an indeterminate analysis:

- Linear-elastic behaviour -
- Equilibrium – i.e. sums of forces and moments = 0 – it's not moving
- Compatibility – no translational or rotational discontinuities that are not allowed by the actual structure

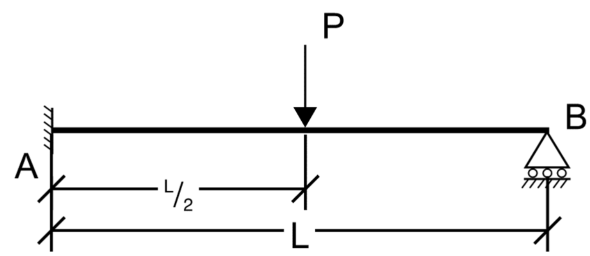
Due to the assumption of linear-elastic behaviour, superposition may be used in the analysis, and therefore two general approaches may be used for the analysis:

- Force Method (Flexibility Method): Force x Flexibility = Displacement
    - Redundant forces are determined and then remaining forces are solved for using equilibrium
    - Compatibility is explicitly satisfied
    - Equilibrium is implicitly satisfied
- Displacement Method (Stiffness Method): Force = Stiffness x Displacement (F=kx)
    - Equilibrium and force-displacement relationships are used to solve for independent nodal displacements equal to the degree of kinematic indeterminacy. Forces are then determined using compatibility and force-displacement relationships
    - Equilibrium is explicitly satisfied
    - Compatibility is implicitly satisfied

Force Method (Flexibility Method/Consistent Deformations) (Leet *et al*. Chapter 9)

Statically Indeterminate Beam/Frame

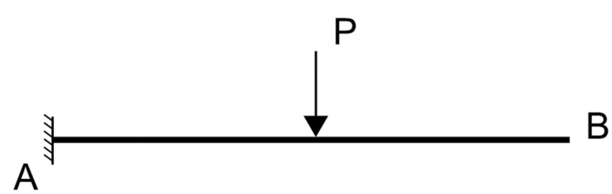
Consider:



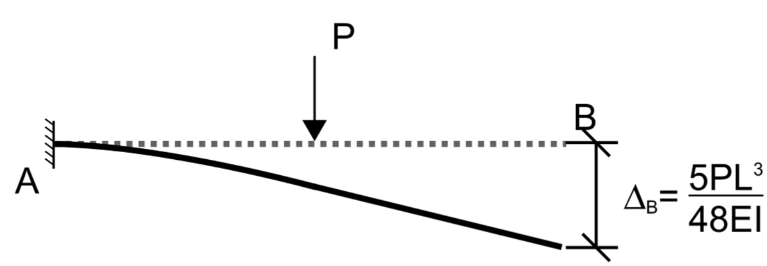
1. Determine the degree of static indeterminacy

$$R - (3 + C) = 4 - (3 + 0) = 1$$

2. Remove the redundant force(s) to create a statically determinate primary structure.



3. Perform displacement analysis of the primary structure – determine displacements at the locations of the redundant force(s).



4. Remove the applied loads and then apply a unit load at the location of, and in the direction of, the redundant force(s). Compute the displacement in the direction of the redundant unit load (this step is determining the flexibility coefficient).

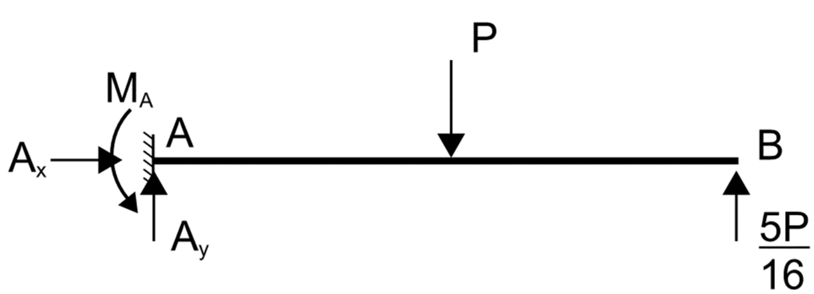
5. Use superposition to impose compatibility constraints onto the system in order to solve for the redundant force.

$$\Delta_B + \Delta_B' = 0$$

$$-\frac{5PL^3}{48EI} + \frac{B_y L^3}{3EI} = 0$$

$$B_y = \frac{5P}{16}$$

6. The structure is now determinate. The system can be solved using equilibrium.



$\Sigma F_x = 0;\ A_x = 0$

$\Sigma F_y = 0;\ A_y + {}^{5P}/_{16} = P$
$A_y = {}^{11P}/_{16}$

$\Sigma M_A = 0;\ M_A = P\,{}^{L}/_{2} - {}^{5PL}/_{16}$
$M_A = {}^{3PL}/_{16}$

In more general terms:

1. Determine the degree of static indeterminacy, n.
2. Select the n redundant forces to be solved by the analysis (X_1, X_2,....X_n) and identify the corresponding compatibility conditions (i.e. known displacements, rotations, etc.)
3. Remove the redundant forces to produce the primary structure.
4. Analyse the primary structure. Perform a displacement analysis under the applied loads where displacements at the locations and in the directions of the redundant forces are determined (Δ_1, Δ_2,...Δ_n).
5. Remove the applied loads and sequentially apply a unit load at the location and in the direction of the redundant forces determined in step 2. The resulting deflections due to these unit loads are the flexibility coefficients, f_{ij}, in which i is the location of interest and j is the location of the unit force applied. For instance:

    f_{12} = the deflection at the location of redundant 1 due to the unit load applied at redundant 2
6. Assuming linear elastic behaviour and applying the principle of superposition, compatibility must then be evaluated:

$$\Delta_1 + f_{11}X_1 + f_{12}X_2 + \cdots + f_{1n}X_n = (\Delta_1)_0$$
$$\Delta_2 + f_{21}X_1 + f_{22}X_2 + \cdots + f_{2n}X_n = (\Delta_2)_0$$
$$\Delta_n + f_{n1}X_1 + f_{n2}X_2 + \cdots + f_{nn}X_n = (\Delta_n)_0$$

    This can be expressed in matrix form:

$$[\Delta] + [f][X] = [\Delta_0]$$

    in which:

    $[\Delta]$ = displacement vector due to the applied loads on the primary structure

[f] = displacement (or flexibility) matrix due to unit loads applied to the primary structure in the direction of the redundant forces

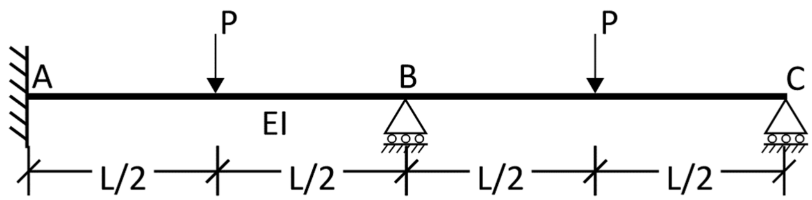
[X] = redundant force vector

[Δ_0] = displacement vector at the locations of the redundants (generally known; not necessarily zero)

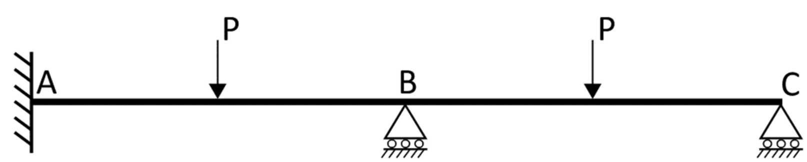
7. Solving the system of equations yields the n redundant forces.
8. Through the determination of the redundant forces, the structure is now statically determinate. Therefore, the remaining information that is desired may be determined through conventional static analysis.

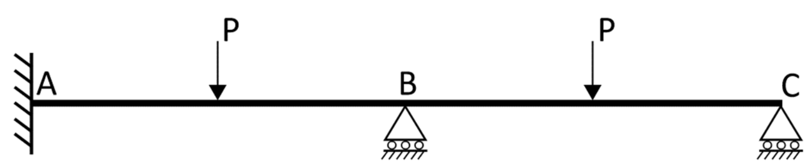
Example: Indeterminate Beam

Determine all support reactions and derive the shear and bending moment diagrams for the beam shown below:



First things first. Draw the moment diagram/deflected shape. They won't be precisely correct without doing an analysis, but knowing an approximate result before starting is a means of checking the detailed results:





The degree of redundancy is:

$$R - (3 + C) = 5 - (3 + 0) = 2$$

Select the redundant forces and identify the corresponding compatibility conditions:

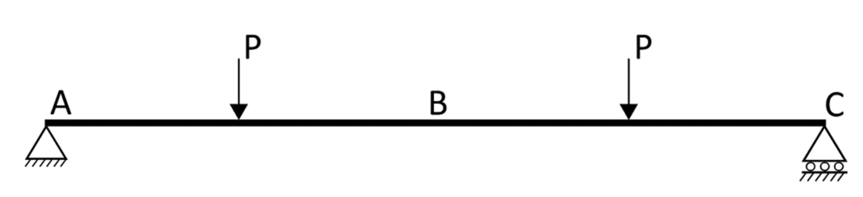
Redundant forces:

1. Select the vertical support reaction at B
2. Rotational restraint at A

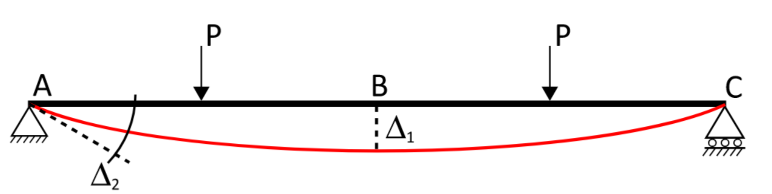
Compatibility conditions:

1. Vertical displacement at B = 0
2. Rotational displacement at A = 0

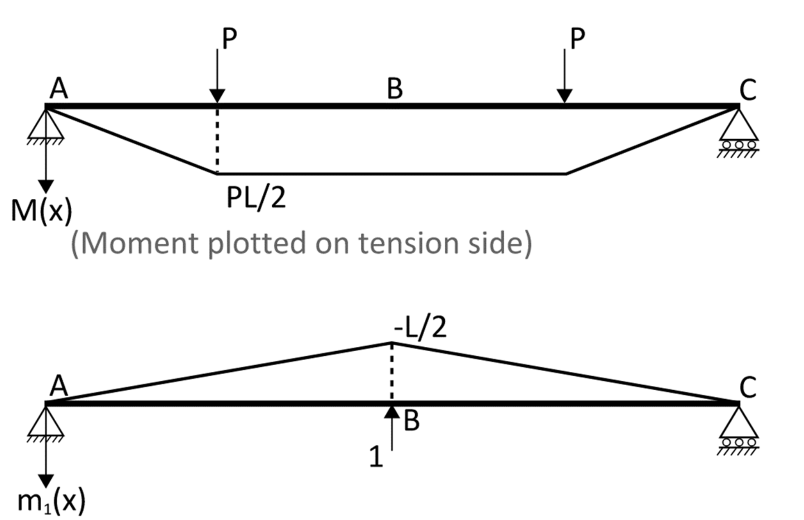
Remove the redundant forces and produce the primary structure:



Perform a displacement analysis of the primary structure under the applied loads, determining the displacements corresponding to the redundant forces:



For Δ_1:



Using virtual work:

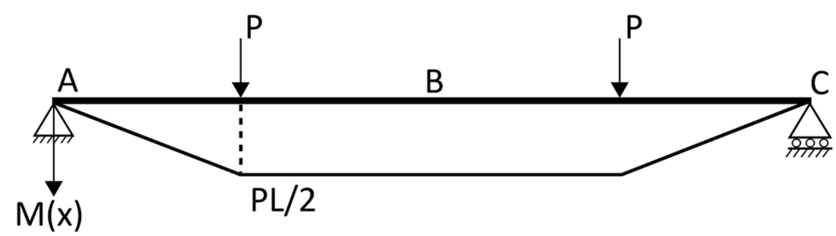
$$1 \times \Delta_1 = \int_0^L m_1(x)\left(\frac{M(x)}{EI}\right)dx$$

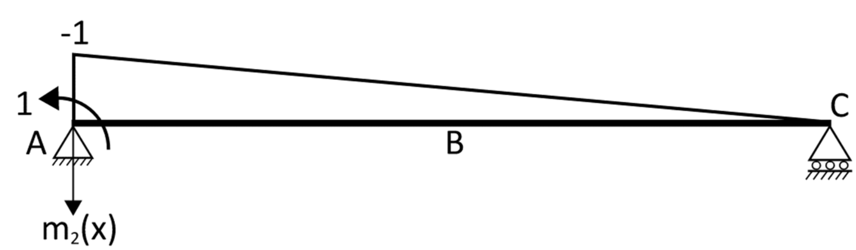
Recall that $\int f(x)g(x)dx$ equals the area under f(x) multiplied by the magnitude fo g(x) evaluated at the location of the centroid of f(x), provided g(x) is linear, and, f(x) and g(x) must be continuous. If not continuous, the sections must be sub-divided between points of discontinuity.

$$1 \times \Delta_1 = \int_0^{2L} m_1(x)\left(\frac{M(x)}{EI}\right)dx$$
$$= \frac{1}{EI}\left[\frac{1}{2}\frac{PL}{2}\left(\frac{L}{2}\right)\left(\frac{1}{3}\left(-\frac{L}{2}\right)\right) + \frac{PL}{2}\left(\frac{L}{2}\right)\left(\frac{3}{4}\left(-\frac{L}{2}\right)\right)\right] \times 2$$
$$= -\frac{11PL^3}{48EI}$$

Note that the deflection is negative – i.e. opposite of the virtual load that was applied. This is the expected result under the **applied loads.**

For Δ_2:





$$1 \times \Delta_2 = \int_0^{2L} m_2(x)\left(\frac{M(x)}{EI}\right)dx$$

$$= \frac{1}{EI}\left[\frac{1}{2}\frac{PL}{2}\left(\frac{L}{2}\right)\left(\frac{5}{6}(-1)\right) + \frac{PL}{2}(L)\left(\frac{1}{2}(-1)\right) + \frac{1}{2}\frac{PL}{2}\left(\frac{L}{2}\right)\left(\frac{1}{6}(-1)\right)\right] = -\frac{3PL^2}{8EI}$$

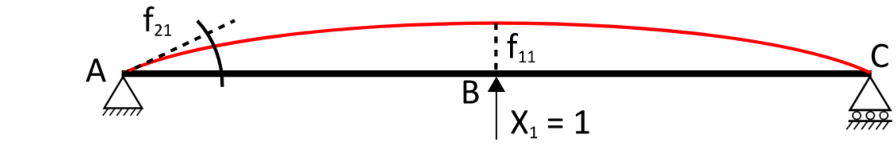
The negative sign indicates the direction of the rotation is opposite the virtual force that was applied, again this is the expected result based on the applied loads.

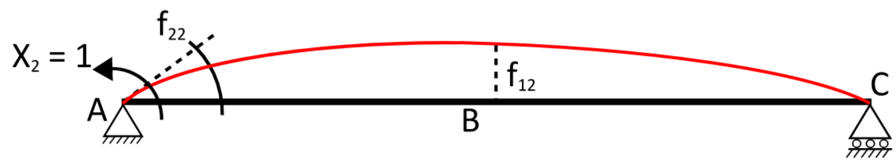
Now apply unit values of the redundant reactions to the primary structure to determine the displacement in the direction of each redundant. These “displacements” are the flexibility coefficients related to the redundant reactions.

f_{ij} = flexibility coefficient
= displacement per unit force
= displacement at “i” due to a unit force applied at “j”

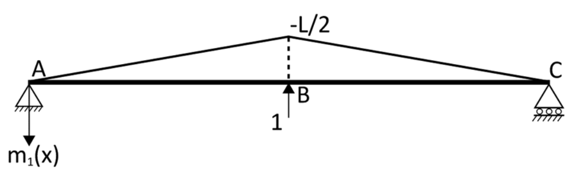
Units will work out to a distance or a rotation, however it is based on the context of a unit load being applied. Therefore it can be considered as a distance/unit load. These coefficients will subsequently be multiplied by load X_i.

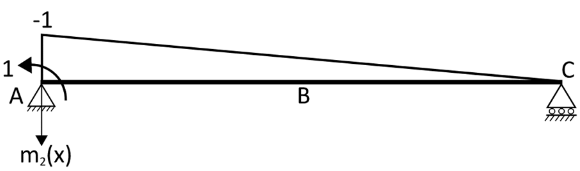
Applying the redundant reactions:





These coefficients are determined using the force method in conjunction with the moment diagrams previously used for determining Δ_1 and Δ_2.





$$1 \times f_{11} = \int_0^{2L} m_1(x)\left(\frac{m_1(x)}{EI}\right)dx = \frac{1}{EI}\left[\frac{1}{2}\left(-\frac{L}{2}\right)\left(\frac{L}{2}\right)\left(\frac{2}{3}\left(-\frac{L}{2}\right)\right)\right]\times 2$$

$$f_{11} = \frac{L^3}{6EI}$$

$$1 \times f_{12} = \frac{1}{EI}\left[\frac{1}{2}(-1)(L)\left(\frac{1}{3}\left(-\frac{L}{2}\right)\right) + \frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{L}{2}\right)\left(\frac{2}{3}\left(-\frac{L}{2}\right)\right) + \frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{L}{2}\right)\left(\frac{2}{3}\left(-\frac{L}{2}\right)\right)\right]$$

$$f_{12} = \frac{L^2}{4EI}$$

$$1 \times f_{21} = \frac{1}{EI}\left[\frac{1}{2}\left(-\frac{L}{2}\right)(L)\left(\frac{2}{3}(-1)\right) + \frac{1}{2}\left(-\frac{L}{2}\right)(L)\left(\frac{1}{3}(-1)\right)\right]$$

$$f_{21} = \frac{L^2}{4EI} = f_{12}$$

*Maxwell-Betti Law of reciprocal deflections

$$1 \times f_{22} = \frac{1}{EI}\left[\frac{1}{2}(-1)(2L)\left(\frac{2}{3}(-1)\right)\right]$$

$$f_{22} = \frac{2L}{3EI}$$

Evaluate the compatibility for the actual structure:

$$\Delta_B = 0 = \Delta_1 + f_{11}X_1 + f_{12}X_2$$

$$\Theta_A = 0 = \Delta_2 + f_{21}X_1 + f_{22}X_2$$

Δ_B:

$$-\frac{11PL^3}{48EI} + \frac{L^3}{6EI}X_1 + \frac{L^2}{4EI}X_2 = 0$$

Θ_A:

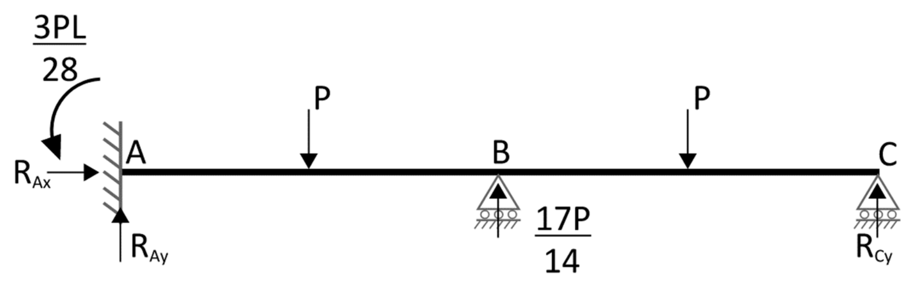
$$-\frac{3PL^2}{8EI} + \frac{L^2}{4EI}X_1 + \frac{2L}{3EI}X_2 = 0$$

We now have two equations and two unknowns. Solving for X_1 and X_2:

$$X_1 = \frac{17P}{14}$$

$$X_2 = \frac{3PL}{28}$$

Having found the redundant reactions, the balance of the support reactions can now be determined through equilibrium.



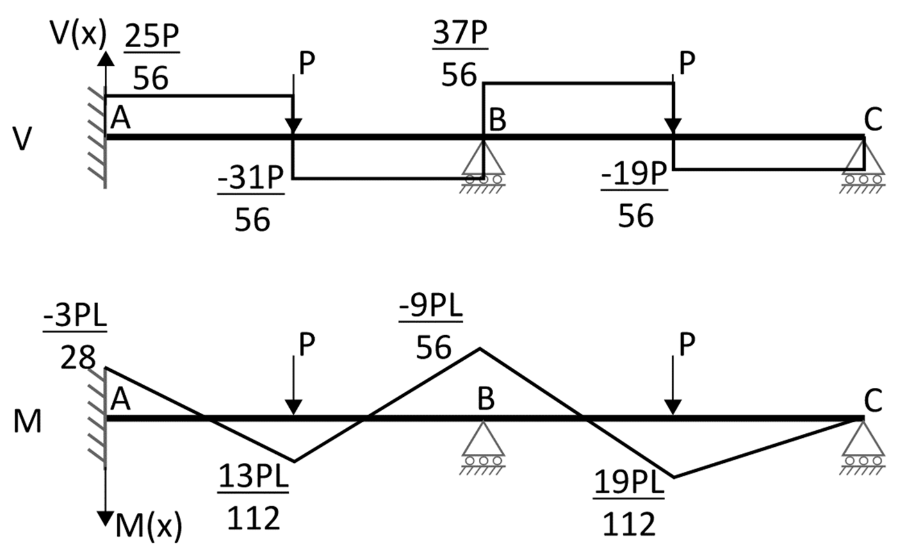
$$\sum F_x = 0 \rightarrow R_{Ax} = 0$$

$$\sum M_A = 0 \rightarrow \frac{3PL}{28} + \frac{17P}{14}(L) - P\frac{L}{2} - P\frac{3L}{2} + R_{Cy}2L = 0$$

$$R_{Cy} = \frac{19P}{56}$$

$$\sum F_y = 0 \rightarrow R_{Ay} = \frac{25P}{56}$$

Finally, the shear and moment diagrams can be plotted.

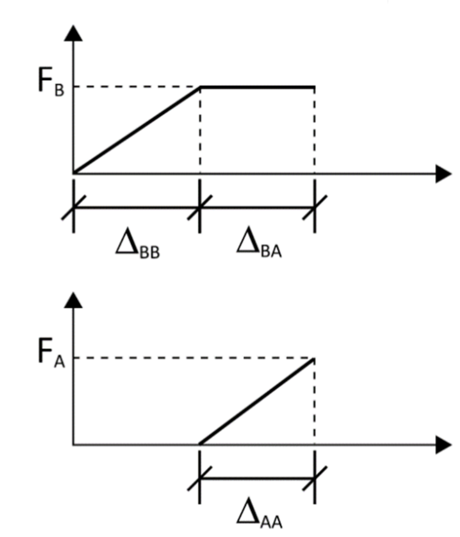


Maxwell Betti Law of Reciprocal Deflections

(reducing calculations since 1872!)

The displacement of point A due to a unit load applied at point B is equal to the displacement of point B due to a unit load applied at point A.

Consider F_B applied followed by F_A:



Work done when F_B is applied:

$$W_B = ½F_B\Delta_{BB}$$

Additional work when F_A is applied:

$$W_A = ½F_A\Delta_{AA} + F_B\Delta_{BA}$$

$$W_{TOTAL} = W_A + W_B = ½F_B\Delta_{BB} + ½F_A\Delta_{AA} + F_B\Delta_{BA}$$

Consider F_A is applied first, then F_B. Similarly:

$$W_{TOTAL}^* = W_A + W_B = ½F_A\Delta_{AA} + ½F_B\Delta_{BB} + F_A\Delta_{AB}$$

Equating the work equations:

$$W_{TOTAL} = W_{TOTAL}^*$$

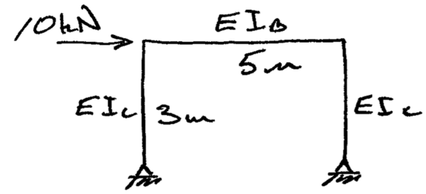
And simplifying:

$$F_B\Delta_{BA} = F_A\Delta_{AB}$$

If $F_B = F_A$ = unit load:

$$\boldsymbol{\Delta_{BA} = \Delta_{AB}}$$

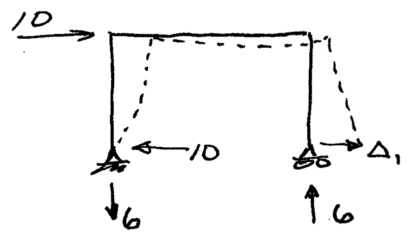
EXAMPLE — MOMENT FRAME



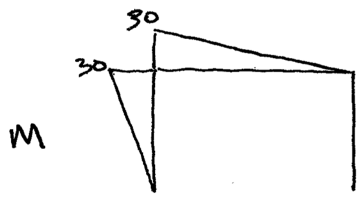
→ DETERMINE THE DEGREE OF INDETERMINACY

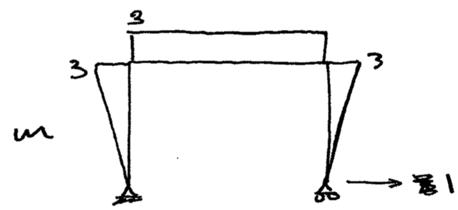
$$R-(3+C) = 4-3 = 1$$

→ CONSIDER THE PRIMARY STRUCTURE



MOMENTS PLOTTED ON THE COMPRESSION SIDE





→ DETERMINE DEFN AT LOCATION OF REDUNDANT FORCE

$$1 \times \Delta_1 = \int m \frac{M}{EI} dx$$

$$\Delta_1 = \frac{1}{EI_B}\left(\frac{1}{2}(5)(30)(3)\right) + \frac{1}{EI_c}\left(\frac{1}{2}(3)(30)(2)\right)$$

$$\Delta_1 = \frac{225}{EI_B} + \frac{90}{EI_c}$$

→ APPLY UNIT LOAD AT LOCATION OF REDUNDANT FORCE TO DETERMINE FLEXIBILITY COEFF.

$$1 \times f_{11} = \int m \frac{m}{EI} dx$$

2 columns

$$= \frac{1}{EI_B}\left(3(5)(3)\right) + \frac{2}{EI_c}\left(\frac{1}{2}(3)(3)(2)\right)$$

$$= \frac{45}{EI_B} + \frac{18}{EI_c}$$

↳ IMPOSED COMPATIBILITY

$$\Delta_1 + f_{11} X_1 = \cancelto{0}{(\Delta_1)_0}$$

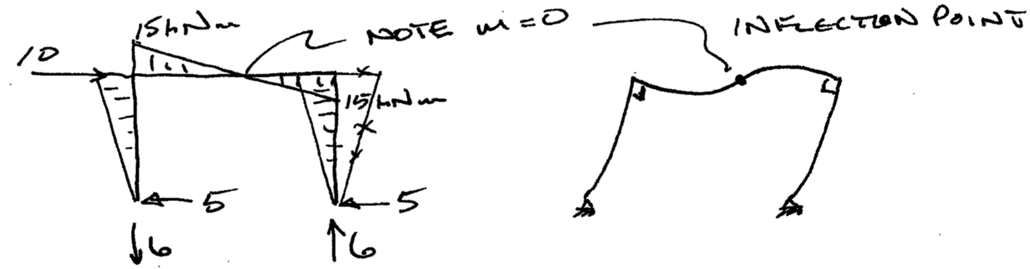
SUPPORT - DOESN'T MOVE

$$\therefore \frac{225}{EI_B} + \frac{90}{EI_C} + X_1\left(\frac{45}{EI_B} + \frac{18}{EI_C}\right) = 0$$

$$5\cancelto{1}{\left(\frac{45}{EI_B} + \frac{18}{EI_C}\right)} + X_1 \cancelto{1}{\left(\frac{45}{EI_B} + \frac{18}{EI_C}\right)} = 0$$

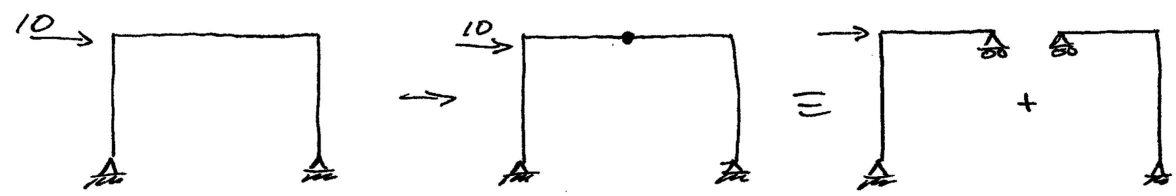
$$X_1 = -5$$

THE FRAME IS NOW DETERMINATE



⚹ NOW THE LATERAL DISPLACEMENT MAY BE DETERMINED AT THE TOP OF THE FRAME USING VIRT. WORK.

THIS STRUCTURE COULD HAVE BEEN MADE DETERMINATE IF SYMMETRY WAS CONSIDERED



$R - (3 + C) = 1$

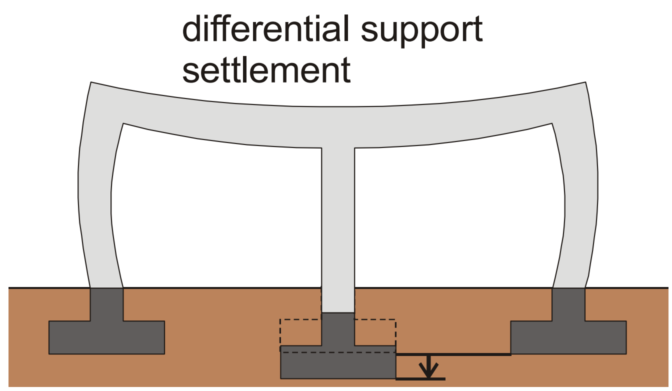
$R - (3 + C)$
$4 - (3 + 1) = 0$
DETERMINATE

↳ CAN PERFORM QUICK APPROXIMATE ANALYSIS BY MAKING REASONABLE VALID ASSUMPTIONS TO ADD RELEASES TO THE SYSTEM

But what if Δ_0 is not zero?

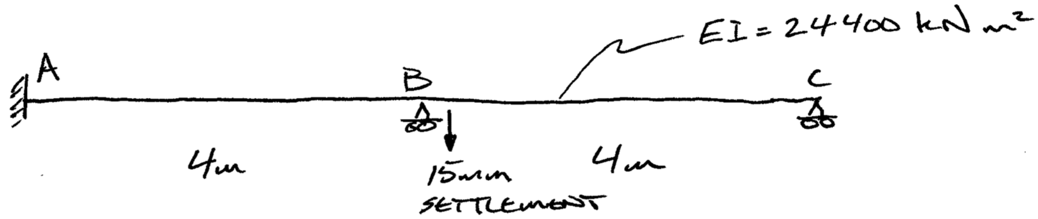
Generally, supports are assumed to not deflect, however, only relatively rigid supports exist in reality—infinitely rigid supports do not occur.

A common provision in geotechnical investigations states that at the determined bearing pressures, total settlements will not exceed 25mm and differential settlements will not exceed 19mm. That is, overall a building should not settle more than 25mm, and no portion of a building will settle more than 19mm more than another portion.



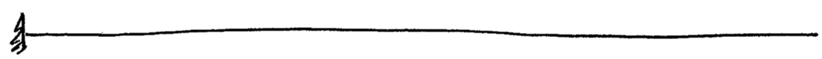
Settlement of one location relative to another location results in an imposed displacement. The Force Method can be used to determine the resulting reactions and stresses in the structure due to such an imposed deflection.

EXAMPLE: DETERMINE REACTIONS AND INTERNAL FORCES IN A BEAM SUBJECTED TO AN IMPOSED SUPPORT DISPLACEMENT.



→ DEGREE OF REDUNDANCY $R-3+C = 5-3+0 = \underline{2}$

→ PRIMARY STRUCTURE



→ DETERMINE DEFLECTIONS OF PRIMARY STRUCTURE DUE TO APPLIED LOADS

→ THERE ARE NO LOADS!

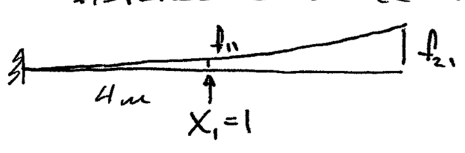
$\therefore\ \Delta_1 = 0$ - DEFLECTION AT REDUNDANT 1 (@ B)

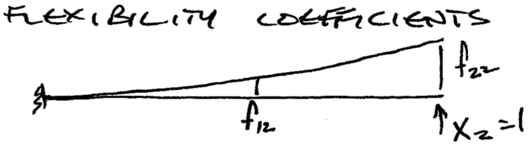
$\Delta_2 = 0$ - " " " 2 (@ C)

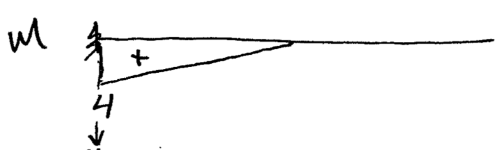
→ NOTE - IN PREVIOUS EXAMPLE WE CHOSE THE MOMENT @ A AS A REDUNDANT.

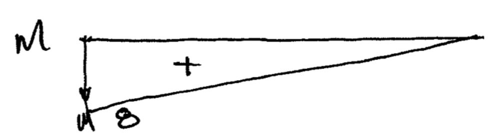
→ ~~WE~~ GET TO CHOOSE - ARBITRARY BUT BEST TO CHOOSE WISELY TO MAKE CALCS. EASY

→ NOW, APPLY UNIT VALUES OF THE REDUNDANT REACTIONS TO THE PRIMARY STRUCTURE TO DETERMINE THE DISPLACEMENTS ie THE FLEXIBILITY COEFFICIENTS









↳ USING VIRT. WORK

$$1 \times \Delta = \int m \frac{M}{EI} dx$$

USING TABLE A.2

f_{11} - DEFLECTION @ 1 DUE TO UNIT LOAD APPLIED AT 1

$$= \frac{1}{EI}\left(\frac{1}{3}(4)(4)(4)\right) = \frac{64}{3EI}$$

f_{21} - DEFL^n @ 2 DUE TO UNIT LOAD APPLIED AT 1

= NEED TO BREAK UP INTO CONTINUOUS SECTIONS

$$= \frac{1}{EI}\left(\underbrace{\frac{1}{6}(4(4+2(8))4}_{A \to B} + \underbrace{0}_{B \to C}\right) = \frac{160}{3EI}$$

$f_{12} = f_{21} = \frac{160}{3EI}$ BASED ON MAXWELL-BETTI LAW OF RECIPROCAL DEFLECTIONS.

$$f_{22} = \frac{1}{EI}\left(\frac{1}{3}(8)(8)(8)\right) = \frac{512}{3EI}$$

NOTE: UNITS HAVE BEEN ALL DONE IN <u>kN + m</u>

↳ IMPOSE COMPATIBILITY

@ B $\quad \Delta_1 + f_{11}X_1 + f_{12}X_2 = (\Delta_1)_0$

IN THIS CASE $(\Delta_1)_0 \neq 0$ → DOWNWARD IMPOSED DISPLACEMENT OF 15mm

$\therefore (\Delta_1)_0 = -0.015\,\underline{\underline{m}}$

@ C $\quad (\Delta_2)_0 = 0$ → NO SETTLEMENT

THEREFORE:

$$\cancel{\Delta_1}^{0} + \frac{64}{3EI}X_1 + \frac{160}{3EI}X_2 = -0.015 \qquad (1)$$

$$\cancel{\Delta_2}^{0} + \frac{160}{3EI}X_1 + \frac{512}{3EI}X_2 = 0 \qquad (2)$$

FROM (2) $X_1 = -\frac{512}{160}X_2 = -3.2X_2$

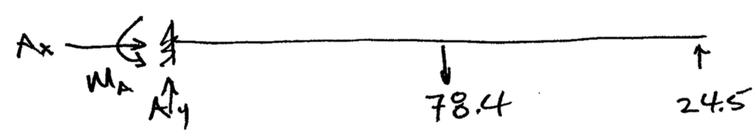
IN (1) $64(-3.2X_2) + 160X_2 = -0.015(3EI)$

$$44.8X_2 = 1098$$

$$X_2 = 24.5 \text{ kN}$$

$$\therefore X_1 = -78.4 \text{ kN}$$

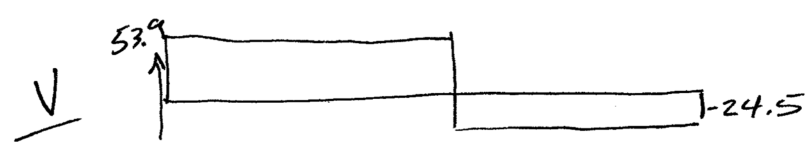
NOW THE SYSTEM IS DETERMINATE $\therefore$ SOLVE FOR THE OTHER UNKNOWNS USING STATIC EQUIL^{m}

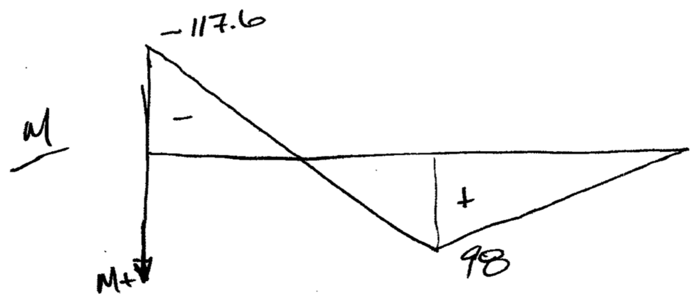


$\Sigma F_x = 0 \therefore A_x = 0$

$\Sigma F_y = 0 \therefore A_y = 78.4 - 24.5 = 53.9$ kN

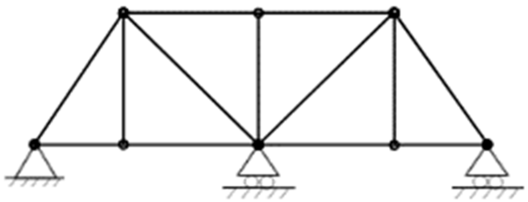
$\Sigma M = 0 \therefore M_A = 78.4(4) - 24.5(8) = 117.6$ kNm

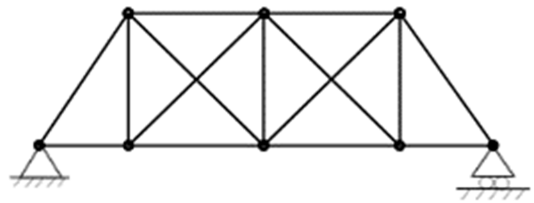




Statically Indeterminate Truss

The force (flexibility) method can also be used to analyse an indeterminate truss. Trusses can be internally indeterminate or externally indeterminate:





Externally Indeterminate

m+r-2j = 13 + 4 – 2(8) = 1

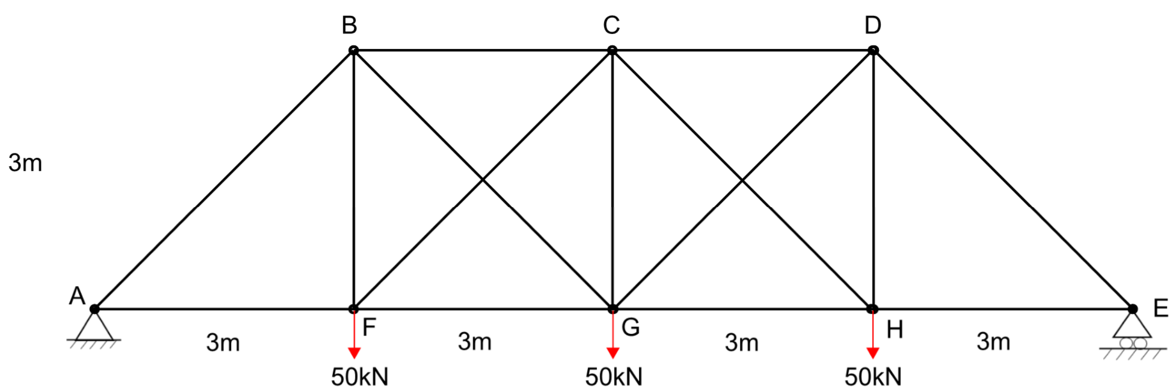
1 redundant external reaction

Internally Indeterminate

m+r-2j = 15 + 3 – 2(8) = 2

2 redundant internal members

Example: Statically Indeterminate Truss



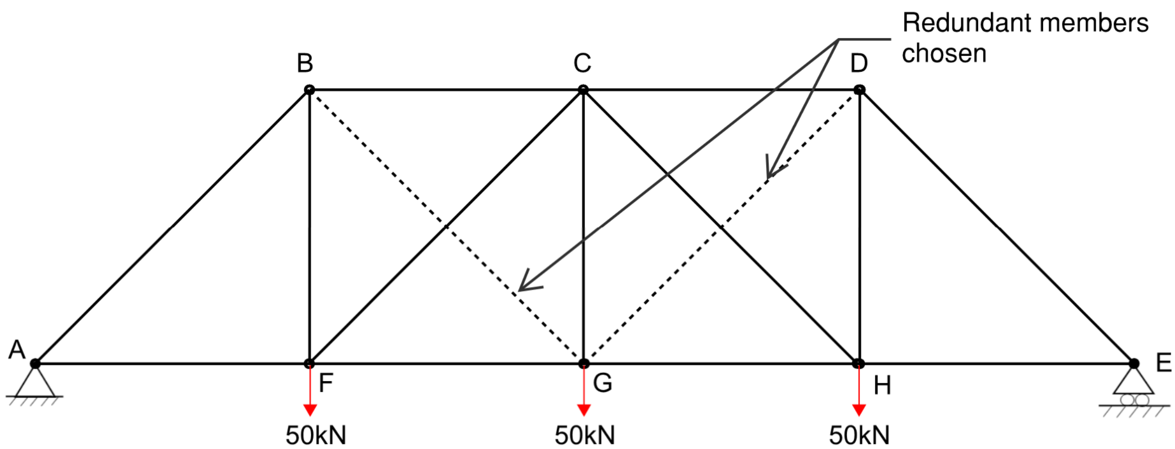
Assume all members in the truss have the same AE.

Determine the degree of indeterminacy:

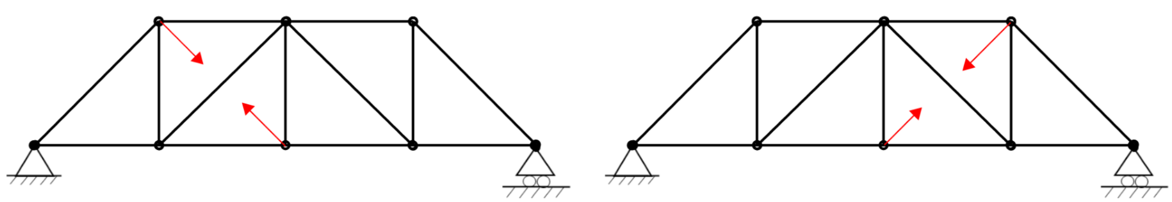
$$m + r - 2j = 15 + 3 - 2(8) = 2$$

Therefore there are 2 degrees of indeterminacy.

Select redundant members in order to render the truss determinate and achieve a primary structure. Now the truss can be analysed using conventional nodal analysis.



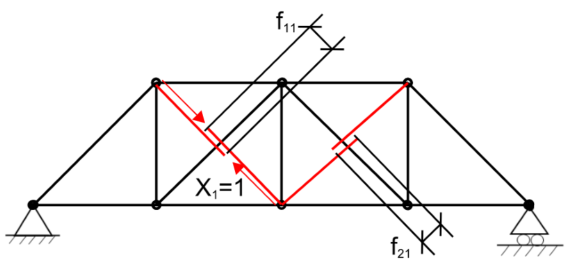
Now virtual unit loads are applied in turn to the primary structure, and a displacement analysis can be conducted to determine Δ_1 and Δ_2—the relative displacements between the cut ends of the redundant members under the applied loads.

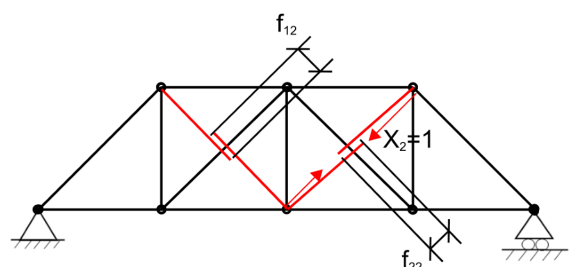


Recall:

$$1\ x\ \Delta = \sum n\frac{NL}{AE}$$

Now applying unit values of X_1 and X_2, the displacements in the direction of the redundant loads can be determined. These are the flexibility coefficients, f_{ij}, in which "i" is the location of the displacement, and "j" is the location of the unit force causing the displacement.





The use of a spreadsheet organizes and simplifies the calculations:

Member	L (m)	N (kN)	n_1 for Δ_1	n_2 for Δ_2	For Δ_1 n(NL)	For Δ_2 n(NL)	For f_{11} $n_1(n_1L)$	For f_{12} $n_1(n_2L)$	For f_{21} $n_2(n_1L)$	For f_{22} $n_2(n_2L)$
AB	4.24	-106	0	0	0	0	0	0	0	0
BC	3	-75	-0.707	0	159	0	1.50	0	0	0
CD	3	-75	0	-0.707	0	159	0	0	0	1.50
DE	4.24	-106	0	0	0	0	0	0	0	0
AF	3	75	0	0	0	0	0	0	0	0
FG	3	100	-0.707	0	-212	0	1.50	0	0	0
GH	3	100	0	-0.707	0	-212	0	0	0	1.50
HE	3	75	0	0	0	0	0	0	0	0
BF	3	75	-0.707	0	-159	0	1.50	0	0	0
CG	3	50	-0.707	-0.707	-106	-106	1.50	1.50	1.50	1.50
DH	3	75	0	-0.707	0	-159	0	0	0	1.50
BG	4.24	0	1	0	0	0	4.24	0	0	0
CF	4.24	-35.4	1	0	-150	0	4.24	0	0	0
DG	4.24	0	0	1	0	0	0	0	0	4.24
CH	4.24	-35.4	0	1	0	-150	0	0	0	4.24
				Sum (kN^2m)	-468.3	-468.3	14.48	1.50	1.50	14.48

Now imposing the constraints of compatibility:

$$\Delta_1 + f_{11}X_1 + f_{12}X_2 = 0$$

$$\Delta_2 + f_{21}X_1 + f_{22}X_2 = 0$$

Substituting values from the table above:

$$\frac{-468.3}{AE} + \frac{14.48}{AE}X_1 + \frac{1.5}{AE}X_2 = 0$$

$$\frac{-468.3}{AE} + \frac{1.5}{AE}X_1 + \frac{14.48}{AE}X_2 = 0$$

Solving the two equations, we get the forces in the redundant members due to the imposed loading:

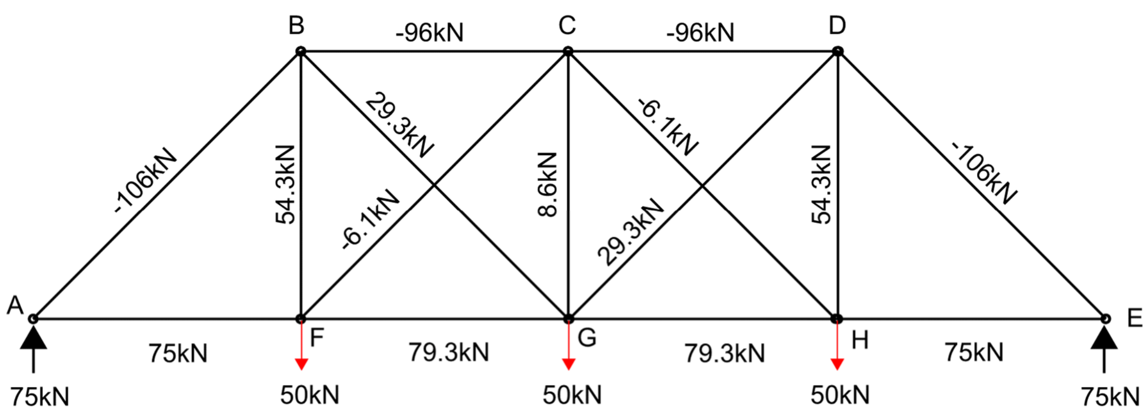
$X_1 = X_2 = 29.3$kN

As expected, they are equal due to the symmetry of the truss.

Now that the redundant member forces have been determined, the indeterminate truss has been rendered determinate. The balance of the member forces can now be determined using:

$$Member\ force = N + n_1X_1 + n_2X_2$$

Member	N (kN)	n_1 for Δ_1	n_2 for Δ_2	Final Member Forces (kN)
AB	-106	0	0	-106.0
BC	-75	-0.707	0	-95.7
CD	-75	0	-0.707	-95.7
DE	-106	0	0	-106.0
AF	75	0	0	75.0
FG	100	-0.707	0	79.3
GH	100	0	-0.707	79.3
HE	75	0	0	75.0
BF	75	-0.707	0	54.3
CG	50	-0.707	-0.707	8.6
DH	75	0	-0.707	54.3
BG	0	1	0	29.3
CF	-35.4	1	0	-6.1
DG	0	0	1	29.3
CH	-35.4	0	1	-6.1



We assumed that all the truss members had the same AE. However the analysis shows that the forces vary widely. If member CG, for example, was made smaller due to the rather small force it needs to carry, the force distribution would also change due to CG essentially being more flexible and therefore taking less load. As a result, the analysis would have to be performed again to establish the revised force distribution.

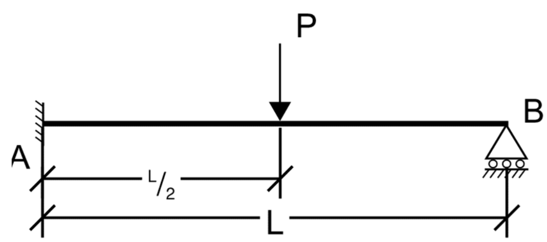
Deflection Analysis of an Indeterminate Structure

Once the final bending moment distribution is known using the Force Method and the Principle of Virtual Work, deflections at any point can then be determined again using the principle of virtual work.

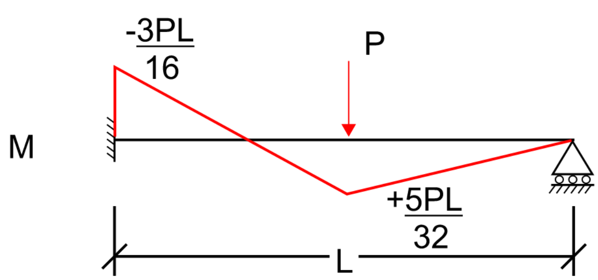
$$1 \times \delta_i^P = \int m_{i0} \frac{M}{EI} dx$$

The displacement at point, i, of an indeterminate system is equivalent to the displacement calculated using virtual work with the actual moment diagram, M, and the moment diagram, m_{i0}, corresponding to Q_i*=1 on <u>any related primary structure</u>.

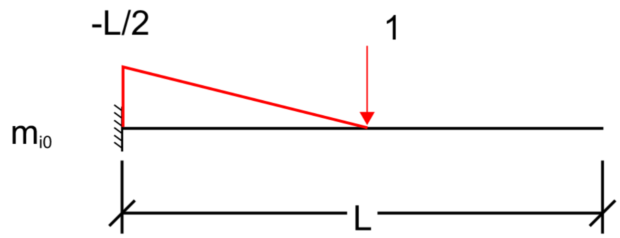
Consider the simple case previously investigated:



We have solved this propped cantilever beam, yielding a moment diagram:

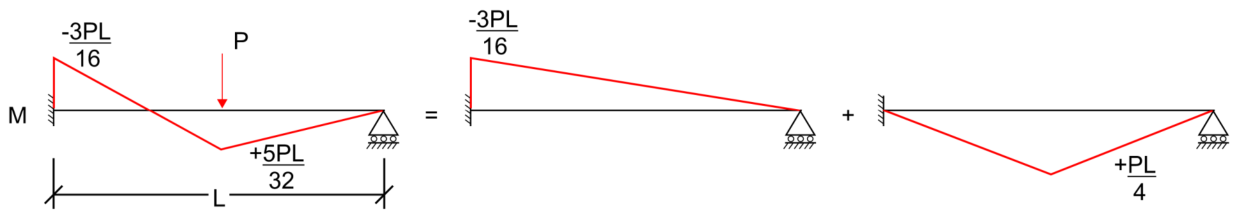


To find the deflection at the mid-span of the beam, we apply a unit load to the mid-span of the primary structure, and determine the resulting moment diagram:



The deflection can then be calculated:

Simplify M for calculating product integrals using superposition:



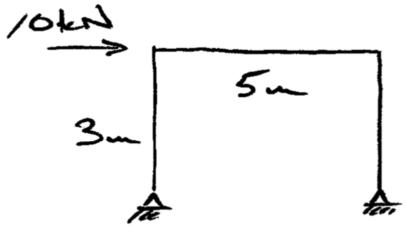
$$1 \times \Delta = \int m_{i0} \frac{M}{EI} dx$$

Using Table A2:

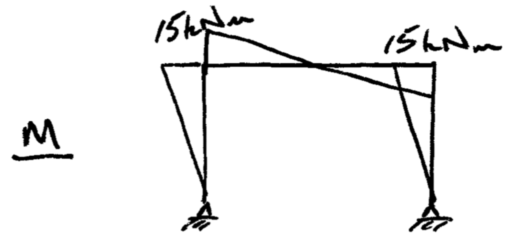
$$\Delta = \frac{1}{6}\left(\frac{-3PL}{32} + 2\left(-\frac{3PL}{16}\right)\right)\left(-\frac{L}{2}\right)\left(\frac{L}{2}\right)\frac{1}{EI} + \frac{1}{6}\left(-\frac{L}{2}\right)\left(\frac{PL}{4}\right)\left(\frac{L}{2}\right)\frac{1}{EI}$$

$$= \frac{15-8}{768}\frac{PL^3}{EI} = \frac{7PL^3}{768EI}$$

RECALL PORTAL FRAME EXAMPLE



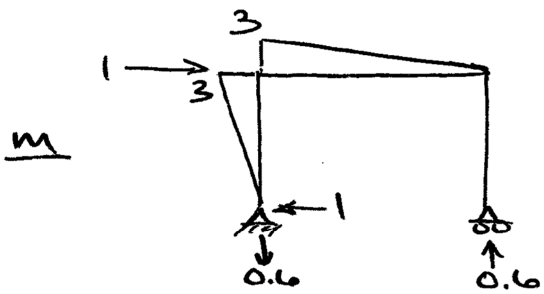
EI = CONSTANT



MOMENT PLOTTED ON COMPRESSION SIDE

WHAT IS THE HORIZONTAL DEFLECTION?

→ PRIMARY STRUCTURE



$$1 \times \Delta = \int m \frac{M}{EI} dx$$

$$= \frac{1}{EI}\left[\frac{1}{3}(3)(15)(3) + \frac{1}{6}(1.5+2(3))(15)(2.5) + \frac{1}{6}(-15)(1.5)(2.5)\right]$$

$$= \frac{1}{EI}(45 + 46.9 - 9.4)$$

$$= \frac{41.5}{EI}$$

TAKE $EI = 5000\,kNm^2$

$$\Delta = \frac{41.5}{5000} = 8.3 \times 10^{-3}\,m$$

$$= \underline{8.3\,mm}$$

Influence Lines (Leet *et al*., Chapter 12)

Loads acting on a structure are divided into two general types: permanent and transient.

Permanent loads (or dead loads) consist of the self-weight of the structure, itself, as well as superimposed dead loads such as finishes, mechanical equipment, partitions, soil loads, etc.

Transient loads (or live loads) consist of occupancy loads (people), climatic loads (snow, wind), moving loads (trucks), loads from storage of goods, etc. They may or may not act on a structure at any given time, and may or may not act on any given part of a structure.

Influence lines are useful tools to determine a reaction, internal force/moment, or displacement at a specific point in a structure. Influence lines are constructed to reflect the effects of a moving unit load acting on the structure. (Compare this to a moment or shear diagram which represents the effects at all points along a member of a fixed load distribution.)

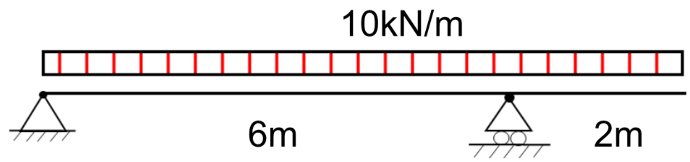
There are two types of influence lines: quantitative and qualitative.

Quantitative influence lines assist a designer in determining the magnitude of the effect of a given loading distribution.

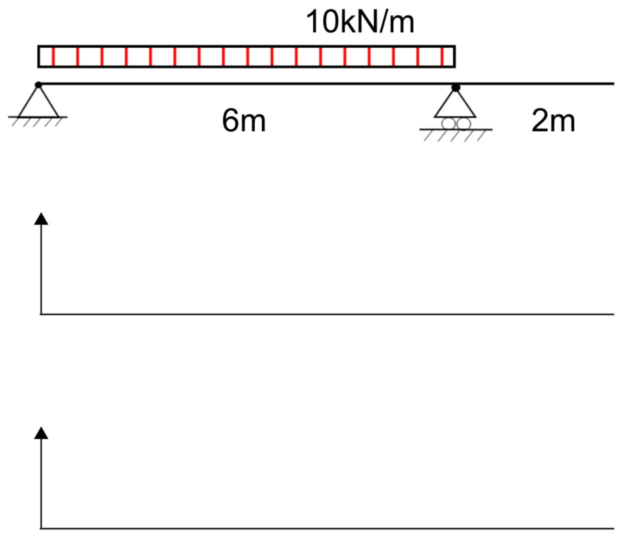
Qualitative influence lines assist a designer in determining the arrangement of loads to achieve the controlling load distribution for further analysis.

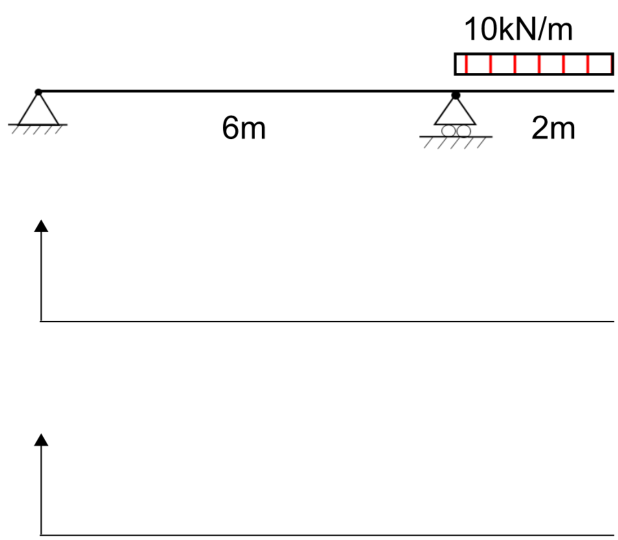
Quantative Influence Lines

Consider a simple propped cantilever beam supporting a uniformly distributed load:



Now consider if the load was only applied to part of the beam:



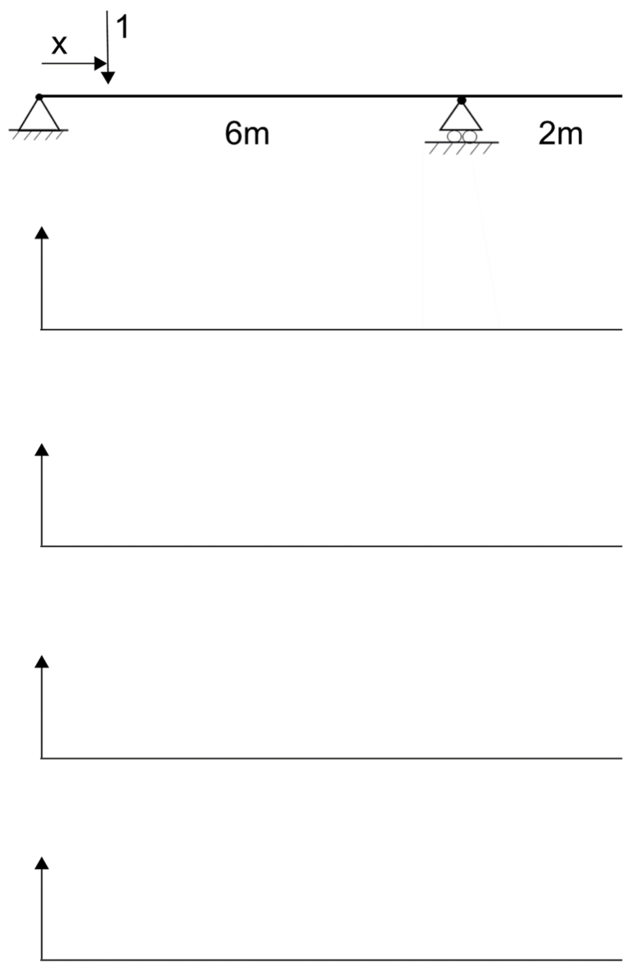


Remember, engineers do not normally care about "average" or "regular" or "everyday" conditions. They care about the worst reasonably possible condition that may produce a structural failure.

How does one determine the distribution of loads so that the worst case can be evaluated? There are two options:

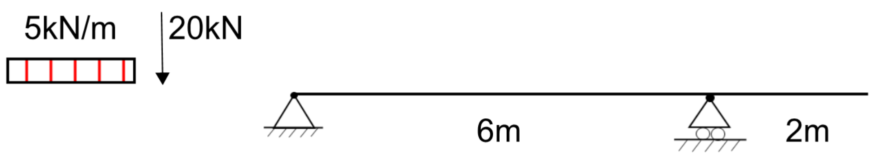
- Analyze all possible load distributions (sounds like a lot of work)
- Use influence lines

Consider a moving unit load on the same cantilevered beam:



With the influence lines drawn for various effects on the beam, they can now be used to calculate load effects under different loading conditions.

Consider a load of 5kN/m of variable length applied to some or all of the beam, and also an independent point load of 20kN applied anywhere along the beam. What configuration of these loads will produce the largest positive moment at the mid-span and the largest negative moment at the right support?



For positive moment at mid-span, place loads only where the influence line for the midspan moment is positive, and arrange to achieve the maximum magnitude.

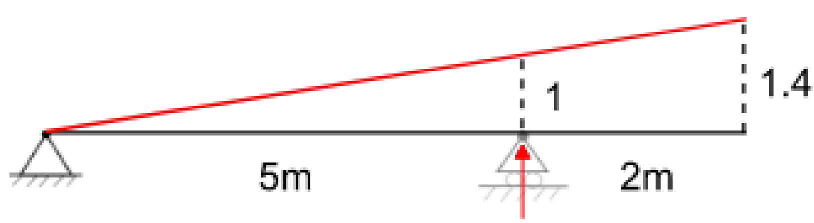
For the negative moment at the right support, likewise apply loads to the influence line to achieve the maximum negative result.

Müller-Breslau Principle

> *"The influence line for any reaction or internal force (shear, moment) corresponds to the deflected shape of the structure produced by removing the capacity of the structure to carry that force and then introducing into the modified (or released) structure a unit deformation that corresponds to the restraint removed."*

For determinate systems, the Müller-Breslau Principle can be used to rapidly determine the influence line.

To construct the influence line for a determinate system, remove the ability of the system to resist the internal force/external reaction to be determined. Apply the force/reaction to produce a unit displacement/rotation in the positive direction. The influence line is the resulting deflected shape.

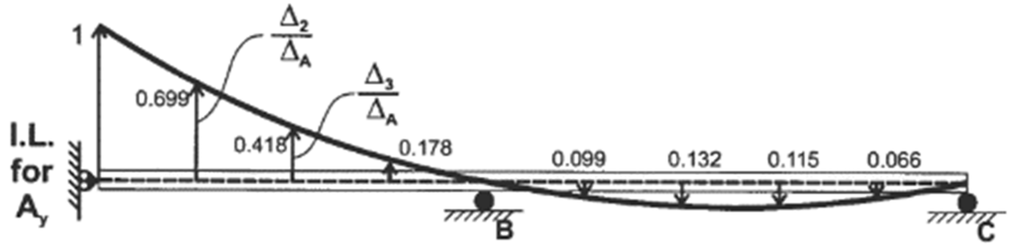


Influence Line for Support Reaction

For determinate systems, the structure is strictly stable—no excess constraints are present. In removing the structure's ability to resist the force/reaction to be determined, the structure is rendered unstable—you are creating a mechanism. The influence lines for statically determinate structures are comprised of straight line segments.

Qualitative Influence Lines

Indeterminate structures would require multiple indeterminate analysis in order to construct a quantitative influence line.



Qualitative influence lines are shape only—they do not include magnitude of internal forces and cannot be used to calculate specific forces. They can however be used to determine where a transient load will create the most significant effect on a structural system. Based on an intuitive understanding of the structural behaviour, analysis can then be efficiently focused on the most critical conditions.

The Müller-Breslau Principle can be used to rapidly produce the qualitative influence line of an indeterminate structure. Since an indeterminate structure has more internal or external constraints than needed to be strictly stable, mechanisms are not necessarily created and so the deflected shape—and the influence line—is frequently made of curved segments rather than straight lines.