CHAPTER 8

EXERCISES 8.1

1. $\displaystyle\int \frac{x^2}{5-3x^3}\,dx = -\frac{1}{9}\ln|5-3x^3| + C$

2. $\displaystyle\int xe^{-2x^2}\,dx = -\frac{1}{4}e^{-2x^2} + C$

3. $\displaystyle\int \frac{x}{(x^2+2)^{1/3}}\,dx = \frac{3}{4}(x^2+2)^{2/3} + C$

4. $\displaystyle\int \frac{e^x}{1+e^x}\,dx = \ln(1+e^x) + C$

5. $\displaystyle\int \frac{4t+8}{t^2+4t+5}\,dt = 2\ln|t^2+4t+5| + C$

6. $\displaystyle\int x^2\sqrt{1-3x^3}\,dx = -\frac{2}{27}(1-3x^3)^{3/2} + C$

7. $\displaystyle\int (x+1)(x^2+2x)^{1/3}\,dx = \frac{3}{8}(x^2+2x)^{4/3} + C$

8. $\displaystyle\int \frac{x^2}{(1+x^3)^3}\,dx = \frac{-1}{6(1+x^3)^2} + C$

9. If we set $u = 1-\sqrt{x}$, then $du = \dfrac{-1}{2\sqrt{x}}dx$, and

$$\begin{aligned}\int \frac{\sqrt{x}}{1-\sqrt{x}}\,dx &= \int \frac{1-u}{u}[-2(1-u)\,du] = 2\int\left(-\frac{1}{u}+2-u\right)du = 2\left(-\ln|u| + 2u - \frac{u^2}{2}\right) + C\\ &= -2\ln|1-\sqrt{x}| + 4(1-\sqrt{x}) - (1-\sqrt{x})^2 + C = -2\ln|1-\sqrt{x}| - 2\sqrt{x} - x + D.\end{aligned}$$

10. $\displaystyle\int \frac{1-\sqrt{x}}{\sqrt{x}}\,dx = \int (x^{-1/2}-1)\,dx = 2\sqrt{x} - x + C$

11. $\displaystyle\int \frac{x+2}{x+1}\,dx = \int\left(1+\frac{1}{x+1}\right)dx = x + \ln|x+1| + C$

12. $\displaystyle\int \frac{x^2+2}{x^2+1}\,dx = \int\left(1+\frac{1}{x^2+1}\right)dx = x + \operatorname{Tan}^{-1}x + C$

13. If we set $u = \cos\theta - 1$, then $du = -\sin\theta\,d\theta$, and

$$\int \frac{\sin\theta}{\cos\theta-1}\,d\theta = \int \frac{1}{u}(-du) = -\ln|u| + C = -\ln|\cos\theta-1| + C = -\ln(1-\cos\theta) + C.$$

14. If we set $u = 2x+4$, then $du = 2\,dx$, and

$$\begin{aligned}\int \frac{x+3}{\sqrt{2x+4}}\,dx &= \int \frac{(u-4)/2+3}{\sqrt{u}}\frac{du}{2} = \frac{1}{4}\int\left(\sqrt{u}+\frac{2}{\sqrt{u}}\right)du\\ &= \frac{1}{4}\left(\frac{2}{3}u^{3/2} + 4\sqrt{u}\right) + C = \frac{1}{6}(2x+4)^{3/2} + \sqrt{2x+4} + C.\end{aligned}$$

15. If we set $u = e^x$, then $du = e^x\,dx$, and $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{1}{1+u^2}\,du = \operatorname{Tan}^{-1}u + C = \operatorname{Tan}^{-1}(e^x) + C.$

16. $\displaystyle\int \sin^3 2x\cos 2x\,dx = \frac{1}{8}\sin^4 2x + C$

17. If we set $u = 2x^2-5$, then $du = 4x\,dx$, and

$$\begin{aligned}\int x^5(2x^2-5)^4\,dx &= \int (x^2)^2(2x^2-5)^4\,x\,dx = \int\left(\frac{u+5}{2}\right)^2 u^4\left(\frac{du}{4}\right) = \frac{1}{16}\int (u^6+10u^5+25u^4)\,du\\ &= \frac{1}{16}\left(\frac{u^7}{7}+\frac{5u^6}{3}+5u^5\right) + C = \frac{1}{112}(2x^2-5)^7 + \frac{5}{48}(2x^2-5)^6 + \frac{5}{16}(2x^2-5)^5 + C.\end{aligned}$$

18. If we set $u = x+5$, then $du = dx$, and

$$\begin{aligned}\int \frac{x^3}{(x+5)^2}\,dx &= \int \frac{(u-5)^3}{u^2}\,du = \int\left(u - 15 + \frac{75}{u} - \frac{125}{u^2}\right)du = \frac{u^2}{2} - 15u + 75\ln|u| + \frac{125}{u} + C\\ &= \frac{1}{2}(x+5)^2 - 15(x+5) + 75\ln|x+5| + \frac{125}{x+5} + C\\ &= \frac{x^2}{2} - 10x + 75\ln|x+5| + \frac{125}{x+5} + D.\end{aligned}$$

19. If we set $u = 3 - z$, then $du = -dz$, and

$$\int z^2\sqrt{3-z}\,dz = \int (3-u)^2\sqrt{u}(-du) = \int (-9\sqrt{u} + 6u^{3/2} - u^{5/2})\,du$$
$$= -6u^{3/2} + \frac{12u^{5/2}}{5} - \frac{2u^{7/2}}{7} + C = -6(3-z)^{3/2} + \frac{12}{5}(3-z)^{5/2} - \frac{2}{7}(3-z)^{7/2} + C.$$

20. If we set $u = \cos 3x$, then $du = -3\sin 3x\,dx$, and

$$\int \tan 3x\,dx = \int \frac{\sin 3x}{\cos 3x}dx = \int \frac{1}{u}\left(-\frac{du}{3}\right) = -\frac{1}{3}\ln|u| + C = -\frac{1}{3}\ln|\cos 3x| + C = \frac{1}{3}\ln|\sec 3x| + C.$$

21. If we set $u = (x-3)^{1/3}$, then $x = 3 + u^3$, from which $dx = 3u^2\,du$, and

$$\int \frac{(x-3)^{2/3}}{(x-3)^{2/3}+1}dx = \int \frac{u^2}{u^2+1}(3u^2\,du) = 3\int \frac{u^4}{u^2+1}du = 3\int \left(u^2 - 1 + \frac{1}{u^2+1}\right)du$$
$$= 3\left(\frac{u^3}{3} - u + \mathrm{Tan}^{-1}u\right) + C = (x-3) - 3(x-3)^{1/3} + 3\,\mathrm{Tan}^{-1}(x-3)^{1/3} + C$$
$$= x - 3(x-3)^{1/3} + 3\,\mathrm{Tan}^{-1}(x-3)^{1/3} + D.$$

22. If we set $u = x^{1/4}$, or, $x = u^4$, then $dx = 4u^3\,du$, and

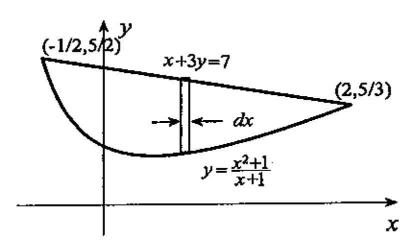
$$\int \frac{\sqrt{x}}{1+x^{1/4}}dx = \int \frac{u^2}{1+u}4u^3\,du = 4\int \frac{u^5}{u+1}du = 4\int \left(u^4 - u^3 + u^2 - u + 1 - \frac{1}{u+1}\right)du$$
$$= 4\left(\frac{u^5}{5} - \frac{u^4}{4} + \frac{u^3}{3} - \frac{u^2}{2} + u - \ln|u+1|\right) + C$$
$$= \frac{4}{5}x^{5/4} - x + \frac{4}{3}x^{3/4} - 2\sqrt{x} + 4x^{1/4} - 4\ln(x^{1/4}+1) + C.$$

23. Since small lengths along the curve can be approximated by

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \left(\frac{-\sin x}{\cos x}\right)^2}\,dx = \sqrt{1+\tan^2 x}\,dx = |\sec x|\,dx,$$

the total length of the curve is $\displaystyle\int_0^{\pi/4} |\sec x|\,dx = \int_0^{\pi/4} \sec x\,dx = \Big\{\ln|\sec x + \tan x|\Big\}_0^{\pi/4} = \ln(\sqrt{2}+1).$

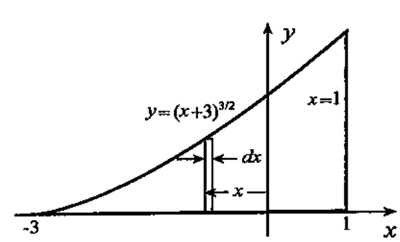
24.
$$A = \int_{-1/2}^{2}\left(\frac{7-x}{3} - \frac{x^2+1}{x+1}\right)dx$$
$$= \int_{-1/2}^{2}\left(\frac{7}{3} - \frac{x}{3} - x + 1 - \frac{2}{x+1}\right)dx$$
$$= \int_{-1/2}^{2}\left(\frac{10}{3} - \frac{4x}{3} - \frac{2}{x+1}\right)dx$$
$$= \left\{\frac{10x}{3} - \frac{2x^2}{3} - 2\ln|x+1|\right\}_{-1/2}^{2} = \frac{35 - 12\ln 6}{6}$$



25. (a) $A = \displaystyle\int_{-3}^{1}(3+x)^{3/2}\,dx = \left\{\frac{2(3+x)^{5/2}}{5}\right\}_{-3}^{1} = \frac{64}{5}$

If we set $u = 3 + x$ and $du = dx$, then

$$A\overline{x} = \int_{-3}^{1} x(3+x)^{3/2}\,dx = \int_0^4 (u-3)u^{3/2}\,du$$
$$= \left\{\frac{2u^{7/2}}{7} - \frac{6u^{5/2}}{5}\right\}_0^4 = \frac{-64}{35}.$$



Thus, $\overline{x} = -(64/35)(5/64) = -1/7$. Since

$$A\overline{y} = \int_{-3}^{1} \frac{1}{2}(3+x)^{3/2}(3+x)^{3/2}\,dx = \frac{1}{2}\int_{-3}^{1}(3+x)^3\,dx = \frac{1}{2}\left\{\frac{(3+x)^4}{4}\right\}_{-3}^{1} = 32,$$

we find $\overline{y} = 32(5/64) = 5/2$.
(b) If we again set $u = 3 + x$ and $du = dx$, then

$$\begin{aligned} I &= \int_{-3}^{1}(x-1)^2(3+x)^{3/2}\,dx = \int_0^4 (u-4)^2u^{3/2}\,du \\ &= \int_0^4 (u^{7/2} - 8u^{5/2} + 16u^{3/2})\,du = \left\{\frac{2u^{9/2}}{9} - \frac{16u^{7/2}}{7} + \frac{32u^{5/2}}{5}\right\}_0^4 = \frac{2^{13}}{315}. \end{aligned}$$

26. $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$

If we set $u = -x$ and $du = -dx$ in the first integral on the right, then when $f(x)$ is an even function,

$$\int_{-a}^{a} f(x)\,dx = \int_a^0 f(-u)(-du) + \int_0^a f(x)\,dx = \int_0^a f(u)\,du + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx;$$

and when $f(x)$ is an odd function,

$$\int_{-a}^{a} f(x)\,dx = \int_a^0 f(-u)(-du) + \int_0^a f(x)\,dx = \int_0^a -f(u)\,du + \int_0^a f(x)\,dx = 0.$$

27. (a) Certainly, $f(x) > 0$ for $x \ge 0$. Furthermore,

$$\int_0^\infty \lambda e^{-\lambda x}\,dx = \lim_{b\to\infty}\int_0^b \lambda e^{-\lambda x}\,dx = \lim_{b\to\infty}\left\{-e^{-\lambda x}\right\}_0^b = \lim_{b\to\infty}(1 - e^{-\lambda b}) = 1.$$

Thus, the function qualifies as a pdf.
(b) The probability that $x \ge 3$ is

$$0.5 = \int_3^\infty \lambda e^{-\lambda x}\,dx = \lim_{b\to\infty}\left\{-e^{-\lambda x}\right\}_3^b = \lim_{b\to\infty}(e^{-3\lambda} - e^{-b\lambda}) = e^{-3\lambda}.$$

The solution of this equation is $\lambda = -(1/3)\ln 0.5 = (1/3)\ln 2$.

28. (a) When we separate variables $\dfrac{dv}{1962 - v} = \dfrac{dt}{200}$, solutions are defined implicitly by

$$-\ln|1962 - v| = \frac{t}{200} + C \quad\Longrightarrow\quad \ln|1962 - v| = -\frac{t}{200} - C.$$

Exponentiation gives

$$|1962 - v| = e^{-C}e^{-t/200} \quad\Longrightarrow\quad 1962 - v = \pm e^{-C}e^{-t/200} = De^{-t/200},$$

where $D = \pm e^{-C}$. If we choose time $t = 0$ when descent begins, then $v(0) = 0$, and this requires $D = 1962$. Hence, $v = 1962 - 1962e^{-t/200} = 1962(1 - e^{-t/200})$ m/s.
(b) We set the velocity equal to dx/dt and integrate again,

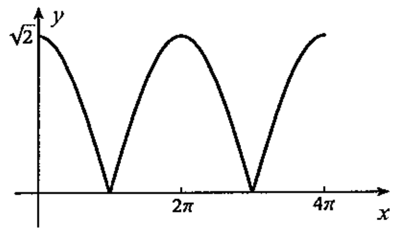
$$x = 1962(t + 200e^{-t/200}) + E.$$

Since $x(0) = 0$, we find $E = -1962(200)$, and therefore

$$x = 1962(t + 200e^{-t/200}) - 1962(200) = 1962t + 392\,400(e^{-t/200} - 1)\text{ m}.$$

29. Because of the symmetry of the function, we may integrate over $0 \le x \le \pi$ and quadruple the result. This saves later difficulties.

$$\begin{aligned}\int_0^{4\pi} \sqrt{1+\cos x}\,dx &= 4\int_0^{\pi} \sqrt{1+[2\cos^2(x/2)-1]}\,dx\\ &= 4\sqrt{2}\int_0^{\pi} |\cos(x/2)|\,dx\\ &= 4\sqrt{2}\int_0^{\pi} \cos(x/2)\,dx\\ &= 4\sqrt{2}\Big\{2\sin(x/2)\Big\}_0^{\pi} = 8\sqrt{2}.\end{aligned}$$



30. If we write $\dfrac{1}{x(3+2x^n)} = \dfrac{A}{x} + \dfrac{Bx^{n-1}}{3+2x^n} = \dfrac{3A+2Ax^n+Bx^n}{x(3+2x^n)}$, and equate numerators, then $3A+(2A+B)x^n = 1$. This equation is satisfied for all x if we choose $A = 1/3$ and $B = -2/3$. Then

$$\int \frac{1}{x(3+2x^n)}dx = \int \left[\frac{1}{3x} - \frac{2x^{n-1}}{3(3+2x^n)}\right]dx.$$

In the second integral on the right we set $u = 3+2x^n$ and $du = 2nx^{n-1}\,dx$,

$$\begin{aligned}\int \frac{1}{x(3+2x^n)}dx &= \frac{1}{3}\ln|x| - \frac{2}{3}\int \frac{1}{u}\left(\frac{du}{2n}\right) = \frac{1}{3}\ln|x| - \frac{1}{3n}\ln|u| + C\\ &= \frac{1}{3}\ln|x| - \frac{1}{3n}\ln|3+2x^n| + C = \frac{1}{3n}\ln\left|\frac{x^n}{3+2x^n}\right| + C.\end{aligned}$$

31. If $u - x = \sqrt{x^2+3x+4}$, then $(u-x)^2 = x^2+3x+4$, and when this equation is solved for x, the result is $x = (u^2-4)/(2u+3)$. Thus,

$$dx = \frac{(2u+3)(2u)-(u^2-4)(2)}{(2u+3)^2}du = \frac{2(u^2+3u+4)}{(2u+3)^2}du.$$

Since $\sqrt{x^2+3x+4} = u - x = u - \dfrac{u^2-4}{2u+3} = \dfrac{u^2+3u+4}{2u+3}$,

$$\begin{aligned}\int \frac{1}{(x^2+3x+4)^{3/2}}dx &= \int \frac{(2u+3)^3}{(u^2+3u+4)^3}\frac{2(u^2+3u+4)}{(2u+3)^2}du\\ &= 2\int \frac{2u+3}{(u^2+3u+4)^2}du = \frac{-2}{u^2+3u+4} + C\\ &= \frac{-2}{\left(x+\sqrt{x^2+3x+4}\right)^2 + 3\left(x+\sqrt{x^2+3x+4}\right)+4} + C\\ &= \frac{-2}{2(x^2+3x+4)+(2x+3)\sqrt{x^2+3x+4}} + C\\ &= \frac{-2}{\sqrt{x^2+3x+4}\left(2x+3+2\sqrt{x^2+3x+4}\right)}\,\frac{2x+3-2\sqrt{x^2+3x+4}}{2x+3-2\sqrt{x^2+3x+4}} + C\\ &= \frac{2(2\sqrt{x^2+3x+4}-2x-3)}{\sqrt{x^2+3x+4}(-7)} + C = \frac{2(2x+3)}{7\sqrt{x^2+3x+4}} + D.\end{aligned}$$

32. If $u - x = \sqrt{x^2+bx+c}$, then $(u-x)^2 = x^2+bx+c$, and when this equation is solved for x, the result is $x = (u^2-c)/(2u+b)$. Thus,

$$dx = \frac{(2u+b)(2u)-(u^2-c)(2)}{(2u+b)^2}du = \frac{2(u^2+bu+c)}{(2u+b)^2}du.$$

Since $\sqrt{x^2+bx+c} = u - x = u - \dfrac{u^2-c}{2u+b} = \dfrac{u^2+bu+c}{2u+b}$,

$$\int \sqrt{x^2+bx+c}\,dx = \int \frac{u^2+bu+c}{2u+b}\frac{2(u^2+bu+c)}{(2u+b)^2}du = 2\int \frac{(u^2+bu+c)^2}{(2u+b)^3}du.$$

The integrand is a rational function of u.

33. If $(p+x)u = \sqrt{c+bx-x^2} = \sqrt{(p+x)(q-x)}$, then $u = \sqrt{(q-x)/(p+x)}$. If we square, $u^2 = (q-x)/(p+x)$, and when this is solved for x, the result is $x = (q-pu^2)/(u^2+1)$. Thus,

$$dx = \frac{(u^2+1)(-2pu)-(q-pu^2)(2u)}{(u^2+1)^2}du = \frac{-2(p+q)u}{(u^2+1)^2}du.$$

Since $\sqrt{c+bx-x^2} = (p+x)u = u\left(p + \dfrac{q-pu^2}{u^2+1}\right) = \dfrac{(p+q)u}{u^2+1}$,

$$\begin{aligned}\int \frac{1}{\sqrt{c+bx-x^2}}dx &= \int \frac{u^2+1}{(p+q)u}\left[\frac{-2(p+q)u}{(u^2+1)^2}du\right] = -2\int\frac{1}{u^2+1}du\\ &= -2\,\mathrm{Tan}^{-1}u + C = -2\,\mathrm{Tan}^{-1}\sqrt{\frac{q-x}{p+x}} + C.\end{aligned}$$

A similar derivation with the substitution $(q-x)u = \sqrt{c+bx-x^2}$ leads to

$$\int \frac{1}{\sqrt{c+bx-x^2}}dx = 2\,\mathrm{Tan}^{-1}\sqrt{\frac{p+x}{q-x}} + C.$$

34. If we set $u = a+bx$, then $du = b\,dx$, and

$$\begin{aligned}\int x(a+bx)^n\,dx &= \int\left(\frac{u-a}{b}\right)u^n\,\frac{du}{b} = \frac{1}{b^2}\int(u^{n+1}-au^n)\,du\\
&= \begin{cases}\dfrac{1}{b^2}\left(\dfrac{u^{n+2}}{n+2} - \dfrac{au^{n+1}}{n+1}\right) + C, & n \neq -2,\ -1\\ \dfrac{1}{b^2}(u - a\ln|u|) + C, & n=-1\\ \dfrac{1}{b^2}\left(\ln|u| + \dfrac{a}{u}\right) + C, & n=-2\end{cases}\\
&= \begin{cases}\dfrac{1}{b^2}\left[\dfrac{(a+bx)^{n+2}}{n+2} - \dfrac{a(a+bx)^{n+1}}{n+1}\right] + C, & n \neq -2,\ -1\\ \dfrac{1}{b^2}(bx - a\ln|a+bx|) + D, & n=-1\\ \dfrac{1}{b^2}\left(\ln|a+bx| + \dfrac{a}{a+bx}\right) + C, & n=-2.\end{cases}\end{aligned}$$

35. If we take the liberty of understanding the limiting procedure for the limits of $\pm\infty$, then

$$\begin{aligned}\int_{-\infty}^{\infty} x\,f(x)\,dx &= \int_{-\infty}^{\infty}\frac{x}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/(2\sigma^2)}dx = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}[(x-\mu)+\mu]e^{-(x-\mu)^2/(2\sigma^2)}dx\\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(x-\mu)e^{-(x-\mu)^2/(2\sigma^2)}dx + \mu\int_{-\infty}^{\infty}f(x)\,dx\\
&= \frac{1}{\sqrt{2\pi}\sigma}\left\{-\sigma^2e^{-(x-\mu)^2/(2\sigma^2)}\right\}_{-\infty}^{\infty} + \mu = \mu.\end{aligned}$$

36. We take the liberty of understanding limiting procedures for the limits of $\pm\infty$. When $a \ge 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-|a-x|}e^{-|x|}\,dx &= \int_{-\infty}^{0} e^{-(a-x)}e^{x}\,dx + \int_0^a e^{-(a-x)}e^{-x}\,dx + \int_a^{\infty} e^{a-x}e^{-x}\,dx\\ &= \int_{-\infty}^{0} e^{2x-a}\,dx + \int_0^a e^{-a}\,dx + \int_a^{\infty} e^{a-2x}\,dx\\ &= \left\{\frac{e^{2x-a}}{2}\right\}_{-\infty}^{0} + \left\{e^{-a}x\right\}_0^a + \left\{\frac{e^{a-2x}}{-2}\right\}_a^{\infty} = e^{-a}(a+1).\end{aligned}$$

When $a < 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-|a-x|}e^{-|x|}\,dx &= \int_{-\infty}^{a} e^{-(a-x)}e^{x}\,dx + \int_a^0 e^{a-x}e^{x}\,dx + \int_0^{\infty} e^{a-x}e^{-x}\,dx\\ &= \int_{-\infty}^{a} e^{2x-a}\,dx + \int_a^0 e^{a}\,dx + \int_0^{\infty} e^{a-2x}\,dx\\ &= \left\{\frac{e^{2x-a}}{2}\right\}_{-\infty}^{a} + \left\{e^{a}x\right\}_a^0 + \left\{\frac{e^{a-2x}}{-2}\right\}_0^{\infty} = e^{a}(1-a).\end{aligned}$$

These may be combined into $\displaystyle\int_{-\infty}^{\infty} e^{-|a-x|}e^{-|x|}\,dx = e^{-|a|}(1+|a|)$.

EXERCISES 8.2

1. When we set $u = x$, $dv = \sin x\,dx$, then $du = dx$, $v = -\cos x$, and

$$\int x\sin x\,dx = -x\cos x - \int -\cos x\,dx = -x\cos x + \sin x + C.$$

2. When we set $u = x^2$, $dv = e^{2x}\,dx$, then $du = 2x\,dx$, $v = e^{2x}/2$, and

$$\int x^2e^{2x}\,dx = \frac{x^2}{2}e^{2x} - \int 2x\frac{e^{2x}}{2}dx.$$

We now set $u = x$, $dv = e^{2x}\,dx$, in which case $du = dx$, $v = e^{2x}/2$, and

$$\int x^2e^{2x}\,dx = \frac{x^2}{2}e^{2x} - \left(\frac{x}{2}e^{2x} - \int\frac{e^{2x}}{2}dx\right) = \frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{1}{4}e^{2x} + C.$$

3. When we set $u = \ln x$, $dv = x^4\,dx$, then $du = (1/x)dx$, $v = x^5/5$, and

$$\int x^4\ln x\,dx = \frac{x^5}{5}\ln x - \int\frac{x^4}{5}dx = \frac{x^5}{5}\ln x - \frac{x^5}{25} + C.$$

4. When we set $u = \ln(2x)$, $dv = \sqrt{x}\,dx$, then $du = (1/x)\,dx$, $v = (2/3)x^{3/2}$, and

$$\int\sqrt{x}\ln(2x)\,dx = \frac{2}{3}x^{3/2}\ln(2x) - \int\frac{2}{3}x^{3/2}\frac{1}{x}dx = \frac{2}{3}x^{3/2}\ln(2x) - \frac{4}{9}x^{3/2} + C.$$

5. When we set $u = z$, $dv = \sec^2(z/3)\,dz$, then $du = dz$, $v = 3\tan(z/3)$, and

$$\int z\sec^2(z/3)\,dz = 3z\tan(z/3) - \int 3\tan(z/3)\,dz = 3z\tan(z/3) + 9\ln|\cos(z/3)| + C.$$

6. When we set $u = x$, $dv = \sqrt{3-x}\,dx$, then $du = dx$, $v = -\dfrac{2}{3}(3-x)^{3/2}$, and

$$\int x\sqrt{3-x}\,dx = -\frac{2x}{3}(3-x)^{3/2} - \int -\frac{2}{3}(3-x)^{3/2}\,dx = -\frac{2x}{3}(3-x)^{3/2} - \frac{4}{15}(3-x)^{5/2} + C.$$

7. When we set $u = \mathrm{Sin}^{-1}x$, $dv = dx$, then $du = \dfrac{1}{\sqrt{1-x^2}}dx$, $v = x$, and

$$\int \mathrm{Sin}^{-1}x\,dx = x\,\mathrm{Sin}^{-1}x - \int \frac{x}{\sqrt{1-x^2}}dx = x\,\mathrm{Sin}^{-1}x + \sqrt{1-x^2} + C.$$

8. When we set $u = x^2$, $dv = \sqrt{x+5}\,dx$, then $du = 2x\,dx$, $v = \dfrac{2}{3}(x+5)^{3/2}$, and

$$\int x^2\sqrt{x+5}\,dx = \frac{2}{3}x^2(x+5)^{3/2} - \int \frac{4}{3}x(x+5)^{3/2}\,dx.$$

We now set $u = x$, $dv = (x+5)^{3/2}\,dx$, in which case $du = dx$, $v = \dfrac{2}{5}(x+5)^{5/2}$, and

$$\begin{aligned}\int x^2\sqrt{x+5}\,dx &= \frac{2}{3}x^2(x+5)^{3/2} - \frac{4}{3}\left[\frac{2}{5}x(x+5)^{5/2} - \int \frac{2}{5}(x+5)^{5/2}\,dx\right]\\ &= \frac{2}{3}x^2(x+5)^{3/2} - \frac{8}{15}x(x+5)^{5/2} + \frac{16}{105}(x+5)^{7/2} + C.\end{aligned}$$

9. When we set $u = x$, $dv = \dfrac{1}{\sqrt{2+x}}dx$, then $du = dx$ and $v = 2\sqrt{2+x}$, and

$$\int \frac{x}{\sqrt{2+x}}dx = 2x\sqrt{2+x} - \int 2\sqrt{2+x}\,dx = 2x\sqrt{2+x} - \frac{4}{3}(2+x)^{3/2} + C.$$

10. When we set $u = x^2$, $dv = \dfrac{1}{\sqrt{2+x}}dx$, then $du = 2x\,dx$, $v = 2\sqrt{2+x}$, and

$$\int \frac{x^2}{\sqrt{2+x}}dx = 2x^2\sqrt{2+x} - \int 4x\sqrt{2+x}\,dx.$$

We now set $u = x$, $dv = \sqrt{2+x}\,dx$, in which case $du = dx$, $v = \dfrac{2}{3}(2+x)^{3/2}$, and

$$\begin{aligned}\int \frac{x^2}{\sqrt{2+x}}dx &= 2x^2\sqrt{2+x} - 4\left[\frac{2x}{3}(2+x)^{3/2} - \int \frac{2}{3}(2+x)^{3/2}\,dx\right]\\ &= 2x^2\sqrt{2+x} - \frac{8}{3}x(2+x)^{3/2} + \frac{16}{15}(2+x)^{5/2} + C.\end{aligned}$$

11. $\displaystyle\int \frac{x}{\sqrt{2+x^2}}dx = \sqrt{2+x^2} + C$

12. When we set $u = \ln x$, $dv = (x-1)^2\,dx$, then $du = \dfrac{1}{x}dx$, $v = \dfrac{1}{3}(x-1)^3$, and

$$\begin{aligned}\int (x-1)^2\ln x\,dx &= \frac{(x-1)^3}{3}\ln x - \int \frac{(x-1)^3}{3}\frac{1}{x}dx = \frac{(x-1)^3}{3}\ln x - \frac{1}{3}\int\left(x^2 - 3x + 3 - \frac{1}{x}\right)dx\\ &= \frac{(x-1)^3}{3}\ln x - \frac{1}{3}\left(\frac{x^3}{3} - \frac{3x^2}{2} + 3x - \ln x\right) + C.\end{aligned}$$

13. When we set $u = e^x$, $dv = \cos x\,dx$, then $du = e^x\,dx$, $v = \sin x$, and

$$\int e^x\cos x\,dx = e^x\sin x - \int e^x\sin x\,dx.$$

We now set $u = e^x$, $dv = \sin x\,dx$, in which case $du = e^x\,dx$, $v = -\cos x$, and

$$\int e^x\cos x\,dx = e^x\sin x - \left(-e^x\cos x - \int -e^x\cos x\,dx\right).$$

If we bring both integrals to the left,

$$2\int e^x\cos x\,dx = e^x\sin x + e^x\cos x \quad\Longrightarrow\quad \int e^x\cos x\,dx = \frac{e^x}{2}(\sin x + \cos x) + C.$$

14. When we set $u = \text{Tan}^{-1}x$, $dv = dx$, then $du = \dfrac{1}{1+x^2}dx$, $v = x$, and

$$\int \text{Tan}^{-1}x\,dx = x\,\text{Tan}^{-1}x - \int \frac{x}{1+x^2}dx = x\,\text{Tan}^{-1}x - \frac{1}{2}\ln(1+x^2) + C.$$

15. When we set $u = \cos(\ln x)$, $dv = dx$, then $du = -\dfrac{1}{x}\sin(\ln x)\,dx$, $v = x$, and

$$\int \cos(\ln x)\,dx = x\,\cos(\ln x) - \int -\sin(\ln x)\,dx.$$

We now set $u = \sin(\ln x)$, $dv = dx$, in which case $du = \dfrac{1}{x}\cos(\ln x)\,dx$, $v = x$, and

$$\int \cos(\ln x)\,dx = x\,\cos(\ln x) + x\,\sin(\ln x) - \int \cos(\ln x)\,dx.$$

If we bring both integrals to the left,

$$2\int \cos(\ln x)\,dx = x[\cos(\ln x) + \sin(\ln x)] \quad \Longrightarrow \quad \int \cos(\ln x)\,dx = \frac{x}{2}[\cos(\ln x) + \sin(\ln x)] + C.$$

16. When we set $u = e^{2x}$, $dv = \cos 3x\,dx$, then $du = 2e^{2x}\,dx$, $v = \dfrac{1}{3}\sin 3x$, and

$$\int e^{2x}\cos 3x\,dx = \frac{1}{3}e^{2x}\sin 3x - \int \frac{2}{3}e^{2x}\sin 3x\,dx.$$

We now set $u = e^{2x}$, $dv = \sin 3x\,dx$, in which case $du = 2e^{2x}\,dx$, $v = -\dfrac{1}{3}\cos 3x$, and

$$\int e^{2x}\cos 3x\,dx = \frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\left(-\frac{1}{3}e^{2x}\cos 3x - \int -\frac{2}{3}e^{2x}\cos 3x\,dx\right).$$

If we bring both integrals to the left,

$$\left(1 + \frac{4}{9}\right)\int e^{2x}\cos 3x\,dx = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x,$$

and therefore $\displaystyle\int e^{2x}\cos 3x\,dx = \frac{1}{13}e^{2x}(3\sin 3x + 2\cos 3x) + C.$

17. When we set $u = x^2$, $dv = \dfrac{x}{\sqrt{5+3x^2}}dx$, then $du = 2x\,dx$, $v = (1/3)\sqrt{5+3x^2}$, and

$$\begin{aligned}\int \frac{x^3}{\sqrt{5+3x^2}}dx &= \frac{x^2}{3}\sqrt{5+3x^2} - \int \frac{2x}{3}\sqrt{5+3x^2}\,dx = \frac{x^2}{3}\sqrt{5+3x^2} - \frac{2}{3}\left[\frac{1}{9}(5+3x^2)^{3/2}\right] + C\\ &= \frac{x^2}{3}\sqrt{5+3x^2} - \frac{2}{27}(5+3x^2)^{3/2} + C.\end{aligned}$$

18. When we set $u = \ln(x^2+4)$, $dv = dx$, then $du = \dfrac{2x}{x^2+4}dx$, $v = x$, and

$$\int \ln(x^2+4)\,dx = x\ln(x^2+4) - \int \frac{2x^2}{x^2+4}dx = x\ln(x^2+4) - 2\int\left(1 - \frac{4}{x^2+4}\right)dx.$$

We now set $u = x/2$ and $du = dx/2$,

$$\begin{aligned}\int \ln(x^2+4)\,dx &= x\ln(x^2+4) - 2x + 8\int \frac{1}{4u^2+4}(2\,du) = x\ln(x^2+4) - 2x + 4\int \frac{1}{u^2+1}du\\ &= x\ln(x^2+4) - 2x + 4\,\text{Tan}^{-1}u + C = x\ln(x^2+4) - 2x + 4\,\text{Tan}^{-1}\left(\frac{x}{2}\right) + C.\end{aligned}$$

19. If we differentiate the equation,

$$\begin{aligned} x^5e^x &= (Ax^5e^x + 5Ax^4e^x) + (Bx^4e^x + 4Bx^3e^x) + (Cx^3e^x + 3Cx^2e^x) + (Dx^2e^x + 2Dxe^x) \\ &\quad + (Exe^x + Ee^x) + Fe^x \\ &= Ax^5e^x + (5A+B)x^4e^x + (4B+C)x^3e^x + (3C+D)x^2e^x + (2D+E)xe^x + (E+F)e^x. \end{aligned}$$

When we equate coefficients of like terms, left and right, we obtain

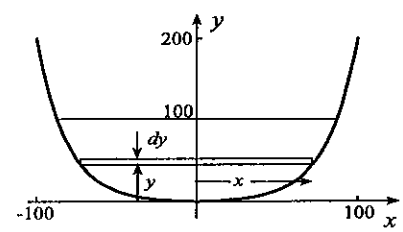
$$A = 1,\ 5A + B = 0,\ 4B + C = 0,\ 3C + D = 0,\ 2D + E = 0,\ E + F = 0.$$

These gives $B = -5$, $C = 20$, $D = -60$, $E = 120$, and $F = -120$. Thus,

$$\int x^5e^x\,dx = (x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120)e^x + G.$$

20. $F = \displaystyle\int_0^{100} 9.81(1000)(100-y)(2x)\,dy$

$\quad = 19\,620 \displaystyle\int_0^{100} (100-y)\frac{1}{k}\ln(y+1)\,dy.$



If we set $u = \ln(y+1)$, $dv = (100-y)\,dy$, then $du = \dfrac{1}{y+1}dy$, $v = -\dfrac{1}{2}(100-y)^2$, and

$$\begin{aligned} F &= \frac{19\,620}{k}\left[-\frac{1}{2}(100-y)^2\ln(y+1)\Big|_0^{100} - \int_0^{100} -\frac{1}{2}(100-y)^2\frac{1}{y+1}dy\right] \\ &= \frac{9810}{k}\int_0^{100}\frac{10\,000 - 200y + y^2}{y+1}dy = \frac{9810}{k}\int_0^{100}\left(y - 201 + \frac{10\,201}{y+1}\right)dy \\ &= \frac{9810}{k}\left\{\frac{y^2}{2} - 201y + 10\,201\ln|y+1|\right\}_0^{100} = 5.92\times 10^9\ \text{N}. \end{aligned}$$

21. When a constant of integration is included, there is no contradiction.

22. If we set $u = x$, $dv = \sin\dfrac{n\pi x}{L}dx$, $du = dx$, and $v = \dfrac{-L}{n\pi}\cos\dfrac{n\pi x}{L}$, then

$$\begin{aligned} \int_{-L}^{L} x\sin\frac{n\pi x}{L}\,dx &= \left\{\frac{-Lx}{n\pi}\cos\frac{n\pi x}{L}\right\}_{-L}^{L} - \int_{-L}^{L}\frac{-L}{n\pi}\cos\frac{n\pi x}{L}dx \\ &= -\frac{L^2}{n\pi}\cos n\pi - \frac{L^2}{n\pi}\cos(-n\pi) + \left\{\frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\right\}_{-L}^{L} = \frac{2(-1)^{n+1}L^2}{n\pi}. \end{aligned}$$

If we set $u = x$, $dv = \cos\dfrac{n\pi x}{L}dx$, $du = dx$, and $v = \dfrac{L}{n\pi}\sin\dfrac{n\pi x}{L}$, then

$$\int_{-L}^{L} x\cos\frac{n\pi x}{L}\,dx = \left\{\frac{Lx}{n\pi}\sin\frac{n\pi x}{L}\right\}_{-L}^{L} - \int_{-L}^{L}\frac{L}{n\pi}\sin\frac{n\pi x}{L}dx = -\left\{-\frac{L^2}{n^2\pi^2}\cos\frac{n\pi x}{L}\right\}_{-L}^{L} = 0.$$

23. If we set $u = x^2$, $dv = \sin\dfrac{n\pi x}{L}dx$, $du = 2x\,dx$, and $v = \dfrac{-L}{n\pi}\cos\dfrac{n\pi x}{L}$, then

$$\int_{-L}^{L} x^2\sin\frac{n\pi x}{L}\,dx = \left\{\frac{-Lx^2}{n\pi}\cos\frac{n\pi x}{L}\right\}_{-L}^{L} - \int_{-L}^{L}\frac{-2Lx}{n\pi}\cos\frac{n\pi x}{L}dx = \frac{2L}{n\pi}\int_{-L}^{L} x\cos\frac{n\pi x}{L}dx.$$

We now set $u = x$, $dv = \cos\dfrac{n\pi x}{L}dx$, $du = dx$, and $v = \dfrac{L}{n\pi}\sin\dfrac{n\pi x}{L}$, in which case

$$\int_{-L}^{L} x^2 \sin\frac{n\pi x}{L}\,dx = \frac{2L}{n\pi}\left\{\frac{Lx}{n\pi}\sin\frac{n\pi x}{L}\right\}_{-L}^{L} - \frac{2L}{n\pi}\int_{-L}^{L}\frac{L}{n\pi}\sin\frac{n\pi x}{L}dx$$
$$= -\frac{2L^2}{n^2\pi^2}\left\{\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\right\}_{-L}^{L} = 0.$$

If we set $u = x^2$, $dv = \cos\dfrac{n\pi x}{L}dx$, $du = 2x\,dx$, and $v = \dfrac{L}{n\pi}\sin\dfrac{n\pi x}{L}$, then

$$\int_{-L}^{L} x^2 \cos\frac{n\pi x}{L}\,dx = \left\{\frac{Lx^2}{n\pi}\sin\frac{n\pi x}{L}\right\}_{-L}^{L} - \int_{-L}^{L}\frac{2Lx}{n\pi}\sin\frac{n\pi x}{L}dx = -\frac{2L}{n\pi}\int_{-L}^{L} x\sin\frac{n\pi x}{L}dx.$$

We now set $u = x$, $dv = \sin\dfrac{n\pi x}{L}dx$, $du = dx$, and $v = \dfrac{-L}{n\pi}\cos\dfrac{n\pi x}{L}$, in which case

$$\int_{-L}^{L} x^2 \cos\frac{n\pi x}{L}\,dx = -\frac{2L}{n\pi}\left\{\frac{-Lx}{n\pi}\cos\frac{n\pi x}{L}\right\}_{-L}^{L} + \frac{2L}{n\pi}\int_{-L}^{L}\frac{-L}{n\pi}\cos\frac{n\pi x}{L}dx$$
$$= \frac{2L^2}{n^2\pi^2}[L\cos n\pi + L\cos(-n\pi)] - \frac{2L^2}{n^2\pi^2}\left\{\frac{L}{n\pi}\sin\frac{n\pi x}{L}\right\}_{-L}^{L} = \frac{4(-1)^nL^3}{n^2\pi^2}.$$

24. If we set $u = 1-2x$, $dv = \sin\dfrac{n\pi x}{L}dx$, $du = -2\,dx$, and $v = \dfrac{-L}{n\pi}\cos\dfrac{n\pi x}{L}$, then

$$\int_{-L}^{L}(1-2x)\sin\frac{n\pi x}{L}\,dx = \left\{\frac{-L(1-2x)}{n\pi}\cos\frac{n\pi x}{L}\right\}_{-L}^{L} - \int_{-L}^{L}\frac{2L}{n\pi}\cos\frac{n\pi x}{L}dx$$
$$= \frac{-L}{n\pi}[(1-2L)\cos n\pi - (1+2L)\cos(-n\pi)] - \frac{2L}{n\pi}\left\{\frac{L}{n\pi}\sin\frac{n\pi x}{L}\right\}_{-L}^{L} = \frac{4(-1)^nL^2}{n\pi}.$$

If we set $u = 1-2x$, $dv = \cos\dfrac{n\pi x}{L}dx$, $du = -2\,dx$, and $v = \dfrac{L}{n\pi}\sin\dfrac{n\pi x}{L}$, then

$$\int_{-L}^{L}(1-2x)\cos\frac{n\pi x}{L}\,dx = \left\{\frac{L(1-2x)}{n\pi}\sin\frac{n\pi x}{L}\right\}_{-L}^{L} - \int_{-L}^{L}\frac{-2L}{n\pi}\sin\frac{n\pi x}{L}dx = \frac{2L}{n\pi}\left\{\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\right\}_{-L}^{L} = 0.$$

25. If we set $u = 2x^2-3x$, $dv = \sin\dfrac{n\pi x}{L}dx$, $du = (4x-3)\,dx$, and $v = \dfrac{-L}{n\pi}\cos\dfrac{n\pi x}{L}$, then

$$\int_{-L}^{L}(2x^2-3x)\sin\frac{n\pi x}{L}\,dx = \left\{\frac{-L(2x^2-3x)}{n\pi}\cos\frac{n\pi x}{L}\right\}_{-L}^{L} - \int_{-L}^{L} -\frac{(4x-3)L}{n\pi}\cos\frac{n\pi x}{L}dx$$
$$= -\frac{L}{n\pi}[(2L^2-3L)\cos n\pi - (2L^2+3L)\cos(-n\pi)] + \frac{L}{n\pi}\int_{-L}^{L}(4x-3)\cos\frac{n\pi x}{L}dx$$
$$= \frac{6(-1)^nL^2}{n\pi} + \frac{L}{n\pi}\int_{-L}^{L}(4x-3)\cos\frac{n\pi x}{L}dx.$$

If we now set $u = 4x-3$, $dv = \cos\dfrac{n\pi x}{L}dx$, $du = 4\,dx$, and $v = \dfrac{L}{n\pi}\sin\dfrac{n\pi x}{L}$, then

$$\int_{-L}^{L}(2x^2-3x)\sin\frac{n\pi x}{L}\,dx = \frac{6(-1)^nL^2}{n\pi} + \frac{L}{n\pi}\left\{\frac{(4x-3)L}{n\pi}\sin\frac{n\pi x}{L}\right\}_{-L}^{L} - \frac{L}{n\pi}\int_{-L}^{L}\frac{4L}{n\pi}\sin\frac{n\pi x}{L}dx$$
$$= \frac{6(-1)^nL^2}{n\pi} - \frac{4L^2}{n^2\pi^2}\left\{\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\right\}_{-L}^{L} = \frac{6(-1)^nL^2}{n\pi}.$$

If we set $u = 2x^2-3x$, $dv = \cos\dfrac{n\pi x}{L}dx$, $du = (4x-3)\,dx$, and $v = \dfrac{L}{n\pi}\sin\dfrac{n\pi x}{L}$, then

$$\int_{-L}^{L}(2x^2-3x)\cos\frac{n\pi x}{L}\,dx = \left\{\frac{L(2x^2-3x)}{n\pi}\sin\frac{n\pi x}{L}\right\}_{-L}^{L} - \int_{-L}^{L}\frac{(4x-3)L}{n\pi}\sin\frac{n\pi x}{L}dx$$
$$= -\frac{L}{n\pi}\int_{-L}^{L}(4x-3)\sin\frac{n\pi x}{L}dx.$$

If we now set $u=4x-3$, $dv=\sin\frac{n\pi x}{L}dx$, $du=4\,dx$, and $v=\frac{-L}{n\pi}\cos\frac{n\pi x}{L}$, then

$$\int_{-L}^{L}(2x^2-3x)\cos\frac{n\pi x}{L}\,dx = -\frac{L}{n\pi}\left\{\frac{-(4x-3)L}{n\pi}\cos\frac{n\pi x}{L}\right\}_{-L}^{L} + \frac{L}{n\pi}\int_{-L}^{L}\frac{-4L}{n\pi}\cos\frac{n\pi x}{L}dx$$
$$= \frac{L^2}{n^2\pi^2}[(4L-3)\cos n\pi - (-4L-3)\cos(-n\pi)] - \frac{4L^2}{n^2\pi^2}\left\{\frac{L}{n\pi}\sin\frac{n\pi x}{L}\right\}_{-L}^{L}$$
$$= \frac{8(-1)^nL^3}{n^2\pi^2}.$$

26. When we set $u=x^{n-1}$, $dv=e^{-x}\,dx$, then $du=(n-1)x^{n-2}\,dx$, $v=-e^{-x}$, and

$$\Gamma(n) = \lim_{b\to\infty}\int_0^b x^{n-1}e^{-x}\,dx = \lim_{b\to\infty}\left[\left\{-x^{n-1}e^{-x}\right\}_0^b - \int_0^b -(n-1)x^{n-2}e^{-x}\,dx\right]$$
$$= \lim_{b\to\infty}\left[-b^{n-1}e^{-b} + (n-1)\int_0^b x^{n-2}e^{-x}\,dx\right] = (n-1)\int_0^\infty x^{n-2}e^{-x}\,dx.$$

Further integrations by parts lead to

$$\Gamma(n) = (n-1)(n-2)(n-3)\cdots(2)(1)\int_0^\infty e^{-x}\,dx = (n-1)!\lim_{b\to\infty}\int_0^b e^{-x}\,dx$$
$$= (n-1)!\lim_{b\to\infty}\left\{-e^{-x}\right\}_0^b = (n-1)!\lim_{b\to\infty}(1-e^{-b}) = (n-1)!.$$

27. $F(s) = \int_0^\infty e^{-st}e^{3t}\,dt = \lim_{b\to\infty}\int_0^b e^{(3-s)t}\,dt = \lim_{b\to\infty}\left\{\frac{e^{(3-s)t}}{3-s}\right\}_0^b = \lim_{b\to\infty}\left[\frac{e^{(3-s)b}-1}{3-s}\right]$
$= \frac{1}{s-3}$ (provided $s>3$)

28. $F(s) = \int_0^\infty t^2\,e^{-st}\,dt = \lim_{b\to\infty}\int_0^b t^2\,e^{-st}\,dt$ If we set $u=t^2$, $dv=e^{-st}\,dt$, $du=2t\,dt$, and $v=-e^{-st}/s$, then

$$F(s) = \lim_{b\to\infty}\left[\left\{\frac{t^2e^{-st}}{-s}\right\}_0^b - \int_0^b\frac{2te^{-st}}{-s}dt\right] = \lim_{b\to\infty}\frac{b^2e^{-bs}}{-s} + \frac{2}{s}\lim_{b\to\infty}\int_0^b te^{-st}\,dt.$$

The first limit is zero provided $s>0$. In the integral we set $u=t$, $dv=e^{-st}\,dt$, $du=dt$, and $v=-e^{-st}/s$, in which case

$$F(s) = \frac{2}{s}\lim_{b\to\infty}\left[\left\{\frac{te^{-st}}{-s}\right\}_0^b - \int_0^b -\frac{e^{-st}}{s}dt\right] = \frac{2}{s}\lim_{b\to\infty}\left(\frac{be^{-bs}}{-s}\right) + \frac{2}{s^2}\lim_{b\to\infty}\int_0^b e^{-st}\,dt$$
$$= \frac{2}{s^2}\lim_{b\to\infty}\left\{\frac{e^{-st}}{-s}\right\}_0^b = -\frac{2}{s^3}\lim_{b\to\infty}(e^{-bs}-1) = \frac{2}{s^3}.$$

29. If we set $u = e^{-st}$, $dv = \sin t\,dt$, $du = -se^{-st}\,dt$, and $v = -\cos t$, then

$$\int e^{-st}\sin t\,dt = -e^{-st}\cos t - \int se^{-st}\cos t\,dt.$$

If we now set $u = e^{-st}$, $dv = \cos t\,dt$, $du = -se^{-st}\,dt$, and $v = \sin t$, then

$$\int e^{-st}\sin t\,dt = -e^{-st}\cos t - s\left(e^{-st}\sin t - \int -se^{-st}\sin t\,dt\right)$$
$$= -e^{-st}(\cos t + s\sin t) - s^2\int e^{-st}\sin t\,dt.$$

When we solve for the integral, we obtain $\displaystyle\int e^{-st}\sin t\,dt = \frac{-e^{-st}(\cos t + s\sin t)}{1+s^2}$. Thus,

$$F(s) = \int_0^\infty e^{-st}\sin t\,dt = \lim_{b\to\infty}\int_0^b e^{-st}\sin t\,dt = \lim_{b\to\infty}\left\{\frac{-e^{-st}(\cos t + s\sin t)}{1+s^2}\right\}_0^b$$
$$= -\frac{1}{1+s^2}\lim_{b\to\infty}[e^{-bs}(\cos b + s\sin b) - 1] = \frac{1}{1+s^2} \quad (\text{provided } s > 0)$$

30. If we set $u = t$, $dv = e^{-(s+1)t}\,dt$, $du = dt$, and $v = \dfrac{e^{-(s+1)t}}{-(s+1)}$, then

$$\int te^{-t}e^{-st}\,dt = \int te^{-(s+1)t}\,dt = -\frac{te^{-(s+1)t}}{s+1} - \int\frac{e^{-(s+1)t}}{-(s+1)}dt = -\frac{te^{-(s+1)t}}{s+1} - \frac{e^{-(s+1)t}}{(s+1)^2}.$$

Thus,

$$F(s) = \int_0^\infty te^{-t}e^{-st}\,dt = \lim_{b\to\infty}\int_0^b te^{-(s+1)t}\,dt = \lim_{b\to\infty}\left\{-\frac{te^{-(s+1)t}}{s+1} - \frac{e^{-(s+1)t}}{(s+1)^2}\right\}_0^b$$
$$= \lim_{b\to\infty}\left[-\frac{be^{-(s+1)b}}{s+1} - \frac{e^{-(s+1)b}}{(s+1)^2} + \frac{1}{(s+1)^2}\right] = \frac{1}{(s+1)^2} \quad (\text{provided } s > -1)$$

31. Certainly $f(x) \geq 0$, and $\displaystyle\int_0^\infty f(x)\,dx = \int_0^\infty \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}dx$. If we set $u = x/\beta$, then $du = (1/\beta)\,dx$, and

$$\int_0^\infty f(x)\,dx = \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty(\beta u)^{\alpha-1}e^{-u}(\beta\,du) = \frac{1}{\Gamma(\alpha)}\int_0^\infty u^{\alpha-1}e^{-u}\,du = \frac{1}{\Gamma(\alpha)}\Gamma(\alpha) = 1.$$

32. $$F(\omega) = \int_{-\infty}^\infty f(t)e^{-i\omega t}\,dt = \int_{-L/2}^{L/2} e^{-i\omega t}dt = \left\{\frac{e^{-i\omega t}}{-i\omega}\right\}_{-L/2}^{L/2}$$
$$= \frac{-1}{i\omega}(e^{-i\omega L/2} - e^{i\omega L/2}) = \frac{1}{i\omega}\left(e^{i\omega L/2} - e^{-i\omega L/2}\right) = \frac{2}{i\omega}\sinh\left(\frac{i\omega L}{2}\right).$$

33. $$F(\omega) = \int_{-\infty}^\infty f(t)e^{-i\omega t}\,dt = \int_{-T}^0\left(1+\frac{t}{T}\right)e^{-i\omega t}\,dt + \int_0^T\left(1-\frac{t}{T}\right)e^{-i\omega t}\,dt$$

If we set $u = 1 + t/T$, $dv = e^{-i\omega t}\,dt$, $du = dt/T$, $v = \dfrac{e^{-i\omega t}}{-i\omega}$ in the first integral, and $u = 1 - t/T$, $dv = e^{-i\omega t}\,dt$, $du = -dt/T$, $v = \dfrac{e^{-i\omega t}}{-i\omega}$ in the second,

$$F(\omega) = \left\{\left(1+\frac{t}{T}\right)\left(\frac{e^{-i\omega t}}{-i\omega}\right)\right\}_{-T}^0 - \int_{-T}^0\frac{e^{-i\omega t}}{-i\omega T}dt + \left\{\left(1-\frac{t}{T}\right)\left(\frac{e^{-i\omega t}}{-i\omega}\right)\right\}_0^T - \int_0^T\frac{e^{-i\omega t}}{i\omega T}dt$$
$$= -\frac{1}{i\omega} + \left\{\frac{e^{-i\omega t}}{-i^2\omega^2 T}\right\}_{-T}^0 + \frac{1}{i\omega} - \left\{\frac{e^{-i\omega t}}{-i^2\omega^2 T}\right\}_0^T = \frac{1}{i^2\omega^2 T}\left(-1 + e^{i\omega T} + e^{-i\omega T} - 1\right)$$
$$= \frac{1}{\omega^2 T}\left[2 - 2\left(\frac{e^{i\omega T} + e^{-i\omega T}}{2}\right)\right] = \frac{2}{\omega^2 T}(1 - \cosh i\omega T).$$

34. $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt = \int_{-\infty}^{\infty} e^{-a|t|}e^{-i\omega t}\,dt = \int_{-\infty}^{0} e^{(a-i\omega)t}\,dt + \int_{0}^{\infty} e^{-(a+i\omega)t}\,dt$

$$= \lim_{b\to-\infty}\left\{\frac{e^{(a-i\omega)t}}{a-i\omega}\right\}_b^0 + \lim_{b\to\infty}\left\{\frac{e^{-(a+i\omega)t}}{-(a+i\omega)}\right\}_0^b$$
$$= \frac{1}{a-i\omega}\lim_{b\to-\infty}[1-e^{(a-i\omega)b}] - \frac{1}{a+i\omega}\lim_{b\to\infty}[e^{-(a+i\omega)b}-1] = \frac{1}{a-i\omega}+\frac{1}{a+i\omega} = \frac{2a}{\omega^2+a^2}.$$

35. $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt = \int_a^b e^{-i\omega t}\,dt = \left\{\frac{e^{-i\omega t}}{-i\omega}\right\}_a^b = \frac{e^{-i\omega a}-e^{-i\omega b}}{i\omega}$

36. If we set $u = \mathrm{Tan}^{-1}\sqrt{x}$, $dv = dx$, $du = \dfrac{1}{2\sqrt{x}(1+x)}dx$, and $v = x$,

$$\int \mathrm{Tan}^{-1}\sqrt{x}\,dx = x\,\mathrm{Tan}^{-1}\sqrt{x} - \int \frac{x}{2\sqrt{x}(1+x)}dx = x\,\mathrm{Tan}^{-1}\sqrt{x} - \frac{1}{2}\int\frac{\sqrt{x}}{1+x}dx.$$

If we now set $u = \sqrt{x}$, from which $x = u^2$, and $dx = 2u\,du$, then

$$\int \mathrm{Tan}^{-1}\sqrt{x}\,dx = x\,\mathrm{Tan}^{-1}\sqrt{x} - \frac{1}{2}\int\frac{u}{1+u^2}(2u\,du) = x\,\mathrm{Tan}^{-1}\sqrt{x} - \int\left(1-\frac{1}{1+u^2}\right)du$$
$$= x\,\mathrm{Tan}^{-1}\sqrt{x} - u + \mathrm{Tan}^{-1}u + C = x\,\mathrm{Tan}^{-1}\sqrt{x} - \sqrt{x} + \mathrm{Tan}^{-1}\sqrt{x} + C.$$

37. $\int x^2\cos^2 x\,dx = \int x^2\left(\dfrac{1+\cos 2x}{2}\right)dx = \dfrac{x^3}{6} + \dfrac{1}{2}\int x^2\cos 2x\,dx$

When we set $u = x^2$, $dv = \cos 2x\,dx$, then $du = 2x\,dx$, $v = \dfrac{1}{2}\sin 2x$, and

$$\int x^2\cos^2 x\,dx = \frac{x^3}{6} + \frac{1}{2}\left(\frac{x^2}{2}\sin 2x - \int x\sin 2x\,dx\right).$$

We now set $u = x$, $dv = \sin 2x\,dx$, in which case $du = dx$, $v = -\dfrac{1}{2}\cos 2x$, and

$$\int x^2\cos^2 x\,dx = \frac{x^3}{6} + \frac{x^2}{4}\sin 2x - \frac{1}{2}\left(-\frac{x}{2}\cos 2x - \int -\frac{1}{2}\cos 2x\,dx\right)$$
$$= \frac{x^3}{6} + \frac{x^2}{4}\sin 2x + \frac{x}{4}\cos 2x - \frac{1}{8}\sin 2x + C.$$

38. First we evaluate the integral of $e^x\sin x$. When we set $u = e^x$, $dv = \sin x\,dx$, then $du = e^x\,dx$, $v = -\cos x$, and

$$\int e^x\sin x\,dx = -e^x\cos x - \int -e^x\cos x\,dx.$$

We now set $u = e^x$, $dv = \cos x\,dx$, in which case $du = e^x\,dx$, $v = \sin x$, and

$$\int e^x\sin x\,dx = -e^x\cos x + \left(e^x\sin x - \int e^x\sin x\,dx\right).$$

If we bring both integrals to the left,

$$2\int e^x\sin x\,dx = e^x\sin x - e^x\cos x \quad\Longrightarrow\quad \int e^x\sin x\,dx = \frac{e^x}{2}(\sin x - \cos x) + C.$$

In the given integral we now set $u = x$, $dv = e^x\sin x\,dx$, $du = dx$, and $v = e^x(\sin x - \cos x)/2$, in which case

$$\int x\,e^x\sin x\,dx = \frac{x\,e^x(\sin x - \cos x)}{2} - \int\frac{e^x(\sin x-\cos x)}{2}dx.$$

We now need the integral of $e^x\cos x$. When we set $u = e^x$, $dv = \cos x\,dx$, $du = e^x\,dx$, $v = \sin x$, then

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$

We now set $u = e^x$, $dv = \sin x\,dx$, in which case $du = e^x\,dx$, $v = -\cos x$, and

$$\int e^x \cos x\,dx = e^x \sin x - \left(-e^x \cos x - \int -e^x \cos x\,dx\right).$$

If we bring both integrals to the left,

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x \quad \Longrightarrow \quad \int e^x \cos x\,dx = \frac{e^x}{2}(\sin x + \cos x) + C.$$

We can now say that

$$\frac{1}{2}\int e^x(\sin x - \cos x)\,dx = \frac{1}{2}\left[\frac{e^x(\sin x - \cos x)}{2} - \frac{e^x(\sin x + \cos x)}{2}\right] = -\frac{e^x \cos x}{2}.$$

Finally,

$$\int x\,e^x \sin x\,dx = \frac{x\,e^x(\sin x - \cos x)}{2} + \frac{e^x \cos x}{2} + C.$$

39. If we set $x = \sin^2\theta$, then $dx = 2\sin\theta\cos\theta\,d\theta$, and

$$\int_0^1 x^n(1-x)^m\,dx = \int_0^{\pi/2} \sin^{2n}\theta\,(\cos^2\theta)^m\,2\sin\theta\cos\theta\,d\theta = 2\int_0^{\pi/2}\sin^{2n+1}\theta\cos^{2m+1}\theta\,d\theta.$$

Consider now using integration by parts on the integral of $\sin^p\theta\cos^q\theta$ where $p \ge 2$ and $q \ge 1$ are integers. With $u = \sin^{p-1}\theta$, $dv = \cos^q\theta\sin\theta\,d\theta$, $du = (p-1)\sin^{p-2}\theta\cos\theta\,d\theta$, and $v = -\dfrac{\cos^{q+1}\theta}{q+1}$,

$$\begin{aligned}\int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta &= \left\{\frac{-1}{q+1}\sin^{p-1}\theta\cos^{q+1}\theta\right\}_0^{\pi/2} + \int_0^{\pi/2}\frac{p-1}{q+1}\sin^{p-2}\theta\cos^{q+2}\theta\,d\theta\\ &= \frac{p-1}{q+1}\int_0^{\pi/2}\sin^{p-2}\theta(1-\sin^2\theta)\cos^q\theta\,d\theta\\ &= \frac{p-1}{q+1}\int_0^{\pi/2}\sin^{p-2}\theta\cos^q\theta\,d\theta - \frac{p-1}{q+1}\int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta.\end{aligned}$$

If we combine the integrals,

$$\left(1 + \frac{p-1}{q+1}\right)\int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta = \frac{p-1}{q+1}\int_0^{\pi/2}\sin^{p-2}\theta\cos^q\theta\,d\theta,$$

from which

$$\int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta = \frac{p-1}{p+q}\int_0^{\pi/2}\sin^{p-2}\theta\cos^q\theta\,d\theta.$$

We use this as a formula on the integral of $\sin^{2n+1}\theta\cos^{2m+1}\theta$ to eliminate the power on $\sin\theta$,

$$\begin{aligned}\int_0^1 x^n(1-x)^m\,dx = 2\int_0^{\pi/2}\sin^{2n+1}\theta\cos^{2m+1}\theta\,d\theta &= \frac{2(2n)}{2n+2m+2}\int_0^{\pi/2}\sin^{2n-1}\theta\cos^{2m+1}\theta\,d\theta\\ &= \frac{2n}{n+m+1}\,\frac{2n-2}{2n+2m}\int_0^{\pi/2}\sin^{2n-3}\theta\cos^{2m+1}\theta\,d\theta\\ &= \frac{2n(n-1)}{(n+m+1)(n+m)}\,\frac{2n-4}{2n+2m-2}\int_0^{\pi/2}\sin^{2n-5}\theta\cos^{2m+1}\theta\,d\theta\\ &\quad\vdots \qquad \vdots \qquad \vdots \qquad \vdots\end{aligned}$$

$$= \frac{2n(n-1)(n-2)\cdots(1)}{(n+m+1)(n+m)(n+m-1)\cdots(m+2)}\int_0^{\pi/2}\sin\theta\,\cos^{2m+1}\theta\,d\theta$$
$$= \frac{2\,n!}{(n+m+1)(n+m)(n+m-1)\cdots(m+2)}\left\{-\frac{\cos^{2m+2}\theta}{2m+2}\right\}_0^{\pi/2}$$
$$= \frac{n!}{(n+m+1)(n+m)(n+m-1)\cdots(m+1)} = \frac{n!\,m!}{(n+m+1)!}.$$

40. We omit constants of integration in the following proof by mathematical induction. When $n=1$, the left side is the integral of $x\cos x$. If we set $u=x$, $dv=\cos x\,dx$, $du=dx$, and $v=\sin x$, then

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x.$$

When $n=1$, the right side of the formula gives

$$\sin x\sum_{r=0}^{\lfloor 1/2\rfloor}\frac{(-1)^r}{(1-2r)!}x^{1-2r} + \cos x\sum_{r=0}^{\lfloor 0/2\rfloor}\frac{(-1)^r}{(1-2r-1)!}x^{-2r} = \sin x\left[\frac{(-1)^0x}{1!}\right] + \cos x\left[\frac{(-1)^0}{0!}\right] = x\sin x + \cos x.$$

Thus, the formula is correct for $n=1$. When $n=2$, the left side is the integral of $x^2\cos x$. If we set $u=x^2$, $dv=\cos x\,dx$, $du=2x\,dx$, and $v=\sin x$, then

$$\int x^2\cos x\,dx = x^2\sin x - \int 2x\sin x\,dx.$$

If we set $u=x$, $dv=\sin x\,dx$, $du=dx$, and $v=-\cos x$, then

$$\int x^2\cos x\,dx = x^2\sin x - 2\left(-x\cos x - \int -\cos x\,dx\right) = x^2\sin x + 2x\cos x - 2\sin x.$$

When $n=2$, the right side of the formula gives

$$\sin x\sum_{r=0}^{\lfloor 1\rfloor}\frac{(-1)^r2!}{(2-2r)!}x^{2-2r} + \cos x\sum_{r=0}^{\lfloor 1/2\rfloor}\frac{(-1)^r2!}{(2-2r-1)!}x^{2-2r-1} = \sin x\left[\frac{2x^2}{2}+\frac{(-1)2}{0!}\right] + \cos x\left(\frac{2x}{1!}\right)$$
$$= (x^2-2)\sin x + 2x\cos x.$$

The formula is valid for $n=2$ also. Suppose the result is valid for some integer k; that is, assume that

$$\int x^k\cos x\,dx = \sin x\sum_{r=0}^{\lfloor k/2\rfloor}\frac{(-1)^rk!}{(k-2r)!}x^{k-2r} + \cos x\sum_{r=0}^{\lfloor (k-1)/2\rfloor}\frac{(-1)^rk!}{(k-2r-1)!}x^{k-2r-1}.$$

If we set $u=x^{k+2}$, $dv=\cos x\,dx$, $du=(k+2)x^{k+1}\,dx$, and $v=\sin x$, then

$$\int x^{k+2}\cos x\,dx = x^{k+2}\sin x - \int (k+2)x^{k+1}\sin x\,dx.$$

We now set $u=x^{k+1}$, $dv=\sin x\,dx$, $du=(k+1)x^k\,dx$, and $v=-\cos x$, in which case

$$\int x^{k+2}\cos x\,dx = x^{k+2}\sin x - (k+2)\left[-x^{k+1}\cos x - \int -(k+1)x^k\cos x\,dx\right]$$
$$= x^{k+2}\sin x + (k+2)x^{k+1}\cos x - (k+2)(k+1)\int x^k\cos x\,dx$$
$$= x^{k+2}\sin x + (k+2)x^{k+1}\cos x - (k+2)(k+1)\left[\sin x\sum_{r=0}^{\lfloor k/2\rfloor}\frac{(-1)^rk!}{(k-2r)!}x^{k-2r}\right.$$
$$\left. + \cos x\sum_{r=0}^{\lfloor (k-1)/2\rfloor}\frac{(-1)^rk!}{(k-2r-1)!}x^{k-2r-1}\right]$$

$$= x^{k+2}\sin x + (k+2)x^{k+1}\cos x - \sin x \sum_{r=0}^{\lfloor k/2 \rfloor} \frac{(-1)^r (k+2)!}{(k-2r)!} x^{k-2r}$$
$$- \cos x \sum_{r=0}^{\lfloor (k-1)/2 \rfloor} \frac{(-1)^r (k+2)!}{(k-2r-1)!} x^{k-2r-1}$$
$$= x^{k+2}\sin x + (k+2)x^{k+1}\cos x - \sin x \sum_{r=1}^{1+\lfloor k/2 \rfloor} \frac{(-1)^{r-1} (k+2)!}{[k-2(r-1)]!} x^{k-2(r-1)}$$
$$- \cos x \sum_{r=1}^{1+\lfloor (k-1)/2 \rfloor} \frac{(-1)^{r-1} (k+2)!}{[k-2(r-1)-1]!} x^{k-2(r-1)-1}$$
$$= x^{k+2}\sin x + (k+2)x^{k+1}\cos x + \sin x \sum_{r=1}^{\lfloor (k+2)/2 \rfloor} \frac{(-1)^r (k+2)!}{[k+2-2r]!} x^{k+2-2r}$$
$$+ \cos x \sum_{r=1}^{\lfloor [(k+2)-1]/2 \rfloor} \frac{(-1)^r (k+2)!}{[(k+2)-2r-1]!} x^{k+2-2r-1}$$
$$= \sin x \sum_{r=0}^{\lfloor (k+2)/2 \rfloor} \frac{(-1)^r (k+2)!}{[k+2-2r]!} x^{k+2-2r} + \cos x \sum_{r=0}^{\lfloor [(k+2)-1]/2 \rfloor} \frac{(-1)^r (k+2)!}{[(k+2)-2r-1]!} x^{k+2-2r-1}.$$

But this is the formula for $n = k+2$. By mathematical induction then, the result is valid for all $n \geq 1$.

EXERCISES 8.3

1. $\displaystyle\int \cos^3 x \sin x\,dx = -\frac{1}{4}\cos^4 x + C$

2. $\displaystyle\int \frac{\cos x}{\sin^3 x}dx = -\frac{1}{2\sin^2 x} + C$

3. $\displaystyle\int \tan^5 x \sec^2 x\,dx = \frac{1}{6}\tan^6 x + C$

4. $\displaystyle\int \csc^3 x \cot x\,dx = -\frac{1}{3}\csc^3 x + C$

5. $\displaystyle\int \cos^3(x+2)\,dx = \int [1 - \sin^2(x+2)]\cos(x+2)\,dx = \sin(x+2) - \frac{1}{3}\sin^3(x+2) + C$

6. $\displaystyle\int \sqrt{\tan x}\sec^4 x\,dx = \int \sqrt{\tan x}(1+\tan^2 x)\sec^2 x\,dx$

$$= \int (\tan^{1/2} x + \tan^{5/2} x)\sec^2 x\,dx = \frac{2}{3}\tan^{3/2} x + \frac{2}{7}\tan^{7/2} x + C$$

7. $\displaystyle\int \frac{1}{\sin^4 t}dt = \int \csc^4 t\,dt = \int \csc^2 t(1+\cot^2 t)\,dt = -\cot t - \frac{1}{3}\cot^3 t + C$

8. $\displaystyle\int \sec^6 3x \tan 3x\,dx = \frac{1}{18}\sec^6 3x + C$

9. $\displaystyle\int \cos^2 x\,dx = \int \left(\frac{1+\cos 2x}{2}\right)dx = \frac{x}{2} + \frac{1}{4}\sin 2x + C$

10. $\displaystyle\int \frac{\tan^3 x \sec^2 x}{\sin^2 x}dx = \int \frac{\sin x}{\cos^5 x}dx = \frac{1}{4\cos^4 x} + C = \frac{1}{4}\sec^4 x + C$

11. $\displaystyle\int \sin^3 y \cos^2 y\,dy = \int \sin y(1-\cos^2 y)\cos^2 y\,dy = -\frac{1}{3}\cos^3 y + \frac{1}{5}\cos^5 y + C$

12. $\displaystyle\int \frac{\csc^2\theta}{\cot^2\theta}d\theta = \frac{1}{\cot\theta} + C = \tan\theta + C$

13. $\displaystyle\int \frac{\sin\theta}{1+\cos\theta}d\theta = -\ln|1+\cos\theta| + C = -\ln(1+\cos\theta) + C$

14. $\displaystyle\int \frac{\sec^2 x}{\sqrt{1+\tan x}}dx = 2\sqrt{1+\tan x} + C$

15. $\displaystyle\int \cos\theta \sin 2\theta\, d\theta = \int 2\sin\theta\cos^2\theta\, d\theta = -\frac{2}{3}\cos^3\theta + C$

16. $\displaystyle\int \frac{3+4\csc^2 x}{\cot^2 x}dx = \int (3\tan^2 x + 4\sec^2 x)\,dx = \int (3\sec^2 x - 3 + 4\sec^2 x)\,dx = 7\tan x - 3x + C$

17. $\displaystyle\int \sin^5 x\cos^5 x\,dx = \int \sin^5 x(1-\sin^2 x)^2\cos x\,dx = \int \sin^5 x(1-2\sin^2 x+\sin^4 x)\cos x\,dx$

$$= \frac{1}{6}\sin^6 x - \frac{1}{4}\sin^8 x + \frac{1}{10}\sin^{10} x + C$$

18. $\displaystyle\int \sin^4 x\,dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx$

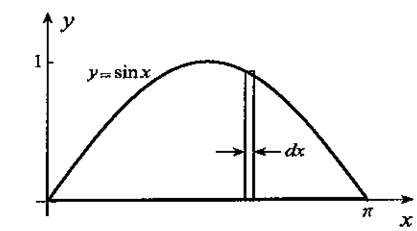
$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right) dx = \frac{1}{4}\left(\frac{3x}{2} - \sin 2x + \frac{1}{8}\sin 4x\right) + C$$

19. $\displaystyle\int \frac{\tan^3 x}{\sec^4 x}dx = \int \sin^3 x\cos x\,dx = \frac{1}{4}\sin^4 x + C$

20. $\displaystyle\int \frac{\csc^4 x}{\cot^3 x}dx = \int \frac{(1+\cot^2 x)\csc^2 x}{\cot^3 x}dx = \int \left(\frac{1}{\cot^3 x} + \frac{1}{\cot x}\right)\csc^2 x\,dx$

$$= \frac{1}{2\cot^2 x} - \ln|\cot x| + C = \frac{1}{2}\tan^2 x + \ln|\tan x| + C$$

21. $\displaystyle A = \int_0^\pi \sin x\,dx = \Big\{-\cos x\Big\}_0^\pi = 2$

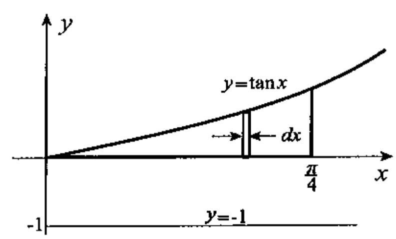


22. $\displaystyle V = \int_0^{\pi/4} [\pi(1+\tan x)^2 - \pi(1)^2]\,dx$

$$= \pi\int_0^{\pi/4} (\tan^2 x + 2\tan x)\,dx$$

$$= \pi\int_0^{\pi/4} (\sec^2 x - 1 + 2\tan x)\,dx$$

$$= \pi\Big\{\tan x - x + 2\ln|\sec x|\Big\}_0^{\pi/4} = \pi(1 - \pi/4 + \ln 2)$$



23. $\displaystyle\int_0^\pi \sqrt{1-\sin^2 x}\,dx = \int_0^\pi |\cos x|\,dx = \int_0^{\pi/2}\cos x\,dx + \int_{\pi/2}^\pi -\cos x\,dx = \Big\{\sin x\Big\}_0^{\pi/2} - \Big\{\sin x\Big\}_{\pi/2}^\pi = 2$

24. $\displaystyle\int \cot^4 z\,dz = \int \cot^2 z(\csc^2 z - 1)\,dz = \int (\cot^2 z\csc^2 z - \csc^2 z + 1)\,dz = -\frac{1}{3}\cot^3 z + \cot z + z + C$

25. If we set $u = 3+\sin\theta$ and $du = \cos\theta\,d\theta$, then

$$\int \frac{\cos^3\theta}{3+\sin\theta}d\theta = \int \frac{(1-\sin^2\theta)\cos\theta}{3+\sin\theta}d\theta = \int \frac{1-(u-3)^2}{u}du = \int\left(-\frac{8}{u} + 6 - u\right)du$$

$$= -8\ln|u| + 6u - \frac{u^2}{2} + C = -8\ln|3+\sin\theta| + 6(3+\sin\theta) - \frac{1}{2}(3+\sin\theta)^2 + C$$

$$= 3\sin\theta - 8\ln(3+\sin\theta) - \frac{1}{2}\sin^2\theta + D.$$

26. $\displaystyle\int \frac{\cos^4\theta}{1+\sin\theta}d\theta = \int \frac{(1-\sin^2\theta)\cos^2\theta}{1+\sin\theta}d\theta = \int \frac{(1-\sin\theta)(1+\sin\theta)\cos^2\theta}{1+\sin\theta}d\theta = \int (1-\sin\theta)\cos^2\theta\,d\theta$

$$= \int\left(\frac{1+\cos 2\theta}{2} - \cos^2\theta\sin\theta\right)d\theta = \frac{\theta}{2} + \frac{1}{4}\sin 2\theta + \frac{1}{3}\cos^3\theta + C$$

27. $$\int \sin^4 x \cos^2 x\,dx = \int \sin^2 x(\sin x \cos x)^2\,dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{\sin 2x}{2}\right)^2 dx$$
$$= \frac{1}{8}\int (\sin^2 2x - \sin^2 2x \cos 2x)\,dx = \frac{1}{8}\int \left(\frac{1-\cos 4x}{2} - \sin^2 2x \cos 2x\right) dx$$
$$= \frac{x}{16} - \frac{1}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C$$

28. Using identity 1.48c, $\int \cos 6x \cos 2x\,dx = \int \frac{1}{2}(\cos 8x + \cos 4x)\,dx = \frac{1}{16}\sin 8x + \frac{1}{8}\sin 4x + C$

29. $\int \cos^2 2x \sin 3x\,dx = \int \left(\frac{1+\cos 4x}{2}\right)\sin 3x\,dx$ We use identity 1.48b on the second term,

$$\int \cos^2 2x \sin 3x\,dx = \frac{1}{2}\int \left(\sin 3x + \frac{1}{2}\sin 7x - \frac{1}{2}\sin x\right) dx = -\frac{1}{6}\cos 3x - \frac{1}{28}\cos 7x + \frac{1}{4}\cos x + C.$$

30. $$\int \frac{1}{\sin x \cos^2 x}dx = \int \csc x \sec^2 x\,dx = \int \csc x(1+\tan^2 x)\,dx$$
$$= \int (\csc x + \sec x \tan x)\,dx = \ln|\csc x - \cot x| + \sec x + C$$

31. If we set $u = \sec^3 x$, $dv = \sec^2 x\,dx$, $du = 3\sec^3 x \tan x\,dx$, and $v = \tan x$, then

$$\int \sec^5 x\,dx = \sec^3 x \tan x - \int 3\sec^3 x \tan^2 x\,dx = \sec^3 x \tan x - 3\int \sec^3 x(\sec^2 x - 1)\,dx$$
$$= \sec^3 x \tan x - 3\int \sec^5 x\,dx + 3\int \sec^3 x\,dx.$$

If we solve for the integral of $\sec^5 x$, we obtain

$$\int \sec^5 x\,dx = \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}\int \sec^3 x\,dx.$$

We now use the result of Example 8.9,

$$\int \sec^5 x\,dx = \frac{1}{4}\sec^3 x \tan x + \frac{3}{8}\sec x \tan x + \frac{3}{8}\ln|\sec x + \tan x| + C.$$

32. The average power is the integral of Vi over one period $2\pi/\omega$ divided by the period,

$$P_{av} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} V_m \cos(\omega t + \phi_2)\, i_m \cos(\omega t + \phi_1)\,dt$$
$$= \frac{\omega V_m i_m}{2\pi}\int_0^{2\pi/\omega} \frac{1}{2}[\cos(2\omega t + \phi_1 + \phi_2) + \cos(\phi_1 - \phi_2)]\,dt$$
$$= \frac{\omega V_m i_m}{4\pi}\left\{\frac{1}{2\omega}\sin(2\omega t + \phi_1 + \phi_2) + t\cos(\phi_1 - \phi_2)\right\}_0^{2\pi/\omega}$$
$$= \frac{\omega V_m i_m}{4\pi}\left[\frac{1}{2\omega}\sin(4\pi + \phi_1 + \phi_2) - \frac{1}{2\omega}\sin(\phi_1 + \phi_2) + \frac{2\pi}{\omega}\cos(\phi_1 - \phi_2)\right]$$
$$= \frac{V_m i_m}{2}\cos(\phi_1 - \phi_2).$$

33. The current can be expressed in the form $R\sin(\omega t + \phi)$ by setting

$$A\cos\omega t + B\sin\omega t = R\sin(\omega t + \phi) = R(\sin\omega t\cos\phi + \cos\omega t\sin\phi).$$

This equation is satisfied for all t if we choose R and ϕ such that $R\cos\phi = B$ and $R\sin\phi = A$. Squaring and adding these gives $R^2 = A^2 + B^2$, and therefore the amplitude of the current is $R = \sqrt{A^2+B^2}$. If we choose $T = 2\pi/\omega$, the I_{rms} current is given by

$$\begin{aligned}(I_{\text{rms}})^2 &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega}(A\cos\omega t + B\sin\omega t)^2\,dt\\
&= \frac{\omega}{2\pi}\int_0^{2\pi/\omega}(A^2\cos^2\omega t + 2AB\,\sin\omega t\,\cos\omega t + B^2\sin^2\omega t)\,dt\\
&= \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\left[A^2\left(\frac{1+\cos 2\omega t}{2}\right) + AB\,\sin 2\omega t + B^2\left(\frac{1-\cos 2\omega t}{2}\right)\right]dt\\
&= \frac{\omega}{2\pi}\left\{\frac{A^2}{2}\left(t+\frac{1}{2\omega}\sin 2\omega t\right) - \frac{AB}{2\omega}\cos 2\omega t + \frac{B^2}{2}\left(t - \frac{1}{2\omega}\sin 2\omega t\right)\right\}\Bigg|_0^{2\pi/\omega}\\
&= \frac{1}{2}(A^2+B^2).\end{aligned}$$

Thus, $I_{\text{rms}} = \sqrt{A^2+B^2}/\sqrt{2} = R/\sqrt{2}$.

34. $F_{dc} = \dfrac{\omega}{2\pi}\displaystyle\int_{-\pi/\omega}^{\pi/\omega}(A\,\cos\omega t + B\,\sin\omega t)\,dt = \frac{\omega}{2\pi}\left\{\frac{A}{\omega}\sin\omega t - \frac{B}{\omega}\cos\omega t\right\}\Bigg|_{-\pi/\omega}^{\pi/\omega} = 0$

35. $F_{dc} = \dfrac{\omega}{2\pi}\displaystyle\int_{-\pi/\omega}^{\pi/\omega}(A + B\,\cos\omega t)\,dt = \frac{\omega}{2\pi}\left\{At + \frac{B}{\omega}\sin\omega t\right\}\Bigg|_{-\pi/\omega}^{\pi/\omega} = A$

36. $F_{dc} = \dfrac{\omega}{\pi}\displaystyle\int_{-\pi/(2\omega)}^{\pi/(2\omega)}\sin^2\omega t\,dt = \frac{\omega}{\pi}\int_{-\pi/(2\omega)}^{\pi/(2\omega)}\left(\frac{1-\cos 2\omega t}{2}\right)dt = \frac{\omega}{2\pi}\left\{t - \frac{\sin 2\omega t}{2\omega}\right\}\Bigg|_{-\pi/(2\omega)}^{\pi/(2\omega)} = \frac{1}{2}$

37. Since we must integrate the function over one full period, we choose to integrate over $0 \le t \le 1$,

$$F_{dc} = \frac{1}{1}\int_0^1 t\,dt = \left\{\frac{t^2}{2}\right\}_0^1 = \frac{1}{2}.$$

38. First we consider summing $1 + \sin\theta + \sin^2\theta + \cdots$. If we denote the sum of the first n terms by $S_n = 1 + \sin\theta + \cdots + \sin^{n-1}\theta$, then multiplication of this by $\sin\theta$ gives $(\sin\theta)S_n = \sin\theta + \sin^2\theta + \cdots + \sin^n\theta$. If we subtract, many cancellations occur, leaving

$$S_n - (\sin\theta)S_n = 1 - \sin^n\theta \quad\Longrightarrow\quad S_n = \frac{1-\sin^n\theta}{1-\sin\theta},\ \text{provided } \sin\theta \ne 1.$$

If we take limits as $n \to \infty$, we obtain $1 + \sin\theta + \cdots = \dfrac{1}{1-\sin\theta}$. Hence,

$$\begin{aligned}\int(1+\sin\theta+\sin^2\theta+\cdots)\,d\theta &= \int\frac{1}{1-\sin\theta}d\theta = \int\frac{1+\sin\theta}{1-\sin^2\theta}d\theta = \int\frac{1+\sin\theta}{\cos^2\theta}d\theta\\
&= \int(\sec^2\theta + \sec\theta\,\tan\theta)\,d\theta = \tan\theta + \sec\theta + C.\end{aligned}$$

39.
$$\begin{aligned}\int\sec^n x\,dx &= \int\sec^{n-2}x\,\sec^2 x\,dx = \int(1+\tan^2 x)^{n/2-1}\sec^2 x\,dx\\
&= \int\sec^2 x\left[\sum_{r=0}^{n/2-1}\binom{n/2-1}{r}\tan^{2r}x\right]dx = \sum_{r=0}^{n/2-1}\binom{n/2-1}{r}\int\tan^{2r}x\,\sec^2 x\,dx\\
&= \sum_{r=0}^{n/2-1}\binom{n/2-1}{r}\frac{\tan^{2r+1}x}{2r+1} + C\end{aligned}$$

40. We integrate the functions in pairs:

$$\int_0^{2\pi}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}dx = \frac{1}{2\pi}\left\{x\right\}_0^{2\pi} = 1;$$

$$\int_0^{2\pi}\left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{1}{\sqrt{\pi}}\sin nx\right)dx = \frac{1}{\sqrt{2}\pi}\left\{-\frac{1}{n}\cos nx\right\}_0^{2\pi} = 0;$$

$$\int_0^{2\pi}\left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{1}{\sqrt{\pi}}\cos nx\right)dx = \frac{1}{\sqrt{2}\pi}\left\{\frac{1}{n}\sin nx\right\}_0^{2\pi} = 0;$$

$$\int_0^{2\pi}\left(\frac{1}{\sqrt{\pi}}\sin nx\right)\left(\frac{1}{\sqrt{\pi}}\cos mx\right)dx = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}[\sin(n+m)x+\sin(n-m)x]\,dx$$
$$= \frac{1}{2\pi}\begin{cases}\left\{-\frac{1}{2n}\cos 2nx\right\}_0^{2\pi}, & m=n\\ \left\{\frac{-1}{n+m}\cos(n+m)x-\frac{1}{n-m}\cos(n-m)x\right\}_0^{2\pi}, & m\neq n\end{cases}$$
$$= 0;$$

$$\int_0^{2\pi}\left(\frac{1}{\sqrt{\pi}}\sin nx\right)\left(\frac{1}{\sqrt{\pi}}\sin mx\right)dx = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}[-\cos(n+m)x+\cos(n-m)x]\,dx$$
$$= \frac{1}{2\pi}\begin{cases}\left\{x-\frac{1}{2n}\sin 2nx\right\}_0^{2\pi}, & m=n\\ \left\{\frac{-1}{n+m}\sin(n+m)x+\frac{1}{n-m}\sin(n-m)x\right\}_0^{2\pi}, & m\neq n\end{cases}$$
$$= \begin{cases}1, & m=n\\ 0, & m\neq n\end{cases};$$

$$\int_0^{2\pi}\left(\frac{1}{\sqrt{\pi}}\cos nx\right)\left(\frac{1}{\sqrt{\pi}}\cos mx\right)dx = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}[\cos(n+m)x+\cos(n-m)x]\,dx$$
$$= \frac{1}{2\pi}\begin{cases}\left\{x+\frac{1}{2n}\sin 2nx\right\}_0^{2\pi}, & m=n\\ \left\{\frac{1}{n+m}\sin(n+m)x+\frac{1}{n-m}\sin(n-m)x\right\}_0^{2\pi}, & m\neq n\end{cases}$$
$$= \begin{cases}1, & m=n\\ 0, & m\neq n\end{cases}.$$

EXERCISES 8.4

1. If we set $x=\sqrt{2}\sec\theta$, then $dx=\sqrt{2}\sec\theta\,\tan\theta\,d\theta$, and

$$\int\frac{1}{x\sqrt{2x^2-4}}dx = \int\frac{1}{\sqrt{2}\sec\theta\,2\tan\theta}\sqrt{2}\sec\theta\,\tan\theta\,d\theta = \frac{\theta}{2}+C = \frac{1}{2}\,\mathrm{Sec}^{-1}\!\left(\frac{x}{\sqrt{2}}\right)+C.$$

2. If we set $x=\frac{3}{\sqrt{5}}\sin\theta$, then $dx=\frac{3}{\sqrt{5}}\cos\theta\,d\theta$, and

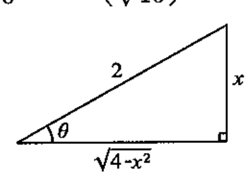
$$\int\frac{1}{\sqrt{9-5x^2}}dx = \int\frac{1}{3\cos\theta}\left(\frac{3}{\sqrt{5}}\right)\cos\theta\,d\theta = \frac{\theta}{\sqrt{5}}+C = \frac{1}{\sqrt{5}}\,\mathrm{Sin}^{-1}\!\left(\frac{\sqrt{5}x}{3}\right)+C.$$

3. If we set $x=\sqrt{10}\tan\theta$, then $dx=\sqrt{10}\sec^2\theta\,d\theta$, and

$$\int\frac{1}{10+x^2}dx = \int\frac{1}{10\sec^2\theta}\sqrt{10}\sec^2\theta\,d\theta = \frac{\theta}{\sqrt{10}}+C = \frac{1}{\sqrt{10}}\,\mathrm{Tan}^{-1}\!\left(\frac{x}{\sqrt{10}}\right)+C.$$

4. If we set $x=2\sin\theta$, then $dx=2\cos\theta\,d\theta$, and

$$\int\frac{1}{x^2\sqrt{4-x^2}}dx = \int\frac{1}{4\sin^2\theta\,2\cos\theta}2\cos\theta\,d\theta = \frac{1}{4}\int\csc^2\theta\,d\theta$$
$$= -\frac{1}{4}\cot\theta+C = -\frac{1}{4}\frac{\sqrt{4-x^2}}{x}+C.$$



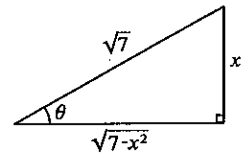
5. If we set $x = \sqrt{7}\sin\theta$, then $dx = \sqrt{7}\cos\theta\, d\theta$, and

$$\int \sqrt{7-x^2}\,dx = \int \sqrt{7}\cos\theta\,\sqrt{7}\cos\theta\,d\theta = 7\int\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{7}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C = \frac{7\theta}{2} + \frac{7}{2}\sin\theta\,\cos\theta + C$$

$$= \frac{7}{2}\,\mathrm{Sin}^{-1}\!\left(\frac{x}{\sqrt{7}}\right) + \frac{7}{2}\left(\frac{x}{\sqrt{7}}\right)\left(\frac{\sqrt{7-x^2}}{\sqrt{7}}\right) + C$$

$$= \frac{7}{2}\,\mathrm{Sin}^{-1}\!\left(\frac{x}{\sqrt{7}}\right) + \frac{x}{2}\sqrt{7-x^2} + C.$$



6. $\displaystyle\int x\sqrt{5x^2+3}\,dx = \frac{1}{15}(5x^2+3)^{3/2} + C$

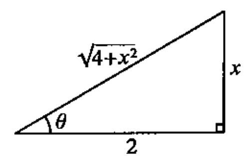
7. If we set $x = 2\tan\theta$, then $dx = 2\sec^2\theta\,d\theta$, and

$$\int x^3\sqrt{4+x^2}\,dx = \int 8\tan^3\theta\,2\sec\theta\,2\sec^2\theta\,d\theta = 32\int \sec^3\theta(\sec^2\theta - 1)\tan\theta\,d\theta$$

$$= 32\left(\frac{1}{5}\sec^5\theta - \frac{1}{3}\sec^3\theta\right) + C$$

$$= \frac{32}{5}\left(\frac{\sqrt{4+x^2}}{2}\right)^5 - \frac{32}{3}\left(\frac{\sqrt{4+x^2}}{2}\right)^3 + C$$

$$= \frac{1}{5}(4+x^2)^{5/2} - \frac{4}{3}(4+x^2)^{3/2} + C.$$

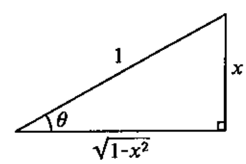


8. If we set $x = \sin\theta$, then $dx = \cos\theta\,d\theta$, and

$$\int \frac{1}{1-x^2}dx = \int \frac{1}{\cos^2\theta}\cos\theta\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\right| + C = \ln\left|\frac{1+x}{\sqrt{(1-x)(1+x)}}\right| + C$$

$$= \ln\left|\frac{\sqrt{1+x}}{\sqrt{1-x}}\right| + C = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C.$$

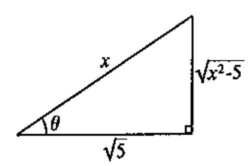


9. If we set $x = \sqrt{5}\sec\theta$, then $dx = \sqrt{5}\sec\theta\,\tan\theta\,d\theta$, and

$$\int \frac{1}{\sqrt{x^2-5}}dx = \int \frac{1}{\sqrt{5}\tan\theta}\sqrt{5}\sec\theta\,\tan\theta\,d\theta = \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{x}{\sqrt{5}} + \frac{\sqrt{x^2-5}}{\sqrt{5}}\right| + C$$

$$= \ln\left|x + \sqrt{x^2-5}\right| + D.$$



10. $\displaystyle\int \frac{x+5}{10x^2+2}dx = \int \frac{x}{10x^2+2}dx + \frac{5}{2}\int \frac{1}{5x^2+1}dx$

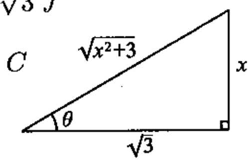
In the second term we set $x = \dfrac{1}{\sqrt{5}}\tan\theta$ and $dx = \dfrac{1}{\sqrt{5}}\sec^2\theta\,d\theta$,

$$\int \frac{x+5}{10x^2+2}dx = \frac{1}{20}\ln(10x^2+2) + \frac{5}{2}\int \frac{1}{\sec^2\theta}\left(\frac{1}{\sqrt{5}}\right)\sec^2\theta\,d\theta$$

$$= \frac{1}{20}\ln(10x^2+2) + \frac{\sqrt{5}}{2}\theta + C = \frac{1}{20}\ln(5x^2+1) + \frac{\sqrt{5}}{2}\,\mathrm{Tan}^{-1}(\sqrt{5}x) + D.$$

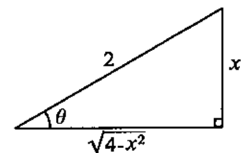
11. If we set $x = \sqrt{3}\tan\theta$, then $dx = \sqrt{3}\sec^2\theta\, d\theta$, and

$$\int \frac{1}{x\sqrt{x^2+3}}dx = \int \frac{1}{\sqrt{3}\tan\theta\,\sqrt{3}\sec\theta}\sqrt{3}\sec^2\theta\, d\theta = \frac{1}{\sqrt{3}}\int \frac{\sec\theta}{\tan\theta}d\theta = \frac{1}{\sqrt{3}}\int \csc\theta\, d\theta$$
$$= \frac{1}{\sqrt{3}}\ln|\csc\theta - \cot\theta| + C = \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{x^2+3}}{x} - \frac{\sqrt{3}}{x}\right| + C$$
$$= \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{x^2+3}-\sqrt{3}}{x}\right| + C.$$



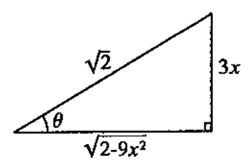
12. If we set $x = 2\sin\theta$, then $dx = 2\cos\theta\, d\theta$, and

$$\int \frac{\sqrt{4-x^2}}{x}dx = \int \frac{2\cos\theta}{2\sin\theta}2\cos\theta\, d\theta = 2\int \frac{1-\sin^2\theta}{\sin\theta}d\theta$$
$$= 2\int(\csc\theta - \sin\theta)\, d\theta = 2(\ln|\csc\theta - \cot\theta| + \cos\theta) + C$$
$$= 2\left(\ln\left|\frac{2}{x} - \frac{\sqrt{4-x^2}}{x}\right| + \frac{\sqrt{4-x^2}}{2}\right) + C = 2\ln\left|\frac{2-\sqrt{4-x^2}}{x}\right| + \sqrt{4-x^2} + C.$$



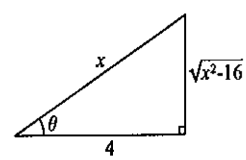
13. If we set $x = (\sqrt{2}/3)\sin\theta$, then $dx = (\sqrt{2}/3)\cos\theta\, d\theta$, and

$$\int \frac{x^2}{(2-9x^2)^{3/2}}dx = \int \frac{(2/9)\sin^2\theta}{2\sqrt{2}\cos^3\theta}\frac{\sqrt{2}}{3}\cos\theta\, d\theta = \frac{1}{27}\int \frac{\sin^2\theta}{\cos^2\theta}d\theta = \frac{1}{27}\int \tan^2\theta\, d\theta$$
$$= \frac{1}{27}\int(\sec^2\theta - 1)\, d\theta = \frac{1}{27}(\tan\theta - \theta) + C$$
$$= \frac{1}{27}\left(\frac{3x}{\sqrt{2-9x^2}}\right) - \frac{1}{27}\mathrm{Sin}^{-1}\left(\frac{3x}{\sqrt{2}}\right) + C$$
$$= \frac{x}{9\sqrt{2-9x^2}} - \frac{1}{27}\mathrm{Sin}^{-1}\left(\frac{3x}{\sqrt{2}}\right) + C.$$



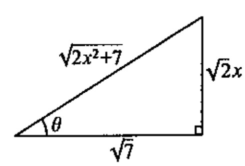
14. If we set $x = 4\sec\theta$, then $dx = 4\sec\theta\,\tan\theta\, d\theta$, and

$$\int \frac{\sqrt{x^2-16}}{x^2}dx = \int \frac{4\tan\theta}{16\sec^2\theta}4\sec\theta\,\tan\theta\, d\theta = \int \frac{\sec^2\theta - 1}{\sec\theta}d\theta$$
$$= \int(\sec\theta - \cos\theta)\, d\theta = \ln|\sec\theta + \tan\theta| - \sin\theta + C$$
$$= \ln\left|\frac{x}{4} + \frac{\sqrt{x^2-16}}{4}\right| - \frac{\sqrt{x^2-16}}{x} + C = \ln\left|x + \sqrt{x^2-16}\right| - \frac{\sqrt{x^2-16}}{x} + D.$$



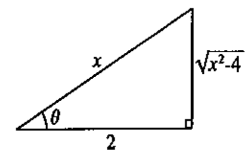
15. If we set $x = \sqrt{7/2}\tan\theta$, then $dx = \sqrt{7/2}\sec^2\theta\, d\theta$, and

$$\int \frac{1}{x^2\sqrt{2x^2+7}}dx = \int \frac{1}{(7/2)\tan^2\theta\,\sqrt{7}\sec\theta}\sqrt{\frac{7}{2}}\sec^2\theta\, d\theta = \frac{\sqrt{2}}{7}\int \frac{\sec\theta}{\tan^2\theta}d\theta$$
$$= \frac{\sqrt{2}}{7}\int \csc\theta\,\cot\theta\, d\theta = \frac{\sqrt{2}}{7}(-\csc\theta) + C$$
$$= -\frac{\sqrt{2}}{7}\left(\frac{\sqrt{2x^2+7}}{\sqrt{2}x}\right) + C = -\frac{\sqrt{2x^2+7}}{7x} + C.$$



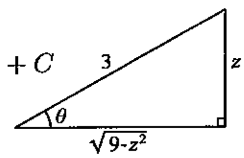
16. If we set $x = 2\sec\theta$, then $dx = 2\sec\theta\,\tan\theta\,d\theta$, and

$$\int \frac{1}{x^3\sqrt{x^2-4}}dx = \int \frac{1}{8\sec^3\theta\,2\tan\theta}2\sec\theta\,\tan\theta\,d\theta = \frac{1}{8}\int \cos^2\theta\,d\theta$$
$$= \frac{1}{8}\int\left(\frac{1+\cos 2\theta}{2}\right)d\theta = \frac{1}{16}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$
$$= \frac{\theta}{16} + \frac{1}{16}\sin\theta\,\cos\theta + C$$
$$= \frac{1}{16}\,\mathrm{Sec}^{-1}\!\left(\frac{x}{2}\right) + \frac{1}{16}\frac{\sqrt{x^2-4}}{x}\frac{2}{x} + C = \frac{1}{16}\,\mathrm{Sec}^{-1}\!\left(\frac{x}{2}\right) + \frac{\sqrt{x^2-4}}{8x^2} + C.$$



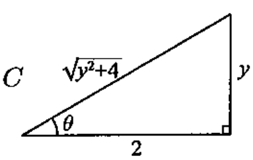
17. If we set $z = 3\sin\theta$, then $dz = 3\cos\theta\,d\theta$, and

$$\int \frac{\sqrt{9-z^2}}{z^4}dz = \int \frac{3\cos\theta}{81\sin^4\theta}3\cos\theta\,d\theta = \frac{1}{9}\int \cot^2\theta\,\csc^2\theta\,d\theta = \frac{1}{9}\left(-\frac{1}{3}\cot^3\theta\right) + C$$
$$= -\frac{1}{27}\left(\frac{\sqrt{9-z^2}}{z}\right)^3 + C = -\frac{(9-z^2)^{3/2}}{27z^3} + C.$$



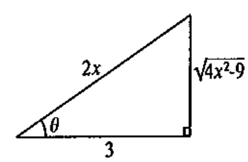
18. If we set $y = 2\tan\theta$, then $dy = 2\sec^2\theta\,d\theta$, and

$$\int \frac{y^3}{\sqrt{y^2+4}}dy = \int \frac{8\tan^3\theta}{2\sec\theta}2\sec^2\theta\,d\theta = 8\int \tan\theta\,(\sec^2\theta - 1)\,\sec\theta\,d\theta$$
$$= 8\left(\frac{\sec^3\theta}{3} - \sec\theta\right) + C = \frac{8}{3}\left(\frac{\sqrt{y^2+4}}{2}\right)^3 - \frac{8\sqrt{y^2+4}}{2} + C$$
$$= \frac{1}{3}(y^2+4)^{3/2} - 4\sqrt{y^2+4} + C.$$



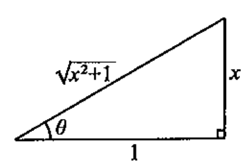
19. If we set $x = (3/2)\sec\theta$, then $dx = (3/2)\sec\theta\,\tan\theta\,d\theta$, and

$$\int \frac{1}{(4x^2-9)^{3/2}}dx = \int \frac{1}{27\tan^3\theta}\frac{3}{2}\sec\theta\,\tan\theta\,d\theta = \frac{1}{18}\int \frac{\sec\theta}{\tan^2\theta}d\theta$$
$$= \frac{1}{18}\int \csc\theta\,\cot\theta\,d\theta = \frac{1}{18}(-\csc\theta) + C$$
$$= -\frac{1}{18}\left(\frac{2x}{\sqrt{4x^2-9}}\right) + C = \frac{-x}{9\sqrt{4x^2-9}} + C.$$



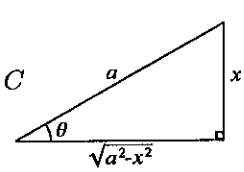
20. If we set $x = \tan\theta$, then $dx = \sec^2\theta\,d\theta$, and

$$\int \frac{x^2+2}{x^3+x}dx = \int \frac{x^2+2}{x(x^2+1)}dx = \int \frac{\tan^2\theta + 2}{\tan\theta\,\sec^2\theta}\sec^2\theta\,d\theta$$
$$= \int (\tan\theta + 2\cot\theta)\,d\theta = \ln|\sec\theta| + 2\ln|\sin\theta| + C$$
$$= \ln|\sqrt{x^2+1}| + 2\ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C = 2\ln|x| - \frac{1}{2}\ln(x^2+1) + C.$$



21. If we set $x = a\sin\theta$, then $dx = a\cos\theta\,d\theta$, and

$$\int \frac{1}{a^2-x^2}dx = \int \frac{1}{a^2\cos^2\theta}a\cos\theta\,d\theta = \frac{1}{a}\int \sec\theta\,d\theta = \frac{1}{a}\ln|\sec\theta + \tan\theta| + C$$
$$= \frac{1}{a}\ln\left|\frac{a}{\sqrt{a^2-x^2}} + \frac{x}{\sqrt{a^2-x^2}}\right| + C = \frac{1}{a}\ln\left|\frac{a+x}{\sqrt{(a-x)(a+x)}}\right| + C$$
$$= \frac{1}{a}\ln\left|\frac{\sqrt{a+x}}{\sqrt{a-x}}\right| + C = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

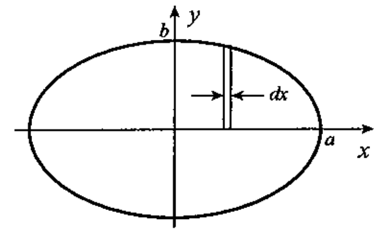


22. $A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$

If we set $x = a\sin\theta$, then $dx = a\cos\theta\,d\theta$, and

$$A = \frac{4b}{a}\int_0^{\pi/2} a\cos\theta\, a\cos\theta\,d\theta = 4ab\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= 2ab\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = \pi ab.$$

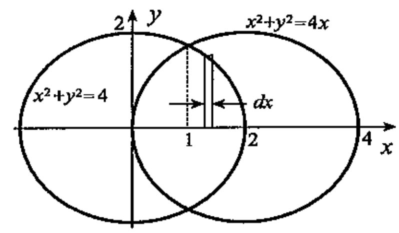


23. $A = 4\int_1^2 \sqrt{4 - x^2}\,dx$

If we set $x = 2\sin\theta$ and $dx = 2\cos\theta\,d\theta$, then

$$A = 4\int_{\pi/6}^{\pi/2} 2\cos\theta\, 2\cos\theta\,d\theta = 16\int_{\pi/6}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= 8\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_{\pi/6}^{\pi/2} = \frac{8\pi}{3} - 2\sqrt{3}.$$



24. A plot of the ellipse is shown to the right. If we solve

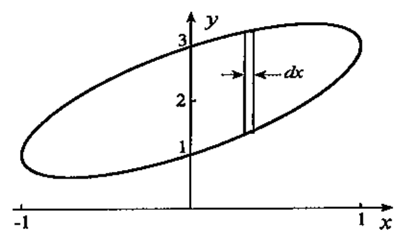
$$y^2 - (2x+4)y + (2x^2 + 4x + 3) = 0$$

for y in terms of x, we obtain

$$y = \frac{2x + 4 \pm \sqrt{(2x+4)^2 - 4(2x^2 + 4x + 3)}}{2}$$

$$= \frac{2x + 4 \pm \sqrt{4 - 4x^2}}{2}$$

$$= x + 2 \pm \sqrt{1 - x^2}.$$



It follows that a rectangle of width dx at position x between the top and bottom of the ellipse has area

$$[(x + 2 + \sqrt{1-x^2}) - (x + 2 - \sqrt{1-x^2})]dx = 2\sqrt{1-x^2}\,dx.$$

Since the ellipse extends from $x = -1$ to $x = 1$, its area must be $\quad A = \int_{-1}^1 2\sqrt{1-x^2}\,dx.$

Setting $x = \sin\theta$ and $dx = \cos\theta\,d\theta$,

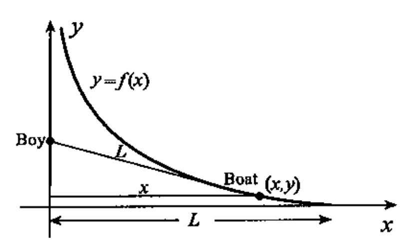
$$A = 2\int_{-\pi/2}^{\pi/2}\cos\theta\cos\theta\,d\theta = 2\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta = \left\{\theta + \frac{1}{2}\sin 2\theta\right\}_{-\pi/2}^{\pi/2} = \pi.$$

25. (a) Since the slope of the curve is also the slope of the string,

$$\frac{dy}{dx} = -\frac{\sqrt{L^2 - x^2}}{x}.$$

(b) If we integrate with respect to x,

$$y = -\int\frac{\sqrt{L^2 - x^2}}{x}dx.$$



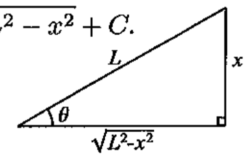
We now set $x = L\sin\theta$ and $dx = L\cos\theta\,d\theta$,

$$y = -\int\frac{L\cos\theta}{L\sin\theta}L\cos\theta\,d\theta = -L\int\frac{1-\sin^2\theta}{\sin\theta}d\theta = L\int(\sin\theta - \csc\theta)\,d\theta$$

$$= L[-\cos\theta - \ln|\csc\theta - \cot\theta|] + C = -L\left(\frac{\sqrt{L^2-x^2}}{L} + \ln\left|\frac{L}{x} - \frac{\sqrt{L^2-x^2}}{x}\right|\right) + C$$

$$= -\sqrt{L^2-x^2} - L\ln\left|\frac{L - \sqrt{L^2-x^2}}{x}\right| + C = -\sqrt{L^2-x^2} - L\ln\left|\frac{L-\sqrt{L^2-x^2}}{x}\,\frac{L+\sqrt{L^2-x^2}}{L+\sqrt{L^2-x^2}}\right| + C$$

$$= -\sqrt{L^2 - x^2} - L\ln\left|\frac{x}{L+\sqrt{L^2-x^2}}\right| + C = L\ln\left|\frac{L+\sqrt{L^2-x^2}}{x}\right| - \sqrt{L^2-x^2} + C.$$



Since $y = 0$ when $x = L$, it follows that $C = 0$, and

$$y = L\ln\left(\frac{L+\sqrt{L^2-x^2}}{x}\right) - \sqrt{L^2-x^2}.$$

26. $I = 2\int_0^{\tilde{y}} y^2(\sqrt{4-y^2} - y^2)\,dy$

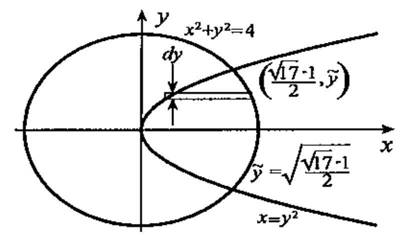
In the first integral we set $y = 2\sin\theta$ and $dy = 2\cos\theta\,d\theta$. If $\tilde{\theta} = \mathrm{Sin}^{-1}(\tilde{y}/2)$, then

$$I = 2\int_0^{\tilde{\theta}} 4\sin^2\theta\, 2\cos\theta\, 2\cos\theta\,d\theta - 2\left\{\frac{y^5}{5}\right\}_0^{\tilde{y}}$$

$$= 32\int_0^{\tilde{\theta}} \sin^2\theta\,\cos^2\theta\,d\theta - \frac{2\tilde{y}^5}{5}$$

$$= 32\int_0^{\tilde{\theta}} \frac{1}{4}\sin^2 2\theta\,d\theta - \frac{2\tilde{y}^5}{5} = 8\int_0^{\tilde{\theta}}\left(\frac{1-\cos 4\theta}{2}\right)d\theta - \frac{2\tilde{y}^5}{5}$$

$$= 4\left\{\theta - \frac{1}{4}\sin 4\theta\right\}_0^{\tilde{\theta}} - \frac{2\tilde{y}^5}{5} = 4\tilde{\theta} - \sin 4\tilde{\theta} - \frac{2\tilde{y}^5}{5} = 1.053.$$



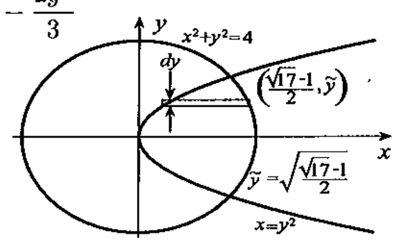
27. By symmetry, $\overline{y} = 0$. The area is $A = 2\int_0^{\tilde{y}}(\sqrt{4-y^2} - y^2)\,dy$.

We set $y = 2\,\sin\theta$ and $dy = 2\,\cos\theta\,d\theta$ in the first term. If $\tilde{\theta} = \mathrm{Sin}^{-1}(\tilde{y}/2)$, then

$$A = 2\int_0^{\tilde{\theta}} 2\,\cos\theta\, 2\,\cos\theta\,d\theta - 2\left\{\frac{y^3}{3}\right\}_0^{\tilde{y}} = 8\int_0^{\tilde{\theta}}\left(\frac{1+\cos 2\theta}{2}\right)d\theta - \frac{2\tilde{y}^3}{3}$$

$$= 4\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_0^{\tilde{\theta}} - \frac{2\tilde{y}^3}{3}$$

$$= 4\tilde{\theta} + 2\,\sin 2\tilde{\theta} - \frac{2\tilde{y}^3}{3} = 3.3500.$$



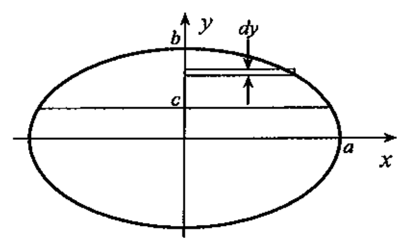
Since

$$A\overline{x} = 2\int_0^{\tilde{y}} \frac{1}{2}(\sqrt{4-y^2} + y^2)(\sqrt{4-y^2} - y^2)\,dy = \int_0^{\tilde{y}}(4 - y^2 - y^4)\,dy = \left\{4y - \frac{y^3}{3} - \frac{y^5}{5}\right\}_0^{\tilde{y}} = 3.7386,$$

it follows that $\overline{x} = 3.7386/3.3500 = 1.116$.

28. According to Exercise 22, the area of the ellipse is πab. If we let the required line be $y = c$, then c must satisfy the equation

$$\frac{\pi ab}{3} = 2\int_c^b \frac{a}{b}\sqrt{b^2-y^2}\,dy.$$



We let $y = b\sin\theta$ and $dy = b\cos\theta\,d\theta$. If $\tilde{\theta} = \mathrm{Sin}^{-1}(c/b)$, then

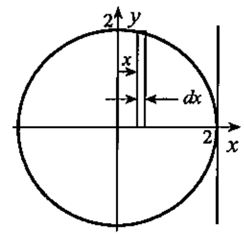
$$\frac{\pi ab}{3} = \frac{2a}{b}\int_{\tilde{\theta}}^{\pi/2} b\cos\theta\, b\cos\theta\,d\theta$$

$$= 2ab\int_{\tilde{\theta}}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta = ab\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_{\tilde{\theta}}^{\pi/2} = ab\left(\frac{\pi}{2} - \tilde{\theta} - \frac{1}{2}\sin 2\tilde{\theta}\right).$$

Thus, $\tilde{\theta}$ must satisfy the equation $\dfrac{\pi}{3} = \dfrac{\pi}{2} - \tilde{\theta} - \dfrac{1}{2}\sin 2\tilde{\theta}$, or, $6\tilde{\theta} + 3\sin 2\tilde{\theta} - \pi = 0$. Newton's iterative procedure with $\tilde{\theta}_1 = 0.25$, $\tilde{\theta}_{n+1} = \tilde{\theta}_n - \dfrac{6\tilde{\theta}_n + 3\sin 2\tilde{\theta}_n - \pi}{6 + 6\cos 2\tilde{\theta}_n}$ gives the iterations $\tilde{\theta}_2 = 0.268$, $\tilde{\theta}_3 = 0.268\,133$, $\tilde{\theta}_4 = 0.268\,133$. Hence, the required line is $y = b\sin\tilde{\theta} = 0.265b$.

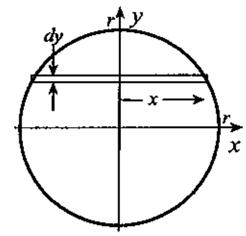
29. We find the moment of inertia about the line $x = 2$, $\quad I = 2\int_{-2}^{2}(x-2)^2\rho\sqrt{4-x^2}\,dx.$
If we set $x = 2\sin\theta$ and $dx = 2\cos\theta\,d\theta$,

$$\begin{aligned}
I &= 2\rho\int_{-\pi/2}^{\pi/2}(2\sin\theta-2)^2\,2\cos\theta\,2\cos\theta\,d\theta = 32\rho\int_{-\pi/2}^{\pi/2}(\sin^2\theta-2\sin\theta+1)\cos^2\theta\,d\theta\\
&= 32\rho\int_{-\pi/2}^{\pi/2}\left[\left(\frac{\sin 2\theta}{2}\right)^2 - 2\cos^2\theta\sin\theta+\cos^2\theta\right]d\theta\\
&= 8\rho\int_{-\pi/2}^{\pi/2}\left[\frac{1-\cos 4\theta}{2} - 8\cos^2\theta\sin\theta + 2(1+\cos 2\theta)\right]d\theta\\
&= 8\rho\left\{\frac{5\theta}{2} - \frac{1}{8}\sin 4\theta + \frac{8}{3}\cos^3\theta + \sin 2\theta\right\}_{-\pi/2}^{\pi/2} = 20\rho\pi.
\end{aligned}$$



30. $F = \int_{-r}^{r} 9.81\rho(r-y)2x\,dy = 19.62\rho\int_{-r}^{r}(r-y)\sqrt{r^2-y^2}\,dy$
If we set $y = r\sin\theta$, then $dy = r\cos\theta\,d\theta$, and

$$\begin{aligned}
F &= 19.62\rho\int_{-\pi/2}^{\pi/2}(r-r\sin\theta)r\cos\theta\,r\cos\theta\,d\theta\\
&= 19.62\rho r^3\int_{-\pi/2}^{\pi/2}(1-\sin\theta)\cos^2\theta\,d\theta\\
&= 19.62\rho r^3\int_{-\pi/2}^{\pi/2}\left[\frac{1}{2}(1+\cos 2\theta) - \cos^2\theta\sin\theta\right]d\theta\\
&= 19.62\rho r^3\left\{\frac{\theta}{2} + \frac{1}{4}\sin 2\theta + \frac{1}{3}\cos^3\theta\right\}_{-\pi/2}^{\pi/2} = 9.81\pi\rho r^3.
\end{aligned}$$

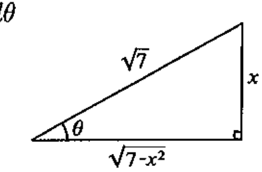


31. It is advantageous first to divide, $\int\frac{2x^4-x^2}{2x^2+1}dx = \int\left(x^2-1+\frac{1}{2x^2+1}\right)dx$. If we set $x = (1/\sqrt{2})\tan\theta$ and $dx = (1/\sqrt{2})\sec^2\theta\,d\theta$ in the last term,

$$\int\frac{2x^4-x^2}{2x^2+1}dx = \frac{x^3}{3} - x + \int\frac{1}{\sec^2\theta}\frac{1}{\sqrt{2}}\sec^2\theta\,d\theta = \frac{x^3}{3} - x + \frac{\theta}{\sqrt{2}} + C = \frac{x^3}{3} - x + \frac{1}{\sqrt{2}}\mathrm{Tan}^{-1}(\sqrt{2}x) + C.$$

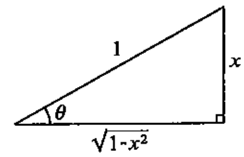
32. If we set $x = \sqrt{7}\sin\theta$, then $dx = \sqrt{7}\cos\theta\,d\theta$, and

$$\begin{aligned}
\int(7-x^2)^{3/2}\,dx &= \int 7\sqrt{7}\cos^3\theta\,\sqrt{7}\cos\theta\,d\theta = 49\int\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta\\
&= \frac{49}{4}\int\left[1+2\cos 2\theta+\frac{1}{2}(1+\cos 4\theta)\right]d\theta\\
&= \frac{49}{4}\left(\frac{3\theta}{2}+\sin 2\theta+\frac{1}{8}\sin 4\theta\right)+C\\
&= \frac{147}{8}\theta + \frac{49}{2}\sin\theta\cos\theta + \frac{49}{16}\sin 2\theta\cos 2\theta + C\\
&= \frac{147}{8}\theta + \frac{49}{2}\sin\theta\cos\theta + \frac{49}{8}\sin\theta\cos\theta(1-2\sin^2\theta) + C\\
&= \frac{147}{8}\mathrm{Sin}^{-1}\left(\frac{x}{\sqrt{7}}\right) + \frac{245}{8}\frac{x}{\sqrt{7}}\frac{\sqrt{7-x^2}}{\sqrt{7}} - \frac{49}{4}\left(\frac{x}{\sqrt{7}}\right)^3\frac{\sqrt{7-x^2}}{\sqrt{7}} + C\\
&= \frac{147}{8}\mathrm{Sin}^{-1}\left(\frac{x}{\sqrt{7}}\right) + \frac{x}{8}(35-2x^2)\sqrt{7-x^2} + C.
\end{aligned}$$



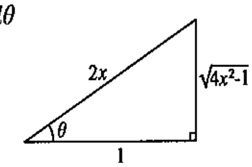
33. If we set $x = \sin\theta$ and $dx = \cos\theta\, d\theta$, then

$$\begin{aligned}\int \frac{1}{x - x^3}dx &= \int \frac{1}{x(1-x^2)}dx = \int \frac{1}{\sin\theta\, \cos^2\theta}\cos\theta\, d\theta = \int \frac{1}{\sin\theta\, \cos\theta}d\theta \\ &= \int 2\, \csc 2\theta\, d\theta = \ln|\csc 2\theta - \cot 2\theta| + C \\ &= \ln\left|\frac{1-\cos 2\theta}{\sin 2\theta}\right| + C = \ln\left|\frac{1-(1-2\,\sin^2\theta)}{2\,\sin\theta\,\cos\theta}\right| + C \\ &= \ln|\tan\theta| + C = \ln\left|\frac{x}{\sqrt{1-x^2}}\right| + C.\end{aligned}$$



34. If we set $x = (1/2)\sec\theta$, then $dx = (1/2)\sec\theta\, \tan\theta\, d\theta$, and

$$\begin{aligned}\int \frac{1}{x^3(4x^2-1)^{3/2}}dx &= \int \frac{1}{(1/8)\sec^3\theta\, \tan^3\theta}(1/2)\sec\theta\, \tan\theta\, d\theta = 4\int \frac{\cos^4\theta}{\sin^2\theta}d\theta \\ &= 4\int \frac{\cos^2\theta(1-\sin^2\theta)}{\sin^2\theta}d\theta = 4\int(\cot^2\theta - \cos^2\theta)\, d\theta \\ &= 4\int\left(\csc^2\theta - 1 - \frac{1+\cos 2\theta}{2}\right)d\theta \\ &= 4\left(-\cot\theta - \frac{3\theta}{2} - \frac{1}{4}\sin 2\theta\right) + C \\ &= 4\left[-\frac{1}{\sqrt{4x^2-1}} - \frac{3}{2}\,\mathrm{Sec}^{-1}(2x) - \frac{1}{2}\frac{\sqrt{4x^2-1}}{2x}\frac{1}{2x}\right] + C \\ &= -6\,\mathrm{Sec}^{-1}(2x) + \frac{1-12x^2}{2x^2\sqrt{4x^2-1}} + C.\end{aligned}$$

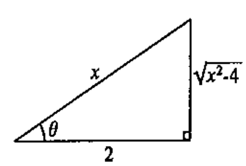


35. If we set $x = 2\,\sec\theta$ and $dx = 2\,\sec\theta\, \tan\theta\, d\theta$, then

$$\begin{aligned}\int\sqrt{x^2-4}\,dx &= \int 2\,\tan\theta\, 2\,\sec\theta\, \tan\theta\, d\theta = 4\int \tan^2\theta\, \sec\theta\, d\theta \\ &= 4\int(\sec^2\theta - 1)\sec\theta\, d\theta = 4\int(\sec^3\theta - \sec\theta)\, d\theta.\end{aligned}$$

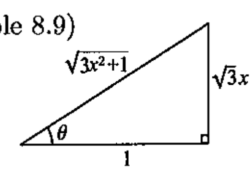
We use Example 8.9 to write

$$\begin{aligned}\int\sqrt{x^2-4}\,dx &= 2\,\sec\theta\,\tan\theta + 2\ln|\sec\theta + \tan\theta| - 4\ln|\sec\theta + \tan\theta| + C \\ &= 2\,\sec\theta\,\tan\theta - 2\ln|\sec\theta + \tan\theta| + C \\ &= 2\left(\frac{x}{2}\right)\left(\frac{\sqrt{x^2-4}}{2}\right) - 2\ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C \\ &= \frac{x\sqrt{x^2-4}}{2} - 2\ln|x + \sqrt{x^2-4}| + D.\end{aligned}$$



36. If we set $x = (1/\sqrt{3})\tan\theta$, then $dx = (1/\sqrt{3})\sec^2\theta\, d\theta$, and

$$\begin{aligned}\int\sqrt{1+3x^2}\,dx &= \int\sec\theta\left(\frac{1}{\sqrt{3}}\right)\sec^2\theta\, d\theta = \frac{1}{\sqrt{3}}\int\sec^3\theta\, d\theta \\ &= \frac{1}{2\sqrt{3}}(\sec\theta\,\tan\theta + \ln|\sec\theta + \tan\theta|) + C \quad \text{(using Example 8.9)} \\ &= \frac{1}{2\sqrt{3}}(\sqrt{1+3x^2}\,\sqrt{3}x + \ln|\sqrt{1+3x^2} + \sqrt{3}x|) + C \\ &= \frac{x}{2}\sqrt{1+3x^2} + \frac{1}{2\sqrt{3}}\ln|\sqrt{1+3x^2} + \sqrt{3}x| + C.\end{aligned}$$

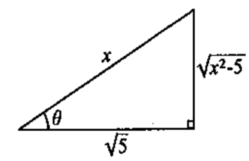


37. If we set $x = \sqrt{5}\sec\theta$ and $dx = \sqrt{5}\sec\theta\,\tan\theta\,d\theta$, then

$$\int \frac{x^2}{\sqrt{x^2-5}}\,dx = \int \frac{5\sec^2\theta}{\sqrt{5}\,\tan\theta}\sqrt{5}\sec\theta\,\tan\theta\,d\theta = 5\int \sec^3\theta\,d\theta.$$

We use Example 8.9 to write

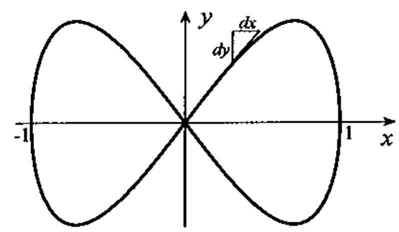
$$\begin{aligned}\int \frac{x^2}{\sqrt{x^2-5}}\,dx &= \frac{5}{2}\sec\theta\,\tan\theta + \frac{5}{2}\ln|\sec\theta+\tan\theta| + C\\ &= \frac{5}{2}\left(\frac{x}{\sqrt{5}}\right)\left(\frac{\sqrt{x^2-5}}{\sqrt{5}}\right) + \frac{5}{2}\ln\left|\frac{x}{\sqrt{5}} + \frac{\sqrt{x^2-5}}{\sqrt{5}}\right| + C\\ &= \frac{x\sqrt{x^2-5}}{2} + \frac{5}{2}\ln|x+\sqrt{x^2-5}| + D.\end{aligned}$$



38. Differentiation of $8y^2 = x^2 - x^4$ gives

$$16y\frac{dy}{dx} = 2x - 4x^3.$$

Therefore, $\dfrac{dy}{dx} = \dfrac{x-2x^3}{8y}$. Small lengths along that part of the curve in the first quadrant are approximated by



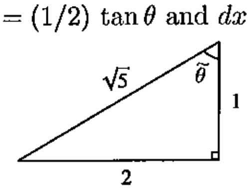
$$\begin{aligned}\sqrt{1+\left(\frac{x-2x^3}{8y}\right)^2}\,dx &= \sqrt{1+\frac{(x-2x^3)^2}{64y^2}}\,dx = \sqrt{1+\frac{x^2(1-2x^2)^2}{8x^2(1-x^2)}}\,dx\\ &= \sqrt{\frac{9-12x^2+4x^4}{8(1-x^2)}}\,dx = \frac{3-2x^2}{2\sqrt{2}\sqrt{1-x^2}}\,dx.\end{aligned}$$

The length of the curve is therefore $L = 4\displaystyle\int_0^1 \frac{3-2x^2}{2\sqrt{2}\sqrt{1-x^2}}\,dx$. When we set $x=\sin\theta$, and $dx = \cos\theta\,d\theta$,

$$L = \sqrt{2}\int_0^{\pi/2}\frac{3-2\sin^2\theta}{\cos\theta}\cos\theta\,d\theta = \sqrt{2}\int_0^{\pi/2}[3-(1-\cos 2\theta)]\,d\theta = \sqrt{2}\left\{2\theta + \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = \sqrt{2}\pi.$$

39. The length of the parabola is $L = \displaystyle\int_0^1 \sqrt{1+4x^2}\,dx$. We set $x = (1/2)\tan\theta$ and $dx = (1/2)\sec^2\theta\,d\theta$. If $\tilde{\theta} = \mathrm{Tan}^{-1}2$, then

$$L = \int_0^{\tilde{\theta}} \sec\theta\frac{1}{2}\sec^2\theta\,d\theta = \frac{1}{2}\int_0^{\tilde{\theta}}\sec^3\theta\,d\theta.$$

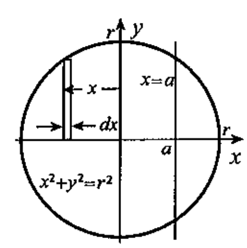


We now use Example 8.9 to write

$$L = \frac{1}{4}\left\{\sec\theta\,\tan\theta + \ln|\sec\theta+\tan\theta|\right\}_0^{\tilde{\theta}} = \frac{1}{4}(\sec\tilde{\theta}\,\tan\tilde{\theta} + \ln|\sec\tilde{\theta}+\tan\tilde{\theta}|) = \frac{1}{4}[2\sqrt{5}+\ln(2+\sqrt{5})].$$

40. Let the radius of the circle be r, and let the position of the line be denoted by $x=a$. Then the requirement that the second moment of area about $x=a$ be twice that about $x=0$ can be expressed as

$$\begin{aligned}&2\int_{-r}^{r}(x-a)^2\sqrt{r^2-x^2}\,dx\\ &\quad = 2(2)\int_{-r}^{r}x^2\sqrt{r^2-x^2}\,dx.\end{aligned}$$



If we set $x = r\sin\theta$ and $dx = r\cos\theta\,d\theta$ in these integrals, then

$$\int_{-\pi/2}^{\pi/2} (r\sin\theta - a)^2 r\cos\theta\, r\cos\theta\, d\theta = 2\int_{-\pi/2}^{\pi/2} r^2 \sin^2\theta\, r\cos\theta\, r\cos\theta\, d\theta,$$

or,

$$\begin{aligned}
0 &= r^2 \int_{-\pi/2}^{\pi/2} (2r^2\sin^2\theta\,\cos^2\theta - r^2\sin^2\theta\,\cos^2\theta + 2ar\cos^2\theta\,\sin\theta - a^2\cos^2\theta)\,d\theta \\
&= r^2 \int_{-\pi/2}^{\pi/2} \left(\frac{r^2}{4}\sin^2 2\theta + 2ar\cos^2\theta\,\sin\theta - a^2\cos^2\theta\right) d\theta \\
&= r^2 \int_{-\pi/2}^{\pi/2} \left[\frac{r^2}{4}\left(\frac{1-\cos 4\theta}{2}\right) + 2ar\cos^2\theta\,\sin\theta - a^2\left(\frac{1+\cos 2\theta}{2}\right)\right] d\theta \\
&= r^2 \left\{\frac{r^2}{8}\left(\theta - \frac{1}{4}\sin 4\theta\right) - \frac{2ar}{3}\cos^3\theta - \frac{a^2}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\right\}_{-\pi/2}^{\pi/2} \\
&= r^2\left(\frac{\pi r^2}{8} - \frac{\pi a^2}{2}\right).
\end{aligned}$$

Thus, $a = r/2$.

41. If we substitute $K(x) = k/(cx^2+1)$,

$$z(x) = -\int \frac{1}{1 + \dfrac{V(cx^2+1)}{k}}\,dx = -k\int \frac{1}{(k+V) + Vcx^2}\,dx.$$

If we set $x = \sqrt{\dfrac{k+V}{cV}}\tan\theta$ and $dx = \sqrt{\dfrac{k+V}{cV}}\sec^2\theta\,d\theta$, then

$$z(x) = -k\int \frac{\sqrt{\dfrac{k+V}{cV}}\sec^2\theta}{(k+V)\sec^2\theta}\,d\theta = -\frac{k}{\sqrt{Vc(k+V)}}\theta + C = \frac{-k}{\sqrt{Vc(k+V)}}\mathrm{Tan}^{-1}\sqrt{\frac{Vc}{k+V}}x + C.$$

When $z(0) = H$, we find that $H = C$, and $z(x) = H - \dfrac{k}{\sqrt{Vc(k+V)}}\mathrm{Tan}^{-1}\sqrt{\dfrac{Vc}{k+V}}x$. On other hand, when $z(H_w - L) = L$,

$$L = \frac{-k}{\sqrt{Vc(k+L)}}\mathrm{Tan}^{-1}\sqrt{\frac{Vc}{k+V}}(H_w - L) + C \quad\Longrightarrow\quad C = L + \frac{k}{\sqrt{Vc(k+L)}}\mathrm{Tan}^{-1}\sqrt{\frac{Vc}{k+V}}(H_w - L).$$

Thus, $z(x) = L + \dfrac{k}{\sqrt{Vc(k+L)}}\mathrm{Tan}^{-1}\sqrt{\dfrac{Vc}{k+V}}(H_w - L) - \dfrac{k}{\sqrt{Vc(k+V)}}\mathrm{Tan}^{-1}\sqrt{\dfrac{Vc}{k+V}}x.$

42. (a) If we set $p = \tan\theta$ and $dp = \sec^2\theta\,d\theta$, then

$$kx + C = \int \frac{1}{\sec\theta}\sec^2\theta\,d\theta = \ln|\sec\theta + \tan\theta| = \ln|\sqrt{1+p^2} + p|.$$

Exponentiation gives $\sqrt{1+p^2} + p = De^{kx}$, where $D = e^C$. Since $p(0) = f'(0) = 0$, we obtain $D = 1$. Hence,

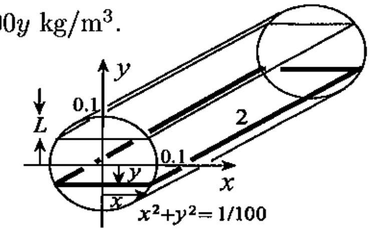
$$\sqrt{1+p^2} = e^{kx} - p \quad\Longrightarrow\quad 1 + p^2 = e^{2kx} - 2pe^{kx} + p^2.$$

This can be solved for $\quad p = \dfrac{dy}{dx} = \dfrac{e^{2kx}-1}{2e^{kx}} = \dfrac{1}{2}(e^{kx} - e^{-kx}).$

(b) Integration now yields $y = \dfrac{1}{2k}(e^{kx} + e^{-kx}) + C.$

43. Suppose L denotes the height of log above water. The density of the log is

$$\rho(y) = 1000 - \frac{500}{0.2}(y + 0.1) = 750 - 2500y \text{ kg/m}^3.$$



The weight of the log is

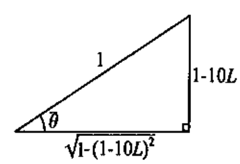
$$\begin{aligned} W_{\text{log}} &= \int_{-0.1}^{0.1} (750 - 2500y)g(2x)(2)\,dy \\ &= 1000g \int_{-0.1}^{0.1} (3 - 10y)\sqrt{1/100 - y^2}\,dy \\ &= 3000g \int_{-0.1}^{0.1} \sqrt{1/100 - y^2}\,dy + 10\,000g \left\{ \frac{1}{3}\left(\frac{1}{100} - y^2\right)^{3/2} \right\}_{-0.1}^{0.1}. \end{aligned}$$

Since the integral represents half the area of the end of the log, $W = 3000g(1/2)\pi(1/100) = 15\pi g$ N. The weight of the water displaced by the log is

$$W_{\text{water}} = \int_{-1/10}^{1/10-L} 1000g(2x)(2)\,dy = 4000g \int_{-1/10}^{1/10-L} \sqrt{1/100 - y^2}\,dy.$$

If we set $y = (1/10)\sin\theta$ and $dy = (1/10)\cos\theta d\theta$, then

$$\begin{aligned} W_{\text{water}} &= 4000g \int_{-\pi/2}^{\overline{\theta}} \frac{1}{10}\cos\theta \frac{1}{10}\cos\theta\,d\theta \qquad \text{where } \overline{\theta} = \text{Sin}^{-1}(1 - 10L) \\ &= 40g \int_{-\pi/2}^{\overline{\theta}} \left(\frac{1 + \cos 2\theta}{2}\right) d\theta = 20g \left\{ \theta + \frac{1}{2}\sin 2\theta \right\}_{-\pi/2}^{\overline{\theta}} \\ &= 20g \left\{ \theta + \sin\theta\cos\theta \right\}_{-\pi/2}^{\overline{\theta}} = 20g\left(\overline{\theta} + \sin\overline{\theta}\cos\overline{\theta} + \frac{\pi}{2}\right) \\ &= 20g[\text{Sin}^{-1}(1 - 10L) + (1 - 10L)\sqrt{20L - 100L^2}] \text{ N}. \end{aligned}$$



Archimedes' principle requires $W_{\text{log}} = W_{\text{water}}$ so that

$$15\pi g = 20g[\text{Sin}^{-1}(1 - 10L) + (1 - 10L)\sqrt{20L - 100L^2}].$$

Instead of solving this equation for L, we return to the expression for W_{water} in terms of $\overline{\theta}$, drop the overbars, and equate

$$15\pi g = 20g\left(\theta + \frac{1}{2}\sin 2\theta + \frac{\pi}{2}\right) \quad \Longrightarrow \quad 4\theta + 2\sin 2\theta - \pi = 0.$$

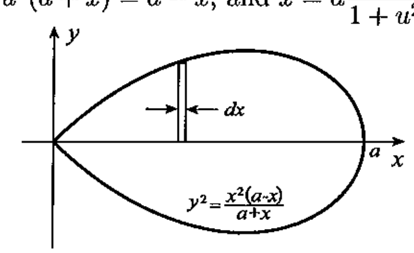
We use Newton's iterative procedure to solve this equation numerically,

$$\theta_1 = 0.5, \qquad \theta_{n+1} = \theta_n - \frac{4\theta_n + 2\sin 2\theta_n - \pi}{4 + 4\cos 2\theta_n}.$$

Iteration gives $\theta_2 = 0.412$, $\theta_3 = 0.415\,849$, $\theta_4 = 0.415\,856$, $\theta_5 = 0.415\,856$. Using $\theta = \overline{\theta} = 0.415\,856$ in $\overline{\theta} = \text{Sin}^{-1}(1 - 10L)$, we obtain $L = (1 - \sin\overline{\theta})/10 = 0.06$; that is, only 6 cm of the log is above water.

44. $A = 2\displaystyle\int_0^a x\sqrt{\frac{a - x}{a + x}}\,dx$ If we set $u = \sqrt{(a - x)/(a + x)}$, then $u^2(a + x) = a - x$, and $x = a\dfrac{1 - u^2}{1 + u^2}$.

Thus,
$$\begin{aligned} dx &= a\frac{(1 + u^2)(-2u) - (1 - u^2)(2u)}{(1 + u^2)^2}du \\ &= \frac{-4au}{(1 + u^2)^2}du, \end{aligned}$$



and

$$A = 2\int_1^0 \frac{a(1 - u^2)}{1 + u^2} u \frac{-4au}{(1 + u^2)^2}du = 8a^2 \int_0^1 \frac{u^2(1 - u^2)}{(1 + u^2)^3}du.$$

We now set $u = \tan\theta$, and $du = \sec^2\theta\, d\theta$,

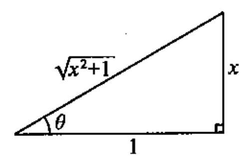
$$\begin{aligned}A &= 8a^2 \int_0^{\pi/4} \frac{\tan^2\theta\,(1-\tan^2\theta)}{\sec^6\theta}\sec^2\theta\, d\theta = 8a^2\int_0^{\pi/4}(\sin^2\theta\,\cos^2\theta - \sin^4\theta)\, d\theta\\ &= 8a^2\int_0^{\pi/4}\left[\frac{\sin^2 2\theta}{4} - \left(\frac{1-\cos 2\theta}{2}\right)^2\right] d\theta\\ &= 2a^2\int_0^{\pi/4}\left[\frac{1}{2}(1-\cos 4\theta) - 1 + 2\cos 2\theta - \frac{1}{2}(1+\cos 4\theta)\right] d\theta\\ &= 2a^2\left\{-\theta + \sin 2\theta - \frac{1}{4}\sin 4\theta\right\}_0^{\pi/4} = \frac{a^2(4-\pi)}{2}.\end{aligned}$$

45. (a) $\dfrac{1}{x+1+\sqrt{x^2+1}} = \dfrac{1}{x+1+\sqrt{x^2+1}}\,\dfrac{x+1-\sqrt{x^2+1}}{x+1-\sqrt{x^2+1}} = \dfrac{x+1-\sqrt{x^2+1}}{2x}$

(b) $\displaystyle\int_0^1 \frac{1}{x+1+\sqrt{x^2+1}}dx = \int_0^1 \frac{x+1-\sqrt{x^2+1}}{2x}dx = \frac{1}{2}\int_0^1\left(1+\frac{1}{x}-\frac{\sqrt{x^2+1}}{x}\right)dx$

In the last term we set $x = \tan\theta$ and $dx = \sec^2\theta\, d\theta$,

$$\begin{aligned}\int\frac{\sqrt{x^2+1}}{x}dx &= \int\frac{\sec\theta}{\tan\theta}\sec^2\theta\, d\theta = \int\frac{\sec\theta}{\tan\theta}(1+\tan^2\theta)\, d\theta = \int(\csc\theta + \tan\theta\,\sec\theta)\, d\theta\\ &= \ln|\csc\theta - \cot\theta| + \sec\theta + C\\ &= \ln\left|\frac{\sqrt{x^2+1}}{x} - \frac{1}{x}\right| + \sqrt{x^2+1} + C.\end{aligned}$$



Thus,

$$\begin{aligned}\int_0^1 \frac{1}{x+1+\sqrt{x^2+1}}dx &= \frac{1}{2}\left\{x + \ln|x| - \ln|\sqrt{x^2+1}-1| + \ln|x| - \sqrt{x^2+1}\right\}_0^1\\ &= \frac{1}{2}\left\{x - \sqrt{x^2+1} + \ln\left(\frac{x^2}{\sqrt{x^2+1}-1}\,\frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}+1}\right)\right\}_0^1\\ &= \frac{1}{2}\left\{x - \sqrt{x^2+1} + \ln(\sqrt{x^2+1}+1)\right\}_0^1 = 1 - \frac{\sqrt{2}}{2} + \frac{1}{2}\ln\left(\frac{1+\sqrt{2}}{2}\right).\end{aligned}$$

46. If we set $u = \sqrt{(1+x)/(1-x)}$, then $(1-x)u^2 = 1+x$, and $x = (u^2-1)/(u^2+1)$. Thus,

$$dx = \frac{(u^2+1)(2u)-(u^2-1)(2u)}{(u^2+1)^2}du = \frac{4u}{(u^2+1)^2}du,$$

and

$$\int_{-1}^1\sqrt{\frac{1+x}{1-x}}dx = \int_0^\infty \frac{u(4u)}{(u^2+1)^2}du.$$

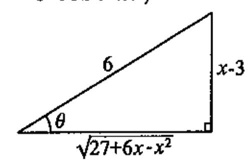
We now set $u = \tan\theta$ and $du = \sec^2\theta\, d\theta$,

$$\begin{aligned}\int_{-1}^1\sqrt{\frac{1+x}{1-x}}dx &= 4\int_0^{\pi/2}\frac{\tan^2\theta}{\sec^4\theta}\sec^2\theta\, d\theta = 4\int_0^{\pi/2}\sin^2\theta\, d\theta = 4\int_0^{\pi/2}\frac{1}{2}(1-\cos 2\theta)\, d\theta\\ &= 2\left\{\theta - \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = \pi.\end{aligned}$$

EXERCISES 8.5

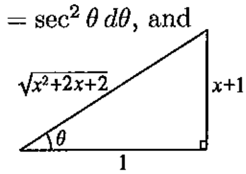
1. Since $27+6x-x^2 = 36-(x-3)^2$, we set $x-3 = 6\sin\theta$, in which case $dx = 6\cos\theta\, d\theta$, and

$$\int \frac{x}{\sqrt{27+6x-x^2}}dx = \int \frac{x}{\sqrt{36-(x-3)^2}}dx = \int \frac{3+6\sin\theta}{6\cos\theta}6\cos\theta\, d\theta$$
$$= 3\theta - 6\cos\theta + C$$
$$= 3\,\mathrm{Sin}^{-1}\left(\frac{x-3}{6}\right) - \sqrt{27+6x-x^2} + C.$$



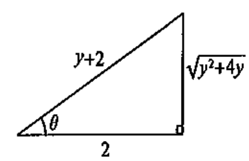
2. Since $x^2+2x+2 = (x+1)^2+1$, we set $x+1 = \tan\theta$, in which case $dx = \sec^2\theta\, d\theta$, and

$$\int \frac{1}{\sqrt{x^2+2x+2}}dx = \int \frac{1}{\sec\theta}\sec^2\theta\, d\theta$$
$$= \ln|\sec\theta + \tan\theta| + C$$
$$= \ln|\sqrt{x^2+2x+2}+x+1| + C.$$



3. Since $y^2+4y = (y+2)^2-4$, we set $y+2 = 2\sec\theta$, in which case $dy = 2\sec\theta\tan\theta\, d\theta$, and

$$\int \frac{1}{(y^2+4y)^{3/2}}dy = \int \frac{1}{[(y+2)-4]^{3/2}}dy = \int \frac{1}{8\tan^3\theta}2\sec\theta\tan\theta\, d\theta$$
$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}d\theta = \frac{1}{4}\int \csc\theta\cot\theta\, d\theta$$
$$= \frac{1}{4}(-\csc\theta) + C = -\frac{y+2}{4\sqrt{y^2+4y}} + C.$$

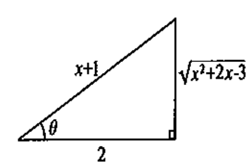


4. Since $-x^2+3x-4 = -(x-3/2)^2-7/4$, we set $x-3/2 = (\sqrt{7}/2)\tan\theta$, in which case $dx = (\sqrt{7}/2)\sec^2\theta\, d\theta$, and

$$\int \frac{1}{3x-x^2-4}dx = \int \frac{1}{-(7/4)\sec^2\theta}(\sqrt{7}/2)\sec^2\theta\, d\theta = -\frac{2}{\sqrt{7}}\theta + C = -\frac{2}{\sqrt{7}}\,\mathrm{Tan}^{-1}\left(\frac{2x-3}{\sqrt{7}}\right) + C.$$

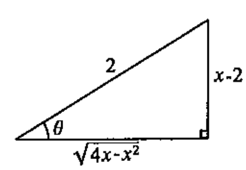
5. Since $x^2+2x-3 = (x+1)^2-4$, we set $x+1 = 2\sec\theta$, in which case $dx = 2\sec\theta\tan\theta\, d\theta$, and

$$\int \frac{\sqrt{x^2+2x-3}}{x+1}dx = \int \frac{\sqrt{(x+1)^2-4}}{x+1}dx = \int \frac{2\tan\theta}{2\sec\theta}2\sec\theta\tan\theta\, d\theta$$
$$= 2\int \tan^2\theta\, d\theta = 2\int (\sec^2\theta - 1)\, d\theta = 2(\tan\theta - \theta) + C$$
$$= \sqrt{x^2+2x-3} - 2\,\mathrm{Sec}^{-1}\left(\frac{x+1}{2}\right) + C.$$



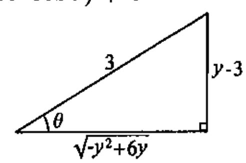
6. Since $4x-x^2 = -(x-2)^2+4$, we set $x-2 = 2\sin\theta$, in which case $dx = 2\cos\theta\, d\theta$, and

$$\int \frac{x}{(4x-x^2)^{3/2}}dx = \int \frac{2+2\sin\theta}{8\cos^3\theta}2\cos\theta\, d\theta = \frac{1}{2}\int \frac{1+\sin\theta}{\cos^2\theta}d\theta$$
$$= \frac{1}{2}\int \left(\sec^2\theta + \frac{\sin\theta}{\cos^2\theta}\right)d\theta = \frac{1}{2}\left(\tan\theta + \frac{1}{\cos\theta}\right) + C$$
$$= \frac{1}{2}\left(\frac{x-2}{\sqrt{4x-x^2}} + \frac{2}{\sqrt{4x-x^2}}\right) + C$$
$$= \frac{x}{2\sqrt{4x-x^2}} + C.$$



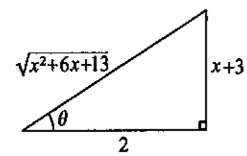
7. Since $-y^2+6y = 9-(y-3)^2$, we set $y-3 = 3\sin\theta$, in which case $dy = 3\cos\theta\,d\theta$, and

$$\begin{aligned}\int\sqrt{-y^2+6y}\,dy &= \int\sqrt{9-(y-3)^2}\,dy = \int 3\cos\theta\,3\cos\theta\,d\theta\\ &= 9\int\left(\frac{1+\cos 2\theta}{2}\right)d\theta = \frac{9}{2}\left(\theta+\frac{1}{2}\sin 2\theta\right)+C = \frac{9}{2}(\theta+\sin\theta\cos\theta)+C\\ &= \frac{9}{2}\,\mathrm{Sin}^{-1}\!\left(\frac{y-3}{3}\right)+\frac{9}{2}\left(\frac{y-3}{3}\right)\left(\frac{\sqrt{-y^2+6y}}{3}\right)+C\\ &= \frac{9}{2}\,\mathrm{Sin}^{-1}\!\left(\frac{y-3}{3}\right)+\frac{1}{2}(y-3)\sqrt{-y^2+6y}+C.\end{aligned}$$



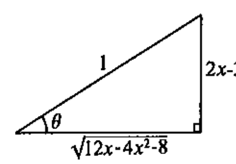
8. Since $x^2+6x+13 = (x+3)^2+4$, we set $x+3 = 2\tan\theta$, in which case $dx = 2\sec^2\theta\,d\theta$, and

$$\begin{aligned}\int\frac{2x-3}{x^2+6x+13}dx &= \int\frac{2(2\tan\theta-3)-3}{4\sec^2\theta}2\sec^2\theta\,d\theta = \frac{1}{2}(4\ln|\sec\theta|-9\theta)+C\\ &= 2\ln\left|\frac{\sqrt{x^2+6x+13}}{2}\right|-\frac{9}{2}\,\mathrm{Tan}^{-1}\!\left(\frac{x+3}{2}\right)+C\\ &= \ln(x^2+6x+13)-\frac{9}{2}\,\mathrm{Tan}^{-1}\!\left(\frac{x+3}{2}\right)+D.\end{aligned}$$



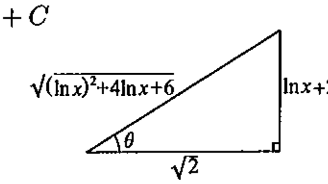
9. Since $12x-4x^2-8 = 4(-x^2+3x-2) = 4[-(x-3/2)^2+1/4]$, we set $x-3/2 = (1/2)\sin\theta$, in which case $dx = (1/2)\cos\theta\,d\theta$, and

$$\begin{aligned}\int\frac{5-4x}{\sqrt{12x-4x^2-8}}dx &= \int\frac{5-4x}{2\sqrt{1/4-(x-3/2)^2}}dx = \int\frac{5-2(3+\sin\theta)}{\cos\theta}\frac{1}{2}\cos\theta\,d\theta\\ &= -\frac{1}{2}\int(1+2\sin\theta)\,d\theta\\ &= -\frac{1}{2}(\theta-2\cos\theta)+C\\ &= -\frac{1}{2}\,\mathrm{Sin}^{-1}(2x-3)+\sqrt{12x-4x^2-8}+C.\end{aligned}$$



10. Since $6+4\ln x+(\ln x)^2 = (\ln x+2)^2+2$, we set $\ln x+2 = \sqrt{2}\tan\theta$. Then $(1/x)dx = \sqrt{2}\sec^2\theta\,d\theta$ and

$$\begin{aligned}\int\frac{1}{x\sqrt{6+4\ln x+(\ln x)^2}}dx &= \int\frac{1}{\sqrt{2}\sec\theta}\sqrt{2}\sec^2\theta\,d\theta = \ln|\sec\theta+\tan\theta|+C\\ &= \ln\left|\frac{\sqrt{(\ln x)^2+4\ln x+6}}{\sqrt{2}}+\frac{\ln x+2}{\sqrt{2}}\right|+C\\ &= \ln\left|\sqrt{(\ln x)^2+4\ln x+6}+\ln x+2\right|+D.\end{aligned}$$



11. If we set $z = 1/x$ and $dx = -(1/z^2)dz$, then

$$\begin{aligned}\int\frac{1}{x\sqrt{x^2+6x+3}}dx &= \int\frac{1}{\dfrac{1}{z}\sqrt{\dfrac{1}{z^2}+\dfrac{6}{z}+3}}\left(\frac{dz}{-z^2}\right) = -\int\frac{|z|}{z\sqrt{3z^2+6z+1}}dz\\ &= \frac{-1}{\sqrt{3}}\int\frac{|z|}{z\sqrt{z^2+2z+1/3}}dz = \frac{-1}{\sqrt{3}}\int\frac{|z|}{z\sqrt{(z+1)^2-2/3}}dz.\end{aligned}$$

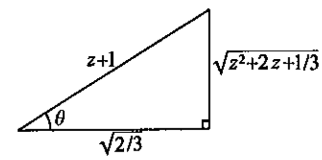
When $z>0$, $|z|/z = 1$, and we set $z+1 = \sqrt{2/3}\sec\theta$ and $dz = \sqrt{2/3}\sec\theta\tan\theta\,d\theta$, in which case

$$\int \frac{1}{x\sqrt{x^2+6x+3}}dx = \frac{-1}{\sqrt{3}}\int \frac{1}{\sqrt{2/3}\tan\theta}\sqrt{\frac{2}{3}}\sec\theta\,\tan\theta\,d\theta = \frac{-1}{\sqrt{3}}\int \sec\theta\,d\theta$$

$$= \frac{-1}{\sqrt{3}}\ln|\sec\theta+\tan\theta| + C = \frac{-1}{\sqrt{3}}\ln\left|\frac{z+1}{\sqrt{2/3}} + \frac{\sqrt{z^2+2z+1/3}}{\sqrt{2/3}}\right| + C$$

$$= \frac{-1}{\sqrt{3}}\ln\left|\frac{1}{x}+1+\sqrt{\frac{1}{x^2}+\frac{2}{x}+\frac{1}{3}}\right| + D = \frac{-1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}(x+1)+\sqrt{x^2+6x+3}}{\sqrt{3}x}\right| + D$$

$$= \frac{-1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}(x+1)+\sqrt{x^2+6x+3}}{x}\right| + E$$



When $z < 0$, $|z|/z = -1$. We make the same substitution as above, in which case

$$\int \frac{1}{x\sqrt{x^2+6x+3}}dx = \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{2/3}\tan\theta}\sqrt{\frac{2}{3}}\sec\theta\,\tan\theta\,d\theta = \frac{1}{\sqrt{3}}\int \sec\theta\,d\theta$$

$$= \frac{1}{\sqrt{3}}\ln|\sec\theta+\tan\theta| + C = \frac{1}{\sqrt{3}}\ln\left|\frac{z+1}{\sqrt{2/3}} + \frac{\sqrt{z^2+2z+1/3}}{\sqrt{2/3}}\right| + C$$

$$= \frac{1}{\sqrt{3}}\ln\left|\frac{1}{x}+1+\sqrt{\frac{1}{x^2}+\frac{2}{x}+\frac{1}{3}}\right| + D = \frac{1}{\sqrt{3}}\ln\left|\frac{1}{x}+1+\frac{\sqrt{x^2+6x+3}}{-\sqrt{3}x}\right| + D$$

$$= \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}(x+1)-\sqrt{x^2+6x+3}}{x}\right| + E$$

$$= \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}(x+1)-\sqrt{x^2+6x+3}}{x}\,\frac{\sqrt{3}(x+1)+\sqrt{x^2+6x+3}}{\sqrt{3}(x+1)+\sqrt{x^2+6x+3}}\right| + E$$

$$= \frac{1}{\sqrt{3}}\ln\left|\frac{3(x+1)^2-(x^2+6x+3)}{x[\sqrt{3}(x+1)+\sqrt{x^2+6x+3}]}\right| + E$$

$$= \frac{1}{\sqrt{3}}\ln\left|\frac{2x}{\sqrt{3}(x+1)+\sqrt{x^2+6x+3}}\right| + E$$

$$= \frac{-1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}(x+1)+\sqrt{x^2+6x+3}}{x}\right| + F,$$

the same antiderivative as when $x > 0$.

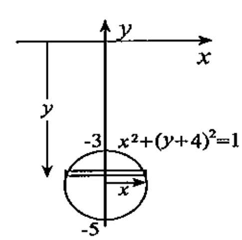
12. $F = \int_{-5}^{-3} 9810(-y)2x\,dy = -19\,620\int_{-5}^{-3} y\sqrt{1-(y+4)^2}\,dy$

If we set $y+4 = \sin\theta$, then $dy = \cos\theta\,d\theta$, and

$$F = -19\,620\int_{-\pi/2}^{\pi/2}(\sin\theta-4)\cos\theta\,\cos\theta\,d\theta$$

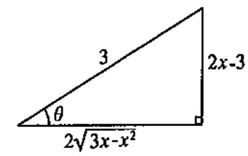
$$= -19\,620\int_{-\pi/2}^{\pi/2}[\cos^2\theta\,\sin\theta - 2(1+\cos 2\theta)]\,d\theta$$

$$= -19\,620\left\{-\frac{1}{3}\cos^3\theta - 2\theta - \sin 2\theta\right\}_{-\pi/2}^{\pi/2} = 39\,240\pi \text{ N}.$$



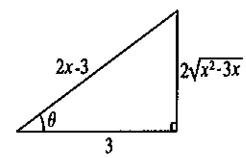
13. (a) With $3x - x^2 = -(x - 3/2)^2 + 9/4$, we set $x - 3/2 = (3/2)\sin\theta$ and $dx = (3/2)\cos\theta\, d\theta$. Then

$$\begin{aligned}\int \frac{1}{3x - x^2}dx &= \int \frac{1}{-(x-3/2)^2 + 9/4}dx = \int \frac{1}{(9/4)\cos^2\theta}\frac{3}{2}\cos\theta\, d\theta\\ &= \frac{2}{3}\int \sec\theta\, d\theta = \frac{2}{3}\ln|\sec\theta + \tan\theta| + C\\ &= \frac{2}{3}\ln\left|\frac{3}{2\sqrt{3x - x^2}} + \frac{2x-3}{2\sqrt{3x-x^2}}\right| + C = \frac{2}{3}\ln\left|\frac{x}{\sqrt{3x-x^2}}\right| + C.\end{aligned}$$



(b) With $x^2 - 3x = (x - 3/2)^2 - 9/4$, we set $x - 3/2 = (3/2)\sec\theta$ and $dx = (3/2)\sec\theta\,\tan\theta\, d\theta$. Then

$$\begin{aligned}\int \frac{1}{3x - x^2}dx &= \int \frac{-1}{x^2 - 3x}dx = \int \frac{-1}{(x-3/2)^2 - 9/4}dx\\ &= \int \frac{-1}{(9/4)\tan^2\theta}\frac{3}{2}\sec\theta\,\tan\theta\, d\theta = -\frac{2}{3}\int \frac{\sec\theta}{\tan\theta}d\theta = -\frac{2}{3}\int \csc\theta\, d\theta\\ &= -\frac{2}{3}\ln|\csc\theta - \cot\theta| + C\\ &= -\frac{2}{3}\ln\left|\frac{2x-3}{2\sqrt{x^2-3x}} - \frac{3}{2\sqrt{x^2-3x}}\right| + C\\ &= -\frac{2}{3}\ln\left|\frac{x-3}{\sqrt{x^2-3x}}\right| + C.\end{aligned}$$



(c) The first answer should only be used when $3x - x^2 > 0$; that is, when $0 < x < 3$. The second answer should be used when $x < 0$ or $x > 3$. We can find a single expression combining both answers, valid for all x except $x = 0$ and $x = 3$. The solution in part (a) can be rewritten

$$\frac{2}{3}\ln\left|\frac{x}{\sqrt{x(3-x)}}\right| + C = \frac{2}{3}\ln\left|\sqrt{\frac{x}{3-x}}\right| + C = \frac{1}{3}\ln\left(\frac{x}{3-x}\right) + C,$$

valid for $0 < x < 3$. For the solution in part (b), we write

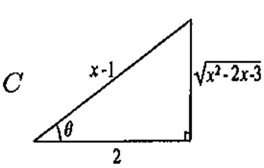
$$-\frac{2}{3}\ln\left|\frac{x-3}{\sqrt{x(x-3)}}\right| + C = -\frac{2}{3}\ln\left|\sqrt{\frac{x-3}{x}}\right| + C = -\frac{1}{3}\ln\left(\frac{x-3}{x}\right) + C = \frac{1}{3}\ln\left(\frac{x}{x-3}\right) + C,$$

valid for $x < 0$ and $x > 3$. Both of these can be combined into

$$\frac{1}{3}\ln\left|\frac{x}{x-3}\right| + C.$$

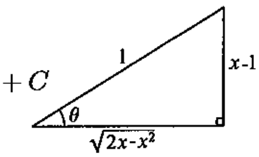
14. Since $x^2 - 2x - 3 = (x-1)^2 - 4$, we set $x - 1 = 2\sec\theta$ and $dx = 2\sec\theta\,\tan\theta\, d\theta$,

$$\begin{aligned}\int \sqrt{x^2 - 2x - 3}\, dx &= \int 2\tan\theta\, 2\sec\theta\,\tan\theta\, d\theta = 4\int \tan^2\theta\,\sec\theta\, d\theta\\ &= 4\int (\sec^2\theta - 1)\sec\theta\, d\theta = 4\int(\sec^3\theta - \sec\theta)\, d\theta\\ &= 4\left[\frac{1}{2}\ln|\sec\theta + \tan\theta| + \frac{1}{2}\sec\theta\,\tan\theta - \ln|\sec\theta + \tan\theta|\right] + C \text{ (see Example 8.9)}\\ &= 2[\sec\theta\,\tan\theta - \ln|\sec\theta + \tan\theta|] + C\\ &= 2\left(\frac{x-1}{2}\right)\frac{\sqrt{x^2-2x-3}}{2} - 2\ln\left|\frac{x-1}{2} + \frac{\sqrt{x^2-2x-3}}{2}\right| + C\\ &= \frac{1}{2}(x-1)\sqrt{x^2-2x-3} - 2\ln|x - 1 + \sqrt{x^2-2x-3}| + D.\end{aligned}$$



15. Since $2x - x^2 = 1 - (x-1)^2$, we set $x - 1 = \sin\theta$, in which case $dx = \cos\theta\, d\theta$, and

$$\int \frac{1}{x\sqrt{2x-x^2}}dx = \int \frac{1}{(1+\sin\theta)\cos\theta}\cos\theta\, d\theta = \int \frac{1}{1+\sin\theta}\frac{1-\sin\theta}{1-\sin\theta}d\theta$$
$$= \int \frac{1-\sin\theta}{\cos^2\theta}d\theta = \int (\sec^2\theta - \sec\theta\,\tan\theta)\,d\theta = \tan\theta - \sec\theta + C$$
$$= \frac{x-1}{\sqrt{2x-x^2}} - \frac{1}{\sqrt{2x-x^2}} + C = \frac{x-2}{\sqrt{2x-x^2}} + C.$$



16. If we set $u = \sqrt{2x-3}$, then $du = \dfrac{1}{\sqrt{2x-3}}dx$, and

$$\int \frac{1}{(2x+5)\sqrt{2x-3}+8x-12}dx = \int \frac{1}{(u^2+8)u+4u^2}u\,du = \int \frac{1}{u^2+4u+8}du = \int \frac{1}{(u+2)^2+4}du.$$

If we now set $u + 2 = 2\tan\theta$, then $du = 2\sec^2\theta\, d\theta$, and

$$\int \frac{1}{(2x+5)\sqrt{2x-3}+8x-12}dx = \int \frac{1}{4\sec^2\theta}2\sec^2\theta\, d\theta = \frac{1}{2}\theta + C$$
$$= \frac{1}{2}\,\mathrm{Tan}^{-1}\left(\frac{u+2}{2}\right) + C = \frac{1}{2}\,\mathrm{Tan}^{-1}\left(\frac{\sqrt{2x-3}+2}{2}\right) + C.$$

EXERCISES 8.6

1. If we set $\dfrac{x+2}{(x-1)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2}$, then $A = 1$ and $B = 3$, and

$$\int \frac{x+2}{x^2-2x+1}dx = \int \left[\frac{1}{x-1} + \frac{3}{(x-1)^2}\right]dx = \ln|x-1| - \frac{3}{x-1} + C.$$

2. $\displaystyle\int \frac{1}{y^3+3y^2+3y+1}dy = \int \frac{1}{(y+1)^3}dy = \frac{-1}{2(y+1)^2} + C$

3. If we set $\dfrac{1}{z(z^2+1)} = \dfrac{A}{z} + \dfrac{Bz+C}{z^2+1}$, then $A = 1$, $B = -1$, and $C = 0$, and

$$\int \frac{1}{z^3+z}dz = \int \left(\frac{1}{z} - \frac{z}{z^2+1}\right)dz = \ln|z| - \frac{1}{2}\ln(z^2+1) + C.$$

4. $\displaystyle\int \frac{x^2+2x-4}{x^2-2x-8}dx = \int \left(1 + \frac{4x+4}{x^2-2x-8}\right)dx = x + 4\int \frac{x+1}{(x-4)(x+2)}dx$

If we set $\dfrac{x+1}{(x-4)(x+2)} = \dfrac{A}{x-4} + \dfrac{B}{x+2}$, then $A = 5/6$ and $B = 1/6$, and

$$\int \frac{x^2+2x-4}{x^2-2x-8}dx = x + 4\int \left(\frac{5/6}{x-4} + \frac{1/6}{x+2}\right)dx = x + \frac{10}{3}\ln|x-4| + \frac{2}{3}\ln|x+2| + C.$$

5. $\displaystyle\int \frac{x}{(x-4)^2}dx = \int \frac{(x-4)+4}{(x-4)^2}dx = \int \left[\frac{1}{x-4} + \frac{4}{(x-4)^2}\right]dx = \ln|x-4| - \frac{4}{x-4} + C.$

6. If we set $\dfrac{y+1}{y(y+3)(y-2)} = \dfrac{A}{y} + \dfrac{B}{y+3} + \dfrac{C}{y-2}$, then $A = -1/6$, $B = -2/15$, $C = 3/10$, and

$$\int \frac{y+1}{y^3+y^2-6y}dy = \int \left(\frac{-1/6}{y} - \frac{2/15}{y+3} + \frac{3/10}{y-2}\right)dy = -\frac{1}{6}\ln|y| - \frac{2}{15}\ln|y+3| + \frac{3}{10}\ln|y-2| + C.$$

7. If we set $\dfrac{3x+5}{(x-1)^2(x+1)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+1}$, then $A = -1/2$, $B = 4$, and $C = 1/2$, and

$$\int \frac{3x+5}{x^3-x^2-x+1}\,dx = \int \left(\frac{-1/2}{x-1} + \frac{4}{(x-1)^2} + \frac{1/2}{x+1}\right) dx = -\frac{1}{2}\ln|x-1| - \frac{4}{x-1} + \frac{1}{2}\ln|x+1| + C.$$

8. If we set $\dfrac{x^3}{(x^2+2)^2} = \dfrac{Ax+B}{x^2+2} + \dfrac{Cx+D}{(x^2+2)^2}$, then $A=1$, $B=0$, $C=-2$, $D=0$, and

$$\int \frac{x^3}{(x^2+2)^2}\,dx = \int \left[\frac{x}{x^2+2} - \frac{2x}{(x^2+2)^2}\right] dx = \frac{1}{2}\ln(x^2+2) + \frac{1}{x^2+2} + C.$$

9. If we set $\dfrac{1}{x^2-3} = \dfrac{A}{x+\sqrt{3}} + \dfrac{B}{x-\sqrt{3}}$, then $A = -1/(2\sqrt{3})$ and $b = 1/(2\sqrt{3})$, and

$$\int \frac{1}{x^2-3}\,dx = \int \left(\frac{-1/(2\sqrt{3})}{x+\sqrt{3}} + \frac{1/(2\sqrt{3})}{x-\sqrt{3}}\right) dx = \frac{1}{2\sqrt{3}}(\ln|x-\sqrt{3}| - \ln|x+\sqrt{3}|) + C.$$

10. $\displaystyle\int \frac{y^2}{y^2+3y+2}\,dy = \int \left(1 + \frac{-3y-2}{y^2+3y+2}\right) dy$ If we set $\dfrac{3y+2}{y^2+3y+2} = \dfrac{A}{y+2} + \dfrac{B}{y+1}$, then $A=4$, $B = -1$, and

$$\int \frac{y^2}{y^2+3y+2}\,dy = y + \int \left(\frac{-4}{y+2} + \frac{1}{y+1}\right) dy = y - 4\ln|y+2| + \ln|y+1| + C.$$

11. If we set $\dfrac{z^2+3z-2}{z^3+5z} = \dfrac{A}{z} + \dfrac{Bz+C}{z^2+5}$, then $A = -2/5$, $B = 7/5$ and $C = 3$, and

$$\int \frac{z^2+3z-2}{z^3+5z}\,dz = \int \left(\frac{-2/5}{z} + \frac{7z/5+3}{z^2+5}\right) dz.$$

In the term $3/(z^2+5)$, we set $z = \sqrt{5}\tan\theta$ and $dz = \sqrt{5}\sec^2\theta\,d\theta$,

$$\begin{aligned}\int \frac{z^2+3z-2}{z^3+5z}\,dz &= -\frac{2}{5}\ln|z| + \frac{7}{10}\ln(z^2+5) + 3\int \frac{1}{5\sec^2\theta}\sqrt{5}\sec^2\theta\,d\theta \\ &= -\frac{2}{5}\ln|z| + \frac{7}{10}\ln(z^2+5) + \frac{3}{\sqrt{5}}\theta + C \\ &= -\frac{2}{5}\ln|z| + \frac{7}{10}\ln(z^2+5) + \frac{3}{\sqrt{5}}\operatorname{Tan}^{-1}\left(\frac{z}{\sqrt{5}}\right) + C.\end{aligned}$$

12. If we set $\dfrac{y^2+6y+4}{(y^2+4)(y^2+1)} = \dfrac{Ay+B}{y^2+4} + \dfrac{Cy+D}{y^2+1}$, then $A=-2$, $B=0$, $C=2$, $D=1$, and

$$\int \frac{y^2+6y+4}{y^4+5y^2+4}\,dy = \int \left(\frac{-2y}{y^2+4} + \frac{2y+1}{y^2+1}\right) dy = -\ln(y^2+4) + \ln(y^2+1) + \operatorname{Tan}^{-1}y + C.$$

13. If we set $\dfrac{x}{(x^2+6)(x^2+1)} = \dfrac{Ax+B}{x^2+6} + \dfrac{Cx+D}{x^2+1}$, then $A = -1/5$, $B = 0$, $C = 1/5$, and $D = 0$, and

$$\int \frac{x}{x^4+7x^2+6}\,dx = \int \left(\frac{x/5}{x^2+1} - \frac{x/5}{x^2+6}\right) dx = \frac{1}{10}\ln(x^2+1) - \frac{1}{10}\ln(x^2+6) + C.$$

14. If we set $\dfrac{x^2+3}{(x^2+2)(x-1)(x+1)} = \dfrac{Ax+B}{x^2+2} + \dfrac{C}{x-1} + \dfrac{D}{x+1}$, then $A = 0$, $B = -1/3$, $C = 2/3$, $D = -2/3$, and

$$\int \frac{x^2+3}{x^4+x^2-2}\,dx = \int \left(\frac{-1/3}{x^2+2} + \frac{2/3}{x-1} - \frac{2/3}{x+1}\right) dx.$$

In the first term we set $x = \sqrt{2}\tan\theta$ and $dx = \sqrt{2}\sec^2\theta\, d\theta$,

$$\int \frac{x^2+3}{x^4+x^2-2}dx = -\frac{1}{3}\int \frac{1}{2\sec^2\theta}\sqrt{2}\sec^2\theta\, d\theta + \frac{2}{3}\ln|x-1| - \frac{2}{3}\ln|x+1|$$
$$= -\frac{1}{3\sqrt{2}}\theta + \frac{2}{3}\ln\left|\frac{x-1}{x+1}\right| + C = -\frac{1}{3\sqrt{2}}\mathrm{Tan}^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{2}{3}\ln\left|\frac{x-1}{x+1}\right| + C.$$

15. If we set $\dfrac{3t+4}{t(t-1)^3} = \dfrac{A}{t} + \dfrac{B}{t-1} + \dfrac{C}{(t-1)^2} + \dfrac{D}{(t-1)^3}$, then $A = -4$, $B = 4$, $C = -4$, and $D = 7$, and

$$\int \frac{3t+4}{t^4-3t^3+3t^2-t}dt = \int\left[-\frac{4}{t} + \frac{4}{t-1} - \frac{4}{(t-1)^2} + \frac{7}{(t-1)^3}\right]dt$$
$$= -4\ln|t| + 4\ln|t-1| + \frac{4}{t-1} - \frac{7}{2(t-1)^2} + C.$$

16. If we set $\dfrac{x^3+6}{(x-1)^2(x+2)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+2} + \dfrac{D}{(x+2)^2}$, then $A = -5/27$, $B = 7/9$, $C = 32/27$, $D = -2/9$, and

$$\int \frac{x^3+6}{x^4+2x^3-3x^2-4x+4}dx = \int\left[\frac{-5/27}{x-1} + \frac{7/9}{(x-1)^2} + \frac{32/27}{x+2} - \frac{2/9}{(x+2)^2}\right]dx$$
$$= -\frac{5}{27}\ln|x-1| - \frac{7}{9(x-1)} + \frac{32}{27}\ln|x+2| + \frac{2}{9(x+2)} + C.$$

17. The length of the curve is

$$L = \int_0^{1/2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = \int_0^{1/2}\sqrt{1+\left(\frac{-2x}{1-x^2}\right)^2}dx = \int_0^{1/2}\sqrt{\frac{1-2x^2+x^4+4x^2}{(1-x^2)^2}}dx$$
$$= \int_0^{1/2}\sqrt{\frac{(1+x^2)^2}{(1-x^2)^2}}dx = \int_0^{1/2}\frac{1+x^2}{1-x^2}dx = \int_0^{1/2}\left(-1+\frac{2}{1-x^2}\right)dx$$
$$= \int_0^{1/2}\left(-1+\frac{1}{1-x}+\frac{1}{1+x}\right)dx = \Big\{-x-\ln|1-x|+\ln|1+x|\Big\}_0^{1/2} = \ln 3 - \frac{1}{2}.$$

18. Separation of variables and partial fractions give

$$\int -\frac{dt}{1500} = \int \frac{dv}{v^2-2500} = \int\left(\frac{1/100}{v-50} - \frac{1/100}{v+50}\right)dv = \frac{1}{100}\int\left(\frac{1}{v-50} - \frac{1}{v+50}\right)dv.$$

Thus,

$$\frac{-t}{1500} + C = \frac{1}{100}\left(\ln|v-50| - \ln|v+50|\right) \quad\Longrightarrow\quad -\frac{t}{15} + 100C = \ln\left|\frac{v-50}{v+50}\right|.$$

When we exponentiate,

$$\left|\frac{v-50}{v+50}\right| = e^{100C-t/15} \quad\Longrightarrow\quad v-50 = (v+50)De^{-t/15},$$

where $D = \pm e^{100C}$. When we solve this for v, the result is $v(t) = \dfrac{50(1+De^{-t/15})}{1-De^{-t/15}}$. If we choose time $t = 0$ when the car begins motion, then $v(0) = 0$, and this requires $0 = 50(1+D)/(1-D) \Longrightarrow D = -1$. Hence, $v(t) = 50(1-e^{-t/15})/(1+e^{-t/15})$.
(b) If we set $u = 1 + e^{-t/15}$ and $du = -(1/15)e^{-t/15}dt$,

$$x(t) = \int \frac{50(1-e^{-t/15})}{1+e^{-t/15}}dt = 50\int \frac{1-(u-1)}{u}\left(\frac{-15\,du}{u-1}\right) = 750\int \frac{u-2}{u(u-1)}du$$
$$= 750\int\left(\frac{2}{u}-\frac{1}{u-1}\right)du = 750(2\ln|u| - \ln|u-1|) + E$$
$$= 750[2\ln(1+e^{-t/15}) - \ln(e^{-t/15})] + E = 750\left[\frac{t}{15} + 2\ln(1+e^{-t/15})\right] + E.$$

If we choose $x(0) = 0$, then $0 = 750(2\ln 2) + E \Longrightarrow E = -1500\ln 2$, and

$$x(t) = 750\left[\frac{t}{15} + 2\ln(1+e^{-t/15})\right] - 1500\ln 2 = 750\left[\frac{t}{15} + 2\ln\left(\frac{1+e^{-t/15}}{2}\right)\right].$$

19. If we set $V = \sqrt{mg/k}$, the differential equation can be expressed in the form

$$\frac{m}{k}\frac{dv}{dt} = \frac{mg}{k} - v^2 = V^2 - v^2 \quad\Longrightarrow\quad \int\frac{dv}{v^2-V^2} = \int -\frac{k}{m}dt.$$

Partial fractions gives

$$-\frac{kt}{m} + C = \int\left[\frac{1/(2V)}{v-V} - \frac{1/(2V)}{v+V}\right]dv = \frac{1}{2V}(\ln|v-V| - \ln|v+V|) = \frac{1}{2V}\ln\left|\frac{v-V}{v+V}\right|.$$

When we exponentiate,

$$\left|\frac{v-V}{v+V}\right| = e^{2VC-2kVt/m} \quad\Longrightarrow\quad v - V = (v+V)De^{-2kVt/m},$$

where $D = \pm e^{2VC}$. When we solve this for v, the result is $v(t) = \dfrac{V(1+De^{-2kVt/m})}{1-De^{-2kVt/m}}$. If we choose time $t = 0$ when the raindrop exits the cloud, then $v(0) = v_0$, and this requires

$$v_0 = \frac{V(1+D)}{1-D} \quad\Longrightarrow\quad v_0(1-D) = V(1+D) \quad\Longrightarrow\quad D = \frac{v_0-V}{v_0+V}.$$

Hence, $v(t) = \dfrac{V\left[1-\left(\dfrac{V-v_0}{V+v_0}\right)e^{-2kVt/m}\right]}{1+\left(\dfrac{V-v_0}{V+v_0}\right)e^{-2kVt/m}}$. Since $V = \lim_{t\to\infty} v(t)$, it follows that V is the limiting velocity of the raindrop.

20. With $mv\dfrac{dv}{dy} = mg - kv^2$ expressed in the form $\dfrac{v\,dv}{mg-kv^2} = \dfrac{dy}{m}$, solutions are defined implicitly by

$$\frac{y}{m} + C = \int\frac{v\,dv}{mg-kv^2} = -\frac{1}{2k}\ln|mg-kv^2|.$$

When we multiply by $-2k$ and exponentiate,

$$|mg-kv^2| = e^{-2kC-2ky/m} \quad\Longrightarrow\quad mg - kv^2 = De^{-2ky/m} \quad\Longrightarrow\quad v = \sqrt{\frac{mg}{k} - \frac{D}{k}e^{-2ky/m}},$$

where $D = \pm e^{-2kC}$. Since $v(0) = v_0$, we have $v_0 = \sqrt{\dfrac{mg}{k} - \dfrac{D}{k}} \Longrightarrow \dfrac{D}{k} = \dfrac{mg}{k} - v_0^2$. Hence,

$$v(y) = \sqrt{\frac{mg}{k} - \left(\frac{mg}{k} - v_0^2\right)e^{-2ky/m}}.$$

The velocity of the raindrop when it strikes the earth is $\sqrt{\dfrac{mg}{k} - \left(\dfrac{mg}{k} - v_0^2\right)e^{-2kh/m}}$.

21. With $k = 1$ and $C = 10^6$, the differential equation can be expressed in the form

$$\frac{dN}{dt} = N\left(1 - \frac{N}{10^6}\right) = 10^{-6}N(10^6 - N) \quad \Longrightarrow \quad \frac{dN}{N(10^6 - N)} = 10^{-6}dt.$$

Partial fractions gives

$$\int \left(\frac{10^{-6}}{N} + \frac{10^{-6}}{10^6 - N}\right) dN = \int 10^{-6}\, dt.$$

If we divide by 10^{-6}, solutions are defined implicitly by

$$t + D = \ln|N| - \ln|10^6 - N| = \ln\left|\frac{N}{10^6 - N}\right|.$$

Exponentiation gives $\left|\dfrac{N}{10^6 - N}\right| = e^{t+D} \quad \Longrightarrow \quad N = (10^6 - N)Ee^t \quad \Longrightarrow \quad N = \dfrac{10^6 Ee^t}{1 + Ee^t}$, where $E = \pm e^D$. For $N(0) = 100$, we must have $100 = \dfrac{10^6 E}{1 + E} \Longrightarrow E = \dfrac{1}{9999}$. Hence, $N(t) = 10^6/(1 + 9999e^{-t})$.

22. The differential equation can be expressed in the form $\dfrac{dN}{N(C - N)} = \dfrac{k}{C}\, dt$. Partial fractions gives

$$\int \frac{k}{C} dt = \int \left(\frac{1/C}{N} + \frac{1/C}{C - N}\right) dN = \frac{1}{C}\int \left(\frac{1}{N} + \frac{1}{C - N}\right) dN.$$

When we multiply by C, solutions are defined implicitly by

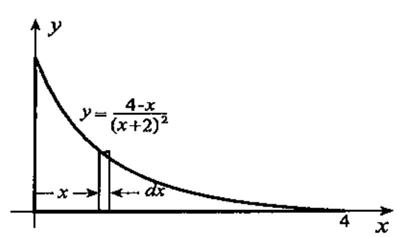
$$kt + D = \ln|N| - \ln|C - N| = \ln\left|\frac{N}{C - N}\right|.$$

Exponentiation gives $\left|\dfrac{N}{C - N}\right| = e^{kt+D} \Longrightarrow N = (C - N)Ee^{kt} \Longrightarrow N = \dfrac{CEe^{kt}}{1 + Ee^{kt}}$, where $E = \pm e^D$. For $N(0) = N_0$, we must have $N_0 = \dfrac{CE}{1 + E} \Longrightarrow N_0(1 + E) = CE \Longrightarrow E = \dfrac{N_0}{C - N_0}$. Hence,

$$N(t) = \frac{C\left(\dfrac{N_0}{C - N_0}\right)e^{kt}}{1 + \left(\dfrac{N_0}{C - N_0}\right)e^{kt}} = \frac{C}{1 + \left(\dfrac{C - N_0}{N_0}\right)e^{-kt}}.$$

23. $A = \displaystyle\int_0^4 \frac{4 - x}{(x + 2)^2} dx = \int_0^4 \left[\frac{6}{(x + 2)^2} - \frac{1}{x + 2}\right] dx = \left\{\frac{-6}{x + 2} - \ln|x + 2|\right\}_0^4 = 2 - \ln 3$

Since
$$A\overline{x} = \int_0^4 \frac{x(4 - x)}{(x + 2)^2} dx = \int_0^4 \left[-1 + \frac{8x + 4}{(x + 2)^2}\right] dx$$
$$= \int_0^4 \left[-1 + \frac{8}{x + 2} - \frac{12}{(x + 2)^2}\right] dx$$
$$= \left\{-x + 8\ln|x + 2| + \frac{12}{x + 2}\right\}_0^4 = 8(-1 + \ln 3),$$



it follows that $\overline{x} = 8(-1 + \ln 3)/(2 - \ln 3) = 0.875$. Since

$$A\overline{y} = \int_0^4 \frac{1}{2}\left[\frac{4 - x}{(x + 2)^2}\right]^2 dx = \frac{1}{2}\int_0^4 \frac{(4 - x)^2}{(x + 2)^4} dx = \frac{1}{2}\int_0^4 \left[\frac{1}{(x + 2)^2} - \frac{12}{(x + 2)^3} + \frac{36}{(x + 2)^4}\right] dx$$
$$= \frac{1}{2}\left\{-\frac{1}{x + 2} + \frac{6}{(x + 2)^2} - \frac{12}{(x + 2)^3}\right\}_0^4 = \frac{2}{9},$$

we obtain $\overline{y} = (2/9)/(2 - \ln 3) = 0.247$.

24. We can separate the differential equation,

$$k\,dt = \frac{1}{x(N-x)}\,dx \qquad \Longrightarrow \qquad k\,dt = \left(\frac{1/N}{x} + \frac{1/N}{N-x}\right)dx.$$

Solutions are defined implicitly by

$$\frac{1}{N}\left(\ln|x| - \ln|N-x|\right) = kt + C.$$

Since x and $N-x$ are both positive, we may drop the absolute values and write

$$\ln\left(\frac{x}{N-x}\right) = N(kt+C) \qquad \Longrightarrow \qquad \frac{x}{N-x} = De^{Ft},$$

where we have substituted $D = e^{NC}$ and $F = kN$. Multiplication by $N-x$ gives

$$x = De^{Ft}(N-x) \quad \Longrightarrow \quad x\left(1 + De^{Ft}\right) = NDe^{Ft}.$$

Thus,

$$x(t) = \frac{NDe^{Ft}}{1+De^{Ft}} = \frac{ND}{D+e^{-Ft}}.$$

Since $x(0) = 1$, it follows that $1 = \dfrac{ND}{1+D} \Longrightarrow D = 1/(N-1)$, and

$$x(t) = \frac{\dfrac{N}{N-1}}{\dfrac{1}{N-1} + e^{-Ft}} = \frac{N}{1+(N-1)e^{-Ft}}.$$

25. (a) When $a = b$, the differential equation becomes

$$\frac{dx}{dt} = k(a-x)^2 \quad \Longrightarrow \quad \frac{1}{(a-x)^2}\,dx = k\,dt,$$

a separated equation. Solutions are defined implicitly by

$$\frac{1}{a-x} = kt + C \quad \Longrightarrow \quad x - a = -\frac{1}{kt+C} \quad \Longrightarrow \quad x(t) = a - \frac{1}{kt+C}.$$

(b) When $a \neq b$, we again separate the differential equation, but use partial fractions to write,

$$\frac{1}{(a-x)(b-x)}\,dx = k\,dt \quad \Longrightarrow \quad \left[\frac{-1/(a-b)}{a-x} + \frac{1/(a-b)}{b-x}\right]dx = k\,dt.$$

Solutions are defined implicitly by

$$\frac{1}{a-b}\left[\ln|a-x| - \ln|b-x|\right] = kt + C.$$

To find explicit solutions we write

$$\ln\left|\frac{a-x}{b-x}\right| = (a-b)kt + C(a-b) \quad \Longrightarrow \quad \left|\frac{a-x}{b-x}\right| = e^{(a-b)kt+C(a-b)} \quad \Longrightarrow \quad \frac{a-x}{b-x} = De^{(a-b)kt},$$

where $D = \pm e^{C(a-b)}$. Multiplication by $b-x$ gives

$$a - x = (b-x)De^{(a-b)kt} \quad \Longrightarrow \quad x(t) = \frac{a - bDe^{(a-b)kt}}{1 - De^{(a-b)kt}}.$$

26. We separate the differential equation and use partial fractions to write

$$\frac{1}{v_0^2 - v^2}\,dv = \frac{1}{a}\,dt \quad \Longrightarrow \quad \left[\frac{1/(2v_0)}{v_0 - v} + \frac{1/(2v_0)}{v_0 + v}\right] dv = \frac{1}{a}\,dt.$$

Solutions are defined implicitly by

$$\frac{1}{2v_0}\left[-\ln(v_0 - v) + \ln(v_0 + v)\right] = \frac{t}{a} + C \quad \Longrightarrow \quad \ln\left(\frac{v_0 + v}{v_0 - v}\right) = \frac{2v_0 t}{a} + 2v_0 C.$$

Exponentiation gives

$$\frac{v_0 + v}{v_0 - v} = e^{2v_0 t/a + 2v_0 C} = De^{2v_0 t/a},$$

where $D = e^{2v_0 C}$. We can now solve for $v(t)$,

$$v_0 + v = (v_0 - v)De^{2v_0 t/a} \quad \Longrightarrow \quad v = \frac{v_0(De^{2v_0 t/a} - 1)}{De^{2v_0 t/a} + 1}.$$

The initial condition $v(0) = 0$ requires $D = 1$, and therefore

$$v(t) = \frac{v_0(e^{2v_0 t/a} - 1)}{e^{2v_0 t/a} + 1}.$$

27. If we set $\dfrac{x^3 + x + 2}{x(x^2+1)^2} = \dfrac{A}{x} + \dfrac{Bx + C}{x^2 + 1} + \dfrac{Dx + E}{(x^2+1)^2}$, then $A = 2$, $B = -2$, $C = 1$, $D = -2$, and $E = 0$, so that

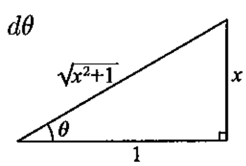
$$\int \frac{x^3 + x + 2}{x^5 + 2x^3 + x}dx = \int \left[\frac{2}{x} + \frac{-2x + 1}{x^2 + 1} - \frac{2x}{(1 + x^2)^2}\right] dx = 2\ln|x| - \ln(x^2 + 1) + \mathrm{Tan}^{-1}x + \frac{1}{x^2 + 1} + C.$$

28. If we set $\dfrac{1}{x^5 + x^4 + 2x^3 + 2x^2 + x + 1} = \dfrac{1}{(x+1)(x^2+1)^2} = \dfrac{A}{x+1} + \dfrac{Bx + C}{x^2 + 1} + \dfrac{Dx + E}{(x^2+1)^2}$, then $A = 1/4$, $B = -1/4$, $C = 1/4$, $D = -1/2$, $E = 1/2$, and

$$\int \frac{1}{x^5 + x^4 + 2x^3 + 2x^2 + x + 1}dx = \int \left[\frac{1/4}{x+1} + \frac{-x/4 + 1/4}{x^2 + 1} + \frac{-x/2 + 1/2}{(x^2+1)^2}\right] dx.$$

In the very last term we set $x = \tan\theta$ and $dx = \sec^2\theta\,d\theta$, in which case

$$\begin{aligned}\int \frac{1}{(x^2+1)^2}dx &= \int \frac{1}{\sec^4\theta}\sec^2\theta\,d\theta = \int \cos^2\theta\,d\theta = \int \left(\frac{1 + \cos 2\theta}{2}\right) d\theta \\ &= \frac{\theta}{2} + \frac{1}{4}\sin 2\theta + C = \frac{\theta}{2} + \frac{1}{2}\sin\theta\,\cos\theta + C \\ &= \frac{1}{2}\mathrm{Tan}^{-1}x + \frac{1}{2}\frac{x}{x^2+1} + C.\end{aligned}$$



Consequently,

$$\begin{aligned}\int \frac{1}{x^5 + x^4 + 2x^3 + 2x^2 + x + 1}dx &= \frac{1}{4}\ln|x+1| - \frac{1}{8}\ln(x^2+1) + \frac{1}{4}\mathrm{Tan}^{-1}x \\ &\quad + \frac{1}{4(x^2+1)} + \frac{1}{4}\mathrm{Tan}^{-1}x + \frac{x}{4(x^2+1)} + C \\ &= \frac{1}{4}\ln|x+1| - \frac{1}{8}\ln(x^2+1) + \frac{1}{2}\mathrm{Tan}^{-1}x + \frac{x+1}{4(x^2+1)} + C.\end{aligned}$$

29. If we set $\dfrac{1}{(x^2+5)(x^2+2x+3)} = \dfrac{Ax + B}{x^2 + 5} + \dfrac{Cx + D}{x^2 + 2x + 3}$, then $A = -1/12$, $B = -1/12$, $C = 1/12$, and $D = 1/4$, so that

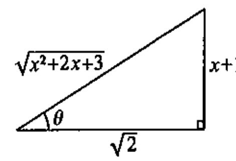
$$\int \frac{1}{(x^2+5)(x^2+2x+3)}dx = \frac{1}{12}\int \left(\frac{-x-1}{x^2+5} + \frac{x+3}{x^2+2x+3}\right) dx.$$

If we set $x = \sqrt{5}\tan\theta$ and $dx = \sqrt{5}\sec^2\theta\,d\theta$, then

$$\int \frac{1}{x^2+5}dx = \int \frac{1}{5\sec^2\theta}\sqrt{5}\sec^2\theta\,d\theta = \frac{\theta}{\sqrt{5}} + C = \frac{1}{\sqrt{5}}\,\mathrm{Tan}^{-1}\left(\frac{x}{\sqrt{5}}\right) + C.$$

Since $x^2+2x+3 = (x+1)^2+2$, we set $x+1 = \sqrt{2}\tan\theta$ and $dx = \sqrt{2}\sec^2\theta\,d\theta$ in

$$\begin{aligned}\int \frac{x+3}{(x+1)^2+2}dx &= \int \frac{2+\sqrt{2}\tan\theta}{2\sec^2\theta}\sqrt{2}\sec^2\theta\,d\theta = \int(\sqrt{2}+\tan\theta)\,d\theta \\ &= \sqrt{2}\,\theta + \ln|\sec\theta| + C \\ &= \sqrt{2}\,\mathrm{Tan}^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + \ln\left|\frac{\sqrt{x^2+2x+3}}{\sqrt{2}}\right| + C \\ &= \sqrt{2}\,\mathrm{Tan}^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + \frac{1}{2}\ln(x^2+2x+3) + D.\end{aligned}$$



Thus,

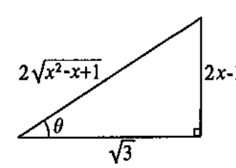
$$\begin{aligned}\int \frac{1}{(x^2+5)(x^2+2x+3)}dx &= \frac{1}{12}\Big[-\frac{1}{2}\ln(x^2+5) - \frac{1}{\sqrt{5}}\,\mathrm{Tan}^{-1}\left(\frac{x}{\sqrt{5}}\right) + \sqrt{2}\,\mathrm{Tan}^{-1}\left(\frac{x+1}{\sqrt{2}}\right) \\ &\qquad + \frac{1}{2}\ln(x^2+2x+3)\Big] + C \\ &= -\frac{1}{24}\ln(x^2+5) - \frac{1}{12\sqrt{5}}\,\mathrm{Tan}^{-1}\left(\frac{x}{\sqrt{5}}\right) + \frac{\sqrt{2}}{12}\,\mathrm{Tan}^{-1}\left(\frac{x+1}{\sqrt{2}}\right) \\ &\qquad + \frac{1}{24}\ln(x^2+2x+3) + C.\end{aligned}$$

30. If we set $\dfrac{1}{x^3+1} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2-x+1}$, then $A = 1/3$, $B = -1/3$, $C = 2/3$, and

$$\int \frac{1}{x^3+1}dx = \int\left(\frac{1/3}{x+1} + \frac{-x/3+2/3}{x^2-x+1}\right)dx = \frac{1}{3}\ln|x+1| + \frac{1}{3}\int\frac{-x+2}{(x-1/2)^2+3/4}dx.$$

In the remaining integral we set $x - 1/2 = (\sqrt{3}/2)\tan\theta$, and $dx = (\sqrt{3}/2)\sec^2\theta\,d\theta$,

$$\begin{aligned}\int \frac{1}{x^3+1}dx &= \frac{1}{3}\ln|x+1| + \frac{1}{3}\int\frac{-1/2-(\sqrt{3}/2)\tan\theta+2}{(3/4)\sec^2\theta}\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta \\ &= \frac{1}{3}\ln|x+1| + \frac{1}{3}(\sqrt{3}\theta + \ln|\cos\theta|) + C \\ &= \frac{1}{3}\ln|x+1| + \frac{1}{\sqrt{3}}\,\mathrm{Tan}^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + \frac{1}{3}\ln\left|\frac{\sqrt{3}}{2\sqrt{x^2-x+1}}\right| + C \\ &= \frac{1}{3}\ln|x+1| + \frac{1}{\sqrt{3}}\,\mathrm{Tan}^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) - \frac{1}{6}\ln(x^2-x+1) + D.\end{aligned}$$



31. If we set $u = \cos x$ and $du = -\sin x\,dx$, then

$$\begin{aligned}\int \frac{\sin x}{\cos x(1+\cos^2 x)}dx &= \int\frac{1}{u(1+u^2)}(-du) = -\int\left(\frac{1}{u} - \frac{u}{1+u^2}\right)du \\ &= -\ln|u| + \frac{1}{2}\ln(1+u^2) + C = \frac{1}{2}\ln(1+\cos^2 x) - \ln|\cos x| + C.\end{aligned}$$

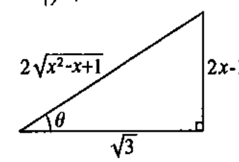
32. If we set $\dfrac{x^4+8x^3-x^2+2x+1}{x^5+x^4+x^2+x} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2} + \dfrac{Dx+E}{x^2-x+1}$, then $A = 1$, $B = -2$, $C = 3$, $D = 2$, $E = 0$, and

$$\int\frac{x^4+8x^3-x^2+2x+1}{x^5+x^4+x^2+x}dx = \int\left[\frac{1}{x} - \frac{2}{x+1} + \frac{3}{(x+1)^2} + \frac{2x}{(x-1/2)^2+3/4}\right]dx.$$

In the last term we set $x - 1/2 = (\sqrt{3}/2)\tan\theta$ and $dx = (\sqrt{3}/2)\sec^2\theta\, d\theta$, in which case

$$\int \frac{2x}{(x-1/2)^2+3/4}dx = 2\int \frac{1/2+(\sqrt{3}/2)\tan\theta}{(3/4)\sec^2\theta}\frac{\sqrt{3}}{2}\sec^2\theta\, d\theta = \frac{2}{\sqrt{3}}(\theta + \sqrt{3}\ln|\sec\theta|) + C$$

$$= \frac{2}{\sqrt{3}}\operatorname{Tan}^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + 2\ln\left|\frac{2\sqrt{x^2-x+1}}{\sqrt{3}}\right| + C$$



$$= \frac{2}{\sqrt{3}}\operatorname{Tan}^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + \ln(x^2-x+1) + D.$$

Thus,

$$\int \frac{x^4+8x^3-x^2+2x+1}{x^5+x^4+x^2+x}dx = \ln|x| - 2\ln|x+1| - \frac{3}{x+1} + \frac{2}{\sqrt{3}}\operatorname{Tan}^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + \ln(x^2-x+1) + D.$$

33. (a) $\dfrac{dv}{dt} = \dfrac{gH}{L}\left(1 - \dfrac{v^2}{v_f^2}\right) = \dfrac{gH}{Lv_f^2}(v_f^2 - v^2)$, a separable differential equation $\dfrac{dv}{v_f^2 - v^2} = \dfrac{gH}{Lv_f^2}dt$. Partial fractions gives

$$\frac{gHt}{Lv_f^2} + C = \int \frac{1}{v_f^2 - v^2}dv = \frac{1}{2v_f}\int\left(\frac{1}{v_f+v} + \frac{1}{v_f-v}\right)dv$$

$$= \frac{1}{2v_f}[\ln(v_f+v) - \ln(v_f-v)] = \frac{1}{2v_f}\ln\left(\frac{v_f+v}{v_f-v}\right).$$

The initial condition $v(0) = 0$ implies that $C = 0$, and therefore

$$\frac{gHt}{Lv_f^2} = \frac{1}{2v_f}\ln\left(\frac{v_f+v}{v_f-v}\right) \quad\Longrightarrow\quad t = \frac{Lv_f}{2gH}\ln\left(\frac{v_f+v}{v_f-v}\right).$$

(b) If we exponentiate $\ln\left(\dfrac{v_f+v}{v_f-v}\right) = \dfrac{2gHt}{Lv_f}$, we obtain

$$\frac{v_f+v}{v_f-v} = e^{2gHt/(Lv_f)} \quad\Longrightarrow\quad v_f + v = (v_f - v)e^{2gHt/(Lv_f)}.$$

Hence, $v = v_f\left[\dfrac{e^{2gHt/(Lv_f)} - 1}{e^{2gHt/(Lv_f)} + 1}\right] = v_f\tanh\left(\dfrac{gHt}{Lv_f}\right).$

34. We could use partial fractions on the integrand as it now stands, but it is easier if we first substitute $x = M^2$ and $dx = 2M\,dM$. At the same time, let us set $a = (k-1)/2$ and denote the integral by I:

$$I = \int \frac{M(1-M^2)}{M^4\left(1 + \dfrac{k-1}{2}M^2\right)}dM = \frac{1}{2}\int\frac{1-x}{x^2(1+ax)}dx = \frac{1}{2}\int\left(\frac{-a-1}{x} + \frac{1}{x^2} + \frac{a+a^2}{1+ax}\right)dx$$

$$= \frac{1}{2}\left[-(a+1)\ln x - \frac{1}{x} + (1+a)\ln(1+ax)\right] + C = -\frac{1}{2x} + \left(\frac{a+1}{2}\right)\ln\left(\frac{1+ax}{x}\right) + C$$

$$= -\frac{1}{2M^2} + \left(\frac{k+1}{4}\right)\ln\left[\frac{1 + \left(\dfrac{k-1}{2}\right)M^2}{M^2}\right] + C.$$

35. If $t = \tan(x/2)$, then $x = 2\operatorname{Tan}^{-1}t$, from which $dx = \dfrac{2}{1+t^2}dt$. Since $t = \sin(x/2)/\cos(x/2)$, it follows that $\sin(x/2) = t\cos(x/2)$. Using the fact that $\sin^2(x/2) + \cos^2(x/2) = 1$, we obtain

$$1 = t^2\cos^2(x/2) + \cos^2(x/2) \quad\Longrightarrow\quad \cos^2(x/2) = \frac{1}{1+t^2}.$$

Thus, $\cos x = 2\cos^2(x/2) - 1 = \dfrac{2}{1+t^2} - 1 = \dfrac{1-t^2}{1+t^2}$. Furthermore, $\sin x = 2\sin(x/2)\cos(x/2)$ $= 2t\cos^2(x/2) = \dfrac{2t}{1+t^2}$.

36. With the substitution from Exercise 35,

$$\int \sec x\,dx = \int \frac{1}{\cos x}dx = \int \frac{1+t^2}{1-t^2}\frac{2}{1+t^2}dt = 2\int \frac{1}{1-t^2}dt = 2\int\left(\frac{1/2}{1-t} + \frac{1/2}{1+t}\right)dt$$
$$= -\ln|1-t| + \ln|1+t| + C = \ln\left|\frac{1+t}{1-t}\right| + C = \ln\left|\frac{1+\tan(x/2)}{1-\tan(x/2)}\right| + C.$$

37. With the substitution from Exercise 35,

$$\int \frac{1}{3+5\sin x}dx = \int \frac{1}{3 + \dfrac{10t}{1+t^2}}\frac{2}{1+t^2}dt = 2\int \frac{1}{3t^2+10t+3}dt$$
$$= 2\int \frac{1}{(3t+1)(t+3)}dt = 2\int\left(\frac{3/8}{3t+1} - \frac{1/8}{t+3}\right)dt$$
$$= \frac{1}{4}\ln|3t+1| - \frac{1}{4}\ln|t+3| + C = \frac{1}{4}\ln|3\tan(x/2)+1| - \frac{1}{4}\ln|\tan(x/2)+3| + C.$$

38. With the substitution from Exercise 35,

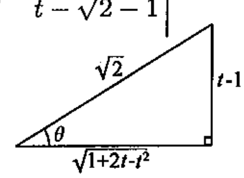
$$\int \frac{1}{1-2\cos x}dx = \int \frac{1}{1-2\left(\dfrac{1-t^2}{1+t^2}\right)}\frac{2}{1+t^2}dt = 2\int \frac{1}{3t^2-1}dt$$
$$= 2\int\left(\frac{-1/2}{\sqrt{3}t+1} + \frac{1/2}{\sqrt{3}t-1}\right)dt = -\frac{1}{\sqrt{3}}\ln|\sqrt{3}t+1| + \frac{1}{\sqrt{3}}\ln|\sqrt{3}t-1| + C$$
$$= \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}t-1}{\sqrt{3}t+1}\right| + C = \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}\tan(x/2)-1}{\sqrt{3}\tan(x/2)+1}\right| + C.$$

39. With the substitution from Exercise 35,

$$\int \frac{1}{\sin x + \cos x}dx = \int \frac{1}{\dfrac{2t}{1+t^2} + \dfrac{1-t^2}{1+t^2}}\frac{2}{1+t^2}dt = 2\int \frac{1}{1+2t-t^2}dt = 2\int \frac{1}{2-(t-1)^2}dt.$$

We now set $t-1 = \sqrt{2}\sin\theta$ and $dt = \sqrt{2}\cos\theta\,d\theta$,

$$\int \frac{1}{\sin x + \cos x}dx = 2\int \frac{1}{2\cos^2\theta}\sqrt{2}\cos\theta\,d\theta = \sqrt{2}\int \sec\theta\,d\theta$$
$$= \sqrt{2}\ln|\sec\theta + \tan\theta| + C = \sqrt{2}\ln\left|\frac{\sqrt{2}}{\sqrt{1+2t-t^2}} + \frac{t-1}{\sqrt{1+2t-t^2}}\right| + C$$
$$= \sqrt{2}\ln\left|\frac{t+\sqrt{2}-1}{\sqrt{-(t+\sqrt{2}-1)(t-\sqrt{2}-1)}}\right| + C = \sqrt{2}\ln\left|\sqrt{-\frac{t+\sqrt{2}-1}{t-\sqrt{2}-1}}\right| + C$$
$$= \frac{1}{\sqrt{2}}\ln\left|\frac{\tan(x/2)+\sqrt{2}-1}{\tan(x/2)-\sqrt{2}-1}\right| + C.$$



40. We must show that $\dfrac{1+\tan(x/2)}{1-\tan(x/2)} = \sec x + \tan x$.

$$\frac{1+\tan(x/2)}{1-\tan(x/2)} = \frac{1+\dfrac{\sin(x/2)}{\cos(x/2)}}{1-\dfrac{\sin(x/2)}{\cos(x/2)}} = \frac{\cos(x/2)+\sin(x/2)}{\cos(x/2)-\sin(x/2)} = \frac{\cos(x/2)+\sin(x/2)}{\cos(x/2)-\sin(x/2)}\frac{\cos(x/2)+\sin(x/2)}{\cos(x/2)+\sin(x/2)}$$
$$= \frac{\cos^2(x/2)+2\sin(x/2)\cos(x/2)+\sin^2(x/2)}{\cos^2(x/2)-\sin^2(x/2)} = \frac{1+\sin x}{\cos x} = \sec x + \tan x.$$

41. (a) With the substitution from Exercise 35,

$$\int \frac{1}{5-4\cos x}dx = \int \frac{1}{5-\dfrac{4(1-t^2)}{1+t^2}}\frac{2}{1+t^2}dt = 2\int \frac{1}{1+9t^2}dt.$$

If we now set $t=(1/3)\tan\theta$ and $dt=(1/3)\sec^2\theta\,d\theta$, then

$$\int \frac{1}{5-4\cos x}dx = 2\int \frac{1}{\sec^2\theta}\frac{1}{3}\sec^2\theta\,d\theta = \frac{2\theta}{3}+C = \frac{2}{3}\,\mathrm{Tan}^{-1}3t + C = \frac{2}{3}\,\mathrm{Tan}^{-1}\left[3\tan\left(\frac{x}{2}\right)\right]+C.$$

(b) $\displaystyle\int_0^{2\pi} \frac{1}{5-4\cos x}dx = \left\{\frac{2}{3}\,\mathrm{Tan}^{-1}\left[3\tan\left(\frac{x}{2}\right)\right]\right\}_0^{2\pi} = 0$

This cannot be correct because the integrand is always positive.

(c) To verify that the function is an antiderivative, we differentiate it, obtaining

$$\frac{1}{3}+\frac{2/3}{1+\dfrac{\sin^2 x}{(2-\cos x)^2}}\left[\frac{(2-\cos x)(\cos x)-\sin x(\sin x)}{(2-\cos x)^2}\right],$$

and this simplifies to $1/(5-4\cos x)$. When we use this antiderivative,

$$\int_0^{2\pi}\frac{1}{5-4\cos x}dx = \left\{\frac{x}{3}+\frac{2}{3}\,\mathrm{Tan}^{-1}\left(\frac{\sin x}{2-\cos x}\right)\right\}_0^{2\pi} = \frac{2\pi}{3}.$$

42. If we set $x^4+x^3+2x^2+11x-5=(x^2+bx+c)(x^2+dx+e)$, multiply the right side out, and equate coefficients, we obtain the equations

$$b+d=1,\quad c+bd+e=2,\quad be+cd=11,\quad ce=-5.$$

Solutions of these are $b=-1$, $c=5$, $d=2$, and $e=-1$. The partial fraction decomposition of the integrand therefore takes the form

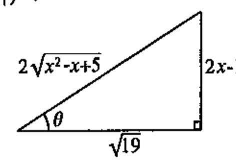
$$\frac{x^2+x+3}{(x^2-x+5)(x^2+2x-1)} = \frac{Ax+B}{x^2-x+5}+\frac{Cx+D}{x^2+2x-1}.$$

We find that $A=-2/21$, $B=4/7$, $C=2/21$, and $D=5/7$. Hence

$$\int \frac{x^2+x+3}{x^4+x^3+2x^2+11x-5}dx = \frac{1}{21}\int\left(\frac{-2x+12}{x^2-x+5}+\frac{2x+15}{x^2+2x-1}\right)dx$$
$$= \frac{2}{21}\int\frac{-x+6}{(x-1/2)^2+19/4}dx+\frac{1}{21}\int\frac{2x+15}{(x+1)^2-2}dx.$$

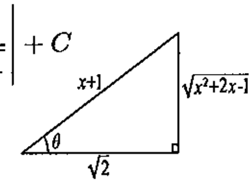
In the first integral we set $x-1/2=(\sqrt{19}/2)\tan\theta$ and $dx=(\sqrt{19}/2)\sec^2\theta\,d\theta$,

$$\int \frac{-x+6}{(x-1/2)^2+19/4}dx = \int \frac{-1/2-(\sqrt{19}/2)\tan\theta+6}{(19/4)\sec^2\theta}\frac{\sqrt{19}}{2}\sec^2\theta\,d\theta$$
$$= \frac{1}{\sqrt{19}}\int(11-\sqrt{19}\tan\theta)\,d\theta = \frac{1}{\sqrt{19}}(11\theta+\sqrt{19}\ln|\cos\theta|)+C$$
$$= \frac{11}{\sqrt{19}}\mathrm{Tan}^{-1}\left(\frac{2x-1}{\sqrt{19}}\right)+\ln\left|\frac{\sqrt{19}}{2\sqrt{x^2-x+5}}\right|+C$$
$$= \frac{11}{\sqrt{19}}\mathrm{Tan}^{-1}\left(\frac{2x-1}{\sqrt{19}}\right)-\frac{1}{2}\ln(x^2-x+5)+D.$$



In the second integral we set $x+1=\sqrt{2}\sec\theta$ and $dx=\sqrt{2}\sec\theta\tan\theta\,d\theta$,

$$\int \frac{2x+15}{(x+1)^2-2}dx = \int \frac{2(\sqrt{2}\sec\theta-1)+15}{2\tan^2\theta}\sqrt{2}\sec\theta\tan\theta\,d\theta = \frac{1}{\sqrt{2}}\int\frac{\sec\theta}{\tan\theta}(2\sqrt{2}\sec\theta+13)\,d\theta$$
$$= \frac{1}{\sqrt{2}}\int\left(\frac{2\sqrt{2}\sec^2\theta}{\tan\theta}+13\csc\theta\right)d\theta = 2\ln|\tan\theta|+\frac{13}{\sqrt{2}}\ln|\csc\theta-\cot\theta|+C$$
$$= 2\ln\left|\frac{\sqrt{x^2+2x-1}}{\sqrt{2}}\right|+\frac{13}{\sqrt{2}}\ln\left|\frac{x+1}{\sqrt{x^2+2x-1}}-\frac{\sqrt{2}}{\sqrt{x^2+2x-1}}\right|+C$$
$$= \frac{13}{\sqrt{2}}\ln|x+1-\sqrt{2}|+\frac{2\sqrt{2}-13}{2\sqrt{2}}\ln(x^2+2x-1)+D.$$



Thus,
$$\int\frac{x^2+x+3}{x^4+x^3+2x^2+11x-5}dx = \frac{22}{21\sqrt{19}}\mathrm{Tan}^{-1}\left(\frac{2x-1}{\sqrt{19}}\right)-\frac{1}{21}\ln(x^2-x+5)$$
$$+\frac{13}{21\sqrt{2}}\ln|x+1-\sqrt{2}|+\frac{2\sqrt{2}-13}{42\sqrt{2}}\ln(x^2+2x-1)+G.$$

43. If we set $x^4+3x^3+x^2+2x-12=(x^2+bx+c)(x^2+dx+e)$, multiply the right side out, and equate coefficients, we obtain the equations

$$b+d=3,\quad c+bd+e=1,\quad be+cd=2,\quad ce=-12.$$

Solutions of these are $b=1$, $c=3$, $d=2$, and $e=-4$. The partial fraction decomposition of the integrand therefore takes the form

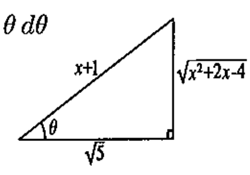
$$\frac{2x^3+8x^2-3x+5}{x^4+3x^3+x^2+2x-12}=\frac{Ax+B}{x^2+x+3}+\frac{Cx+D}{x^2+2x-4}.$$

We find that $A=2$, $B=1$, $C=0$, and $D=3$, so that

$$\int\frac{2x^3+8x^2-3x+5}{x^4+3x^3+x^2+2x-12}dx=\int\left[\frac{2x+1}{x^2+x+3}+\frac{3}{(x+1)^2-5}\right]dx.$$

In the second integral we set $x+1=\sqrt{5}\sec\theta$ and $dx=\sqrt{5}\sec\theta\tan\theta\,d\theta$,

$$\int\frac{2x^3+8x^2-3x+5}{x^4+3x^3+x^2+2x-12}dx=\ln(x^2+x+3)+3\int\frac{1}{5\tan^2\theta}\sqrt{5}\sec\theta\tan\theta\,d\theta$$
$$=\ln(x^2+x+3)+\frac{3}{\sqrt{5}}\int\csc\theta\,d\theta$$
$$=\ln(x^2+x+3)+\frac{3}{\sqrt{5}}\ln|\csc\theta-\cot\theta|+C$$
$$=\ln(x^2+x+3)+\frac{3}{\sqrt{5}}\ln\left|\frac{x+1}{\sqrt{x^2+2x-4}}-\frac{\sqrt{5}}{\sqrt{x^2+2x-4}}\right|+C$$
$$=\ln(x^2+x+3)+\frac{3}{\sqrt{5}}\ln|x+1-\sqrt{5}|-\frac{3}{2\sqrt{5}}\ln(x^2+2x-4)+C.$$



EXERCISES 8.7

1. With the trapezoidal rule,

$$\int_1^2 \frac{1}{x}dx \approx \frac{1/10}{2}\left(\frac{1}{1}+2\sum_{i=1}^{9}\frac{1}{1+i/10}+\frac{1}{2}\right)=\frac{1}{20}\left(1+20\sum_{i=1}^{9}\frac{1}{10+i}+\frac{1}{2}\right)=0.693\,77.$$

With Simpson's rule, $\displaystyle\int_1^2 \frac{1}{x}dx \approx \frac{1/10}{3}\left(1+\frac{4}{1.1}+\frac{2}{1.2}+\cdots+\frac{2}{1.8}+\frac{4}{1.9}+\frac{1}{2}\right)=0.693\,15.$

Analytically, $\displaystyle\int_1^2 \frac{1}{x}dx = \Big\{\ln x\Big\}_1^2 = \ln 2 \approx 0.693\,15.$

2. With the trapezoidal rule,

$$\int_2^3 \frac{1}{\sqrt{x+2}}dx \approx \frac{1/10}{2}\left(\frac{1}{\sqrt{4}}+2\sum_{i=1}^{9}\frac{1}{\sqrt{\frac{i}{10}+4}}+\frac{1}{\sqrt{5}}\right)=\frac{1}{20}\left(\frac{1}{2}+2\sqrt{10}\sum_{i=1}^{9}\frac{1}{\sqrt{40+i}}+\frac{1}{\sqrt{5}}\right)=0.472\,15.$$

With Simpson's rule,

$$\int_2^3 \frac{1}{\sqrt{x+2}}dx \approx \frac{1/10}{3}\left(\frac{1}{2}+\frac{4}{\sqrt{4.1}}+\frac{2}{\sqrt{4.2}}+\cdots+\frac{2}{\sqrt{4.8}}+\frac{4}{\sqrt{4.9}}+\frac{1}{\sqrt{5}}\right)=0.472\,14.$$

Analytically, $\displaystyle\int_2^3 \frac{1}{\sqrt{x+2}}dx = \Big\{2\sqrt{x+2}\Big\}_2^3 = 2\sqrt{5}-4 \approx 0.472\,14.$

3. With the trapezoidal rule, $\displaystyle\int_0^1 \tan x\,dx \approx \frac{1/10}{2}\left(\tan 0+2\sum_{i=1}^{9}\tan(i/10)+\tan 1\right)=0.617\,64.$

With Simpson's rule,

$$\int_0^1 \tan x\,dx \approx \frac{1/10}{3}\left[\tan 0+4\,\tan(0.1)+2\,\tan(0.2)+\cdots+2\,\tan(0.8)+4\,\tan(0.9)+\tan 1\right]=0.615\,65.$$

Analytically, $\displaystyle\int_0^1 \tan x\,dx = \Big\{\ln(\sec x)\Big\}_0^1 = \ln(\sec 1) \approx 0.615\,63.$

4. With the trapezoidal rule,

$$\int_0^{1/2} e^x\,dx \approx \frac{1/20}{2}\left(e^0+2\sum_{i=1}^{9}e^{i/20}+e^{1/2}\right)=\frac{1}{40}\left(1+2\sum_{i=1}^{9}e^{i/20}+\sqrt{e}\right)=0.648\,86.$$

With Simpson's rule, $\displaystyle\int_0^{1/2} e^x\,dx \approx \frac{1/20}{3}\left(e^0+4e^{1/20}+2e^{1/10}+\cdots+2e^{2/5}+4e^{9/20}+e^{1/2}\right)=0.648\,72.$

Analytically, $\displaystyle\int_0^{1/2} e^x\,dx = \Big\{e^x\Big\}_0^{1/2} = \sqrt{e}-1 \approx 0.648\,72.$

5. With the trapezoidal rule,

$$\int_{-1}^1 \sqrt{x+1}\,dx \approx \frac{1/5}{2}\left(0+2\sum_{i=1}^{9}\sqrt{(-1+i/5)+1}+\sqrt{2}\right)=\frac{1}{10}\left(\frac{2}{\sqrt{5}}\sum_{i=1}^{9}\sqrt{i}+\sqrt{2}\right)=1.868\,2.$$

With Simpson's rule,

$$\int_{-1}^1 \sqrt{x+1}\,dx \approx \frac{1/5}{3}\left(0+4\sqrt{1-4/5}+2\sqrt{1-3/5}+\cdots+2\sqrt{1+3/5}+4\sqrt{1+4/5}+\sqrt{2}\right)=1.878\,4.$$

Analytically, $\displaystyle\int_{-1}^1 \sqrt{x+1}\,dx = \left\{\frac{2(x+1)^{3/2}}{3}\right\}_{-1}^1 = \frac{4\sqrt{2}}{3} \approx 1.885\,6.$

6. With the trapezoidal rule,

$$\int_{-3}^{-2} \frac{1}{x^3}dx \approx \frac{1/10}{2}\left[-\frac{1}{27}+2\sum_{i=1}^{9}\frac{1}{(-3+i/10)^3}-\frac{1}{8}\right] = \frac{1}{20}\left[-\frac{1}{27}-2000\sum_{i=1}^{9}\frac{1}{(30-i)^3}-\frac{1}{8}\right] = -0.069\,570.$$

With Simpson's rule,

$$\int_{-3}^{-2}\frac{1}{x^3}dx \approx \frac{1/10}{3}\left[-\frac{1}{27}+\frac{4}{(-2.9)^3}+\frac{2}{(-2.8)^3}+\cdots+\frac{2}{(-2.2)^3}+\frac{4}{(-2.1)^3}-\frac{1}{8}\right] = -0.069\,445.$$

Analytically, $\int_{-3}^{-2}\frac{1}{x^3}dx = \left\{-\frac{1}{2x^2}\right\}_{-3}^{-2} = -\frac{5}{72} \approx -0.069\,444.$

7. With the trapezoidal rule, $\int_{1/2}^{1}\cos x\,dx \approx \frac{1/20}{2}\left[\cos(1/2)+2\sum_{i=1}^{9}\cos(1/2+i/20)+\cos 1\right] = 0.361\,97.$

With Simpson's rule,

$$\int_{1/2}^{1}\cos x\,dx \approx \frac{1/20}{3}[\cos(0.5)+4\cos(0.55)+2\cos(0.6)+\cdots+2\cos(0.9)+4\cos(0.95)+\cos 1] = 0.362\,05.$$

Analytically, $\int_{1/2}^{1}\cos x\,dx = \left\{\sin x\right\}_{1/2}^{1} = \sin 1 - \sin(1/2) \approx 0.362\,05.$

8. With the trapezoidal rule,

$$\int_0^1 \frac{1}{3+x^2}dx \approx \frac{1/10}{2}\left[\frac{1}{3}+2\sum_{i=1}^{9}\frac{1}{3+(i/10)^2}+\frac{1}{4}\right] = \frac{1}{20}\left(\frac{7}{12}+200\sum_{i=1}^{9}\frac{1}{300+i^2}\right) = 0.302\,20.$$

With Simpson's rule,

$$\int_0^1\frac{1}{3+x^2}dx \approx \frac{1/10}{3}\left[\frac{1}{3}+\frac{4}{3+(1/10)^2}+\frac{2}{3+(2/10)^2}+\cdots+\frac{2}{3+(8/10)^2}+\frac{4}{3+(9/10)^2}+\frac{1}{4}\right] = 0.302\,30.$$

Analytically, we set $x=\sqrt{3}\tan\theta$ and $dx=\sqrt{3}\sec^2\theta\,d\theta$,

$$\int_0^1\frac{1}{3+x^2}dx = \int_0^{\pi/6}\frac{1}{3\sec^2\theta}\sqrt{3}\sec^2\theta\,d\theta = \frac{1}{\sqrt{3}}\left\{\theta\right\}_0^{\pi/6} = \frac{\pi}{6\sqrt{3}} \approx 0.302\,30.$$

9. With the trapezoidal rule, $\int_1^3\frac{1}{x^2+x}dx \approx \frac{1/5}{2}\left[\frac{1}{2}+2\sum_{i=1}^{9}\frac{1}{(1+i/5)^2+(1+i/5)}+\frac{1}{12}\right] = 0.407\,79.$

With Simpson's rule,

$$\int_1^3\frac{1}{x^2+x}dx \approx \frac{1/5}{3}\left[\frac{1}{2}+\frac{4}{(1.2)^2+1.2}+\frac{2}{(1.4)^2+1.4}+\cdots+\frac{2}{(2.6)^2+2.6}+\frac{4}{(2.8)^2+2.8}+\frac{1}{12}\right] = 0.405\,51.$$

Analytically, $\int_1^3\frac{1}{x^2+x}dx = \int_1^3\left(\frac{1}{x}-\frac{1}{x+1}\right)dx = \left\{\ln x - \ln(x+1)\right\}_1^3 = \ln(3/2) \approx 0.405\,47.$

10. With the trapezoidal rule,

$$\int_0^{1/2} xe^{x^2}\,dx \approx \frac{1/20}{2}\left[0+2\sum_{i=1}^{9}(i/20)e^{(i/20)^2}+\frac{1}{2}e^{1/4}\right] = \frac{1}{40}\left(\frac{1}{10}\sum_{i=1}^{9}ie^{i^2/400}+\frac{1}{2}e^{1/4}\right) = 0.142\,21.$$

With Simpson's rule,

$$\int_0^{1/2} xe^{x^2}\,dx \approx \frac{1/20}{3}\Big[0+4(1/20)e^{1/400}+2(1/10)e^{1/100}+\cdots + 2(2/5)e^{4/25}+4(9/20)e^{81/400}+(1/2)e^{1/4}\Big] = 0.142\,01.$$

Analytically, $\displaystyle\int_0^{1/2} xe^{x^2}\,dx = \left\{\frac{e^{x^2}}{2}\right\}_0^{1/2} = \frac{e^{1/4}-1}{2} \approx 0.142\,01.$

11. With the trapezoidal rule, $\displaystyle\int_0^2 \frac{1}{1+x^3}dx \approx \frac{1/5}{2}\left[1+2\sum_{i=1}^{9}\frac{1}{1+(i/5)^3}+\frac{1}{9}\right] = 1.0895.$

With Simpson's rule,

$$\int_0^2 \frac{1}{1+x^3}dx \approx \frac{1/5}{3}\left[1+\frac{4}{1+(0.2)^3}+\frac{2}{1+(0.4)^3}+\cdots+\frac{2}{1+(1.6)^3}+\frac{4}{1+(1.8)^3}+\frac{1}{9}\right] = 1.0900.$$

12. With the trapezoidal rule, $\displaystyle\int_0^1 e^{x^2}\,dx \approx \frac{1/10}{2}\left[1+2\sum_{i=1}^{9}e^{(i/10)^2}+e\right] = 1.4672.$

With Simpson's rule, $\displaystyle\int_0^1 e^{x^2}\,dx \approx \frac{1/10}{3}\left(1+4e^{0.01}+2e^{0.04}+\cdots+2e^{0.64}+4e^{0.81}+e\right) = 1.4627.$

13. With the trapezoidal rule, $\displaystyle\int_1^2 \sqrt{1+x^4}\,dx \approx \frac{1/10}{2}\left[\sqrt{2}+2\sum_{i=1}^{9}\sqrt{1+(1+i/10)^4}+\sqrt{17}\right] = 2.5661.$

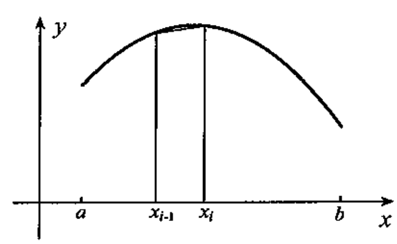
With Simpson's rule,
$$\int_1^2 \sqrt{1+x^4}\,dx \approx \frac{1/10}{3}\left[\sqrt{2}+4\sqrt{1+(1.1)^4}+2\sqrt{1+(1.2)^4}+\cdots\right.$$
$$\left.+2\sqrt{1+(1.8)^4}+4\sqrt{1+(1.9)^4}+\sqrt{17}\right] = 2.5641.$$

14. With the trapezoidal rule, and noting that $\sin(x^2)$ is an even function,

$$\int_{-1}^0 \sin(x^2)\,dx = \int_0^1 \sin(x^2)\,dx \approx \frac{1/10}{2}\left[0+2\sum_{i=1}^{9}\sin(i/10)^2+\sin 1\right] = 0.311\,17.$$

With Simpson's rule,
$$\int_{-1}^0 \sin(x^2)\,dx = \int_0^1 \sin(x^2)\,dx \approx \frac{1/10}{3}\left[0+4\sin(0.01)+2\sin(0.04)+\cdots\right.$$
$$\left.+2\sin(0.64)+4\sin(0.81)+\sin 1\right] = 0.310\,26.$$

15. The trapezoid on the i^{th} interval between x_{i-1} and x_i underestimates the area under the graph on that subinterval.



16. In equation 8.15 the error is reduced by a factor of 1/4; in equation 8.16 it is reduced by 1/16.

17. With 16 subdivisions, Simpson's rule gives

$$\int_0^1 e^{-x^2}\,dx \approx \frac{1/16}{3}\left[e^0+4e^{-(1/16)^2}+2e^{-(1/8)^2}+\cdots+2e^{-(7/8)^2}+4e^{-(15/16)^2}+e^{-1}\right] = 0.746\,82.$$

18. The length of the parabola is
$$\int_0^1 \sqrt{1+(2x)^2}\,dx = \int_0^1 \sqrt{1+4x^2}\,dx \approx \frac{1/10}{3}\left[1+4\sqrt{1+4(1/10)^2}+2\sqrt{1+4(1/5)^2}+\cdots\right.$$
$$\left.+2\sqrt{1+4(4/5)^2}+4\sqrt{1+4(9/10)^2}+\sqrt{5}\right] = 1.4789.$$

According to Exercise 8.4-39, the length of the parabola is $[2\sqrt{5}+\ln(2+\sqrt{5})]/4$, which to four decimals is also 1.4789.

19. The length of the curve is given by $L = \int_0^{\pi/2} \sqrt{1+\cos^2 x}\,dx$. When we use the trapezoidal rule with 10 subdivisions, $L \approx \dfrac{\pi/20}{2}\left[\sqrt{2} + 2\sum_{i=1}^{9}\sqrt{1+\cos^2(\pi i/20)} + 1\right] = 1.910.$

With Simpson's rule, $L \approx \dfrac{\pi/20}{3}\Big[\sqrt{2} + 4\sqrt{1+\cos^2(\pi/20)} + 2\sqrt{1+\cos^2(\pi/10)} + \cdots$

$+2\sqrt{1+\cos^2(2\pi/5)} + 4\sqrt{1+\cos^2(9\pi/20)} + 1\Big] = 1.910.$

20. Using Simpson's rule, the volume in cubic metres is approximately

$$(1.8)\left(\frac{1}{3}\right)[0 + 4(6.0) + 2(7.0) + 4(6.8) + 2(5.8) + 4(4.6) + 2(3.8) + 4(3.6) + 2(3.6) + 4(3.8) + 0] = 83.76.$$

21. Since there is an odd number of subdivisions, we use the trapezoidal rule to approximate the area of the spill

$$\frac{50}{2}[0 + 2(180) + 2(190) + 2(200) + 2(440) + 2(210) + 2(180) + 0] = 70\,000.$$

The volume of oil is approximately 700 m^3.

22. (a) Both rules require the value of the integrand at the lower limit of integration, but $e^x/\sqrt{x}$ is undefined at $x = 0$.

(b) If we set $u = \sqrt{x}$ and $du = 1/(2\sqrt{x})\,dx$, then

$$\int_0^4 \frac{e^x}{\sqrt{x}}dx = \int_0^2 e^{u^2} 2\,du = 2\int_0^2 e^{u^2}du,$$

and this integral is no longer improper. With Simpson's rule and 20 equal subdivisions,

$$2\int_0^2 e^{u^2}du \approx 2\left(\frac{1/10}{3}\right)\left[e^0 + 4e^{(0.1)^2} + 2e^{(0.2)^2} + \cdots + 2e^{(1.8)^2} + 4e^{(1.9)^2} + e^4\right] = 32.91.$$

(c) Rectangular rule 8.11 can be used since it does not require the value of $e^x/\sqrt{x}$ at $x = 0$.

23. (a) Since $y = (2/3)\sqrt{9-x^2}$ on the first quadrant part of the ellipse, small lengths thereon can be approximated by

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = \sqrt{1+\left[\frac{(2/3)(-x)}{\sqrt{9-x^2}}\right]^2}dx = \sqrt{\frac{81-9x^2+4x^2}{9(9-x^2)}}dx = \frac{1}{3}\sqrt{\frac{81-5x^2}{9-x^2}}dx.$$

The total length of the ellipse is therefore $L = \dfrac{4}{3}\displaystyle\int_0^3 \sqrt{\frac{81-5x^2}{9-x^2}}dx.$

(b) If we set $x = 3\sin\theta$ and $dx = 3\cos\theta\,d\theta$, then

$$L = \frac{4}{3}\int_0^{\pi/2} \frac{\sqrt{81-45\sin^2\theta}}{3\cos\theta} 3\cos\theta\,d\theta = 4\int_0^{\pi/2}\sqrt{9-5\sin^2\theta}\,d\theta.$$

(c) If we use the trapezoidal rule with 8 subdivisions to approximate the integral,

$$L \approx \frac{4(\pi/16)}{2}\left[3 + 2\sum_{i=1}^{7}\sqrt{9-5\sin^2(\pi i/16)} + 2\right] = 15.865.$$

With Simpson's rule, $L \approx \dfrac{4(\pi/16)}{3}\Big[3 + 4\sqrt{9-5\sin^2(\pi/16)} + 2\sqrt{9-5\sin^2(\pi/8)} + \cdots$

$+2\sqrt{9-5\sin^2(3\pi/8)} + 4\sqrt{9-5\sin^2(7\pi/16)} + 2\Big] = 15.865.$

24. If we set $x = 1/t$, then $dx = -(1/t^2)\,dt$, and

$$\int_1^{\infty} \frac{1}{1+x^4}dx = \int_1^0 \frac{1}{1+t^{-4}}\left(-\frac{1}{t^2}dt\right) = \int_0^1 \frac{t^2}{1+t^4}dt.$$

The trapezoidal rule with 10 subdivisions gives

$$\int_0^1 \frac{t^2}{1+t^4}dt \approx \frac{1/10}{2}\left[0 + 2\sum_{i=1}^{9}\frac{(i/10)^2}{1+(i/10)^4} + \frac{1}{2}\right] = 0.2437.$$

Simpson's rule with the same subdivision yields

$$\int_0^1 \frac{t^2}{1+t^4}dt \approx \frac{1/10}{3}\left[0 + \frac{4(0.1)^2}{1+(0.1)^4} + \frac{2(0.2)^2}{1+(0.2)^4} + \cdots + \frac{4(0.9)^2}{1+(0.9)^4} + \frac{1}{2}\right] = 0.2438.$$

25. If we set $x = 1/t$ and $dx = -dt/t^2$, then

$$\int_1^{\infty} \frac{x^2}{x^4+x^2+1}dx = \int_1^0 \frac{1/t^2}{1/t^4+1/t^2+1}\left(\frac{dt}{-t^2}\right) = \int_0^1 \frac{1}{t^4+t^2+1}dt.$$

If we use Simpson's rule with 10 subdivisions to approximate this integral,

$$\int_0^1 \frac{1}{t^4+t^2+1}dt \approx \frac{1/10}{3}\left[1 + \frac{4}{(1/10)^4+(1/10)^2+1} + \frac{2}{(1/5)^4+(1/5)^2+1} + \cdots \right.$$
$$\left. + \frac{2}{(4/5)^4+(4/5)^2+1} + \frac{4}{(9/10)^4+(9/10)^2+1} + \frac{1}{3}\right] = 0.728.$$

26. If $f(x)$ is the function, then the trapezoidal rule gives

$$\int_{-1}^{4} f(x)\,dx \approx \frac{1/2}{2}\{f(-1) + 2[f(-0.5) + f(0) + \cdots + f(3.5)] + f(4)\} = 2.113.$$

Simpson's rule gives

$$\int_{-1}^{4} f(x)\,dx \approx \frac{1/2}{3}[f(-1) + 4f(-0.5) + 2f(0) + \cdots + 2f(3) + 4f(3.5) + f(4)] = 1.729.$$

27. If $f(x)$ is the function, then the trapezoidal rule gives

$$\int_1^3 f(x)\,dx \approx \frac{1/5}{2}\{f(1.0) + 2[f(1.2) + f(1.4) + \cdots + f(2.6) + f(2.8)] + f(3.0)\} = 2.80.$$

Simpson's rule gives

$$\int_1^3 f(x)\,dx \approx \frac{1/5}{3}[f(1.0) + 4f(1.2) + 2f(1.4) + \cdots + 2f(2.6) + 4f(2.8) + f(3.0)] = 2.81.$$

28. According to equation 8.16, the maximum error in approximating the definite integral of $f(x)$ from $x = a$ to $x = b$ with n equal subdivisions is given by $|S_n| = M(b-a)^5/(180n^4)$ where M is the maximum value of $|f''''(x)|$ on $a \le x \le b$. But if $f(x)$ is a cubic polynomial, $f''''(x) = 0$ for all x. Hence, $S_n = 0$. For example,

$$\int_1^2 (x^3+1)\,dx = \left\{\frac{x^4}{4} + x\right\}_1^2 = \frac{19}{4}.$$

Simpson's rule with 10 equal subdivisions, and $f(x) = x^3 + 1$, gives

$$\int_1^2 (x^3+1)\,dx \approx \frac{1/10}{3}[f(1) + 4f(1.1) + 2f(1.2) + \cdots + 2f(1.8) + 4f(1.9) + f(2)] = 4.75.$$

29. (a) According to equation 8.15, the error in using the trapezoidal rule with n equal partitions is $|T_n| \le M(3)^3/(12n^2)$ where M is the maximum of the absolute value of the second derivative of $1/x$ on $1 \le x \le 4$. Since $d^2(1/x)/dx^2 = 2/x^3$, it follows that $M = 2$, and $|T_n| \le 2(3)^3/(12n^2) = 9/(2n^2)$. For $|T_n|$ to be less than 10^{-4}, we require

$$\frac{9}{2n^2} < 10^{-4} \qquad \Longrightarrow \qquad n > \sqrt{\frac{9(10^4)}{2}} = 212.1.$$

At least 213 subdivisions should be used.

(b) According to equation 8.16, the error in using Simpson's rule with n equal partitions is $|S_n| \le M(3)^5/(180n^4)$ where M is the maximum of the absolute value of the fourth derivative of $1/x$ on $1 \le x \le 4$. Since $d^4(1/x)/dx^4 = 24/x^5$, it follows that $M = 24$, and $|S_n| \le 24(3)^5/(180n^4) = 162/(5n^4)$. For $|S_n|$ to be less than 10^{-4}, we require

$$\frac{162}{5n^4} < 10^{-4} \qquad \Longrightarrow \qquad n > \sqrt[4]{\frac{162(10^4)}{5}} = 23.9.$$

We should use at least 24 subdivisions.

30. (a) According to equation 8.15, the error in using the trapezoidal rule with n equal partitions is $|T_n| \le M(\pi/4)^3/(12n^2)$ where M is the maximum of the absolute value of the second derivative of $\cos x$ on $0 \le x \le \pi/4$. Since $d^2(\cos x)/dx^2 = -\cos x$, it follows that $M = 1$, and $|T_n| \le \pi^3/(768n^2)$. For $|T_n|$ to be less than 10^{-4}, we require

$$\frac{\pi^3}{768n^2} < 10^{-4} \qquad \Longrightarrow \qquad n > \sqrt{\frac{10^4\pi^3}{768}} = 20.09.$$

Thus, at least 21 subdivisions should be used.

(b) According to equation 8.16, the error in using Simpson's rule with n equal partitions is $|S_n| \le M(\pi/4)^5/(180n^4)$ where M is the maximum of the absolute value of the fourth derivative of $\cos x$ on $0 \le x \le \pi/4$. Since $d^4(\cos x)/dx^4 = \cos x$, it follows that $M = 1$, and $|S_n| \le (\pi/4)^5/(180n^4)$. For $|S_n|$ to be less than 10^{-4}, we require

$$\frac{\pi^5}{180(4)^5n^4} < 10^{-4} \qquad \Longrightarrow \qquad n > \sqrt[4]{\frac{10^4\pi^5}{180(4)^5}} = 2.02.$$

Since n must be even, we should use at least 4 subdivisions.

31. (a) According to equation 8.15, the error in using the trapezoidal rule with n equal partitions is $|T_n| \le M(1/3)^3/(12n^2)$ where M is the maximum of the absolute value of the second derivative of e^{2x} on $0 \le x \le 1/3$. Since $d^2(e^{2x})/dx^2 = 4e^{2x}$, it follows that $M = 4e^{2/3}$, and $|T_n| \le 4e^{2/3}(1/3)^3/(12n^2) = e^{2/3}/(81n^2)$. For $|T_n|$ to be less than 10^{-4}, we require

$$\frac{e^{2/3}}{81n^2} < 10^{-4} \qquad \Longrightarrow \qquad n > \sqrt{\frac{10^4(e^{2/3})}{81}} = 15.5.$$

At least 16 subdivisions should be used.

(b) According to equation 8.16, the error in using Simpson's rule with n equal partitions is $|S_n| \le M(1/3)^5/(180n^4)$ where M is the maximum of the absolute value of the fourth derivative of e^{2x} on $0 \le x \le 1/3$. Since $d^4(e^{2x})/dx^4 = 16e^{2x}$, it follows that $M = 16e^{2/3}$, and $|S_n| \le 16e^{2/3}(1/3)^5/(180n^4) = 4e^{2/3}/[45(3^5)n^4]$. For $|S_n|$ to be less than 10^{-4}, we require

$$\frac{4e^{2/3}}{45(3)^5n^4} < 10^{-4} \qquad \Longrightarrow \qquad n > \sqrt[4]{\frac{4(10^4)e^{2/3}}{45(3)^5}} = 1.6.$$

We need only use 2 subdivisions.

32. (a) According to equation 8.15, the error in using the trapezoidal rule with n equal partitions is $|T_n| \le M(5-4)^3/(12n^2)$ where M is the maximum of the absolute value of the second derivative of $1/\sqrt{x+2}$ on $4 \le x \le 5$. Since $d^2(1/\sqrt{x+2})/dx^2 = (3/4)(x+2)^{-5/2}$, it follows that $M = (3/4)6^{-5/2}$, and $|T_n| \le (3/4)6^{-5/2}/(12n^2)$. For $|T_n|$ to be less than 10^{-4}, we require

$$\frac{1}{16(6^{5/2})n^2} < 10^{-4} \qquad \Longrightarrow \qquad n > \sqrt{\frac{10^4}{16(6^{5/2})}} = 2.66.$$

Thus, at least 3 subdivisions should be used.

(b) According to equation 8.16, the error in using Simpson's rule with n equal partitions is $|S_n| \le M(5-4)^5/(180n^4)$ where M is the maximum of the absolute value of the fourth derivative of $1/\sqrt{x+2}$ on $4 \le x \le 5$. Since $d^4(1/\sqrt{x+2})/dx^4 = (105/16)(x+2)^{-9/2}$, it follows that $M = (105/16)6^{-9/2}$, and $|S_n| \le (105/16)6^{-9/2}/(180n^4)$. For $|S_n|$ to be less than 10^{-4}, we require

$$\frac{105}{16(180)(6^{9/2})n^4} < 10^{-4} \qquad \Longrightarrow \qquad n > \sqrt[4]{\frac{105(10^4)}{16(180)(6^{9/2})}} = 0.58.$$

Since n must be even, only 2 subdivisions are needed.

REVIEW EXERCISES

1. $\displaystyle\int \sqrt{2-x}\,dx = -\frac{2}{3}(2-x)^{3/2} + C$

2. $\displaystyle\int \frac{1}{(x+3)^2}dx = -\frac{1}{x+3} + C$

3. $\displaystyle\int \frac{x^2+3}{x}dx = \int \left(x + \frac{3}{x}\right)dx = \frac{x^2}{2} + 3\ln|x| + C$

4. $\displaystyle\int \frac{x^2+3}{x+1}dx = \int \left(x - 1 + \frac{4}{x+1}\right)dx = \frac{x^2}{2} - x + 4\ln|x+1| + C$

5. $\displaystyle\int \frac{x^2+3}{x^2+1}dx = \int \left(1 + \frac{2}{1+x^2}\right)dx = x + 2\,\mathrm{Tan}^{-1}x + C$

6. If we set $u = x+3$ and $du = dx$, then

$$\int \frac{x}{\sqrt{x+3}}dx = \int \frac{u-3}{\sqrt{u}}du = \int (\sqrt{u} - 3u^{-1/2})\,du = \frac{2}{3}u^{3/2} - 6u^{1/2} + C = \frac{2}{3}(x+3)^{3/2} - 6\sqrt{x+3} + C.$$

7. $\displaystyle\int \sin^2 x\cos^3 x\,dx = \int \sin^2 x(1-\sin^2 x)\cos x\,dx = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$

8. If we set $u = x$, $dv = \sin x\,dx$, then $du = dx$, $v = -\cos x$, and

$$\int x\sin x\,dx = -x\cos x - \int -\cos x\,dx = -x\cos x + \sin x + C.$$

9. $\displaystyle\int \tan^2(2x)\,dx = \int [\sec^2(2x) - 1]\,dx = \frac{1}{2}\tan(2x) - x + C$

10. $\displaystyle\int \frac{x}{x^2+2x-3}dx = \int \left(\frac{3/4}{x+3} + \frac{1/4}{x-1}\right)dx = \frac{3}{4}\ln|x+3| + \frac{1}{4}\ln|x-1| + C$

11. If we set $x = (2/\sqrt{3})\sin\theta$ and $dx = (2/\sqrt{3})\cos\theta\,d\theta$, then

$$\int \frac{1}{\sqrt{4-3x^2}}dx = \int \frac{1}{2\cos\theta}\frac{2}{\sqrt{3}}\cos\theta\,d\theta = \frac{\theta}{\sqrt{3}} + C = \frac{1}{\sqrt{3}}\mathrm{Sin}^{-1}\left(\frac{\sqrt{3}x}{2}\right) + C.$$

12. If we set $u = \sqrt{x} + 5$, then $du = 1/(2\sqrt{x})dx$, and

$$\int \frac{2-\sqrt{x}}{\sqrt{x}+5}dx = \int \frac{2-(u-5)}{u}(2)(u-5)\,du = 2\int \frac{(7-u)(u-5)}{u}du$$

$$= 2\int\left(-\frac{35}{u} + 12 - u\right)du = 2\left(-35\ln|u| + 12u - \frac{u^2}{2}\right) + C$$
$$= -70\ln|\sqrt{x}+5| + 24(\sqrt{x}+5) - (\sqrt{x}+5)^2 + C$$
$$= -70\ln(\sqrt{x}+5) + 14\sqrt{x} - x + D.$$

13. $\displaystyle\int \frac{x}{3x^2+4}dx = \frac{1}{6}\ln(3x^2+4) + C$

14. If we set $u = e^x$, then $du = e^x\,dx$, and $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}dx = \int\frac{1}{\sqrt{1-u^2}}du = \mathrm{Sin}^{-1}u + C = \mathrm{Sin}^{-1}(e^x) + C.$

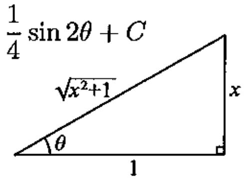
15. If we set $u = \ln x$, $dv = x^2\,dx$, $du = (1/x)\,dx$, and $v = x^3/3$, then

$$\int x^2\ln x\,dx = \frac{x^3}{3}\ln x - \int\frac{x^2}{3}dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C.$$

16. $\displaystyle\int\frac{x}{(x^2+1)^2}dx = -\frac{1}{2(x^2+1)} + C$

17. If we set $x = \tan\theta$ and $dx = \sec^2\theta\,d\theta$, then

$$\int\frac{x^2}{(1+x^2)^2}dx = \int\frac{\tan^2\theta}{\sec^4\theta}\sec^2\theta\,d\theta = \int\frac{\tan^2\theta}{\sec^2\theta}d\theta = \int\frac{\sec^2\theta-1}{\sec^2\theta}d\theta$$
$$= \int(1-\cos^2\theta)\,d\theta = \int\sin^2\theta\,d\theta = \int\left(\frac{1-\cos 2\theta}{2}\right)d\theta = \frac{\theta}{2} - \frac{1}{4}\sin 2\theta + C$$
$$= \frac{1}{2}\mathrm{Tan}^{-1}x - \frac{1}{2}\left(\frac{x}{\sqrt{x^2+1}}\right)\left(\frac{1}{\sqrt{x^2+1}}\right) + C$$
$$= \frac{1}{2}\mathrm{Tan}^{-1}x - \frac{x}{2(x^2+1)} + C.$$



18. If we set $u = x^2+1$, then $du = 2x\,dx$, and

$$\int\frac{x^3}{(x^2+1)^2}dx = \int\frac{u-1}{u^2}\left(\frac{du}{2}\right) = \frac{1}{2}\int\left(\frac{1}{u} - \frac{1}{u^2}\right)du = \frac{1}{2}\left(\ln|u| + \frac{1}{u}\right) + C$$
$$= \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + C.$$

19. If we set $\dfrac{x+1}{x(x-2)(x+2)} = \dfrac{A}{x} + \dfrac{B}{x-2} + \dfrac{C}{x+2}$, then $A = -1/4$, $B = 3/8$, and $C = -1/8$. Hence,

$$\int\frac{x+1}{x^3-4x}dx = \int\left(\frac{-1/4}{x} + \frac{3/8}{x-2} - \frac{1/8}{x+2}\right)dx = -\frac{1}{4}\ln|x| + \frac{3}{8}\ln|x-2| - \frac{1}{8}\ln|x+2| + C.$$

20. $\displaystyle\int\left(\frac{x+1}{x-1}\right)^2dx = \int\left(1+\frac{2}{x-1}\right)^2dx = \int\left[1 + \frac{4}{x-1} + \frac{4}{(x-1)^2}\right]dx = x + 4\ln|x-1| - \frac{4}{x-1} + C$

21. $\displaystyle\int\frac{x^2}{(1+3x^3)^4}dx = \frac{-1}{27(1+3x^3)^3} + C$

22. If we set $u = \mathrm{Cos}^{-1}x$, $dv = dx$, then $du = (-1/\sqrt{1-x^2})dx$, $v = x$, and

$$\int\mathrm{Cos}^{-1}x\,dx = x\,\mathrm{Cos}^{-1}x - \int\frac{-x}{\sqrt{1-x^2}}dx = x\,\mathrm{Cos}^{-1}x - \sqrt{1-x^2} + C.$$

23. $\displaystyle\int\sin x\,\cos 2x\,dx = \int\sin x(2\cos^2 x - 1)\,dx = -\frac{2}{3}\cos^3 x + \cos x + C$

24. Using identity 1.48b, $\int \sin x \cos 5x\, dx = \frac{1}{2}\int (\sin 6x - \sin 4x)\, dx = -\frac{1}{12}\cos 6x + \frac{1}{8}\cos 4x + C.$

25. If we set $u = e^{3x}$, $dv = \cos 2x\, dx$, $du = 3e^{3x}\, dx$, and $v = (1/2)\sin 2x$,

$$\int e^{3x}\cos 2x\, dx = \frac{1}{2}e^{3x}\sin 2x - \int \frac{3}{2}e^{3x}\sin 2x\, dx.$$

We now set $u = e^{3x}$, $dv = \sin 2x\, dx$, $du = 3e^{3x}\, dx$, and $v = -(1/2)\cos 2x$,

$$\int e^{3x}\cos 2x\, dx = \frac{1}{2}e^{3x}\sin 2x - \frac{3}{2}\left(-\frac{1}{2}e^{3x}\cos 2x - \int -\frac{3}{2}e^{3x}\cos 2x\, dx\right).$$

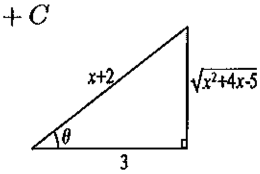
When we combine both integrals of $e^{3x}\cos 2x$, we obtain

$$\left(1 + \frac{9}{4}\right)\int e^{3x}\cos 2x\, dx = \frac{1}{2}e^{3x}\sin 2x + \frac{3}{4}e^{3x}\cos 2x,$$

from which $\int e^{3x}\cos 2x\, dx = \frac{e^{3x}}{13}(2\sin 2x + 3\cos 2x) + C.$

26. Since $x^2 + 4x - 5 = (x+2)^2 - 9$, we set $x + 2 = 3\sec\theta$, in which case $dx = 3\sec\theta\tan\theta\, d\theta$, and

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+4x-5}}dx &= \int \frac{1}{3\tan\theta}3\sec\theta\tan\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C\\ &= \ln\left|\frac{x+2}{3} + \frac{\sqrt{x^2+4x-5}}{3}\right| + C\\ &= \ln|x + 2 + \sqrt{x^2+4x-5}| + D.\end{aligned}$$



27. If we set $\frac{1}{(x+5)(x-1)} = \frac{A}{x+5} + \frac{B}{x-1}$, then $A = -1/6$ and $B = 1/6$. Hence,

$$\int \frac{1}{x^2+4x-5}dx = \int\left(\frac{-1/6}{x+5} + \frac{1/6}{x-1}\right)dx = -\frac{1}{6}\ln|x+5| + \frac{1}{6}\ln|x-1| + C.$$

28. If we set $u = 4 - x^2$, then $du = -2x\, dx$, and

$$\begin{aligned}\int x^3\sqrt{4-x^2}\, dx &= \int (4-u)\sqrt{u}\left(-\frac{du}{2}\right) = \frac{1}{2}\int (u^{3/2} - 4\sqrt{u})\, du\\ &= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{8}{3}u^{3/2}\right) + C = \frac{1}{5}(4-x^2)^{5/2} - \frac{4}{3}(4-x^2)^{3/2} + C.\end{aligned}$$

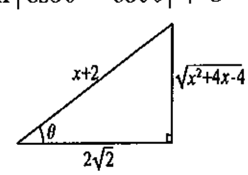
29. $\int \frac{\cos 2x}{1 - \sin 2x}dx = -\frac{1}{2}\ln(1 - \sin 2x) + C$

30. $\int \frac{6x}{4 - x^2}dx = -3\ln|4 - x^2| + C$

31. If we set $u = \ln x$, then $du = (1/x)\, dx$, and $\int \frac{1}{x\sqrt{\ln x}}dx = \int \frac{1}{\sqrt{u}}du = 2\sqrt{u} + C = 2\sqrt{\ln x} + C.$

32. Since $x^2 + 4x - 4 = (x+2)^2 - 8$, we set $x + 2 = 2\sqrt{2}\sec\theta$, in which case $dx = 2\sqrt{2}\sec\theta\tan\theta\, d\theta$, and

$$\begin{aligned}\int \frac{1}{x^2+4x-4}dx &= \int \frac{1}{8\tan^2\theta}2\sqrt{2}\sec\theta\tan\theta\, d\theta = \frac{1}{2\sqrt{2}}\int \csc\theta\, d\theta = \frac{1}{2\sqrt{2}}\ln|\csc\theta - \cot\theta| + C\\ &= \frac{1}{2\sqrt{2}}\ln\left|\frac{x+2}{\sqrt{x^2+4x-4}} - \frac{2\sqrt{2}}{\sqrt{x^2+4x-4}}\right| + C\\ &= \frac{1}{2\sqrt{2}}\ln\left|\frac{x+2-2\sqrt{2}}{\sqrt{x^2+4x-4}}\right| + C.\end{aligned}$$



33. If we set $u = \cos x$, then $du = -\sin x\, dx$, and

$$\int \frac{\sin x}{1 + \cos^2 x}dx = \int \frac{1}{1+u^2}(-du) = -\mathrm{Tan}^{-1}u + C = -\mathrm{Tan}^{-1}(\cos x) + C.$$

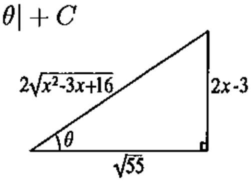
34. When we set $\dfrac{1}{x^4+x^3} = \dfrac{1}{x^3(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{D}{x+1}$, we find that $A = 1$, $B = -1$, $C = 1$, and $D = -1$. Thus, $\displaystyle\int \frac{1}{x^4+x^3}dx = \int \left(\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{x+1}\right)dx = \ln|x| + \frac{1}{x} - \frac{1}{2x^2} - \ln|x+1| + C.$

35. If we set $u = x$, $dv = \sec^2(3x)\,dx$, $du = dx$, and $v = (1/3)\tan(3x)$, then

$$\int x\sec^2(3x)\,dx = \frac{x}{3}\tan(3x) - \int \frac{1}{3}\tan(3x)\,dx = \frac{x}{3}\tan(3x) - \frac{1}{9}\ln|\sec(3x)| + C.$$

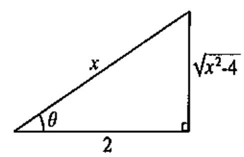
36. Since $16 - 3x + x^2 = (x-3/2)^2 + 55/4$, we set $x - 3/2 = (\sqrt{55}/2)\tan\theta$. Then $dx = (\sqrt{55}/2)\sec^2\theta\,d\theta$, and

$$\begin{aligned}\int \frac{1}{\sqrt{16-3x+x^2}}dx &= \int \frac{1}{(\sqrt{55}/2)\sec\theta}(\sqrt{55}/2)\sec^2\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C\\ &= \ln\left|\frac{2\sqrt{x^2-3x+16}}{\sqrt{55}} + \frac{2x-3}{\sqrt{55}}\right| + C\\ &= \ln|2\sqrt{x^2-3x+16} + 2x - 3| + D.\end{aligned}$$



37. If we set $x = 2\sec\theta$, then $dx = 2\sec\theta\tan\theta\,d\theta$, and

$$\begin{aligned}\int \frac{\sqrt{x^2-4}}{x^2}dx &= \int \frac{2\tan\theta}{4\sec^2\theta}2\sec\theta\tan\theta\,d\theta = \int \frac{\tan^2\theta}{\sec\theta}d\theta = \int \frac{\sec^2\theta - 1}{\sec\theta}d\theta\\ &= \int (\sec\theta - \cos\theta)\,d\theta = \ln|\sec\theta + \tan\theta| - \sin\theta + C\\ &= \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| - \frac{\sqrt{x^2-4}}{x} + C\\ &= \ln|x + \sqrt{x^2-4}| - \frac{\sqrt{x^2-4}}{x} + D.\end{aligned}$$



38. When we set $u = \text{Tan}^{-1}x$, $dv = x^2\,dx$, then $du = \dfrac{1}{1+x^2}dx$, $v = x^3/3$, and

$$\begin{aligned}\int x^2\,\text{Tan}^{-1}x\,dx &= \frac{x^3}{3}\text{Tan}^{-1}x - \frac{1}{3}\int \frac{x^3}{1+x^2}dx = \frac{x^3}{3}\text{Tan}^{-1}x - \frac{1}{3}\int\left(x - \frac{x}{1+x^2}\right)dx\\ &= \frac{x^3}{3}\text{Tan}^{-1}x - \frac{x^2}{6} + \frac{1}{6}\ln(1+x^2) + C.\end{aligned}$$

39. If we set $u = x + 1$ and $du = dx$, then

$$\begin{aligned}\int \frac{x^2}{x^3+3x^2+3x+1}dx &= \int \frac{x^2}{(x+1)^3}dx = \int \frac{(u-1)^2}{u^3}du = \int\left(\frac{1}{u} - \frac{2}{u^2} + \frac{1}{u^3}\right)du\\ &= \ln|u| + \frac{2}{u} - \frac{1}{2u^2} + C = \ln|x+1| + \frac{2}{x+1} - \frac{1}{2(x+1)^2} + C.\end{aligned}$$

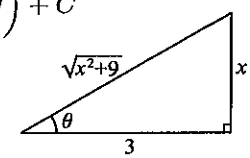
40. If we set $u = \ln x$, then $du = (1/x)\,dx$, and $\displaystyle\int \frac{\ln x}{x}dx = \int u\,du = \frac{u^2}{2} + C = \frac{1}{2}(\ln x)^2 + C.$

41. If we set $2x^3 = \tan\theta$, then $6x^2\,dx = \sec^2\theta\,d\theta$, and

$$\int \frac{x^2}{1+4x^6}dx = \int \frac{1}{\sec^2\theta}\frac{1}{6}\sec^2\theta\,d\theta = \frac{\theta}{6} + C = \frac{1}{6}\text{Tan}^{-1}(2x^3) + C.$$

42. If we set $x = 3\tan\theta$, then $dx = 3\sec^2\theta\, d\theta$, and

$$\int \frac{1}{x(9+x^2)^2}dx = \int \frac{1}{3\tan\theta(81\sec^4\theta)} 3\sec^2\theta\, d\theta = \frac{1}{81}\int \frac{\cos^3\theta}{\sin\theta}d\theta$$
$$= \frac{1}{81}\int \frac{\cos\theta(1-\sin^2\theta)}{\sin\theta}d\theta = \frac{1}{81}\left(\ln|\sin\theta| + \frac{1}{2}\cos^2\theta\right) + C$$
$$= \frac{1}{81}\ln\left|\frac{x}{\sqrt{x^2+9}}\right| + \frac{1}{162}\left(\frac{3}{\sqrt{x^2+9}}\right)^2 + C$$
$$= \frac{1}{81}\ln|x| - \frac{1}{162}\ln(x^2+9) + \frac{1}{18(x^2+9)} + C.$$



43. If we set $\dfrac{x^2+2}{x(x+1)(x+4)} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x+4}$, then $A = 1/2$, $B = -1$, and $C = 3/2$. Hence,

$$\int \frac{x^2+2}{x^3+5x^2+4x}dx = \int\left(\frac{1/2}{x} - \frac{1}{x+1} + \frac{3/2}{x+4}\right)dx = \frac{1}{2}\ln|x| - \ln|x+1| + \frac{3}{2}\ln|x+4| + C.$$

44. If we set $\dfrac{x^2+2}{x^3+4x^2+4x} = \dfrac{x^2+2}{x(x+2)^2} = \dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{(x+2)^2}$, we find that $A = 1/2$, $B = 1/2$, and $C = -3$. Thus,

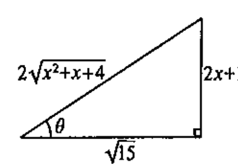
$$\int \frac{x^2+2}{x^3+4x^2+4x}dx = \int\left[\frac{1/2}{x} + \frac{1/2}{x+2} - \frac{3}{(x+2)^2}\right]dx = \frac{1}{2}\ln|x| + \frac{1}{2}\ln|x+2| + \frac{3}{x+2} + C.$$

45. If we set $\dfrac{x^2+2}{x(x^2+x+4)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+x+4}$, then $A = 1/2$, $B = 1/2$, and $C = -1/2$. Hence

$$\int \frac{x^2+2}{x^3+x^2+4x}dx = \frac{1}{2}\int\left(\frac{1}{x} + \frac{x-1}{x^2+x+4}\right)dx = \frac{1}{2}\ln|x| + \frac{1}{2}\int \frac{x-1}{(x+1/2)^2+15/4}dx.$$

We set $x + 1/2 = (\sqrt{15}/2)\tan\theta$ and $dx = (\sqrt{15}/2)\sec^2\theta\, d\theta$, in which case

$$\int \frac{x^2+2}{x^3+x^2+4x}dx = \frac{1}{2}\ln|x| + \frac{1}{2}\int \frac{-3/2 + (\sqrt{15}/2)\tan\theta}{(15/4)\sec^2\theta}\frac{\sqrt{15}}{2}\sec^2\theta\, d\theta$$
$$= \frac{1}{2}\ln|x| + \frac{1}{2\sqrt{15}}\int(\sqrt{15}\tan\theta - 3)\,d\theta$$
$$= \frac{1}{2}\ln|x| + \frac{1}{2\sqrt{15}}(\sqrt{15}\ln|\sec\theta| - 3\theta) + C$$
$$= \frac{1}{2}\ln|x| + \frac{1}{2}\ln\left|\frac{2\sqrt{x^2+x+4}}{\sqrt{15}}\right| - \frac{3}{2\sqrt{15}}\operatorname{Tan}^{-1}\left(\frac{2x+1}{\sqrt{15}}\right) + C$$
$$= \frac{1}{2}\ln|x| + \frac{1}{4}\ln(x^2+x+4) - \frac{3}{2\sqrt{15}}\operatorname{Tan}^{-1}\left(\frac{2x+1}{\sqrt{15}}\right) + D.$$



46. If we set $\dfrac{3x^2+2x+4}{x^3+x^2+4x} = \dfrac{3x^2+2x+4}{x(x^2+x+4)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+x+4}$, we find that $A = 1$, $B = 2$, and $C = 1$. Thus,

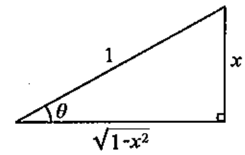
$$\int \frac{3x^2+2x+4}{x^3+x^2+4x}dx = \int\left(\frac{1}{x} + \frac{2x+1}{x^2+x+4}\right)dx = \ln|x| + \ln(x^2+x+4) + C.$$

47. If we set $u = \operatorname{Sin}^{-1}x$, $dv = x\,dx$, $du = \dfrac{1}{\sqrt{1-x^2}}dx$, and $v = \dfrac{x^2}{2}$, then

$$\int x\operatorname{Sin}^{-1}x\,dx = \frac{x^2}{2}\operatorname{Sin}^{-1}x - \int \frac{x^2}{2\sqrt{1-x^2}}dx.$$

We now set $x = \sin\theta$ and $dx = \cos\theta\, d\theta$,

$$\int x\,\mathrm{Sin}^{-1}x\,dx = \frac{x^2}{2}\,\mathrm{Sin}^{-1}x - \frac{1}{2}\int \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta = \frac{x^2}{2}\,\mathrm{Sin}^{-1}x - \frac{1}{2}\int\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$
$$= \frac{x^2}{2}\,\mathrm{Sin}^{-1}x - \frac{1}{4}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C$$
$$= \frac{x^2}{2}\,\mathrm{Sin}^{-1}x - \frac{\theta}{4} + \frac{1}{4}\sin\theta\,\cos\theta + C$$
$$= \frac{x^2}{2}\,\mathrm{Sin}^{-1}x - \frac{1}{4}\,\mathrm{Sin}^{-1}x + \frac{1}{4}(x)\sqrt{1-x^2} + C.$$



48. $\displaystyle\int \sqrt{\cot x}\,\csc^4 x\,dx = \int \sqrt{\cot x}(1+\cot^2 x)\csc^2 x\,dx = -\frac{2}{3}\cot^{3/2}x - \frac{2}{7}\cot^{7/2}x + C$

49. If we set $u = \ln(\sqrt{x}+1)$, $dv = dx$, $du = \dfrac{1}{2\sqrt{x}(\sqrt{x}+1)}dx$, and $v = x$, then

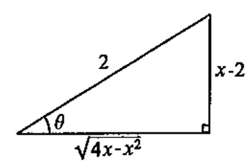
$$\int \ln(\sqrt{x}+1)\,dx = x\,\ln(\sqrt{x}+1) - \int \frac{x}{2\sqrt{x}(\sqrt{x}+1)}dx = x\,\ln(\sqrt{x}+1) - \frac{1}{2}\int\frac{\sqrt{x}}{\sqrt{x}+1}dx.$$

We now set $u = \sqrt{x} \implies x = u^2$, and $dx = 2u\,du$,

$$\int \ln(\sqrt{x}+1)\,dx = x\,\ln(\sqrt{x}+1) - \frac{1}{2}\int\frac{u}{u+1}(2u\,du) = x\,\ln(\sqrt{x}+1) - \int\left(u - 1 + \frac{1}{u+1}\right)du$$
$$= x\,\ln(\sqrt{x}+1) - \frac{u^2}{2} + u - \ln|u+1| + C = x\,\ln(\sqrt{x}+1) - \frac{x}{2} + \sqrt{x} - \ln(\sqrt{x}+1) + C.$$

50. Since $4x - x^2 = -(x-2)^2 + 4$, we set $x - 2 = 2\sin\theta$. Then $dx = 2\cos\theta\,d\theta$, and

$$\int\frac{1}{(4x-x^2)^{3/2}}dx = \int\frac{1}{8\cos^3\theta}2\cos\theta\,d\theta = \frac{1}{4}\int\sec^2\theta\,d\theta = \frac{1}{4}\tan\theta + C$$
$$= \frac{1}{4}\frac{x-2}{\sqrt{4x-x^2}} + C.$$



51. With the trapezoidal rule and 10 equal partitions,

$$\int_1^2 \frac{\sin x}{x}dx \approx \frac{1/10}{2}\left[\sin 1 + 2\sum_{i=1}^{9}\frac{\sin(1+i/10)}{1+i/10} + \frac{\sin 2}{2}\right] = 0.659\,22.$$

With Simpson's rule, $\displaystyle\int_1^2\frac{\sin x}{x}dx \approx \frac{1/10}{3}\left[\sin 1 + \frac{4\,\sin(1.1)}{1.1} + \frac{2\,\sin(1.2)}{1.2} + \cdots\right.$

$$\left. + \frac{2\,\sin(1.8)}{1.8} + \frac{4\,\sin(1.9)}{1.9} + \frac{\sin 2}{2}\right] = 0.659\,33.$$

52. With the trapezoidal rule and 10 equal partitions,

$$\int_0^1 \sqrt{\sin x}\,dx \approx \frac{1/10}{2}\left[\sqrt{\sin 0} + 2\sum_{i=1}^{9}\sqrt{\sin(i/10)} + \sqrt{\sin 1}\right] = 0.636\,65.$$

With Simpson's rule, $\displaystyle\int_0^1\sqrt{\sin x}\,dx \approx \frac{1/10}{3}[\sqrt{\sin 0} + 4\sqrt{\sin(1/10)} + 2\sqrt{\sin(1/5)} + \cdots$

$$+2\sqrt{\sin(4/5)} + 4\sqrt{\sin(9/10)} + \sqrt{\sin 1}] = 0.640\,41.$$

53. With the trapezoidal rule and 10 equal partitions,

$$\int_2^4\frac{1}{\ln x}dx \approx \frac{1/5}{2}\left[\frac{1}{\ln 2} + 2\sum_{i=1}^{9}\frac{1}{\ln(2+i/5)} + \frac{1}{\ln 4}\right] = 1.9254.$$

With Simpson's rule, $\displaystyle\int_2^4\frac{1}{\ln x}dx \approx \frac{1/5}{3}\left[\frac{1}{\ln 2} + \frac{4}{\ln 2.2} + \frac{2}{\ln 2.4} + \cdots + \frac{2}{\ln 3.6} + \frac{4}{\ln 3.8} + \frac{1}{\ln 4}\right] = 1.9225.$

54. With the trapezoidal rule and 10 equal subdivisions,

$$\int_{-1}^{3} \frac{1}{1+e^x}dx \approx \frac{2/5}{2}\left[\frac{1}{1+e^{-1}} + 2\sum_{i=1}^{9}\frac{1}{1+e^{-1+2i/5}} + \frac{1}{1+e^3}\right] = 1.2667.$$

With Simpson's rule, $$\int_{-1}^{3}\frac{1}{1+e^x}dx \approx \frac{2/5}{3}\left(\frac{1}{1+e^{-1}} + \frac{4}{1+e^{-3/5}} + \frac{2}{1+e^{-1/5}} + \cdots \right.$$
$$\left. + \frac{2}{1+e^{11/5}} + \frac{4}{1+e^{13/5}} + \frac{1}{1+e^3}\right) = 1.2647.$$

55. With the trapezoidal rule and 10 equal partitions,

$$\int_0^1 \frac{1}{(1+x^4)^2}\,dx \approx \frac{1/10}{2}\left[1 + 2\sum_{i=1}^{9}\frac{1}{[1+(i/10)^4]^2} + \frac{1}{(1+1^4)^2}\right] = 0.77440.$$

With Simpson's rule, $$\int_0^1 \frac{1}{(1+x^4)^2}\,dx \approx \frac{1/10}{3}\left\{1 + 4\left[\frac{1}{(1+0.1^4)^2}\right] + 2\left[\frac{1}{(1+0.2^4)^2}\right] + \cdots \right.$$
$$\left. +2\left[\frac{1}{(1+0.8^4)^2}\right] + 4\left[\frac{1}{(1+0.9^4)^2}\right] + \frac{1}{(1+1^4)^2}\right\}$$
$$= 0.77523.$$

56. If we set $u = x^{1/6}$, or, $x = u^6$, then $dx = 6u^5\,du$, and

$$\int \frac{1}{x^{1/3}-\sqrt{x}}dx = \int \frac{1}{u^2-u^3}6u^5\,du = 6\int \frac{u^3}{1-u}du = 6\int\left(-u^2-u-1+\frac{1}{1-u}\right)du$$
$$= 6\left(-\frac{u^3}{3} - \frac{u^2}{2} - u - \ln|1-u|\right) + C = -2\sqrt{x} - 3x^{1/3} - 6x^{1/6} - 6\ln|1-x^{1/6}| + C.$$

57. If we set $u = \ln(1+x^2)$, $dv = dx$, $du = \dfrac{2x}{1+x^2}dx$, and $v = x$, then

$$\int \ln(1+x^2)\,dx = x\ln(1+x^2) - \int \frac{2x^2}{1+x^2}dx = x\ln(1+x^2) - 2\int\left(1 - \frac{1}{1+x^2}\right)dx$$
$$= x\ln(1+x^2) - 2x + 2\,\mathrm{Tan}^{-1}x + C.$$

58. If we set $x^2 = 4\tan\theta$, then $2x\,dx = 4\sec^2\theta\,d\theta$, and

$$\int \frac{x}{x^4+16}dx = \int \frac{1}{16\sec^2\theta}2\sec^2\theta\,d\theta = \frac{\theta}{8} + C = \frac{1}{8}\,\mathrm{Tan}^{-1}\left(\frac{x^2}{4}\right) + C.$$

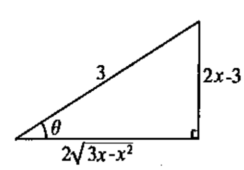
59. If we set $u = \csc x$, $dv = \csc^2 x\,dx$, $du = -\csc x\cot x\,dx$, $v = -\cot x$, then

$$\int \csc^3 x\,dx = -\csc x\cot x - \int \csc x\cot^2 x\,dx = -\csc x\cot x - \int \csc x(\csc^2 x - 1)\,dx$$
$$= -\csc x\cot x - \int \csc^3 x\,dx + \ln|\csc x - \cot x|.$$

We now solve for $\displaystyle\int \csc^3 x\,dx = \frac{1}{2}\ln|\csc x - \cot x| - \frac{1}{2}\csc x\cot x + C.$

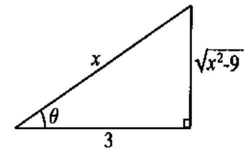
60. Since $3x - x^2 = -(x-3/2)^2 + 9/4$, we set $x - 3/2 = (3/2)\sin\theta$, in which case $dx = (3/2)\cos\theta\,d\theta$, and

$$\int \frac{1}{(3x-x^2)^{3/2}}dx = \int \frac{1}{(27/8)\cos^3\theta}(3/2)\cos\theta\,d\theta = \frac{4}{9}\int \sec^2\theta\,d\theta$$
$$= \frac{4}{9}\tan\theta + C = \frac{4}{9}\left(\frac{2x-3}{2\sqrt{3x-x^2}}\right) + C$$
$$= \frac{4x-6}{9\sqrt{3x-x^2}} + C.$$



61. If we set $x = 3\sec\theta$ and $dx = 3\sec\theta\tan\theta\,d\theta$, then

$$\int \frac{1}{x^3\sqrt{x^2-9}}dx = \int \frac{1}{27\sec^3\theta(3\tan\theta)}3\sec\theta\tan\theta\,d\theta = \frac{1}{27}\int \cos^2\theta\,d\theta = \frac{1}{27}\int\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$
$$= \frac{1}{54}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C = \frac{1}{54}(\theta + \sin\theta\cos\theta) + C$$
$$= \frac{1}{54}\,\mathrm{Sec}^{-1}\left(\frac{x}{3}\right) + \frac{1}{54}\left(\frac{\sqrt{x^2-9}}{x}\right)\left(\frac{3}{x}\right) + C$$
$$= \frac{1}{54}\,\mathrm{Sec}^{-1}\left(\frac{x}{3}\right) + \frac{\sqrt{x^2-9}}{18x^2} + C.$$



62. If we set $y = \sqrt{x}$ and $dy = 1/(2\sqrt{x})\,dx$, then $\int \sin\sqrt{x}\,dx = \int \sin y\,(2y\,dy)$. Now we set $u = y$, $dv = \sin y\,dy$, $du = dy$, $v = -\cos y$, and use integration by parts,

$$\int \sin\sqrt{x}\,dx = 2\left(-y\cos y - \int -\cos y\,dy\right) = -2y\cos y + 2\sin y + C = -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C.$$

63. If we set $u = \sin(\ln x)$, $dv = dx$, $du = \frac{1}{x}\cos(\ln x)$, and $v = x$, then

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - \int \cos(\ln x)\,dx.$$

We now set $u = \cos(\ln x)$, $dv = dx$, $du = -\frac{1}{x}\sin(\ln x)$, and $v = x$,

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - \left[x\cos(\ln x) - \int -\sin(\ln x)\,dx\right].$$

We can now solve for $\int \sin(\ln x)\,dx = \frac{x}{2}[\sin(\ln x) - \cos(\ln x)] + C.$

64. Using identity 1.48b, $\int x\cos x\sin 3x\,dx = \int \frac{x}{2}(\sin 2x + \sin 4x)\,dx$. We now set $u = x$, $dv = (\sin 2x + \sin 4x)dx$, in which case $du = dx$, $v = -(1/2)\cos 2x - (1/4)\cos 4x$, and

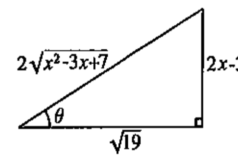
$$\int x\cos x\sin 3x\,dx = \frac{1}{2}\left[x\left(-\frac{1}{2}\cos 2x - \frac{1}{4}\cos 4x\right) - \int\left(-\frac{1}{2}\cos 2x - \frac{1}{4}\cos 4x\right)dx\right]$$
$$= -\frac{x}{8}(2\cos 2x + \cos 4x) + \frac{1}{2}\left(\frac{1}{4}\sin 2x + \frac{1}{16}\sin 4x\right) + C.$$

65. $\displaystyle\int \frac{x^4+3x^2+1}{x(x^2+1)^2}dx = \int \frac{(x^4+2x^2+1)+x^2}{x(x^2+1)^2}dx = \int\left[\frac{1}{x} + \frac{x}{(x^2+1)^2}\right]dx = \ln|x| - \frac{1}{2(x^2+1)} + C$

66. $\displaystyle\int \frac{1}{1+\cos 2x}dx = \int \frac{1}{1+(2\cos^2 x - 1)}dx = \frac{1}{2}\int \sec^2 x\,dx = \frac{1}{2}\tan x + C$

67. Long division gives $\frac{x^4+3x^2-2x+5}{x^2-3x+7} = x^2 + 3x + 5 - \frac{8x+30}{x^2-3x+7}$. Consider the integral of $\frac{4x+15}{x^2-3x+7} = \frac{4x+15}{(x-3/2)^2+19/4}$. If we set $x - 3/2 = (\sqrt{19}/2)\tan\theta$ and $dx = (\sqrt{19}/2)\sec^2\theta\,d\theta$, then

$$\int \frac{4x+15}{(x-3/2)^2+19/4}dx = \int \frac{21+2\sqrt{19}\,\tan\theta}{(19/4)\sec^2\theta}\frac{\sqrt{19}}{2}\sec^2\theta\,d\theta = \frac{2}{\sqrt{19}}\int (21+2\sqrt{19}\,\tan\theta)\,d\theta$$
$$= \frac{2}{\sqrt{19}}(21\theta + 2\sqrt{19}\,\ln|\sec\theta|) + C$$
$$= \frac{42}{\sqrt{19}}\,\mathrm{Tan}^{-1}\left(\frac{2x-3}{\sqrt{19}}\right) + 4\ln\left|\frac{2\sqrt{x^2-3x+7}}{\sqrt{19}}\right| + C$$
$$= \frac{42}{\sqrt{19}}\,\mathrm{Tan}^{-1}\left(\frac{2x-3}{\sqrt{19}}\right) + 2\ln(x^2-3x+7) + D.$$



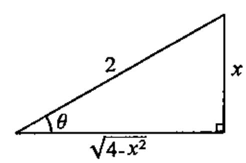
Finally then,

$$\int \frac{x^4+3x^2-2x+5}{x^2-3x+7}dx = \frac{x^3}{3} + \frac{3x^2}{2} + 5x - \frac{84}{\sqrt{19}}\,\mathrm{Tan}^{-1}\left(\frac{2x-3}{\sqrt{19}}\right) - 4\ln(x^2-3x+7) + C.$$

68. $$\int \sin^2 x\,\cos 3x\,dx = \int \left(\frac{1-\cos 2x}{2}\right)\cos 3x\,dx = \frac{1}{2}\int \left[\cos 3x - \frac{1}{2}(\cos 5x + \cos x)\right] dx$$
$$= \frac{1}{6}\sin 3x - \frac{1}{20}\sin 5x - \frac{1}{4}\sin x + C.$$

69. If we set $x = 2\sin\theta$ and $dx = 2\cos\theta\,d\theta$, then

$$\int \frac{1}{x^3(4-x^2)^{3/2}}dx = \int \frac{1}{8\sin^3\theta(8\cos^3\theta)}2\cos\theta\,d\theta = \frac{1}{32}\int \frac{\sec^5\theta}{\tan^3\theta}d\theta$$
$$= \frac{1}{32}\int \frac{(\tan^2\theta+1)^2\sec\theta}{\tan^3\theta}d\theta = \frac{1}{32}\int \left(\tan\theta + \frac{2}{\tan\theta} + \frac{1}{\tan^3\theta}\right)\sec\theta\,d\theta$$
$$= \frac{1}{32}\int (\sec\theta\,\tan\theta + 2\csc\theta + \cot^2\theta\,\csc\theta)\,d\theta$$
$$= \frac{1}{32}\int [\sec\theta\,\tan\theta + (\csc^2\theta + 1)\csc\theta]\,d\theta$$
$$= \frac{1}{32}\int (\sec\theta\,\tan\theta + \csc\theta + \csc^3\theta)\,d\theta.$$



For the integral of $\csc^3\theta$, we use Exercise 59,

$$\int \frac{1}{x^3(4-x^2)^{3/2}}dx = \frac{1}{32}\left(\sec\theta + \ln|\csc\theta - \cot\theta| + \frac{1}{2}\ln|\csc\theta - \cot\theta| - \frac{1}{2}\csc\theta\,\cot\theta\right) + C$$
$$= \frac{1}{32}\left[\frac{2}{\sqrt{4-x^2}} + \frac{3}{2}\ln\left|\frac{2}{x} - \frac{\sqrt{4-x^2}}{x}\right| - \frac{1}{2}\left(\frac{2}{x}\right)\left(\frac{\sqrt{4-x^2}}{x}\right)\right] + C$$
$$= \frac{1}{16\sqrt{4-x^2}} + \frac{3}{64}\ln\left|\frac{2-\sqrt{4-x^2}}{x}\right| - \frac{\sqrt{4-x^2}}{32x^2} + C.$$

70. If we set $y = \mathrm{Sin}^{-1}x$, then $x = \sin y$ and $dx = \cos y\,dy$. With these,

$$\int \sqrt{1-x^2}\,\mathrm{Sin}^{-1}x\,dx = \int \cos y\,(y)\,\cos y\,dy = \int y\left(\frac{1+\cos 2y}{2}\right)dy = \frac{y^2}{4} + \frac{1}{2}\int y\,\cos 2y\,dy.$$

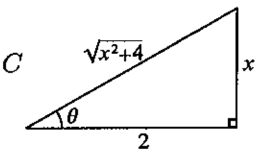
We now set $u = y$, $dv = \cos 2y\,dy$, $du = dy$, $v = (1/2)\sin 2y$, and use integration by parts,

$$\int \sqrt{1-x^2}\,\mathrm{Sin}^{-1}x\,dx = \frac{y^2}{4} + \frac{1}{2}\left(\frac{y}{2}\sin 2y - \int \frac{1}{2}\sin 2y\,dy\right)$$
$$= \frac{y^2}{4} + \frac{y}{4}\sin 2y + \frac{1}{8}\cos 2y + C$$
$$= \frac{y^2}{4} + \frac{y}{2}\sin y\,\cos y + \frac{1}{8}(1-2\sin^2 y) + C$$
$$= \frac{1}{4}(\mathrm{Sin}^{-1}x)^2 + \frac{1}{2}(\mathrm{Sin}^{-1}x)x\sqrt{1-x^2} - \frac{1}{4}x^2 + D.$$

71. $\displaystyle\int \frac{1}{x+\sqrt{x^2+4}}dx = \int \frac{1}{x+\sqrt{x^2+4}}\,\frac{x-\sqrt{x^2+4}}{x-\sqrt{x^2+4}}dx = \int \frac{x-\sqrt{x^2+4}}{x^2-(x^2+4)}dx = \frac{1}{4}\int(-x+\sqrt{x^2+4})\,dx$

If we set $x = 2\tan\theta$ and $dx = 2\sec^2\theta\,d\theta$, then

$$\begin{aligned}\int \frac{1}{x+\sqrt{x^2+4}}dx &= -\frac{x^2}{8} + \frac{1}{4}\int 2\sec\theta\, 2\sec^2\theta\,d\theta \quad \text{(and using Example 8.9)}\\ &= -\frac{x^2}{8} + \frac{1}{2}\sec\theta\,\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C\\ &= -\frac{x^2}{8} + \frac{1}{2}\left(\frac{\sqrt{x^2+4}}{2}\right)\left(\frac{x}{2}\right) + \frac{1}{2}\ln\left|\frac{\sqrt{x^2+4}}{2}+\frac{x}{2}\right| + C\\ &= -\frac{x^2}{8} + \frac{x\sqrt{x^2+4}}{8} + \frac{1}{2}\ln|\sqrt{x^2+4}+x| + D.\end{aligned}$$



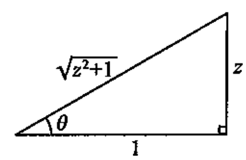
72. (a) If $z^2 = (1+x)/(1-x)$, then $z^2(1-x) = 1+x \Longrightarrow x = (z^2-1)/(z^2+1)$, and

$$dx = \frac{(z^2+1)(2z)-(z^2-1)(2z)}{(z^2+1)^2}dz = \frac{4z}{(z^2+1)^2}dz.$$

Thus, $\displaystyle\int\sqrt{\frac{1+x}{1-x}}dx = \int z\frac{4z}{(z^2+1)^2}dz = 4\int\frac{z^2}{(z^2+1)^2}dz.$

We now set $z = \tan\theta$ and $dz = \sec^2\theta\,d\theta$,

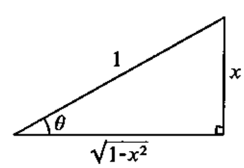
$$\begin{aligned}\int\sqrt{\frac{1+x}{1-x}}dx &= 4\int\frac{\tan^2\theta}{\sec^4\theta}\sec^2\theta\,d\theta = 4\int\sin^2\theta\,d\theta = 2\int(1-\cos 2\theta)\,d\theta\\ &= 2\left(\theta - \frac{1}{2}\sin 2\theta\right) + C = 2\theta - 2\sin\theta\,\cos\theta + C\\ &= 2\,\mathrm{Tan}^{-1}z - 2\frac{z}{\sqrt{z^2+1}}\frac{1}{\sqrt{z^2+1}} + C\\ &= 2\,\mathrm{Tan}^{-1}z - \frac{2z}{z^2+1} + C\\ &= 2\,\mathrm{Tan}^{-1}\sqrt{\frac{1+x}{1-x}} - \frac{2\sqrt{\dfrac{1+x}{1-x}}}{\dfrac{1+x}{1-x}+1} + C = 2\,\mathrm{Tan}^{-1}\sqrt{\frac{1+x}{1-x}} - \sqrt{1-x^2} + C.\end{aligned}$$



(b) $\displaystyle\int\sqrt{\frac{1+x}{1-x}}dx = \int\sqrt{\frac{1+x}{1-x}}\frac{\sqrt{1+x}}{\sqrt{1+x}}dx = \int\frac{1+x}{\sqrt{1-x^2}}dx$

We now set $x = \sin\theta$ and $dx = \cos\theta\,d\theta$,

$$\begin{aligned}\int\sqrt{\frac{1+x}{1-x}}dx &= \int\frac{1+\sin\theta}{\cos\theta}\cos\theta\,d\theta\\ &= \theta - \cos\theta + C\\ &= \mathrm{Sin}^{-1}x - \sqrt{1-x^2} + C.\end{aligned}$$



If we set $\phi = 2\,\mathrm{Tan}^{-1}\sqrt{\dfrac{1+x}{1-x}}$, then $\dfrac{1+x}{1-x} = \tan^2(\phi/2)$. When we solve this equation for x,

$$x = \frac{\tan^2(\phi/2)-1}{\tan^2(\phi/2)+1} = \frac{\sin^2(\phi/2)-\cos^2(\phi/2)}{\sin^2(\phi/2)+\cos^2(\phi/2)} = -\cos\phi = -\sin\left(\frac{\pi}{2}-\phi\right).$$

Thus, $\dfrac{\pi}{2} - \phi = \mathrm{Sin}^{-1}(-x) = -\mathrm{Sin}^{-1}x$, or $\phi = \mathrm{Sin}^{-1}x + \pi/2$, and it follows that

$$2\,\mathrm{Tan}^{-1}\sqrt{\frac{1+x}{1-x}} - \sqrt{1-x^2} + C = \mathrm{Sin}^{-1}x + \frac{\pi}{2} - \sqrt{1-x^2} + C = \mathrm{Sin}^{-1}x - \sqrt{1-x^2} + D.$$