CHAPTER 3

EXERCISES 3.1

1. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h+2) - (x+2)}{h} = \lim_{h \to 0} \frac{h}{h} = 1$

2. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[3(x+h)^2 + 5] - (3x^2 + 5)}{h}$

$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 + 5 - 3x^2 - 5}{h} = \lim_{h \to 0} (6x + 3h) = 6x$

3. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[1 + 2(x+h) - (x+h)^2] - (1 + 2x - x^2)}{h}$

$= \lim_{h \to 0} \frac{1 + 2x + 2h - x^2 - 2xh - h^2 - 1 - 2x + x^2}{h} = \lim_{h \to 0} \frac{h(2 - 2x - h)}{h} = 2 - 2x$

4. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^3 + 2(x+h)^2] - (x^3 + 2x^2)}{h}$

$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 + 2x^2 + 4xh + 2h^2 - x^3 - 2x^2}{h} = \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2 + 4x + 2h)}{h}$

$= 3x^2 + 4x$

5. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^4 + 4(x+h) - 12] - (x^4 + 4x - 12)}{h}$

$= \lim_{h \to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 + 4x + 4h - 12 - x^4 - 4x + 12}{h}$

$= \lim_{h \to 0} \frac{h(4x^3 + 6x^2h + 4xh^2 + h^3 + 4)}{h} = 4x^3 + 4$

6. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{h}\left(\frac{x+h+4}{x+h-5} - \frac{x+4}{x-5}\right)$

$= \lim_{h \to 0} \frac{1}{h}\left[\frac{(x^2 + xh + 4x - 5x - 5h - 20) - (x^2 + xh - 5x + 4x + 4h - 20)}{(x+h-5)(x-5)}\right]$

$= \lim_{h \to 0} \frac{-9}{(x+h-5)(x-5)} = \frac{-9}{(x-5)^2}$

7. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{h}\left[\frac{(x+h)^2 + 2}{x+h+3} - \frac{x^2 + 2}{x+3}\right]$

$= \lim_{h \to 0} \frac{(x+3)(x^2 + 2xh + h^2 + 2) - (x^2 + 2)(x+h+3)}{h(x+h+3)(x+3)} = \lim_{h \to 0} \frac{h(x^2 + xh + 6x + 3h - 2)}{h(x+h+3)(x+3)}$

$= \frac{x^2 + 6x - 2}{(x+3)^2}$

8. If we write $f(x) = x^3 + 2x^2$, then this is the same function as in Exercise 4. Consequently, $f'(x) = 3x^2 + 4x$.

9. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{h}\left[\frac{3(x+h) - 2}{4 - (x+h)} - \frac{3x - 2}{4 - x}\right]$

$= \lim_{h \to 0} \frac{(3x + 3h - 2)(4 - x) - (3x - 2)(4 - x - h)}{h(4 - x - h)(4 - x)} = \lim_{h \to 0} \frac{10h}{h(4 - x - h)(4 - x)} = \frac{10}{(4 - x)^2}$

10. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{h}\left\{\frac{(x+h)^2 - (x+h) + 1}{(x+h)^2 + (x+h) + 1} - \frac{x^2 - x + 1}{x^2 + x + 1}\right\}$

$= \lim_{h \to 0} \frac{1}{h}\left\{\frac{(x^2 + x + 1)[(x+h)^2 - (x+h) + 1] - (x^2 - x + 1)[(x+h)^2 + (x+h) + 1]}{(x^2 + x + 1)[(x+h)^2 + (x+h) + 1]}\right\}$

$= \lim_{h \to 0} \left\{\frac{h(2x^2 + 2xh - 2)}{h(x^2 + x + 1)[(x+h)^2 + (x+h) + 1]}\right\} = \frac{2x^2 - 2}{(x^2 + x + 1)^2}$

11. Since $C = f(r) = 2\pi r$, $\quad \dfrac{dC}{dr} = \lim_{h\to 0} \dfrac{f(r+h)-f(r)}{h} = \lim_{h\to 0} \dfrac{2\pi(r+h)-2\pi r}{h} = \lim_{h\to 0} \dfrac{2\pi h}{h} = 2\pi.$

12. Since $A = f(r) = \pi r^2$, $\quad \dfrac{dA}{dr} = \lim_{h\to 0} \dfrac{f(r+h)-f(r)}{h} = \lim_{h\to 0} \dfrac{\pi(r+h)^2-\pi r^2}{h} = \lim_{h\to 0} \dfrac{\pi h(2r+h)}{h} = 2\pi r.$

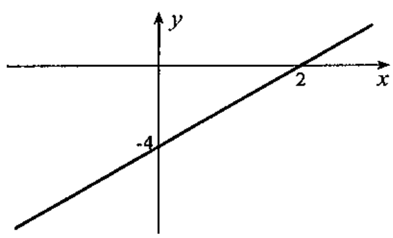
13. Since $A = f(r) = 4\pi r^2$,

$$\frac{dA}{dr} = \lim_{h\to 0} \frac{f(r+h)-f(r)}{h} = \lim_{h\to 0} \frac{4\pi(r+h)^2-4\pi r^2}{h} = \lim_{h\to 0} \frac{4\pi h(2r+h)}{h} = 8\pi r.$$

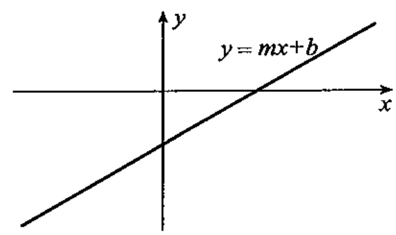
14. Since $V = f(r) = (4/3)\pi r^3$,

$$\begin{aligned}\frac{dV}{dr} &= \lim_{h\to 0} \frac{f(r+h)-f(r)}{h} = \lim_{h\to 0} \frac{(4/3)\pi(r+h)^3-(4/3)\pi r^3}{h} \\ &= \frac{4\pi}{3}\lim_{h\to 0} \frac{r^3+3r^2h+3rh^2+h^3-r^3}{h} = \frac{4\pi}{3}\lim_{h\to 0}(3r^2+3rh+h^2) = 4\pi r^2.\end{aligned}$$

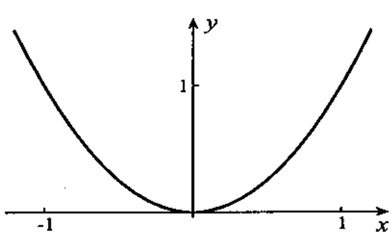
15. Since $f'(x)$ is the slope of the tangent line to the straight line, and the tangent line is the line itself, $f'(x) = 2$.



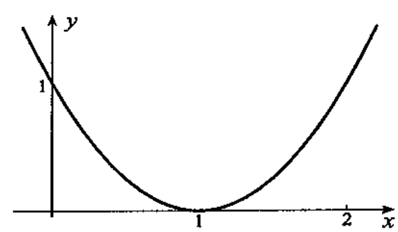
16. Since $f'(x)$ is the slope of the tangent line to the straight line, and the tangent line is the line itself, $f'(x) = m$.



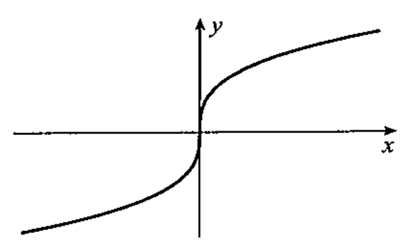
17. Since $f'(0)$ is the slope of the tangent line to the parabola at $x = 0$, it follows that $f'(0) = 0$.



18. Since $f'(1)$ is the slope of the tangent line to the parabola at $x = 1$, it follows that $f'(1) = 0$.



19. Since the tangent to the graph at $(0,0)$ is the y-axis, the derivative does not exist at $x = 0$.



20. Since $\dfrac{dy}{dx} = \lim_{h\to 0} \dfrac{[(x+h)^2+3]-[x^2+3]}{h} = \lim_{h\to 0} \dfrac{h(2x+h)}{h} = 2x$, the slope of the tangent line to the parabola at $(1,4)$ is $2(1) = 2$. The equation of the tangent line is $y - 4 = 2(x-1) \Longrightarrow y = 2x+2$.

21. Since $\dfrac{dy}{dx} = \lim_{h\to 0} \dfrac{[3-2(x+h)-(x+h)^2]-[3-2x-x^2]}{h} = \lim_{h\to 0} \dfrac{h(-2-2x-h)}{h} = -2-2x$, the slope of the tangent line to the parabola at $(4,-21)$ is $-2-2(4) = -10$. The equation of the tangent line is $y+21 = -10(x-4) \Longrightarrow 10x+y = 19$.

22. Since $\dfrac{dy}{dx} = \lim_{h\to 0} \dfrac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h} = \lim_{h\to 0} \dfrac{1}{h}\left[\dfrac{x^2-(x+h)^2}{x^2(x+h)^2}\right] = \lim_{h\to 0} \dfrac{-(2x+h)}{x^2(x+h)^2} = -\dfrac{2}{x^3}$, the slope of the tangent line at $(2, 1/4)$ is $-2/(2^3) = -1/4$. The equation of the tangent line is $y - 1/4 = -(1/4)(x-2) \Longrightarrow x + 4y = 3$.

23. Since
$$\frac{dy}{dx} = \lim_{h\to 0} \frac{1}{h}\left(\frac{x+h+1}{x+h+2} - \frac{x+1}{x+2}\right) = \lim_{h\to 0} \frac{(x+h+1)(x+2)-(x+h+2)(x+1)}{h(x+h+2)(x+2)}$$
$$= \lim_{h\to 0} \frac{h}{h(x+h+2)(x+2)} = \frac{1}{(x+2)^2},$$
the slope of the tangent line at $(0, 1/2)$ is $1/4$. The equation of the tangent line is $y - 1/2 = (1/4)x \Longrightarrow x = 4y - 2$.

24. Since $\dfrac{dy}{dx} = \lim_{h\to 0} \dfrac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \dfrac{h(2x+h)}{h} = 2x$, the slope of the tangent line to the parabola at $(1,1)$ is 2. The inclination is defined by $\tan\phi = 2$, and is therefore equal to 1.107 radians.

25. Since $\dfrac{dy}{dx} = \lim_{h\to 0} \dfrac{[(x+h)^3 - 6(x+h)] - (x^3-6x)}{h} = \lim_{h\to 0} \dfrac{h(3x^2+3xh+h^2-6)}{h} = 3x^2 - 6$, the slope of the tangent line to the curve at $(2, -4)$ is 6. The inclination is defined by $\tan\phi = 6$, and is therefore equal to 1.406 radians.

26. According to Exercise 22, the slope of the tangent line at $(2, 1/4)$ is $-1/4$. Since the inclination satisfies $\tan\phi = -1/4$, it follows that $\phi = 2.897$ radians.

27. Since $\dfrac{dy}{dx} = \lim_{h\to 0} \dfrac{1}{h}\left[\dfrac{1}{x+h+1} - \dfrac{1}{x+1}\right] = \lim_{h\to 0} \dfrac{-h}{h(x+h+1)(x+1)} = -\dfrac{1}{(x+1)^2}$, the slope of the tangent line to the curve at $(0, 1)$ is -1. The inclination is defined by $\tan\phi = -1$, and is therefore equal to $3\pi/4$ radians.

28.
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^8 - x^8}{h}$$
$$= \lim_{h\to 0} \frac{(x^8 + 8x^7h + 28x^6h^2 + \cdots + h^8) - x^8}{h} \qquad \text{(using the binomial theorem)}$$
$$= \lim_{h\to 0} (8x^7 + 28x^6h + \cdots + h^7) = 8x^7$$

29.
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h}\,\frac{\sqrt{x+h+1} + \sqrt{x+1}}{\sqrt{x+h+1} + \sqrt{x+1}} = \lim_{h\to 0} \frac{(x+h+1)-(x+1)}{h(\sqrt{x+h+1} + \sqrt{x+1})} = \frac{1}{2\sqrt{x+1}}$$

30.
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left\{\frac{1}{(x+h-2)^4} - \frac{1}{(x-2)^4}\right\}$$
$$= \lim_{h\to 0} \left\{\frac{(x-2)^4 - [(x-2)^4 + 4(x-2)^3h + 6(x-2)^2h^2 + 4(x-2)h^3 + h^4]}{h(x-2)^4(x+h-2)^4}\right\}$$
$$= \lim_{h\to 0} \left\{\frac{-4(x-2)^3 - 6(x-2)^2h - 4(x-2)h^2 - h^3}{(x-2)^4(x+h-2)^4}\right\} = \frac{-4(x-2)^3}{(x-2)^8} = \frac{-4}{(x-2)^5}$$

31.
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{\sqrt{x+h-3}} - \frac{1}{\sqrt{x-3}}\right)$$
$$= \lim_{h\to 0} \frac{\sqrt{x-3} - \sqrt{x+h-3}}{h\sqrt{x+h-3}\sqrt{x-3}}\,\frac{\sqrt{x-3} + \sqrt{x+h-3}}{\sqrt{x-3} + \sqrt{x+h-3}} = \lim_{h\to 0} \frac{(x-3)-(x+h-3)}{h\sqrt{x+h-3}\sqrt{x-3}(\sqrt{x-3} + \sqrt{x+h-3})}$$
$$= \frac{-1}{\sqrt{x-3}\sqrt{x-3}(2\sqrt{x-3})} = \frac{-1}{2(x-3)^{3/2}}$$

32. $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)\sqrt{(x+h)+1} - x\sqrt{x+1}}{h}$$
$$= \lim_{h\to 0}\left[\frac{(x+h)\sqrt{x+h+1} - x\sqrt{x+1}}{h} \frac{(x+h)\sqrt{x+h+1} + x\sqrt{x+1}}{(x+h)\sqrt{x+h+1} + x\sqrt{x+1}}\right]$$
$$= \lim_{h\to 0} \frac{(x+h)^2(x+h+1) - x^2(x+1)}{h[(x+h)\sqrt{x+h+1} + x\sqrt{x+1}]}$$
$$= \lim_{h\to 0} \frac{x^2 + 2x(x+h+1) + h(x+h+1)}{(x+h)\sqrt{x+h+1} + x\sqrt{x+1}} = \frac{3x^2+2x}{2x\sqrt{x+1}} = \frac{3x+2}{2\sqrt{x+1}}$$

33. Since $A = \pi r^2$, we have $r = \sqrt{A}/\sqrt{\pi}$. Then,
$$\frac{dr}{dA} = \lim_{h\to 0} \frac{\sqrt{A+h}/\sqrt{\pi} - \sqrt{A}/\sqrt{\pi}}{h} = \frac{1}{\sqrt{\pi}} \lim_{h\to 0} \frac{\sqrt{A+h} - \sqrt{A}}{h} \frac{\sqrt{A+h} + \sqrt{A}}{\sqrt{A+h} + \sqrt{A}}$$
$$= \frac{1}{\sqrt{\pi}} \lim_{h\to 0} \frac{(A+h) - A}{h(\sqrt{A+h} + \sqrt{A})} = \frac{1}{2\sqrt{\pi A}}.$$

34. Since $V = (4/3)\pi r^3$ and $A = 4\pi r^2$, we have $V = f(A) = \dfrac{4\pi}{3}\left(\dfrac{A}{4\pi}\right)^{3/2} = \dfrac{1}{6\sqrt{\pi}}A^{3/2}$. The derivative of this function is
$$\frac{dV}{dA} = \lim_{h\to 0} \frac{f(A+h)-f(A)}{h} = \lim_{h\to 0} \frac{(A+h)^{3/2}/(6\sqrt{\pi}) - A^{3/2}/(6\sqrt{\pi})}{h}$$
$$= \frac{1}{6\sqrt{\pi}} \lim_{h\to 0}\left[\frac{(A+h)^{3/2} - A^{3/2}}{h} \frac{(A+h)^{3/2} + A^{3/2}}{(A+h)^{3/2} + A^{3/2}}\right]$$
$$= \frac{1}{6\sqrt{\pi}} \lim_{h\to 0} \frac{A^3 + 3A^2h + 3Ah^2 + h^3 - A^3}{h[(A+h)^{3/2} + A^{3/2}]} = \frac{1}{6\sqrt{\pi}} \lim_{h\to 0} \frac{3A^2 + 3Ah + h^2}{(A+h)^{3/2} + A^{3/2}}$$
$$= \frac{1}{6\sqrt{\pi}} \frac{3A^2}{2A^{3/2}} = \frac{\sqrt{A}}{4\sqrt{\pi}}.$$

35. The slope of the tangent line to $y = x^2$ at any point is
$$\frac{dy}{dx} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} = 2x.$$
Hence, the slope of the tangent line to $y = x^2$ at the point of intersection $(1,1)$ is $m_1 = 2$. The slope of $y = \sqrt{x}$ at any point is
$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{2\sqrt{x}}.$$
The slope of the tangent line to $y = \sqrt{x}$ at $(1,1)$ is therefore $m_2 = 1/2$. Using formula 1.60, the angle θ between these tangent lines is $\theta = \mathrm{Tan}^{-1}\left|\dfrac{2 - 1/2}{1 + 2(1/2)}\right| = 0.644$ radians.

36. If $x > 0$, then $f(x) = x$, and $\quad f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{(x+h)-x}{h} = \lim_{h\to 0} 1 = 1.$

When $x < 0$, $f(x) = -x$, and $\quad f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{-(x+h)-(-x)}{h} = \lim_{h\to 0} (-1) = -1.$

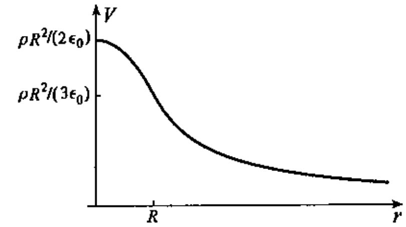
Finally, $\quad f'(0) = \lim_{h\to 0} \dfrac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \dfrac{|0+h| - 0}{h} = \lim_{h\to 0} \dfrac{|h|}{h}.$

Since this limit does not exist, $f'(0)$ does not exist. These three results are combined in the one formula $f'(x) = |x|/x$.

37. The graph of the function suggests that there is a tangent line when $r = R$. To verify this we calculate

$$f'(R) = \lim_{h\to 0} \frac{f(R+h) - f(R)}{h}.$$

When $h < 0$, we find that



$$\lim_{h\to 0^-} \frac{1}{h}\left\{\frac{\rho}{6\epsilon_0}[3R^2 - (R+h)^2] - \frac{\rho}{6\epsilon_0}(3R^2 - R^2)\right\} = \frac{\rho}{6\epsilon_0}\lim_{h\to 0^-}\frac{-h(2R+h)}{h} = -\frac{\rho R}{3\epsilon_0}.$$

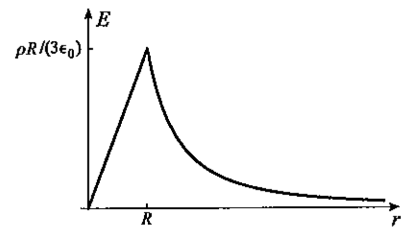
When $h > 0$, we obtain

$$\lim_{h\to 0^+}\frac{1}{h}\left[\frac{R^3\rho}{3\epsilon_0(R+h)} - \frac{R^3\rho}{3\epsilon_0 R}\right] = \frac{R^3\rho}{3\epsilon_0}\lim_{h\to 0^+}\frac{R-(R+h)}{hR(R+h)} = -\frac{R\rho}{3\epsilon_0}.$$

Since these limits are the same, $f'(R) = -\rho R/(3\epsilon_0)$.

38. The graph of the function suggests that $f'(R)$ does not exist since there appears to be no tangent line when $r = R$. To verify this we calculate

$$f'(R) = \lim_{h\to 0}\frac{f(R+h) - f(R)}{h}.$$



When $h < 0$, we find that $\displaystyle\lim_{h\to 0^-}\frac{1}{h}\left[\frac{\rho(R+h)}{3\epsilon_0} - \frac{\rho R}{3\epsilon_0}\right] = \lim_{h\to 0^-}\frac{\rho}{3\epsilon_0} = \frac{\rho}{3\epsilon_0}.$ When $h > 0$,

$$\lim_{h\to 0^+}\frac{1}{h}\left[\frac{\rho R^3}{3\epsilon_0(R+h)^2} - \frac{\rho R}{3\epsilon_0}\right] = \frac{\rho R}{3\epsilon_0}\lim_{h\to 0^+}\frac{1}{h}\left[\frac{R^2-(R+h)^2}{(R+h)^2}\right] = \frac{\rho R}{3\epsilon_0}\lim_{h\to 0^+}\frac{-2R-h}{(R+h)^2} = \frac{-2\rho}{3\epsilon_0}.$$

Since these limits are not the same, $f'(R)$ does not exist.

39.
$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^{1/3} - x^{1/3}}{h}\\ &= \lim_{h\to 0}\frac{(x+h)^{1/3}-x^{1/3}}{h}\,\frac{(x+h)^{2/3}+(x+h)^{1/3}x^{1/3}+x^{2/3}}{(x+h)^{2/3}+(x+h)^{1/3}x^{1/3}+x^{2/3}}\\ &= \lim_{h\to 0}\frac{(x+h)-x}{h[(x+h)^{2/3}+(x+h)^{1/3}x^{1/3}+x^{2/3}]} = \frac{1}{3x^{2/3}} \end{aligned}$$

40. By equation 3.3, $\displaystyle f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f(x)f(h)-f(x)}{h} = f(x)\lim_{h\to 0}\frac{f(h)-1}{h}.$

Since $f'(0) = 1$, it follows that $\displaystyle 1 = f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)-1}{h}.$

Consequently, $f'(x) = f(x)\cdot 1 = f(x)$.

EXERCISES 3.2

1. $f'(x) = 4x$
2. $f'(x) = 9x^2 + 4$
3. $f'(x) = 20x - 3$
4. $f'(x) = 20x^4 - 30x^2 + 3$
5. $f'(x) = -2x^{-3} = -2/x^3$
6. $f'(x) = 2(-3x^{-4}) = -6/x^4$
7. $f'(x) = 20x^3 - 9x^2 + (-1)x^{-2} = 20x^3 - 9x^2 - 1/x^2$
8. $f'(x) = -(1/2)(-2x^{-3}) + 3(-4x^{-5}) = 1/x^3 - 12/x^5$
9. $f'(x) = 10x^9 - (-10)x^{-11} = 10x^9 + 10/x^{11}$
10. $f'(x) = 20x^3 + \dfrac{1}{4}(-5x^{-6}) = 20x^3 - \dfrac{5}{4x^6}$

11. $f'(x) = -20x^{-5} + (1/4)(5x^4) = -\dfrac{20}{x^5} + \dfrac{5x^4}{4}$

12. $f'(x) = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$

13. $f'(x) = -6x^{-3} + 2(-1/2)x^{-3/2} = -\dfrac{6}{x^3} - \dfrac{1}{x^{3/2}}$

14. $f'(x) = -(3/2)x^{-5/2} + (3/2)x^{1/2} = -\dfrac{3}{2x^{5/2}} + \dfrac{3\sqrt{x}}{2}$

15. $f'(x) = 2(1/3)x^{-2/3} - 3(2/3)x^{-1/3} = \dfrac{2}{3x^{2/3}} - \dfrac{2}{x^{1/3}}$

16. $f'(x) = \pi(\pi x^{\pi-1}) = \pi^2 x^{\pi-1}$

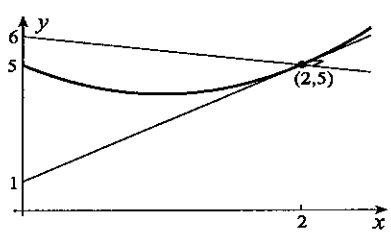
17. Since $f(x) = x^4 + 4x^2 + 4$, we find that $f'(x) = 4x^3 + 8x$.

18. Since $f(x) = 4x - x^{-3}$, we obtain $f'(x) = 4 + 3x^{-4} = 4 + 3/x^4$.

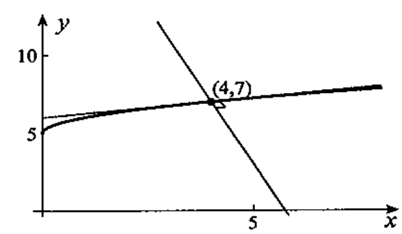
19. $f'(x) = (5/3)x^{2/3} - (2/3)x^{-1/3} = \dfrac{5x^{2/3}}{3} - \dfrac{2}{3x^{1/3}}$

20. Since $f(x) = 8x^3 + 60x^2 + 150x + 125$, we obtain $f'(x) = 24x^2 + 120x + 150 = 6(2x+5)^2$.

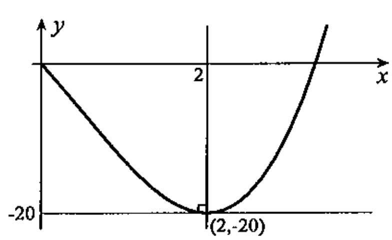
21. Since $f'(x) = 2x - 2$, the slope of the tangent line at $(2,5)$ is $2(2) - 2 = 2$. Equations for the tangent and normal lines are $y - 5 = 2(x-2)$ and $y - 5 = -(1/2)(x-2)$, or, $y = 2x + 1$ and $x + 2y = 12$. The tangent and normal lines do not appear to intersect at right angles because there are different scales along the axes.



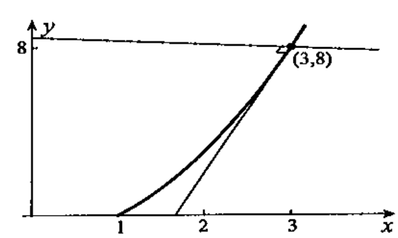
22. Since $f'(x) = (1/2)x^{-1/2}$, the slope of the tangent line at $(4,7)$ is $(1/2)4^{-1/2} = 1/4$. Equations for the tangent and normal lines are $y - 7 = (1/4)(x-4)$ and $y - 7 = -4(x-4)$, or, $4y = x + 24$ and $4x + y = 23$. The tangent and normal lines do not appear to intersect at right angles because there are different scales along the axes.



23. Since $f'(x) = 6x^2 - 6x - 12$, the slope of the tangent line at $(2,-20)$ is $6(4) - 6(2) - 12 = 0$. Equations for the tangent and normal lines are $y = -20$ and $x = 2$.



24. Since $y = x^2 - 1$, $dy/dx = 2x$. The slope of the the tangent line is therefore 6, and equations for the tangent and normal lines are $y - 8 = 6(x-3)$ and $y - 8 = -(1/6)(x-3)$, or, $y = 6x - 10$ and $x + 6y = 51$. The tangent and normal lines do not appear to intersect at right angles because the axes have different scales.



25. Points at which the slope of the tangent line is equal to 2 are defined by

$$2 = \frac{dy}{dx} = x^3 - 2x^2 - 19x + 22 \quad \Longrightarrow \quad 0 = x^3 - 2x^2 - 19x + 20 = (x-1)(x+4)(x-5).$$

The points are $(1, 145/12)$, $(-4, -400/3)$, and $(5, -655/12)$.

26. Since the slope of the tangent line at the point (x_0, y_0) is $2ax_0$, the equation of the tangent line is $y - y_0 = 2ax_0(x - x_0)$. The x-intercept can be found by setting $y = 0$ and solving for x,

$$-y_0 = 2ax_0(x - x_0) \quad\Longrightarrow\quad x = x_0 - \frac{y_0}{2ax_0}.$$

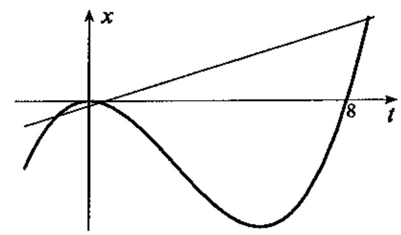
Since $y_0 = ax_0^2$, it follows that the x-intercept is

$$x = x_0 - \frac{ax_0^2}{2ax_0} = x_0 - \frac{x_0}{2} = \frac{x_0}{2},$$

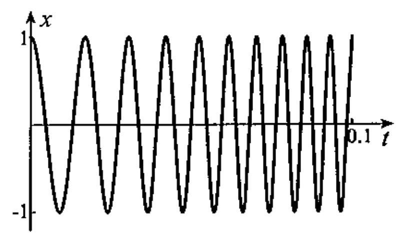
which is halfway between the origin and the point $x = x_0$ on the x-axis.

27. The slope of the tangent line to $x = t^3 - 8t^2$ is $f'(t) = 3t^2 - 16t$. For the tangent line to be parallel to $x = 6t - 3$, we set $3t^2 - 16t = 6$ and this implies that

$$t = \frac{16 \pm \sqrt{256 + 72}}{6} = \frac{8 \pm \sqrt{82}}{3}.$$

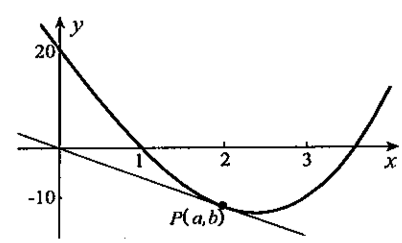


28. (a) The graph is to the right.
(b) Since $\frac{d}{dt}(1000\pi t^2 + 100\pi t) = 2000\pi t + 100\pi$, the frequencies at $t = 0$ and $t = 0.1$ are $100\pi/(2\pi) = 50$ Hz and $300\pi/(2\pi) = 150$ Hz.



29. Since $\frac{d}{dt}(\alpha t^2 + \beta t + \phi) = 2\alpha t + \beta$, initial and final frequencies are $(2\alpha t_1 + \beta)/(2\pi)$ and $(2\alpha t_2 + \beta)/(2\pi)$. The change is therefore $(2\alpha t_2 + \beta)/(2\pi) - (2\alpha t_1 + \beta)/(2\pi) = \alpha(t_2 - t_1)/\pi$.

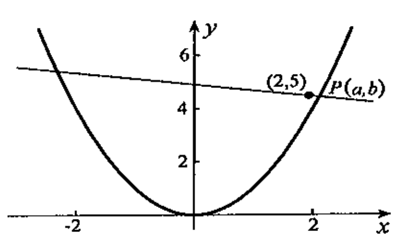
30. If $P(a, b)$ is any point on the curve, the slope of the tangent line at P is $f'(a) = 3a^2 + 2a - 22$. The equation of the tangent line at P is $y - b = (3a^2 + 2a - 22)(x - a)$, and this line will pass through the origin if $-b = (3a^2 + 2a - 22)(-a)$. Since (a, b) is on the curve, its coordinates must satisfy the equation of the curve, $b = a^3 + a^2 - 22a + 20$. When we equate these two expressions for b,



$$a^3 + a^2 - 22a + 20 = 3a^3 + 2a^2 - 22a$$

which simplifies to $0 = 2a^3 + a^2 - 20 = (a - 2)(2a^2 + 5a + 10)$. The only solution is $a = 2$, giving the point $(2, -12)$.

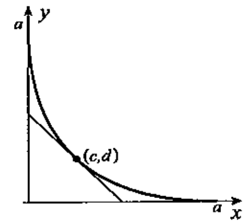
31. If $P(a, b)$ is any point on the curve, the slope of the tangent line at P is $f'(a) = 2a$. The equation of the normal line at P is $y - b = -1/(2a)(x - a)$, and this line will pass through the point $(2, 5)$ if $5 - b = -1/(2a)(2 - a)$. Since (a, b) is on the curve, its coordinates must satisfy the equation of the curve, $b = a^2$. When we equate these two expressions for b,



$$a^2 = 5 + \frac{1}{2a}(2 - a),$$

and this simplifies to $0 = 2a^3 - 9a - 2 = (a + 2)(2a^2 - 4a - 1)$. Solutions are $a = -2, (2 \pm \sqrt{6})/2$. The required points are therefore $(-2, 4)$ and $\left((2 \pm \sqrt{6})/2, (5 \pm 2\sqrt{6})/2\right)$. This could be used to find the shortest distance from the point $(2, 5)$ to the parabola.

32. The slope of the tangent line to $\sqrt{x}+\sqrt{y}=\sqrt{a}$, or, $y=f(x)=(\sqrt{a}-\sqrt{x})^2=a-2\sqrt{a}\sqrt{x}+x$ at any point (c,d) is $f'(c)=-\dfrac{\sqrt{a}}{\sqrt{c}}+1$.



The equation of the tangent line at this point is

$$y-d=\left(1-\frac{\sqrt{a}}{\sqrt{c}}\right)(x-c),$$

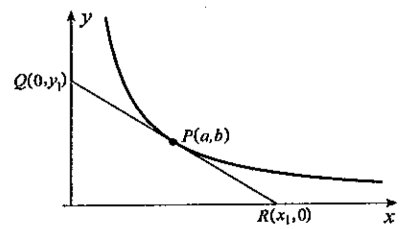
and its x- and y-intercepts are

$$c-\frac{d}{1-\sqrt{a}/\sqrt{c}} \quad \text{and} \quad d-c(1-\sqrt{a}/\sqrt{c}).$$

Since (c,d) is on the curve, it follows that $d=(\sqrt{a}-\sqrt{c})^2$, and the sum of the intercepts is

$$\begin{aligned} c-\frac{\sqrt{c}d}{\sqrt{c}-\sqrt{a}}+d-\sqrt{c}(\sqrt{c}-\sqrt{a}) &= c-\frac{\sqrt{c}(\sqrt{a}-\sqrt{c})^2}{\sqrt{c}-\sqrt{a}}+(\sqrt{a}-\sqrt{c})^2-\sqrt{c}(\sqrt{c}-\sqrt{a}) \\ &= c+\sqrt{c}(\sqrt{a}-\sqrt{c})+(\sqrt{a}-\sqrt{c})^2-\sqrt{c}(\sqrt{c}-\sqrt{a}) \\ &= c+2\sqrt{c}(\sqrt{a}-\sqrt{c})+a-2\sqrt{c}\sqrt{a}+c=a. \end{aligned}$$

33. If $P(a,b)$ is any point on the curve, the slope of the tangent line at P is $f'(a)=-1/a^2$. The equation of the tangent line at P is $y-b=(-1/a^2)(x-a)$, and x- and y-intercepts of this line are $x_1=a+a^2b$, $\quad y_1=b+1/a$.

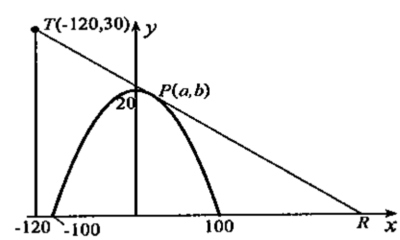


Since $\|PQ\|^2=a^2+(b-y_1)^2=a^2+\left(\dfrac{-1}{a}\right)^2=a^2+\dfrac{1}{a^2}$, and

$$\|PR\|^2=(a-x_1)^2+b^2=(-a^2b)^2+b^2=a^4b^2+b^2=a^4\left(\frac{1}{a^2}\right)+\frac{1}{a^2}=a^2+\frac{1}{a^2},$$

point P does indeed bisect the line segment joining Q and R.

34. The required position occurs at the point R where the tangent line from $T(-120,30)$ to the parabola representing the hill intersects the x-axis. Let the point of tangency be $P(a,b)$. If $y=cx^2+d$ is the equation of the parabola, then using the points $(100,0)$ and $(0,20)$, we obtain $0=100^2c+d$ and $20=c(0)^2+d$. These give $c=-1/500$ and $d=20$, so that the equation of the parabola is $y=f(x)=20-x^2/500$, $-100\le x\le 100$. Since $f'(x)=-x/250$, the slope of the tangent line at $P(a,b)$ is $f'(a)=-a/250$. The slope of this line is also the slope of PT, namely, $(b-30)/(a+120)$, and therefore



$$\frac{b-30}{a+120}=-\frac{a}{250} \quad\Longrightarrow\quad b=30-\frac{a(a+120)}{250}.$$

Since $P(a,b)$ is on the parabola, it also follows that $b=20-a^2/500$. When we equate these expressions for b,

$$30-\frac{a(a+120)}{250}=20-\frac{a^2}{500} \quad\Longrightarrow\quad a^2+240a-5000=0.$$

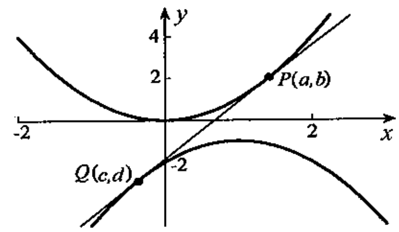
Of the two solutions $-120\pm 10\sqrt{194}$, only $a=10\sqrt{194}-120$ is positive. The y-coordinate of P is $b=20-(10\sqrt{194}-120)^2/500=(24\sqrt{194}-238)/5$. The equation of the tangent line is therefore $y-b=(-a/250)(x-a)$, and its x-intercept is given by $-b=(-a/250)(x-a) \Longrightarrow x=250b/a+a$. When we substitute the calculated values for a and b, the result is

$$x=\frac{250(24\sqrt{194}-238)/5}{10\sqrt{194}-120}+10\sqrt{194}-120=15(4+\sqrt{194})\text{ m}.$$

35. If $f(x) = x^n$, then

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \quad \text{and using the given identity}$$
$$= \lim_{h\to 0} \frac{h[(x+h)^{n-1} + (x+h)^{n-2}x + \cdots + (x+h)x^{n-2} + x^{n-1}]}{h} = nx^{n-1}.$$

36. If we denote coordinates of the two points by $P(a,b)$ and $Q(c,d)$, then the fact that P and Q are on the curves requires



$$b = a^2, \quad d = -c^2 + 2c - 2.$$

Since $d(x^2)/dx = 2x$, and $d(-x^2+2x-2)/dx = -2x+2$, and P and Q share a common tangent line, it follows that

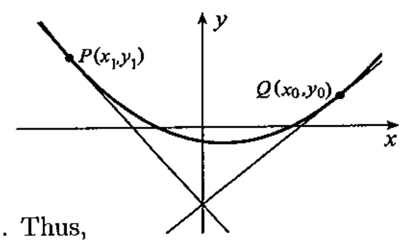
$$2a = -2c + 2, \quad \frac{d-b}{c-a} = 2a.$$

To find P and Q, we solve these four equations in a, b, c, and d. If we substitute from the first and second equations into the fourth,

$$a^2 - (-c^2 + 2c - 2) = 2a(a-c) \qquad \Longrightarrow \qquad c^2 - 2c + 2 - a^2 + 2ac = 0.$$

From the third equation, $a = 1 - c$, and therefore $c^2 - 2c + 2 - (1-c)^2 + 2c(1-c) = 0$. This reduces to $2c^2 - 2c - 1 = 0$ with solutions $c = (1 \pm \sqrt{3})/2$. These give the pairs of points $P\big((1+\sqrt{3})/2, (2+\sqrt{3})/2\big)$ and $Q\big((1-\sqrt{3})/2, (-4-\sqrt{3})/2\big)$, and $P\big((1-\sqrt{3})/2, (2-\sqrt{3})/2\big)$ and $Q\big((1+\sqrt{3})/2, (-4+\sqrt{3})/2\big)$.

37. Let $P(x_1, y_1)$ and $Q(x_0, y_0)$ be any two non-vertex points on the parabola. Since $f'(x) = 2ax + b$, equations of the tangent lines at P and Q are respectively



$$y - y_1 = (2ax_1 + b)(x - x_1),$$

and

$$y - y_0 = (2ax_0 + b)(x - x_0).$$

When we solve these for y and equate results,

$$y_1 + (2ax_1 + b)(x - x_1) = y_0 + (2ax_0 + b)(x - x_0),$$

or, $(2ax_0 + b - 2ax_1 - b)x = y_1 - y_0 - x_1(2ax_1 + b) + x_0(2ax_0 + b)$. Thus,

$$x = \frac{y_1 - y_0 - x_1(2ax_1+b) + x_0(2ax_0+b)}{2a(x_0 - x_1)}$$
$$= \frac{(ax_1^2 + bx_1 + c) - (ax_0^2 + bx_0 + c) - x_1(2ax_1 + b) + x_0(2ax_0 + b)}{2a(x_0 - x_1)}$$
$$= \frac{a(x_1^2 - x_0^2) + b(x_1 - x_0) - 2a(x_1^2 - x_0^2) - b(x_1 - x_0)}{2a(x_0 - x_1)} = -\frac{a(x_1 + x_0)(x_1 - x_0)}{2a(x_0 - x_1)} = \frac{x_0 + x_1}{2}.$$

Thus, the point of intersection is on the vertical line half way between P and Q.

38. If $x > 0$, then $\dfrac{d}{dx}|x|^n = \dfrac{d}{dx}x^n = nx^{n-1} = n|x|^{n-1}$. On the other hand, if $x < 0$, then

$$\frac{d}{dx}|x|^n = \frac{d}{dx}(-x)^n = (-1)^n \frac{d}{dx}x^n = (-1)^n nx^{n-1} = -n(-x)^{n-1} = -n|x|^{n-1}.$$

Graphs of $y = f(x) = |x|^n$ indicate that $f'(0) = 0$ when $n > 1$. This can also be verified algebraically,

$$f'(0) = \lim_{h\to 0} \frac{|0+h|^n - |0|^n}{h} = \lim_{h\to 0} \frac{|h|^n}{h} = 0 \quad \text{when } n > 1.$$

These three situations are all encompassed by the formula $\dfrac{d}{dx}|x|^n = n|x|^{n-1}\text{sgn}(x)$, where $\text{sgn}(x)$ is the signum function of Exercise 47 in Section 2.4.

39. We use the binomial theorem to write that

$$\begin{aligned}\frac{d}{dx}(ax+b)^n &= \frac{d}{dx}\left[a^nx^n + \binom{n}{1}a^{n-1}x^{n-1}b + \binom{n}{2}a^{n-2}x^{n-2}b^2 + \cdots + \binom{n}{n-1}axb^{n-1} + b^n\right]\\ &= a^n(nx^{n-1}) + \binom{n}{1}a^{n-1}(n-1)x^{n-2}b + \binom{n}{2}a^{n-2}(n-2)x^{n-3}b^2 + \cdots + \binom{n}{n-1}ab^{n-1}.\end{aligned}$$

Now, for $1 \le r \le n-1$, we note that

$$\binom{n}{r}(n-r) = \frac{n!\,(n-r)}{r!\,(n-r)!} = \frac{n(n-1)!\,(n-r)}{r!\,(n-r)(n-r-1)!} = n\binom{n-1}{r}.$$

Consequently,

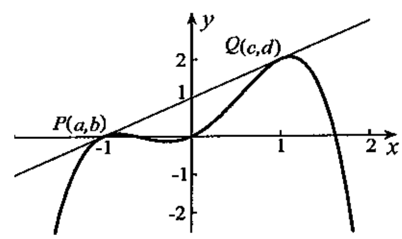
$$\begin{aligned}\frac{d}{dx}(ax+b)^n &= a^n(nx^{n-1}) + n\binom{n-1}{1}a^{n-1}x^{n-2}b + n\binom{n-1}{2}a^{n-2}x^{n-3}b^2 + \cdots + n\binom{n-1}{n-1}ab^{n-1}\\ &= an\left[a^{n-1}x^{n-1} + \binom{n-1}{1}a^{n-2}x^{n-2}b + \binom{n-1}{2}a^{n-3}x^{n-3}b^2 + \cdots + b^{n-1}\right]\\ &= an(ax+b)^{n-1}.\end{aligned}$$

40. If we denote coordinates of the two points by $P(a,b)$ and $Q(c,d)$, then the fact that P and Q are on the curves requires

$$b = a + 2a^2 - a^4, \quad d = c + 2c^2 - c^4.$$

Since $dy/dx = 1 + 4x - 4x^3$, and P and Q share a common tangent line, it follows that

$$1 + 4a - 4a^3 = 1 + 4c - 4c^3, \quad \frac{d-b}{c-a} = 1 + 4a - 4a^3.$$



To find P and Q, we solve these four equations in a, b, c, and d. The third equation implies that

$$0 = 4a - 4c + 4c^3 - 4a^3 = 4[(a-c) - (a^3 - c^3)] = 4(a-c)(1 - a^2 - ac - c^2).$$

Either $a = c$, which is unacceptable, or $1 = a^2 + ac + c^2$. When we subtract the first two equations, the result is $d - b = c - a + 2(c^2 - a^2) - (c^4 - a^4)$. Substitution of this into the fourth equation gives

$$\frac{c - a + 2(c^2 - a^2) - (c^4 - a^4)}{c - a} = 1 + 4a - 4a^3$$

or,

$$1 + 2(a+c) - (a^3 + a^2c + ac^2 + c^3) = 1 + 4a - 4a^3.$$

We now have two equations in a and c,

$$1 = a^2 + ac + c^2, \quad 0 = 2c - 2a + 4a^3 - (a^3 + a^2c + ac^2 + c^3) = 2c - 2a + 4a^3 - a(a^2 + ac + c^2) - c^3.$$

These imply that
$$\begin{aligned}0 &= 2c - 2a + 4a^3 - a - c^3 = 2c - 3a + 4a^3 - c(1 - a^2 - ac)\\ &= c - 3a + 4a^3 + a^2c + ac^2 = c - 3a + 4a^3 + a^2c + a(1 - a^2 - ac) = c - 2a + 3a^3.\end{aligned}$$

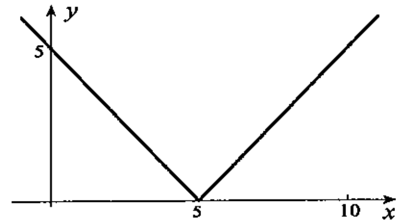
Thus, $c = 2a - 3a^3$, and therefore $1 = a^2 + a(2a - 3a^3) + (2a - 3a^3)^2$, from which

$$0 = 9a^6 - 15a^4 + 7a^2 - 1 = (a-1)(a+1)(3a^2-1)^2.$$

Thus, $a = \pm 1$ or $a = \pm 1/\sqrt{3}$. The first two lead to the points $(-1, 0)$ and $(1, 2)$. The second pair of values do not lead to distinct points.

EXERCISES 3.3

1. The graph of the function shows that $f(x)$ has left-hand derivative equal to -1 at $x = 5$, and right-hand derivative equal to 1. It therefore has no derivative at $x = 5$.



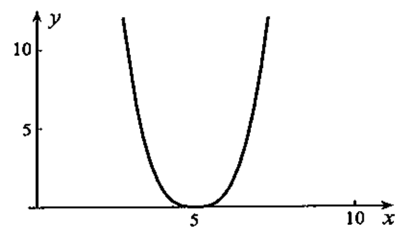
2. Since $f(x)$ is not defined for $x < 0$, it cannot have a left-hand derivative at $x = 0$. Its right-hand derivative at $x = 0$ is defined by

$$f'_+(0) = \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{h^{3/2} - 0}{h} = \lim_{h\to 0^+} \sqrt{h} = 0.$$

3. The graph of the function indicates that all three derivatives have value 0 at $x = 5$. We verify this algebraically.

$$\begin{aligned} f'_+(5) &= \lim_{h\to 0^+} \frac{f(5+h) - f(5)}{h} \\ &= \lim_{h\to 0^+} \frac{|5+h-5|^3 - (0)}{h} \\ &= \lim_{h\to 0^+} (h^2) = 0. \end{aligned}$$

Similarly, $f'_-(5) = 0$, and therefore $f'(5) = 0$.

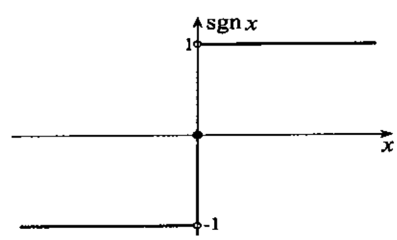


4. The graph of $f(x) = \operatorname{sgn} x$ makes it clear that the function does not have a left-hand derivative at $x = 0$ or a right-hand derivative, and cannot therefore have a derivative. We can also verify this algebraically.

$$f'_-(0) = \lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{-1-0}{h} = \infty,$$

and

$$f'_+(0) = \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{1-0}{h} = \infty.$$



5. Since the function is undefined at $x = 1$, it cannot have a derivative there.

6. The graph of $\lfloor x \rfloor$ in Exercise 68 of Section 1.5 shows that the right derivative is equal to 0, but there is no left derivative, and therefore no derivative.

7. If $h(a)$ is defined as 0, then $h'_-(a) = 0$, but $h'_+(a)$ and $h'(a)$ are still undefined.

8. If $h(a)$ is defined as 1, then $h'_+(a) = 0$, but $h'_-(a)$ and $h'(a)$ are still undefined.

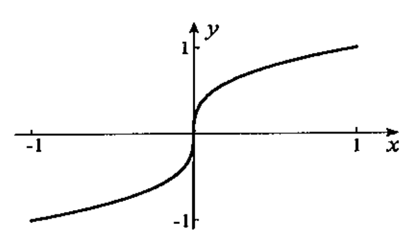
9. If $\operatorname{sgn} x$ has no value at $x = 0$, then it cannot have a left, a right, or a full derivative at $x = 0$.

10. True **11.** False **12.** True **13.** False

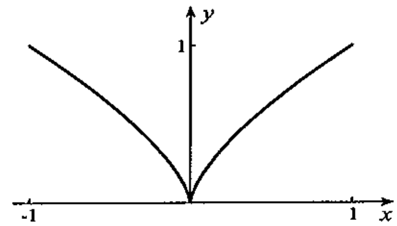
14. By equation 3.2,

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{h^{1/3} - 0}{h} = \lim_{h\to 0} \frac{1}{h^{2/3}} = \infty. \end{aligned}$$

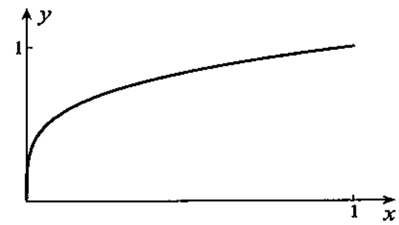
The graph of the function confirms this.



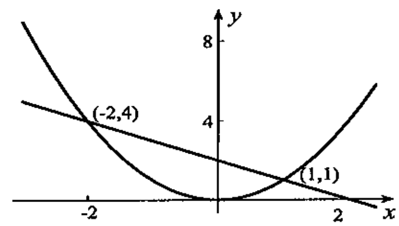
15. By equation 3.2, $f'(0) = \lim_{h\to 0} \dfrac{f(0+h)-f(0)}{h}$
$$= \lim_{h\to 0} \frac{h^{2/3}-0}{h} = \lim_{h\to 0} \frac{1}{h^{1/3}},$$
and this limit does not exist. The graph of the function confirms this.



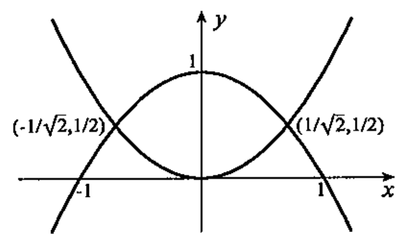
16. Since $f(x)$ is not defined for $x < 0$, it cannot have a derivative at $x = 0$. We can show that it does not have a right-hand derivative at $x = 0$ by calculating
$$f'_+(0) = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h}$$
$$= \lim_{h\to 0^+} \frac{h^{1/4}-0}{h} = \lim_{h\to 0^+} \frac{1}{h^{3/4}} = \infty.$$
The graph of the function confirms this.



17. To find point(s) of intersection, we set $x^2 = 2 - x$, solutions of which are $x = 1, -2$. These give the points of intersection $(1,1)$ and $(-2,4)$. The slope of the tangent line to $y = x^2$ at $(1,1)$ is 2 and that of $y = 2 - x$ is -1. According to formula 1.60, the angle between the tangent lines at $(1,1)$ is
$$\theta = \mathrm{Tan}^{-1}\left|\frac{2+1}{1+(2)(-1)}\right| = 1.249 \text{ radians.}$$
A similar calculation for the point $(-2,4)$ gives the angle 0.540 radians.



18. To find point(s) of intersection, we set $x^2 = 1 - x^2$, solutions of which are $x = \pm 1/\sqrt{2}$. These give the points of intersection $(\pm 1/\sqrt{2}, 1/2)$. Slopes of the curves $y = f(x) = x^2$ and $y = g(x) = 1 - x^2$ at $(1/\sqrt{2}, 1/2)$ are
$$f'(1/\sqrt{2}) = \{2x\}_{|x=1/\sqrt{2}} = \sqrt{2}$$
and
$$g'(1/\sqrt{2}) = \{-2x\}_{|x=1/\sqrt{2}} = -\sqrt{2}.$$
Equation 1.60 gives for the angle θ between the curves at this point
$$\theta = \mathrm{Tan}^{-1}\left|\frac{\sqrt{2}+\sqrt{2}}{1+(\sqrt{2})(-\sqrt{2})}\right| = 1.231 \text{ radians.}$$
The same angle is obtained at the other point of intersection.



19. To find point(s) of intersection of the curves we set $2 - x^2 = (1+x)/2$, solutions of which are $x = -3/2, 1$. Points of intersection are therefore $(1,1)$ and $(-3/2, -1/4)$. Slopes of the curves $y = f(x) = (x+1)/2$ and $y = g(x) = 2 - x^2$ at $(1,1)$ are
$$f'(1) = 1/2 \qquad \text{and} \qquad g'(1) = \{-2x\}_{|x=1} = -2.$$
Since these slopes are negative reciprocals, the curves intersect orthogonally at the point $(1,1)$. A similar calculation shows that the curves are not orthogonal at $(-3/2, -1/4)$.

20. To find point(s) of intersection of the curves we set $3 - x^2 = (7 + x^2)/4$, solutions of which are $x = \pm 1$. Points of intersection are therefore $(\pm 1, 2)$. Slopes of the curves $y = f(x) = 3 - x^2$ and $y = g(x) = (x^2+7)/4$ at $(1,2)$ are
$$f'(1) = \{-2x\}_{|x=1} = -2 \qquad \text{and} \qquad g'(1) = \{x/2\}_{|x=1} = 1/2.$$
Since these slopes are negative reciprocals, the curves intersect orthogonally at the point $(1,2)$. The curves are also orthogonal at $(-1,2)$.

21. Slopes of the curves $y = f(x) = x - 2x^2$ and $y = g(x) = x^3 + 2x$ at the point $(-1, -3)$ are

$$f'(-1) = \{1 - 4x\}_{|x=-1} = 5 \qquad \text{and} \qquad g'(-1) = \{3x^2 + 2\}_{|x=-1} = 5.$$

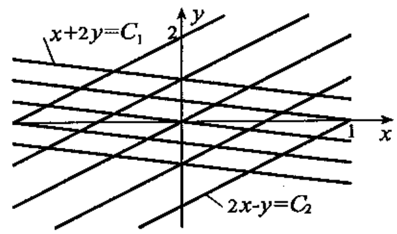
The curves are therefore tangent at the point.

22. Slopes of the curves $y = f(x) = x^3$ and $y = g(x) = x^2 + x - 1$ at the point $(1, 1)$ are

$$f'(1) = \{3x^2\}_{|x=1} = 3 \qquad \text{and} \qquad g'(1) = \{2x + 1\}_{|x=1} = 3.$$

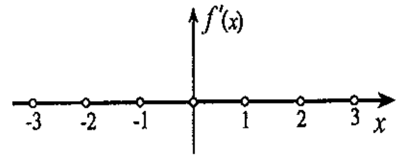
The curves are therefore tangent at the point.

23. (a),(b) Lines are shown to the right.
(c) Yes The slope of every line in the first family is $-1/2$, and the slope of every line in the second family is 2. They do not appear to intersect at right angles because scales are different on the x- and y-axes.



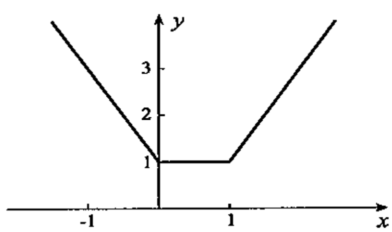
24. Since each straight line in the first family has slope $2/3$ and each line in the second family has slope $-1/k$, the families are orthogonal trajectories if $(2/3)(-1/k) = -1 \Longrightarrow k = 2/3$.

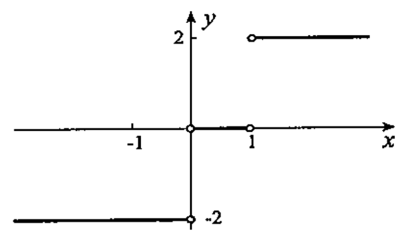
25. $f'(x) = 0$ for all x except for integer values of x for which there is no derivative. The graph is therefore a horizontal line with holes at integer values.



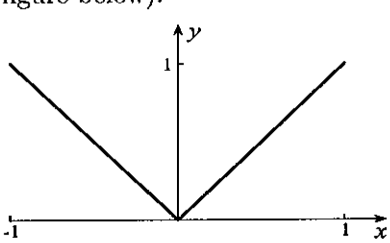
26. According to equation 3.2, $f'(0) = \lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \dfrac{h|h| - 0}{h} = \lim_{h \to 0} |h| = 0.$

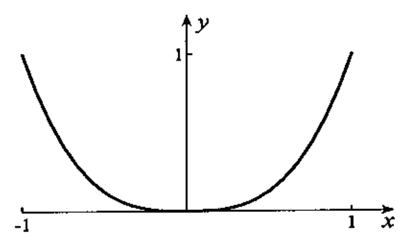
27. Using formula 3.13, $f'(x) = \dfrac{|x|}{x} + \dfrac{|x-1|}{x-1}.$



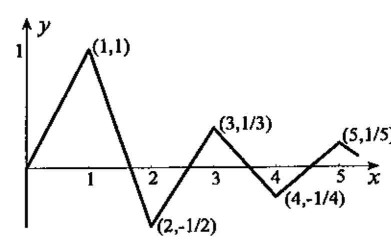


28. Sometimes. The function $f(x) = x$ is differentiable, but $|x|$ does not have a derivative at $x = 0$ (left figure below). On the other hand, $f(x) = x^3$ is differentiable, and $|x^3|$ does have a derivative at $x = 0$ (right figure below).





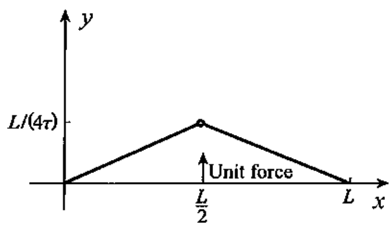
29. No. Consider the function $f(x)$ shown in the figure. It is asymptotic to the x-axis, but because the derivative is undefined at integer values of x, we cannot say that $\lim_{x \to \infty} f'(x) = 0$.

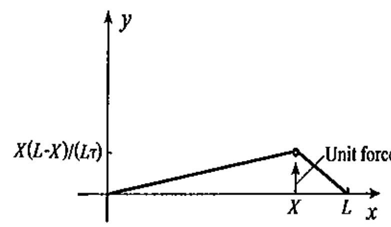


30. (a) For $X = L/2$,

$$G(x; L/2) = \frac{1}{L\tau}[x(L - L/2)h(L/2 - x) + (L/2)(L - x)h(x - L/2)]$$
$$= \frac{1}{\tau}\begin{cases} x/2 & 0 \le x < L/2 \\ (L-x)/2, & L/2 < x \le L \end{cases}.$$

Its graph in the left figure below is symmetric about $x = L/2$ as would be expected.





(b) When $L/2 < X < L$,

$$G(x; X) = \frac{1}{L\tau}\begin{cases} x(L - X), & 0 \le x < X \\ X(L - x), & X < x \le L \end{cases}.$$

This is shown in the right figure above.
(c) Since the functions x, $h(x - X)$, $L - x$, and $h(X - x)$ are continuous for $0 \le x \le L$, except at $x = X$ for the Heaviside functions, it follows that $G(x; X)$ is continuous for all $x \ne X$. Since

$$\lim_{x \to X^+} G(x; X) = \frac{X(L - X)}{L\tau} = \lim_{x \to X^-} G(x; X),$$

the discontinuity at $x = X$ is removable. The graphs in parts (a) and (b) illustrate this.
(d) The jump in the discontinuity of dG/dx at $x = X$ is

$$\lim_{x \to X^+} \frac{dG}{dx} - \lim_{x \to X^-} \frac{dG}{dx} = \frac{1}{L\tau} \lim_{x \to X^+} [(L - X)\,h(X - x) - X\,h(x - X)]$$
$$- \frac{1}{L\tau} \lim_{x \to X^-} [(L - X)\,h(X - x) - X\,h(x - X)]$$
$$= \frac{1}{L\tau}[-X] - \frac{1}{L\tau}[L - X] = -\frac{1}{\tau}.$$

This is the change in the slope of the graph of $G(x; X)$ at X.

31. They are not always the same. For the Heaviside function $h(x - a)$ in Figure 2.35, there is no right-hand derivative $f'_+(a)$ at $x = a$. On the other hand, $\lim_{x \to a^+} f'(x) = 0$.

32. According to equation 3.2, $f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{h \sin(1/h) - 0}{h} = \lim_{h \to 0} \sin\left(\frac{1}{h}\right)$. Since this limit does not exist, there is no derivative at $x = 0$.

33. According to equation 3.2, $f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{h^n \sin(1/h) - 0}{h} = \lim_{h \to 0} h^{n-1} \sin\left(\frac{1}{h}\right)$. This limit will be 0 when $n > 1$.

34. According to equation 3.2, $f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - 0}{h} = \lim_{h \to 0} \frac{f(h)}{h}$.
When h is a rational number, $f(h)/h = h^2/h = h$, and when h is irrational, $f(h)/h = 0/h = 0$. It follows that $f'(0) = \lim_{h \to 0} \frac{f(h)}{h} = 0$.

EXERCISES 3.4

1. $f'(x) = 2x(x+3) + (x^2+2)(1) = 3x^2 + 6x + 2$

2. $f'(x) = (2-x^2)(2x+4) + (-2x)(x^2+4x+2) = 8 - 12x^2 - 4x^3$

3. $f'(x) = \dfrac{(3x+2)(1) - x(3)}{(3x+2)^2} = \dfrac{2}{(3x+2)^2}$

4. $f'(x) = \dfrac{(4x^2-5)(2x) - x^2(8x)}{(4x^2-5)^2} = \dfrac{-10x}{(4x^2-5)^2}$

5. $f'(x) = \dfrac{(2x-1)(2x) - x^2(2)}{(2x-1)^2} = \dfrac{2x^2-2x}{(2x-1)^2}$

6. $f'(x) = \dfrac{(4x^2+1)(3x^2) - x^3(8x)}{(4x^2+1)^2} = \dfrac{x^2(4x^2+3)}{(4x^2+1)^2}$

7. With $f(x) = x^{3/2} + \sqrt{x}$, we find $f'(x) = (3/2)\sqrt{x} + 1/(2\sqrt{x})$.

8. $f'(x) = \dfrac{(3x+2)\left(\dfrac{1}{2\sqrt{x}}\right) - \sqrt{x}(3)}{(3x+2)^2} = \dfrac{2-3x}{2\sqrt{x}(3x+2)^2}$

9. $f'(x) = \dfrac{(3x+4)(4x) - (2x^2-5)(3)}{(3x+4)^2} = \dfrac{6x^2+16x+15}{(3x+4)^2}$

10. $f'(x) = \dfrac{(2x^2-1)(1) - (x+5)(4x)}{(2x^2-1)^2} = -\dfrac{2x^2+20x+1}{(2x^2-1)^2}$

11. $f'(x) = \dfrac{(1-3x)(2x+1) - (x^2+x)(-3)}{(1-3x)^2} = \dfrac{1+2x-3x^2}{(1-3x)^2}$

12. $f'(x) = \dfrac{(x^2-5x+1)(2x+2) - (x^2+2x+3)(2x-5)}{(x^2-5x+1)^2} = \dfrac{-7x^2-4x+17}{(x^2-5x+1)^2}$

13. $f'(x) = \dfrac{(x^3-3x^2+2x+5)(0) - (3x^2-6x+2)}{(x^3-3x^2+2x+5)^2} = -\dfrac{3x^2-6x+2}{(x^3-3x^2+2x+5)^2}$

14. If we write the function in the form $f(x) = \dfrac{(x+1)^3+9}{(x+1)^3} = 1 + \dfrac{9}{(x+1)^3}$, its derivative is

$f'(x) = \dfrac{(x+1)^3(0) - 9\dfrac{d}{dx}(x+1)^3}{(x+1)^6}$. Since $\dfrac{d}{dx}(x+1)^3 = \dfrac{d}{dx}(x^3+3x^2+3x+1) = 3x^2+6x+3$

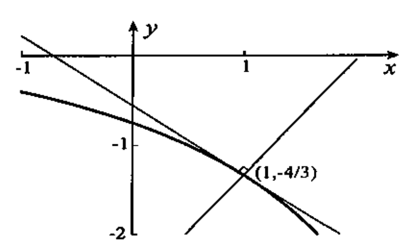
$= 3(x+1)^2$, we obtain $f'(x) = \dfrac{-9(3)(x+1)^2}{(x+1)^6} = \dfrac{-27}{(x+1)^4}$.

15. $f'(x) = \dfrac{(1-\sqrt{x})(1/3)x^{-2/3} - x^{1/3}(-1/2)x^{-1/2}}{(1-\sqrt{x})^2} = \dfrac{2(1-\sqrt{x})+3\sqrt{x}}{6x^{2/3}(1-\sqrt{x})^2} = \dfrac{2+\sqrt{x}}{6x^{2/3}(1-\sqrt{x})^2}$

16. $f'(x) = \dfrac{(\sqrt{x}-4)\left(\dfrac{1}{2\sqrt{x}}+2\right) - (\sqrt{x}+2x)\left(\dfrac{1}{2\sqrt{x}}\right)}{(\sqrt{x}-4)^2}$

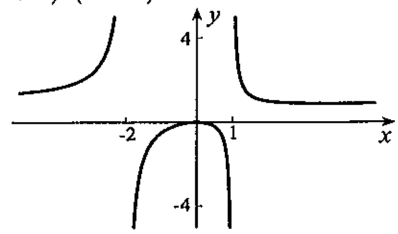
$= \dfrac{(\sqrt{x}-4)(1+4\sqrt{x}) - (\sqrt{x}+2x)}{2\sqrt{x}(\sqrt{x}-4)^2} = \dfrac{2x-16\sqrt{x}-4}{2\sqrt{x}(\sqrt{x}-4)^2} = \dfrac{x-8\sqrt{x}-2}{\sqrt{x}(\sqrt{x}-4)^2}$

17. Since
$f'(x) = \dfrac{(x-4)(1) - (x+3)(1)}{(x-4)^2} = \dfrac{-7}{(x-4)^2}$,
the slope of the tangent line at $(1,-4/3)$ is $f'(1) = -7/9$. Equations for the tangent and normal lines are $y + 4/3 = -(7/9)(x-1)$ and $y + 4/3 = (9/7)(x-1)$, or, $7x + 9y + 5 = 0$ and $27x - 21y = 55$.



18. (a) The plot is shown to the right.

(b) $f'(x) = \dfrac{(x^2+x-2)(2x) - x^2(2x+1)}{(x^2+x-2)^2} = \dfrac{x^2-4x}{(x^2+x-2)^2} = \dfrac{x(x-4)}{(x+2)^2(x-1)^2}$



This expression shows that:

$f'(x) > 0$ for $x < -2$,
$f'(x) > 0$ for $-2 < x < 0$,
$f'(x) < 0$ for $0 < x < 1$,
$f'(x) < 0$ for $1 < x < 4$,
$f'(x) > 0$ for $x > 4$.

We can see the first three of these on the graph, but not the last two. The expression for $f'(x)$ also indicates that $f'(x) = 0$ at $x = 0$ and $x = 4$. This is clear on the graph when $x = 0$, but not when $x = 4$.

19. With $C(x) = a\left(\dfrac{x^2+bx}{x+c}\right)$,
$$C'(x) = a\left[\frac{(x+c)(2x+b)-(x^2+bx)(1)}{(x+c)^2}\right] = a\left[\frac{x^2+2cx+bc}{(x+c)^2}\right]$$
$$= a\left[\frac{(x^2+2cx+c^2)+(bc-c^2)}{(x+c)^2}\right] = a\left[1+\frac{c(b-c)}{(x+c)^2}\right].$$

20.
$$\begin{aligned}\frac{d}{dx}[f(x)g(x)h(x)] &= f(x)\frac{d}{dx}[g(x)h(x)] + f'(x)[g(x)h(x)]\\ &= f(x)[g(x)h'(x)+g'(x)h(x)] + f'(x)[g(x)h(x)]\\ &= f(x)g(x)h'(x) + f(x)g'(x)h(x) + f'(x)g(x)h(x)\end{aligned}$$

21. To find point(s) of intersection, we set $x^3 = 2/(1+x^2)$, the only solution of which is $x = 1$. The only point of intersection is therefore $(1,1)$. Slopes of the curves $y = f(x) = x^3$ and $y = g(x) = 2/(1+x^2)$ at $(1,1)$ are
$$f'(1) = \{3x^2\}_{|x=1} = 3 \qquad \text{and} \qquad g'(1) = \left\{\frac{(1+x^2)(0)-2(2x)}{(1+x^2)^2}\right\}_{|x=1} = -1.$$
Angle θ between the curves at $(1,1)$ is given by equation 1.60,
$$\theta = \operatorname{Tan}^{-1}\left|\frac{3+1}{1+(3)(-1)}\right| = 1.107 \text{ radians.}$$

22. To find the point(s) of intersection, we set $2x+2 = x^2/(x-1)$ from which $x = \pm\sqrt{2}$. These give the points of intersection $(\pm\sqrt{2}, 2\pm 2\sqrt{2})$. Slopes of the curves $y = f(x) = 2x+2$ and $y = g(x) = x^2/(x-1)$ at the point $(\sqrt{2}, 2+2\sqrt{2})$ are
$$f'(\sqrt{2}) = 2 \qquad \text{and} \qquad g'(\sqrt{2}) = \left[\frac{(x-1)(2x)-x^2(1)}{(x-1)^2}\right]_{|x=\sqrt{2}} = \frac{2}{1-\sqrt{2}}.$$
Angle θ between the curves at this point is given by equation 1.60,
$$\theta = \operatorname{Tan}^{-1}\left|\frac{2-\left(\dfrac{2}{1-\sqrt{2}}\right)}{1+2\left(\dfrac{2}{1-\sqrt{2}}\right)}\right| = 0.668 \text{ radians.}$$
Slopes of the curves at the other point of intersection $(-\sqrt{2}, 2-2\sqrt{2})$ are
$$f'(-\sqrt{2}) = 2 \qquad \text{and} \qquad g'(-\sqrt{2}) = \left[\frac{(x-1)(2x)-x^2(1)}{(x-1)^2}\right]_{|x=-\sqrt{2}} = \frac{2}{1+\sqrt{2}}.$$
The angle between the curves at this point is

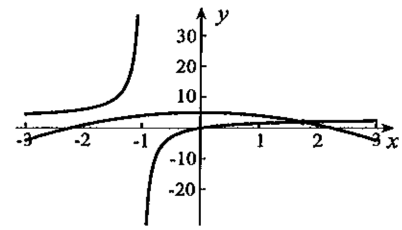
$$\theta = \text{Tan}^{-1}\left|\frac{2-\left(\dfrac{2}{1+\sqrt{2}}\right)}{1+2\left(\dfrac{2}{1+\sqrt{2}}\right)}\right| = 0.415 \text{ radians.}$$

23. Graphs of the curves suggest that they intersect near the point $(2,2)$. To confirm this, we solve

$$5 - x^2 = \frac{3x}{x+1}.$$

A calculator yields the only (real) solution as $1.757\,28$. The slope of $y = f(x) = 5 - x^2$ at this value of x is

$f'(1.757\,28) = -2(1.757\,28) = -3.514\,56$.



Because the slope of $y = g(x) = 3x/(x+1)$ at any value of x is

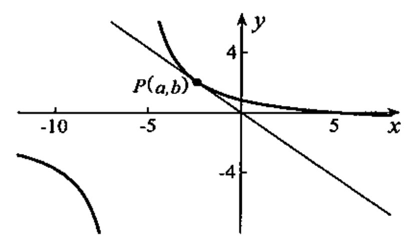
$$g'(x) = \frac{(x+1)(3) - 3x(1)}{(x+1)^2} = \frac{3}{(x+1)^2},$$

slope at $x = 1.757\,28$ is $g'(1.757\,28) = 3/(1.757\,28+1)^2 = 0.394\,602$. Using formula 1.60 with $m_1 = -3.514\,56$ and $m_2 = 0.394\,602$, the angle θ between the curves at their point of intersection is

$$\theta = \text{Tan}^{-1}\left|\frac{-3.514\,56 - 0.394\,602}{1 + (-3.514\,56)(0.394\,602)}\right| = 1.47 \text{ radians.}$$

24. The slope of the tangent line to $y = f(x) = (5-x)/(6+x)$ at $P(a,b)$ is

$$f'(a) = \left[\frac{(6+x)(-1) - (5-x)(1)}{(6+x)^2}\right]_{|x=a} = \frac{-11}{(6+a)^2}.$$



The equation of the tangent line at P is

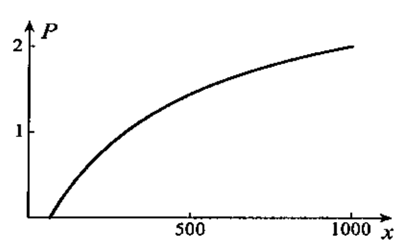
$$y - b = [-11/(6+a)^2](x-a),$$

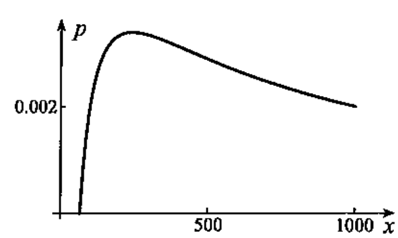
and this line passes through $(0,0)$ if

$$-b = [-11/(6+a)^2(-a).$$

Since $b = (5-a)/(6+a)$, it follows that $-\dfrac{5-a}{6+a} = \dfrac{11a}{(6+a)^2} \implies a^2 - 10a - 30 = 0$. The two solutions of this equation are $a = 5 \pm \sqrt{55}$. The points at which the tangent line passes through $(0,0)$ are therefore $\left(5 \pm \sqrt{55}, \mp\sqrt{55}/(11 \pm \sqrt{55})\right)$.

25. (a),(b) Plots of $P(x)$ and $p(x)$ are shown below.



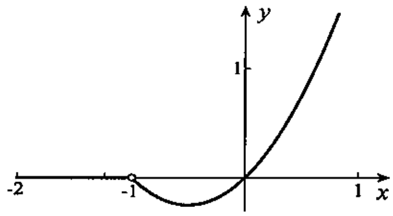
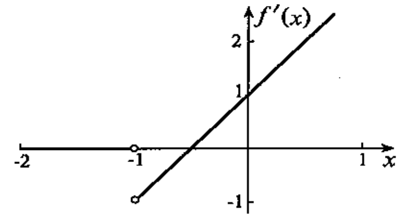


(c) Since average profit $p(x)$ is the slope of the line joining (x,P) to the origin, $p(x)$ is maximized when this line is tangent to the graph. This occurs when

$$\frac{P}{x} = P'(x) \quad\implies\quad \frac{3x - 200}{x(x+400)} = \frac{(x+400)(3) - (3x-200)(1)}{(x+400)^2} = \frac{1400}{(x+400)^2}.$$

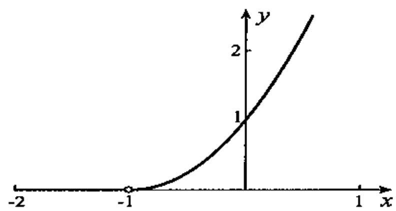
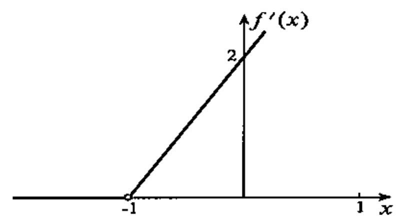
This equation implies that $1400x = (3x-200)(x+400) \implies 3x^2 - 400x - 80\,000 = 0$. The positive solution of this quadratic is approximately 243.

26. (a) $f'(x) = \begin{cases} 0, & x < -1 \\ 2x+1, & x > -1 \end{cases}$

(b) No. The left-hand derivative would be 0 and the right-hand derivative would be -1.

27. (a) $f'(x) = \begin{cases} 0, & x < -1 \\ 2(x+1), & x > -1 \end{cases}$

(b) Yes. $f'(-1) = 0$

EXERCISES 3.5

1. Since $f'(x) = 3x^2 + 20x^3$, $f''(x) = 6x + 60x^2$.

2. Since $f'(x) = 3x^2 - 6x + 2$, $f''(x) = 6x - 6$, and $f'''(x) = 6$.

3. Since $f'(x) = x^3 + 3x + 2 + (x+1)(3x^2+3) = 4x^3 + 3x^2 + 6x + 5$, $f''(x) = 12x^2 + 6x + 6$. Hence, $f''(2) = 12(4) + 6(2) + 6 = 66$.

4. Since $f'(x) = 4x^3 - 6x - 1/x^2$, $f''(x) = 12x^2 - 6 + 2/x^3$, and $f'''(x) = 24x - 6/x^4$. Hence, $f'''(1) = 24(1) - 6/(1)^4 = 18$.

5. Since $f(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}}$, $f'(x) = \dfrac{1}{2\sqrt{x}} - \dfrac{1}{2x^{3/2}}$, and $f''(x) = -\dfrac{1}{4x^{3/2}} + \dfrac{3}{4x^{5/2}}$.

6. Since $f'(t) = 3t^2 + 3/t^4$ and $f''(t) = 6t - 12/t^5$, we obtain $f'''(t) = 6 + 60/t^6$.

7. $\dfrac{d^9}{dx^9}(x^{10}) = \dfrac{d^8}{dx^8}(10x^9) = \dfrac{d^7}{dx^7}(10\cdot 9x^8) = \cdots = 10\cdot 9\cdots 2\,x = 10!\,x$

8. Since $f'(u) = \dfrac{(u+1)\left(\dfrac{1}{2\sqrt{u}}\right) - \sqrt{u}(1)}{(u+1)^2} = \dfrac{1-u}{2\sqrt{u}(u+1)^2}$, it follows that

$$f''(u) = \frac{1}{2}\left\{\frac{\sqrt{u}(u+1)^2(-1) - (1-u)\left[\dfrac{1}{2\sqrt{u}}(u+1)^2 + \sqrt{u}\dfrac{d}{du}(u^2+2u+1)\right]}{u(u+1)^4}\right\}$$

$$= \frac{\dfrac{-2u(u+1)^2 - (1-u)[(u+1)^2 + 2u(2u+2)]}{2\sqrt{u}}}{2u(u+1)^4} = \frac{3u^2 - 6u - 1}{4u^{3/2}(u+1)^3}.$$

9. Since $\dfrac{dt}{dx} = \dfrac{(2x-6)(1) - x(2)}{(2x-6)^2} = \dfrac{-6}{(2x-6)^2}$,

$$\frac{d^2t}{dx^2} = -6\left[\frac{(2x-6)^2(0) - \dfrac{d}{dx}(2x-6)^2}{(2x-6)^4}\right] = 6\left[\frac{\dfrac{d}{dx}(4x^2 - 24x + 36)}{(2x-6)^4}\right] = \frac{6(8x-24)}{(2x-6)^4} = \frac{3}{(x-3)^3}.$$

10. Since $f'(x) = \dfrac{(\sqrt{x}+1)(1) - x\left(\dfrac{1}{2\sqrt{x}}\right)}{(\sqrt{x}+1)^2} = \dfrac{\sqrt{x}+2}{2\left(\sqrt{x}+1\right)^2}$, it follows that

$$f''(x) = \frac{1}{2}\left\{\frac{(\sqrt{x}+1)^2\left(\dfrac{1}{2\sqrt{x}}\right) - (\sqrt{x}+2)\dfrac{d}{dx}(x+2\sqrt{x}+1)}{\left(\sqrt{x}+1\right)^4}\right\}$$

$$= \frac{\dfrac{\left(\sqrt{x}+1\right)^2 - 2\sqrt{x}(\sqrt{x}+2)\left(1+\frac{1}{\sqrt{x}}\right)}{2\sqrt{x}}}{2\left(\sqrt{x}+1\right)^4} = -\frac{\sqrt{x}+3}{4\sqrt{x}\left(\sqrt{x}+1\right)^3}.$$

11. (a) $\dfrac{2}{r}\dfrac{dT}{dr} + \dfrac{d^2T}{dr^2} = \dfrac{2}{r}\left(-\dfrac{d}{r^2}\right) + \dfrac{2d}{r^3} = 0$

(b) Since $f(a) = T_a$ and $f(b) = T_b$, it follows that $T_a = c + \dfrac{d}{a}$, $\quad T_b = c + \dfrac{d}{b}$. The solution of these equations is $c = \dfrac{bT_b - aT_a}{b-a}$ and $d = \dfrac{ab(T_a - T_b)}{b-a}$.

12. (a) $\dfrac{d}{dr}\left(r^2\dfrac{dT}{dr}\right) = \dfrac{d}{dr}\left[r^2\left(\dfrac{kr}{3} - \dfrac{c}{r^2}\right)\right] = \dfrac{d}{dr}\left[\dfrac{kr^3}{3} - c\right] = kr^2$

(b) Since $f(a) = T_a$ and $f(b) = T_b$, it follows that $T_a = \dfrac{ka^2}{6} + \dfrac{c}{a} + d$, $\qquad T_b = \dfrac{kb^2}{6} + \dfrac{c}{b} + d$. The solution of these equations is

$$c = \frac{ab}{b-a}\left[(T_a - T_b) + \frac{k}{6}(b^2 - a^2)\right], \qquad d = \frac{bT_b - aT_a}{b-a} - \frac{k}{6}(a^2 + ab + b^2).$$

13. Since $f'(x) = 3ax^2 + 2bx + c$, and $f''(x) = 6ax + 2b$, we must have

$$4 = 3a + 2b + c, \qquad 5 = 12a + 2b.$$

Because $f(1) = 0$ and $f(2) = 4$,

$$0 = a + b + c + d, \qquad 4 = 8a + 4b + 2c + d.$$

The solution of these four equations is $a = 5/4$, $b = -5$, $c = 41/4$, and $d = -13/2$.

14. (a) Using the result in Example 3.19,

$$\frac{d^3}{dx^3}(uv) = \frac{d}{dx}\left(v\frac{d^2u}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}\right)$$

$$= \frac{dv}{dx}\frac{d^2u}{dx^2} + v\frac{d^3u}{dx^3} + 2\frac{d^2u}{dx^2}\frac{dv}{dx} + 2\frac{du}{dx}\frac{d^2v}{dx^2} + \frac{du}{dx}\frac{d^2v}{dx^2} + u\frac{d^3v}{dx^3}$$

$$= v\frac{d^3u}{dx^3} + 3\frac{dv}{dx}\frac{d^2u}{dx^2} + 3\frac{d^2v}{dx^2}\frac{du}{dx} + u\frac{d^3v}{dx^3}.$$

(b) $\dfrac{d^4}{dx^4}(uv) = v\dfrac{d^4u}{dx^4} + 4\dfrac{dv}{dx}\dfrac{d^3u}{dx^3} + 6\dfrac{d^2v}{dx^2}\dfrac{d^2u}{dx^2} + 4\dfrac{d^3v}{dx^3}\dfrac{du}{dx} + v\dfrac{d^4u}{dx^4}$

15. If $(x^2-1)^n$ were expanded, the result would be a polynomial of degree $2n$. When $2n$ differentiations are performed, the only nonzero term is the $(2n)^{\text{th}}$-derivative of x^{2n}, and this gives $(2n)!$.

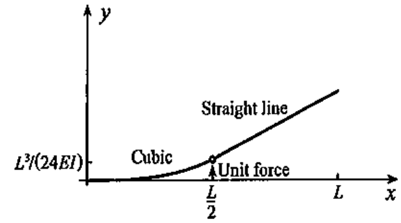
16. (a) When $X = L/2$,

$$G(x; L/2) = \frac{1}{6EI}(x - L/2)^3 h(x - L/2) - \frac{x^3}{6EI} + \frac{Lx^2}{4EI}$$

$$= \frac{1}{12EI}\begin{cases} -2x^3 + 3Lx^2, & 0 \le x < L/2 \\ 2(x - L/2)^3 - 2x^3 + 3Lx^2, & L/2 < x \le L \end{cases}$$

$$= \frac{1}{12EI}\begin{cases} -2x^3 + 3Lx^2, & 0 \le x < L/2 \\ 3xL^2/2 - L^3/4, & L/2 < x \le L \end{cases}.$$

A plot is shown to the right. It has a removable discontinuity at $x = L/2$. It is straight for $L/2 < x \le L$, as would be expected since no forces act on the board for $x > L/2$.



(b) For $x > X$,

$$\begin{aligned} G(x;X) &= \frac{1}{6EI}(x-X)^3 - \frac{x^3}{6EI} + \frac{Xx^2}{2EI} \\ &= \frac{1}{6EI}(x^3 - 3x^2X + 3xX^2 - X^3) - \frac{x^3}{6EI} + \frac{Xx^2}{2EI} = \frac{X^2}{2EI}x - \frac{X^3}{6EI}, \end{aligned}$$

and this is a straight line.
(c) Since the functions $(x-X)^3$, $h(x-X)$, x^3, and x^2 are continuous for $0 \le x \le L$, except at $x = X$ for the Heaviside function, it follows that $G(x;X)$ is continuous for all $x \ne X$. Since

$$\lim_{x \to X^+} G(x;X) = -\frac{X^3}{6EI} + \frac{X^3}{2EI} = \frac{X^3}{3EI} = \lim_{x \to X^-} G(x;X),$$

the discontinuity at $x = X$ is removable. Limits of $dG/dx = [(x-X)^2h(x-X) - x^2 + 2Xx]/(2EI)$ and $d^2G/dx^2 = [(x-X)h(x-X) - x + X]/(EI)$, show that they are also continuous except for a removable discontinuity at $x = X$.
(d) Since $d^3G/dx^3 = [h(x-X) - 1]/(EI)$, the jump in d^3G/dx^3 at $x = X$ is

$$\lim_{x \to X^+} \frac{d^3G}{dx^3} - \lim_{x \to X^-} \frac{d^3G}{dx^3} = 0 - \left(-\frac{1}{EI}\right) = \frac{1}{EI}.$$

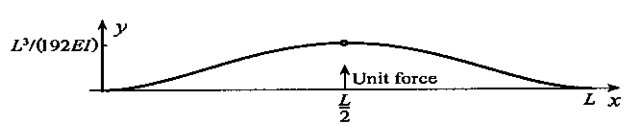
17. (a) When $X = L/2$,

$$\begin{aligned} G(x;L/2) &= \frac{1}{6EI}(x-L/2)^3h(x-L/2) + \frac{x^3}{6EIL^3}\left(-L^3 + \frac{3L^3}{4} - \frac{2L^3}{8}\right) \\ &\quad + \frac{x^2}{2EIL^2}\left[\frac{L^3}{8} - 2L\left(\frac{L^2}{4}\right) + L^2\left(\frac{L}{2}\right)\right] \\ &= \frac{1}{48EI}\begin{cases} -4x^3 + 3Lx^2, & 0 \le x < L/2 \\ 4x^3 - 9Lx^2 + 6L^2x - L^3, & L/2 < x \le L \end{cases}. \end{aligned}$$

It is straightforward to check that values of this function and its derivative are zero at $x = 0$ and $x = L$. We can show that it is symmetric about $x = L/2$ by replacing each x by $L - x$ in that part of $G(x;L/2)$ for $x > L/2$ to show that we get $-4x^3 + 3Lx^2$,

$$\begin{aligned} 4(L-x)^3 - 9L(L-x)^2 + 6L^2(L-x) - L^3 &= 4(L^3 - 3L^2x + 3Lx^2 - x^3) \\ &\quad - 9L(L^2 - 2Lx + x^2) + 6L^2(L-x) - L^3 \\ &= 3Lx^2 - 4x^3. \end{aligned}$$

The graph is shown below with a removable discontinuity at $x = L/2$.



(b) Only the first term of $G(x;X)$ has a discontinuity, and it is at $x = X$. Since

$$\lim_{x \to X^+} G(x;X) = \lim_{x \to X^-} G(x;X),$$

the discontinuity at $x = X$ is removable. Limits of

$$\frac{dG}{dx} = \frac{1}{2EI}(x-X)^2h(x-X) + \frac{x^2}{2EIL^3}(-L^3+3LX^2-2X^3) + \frac{x}{EIL^2}(X^3-2LX^2+L^2X)$$

and

$$\frac{d^2G}{dx^2} = \frac{1}{EI}(x-X)h(x-X) + \frac{x}{EIL^3}(-L^3+3LX^2-2X^3) + \frac{1}{EIL^2}(X^3-2LX^2+L^2X)$$

show that they are also continuous except for a removable discontinuity at $x = X$.

(c) Since $\dfrac{d^3G}{dx^3} = \dfrac{1}{EI}h(x-X) + \dfrac{1}{EIL^3}(-L^3+3LX^2-2X^3)$, the jump in d^3G/dx^3 at $x = X$ is

$$\lim_{x\to X^+}\frac{d^3G}{dx^3} - \lim_{x\to X^-}\frac{d^3G}{dx^3} = \frac{1}{EI}.$$

18. When $n = 1$: Left side $= \dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ and

Right side $= \displaystyle\sum_{r=0}^{1}\binom{1}{r}\frac{d^ru}{dx^r}\frac{d^{1-r}v}{dx^{1-r}} = \binom{1}{0}\frac{d^0u}{dx^0}\frac{dv}{dx} + \binom{1}{1}\frac{du}{dx}\frac{d^0v}{dx^0} = u\frac{dv}{dx} + v\frac{du}{dx}.$

The result is therefore valid for $n = 1$. Suppose k is some integer for which the result is valid; that is,

$$\frac{d^k}{dx^k}(uv) = \sum_{r=0}^{k}\binom{k}{r}\frac{d^ru}{dx^r}\frac{d^{k-r}v}{dx^{k-r}}.$$

Then,

$$\begin{aligned}
\frac{d^{k+1}}{dx^{k+1}}(uv) &= \frac{d}{dx}\left[\frac{d^k}{dx^k}(uv)\right] = \sum_{r=0}^{k}\binom{k}{r}\left[\frac{d^{r+1}u}{dx^{r+1}}\frac{d^{k-r}v}{dx^{k-r}} + \frac{d^ru}{dx^r}\frac{d^{k-r+1}v}{dx^{k-r+1}}\right]\\
&= \sum_{r=0}^{k}\binom{k}{r}\frac{d^{r+1}u}{dx^{r+1}}\frac{d^{k-r}v}{dx^{k-r}} + \sum_{r=0}^{k}\binom{k}{r}\frac{d^ru}{dx^r}\frac{d^{k-r+1}v}{dx^{k-r+1}}\\
&= \sum_{r=1}^{k+1}\binom{k}{r-1}\frac{d^ru}{dx^r}\frac{d^{k-r+1}v}{dx^{k-r+1}} + \sum_{r=0}^{k}\binom{k}{r}\frac{d^ru}{dx^r}\frac{d^{k-r+1}v}{dx^{k-r+1}}\\
&= \binom{k}{0}\frac{d^0u}{dx^0}\frac{d^{k+1}v}{dx^{k+1}} + \sum_{r=1}^{k}\left[\binom{k}{r-1} + \binom{k}{r}\right]\frac{d^ru}{dx^r}\frac{d^{k-r+1}v}{dx^{k-r+1}} + \binom{k}{k}\frac{d^{k+1}u}{dx^{k+1}}\frac{d^0v}{dx^0}\\
&= \binom{k+1}{0}\frac{d^0u}{dx^0}\frac{d^{k+1}v}{dx^{k+1}} + \sum_{r=1}^{k}\binom{k+1}{r}\frac{d^ru}{dx^r}\frac{d^{k-r+1}v}{dx^{k-r+1}} + \binom{k+1}{k+1}\frac{d^{k+1}u}{dx^{k+1}}\frac{d^0v}{dx^0}\\
&= \sum_{r=0}^{k+1}\binom{k+1}{r}\frac{d^ru}{dx^r}\frac{d^{k-r+1}v}{dx^{k-r+1}}.
\end{aligned}$$

Since this is the result for $k+1$, the formula is correct for all n by mathematical induction.

EXERCISES 3.6

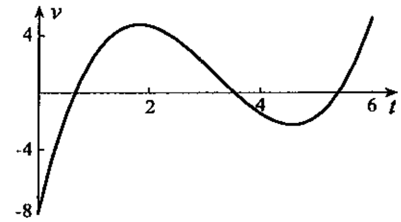
1. (a) Since x is negative at $t = 1$, the particle is to the left of the origin at this time. Since x is positive at $t = 4$, the particle is to the right of the origin at this time.
(b) Since the slope of the graph is negative at $t = 1/2$, the particle is moving to the left at this time. Since the slope of the graph is positive at $t = 3$, the particle is moving to the right at this time.
(c) Since the slope changes sign three times, the particle changes direction three times.
(d) At $t = 7/2$ the slope of the graph is close to zero, whereas at $t = 9/2$ it is clearly negative. Hence the velocity is greater at $t = 7/2$.
(e) Since speed is the magnitude of velocity, it is greater at $t = 9/2$.

2. The velocity of the particle is $v(t) = \dfrac{2t^3}{3} - \dfrac{32t^2}{5} + \dfrac{50t}{3} - \dfrac{251}{30}$.

(a) Since $x(1) = -2$, the particle is to the left of the origin at $t = 1$. Since $x(4) = 6$, the particle is to the right of the origin at $t = 4$.

(b) Since $v(1/2) = -31/20$, the particle is moving to the left at $t = 1/2$. Since $v(3) = 61/30$, the particle is moving to the right at $t = 3$.

(c) The graph of $v(t)$ to the right shows that $v(t)$ changes sign three times. Hence, the particle changes direction three times.



(d) Since $v(7/2) = 3/20$ and $v(9/2) = -133/60$, the velocity is greater at $t = 7/2$.

(e) Since $|v(7/2)| = 3/20$ and $|v(9/2)| = 133/60$, the speed is greater at $t = 9/2$.

3. (a) Since x is positive at $t = 1$, the particle is to the right of the origin at this time. Since x is negative at $t = 4$, the particle is to the left of the origin at this time.

(b) Since the slope of the graph is positive at $t = 1/2$, the particle is moving to the right at this time. Since the slope of the graph is negative at $t = 3$, the particle is moving to the left at this time.

(c) Since the slope changes sign twice, the particle changes direction twice.

(d) At $t = 7/2$ the slope of the graph is negative, whereas at $t = 9/2$ it is positive. Hence the velocity is greater at $t = 9/2$.

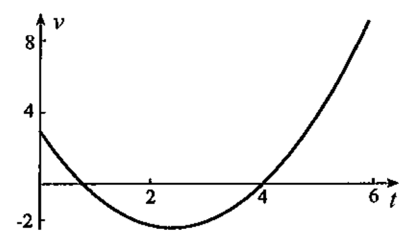
(e) Since speed is the magnitude of velocity, it would appear to be slightly greater at $t = 9/2$.

4. The velocity of the particle is $v(t) = \dfrac{14t^2}{15} - \dfrac{202t}{45} + \dfrac{132}{45}$.

(a) Since $x(1) = 3$, the particle is to the right of the origin at $t = 1$. Since $x(4) = -34/15$, the particle is to the left of the origin at $t = 4$.

(b) Since $v(1/2) = 83/90$, the particle is moving to the right at $t = 1/2$. Since $v(3) = -32/15$, the particle is moving to the left at $t = 3$.

(c) The graph of $v(t)$ to the right shows that $v(t)$ changes sign twice. Hence, the particle changes direction twice.



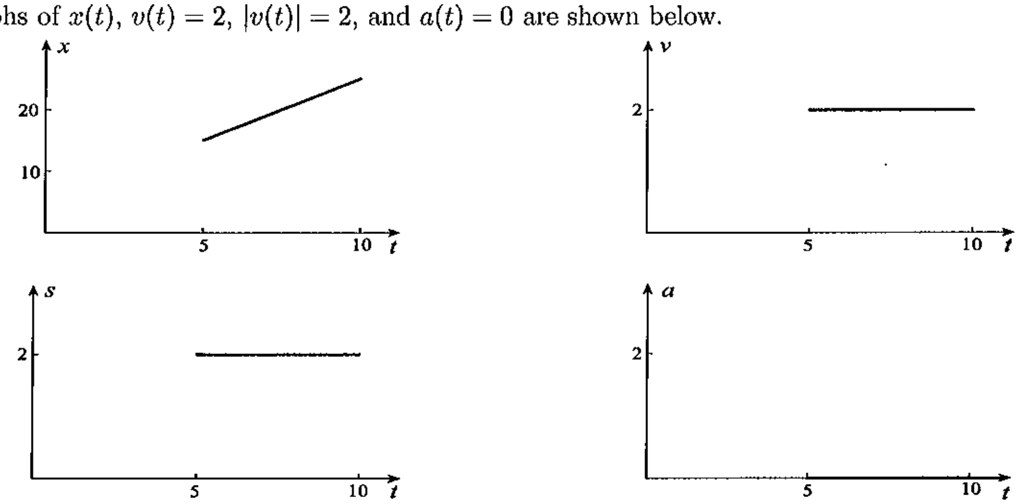
(d) Since $v(7/2) = -121/90$ and $v(9/2) = 49/30$, the velocity is greater at $t = 9/2$.

(e) Since $|v(7/2)| = 121/90$ and $|v(9/2)| = 49/30$, the speed is greater at $t = 9/2$.

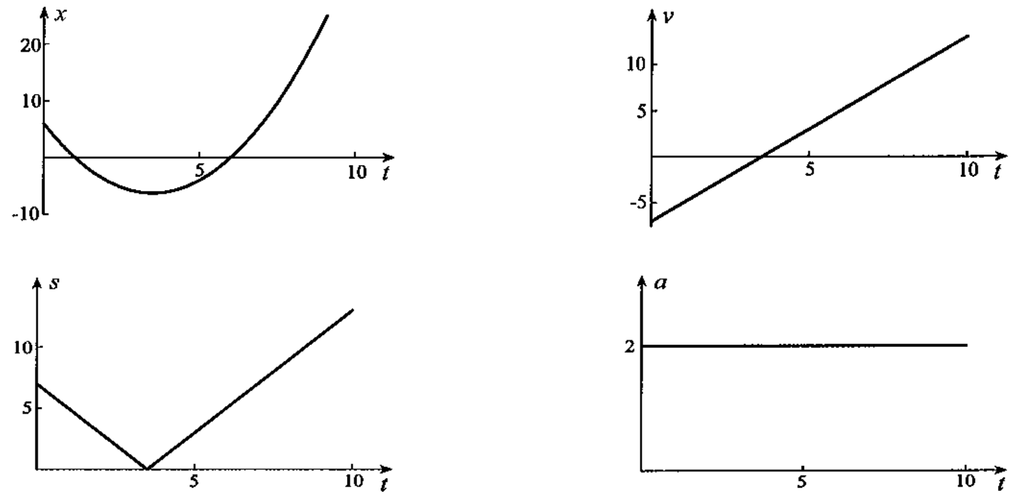
5. If the acceleration of a car changes rapidly (perhaps because the clutch is let out too fast), the car experiences jerky motion. The jerk for the dispacement function in Exercise 2 is $x'''(t) = 4t - 64/5$.

6. The jerk is $x'''(t) = 28/15$.

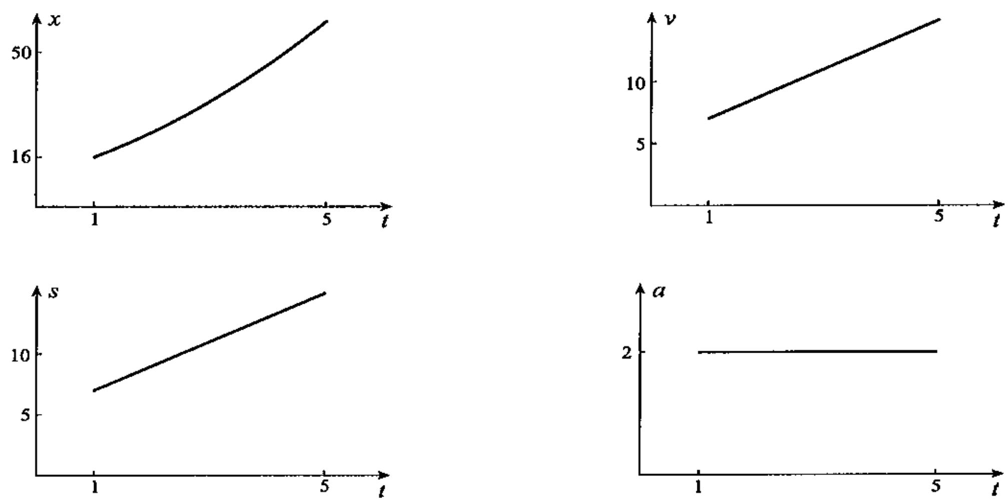
7. Graphs of $x(t)$, $v(t) = 2$, $|v(t)| = 2$, and $a(t) = 0$ are shown below.



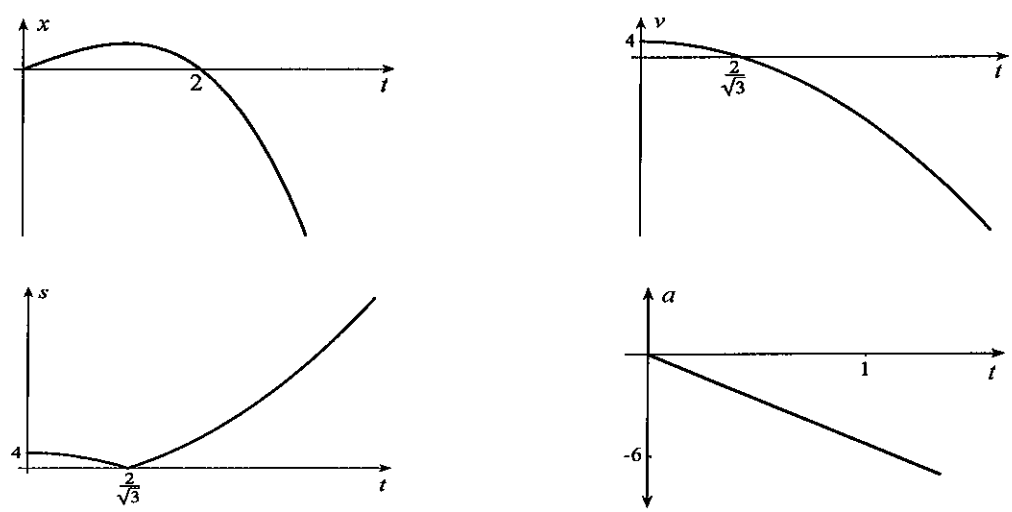
8. Graphs of $x(t)$, $v(t) = 2t - 7$, $|v(t)| = |2t - 7|$, and $a(t) = 2$ are shown below.



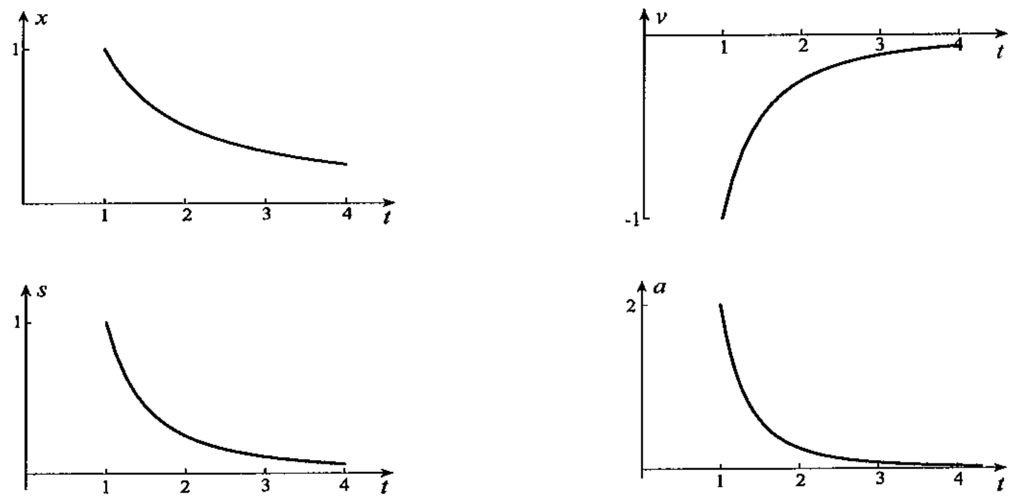
9. Graphs of $x(t)$, $v(t) = 2t + 5$, $|v(t)| = 2t + 5$, and $a(t) = 2$ are shown below.



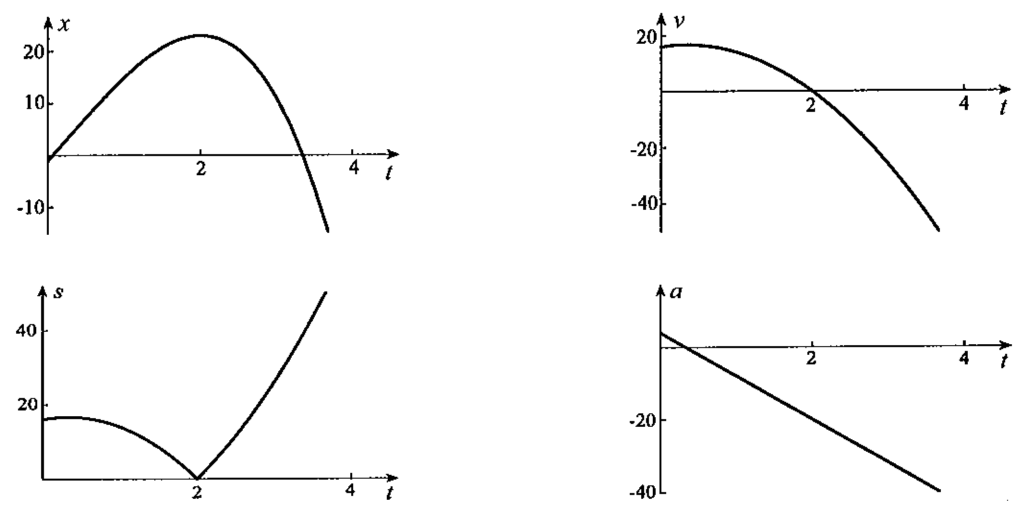
10. Graphs of $x(t)$, $v(t) = 4 - 3t^2$, $|v(t)| = |4 - 3t^2|$, and $a(t) = -6t$ are shown below.



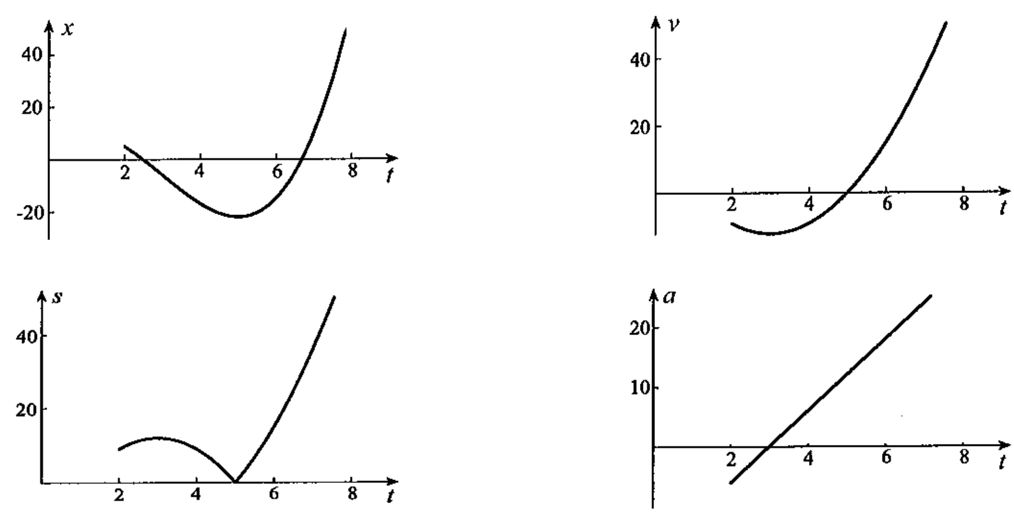
11. Graphs of $x(t)$, $v(t) = -1/t^2$, $|v(t)| = 1/t^2$, and $a(t) = 2/t^3$ are shown below.



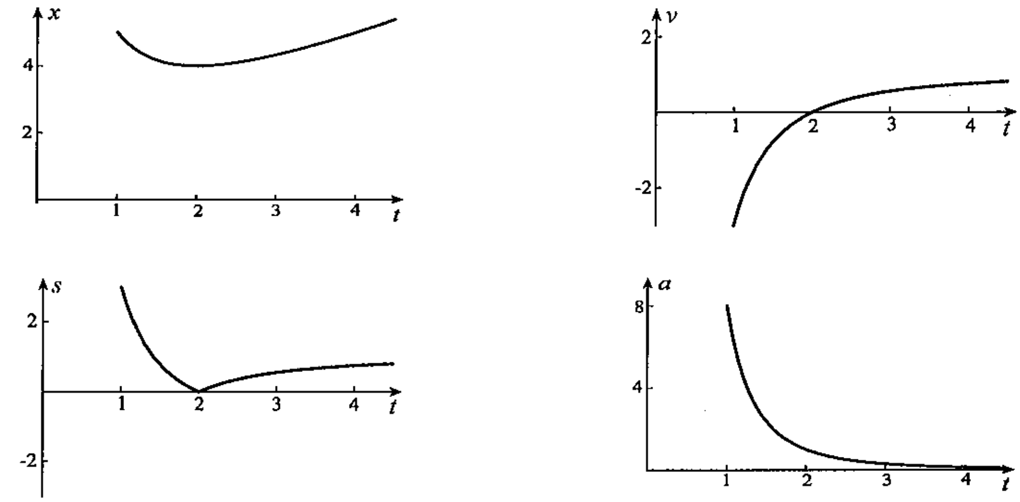
12. Plots of $x(t)$, $v(t) = -6t^2 + 4t + 16$, $|v(t)| = |-6t^2 + 4t + 16|$, and $a(t) = -12t + 4$ are shown below.



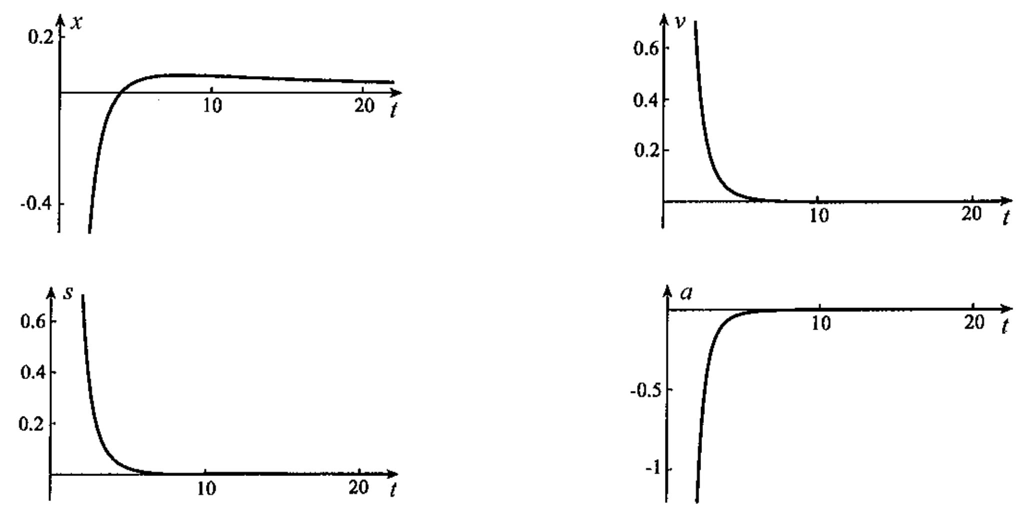
13. Plots of $x(t)$, $v(t) = 3t^2 - 18t + 15$, $|v(t)| = |3t^2 - 18t + 15|$, and $a(t) = 6t - 18$ are shown below.



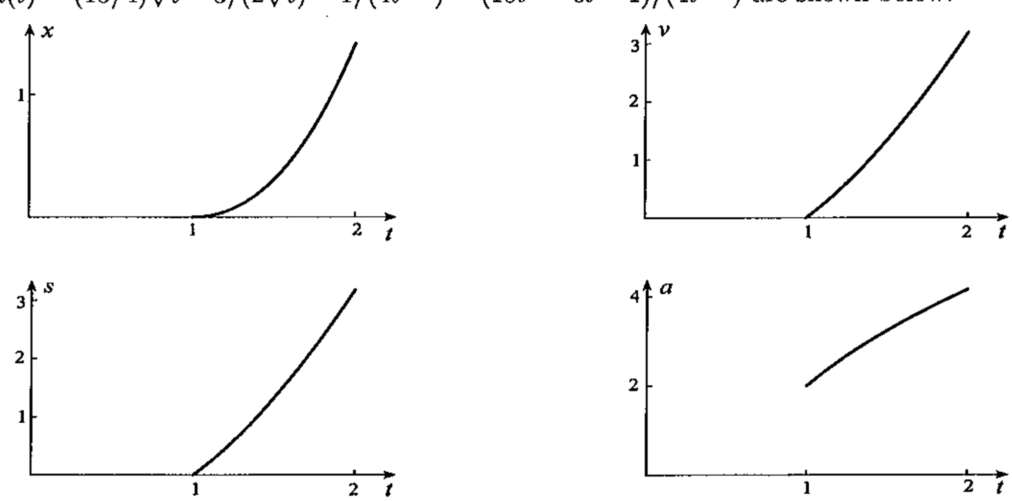
14. Plots of $x(t)$, $v(t) = 1 - 4/t^2$, $|v(t)| = |1 - 4/t^2|$, and $a(t) = 8/t^3$ are shown below.



15. Plots of $x(t)$, $v(t) = -1/t^2 + 8/t^3$, $|v(t)| = |-1/t^2 + 8/t^3|$, and $a(t) = 2/t^3 - 24/t^4$ are shown below.



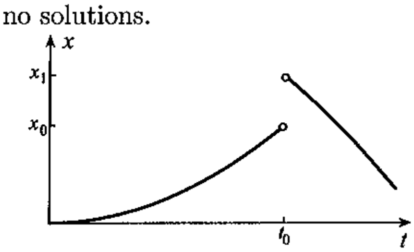
16. Plots of $x(t)$, $v(t) = (5/2)t^{3/2} - 3\sqrt{t} + (1/2)/\sqrt{t} = (5t^2 - 6t + 1)/(2\sqrt{t})$, $|v(t)| = |(5t^2 - 6t + 1)/(2\sqrt{t})|$, and $a(t) = (15/4)\sqrt{t} - 3/(2\sqrt{t}) - 1/(4t^{3/2}) = (15t^2 - 6t - 1)/(4t^{3/2})$ are shown below.



17. We calculate that $v(t) = 3t^2 - 18t + 24 = 3(t-2)(t-4)$ and $a(t) = 6t - 18 = 6(t-3)$.
(a) $x(3) = 19$ m, $v(3) = -3$ m/s, $|v(3)| = 3$ m/s, $a(3) = 0$ m/s^2
(b) The object is at rest when $v(t) = 0$ and this occurs at $t = 2$ s and $t = 4$ s.
(c) Acceleration vanishes at $t = 3$ s.
(d) Since $v(t) > 0$ for $0 \le t < 2$ and $t > 4$, the object is moving to the right for these times. It moves to the left for $2 < t < 4$.
(e) The velocity is 1 m/s if $1 = 3t^2 - 18t + 24 \implies 3t^2 - 18t + 23 = 0$. There are two solutions of this equation $t = (9 \pm 2\sqrt{3})/3$.
(f) The speed is 1 m/s if $|3t^2 - 18t + 24| = 1$. This implies that $3t^2 - 18t + 24 = 1$ or $3t^2 - 18t + 24 = -1$. The first gives the times in part (e). For the second possibility, we solve $3t^2 - 18t + 25 = 0$ for the times $t = (9 \pm \sqrt{6})/3$.
(g) The velocity is 20 m/s if $20 = 3t^2 - 18t + 24 \implies 3t^2 - 18t + 4 = 0$. There are two solutions $t = (9 \pm \sqrt{69})/3$.
(h) The speed is 20 m/s if $|3t^2 - 18t + 24| = 20$. This implies that $3t^2 - 18t + 24 = 20$ or $3t^2 - 18t + 24 = -20$. The first gives the times in part (g). The second quadratic has no solutions.

18. We calulate that $v(t) = 3t^2 - 18t + 15 = 3(t-1)(t-5)$ and $a(t) = 6t - 18 = 6(t-3)$.
(a) $x(3) = -11$ m, $v(3) = -12$ m/s, $|v(3)| = 12$ m/s, $a(3) = 0$ m/s^2
(b) The object is at rest when $v(t) = 0$ and this occurs at $t = 1$ s and $t = 5$ s.
(c) Acceleration vanishes at $t = 3$ s.
(d) Since $v(t) > 0$ for $0 \le t < 1$ and $t > 5$, the object is moving to the right for these times. It moves to the left for $1 < t < 5$.
(e) The velocity is 1 m/s if $1 = 3t^2 - 18t + 15 \implies 3t^2 - 18t + 14 = 0$. There are two solutions $t = (9 \pm \sqrt{39})/3$.
(f) The speed is 1 m/s if $|3t^2 - 18t + 15| = 1$. This implies that $3t^2 - 18t + 15 = 1$ or $3t^2 - 18t + 15 = -1$. The first gives the times in part (e). For the second possibility, we solve $3t^2 - 18t + 16 = 0$ for $t = (9 \pm \sqrt{33})/3$.
(g) The velocity is 20 m/s if $20 = 3t^2 - 18t + 15 \implies 3t^2 - 18t - 5 = 0$. Of the two solutions $t = (9 \pm 4\sqrt{6})/3$, only $t = (9 + 4\sqrt{6})/3$ is positive.
(h) The speed is 20 m/s if $|3t^2 - 18t + 15| = 20$. This implies that $3t^2 - 18t + 15 = 20$ or $3t^2 - 18t + 15 = -20$. The first gives the time in part (g). The second quadratic has no solutions.

19. No. If it were to have a discontinuity at $t = t_0$ as in the figure to the right, the particle would disappear at position x_0 and reappear instantaneously at position x_1. At a removable discontinuity, it reappears at the same position.



20. (a) Its average velocity is

$$\frac{x_2 - x_1}{t_2 - t_1} = \frac{(at_2^2/2 + bt_2 + c) - (at_1^2/2 + bt_1 + c)}{t_2 - t_1} = \frac{(a/2)(t_2^2 - t_1^2) + b(t_2 - t_1)}{t_2 - t_1} = \frac{a}{2}(t_1 + t_2) + b.$$

(b) The instantaneous velocity is equal to the average velocity when

$$at + b = \frac{a}{2}(t_1 + t_2) + b \quad \implies \quad t = \frac{t_1 + t_2}{2}.$$

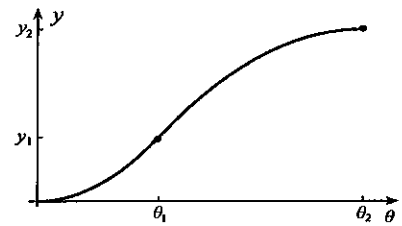
The position of the object at this time is $x\big((t_1 + t_2)/2\big) = \dfrac{a}{2}\left(\dfrac{t_1 + t_2}{2}\right)^2 + b\left(\dfrac{t_1 + t_2}{2}\right) + c$.
If we subtract this from $(x_1 + x_2)/2$, we obtain

$$\begin{aligned}\frac{x_1 + x_2}{2} - x\big((t_1 + t_2)/2\big) &= \frac{1}{2}\left(\frac{1}{2}at_1^2 + bt_1 + c + \frac{1}{2}at_2^2 + bt_2 + c\right) - \frac{a}{2}\left(\frac{t_1 + t_2}{2}\right)^2 - b\left(\frac{t_1 + t_2}{2}\right) - c \\ &= \frac{a}{8}(t_1 - t_2)^2.\end{aligned}$$

Since this quantity is positive, the object is closer to x_1.

21. Equations of the parabolas must be

$$y(\theta) = \begin{cases} a\theta^2, & 0 \le \theta \le \theta_1 \\ A(\theta - \theta_2)^2 + y_2, & \theta_1 \le \theta \le \theta_2 \end{cases}.$$



For (θ_1, y_1) to be a point on both parabolas,

$y_1 = a\theta_1^2, \qquad y_1 = A(\theta_1 - \theta_2)^2 + y_2.$

For continuity of the slope at (θ_1, y_1), the left-hand derivative of $a\theta^2$ and the right-hand derivative of $A(\theta - \theta_2)^2 + y_2$ must be equal at (θ_1, y_1),

$$2a\theta_1 = 2A(\theta_1 - \theta_2).$$

If we solve the first two equations for a and A, and substitute these into the last equation,

$$2\left(\frac{y_1}{\theta_1^2}\right)\theta_1 = 2\left[\frac{y_1 - y_2}{(\theta_1 - \theta_2)^2}\right](\theta_1 - \theta_2) \quad \Longrightarrow \quad y_1(\theta_1 - \theta_2) = \theta_1(y_1 - y_2) \quad \Longrightarrow \quad \frac{\theta_1}{\theta_2} = \frac{y_1}{y_2}.$$

The equation of the line through $(0,0)$ and (θ_2, y_2) is $y = y_2\theta/\theta_2$. Since (θ_1, y_1) satisfies this equation by virtue of the fact that $\theta_1/\theta_2 = y_1/y_2$, (θ_1, y_1) is on the line. Since

$$a = y_1\left(\frac{y_2}{y_1\theta_2}\right)^2 = \frac{y_2^2}{y_1\theta_2^2}, \qquad A = \frac{y_1 - y_2}{\left(\dfrac{y_1\theta_2}{y_2} - \theta_2\right)^2} = \frac{y_2^2}{(y_1 - y_2)\theta_2^2},$$

the equation of the curve is

$$y(\theta) = \begin{cases} \dfrac{y_2^2\theta^2}{y_1\theta_2^2}, & 0 \le \theta \le y_1\theta_2/y_2 \\ \dfrac{y_2^2(\theta - \theta_2)^2}{(y_1 - y_2)\theta_2^2} + y_2, & y_1\theta_2/y_2 \le \theta \le \theta_2 \end{cases}.$$

22. At point A, we must have

$$y_1 = a\theta_1^2, \qquad y_1 = m\theta_1 + b, \qquad m = 2a\theta_1.$$

From these,

$$a = \frac{y_1}{\theta_1^2}, \qquad m = 2\left(\frac{y_1}{\theta_1^2}\right)\theta_1 = \frac{2y_1}{\theta_1}, \qquad b = y_1 - 2\theta_1\left(\frac{y_1}{\theta_1^2}\right)\theta_1 = -y_1.$$

At point B, we must have

$$y_2 = A(\theta_2 - \theta_3)^2 + y_3, \qquad y_2 = m\theta_2 + b, \qquad m = 2A(\theta_2 - \theta_3).$$

From these,

$$A = \frac{y_2 - y_3}{(\theta_2 - \theta_3)^2}, \qquad m = 2\left[\frac{y_2 - y_3}{(\theta_2 - \theta_3)^2}\right](\theta_2 - \theta_3) = \frac{2(y_2 - y_3)}{\theta_2 - \theta_3},$$

and

$$b = y_3 + \frac{(y_2 - y_3)}{(\theta_2 - \theta_3)^2}(\theta_2 - \theta_3)^2 - \frac{2(y_2 - y_3)\theta_2}{\theta_2 - \theta_3} = \frac{y_2(\theta_2 - \theta_3) - 2(y_2 - y_3)\theta_2}{\theta_2 - \theta_3} = \frac{2y_3\theta_2 - y_2\theta_2 - y_2\theta_3}{\theta_2 - \theta_3}.$$

If we equate the values of b,

$$-y_1 = \frac{2y_3\theta_2 - y_2\theta_2 - y_2\theta_3}{\theta_2 - \theta_3} \quad \Longrightarrow \quad \theta_2 = \frac{(y_1 + y_2)\theta_3}{y_1 - y_2 + 2y_3}.$$

When we equate values of m, $\quad \dfrac{2y_1}{\theta_1} = \dfrac{2(y_2 - y_3)}{\theta_2 - \theta_3}, \quad$ from which

$$\theta_1 = \frac{y_1(\theta_2-\theta_3)}{y_2-y_3} = \frac{y_1}{y_2-y_3}\left[\frac{(y_1+y_2)\theta_3}{y_1-y_2+2y_3} - \theta_3\right] = \frac{y_1}{y_2-y_3}\left[\frac{(y_1+y_2)\theta_3 - \theta_3(y_1-y_2+2y_3)}{y_1-y_2+2y_3}\right]$$
$$= \frac{y_1}{y_2-y_3}\left[\frac{2\theta_3(y_2-y_3)}{y_1-y_2+2y_3}\right] = \frac{2y_1\theta_3}{y_1-y_2+2y_3}.$$

These now give

$$a = \frac{y_1}{\theta_1^2} = \frac{y_1(y_1-y_2+2y_3)^2}{4y_1^2\theta_3^2} = \frac{(y_1-y_2+2y_3)^2}{4y_1\theta_3^2},$$
$$m = \frac{2y_1}{\theta_1} = \frac{2y_1(y_1-y_2+2y_3)}{2y_1\theta_3} = \frac{y_1-y_2+2y_3}{\theta_3},$$
$$b = -y_1,$$
$$A = \frac{y_2-y_3}{(\theta_2-\theta_3)^2} = \frac{y_2-y_3}{\left[\dfrac{(y_1+y_2)\theta_3}{y_1-y_2+2y_3} - \theta_3\right]^2} = \frac{(y_2-y_3)(y_1-y_2+2y_3)^2}{[(y_1+y_2)\theta_3 - \theta_3(y_1-y_2+2y_3)]^2}$$
$$= \frac{(y_2-y_3)(y_1-y_2+2y_3)^2}{4\theta_3^2(y_2-y_3)^2} = \frac{(y_1-y_2+2y_3)^2}{4\theta_3^2(y_2-y_3)}.$$

EXERCISES 3.7

1. $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \left(2t - \dfrac{1}{t^2}\right)(2x)$

2. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \left[\dfrac{(u+1)(1)-u(1)}{(u+1)^2}\right]\left(\dfrac{1}{2\sqrt{x}}\right) = \dfrac{1}{2\sqrt{x}(u+1)^2}$

3. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (3u^2+2u+1)\left[\dfrac{-1/(2\sqrt{x})}{(\sqrt{x}-4)^2}\right] = -\dfrac{3u^2+2u+1}{2\sqrt{x}(\sqrt{x}-4)^2}$

4. $\dfrac{dy}{dx} = \dfrac{dy}{ds}\dfrac{ds}{dx} = \left[\dfrac{(s^2-2)(1)-s(2s)}{(s^2-2)^2}\right](2x-2) = \dfrac{2(1-x)(s^2+2)}{(s^2-2)^2}$

5. $\dfrac{dy}{dx} = \dfrac{dy}{dv}\dfrac{dv}{dx} = \left[\dfrac{5v^{3/2}}{2} + \dfrac{3v^{1/2}}{2} + 2v + 1\right]\left[\dfrac{(x^2-1)(1)-x(2x)}{(x^2-1)^2}\right] = \dfrac{-(5v^{3/2}+3\sqrt{v}+4v+2)(x^2+1)}{2(x^2-1)^2}$

6. $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \left[\dfrac{(t-4)(1)-(t+3)(1)}{(t-4)^2}\right]\left[\dfrac{(x+1)(1)-(x-2)(1)}{(x+1)^2}\right] = \dfrac{-21}{(t-4)^2(x+1)^2}$

7. $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \left[\dfrac{(t-4)(2t)-(t^2+3)(1)}{(t-4)^2}\right](9x^2+28x+8) = \dfrac{(t^2-8t-3)(9x^2+28x+8)}{(t-4)^2}$

8. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \left(2u + \dfrac{5}{2}u^{3/2}\right)\left[\dfrac{(x-x^2)(1)-(x+1)(1-2x)}{(x-x^2)^2}\right] = \dfrac{u(4+5\sqrt{u})(x^2+2x-1)}{2(x-x^2)^2}$

9. $f'(x) = (x^3+3)^4 + x(4)(x^3+3)^3(3x^2) = (13x^3+3)(x^3+3)^3$

10. $f'(x) = (1)\sqrt{x+1} + x\left(\dfrac{1}{2\sqrt{x+1}}\right) = \dfrac{3x+2}{2\sqrt{x+1}}$

11. $f'(x) = 2x(2x+1)^2 + x^2(2)(2x+1)(2) = 2x(4x+1)(2x+1)$

12. $f'(x) = \dfrac{\sqrt{2x+1}(1) - x\left(\dfrac{2}{2\sqrt{2x+1}}\right)}{2x+1} = \dfrac{x+1}{(2x+1)^{3/2}}$

13. $f'(x) = 2(x+2)(x^2+3) + (x+2)^2(2x) = 2(x+2)(2x^2+2x+3)$

14. $f'(x) = \dfrac{(3x+5)(2)(2x-1)(2) - (2x-1)^2(3)}{(3x+5)^2} = \dfrac{(2x-1)(6x+23)}{(3x+5)^2}$

15. $f'(x) = \dfrac{(2x-1)^2(3) - (3x+5)(2)(2x-1)(2)}{(2x-1)^4} = -\dfrac{6x+23}{(2x-1)^3}$

16. $f'(x) = 3x^2(2-5x^2)^{1/3} + x^3(1/3)(2-5x^2)^{-2/3}(-10x)$
$= 3x^2(2-5x^2)^{1/3} - \dfrac{10x^4}{3(2-5x^2)^{2/3}} = \dfrac{x^2(18-55x^2)}{3(2-5x^2)^{2/3}}$

17. $f'(x) = \dfrac{(2-5x^2)^{1/3}(3x^2) - x^3(1/3)(2-5x^2)^{-2/3}(-10x)}{(2-5x^2)^{2/3}} = \dfrac{x^2(18-35x^2)}{3(2-5x^2)^{4/3}}$

18. $f'(x) = (x+1)^2(3)(3x+1)^2(3) + 2(x+1)(3x+1)^3 = (3x+1)^2(x+1)[9(x+1)+2(3x+1)]$
$= (3x+1)^2(x+1)(15x+11)$

19. $f'(x) = \dfrac{(1-\sqrt{x})(1/3)x^{-2/3} - x^{1/3}(-1/2)x^{-1/2}}{(1-\sqrt{x})^2} = \dfrac{2(1-\sqrt{x})+3\sqrt{x}}{6x^{2/3}(1-\sqrt{x})^2} = \dfrac{2+\sqrt{x}}{6x^{2/3}(1-\sqrt{x})^2}$

20. $f'(x) = \dfrac{x^2(1/2)(2-3x)^{-1/2}(-3) - \sqrt{2-3x}(2x)}{x^4} = \dfrac{\dfrac{-3x^2}{2\sqrt{2-3x}} - 2x\sqrt{2-3x}}{x^4}$
$= \dfrac{-3x-4(2-3x)}{2x^3\sqrt{2-3x}} = \dfrac{9x-8}{2x^3\sqrt{2-3x}}$

21. $f'(x) = 4\left(\dfrac{x^3-1}{2x^3+1}\right)^3\left[\dfrac{(2x^3+1)(3x^2)-(x^3-1)(6x^2)}{(2x^3+1)^2}\right] = \dfrac{36x^2(x^3-1)^3}{(2x^3+1)^5}$

22. $f'(x) = \dfrac{1}{4}\left(\dfrac{2-x}{2+x}\right)^{-3/4}\left[\dfrac{(2+x)(-1)-(2-x)(1)}{(2+x)^2}\right] = \dfrac{1}{4}\left(\dfrac{2+x}{2-x}\right)^{3/4}\left[\dfrac{-4}{(2+x)^2}\right] = \dfrac{-1}{(2-x)^{3/4}(2+x)^{5/4}}$

23. $f'(x) = 3(x^3-2x^2)^2(3x^2-4x)(x^4-2x)^5 + (x^3-2x^2)^3(5)(x^4-2x)^4(4x^3-2)$
$= (x^3-2x^2)^2(x^4-2x)^4[3(3x^2-4x)(x^4-2x)+5(x^3-2x^2)(4x^3-2)]$
$= x^2(x^3-2x^2)^2(x^4-2x)^4(29x^4-52x^3-28x+44)$

24. $f'(x) = 4(x+5)^3\sqrt{1+x^3} + (x+5)^4(1/2)(1+x^3)^{-1/2}(3x^2)$
$= (x+5)^3\left[4\sqrt{1+x^3} + \dfrac{3x^2(x+5)}{2\sqrt{1+x^3}}\right] = (x+5)^3\left[\dfrac{8(1+x^3)+3x^2(x+5)}{2\sqrt{1+x^3}}\right]$
$= \dfrac{(x+5)^3(11x^3+15x^2+8)}{2\sqrt{1+x^3}}$

25. $f'(x) = \dfrac{(3+x)^{1/3}(\sqrt{1-x^2} - x^2/\sqrt{1-x^2}) - x\sqrt{1-x^2}(1/3)(3+x)^{-2/3}}{(3+x)^{2/3}}$
$= \dfrac{(3+x)^{1/3}\left(\dfrac{1-2x^2}{\sqrt{1-x^2}}\right) - \dfrac{x\sqrt{1-x^2}}{3(3+x)^{2/3}}}{(3+x)^{2/3}} = \dfrac{3(3+x)(1-2x^2)-x(1-x^2)}{3(3+x)^{4/3}\sqrt{1-x^2}}$
$= \dfrac{9+2x-18x^2-5x^3}{3(3+x)^{4/3}\sqrt{1-x^2}}$

26. With the product rule of Exercise 20 in Section 3.4,
$f'(x) = (1)(x+5)^4\sqrt{1+x^3} + x(4)(x+5)^3\sqrt{1+x^3} + x(x+5)^4(1/2)(1+x^3)^{-1/2}(3x^2)$
$= (x+5)^3\left[(x+5)\sqrt{1+x^3} + 4x\sqrt{1+x^3} + \dfrac{3x^3(x+5)}{2\sqrt{1+x^3}}\right]$
$= (x+5)^3\left[\dfrac{2(x+5)(1+x^3)+8x(1+x^3)+3x^3(x+5)}{2\sqrt{1+x^3}}\right] = \dfrac{(x+5)^3(13x^4+25x^3+10x+10)}{2\sqrt{1+x^3}}$

27. $f'(x) = \dfrac{(x-2)(x+5)^2[2x(x^3+3)^2+x^2(2)(x^3+3)(3x^2)] - x^2(x^3+3)^2[(x+5)^2+(x-2)(2)(x+5)]}{(x-2)^2(x+5)^4}$
$= \dfrac{x(x^3+3)[(x-2)(x+5)(8x^3+6)-x(x^3+3)(3x+1)]}{(x-2)^2(x+5)^3}$
$= \dfrac{x(x^3+3)(5x^5+23x^4-80x^3-3x^2+15x-60)}{(x-2)^2(x+5)^3}$

28. $f'(x) = (1)\sqrt{1+x\sqrt{1+x}} + x(1/2)\left(1+x\sqrt{1+x}\right)^{-1/2}[\sqrt{1+x} + x(1/2)(1+x)^{-1/2}]$

$$= \sqrt{1+x\sqrt{1+x}} + \frac{x}{2\sqrt{1+x\sqrt{1+x}}}\left[\frac{2(1+x)+x}{2\sqrt{1+x}}\right]$$

$$= \frac{4\sqrt{1+x}\left(1+x\sqrt{1+x}\right)+3x^2+2x}{4\sqrt{1+x}\sqrt{1+x\sqrt{1+x}}} = \frac{7x^2+6x+4\sqrt{1+x}}{4\sqrt{1+x}\sqrt{1+x\sqrt{1+x}}}$$

29. $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \left(2t - \dfrac{3}{t^4}\right)\left(\dfrac{-x}{\sqrt{4-x^2}}\right)$

30. $\dfrac{dy}{dx} = \dfrac{dy}{ds}\dfrac{ds}{dx} = \left[\dfrac{1}{3}(2s-s^2)^{-2/3}(2-2s)\right]\left[\dfrac{-2x}{(x^2+5)^2}\right] = \dfrac{4x(s-1)}{3(2s-s^2)^{2/3}(x^2+5)^2}$

31. $\dfrac{dy}{dx} = \dfrac{dy}{dv}\dfrac{dv}{dx} = \left[\dfrac{(v^3-1)(2v)-v^2(3v^2)}{(v^3-1)^2}\right]\left(\sqrt{x^2-1} + \dfrac{x^2}{\sqrt{x^2-1}}\right) = \dfrac{(v^4+2v)(1-2x^2)}{(v^3-1)^2\sqrt{x^2-1}}$

32. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \left[\dfrac{(u+5)(1)-u(1)}{(u+5)^2}\right]\left[\dfrac{x(1/2)(x-1)^{-1/2}-\sqrt{x-1}(1)}{x^2}\right]$

$$= \frac{5}{(u+5)^2}\left[\frac{x-2(x-1)}{2x^2\sqrt{x-1}}\right] = \frac{5(2-x)}{2x^2\sqrt{x-1}(u+5)^2}$$

33. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = [4u^3(u^3-2u)^2 + u^4(2)(u^3-2u)(3u^2-2)]\left(\dfrac{1-4x}{2\sqrt{x-2x^2}}\right)$

$$= 2u^3(u^3-2u)[2(u^3-2u)+u(3u^2-2)]\left(\frac{1-4x}{2\sqrt{x-2x^2}}\right) = \frac{u^5(u^2-2)(5u^2-6)(1-4x)}{\sqrt{x-2x^2}}$$

34. $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \left[1 + \dfrac{1}{2\sqrt{t+\sqrt{t}}}\left(1+\dfrac{1}{2\sqrt{t}}\right)\right]\left[\dfrac{(x^2-1)(2x)-(x^2+1)(2x)}{(x^2-1)^2}\right]$

$$= \left[1 + \frac{2\sqrt{t}+1}{4\sqrt{t}\sqrt{t+\sqrt{t}}}\right]\left[\frac{-4x}{(x^2-1)^2}\right]$$

35. $\dfrac{dy}{dx} = \dfrac{dy}{dv}\dfrac{dv}{dx} = \left\{3\left(\dfrac{v^2+1}{1-v^3}\right)^2\left[\dfrac{(1-v^3)(2v)-(v^2+1)(-3v^2)}{(1-v^3)^2}\right]\right\}\left[\dfrac{-(3x^2+6x)}{(x^3+3x^2+2)^2}\right]$

$$= -\frac{9xv(v^2+1)^2(v^3+3v+2)(x+2)}{(1-v^3)^4(x^3+3x^2+2)^2}$$

36. $\dfrac{dy}{dx} = \dfrac{dy}{dk}\dfrac{dk}{dx} = \left[\dfrac{(1+k+k^2)(1/2)k^{-1/2} - \sqrt{k}(1+2k)}{(1+k+k^2)^2}\right][(x^2+5)^5 + 5x(x^2+5)^4(2x)]$

$$= \left[\frac{1+k+k^2-2k(1+2k)}{2\sqrt{k}(1+k+k^2)^2}\right][(x^2+5)^4(x^2+5+10x^2)]$$

$$= \frac{(1-k-3k^2)(x^2+5)^4(11x^2+5)}{2\sqrt{k}(1+k+k^2)^2}$$

37. Since y is a function of s, namely, $f[g(s)]$, and s is a function of x, we can write that $\dfrac{dy}{dx} = \dfrac{dy}{ds}\dfrac{ds}{dx}$. But $\dfrac{dy}{ds} = \dfrac{dy}{du}\dfrac{du}{ds}$. Consequently, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{ds}\dfrac{ds}{dx}$.

38. Since F is a function of r, and r is a function of t, the chain rule gives $\dfrac{dF}{dt} = \dfrac{dF}{dr}\dfrac{dr}{dt} = \dfrac{-2Qq}{4\pi\epsilon_0 r^3}\dfrac{dr}{dt}$.
Since $dr/dt = 2$ m/s, we obtain when $r = 2$, $\dfrac{dF}{dt} = -\dfrac{2(3\times10^{-6})(5\times10^{-6})}{4\pi(8.85\times10^{-12})(2)^3}(2) = -0.067$ N/s.

39. Since F is a function of r, and r is a function of t, the chain rule gives $\dfrac{dF}{dt} = \dfrac{dF}{dr}\dfrac{dr}{dt} = \dfrac{-2GMm}{r^3}\dfrac{dr}{dt}$.
Since $dr/dt = -250/9$ m/s when $r = 5$,

$$\frac{dF}{dt} = -\left[\frac{2(6.67\times10^{-11})(5)(4/3)\pi(6.37\times10^6)^3(5.52\times10^3)}{(6.375\times10^6)^3}\right]\left(-\frac{250}{9}\right) = 4.27\times10^{-4} \text{ N/s.}$$

40. If we use the chain rule on $y = f(x)$, $x = g(y)$, then $\dfrac{dy}{dy} = 1 = \dfrac{dy}{dx}\dfrac{dx}{dy} \implies \dfrac{dy}{dx} = \dfrac{1}{\dfrac{dx}{dy}}$.

41. If $f(x)$ is an odd function, then $f(-x) = -f(x)$. To differentiate this equation with respect to x, we set $u = -x$ on the left. Differentiation of $f(u) = -f(x)$ with respect to x then gives $f'(u)\dfrac{du}{dx} = -f'(x) \implies f'(-x)(-1) = -f'(x) \implies f'(-x) = f'(x)$. This shows that $f'(x)$ is an even function. A similar proof shows that $f'(x)$ is odd when $f(x)$ is even.

42. The proof relies on the fact that $\Delta u \neq 0$. If u is a constant function, then $\Delta u \equiv 0$. In addition, even when u is not a constant function, we might have $\Delta u = 0$ for arbitrarily small Δx. A complete proof of the chain rule must account for both these possibilities.

43. Since $\dfrac{dy}{dx} = \dfrac{dy}{dv}\dfrac{dv}{dx} = (2v+1)\left[\dfrac{(x+1)(1) - x(1)}{(x+1)^2}\right] = \dfrac{2v+1}{(x+1)^2}$, it follows that

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{(x+1)^2(2dv/dx) - (2v+1)(2)(x+1)}{(x+1)^4}\\ &= \frac{2(x+1)^2\left[\dfrac{1}{(x+1)^2}\right] - 2(2v+1)(x+1)}{(x+1)^4} = -\frac{2(2vx+2v+x)}{(x+1)^4}.\end{aligned}$$

44. Since $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \left[3(u+1)^2 + \dfrac{1}{u^2}\right]\left[1 + \dfrac{1}{2\sqrt{x+1}}\right]$, it follows that

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left[3(u+1)^2 + \frac{1}{u^2}\right]\left[1 + \frac{1}{2\sqrt{x+1}}\right] + \left[3(u+1)^2 + \frac{1}{u^2}\right]\frac{d}{dx}\left[1 + \frac{1}{2\sqrt{x+1}}\right]\\ &= \left[6(u+1) - \frac{2}{u^3}\right]\frac{du}{dx}\left[1 + \frac{1}{2\sqrt{x+1}}\right] + \left[3(u+1)^2 + \frac{1}{u^2}\right]\left[\frac{-1}{4(x+1)^{3/2}}\right]\\ &= \left[\frac{6u^3(u+1) - 2}{u^3}\right]\left[1 + \frac{1}{2\sqrt{x+1}}\right]^2 - \frac{3u^2(u+1)^2 + 1}{4u^2(x+1)^{3/2}}.\end{aligned}$$

45. Since $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \left(\dfrac{1}{2\sqrt{t-1}}\right)[2(x+x^2)(1+2x)] = \dfrac{2x^3+3x^2+x}{\sqrt{t-1}}$, it follows that

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{\sqrt{t-1}(6x^2+6x+1) - (2x^3+3x^2+x)(1/2)(t-1)^{-1/2}(dt/dx)}{t-1}\\ &= \frac{\sqrt{t-1}(6x^2+6x+1) - (2x^3+3x^2+x)\left(\dfrac{1}{2\sqrt{t-1}}\right)(4x^3+6x^2+2x)}{t-1}\\ &= \frac{(t-1)(6x^2+6x+1) - (2x^3+3x^2+x)^2}{(t-1)^{3/2}}.\end{aligned}$$

46. Since
$$\begin{aligned}\frac{dy}{dx} = \frac{dy}{ds}\frac{ds}{dx} &= \left[\frac{(s+6)(1) - s(1)}{(s+6)^2}\right]\left[\frac{(1+\sqrt{x})(1/2)x^{-1/2} - \sqrt{x}(1/2)x^{-1/2}}{(1+\sqrt{x})^2}\right]\\ &= \left[\frac{6}{(s+6)^2}\right]\left[\frac{1}{2\sqrt{x}(1+\sqrt{x})^2}\right],\end{aligned}$$
it follows that
$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left[\frac{3}{(s+6)^2}\right]\left[\frac{1}{\sqrt{x}(1+\sqrt{x})^2}\right] + \left[\frac{3}{(s+6)^2}\right]\frac{d}{dx}\left[\frac{1}{\sqrt{x}(1+\sqrt{x})^2}\right]\\ &= \left[\frac{-6}{(s+6)^3}\frac{ds}{dx}\right]\left[\frac{1}{\sqrt{x}(1+\sqrt{x})^2}\right] + \left[\frac{3}{(s+6)^2}\right]\left[\frac{-1}{2x^{3/2}(1+\sqrt{x})^2} - \frac{2}{\sqrt{x}(1+\sqrt{x})^3 2\sqrt{x}}\right]\end{aligned}$$

$$= \left[\frac{-6}{(s+6)^3\sqrt{x}(1+\sqrt{x})^2}\right]\left[\frac{1}{2\sqrt{x}(1+\sqrt{x})^2}\right] + \left[\frac{3}{(s+6)^2}\right]\left[\frac{-1-\sqrt{x}-2\sqrt{x}}{2x^{3/2}(1+\sqrt{x})^3}\right]$$
$$= \frac{-6\sqrt{x}-3(s+6)(1+\sqrt{x})(1+3\sqrt{x})}{2x^{3/2}(s+6)^3(1+\sqrt{x})^4} = \frac{-3[2\sqrt{x}+(s+6)(1+4\sqrt{x}+3x)]}{2x^{3/2}(1+\sqrt{x})^4(s+6)^3}.$$

47. If we set $u = 2x+3$, then the chain rule gives

$$\frac{d}{dx}f(2x+3) = \frac{d}{du}f(u)\frac{du}{dx} = f'(u)(2) = 2f'(2x+3).$$

When $f(u) = u^3 - 2u$, then $f'(u) = 3u^2 - 2$, and

$$\frac{d}{dx}f(2x+3) = 2[3(2x+3)^2 - 2] = 2(12x^2 + 36x + 25).$$

48. If we set $u = 3 - 4x$, then the chain rule gives

$$\frac{d}{dx}[f(3-4x)]^2 = \frac{d}{du}[f(u)]^2\frac{du}{dx} = 2f(u)f'(u)(-4) = -8f(3-4x)f'(3-4x).$$

When $f(u) = u^3 - 2u$, then $f'(u) = 3u^2 - 2$, and

$$\frac{d}{dx}[f(3-4x)]^2 = -8[(3-4x)^3 - 2(3-4x)][3(3-4x)^2 - 2]$$
$$= 8(4x-3)(16x^2 - 24x + 7)(48x^2 - 72x + 25).$$

49. If we set $u = 1 - x^2$, then the chain rule gives

$$\frac{d}{dx}f(1-x^2) = \frac{d}{du}f(u)\frac{du}{dx} = f'(u)(-2x) = -2xf'(1-x^2).$$

When $f(u) = u^3 - 2u$, then $f'(u) = 3u^2 - 2$, and

$$\frac{d}{dx}f(1-x^2) = -2x[3(1-x^2)^2 - 2] = -2x(1 - 6x^2 + 3x^4).$$

50. If we set $u = x + 1/x$, then

$$\frac{d}{dx}f\left(x+\frac{1}{x}\right) = f'(u)\frac{du}{dx} = f'(u)\left(1-\frac{1}{x^2}\right) = \left(1-\frac{1}{x^2}\right)f'\left(x+\frac{1}{x}\right).$$

When $f(u) = u^3 - 2u$, then $f'(u) = 3u^2 - 2$, and

$$\frac{d}{dx}f\left(x+\frac{1}{x}\right) = \left(1-\frac{1}{x^2}\right)\left[3\left(x+\frac{1}{x}\right)^2 - 2\right] = \frac{(x^2-1)(3x^4+4x^2+3)}{x^4}.$$

51. If we set $u = f(x)$, then

$$\frac{d}{dx}f\big(f(x)\big) = \frac{d}{du}f(u)\frac{du}{dx} = f'(u)f'(x) = f'\big(f(x)\big)f'(x).$$

When $f(u) = u^3 - 2u$, then $f'(u) = 3u^2 - 2$, and

$$\frac{d}{dx}f\big(f(x)\big) = [3(x^3-2x)^2 - 2](3x^2 - 2) = (3x^2-2)(3x^6 - 12x^4 + 12x^2 - 2).$$

52. If we set $u = 1 - 3x$, then

$$\frac{d}{dx}\sqrt{3-4[f(1-3x)]^2} = \frac{d}{du}\sqrt{3-4[f(u)]^2}\frac{du}{dx} = \frac{1}{2\sqrt{3-4[f(u)]^2}}[-8f(u)f'(u)](-3)$$
$$= \frac{12f(u)f'(u)}{\sqrt{3-4[f(u)]^2}} = \frac{12f(1-3x)f'(1-3x)}{\sqrt{3-4[f(1-3x)]^2}}.$$

When $f(u) = u^3 - 2u$, then $f'(u) = 3u^2 - 2$, and

$$\begin{aligned}\frac{d}{dx}\sqrt{3 - 4[f(1-3x)]^2} &= \frac{12[(1-3x)^3 - 2(1-3x)][3(1-3x)^2 - 2]}{\sqrt{3 - 4[(1-3x)^3 - 2(1-3x)]^2}}\\ &= \frac{12(1-3x)(9x^2 - 6x - 1)(27x^2 - 18x + 1)}{\sqrt{3 - 4(1-3x)^2(9x^2 - 6x - 1)^2}}.\end{aligned}$$

53. If we set $u = -x$ and $v = x^2$, then

$$\begin{aligned}\frac{d}{dx}\left[\frac{f(-x)}{3 + 2f(x^2)}\right] = \frac{d}{dx}\left[\frac{f(u)}{3 + 2f(v)}\right] &= \frac{[3 + 2f(v)]f'(u)du/dx - f(u)[2f'(v)dv/dx]}{[3 + 2f(v)]^2}\\ &= \frac{[3 + 2f(x^2)]f'(-x)(-1) - f(-x)[2f'(x^2)(2x)]}{[3 + 2f(x^2)]^2}\\ &= -\frac{f'(-x)[3 + 2f(x^2)] + 4xf(-x)f'(x^2)}{[3 + 2f(x^2)]^2}\end{aligned}$$

When $f(u) = u^3 - 2u$, then $f'(u) = 3u^2 - 2$, and

$$\begin{aligned}\frac{d}{dx}\left[\frac{f(-x)}{3 + 2f(x^2)}\right] &= -\frac{(3x^2 - 2)[3 + 2(x^6 - 2x^2)] + 4x(-x^3 + 2x)(3x^4 - 2)}{[3 + 2(x^6 - 2x^2)]^2}\\ &= \frac{6x^8 - 20x^6 + 4x^4 - x^2 + 6}{(2x^6 - 4x^2 + 3)^2}\end{aligned}$$

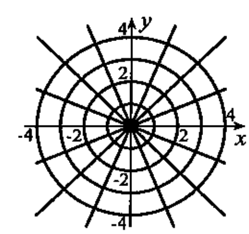
54. If we set $u = x - f(x)$, then

$$\frac{d}{dx}[f(x - f(x))] = f'(u)\frac{du}{dx} = f'(u)[1 - f'(x)] = f'(x - f(x))[1 - f'(x)].$$

When $f(u) = u^3 - 2u$, then $f'(u) = 3u^2 - 2$, and

$$\begin{aligned}\frac{d}{dx}[f(x - f(x))] &= \{3[x - f(x)]^2 - 2\}[1 - (3x^2 - 2)] = \{3[x - (x^3 - 2x)]^2 - 2\}(3 - 3x^2)\\ &= \{3[3x - x^3]^2 - 2\}(3 - 3x^2) = [27x^2 - 18x^4 + 3x^6 - 2](3 - 3x^2)\\ &= 3(1 - x^2)(3x^6 - 18x^4 + 27x^2 - 2).\end{aligned}$$

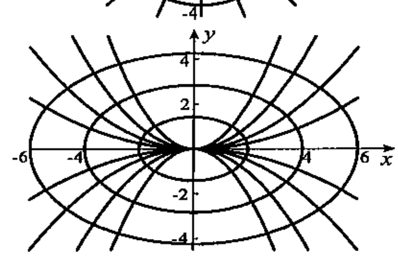
55. Suppose the curves intersect at a point (x, y). The straight line has slope m where $m = y/x$. When we solve $x^2 + y^2 = r^2$ for y, we obtain $y = \pm\sqrt{r^2 - x^2}$. When we differentiate $y = \pm\sqrt{r^2 - x^2}$, we obtain $dy/dx = \mp x/\sqrt{r^2 - x^2} = -x/y$. This is the negative reciprocal of the slope of the line at (x, y).



56. The slope of the tangent line at any point on any of the parabolas in the first family is

$$\frac{dy}{dx} = 2ax = 2\left(\frac{y}{x^2}\right)x = \frac{2y}{x}.$$

When we solve for y in the family of ellipses, $y = \pm(1/\sqrt{2})\sqrt{c^2 - x^2}$. The slope of the tangent line at any point on any ellipse is

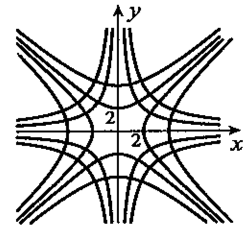


$$\frac{dy}{dx} = \pm\frac{1}{\sqrt{2}}\frac{-2x}{2\sqrt{c^2 - x^2}} = \mp\frac{x}{\sqrt{2}\sqrt{c^2 - x^2}} = \mp\frac{x}{\sqrt{2}(\pm\sqrt{2}y)} = -\frac{x}{2y}.$$

Since these slopes are negative reciprocals, the families are orthogonal.

57. When we solve for y in the first family of hyperbolas, $y = \pm\sqrt{x^2 - C_1}$. The slope of the tangent line at any point on any hyperbola is

$$\frac{dy}{dx} = \pm x/\sqrt{x^2 - C_1} = \frac{\pm x}{(\pm y)} = \frac{x}{y}.$$



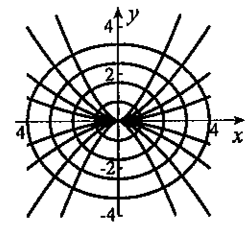
The slope of the tangent line at any point on $y = C_2/x$ is

$$\frac{dy}{dx} = -\frac{C_2}{x^2} = -\left(\frac{C_2}{x}\right)\left(\frac{1}{x}\right) = -\frac{y}{x}.$$

Slopes of the families are negative reciprocals, and the families are orthogonal trajectories.

58. When we solve for y in the family of ellipses, $y = \pm\frac{1}{\sqrt{3}}\sqrt{C^2 - 2x^2}$. The slope of the tangent line at any point on any ellipse is

$$\frac{dy}{dx} = \pm\frac{1}{\sqrt{3}}\frac{-4x}{2\sqrt{C^2 - 2x^2}} = \frac{\mp 2x}{\sqrt{3}\sqrt{C^2 - 2x^2}}$$
$$= \frac{\mp 2x}{\sqrt{3}(\pm\sqrt{3}y)} = -\frac{2x}{3y}.$$



If $a > 0$ in the second family, then $y = \sqrt{a}x^{3/2}$, and $\dfrac{dy}{dx} = \dfrac{3}{2}\sqrt{a}x^{1/2} = \dfrac{3}{2}\left(\dfrac{y}{x^{3/2}}\right)x^{1/2} = \dfrac{3y}{2x}$.

If $a < 0$, then $y = \sqrt{-a}(-x)^{3/2}$, and $\dfrac{dy}{dx} = \dfrac{3}{2}\sqrt{-a}(-x)^{1/2}(-1) = -\dfrac{3}{2}\left[\dfrac{y}{(-x)^{3/2}}\right](-x)^{1/2} = \dfrac{3y}{2x}$.

In either case, slopes of the families are negative reciprocals, and the families are orthogonal trajectories.

59. If we differentiate $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ with respect to x,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{du}\frac{du}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{du}\right)\frac{du}{dx} + \frac{dy}{du}\frac{d^2u}{dx^2}$$
$$= \left[\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}\right]\frac{du}{dx} + \frac{dy}{du}\frac{d^2u}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\frac{d^2u}{dx^2}.$$

60. Using the result of Exercise 59, $\dfrac{d^2y}{dx^2} = -\dfrac{12}{x^4}3(u+1)^2 + 6(u+1)\left(3 + \dfrac{4}{x^3}\right)^2$. When $x = 1$, we find that $u = 1$ also, and $\dfrac{d^2y}{dx^2} = -36(2)^2 + 6(2)(7)^2 = 444$.

61. If we set $u = x^3$, then $x = u^{1/3}$, and

$$\frac{dy}{du} = \frac{dy}{dx}\frac{dx}{du} = (9x^8 + 6x^5)(1/3)u^{-2/3} = (3x^8 + 2x^5)x^{-2} = 3x^6 + 2x^3.$$

62. If we set $v = x/(x+1)$, then $x = v/(1-v)$, and

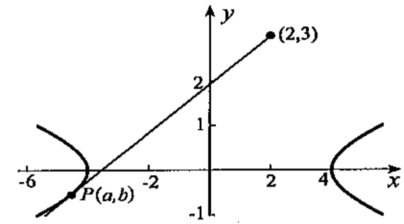
$$\frac{dy}{dv} = \frac{dy}{dx}\frac{dx}{dv} = \left[\frac{-x}{\sqrt{1-x^2}}\right]\left[\frac{(1-v)(1) - v(-1)}{(1-v)^2}\right] = \frac{-x}{\sqrt{1-x^2}(1-v)^2}$$
$$= \frac{-x}{\sqrt{1-x^2}\left(1 - \dfrac{x}{x+1}\right)^2} = -\frac{x(x+1)^2}{\sqrt{1-x^2}}.$$

63. If we differentiate the result in Exercise 59 with respect to x,

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2u}{dx^2}\right)\frac{dy}{du} + \frac{d^2u}{dx^2}\frac{d}{dx}\left(\frac{dy}{du}\right) + \frac{d}{dx}\left(\frac{d^2y}{du^2}\right)\left(\frac{du}{dx}\right)^2 + \frac{d^2y}{du^2}\frac{d}{dx}\left(\frac{du}{dx}\right)^2$$
$$= \frac{d^3u}{dx^3}\frac{dy}{du} + \frac{d^2u}{dx^2}\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx} + \frac{d}{du}\left(\frac{d^2y}{du^2}\right)\frac{du}{dx}\left(\frac{du}{dx}\right)^2 + \frac{d^2y}{du^2}(2)\left(\frac{du}{dx}\right)\frac{d^2u}{dx^2}$$
$$= \frac{d^3u}{dx^3}\frac{dy}{du} + \frac{d^2u}{dx^2}\frac{d^2y}{du^2}\frac{du}{dx} + \frac{d^3y}{du^3}\left(\frac{du}{dx}\right)^3 + 2\frac{d^2y}{du^2}\frac{du}{dx}\frac{d^2u}{dx^2} = \frac{d^3u}{dx^3}\frac{dy}{du} + 3\frac{d^2u}{dx^2}\frac{d^2y}{du^2}\frac{du}{dx} + \frac{d^3y}{du^3}\left(\frac{du}{dx}\right)^3.$$

64. The diagram makes it clear that there are two such points and they both have negative y-coordinates. We therefore take the lower half of the hyperbola in the form $y = f(x) = -\sqrt{x^2-16}/4$. The slope of the tangent line to the hyperbola at $P(a,b)$ is

$$f'(a) = \frac{-x}{4\sqrt{x^2-16}}_{|x=a} = \frac{-a}{4\sqrt{a^2-16}}.$$



Since the tangent line must pass through the point $(2,3)$ the slope of the tangent line is also given by $(b-3)/(a-2)$. Consequently,

$$\frac{b-3}{a-2} = \frac{-a}{4\sqrt{a^2-16}}.$$

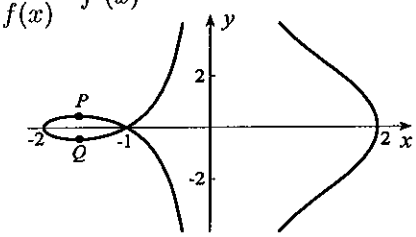
We combine this with $b = -\frac{1}{4}\sqrt{a^2-16}$, since $P(a,b)$ is on the hyperbola. Substitution from the second into the first gives

$$-\frac{1}{4}\sqrt{a^2-16} - 3 = \frac{-a(a-2)}{4\sqrt{a^2-16}}, \qquad \text{or,} \qquad (a^2-16) + 12\sqrt{a^2-16} = a(a-2).$$

This equation simplifies to $6\sqrt{a^2-16} = 8-a$, and squaring leads to the quadratic $35a^2+16a-640 = 0$. The two solutions are $a = (-8 \pm 24\sqrt{39})/35$. The y-coordinates of these points are $(-12 \pm \sqrt{39})/35$.

65. $\displaystyle \frac{d}{dx}|f(x)|^n = n|f(x)|^{n-1}\frac{d}{dx}|f(x)| = n|f(x)|^{n-1}\frac{|f(x)|}{f(x)}f'(x) = \frac{n|f(x)|^n}{f(x)}f'(x)$

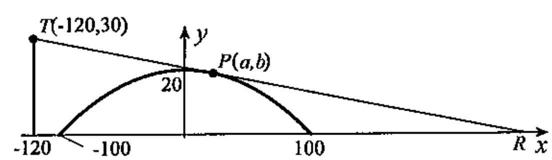
66. The conchoid is shown to the right. There are two points P and Q at which the tangent line is horizontal. To find P the x-coordinate of which is between -2 and -1, we first solve the equation $x^2y^2 = (x+1)^2(4-x^2)$ for $y = [(x+1)/x]\sqrt{4-x^2}$, and set the derivative equal to zero,



$$0 = \frac{d}{dx}\left[\left(1+\frac{1}{x}\right)\sqrt{4-x^2}\right] = \frac{-1}{x^2}\sqrt{4-x^2} + \left(1+\frac{1}{x}\right)\frac{-x}{\sqrt{4-x^2}} = \frac{-(4-x^2)-x^2(x+1)}{x^2\sqrt{4-x^2}} = \frac{-(x^3+4)}{x^2\sqrt{4-x^2}}.$$

The only solution of this equation is $x = -4^{1/3}$. The coordinates of P are therefore $(-4^{1/3}, (4^{1/3}-1)^{3/2})$. Point Q has coordinates $(-4^{1/3}, -(4^{1/3}-1)^{3/2})$.

67. The required position occurs at the point R where the tangent line from $T(-120,30)$ to the arc of the circle representing the hill intersects the x-axis. Let the point of tangency be $P(a,b)$. If $x^2+(y-k)^2 = r^2$ is the equation of the circle, then using the points $(100,0)$



and $(0,20)$, we obtain $100^2+k^2 = r^2$ and $(20-k)^2 = r^2$. These give $k = -240$ and $r^2 = 67\,600$, so that the equation of the circular arc is $x^2+(y+240)^2 = 67\,600$, or, $y = f(x) = -240+\sqrt{67\,600-x^2}$, $-100 \le x \le 100$. Since $f'(x) = (1/2)(67\,600-x^2)^{-1/2}(-2x) = -x/\sqrt{67\,600-x^2}$, the slope of the tangent line at $P(a,b)$ is $f'(a) = -a/\sqrt{67\,600-a^2}$. The slope of this line is also the slope of PT, namely, $(b-30)/(a+120)$, and therefore

$$\frac{b-30}{a+120} = -\frac{a}{\sqrt{67\,600-a^2}} \quad\Longrightarrow\quad b = 30 - \frac{a(a+120)}{\sqrt{67\,600-a^2}}.$$

Since $P(a,b)$ is on the circular arc, it also follows that $b = -240+\sqrt{67\,600-a^2}$. When we equate these expressions for b,

$$30 - \frac{a(a+120)}{\sqrt{67\,600 - a^2}} = -240 + \sqrt{67\,600 - a^2} \quad \Longrightarrow \quad 873a^2 + 162\,240a - 3\,582\,800 = 0.$$

The positive solution is $a = 19.9432$. The y-coordinate of P is $b = 19.2340$. The equation of the tangent line is therefore $y - b = (-a/\sqrt{67\,600 - a^2})(x - a)$, and its x-intercept is given by $-b = (-a/\sqrt{67\,600 - a^2})(x - a) \Longrightarrow x = (b/a)\sqrt{67\,600 - a^2} + a$. When we substitute the calculated values for a and b, the result is $x \approx 270$ m.

EXERCISES 3.8

1. If we differentiate with respect to x, we find $4y^3\dfrac{dy}{dx} + \dfrac{dy}{dx} = 12x^2 \Longrightarrow \dfrac{dy}{dx} = \dfrac{12x^2}{4y^3+1}$.

2. If we differentiate with respect to x, we find $4x^3 + 2y\dfrac{dy}{dx} + 3y^2\dfrac{dy}{dx} = 0 \Longrightarrow \dfrac{dy}{dx} = \dfrac{-4x^3}{3y^2+2y}$.

3. If we differentiate with respect to x, we find $x\dfrac{dy}{dx} + y + 2 = 8y\dfrac{dy}{dx} \Longrightarrow \dfrac{dy}{dx} = \dfrac{y+2}{8y-x}$.

4. Differentiation with respect to x gives $6x^2 - 3y^4 - 12xy^3\dfrac{dy}{dx} + 5y + 5x\dfrac{dy}{dx} = 0$, and therefore $\dfrac{dy}{dx} = \dfrac{6x^2 - 3y^4 + 5y}{12xy^3 - 5x}$.

5. If we differentiate with respect to x, we find $1 + y^5 + 5xy^4\dfrac{dy}{dx} + 2xy^3 + 3x^2y^2\dfrac{dy}{dx} = 0$, and therefore $\dfrac{dy}{dx} = -\dfrac{1 + 2xy^3 + y^5}{5xy^4 + 3x^2y^2}$.

6. When we differentiate with respect to x, we obtain $2(x+y)\left(1 + \dfrac{dy}{dx}\right) = 2 \Longrightarrow \dfrac{dy}{dx} = \dfrac{1-x-y}{x+y}$.

7. If we differentiate with respect to x, we find $2x - y - x\dfrac{dy}{dx} - 12y^2\dfrac{dy}{dx} = 2 \Longrightarrow \dfrac{dy}{dx} = \dfrac{2x-y-2}{x+12y^2}$.

8. Differentiation with respect to x gives $\dfrac{1}{2\sqrt{x+y}}\left(1 + \dfrac{dy}{dx}\right) + 2y\dfrac{dy}{dx} = 24x + \dfrac{dy}{dx}$, from which $\left(\dfrac{1}{2\sqrt{x+y}} + 2y - 1\right)\dfrac{dy}{dx} = 24x - \dfrac{1}{2\sqrt{x+y}}$. Thus,

$$\frac{dy}{dx} = \left(\frac{48x\sqrt{x+y} - 1}{2\sqrt{x+y}}\right)\left(\frac{2\sqrt{x+y}}{1 + 2(2y-1)\sqrt{x+y}}\right) = \frac{48x\sqrt{x+y} - 1}{1 + 2(2y-1)\sqrt{x+y}}.$$

9. If we differentiate with respect to x, we find $\dfrac{1}{2\sqrt{1+xy}}\left(y + x\dfrac{dy}{dx}\right) - y - x\dfrac{dy}{dx} = 0$, and therefore

$$\frac{dy}{dx} = \frac{y - \dfrac{y}{2\sqrt{1+xy}}}{-x + \dfrac{x}{2\sqrt{1+xy}}} = \frac{y(2\sqrt{1+xy} - 1)}{x(1 - 2\sqrt{1+xy})} = -\frac{y}{x}.$$

10. If we write $x^2 - xy - y^2 = 4x(x+y)$, or, $3x^2 + 5xy + y^2 = 0$, and differentiate with respect to x, we obtain $6x + 5y + 5x\dfrac{dy}{dx} + 2y\dfrac{dy}{dx} = 0$. Therefore, $\dfrac{dy}{dx} = -\dfrac{6x+5y}{5x+2y}$.

11. If we differentiate the equation of the curve with respect to x, we find $y^2 + 2xy\dfrac{dy}{dx} + 3y^2\dfrac{dy}{dx} = 0$, and therefore $\dfrac{dy}{dx}_{|(1,1)} = -\dfrac{y^2}{2xy+3y^2}_{|(1,1)} = -\dfrac{1}{5}$. Equations for the tangent and normal lines are $y - 1 = -(1/5)(x-1)$ and $y - 1 = 5(x-1)$, or, $x + 5y = 6$ and $5x - y = 4$.

12. When we differentiate with respect to x, we find $2x + 3y^2\dfrac{dy}{dx} + \dfrac{dy}{dx} = 0$, and therefore $\dfrac{dy}{dx} = -\dfrac{2x}{3y^2+1}$. Differentiation of this equation gives

$$\frac{d^2y}{dx^2} = -\frac{(3y^2+1)(2) - 2x\left(6y\dfrac{dy}{dx}\right)}{(3y^2+1)^2} = -\frac{2(3y^2+1) + 12xy\left(\dfrac{2x}{3y^2+1}\right)}{(3y^2+1)^2} = -\frac{2(3y^2+1)^2 + 24x^2y}{(3y^2+1)^3}.$$

13. When we differentiate with respect to x, we find $4x - 3y^2\dfrac{dy}{dx} = -y - x\dfrac{dy}{dx}$, and therefore $\dfrac{dy}{dx} = \dfrac{4x+y}{3y^2-x}$. Differentiation of this equation gives

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{(3y^2-x)\left(4+\dfrac{dy}{dx}\right) - (4x+y)\left(6y\dfrac{dy}{dx}-1\right)}{(3y^2-x)^2}\\ &= \frac{(3y^2-x)\left(4+\dfrac{4x+y}{3y^2-x}\right) - (4x+y)\left[6y\left(\dfrac{4x+y}{3y^2-x}\right)-1\right]}{(3y^2-x)^2}\\ &= \frac{(3y^2-x)(12y^2+y) - (4x+y)(3y^2+24xy+x)}{(3y^2-x)^3}\\ &= \frac{36y^4 - 48xy^2 - 2xy - 96x^2y - 4x^2}{(3y^2-x)^3}.\end{aligned}$$

14. If we differentiate with respect to x, we obtain $2y\dfrac{dy}{dx} + 2\dfrac{dy}{dx} = 5$, and this equation can be solved for $\dfrac{dy}{dx} = \dfrac{5}{2(y+1)}$. A second differentiation gives

$$\frac{d^2y}{dx^2} = \frac{-5}{2(y+1)^2}\frac{dy}{dx} = \frac{-5}{2(y+1)^2}\left[\frac{5}{2(y+1)}\right] = \frac{-25}{4(y+1)^3}.$$

15. If we differentiate with respect to x, we obtain $2(x+y)\left(1+\dfrac{dy}{dx}\right) = 1$, from which $\dfrac{dy}{dx} = \dfrac{1}{2(x+y)} - 1$. A second differentiation gives

$$\frac{d^2y}{dx^2} = \frac{-1}{2(x+y)^2}\left(1+\frac{dy}{dx}\right) = \frac{-1}{2(x+y)^2}\left[1+\frac{1}{2(x+y)}-1\right] = \frac{-1}{4(x+y)^3}.$$

16. If we differentiate with respect to x, we obtain $3x^2y + x^3\dfrac{dy}{dx} + y^3 + 3xy^2\dfrac{dy}{dx} = 0$. Therefore, $\dfrac{dy}{dx} = -\dfrac{3x^2y+y^3}{x^3+3xy^2}$. When $x = 1$, we have $y + y^3 = 2$, and the only solution of this equation is $y = 1$. Thus, $\dfrac{dy}{dx}_{|x=1} = -\dfrac{3+1}{1+3} = -1$.

17. The equation implies that $x^2 + 2xy + y^2 = x^2 + y^2 \implies 2xy = 0 \implies y = 0$. Consequently, dy/dx and d^2y/dx^2 are both 0.

18. Differentiation with respect to x leads to $2xy^3 + 3x^2y^2\dfrac{dy}{dx} + 2 + 4\dfrac{dy}{dx} = 0$, from which $\dfrac{dy}{dx} = -\dfrac{2+2xy^3}{3x^2y^2+4}$. A second differentiation gives

$$\frac{d^2y}{dx^2} = -\frac{(3x^2y^2+4)\left(2y^3+6xy^2\dfrac{dy}{dx}\right) - (2+2xy^3)\left(6xy^2+6x^2y\dfrac{dy}{dx}\right)}{(3x^2y^2+4)^2}$$

$$= -\frac{(3x^2y^2+4)\left[2y^3 - 6xy^2\left(\dfrac{2+2xy^3}{3x^2y^2+4}\right)\right] - (2+2xy^3)\left[6xy^2 - 6x^2y\left(\dfrac{2+2xy^3}{3x^2y^2+4}\right)\right]}{(3x^2y^2+4)^2}$$
$$= \frac{-2y^3(3x^2y^2+4)^2 + 12xy^2(2+2xy^3)(3x^2y^2+4) - 6x^2y(2+2xy^3)^2}{(3x^2y^2+4)^3}$$

19. Differentiation with respect to x gives $y^2 + 2xy\dfrac{dy}{dx} - 6xy - 3x^2\dfrac{dy}{dx} = 1$, from which $\dfrac{dy}{dx} = \dfrac{1+6xy-y^2}{2xy-3x^2}$. A second differentiation gives

$$\frac{d^2y}{dx^2} = \frac{(2xy-3x^2)\left(6y + 6x\dfrac{dy}{dx} - 2y\dfrac{dy}{dx}\right) - (1+6xy-y^2)\left(2y + 2x\dfrac{dy}{dx} - 6x\right)}{(2xy-3x^2)^2}$$
$$= \frac{(2xy-3x^2)\left[6y + (6x-2y)\left(\dfrac{1+6xy-y^2}{2xy-3y^2}\right)\right] - (1+6xy-y^2)\left[2y - 6x + 2x\left(\dfrac{1+6xy-y^2}{2xy-3x^2}\right)\right]}{(2xy-3x^2)^2}$$
$$= \frac{6y(2xy-3x^2)^2 + (12x-4y)(2xy-3x^2)(1+6xy-y^2) - 2x(1+6xy-y^2)^2}{(2xy-3x^2)^3}.$$

20. Differentiation with respect to x gives $1 = \dfrac{dy}{dx}\sqrt{1-y^2} + y\dfrac{-y}{\sqrt{1-y^2}}\dfrac{dy}{dx} = \dfrac{1-y^2-y^2}{\sqrt{1-y^2}}\dfrac{dy}{dx} = \dfrac{1-2y^2}{\sqrt{1-y^2}}\dfrac{dy}{dx}$.

Thus, $\dfrac{dy}{dx} = \dfrac{\sqrt{1-y^2}}{1-2y^2}$. Another differentiation gives

$$\frac{d^2y}{dx^2} = \frac{(1-2y^2)(1/2)(1-y^2)^{-1/2}\left(-2y\dfrac{dy}{dx}\right) - \sqrt{1-y^2}\left(-4y\dfrac{dy}{dx}\right)}{(1-2y^2)^2}$$
$$= \frac{-y(1-2y^2) + 4y(1-y^2)}{\sqrt{1-y^2}(1-2y^2)^2}\frac{dy}{dx} = \frac{3y-2y^3}{\sqrt{1-y^2}(1-2y^2)^2}\frac{\sqrt{1-y^2}}{1-2y^2} = \frac{3y-2y^3}{(1-2y^2)^3}.$$

21. If we differentiate $1 - xy = (4-3y)^2 = 16 - 24y + 9y^2$, then $-y - x\dfrac{dy}{dx} = -24\dfrac{dy}{dx} + 18y\dfrac{dy}{dx}$, from which $\dfrac{dy}{dx} = \dfrac{y}{-x-18y+24}$. When $x = 0$, we find that $y = 1$, and therefore $\dfrac{dy}{dx}_{|x=0} = \dfrac{1}{-18+24} = \dfrac{1}{6}$.

22. When we differentiate with respect to x, we obtain $2xy^3 + 3x^2y^2\dfrac{dy}{dx} + y + x\dfrac{dy}{dx} = 0$, and therefore $\dfrac{dy}{dx} = -\dfrac{2xy^3+y}{3x^2y^2+x}$. When $x = 1$, we find that $y^3 + y = 2$, the only solution of which is $y = 1$. Thus, $\dfrac{dy}{dx}_{|x=1} = -\dfrac{2+1}{3+1} = -\dfrac{3}{4}$.

23. If we differentiate with respect to x, then $5y^4\dfrac{dy}{dx} + y + x\dfrac{dy}{dx} - 2\dfrac{dy}{dx} = 0$, from which $\dfrac{dy}{dx} = \dfrac{y}{2-x-5y^4}$. When $x = 2$, we find that $y = 1$, and therefore $\dfrac{dy}{dx}_{|x=2} = \dfrac{1}{2-2-5} = -\dfrac{1}{5}$. Differentiation of the first derivative gives

$$\frac{d^2y}{dx^2} = \frac{(2-x-5y^4)\dfrac{dy}{dx} - y\left(-1 - 20y^3\dfrac{dy}{dx}\right)}{(2-x-5y^4)^2},$$

and when we substitute $x = 2$, $y = 1$, and $dy/dx = -1/5$,

$$\frac{d^2y}{dx^2}_{|x=2} = \frac{(2-2-5)(-1/5) - (1)[-1 - 20(1)^3(-1/5)]}{(2-2-5)^2} = -\frac{2}{25}.$$

24. If we differentiate with respect to x, we find $2x+2y+2x\dfrac{dy}{dx}+6y\dfrac{dy}{dx}=0$. Therefore, $\dfrac{dy}{dx}=-\dfrac{2x+2y}{2x+6y}=-\dfrac{x+y}{x+3y}$. When $y=1$, we find that $x^2+2x+1=0$, and the only solution of this equation is $x=-1$. Thus, $dy/dx_{|x=-1}=-(-1+1)/(-1+3)=0$. A second differentiation gives

$$\frac{d^2y}{dx^2}=-\frac{(x+3y)\left(1+\frac{dy}{dx}\right)-(x+y)\left(1+3\frac{dy}{dx}\right)}{(x+3y)^2},$$

and when we substitute $x=-1$, $y=1$, and $dy/dx=0$, $\dfrac{d^2y}{dx^2}_{|x=-1}=-\dfrac{(-1+3)(1)-(-1+1)(1)}{(-1+3)^2}=-\dfrac{1}{2}$.

25. If we differentiate with respect to x, then $y^2+2xy\dfrac{dy}{dx}+2xy+x^2\dfrac{dy}{dx}=0$, from which $\dfrac{dy}{dx}=-\dfrac{2xy+y^2}{2xy+x^2}$. The slope of the tangent line is 0 when $0=2xy+y^2=y(2x+y)$. Thus, $y=0$ or $y=-2x$. The first is impossible since x and y must also satisfy $xy^2+x^2y=16$. When we substitute $y=-2x$ into this equation, $16=x(-2x)^2+x^2(-2x)=2x^3 \implies x=2$. Thus, the only point is $(2,-4)$.

26. If we differentiate with respect to x, then $2x+\dfrac{2}{3}y^{-1/3}\dfrac{dy}{dx}=0 \quad\implies\quad \dfrac{dy}{dx}=-3xy^{1/3}$. Since

$$\frac{d^2y}{dx^2}=-3y^{1/3}-xy^{-2/3}\frac{dy}{dx}=-3y^{1/3}-xy^{-2/3}(-3xy^{1/3})=-3y^{1/3}+3x^2y^{-1/3}=3y^{-1/3}(-y^{2/3}+x^2),$$

the second derivative vanishes if $x^2=y^{2/3}$. When we substitute this into $x^2+y^{2/3}=2$, we obtain $x^2+x^2=2$, and therefore $x=\pm 1$. The corresponding y-values are $y=\pm 1$, and the required points are $(1,\pm 1)$ and $(-1,\pm 1)$.

27. If we differentiate the equation with respect to h,

$$-\frac{2(2.4048)^2}{r^3}\frac{dr}{dh}-\frac{2\pi^2}{h^3}=0 \quad\implies\quad \frac{dr}{dh}=-\left(\frac{\pi}{2.4048}\right)^2\left(\frac{r}{h}\right)^3.$$

28. (a) Since $\dfrac{dy}{dx}=\dfrac{(x+2)(2x+1)-(x^2+x)(1)}{(x+2)^2}=\dfrac{x^2+4x+2}{(x+2)^2}$, it follows that

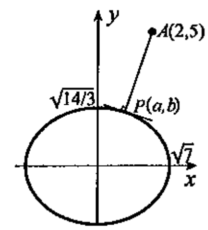
$$\frac{Ey}{Ex}=\frac{x}{\frac{x(x+1)}{x+2}}\frac{x^2+4x+2}{(x+2)^2}=\frac{x^2+4x+2}{(x+1)(x+2)}.$$

(b) If we differentiate with respect to x, $1=\dfrac{(3-y)\left(400\frac{dy}{dx}\right)-(400y+200)\left(-\frac{dy}{dx}\right)}{(3-y)^2}=\dfrac{1400\frac{dy}{dx}}{(3-y)^2}$, from which $\dfrac{dy}{dx}=\dfrac{(3-y)^2}{1400}$. Thus, $\dfrac{Ey}{Ex}=\left[\dfrac{400y+200}{y(3-y)}\right]\dfrac{(3-y)^2}{1400}=\dfrac{(2y+1)(3-y)}{7y}$.

29. The elasticity of a function is equal to one if and only if $1=\dfrac{x}{y}\dfrac{dy}{dx} \implies \dfrac{dy}{dx}=\dfrac{y}{x}$. But y/x is the slope of the line joining the point (x,y) on the graph of the function to the origin. Hence elasticity is equal to one if and only if slope dy/dx of the graph is equal to slope of the line joining the point to the origin.

30. If we differentiate $2x^2 + 3y^2 = 14$ with respect to x, $4x + 6y\dfrac{dy}{dx} = 0$.
The slope of the tangent line at P is therefore

$$\frac{dy}{dx}_{|(a,b)} = -\frac{2a}{3b}.$$



Since the slope of AP is $(b-5)/(a-2)$, and this line is perpendicular to the tangent line at P,

$$-\frac{2a}{3b} = -\frac{a-2}{b-5}.$$

When this equation is solved for b, the result is $b = 10a/(6-a)$. Because P is on the ellipse, we may write that

$$14 = 2a^2 + 3b^2 = 2a^2 + 3\left(\frac{10a}{6-a}\right)^2 \implies (14-2a^2)(6-a)^2 = 300a^2.$$

Our diagram makes it clear that there is one and only one point in the first quadrant which satisfies the requirements, and $a = 1$ is a solution of the above equation. Thus, the required point is $(1, 2)$.

31. If we differentiate the equation of the hyperbola with respect to x, then $2b^2x - 2a^2y(dy/dx) = 0$, from which $dy/dx = b^2x/(a^2y)$. The slope of the tangent line at (x_0, y_0) is therefore $b^2x_0/(a^2y_0)$, and the equation of the tangent line is

$$y - y_0 = \frac{b^2x_0}{a^2y_0}(x - x_0) \implies b^2x_0x - a^2y_0y = b^2x_0^2 - a^2y_0^2.$$

Since (x_0, y_0) is on the hyperbola, $b^2x_0^2 - a^2y_0^2 = a^2b^2$, and the equation of the tangent line simplifies to $b^2x_0x - a^2y_0y = a^2b^2$.

32. If we differentiate the equation of the circle with respect to x,

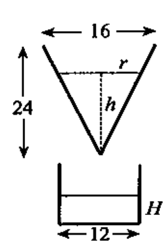
$$2(x-h) + 2(y-k)\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x-h}{y-k}.$$

Thus,
$$\frac{d^2y}{dx^2} = -\frac{(y-k)(1) - (x-h)dy/dx}{(y-k)^2} = \frac{-1}{(y-k)^2}\left[(y-k) + (x-h)\left(\frac{x-h}{y-k}\right)\right]$$
$$= \frac{-1}{(y-k)^3}\left[(y-k)^2 + (x-h)^2\right] = \frac{-r^2}{(y-k)^3}.$$

We now calculate that
$$\left|\frac{d^2y/dx^2}{[1+(dy/dx)^2]^{3/2}}\right| = \left|\frac{-r^2/(y-k)^3}{\left[1 + \left(-\dfrac{x-h}{y-k}\right)^2\right]^{3/2}}\right| = \left|\frac{r^2/(y-k)^3}{\left[\dfrac{(y-k)^2 + (x-h)^2}{(y-k)^2}\right]^{3/2}}\right|$$
$$= \left|\frac{r^2/(y-k)^3}{\left[\dfrac{r^2}{(y-k)^2}\right]^{3/2}}\right| = \frac{r^2}{|y-k|^3}\,\frac{|y-k|^3}{r^3} = \frac{1}{r}.$$

33. Since the amount of solution in the two containers is always the same, call it C, we can write that

$$C = \frac{1}{3}\pi r^2 h + \pi(6)^2 H.$$



Similar triangles for the cone give $r/h = 8/24 = 1/3$, and therefore

$$C = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h + 36\pi H = \frac{\pi h^3}{27} + 36\pi H.$$

If we differentiate this equation with respect to H,

$$\frac{\pi h^2}{9}\frac{dh}{dH} + 36\pi = 0 \implies \frac{dh}{dH} = -\frac{324}{h^2}.$$

34. The product rule gives $R'(x) = r(x) + xr'(x)$. To obtain $r'(x)$ we differentiate $x = 4a^3 - 3ar^2 + r^3$ with respect to x,

$$1 = -6ar\frac{dr}{dx} + 3r^2\frac{dr}{dx}.$$

Thus, $\dfrac{dr}{dx} = \dfrac{1}{3r^2 - 6ar}$, and $R'(x) = r + \dfrac{x}{3r^2 - 6ar} = \dfrac{3r^3 - 6ar^2 + x}{3r^2 - 6ar}$.

35. If the equation is to be valid for all x, then it must be valid for $x = 0$. Substitution of $x = 0$ yields $a_0 = b_0$. The equation now reads

$$a_1x + a_2x^2 + \cdots + a_nx^n = b_1x + b_2x^2 + \cdots + b_nx^n.$$

If we differentiate this equation with respect to x,

$$a_1 + 2a_2x + \cdots + na_nx^{n-1} = b_1 + 2b_2x + \cdots + nb_nx^{n-1}.$$

If we set $x = 0$, we obtain $a_1 = b_1$. The equation now reads

$$2a_2x + \cdots + na_nx^{n-1} = 2b_2x + \cdots + nb_nx^{n-1}.$$

If we differentiate this equation with respect to x,

$$2a_2 + (3)(2)a_3x + \cdots + n(n-1)a_nx^{n-2} = 2b_2 + (3)(2)b_3x + \cdots + n(n-1)b_nx^{n-1}.$$

If we set $x = 0$, we obtain $a_2 = b_2$. Continued differentiations lead to equality of all coefficients.

36. (a) When we differentiate with respect to x, we obtain $\sqrt{1+2y} + \dfrac{x}{\sqrt{1+2y}}\dfrac{dy}{dx} = 2x - \dfrac{dy}{dx}$, and therefore $\dfrac{dy}{dx} = \dfrac{2x - \sqrt{1+2y}}{\dfrac{x}{\sqrt{1+2y}} + 1}$. Since $y = 0$ when $x = 0$, we obtain $f'(0) = -1/1 = -1$.

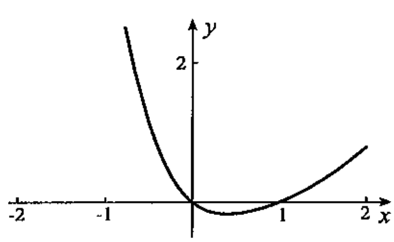
(b) When the equation is squared, $x^2(1+2y) = x^4 - 2x^2y + y^2 \Longrightarrow x^2 + 4x^2y = x^4 + y^2$. If we differentiate this equation with respect to x,

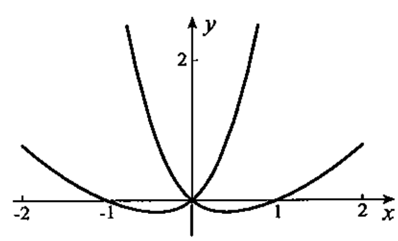
$$2x + 8xy + 4x^2\frac{dy}{dx} = 4x^3 + 2y\frac{dy}{dx} \quad\Longrightarrow\quad \frac{dy}{dx} = \frac{2x + 8xy - 4x^3}{2y - 4x^2} = \frac{x + 4xy - 2x^3}{y - 2x^2}.$$

In this case we cannot simply set $x = 0$ and $y = 0$ to obtain $f'(0)$. To see why $x^2 + 4x^2y = x^4 + y^2$ cannot be used to find $f'(0)$, we write the equation in the form $y^2 - 4x^2y + x^4 - x^2 = 0$. This is a quadratic equation in y with solutions

$$y = \frac{4x^2 \pm \sqrt{16x^4 - 4(x^4 - x^2)}}{2} = 2x^2 \pm \sqrt{3x^4 + x^2}.$$

Thus, the equation $x^2 + 4x^2y = x^4 + y^2$ does not define y as a function of x; there are two values of y satisfying the equation for each value of x. We can also see this graphically. A plot of $x\sqrt{1+2y} = x^2 - y$ is shown in the left figure below. It defines y as a function of x near $x = 0$. A plot of $x^2 + 4x^2y = x^4 + y^2$ in the right figure shows that it does not define y as a function of x near $x = 0$.





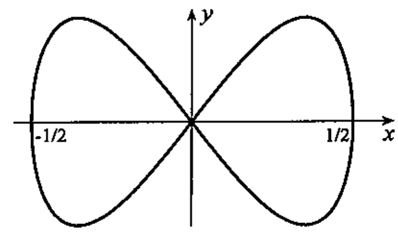
37. The straight lines have slope m where $m = y/x$. Diferentiation of $x^2 + y^2 = r^2$ with respect to x gives $2x + 2y(dy/dx) = 0 \Longrightarrow dy/dx = -x/y$. These slopes are negative reciprocals.

38. The slope of the tangent line at any point on any of the parabolas in the first family is $dy/dx = 2ax = 2x(y/x^2) = 2y/x$. To find the slope of the tangent line at any point on any of the ellipses, we differentiate with respect to x, getting $2x+4y(dy/dx) = 0 \Longrightarrow dy/dx = -x/(2y)$. Since this is the negative reciprocal of the slope of the parabola, the families are orthogonal trajectories.

39. When we differentiate $x^2 - y^2 = C_1$ with respect to x, we obtain $2x - 2y(dy/dx) = 0 \Longrightarrow dy/dx = x/y$. When we differentiate $xy = C_2$ with resepct to x, the result is $y + x(dy/dx) = 0 \Longrightarrow dy/dx = -y/x$. Slopes of the families are negative reciprocals, and the families are orthogonal trajectories.

40. Differentiation of $2x^2 + 3y^2 = C^2$ with respect to x gives $4x + 6y(dy/dx) = 0 \Longrightarrow dy/dx = -2x/(3y)$. Differentiation of $y^2 = ax^3$ gives $2y\dfrac{dy}{dx} = 3ax^2 \Longrightarrow \dfrac{dy}{dx} = \dfrac{3ax^2}{2y} = \dfrac{3x^2(y^2/x^3)}{2y} = \dfrac{3y}{2x}$. Since these slopes are negative reciprocals, the families are orthogonal trajectories.

41. (a) If we differentiate $y^2 = x^2 - 4x^4$ with respect to x, the result is $2y(dy/dx) = 2x - 16x^3$, from which $dy/dx = x(1 - 8x^2)/y$.
(b) Since $y = 0$ when $x = 0$, we cannot use the formula in part (a) to calculate dy/dx when $x = 0$. The plot indicates the reason. The curve crosses itself at $(0,0)$, and therefore the slope is not uniquely defined.

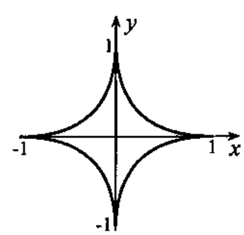


42. (a) If we differentiate with respect to x, we obtain $\dfrac{1}{2\sqrt{x}} + \dfrac{1}{2\sqrt{y}}\dfrac{dy}{dx} = 0$, from which $\dfrac{dy}{dx} = -\dfrac{\sqrt{y}}{\sqrt{x}}$.

(b) When $x \geq 0$ and $y \geq 0$, the equation of the curve reduces to $\sqrt{x} + \sqrt{y} = 1$. According to the formula in part (a),

$$\lim_{x\to 1^-} \frac{dy}{dx} = -\frac{0}{1} = 0^-, \quad \text{and} \quad \lim_{x\to 0^+} \frac{dy}{dx} = -\infty.$$

This enables us to sketch the first quadrant part of the curve. The remaining parts are obtained by symmetry.



43. There is no derivative because the equation does not define y as a function of x. To see this we complete squares on x- and y-terms, $(x+2)^2 + (y+3)^2 = -2$.

44. The chain rule gives

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{\sqrt{u^2-1}(1) - u(1/2)(u^2-1)^{-1/2}(2u)}{u^2-1}\right]\frac{du}{dx} = \frac{u^2-1-u^2}{(u^2-1)^{3/2}}\frac{du}{dx} = \frac{-1}{(u^2-1)^{3/2}}\frac{du}{dx}.$$

To obtain du/dx we differentiate $x^2u^2 + \sqrt{u^2-1} = 4$ with respect to x,

$$2xu^2 + 2x^2u\frac{du}{dx} + \frac{u}{\sqrt{u^2-1}}\frac{du}{dx} = 0 \quad \Longrightarrow \quad \frac{du}{dx} = \frac{-2xu^2}{2x^2u + \frac{u}{\sqrt{u^2-1}}} = \frac{-2xu\sqrt{u^2-1}}{1+2x^2\sqrt{u^2-1}}.$$

Thus, $\dfrac{dy}{dx} = \left[\dfrac{-1}{(u^2-1)^{3/2}}\right]\left[\dfrac{-2xu\sqrt{u^2-1}}{1+2x^2\sqrt{u^2-1}}\right] = \dfrac{2xu}{(u^2-1)\left(1+2x^2\sqrt{u^2-1}\right)}$.

45. If we differentiate the equations with respect to x,

$$4y^3\frac{dy}{dx} + v^3\frac{dy}{dx} + 3yv^2\frac{dv}{dx} = 0, \qquad 2xv + x^2\frac{dv}{dx} + 3v^2 + 6xv\frac{dv}{dx} = 6x^2y + 2x^3\frac{dy}{dx}.$$

When we solve the first for $\dfrac{dv}{dx} = -\left(\dfrac{4y^3+v^3}{3yv^2}\right)\dfrac{dy}{dx}$, and substitute into the second,

$$2xv + 3v^2 - (x^2+6xv)\left(\frac{4y^3+v^3}{3yv^2}\right)\frac{dy}{dx} = 6x^2y + 2x^3\frac{dy}{dx} \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{3yv^2(2xv+3v^2-6x^2y)}{6x^3yv^2 + (x^2+6xv)(4y^3+v^3)}.$$

46. If we differentiate the equation with respect to x,

$$\frac{2xy^3 - 3x^2y^2\dfrac{dy}{dx}}{y^6} - 1 = 0 \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{2xy^3 - y^6}{3x^2y^2} = \frac{2xy - y^4}{3x^2}.$$

Consequently, $3x^2\dfrac{dy}{dx} + y^4 = 3x^2\left(\dfrac{2xy - y^4}{3x^2}\right) + y^4 = 2xy.$

47. If we differentiate the equation in the form $\dfrac{2}{y^3} - \dfrac{3}{x^2y^2} = C$ with respect to x, the result is $\dfrac{-6}{y^4}\dfrac{dy}{dx} + \dfrac{6}{(xy)^3}\left(y + x\dfrac{dy}{dx}\right) = 0$. Multiplication by $x^3y^4/6$ gives

$$-x^3\frac{dy}{dx} + y\left(y + x\frac{dy}{dx}\right) = 0 \qquad \Longrightarrow \qquad (xy - x^3)\frac{dy}{dx} + y^2 = 0.$$

48. If $n = a/b$, where a and b are integers, then when $y = x^n = x^{a/b}$, we may set $y^b = x^a$. Differentiation with respect to x gives

$$by^{b-1}\frac{dy}{dx} = ax^{a-1} \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{a}{b}\frac{x^{a-1}}{y^{b-1}} = \frac{a}{b}\frac{x^{a-1}}{(x^{a/b})^{b-1}} = \frac{a}{b}x^{a-1-a+a/b} = nx^{n-1}.$$

49. If we differentiate with respect to x, then $y^2 + 2xy\dfrac{dy}{dx} + 2xy + x^2\dfrac{dy}{dx} = 0$, from which $\dfrac{dy}{dx} = -\dfrac{2xy + y^2}{2xy + x^2}$. The slope of the tangent line is 1 when

$$1 = -\frac{2xy + y^2}{2xy + x^2} \qquad \Longrightarrow \qquad y^2 + 4xy + x^2 = 0 \qquad \Longrightarrow \qquad (x + y)^2 = -2xy.$$

But the equation $x^2y + xy^2 = 2 \Longrightarrow xy(x + y) = 2 \Longrightarrow (x + y)^2 = 4/(xy)^2$. When these expressions for $(x + y)^2$ are equated, the result is $-2xy = 4/(xy)^2 \Longrightarrow (xy)^3 = -2 \Longrightarrow y = -2^{1/3}/x$. We now substitute this into $xy(x + y) = 2$,

$$x\left(\frac{-2^{1/3}}{x}\right)\left(x - \frac{2^{1/3}}{x}\right) = 2 \Longrightarrow x^2 + 2^{2/3}x - 2^{1/3} = 0 \Longrightarrow x = \frac{-2^{2/3} \pm \sqrt{2^{4/3} + 4(2^{1/3})}}{2} = \frac{-1 \pm \sqrt{3}}{2^{1/3}}.$$

Corresponding y-values are $y = -2^{1/3}\left(\dfrac{2^{1/3}}{-1 \pm \sqrt{3}}\right) = \dfrac{2^{2/3}}{1 \mp \sqrt{3}}$.

50. If we express the first family in the form $y^2(a - x) = x^3$, we may solve for $a = x + x^3/y^2$. Differentiation of this equation with respect to x gives

$$0 = 1 + \frac{3y^2x^2 - 2x^3y\dfrac{dy}{dx}}{y^4} \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{3x^2y + y^3}{2x^3}.$$

When we solve the second family for $b = (x^2 + y^2)^2/(2x^2 + y^2)$, and differentiate

$$0 = \frac{(2x^2 + y^2)(2)(x^2 + y^2)\left(2x + 2y\dfrac{dy}{dx}\right) - (x^2 + y^2)^2\left(4x + 2y\dfrac{dy}{dx}\right)}{(2x^2 + y^2)^2}.$$

This implies that $4(2x^2 + y^2)\left(x + y\dfrac{dy}{dx}\right) - 2(x^2 + y^2)\left(2x + y\dfrac{dy}{dx}\right) = 0$, and we can solve for

$$\frac{dy}{dx} = \frac{4x(2x^2 + y^2) - 4x(x^2 + y^2)}{2y(x^2 + y^2) - 4y(2x^2 + y^2)} = \frac{-2x^3}{3x^2y + y^3}.$$

Since these expressions for dy/dx are negative reciprocals, the families are orthogonal trajectories.

51. If we differentiate the equation with respect to x, $\dfrac{2}{3x^{1/3}} + \dfrac{2}{3y^{1/3}}\dfrac{dy}{dx} = 0$. When we solve this for dy/dx, we obtain $\dfrac{dy}{dx} = -\dfrac{y^{1/3}}{x^{1/3}}$. The equation of the tangent line at a point (x_0, y_0) is $y - y_0 = -\dfrac{y_0^{1/3}}{x_0^{1/3}}(x - x_0)$. The x- and y-intercepts of this line are $x = \dfrac{x_0^{1/3}y_0}{y_0^{1/3}} + x_0 = x_0^{1/3}y_0^{2/3} + x_0$ and $y = \dfrac{y_0^{1/3}}{x_0^{1/3}}x_0 + y_0 = x_0^{2/3}y_0^{1/3}$ $+y_0$. The length L of that part of the tangent line between the coordinate axes is therefore given by

$$\begin{aligned} L^2 &= (x_0^{2/3}y_0^{1/3} + y_0)^2 + (x_0^{1/3}y_0^{2/3} + x_0)^2 = y_0^{2/3}(x_0^{2/3} + y_0^{2/3})^2 + x_0^{2/3}(y_0^{2/3} + x_0^{2/3})^2 \\ &= (x_0^{2/3} + y_0^{2/3})^3 = (a^{2/3})^3 = a^2. \end{aligned}$$

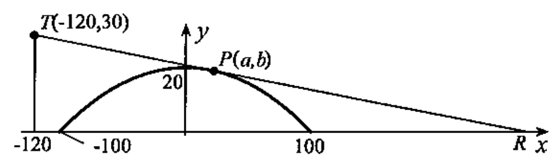
Thus, the length is always equal to a.

52. If we differentiate the equation with respect to x, $3x^2 + 3y^2(dy/dx) = 3ay + 3ax(dy/dx)$. When we solve this for dy/dx and equate the result to -1,

$$-1 = \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax} \quad\Longrightarrow\quad a(y - x) = x^2 - y^2 = (x - y)(x + y).$$

Thus, $y = x$ or $x + y = -a$. When $y = x$, the equation of the folium implies that $x^3 + x^3 = 3ax^2 \Longrightarrow$ $x = 3a/2$. A contradiction is obtained when we put $y = -a - x$ into the equation of the folium. Thus, the only point at which the slope of the tangent line is equal to -1 is $(3a/2, 3a/2)$.

53. The required position occurs at the point R where the tangent line from $T(-120, 30)$ to the arc of the circle representing the hill intersects the x-axis. Let the point of tangency be $P(a, b)$. If $x^2 + (y - k)^2 = r^2$ is the equation of the circle, then using the points $(100, 0)$ and $(0, 20)$, we obtain $100^2 + k^2 = r^2$ and $(20 - k)^2 = r^2$. These give $k = -240$ and $r^2 = 67\,600$, so that the equation of the circular arc is $x^2 + (y + 240)^2 = 67\,600$. The slope of the tangent line to this curve is defined by



$$2x + 2(y + 240)\frac{dy}{dx} = 0 \Longrightarrow \frac{dy}{dx} = \frac{-x}{y + 240},$$

and hence at P, $\left.\dfrac{dy}{dx}\right|_{(a,b)} = \dfrac{-a}{b + 240}$. The slope of this line is also the slope of PT, namely, $(b - 30)/(a + 120)$, and therefore

$$\frac{b - 30}{a + 120} = \frac{-a}{b + 240} \quad\Longrightarrow\quad a^2 + b^2 + 210b + 120a = 7200.$$

Since $P(a, b)$ is on the circular arc, it also follows that $a^2 + (b+240)^2 = 67\,600 \Longrightarrow a^2 + b^2 + 480b = 10\,000$. When we solve each of these equations for $a^2 + b^2$ and equate results, we obtain

$$7200 - 210b - 120a = 10\,000 - 480b \quad\Longrightarrow\quad b = \frac{12a + 280}{27}.$$

Substituting this into $a^2 + b^2 + 480b = 10\,000$ gives

$$a^2 + \left(\frac{12a + 280}{27}\right)^2 + 480\left(\frac{12a + 280}{27}\right) = 10\,000 \quad\Longrightarrow\quad 873a^2 + 162\,240a - 3\,582\,800 = 0.$$

The positive solution is $a = 19.9432$. The y-coordinate of P is $b = 19.2340$. The equation of the tangent line is therefore $y - b = [-a/(b+240)](x - a)$, and its x-intercept is given by $-b = [-a/(b+240)](x - a) \Longrightarrow$ $x = b(b + 240)/a + a$. When we substitute the calculated values for a and b, the result is $x \approx 270$ m.

54. (a) The equation for the ovals must be $c^2 = \sqrt{(x-a)^2+y^2}\sqrt{(x+a)^2+y^2}$. When this equation is squared and rearranged,

$$\begin{aligned} c^4 &= [(x-a)^2+y^2][(x+a)^2+y^2] = (x^2-a^2)^2+y^2[(x-a)^2+(x+a)^2]+y^4 \\ &= x^4+y^4+a^4-2a^2x^2+2x^2y^2+2a^2y^2 = (x^2+y^2+a^2)^2-4a^2x^2. \end{aligned}$$

(b) When the equation for the ovals is differentiated with respect to x,

$$2(x^2+y^2+a^2)\left(2x+2y\frac{dy}{dx}\right) = 8a^2x \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{2a^2x - x(x^2+y^2+a^2)}{y(x^2+y^2+a^2)}.$$

The tangent line is horizontal when $dy/dx = 0$, and this requires $2a^2x = x(x^2+y^2+a^2)$. To satisfy this equation we must set $x = 0$ or $x^2+y^2 = a^2$. When $x = 0$, the equation of the ovals gives $(y^2+a^2)^2 = c^4$, from which $y = \pm\sqrt{c^2-a^2}$. Thus, two points at which the tangent line is horizontal are $(0, \pm\sqrt{c^2-a^2})$.

When $x^2+y^2 = a^2$, the equation of the ovals gives $c^4 = 4a^4-4a^2x^2$, from which $x^2 = (4a^4-c^4)/(4a^2)$. If $c \geq \sqrt{2}a$, then no more solutions are obtained. If $c < \sqrt{2}a$, then $x = \pm\sqrt{4a^4-c^4}/(2a)$, and these give the four points $\left(\dfrac{\sqrt{4a^4-c^4}}{2a}, \pm\dfrac{c^2}{2a}\right)$ and $\left(-\dfrac{\sqrt{4a^4-c^4}}{2a}, \pm\dfrac{c^2}{2a}\right)$.

55. When we differentiate the equation of the ellipse with respect to x, the result is $2b^2x + 2a^2y(dy/dx) = 0 \Longrightarrow dy/dx = -b^2x/(a^2y)$. When we differentiate the equation of the hyperbola with respect to x, the result is $2d^2x - 2c^2y(dy/dx) = 0 \Longrightarrow dy/dx = d^2x/(c^2y)$. These curves intersect orthogonally if

$$-1 = \left(-\frac{b^2x}{a^2y}\right)\left(\frac{d^2x}{c^2y}\right) \quad \Longleftrightarrow \quad y = \pm\frac{bdx}{ac}.$$

When we substitute this into the equation of the ellipse,

$$b^2x^2 + \frac{a^2b^2d^2x^2}{a^2c^2} = a^2b^2 \quad \Longleftrightarrow \quad x^2 = \frac{a^2c^2}{c^2+d^2}.$$

Substitution of $y = \pm bdx/(ac)$ into the equation of the hyperbola gives

$$d^2x^2 - \frac{c^2b^2d^2x^2}{a^2c^2} = c^2d^2 \quad \Longleftrightarrow \quad x^2 = \frac{a^2c^2}{a^2-b^2}.$$

Thus, the curves intersect orthogonally if $\dfrac{a^2c^2}{c^2+d^2} = \dfrac{a^2c^2}{a^2-b^2} \Longleftrightarrow c^2+d^2 = a^2-b^2$.

EXERCISES 3.9

1. $\dfrac{dy}{dx} = 2\cos 3x(3) = 6\cos 3x$

2. $\dfrac{dy}{dx} = -\sin x - 4\cos 5x\,(5) = -\sin x - 20\cos 5x$

3. $\dfrac{dy}{dx} = 2\sin x\cos x = \sin 2x$

4. $\dfrac{dy}{dx} = -3\tan^{-4}3x\sec^2 3x\,(3) = -\dfrac{9\sec^2 3x}{\tan^4 3x}$

5. $\dfrac{dy}{dx} = 4\sec^3 10x(\sec 10x\tan 10x)(10) = 40\sec^4 10x\tan 10x$

6. $\dfrac{dy}{dx} = -\csc(4-2x)\cot(4-2x)(-2) = 2\csc(4-2x)\cot(4-2x)$

7. $\dfrac{dy}{dx} = 2\sin(3-2x^2)\cos(3-2x^2)(-4x) = -8x\sin(3-2x^2)\cos(3-2x^2)$

8. $\dfrac{dy}{dx} = \cot x^2 - x\csc^2 x^2\,(2x) = \cot x^2 - 2x^2\csc^2 x^2$

9. $\dfrac{dy}{dx} = \dfrac{(\cos 5x)(2\cos 2x) - (\sin 2x)(-5\sin 5x)}{(\cos 5x)^2} = \dfrac{2\cos 5x\cos 2x + 5\sin 5x\sin 2x}{\cos^2 5x}$

10. $\dfrac{dy}{dx} = \dfrac{(x+1)(\sin x + x\cos x) - x\sin x\,(1)}{(x+1)^2} = \dfrac{\sin x + x(1+x)\cos x}{(x+1)^2}$

11. $\dfrac{dy}{dx} = 3\sin^2 x\cos x - \sin x$

12. Since $y = \dfrac{1}{2}\sin 4x$, it follows that $\dfrac{dy}{dx} = \dfrac{1}{2}\cos 4x\,(4) = 2\cos 4x$.

13. $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{\sin 3x}} 3\cos 3x = \dfrac{3\cos 3x}{2\sqrt{\sin 3x}}$

14. $\dfrac{dy}{dx} = \dfrac{1}{4}(1+\tan^3 x)^{-3/4}(3\tan^2 x\sec^2 x) = \dfrac{3\tan^2 x\sec^2 x}{4(1+\tan^3 x)^{3/4}}$

15. If we differentiate with respect to x, we find $2\cos y\dfrac{dy}{dx} - 3\sin x = 0 \Longrightarrow \dfrac{dy}{dx} = \dfrac{3\sin x}{2\cos y}$.

16. Differentiation with respect to x gives $\cos y - x\sin y\dfrac{dy}{dx} + y\sin x - \cos x\dfrac{dy}{dx} = 0$, and therefore $\dfrac{dy}{dx} = \dfrac{\cos y + y\sin x}{\cos x + x\sin y}$.

17. Differentiation with respect to x gives $8\sin x\cos x - 9\cos^2 y\,(-\sin y)\dfrac{dy}{dx} = 0$, and therefore $\dfrac{dy}{dx} = -\dfrac{8\sin x\cos x}{9\cos^2 y\sin y}$.

18. If we differentiate with respect to x, we find $\sec^2(x+y)\left(1+\dfrac{dy}{dx}\right) = \dfrac{dy}{dx}$, and therefore $\dfrac{dy}{dx} = \dfrac{\sec^2(x+y)}{1-\sec^2(x+y)}$.

19. If we differentiate with respect to x, we obtain $1 + \sec xy\tan xy\left(y + x\dfrac{dy}{dx}\right) = 0$, and therefore $\dfrac{dy}{dx} = -\dfrac{1 + y\sec xy\tan xy}{x\sec xy\tan xy}$.

20. Differentiation with respect to x gives $3x^2y + x^3\dfrac{dy}{dx} + 2\tan y\sec^2 y\dfrac{dy}{dx} = 3$, and therefore $\dfrac{dy}{dx} = \dfrac{3(1-x^2y)}{x^3 + 2\tan y\sec^2 y}$.

21. Differentiation with respect to x gives $1 = \left(3y^2\csc^3 y - 3y^3\csc^2 y\csc y\cot y\right)\dfrac{dy}{dx}$, from which $\dfrac{dy}{dx} = \dfrac{1}{3y^2\csc^3 y - 3y^3\csc^3 y\cot y}$.

22. $\dfrac{dy}{dx} = -\sin(\tan x)\sec^2 x = -\sec^2 x\sin(\tan x)$

23. $\dfrac{dy}{dx} = 3x^2 - 2x\cos x + x^2\sin x + 2\sin x + 2x\cos x - 2\sin x = 3x^2 + x^2\sin x$

24. Since $y = (\sin^2 x^2 + \cos^2 x^2)(\sin^2 x^2 - \cos^2 x^2) = -(\cos^2 x^2 - \sin^2 x^2) = -\cos 2x^2$, we find that $dy/dx = \sin 2x^2\,(4x) = 4x\sin 2x^2$.

25. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = [3u^2\sec u + u^3\sec u\tan u][\tan(x+1) + x\sec^2(x+1)]$

26. $\dfrac{dy}{dx} = \dfrac{dy}{dv}\dfrac{dv}{dx} = \left(\dfrac{-\sec v\tan v}{2\sqrt{3-\sec v}}\right)\left(\sec^2\sqrt{x}\dfrac{1}{2\sqrt{x}}\right) = -\dfrac{\sec v\tan v\sec^2\sqrt{x}}{4\sqrt{x}\sqrt{3-\sec v}}$.

27. $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \left(\dfrac{t}{\sqrt{t^2+1}}\right)[\cos(\sin x)\cos x]$

28.
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{1}{3}(1+\sec^3 u)^{-2/3}3\sec^2 u\sec u\tan u\right]\left[\frac{1}{2}(1+\cos x^2)^{-1/2}(-\sin x^2\,(2x))\right]$$
$$= -\frac{x\sin x^2\sec^3 u\tan u}{\sqrt{1+\cos x^2}(1+\sec^3 u)^{2/3}}$$

29. $\dfrac{dy}{dx} = \dfrac{(1+\cos^3 x)(2\sin x\cos x) - \sin^2 x(-3\cos^2 x\sin x)}{(1+\cos^3 x)^2} = \dfrac{\sin x\cos x(2 + 2\cos^3 x + 3\sin^2 x\cos x)}{(1+\cos^3 x)^2}$

30. $\dfrac{dy}{dx} = \dfrac{x^2\sin x[3\tan^2(3x^2-4)\sec^2(3x^2-4)(6x)] - [1+\tan^3(3x^2-4)][2x\sin x + x^2\cos x]}{x^4\sin^2 x}$

$= \dfrac{18x^2\tan^2(3x^2-4)\sec^2(3x^2-4) - (2 + x\cot x)[1+\tan^3(3x^2-4)]}{x^3\sin x}$

31. If we differentiate with respect to x, we obtain $\cos y\dfrac{dy}{dx} = 2x + \dfrac{dy}{dx} \Longrightarrow \dfrac{dy}{dx} = \dfrac{2x}{\cos y - 1}$. A second differentiation gives

$$\frac{d^2y}{dx^2} = \frac{(\cos y - 1)(2) - 2x\left(-\sin y\dfrac{dy}{dx}\right)}{(\cos y - 1)^2} = \frac{2(\cos y - 1) + 2x\sin y\left(\dfrac{2x}{\cos y - 1}\right)}{(\cos y - 1)^2} = \frac{2(\cos y - 1)^2 + 4x^2\sin y}{(\cos y - 1)^3}.$$

32. If we differentiate with respect to x, we obtain $\sec^2 y\dfrac{dy}{dx} = 1 + x\dfrac{dy}{dx} + y \Longrightarrow \dfrac{dy}{dx} = \dfrac{1+y}{\sec^2 y - x}$. A second differentiation now gives

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{(\sec^2 y - x)(dy/dx) - (1+y)[2\sec^2 y\tan y\,(dy/dx) - 1]}{(\sec^2 y - x)^2}\\ &= \frac{\left[\sec^2 y - x - 2(1+y)\sec^2 y\tan y\right]\left(\dfrac{1+y}{\sec^2 y - x}\right) + (1+y)}{(\sec^2 y - x)^2}\\ &= \frac{(1+y)[(\sec^2 y - x) - 2(1+y)\sec^2 y\tan y + (\sec^2 y - x)]}{(\sec^2 y - x)^3}\\ &= \frac{2(1+y)[\sec^2 y - x - (1+y)\sec^2 y\tan y]}{(\sec^2 y - x)^3}.\end{aligned}$$

33. $\displaystyle\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{\sin x}{x}\frac{1}{\cos x} = (1)(1) = 1$

34. $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{1 - [1 - 2\sin^2(x/2)]}{x} = \lim_{x\to 0}\left[\frac{\sin(x/2)}{x/2}\sin(x/2)\right] = (1)(0) = 0$

35. $\displaystyle\lim_{x\to 0}\frac{\sin 2x}{x} = \lim_{x\to 0}\frac{2\sin x\cos x}{x} = 2\lim_{x\to 0}\left[\left(\frac{\sin x}{x}\right)\cos x\right] = 2(1)(1) = 2$

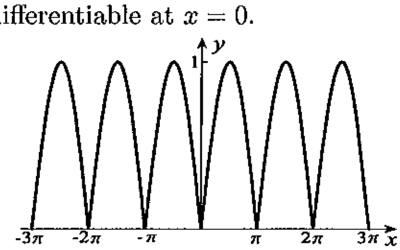
36. $\displaystyle\lim_{x\to\infty}\frac{\sin(2/x)}{\sin(1/x)} = \lim_{x\to\infty}\frac{2\sin(1/x)\cos(1/x)}{\sin(1/x)} = \lim_{x\to\infty}[2\cos(1/x)] = 2$

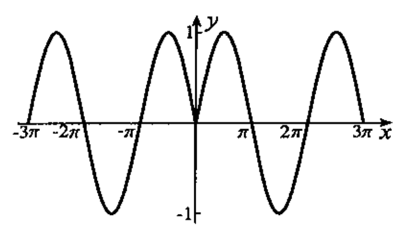
37. $\displaystyle\lim_{x\to 0}\frac{(x+1)^2\sin x}{3x^3} = \lim_{x\to 0}\frac{(x+1)^2}{3x^2}\frac{\sin x}{x}$. Since $\displaystyle\lim_{x\to 0}\frac{(x+1)^2}{3x^2} = \infty$ and $\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1$, it follows that $\displaystyle\lim_{x\to 0}\frac{(x+1)^2\sin x}{3x^3} = \infty$.

38. $\displaystyle\lim_{x\to\pi/2}\frac{\cos x}{(x-\pi/2)^2} = \lim_{x\to\pi/2}\frac{\sin(\pi/2 - x)}{(x-\pi/2)^2} = \lim_{x\to\pi/2}\left[\frac{\sin(\pi/2 - x)}{\pi/2 - x}\frac{1}{\pi/2 - x}\right]$

Since $\displaystyle\lim_{x\to\pi/2}\frac{\sin(\pi/2 - x)}{\pi/2 - x} = 1$ and $\displaystyle\lim_{x\to\pi/2}\frac{1}{\pi/2 - x}$ does not exist, the original limit does not exist.

39. Graphs are shown below. $|\sin x|$ is not differentiable at $x = n\pi$, where n is an integer; whereas $\sin|x|$ is not differentiable at $x = 0$.





40. For $x \neq 0$, $g'(x) = \sin\left(\frac{1}{x}\right) + x\cos\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right) = \sin\left(\frac{1}{x}\right) - \frac{1}{x}\cos\left(\frac{1}{x}\right)$. The limit of this function as x approaches 0 does not exist.

41. Since $d\theta/dt = -\omega A\,\sin(\omega t + \phi)$, it follows that

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = -\omega^2 A\,\cos(\omega t + \phi) + \omega^2[A\,\cos(\omega t + \phi)] = 0.$$

42. Since $\frac{dy}{dt} = A\cos\left(\sqrt{\frac{k}{m}}t\right)\sqrt{\frac{k}{m}} - B\sin\left(\sqrt{\frac{k}{m}}t\right)\sqrt{\frac{k}{m}} = \sqrt{\frac{k}{m}}\left[A\cos\left(\sqrt{\frac{k}{m}}t\right) - B\sin\left(\sqrt{\frac{k}{m}}t\right)\right]$, a second differentiation gives

$$\frac{d^2y}{dt^2} = \sqrt{\frac{k}{m}}\left[-A\sin\left(\sqrt{\frac{k}{m}}t\right)\sqrt{\frac{k}{m}} - B\cos\left(\sqrt{\frac{k}{m}}t\right)\sqrt{\frac{k}{m}}\right] = -\frac{k}{m}\left[A\sin\left(\sqrt{\frac{k}{m}}t\right) + B\cos\left(\sqrt{\frac{k}{m}}t\right)\right].$$

Hence,

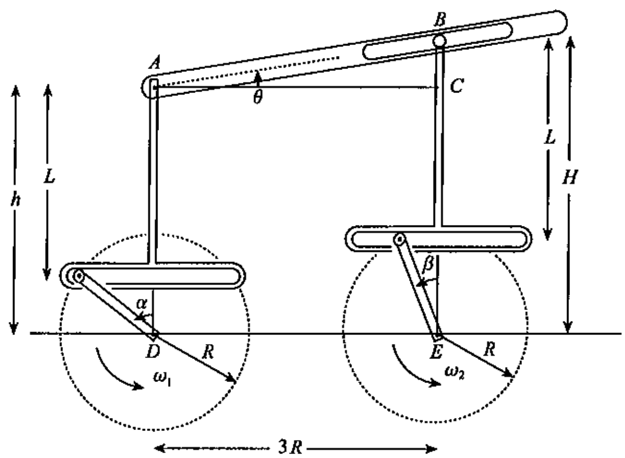
$$m\frac{d^2y}{dt^2} + ky = -k\left[A\sin\left(\sqrt{\frac{k}{m}}t\right) + B\cos\left(\sqrt{\frac{k}{m}}t\right)\right] + k\left[A\sin\left(\sqrt{\frac{k}{m}}t\right) + B\cos\left(\sqrt{\frac{k}{m}}t\right)\right] = 0.$$

43. From triangle ABC, we can write that $\|BC\| = 3R\tan\theta$. But

$$\begin{aligned}\|BC\| &= \|BE\| - \|CE\| = H - h\\ &= (L + R\cos\beta) - (L + R\cos\alpha)\\ &= R(\cos\beta - \cos\alpha).\end{aligned}$$

Hence, $R(\cos\beta - \cos\alpha) = 3R\tan\theta$. Division by R and differentiation with respect to t gives

$$-\sin\beta\frac{d\beta}{dt} + \sin\alpha\frac{d\alpha}{dt} = 3\sec^2\theta\frac{d\theta}{dt}$$
$$\Longrightarrow$$
$$\frac{d\theta}{dt} = \frac{1}{3}\cos^2\theta(\omega_1\sin\alpha - \omega_2\sin\beta).$$

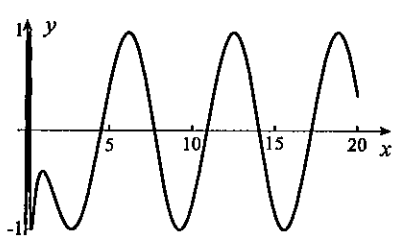


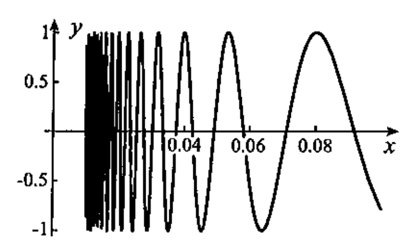
Since $\tan\theta = (\cos\beta - \cos\alpha)/3$, it follows that

$$\cos^2\theta = \frac{1}{\sec^2\theta} = \frac{1}{1 + \tan^2\theta} = \frac{1}{1 + (\cos\beta - \cos\alpha)^2/9}.$$

Thus, $\frac{d\theta}{dt} = \frac{1}{3}\left[\frac{9}{9 + (\cos\beta - \cos\alpha)^2}\right](\omega_1\sin\alpha - \omega_2\sin\beta) = \frac{3(\omega_1\sin\alpha - \omega_2\sin\beta)}{9 + (\cos\beta - \cos\alpha)^2}$.

44. To get an idea of how many values of x satisfy $f'(x) = 0$ and approximations to them, we plot a graph of $f(x)$. The plot on the interval $0.1 \leq x \leq 20$ in the left figure below suggests regular behaviour of the function for large values of x, but wild oscillations as $x \to 0$. This is consistent with the fact that for large x, the term $1/x$ becomes less and less significant, and $f(x)$ can be approximated by $\cos x$. On the other hand, when x is close to 0, $f(x)$ should behave much like $\sin(1/x)$ in Figure 2.8b. This is illustrated in the right figure below where we have plotted the function on the interval $0.01 \leq x \leq 0.1$.





With these ideas in mind, we now solve $0 = f'(x) = -\left(1 - \dfrac{1}{x^2}\right)\sin\left(x + \dfrac{1}{x}\right)$. There are two possibilities:

$$1 - \frac{1}{x^2} = 0 \qquad \text{or} \qquad \sin\left(x + \frac{1}{x}\right) = 0.$$

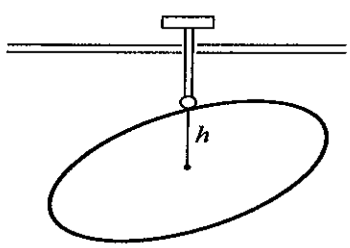
The first gives $x = \pm 1$, and it is clear that the tangent line at $x = 1$ in the left figure is indeed horizontal. The second equation above implies that $x + \dfrac{1}{x} = n\pi$, where n is an integer. Multiplication by x leads to the quadratic equation

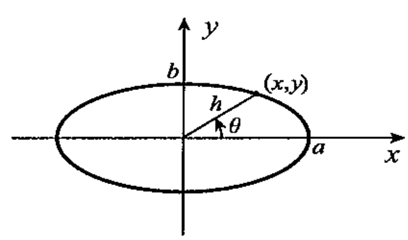
$$x^2 - n\pi x + 1 = 0 \quad \Longrightarrow \quad x = \frac{n\pi \pm \sqrt{n^2\pi^2 - 4}}{2}.$$

Clearly, we must choose $n > 0$ (else $n^2\pi^2 - 4 < 0$ when $n = 0$, and $x < 0$ when $n < 0$). When $n = 1$, the solution is $x = (\pi + \sqrt{\pi^2 - 4})/2 \approx 2.8$. This is the first point to the right of $x = 1$ at which the tangent line is horizontal in the left figure. As n increases, the 4 becomes less and less significant and $x \approx (n\pi \pm n\pi)/2$. When we choose the positive sign, we obtain $x \approx n\pi$, and we can indeed see that points on the graph in the left figure where the tangent line is horizontal do indeed seem to be multiples of π. When we choose the negative sign, we obtain points to the left of $x = 1$ at which the tangent line is horizontal (right figure). For large n, they can be approximated by $1/(n\pi)$, but this is not an easy fact to show.

45. The velocity of the follower is the rate of change of the length h from the end of the follower on the cam to the centre of the ellipse (left figure below). We can calculate the rate of change of h by fixing the ellipse (right figure) and letting the line joining the origin to a point $P(x, y)$ on the ellipse rotate at 600 rpm counterclockwise around the ellipse. The x- and y-coordinates of P can be expressed in terms of h and θ by $x = h\cos\theta$ and $y = h\sin\theta$. We substitute these into the equation $b^2x^2 + a^2y^2 = a^2b^2$ of the ellipse,

$$b^2h^2\cos^2\theta + a^2h^2\sin^2\theta = a^2b^2 \qquad \Longrightarrow \qquad h^2 = \frac{a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta}.$$





Differentiation with respect to t gives

$$2h\frac{dh}{dt} = \frac{-a^2b^2(-2b^2\cos\theta\sin\theta + 2a^2\sin\theta\cos\theta)}{(b^2\cos^2\theta + a^2\sin^2\theta)^2}\frac{d\theta}{dt} \quad \Longrightarrow \quad \frac{dh}{dt} = -\frac{a^2b^2(a^2 - b^2)\sin 2\theta}{2h(b^2\cos^2\theta + a^2\sin^2\theta)^2}\frac{d\theta}{dt}.$$

Since the cam rotates at 600 rpm, $\dfrac{d\theta}{dt} = \dfrac{2\pi(600)}{60} = 20\pi$ radians per second. Thus,

$$\frac{dh}{dt} = -\frac{a^2b^2(a^2 - b^2)\sin 2\theta}{\dfrac{2ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}(b^2\cos^2\theta + a^2\sin^2\theta)^2}(20\pi) = \frac{10\pi ab(b^2 - a^2)\sin 2\theta}{(b^2\cos^2\theta + a^2\sin^2\theta)^{3/2}}.$$

46. According to the extended power rule and formula 3.13,

$$\frac{d}{dx}|\sin x|^n = n|\sin x|^{n-1}\frac{|\sin x|}{\sin x}\cos x$$

valid whenever $x \neq n\pi$. Graphs of $|\sin x|^n$ quickly show that its derivative is zero when $x = n\pi$. Now,

$|\sin x|/\sin x$ is equal to 1 when $\sin x > 0$ and equal to -1 when $\sin x < 0$. It is undefined when $x = n\pi$. On the other hand, when $\operatorname{sgn}(x)$ is the signum function of Exercise 47 in Section 2.4,

$$\operatorname{sgn}(\sin x) = \begin{cases} -1, & \sin x < 0 \\ 0, & \sin x = 0 \\ 1, & \sin x > 0. \end{cases}$$

It follows that we can express the derivative as

$$\frac{d}{dx}|\sin x|^n = n|\sin x|^{n-1}\cos x \operatorname{sgn}(\sin x),$$

and this is valid for all x.

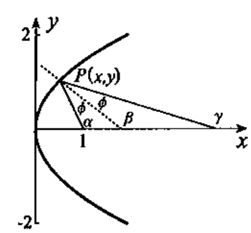
47. According to equation 3.2, $f'(0) = \lim_{h\to 0} \dfrac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \dfrac{h^2 \sin(1/h) - 0}{h} = \lim_{h\to 0} h\sin(1/h) = 0$ (see Example 2.9 in Section 2.1). On the other hand, when $x \neq 0$,

$$f'(x) = 2x\sin\left(\frac{1}{x}\right) + x^2\cos\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right).$$

Since this derivative does not have a limit as $x \to 0$, it follows that $f'(x)$ is not continuous at $x = 0$.

48. Suppose a ray of light from the bulb strikes the reflector at $P(x, y)$ making an angle ϕ with the normal to the reflector at P. Because $\gamma = 2\phi + \alpha$ and $\beta = \alpha + \phi$, it follows that $\gamma = 2\beta - \alpha$, and hence

$$\tan\gamma = \frac{\tan 2\beta - \tan\alpha}{1 + \tan 2\beta\, \tan\alpha} = \frac{\dfrac{2\tan\beta}{1 - \tan^2\beta} - \tan\alpha}{1 + \tan 2\beta\, \tan\alpha}.$$



Now the slope dy/dx of the reflector is given by $2y(dy/dx) = 4$, from which $dy/dx = 2/y$, provided $y \neq 0$. Consequently, $\tan\beta = -y/2$ (the slope of the normal line), and

$$\frac{2\tan\beta}{1 - \tan^2\beta} - \tan\alpha = \frac{2(-y/2)}{1 - (-y/2)^2} - \frac{y}{x - 1} = \frac{-y}{1 - y^2/4} - \frac{y}{y^2/4 - 1} = 0.$$

In other words, $\tan\gamma = 0$ which implies that either $\gamma = 0$ or π; that is, rays are reflected parallel to the x-axis. The only ray still in question is that for which $y = 0$. In this case the ray is along the x-axis, and is reflected back along the x-axis.

EXERCISES 3.10

1. $\dfrac{dy}{dx} = \dfrac{-1}{\sqrt{1 - (2x+3)^2}}(2) = \dfrac{-2}{\sqrt{1 - (2x+3)^2}}$

2. $\dfrac{dy}{dx} = \dfrac{-1}{1 + (x^2+2)^2}(2x) = \dfrac{-2x}{1 + (x^2+2)^2}$

3. $\dfrac{dy}{dx} = \dfrac{-1}{(3-4x)\sqrt{(3-4x)^2 - 1}}(-4) = \dfrac{4}{(3-4x)\sqrt{(3-4x)^2 - 1}}$

4. $\dfrac{dy}{dx} = \dfrac{1}{1 + (2-x^2)^2}(-2x) = \dfrac{-2x}{1 + (2-x^2)^2}$

5. $\dfrac{dy}{dx} = \dfrac{1}{(3-2x^2)\sqrt{(3-2x^2)^2 - 1}}(-4x) = \dfrac{-4x}{(3-2x^2)\sqrt{(3-2x^2)^2 - 1}}$

6. $\dfrac{dy}{dx} = \operatorname{Csc}^{-1}(x^2+5) - \dfrac{x}{(x^2+5)\sqrt{(x^2+5)^2 - 1}}(2x) = \operatorname{Csc}^{-1}(x^2+5) - \dfrac{2x^2}{(x^2+5)\sqrt{(x^2+5)^2 - 1}}$

7. $\dfrac{dy}{dx} = 2x\operatorname{Sin}^{-1}(2x) + \dfrac{(x^2+2)}{\sqrt{1 - (2x)^2}}(2) = 2x\operatorname{Sin}^{-1}(2x) + \dfrac{2(x^2+2)}{\sqrt{1 - 4x^2}}$

8. $\dfrac{dy}{dx} = \dfrac{1}{1+(x+2)}\dfrac{1}{2\sqrt{x+2}} = \dfrac{1}{2\sqrt{x+2}(x+3)}$

9. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-(1-x^2)}}\dfrac{-x}{\sqrt{1-x^2}} = \dfrac{-x}{|x|\sqrt{1-x^2}}$

10. $\dfrac{dy}{dx} = \dfrac{-1}{1+x^2-1}\dfrac{x}{\sqrt{x^2-1}} = \dfrac{-1}{x\sqrt{x^2-1}}$

11. $\dfrac{dy}{dx} = 2\,\mathrm{Tan}^{-1}(x^2)\,\dfrac{2x}{1+x^4} = \dfrac{4x\,\mathrm{Tan}^{-1}(x^2)}{1+x^4}$

12. $\dfrac{dy}{dx} = 2x\,\mathrm{Sec}^{-1}x + \dfrac{x^2}{x\sqrt{x^2-1}} = 2x\,\mathrm{Sec}^{-1}x + \dfrac{x}{\sqrt{x^2-1}}$

13. $\dfrac{dy}{dx} = \sec^2(3\,\mathrm{Sin}^{-1}x)\,\dfrac{3}{\sqrt{1-x^2}} = \dfrac{3\sec^2(3\,\mathrm{Sin}^{-1}x)}{\sqrt{1-x^2}}$

14. $\dfrac{dy}{dx} = \dfrac{-1}{1+\left(\dfrac{1+x}{1-x}\right)^2}\left[\dfrac{(1-x)(1)-(1+x)(-1)}{(1-x)^2}\right] = \dfrac{-(1-x)^2}{(1-x)^2+(1+x)^2}\left[\dfrac{2}{(1-x)^2}\right] = \dfrac{-2}{2+2x^2} = \dfrac{-1}{1+x^2}$

15. $\dfrac{dy}{dx} = \dfrac{-1}{\dfrac{1}{x}\sqrt{\dfrac{1}{x^2}-1}}\left(\dfrac{-1}{x^2}\right) = \dfrac{1}{x\sqrt{\dfrac{1-x^2}{x^2}}} = \dfrac{|x|}{x\sqrt{1-x^2}}$

16. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-\left(\dfrac{1-x}{1+x}\right)^2}}\left[\dfrac{(1+x)(-1)-(1-x)(1)}{(1+x)^2}\right] = \dfrac{|x+1|}{\sqrt{(1+x)^2-(1-x)^2}}\left[\dfrac{-2}{(x+1)^2}\right]$ Since x must be greater than 0 in order that $-1 \le \dfrac{1-x}{1+x} \le 1$, it follows that $\dfrac{dy}{dx} = \dfrac{(x+1)(-2)}{2\sqrt{x}(x+1)^2} = \dfrac{-1}{\sqrt{x}(x+1)}$.

17. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \left[\dfrac{1}{1+(u^2+1/u)^2}\left(2u-\dfrac{1}{u^2}\right)\right][\sec^2(x^2+4)(2x)] = \dfrac{2x(2u^3-1)\sec^2(x^2+4)}{u^6+2u^3+u^2+1}$

18. $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \left(\mathrm{Cos}^{-1}t - \dfrac{t}{\sqrt{1-t^2}}\right)\left(\dfrac{-x}{\sqrt{1-x^2}}\right) = \dfrac{x(t-\sqrt{1-t^2}\,\mathrm{Cos}^{-1}t)}{\sqrt{1-t^2}\sqrt{1-x^2}}$

19. If we differentiate with respect to x, we obtain $y^2\cos x + 2y\dfrac{dy}{dx}\sin x + \dfrac{dy}{dx} = \dfrac{1}{1+x^2}$, from which

$$\frac{dy}{dx} = \frac{\dfrac{1}{1+x^2} - y^2\cos x}{1+2y\sin x}.$$

20. If we differentiate with respect to x, we obtain $\dfrac{1}{\sqrt{1-x^2y^2}}\left(x\dfrac{dy}{dx}+y\right) = 5 + 2\dfrac{dy}{dx}$, and therefore

$$\frac{dy}{dx} = \frac{5 - \dfrac{y}{\sqrt{1-x^2y^2}}}{\dfrac{x}{\sqrt{1-x^2y^2}} - 2} = \frac{5\sqrt{1-x^2y^2}-y}{x-2\sqrt{1-x^2y^2}}.$$

21. $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2-1}} - \dfrac{1}{x\sqrt{x^2-1}} = \dfrac{x^2-1}{x\sqrt{x^2-1}} = \dfrac{\sqrt{x^2-1}}{x}$

22. $\dfrac{dy}{dx} = \dfrac{-1/3}{\dfrac{x}{3}\sqrt{\dfrac{x^2}{9}-1}}\left(\dfrac{1}{3}\right) - \dfrac{x^2(1/2)(x^2-9)^{-1/2}(2x) - \sqrt{x^2-9}(2x)}{x^4}$

$= \dfrac{-1}{x\sqrt{x^2-9}} - \dfrac{x^2-2(x^2-9)}{x^3\sqrt{x^2-9}} = \dfrac{-x^2-(-x^2+18)}{x^3\sqrt{x^2-9}} = \dfrac{-18}{x^3\sqrt{x^2-9}}$

23. $\dfrac{dy}{dx} = \mathrm{Cos}^{-1}(x/2) - \dfrac{x/2}{\sqrt{1-x^2/4}} + \dfrac{x}{\sqrt{4-x^2}} = \mathrm{Cos}^{-1}(x/2)$

24. $\dfrac{dy}{dx} = -\dfrac{1}{x^2}\mathrm{Csc}^{-1}(3x) + \dfrac{1}{x}\dfrac{(-1)(3)}{3x\sqrt{9x^2-1}} + \dfrac{\sqrt{9x^2-1}}{x^2} - \dfrac{9x}{x\sqrt{9x^2-1}}$

$$= -\frac{1}{x^2}\mathrm{Csc}^{-1}(3x) + \frac{-1+(9x^2-1)-9x^2}{x^2\sqrt{9x^2-1}} = -\frac{1}{x^2}\mathrm{Csc}^{-1}(3x) - \frac{2}{x^2\sqrt{9x^2-1}}$$

25. $\dfrac{dy}{dx} = 2x\,\mathrm{Sec}^{-1}x + \dfrac{x^2}{x\sqrt{x^2-1}} - \dfrac{x}{\sqrt{x^2-1}} = 2x\,\mathrm{Sec}^{-1}x$

26. $\dfrac{dy}{dx} = (\mathrm{Cos}^{-1}x)^2 - \dfrac{2x\,\mathrm{Cos}^{-1}x}{\sqrt{1-x^2}} - 2 + \dfrac{2x}{\sqrt{1-x^2}} = (\mathrm{Cos}^{-1}x)^2 - 2 + \dfrac{2x}{\sqrt{1-x^2}}(1-\mathrm{Cos}^{-1}x)$

27. $\dfrac{dy}{dx} = 18x\,\mathrm{Cot}^{-1}(3x) - \dfrac{(1+9x^2)(3)}{1+9x^2} + 3 = 18x\,\mathrm{Cot}^{-1}(3x)$

28. $\dfrac{dy}{dx} = \sqrt{4x-x^2} + \dfrac{x-2}{2\sqrt{4x-x^2}}(4-2x) + \dfrac{4}{\sqrt{1-\left(\dfrac{x-2}{2}\right)^2}}\left(\dfrac{1}{2}\right)$

$$= \frac{(4x-x^2)+(x-2)(2-x)}{\sqrt{4x-x^2}} + \frac{4}{\sqrt{4x-x^2}} = \frac{4x-x^2-x^2+4x-4+4}{\sqrt{4x-x^2}} = 2\sqrt{4x-x^2}$$

29. $\dfrac{dy}{dx} = 4x\,\mathrm{Sin}^{-1}x + \dfrac{2x^2-1}{\sqrt{1-x^2}} + \sqrt{1-x^2} - \dfrac{x^2}{\sqrt{1-x^2}} = 4x\,\mathrm{Sin}^{-1}x$

30. $\dfrac{dy}{dx} = \dfrac{1}{1+\dfrac{2x^2}{1+x^4}}\left[\dfrac{\sqrt{2}}{\sqrt{1+x^4}} - \dfrac{\sqrt{2}x}{2(1+x^4)^{3/2}}(4x^3)\right]$

$$= \frac{1+x^4}{1+x^4+2x^2}\left[\frac{\sqrt{2}(1+x^4)-2\sqrt{2}x^4}{(1+x^4)^{3/2}}\right] = \frac{\sqrt{2}(1-x^4)}{(1+x^2)^2\sqrt{1+x^4}} = \frac{\sqrt{2}(1-x^2)}{(1+x^2)\sqrt{1+x^4}}$$

31. $\dfrac{dy}{dx} = \dfrac{1}{x\sqrt{x^2-1}} - \dfrac{1}{1+x^2-1}\dfrac{x}{\sqrt{x^2-1}} = 0$ Because the derivative always vanishes, the function must be equal to a constant. When $x \geq 1$, we set $\mathrm{Sec}^{-1}x + \mathrm{Cot}^{-1}\sqrt{x^2-1} = C$. Substituting $x = 1$ gives $0+\pi/2 = C$. Thus, $\mathrm{Sec}^{-1}x + \mathrm{Cot}^{-1}\sqrt{x^2-1} = \pi/2$ when $x \geq 1$. When we set $\mathrm{Sec}^{-1}x + \mathrm{Cot}^{-1}\sqrt{x^2-1} = D$ for $x \leq -1$, and substitute $x = -1$, we obtain $-\pi + \pi/2 = D$. Thus, $\mathrm{Sec}^{-1}x + \mathrm{Cot}^{-1}\sqrt{x^2-1} = -\pi/2$ when $x \leq -1$.

32. (a) If we set $y = \mathrm{Sec}^{-1}x$, then $x = \sec y$, and implicit differentiation gives

$$1 = \sec y\,\tan y\,\frac{dy}{dx} \quad\Longrightarrow\quad \frac{dy}{dx} = \frac{1}{\sec y\,\tan y}.$$

Now, $\tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2-1}$. If $x > 0$, then $0 < y < \pi/2$, and $\tan y = \sqrt{x^2-1}$. If $x < 0$, and principal values are chosen as stated, then $\pi/2 < y < \pi$, and $\tan y = -\sqrt{x^2-1}$. Thus,

$$\frac{dy}{dx} = \begin{cases} \dfrac{1}{x\sqrt{x^2-1}}, & x > 0 \\ \dfrac{-1}{x\sqrt{x^2-1}}, & x < 0 \end{cases} = \frac{1}{|x|\sqrt{x^2-1}}.$$

(b) A similar analysis gives $\dfrac{d}{dx}\mathrm{Csc}^{-1}x = \dfrac{-1}{|x|\sqrt{x^2-1}}$.

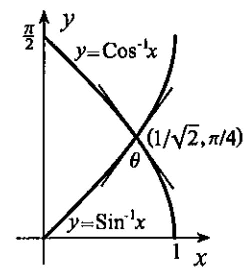
33. The slope of $y = f(x) = \mathrm{Sin}^{-1}x$ at $x = 1/\sqrt{2}$ is $f'(1/\sqrt{2}) = \dfrac{1}{\sqrt{1-(1/\sqrt{2})^2}} = \sqrt{2}$.

The slope of $y = g(x) = \mathrm{Cos}^{-1}x$ at $x = 1/\sqrt{2}$ is

$$g'(1/\sqrt{2}) = \frac{-1}{\sqrt{1-(1/\sqrt{2})^2}} = -\sqrt{2}.$$

According to equation 1.60, the angle θ between the tangent lines is

$$\theta = \mathrm{Tan}^{-1}\left|\frac{\sqrt{2}+\sqrt{2}}{1+(\sqrt{2})(-\sqrt{2})}\right| = 1.23 \text{ radians.}$$



34. To verify 3.37c, we set $y = \mathrm{Tan}^{-1}x$, in which case $x = \tan y$. Differentiation gives $1 = \sec^2 y\,(dy/dx) \Longrightarrow dy/dx = 1/\sec^2 y = 1/(1+\tan^2 y) = 1/(1+x^2)$. Verification of 3.37d is similar. To verify 3.37e, we set $y = \mathrm{Sec}^{-1}x$, in which case $x = \sec y$. Differentiation gives $1 = \sec y \tan y\,(dy/dx) \Longrightarrow dy/dx = 1/(\sec y \tan y)$. Now $\tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2-1}$. When $x > 0$, we obtain $0 < y < \pi/2$, so that $\tan y > 0$. On the other hand, when $x < 0$, principal values give $-\pi < y < -\pi/2$, but $\tan y$ is again positive. Thus, in either case, $\tan y = \sqrt{x^2-1}$, and $dy/dx = 1/(x\sqrt{x^2-1})$. Verification of 3.37f is similar.

35. When the crank rotates at 300 rpm, $\dfrac{d\phi}{dt} = \dfrac{2\pi(300)}{60} = 10\pi$ radians per second. Differentiation of

$$\theta = \mathrm{Tan}^{-1}\left(\frac{R\sin\phi}{L+R\cos\phi}\right),$$

with respect to t gives

$$\frac{d\theta}{dt} = \frac{1}{1+\dfrac{R^2\sin^2\phi}{(L+R\cos\phi)^2}}\left[\frac{(L+R\cos\phi)R\cos\phi\dfrac{d\phi}{dt} - R\sin\phi\left(-R\sin\phi\dfrac{d\phi}{dt}\right)}{(L+R\cos\phi)^2}\right]$$

$$= \frac{(L+R\cos\phi)^2}{(L+R\cos\phi)^2+R^2\sin^2\phi}\left[\frac{R^2+LR\cos\phi}{(L+R\cos\phi)^2}\right](10\pi) = \frac{10\pi(R^2+LR\cos\phi)}{R^2+L^2+2LR\cos\phi}.$$

EXERCISES 3.11

1. $\dfrac{dy}{dx} = 3^{2x}(2)(\ln 3)$

2. $\dfrac{dy}{dx} = \dfrac{1}{3x^2+1}(6x) = \dfrac{6x}{3x^2+1}$

3. $\dfrac{dy}{dx} = \dfrac{1}{2x+1}(2)(\log_{10} e) = \dfrac{2\log_{10} e}{2x+1}$

4. $\dfrac{dy}{dx} = e^{1-2x}(-2) = -2e^{1-2x}$

5. $\dfrac{dy}{dx} = e^{2x} + xe^{2x}(2) = (2x+1)e^{2x}$

6. $\dfrac{dy}{dx} = \ln x + x\left(\dfrac{1}{x}\right) = 1 + \ln x$

7. Since $y = x^2$, we have $\dfrac{dy}{dx} = 2x$.

8. $\dfrac{dy}{dx} = \dfrac{1}{3-4x}(-4)\log_{10} e = \dfrac{-4\log_{10} e}{3-4x}$

9. $\dfrac{dy}{dx} = \dfrac{1}{\sin x}(\cos x) = \cot x$

10. $\dfrac{dy}{dx} = \dfrac{1}{3\cos x}(-3\sin x) = -\tan x$

11. $\dfrac{dy}{dx} = \ln(x+1) + \dfrac{x}{x+1}$

12. $\dfrac{dy}{dx} = 2x + 3x^2e^{4x} + x^3e^{4x}(4) = 2x + (3x^2+4x^3)e^{4x}$

13. $\dfrac{dy}{dx} = \dfrac{(1-x)e^{1-x}(-1) - e^{1-x}(-1)}{(1-x)^2} = \dfrac{xe^{1-x}}{(1-x)^2}$

14. $\dfrac{dy}{dx} = \cos(e^{2x})(2e^{2x}) = 2e^{2x}\cos(e^{2x})$

15. $\dfrac{dy}{dx} = \dfrac{1}{\ln x}\left(\dfrac{1}{x}\right) = \dfrac{1}{x\ln x}$

16. $\dfrac{dy}{dx} = -2e^{-2x}\sin 3x + e^{-2x}\cos 3x\,(3) = e^{-2x}(3\cos 3x - 2\sin 3x)$

17. $\dfrac{dy}{dx} = \dfrac{1}{x^2e^{4x}}(2xe^{4x} + 4x^2e^{4x}) = \dfrac{2}{x} + 4$

18. $\dfrac{dy}{dx} = \dfrac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2} = \dfrac{4}{(e^x + e^{-x})^2}$

19. $\dfrac{dy}{dx} = -e^{-x}\ln x + e^{-x}(1/x) = e^{-x}(1/x - \ln x)$

20. $\dfrac{dy}{dx} = \sin(\ln x) - \cos(\ln x) + x\left[\dfrac{1}{x}\cos(\ln x) + \dfrac{1}{x}\sin(\ln x)\right] = 2\sin(\ln x)$

21. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = e^{\sin u}(\cos u)[e^{1/x}(-1/x^2)] = -\dfrac{u\,e^{\sin u}\cos u}{x^2}$

22. $\dfrac{dy}{dx} = \dfrac{dy}{dv}\dfrac{dv}{dx} = \dfrac{1}{\cos v}(-\sin v)\,2\sin x\,\cos x = -\tan v\,\sin 2x$

23. If we differentiate with respect to x, we obtain $\dfrac{1}{x+y}\left(1 + \dfrac{dy}{dx}\right) = 2xy + x^2\dfrac{dy}{dx}$, from which

$$\frac{dy}{dx} = \frac{2xy - \dfrac{1}{x+y}}{\dfrac{1}{x+y} - x^2} = \frac{2x^2y + 2xy^2 - 1}{1 - x^3 - x^2y}.$$

24. Differentiation with respect to x gives $e^y + xe^y\dfrac{dy}{dx} + 2x\ln y + \dfrac{x^2}{y}\dfrac{dy}{dx} + \sin x\dfrac{dy}{dx} + y\cos x = 0$, and therefore $\dfrac{dy}{dx} = -\dfrac{y\cos x + 2x\ln y + e^y}{\sin x + x^2/y + xe^y}$.

25. $\dfrac{dy}{dx} = \dfrac{1}{\sec x + \tan x}(\sec x\,\tan x + \sec^2 x) = \sec x$

26.
$$\begin{aligned}\frac{dy}{dx} &= \sqrt{x^2+1} + \frac{x^2}{\sqrt{x^2+1}} - \frac{1}{x + \sqrt{x^2+1}}\left(1 + \frac{x}{\sqrt{x^2+1}}\right)\\ &= \frac{x^2 + 1 + x^2}{\sqrt{x^2+1}} - \frac{1}{x + \sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1}}\right) = \frac{2x^2}{\sqrt{x^2+1}}\end{aligned}$$

27. $\dfrac{dy}{dx} = \dfrac{1}{x + 4 + \sqrt{8x + x^2}}\left(1 + \dfrac{8 + 2x}{2\sqrt{8x + x^2}}\right) = \dfrac{1}{\sqrt{8x + x^2}}$

28. $\dfrac{dy}{dx} = 1 - \dfrac{1}{4(1 + 5e^{4x})}(5e^{4x})(4) = \dfrac{1 + 5e^{4x} - 5e^{4x}}{1 + 5e^{4x}} = \dfrac{1}{1 + 5e^{4x}}$

29. If we differentiate with respect to x, we obtain $e^{xy}\left(y + x\dfrac{dy}{dx}\right) = 2(x+y)\left(1 + \dfrac{dy}{dx}\right)$, from which $\dfrac{dy}{dx} = \dfrac{2x + 2y - y\,e^{xy}}{x\,e^{xy} - 2x - 2y}$.

30. If we differentiate with respect to x, we obtain $e^{1/x}\left(-\dfrac{1}{x^2}\right) + e^{1/y}\left(-\dfrac{1}{y^2}\right)\dfrac{dy}{dx} = -\dfrac{1}{x^2} - \dfrac{1}{y^2}\dfrac{dy}{dx}$, and therefore $\dfrac{dy}{dx} = \dfrac{e^{1/x}(1/x^2) - 1/x^2}{1/y^2 + e^{1/y}(-1/y^2)} = \dfrac{e^{1/x} - 1}{x^2}\dfrac{y^2}{1 - e^{1/y}} = \dfrac{y^2(e^{1/x} - 1)}{x^2(1 - e^{1/y})}$.

31. (a) Since $dT/dr = c/r$, it follows that $\dfrac{d}{dr}\left(r\dfrac{dT}{dr}\right) = \dfrac{d}{dr}(c) = 0$.

(b) If temperatures on inner and outer edges of the cylinders are T_a and T_b, then

$$T_a = c\ln a + d, \qquad T_b = c\ln b + d.$$

These can be solved for $c = \dfrac{T_b - T_a}{\ln(b/a)}$ and $d = \dfrac{T_a\ln b - T_b\ln a}{\ln(b/a)}$.

32. (a) Since $dT/dr = k + c/r$, it follows that $\dfrac{d}{dr}\left(r\dfrac{dT}{dr}\right) = \dfrac{d}{dr}(kr + c) = k$.

(b) If temperatures on inner and outer edges of the insulation are T_a and T_b, then

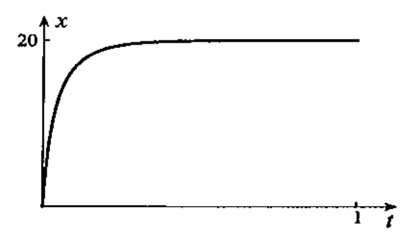
$$T_a = ka + c\ln a + d, \qquad T_b = kb + c\ln b + d.$$

These can be solved for $c = \dfrac{T_b - T_a + k(a-b)}{\ln(b/a)}$ and $d = \dfrac{T_a\ln b - T_b\ln a + k(b\ln a - a\ln b)}{\ln(b/a)}$.

33. (a) If we substitute $x(t)$ into the left and right sides of the differential equation, we obtain

$$\begin{aligned}\frac{dx}{dt} &= \frac{(3-2e^{-10t})(-60e^{-10t})(-10) - 60(1-e^{-10t})(-2e^{-10t})(-10)}{(3-2e^{-10t})^2}\\ &= \frac{600e^{-10t}(3-2e^{-10t}-2+2e^{-10t})}{(3-2e^{-10t})^2} = \frac{600e^{-10t}}{(3-2e^{-10t})^2}\\ (20-x)(30-x) &= \left[20 - \frac{60(1-e^{-10t})}{3-2e^{-10t}}\right]\left[30 - \frac{60(1-e^{-10t})}{3-2e^{-10t}}\right]\\ &= \frac{[60-40e^{-10t}-60(1-e^{-10t})][90-60e^{-10t}-60(1-e^{-10t})]}{(3-2e^{-10t})^2}\\ &= \frac{(20e^{-10t})(30)}{(3-2e^{-10t})^2} = \frac{600e^{-10t}}{(3-2e^{-10t})^2}\end{aligned}$$

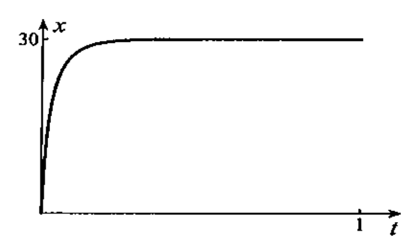
(b) $\lim_{t\to\infty} x(t) = 20$ This is reasonable since 10 g of A will eventually combine with 10 g of B to produce 20 g of C.



34. (a) If we substitute $x(t)$ into the left and right sides of the differential equation, we obtain

$$\begin{aligned}\frac{dx}{dt} &= \frac{(3-2e^{-15t})(-90e^{-15t})(-15) - 90(1-e^{-15t})(-2e^{-15t})(-15)}{(3-2e^{-15t})^2}\\ &= \frac{1350e^{-15t}(3-2e^{-15t}-2+2e^{-15t})}{(3-2e^{-15t})^2} = \frac{1350e^{-15t}}{(3-2e^{-15t})^2}\\ (30-x)(45-x) &= \left[30 - \frac{90(1-e^{-15t})}{3-2e^{-15t}}\right]\left[45 - \frac{90(1-e^{-15t})}{3-2e^{-15t}}\right]\\ &= \frac{[90-60e^{-15t}-90(1-e^{-15t})][135-90e^{-15t}-90(1-e^{-15t})]}{(3-2e^{-15t})^2}\\ &= \frac{(30e^{-15t})(45)}{(3-2e^{-15t})^2} = \frac{1350e^{-15t}}{(3-2e^{-15t})^2}\end{aligned}$$

(b) $\lim_{t\to\infty} x(t) = 30$ This is reasonable since 10 g of A will eventually combine with 20 g of B to produce 30 g of C.

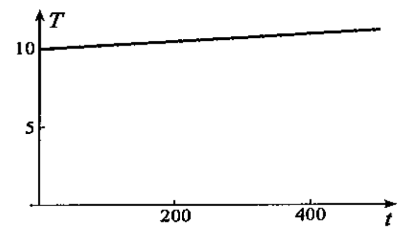


35. (a) The energy balance equation is

$$(4190)(10)\left(\frac{100}{t+1}\right) + 2000 = (4190)(T)\left(\frac{100}{t+1}\right) + (4190)(100)\frac{dT}{dt} \implies \frac{dT}{dt} + \frac{T}{t+1} = \frac{10}{t+1} + \frac{2}{419}.$$

(b) If we substitute $T(t)$ into the left side of the differential equation,

$$\frac{dT}{dt}+\frac{T}{t+1}=\left[\frac{(t+1)(4190)-(4190t+4189)(1)}{419(t+1)^2}+\frac{1}{419}\right]$$
$$+\frac{1}{t+1}\left[\frac{4190t+4189}{419(t+1)}+\frac{t+1}{419}\right]$$
$$=\frac{1}{419(t+1)^2}+\frac{1}{419}+\frac{4190t+4189}{419(t+1)^2}+\frac{1}{419}$$
$$=\frac{10}{t+1}+\frac{2}{419}.$$



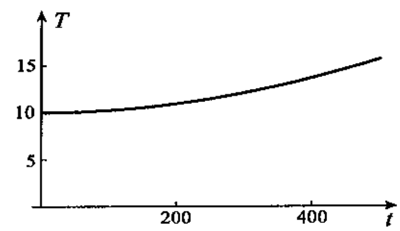
The value of the function at $t=0$ is $T(0)=10$.

36. (a)The energy balance equation is

$$1257+20t=\frac{1257T}{10}+419000\frac{dT}{dt} \implies \frac{dT}{dt}+\frac{3T}{10\,000}=\frac{3}{1000}+\frac{t}{20\,950}.$$

(b) If we substitute $T(t)$ into the left side of the differential equation,

$$\frac{dT}{dt}+\frac{3T}{10\,000}=\frac{200}{1257}+\frac{2\,000\,000}{3771}\left(-\frac{3}{10\,000}\right)e^{-3t/10\,000}$$
$$+\frac{3}{10\,000}\left(\frac{200t}{1257}-\frac{1\,962\,290}{3771}+\frac{2\,000\,000}{3771}e^{-3t/10\,000}\right)$$
$$=\frac{3}{1000}+\frac{t}{20950}.$$



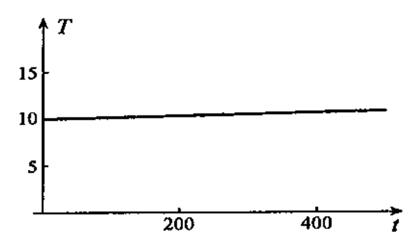
The value of the function at $t=0$ is $T(0)=10$.

37. (a) The energy balance equation is

$$(4190)\left(\frac{3}{100}\right)(10e^{-t})+2000=(4190)(T)\left(\frac{3}{100}\right)+(4190)(100)\frac{dT}{dt} \implies \frac{dT}{dt}+\frac{3T}{10\,000}=\frac{2}{419}+\frac{3}{1000}e^{-t}.$$

(b) If we substitute $T(t)$ into the left side of the differential equation,

$$\frac{dT}{dt}+\frac{3T}{10\,000}=-\frac{74\,240\,000}{12\,566\,229}\left(-\frac{3}{10\,000}\right)e^{-3t/10\,000}+\frac{30}{9997}e^{-t}$$
$$+\frac{3}{10\,000}\left(\frac{20\,000}{1257}-\frac{74\,240\,000}{12\,566\,229}e^{-3t/10\,000}-\frac{30}{9997}e^{-t}\right)$$
$$=\frac{2}{419}+\frac{3}{1000}e^{-t}.$$



The value of the function at $t=0$ is $T(0)=10$.

38. We calculate:

$$\frac{dx}{dt}=-e^{-t}(\sin 2t-\cos 2t)+e^{-t}(2\cos 2t+2\sin 2t)=e^{-t}(\sin 2t+3\cos 2t)$$

and

$$\frac{d^2x}{dt^2}=-e^{-t}(\sin 2t+3\cos 2t)+e^{-t}(2\cos 2t-6\sin 2t)=e^{-t}(-7\sin 2t-\cos 2t).$$

Consequently,

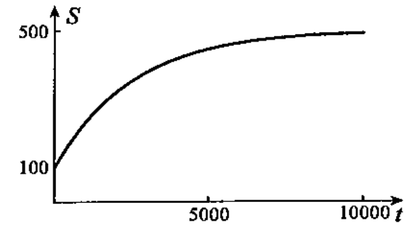
$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+5x=e^{-t}(-7\sin 2t-\cos 2t)+2e^{-t}(\sin 2t+3\cos 2t)+5e^{-t}(\sin 2t-\cos 2t)=0.$$

In addition, $x(0)=-1$ and $x'(0)=3$.

39. The function satisfies the differential equation since

$$\frac{dS}{dt}+\frac{S}{2500}=\frac{-400}{-2500}e^{-t/2500}+\frac{100}{2500}(5-4e^{-t/2500})=\frac{1}{5}$$

It also satisfies $S(0) = 100$.
The graph is asymptotic to the line $S = 500$.
We would expect this since the concentration of salt in the tank should approach that of the incoming solution, namely, 0.05 kg/L.
In a tank with 10 000 L, this would imply 500 kg of salt.

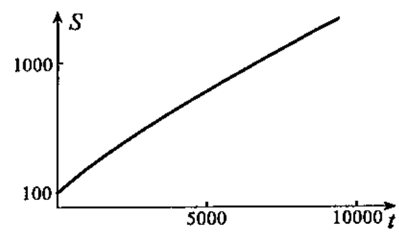


40. (a) Since solution is added at 4 L/s and removed at 2 L/s, the amount of solution in the tank is $10\,000+2t$.
In this case,

$$\frac{dS}{dt} = \frac{1}{5} - \frac{2S}{10\,000 + 2t},$$

subject to the condition that $S(0) = 100$.

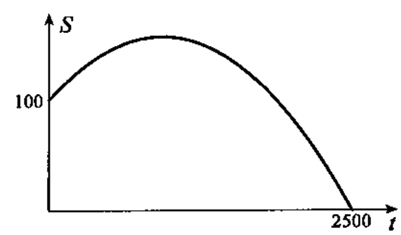
(b) $$\frac{dS}{dt} + \frac{S}{5000+t} = \frac{1}{10} + \frac{2\times 10^6}{(5000+t)^2} + \frac{1}{5000+t}\left(500 + \frac{t}{10} - \frac{2\times 10^6}{5000+t}\right) = \frac{1}{5}$$



The graph is asymptotic to the line $S = 500 + t/10$.

41. (a) Since solution is added at 4 L/s and removed at 8 L/s, the amount of solution in the tank is $10\,000-4t$.
In this case, $\dfrac{dS}{dt} = \dfrac{1}{5} - \dfrac{8S}{10\,000 - 4t}$, subject to the condition that $S(0) = 100$.

(b) $$\begin{aligned}\frac{dS}{dt} + \frac{2S}{2500-t} &= \frac{3}{25} - \frac{2t}{15\,625} + \frac{2}{2500-t}\left(100 + \frac{3t}{25} - \frac{t^2}{15\,625}\right)\\ &= \frac{3}{25} - \frac{2t}{15\,625} + \frac{2}{2500-t}\left(\frac{1\,562\,500 + 1875t - t^2}{15\,625}\right)\\ &= \frac{3}{25} - \frac{2t}{15\,625} + \frac{2(625+t)(2500-t)}{15\,625(2500-t)} = \frac{1}{5}\end{aligned}$$



This solution is valid as long as there is solution in the tank. This is for $0 \le t \le 2500$. The parabola has maximum value 625/4 kg at $t = 1875/2$.

42. (a) Since alcohol enters the vat at the rate of 4/25 L/s and leaves at the rate of $2(A/2000)$, it follows that $\dfrac{dA}{dt} = \dfrac{4}{25} - \dfrac{A}{1000}$. We must also have $A(0) = 80$.

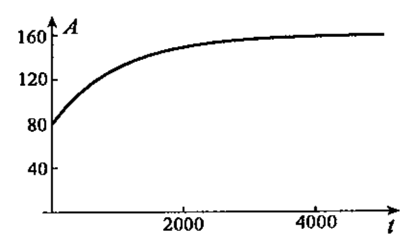
(b) $$\frac{dA}{dt} + \frac{A}{1000} = \frac{80}{1000}e^{-t/1000} + \frac{1}{1000}(160 - 80e^{-t/1000}) = \frac{4}{25}$$

A plot of the function is shown to the right.
It is asymptotic to the line $A = 160$.

(c) The beer is 5% alcohol when

$$\frac{5}{100}(2000) = 160 - 80e^{-t/1000}.$$

This implies that $e^{-t/1000} = 3/4$, from which $t = 1000\ln(4/3)$.

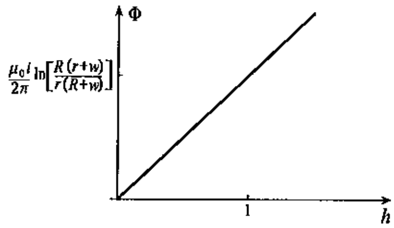


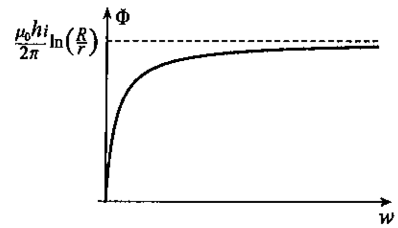
43. First we express M in the form $M = \dfrac{\mu_0 l}{2\pi}\ln\left[\dfrac{(s+w)(s+W)}{s(s+w+W)}\right]$.

Then,

$$\begin{aligned}\frac{dM}{ds} &= \frac{\mu_0 l}{2\pi}\frac{s(s+w+W)}{(s+w)(s+W)}\left[\frac{s(s+w+W)(2s+w+W) - (s+w)(s+W)(2s+w+W)}{s^2(s+w+W)^2}\right]\\ &= \frac{-\mu_0 lwW(2s+w+W)}{2\pi s(s+w)(s+W)(s+w+W)} < 0.\end{aligned}$$

44. The function is linear in h (left figure below). To draw the right graph, we express the function in the form $\Phi = \dfrac{\mu_0 hi}{2\pi}\left[\ln\left(\dfrac{R}{r}\right) + \ln\left(\dfrac{r+w}{R+w}\right)\right]$. It has value 0 at $w = 0$; it is increasing; and $\displaystyle\lim_{w\to\infty}\Phi = \frac{\mu_0 hi}{2\pi}\ln\left(\frac{R}{r}\right)$.





45. Since $\frac{dy}{dx} = A + B\left\{\frac{1}{2}\ln\left(\frac{x-1}{x+1}\right) + \frac{x}{2}\left(\frac{x+1}{x-1}\right)\left[\frac{(x+1)-(x-1)}{(x+1)^2}\right]\right\} = A + \frac{B}{2}\ln\left(\frac{x-1}{x+1}\right) + \frac{Bx}{x^2-1}$,
and $\frac{d^2y}{dx^2} = \frac{B}{2}\left(\frac{x+1}{x-1}\right)\left[\frac{(x+1)-(x-1)}{(x+1)^2}\right] + B\frac{(x^2-1)-x(2x)}{(x^2-1)^2} = \frac{B}{x^2-1} - \frac{B(x^2+1)}{(x^2-1)^2} = \frac{-2B}{(x^2-1)^2}$,
it follows that

$$\begin{aligned}(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y &= (1-x^2)\left[\frac{-2B}{(x^2-1)^2}\right] - 2x\left[A + \frac{B}{2}\ln\left(\frac{x-1}{x+1}\right) + \frac{Bx}{x^2-1}\right]\\ &\quad + 2\left[Ax + \frac{Bx}{2}\ln\left(\frac{x-1}{x+1}\right) + B\right]\\ &= \frac{2B}{x^2-1} - 2Ax - Bx\ln\left(\frac{x-1}{x+1}\right) - \frac{2Bx^2}{x^2-1}\\ &\quad + 2Ax + Bx\ln\left(\frac{x-1}{x+1}\right) + 2B\\ &= 0.\end{aligned}$$

46. (a) If we multiply the equation by e^y, then $(e^y)^2 - x\,e^y - 1 = 0 \implies e^y = \frac{x \pm \sqrt{x^2+4}}{2}$. Since e^y must be positive, we choose $e^y = \frac{x+\sqrt{x^2+4}}{2} \implies y = \ln\left(\frac{x+\sqrt{x^2+4}}{2}\right)$. Consequently,

$$\frac{dy}{dx} = \frac{2}{x+\sqrt{x^2+4}}\left(\frac{1}{2}\right)\left(1 + \frac{x}{\sqrt{x^2+4}}\right) = \frac{1}{x+\sqrt{x^2+4}}\left(\frac{\sqrt{x^2+4}+x}{\sqrt{x^2+4}}\right) = \frac{1}{\sqrt{x^2+4}}.$$

(b) If we differentiate implicity with respect to x, then $1 = e^y\frac{dy}{dx} + e^{-y}\frac{dy}{dx}$, from which $\frac{dy}{dx} = \frac{1}{e^y + e^{-y}}$. To show that this is the same derivative as in part (a), we note that

$$\begin{aligned}e^y + e^{-y} &= \frac{x+\sqrt{x^2+4}}{2} + \frac{2}{x+\sqrt{x^2+4}} = \frac{x+\sqrt{x^2+4}}{2} + \frac{2}{x+\sqrt{x^2+4}}\left(\frac{x-\sqrt{x^2+4}}{x-\sqrt{x^2+4}}\right)\\ &= \frac{x+\sqrt{x^2+4}}{2} + \frac{2(x-\sqrt{x^2+4})}{x^2-(x^2+4)} = \sqrt{x^2+4}.\end{aligned}$$

This shows that $1/(e^y + e^{-y}) = 1/\sqrt{x^2+4}$.

47. (a) The rate at which energy enters the tank in incoming liquid is $\dot{m}c_pT_0$. The rate at which it leaves in outgoing liquid is $\dot{m}c_pT$. The rate at which energy is used to raise the temperature of the liquid in the tank is $Mc_p(dT/dt)$. Consequently, the energy balance equation becomes

$$\dot{m}c_pT_0 + q = \dot{m}c_pT + Mc_p\frac{dT}{dt}.$$

(b) When $\dot{m}$, T_0 and q are all constants,

$$Mc_p\frac{dT}{dt} + \dot{m}c_pT = Mc_p\left[-\frac{\dot{m}T_0}{M}e^{-\dot{m}t/M} + \left(T_0 + \frac{q}{c_p\dot{m}}\right)\left(\frac{\dot{m}}{M}\right)e^{-\dot{m}t/M}\right]$$

$$+ \dot{m}c_p\left[T_0 e^{-\dot{m}t/M} + \left(T_0 + \frac{q}{c_p\dot{m}}\right)\left(1 - e^{-\dot{m}t/M}\right)\right]$$
$$= e^{-\dot{m}t/M}\left[-\dot{m}c_pT_0 + \dot{m}c_p\left(T_0 + \frac{q}{c_p\dot{m}}\right) + \dot{m}c_pT_0 - \dot{m}c_p\left(T_0 + \frac{q}{c_p\dot{m}}\right)\right]$$
$$+ \dot{m}c_p\left(T_0 + \frac{q}{c_p\dot{m}}\right)$$
$$= q + \dot{m}c_pT_0.$$

48. Implicit differentiation gives $\dfrac{1}{x^2+y^2}\left(2x + 2y\dfrac{dy}{dx}\right) = \dfrac{2}{1+(y^2/x^2)}\left(\dfrac{1}{x}\dfrac{dy}{dx} - \dfrac{y}{x^2}\right)$, from which

$$\frac{1}{x^2+y^2}\left(x + y\frac{dy}{dx}\right) = \frac{1}{x^2+y^2}\left(x\frac{dy}{dx} - y\right) \implies \frac{dy}{dx} = \frac{x+y}{x-y}.$$

A second differentiation now gives

$$\frac{d^2y}{dx^2} = \frac{(x-y)\left(1+\dfrac{dy}{dx}\right) - (x+y)\left(1-\dfrac{dy}{dx}\right)}{(x-y)^2} = \frac{(x-y)\left(1+\dfrac{x+y}{x-y}\right) - (x+y)\left(1-\dfrac{x+y}{x-y}\right)}{(x-y)^2}$$
$$= \frac{2(x^2+y^2)}{(x-y)^3}.$$

49. (a)We divide the discussion into two cases. Since n is rational, we set $n = a/b$, where a and b are relatively prime. For x^n to be defined for $x < 0$, b must be odd.
Case 1 (a is even): In this case, $y = x^n = (-x)^n = e^{n\ln(-x)}$, and

$$\frac{dy}{dx} = e^{n\ln(-x)}\left(\frac{n}{x}\right) = (-x)^n\left(\frac{n}{x}\right) = x^n\left(\frac{n}{x}\right) = nx^{n-1}.$$

Case 11 (a is odd): In this case, $y = -(-x)^n = -e^{n\ln(-x)}$, and

$$\frac{dy}{dx} = -e^{n\ln(-x)}\left(\frac{n}{x}\right) = -(-x)^n\left(\frac{n}{x}\right) = n(-x)^{n-1}.$$

Because $n-1 = a/b-1 = (a-b)/b$, and $a-b$ is even, it follows that $(-x)^{n-1} = x^{n-1}$, and $dy/dx = nx^{n-1}$.
(b) Since x^n is not defined for $x < 0$ when n is irrational, $f'(0)$ does not exist for irrational n. When n is rational and we set $n = a/b$ as in part (a), x^n is still defined for $x < 0$ only when b is odd. In such a case,

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h^{a/b}}{h} = \lim_{h\to 0}h^{a/b-1} = \begin{cases} 0, & \text{if } a/b \ge 1 \text{ or } a = 0 \\ 1, & \text{if } a/b = 1 \\ \text{does not exist}, & \text{if } a/b < 1 \end{cases}.$$

Thus, $f'(0)$ exists and is equal to zero when $n = 0$, or when $n = a/b$, where b is odd and $a/b \ge 1$. When $n = 1$, $f'(0) = 1$, and in all other cases, $f'(0)$ does not exist.

EXERCISES 3.12

1. If we take natural logarithms of $y = x^{-x}$, then $\ln y = -x\ln x$, and differentiation with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = -\ln x - x\left(\frac{1}{x}\right) \implies \frac{dy}{dx} = -x^{-x}(1+\ln x).$$

2. Natural logarithms of $y = x^{4\cos x}$ give $\ln y = 4\cos x\ln x$, and differentiation with respect to x yields

$$\frac{1}{y}\frac{dy}{dx} = 4\left(-\sin x\ln x + \frac{1}{x}\cos x\right) \implies \frac{dy}{dx} = 4x^{4\cos x}\left(\frac{1}{x}\cos x - \sin x\ln x\right).$$

3. If we take natural logarithms of $y = x^{4x}$, then $\ln y = 4x \ln x$, and differentiation with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = 4\ln x + 4x\left(\frac{1}{x}\right) \implies \frac{dy}{dx} = 4x^{4x}(1+\ln x).$$

4. Natural logarithms of $y = (\sin x)^x$ give $\ln y = x \ln(\sin x)$ and differentiation with respect to x yields

$$\frac{1}{y}\frac{dy}{dx} = \ln(\sin x) + x\frac{\cos x}{\sin x} \implies \frac{dy}{dx} = (\sin x)^x[\ln(\sin x) + x\cot x].$$

5. If we take natural logarithms of $y = \left(1+\frac{1}{x}\right)^x$, then $\ln y = x \ln\left(1+\frac{1}{x}\right)$, and differentiation with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \ln\left(1+\frac{1}{x}\right) + x\left(\frac{x}{x+1}\right)\left(\frac{-1}{x^2}\right) \implies \frac{dy}{dx} = \left(1+\frac{1}{x}\right)^x\left[\frac{-1}{x+1} + \ln\left(1+\frac{1}{x}\right)\right].$$

6. Natural logarithms of $y = \left(1+\frac{1}{x}\right)^{x^2}$ give $\ln y = x^2 \ln\left(1+\frac{1}{x}\right) = x^2[\ln(x+1) - \ln x]$, and differentiation with respect to x leads to $\frac{1}{y}\frac{dy}{dx} = 2x[\ln(x+1) - \ln x] + x^2\left(\frac{1}{x+1} - \frac{1}{x}\right)$. Thus,

$$\frac{dy}{dx} = xy\left[2\ln\left(1+\frac{1}{x}\right) + x\left(\frac{x-(x+1)}{x(x+1)}\right)\right] = x\left(1+\frac{1}{x}\right)^{x^2}\left[2\ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1}\right].$$

7. If we take natural logarithms of $y = (1/x)^{1/x}$, then $\ln y = (1/x)\ln(1/x) = -(1/x)\ln x$, and differentiation with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2}\ln x - \left(\frac{1}{x}\right)\left(\frac{1}{x}\right) \implies \frac{dy}{dx} = \left(\frac{1}{x}\right)^{1/x}\left[\frac{1}{x^2}(-1+\ln x)\right] = \left(\frac{1}{x}\right)^{2+1/x}(-1+\ln x).$$

8. If we take natural logarithms of $y = \left(\frac{2}{x}\right)^{3/x}$, then $\ln y = \frac{3}{x}\ln\left(\frac{2}{x}\right) = \frac{3}{x}(\ln 2 - \ln x)$, and differentiation with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = -\frac{3}{x^2}(\ln 2 - \ln x) + \frac{3}{x}\left(-\frac{1}{x}\right) \implies \frac{dy}{dx} = -\frac{3y}{x^2}\left[\ln\left(\frac{2}{x}\right) + 1\right] = -\frac{3}{x^2}\left(\frac{2}{x}\right)^{3/x}\left[1+\ln\left(\frac{2}{x}\right)\right].$$

9. If we take natural logarithms of $y = (\sin x)^{\sin x}$, then $\ln y = \sin x \ln\sin x$, and differentiation with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \cos x \ln\sin x + \sin x\left(\frac{\cos x}{\sin x}\right) \implies \frac{dy}{dx} = (\sin x)^{\sin x}\cos x(1+\ln\sin x).$$

10. When we take natural logarithms of $y = (\ln x)^{\ln x}$, we find $\ln y = \ln x \ln(\ln x)$, and differentiation with respect to x yields

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\ln(\ln x) + \ln x\left(\frac{1}{x\ln x}\right) \implies \frac{dy}{dx} = \frac{y}{x}[\ln(\ln x) + 1] = \frac{1}{x}(\ln x)^{\ln x}[1+\ln(\ln x)].$$

11. If we take natural logarithms of $y = (x^2+3x^4)^3(x^2+5)^4$, then $\ln y = 3\ln(x^2+3x^4) + 4\ln(x^2+5)$, and differentiation with respect to x gives $\frac{1}{y}\frac{dy}{dx} = \frac{3(2x+12x^3)}{x^2+3x^4} + \frac{4(2x)}{x^2+5}$. Thus,

$$\begin{aligned}\frac{dy}{dx} &= (x^2+3x^4)^3(x^2+5)^4\left[\frac{3(12x^5+62x^3+10x)+(24x^5+8x^3)}{(x^2+5)(x^2+3x^4)}\right]\\ &= 2x(x^2+3x^4)^2(x^2+5)^3(30x^4+97x^2+15).\end{aligned}$$

12. If we take natural logarithms of $y = \dfrac{\sqrt{x}(1+2x^2)}{\sqrt{1+x^2}}$, then $\ln y = \dfrac{1}{2}\ln x + \ln(1+2x^2) - \dfrac{1}{2}\ln(1+x^2)$. Differentiation with respect to x gives $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{2x} + \dfrac{4x}{1+2x^2} - \dfrac{x}{1+x^2}$, and therefore

$$\frac{dy}{dx} = y\left[\frac{(1+2x^2)(1+x^2)+8x^2(1+x^2)-2x^2(1+2x^2)}{2x(1+2x^2)(1+x^2)}\right] = \frac{6x^4+9x^2+1}{2\sqrt{x}(1+x^2)^{3/2}}.$$

13. If we take natural logarithms of $|y| = |x|\sqrt[3]{1-\sin x}$, then $\ln|y| = \ln|x| + (1/3)\ln(1-\sin x)$. Differentiation with respect to x using formula 3.46 gives $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{x} + \dfrac{-\cos x}{3(1-\sin x)}$. Thus,

$$\frac{dy}{dx} = x\sqrt[3]{1-\sin x}\left[\frac{1}{x} - \frac{\cos x}{3(1-\sin x)}\right] = \frac{3-3\sin x - x\cos x}{3(1-\sin x)^{2/3}}.$$

14. If we take natural logarithms of $|y| = |x^2+3x|^3(x^2+5)^4$, then $\ln|y| = 3\ln|x^2+3x| + 4\ln(x^2+5)$. Differentiation with respect to x using formula 3.46 gives $\dfrac{1}{y}\dfrac{dy}{dx} = 3\left(\dfrac{2x+3}{x^2+3x}\right) + 4\left(\dfrac{2x}{x^2+5}\right)$, and therefore

$$\begin{aligned}\frac{dy}{dx} &= y\left[\frac{3(2x+3)(x^2+5)+8x(x^2+3x)}{(x^2+3x)(x^2+5)}\right] = (x^2+3x)^3(x^2+5)^4\left[\frac{14x^3+33x^2+30x+45}{(x^2+3x)(x^2+5)}\right]\\ &= (x^2+3x)^2(x^2+5)^3(14x^3+33x^2+30x+45).\end{aligned}$$

15. If we take natural logarithms of $|y| = |x|^2e^{4x}$, then $\ln|y| = 2\ln|x| + 4x$. Differentiation with respect to x using formula 3.46 gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + 4 \quad\Longrightarrow\quad \frac{dy}{dx} = x^2e^{4x}\left(\frac{2}{x}+4\right) = 2x(2x+1)e^{4x}.$$

16. When we take natural logarithms of $y = x^{3/2}e^{-2x}$, we have $\ln y = (3/2)\ln x - 2x$, and differentiation with respect to x results in

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{2x} - 2 \quad\Longrightarrow\quad \frac{dy}{dx} = x^{3/2}e^{-2x}\left(\frac{3-4x}{2x}\right) = \frac{1}{2}\sqrt{x}(3-4x)e^{-2x}.$$

17. If we take natural logarithms of $|y| = x^2|\ln x|$, then $\ln|y| = 2\ln x + \ln|\ln x|$. Differentiation with respect to x using formula 3.46 gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{1}{\ln x}\left(\frac{1}{x}\right) \quad\Longrightarrow\quad \frac{dy}{dx} = x^2\ln x\left(\frac{2}{x} + \frac{1}{x\ln x}\right) = x(1+2\ln x).$$

18. Natural logarithms of $|y| = \dfrac{e^x}{|\ln(x-1)|}$ give $\ln|y| = x - \ln|\ln(x-1)|$. Differentiation with respect to x now gives $\dfrac{1}{y}\dfrac{dy}{dx} = 1 - \dfrac{1}{(x-1)\ln(x-1)}$. Thus, $\dfrac{dy}{dx} = \dfrac{e^x}{\ln(x-1)}\left[1 - \dfrac{1}{(x-1)\ln(x-1)}\right]$.

19. If we take natural logarithms of $|y| = |x^3+3|^3|x^2-2x|$, then $\ln|y| = 3\ln|x^3+3| + \ln|x^2-2x|$. Differentiation with respect to x using formula 3.46 gives $\dfrac{1}{y}\dfrac{dy}{dx} = 3\left(\dfrac{3x^2}{x^3+3}\right) + \dfrac{2x-2}{x^2-2x}$, and therefore

$$\frac{dy}{dx} = (x^3+3)^3(x^2-2x)\left[\frac{9x^2(x^2-2x)+(2x-2)(x^3+3)}{(x^3+3)(x^2-2x)}\right] = (x^3+3)^2(11x^4-20x^3+6x-6).$$

20. If we take natural logarithms of $|y| = \dfrac{\sqrt{x}|1-x^2|}{\sqrt{1+x^2}}$, then $\ln|y| = \dfrac{1}{2}\ln x + \ln|1-x^2| - \dfrac{1}{2}\ln(1+x^2)$. Differentiation with respect to x using formula 3.46 gives $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{2x} - \dfrac{2x}{1-x^2} - \dfrac{x}{1+x^2}$, and therefore

$$\frac{dy}{dx} = \frac{\sqrt{x}(1-x^2)}{\sqrt{1+x^2}}\left[\frac{(1-x^2)(1+x^2) - 4x^2(1+x^2) - 2x^2(1-x^2)}{2x(1-x^2)(1+x^2)}\right] = \frac{1-6x^2-3x^4}{2\sqrt{x}(1+x^2)^{3/2}}.$$

21. Natural logarithms of $|y| = \dfrac{|x^2-1|}{|x|\sqrt{1-4\tan^2 x}}$ give $\ln|y| = \ln|x^2-1| - \ln|x| - \dfrac{1}{2}\ln(1-4\tan^2 x)$. Differentiation with respect to x using formula 3.46 gives $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{2x}{x^2-1} - \dfrac{1}{x} + \dfrac{8\tan x\sec^2 x}{2(1-4\tan^2 x)}$, and therefore $\dfrac{dy}{dx} = \left(\dfrac{x^2-1}{x\sqrt{1-4\tan^2 x}}\right)\left(\dfrac{2x}{x^2-1} - \dfrac{1}{x} + \dfrac{4\tan x\sec^2 x}{1-4\tan^2 x}\right)$.

22. Natural logarithms of $|y| = |x|^3|x^2-4x|\sqrt{1+x^3}$ give $\ln|y| = 3\ln|x| + \ln|x^2-4x| + \dfrac{1}{2}\ln(1+x^3)$. Differentiation with respect to x using formula 3.46 results in $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{3}{x} + \dfrac{2x-4}{x^2-4x} + \dfrac{3x^2}{2(1+x^3)}$, and therefore

$$\begin{aligned}\frac{dy}{dx} &= x^3(x^2-4x)\sqrt{1+x^3}\left[\frac{6(x^2-4x)(1+x^3) + 2x(2x-4)(1+x^3) + 3x^3(x^2-4x)}{2x(x^2-4x)(1+x^3)}\right]\\ &= \frac{x^3(13x^4 - 44x^3 + 10x - 32)}{2\sqrt{1+x^3}}.\end{aligned}$$

23. If we take natural logarithms of $|y| = \dfrac{|\sin 3x|^3}{|\tan 2x|^5}$, then $\ln|y| = 3\ln|\sin 3x| - 5\ln|\tan 2x|$. Differentiation with respect to x using formula 3.46 gives $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{9\cos 3x}{\sin 3x} - \dfrac{10\sec^2 2x}{\tan 2x}$. Thus,

$$\frac{dy}{dx} = \left(\frac{\sin^3 3x}{\tan^5 2x}\right)(9\cot 3x - 10\sec^2 2x\cot 2x).$$

24. Natural logarithms of $|y| = \dfrac{|\sin 2x||\sec 5x|}{|1-2\cot x|^3}$ give $\ln|y| = \ln|\sin 2x| + \ln|\sec 5x| - 3\ln|1-2\cot x|$. Differentiation with respect to x using 3.46 yields $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{2\cos 2x}{\sin 2x} + \dfrac{5\sec 5x\tan 5x}{\sec 5x} - \dfrac{3(2\csc^2 x)}{1-2\cot x}$, from which $\dfrac{dy}{dx} = \dfrac{\sin 2x\sec 5x}{(1-2\cot x)^3}\left[2\cot 2x + 5\tan 5x - \dfrac{6\csc^2 x}{1-2\cot x}\right]$.

25. If we set $y = u^u$ and take natural logarithms, $\ln y = u\ln u$. Differentiation with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{du}{dx}\ln u + u\left(\frac{1}{u}\frac{du}{dx}\right) \implies \frac{dy}{dx} = u^u(1+\ln u)\frac{du}{dx}.$$

26. (a) If we take natural logarithms, $\ln x = a\ln r - b(r+c)$. Differentiation with respect to r gives $\dfrac{1}{x}\dfrac{dx}{dr} = \dfrac{a}{r} - b \implies \dfrac{dx}{dr} = \dfrac{x}{r}(a-br)$. Since $r > a/b$, it follows that the derivative $dx/dr < 0$. But if the slope of the tangent line is always negative, x must decrease as r increases. Therefore x increases as r decreases.

(b) If we differentiate $\ln x = a\ln r - b(r+c)$ with respect to x, $\dfrac{1}{x} = \dfrac{a}{r}\dfrac{dr}{dx} - b\dfrac{dr}{dx} = \left(\dfrac{a-br}{r}\right)\dfrac{dr}{dx}$. Thus,

$$\frac{dr}{dx} = \frac{r}{x(a-br)} \qquad \implies \qquad \frac{Er}{Ex} = \frac{x}{r}\frac{r}{x(a-br)} = \frac{1}{a-br}.$$

EXERCISES 3.13

1. $\dfrac{dy}{dx} = -\text{csch}(2x+3)\coth(2x+3)(2) = -2\,\text{csch}(2x+3)\coth(2x+3)$

2. $\dfrac{dy}{dx} = \sinh(x/2) + x\cosh(x/2)(1/2) = \sinh(x/2) + (x/2)\cosh(x/2)$

3. $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{1-\text{sech}x}}(\text{sech}x\,\tanh x) = \dfrac{\text{sech}x\,\tanh x}{2\sqrt{1-\text{sech}x}}$

4. $\dfrac{dy}{dx} = \text{sech}^2(\ln x)\left(\dfrac{1}{x}\right)$

5. If we differentiate with respect to x, we obtain $\sinh(x+y)\left(1+\dfrac{dy}{dx}\right) = 2 \Longrightarrow \dfrac{dy}{dx} = \dfrac{2-\sinh(x+y)}{\sinh(x+y)}$.

6. Differentiation with respect to x gives $\dfrac{dy}{dx} - \text{csch}^2x = \dfrac{1}{2\sqrt{1+y}}\dfrac{dy}{dx}$, from which

$$\frac{dy}{dx} = \frac{\text{csch}^2x}{1-\dfrac{1}{2\sqrt{1+y}}} = \frac{2\sqrt{1+y}\,\text{csch}^2x}{2\sqrt{1+y}-1}.$$

7. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (u\sinh u + \cosh u)(e^x - e^{-x}) = 2\sinh x(u\sinh u + \cosh u)$

8. $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = [\sec^2(\cosh t)\sinh t][-\sin(\tanh x)\text{sech}^2x] = -\sinh t\,\text{sech}^2x\,\sec^2(\cosh t)\sin(\tanh x)$

9. $\dfrac{dy}{dx} = \dfrac{1}{1+\sinh^2x}(\cosh x) = \dfrac{\cosh x}{\cosh^2x} = \text{sech}x$

10. Since $y = (1/2)\ln(\tanh 2x)$,

$$\frac{dy}{dx} = \frac{1}{2\tanh 2x}\text{sech}^2 2x(2) = \frac{1}{\cosh^2 2x}\frac{\cosh 2x}{\sinh 2x} = \frac{1}{\sinh 2x\,\cosh 2x} = \frac{1}{(\sinh 4x)/2} = 2\,\text{csch}4x.$$

11.
$$\frac{d}{dx}\sinh u = \frac{d}{du}\left(\frac{e^u-e^{-u}}{2}\right)\frac{du}{dx} = \frac{e^u+e^{-u}}{2}\frac{du}{dx} = \cosh u\frac{du}{dx}$$
$$\frac{d}{dx}\cosh u = \frac{d}{du}\left(\frac{e^u+e^{-u}}{2}\right)\frac{du}{dx} = \frac{e^u-e^{-u}}{2}\frac{du}{dx} = \sinh u\frac{du}{dx}$$
$$\frac{d}{du}\tanh u = \frac{d}{du}\left(\frac{\sinh u}{\cosh u}\right)\frac{du}{dx} = \frac{\cosh u\cosh u - \sinh u\sinh u}{\cosh^2 u}\frac{du}{dx} = \frac{1}{\cosh^2 u}\frac{du}{dx} = \text{sech}^2u\frac{du}{dx}$$

Similarly, $\dfrac{d}{dx}\text{coth}u = -\text{csch}^2u\dfrac{du}{dx}$.

$$\frac{d}{dx}\text{sech}u = \frac{d}{du}\left(\frac{1}{\cosh u}\right)\frac{du}{dx} = \frac{-1}{\cosh^2u}\sinh u\frac{du}{dx} = -\frac{\sinh u}{\cosh u}\frac{1}{\cosh u}\frac{du}{dx} = -\text{sech}u\,\tanh u\frac{du}{dx}$$

Similarly, $\dfrac{d}{dx}\text{csch}u = -\text{csch}u\,\text{coth}u\dfrac{du}{dx}$.

12. (a)
$$\begin{aligned} f'(x) &= -Ak\sin kx + Bk\cos kx + Ck\sinh kx + Dk\cosh kx \\ f''(x) &= -Ak^2\cos kx - Bk^2\sin kx + Ck^2\cosh kx + Dk^2\sinh kx \\ f'''(x) &= Ak^3\sin kx - Bk^3\cos kx + Ck^3\sinh kx + Dk^3\cosh kx \\ f''''(x) &= Ak^4\cos kx + Bk^4\sin kx + Ck^4\cosh kx + Dk^4\sinh kx \end{aligned}$$

Thus, $\dfrac{d^4y}{dx^4} - k^4y = 0$.

(b) These conditions imply that

$$\begin{aligned}
0 &= f(0) = A + C,\\
0 &= f'(0) = Bk + Dk = k(B + D),\\
0 &= f(L) = A\cos kL + B\sin kL + C\cosh kL + D\sinh kL,\\
0 &= f''(L) = -Ak^2\cos kL - Bk^2\sin kL + Ck^2\cosh kL + Dk^2\sinh kL.
\end{aligned}$$

Thus, $C = -A$, $D = -B$, and

$$\begin{aligned}
0 &= A\cos kL + B\sin kL - A\cosh kL - B\sinh kL = A(\cos kL - \cosh kL) + B(\sin kL - \sinh kL);\\
0 &= -A\cos kL - B\sin kL - A\cosh kL - B\sinh kL = A(\cos kL + \cosh kL) + B(\sin kL + \sinh kL).
\end{aligned}$$

(c) If we write the conditions in part (b) in the form

$$A(\cos kL - \cosh kL) = -B(\sin kL - \sinh kL), \quad A(\cos kL + \cosh kL) = -B(\sin kL + \sinh kL),$$

and divide one by the other,

$$\frac{\cos kL - \cosh kL}{\cos kL + \cosh kL} = \frac{\sin kL - \sinh kL}{\sin kL + \sinh kL}.$$

Hence,

$$(\cos kL - \cosh kL)(\sin kL + \sinh kL) = (\cos kL + \cosh kL)(\sin kL - \sinh kL)$$

or, $2\cos kL \sinh kL = 2\sin kL \cosh kL$. Division by $2\cos kL \cosh kL$ gives $\tanh kL = \tan kL$.

13. If we set $y = \text{Sinh}^{-1}x$, then $x = \sinh y$, and differentiation gives $1 = \cosh y\,\dfrac{dy}{dx}$, from which

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}.$$

If we set $y = \text{Tanh}^{-1}x$, then $x = \tanh y$, and differentiation gives $1 = \text{sech}^2 y\,\dfrac{dy}{dx}$, from which

$$\frac{dy}{dx} = \frac{1}{\text{sech}^2 y} = \frac{1}{1-\tanh^2 y} = \frac{1}{1-x^2}.$$

If we set $y = \text{Sech}^{-1}x$, then $x = \text{sech}y$, and differentiation gives $1 = -\text{sech}y\,\tanh y\,\dfrac{dy}{dx}$, from which

$$\frac{dy}{dx} = \frac{-1}{\text{sech}y\,\tanh y} = \frac{-1}{\text{sech}y\sqrt{1-\text{sech}^2 y}} = \frac{-1}{x\sqrt{1-x^2}}.$$

Derivations of the other derivatives are similar.

EXERCISES 3.14

1. Since $f(x)$ and $f'(x)$ are continuous on $-3 \le x \le 2$, we may apply the mean value theorem on this interval. Equation 3.49 states that $2c + 2 = \dfrac{8-3}{2+3} = 1$. The only solution is $c = -1/2$.

2. Since $f(x)$ and $f'(x)$ are continuous on $1 \le x \le 3$, we may apply the mean value theorem on this interval. Equation 3.49 states that $3 - 4c = \dfrac{-5-5}{3-1} = -5$. The only solution is $c = 2$.

3. Since $f(x)$ and $f'(x)$ are continuous on $2 \le x \le 3$, we may apply the mean value theorem on this interval. Equation 3.49 states that $1 = \dfrac{8-7}{3-2} = 1$. All x's in the interval satisfy this equation.

4. Since $f'(x)$ is not defined at $x = 0$, we cannot apply the mean value theorem on the interval $-1 \le x \le 1$.

5. Since $f(x)$ is continuous on $0 \le x \le 1$, and $f'(x)$ is continuous on $0 < x < 1$, we may apply the mean value theorem on this interval. Equation 3.49 states that $1 = \dfrac{1-0}{1-0} = 1$. All x's in the interval satisfy this equation.

6. Since $f(x)$ and $f'(x)$ are continuous on $-3 \le x \le 2$, we may apply the mean value theorem on this interval. Equation 3.49 states that $3c^2 + 4c - 1 = \dfrac{12+8}{2+3} = 4$. Of the two solutions $c = (-2 \pm \sqrt{19})/3$, only $(-2+\sqrt{19})/3$ is within the interval $-3 \le x \le 2$.

7. Since $f(x)$ and $f'(x)$ are continuous on $-1 \le x \le 2$, we may apply the mean value theorem on this interval. Equation 3.49 states that $3c^2 + 4c - 1 = \dfrac{12-0}{2+1} = 4$. Of the two solutions $c = (-2 \pm \sqrt{19})/3$ of this equation, only $c = (\sqrt{19}-2)/3$ lies in the given interval.

8. Since $f(x)$ and $f'(x)$ are continuous on $2 \le x \le 4$, we may apply the mean value theorem on this interval. Equation 3.49 states that

$$\left[\frac{(x-1)(1)-(x+2)(1)}{(x-1)^2}\right]_{|x=c} = \frac{2-4}{4-2}; \text{ that is, } \qquad \frac{-3}{(c-1)^2} = -1.$$

Of the two solutions $c = 1 \pm \sqrt{3}$ of this equation, only $c = 1+\sqrt{3}$ lies in the given interval.

9. Since $f(x)$ is not defined at $x = -2$, the mean value theorem cannot be applied on the interval $-3 \le x \le 2$.

10. Since $f(x)$ and $f'(x)$ are continuous on $-2 \le x \le 3$, we may apply the mean value theorem on this interval. Equation 3.49 states that

$$\left[\frac{(x+3)(2x)-x^2(1)}{(x+3)^2}\right]_{|x=c} = \frac{3/2-4}{3+2}; \text{ that is, } \qquad \frac{c^2+6c}{(c+3)^2} = -\frac{1}{2}.$$

Of the two solutions $c = -3 \pm \sqrt{6}$ of this equation, only $c = -3+\sqrt{6}$ lies in the given interval.

11. Since $f(x)$ and $f'(x)$ are continuous on $0 \le x \le 2\pi$, we may apply the mean value theorem on this interval. Equation 3.49 states that $\cos c = \dfrac{0-0}{2\pi - 0} = 0$. Solutions of this equation in the given interval are $c = \pi/2, 3\pi/2$.

12. Since $\ln(2x+1)$ and its derivative are continuous on $0 \le x \le 2$, we may apply the mean value theorem. Equation 3.49 states that $\dfrac{2}{2c+1} = \dfrac{\ln 5 - \ln 1}{2} = \dfrac{1}{2}\ln 5$. The solution of this equation is $c = (4 - \ln 5)/(2\ln 5)$.

13. Since $f(x)$ and $f'(x)$ are continuous on $-1 \le x \le 1$, we may apply the mean value theorem on this interval. Equation 3.49 states that $-e^{-c} = \dfrac{e^{-1}-e^{1}}{2}$. The only solution is $c = -\ln[(e^2-1)/(2e)]$.

14. Since $\sec x$ is not defined at $x = \pi/2$, the mean value theorem cannot be applied on the interval $0 \le x \le \pi$.

15. Since $f(x)$, $f'(x)$, $g(x)$, and $g'(x)$ are continuous on $1 \le x \le 2$, and $g'(x) = 1$ is never zero, we may apply Cauchy's generalized mean value theorem. Equation 3.48 states that $\dfrac{4-1}{2-1} = \dfrac{2c}{1}$. The solution is $c = 3/2$.

16. Since $g'(0) = 0$, Cauchy's generalized mean value theorem cannot be applied on the interval $-1 \le x \le 1$.

17. Since $f(x)$, $f'(x)$, $g(x)$, and $g'(x)$ are continuous on $0 \le x \le -2$, and $g'(x) = 3x^2+5$ is never zero, we may apply Cauchy's generalized mean value theorem. Equation 3.48 states that

$$\frac{9+1}{22-4} = \frac{2c+3}{3c^2+5} \quad \Longrightarrow \quad 15c^2 - 18c - 2 = 0.$$

Of the solutions $c = (9 \pm \sqrt{111})/15$ of this equation, only $c = (9+\sqrt{111})/15$ lies in the interval $0 \le x \le 2$.

18. Since $f(x)$, $f'(x)$, $g(x)$, and $g'(x)$ are continuous on $-3 \le x \le -2$, and $g'(x) = -1/(x-1)^2$ is never zero, we may apply Cauchy's generalized mean value theorem. Equation 3.48 states that

$$\frac{2-3/2}{2/3-3/4} = \frac{\dfrac{1}{(c+1)^2}}{\dfrac{-1}{(c-1)^2}} \qquad \Longrightarrow \qquad 6 = \left(\frac{c-1}{c+1}\right)^2.$$

Of the two solutions $c = (1 \pm \sqrt{6})/(1 \mp \sqrt{6})$, only $c = (1+\sqrt{6})/(1-\sqrt{6})$ is in the given interval.

19. The fact that $|f'(x)| \le M$ on $a \le x \le b$ implies that $f'(x)$ exists on $a \le x \le b$ and $f(x)$ is continuous for $a \le x \le b$. Consequently, we may apply the mean value theorem to $f(x)$ on the interval, and state that there exists at least one c for which $f(b) - f(a) = f'(c)(b-a)$. If we take absolute values, $|f(b) - f(a)| = |f'(c)||b-a| \le M|b-a|$.

20. Equation 3.49 for $f(x) = dx^2 + ex + g$ on $a \le x \le b$ states that

$$2dc + e = \frac{(db^2+eb+g)-(da^2+ea+g)}{b-a} = \frac{d(b^2-a^2)+e(b-a)}{b-a} = d(b+a)+e,$$

and therefore $c = (a+b)/2$.

21. Since the derivative of $\sin x$ is always less in absolute value than 1, we can use the result of Exercise 19 to write that $|\sin b - \sin a| \le |b-a|$. The same result is true for the cosine function.

22. If we define a function $F(x) = f(x) - g(x)$, then $F(x)$ is continuous and has a derivative at each point in $a \le x \le b$. We may therefore apply the mean value theorem to $F(x)$ and write $F'(c) = \dfrac{F(b)-F(a)}{b-a}$. Since $F'(x) = f'(x) - g'(x)$, this equation states that there exists a point c between a and b such that $f'(c) - g'(c) = \dfrac{[f(b)-g(b)]-[f(a)-g(a)]}{b-a} = 0$; that is, $f'(c) = g'(c)$.

23. Equation 3.49 states that

$$f'(c) = 3dc^2 + 2ec + g = \frac{f(b)-f(a)}{b-a} = \frac{(db^3+eb^2+gb+h)-(da^3+ea^2+ga+h)}{b-a}$$
$$= \frac{d(b^3-a^3)+e(b^2-a^2)+g(b-a)}{b-a} = d(b^2+ab+a^2)+e(b+a)+g.$$

Thus, c must satisfy a quadratic equation $3dc^2 + 2ec - [d(b^2+ab+a^2)+e(b+a)] = 0$, with solutions

$$c = \frac{-2e \pm \sqrt{4e^2+12d[d(b^2+ab+a^2)+e(b+a)]}}{6d} = -\frac{e}{3d} \pm \frac{\sqrt{e^2+3d[d(b^2+ab+a^2)+e(b+a)]}}{3d}.$$

These solutions are equidistant from $-e/(3d)$.

REVIEW EXERCISES

1. $\dfrac{dy}{dx} = 3x^2 - \dfrac{2}{x^3}$

2. $\dfrac{dy}{dx} = 6x + 2 - \dfrac{1}{x^2}$

3. $\dfrac{dy}{dx} = 2 + \dfrac{2}{3x^3} - \dfrac{1}{2x^{3/2}}$

4. $\dfrac{dy}{dx} = \dfrac{1}{3x^{2/3}} - \dfrac{10}{9}x^{2/3}$

5. $\dfrac{dy}{dx} = (x^2+5)^4 + x(4)(x^2+5)^3(2x) = (x^2+5)^3(9x^2+5)$

6. $\dfrac{dy}{dx} = 2(x^2+2)(2x)(x^3-3)^3 + (x^2+2)^2(3)(x^3-3)^2(3x^2) = x(x^2+2)(x^3-3)^2(13x^3+18x-12)$

7. $\dfrac{dy}{dx} = \dfrac{(x^3-5)(6x)-3x^2(3x^2)}{(x^3-5)^2} = \dfrac{-3x(x^3+10)}{(x^3-5)^2}$

8. $\dfrac{dy}{dx} = \dfrac{(x+5)(3)-(3x-2)(1)}{(x+5)^2} = \dfrac{17}{(x+5)^2}$

9. $\dfrac{dy}{dx} = \dfrac{(x^2+2x-1)(2x+2)-(x^2+2x+2)(2x+2)}{(x^2+2x-1)^2} = \dfrac{-6(x+1)}{(x^2+2x-1)^2}$

10. $\dfrac{dy}{dx} = \dfrac{(x^2+5x-2)(4) - 4x(2x+5)}{(x^2+5x-2)^2} = \dfrac{-4(x^2+2)}{(x^2+5x-2)^2}$

11. If we differentiate with respect to x, then $y + x\dfrac{dy}{dx} + 9y^2\dfrac{dy}{dx} = 1$, and therefore $\dfrac{dy}{dx} = \dfrac{1-y}{x+9y^2}$.

12. We first write the equation in the form $x^2 + y^2 = x^2y$, and then differentiate with respect to x, $2x + 2y\dfrac{dy}{dx} = 2xy + x^2\dfrac{dy}{dx}$. Thus, $\dfrac{dy}{dx} = \dfrac{2xy-2x}{2y-x^2}$.

13. If we differentiate with respect to x, then $2xy^2 + 2x^2y\dfrac{dy}{dx} - 3y\cos x - 3\sin x\dfrac{dy}{dx} = 0$, and therefore $\dfrac{dy}{dx} = \dfrac{3y\cos x - 2xy^2}{2x^2y - 3\sin x}$.

14. Differentiation with respect to x gives $2xy + x^2\dfrac{dy}{dx} + \dfrac{dy}{dx}\sqrt{1+x} + \dfrac{y}{2\sqrt{1+x}} = 0$. Thus,

$$\frac{dy}{dx} = -\frac{2xy + \dfrac{y}{2\sqrt{1+x}}}{x^2 + \sqrt{1+x}} = -\frac{4xy\sqrt{1+x} + y}{2\sqrt{1+x}\left(x^2+\sqrt{1+x}\right)}.$$

15. $\dfrac{dy}{dx} = 3\tan^2(3x+2)\sec^2(3x+2)(3) = 9\tan^2(3x+2)\sec^2(3x+2)$

16. $\dfrac{dy}{dx} = 2\sec(1-4x)\sec(1-4x)\tan(1-4x)(-4) = -8\sec^2(1-4x)\tan(1-4x)$

17. $\dfrac{dy}{dx} = \dfrac{(\cos 3x)(2\cos 2x) - (\sin 2x)(-3\sin 3x)}{\cos^2 3x} = \dfrac{2\cos 3x\cos 2x + 3\sin 3x\sin 2x}{\cos^2 3x}$

18. $\dfrac{dy}{dx} = \sec(\tan 2x)\tan(\tan 2x)\sec^2 2x(2) = 2\sec^2 2x\sec(\tan 2x)\tan(\tan 2x)$

19. $\dfrac{dy}{dx} = 2x\cos x^2 - x^2\sin x^2(2x) = 2x(\cos x^2 - x^2\sin x^2)$

20. Since $y = \left(\dfrac{1}{2}\sin 2x\right)^2 = \dfrac{1}{4}\sin^2 2x = \dfrac{1}{4}\left(\dfrac{1-\cos 4x}{2}\right) = \dfrac{1}{8}(1-\cos 4x)$, it follows that $\dfrac{dy}{dx} = \dfrac{1}{8}\sin 4x(4) = \dfrac{1}{2}\sin 4x$.

21. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (2u-2)(5/3)(1+2x)^{2/3}(2) = \dfrac{20}{3}(u-1)(1+2x)^{2/3}$

22. $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = (1-2\sin 2t)(1+2\sin 2x)$

23. $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \left(\dfrac{-3t^2}{2\sqrt{1-t^3}}\right)\left(\dfrac{x}{\sqrt{1+x^2}}\right) = \dfrac{-3xt^2}{2\sqrt{1-t^3}\sqrt{1+x^2}}$

24. $\dfrac{dy}{dx} = \dfrac{dy}{dv}\dfrac{dv}{dx} = [\cos^2 v + v(2\cos v)(-\sin v)]\left(\dfrac{-x}{\sqrt{1-x^2}}\right) = \dfrac{x(2v\sin v\cos v - \cos^2 v)}{\sqrt{1-x^2}} = \dfrac{x(v\sin 2v - \cos^2 v)}{v}$

25. $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{1+\sqrt{1+x}}}\dfrac{1}{2\sqrt{1+x}} = \dfrac{1}{4\sqrt{1+x}\sqrt{1+\sqrt{1+x}}}$

26. If we differentiate the equation with respect to x, the result is $1 = e^{2y}\left(2\dfrac{dy}{dx}\right) \implies \dfrac{dy}{dx} = \dfrac{1}{2}e^{-2y}$.

27. If write the equation in the form $y = 3x^2/2 + 2x + 3 + 4/x$, then $\dfrac{dy}{dx} = 3x + 2 - \dfrac{4}{x^2}$.

28. $\dfrac{dy}{dx} = 2x\ln(x^2-1) + (x^2+1)\dfrac{2x}{x^2-1}$

29. If we differentiate the equation with respect to x, the result is $\sin y + x\cos y\dfrac{dy}{dx} + 2y + 2x\dfrac{dy}{dx} = 0$, and therefore $\dfrac{dy}{dx} = -\dfrac{2y+\sin y}{2x+x\cos y}$.

30. If we differentiate with respect to x, then $-5\sin(x-y)\left(1-\dfrac{dy}{dx}\right)=0 \Longrightarrow \dfrac{dy}{dx}=1$.

31. $\dfrac{dy}{dx}=\dfrac{-3}{\sqrt{1-(2-3x)^2}}$

32. $\dfrac{dy}{dx}=3\cosh(x^2)(2x)=6x\cosh(x^2)$

33. $$\frac{dy}{dx}=\frac{\dfrac{-\mathrm{Sin}^{-1}x}{\sqrt{1-x^2}}-\dfrac{\mathrm{Cos}^{-1}x}{\sqrt{1-x^2}}}{(\mathrm{Sin}^{-1}x)^2}=-\frac{(\mathrm{Sin}^{-1}x+\mathrm{Cos}^{-1}x)}{\sqrt{1-x^2}(\mathrm{Sin}^{-1}x)^2}$$

34. $$\frac{dy}{dx}=\frac{1}{1+\left(\dfrac{1}{x}+x\right)^2}\left(-\frac{1}{x^2}+1\right)=\frac{1}{1+\dfrac{x^4+2x^2+1}{x^2}}\left(\frac{x^2-1}{x^2}\right)=\frac{x^2-1}{x^4+3x^2+1}$$

35. $\dfrac{dy}{dx}=e^{\cosh x}(\sinh x)$

36. If we differentiate with respect to x, we obtain $\cosh y\,\dfrac{dy}{dx}=\cos x \Longrightarrow \dfrac{dy}{dx}=\dfrac{\cos x}{\cosh y}=\cos x\,\mathrm{sech} y$.

37. If we differentiate with respect to x, we obtain $\dfrac{1}{(x+y)\sqrt{(x+y)^2-1}}\left(1+\dfrac{dy}{dx}\right)=y+x\dfrac{dy}{dx}$, and therefore

$$\frac{dy}{dx}=\frac{\dfrac{1}{(x+y)\sqrt{(x+y)^2-1}}-y}{x-\dfrac{1}{(x+y)\sqrt{(x+y)^2-1}}}=\frac{1-y(x+y)\sqrt{(x+y)^2-1}}{x(x+y)\sqrt{(x+y)^2-1}-1}.$$

38. $$\frac{dy}{dx}=\mathrm{Csc}^{-1}\left(\frac{1}{x^2}\right)-\frac{x}{\dfrac{1}{x^2}\sqrt{\dfrac{1}{x^4}-1}}\left(\frac{-2}{x^3}\right)=\mathrm{Csc}^{-1}\left(\frac{1}{x^2}\right)+\frac{2x^2}{\sqrt{1-x^4}}$$

39. Since $y=e^{2x}\left(\dfrac{e^{2x}+e^{-2x}}{2}\right)=\dfrac{1}{2}(e^{4x}+1)$, it follows that $dy/dx=(1/2)e^{4x}(4)=2\,e^{4x}$.

40. If $\ln[\mathrm{Tan}^{-1}(x+y)]=1/10$, then $\mathrm{Tan}^{-1}(x+y)=e^{1/10}$, from which $x+y=\tan e^{1/10}$ or $y=-x+\tan e^{1/10}$. Consequently, $dy/dx=-1$.

41. If we differentiate the equation with respect to x, the result is

$$1=\frac{1}{2\sqrt{1+x\cot y^2}}\left[\cot y^2-x\csc^2 y^2\left(2y\frac{dy}{dx}\right)\right] \Longrightarrow \frac{dy}{dx}=\frac{\cot y^2-2\sqrt{1+x\cot y^2}}{2xy\csc^2 y^2}=\frac{\cot y^2-2x}{2xy\csc^2 y^2}.$$

42. If we square the equation, $x^2=\dfrac{4+y}{4-y}$. This equation can be solved for $y=\dfrac{4(x^2-1)}{x^2+1}$. Differentiation now gives $\dfrac{dy}{dx}=4\left[\dfrac{(x^2+1)(2x)-(x^2-1)(2x)}{(x^2+1)^2}\right]=\dfrac{16x}{(x^2+1)^2}$.

43. $$\frac{dy}{dx}=\frac{1}{2}\left(\frac{4+x^2}{4-x^2}\right)^{-1/2}\left[\frac{(4-x^2)(2x)-(4+x^2)(-2x)}{(4-x^2)^2}\right]=\frac{8x}{\sqrt{4+x^2}(4-x^2)^{3/2}}$$

44. If we take natural logarithms of $|y|=\dfrac{|x|^2\sqrt{1-x}}{|x+5|}$, then $\ln|y|=2\ln|x|+\dfrac{1}{2}\ln(1-x)-\ln|x+5|$. Differentiation with respect to x using formula 3.46 gives $\dfrac{1}{y}\dfrac{dy}{dx}=\dfrac{2}{x}-\dfrac{1}{2(1-x)}-\dfrac{1}{x+5}$. Therefore,

$$\frac{dy}{dx}=\frac{x^2\sqrt{1-x}}{x+5}\left[\frac{4(1-x)(x+5)-x(x+5)-2x(1-x)}{2x(1-x)(x+5)}\right]=\frac{x(20-23x-3x^2)}{2\sqrt{1-x}(x+5)^2}.$$

45. $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{7-\sqrt{7-\sqrt{x}}}}\dfrac{-1}{2\sqrt{7-\sqrt{x}}}\dfrac{-1}{2\sqrt{x}} = \dfrac{1}{8\sqrt{x}\sqrt{7-\sqrt{x}}\sqrt{7-\sqrt{7-\sqrt{x}}}}$

46. If we write $x^2 - xy = xy + y^2$, or, $y^2 + 2xy - x^2 = 0$, and differentiate with respect to x, we obtain $2y\dfrac{dy}{dx} + 2y + 2x\dfrac{dy}{dx} - 2x = 0$. Thus, $\dfrac{dy}{dx} = \dfrac{x-y}{x+y}$.

47. $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \dfrac{1}{2}\left(\dfrac{4+t}{4-t}\right)^{-1/2}\left[\dfrac{(4-t)(1)-(4+t)(-1)}{(4-t)^2}\right](\sec^2 x) = \dfrac{4\sec^2 x}{\sqrt{4+t}(4-t)^{3/2}}$

48. If we use the result of Exercise 40 in Section 3.7,

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}} = \frac{1}{\dfrac{(y^3+4y+6)(2y-2)-(y^2-2y)(3y^2+4)}{(y^3+4y+6)^2}} = \frac{(y^3+4y+6)^2}{-y^4+4y^3+4y^2+12y-12}.$$

49. If we write the function in the form $y = \dfrac{(x-2)^3}{(x-2)^2} = x-2$, the derivative is $dy/dx = 1$, provided $x \neq 2$.

50. Since $y = \dfrac{(\sqrt{x}-\sqrt{2})(\sqrt{x}+\sqrt{2})}{\sqrt{x}-\sqrt{2}} = \sqrt{x}+\sqrt{2}$, it follows that $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$, provided $x \neq 2$.

51. If we take natural logarithms of $y = x^{2x}$, then $\ln y = 2x\ln x$, and differentiation with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = 2\ln x + \frac{2x}{x} \implies \frac{dy}{dx} = 2x^{2x}(1+\ln x).$$

52. If we take natural logarithms, then $\ln y = x\ln\cos x$, and differentiation with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \ln(\cos x) + x\left(\frac{-\sin x}{\cos x}\right) \implies \frac{dy}{dx} = (\cos x)^x[\ln(\cos x) - x\tan x].$$

53. $\dfrac{dy}{dx} = \dfrac{(e^x+1)(e^x)-e^x(e^x)}{(e^x+1)^2} = \dfrac{e^x}{(e^x+1)^2}$

54. $\dfrac{dy}{dx} = \dfrac{1}{\log_{10} x}\log_{10} e\dfrac{d}{dx}\log_{10} x = \dfrac{1}{\log_{10} x}\log_{10} e\left(\dfrac{1}{x}\right)\log_{10} e = \dfrac{(\log_{10} e)^2}{x\log_{10} x}$

55. $\dfrac{dy}{dx} = e^x\ln x + e^x\left(\dfrac{1}{x}\right) = \dfrac{e^x(1+x\ln x)}{x}$

56. Differentiation with respect to x gives $1 = e^y\dfrac{dy}{dx} - e^{-y}\dfrac{dy}{dx} \implies \dfrac{dy}{dx} = \dfrac{1}{e^y - e^{-y}}$.

57. Differentiation with respect to x gives $ye^{xy} + xe^{xy}\dfrac{dy}{dx} + xye^{xy}\left(y + x\dfrac{dy}{dx}\right) = 0$, from which $\dfrac{dy}{dx} = -\dfrac{y+xy^2}{x+x^2y} = -\dfrac{y}{x}$.

58. If we differentiate with respect to x, we obtain $2xy + x^2\dfrac{dy}{dx} + \dfrac{1}{x+y}\left(1+\dfrac{dy}{dx}\right) = 1$. Thus,

$$\frac{dy}{dx} = \frac{1-2xy-\dfrac{1}{x+y}}{x^2+\dfrac{1}{x+y}} = \frac{(x+y)(1-2xy)-1}{x^2(x+y)+1}.$$

59. Since $\dfrac{dy}{dx}_{|x=1} = (3x^2+3)_{|x=1} = 6$, equations for the tangent and normal lines are $y-2 = 6(x-1)$ and $y-2 = -(1/6)(x-1)$, or $6x-y=4$ and $x+6y=13$.

60. Since $dy/dx = -1/(x+5)^2$, the slope of the tangent line at $(0,1/5)$ is $-1/25$. Equations for the tangent and normal lines are $y-1/5 = -(1/25)(x-0)$ and $y-1/5 = 25(x-0)$, or $x+25y=5$ and $5y-125x=1$.

61. Since $\dfrac{dy}{dx}_{|x=\pi/2} = (-2\sin 2x)_{|x=\pi/2} = 0$, the tangent and normal lines are $y=-1$ and $x=\pi/2$.

62. Since $\dfrac{dy}{dx} = \dfrac{(2x-5)(2x+3)-(x^2+3x)(2)}{(2x-5)^2} = \dfrac{2x^2-10x-15}{(2x-5)^2}$, the slope of the tangent line at $(1,-4/3)$ is $-23/9$. Equations for the tangent and normal lines are $y+4/3 = -(23/9)(x-1)$ and $y+4/3 = (9/23)(x-1)$, or $23x+9y=11$ and $27x-69y=119$.

63. If we differentiate with respect to x, then $2x - 2y\dfrac{dy}{dx} + 2 - 2\dfrac{dy}{dx} = 3y^2\dfrac{dy}{dx} \implies \dfrac{dy}{dx} = \dfrac{2x+2}{3y^2+2y+2}$. A second differentiation gives

$$\frac{d^2y}{dx^2} = \frac{(3y^2+2y+2)(2)-(2x+2)(6y+2)\dfrac{dy}{dx}}{(3y^2+2y+2)^2}$$
$$= \frac{2(3y^2+2y+2)-4(x+1)(3y+1)\left(\dfrac{2x+2}{3y^2+2y+2}\right)}{(3y^2+2y+2)^2} = \frac{2(3y^2+2y+2)^2-8(x+1)^2(3y+1)}{(3y^2+2y+2)^3}.$$

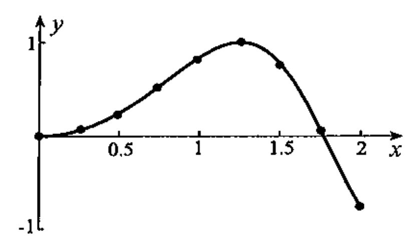
64. If we first expand the equation, we obtain $x^2-5xy+y^2=0$. Differentiation with respect to x now gives $2x-5y-5x\dfrac{dy}{dx}+2y\dfrac{dy}{dx}=0 \implies \dfrac{dy}{dx} = \dfrac{2x-5y}{5x-2y}$. A second differentiation yields

$$\frac{d^2y}{dx^2} = \frac{(5x-2y)\left(2-5\dfrac{dy}{dx}\right)-(2x-5y)\left(5-2\dfrac{dy}{dx}\right)}{(5x-2y)^2}$$
$$= \frac{(5x-2y)\left[2-5\left(\dfrac{2x-5y}{5x-2y}\right)\right]-(2x-5y)\left[5-2\left(\dfrac{2x-5y}{5x-2y}\right)\right]}{(5x-2y)^2}$$
$$= \frac{2(5x-2y)^2-5(2x-5y)(5x-2y)-5(2x-5y)(5x-2y)+2(2x-5y)^2}{(5x-2y)^3} = \frac{-42(x^2-5xy+y^2)}{(5x-2y)^3}.$$

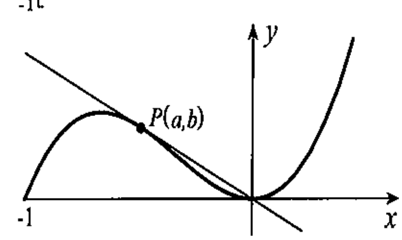
65. If we differentiate with respect to x, then $\cos(x+y)\left(1+\dfrac{dy}{dx}\right) = 1 \implies \dfrac{dy}{dx} = \sec(x+y)-1$. A second differentiation gives

$$\frac{d^2y}{dx^2} = \sec(x+y)\tan(x+y)\left(1+\frac{dy}{dx}\right) = \sec(x+y)\tan(x+y)\left[1+\sec(x+y)-1\right]$$
$$= \sec^2(x+y)\tan(x+y).$$

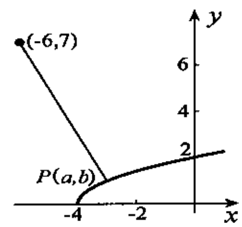
66. The graph to the right was obtained by joining the points shown. The function is not periodic.



67. Let $P(a,b)$ be a required point. If we differentiate the equation of the cubic with respect to x, the result is $dy/dx = 3x^2+2x$. The slope of the tangent line at P is therefore $3a^2+2a$, and the equation of the tangent line is $y-b=(3a^2+2a)(x-a)$. Since this line passes through the origin, a and b must satisfy $-b=(3a^2+2a)(-a)$. Because $P(a,b)$ is on the cubic, a and b must also satisfy $b=a^3+a^2$. When we combine these two equations, we obtain $3a^3+2a^2=a^3+a^2 \implies 0=2a^3+a^2=a^2(2a+1)$. Thus, $a=0$, as expected, and $a=-1/2$. The required points are therefore $(0,0)$ and $(-1/2,1/8)$.



68. Let $P(a,b)$ be the required point. If we differentiate the equation of the parabola with respect to x, the result is $1 = 2y(dy/dx)$. The slope of the tangent line at P is therefore $1/(2b)$, and that of the normal line is $-2b$. Since the slope of the normal line at P is also $(b-7)/(a+6)$, it follows that $(b-7)/(a+6) = -2b$. Because $P(a,b)$ is on the parabola, a and b must also satisfy $a = b^2 - 4$. When we substitute this into the above equation, we obtain $(b-7)/(b^2-4+6) = -2b$. Thus, $0 = 2b^3 + 5b - 7 = (b-1)(2b^2+2b+7)$. The required point is therefore $(-3,1)$. The length of the line segment joining $(-6,7)$ and $(-3,1)$ is the shortest distance from $(-6,7)$ to the parabola.



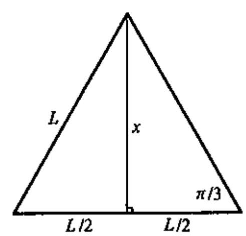
69. (a) There is an infinity of functions defined by the equation.
(b) There are only two continuous functions defined by the equation, $f(x) = \sqrt{1-x^2}$ and $f(x) = -\sqrt{1-x^2}$.

70. Since $x = L\sin(\pi/3) = \sqrt{3}L/2$, the area of the triangle is

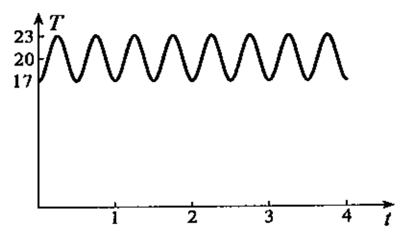
$$A = \frac{xL}{2} = \frac{\sqrt{3}L^2}{4}.$$

Consequently,

$$\frac{dA}{dL} = \frac{\sqrt{3}L}{2}.$$



71. (a) The graph is shown to the right.
(b) Since temperature is rising eight times, the furnace is on eight times.
(c) The rate of change of temperature is $T'(t) = 12\pi\cos(4\pi t - \pi/2)$. The maximum value is 12π degrees per hour.



72. Differentiation with respect to x gives $2(x^2+y^2)\left(2x + 2y\dfrac{dy}{dx}\right) = 2x - 2y\dfrac{dy}{dx}$, and therefore $\dfrac{dy}{dx} = \dfrac{x - 2x(x^2+y^2)}{y + 2y(x^2+y^2)}$. The tangent line is horizontal when $0 = x - 2x(x^2+y^2) = x(1-2x^2-2y^2)$. Since $x = 0$ is not a solution to our problem, we must set $x^2 + y^2 = 1/2$. We substitute this into the equation for the lemniscate, obtaining thereby $1/4 = x^2 - y^2$. Addition of this and $1/2 = x^2 + y^2$ gives $2x^2 = 3/4$, or, $x = \pm\sqrt{3/8} = \pm\sqrt{6}/4$. These values yield the points $(\sqrt{6}/4, \pm\sqrt{2}/4)$ and $(-\sqrt{6}/4, \pm\sqrt{2}/4)$.

73. Equation 3.49 states that $3c^2 + 3 = \dfrac{232-34}{6-3} = 66$. Of the two solutions $c = \pm\sqrt{21}$, only $c = \sqrt{21}$ lies in the interval $3 \le x \le 6$.

74. For this $f(x)$ and $g(x)$, equation 3.48 gives $\dfrac{6c-2}{3c^2+2} = \dfrac{5-9}{3+3} = -\dfrac{2}{3}$. Of the two solutions $c = (-9 \pm \sqrt{93})/6$ of this equation, only $c = (\sqrt{93}-9)/6$ is in the interval $-1 \le x \le 1$.