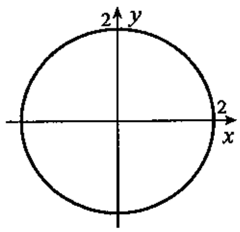
CHAPTER 12

EXERCISES 12.1

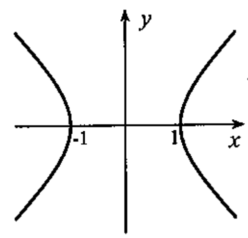
1. (a) $f(1,2) = 1^3(2) + 1\sin 2 = 2 + \sin 2$
   (b) $f(-2,-2) = (-2)^3(-2) - 2\sin(-2) = 16 + 2\sin 2$
   (c) $f(x^2+y, x-y^2) = (x^2+y)^3(x-y^2) + (x^2+y)\sin(x-y^2)$
   (d) $f(x+h,y) - f(x,y) = [(x+h)^3y + (x+h)\sin y] - [x^3y + x\sin y] = y(3x^2h + 3xh^2 + h^3) + h\sin y$

2. $f(a+b, a-b, ab) = (a+b)^2(a-b)^2 - (a+b)^4 + 4ab(a+b)^2 = (a+b)^2[(a-b)^2 - (a+b)^2 + 4ab] = 0$

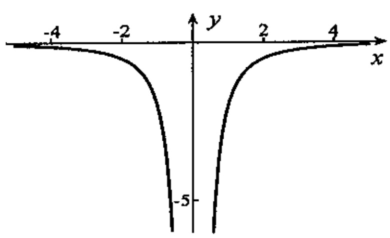
3. For $4 - x^2 - y^2 \geq 0$, we must take $x^2 + y^2 \leq 4$. This inequality describes all points inside and on the circle $x^2 + y^2 = 4$.



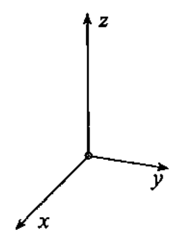
4. For $1 - x^2 + y^2 > 0$, we must take $x^2 - y^2 < 1$. This inequality describes all points between, but not on, the branches of the hyperbola $x^2 - y^2 = 1$



5. For $-1 \leq x^2y + 1 \leq 1$, we require $-2 \leq x^2y \leq 0$ or $-2/x^2 \leq y \leq 0$. Points are below the x-axis and above the curve $y = -2/x^2$. Points on the boundary are also included.



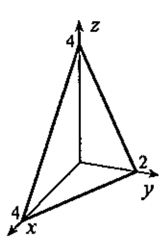
6. This function is defined for all points in space except the origin $(0,0,0)$.



7. For $f(x,y) = \dfrac{12xy - x^2y^2}{2(x+y)} = \dfrac{xy(12-xy)}{2(x+y)} = 0$, we set $xy(12-xy) = 0$. This is satisfied if $x = 0$ or $y = 0$ or $12 - xy = 0$. Thus, the function is equal to zero for all points on the x- and y-axes (except $(0,0)$), and all points on the hyperbola $xy = 12$. The largest domain of the function is all points in the xy-plane except those on the line $y = -x$.

For Exercises 8, 10, 12, 14, 16, 18, and 20, see answers in text.

9.



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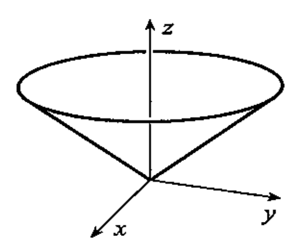

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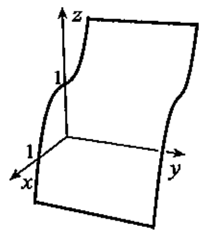

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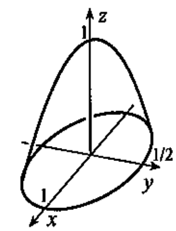

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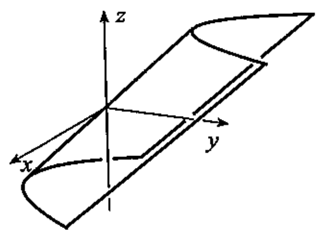

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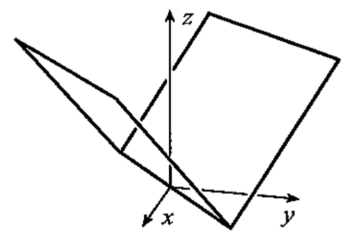

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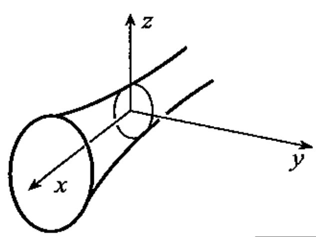
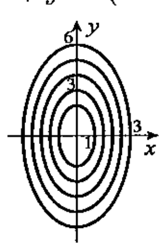
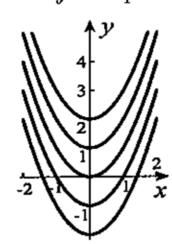


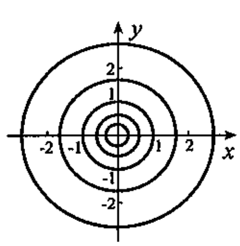
22. Level curves are defined by $4 - \sqrt{4x^2 + y^2} = C$, or, $4x^2 + y^2 = (C - 4)^2$. They are ellipses.



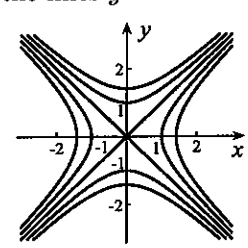
23. Level curves are defined by $y - x^2 = C$, or $y = x^2 + C$. They are parabolas.



24. Level curves are defined by $\ln(x^2 + y^2) = C$, or, $x^2 + y^2 = e^C$. They are circles.



25. Level curves are defined by $x^2 - y^2 = C$. They are hyperbolas, except when $C = 0$ when they are the lines $y = \pm x$.



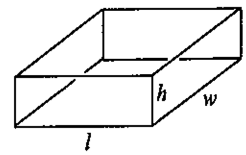
26. The volume of the box is $V = lwh$, where h is its height. Since $30 = 2hl + 2wh + 2wl$, it follows that $h = (15 - wl)/(l + w)$, and therefore $V = lw(15 - wl)/(l + w)$.

27. (a) The cost in cents is

$$C = 125(2wh + 2lh + lw) + 475lw = 600lw + 250h(l + w).$$

(b) If $V = 1000 = lwh$, then $h = 1000/(lw)$, and

$$C = 600lw + 250\left(\frac{1000}{lw}\right)(l + w)$$
$$= 600lw + \frac{250\,000(l + w)}{lw}.$$



(c) If we add the cost of welding to the functions in (a) and (b) we obtain

$$C = 600lw + 250h(l + w) + 750(4l + 4h + 4w) = 600lw + 250h(l + w) + 3000(l + h + w),$$

and

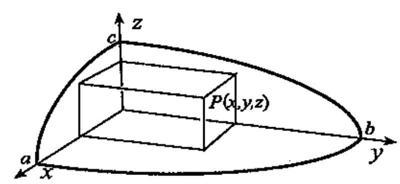
$$C = 600lw + \frac{250\,000(l + w)}{lw} + 3000\left(l + \frac{1000}{lw} + w\right).$$

28. If P is the corner of the ellipsoid in the first octant, then

$$V = 8xyz = 8cxy\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}.$$

When P is not in the first octant,

$$V = 8c|xy|\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}.$$

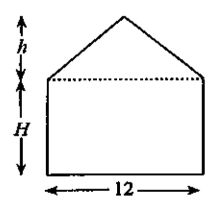


29. (a) The volume of the silo is

$$V = \pi(6)^2H + \tfrac{1}{3}\pi(6)^2h = 12\pi(h + 3H)\ \text{m}^3.$$

(b) If $200 = 2\pi(6)H + \pi(6)\sqrt{36 + h^2}$, then $H = (100 - 3\pi\sqrt{36 + h^2})/(6\pi)$, and

$$V = 12\pi\left(h + \frac{100 - 3\pi\sqrt{36 + h^2}}{2\pi}\right)\ \text{m}^3.$$



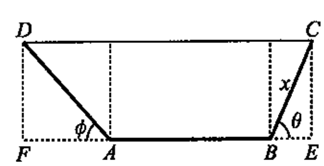
30. (a) $D(0.35, 9.0) = 0.35 + 0.9 + \dfrac{81\cos 0.35}{9.81}\left[\sin 0.35 + \sqrt{\sin^2 0.35 + \dfrac{2(9.81)(0.5)}{81}}\right] = 7.70$ m

(b) Since $D(0.35, 9.9) = 0.35 + 0.9 + \dfrac{(9.9)^2\cos 0.35}{9.81}\left[\sin 0.35 + \sqrt{\sin^2 0.35 + \dfrac{2(9.81)(0.5)}{(9.9)^2}}\right] = 8.847$, the percentage increase is $100(8.847 - 7.70)/7.70 = 14.9$.

(c) Since $D(0.385, 9.0) = 0.35 + 0.9 + \dfrac{81\cos 0.385}{9.81}\left[\sin 0.385 + \sqrt{\sin^2 0.385 + \dfrac{2(9.81)(0.5)}{81}}\right] = 8.042$, the percentage increase is $100(8.042 - 7.70)/7.70 = 4.4$.

31. Since

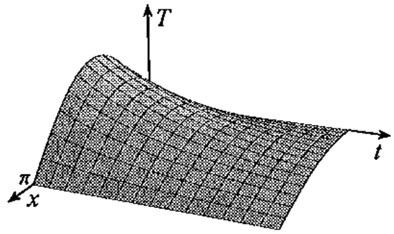
$$\begin{aligned}
\|DF\| &= \|CE\| = x\sin\theta,\\
\|AF\| &= \|DF\|\cot\phi = x\sin\theta\,\cot\phi,\\
\|AB\| &= 1 - \|BC\| - \|AD\|\\
&= 1 - x - x\sin\theta\,\csc\phi,
\end{aligned}$$



the cross-sectional area is

$$\begin{aligned}
\text{Area} &= \|AB\|\|CE\| + \frac{1}{2}\|BE\|\|CE\| + \frac{1}{2}\|FA\|\|DF\|\\
&= \|CE\|\left(\|AB\| + \frac{1}{2}\|BE\| + \frac{1}{2}\|FA\|\right)\\
&= x\sin\theta\left[(1 - x - x\sin\theta\,\csc\phi) + \frac{1}{2}(x\cos\theta) + \frac{1}{2}(x\sin\theta\,\cot\phi)\right].
\end{aligned}$$

32. (a)



(b) The intersection curve with $x = x_0$ is a graphical history of temperature at position x_0 as a function of time. The intersection curve with $t = t_0$ is a graph of the temperature distribution throughout the rod at time t_0.

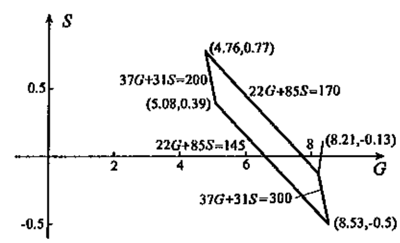
33. The cost per day in cents is $C = f(G,S) = \frac{2750}{1000}(11) + \frac{11\,000}{1000}G + \frac{17\,500}{1000}S = 30.25 + 11G + 17.5S$. Because the cow must have between 9.5 and 11.5 kg of digestive material,

$$9.5 \le 11\left(\frac{1}{2}\right) + G\left(\frac{74}{100}\right) + S\left(\frac{62}{100}\right) \le 11.5 \quad \Longrightarrow \quad 200 \le 37G + 31S \le 300.$$

Because the cow must have between 1.9 and 2.0 kg of protein,

$$1.9 \le 11\left(\frac{12}{100}\right) + G\left(\frac{8.8}{100}\right) + S\left(\frac{34}{100}\right) \le 2.0 \quad \Longrightarrow \quad 145 \le 22G + 85S \le 170.$$

The domain of $f(G,S)$ therefore consists of all non-negative values of G and S satisfying these two inequalities. It is the points in the first quadrant of the parallelogram to the right.



34. If x and y are the numbers of computers of models A and B, then the cost of the 100 computers is

$$C = f(x,y) = 1300x + 1200y + 1000(100 - x - y) = 100\,000 + 300x + 200y.$$

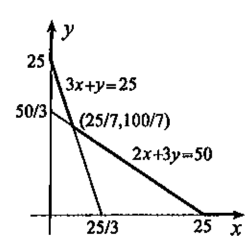
Because the computers must have at least 2000 MB of memory,

$$64x + 32y + 16(100 - x - y) \ge 2000 \quad \Longrightarrow 3x + y \ge 25.$$

Because the computers must have at least 150 GB of disk space,

$$3x + 4y + (100 - x - y) \ge 150 \quad \Longrightarrow \quad 2x + 3y \ge 50.$$

The domain of $f(x,y)$ therefore consists of all non-negative values of x and y satisfying these two inequalities. It is the points in the first quadrant above the lines in the figure to the right.



EXERCISES 12.2

1. $\displaystyle\lim_{(x,y)\to(2,-3)} \frac{x^2-1}{x+y} = -3$

2. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^3+2y^3}{x^3+4y^3} = \frac{3}{5}$

3. $\displaystyle\lim_{(x,y)\to(3,2)} \frac{2x-3y}{x+y} = 0$

4. $\displaystyle\lim_{(x,y,z)\to(2,3,-1)} \frac{xyz}{x^2+y^2+z^2} = -\frac{3}{7}$

5. $\displaystyle\lim_{(x,y)\to(1,0)} \frac{x}{y}$ does not exist.

6. $\displaystyle\lim_{(x,y,z)\to(0,\pi/2,1)} \mathrm{Tan}^{-1}\left(\frac{x}{yz}\right) = 0$

7. $\displaystyle\lim_{(x,y,z)\to(0,\pi/2,1)} \mathrm{Tan}^{-1}\left(\frac{yz}{x}\right)$ does not exist since it depends on whether $x \to 0^-$ or $x \to 0^+$.

8. $\displaystyle\lim_{(x,y,z)\to(0,\pi/2,1)} \mathrm{Tan}^{-1}\left|\frac{yz}{x}\right| = \frac{\pi}{2}$

9. $\displaystyle\lim_{(x,y)\to(3,4)} \frac{|x^2-y^2|}{x^2-y^2} = \frac{|9-16|}{9-16} = -1$

10. $\displaystyle\lim_{(x,y)\to(3,4)} \frac{|x^2+y^2|}{x^2+y^2} = \frac{|9+16|}{9+16} = 1$

11. $\displaystyle\lim_{(x,y)\to(2,1)} \frac{x^2-y^2}{x-y} = 3$

12. $\displaystyle\lim_{(x,y)\to(2,2)} \frac{x^2-y^2}{x-y} = \lim_{(x,y)\to(2,2)} \frac{(x+y)(x-y)}{x-y} = \lim_{(x,y)\to(2,2)} (x+y) = 4$

13. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x-y} = \lim_{(x,y)\to(0,0)} \frac{(x+y)(x-y)}{x-y} = \lim_{(x,y)\to(0,0)} (x+y) = 0$

14. If we approach $(0,0)$ along the line $y = mx$,

$$\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y} = \lim_{x\to 0} \frac{x-mx}{x+mx} = \lim_{x\to 0} \frac{1-m}{1+m} = \frac{1-m}{1+m}.$$

Since this result depends on m, the original limit does not exist.

15. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^2-y^2}{y^2+z^2+1} = 0$

16. $\displaystyle\lim_{(x,y)\to(2,1)} \frac{(x-2)^2(y+1)}{x-2} = \lim_{(x,y)\to(2,1)} (x-2)(y+1) = 0$

17. $\displaystyle\lim_{(x,y,z)\to(1,1,1)} |2x-y-z| = 0$

18. If we approach $(0,0)$ along the line $y = mx$,

$$\lim_{(x,y)\to(0,0)} \frac{3x^3-y^3}{2x^3+4y^3} = \lim_{x\to 0} \frac{3x^3-m^3x^3}{2x^3+4m^3x^3} = \lim_{x\to 0} \frac{3-m^3}{2+4m^3} = \frac{3-m^3}{2+4m^3}.$$

Since this result depends on m, the original limit does not exist.

19. $\displaystyle\lim_{(x,y)\to(0,0)} \mathrm{Sec}^{-1}\left(\frac{-1}{x^2+y^2}\right) = -\frac{\pi}{2}$

20. $\displaystyle\lim_{(x,y)\to(0,0)} \mathrm{Sec}^{-1}(x^2+y^2)$ does not exist.

21. The function is discontinuous at all points on the line $y = -x$.

22. The function is discontinuous at $(0,0)$.

23. The function is discontinuous at all points on the circle $x^2+y^2 = 1$.

24. The function is discontinuous when $x = 0$, or $y = 0$, or $z = 0$.

25. The function has no discontinuities.

26. Since $f(x,y) = \dfrac{x+y}{xy(x+y)}$, the function is discontinuous when $x = 0$, or $y = 0$, or $x+y = 0$.

27.

$$\begin{aligned}\lim_{(x,y)\to(a,a)} \left[\cos(x+y) - \sqrt{1-\sin^2(x+y)}\right] &= \lim_{(x,y)\to(a,a)} [\cos(x+y) - |\cos(x+y)|] \\ &= \cos 2a - |\cos 2a| = \begin{cases} 0, & 0 \le a \le \pi/4 \\ 2\cos 2a, & \pi/4 < a \le \pi/2. \end{cases}\end{aligned}$$

28. If we approach $(0,0)$ along parabola $y = ax^2$,

$$\lim_{(x,y)\to(0,0)} \frac{x^4+y^2}{x^4-y^2} = \lim_{x\to 0} \frac{x^4+a^2x^4}{x^4-a^2x^4} = \lim_{x\to 0} \frac{1+a^2}{1-a^2} = \frac{1+a^2}{1-a^2}.$$

Since this result depends on a, the original limit does not exist.

29. If (x,y) is made to approach $(0,0)$ along the cubic curve $y = ax^3$,

$$\lim_{(x,y)\to(0,0)} \frac{x^6 - 2y^2}{3x^6 + y^2} = \lim_{x\to 0} \frac{x^6 - 2a^2x^6}{3x^6 + a^2x^6} = \lim_{x\to 0} \frac{1-2a^2}{3+a^2} = \frac{1-2a^2}{3+a^2}.$$

Since this result depends on a, the original limit does not exist.

30. If (x,y) is made to approach $(1,0)$ along the straight line $y = m(x-1)$,

$$\lim_{(x,y)\to(1,0)} \frac{(x-1)^2 + y^2}{3(x-1)^2 - 2y^2} = \lim_{x\to 1} \frac{(x-1)^2 + m^2(x-1)^2}{3(x-1)^2 - 2m^2(x-1)^2} = \lim_{x\to 1} \frac{1+m^2}{3-2m^2} = \frac{1+m^2}{3-2m^2}.$$

Since this result depends on m, the original limit does not exist.

31. If (x,y) is made to approach $(0,-2)$ along the straight line $y + 2 = mx$,

$$\lim_{(x,y)\to(0,-2)} \frac{x^3 + 4(y+2)^3}{3x^3 - (y+2)^3} = \lim_{x\to 0} \frac{x^3 + 4m^3x^3}{3x^3 - m^3x^3} = \lim_{x\to 0} \frac{1+4m^3}{3-m^3} = \frac{1+4m^3}{3-m^3}.$$

Since this result depends on m, the original limit does not exist.

32. If (x,y) is made to approach $(1,1)$ along the straight line $y - 1 = m(x-1)$,

$$\lim_{(x,y)\to(1,1)} \frac{x^2 - 2x - y^2 + 2y}{x^2 - 2x + y^2 - 2y + 2} = \lim_{(x,y)\to(1,1)} \frac{(x-1)^2 - (y-1)^2}{(x-1)^2 + (y-1)^2} = \lim_{x\to 1} \frac{(x-1)^2 - m^2(x-1)^2}{(x-1)^2 + m^2(x-1)^2}$$
$$= \lim_{x\to 1} \frac{1-m^2}{1+m^2} = \frac{1-m^2}{1+m^2}.$$

Since this result depends on m, the original limit does not exist.

33. If (x,y) is made to approach $(1,1)$ along the vertical line $x = 1$, then

$$\lim_{(x,y)\to(1,1)} \frac{x^2 - 2x + y^2 + 2y - 2}{x^2 - y^2 - 2x + 2y} = \lim_{y\to 1} \frac{1 - 2 + y^2 + 2y - 2}{1 - y^2 - 2 + 2y} = \lim_{y\to 1} \frac{y^2 + 2y - 3}{-y^2 + 2y - 1} = \lim_{y\to 1} \frac{y+3}{1-y}.$$

Since this limit does not exist, neither does the original limit.

34. $$\lim_{(x,y)\to(1,0)} \frac{\sqrt{x+y} - \sqrt{x-y}}{y} = \lim_{(x,y)\to(1,0)} \left(\frac{\sqrt{x+y} - \sqrt{x-y}}{y} \frac{\sqrt{x+y} + \sqrt{x-y}}{\sqrt{x+y} + \sqrt{x-y}} \right)$$
$$= \lim_{(x,y)\to(1,0)} \frac{(x+y) - (x-y)}{y(\sqrt{x+y} + \sqrt{x-y})} = \lim_{(x,y)\to(1,0)} \frac{2}{\sqrt{x+y} + \sqrt{x-y}} = 1$$

35. If we set $\theta = x^2 + y^2$, then $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$

36. (a) If we set $z = x - y$, then $\displaystyle\lim_{(x,y)\to(1,1)} \frac{\sin(x-y)}{x-y} = \lim_{z\to 0} \frac{\sin z}{z} = 1.$

The function is not continuous at $(1,1)$ (being undefined there).

(b) Since the value of the function along $y = x$ is always equal to 1, its limit as $(x,y) \to (1,1)$ along $y = x$ is also 1. Consequently the limit of the function as $(x,y) \to (1,1)$ is still 1. The function is now continuous at $(1,1)$.

37. $f(x,y,z)$ has limit L as (x,y,z) approaches (x_0,y_0,z_0) if given any $\epsilon > 0$, we can find a $\delta > 0$ such that $|f(x,y,z) - L| < \epsilon$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} < \delta$ and (x,y,z) is in the domain of definition of $f(x,y,z)$.

38. False. In calculating the limit, we consider only those values of (x,y) where $f(x,y)$ is defined. The function in Exercise 36(a) is a counterexample.

39. (a) Since $\lim_{x\to 0} f(x,0) = \lim_{x\to 0} 0 = 0$, and this is the value of $f(0,0)$, the function $f(x,0)$ is continuous at $x = 0$. Similarly, $f(0,y)$ is continuous at $y = 0$.
(b) If we approach $(0,0)$ along straight line $y = mx$, then

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^4} = \lim_{x\to 0} \frac{x^2(mx)^2}{x^4+(mx)^4} = \lim_{x\to 0} \frac{m^2}{1+m^4} = \frac{m^2}{1+m^4}.$$

Since this result depends on m, the limit of $f(x,y)$ as $(x,y) \to (0,0)$ does not exist. The function is therefore discontinuous at $(0,0)$.

40. (a) Suppose $\epsilon > 0$ is given. We must find a $\delta > 0$ such that whenever $0 < x^2 + y^2 < \delta^2$,

$$\epsilon > |(xy+5) - 5| = |xy|.$$

If we choose $\delta = \sqrt{\epsilon}$, then for $x^2 + y^2 < \delta^2 = \epsilon$, it must certainly be true that $|x| < \sqrt{\epsilon}$ and $|y| < \sqrt{\epsilon}$, and hence

$$|xy| = |x||y| < \sqrt{\epsilon}\,\sqrt{\epsilon} = \epsilon.$$

(b) Suppose $\epsilon > 0$ is given. We must find a $\delta > 0$ such that whenever $0 < (x-1)^2 + (y-1)^2 < \delta^2$,

$$\begin{aligned} \epsilon &> |(x^2+2xy+5) - 8| = |x^2+2xy-3| \\ &= |(x-1)^2 + 2(x-1)(y-1) - 6 + 4x + 2y| \\ &= |(x-1)^2 + 2(x-1)(y-1) + 4(x-1) + 2(y-1)|. \end{aligned}$$

Consider the quadratic $Q(z) = 3z^2 + 6z - \epsilon$. It is equal to zero when $z = (-6 \pm \sqrt{36+12\epsilon})/6 = -1 \pm \sqrt{1+\epsilon/3}$. Consequently, we can say that $3z^2 + 6z < \epsilon$ when $0 < z < -1 + \sqrt{1+\epsilon/3}$. If we now choose $\delta = -1 + \sqrt{1+\epsilon/3}$, then for $0 < (x-1)^2 + (y-1)^2 < \delta^2$, we have $|x-1| < \delta = -1 + \sqrt{1+\epsilon/3}$ and $|y-1| < \delta = -1 + \sqrt{1+\epsilon/3}$. Furthermore,

$$|(x-1)^2 + 2(x-1)(y-1) + 4(x-1) + 2(y-1)| < \delta^2 + 2\delta\delta + 4\delta + 2\delta = 3\delta^2 + 6\delta < \epsilon.$$

EXERCISES 12.3

1. $\dfrac{\partial f}{\partial x} = 3x^2y^2 + 2y;\quad \dfrac{\partial f}{\partial y} = 2x^3y + 2x$

2. $\dfrac{\partial f}{\partial x} = 3y - 16x^3y^4;\quad \dfrac{\partial f}{\partial y} = 3x - 16x^4y^3$

3. $\dfrac{\partial f}{\partial x} = 4x^3/y^3;\quad \dfrac{\partial f}{\partial y} = -3x^4/y^4$

4. $\dfrac{\partial f}{\partial x} = \dfrac{(x+y)(1) - x(1)}{(x+y)^2} - \dfrac{1}{y} = \dfrac{y^2 - (x+y)^2}{y(x+y)^2} = \dfrac{-x(x+2y)}{y(x+y)^2};$

$\dfrac{\partial f}{\partial y} = -\dfrac{x}{(x+y)^2} + \dfrac{x}{y^2} = \dfrac{-xy^2 + x(x+y)^2}{y^2(x+y)^2} = \dfrac{x^2(x+2y)}{y^2(x+y)^2}$

5. $\dfrac{\partial f}{\partial x} = \dfrac{(2x^2+y)(1) - x(4x)}{(2x^2+y)^2} = \dfrac{y - 2x^2}{(2x^2+y)^2};\quad \dfrac{\partial f}{\partial y} = -\dfrac{x}{(2x^2+y)^2}$

6. $\dfrac{\partial f}{\partial x} = y\cos(xy);\quad \dfrac{\partial f}{\partial y} = x\cos(xy)$

7. $\dfrac{\partial f}{\partial x} = \cos(x+y) - x\sin(x+y);\quad \dfrac{\partial f}{\partial y} = -x\sin(x+y)$

8. $\dfrac{\partial f}{\partial x} = \dfrac{1}{2}(x^2+y^2)^{-1/2}(2x) = \dfrac{x}{\sqrt{x^2+y^2}};\quad \dfrac{\partial f}{\partial y} = \dfrac{1}{2}(x^2+y^2)^{-1/2}(2y) = \dfrac{y}{\sqrt{x^2+y^2}}$

9. $\dfrac{\partial f}{\partial x} = \sqrt{x^2-y^2} + \dfrac{x^2}{\sqrt{x^2-y^2}} = \dfrac{2x^2-y^2}{\sqrt{x^2+y^2}};\quad \dfrac{\partial f}{\partial y} = \dfrac{-xy}{\sqrt{x^2+y^2}}$

10. $\dfrac{\partial f}{\partial x} = \sec^2(2x^2+y^2)(4x);\quad \dfrac{\partial f}{\partial y} = \sec^2(2x^2+y^2)(2y)$

11. $\dfrac{\partial f}{\partial x} = e^{x+y}; \quad \dfrac{\partial f}{\partial y} = e^{x+y}$

12. $\dfrac{\partial f}{\partial x} = e^{xy}(y); \quad \dfrac{\partial f}{\partial y} = e^{xy}(x)$

13. $\dfrac{\partial f}{\partial x} = ye^{xy} + xye^{xy}(y) = y(1+xy)e^{xy}; \quad \dfrac{\partial f}{\partial y} = xe^{xy} + xye^{xy}(x) = x(1+xy)e^{xy}$

14. $\dfrac{\partial f}{\partial x} = \dfrac{1}{x^2+y^2}(2x); \quad \dfrac{\partial f}{\partial y} = \dfrac{1}{x^2+y^2}(2y)$

15. $\dfrac{\partial f}{\partial x} = \ln(xy) + \dfrac{x+1}{x}; \quad \dfrac{\partial f}{\partial y} = \dfrac{x+1}{y}$

16. $\dfrac{\partial f}{\partial x} = \cos(ye^x)(ye^x); \quad \dfrac{\partial f}{\partial y} = \cos(ye^x)(e^x)$

17. $\dfrac{\partial f}{\partial x} = \dfrac{1/y}{1+(x/y)^2} = \dfrac{y}{x^2+y^2}; \quad \dfrac{\partial f}{\partial y} = \dfrac{-x/y^2}{1+(x/y)^2} = \dfrac{-x}{x^2+y^2}$

18. $\dfrac{\partial f}{\partial x} = \dfrac{1}{3}[1-\cos^3(x^2y)]^{-2/3}[-3\cos^2(x^2y)][-\sin(x^2y)](2xy) = \dfrac{2xy\cos^2(x^2y)\sin(x^2y)}{[1-\cos^3(x^2y)]^{2/3}};$

$\dfrac{\partial f}{\partial y} = \dfrac{1}{3}[1-\cos^3(x^2y)]^{-2/3}[-3\cos^2(x^2y)][-\sin(x^2y)](x^2) = \dfrac{x^2\cos^2(x^2y)\sin(x^2y)}{[1-\cos^3(x^2y)]^{2/3}}$

19. $\dfrac{\partial f}{\partial x} = \dfrac{\cos x}{\cos y}; \quad \dfrac{\partial f}{\partial y} = -\dfrac{\sin x}{\cos^2 y}(-\sin y) = \dfrac{\sin x \sin y}{\cos^2 y}$

20. $\dfrac{\partial f}{\partial x} = \dfrac{1}{\sec\sqrt{x+y}}\sec\sqrt{x+y}\,\tan\sqrt{x+y}\dfrac{1}{2\sqrt{x+y}} = \dfrac{\tan\sqrt{x+y}}{2\sqrt{x+y}};$

$\dfrac{\partial f}{\partial y} = \dfrac{1}{\sec\sqrt{x+y}}\sec\sqrt{x+y}\,\tan\sqrt{x+y}\dfrac{1}{2\sqrt{x+y}} = \dfrac{\tan\sqrt{x+y}}{2\sqrt{x+y}}$

21. $\dfrac{\partial f}{\partial x} = yze^{x^2+y^2} + xyze^{x^2+y^2}(2x) = yz(1+2x^2)e^{x^2+y^2}$

22. $\dfrac{\partial f}{\partial z} = \dfrac{1}{1+\dfrac{1}{(x^2+z^2)^2}}\dfrac{-1}{(x^2+z^2)^2}(2z) = \dfrac{-2z}{(x^2+z^2)^2+1}$

23. Since $\dfrac{\partial f}{\partial y} = x(x^2+y^2+z^2)^{1/3} + (xy/3)(x^2+y^2+z^2)^{-2/3}(2y)$, the partial derivative at $(1,1,0)$ is $2^{1/3} + (1/3)2^{-2/3}(2) = 2^{7/3}/3$.

24. Since $\dfrac{\partial f}{\partial x} = \dfrac{-zt}{(x^2+y^2-t^2)^2}(2x)$, we find $\dfrac{\partial f}{\partial x}_{|(1,-1,1,-1)} = \dfrac{-(1)(-1)(2)}{(1+1-1)^2} = 2$

25.
$$\begin{aligned}\frac{\partial f}{\partial t} &= -\frac{2x}{t^3}\sqrt{t^2-y^2} + \frac{xt}{t^2\sqrt{t^2-y^2}} + \frac{x/y}{(t/3)\sqrt{t^2/9-1}}\left(\frac{1}{3}\right)\\ &= \frac{-2x(t^2-y^2)+xt^2}{t^3\sqrt{t^2-y^2}} + \frac{3x}{ty\sqrt{t^2-9}} = \frac{x(2y^2-t^2)}{t^3\sqrt{t^2-y^2}} + \frac{3x}{ty\sqrt{t^2-9}}\end{aligned}$$

26. $\dfrac{\partial f}{\partial x} = \dfrac{-1}{1+(1+x+y+z)^2}$

27. Since $xyt = 6$, the function and its derivatives are not defined at $(1,2,3)$.

28. $\dfrac{\partial f}{\partial x} = \dfrac{3x^2}{y} + \sin(yz/x) + x\cos(yz/x)\left(-\dfrac{yz}{x^2}\right) = \dfrac{3x^2}{y} + \sin(yz/x) - \dfrac{yz}{x}\cos(yz/x)$

29. The derivative is 0.

30.
$$\begin{aligned}\frac{\partial f}{\partial z} &= z\mathrm{Sin}^{-1}\left(\frac{x}{z}\right) + \frac{z^2}{2}\frac{1}{\sqrt{1-(x/z)^2}}\left(-\frac{x}{z^2}\right) + \frac{x}{2}\left(\frac{1}{2}\right)(z^2-x^2)^{-1/2}(2z)\\ &= z\mathrm{Sin}^{-1}\left(\frac{x}{z}\right) - \frac{x}{2}\frac{|z|}{\sqrt{z^2-x^2}} + \frac{xz}{2\sqrt{z^2-x^2}}\\ &= \begin{cases} z\mathrm{Sin}^{-1}(x/z), & z>0\\ z\mathrm{Sin}^{-1}(x/z) + xz/\sqrt{z^2-x^2}, & z<0\end{cases}\end{aligned}$$

31. $$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=x\left[\frac{(x-y)(3x^2y)-x^3y}{(x-y)^2}\right]+y\left[\frac{(x-y)(x^3)-x^3y(-1)}{(x-y)^2}\right]$$
$$=\frac{3x^4y-3x^3y^2-x^4y+x^4y-x^3y^2+x^3y^2}{(x-y)^2}=\frac{3x^4y-3x^3y^2}{(x-y)^2}=\frac{3x^3y(x-y)}{(x-y)^2}=3f(x,y)$$

32. Since $f(x,y,z)=\dfrac{x^3}{yz}+\dfrac{y^3}{xz}+\dfrac{z^3}{xy}$,
$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=x\left(\frac{3x^2}{yz}-\frac{y^3}{x^2z}-\frac{z^3}{x^2y}\right)+y\left(-\frac{x^3}{y^2z}+\frac{3y^2}{xz}-\frac{z^3}{xy^2}\right)+z\left(-\frac{x^3}{yz^2}-\frac{y^3}{xz^2}+\frac{3z^2}{xy}\right)$$
$$=\frac{x^3}{yz}+\frac{y^3}{xz}+\frac{z^3}{xy}=f(x,y,z).$$

33. $$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=x\left[2x\cos\left(\frac{y+z}{x}\right)-(x^2+y^2)\left(-\frac{y+z}{x^2}\right)\sin\left(\frac{y+z}{x}\right)\right]$$
$$+y\left[2y\cos\left(\frac{y+z}{x}\right)-(x^2+y^2)\left(\frac{1}{x}\right)\sin\left(\frac{y+z}{x}\right)\right]$$
$$+z\left[-(x^2+y^2)\left(\frac{1}{x}\right)\sin\left(\frac{y+z}{x}\right)\right]$$
$$=(2x^2+2y^2)\cos\left(\frac{y+z}{x}\right)+\left[\frac{(x^2+y^2)(y+z)}{x}-\frac{y(x^2+y^2)}{x}-\frac{z(x^2+y^2)}{x}\right]\sin\left(\frac{y+z}{x}\right)$$
$$=2f(x,y,z)$$

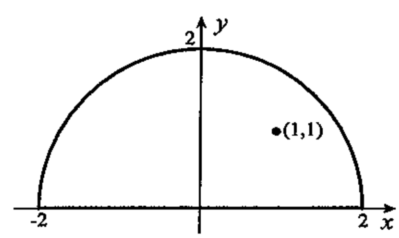
34. (a) This is the normal way to calculate the derivative.

(b) This would lead to an answer of zero. We must always differentiate with respect to a variable and **then** set that variable equal to its prescribed value.

(c) This is acceptable since variables other than the one with respect to which differentiation is being performed can be specified either before or after differentiation.

(d) Same as (b)

35. (a) $T_x(1,1)=(32x-24y)_{|(1,1)}=8$
(b) $T_y(1,1)=(-24x+80y)_{|(1,1)}=56$
(c) $T_x(1,0)=(32x-24y)_{|(1,0)}=32$
(d) $T_y(1,0)$ is not defined
(e) $T_x(0,2)$ is not defined
(f) $T_y(0,2)$ is not defined



36. The derivative vanishes if
$$0=\frac{\partial F}{\partial x}=-\frac{2AE}{L}\left(\frac{L}{\sqrt{L^2-2hx+x^2}}-1\right)+2AE\left(\frac{h-x}{L}\right)\left[\frac{-(L/2)(-2h+2x)}{(L^2-2hx+x^2)^{3/2}}\right]$$
$$=\frac{2AE}{L}\left[\frac{-L}{\sqrt{L^2-2hx+x^2}}+1+\frac{L(h-x)^2}{(L^2-2hx+x^2)^{3/2}}\right]$$
$$=\frac{2AE}{L(L^2-2hx+x^2)^{3/2}}\left[-L(L^2-2hx+x^2)+(L^2-2hx+x^2)^{3/2}+L(h-x)^2\right].$$

This implies that
$$(L^2-2hx+x^2)^{3/2}=L(L^2-2hx+x^2)-L(h^2-2hx+x^2)=L(L^2-h^2).$$

Consequently,
$$x^2-2hx+L^2=(L^3-Lh^2)^{2/3}\quad\Longrightarrow\quad x^2-2hx+L^2-(L^3-Lh^2)^{2/3}=0.$$

Solutions of this quadratic are

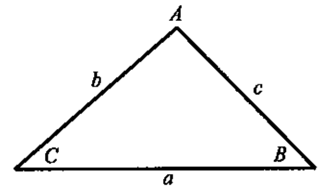
$$x = \frac{2h \pm \sqrt{4h^2 - 4L^2 + 4(L^3 - Lh^2)^{2/3}}}{2} = h \pm (L^3 - Lh^2)^{1/3}\sqrt{1 - \frac{L^2 - h^2}{(L^3 - Lh^2)^{2/3}}}$$
$$= h \pm (L^3 - Lh^2)^{1/3}\sqrt{1 - \frac{L^2 - h^2}{L^{2/3}(L^2 - h^2)^{2/3}}} = h \pm (L^3 - Lh^2)^{1/3}\sqrt{1 - \frac{(L^2 - h^2)^{1/3}}{L^{2/3}}}$$
$$= h \pm (L^3 - Lh^2)^{1/3}\sqrt{1 - \left(1 - \frac{h^2}{L^2}\right)^{1/3}}.$$

Choosing the negative sign gives the required solution.

37. For $f_x(x,y) = 2x - 3y$, the function must be of the form $f(x,y) = x^2 - 3xy + g(y)$ for some function $g(y)$. If we substitute this into the second condition, $-3x + g'(y) = 3x + 4y$ or $g'(y) = 4y + 6x$. This is contradictory, stating that a function of y depends on x.

38. (a) From the cosine law $a = \sqrt{b^2 + c^2 - 2bc\cos A}$,

$$a_A(b,c,A) = \frac{bc\sin A}{\sqrt{b^2 + c^2 - 2bc\cos A}}.$$



(b) From the cosine law in part (a),

$A = \mathrm{Cos}^{-1}\left(\dfrac{b^2 + c^2 - a^2}{2bc}\right)$, and therefore,

$$A_a(a,b,c) = \frac{-1}{\sqrt{1 - (b^2 + c^2 - a^2)^2/(4b^2c^2)}}\left(-\frac{a}{bc}\right)$$
$$= \frac{2a}{\sqrt{4b^2c^2 - (b^2 + c^2 - a^2)^2}}$$

(c) From the cosine law in part (a), $a_b(b,c,A) = \dfrac{b - c\cos A}{\sqrt{b^2 + c^2 - 2bc\cos A}}$.

(d) From the function in part (b),

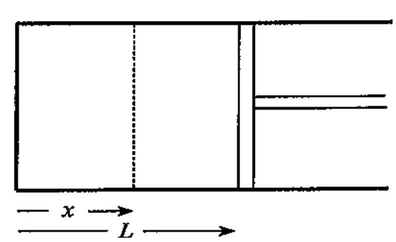
$$A_b(a,b,c) = \frac{-1}{\sqrt{1 - (b^2 + c^2 - a^2)^2/(4b^2c^2)}}\left[\frac{2bc(2b) - (b^2 + c^2 - a^2)(2c)}{4b^2c^2}\right]$$
$$= \frac{-2bc}{\sqrt{4b^2c^2 - (b^2 + c^2 - a^2)^2}}\left[\frac{2c(b^2 + a^2 - c^2)}{4b^2c^2}\right] = \frac{c^2 - a^2 - b^2}{b\sqrt{4b^2c^2 - (b^2 + c^2 - a^2)^2}}$$

39. (a) $\dfrac{\partial\rho}{\partial t} + \dfrac{\partial}{\partial x}(\rho u) + \dfrac{\partial}{\partial y}(\rho v) + \dfrac{\partial}{\partial z}(\rho w) = 0 + \rho(4xt - yt) + \rho(-4xt + 2yt) + \rho(-yt) = 0$

(b)
$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = z + \frac{\partial}{\partial x}(x^3y^2 + xyt + x^2yzt + zt^2)$$
$$+ \frac{\partial}{\partial y}(xy^3z - 2xyt^2 + y^2z^2t - 2zt^3) + \frac{\partial}{\partial z}(5x^2y + 2xyz + 5xzt + 2z^2t)$$
$$= z + (3x^2y^2 + yt + 2xyzt) + (3xy^2z - 2xt^2 + 2yz^2t) + (2xy + 5xt + 4zt) \neq 0$$

40. Let x be the distance from the end of the cylinder to any cross section in the cylinder and let L be the distance from the end of the cylinder to the piston (figure to the right). The x-component (the only component) of the velocity of gas at position x in the cylinder is

$$u = \frac{12x}{L} = \frac{12x}{0.15 + 12t}.$$



Since $\rho = \rho(t)$, the equation of continuity gives

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial \rho}{\partial t} = 0 \quad \Longrightarrow \quad \frac{d\rho}{dt} = -\rho\frac{\partial u}{\partial x} = -\frac{12\rho}{0.15+12t} \quad \Longrightarrow \quad \frac{1}{\rho}d\rho = -\frac{12}{0.15+12t}dt.$$

This is a separated differential equation with solutions defined implicitly by

$$\ln \rho = -\ln(0.15+12t) + C \quad \Longrightarrow \quad \rho = \frac{D}{0.15+12t}, \quad (D = e^C).$$

Since $\rho(0) = 18$, it follows that $18 = D/0.15 \Longrightarrow D = 2.7$. Consequently, $\rho(t) = \dfrac{2.7}{0.15+12t}$ kg/m^3.

41. (a) $\dfrac{\partial u}{\partial x} = -3y^2 + 3x^2 \qquad \dfrac{\partial u}{\partial y} = -6xy + 1 \qquad \dfrac{\partial v}{\partial x} = 6xy - 1 \qquad \dfrac{\partial v}{\partial y} = 3x^2 - 3y^2$

(b) $$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)(2x+1) - (x^2+x+y^2)(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x^2+y^2)(2y) - (x^2+x+y^2)(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial y} = \frac{(x^2+y^2)(-1) + y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

(c) $$\frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x\cos y = e^x[(x+1)\cos y - y\sin y]$$

$$\frac{\partial u}{\partial y} = e^x(-x\sin y - \sin y - y\cos y) = -e^x[(x+1)\sin y + y\cos y]$$

$$\frac{\partial v}{\partial x} = e^x(x\sin y + y\cos y) + e^x\sin y = e^x[(x+1)\sin y + y\cos y]$$

$$\frac{\partial v}{\partial y} = e^x(x\cos y + \cos y - y\sin y) = e^x[(x+1)\cos y - y\sin y]$$

42. (a) $$\frac{\partial u}{\partial r} = \frac{(1+r^2+2r\cos\theta)(2r+\cos\theta) - (r^2+r\cos\theta)(2r+2\cos\theta)}{(1+r^2+2r\cos\theta)^2} = \frac{2r+r^2\cos\theta+\cos\theta}{(1+r^2+2r\cos\theta)^2};$$

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r}\left[\frac{(1+r^2+2r\cos\theta)(r\cos\theta) - r\sin\theta(-2r\sin\theta)}{(1+r^2+2r\cos\theta)^2}\right] = \frac{2r+r^2\cos\theta+\cos\theta}{(1+r^2+2r\cos\theta)^2};$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{1}{r}\left[\frac{(1+r^2+2r\cos\theta)(-r\sin\theta) - (r^2+r\cos\theta)(-2r\sin\theta)}{(1+r^2+2r\cos\theta)^2}\right] = \frac{(r^2-1)\sin\theta}{(1+r^2+2r\cos\theta)^2};$$

$$\frac{\partial v}{\partial r} = \frac{(1+r^2+2r\cos\theta)(\sin\theta) - r\sin\theta(2r+2\cos\theta)}{(1+r^2+2r\cos\theta)^2} = \frac{(1-r^2)\sin\theta}{(1+r^2+2r\cos\theta)^2}$$

(b) $$\frac{\partial u}{\partial r} = \frac{1}{2\sqrt{r}}\cos(\theta/2), \qquad \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{\sqrt{r}}\cos(\theta/2)\left(\frac{1}{2}\right),$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{1}{\sqrt{r}}\sin(\theta/2)\left(\frac{1}{2}\right), \qquad \frac{\partial v}{\partial r} = \frac{1}{2\sqrt{r}}\sin(\theta/2)$$

(c) $$\frac{\partial u}{\partial r} = \frac{1}{r}, \quad \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r}, \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = 0, \quad \frac{\partial v}{\partial r} = 0$$

EXERCISES 12.4

1. $\nabla f = (2xy + z)\hat{\mathbf{i}} + (x^2 + z^2)\hat{\mathbf{j}} + (x + 2yz)\hat{\mathbf{k}}$ **2.** $\nabla f = 2xyz\hat{\mathbf{i}} + x^2z\hat{\mathbf{j}} + x^2y\hat{\mathbf{k}}$

3. $\nabla f = (2xy/z - 2z^6)\hat{\mathbf{i}} + (x^2/z)\hat{\mathbf{j}} - (x^2y/z^2 + 12xz^5)\hat{\mathbf{k}}$

4. $\nabla f = (2xy + y^2)\hat{\mathbf{i}} + (x^2 + 2xy)\hat{\mathbf{j}}$ **5.** $\nabla f = \cos(x+y)\,\hat{\mathbf{i}} + \cos(x+y)\,\hat{\mathbf{j}}$

6. $\nabla f = \dfrac{yz}{1+(xyz)^2}\hat{\mathbf{i}} + \dfrac{xz}{1+(xyz)^2}\hat{\mathbf{j}} + \dfrac{xy}{1+(xyz)^2}\hat{\mathbf{k}} = \dfrac{1}{1+(xyz)^2}(yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}})$

7. $\nabla f = \dfrac{-y/x^2}{1+y^2/x^2}\hat{\mathbf{i}} + \dfrac{1/x}{1+y^2/x^2}\hat{\mathbf{j}} = \dfrac{-y\hat{\mathbf{i}}+x\hat{\mathbf{j}}}{x^2+y^2}$

8. $\nabla f = e^{x+y+z}\hat{\mathbf{i}} + e^{x+y+z}\hat{\mathbf{j}} + e^{x+y+z}\hat{\mathbf{k}} = e^{x+y+z}(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})$

9. $\nabla f = \dfrac{-2x}{(x^2+y^2)^2}\hat{\mathbf{i}} - \dfrac{2y}{(x^2+y^2)^2}\hat{\mathbf{j}} = -\dfrac{2(x\hat{\mathbf{i}}+y\hat{\mathbf{j}})}{(x^2+y^2)^2}$

10. $\nabla f = \dfrac{-x}{(x^2+y^2+z^2)^{3/2}}\hat{\mathbf{i}} + \dfrac{-y}{(x^2+y^2+z^2)^{3/2}}\hat{\mathbf{j}} + \dfrac{-z}{(x^2+y^2+z^2)^{3/2}}\hat{\mathbf{k}} = -\dfrac{x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}}{(x^2+y^2+z^2)^{3/2}}$

11. Since $\nabla f = (y+1)\hat{\mathbf{i}} + (x+1)\hat{\mathbf{j}}$, the gradient at $(1,3)$ is $4\hat{\mathbf{i}}+2\hat{\mathbf{j}}$.

12. Since $\nabla f = -\sin(x+y+z)(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})$, the gradient at the point $(-1,1,1)$ is $= -(\sin 1)(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})$.

13. Since $\nabla f = 2(x^2+y^2+z^2)(2x\hat{\mathbf{i}}+2y\hat{\mathbf{j}}+2z\hat{\mathbf{k}})$, the gradient at $(0,3,6)$ is $4(45)(3\hat{\mathbf{j}}+6\hat{\mathbf{k}}) = 540(\hat{\mathbf{j}}+2\hat{\mathbf{k}})$.

14. $\nabla f_{|(2,2)} = e^{-x^2-y^2}(-2x\hat{\mathbf{i}}-2y\hat{\mathbf{j}})_{|(2,2)} = -2e^{-8}(2\hat{\mathbf{i}}+2\hat{\mathbf{j}}) = -4e^{-8}(\hat{\mathbf{i}}+\hat{\mathbf{j}})$

15. Since $\nabla f = \left[y\ln(x+y) + \dfrac{xy}{x+y}\right]\hat{\mathbf{i}} + \left[x\ln(x+y) + \dfrac{xy}{x+y}\right]\hat{\mathbf{j}}$, the gradient at $(4,-2)$ is $(-2\ln 2 - 4)\hat{\mathbf{i}} + (4\ln 2 - 4)\hat{\mathbf{j}} = -2(2+\ln 2)\hat{\mathbf{i}} + 4(\ln 2 - 1)\hat{\mathbf{j}}$.

16. $\nabla F = A\hat{\mathbf{i}} + B\hat{\mathbf{j}} + C\hat{\mathbf{k}}$ But according to Theorem 11.4, this vector is perpendicular to the plane.

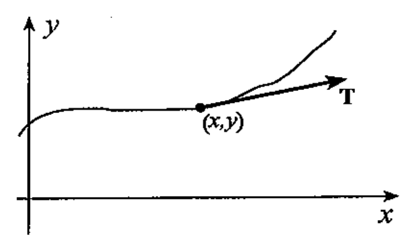
17. Since $\nabla F = (2,3,-2)$ is perpendicular to the plane $2x+3y-2z+4=0$ and $\nabla G = (1,-1,3)$ is perpendicular to the plane $x-y+3z+6=0$, a vector along the line of intersection of the planes is $\nabla F \times \nabla G = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 3 & -2 \\ 1 & -1 & 3 \end{vmatrix} = (7,-8,-5)$. Since $(-22/5, 8/5, 0)$ is a point on the line, parametric equations for the line are $x = -22/5 + 7t$, $y = 8/5 - 8t$, $z = -5t$.

18. $$\nabla(fg) = \frac{\partial}{\partial x}(fg)\hat{\mathbf{i}} + \frac{\partial}{\partial y}(fg)\hat{\mathbf{j}} + \frac{\partial}{\partial z}(fg)\hat{\mathbf{k}} = \left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right)\hat{\mathbf{i}} + \left(f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right)\hat{\mathbf{j}} + \left(f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\right)\hat{\mathbf{k}}$$
$$= f\left(\frac{\partial g}{\partial x}\hat{\mathbf{i}} + \frac{\partial g}{\partial y}\hat{\mathbf{j}} + \frac{\partial g}{\partial z}\hat{\mathbf{k}}\right) + g\left(\frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}\right) = f\,\nabla g + g\,\nabla f$$
It looks like the product rule for differentiation.

19. (a) If we set $y = x$ in the quotient $\dfrac{f(x,y)-f(0,0)}{\sqrt{x^2+y^2}} = \dfrac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}$, and take the limit as $x \to 0^+$, we obtain $\lim\limits_{x\to 0^+}\sqrt{x^2+x^2} = 0$.

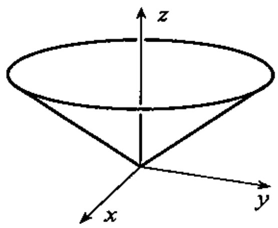
(b) If we set $y = x$ in the quotient $\dfrac{f(x,y)-f(0,0)}{\sqrt{x^2+y^2}} = \dfrac{2x^3-3y}{\sqrt{x^2+y^2}}$, and take the limit as $x \to 0^+$, we obtain $\lim\limits_{x\to 0^+}\dfrac{2x^3-3x}{\sqrt{x^2+x^2}} = \lim\limits_{x\to 0^+}\dfrac{2x^2-3}{\sqrt{2}} = -\dfrac{3}{\sqrt{2}}$.

20. The slope dy/dx of the tangent line at any point (x,y) on this curve is defined by
$$3x^2 + y + x\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 0,$$
or,
$$\frac{dy}{dx} = -\frac{3x^2+y}{x+4y^3}.$$

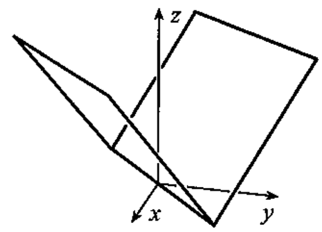


It follows that a vector along the tangent line at (x,y) is $\mathbf{T} = (x+4y^3, -3x^2-y)$. A vector perpendicular to $\mathbf{T}$ is $\mathbf{N} = (3x^2+y, x+4y^3)$. But $\nabla F = (3x^2+y)\hat{\mathbf{i}} + (x+4y^3)\hat{\mathbf{j}}$, and therefore $\nabla F = \mathbf{N}$.

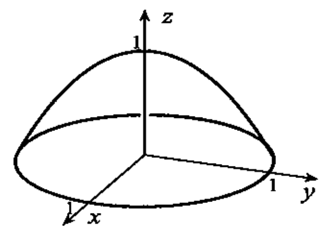
21. The gradient
$$\nabla F = \frac{-x}{\sqrt{x^2+y^2}}\hat{\mathbf{i}} - \frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{j}} + \hat{\mathbf{k}},$$
is undefined at the vertex of the cone.



22. Since $|x-y|$ fails to have derivatives with respect to x and y when $y = x$, it follows that ∇F is not defined at the points $(x, x, 0)$.



23. $\nabla f = -2x\hat{\mathbf{i}} - 2y\hat{\mathbf{j}}$ The gradient is equal to $\mathbf{0}$ at the point $(0, 0)$. This is the highest point on the surface $z = 1 - x^2 - y^2$.



24. If $\nabla f = (2xy - y)\hat{\mathbf{i}} + (x^2 - x)\hat{\mathbf{j}}$, then $\dfrac{\partial f}{\partial x} = 2xy - y$ and $\dfrac{\partial f}{\partial y} = x^2 - x$. From the first equation, we can say that $f(x, y) = x^2y - xy + \phi(y)$, where $\phi(y)$ is any differentiable function of y. To determine $\phi(y)$ we substitute this expression for $f(x, y)$ into the second equation, $x^2 - x + d\phi/dy = x^2 - x$. Consequently, $d\phi/dy = 0$, and therefore $\phi(y) = C$, a constant. Thus, $f(x, y) = x^2y - xy + C$.

25. If $\nabla f = (2x/y + 1)\hat{\mathbf{i}} + (-x^2/y^2 + 2)\hat{\mathbf{j}}$, then $\dfrac{\partial f}{\partial x} = 2x/y + 1$ and $\dfrac{\partial f}{\partial y} = -x^2/y^2 + 2$. From the first equation, we can say that $f(x, y) = x^2/y + x + \phi(y)$, where $\phi(y)$ is any differentiable function of y. To determine $\phi(y)$ we substitute this expression for $f(x, y)$ into the second equation, $-x^2/y^2 + d\phi/dy = -x^2/y^2 + 2$. Consequently, $d\phi/dy = 2$, and therefore $\phi(y) = 2y + C$, a constant. Thus, $f(x, y) = x^2/y + x + 2y + C$.

26. If $\nabla f = yz\hat{\mathbf{i}} + (xz + 2yz)\hat{\mathbf{j}} + (xy + y^2)\hat{\mathbf{k}}$, then $\dfrac{\partial f}{\partial x} = yz$, $\dfrac{\partial f}{\partial y} = xz + 2yz$, $\dfrac{\partial f}{\partial z} = xy + y^2$. From the first, $f(x, y, z) = xyz + \phi(y, z)$, where $\phi(y, z)$ is any function with first partial derivatives. Substitution of this into the second equation gives
$$xz + \frac{\partial\phi}{\partial y} = xz + 2yz \quad\Longrightarrow\quad \frac{\partial\phi}{\partial y} = 2yz.$$
Consequently, $\phi(y, z) = y^2z + \psi(z)$, and $f(x, y, z) = xyz + y^2z + \psi(z)$. Substitution of this into the third equation requires $\psi(z)$ to satisfy
$$xy + y^2 + \frac{d\psi}{dz} = xy + y^2 \quad\Longrightarrow\quad \frac{d\psi}{dz} = 0.$$
Thus, $\psi(z) = C$, and $f(x, y, z) = xyz + y^2z + C$.

27. If $\nabla f = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})/\sqrt{x^2+y^2+z^2}$, then
$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}}, \quad \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}, \quad \frac{\partial f}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}.$$
From the first, $f(x, y, z) = \sqrt{x^2+y^2+z^2} + \phi(y, z)$, where $\phi(y, z)$ is any function with first partial derivatives. Substitution of this into the second equation gives
$$\frac{y}{\sqrt{x^2+y^2+z^2}} + \frac{\partial\phi}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}} \quad\Longrightarrow\quad \frac{\partial\phi}{\partial y} = 0.$$

Consequently, $\phi(y,z)=\psi(z)$, and $f(x,y,z)=\sqrt{x^2+y^2+z^2}+\psi(z)$. Substitution of this into the third equation requires $\psi(z)$ to satisfy

$$\frac{z}{\sqrt{x^2+y^2+z^2}}+\frac{d\psi}{dz}=\frac{z}{\sqrt{x^2+y^2+z^2}} \quad \Longrightarrow \quad \frac{d\psi}{dz}=0.$$

Thus, $\psi(z)=C$, and $f(x,y,z)=\sqrt{x^2+y^2+z^2}+C$.

28. If $\nabla f=\nabla g$, then $\dfrac{\partial f}{\partial x}=\dfrac{\partial g}{\partial x}$ and $\dfrac{\partial f}{\partial y}=\dfrac{\partial g}{\partial y}$. The first requires $f(x,y)=g(x,y)+\phi(y)$, which substituted into the second gives

$$\frac{\partial g}{\partial y}+\frac{d\phi}{dy}=\frac{\partial g}{\partial y} \qquad \text{or} \qquad \frac{d\phi}{dy}=0.$$

Thus, $\phi(y)=C$, a constant, and $f(x,y)=g(x,y)+C$.

29. If $\nabla f=\mathbf{0}$, then $\dfrac{\partial f}{\partial x}=0$, $\dfrac{\partial f}{\partial y}=0$, $\dfrac{\partial f}{\partial z}=0$. From the first, $f(x,y,z)=\phi(y,z)$, where $\phi(y,z)$ is any function with first partial derivatives. Substitution of this function into the second equation gives $\dfrac{\partial \phi}{\partial y}=0 \Longrightarrow \phi(y,z)=\psi(z)$. Substitution of this into the third equation requires $\psi(z)$ to satisfy $d\psi/dz=0$, from which $\psi(z)=C$, and $f(x,y,z)=C$.

30. To find the slope dy/dx of the tangent line to C at any point (x,y), we implicitly differentiate $F(x,y)=0$ with respect to x. This can be accomplished by differentiating $F(x,y)$ partially with respect to x, and adding to this the partial derivative with respect to y multiplied by dy/dx,

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0.$$

Thus, $\dfrac{dy}{dx}=-\dfrac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}}$, and it follows that a tangent vector to C at (x,y) is $\left(\dfrac{\partial F}{\partial y},-\dfrac{\partial F}{\partial x}\right)$. A vector perpendicular to C must therefore be $\left(\dfrac{\partial F}{\partial x},\dfrac{\partial F}{\partial y}\right)$. Since this is ∇F, the proof is complete.

EXERCISES 12.5

1. From $\dfrac{\partial f}{\partial x}=2xy^2-6x^2y$, we obtain $\dfrac{\partial^2 f}{\partial x^2}=2y^2-12xy$.

2. From $\dfrac{\partial f}{\partial y}=-\dfrac{2x}{y^2}+12x^3y^3$ and $\dfrac{\partial^2 f}{\partial y^2}=\dfrac{4x}{y^3}+36x^3y^2$, we obtain $\dfrac{\partial^3 f}{\partial y^3}=-\dfrac{12x}{y^4}+72x^3y$.

3. From $\dfrac{\partial f}{\partial z}=xy\cos(xyz)$, we obtain $\dfrac{\partial^2 f}{\partial z^2}=-x^2y^2\sin(xyz)$.

4. Since $\dfrac{\partial f}{\partial z}=xye^{x+y+z}+xyze^{x+y+z}=xy(1+z)e^{x+y+z}$,

$$\frac{\partial^2 f}{\partial y\partial z}=x(1+z)e^{x+y+z}+xy(1+z)e^{x+y+z}=x(1+z)(1+y)e^{x+y+z}.$$

5. From $\dfrac{\partial f}{\partial x}=\dfrac{x}{\sqrt{x^2+y^2}}$, we obtain $\dfrac{\partial^2 f}{\partial y\partial x}=\dfrac{-xy}{(x^2+y^2)^{3/2}}$.

6. Since $\dfrac{\partial f}{\partial y}=e^{x+y}+\dfrac{2x^2}{y^3}$, we find $\dfrac{\partial^2 f}{\partial x\partial y}=e^{x+y}+\dfrac{4x}{y^3}$, and $\dfrac{\partial^3 f}{\partial x^2\partial y}=e^{x+y}+\dfrac{4}{y^3}$.

7. From $\dfrac{\partial f}{\partial y}=9x^3y^2+\dfrac{3x}{y^2}$, and $\dfrac{\partial^2 f}{\partial y^2}=18x^3y-\dfrac{6x}{y^3}$, we obtain $\dfrac{\partial^3 f}{\partial y^3}=18x^3+\dfrac{18x}{y^4}$. At $(1,3)$, the derivative is $164/9$.

8. Since $\dfrac{\partial f}{\partial z} = 2x^2z + 2y^2z$, we have $\dfrac{\partial^2 f}{\partial y \partial z} = 4yz$, and $\dfrac{\partial^3 f}{\partial x \partial y \partial z} = 0$. This must also be its value at $(1, 0, -1)$.

9. From $\dfrac{\partial f}{\partial x} = \dfrac{-x}{\sqrt{1-x^2-y^2}}$, we obtain $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{-1}{\sqrt{1-x^2-y^2}} - \dfrac{x^2}{(1-x^2-y^2)^{3/2}}$, and this simplifies to $(y^2-1)/(1-x^2-y^2)^{3/2}$.

10. Since $\dfrac{\partial f}{\partial z} = \dfrac{1}{\sqrt{x^2+y^2+z^2}} \dfrac{z}{\sqrt{x^2+y^2+z^2}} = \dfrac{z}{x^2+y^2+z^2}$,

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{x^2+y^2+z^2} - \frac{2z^2}{(x^2+y^2+z^2)^2} = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}.$$

11. From $\dfrac{\partial f}{\partial y} = x^2e^y + 2ye^x$, we obtain $\dfrac{\partial^2 f}{\partial x \partial y} = 2xe^y + 2ye^x$ and $\dfrac{\partial^3 f}{\partial x^2 \partial y} = 2e^y + 2ye^x$.

12. From $\dfrac{\partial f}{\partial x} = \dfrac{1}{1+(y/x)^2}\left(\dfrac{-y}{x^2}\right) = \dfrac{x^2}{x^2+y^2}\left(\dfrac{-y}{x^2}\right) = \dfrac{-y}{x^2+y^2}$, we obtain $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{2xy}{(x^2+y^2)^2}$.

13. From $\dfrac{\partial f}{\partial y} = -2y \csc^2(x^2+y^2+z^2)$, we obtain

$$\begin{aligned}
\frac{\partial^2 f}{\partial y^2} &= -2\csc^2(x^2+y^2+z^2) + 8y^2\csc^2(x^2+y^2+z^2)\cot(x^2+y^2+z^2),\\
\frac{\partial^2 f}{\partial x \partial^2 y} &= 8x\csc^2(x^2+y^2+z^2)\cot(x^2+y^2+z^2) - 32xy^2\csc^2(x^2+y^2+z^2)\cot^2(x^2+y^2+z^2)\\
&\quad - 16xy^2\csc^4(x^2+y^2+z^2).
\end{aligned}$$

14. From $\dfrac{\partial f}{\partial y} = \dfrac{1}{\sqrt{1-(x^2+y^2)^{-2}}}\dfrac{-2y}{(x^2+y^2)^2} = \dfrac{x^2+y^2}{\sqrt{(x^2+y^2)^2-1}}\dfrac{-2y}{(x^2+y^2)^2} = \dfrac{-2y}{(x^2+y^2)\sqrt{(x^2+y^2)^2-1}}$,
we obtain $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{4xy}{(x^2+y^2)^2\sqrt{(x^2+y^2)^2-1}} + \dfrac{y(2)(x^2+y^2)(2x)}{(x^2+y^2)[(x^2+y^2)^2-1]^{3/2}}$.
Thus, $\dfrac{\partial^2 f}{\partial x \partial y}_{|(-2,-2)} = \dfrac{16}{8^2\sqrt{8^2-1}} + \dfrac{(-2)(2)(8)(-4)}{8[8^2-1]^{3/2}} = \dfrac{127}{756\sqrt{7}}$.

15. Three derivatives with respect to y eliminates the first term, and then derivatives with respect to x eliminate the second term; that is, the derivative is 0.

16. $\dfrac{\partial f}{\partial x} = 8x^7y^9z^{10}$; $\quad \dfrac{\partial^2 f}{\partial x^2} = 56x^6y^9z^{10}$; $\quad \cdots$; $\quad \dfrac{\partial^8 f}{\partial x^8} = 8!y^9z^{10}$

17. The derivative is 0.

18. $\dfrac{\partial f}{\partial x} = -\sin(x+y^3)$; $\dfrac{\partial^2 f}{\partial x^2} = -\cos(x+y^3)$; $\dfrac{\partial^3 f}{\partial x^3} = \sin(x+y^3)$; $\dfrac{\partial^4 f}{\partial x^3 \partial y} = 3y^2\cos(x+y^3)$

19. From $\dfrac{\partial f}{\partial t} = \dfrac{-t}{\sqrt{x^2+y^2+z^2-t^2}}$, we obtain $\dfrac{\partial^2 f}{\partial z \partial t} = \dfrac{zt}{(x^2+y^2+z^2-t^2)^{3/2}}$, and

$$\frac{\partial^3 f}{\partial y \partial z \partial t} = \frac{-3yzt}{(x^2+y^2+z^2-t^2)^{5/2}}, \qquad \frac{\partial^4 f}{\partial x \partial y \partial z \partial t} = \frac{15xyzt}{(x^2+y^2+z^2-t^2)^{7/2}}.$$

20. Since $\dfrac{\partial f}{\partial y} = \dfrac{x}{xy\sqrt{x^2y^2-1}} = \dfrac{1}{y\sqrt{x^2y^2-1}}$, we find $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{-xy^2}{y(x^2y^2-1)^{3/2}} = \dfrac{-xy}{(x^2y^2-1)^{3/2}}$.

21.
$$\begin{aligned}
x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} &= x\left[2x + y + y\cos\left(\frac{x}{y}\right)\right] + y\left[x + 2y\sin\left(\frac{x}{y}\right) - x\cos\left(\frac{x}{y}\right)\right]\\
&= 2\left[x^2 + xy + y^2\sin\left(\frac{x}{y}\right)\right] = 2f(x,y)
\end{aligned}$$

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=x^2\left[2-\sin\left(\frac{x}{y}\right)\right]+2xy\left[1+\cos\left(\frac{x}{y}\right)+\frac{x}{y}\sin\left(\frac{x}{y}\right)\right]$$
$$+y^2\left[2\sin\left(\frac{x}{y}\right)-\frac{2x}{y}\cos\left(\frac{x}{y}\right)-\frac{x^2}{y^2}\sin\left(\frac{x}{y}\right)\right]$$
$$=2x^2+2xy+(-x^2+2x^2+2y^2-x^2)\sin\left(\frac{x}{y}\right)+(2xy-2xy)\cos\left(\frac{x}{y}\right)$$
$$=2\left[x^2+xy+y^2\sin\left(\frac{x}{y}\right)\right]=2f(x,y)$$

22. $\dfrac{\partial u}{\partial x}=1+ze^{y/x}(-y/x^2)=1-(yz/x^2)e^{y/x};\quad \dfrac{\partial u}{\partial y}=1+ze^{y/x}(1/x)=1+(z/x)e^{y/x};$

$\dfrac{\partial u}{\partial z}=e^{y/x};\quad \dfrac{\partial^2 u}{\partial x^2}=2(yz/x^3)e^{y/x}-(yz/x^2)e^{y/x}(-y/x^2)=[yz(2x+y)/x^4]e^{y/x};$

$\dfrac{\partial^2 u}{\partial x\partial y}=-(z/x^2)e^{y/x}-(yz/x^2)e^{y/x}(1/x)=-[z(x+y)/x^3]e^{y/x};$

$\dfrac{\partial^2 u}{\partial y^2}=(z/x)e^{y/x}(1/x)=(z/x^2)e^{y/x};\quad \dfrac{\partial^2 u}{\partial y\partial z}=(1/x)e^{y/x};\quad \dfrac{\partial^2 u}{\partial z^2}=0;$

$\dfrac{\partial^2 u}{\partial x\partial z}=e^{y/x}(-y/x^2)=-(y/x^2)e^{y/x}.$

Thus,

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+z^2\frac{\partial^2 u}{\partial z^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+2yz\frac{\partial^2 u}{\partial y\partial z}+2xz\frac{\partial^2 u}{\partial x\partial z}$$
$$=\frac{yz(2x+y)}{x^2}e^{y/x}+\frac{y^2z}{x^2}e^{y/x}+0-\frac{2yz(x+y)}{x^2}e^{y/x}+\frac{2yz}{x}e^{y/x}-\frac{2yz}{x}e^{y/x}=0.$$

23. $\dfrac{\partial f}{\partial x}=2x+2y;\quad \dfrac{\partial f}{\partial y}=-2y+2x+1;\quad \dfrac{\partial^2 f}{\partial x^2}=2;\quad \dfrac{\partial^2 f}{\partial y^2}=-2$ Since $\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}=0$, and second partial derivatives are continuous, $f(x,y)$ is harmonic in the entire xy-plane.

24. From $\dfrac{\partial f}{\partial x}=\dfrac{2x}{x^2+y^2}$, we obtain $\dfrac{\partial^2 f}{\partial x^2}=\dfrac{(x^2+y^2)(2)-2x(2x)}{(x^2+y^2)^2}=\dfrac{2(y^2-x^2)}{(x^2+y^2)^2}$, and similarly, $\dfrac{\partial^2 f}{\partial y^2}=\dfrac{2(x^2-y^2)}{(x^2+y^2)^2}$. Since $\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}=0$, and second partial derivatives are continuous except at $(0,0)$, the function $f(x,y)$ is harmonic in any region not containing $(0,0)$.

25. $\dfrac{\partial f}{\partial x}=3x^2y^2-3y;\quad \dfrac{\partial f}{\partial y}=2x^3y-3x;\quad \dfrac{\partial^2 f}{\partial x^2}=6xy^2;\quad \dfrac{\partial^2 f}{\partial y^2}=2x^3$ Since $\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}=6xy^2+2x^3$, the function is not harmonic.

26. $\dfrac{\partial f}{\partial x}=6xyz+y;\ \dfrac{\partial^2 f}{\partial x^2}=6yz;\ \dfrac{\partial f}{\partial y}=3x^2z-3y^2z+x;\ \dfrac{\partial^2 f}{\partial y^2}=-6yz;\ \dfrac{\partial f}{\partial z}=3x^2y-y^3;\ \dfrac{\partial^2 f}{\partial z^2}=0$

Since $\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}+\dfrac{\partial^2 f}{\partial z^2}=6yz-6yz+0=0$, and all second partial derivatives are continuous, $f(x,y,z)$ is harmonic in all space.

27. From $\dfrac{\partial f}{\partial x}=\dfrac{-x}{(x^2+y^2+z^2)^{3/2}},\quad \dfrac{\partial^2 f}{\partial x^2}=\dfrac{-1}{(x^2+y^2+z^2)^{3/2}}+\dfrac{3x^2}{(x^2+y^2+z^2)^{5/2}}=\dfrac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}.$ With similar results for second derivatives with respect to y and z,

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=\frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}+\frac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}+\frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}=0.$$

Since second partial derivatives are continuous except at $(0,0,0)$, the function is harmonic in any region not containing $(0,0,0)$.

28. From $\dfrac{\partial f}{\partial x} = 3x^2y^3z^3$, we find $\dfrac{\partial^2 f}{\partial x^2} = 6xy^3z^3$. Similarly, $\dfrac{\partial^2 f}{\partial y^2} = 6x^3yz^2$ and $\dfrac{\partial^2 f}{\partial z^2} = 6x^3y^3z$. Since $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} = 6xyz(y^2z^2 + x^2z^2 + x^2y^2)$, and this is not zero in any region of space, the function is not harmonic.

29. According to Example 12.12, potential at (x,y,z) due to point charges q_i at points (x_i, y_i, z_i) is $V(x,y,z) = \displaystyle\sum_{i=1}^{n} \frac{q_i}{4\pi\epsilon_0 r_i} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i}$, where $r_i = \sqrt{(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2}$. Now,
$\dfrac{\partial}{\partial x}\left(\dfrac{1}{r_i}\right) = \dfrac{-(x-x_i)}{[(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2]^{3/2}}$, so that

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\left(\frac{1}{r_i}\right) &= \frac{-1}{[(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2]^{3/2}} + \frac{3(x-x_i)^2}{[(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2]^{5/2}}\\ &= \frac{2(x-x_i)^2 - (y-y_i)^2 - (z-z_i)^2}{[(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2]^{5/2}}.\end{aligned}$$

With similar results for derivatives of $1/r_i$ with respect to y and z,

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\left(\frac{1}{r_i}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{r_i}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{1}{r_i}\right) &= \frac{2(x-x_i)^2 - (y-y_i)^2 - (z-z_i)^2}{[(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2]^{5/2}}\\ &\quad + \frac{2(y-y_i)^2 - (x-x_i)^2 - (z-z_i)^2}{[(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2]^{5/2}}\\ &\quad + \frac{2(z-z_i)^2 - (x-x_i)^2 - (y-y_i)^2}{[(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2]^{5/2}} = 0.\end{aligned}$$

It follows that each term in the sum representing $V(x,y,z)$ satisfies Laplace's equation, so $V(x,y,z)$ itself does.

30. From $\dfrac{\partial V}{\partial x} = GM\left[\dfrac{-x}{(x^2+y^2+z^2)^{3/2}}\right]$, we obtain

$$\frac{\partial^2 V}{\partial x^2} = GM\left[\frac{-1}{(x^2+y^2+z^2)^{3/2}} + \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}\right] = GM\left[\frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}\right].$$

Similarly, $\dfrac{\partial^2 V}{\partial y^2} = GM\left[\dfrac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}\right]$ and $\dfrac{\partial^2 V}{\partial z^2} = GM\left[\dfrac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}\right]$, and therefore

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad \text{(except at } (0,0,0)\text{)}.$$

31. As long as C and D are constants,

$$\begin{aligned}\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} &= [-9\pi^2 C(e^{3\pi y} - e^{-3\pi y})\sin(3\pi x) - 16\pi^2 D(e^{4\pi y} - e^{-4\pi y})\sin(4\pi x)]\\ &\quad + [9\pi^2 C(e^{3\pi y} - e^{-3\pi y})\sin(3\pi x) + 16\pi^2 D(e^{4\pi y} - e^{-4\pi y})\sin(4\pi x)] = 0.\end{aligned}$$

Since second partial derivatives are continuous, $T(x,y)$ is harmonic in the plate. It is obvious that $T(x,y)$ satisfies $T(0,y) = T(1,y) = T(x,0) = 0$. Finally

$$\begin{aligned}T(x,1) &= C(e^{3\pi} - e^{-3\pi})\sin(3\pi x) + D(e^{4\pi} - e^{-4\pi})\sin(4\pi x)\\ &= \sin(3\pi x) - 2\sin(4\pi x).\end{aligned}$$

32. Suppose the cross-sectional area of the rod is A and its density is ρ. Then $F(x) = \rho g A(L-x)$, and $y(x)$ must satisfy

$$0 = E\frac{d^2y}{dx^2} + \rho g A(L-x) \quad\Longrightarrow\quad \frac{d^2y}{dx^2} = -\frac{\rho g A}{E}(L-x) \quad\Longrightarrow\quad \frac{dy}{dx} = \frac{\rho g A}{2E}(L-x)^2 + C.$$

Since $y'(L) = 0$, it follows that $C = 0$. Thus, $\dfrac{dy}{dx} = \dfrac{\rho g A}{2E}(L-x)^2$, and a second integration gives $y = -\dfrac{\rho g A}{6E}(L-x)^3 + D$. The condition $y(0) = 0$ implies that $D = \dfrac{\rho g A L^3}{6E}$, and therefore displacements of cross sections are given by $y(x) = -\dfrac{\rho g A}{6E}(L-x)^3 + \dfrac{\rho g A L^3}{6E}$. Since $y(L) = \dfrac{\rho g A L^3}{6E}$, it follows that the length of the bar is $L + \dfrac{\rho g A L^3}{6E}$.

33. Two integrations of $Ey'' = 0$ give $y = Ax + B$. The conditions $y(0) = 0$ and $y'(L) = F/E$ imply that $y = Fx/E$. Since $y(L) = FL/E$, the length of the bar is $L + FL/E = L(1 + F/E)$.

34. (a) Since second partial derivatives are $\dfrac{\partial^2 y}{\partial x^2} = -\lambda^2 y$ and $\dfrac{\partial^2 y}{\partial t^2} = -c^2\lambda^2 y$, it follows that $y(x,t)$ does indeed satisfy equation 12.13a.

(b) The condition $y(0,t) = 0$ implies that $0 = B(C\sin c\lambda t + D\cos c\lambda t)$ for all t. This requires $B = 0$. With $B = 0$, condition $y_x(L,t) = 0$ implies that $0 = A\lambda\cos\lambda L(C\sin c\lambda t + D\cos c\lambda t)$. Since A cannot be zero, nor can λ, and the term in brackets cannot be equal to 0 for all t, we must set $\cos\lambda L = 0$. But this implies that $\lambda L = (2n-1)\pi/2$, where n is an integer; that is, $\lambda = (2n-1)\pi/(2L)$.

(c) With $B = 0$ and $g(x) = 0$, initial condition 12.13c requires $0 = y_t(x,0) = A\sin\lambda x(Cc\lambda) \implies C = 0$.

(d) With $B = 0$ and $f(x) = 0$, initial condition 12.13b requires $0 = y(x,0) = A\sin\lambda x(D) \implies D = 0$.

35. (a) Since second partial derivatives are $\dfrac{\partial^2 y}{\partial x^2} = -\lambda^2 y$ and $\dfrac{\partial^2 y}{\partial t^2} = -c^2\lambda^2 y$, it follows that $\dfrac{\partial^2 y}{\partial t^2} = c^2\dfrac{\partial^2 y}{\partial x^2}$.

(b) The condition $y(0,t) = 0$ implies that $0 = B(C\sin c\lambda t + D\cos c\lambda t)$ for all t. This requires $B = 0$. With $B = 0$, condition $y(L,t) = 0$ implies that $0 = A\sin\lambda L(C\sin c\lambda t + D\cos c\lambda t)$. Since $A \neq 0$ and the term in brackets cannot be equal to 0 for all t, we must set $\sin\lambda L = 0$. But this implies that $\lambda L = n\pi$, where n is an integer; that is, $\lambda = n\pi/L$.

(c) With $B = 0$ and $g(x) = 0$, initial condition 12.13c requires $0 = y_t(x,0) = A\sin\lambda x(Cc\lambda) \implies C = 0$.

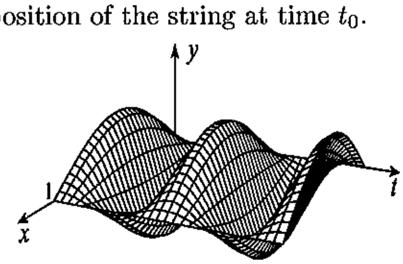
(d) With $B = 0$ and $f(x) = 0$, initial condition 12.13b requires $0 = y(x,0) = A\sin\lambda x(D) \implies D = 0$.

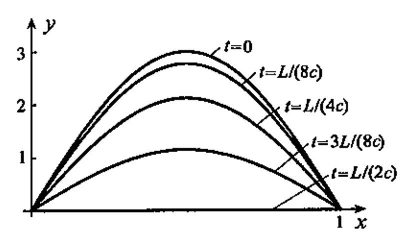
36. With $F(x) = -9.81\rho$, a constant, two integrations of $d^2y/dx^2 = 9.81\rho/T$ give $y(x) = 4.905\rho x^2/T + Ax + B$. The conditions $y(0) = 0 = y(L)$ require $B = 0$ and $A = -4.905\rho L/T$. Thus, $y(x) = 4.905\rho x(x-L)/T$, a parabola. In Exercise 35 it was assumed that that the string experiences only small displacements. This results in a constant force for gravity. No such assumption is made in Example 3.39.

37. Two integrations of $d^2y/dx^2 = 9.81\rho/T$ give $y(x) = 4.905\rho x^2/T + Ax + B$. The conditions $y'(0) = 0 = y(L)$ require $A = 0$ and $B = -4.905\rho L^2/T$. Thus, $y(x) = 4.905\rho(x^2 - L^2)/T$, a parabola.

38. (a) Since $\dfrac{\partial^2 y}{\partial t^2} = -(3\pi^2c^2/L^2)\sin(\pi x/L)\cos(\pi ct/L)$ and $\dfrac{\partial^2 y}{\partial x^2} = -(3\pi^2/L^2)\sin(\pi x/L)\cos(\pi ct/L)$, it follows that $y(x,t)$ satisfies the partial differential equation. It is straightforward to check that it satisfies the remaining conditions.

(b) A plot of the surface is shown to the left below. A cross section of the surface with a plane $x = x_0$ gives a graphical history of the displacement of the point x_0 in the string. A cross section $t = t_0$ gives the position of the string at time t_0.





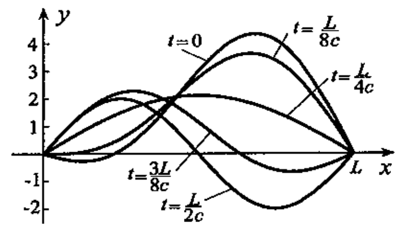
(c) Plots are shown to the right above.

39. Since

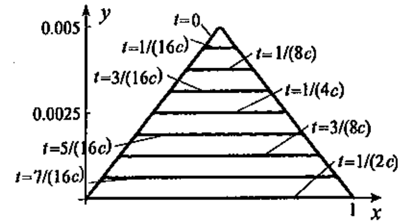
$$\frac{\partial^2 y}{\partial t^2} = -(3\pi^2c^2/L^2)\sin(\pi x/L)\cos(\pi ct/L) + (8\pi^2c^2/L^2)\sin(2\pi x/L)\cos(2\pi ct/L),$$

$$\frac{\partial^2 y}{\partial x^2} = -(3\pi^2/L^2)\sin(\pi x/L)\cos(\pi ct/L) + (8\pi^2/L^2)\sin(2\pi x/L)\cos(2\pi ct/L),$$

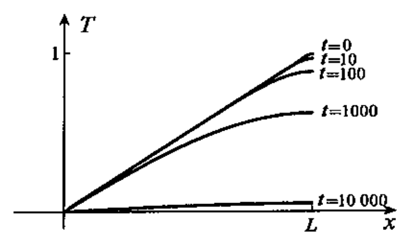
it follows that $y(x,t)$ satisfies the partial differential equation. It is straightforward to check that it satisfies the remaining conditions. Plots are shown to the right.



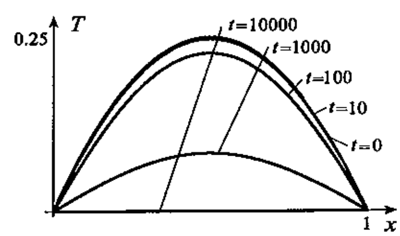
40. (a) If differentiation and summation operations can be interchanged, it is a matter of showing that the function $z(x,t) = \sin(2n-1)\pi x \cos(2n-1)\pi ct$ satisfies the condition for any positive integer n. Since $\partial^2 z/\partial t^2 = -(2n-1)^2\pi^2c^2 z(x,t)$ and $\partial^2 z/\partial x^2 = -(2n-1)^2\pi^2 z(x,t)$, it follows that $z(x,t)$ satisfies the partial differential equation. It is obvious that $z(x,t)$ satisfies the boundary conditions $z(0,t) = z(L,t) = 0$ and the initial condition $z_t(x,0) = 0$.
(b) Plots are shown to the right.



41. (a) Since $\partial T/\partial t = -k\lambda^2 T$ and $\partial^2 T/\partial x^2 = -\lambda^2 T$, it follows that $T(x,t)$ does indeed satisfy the heat conduction equation.
(b) The boundary condition $T(0,t) = 0$ requires $0 = B$. With $B = 0$, the condition $T_x(L,t) = 0$ necessitates $0 = A\lambda\cos\lambda L e^{-k\lambda^2 t}$. Since A cannot be equal to zero, we must set $\cos\lambda L = 0$ and this requires $\lambda L = (2n-1)\pi/2 \implies \lambda = (2n-1)\pi/(2L)$, where n is an integer.
(c) If differentiation and summation operations can be interchanged, it is a matter of showing that the function $z(x,t) = e^{-(2n-1)^2\pi^2 kt/(4L^2)}\sin[(2n-1)\pi x/(2L)]$ satisfies the partial differential equation and boundary conditions. Since this is the function in part (a) with λ as determined in part (b), it must indeed satisfy the heat conduction equation and boundary conditions.
(d) Plots are shown to the right. Notice that the $t=0$ plot approximates $f(x) = x$. Each plot passes through the origin and the slope at $x = L$ is zero, reflecting the boundary conditions.



42. (a) The boundary condition $T(0,t) = 0$ requires $0 = B$. With $B = 0$, the condition $T(L,t) = 0$ necessitates $0 = A\sin\lambda L e^{-k\lambda^2 t}$. Since A cannot be equal to zero, we must set $\sin\lambda L = 0$ and this requires $\lambda L = n\pi \implies \lambda = n\pi/L$, where n is an integer.
(b) Plots are shown to the right. The $t = 0$ plot does appear to be the parabola $x(1-x)$.



43. Two integrations of the differential equation give $T(x) = Ax + B$. The boundary conditions require $T_0 = T(0) = B$ and $T_L = T(L) = AL + B$. Thus, $T(x) = T_0 + (T_L - T_0)x/L$.

44. When $F(x)$ has constant value F, two integrations of the differential equation give $T(x) = -Fx^2/(2k) + Ax + B$. The boundary conditions require $T_0 = T(0) = B$ and $T_L = T(L) = -FL^2/(2k) + AL + B$. These give $T(x) = T_0 + (T_L - T_0)x/L + Fx(L-x)/(2k)$.

45. Two integrations of $d^2y/dx^2 = -x/E$ give $y(x) = -x^3/(6E) + Ax + B$. The boundary conditions give $0 = y(0) = B$ and $0 = y'(L) = A$. Thus, $y(x) = -x^3/(6E)$.

46. Integration of $\dfrac{d^2y}{dx^2} = -\dfrac{\tau(x)}{E}$ gives $\dfrac{dy}{dx} = -\dfrac{1}{E}\displaystyle\int \tau(x)\,dx + C$. To incorporate the boundary condtion at $x = L$, we write this in the form

$$y'(x) = -\frac{1}{E}\int_0^x \tau(u)\,du + C.$$

The condition $y'(L) = 0$ now implies that $0 = -\dfrac{1}{E}\displaystyle\int_0^L \tau(u)\,du + C$. This gives the indicated value for C. A second integration gives

$$y(x) = Cx - \frac{1}{E}\int\left[\int_0^x \tau(u)\,du\right]dx + D.$$

To incorporate the boundary condition at $x = 0$, we write this in the form

$$y(x) = Cx - \frac{1}{E}\int_0^x\left[\int_0^v \tau(u)\,du\right]dv + D.$$

The condition $y(0) = 0$ now implies that $D = 0$, and therefore

$$y(x) = Cx - \frac{1}{E}\int_0^x\int_0^v \tau(u)\,du\,dv.$$

47. Using the Cauchy-Riemann equations from Exercise 41 in Section 12.3,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2}.$$

This shows that $u(x,y)$ satisfies Laplace's equation. Since second partial derivatives are assumed continuous, $u(x,y)$ is harmonic. A similar proof shows that $v(x,y)$ is harmonic.

48. (a) Since $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$, the function is harmonic.

(b) According to Exercise 41 of Section 12.3, $\dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x} = 2x$, $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} = 2y$. From the first $v(x,y) = 2xy + \phi(x)$, which substituted into the second gives $2y + d\phi/dx = 2y$. Thus, $\phi(x) = C$, a constant, and $v(x,y) = 2xy + C$.

49. (a) Since $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = e^x\cos y - e^x\cos y = 0$, the function is harmonic.

(b) According to Exercise 41 of Section 12.3, $\dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x} = e^x\cos y + 1$, $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} = e^x\sin y$. From the first $v(x,y) = e^x\sin y + y + \phi(x)$, which substituted into the second gives $e^x\sin y + d\phi/dx = e^x\sin y$. Thus, $\phi(x) = C$, a constant, and $v(x,y) = e^x\sin y + y + C$.

50. Since $\dfrac{\partial f}{\partial x} = n(x^2+y^2+z^2)^{n-1}(2x)$, we find

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= 2n(x^2+y^2+z^2)^{n-1} + 2nx(n-1)(x^2+y^2+z^2)^{n-2}(2x)\\ &= 2n(x^2+y^2+z^2)^{n-2}[x^2+y^2+z^2+2(n-1)x^2]\\ &= 2n(x^2+y^2+z^2)^{n-2}[(2n-1)x^2+y^2+z^2].\end{aligned}$$

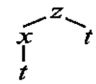
Similarly, $\dfrac{\partial^2 f}{\partial y^2} = 2n(x^2+y^2+z^2)^{n-2}[x^2+(2n-1)y^2+z^2]$, and
$\dfrac{\partial^2 f}{\partial z^2} = 2n(x^2+y^2+z^2)^{n-2}[x^2+y^2+(2n-1)z^2]$. The function satisfies Laplace's equation if

$$\begin{aligned}0 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} &= 2n(x^2+y^2+z^2)^{n-2}[(2n+1)x^2+(2n+1)y^2+(2n+1)z^2]\\ &= 2n(2n+1)(x^2+y^2+z^2)^{n-1}.\end{aligned}$$

Thus, $n = 0$ or $n = -1/2$. When $n = 0$, the function is equal to 1, and it is harmonic in all space. When $n = -1/2$, the function is $1/\sqrt{x^2 + y^2 + z^2}$, and it is harmonic in any region not containing the origin.

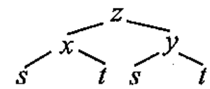
EXERCISES 12.6

1. In general, $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial t}$, and specifically,



$$\frac{dz}{dt} = \left[\frac{(x+t)(t^2) - xt^2}{(x+t)^2}\right](3e^{3t}) + \left[\frac{(x+t)(2xt) - xt^2}{(x+t)^2}\right] = \frac{3t^3e^{3t} + xt(t+2x)}{(x+t)^2}.$$

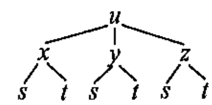
2. In general, $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$, and specifically,



$$\frac{\partial z}{\partial t} = \left(2xe^y + \frac{y}{x}\right)(-s^2 \sin t) + (x^2e^y + \ln x)\left[\frac{4(2t)}{(t^2+2s)\sqrt{(t^2+2s)^2 - 1}}\right].$$

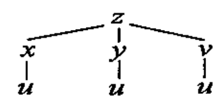
3. In general, $\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial s} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial s}$, and specifically,

$$\frac{\partial u}{\partial s} = \frac{x}{\sqrt{x^2+y^2+z^2}}(2t) + \frac{y}{\sqrt{x^2+y^2+z^2}}(2s) + \frac{z}{\sqrt{x^2+y^2+z^2}}(t)$$
$$= \frac{2xt + 2ys + zt}{\sqrt{x^2+y^2+z^2}}.$$



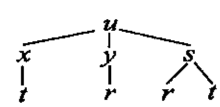
4. In general, $\dfrac{dz}{du} = \dfrac{\partial z}{\partial x}\dfrac{dx}{du} + \dfrac{\partial z}{\partial y}\dfrac{dy}{du} + \dfrac{\partial z}{\partial v}\dfrac{dv}{du}$, and specifically,

$$\frac{dz}{du} = (2xyv^3)(3u^2+2) + (x^2v^3)\left(\frac{2u}{u^2+1}\right) + (3x^2yv^2)(ue^u + e^u)$$
$$= xv^2\left[2yv(3u^2+2) + \frac{2xuv}{u^2+1} + 3xye^u(u+1)\right].$$

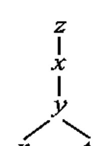


5. In general, $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial y}\dfrac{dy}{dr} + \dfrac{\partial u}{\partial s}\dfrac{\partial s}{\partial r}$, and specifically,

$$\frac{\partial u}{\partial r} = \left(\frac{ys}{\sqrt{x^2+y^2s}}\right)\left[\frac{2r}{\sqrt{1-(r^2+5)^2}}\right] + \left(\frac{y^2}{2\sqrt{x^2+y^2s}}\right)[t\sec^2(rt)]$$
$$= \frac{y}{2\sqrt{x^2+y^2s}}\left[\frac{4rs}{\sqrt{1-(r^2+5)^2}} + yt\sec^2(rt)\right].$$

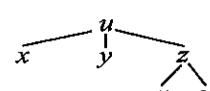


6. In general, $\dfrac{\partial z}{\partial t} = \dfrac{dz}{dx}\dfrac{dx}{dy}\dfrac{\partial y}{\partial t}$, and specifically,



$$\frac{\partial z}{\partial t} = (3^{x+2}\ln 3)(2y)[-\csc(r^2+t)\cot(r^2+t)].$$

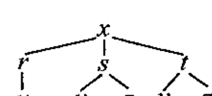
7. In general, $\dfrac{\partial u}{\partial x} = \left.\dfrac{\partial u}{\partial x}\right)_{y,z} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial x}$, and specifically,



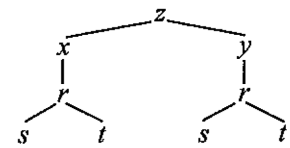
$$\frac{\partial u}{\partial x} = \frac{-xy}{(x^2+y^2+z^2)^{3/2}} - \frac{yz}{(x^2+y^2+z^2)^{3/2}}\left(\frac{1}{y}\right) = \frac{-(xy+z)}{(x^2+y^2+z^2)^{3/2}}$$

8. In general, $\dfrac{\partial x}{\partial y} = \dfrac{\partial x}{\partial r}\dfrac{dr}{dy} + \dfrac{\partial x}{\partial s}\dfrac{\partial s}{\partial y} + \dfrac{\partial x}{\partial t}\dfrac{\partial t}{\partial y}$, and specifically,

$$\frac{\partial x}{\partial y} = (2rs^2t^2)(-5y^{-6}) + (2r^2st^2)\left[\frac{-2y}{(y^2+z^2)^2}\right] + (2r^2s^2t)\left(\frac{-2}{y^3}\right)$$
$$= -2rst\left[\frac{5st}{y^6} + \frac{2rty}{(y^2+z^2)^2} + \frac{2rs}{y^3}\right].$$



9. In general, $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dr}\dfrac{\partial r}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dr}\dfrac{\partial r}{\partial t}$, and specifically,

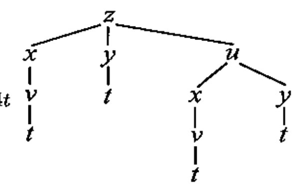


$$\begin{aligned}\frac{\partial z}{\partial t} &= e^{x+y}(2)\left[\ln(s^2+t^2) + \frac{2t^2}{s^2+t^2}\right] + e^{x+y}(2)\left[\ln(s^2+t^2) + \frac{2t^2}{s^2+t^2}\right]\\ &= 4e^{x+y}\left[\ln(s^2+t^2) + \frac{2t^2}{s^2+t^2}\right].\end{aligned}$$

10. In general,

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dv}\frac{dv}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} + \frac{\partial z}{\partial u}\frac{\partial u}{\partial x}\frac{dx}{dv}\frac{dv}{dt} + \frac{\partial z}{\partial u}\frac{\partial u}{\partial y}\frac{dy}{dt}\\ &= \left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial u}\frac{\partial u}{\partial x}\right)\frac{dx}{dv}\frac{dv}{dt} + \left(\frac{\partial z}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial u}{\partial y}\right)\frac{dy}{dt},\end{aligned}$$

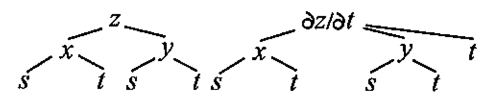
and specifically,



$$\begin{aligned}\frac{dz}{dt} &= \left\{2x + 2u\left[\frac{-2x}{(x^2-y^2)^2}\right]\right\}(3v^2-6v)e^t + \left\{2y + 2u\left[\frac{2y}{(x^2-y^2)^2}\right]\right\}4e^{4t}\\ &= 6xve^t(v-2)\left[1 - \frac{2u}{(x^2-y^2)^2}\right] + 8ye^{4t}\left[1 + \frac{2u}{(x^2-y^2)^2}\right].\end{aligned}$$

11. From $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$

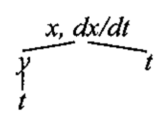
$$\begin{aligned}&= (2xy^2 + e^y)(2t) + (2x^2y + xe^y)(-2t)\\ &= 2t(2xy^2 - 2x^2y + e^y - xe^y),\end{aligned}$$



$$\begin{aligned}\frac{\partial^2 z}{\partial t^2} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial t}\right)\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial t}\right)\frac{\partial y}{\partial t} + \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial t}\right)_{x,y}\\ &= 2t(2y^2 - 4xy - e^y)(2t) + 2t(4xy - 2x^2 + e^y - xe^y)(-2t) + 2(2xy^2 - 2x^2y + e^y - xe^y)\\ &= 4t^2[2(x^2+y^2-4xy) + (x-2)e^y] + 2[2(xy^2 - x^2y) + (1-x)e^y].\end{aligned}$$

12. From $\dfrac{dx}{dt} = \dfrac{\partial x}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial x}{\partial t} = (2y+t)(t^2e^t + 2te^t) + (y-2t)$

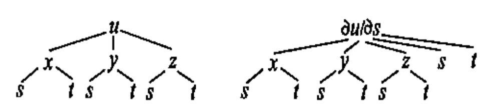
$$= te^t(t^2 + 2yt + 4y + 2t) + y - 2t,$$



$$\begin{aligned}\frac{d^2x}{dt^2} &= \frac{\partial}{\partial y}\left(\frac{dx}{dt}\right)\frac{dy}{dt} + \frac{\partial}{\partial t}\left(\frac{dx}{dt}\right)\\ &= [te^t(2t+4)+1](t^2e^t + 2te^t) + [te^t(2t+2y+2) + (te^t + e^t)(t^2 + 2yt + 4y + 2t) - 2]\\ &= 2t^2(t+2)^2e^{2t} + (6t^2 + 6t + 8ty + t^3 + 2yt^2 + 4y)e^t - 2.\end{aligned}$$

13. From $\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial s} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial s}$

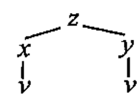
$$\begin{aligned}&= (2x+yz)(2s) + (2y+xz)(2s) + (2z+xy)(t)\\ &= 2s(2x + 2y + xz + yz) + t(2z + xy),\end{aligned}$$

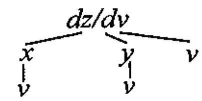


$$\begin{aligned}\frac{\partial^2 u}{\partial s^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial s}\right)\frac{\partial x}{\partial s} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial s}\right)\frac{\partial y}{\partial s} + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial s}\right)\frac{\partial z}{\partial s} + \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial s}\right)_{x,y,z,t}\\ &= (4s + 2sz + ty)(2s) + (4s + 2sz + tx)(2s) + (2sx + 2sy + 2t)(t) + 2(2x + 2y + xz + yz)\\ &= 8s^2(z+2) + 2(x+y)(2st + z + 2) + 2t^2.\end{aligned}$$

14. From $\dfrac{dz}{dv} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dv} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dv}$

$$= y\cos(xy)(-3\sin v) + x\cos(xy)(4\cos v)$$
$$= \cos(xy)(-3y\sin v + 4x\cos v),$$

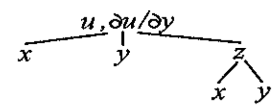




$$\frac{d^2z}{dv^2} = \frac{\partial}{\partial x}\left(\frac{dz}{dv}\right)\frac{dx}{dv} + \frac{\partial}{\partial y}\left(\frac{dz}{dv}\right)\frac{dy}{dv} + \frac{\partial}{\partial v}\left(\frac{dz}{dv}\right)$$
$$= [-y\sin(xy)(-3y\sin v + 4x\cos v) + 4\cos v\ \cos(xy)](-3\sin v)$$
$$+[-x\sin(xy)(-3y\sin v + 4x\cos v) - 3\sin v\ \cos(xy)](4\cos v)$$
$$+\cos(xy)(-3y\cos v - 4x\sin v)$$
$$= (24xy\sin v\ \cos v - 16x^2\cos^2 v - 9y^2\sin^2 v)\sin(xy)$$
$$-(24\sin v\ \cos v + 3y\cos v + 4x\sin v)\cos(xy)$$
$$= -(3y\sin v - 4x\cos v)^2\sin(xy) - (24\sin v\ \cos v + 3y\cos v + 4x\sin v)\cos(xy).$$

15. From $\dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial y} + \left.\dfrac{\partial u}{\partial y}\right)_{x,z}$

$$= \frac{-yz}{(x^2+y^2+z^2)^{3/2}}\left(\frac{-x}{y^2}\right) + \frac{1}{\sqrt{x^2+y^2+z^2}} + \frac{-y^2}{(x^2+y^2+z^2)^{3/2}}$$
$$= \frac{xz}{y(x^2+y^2+z^2)^{3/2}} + \frac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}},$$



$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}\right)\frac{\partial z}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)_{y,z}$$
$$= \left[\frac{x}{y(x^2+y^2+z^2)^{3/2}} - \frac{3xz^2}{y(x^2+y^2+z^2)^{5/2}} + \frac{2z}{(x^2+y^2+z^2)^{3/2}} - \frac{3z(x^2+z^2)}{(x^2+y^2+z^2)^{5/2}}\right]\left(\frac{1}{y}\right)$$
$$+ \frac{z}{y(x^2+y^2+z^2)^{3/2}} - \frac{3x^2z}{y(x^2+y^2+z^2)^{5/2}} + \frac{2x}{(x^2+y^2+z^2)^{3/2}} - \frac{3x(x^2+z^2)}{(x^2+y^2+z^2)^{5/2}}.$$

This simplifies to $\dfrac{x(x^2+y^2+z^2)(1-y^2) + 3(xy+z)(y^3-xz)}{y^2(x^2+y^2+z^2)^{5/2}}$.

16. From the schematic, $\dfrac{\partial u}{\partial x} = \dfrac{du}{dr}\dfrac{\partial r}{\partial x} = \dfrac{du}{dr}\dfrac{x}{\sqrt{x^2+y^2+z^2}}$,

and similarly,

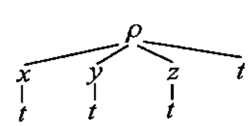
$$\frac{\partial u}{\partial y} = \frac{du}{dr}\frac{y}{\sqrt{x^2+y^2+z^2}},\quad \frac{\partial u}{\partial z} = \frac{du}{dr}\frac{z}{\sqrt{x^2+y^2+z^2}}.$$

Consequently,

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = \left(\frac{du}{dr}\right)^2\left(\frac{x^2}{x^2+y^2+z^2} + \frac{y^2}{x^2+y^2+z^2} + \frac{z^2}{x^2+y^2+z^2}\right) = \left(\frac{du}{dr}\right)^2.$$

17. (a) $\dfrac{d\rho}{dt} = \dfrac{\partial\rho}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial\rho}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial\rho}{\partial z}\dfrac{dz}{dt} + \dfrac{\partial\rho}{\partial t}$

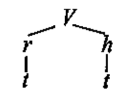
(b) A fixed observer in the gas measures the rate of change of ρ as $\partial\rho/\partial t$. An observer moving with the gas measures the rate of change of ρ as $d\rho/dt$.



18. The volume V of a right circular cone in terms of its radius r and height h is $V = \pi r^2 h/3$. From the schematic,

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = \frac{2}{3}\pi r h\frac{dr}{dt} + \frac{1}{3}\pi r^2\frac{dh}{dt}.$$

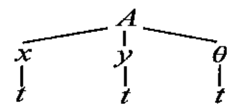
When $r = 10$ and $h = 20$,



$$\frac{dV}{dt} = \frac{2}{3}\pi(10)(20)(1) + \frac{1}{3}\pi(10)^2(-2) = \frac{200\pi}{3}\ \text{cm}^3/\text{min}.$$

Multivariable calculus is not needed since $V = \pi r^2 h/3$ can be differentiated with respect to t using the product and power rules, $\dfrac{dV}{dt} = \dfrac{1}{3}\pi\left(2r\dfrac{dr}{dt}h + r^2\dfrac{dh}{dt}\right)$, and this is the same result as above.

19.
$$\begin{aligned}\frac{dA}{dt} &= \frac{\partial A}{\partial x}\frac{dx}{dt} + \frac{\partial A}{\partial y}\frac{dy}{dt} + \frac{\partial A}{\partial \theta}\frac{d\theta}{dt}\\ &= \frac{1}{2}y\sin\theta\frac{dx}{dt} + \frac{1}{2}x\sin\theta\frac{dy}{dt} + \frac{1}{2}xy\cos\theta\frac{d\theta}{dt}\end{aligned}$$

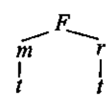


At the instant in question,

$$\frac{dA}{dt} = \frac{1}{2}(2)\sin(1/3)\left(\frac{1}{2}\right) + \frac{1}{2}(1)\sin(1/3)\left(\frac{1}{2}\right) + \frac{1}{2}(1)(2)\cos(1/3)\left(\frac{-1}{10}\right) = 0.151\ \text{m}^2/\text{s}.$$

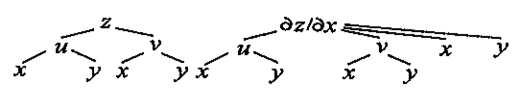
20. Newton's universal law of gravitation states that $F = GMm/r^2$, where $G = 6.67 \times 10^{-11}$, $M =$ mass of earth, $m =$ mass of rocket, and $r =$ distance from centre of earth to rocket.

$$\begin{aligned}\frac{dF}{dt} &= \frac{\partial F}{\partial m}\frac{dm}{dt} + \frac{\partial F}{\partial r}\frac{dr}{dt} = \frac{GM}{r^2}\frac{dm}{dt} - \frac{2GMm}{r^3}\frac{dr}{dt}\\ &= \frac{GM}{r^3}\left(r\frac{dm}{dt} - 2m\frac{dr}{dt}\right).\end{aligned}$$



With $M = (4/3)\pi(\text{radius of earth})^3(\text{density of earth}) = (4/3)\pi(6.37 \times 10^6)^3(5.52 \times 10^3)$, we find that when $r = 6.47 \times 10^6$,

$$\begin{aligned}\frac{dF}{dt} &= 6.67 \times 10^{-11}\left(\frac{4}{3}\right)\pi\frac{(6.37 \times 10^6)^3(5.52 \times 10^3)}{(6.47 \times 10^6)^3}\left[6.47 \times 10^6(-50) - 2(12 \times 10^6)(2 \times 10^3)\right]\\ &= -7.11 \times 10^4\ \text{N/s}.\end{aligned}$$



21. From
$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x},\\ \frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial x}\right)\frac{\partial v}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)_{u,v,y}\\ &= \left(\frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial u\partial v}\frac{\partial v}{\partial x}\right)\frac{\partial u}{\partial x} + \left(\frac{\partial^2 z}{\partial v\partial u}\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}\right)\frac{\partial v}{\partial x}\\ &\quad + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2}\\ &= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial^2 z}{\partial u\partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2}.\end{aligned}$$

22. (a) Since $f(tx, ty) = (tx)^2 + (tx)(ty) + 3(ty)^2 = t^2(x^2 + xy + 3y^2) = t^2 f(x, y)$, the function is positively homogeneous of degree 2.

(b) Since $f(tx, ty) = (tx)^2(ty) + (tx)(ty) - 2(tx)(ty)^2 = t^2(tx^2y + xy - 2txy^2)$, the function is not homogeneous.

(c) Since $f(tx, ty, tz) = (tx)^2\sin[ty/(tz)] + (ty)^2 + (ty)^3/(tz) = t^2[x^2\sin(y/z) + y^2 + y^3/z]$, the function is positively homogeneous of degree 2.

(d) Since $f(tx, ty, tz) = (tx)e^{ty/(tz)} - (tx)(ty)(tz) = t(xe^{y/z} - t^2xyz)$, the function is not homogeneous.

(e) Since $f(ux, uy, uz, ut) = (ux)^4 + (uy)^4 + (uz)^4 + (ut)^4 - (ux)(uy)(uz)(ut) = u^4(x^4 + y^4 + z^4 + t^4 - xyzt)$, the function is positively homogeneous of degree 4.

(f) Since $f(ux,uy,uz,ut) = e^{u^2x^2+u^2y^2}[(uz)^2+(ut)^2] = u^2(z^2+t^2)e^{u^2(x^2+y^2)}$, the function is not homogeneous.

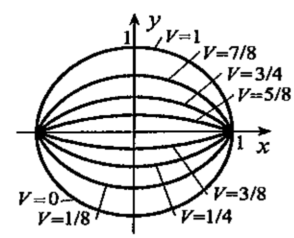
(g) Since $f(tx,ty,tz) = \cos(t^2xy)\sin(t^2yz)$, the function is not homogeneous.

(h) Since $f(tx,ty) = \sqrt{t^2x^2+t^2xy+t^2y^2}e^{ty/(tx)}(2t^2x^2-3t^2y^2) = t^3\sqrt{x^2+xy+y^2}e^{y/x}(2x^2-3y^2)$, the function is positively homogeneous of degree 3.

23.
$$\frac{\partial V}{\partial r} = \frac{1}{\pi\left[1+\left(\dfrac{2Rr\sin\theta}{R^2-r^2}\right)^2\right]}\left[\frac{(R^2-r^2)(2R\sin\theta)-2Rr\sin\theta(-2r)}{(R^2-r^2)^2}\right] = \frac{2R(R^2+r^2)\sin\theta}{\pi[(R^2-r^2)^2+4R^2r^2\sin^2\theta]}$$
$$\frac{\partial^2 V}{\partial r^2} = \frac{4Rr\sin\theta}{\pi[(R^2-r^2)^2+4R^2r^2\sin^2\theta]} - \frac{2R(R^2+r^2)\sin\theta[-4r(R^2-r^2)+8R^2r\sin^2\theta]}{\pi[(R^2-r^2)^2+4R^2r^2\sin^2\theta]^2}$$
$$\frac{\partial V}{\partial \theta} = \frac{1}{\pi\left[1+\left(\dfrac{2Rr\sin\theta}{R^2-r^2}\right)^2\right]}\left(\frac{2Rr\cos\theta}{R^2-r^2}\right) = \frac{2Rr(R^2-r^2)\cos\theta}{\pi[(R^2-r^2)^2+4R^2r^2\sin^2\theta]}$$
$$\frac{\partial^2 V}{\partial \theta^2} = \frac{-2Rr(R^2-r^2)\sin\theta}{\pi[(R^2-r^2)^2+4R^2r^2\sin^2\theta]} - \frac{2Rr(R^2-r^2)\cos\theta(8R^2r^2\sin\theta\cos\theta)}{\pi[(R^2-r^2)^2+4R^2r^2\sin^2\theta]^2}$$

When these are substituted into the left side of equation 12.24, and considerable algebra is performed, the result is zero.

(b) If we solve the equation for r in terms of V and θ, the result is
$$r = \frac{-R\sin\theta+\sqrt{R^2\sin^2\theta+R^2\tan^2[\pi(2V-1)/2]}}{\tan[\pi(2V-1)/2]}.$$
When we set $R=1$,
$$r = \frac{-\sin\theta+\sqrt{\sin^2\theta+\tan^2[\pi(2V-1)/2]}}{\tan[\pi(2V-1)/2]}.$$

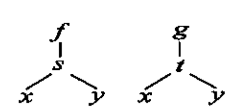


This is plotted to the right for $V = 1/8,\ 1/4,\ 3/8,\ 5/8,\ 3/4,\ 7/8$. For $V=1/2$, the equation in part (a) requires $\sin\theta = 0$, from which $\theta = 0$ or $\theta = \pi$, the x-axis.

24.
$$\frac{\partial V}{\partial r} = \frac{V_1-V_2}{\pi\left[1+\left(\dfrac{2Rr\sin\theta}{R^2-r^2}\right)^2\right]}\left[\frac{(R^2-r^2)(2R\sin\theta)-2Rr\sin\theta(-2r)}{(R^2-r^2)^2}\right] = \frac{2(V_1-V_2)R(R^2+r^2)\sin\theta}{\pi[(R^2-r^2)^2+4R^2r^2\sin^2\theta]}$$
$$\frac{\partial^2 V}{\partial r^2} = \frac{4(V_1-V_2)Rr\sin\theta}{\pi[(R^2-r^2)^2+4R^2r^2\sin^2\theta]} - \frac{2(V_1-V_2)R(R^2+r^2)\sin\theta[-4r(R^2-r^2)+8R^2r\sin^2\theta]}{\pi[(R^2-r^2)^2+4R^2r^2\sin^2\theta]^2}$$
$$\frac{\partial V}{\partial \theta} = \frac{V_1-V_2}{\pi\left[1+\left(\dfrac{2Rr\sin\theta}{R^2-r^2}\right)^2\right]}\left(\frac{2Rr\cos\theta}{R^2-r^2}\right) = \frac{2(V_1-V_2)Rr(R^2-r^2)\cos\theta}{\pi[(R^2-r^2)^2+4R^2r^2\sin^2\theta]}$$
$$\frac{\partial^2 V}{\partial \theta^2} = \frac{-2(V_1-V_2)Rr(R^2-r^2)\sin\theta}{\pi[(R^2-r^2)^2+4R^2r^2\sin^2\theta]} - \frac{2(V_1-V_2)Rr(R^2-r^2)\cos\theta(8R^2r^2\sin\theta\cos\theta)}{\pi[(R^2-r^2)^2+4R^2r^2\sin^2\theta]^2}$$

When these are substituted into the left side of equation 12.24, and considerable algebra is performed, the result is zero.

25. If we set $s = x^2-y^2$ and $t = xy$, then
$$\nabla f(x^2-y^2) = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} = \frac{df}{ds}\frac{\partial s}{\partial x}\hat{\mathbf{i}} + \frac{df}{ds}\frac{\partial s}{\partial y}\hat{\mathbf{j}} = f'(s)(2x\hat{\mathbf{i}} - 2y\hat{\mathbf{j}}),$$
$$\nabla g(xy) = \frac{\partial g}{\partial x}\hat{\mathbf{i}} + \frac{\partial g}{\partial y}\hat{\mathbf{j}} = \frac{dg}{dt}\frac{\partial t}{\partial x}\hat{\mathbf{i}} + \frac{dg}{dt}\frac{\partial t}{\partial y}\hat{\mathbf{j}} = g'(t)(y\hat{\mathbf{i}} + x\hat{\mathbf{j}}),$$

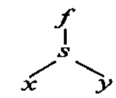


Consequently, $\nabla f(x^2-y^2)\cdot\nabla g(xy) = f'(s)g'(t)(2xy-2xy) = 0$.

26. If we set $s = x - y$, then

$$\frac{\partial f}{\partial y} = \frac{df}{ds}\frac{\partial s}{\partial y} = \frac{df}{ds}(-1) \quad \text{and} \quad \frac{\partial f}{\partial x} = \frac{df}{ds}\frac{\partial s}{\partial x} = \frac{df}{ds}(1).$$

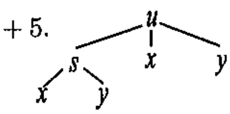
Hence, $\dfrac{\partial f}{\partial y} = -\dfrac{\partial f}{\partial x}$.



27. If we set $s = 4x - 3y$, then

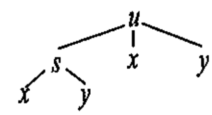
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \left.\frac{\partial u}{\partial x}\right)_{s,y} = f'(s)(4) - 5, \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \left.\frac{\partial u}{\partial y}\right)_{s,x} = f'(s)(-3) + 5.$$

Hence, $3\dfrac{\partial u}{\partial x} + 4\dfrac{\partial u}{\partial y} = 3[4f'(s) - 5] + 4[-3f'(s) + 5] = 5.$



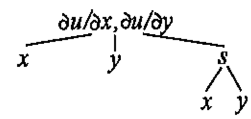
28. If we set $s = x + y$, then $u = xf(s) + yg(s)$. From the schematic,

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \left.\frac{\partial u}{\partial x}\right)_{y,s} \\ &= \left[x\frac{df}{ds} + y\frac{dg}{ds}\right](1) + f(s) = xf'(s) + yg'(s) + f(s),\end{aligned}$$



and similarly, $\dfrac{\partial u}{\partial y} = xf'(s) + yg'(s) + g(s)$. The schematic for $\partial u/\partial x$ and $\partial u/\partial y$ gives

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right)\frac{\partial s}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)_{y,s} \\ &= [xf''(s) + yg''(s) + f'(s)](1) + f'(s); \\ \frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right)\frac{\partial s}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)_{y,s} \\ &= [xf''(s) + yg''(s) + g'(s)](1) + f'(s); \\ \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right)\frac{\partial s}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)_{x,s} = [xf''(s) + yg''(s) + g'(s)](1) + g'(s);\end{aligned}$$



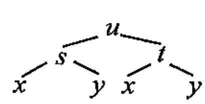
Thus, $\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

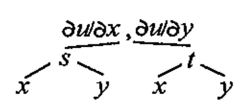
29. If we set $s = x - y$, $t = x + y$, and $u = f(s) + g(t)$, then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} = f'(s) + g'(t), \qquad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} = -f'(s) + g'(t),$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right)\frac{\partial s}{\partial x} + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)\frac{\partial t}{\partial x} = f''(s) + g''(t),$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right)\frac{\partial s}{\partial y} + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)\frac{\partial t}{\partial y} = f''(s) + g''(t),$$



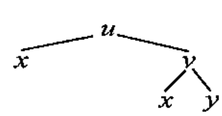


Thus, $\partial^2 u/\partial x^2 = \partial^2 u/\partial y^2$.

30. From the schematic for $u = x^2 f(v)$, $v = y/x$,

$$\frac{\partial u}{\partial x} = \left.\frac{\partial u}{\partial x}\right)_v + \frac{\partial u}{\partial v}\frac{\partial v}{\partial x} = 2xf(v) + x^2 f'(v)\left(\frac{-y}{x^2}\right),$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial y} = x^2 f'(v)\left(\frac{1}{x}\right).$$



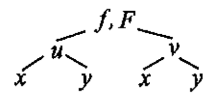
Hence,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x[2xf(v) - yf'(v)] + y[xf'(v)] = 2x^2 f(v) = 2u.$$

This is also an immediate consequence of Euler's theorem since u is positively homogeneous of degree 2.

31. The schematic to the right describes the functional situation $f(x,y) = F[u(x,y), v(x,y)]$ where $u = u(x,y) = (x+y)/2$ and $v = v(x,y) = (x-y)/2$. It gives

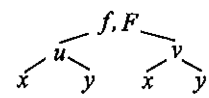
$$\frac{\partial f}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} = \frac{1}{2}\frac{\partial F}{\partial u} + \frac{1}{2}\frac{\partial F}{\partial v},$$
$$\frac{\partial f}{\partial y} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} = \frac{1}{2}\frac{\partial F}{\partial u} - \frac{1}{2}\frac{\partial F}{\partial v}.$$



Hence,

$$0 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{1}{2}\frac{\partial F}{\partial u} + \frac{1}{2}\frac{\partial F}{\partial v}\right)^2 + \left(\frac{1}{2}\frac{\partial F}{\partial u} - \frac{1}{2}\frac{\partial F}{\partial v}\right)^2 = \frac{1}{2}\left[\left(\frac{\partial F}{\partial u}\right)^2 + \left(\frac{\partial F}{\partial v}\right)^2\right].$$

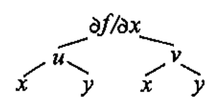
32. The schematic to the right describes the functional situation $f(x,y) = F[u(x,y), v(x,y)]$ where $u = u(x,y) = (x+y)/2$ and $v = v(x,y) = (x-y)/2$. It gives



$$\frac{\partial f}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} = \frac{1}{2}\frac{\partial F}{\partial u} + \frac{1}{2}\frac{\partial F}{\partial v}.$$

The schematic for $\partial f/\partial x$ leads to

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial v}\right)\frac{\partial v}{\partial x} \\ &= \left(\frac{1}{2}\frac{\partial^2 F}{\partial u^2} + \frac{1}{2}\frac{\partial^2 F}{\partial u \partial v}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\frac{\partial^2 F}{\partial v \partial u} + \frac{1}{2}\frac{\partial^2 F}{\partial v^2}\right)\left(\frac{1}{2}\right) \\ &= \frac{1}{4}\left(\frac{\partial^2 F}{\partial u^2} + 2\frac{\partial^2 F}{\partial u \partial v} + \frac{\partial^2 F}{\partial v^2}\right).\end{aligned}$$



Similarly, $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{1}{4}\left(\dfrac{\partial^2 F}{\partial u^2} - 2\dfrac{\partial^2 F}{\partial u \partial v} + \dfrac{\partial^2 F}{\partial v^2}\right)$. Hence, $\quad 0 = \dfrac{\partial^2 f}{\partial x^2} - \dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial^2 F}{\partial u \partial v}$.

33. Since the transformation $x = u\cos v$, $y = u \sin v$ is that due to polar coordinates with u and v replacing r and θ, we use the results of Example 12.19 to write that

$$\frac{\partial f}{\partial x} = \cos v \frac{\partial F}{\partial u} - \frac{\sin v}{u}\frac{\partial F}{\partial v}.$$

A similar derivation gives $\dfrac{\partial f}{\partial y} = \sin v \dfrac{\partial F}{\partial u} + \dfrac{\cos v}{u}\dfrac{\partial F}{\partial v}$. Hence,

$$\begin{aligned}0 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 &= \left(\cos v\frac{\partial F}{\partial u} - \frac{\sin v}{u}\frac{\partial F}{\partial v}\right)^2 + \left(\sin v \frac{\partial F}{\partial u} + \frac{\cos v}{u}\frac{\partial F}{\partial v}\right)^2 \\ &= \left(\frac{\partial F}{\partial u}\right)^2 + \frac{1}{u^2}\left(\frac{\partial F}{\partial v}\right)^2.\end{aligned}$$

34. The biharmonic equation can be expressed in the form

$$0 = \left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right)V = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 V.$$

But according to Example 12.19, the operator equivalent to $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ in polar coordinates is

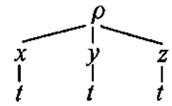
$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Hence, the biharmonic equation in polar coordinates is

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}\right) = 0.$$

35. $\dfrac{d\rho}{dt} = \dfrac{\partial\rho}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial\rho}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial\rho}{\partial z}\dfrac{dz}{dt}$

$= \left(\dfrac{6x}{z^2+5}\right)(2t) + \left(\dfrac{2y}{z^2+5}\right)(9t^2) + \left[-\dfrac{2z(3x^2+y^2)}{(z^2+5)^2}\right](2).$

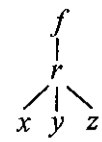


When $t = 2$, we obtain $x = 4$, $y = 25$, $z = 9$, and

$$\frac{d\rho}{dt} = \left(\frac{24}{86}\right)(4) + \left(\frac{50}{86}\right)(36) + \left[-\frac{2(9)(673)}{86^2}\right](2) = 18.77 \text{ kg/m}^3\text{/s}.$$

36. From the schematic,

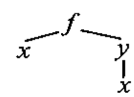
$\dfrac{\partial f}{\partial x} = \dfrac{df}{dr}\dfrac{\partial r}{\partial x} = \dfrac{xf'(r)}{\sqrt{x^2+y^2+z^2}} = \dfrac{xf'(r)}{r}.$



Similarly, $\dfrac{\partial f}{\partial y} = \dfrac{yf'(r)}{r}$ and $\dfrac{\partial f}{\partial z} = \dfrac{zf'(r)}{r}$. Thus,

$$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}} = \frac{xf'(r)}{r}\hat{\mathbf{i}} + \frac{yf'(r)}{r}\hat{\mathbf{j}} + \frac{zf'(r)}{r}\hat{\mathbf{k}} = \frac{f'(r)}{r}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}).$$

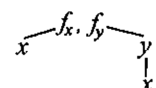
37. If $f(x,y) = 0$ defines y as a function of x, say $y(x)$, then $f[x, y(x)] \equiv 0$ for all x in the domain of the function. The derivative of this function must therefore be zero. To differentiate it with respect to x, we use the schematic to the right,



$$0 = \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}.$$

This equation can be solved for $\dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y} = -\dfrac{f_x}{f_y}$. Differentiation of this equation gives

$$\frac{d^2y}{dx^2} = -\frac{f_y\dfrac{d}{dx}(f_x) - f_x\dfrac{d}{dx}(f_y)}{{f_y}^2}.$$



Now the schematic to the right gives

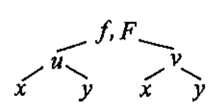
$$\frac{d}{dx}(f_x) = \frac{\partial}{\partial x}(f_x) + \frac{\partial}{\partial y}(f_x)\frac{dy}{dx} = f_{xx} + f_{xy}\frac{dy}{dx} = f_{xx} - \frac{f_{xy}f_x}{f_y},$$

and $\dfrac{d}{dx}(f_y) = \dfrac{\partial}{\partial x}(f_y) + \dfrac{\partial}{\partial y}(f_y)\dfrac{dy}{dx} = f_{yx} + f_{yy}\dfrac{dy}{dx} = f_{xy} - \dfrac{f_{yy}f_x}{f_y}.$

Therefore, $\dfrac{d^2y}{dx^2} = -\dfrac{f_y\left(f_{xx} - \dfrac{f_{xy}f_x}{f_y}\right) - f_x\left(f_{xy} - \dfrac{f_{yy}f_x}{f_y}\right)}{{f_y}^2} = -\dfrac{f_{xx}{f_y}^2 - 2f_{xy}f_xf_y + f_{yy}{f_x}^2}{{f_y}^3}.$

38. If we set $u = x^2 - y^2$ and $v = 2xy$, the schematic to the right gives

$\dfrac{\partial F}{\partial x} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial x} = 2x\dfrac{\partial f}{\partial u} + 2y\dfrac{\partial f}{\partial v}.$

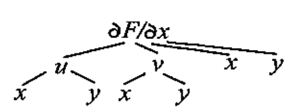


The schematic for $\partial F/\partial x$ now gives

$\dfrac{\partial^2 F}{\partial x^2} = \dfrac{\partial}{\partial u}\left(\dfrac{\partial F}{\partial x}\right)\dfrac{\partial u}{\partial x} + \dfrac{\partial}{\partial v}\left(\dfrac{\partial F}{\partial x}\right)\dfrac{\partial v}{\partial x} + \dfrac{\partial}{\partial x}\left(\dfrac{\partial F}{\partial x}\right)_{u,v,y}$

$= \left(2x\dfrac{\partial^2 f}{\partial u^2} + 2y\dfrac{\partial^2 f}{\partial u\partial v}\right)(2x) + \left(2x\dfrac{\partial^2 f}{\partial v\partial u} + 2y\dfrac{\partial^2 f}{\partial v^2}\right)(2y) + 2\dfrac{\partial f}{\partial u}$

$= 4x^2\dfrac{\partial^2 f}{\partial u^2} + 8xy\dfrac{\partial^2 f}{\partial u\partial v} + 4y^2\dfrac{\partial^2 f}{\partial v^2} + 2\dfrac{\partial f}{\partial u}.$



A similar derivation gives $\dfrac{\partial^2 F}{\partial y^2} = 4y^2\dfrac{\partial^2 f}{\partial u^2} - 8xy\dfrac{\partial^2 f}{\partial u\partial v} + 4x^2\dfrac{\partial^2 f}{\partial v^2} - 2\dfrac{\partial f}{\partial u}$. Hence,

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 4(x^2+y^2)\frac{\partial^2 f}{\partial u^2} + 4(x^2+y^2)\frac{\partial^2 f}{\partial v^2} = 4(x^2+y^2)\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right) = 0,$$

since $f(u,v)$ is harmonic.

39. (a) From $\dfrac{\partial f}{\partial x} = \dfrac{2x}{x^2+y^2}$, we obtain $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{(x^2+y^2)(2) - 2x(2x)}{(x^2+y^2)^2} = \dfrac{2(y^2-x^2)}{(x^2+y^2)^2}$. Similarly, $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{2(x^2-y^2)}{(x^2+y^2)^2}$. Consequently, $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$.

(b) If we change to polar coordinates, and set $F(r) = f(r\cos\theta, r\sin\theta) = \ln r^2 = 2\ln r$, then $\dfrac{\partial F}{\partial r} = \dfrac{2}{r}$, $\dfrac{\partial^2 F}{\partial r^2} = -\dfrac{2}{r^2}$, and $\dfrac{\partial^2 F}{\partial \theta^2} = 0$. Hence, $\dfrac{\partial^2 F}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial F}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 F}{\partial \theta^2} = -\dfrac{2}{r^2} + \dfrac{1}{r}\left(\dfrac{2}{r}\right) = 0$.

40. In the proof of Theorem 12.3, differentiation of equation 12.25 with respect to t gave

$$nt^{n-1}f(x,y,z) = x\frac{\partial f}{\partial u} + y\frac{\partial f}{\partial v} + z\frac{\partial f}{\partial w}.$$

Since the same schematic in the theorem applies to $\partial f/\partial u$, $\partial f/\partial v$, and $\partial f/\partial w$, differentiation of this equation with respect to t gives

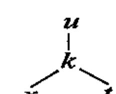
$$\begin{aligned}
n(n-1)t^{n-2}f(x,y,z) &= x\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial u}\right) + y\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial v}\right) + z\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial w}\right)\\
&= x\left[\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\right)\frac{\partial u}{\partial t} + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial u}\right)\frac{\partial v}{\partial t} + \frac{\partial}{\partial w}\left(\frac{\partial f}{\partial u}\right)\frac{\partial w}{\partial t}\right]\\
&\quad + y\left[\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial v}\right)\frac{\partial u}{\partial t} + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial v}\right)\frac{\partial v}{\partial t} + \frac{\partial}{\partial w}\left(\frac{\partial f}{\partial v}\right)\frac{\partial w}{\partial t}\right]\\
&\quad + z\left[\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial w}\right)\frac{\partial u}{\partial t} + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial w}\right)\frac{\partial v}{\partial t} + \frac{\partial}{\partial w}\left(\frac{\partial f}{\partial w}\right)\frac{\partial w}{\partial t}\right]\\
&= x\left[x\frac{\partial^2 f}{\partial u^2} + y\frac{\partial^2 f}{\partial v\partial u} + z\frac{\partial^2 f}{\partial w\partial u}\right] + y\left[x\frac{\partial^2 f}{\partial u\partial v} + y\frac{\partial^2 f}{\partial v^2} + z\frac{\partial^2 f}{\partial w\partial v}\right]\\
&\quad + z\left[x\frac{\partial^2 f}{\partial u\partial w} + y\frac{\partial^2 f}{\partial v\partial w} + z\frac{\partial^2 f}{\partial w^2 u}\right].
\end{aligned}$$

If we set $t = 1$, we obtain the identity

$$n(n-1)f(x,y,z) = x^2\frac{\partial^2 f}{\partial x^2} + y^2\frac{\partial^2 f}{\partial y^2} + z^2\frac{\partial^2 f}{\partial z^2} + 2xy\frac{\partial^2 f}{\partial x\partial y} + 2yz\frac{\partial^2 f}{\partial y\partial z} + 2zx\frac{\partial^2 f}{\partial x\partial z}.$$

41. (a) Since $\dfrac{\partial u}{\partial x} = \dfrac{du}{dk}\dfrac{\partial k}{\partial x}$ and $\dfrac{\partial u}{\partial t} = \dfrac{du}{dk}\dfrac{\partial k}{\partial t}$, it follows that

$$u\frac{du}{dk}\frac{\partial k}{\partial x} + \frac{du}{dk}\frac{\partial k}{\partial t} = -c^2k^n\frac{\partial k}{\partial x} \quad \text{or} \quad \frac{du}{dk}\left(u\frac{\partial k}{\partial x} + \frac{\partial k}{\partial t}\right) = -c^2k^n\frac{\partial k}{\partial x}.$$



(b) If we use the equation of continuity in the form $u\dfrac{\partial k}{\partial x} + \dfrac{\partial k}{\partial t} = -k\dfrac{\partial u}{\partial x}$ to substitute into the equation in part (a),

$$-c^2k^n\frac{\partial k}{\partial x} = \frac{du}{dk}\left(-k\frac{\partial u}{\partial x}\right) = -k\frac{du}{dk}\left(\frac{du}{dk}\frac{\partial k}{\partial x}\right).$$

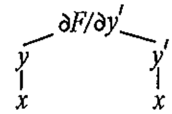
Thus, $c^2k^{n-1} = \left(\dfrac{du}{dk}\right)^2$ or $\dfrac{du}{dk} = \pm ck^{(n-1)/2}$. Since $u(k)$ must be a decreasing function, it follows that $\dfrac{du}{dk} = -ck^{(n-1)/2}$.

(c) Integration gives $u(k) = -\dfrac{2c}{n+1}k^{(n+1)/2} + E$.

42. (a) First we calculate that $\dfrac{\partial F}{\partial y} = -\dfrac{\sqrt{1+(y')^2}}{2y^{3/2}}$ and $\dfrac{\partial F}{\partial y'} = \dfrac{y'}{\sqrt{y}\sqrt{1+(y')^2}}$.

From the schematic,

$$\begin{aligned}
\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) &= \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y'}\right)\frac{dy}{dx} + \frac{\partial}{\partial y'}\left(\frac{\partial F}{\partial y'}\right)\frac{d(y')}{dx} \\
&= \left[\frac{-y'}{2y^{3/2}\sqrt{1+(y')^2}}\right]y' + \left[\frac{1}{\sqrt{y}\sqrt{1+(y')^2}} - \frac{(y')^2}{\sqrt{y}[1+(y')^2]^{3/2}}\right]y'' \\
&= \frac{-(y')^2[1+(y')^2] + 2yy''[1+(y')^2] - 2y(y')^2y''}{2y^{3/2}[1+(y')^2]^{3/2}} \\
&= \frac{2yy'' - (y')^2 - (y')^4}{2y^{3/2}[1+(y')^2]^{3/2}}.
\end{aligned}$$



Consequently, $y(x)$ must satisfy

$$\begin{aligned}
0 &= \frac{2yy'' - (y')^2 - (y')^4}{2y^{3/2}[1+(y')^2]^{3/2}} + \frac{\sqrt{1+(y')^2}}{2y^{3/2}} \\
&= \frac{2yy'' - (y')^2 - (y')^4 + [1+(y')^2]^2}{2y^{3/2}[1+(y')^2]^{3/2}} = \frac{2yy'' + (y')^2 + 1}{2y^{3/2}[1+(y')^2]^{3/2}}.
\end{aligned}$$

Thus, $2yy'' + (y')^2 + 1 = 0$.

(b) Since $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{a\sin\theta}{a(1-\cos\theta)} = \dfrac{\sin\theta}{1-\cos\theta}$, and

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{d\theta}\left(\frac{\sin\theta}{1-\cos\theta}\right)\frac{d\theta}{dx} = \frac{\dfrac{d}{d\theta}\left(\dfrac{\sin\theta}{1-\cos\theta}\right)}{dx/d\theta} = \frac{\dfrac{(1-\cos\theta)\cos\theta - \sin\theta(\sin\theta)}{(1-\cos\theta)^2}}{a(1-\cos\theta)} \\
&= \frac{\cos\theta - 1}{a(1-\cos\theta)^3} = \frac{-1}{a(1-\cos\theta)^2},
\end{aligned}$$

it follows that

$$\begin{aligned}
2yy'' + (y')^2 + 1 &= \frac{-2a(1-\cos\theta)}{a(1-\cos\theta)^2} + \left(\frac{\sin\theta}{1-\cos\theta}\right)^2 + 1 = \frac{-2}{1-\cos\theta} + \frac{\sin^2\theta}{(1-\cos\theta)^2} + 1 \\
&= \frac{-2(1-\cos\theta) + \sin^2\theta + (1 - 2\cos\theta + \cos^2\theta)}{(1-\cos\theta)^2} = 0.
\end{aligned}$$

(c) Suppose that the bead starts at any other point (x_1, y_1) on the cycloid corresponding to parameter value θ_1. Its velocity at any point (x, y) is given by

$$mg(y - y_1) = \frac{1}{2}mv^2 \quad \Longrightarrow \quad v = \sqrt{2g(y-y_1)}.$$

The time to travel from (x_1, y_1) to (x_0, y_0) is

$$\begin{aligned}
t &= \int_{x_1}^{x_0} \frac{\sqrt{(dx)^2 + (dy)^2}}{\sqrt{2g(y-y_1)}} = \frac{1}{\sqrt{2g}}\int_{\theta_1}^{\pi} \frac{1}{\sqrt{a(\cos\theta_1 - \cos\theta)}}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta \\
&= \frac{1}{\sqrt{2ga}}\int_{\theta_1}^{\pi}\sqrt{\frac{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}{\cos\theta_1 - \cos\theta}}\,d\theta = \sqrt{\frac{a}{2g}}\int_{\theta_1}^{\pi}\sqrt{\frac{2 - 2\cos\theta}{\cos\theta_1 - \cos\theta}}\,d\theta \\
&= \sqrt{\frac{a}{g}}\int_{\theta_1}^{\pi}\sqrt{\frac{1 - [1 - 2\sin^2(\theta/2)]}{[2\cos^2(\theta_1/2) - 1] - [2\cos^2(\theta/2) - 1]}}\,d\theta = \sqrt{\frac{a}{g}}\int_{\theta_1}^{\pi}\frac{\sin(\theta/2)}{\sqrt{\cos^2(\theta_1/2) - \cos^2(\theta/2)}}\,d\theta.
\end{aligned}$$

If we set $u = \cos(\theta/2)$ and $du = -(1/2)\sin(\theta/2)\,d\theta$, then

$$t = \sqrt{\frac{a}{g}} \int_{\cos(\theta_1/2)}^{0} \frac{-2}{\sqrt{\cos^2(\theta_1/2) - u^2}}\,du = -2\sqrt{\frac{a}{g}} \left\{ \mathrm{Sin}^{-1}\!\left(\frac{u}{\cos(\theta_1/2)} \right) \right\}_{\cos(\theta_1/2)}^{0} = 2\sqrt{\frac{a}{g}} \left(\frac{\pi}{2}\right) = \pi\sqrt{\frac{a}{g}}.$$

Since this is independent of θ_1, the time taken is the same for all starting points.

43. (a) Newton's second law applied to the two masses gives

$$m\frac{d^2x_1}{dt^2} = -kx_1 + k(x_2 - x_1), \qquad m\frac{d^2x_2}{dt^2} = -kx_2 - k(x_2 - x_1),$$

or, $m\ddot{x}_1 = k(x_2 - 2x_1)$, $m\ddot{x}_2 = k(x_1 - 2x_2)$.

(b) Kinetic energy of a mass is one-half the product of its mass and the square of its speed. Potential energy in a spring is one-half the product of the spring constant and the square of its stretch (or compression). When the masses are displaced as in part (a) then,

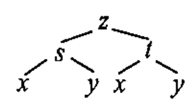
$$L = \frac{1}{2}m\dot{x}_1{}^2 + \frac{1}{2}m\dot{x}_2{}^2 - \frac{1}{2}kx_1{}^2 - \frac{1}{2}k(x_2 - x_1)^2 - \frac{1}{2}kx_2{}^2 = \frac{m}{2}(\dot{x}_1{}^2 + \dot{x}_2{}^2) - k(x_1{}^2 + x_2{}^2 - x_1x_2).$$

(c) Since $\dfrac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1$, $\quad \dfrac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2$, $\quad \dfrac{\partial L}{\partial x_1} = -k(2x_1 - x_2)$, $\quad \dfrac{\partial L}{\partial x_2} = -k(2x_2 - x_1)$,

the Euler-Lagrange equations give

$$0 = m\ddot{x}_1 + k(2x_1 - x_2) \qquad \text{and} \qquad 0 = m\ddot{x}_2 + k(2x_2 - x_1).$$

44. The schematic gives $\quad \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial s}\dfrac{\partial s}{\partial x} + \dfrac{\partial z}{\partial t}\dfrac{\partial t}{\partial x} = a\dfrac{\partial z}{\partial s} + c\dfrac{\partial z}{\partial t}.$



Since the same schematic applies to $\partial z/\partial x$,

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial x}\right)\frac{\partial s}{\partial x} + \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\right)\frac{\partial t}{\partial x} = \left(a\frac{\partial^2 z}{\partial s^2} + c\frac{\partial^2 z}{\partial s \partial t}\right)a + \left(a\frac{\partial^2 z}{\partial t \partial s} + c\frac{\partial^2 z}{\partial t^2}\right)c$$
$$= a^2\frac{\partial^2 z}{\partial s^2} + 2ac\frac{\partial^2 z}{\partial s \partial t} + c^2\frac{\partial^2 z}{\partial t^2}.$$

Similarly, $\dfrac{\partial^2 z}{\partial y^2} = b^2\dfrac{\partial^2 z}{\partial s^2} + 2bd\dfrac{\partial^2 z}{\partial s \partial t} + d^2\dfrac{\partial^2 z}{\partial t^2}$ and $\dfrac{\partial^2 z}{\partial x \partial y} = ab\dfrac{\partial^2 z}{\partial s^2} + (bc + ad)\dfrac{\partial^2 z}{\partial s \partial t} + cd\dfrac{\partial^2 z}{\partial t^2}$. Substituting these into the partial differential equation for $z(x, y)$ gives

$$p\left(a^2\frac{\partial^2 z}{\partial s^2} + 2ac\frac{\partial^2 z}{\partial s \partial t} + c^2\frac{\partial^2 z}{\partial t^2}\right) + q\left(ab\frac{\partial^2 z}{\partial s^2} + (bc + ad)\frac{\partial^2 z}{\partial s \partial t} + cd\frac{\partial^2 z}{\partial t^2}\right)$$
$$+ r\left(b^2\frac{\partial^2 z}{\partial s^2} + 2bd\frac{\partial^2 z}{\partial s \partial t} + d^2\frac{\partial^2 z}{\partial t^2}\right) = F\left(x(s,t), y(s,t), z, a\frac{\partial z}{\partial s} + c\frac{\partial z}{\partial t}, b\frac{\partial z}{\partial s} + d\frac{\partial z}{\partial t}\right).$$

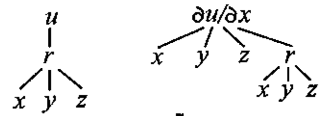
If we define $P = pa^2 + qab + rb^2$, $Q = 2pac + q(bc + ad) + 2rbd$, and $R = pc^2 + qcd + rd^2$, then this equation can be expressed in the form

$$P\frac{\partial^2 z}{\partial s^2} + Q\frac{\partial^2 z}{\partial s \partial t} + R\frac{\partial^2 z}{\partial t^2} = G\left(s, t, z, \frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}\right),$$

where

$$\begin{aligned}
Q^2 - 4PR &= [2pac + q(bc + ad) + 2rbd]^2 - 4(pa^2 + qab + rb^2)(pc^2 + qcd + rd^2) \\
&= 4p^2a^2c^2 + q^2(b^2c^2 + 2abcd + a^2d^2) + 4r^2b^2d^2 + 4pqac(bc + ad) + 4qrbd(bc + ad) + 8prabcd \\
&\qquad - 4pa^2(pc^2 + qcd + rd^2) - 4qab(pc^2 + qcd + rd^2) - 4rb^2(pc^2 + qcd + rd^2) \\
&= q^2(a^2d^2 - 2abcd + b^2c^2) - 4pr(a^2d^2 - 2abcd + b^2c^2) = (q^2 - 4pr)(ad - bc)^2.
\end{aligned}$$

45. Since $\dfrac{\partial u}{\partial x} = \dfrac{du}{dr}\dfrac{\partial r}{\partial x} = \dfrac{du}{dr}\dfrac{x}{\sqrt{x^2+y^2+z^2}}$, it follows that



$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)_{y,z,r} + \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)\frac{\partial r}{\partial x}\\ &= \frac{du}{dr}\left[\frac{1}{\sqrt{x^2+y^2+z^2}} - \frac{x^2}{(x^2+y^2+z^2)^{3/2}}\right] + \left[\frac{x}{\sqrt{x^2+y^2+z^2}}\frac{d^2u}{dr^2}\right]\frac{x}{\sqrt{x^2+y^2+z^2}}\\ &= \frac{1}{(x^2+y^2+z^2)^{3/2}}\left[(x^2+y^2+z^2-x^2)\frac{du}{dr} + x^2\sqrt{x^2+y^2+z^2}\frac{d^2u}{dr^2}\right]\\ &= \frac{1}{r^3}\left[(y^2+z^2)\frac{du}{dr} + rx^2\frac{d^2u}{dr^2}\right].\end{aligned}$$

With similar results for $\partial^2u/\partial y^2$ and $\partial^2u/\partial z^2$, Laplace's equation becomes

$$\begin{aligned}0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} &= \frac{1}{r^3}\Big[(y^2+z^2)\frac{du}{dr} + rx^2\frac{d^2u}{dr^2} + (x^2+z^2)\frac{du}{dr} + ry^2\frac{d^2u}{dr^2}\\ &\qquad + (x^2+y^2)\frac{du}{dr} + rz^2\frac{d^2u}{dr^2}\Big]\\ &= \frac{1}{r^3}\left[2(x^2+y^2+z^2)\frac{du}{dr} + r(x^2+y^2+z^2)\frac{d^2u}{dr^2}\right]\\ &= \frac{2}{r}\frac{du}{dr} + \frac{d^2u}{dr^2}.\end{aligned}$$

When this equation is multiplied by r^2, $0 = 2r\dfrac{du}{dr} + r^2\dfrac{d^2u}{dr^2} = \dfrac{d}{dr}\left(r^2\dfrac{du}{dr}\right)$. Consequently,

$$r^2\frac{du}{dr} = E = \text{constant} \quad \Longrightarrow \quad \frac{du}{dr} = \frac{E}{r^2}.$$

Integration gives $u = -\dfrac{E}{r} + D = \dfrac{C}{r} + D = \dfrac{C}{\sqrt{x^2+y^2+z^2}} + D.$

EXERCISES 12.7

1. If we set $F(x,y) = x^3y^2 - 2xy + 5$, then $\dfrac{dy}{dx} = -\dfrac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(y)}} = -\dfrac{F_x}{F_y} = -\dfrac{3x^2y^2-2y}{2x^3y-2x} = \dfrac{2y-3x^2y^2}{2x^3y-2x}$.

2. If we set $F(x,y) = (x+y)^2 - 2x$, then $\dfrac{dy}{dx} = -\dfrac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(y)}} = -\dfrac{F_x}{F_y} = -\dfrac{2(x+y)-2}{2(x+y)} = \dfrac{1-x-y}{x+y}$.

3. If we set $F(x,y) = x^2 - xy - 4y^3 - 2e^{xy} - 6$, then

$$\frac{dy}{dx} = -\frac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(y)}} = -\frac{F_x}{F_y} = -\frac{2x-y-2ye^{xy}}{-x-12y^2-2xe^{xy}} = \frac{2x-y-2ye^{xy}}{x+12y^2+2xe^{xy}}.$$

4. If we set $F(x,y) = \sin(x+y) + y^2 - 12x^2 - y$, then

$$\frac{dy}{dx} = -\frac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(y)}} = -\frac{F_x}{F_y} = -\frac{\cos(x+y)-24x}{\cos(x+y)+2y-1} = \frac{24x-\cos(x+y)}{\cos(x+y)+2y-1}.$$

5. If we set $F(x,y,z) = x^2\sin z - ye^z - 2x$, then $\dfrac{\partial z}{\partial x} = -\dfrac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(z)}} = -\dfrac{F_x}{F_z} = -\dfrac{2x\sin z - 2}{x^2\cos z - ye^z} = \dfrac{2(1 - x\sin z)}{x^2\cos z - ye^z}$,

and $\dfrac{\partial z}{\partial y} = -\dfrac{\dfrac{\partial(F)}{\partial(y)}}{\dfrac{\partial(F)}{\partial(z)}} = -\dfrac{F_y}{F_z} = -\dfrac{-e^z}{x^2\cos z - ye^z} = \dfrac{e^z}{x^2\cos z - ye^z}$.

6. If we set $F(x,y,z) = x^2z^2 + yz + 3x - 4$, then

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(z)}} = -\frac{F_x}{F_z} = -\frac{2xz^2 + 3}{2x^2z + y}; \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial(F)}{\partial(y)}}{\dfrac{\partial(F)}{\partial(z)}} = -\frac{F_y}{F_z} = -\frac{z}{2x^2z + y}.$$

7. If we set $F(x,y,z) = z\sin^2 y + y\sin^2 x - z^3$, then $\dfrac{\partial z}{\partial x} = -\dfrac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(z)}} = -\dfrac{F_x}{F_z} = -\dfrac{2y\sin x\cos x}{\sin^2 y - 3z^2} = \dfrac{2y\sin x\cos x}{3z^2 - \sin^2 y}$,

and $\dfrac{\partial z}{\partial y} = -\dfrac{\dfrac{\partial(F)}{\partial(y)}}{\dfrac{\partial(F)}{\partial(z)}} = -\dfrac{F_y}{F_z} = -\dfrac{2z\sin y\cos y + \sin^2 x}{\sin^2 y - 3z^2} = \dfrac{2z\sin y\cos y + \sin^2 x}{3z^2 - \sin^2 y}$.

8. If we set $F(x,y,z) = \mathrm{Tan}^{-1}(yz) - xz$, then $\dfrac{\partial z}{\partial x} = -\dfrac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(z)}} = -\dfrac{F_x}{F_z} = -\dfrac{-z}{\dfrac{y}{1 + y^2z^2} - x} = \dfrac{z(1 + y^2z^2)}{y - x(1 + y^2z^2)}$,

and $\dfrac{\partial z}{\partial y} = -\dfrac{\dfrac{\partial(F)}{\partial(y)}}{\dfrac{\partial(F)}{\partial(z)}} = -\dfrac{F_y}{F_z} = -\dfrac{\dfrac{z}{1 + y^2z^2}}{\dfrac{y}{1 + y^2z^2} - x} = \dfrac{z}{x(1 + y^2z^2) - y}$.

9. If we set $F(x,y,u,v) = x^2 - y^2 + u^2 + 2v^2 - 1$ and $G(x,y,u,v) = x^2 + y^2 - u^2 - v^2 - 2$, then

$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(F,G)}{\partial(x,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} 2x & 4v \\ 2x & -2v \end{vmatrix}}{\begin{vmatrix} 2u & 4v \\ -2u & -2v \end{vmatrix}} = \frac{3x}{u},$$

and $\dfrac{\partial v}{\partial y} = -\dfrac{\dfrac{\partial(F,G)}{\partial(u,y)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\dfrac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{4uv} = -\dfrac{\begin{vmatrix} 2u & -2y \\ -2u & 2y \end{vmatrix}}{4uv} = 0$.

10. If we set $F(x,t,z) = \sin(x + t) - \sin(x - t) - z$, then

$$\frac{\partial x}{\partial t} = -\frac{\dfrac{\partial(F)}{\partial(t)}}{\dfrac{\partial(F)}{\partial(x)}} = -\frac{F_t}{F_x} = -\frac{\cos(x + t) + \cos(x - t)}{\cos(x + t) - \cos(x - t)} = \frac{\cos(x + t) + \cos(x - t)}{\cos(x - t) - \cos(x + t)}.$$

11. If we set $F(x,r,\phi,\theta)=r\sin\phi\cos\theta-x$, $G(y,r,\phi,\theta)=r\sin\phi\sin\theta-y$, and $H(z,r,\phi)=r\cos\phi-z$, then

$$\frac{\partial\phi}{\partial x}=-\frac{\dfrac{\partial(F,G,H)}{\partial(r,x,\theta)}}{\dfrac{\partial(F,G,H)}{\partial(r,\phi,\theta)}}=-\frac{\begin{vmatrix}F_r & F_x & F_\theta\\ G_r & G_x & G_\theta\\ H_r & H_x & H_\theta\end{vmatrix}}{\begin{vmatrix}F_r & F_\phi & F_\theta\\ G_r & G_\phi & G_\theta\\ H_r & H_\phi & H_\theta\end{vmatrix}}=-\frac{\begin{vmatrix}\sin\phi\cos\theta & -1 & -r\sin\phi\sin\theta\\ \sin\phi\sin\theta & 0 & r\sin\phi\cos\theta\\ \cos\phi & 0 & 0\end{vmatrix}}{\begin{vmatrix}\sin\phi\cos\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta\\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta\\ \cos\phi & -r\sin\phi & 0\end{vmatrix}}$$

$$=\frac{r\sin\phi\cos\phi\cos\theta}{\cos\phi(r^2\sin\phi\cos\phi)+r\sin\phi(r\sin^2\phi)}=\frac{r\sin\phi\cos\phi\cos\theta}{r^2\sin\phi}=\frac{\cos\phi\cos\theta}{r}.$$

12. If we set $F(x,y,z)=x^2+y^2-z^2+2xy-1$ and $G(x,y)=x^3+y^3-5y-4$, then

$$\frac{dz}{dx}=-\frac{\dfrac{\partial(F,G)}{\partial(y,x)}}{\dfrac{\partial(F,G)}{\partial(y,z)}}=-\frac{\begin{vmatrix}F_y & F_x\\ G_y & G_x\end{vmatrix}}{\begin{vmatrix}F_y & F_z\\ G_y & G_z\end{vmatrix}}=-\frac{\begin{vmatrix}2y+2x & 2x+2y\\ 3y^2-5 & 3x^2\end{vmatrix}}{\begin{vmatrix}2y+2x & -2z\\ 3y^2-5 & 0\end{vmatrix}}$$

$$=-\frac{3x^2(2y+2x)-(2x+2y)(3y^2-5)}{2z(3y^2-5)}=\frac{(x+y)(3x^2-3y^2+5)}{z(5-3y^2)}.$$

13. If we set $F(x,y,u,v,w)=xyu+vw-4$, $G(y,u,v)=y^2+u^2-u^2v-y$, and $H(x,y,u,v,w)=yw+xu+v+4$, then

$$\frac{\partial u}{\partial y}=-\frac{\dfrac{\partial(F,G,H)}{\partial(y,v,w)}}{\dfrac{\partial(F,G,H)}{\partial(u,v,w)}}=-\frac{\begin{vmatrix}F_y & F_v & F_w\\ G_y & G_v & G_w\\ H_y & H_v & H_w\end{vmatrix}}{\begin{vmatrix}F_u & F_v & F_w\\ G_u & G_v & G_w\\ H_u & H_v & H_w\end{vmatrix}}=-\frac{\begin{vmatrix}xu & w & v\\ 2y-1 & -u^2 & 0\\ w & 1 & y\end{vmatrix}}{\begin{vmatrix}xy & w & v\\ 2u(1-v) & -u^2 & 0\\ x & 1 & y\end{vmatrix}}$$

$$=-\frac{v[(2y-1)+wu^2]+y[-xu^3-w(2y-1)]}{v[2u(1-v)+xu^2]+y[-xyu^2-2uw(1-v)]}=\frac{(2y-1)(yw-v)+u^2(xyu-vw)}{2u(1-v)(v-yw)+xu^2(v-y^2)}.$$

14. If we set $F(x,y,z,u,v)=x^2-y\cos(uv)+z^2$, $G(x,y,z,u,v)=x^2+y^2-\sin(uv)+2z^2-2$, and $H(x,y,z,u,v)=xy-\sin u\,\cos v+z$, then

$$\frac{\partial x}{\partial u}=-\frac{\dfrac{\partial(F,G,H)}{\partial(u,y,z)}}{\dfrac{\partial(F,G,H)}{\partial(x,y,z)}}=-\frac{\begin{vmatrix}F_u & F_y & F_z\\ G_u & G_y & G_z\\ H_u & H_y & H_z\end{vmatrix}}{\begin{vmatrix}F_x & F_y & F_z\\ G_x & G_y & G_z\\ H_x & H_y & H_z\end{vmatrix}}=-\frac{\begin{vmatrix}yv\sin(uv) & -\cos(uv) & 2z\\ -v\cos(uv) & 2y & 4z\\ -\cos u\,\cos v & x & 1\end{vmatrix}}{\begin{vmatrix}2x & -\cos(uv) & 2z\\ 2x & 2y & 4z\\ y & x & 1\end{vmatrix}}.$$

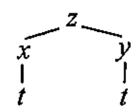
When $x=y=1$, $u=\pi/2$, $v=z=0$, $\displaystyle\frac{\partial x}{\partial u}=-\frac{\begin{vmatrix}0 & -1 & 0\\ 0 & 2 & 0\\ 0 & 1 & 1\end{vmatrix}}{\begin{vmatrix}2 & -1 & 0\\ 2 & 2 & 0\\ 1 & 1 & 1\end{vmatrix}}=0.$

15. $$\left(\frac{\partial z}{\partial x}\right)_y\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x=\left[-\frac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(z)}}\right]\left[-\frac{\dfrac{\partial(F)}{\partial(y)}}{\dfrac{\partial(F)}{\partial(x)}}\right]\left[-\frac{\dfrac{\partial(F)}{\partial(z)}}{\dfrac{\partial(F)}{\partial(y)}}\right]=-1$$

16. $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = e^x \cos y \dfrac{dx}{dt} - e^x \sin y \dfrac{dy}{dt}$ If we set $F(x,t) = x^3 + e^x - t^2 - t - 1$ and

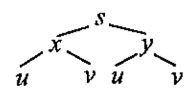
$G(y,t) = yt^2 + y^2t - t + y$, then $\dfrac{dx}{dt} = -\dfrac{\dfrac{\partial(F)}{\partial(t)}}{\dfrac{\partial(F)}{\partial(x)}} = -\dfrac{F_t}{F_x} = -\dfrac{-2t-1}{3x^2+e^x} = \dfrac{2t+1}{3x^2+e^x}$, and

$$\frac{dy}{dt} = -\frac{\dfrac{\partial(G)}{\partial(t)}}{\dfrac{\partial(G)}{\partial(y)}} = -\frac{G_t}{G_y} = -\frac{2yt+y^2-1}{t^2+2yt+1} = \frac{1-2yt-y^2}{1+2yt+t^2}.$$



Consequently, $\dfrac{dz}{dt} = e^x \cos y \left(\dfrac{2t+1}{3x^2+e^x}\right) - e^x \sin y \left(\dfrac{1-2yt-y^2}{1+2yt+t^2}\right)$.

17. The chain rule gives $\dfrac{\partial s}{\partial u} = \dfrac{\partial s}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial s}{\partial y}\dfrac{\partial y}{\partial u} = 2x\dfrac{\partial x}{\partial u} + 2y\dfrac{\partial y}{\partial u}$.
If we set $F(x,y,u) = u - x^2 + y^2$ and
$G(x,y,v) = v - x^2 + y$, then

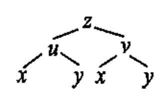


$$\frac{\partial x}{\partial u} = -\frac{\dfrac{\partial(F,G)}{\partial(u,y)}}{\dfrac{\partial(F,G)}{\partial(x,y)}} = -\frac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}} = -\frac{\begin{vmatrix} 1 & 2y \\ 0 & 1 \end{vmatrix}}{\begin{vmatrix} -2x & 2y \\ -2x & 1 \end{vmatrix}} = -\frac{1}{-2x+4xy} = \frac{1}{2x(1-2y)},$$

and $\dfrac{\partial y}{\partial u} = -\dfrac{\dfrac{\partial(F,G)}{\partial(x,u)}}{\dfrac{\partial(F,G)}{\partial(x,y)}} = -\dfrac{\begin{vmatrix} F_x & F_u \\ G_x & G_u \end{vmatrix}}{-2x+4xy} = -\dfrac{\begin{vmatrix} -2x & 1 \\ -2x & 0 \end{vmatrix}}{2x(2y-1)} = \dfrac{-2x}{2x(2y-1)} = \dfrac{1}{1-2y}$.

Thus, $\dfrac{\partial s}{\partial u} = \dfrac{2x}{2x(1-2y)} + \dfrac{2y}{1-2y} = \dfrac{1+2y}{1-2y}$.

18. $\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y} = [3u^2v + v\cos(uv)]\dfrac{\partial u}{\partial y} + [u^3 + u\cos(uv)]\dfrac{\partial v}{\partial y}$
If we set $F(x,u,v) = e^u \cos v - x$, $G(y,u,v) = e^u \sin v - y$, then



$$\frac{\partial u}{\partial y} = -\frac{\dfrac{\partial(F,G)}{\partial(y,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} 0 & -e^u \sin v \\ -1 & e^u \cos v \end{vmatrix}}{\begin{vmatrix} e^u \cos v & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{vmatrix}} = \frac{e^u \sin v}{e^{2u}} = e^{-u} \sin v,$$

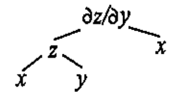
and $\dfrac{\partial v}{\partial y} = -\dfrac{\dfrac{\partial(F,G)}{\partial(u,y)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\dfrac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{e^{2u}} = -\dfrac{\begin{vmatrix} e^u \cos v & 0 \\ e^u \sin v & -1 \end{vmatrix}}{e^{2u}} = e^{-u} \cos v$.

Thus, $\dfrac{\partial z}{\partial y} = [3u^2v + v\cos(uv)]e^{-u} \sin v + [u^3 + u\cos(uv)]e^{-u} \cos v$.

19. If we define $F(x,y,z) = z^3 - xz - y$, then $\dfrac{\partial z}{\partial y} = -\dfrac{\dfrac{\partial(F)}{\partial(y)}}{\dfrac{\partial(F)}{\partial(z)}} = -\dfrac{F_y}{F_z} = -\dfrac{-1}{3z^2 - x} = \dfrac{1}{3z^2 - x}$.

The chain rule now gives

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)_z = -\frac{6z}{(3z^2 - x)^2}\frac{\partial z}{\partial x} + \frac{1}{(3z^2 - x)^2}.$$



Since $\dfrac{\partial z}{\partial x} = -\dfrac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(z)}} = -\dfrac{F_x}{F_z} = -\dfrac{-z}{3z^2 - x} = \dfrac{z}{3z^2 - x}$,

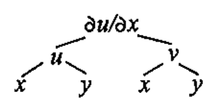
we have $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{-6z^2}{(3z^2 - x)^3} + \dfrac{1}{(3z^2 - x)^2} = -\dfrac{3z^2 + x}{(3z^2 - x)^3}$.

20. If we set $F(x,u,v) = x - u^2 + v^2$, $G(y,u,v) = y - 2uv$, then

$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(F,G)}{\partial(x,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} 1 & 2v \\ 0 & -2u \end{vmatrix}}{\begin{vmatrix} -2u & 2v \\ -2v & -2u \end{vmatrix}} = \frac{2u}{4u^2 + 4v^2} = \frac{u}{2(u^2 + v^2)}.$$

The chain rule now gives

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial u}\left(\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial u}{\partial x}\right)\frac{\partial v}{\partial x} \\ &= \left[\frac{2(u^2 + v^2) - u(4u)}{4(u^2 + v^2)^2}\right]\frac{\partial u}{\partial x} + \left[\frac{-2uv}{2(u^2 + v^2)^2}\right]\frac{\partial v}{\partial x}.\end{aligned}$$



Since $\dfrac{\partial v}{\partial x} = -\dfrac{\dfrac{\partial(F,G)}{\partial(u,x)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\dfrac{\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}}{4(u^2 + v^2)} = -\dfrac{\begin{vmatrix} -2u & 1 \\ -2v & 0 \end{vmatrix}}{4(u^2 + v^2)} = \dfrac{-2v}{4(u^2 + v^2)} = \dfrac{-v}{2(u^2 + v^2)}$,

we obtain $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{v^2 - u^2}{2(u^2 + v^2)^2}\dfrac{u}{2(u^2 + v^2)} - \dfrac{uv}{(u^2 + v^2)^2}\dfrac{-v}{2(u^2 + v^2)} = \dfrac{3uv^2 - u^3}{4(u^2 + v^2)^3}$.

21. (a) If we define $F(x,y,z) = z^4 x + y^3 z + 9x^3 - 2$, then

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(z)}} \quad \text{and} \quad \frac{\partial x}{\partial z} = -\frac{\dfrac{\partial(F)}{\partial(z)}}{\dfrac{\partial(F)}{\partial(x)}}.$$

Therefore, $\partial z/\partial x$ and $\partial x/\partial z$ are reciprocals.

(b) If we define $F(x,y,z) = z^4 x + y^3 z + 9x^3 - 2$ and $G(x,y,z) = x^2 y + xz - 1$, then

$$\frac{dz}{dx} = -\frac{\dfrac{\partial(F,G)}{\partial(y,x)}}{\dfrac{\partial(F,G)}{\partial(y,z)}} \quad \text{and} \quad \frac{dx}{dz} = -\frac{\dfrac{\partial(F,G)}{\partial(y,z)}}{\dfrac{\partial(F,G)}{\partial(y,x)}}.$$

Thus, dz/dx and dx/dz are reciprocals.

(c) If we define $F(x,y,u,v) = u^2 - v - 3x - y$ and $G(x,y,u,v) = u - 2v^2 - x + 2y$, then

$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(F,G)}{\partial(x,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} \quad \text{and} \quad \frac{\partial x}{\partial u} = -\frac{\dfrac{\partial(F,G)}{\partial(u,y)}}{\dfrac{\partial(F,G)}{\partial(x,y)}}.$$

These are not reciprocals.

22. If we set $F(x,y,s,t) = x^2 - 2y^2s^2t - 2st^2 - 1$ and $G(x,y,s,t) = x^2 + 2y^2s^2t + 5st^2 - 1$, then

$$\frac{\partial t}{\partial y} = -\frac{\dfrac{\partial(F,G)}{\partial(s,y)}}{\dfrac{\partial(F,G)}{\partial(s,t)}} = -\frac{\begin{vmatrix} F_s & F_y \\ G_s & G_y \end{vmatrix}}{\begin{vmatrix} F_s & F_t \\ G_s & G_t \end{vmatrix}} = -\frac{\begin{vmatrix} -4y^2st - 2t^2 & -4ys^2t \\ 4y^2st + 5t^2 & 4ys^2t \end{vmatrix}}{\begin{vmatrix} -4y^2st - 2t^2 & -2y^2s^2 - 4st \\ 4y^2st + 5t^2 & 2y^2s^2 + 10st \end{vmatrix}}$$

$$= -\frac{4ys^2t(-4y^2st - 2t^2 + 4y^2st + 5t^2)}{4y^2st(-2y^2s^2 - 10st + 2y^2s^2 + 4st) + t^2(-4y^2s^2 - 20st + 10y^2s^2 + 20st)}$$

$$= \frac{-12ys^2t^3}{-24y^2s^2t^2 + 6y^2s^2t^2} = \frac{2t}{3y}.$$

Thus, $\dfrac{\partial^2 t}{\partial y^2} = \dfrac{2}{3y}\dfrac{\partial t}{\partial y} - \dfrac{2t}{3y^2} = \dfrac{2}{3y}\left(\dfrac{2t}{3y}\right) - \dfrac{2t}{3y^2} = -\dfrac{2t}{9y^2}.$

23. (a) $$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \left(\frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}\right)\left(\frac{\partial v}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial y}\right)$$

$$- \left(\frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}\right)\left(\frac{\partial v}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial x}\right)$$

$$= \frac{\partial u}{\partial s}\frac{\partial s}{\partial x}\left(\frac{\partial v}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial y}\right) + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}\left(\frac{\partial v}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial y}\right)$$

$$- \frac{\partial u}{\partial s}\frac{\partial s}{\partial y}\left(\frac{\partial v}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial x}\right) - \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}\left(\frac{\partial v}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial x}\right)$$

$$= \left(\frac{\partial u}{\partial s}\frac{\partial v}{\partial t} - \frac{\partial u}{\partial t}\frac{\partial v}{\partial s}\right)\left(\frac{\partial s}{\partial x}\frac{\partial t}{\partial y} - \frac{\partial s}{\partial y}\frac{\partial t}{\partial x}\right)$$

$$= \frac{\partial(u,v)}{\partial(s,t)}\frac{\partial(s,t)}{\partial(x,y)}$$

(b) In part (a) we replace s and t by x and y in F, G, H, and I, and x and y by u and v in H and I,

$$\frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial(u,v)}{\partial(u,v)} = 1 \quad \Longrightarrow \quad \frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\dfrac{\partial(x,y)}{\partial(u,v)}}.$$

24. If we set $F_i = \displaystyle\sum_{j=1}^{n} a_{ij}x_j - c_i$, for $i = 1, \ldots, m$, then

$$\frac{\partial x_i}{\partial x_j} = -\frac{\dfrac{\partial(F_1,\ldots,F_{i-1},F_i,F_{i+1},\ldots,F_m)}{\partial(x_1,\ldots,x_{i-1},x_j,x_{i+1},\ldots,x_m)}}{\dfrac{\partial(F_1,\ldots,F_{i-1},F_i,F_{i+1},\ldots,F_m)}{\partial(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_m)}}$$

$$= -\frac{\begin{vmatrix} \partial F_1/\partial x_1 & \cdots & \partial F_1/\partial x_{i-1} & \partial F_1/\partial x_j & \partial F_1/\partial x_{i+1} & \cdots & \partial F_1/\partial x_m \\ \vdots & & & \vdots & & & \vdots \\ \partial F_m/\partial x_1 & \cdots & \partial F_m/\partial x_{i-1} & \partial F_m/\partial x_j & \partial F_m/\partial x_{i+1} & \cdots & \partial F_m/\partial x_m \end{vmatrix}}{\begin{vmatrix} \partial F_1/\partial x_1 & \cdots & \partial F_1/\partial x_{i-1} & \partial F_1/\partial x_i & \partial F_1/\partial x_{i+1} & \cdots & \partial F_1/\partial x_m \\ \vdots & & & \vdots & & & \vdots \\ \partial F_m/\partial x_1 & \cdots & \partial F_m/\partial x_{i-1} & \partial F_m/\partial x_i & \partial F_m/\partial x_{i+1} & \cdots & \partial F_m/\partial x_m \end{vmatrix}}$$

$$= -\frac{\begin{vmatrix} a_{11} & \cdots & a_{1,i-1} & a_{1j} & a_{1,i+1} & \cdots & a_{1m} \\ \vdots & & & \vdots & & & \vdots \\ a_{m1} & \cdots & a_{m,i-1} & a_{mj} & a_{m,i+1} & \cdots & a_{mm} \end{vmatrix}}{\begin{vmatrix} a_{11} & \cdots & a_{1,i-1} & a_{1i} & a_{1,i+1} & \cdots & a_{1m} \\ \vdots & & & \vdots & & & \vdots \\ a_{m1} & \cdots & a_{m,i-1} & a_{mi} & a_{m,i+1} & \cdots & a_{mm} \end{vmatrix}} = -\frac{D_{ij}}{D}.$$

EXERCISES 12.8

1. The vector from $(1,2,3)$ to $(3,5,0)$ is $\mathbf{v} = (2,3,-3)$. At the point $(1,2,3)$,

$$D_{\mathbf{v}}f = \nabla f_{|(1,2,3)} \cdot \hat{\mathbf{v}} = (4x, -2y, 2z)_{|(1,2,3)} \cdot \frac{(2,3,-3)}{\sqrt{4+9+9}} = (4,-4,6) \cdot \frac{(2,3,-3)}{\sqrt{22}} = -\sqrt{22}.$$

2. The vector that joins $(3,2,1)$ to $(3,1,-1)$ is $\mathbf{v} = (0,-1,-2)$. At the point $(-1,1,-1)$,

$$D_{\mathbf{v}}f = \nabla f_{|(-1,1,-1)} \cdot \hat{\mathbf{v}} = (2xy+z, x^2, x)_{|(-1,1,-1)} \cdot \frac{(0,-1,-2)}{\sqrt{1+4}} = (-3,1,-1) \cdot \frac{(0,-1,-2)}{\sqrt{5}} = \frac{1}{\sqrt{5}}.$$

3. The vector from $(3,0)$ to $(-2,-4)$ is $\mathbf{v} = (-5,-4)$. At the point $(3,0)$,

$$D_{\mathbf{v}}f = \nabla f_{|(3,0)} \cdot \hat{\mathbf{v}} = (e^y, xe^y+1)_{|(3,0)} \cdot \frac{(-5,-4)}{\sqrt{25+16}} = (1,4) \cdot \frac{(-5,-4)}{\sqrt{41}} = \frac{-21}{\sqrt{41}}.$$

4. The vector from $(1,1,1)$ to $(-1,-2,3)$ is $\mathbf{v} = (-2,-3,2)$. At the point $(1,1,1)$,

$$\begin{aligned} D_{\mathbf{v}}f = \nabla f_{|(1,1,1)} \cdot \hat{\mathbf{v}} &= \left[\frac{1}{xy+yz+xz}(y+z, x+z, x+y)\right]_{|(1,1,1)} \cdot \frac{(-2,-3,2)}{\sqrt{4+9+4}} \\ &= \frac{1}{3}(2,2,2) \cdot \frac{(-2,-3,2)}{\sqrt{17}} = \frac{-2}{\sqrt{17}}. \end{aligned}$$

5. A vector along the line is $\mathbf{v} = (1,2)$. At the point $(1,2)$,

$$D_{\mathbf{v}}f = \nabla f_{|(1,2)} \cdot \hat{\mathbf{v}} = \left(\frac{y}{1+x^2y^2}, \frac{x}{1+x^2y^2}\right)_{|(1,2)} \cdot \frac{(1,2)}{\sqrt{1+4}} = \frac{(2,1)}{5} \cdot \frac{(1,2)}{\sqrt{5}} = \frac{4}{5\sqrt{5}}.$$

6. A vector along $3x+4y=-2$ in the direction of decreasing y is $\mathbf{v} = (4,-3)$. At the point $(2,-2)$,

$$D_{\mathbf{v}}f = \nabla f_{|(2,-2)} \cdot \hat{\mathbf{v}} = [\cos(x+y)(1,1)]_{|(2,-2)} \cdot \frac{(4,-3)}{\sqrt{16+9}} = (1,1) \cdot \frac{(4,-3)}{5} = \frac{1}{5}.$$

7. A vector along the line is $\mathbf{v} = (-1,-4,-2)$. At the point $(3,-1,-2)$,

$$\begin{aligned} D_{\mathbf{v}}f = \nabla f_{|(3,-1,-2)} \cdot \hat{\mathbf{v}} &= (3x^2y\sin z, x^3\sin z, x^3y\cos z)_{|(3,-1,-2)} \cdot \frac{(-1,-4,-2)}{\sqrt{1+16+4}} \\ &= (27\sin 2, -27\sin 2, -27\cos 2) \cdot \frac{(-1,-4,-2)}{\sqrt{21}} = \frac{81\sin 2 + 54\cos 2}{\sqrt{21}}. \end{aligned}$$

8. Since parametric equations for the line are $x = -2t-1,\ y = t,\ z = \frac{1}{2}(2+2t+1+t) = \frac{3}{2} + \frac{3}{2}t$, a vector along the line in the direction of decreasing z is $\mathbf{v} = (4,-2,-3)$. At the point $(1,-1,0)$,

$$\begin{aligned} D_{\mathbf{v}}f = \nabla f_{|(1,-1,0)} \cdot \hat{\mathbf{v}} &= (2xy+z^2, x^2+2yz, y^2+2xz)_{|(1,-1,0)} \cdot \frac{(4,-2,-3)}{\sqrt{16+4+9}} \\ &= (-2,1,1) \cdot \frac{(4,-2,-3)}{\sqrt{29}} = \frac{-13}{\sqrt{29}}. \end{aligned}$$

9. Since the slope of the curve at $(1,1)$ is 2, a tangent vector at the point is $\mathbf{T} = (1,2)$. The required rate of change is therefore

$$D_{\mathbf{T}}f = \nabla f_{|(1,1)} \cdot \hat{\mathbf{T}} = (2,-3) \cdot \frac{(1,2)}{\sqrt{5}} = \frac{-4}{\sqrt{5}}.$$

10. Since the slope of the curve at $(-1,3)$ is -9, a tangent vector at the point is $\mathbf{T}=(-1,9)$. The required rate of change is therefore

$$D_{\mathbf{T}}f=\nabla f_{|(-1,3)}\cdot\hat{\mathbf{T}}=(2x,1)_{|(-1,3)}\cdot\hat{\mathbf{T}}=(-2,1)\cdot\frac{(-1,9)}{\sqrt{82}}=\frac{11}{\sqrt{82}}.$$

11. Since parametric equations for the curve are $x=t$, $y=t^2-1$, $z=-2t$, a tangent vector to the curve is $\mathbf{T}=(1,2t,-2)$. At the point $(1,0,-2)$, a tangent vector is $(1,2,-2)$, and

$$D_{\mathbf{T}}f=\nabla f_{|(1,0,-2)}\cdot\hat{\mathbf{T}}=(y,x,2z)_{|(1,0,-2)}\cdot\frac{(1,2,-2)}{\sqrt{1+4+4}}=(0,1,-4)\cdot\frac{(1,2,-2)}{3}=\frac{10}{3}.$$

12. Since parametric equations for the curve are $x=t$, $y=-\sqrt{t^2-3}$, $z=t$, a tangent vector at $(2,-1,2)$ is $\mathbf{T}(2)=\left(1,\dfrac{-t}{\sqrt{t^2-3}},1\right)_{|t=2}=(1,-2,1)$. At the point $(2,-1,2)$ then,

$$\begin{aligned}D_{\mathbf{T}}f=\nabla f_{|(2,-1,2)}\cdot\hat{\mathbf{T}}&=(2xy+y^3z,x^2+3xy^2z,xy^3)_{|(2,-1,2)}\cdot\frac{(1,-2,1)}{\sqrt{1+4+1}}\\&=(-6,16,-2)\cdot\frac{(1,-2,1)}{\sqrt{6}}=\frac{-40}{\sqrt{6}}.\end{aligned}$$

13. The function increases most rapidly in the direction

$$\nabla f_{|(1,1,-3)}=(4x^3yz-y^3,x^4z-3xy^2,x^4y+1)_{|(1,1,-3)}=(-13,-6,2).$$

The rate of change in this direction is $\sqrt{169+36+4}=\sqrt{209}$.

14. The function increases most rapidly in the direction

$$\nabla f_{|(2,1/2)}=(2y+1/x,2x+1/y)_{|(2,1/2)}=(3/2,6),$$

or, $(2/3)(3/2,6)=(1,4)$. The rate of change in this direction is $\sqrt{9/4+36}=\sqrt{153}/2$.

15. The function increases most rapidly in the direction

$$\nabla f_{|(1,-3,2)}=\left[\frac{1}{(x^2+y^2+z^2)^{3/2}}(-x,-y,-z)\right]_{|(1,-3,2)}=\frac{1}{14\sqrt{14}}(-1,3,-2),$$

or, $(-1,3,-2)$. The rate of change in this direction is $\dfrac{1}{14\sqrt{14}}\sqrt{1+9+4}=\dfrac{1}{14}$.

16. The function increases most rapidly in the direction

$$\nabla f_{|(1,-3,2)}=\left[\frac{1}{(x^2+y^2+z^2)^{3/2}}(x,y,z)\right]_{|(1,-3,2)}=\frac{1}{14\sqrt{14}}(1,-3,2),$$

or, $(1,-3,2)$. The rate of change in this direction is $\dfrac{1}{14\sqrt{14}}\sqrt{1+9+4}=\dfrac{1}{14}$.

17. The function increases most rapidly in the direction

$$\nabla f_{|(3,2,-4)}=\left(\frac{yz}{1+x^2y^2z^2},\frac{xz}{1+x^2y^2z^2},\frac{xy}{1+x^2y^2z^2}\right)_{|(3,2,-4)}=\frac{(-8,-12,6)}{577},$$

or, $(-4,-6,3)$. The rate of change in this direction is $\dfrac{1}{577}\sqrt{64+144+36}=2\sqrt{61}/577$.

18. The function increases most rapidly in the direction

$$\nabla f_{|(1,1)}=(ye^{xy}+xy^2e^{xy},xe^{xy}+x^2ye^{xy})_{|(1,1)}=(2e,2e),$$

or, $(1,1)$. The rate of change in this direction is $\sqrt{4e^2+4e^2}=2\sqrt{2}e$.

19. The rate of change is smallest in the direction $-\nabla f_{|(2,-1,3)}=-(yz,xz,xy)_{|(2,-1,3)}=(3,-6,2)$.

20. At the point $(1,-1)$ and in the direction $\hat{\mathbf{v}} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$,

$$D_{\hat{\mathbf{v}}}f = \nabla f_{|(1,-1)} \cdot \hat{\mathbf{v}} = (2xy, x^2 + 3y^2)_{|(1,-1)} \cdot \hat{\mathbf{v}} = (-2,4) \cdot (a,b) = -2a + 4b.$$

(a) The directional derivative vanishes if $0 = -2a + 4b$. Because $\hat{\mathbf{v}}$ is a unit vector, we also know that $a^2 + b^2 = 1$. Thus, $1 = (2b)^2 + b^2 = 5b^2$. This implies that $b = \pm 1/\sqrt{5}$ and $a = \pm 2/\sqrt{5}$. The required directions are therefore $\pm(2,1)$. This is to be expected since in a direction perpendicular to ∇f, the rate of change should be zero, and $\pm(2,1)$ are both perpendicular to $(-2,4)$.

(b) The rate of change is 1 if, and when, $1 = -2a + 4b$. Substitution from this equation into $a^2 + b^2 = 1$ gives $1 = \left(\dfrac{4b-1}{2}\right)^2 + b^2 \implies 20b^2 - 8b - 3 = 0$. Solutions of this equation are $b = (2 \pm \sqrt{19})/10$, and these give $a = (-1 \pm 2\sqrt{19})/10$. The required directions are therefore $(-1 \pm 2\sqrt{19}, 2 \pm \sqrt{19})$.

(c) The rate of change is 20, if, and when $20 = -2a + 4b$. Substitution from this equation into $a^2 + b^2 = 1$ gives $1 = (2b - 10)^2 + b^2 \longrightarrow 5b^2 - 40b + 99 = 0$. Since this quadratic equation has no real solutions, there are no directions in which the rate of change of the function is equal to 20. This is also clear from the fact that the maximum rate of change is $|\nabla f| = 2\sqrt{5}$.

21. At the point $(0,1,-2)$ and in the direction $\hat{\mathbf{v}} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$,

$$D_{\hat{\mathbf{v}}}f = \nabla f_{|(0,1,-2)} \cdot \hat{\mathbf{v}} = (y, x, 1)_{|(0,1,-2)} \cdot \hat{\mathbf{v}} = (1,0,1) \cdot (a,b,c) = a + c.$$

(a) The directional derivative vanishes if $0 = a + c$. Because $\hat{\mathbf{v}}$ is a unit vector, we also know that $a^2 + b^2 + c^2 = 1$. Substituting $c = -a$ gives $1 = 2a^2 + b^2 \longrightarrow b = \pm\sqrt{1 - 2a^2}$. Thus, the directional derivative vanishes in the directions $(a, \pm\sqrt{1-2a^2}, -a)$.

(b) The rate of change is 1 if, and when, $1 = a + c$. Because $\hat{\mathbf{v}}$ is a unit vector, we also know that $a^2 + b^2 + c^2 = 1$. Substituting $c = 1 - a$ gives $1 = a^2 + b^2 + (1-a)^2 \implies b = \pm\sqrt{2a(1-a)}$. The required directions are therefore $(a, \pm\sqrt{2a(1-a)}, 1 - a)$.

(c) The rate of change is -20, if, and when $-20 = a + c$. Substitution from this equation into $a^2 + b^2 + c^2 = 1$ gives $1 = a^2 + b^2 + (-20 - a)^2 \implies b^2 = -399 - 40a - 2a^2$. Since the quadratic expression in a is always negative, there are no directions in which the rate of change of the function is equal to -20. This is also clear from the fact that the minimum rate of change is $-|\nabla f| = -\sqrt{2}$.

22. (a) Yes. In any direction perpendicular to the gradient of the function, its rate of change is zero.
(b) Not necessarily. If the gradient of the function at the point has length 2 say, then the maximum rate of change for all directions is 2. Hence, for no direction could it be equal to 3. On the other hand, if the length of the gradient is 4, then values of the rate of change would vary between -4 and 4 and hence there would exist directions in which it is equal to 3.

23. If s were not length along the line then df/ds would not measure the rate of change of f with respect to distance.

24. The distance to the origin is given by $d = \sqrt{x^2 + y^2 + z^2}$. The required derivative is

$$\begin{aligned} D_{\mathbf{T}}d = \nabla d \cdot \hat{\mathbf{T}} &= \frac{1}{\sqrt{x^2+y^2+z^2}}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot \frac{(-2\sin t, 2\cos t, 3)}{\sqrt{4\sin^2 t + 4\cos^2 t + 9}} \\ &= \frac{1}{\sqrt{x^2+y^2+z^2}} \frac{-2x\sin t + 2y\cos t + 3z}{\sqrt{13}} \\ &= \frac{1}{\sqrt{4\cos^2 t + 4\sin^2 t + 9t^2}} \frac{-4\cos t\, \sin t + 4\sin t\, \cos t + 9t}{\sqrt{13}} \\ &= \frac{9t}{\sqrt{13}\sqrt{4+9t^2}}. \end{aligned}$$

When $t = 0$, the rate of change is 0. This is expected since the curve is a helix, and when $t = 0$, the point $(2,0,0)$ is the closest point to the origin. Hence, the distance should have a minimum and its derivative should vanish.

25. Since $\mathbf{T} = (1, -2, 1)$ is a vector along the line, the rate of change of $f(x, y, z)$ with respect to distance travelled along the curve vanishes if

$$0 = D_{\mathbf{T}}f = \nabla f \cdot \hat{\mathbf{T}} = (2x + yz, xz, xy) \cdot \frac{(1, -2, 1)}{\sqrt{6}} \implies 0 = 2x + yz - 2xz + xy.$$

If we substitute the parametric equations of the line into this equation,

$$0 = 2t + (1 - 2t)(t) - 2t(t) + t(1 - 2t) = -6t^2 + 4t = 2t(2 - 3t) \implies t = 0 \text{ or } t = 2/3.$$

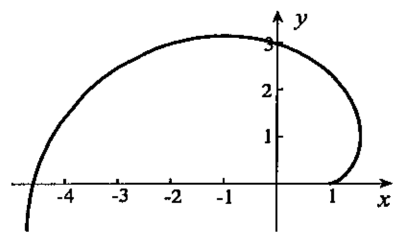
The required points are therfore $(0, 1, 0)$ and $(2/3, -1/3, 2/3)$.

26. A tangent vector along $C: \ x = t^2,\ y = t,\ z = t^2$ is $\mathbf{T} = (2t, 1, 2t)$. At any point on C,

$$D_{\mathbf{T}}f = \nabla f \cdot \hat{\mathbf{T}} = (2x, -2y, 2z) \cdot \frac{(2t, 1, 2t)}{\sqrt{1 + 8t^2}} = \frac{2(2xt - y + 2zt)}{\sqrt{1 + 8t^2}}.$$

For this derivative to vanish, $0 = 2xt - y + 2zt = 2t^3 - t + 2t^3 = t(4t^2 - 1)$. Thus, $t = 0, \pm 1/2$, and these values give the points $(0, 0, 0)$ and $(1/4, \pm 1/2, 1/4)$.

27. A plot of the involute is shown to the right. A tangent vector along the curve is $\mathbf{T} = (-\sin t + \sin t + t\cos t, \cos t - \cos t + t\sin t) = (t\cos t, t\sin t)$. The rate of change of the distance $d = \sqrt{x^2 + y^2}$ from the origin to a point on the involute with respect to distance travelled along the curve is



$$D_{\mathbf{T}}d = \nabla d \cdot \hat{\mathbf{T}} = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}\right) \cdot (\cos t, \sin t) = \frac{x\cos t + y\sin t}{\sqrt{x^2 + y^2}}$$
$$= \frac{\cos t(\cos t + t\sin t) + \sin t(\sin t - t\cos t)}{(\cos t + t\sin t)^2 + (\sin t - t\cos t)^2} = \frac{1}{1 + t^2}.$$

This rate of change is always positive.

28. It is the negative of the rate of change in the other direction.

29. The rate of change is zero (since the component of ∇f perpendicular to ∇f is zero).

30. Let (a, b) be the gradient of $f(x, y)$ at the point (x_0, y_0). Then,

$$3 = D_{\hat{\mathbf{i}}+2\hat{\mathbf{j}}}f_{|(x_0,y_0)} = \nabla f_{|(x_0,y_0)} \cdot \frac{(1, 2)}{\sqrt{5}} = (a, b) \cdot \frac{(1, 2)}{\sqrt{5}} = \frac{a + 2b}{\sqrt{5}}, \quad \text{and}$$
$$-1 = D_{-2\hat{\mathbf{i}}-\hat{\mathbf{j}}}f_{|(x_0,y_0)} = \nabla f_{|(x_0,y_0)} \cdot \frac{(-2, -1)}{\sqrt{5}} = (a, b) \cdot \frac{(-2, -1)}{\sqrt{5}} = \frac{-2a - b}{\sqrt{5}}.$$

These imply that $a = -\sqrt{5}/3$ and $b = 5\sqrt{5}/3$. The rate of change in direction $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ is therefore

$$\left(-\frac{\sqrt{5}}{3}, \frac{5\sqrt{5}}{3}\right) \cdot \frac{(2, 3)}{\sqrt{13}} = \frac{\sqrt{65}}{3}.$$

31. Let (a, b, c) be the gradient of $f(x, y)$ at the point (x_0, y_0, z_0). Then,

$$1 = D_{\hat{\mathbf{i}}+\hat{\mathbf{j}}}f_{|(x_0,y_0,z_0)} = \nabla f_{|(x_0,y_0,z_0)} \cdot \frac{(1, 1, 0)}{\sqrt{2}} = (a, b, c) \cdot \frac{(1, 1, 0)}{\sqrt{2}} = \frac{a + b}{\sqrt{2}},$$
$$2 = D_{2\hat{\mathbf{i}}-\hat{\mathbf{k}}}f_{|(x_0,y_0,z_0)} = \nabla f_{|(x_0,y_0,z_0)} \cdot \frac{(2, 0, -1)}{\sqrt{5}} = (a, b, c) \cdot \frac{(2, 0, -1)}{\sqrt{5}} = \frac{2a - c}{\sqrt{5}},$$
$$-3 = D_{\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}}f_{(x_0,y_0,z_0)} = \nabla f_{|(x_0,y_0,z_0)} \cdot \frac{(1, -1, 1)}{\sqrt{3}} = (a, b, c) \cdot \frac{(1, -1, 1)}{\sqrt{3}} = \frac{a - b + c}{\sqrt{3}}.$$

These imply that $c = (\sqrt{2} - 3\sqrt{3} - 2\sqrt{5})/2$ and this is $\partial f/\partial z$ at the point.

32. Since $\nabla f = (3x^2y^2, 2x^3y)$, the first directional derivative at any point (x, y) in direction $\mathbf{v} = (1, -2)$ is

$$D_{\mathbf{v}} f = (3x^2y^2, 2x^3y) \cdot \frac{(1,-2)}{\sqrt{5}} = \frac{1}{\sqrt{5}}(3x^2y^2 - 4x^3y).$$

The second directional derivative is

$$\begin{aligned} D_{\mathbf{v}}(D_{\mathbf{v}} f)_{|(1,1)} &= \nabla\left[\frac{1}{\sqrt{5}}(3x^2y^2 - 4x^3y)\right]_{|(1,1)} \cdot \frac{(1,-2)}{\sqrt{5}} = \frac{1}{5}(6xy^2 - 12x^2y, 6x^2y - 4x^3)_{|(1,1)} \cdot (1,-2) \\ &= \frac{1}{5}(-6, 2) \cdot (1, -2) = -2. \end{aligned}$$

33. Since $\nabla f = (2x, 4y, 6z)$, the first directional derivative at any point (x, y, z) in direction $\mathbf{v} = (1, 1, -1)$ is

$$D_{\mathbf{v}} f = (2x, 4y, 6z) \cdot \frac{(1,1,-1)}{\sqrt{3}} = \frac{1}{\sqrt{3}}(2x + 4y - 6z).$$

The second directional derivative is

$$D_{\mathbf{v}}(D_{\mathbf{v}} f)_{|(-2,-1,3)} = \nabla\left[\frac{1}{\sqrt{3}}(2x + 4y - 6z)\right]_{|(-2,-1,3)} \cdot \frac{(1,1,-1)}{\sqrt{3}} = \frac{1}{3}(2, 4, -6)_{|(-2,-1,3)} \cdot (1, 1, -1) = 4.$$

34. Since a tangent vector to the curve at any point is $\mathbf{T} = R(1 - \cos\theta, \sin\theta)$, a unit tangent vector is

$$\hat{\mathbf{T}} = \frac{(1 - \cos\theta, \sin\theta)}{\sqrt{(1 - \cos\theta)^2 + \sin^2\theta}} = \frac{(1 - \cos\theta, \sin\theta)}{\sqrt{2 - 2\cos\theta}}.$$

The rate of change of the distance $d = \sqrt{x^2 + y^2}$ from the origin to the stone is

$$D_{\hat{\mathbf{T}}} d = \nabla d \cdot \hat{\mathbf{T}} = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{\sqrt{x^2 + y^2}} \cdot \hat{\mathbf{T}} = \frac{x(1 - \cos\theta) + y\sin\theta}{\sqrt{x^2 + y^2}\sqrt{2 - 2\cos\theta}}.$$

When $\theta = \pi/2$, we have $x = R(\pi/2 - 1)$, $y = R$, and

$$D_{\hat{\mathbf{T}}} d_{|\theta=\pi/2} = \frac{R(\pi/2 - 1)(1) + R(1)}{\sqrt{R^2(\pi/2 - 1)^2 + R^2}\sqrt{2}} = \frac{\pi}{\sqrt{2}\sqrt{8 - 4\pi + \pi^2}}.$$

When $\theta = \pi$, we have $x = \pi R$, $y = 2R$, and

$$D_{\hat{\mathbf{T}}} d_{|\theta=\pi} = \frac{\pi R(2) + 2R(0)}{\sqrt{\pi^2 R^2 + 4R^2}\sqrt{4}} = \frac{\pi}{\sqrt{4 + \pi^2}}.$$

(b) Since $\nabla(y) = \hat{\mathbf{j}}$, the rate of change of the y-coordinate is

$$D_{\hat{\mathbf{T}}} y = \hat{\mathbf{j}} \cdot \frac{(1 - \cos\theta, \sin\theta)}{\sqrt{2 - 2\cos\theta}} = \frac{\sin\theta}{\sqrt{2 - 2\cos\theta}}.$$

When $\theta = \pi/2$, $D_{\hat{\mathbf{T}}} y_{|\theta=\pi/2} = 1/\sqrt{2}$, and when $\theta = \pi$, $D_{\hat{\mathbf{T}}} y_{|\theta=\pi} = 0$.

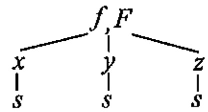
(c) Since $\nabla(x) = \hat{\mathbf{i}}$, the rate of change of the x-coordinate is

$$D_{\hat{\mathbf{T}}} x = \hat{\mathbf{i}} \cdot \frac{(1 - \cos\theta, \sin\theta)}{\sqrt{2 - 2\cos\theta}} = \frac{1 - \cos\theta}{\sqrt{2 - 2\cos\theta}}.$$

When $\theta = \pi/2$, $D_{\hat{\mathbf{T}}} x_{|\theta=\pi/2} = 1/\sqrt{2}$, and when $\theta = \pi$, $D_{\hat{\mathbf{T}}} x_{|\theta=\pi} = 2/\sqrt{4} = 1$.

35. If we apply the mean value theorem (or Taylor's remainder formula) to the function $F(s) = f(x_0 + v_x s, y_0 + v_y s, z_0 + v_z s)$, between $s = 0$ and an arbitrary value of s, we obtain $F(s) = F(0) + F'(c)s$, where $0 < c < s$. Substitution of this into the limit gives

$$D_{\mathbf{v}} f = \lim_{s \to 0^+} \frac{[F(0) + F'(c)s] - F(0)}{s} = \lim_{s \to 0^+} F'(c).$$



The schematic to the right shows that

$$F'(s) = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds} = \frac{\partial f}{\partial x}v_x + \frac{\partial f}{\partial y}v_y + \frac{\partial f}{\partial z}v_z.$$

Consequently, the directional derivative at (x_0, y_0, z_0) is

$$D_{\mathbf{v}} f = \lim_{s \to 0^+} \frac{\partial f}{\partial x}v_x + \frac{\partial f}{\partial y}v_y + \frac{\partial f}{\partial z}v_z = \frac{\partial f}{\partial x}_{|(x_0,y_0,z_0)} v_x + \frac{\partial f}{\partial y}_{|(x_0,y_0,z_0)} v_y + \frac{\partial f}{\partial z}_{|(x_0,y_0,z_0)} v_z.$$

EXERCISES 12.9

1. Since slope of the tangent line at $(-2, 4, 0)$ is -4, equations of the tangent line are $y - 4 = -4(x + 2)$, $z = 0 \implies 4x + y + 4 = 0$, $z = 0$.

2. Since a tangent vector at $(1, 1, 1)$ is $\frac{d\mathbf{r}}{dt}_{|t=1} = (1, 2t, 3t^2)_{|t=1} = (1, 2, 3)$, parametric equations for the tangent line are $x = 1 + u$, $y = 1 + 2u$, $z = 1 + 3u$.

3. Since a tangent vector at $(1, 0, 1)$ is $\frac{d\mathbf{r}}{dt}_{|t=0} = (-\sin t, \cos t, -\sin t)_{|t=0} = (0, 1, 0)$, parametric equations for the tangent line are $x = 1$, $y = u$, $z = 1$.

4. With parametric equations $x = t$, $y = t^2$, $z = t$, a tangent vector at the point $(-2, 4, -2)$ is $\frac{d\mathbf{r}}{dt}_{|t=-2} = (1, 2t, 1)_{|t=-2} = (1, -4, 1)$. Parametric equations for the tangent line are $x = -2 + u$, $y = 4 - 4u$, $z = -2 + u$.

5. With parametric equations $x = t$, $y = t^2$, $z = t^2 - t$, a tangent vector at the point $(1, 1, 0)$ is $\frac{d\mathbf{r}}{dt}_{|t=1} = (1, 2t, 2t - 1)_{|t=1} = (1, 2, 1)$. Parametric equations for the tangent line are $x = 1 + u$, $y = 1 + 2u$, $z = u$.

6. Since a tangent vector at $(1, 5, 1)$ is $\frac{d\mathbf{r}}{dt}_{|t=1} = (-2t, 2, 1)_{|t=1} = (-2, 2, 1)$, parametric equations for the tangent line are $x = 1 - 2u$, $y = 5 + 2u$, $z = 1 + u$.

7. Since a tangent vector at $(\sqrt{2}, -3/\sqrt{2}, 5)$ is $\frac{d\mathbf{r}}{dt}_{|t=-\pi/4} = (-2\sin t, 3\cos t, 0)_{|t=-\pi/4} = (\sqrt{2}, 3/\sqrt{2}, 0)$. Since $(2, 3, 0)$ is also a tangent vector, parametric equations for the tangent line are $x = \sqrt{2} + 2u$, $y = -3/\sqrt{2} + 3u$, $z = 5$.

8. Because a vector normal to the curve at $(1, 4)$ is

$$\nabla(x^2y^3 + xy - 68)_{|(1,4)} = (2xy^3 + y, 3x^2y^2 + x)_{|(1,4)} = (132, 49),$$

a vector tangent to the curve is $(49, -132)$. The slope of the tangent line is is therefore $-132/49$, and its equations are $y - 4 = -\frac{132}{49}(x - 1)$, $z = 0 \implies 132x + 49y = 328$, $z = 0$.

9. Since the curve is a straight line, the tangent line is the line itself.

10. Since a tangent vector at $(1, 0, 0)$ is

$$\frac{d\mathbf{r}}{dt}_{|t=0} = (-e^{-t}\cos t - e^{-t}\sin t, -e^{-t}\sin t + e^{-t}\cos t, 1)_{|t=0} = (-1, 1, 1),$$

parametric equations for the tangent line are $x = 1 - u$, $y = u$, $z = u$.

11. Since a tangent vector at $(2,-6,2)$ is $\dfrac{d\mathbf{r}}{dt}_{|t=-1} = (2t, 2, 3t^2)_{|t=-1} = (-2,2,3)$, parametric equations for the tangent line are $x = 2-2u$, $y = -6+2u$, $z = 2+3u$.

12. Since normals to the surfaces at $(2,-\sqrt{5},-1)$ are $\nabla(y^2+z^2-6)_{|(2,-\sqrt{5},-1)} = (0,2y,2z)_{|(2,-\sqrt{5},-1)} = (0,-2\sqrt{5},-2)$ and $\nabla(x+z-1)_{|(2,-\sqrt{5},-1)} = (1,0,1)$, a vector along the tangent line is

$$(0,\sqrt{5},1)\times(1,0,1) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & \sqrt{5} & 1 \\ 1 & 0 & 1 \end{vmatrix} = (\sqrt{5},1,-\sqrt{5}).$$

Because $(1,1/\sqrt{5},-1)$ is also a tangent vector, parametric equations for the tangent line are $x = 2+t$, $y = -\sqrt{5}+t/\sqrt{5}$, $z = -1-t$.

13. Since normals to the surfaces at $(1,1,-\sqrt{2})$ are $\nabla(x^2+y^2+z^2-4)_{|(1,1,-\sqrt{2})} = (2x,2y,2z)_{|(1,1,-\sqrt{2})} = (2,2,-2\sqrt{2})$ and $\nabla(x^2+y^2-z^2)_{|(1,1,-\sqrt{2})} = (2x,2y,-2z)_{|(1,1,-\sqrt{2})} = (2,2,2\sqrt{2})$, a vector along the tangent line is

$$(1,1,-\sqrt{2})\times(1,1,\sqrt{2}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & -\sqrt{2} \\ 1 & 1 & \sqrt{2} \end{vmatrix} = (2\sqrt{2},-2\sqrt{2},0).$$

Because $(1,-1,0)$ is also a tangent vector, parametric equations for the tangent line are $x = 1+u$, $y = 1-u$, $z = -\sqrt{2}$.

14. Because a tangent vector at $(4,1,\sqrt{17})$ is $\dfrac{d\mathbf{r}}{dt}_{|t=4} = (1,0,t/\sqrt{1+t^2})_{|t=4} = (1,0,4/\sqrt{17})$, as is the vector $(\sqrt{17},0,4)$, parametric equations for the tangent line are $x = 4+\sqrt{17}u$, $y = 1$, $z = \sqrt{17}+4u$.

15. A tangent vector at $(2,2,2)$ is $\dfrac{d\mathbf{r}}{dt}_{|t=0} = (-\sin t, -\cos t, 1/(2\sqrt{4+t}))_{|t=0} = (0,-1,1/4)$. Since $(0,-4,1)$ is also a tangent vector, parametric equations for the tangent line are $x = 2$, $y = 2-4u$, $z = 2+u$.

16. With parametric equations $x = t^2+t^3$, $y = t-t^4$, $z = t$, a tangent vector at the point $(12,-14,2)$ is $\dfrac{d\mathbf{r}}{dt}_{|t=2} = (2t+3t^2, 1-4t^3, 1)_{|t=2} = (16,-31,1)$. Parametric equations for the tangent line are $x = 12+16u$, $y = -14-31u$, $z = 2+u$.

17. With parametric equations $x = t^2+3t^3-2t+5$, $y = t$, $z = 0$, a tangent vector at $(7,1,0)$ is $\dfrac{d\mathbf{r}}{dt}_{|t=1} = (2t+9t^2-2, 1, 0)_{|t=1} = (9,1,0)$. Parametric equations for the tangent line are $x = 7+9u$, $y = 1+u$, $z = 0$.

18. Since normals to the surfaces at $(0,1,1)$ are $\nabla(2x^2+y^2+2y-3)_{|(0,1,1)} = (4x, 2y+2, 0)_{|(0,1,1)} = (0,4,0)$ and $\nabla(x-z+1)_{|(0,1,1)} = (1,0,-1)$, a vector along the tangent line is

$$(0,1,0)\times(1,0,-1) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = (-1,0,-1),$$

as is $(1,0,1)$. Parametric equations for the tangent line are $x = t$, $y = 1$, $z = 1+t$.

19. A tangent vector at the point $(1,1,\sqrt{2})$ is $\dfrac{d\mathbf{r}}{dt}_{|t=1} = (2t, 1, (1+4t^3)/(2\sqrt{t+t^4}))_{|t=1} = (2,1,5/(2\sqrt{2}))$. Since $(8,4,5\sqrt{2})$ is also a tangent vector, parametric equations for the tangent line are $x = 1+8u$, $y = 1+4u$, $z = \sqrt{2}+5\sqrt{2}u$.

20. Since a tangent vector at $(0,2\pi,4\pi)$ is $\dfrac{d\mathbf{r}}{dt}_{|t=2\pi} = (\sin t + t\cos t, \cos t - t\sin t, 2)_{|t=2\pi} = (2\pi,1,2)$, parametric equations for the tangent line are $x = 2\pi u$, $y = 2\pi+u$, $z = 4\pi+2u$.

21. Since a normal to the tangent plane is

$$\nabla(\sqrt{x^2+y^2}-z)_{|(1,1,\sqrt{2})} = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1\right)_{|(1,1,\sqrt{2})} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1\right),$$

as is $(1,1,-\sqrt{2})$, the equation of the tangent plane is

$$0 = (1,1,-\sqrt{2})\cdot(x-1, y-1, z-\sqrt{2}) = x+y-\sqrt{2}z.$$

22. Since a normal to the tangent plane is

$$\nabla(x-x^2+y^3z)_{|(2,-1,-2)} = (1-2x, 3y^2z, y^3)_{|(2,-1,-2)} = (-3,-6,-1),$$

as is $(3,6,1)$, the equation of the tangent plane is

$$0 = (3,6,1)\cdot(x-2, y+1, z+2) = 3x+6y+z+2.$$

23. Since a normal to the tangent plane is

$$\nabla(x^2y+y^2z+z^2x+3)_{|(2,-1,-1)} = (2xy+z^2, x^2+2yz, y^2+2zx)_{|(2,-1,-1)} = (-3,6,-3),$$

as is $(1,-2,1)$, the equation of the tangent plane is

$$0 = (1,-2,1)\cdot(x-2, y+1, z+1) = x-2y+z-3.$$

24. Because the surface is a plane, the tangent plane is the surface itself, $x+y+z=4$.

25. Since a normal to the tangent plane is

$$\nabla(y\sin(\pi z/2)-x)_{|(-1,-1,1)} = (-1, \sin(\pi z/2), (\pi y/2)\cos(\pi z/2))_{|(-1,-1,1)} = (-1,1,0),$$

as is $(1,-1,0)$, the equation of the tangent plane is $0 = (1,-1,0)\cdot(x+1, y+1, z-1) = x-y$.

26. Since a normal to the tangent plane is $\nabla(x^2+y^2+2y-1)_{|(1,0,3)} = (2x, 2y+2, 0)_{|(1,0,3)} = (2,2,0)$, as is $(1,1,0)$, the equation of the tangent plane is $0 = (1,1,0)\cdot(x-1, y, z-3) = x+y-1$.

27. A tangent vector to the curve at $(2,2,1)$ is $(2t^2, 4t, 3)_{|t=1} = (2,4,3)$. A normal vector to the surface at the point is $(2x, 4y, 6z)_{|(2,2,1)} = (4,8,6)$. Since the vectors are in the same direction, the curve intersects the surface at right angles.

28. A vector tangent to the curve at $(1,1,1)$ is

$$\begin{aligned}\mathbf{T} &= \nabla(x^2-y^2+z^2-1)_{|(1,1,1)} \times \nabla(xy+xz-2)_{|(1,1,1)} = (2x,-2y,2z)_{|(1,1,1)} \times (y+z, x, x)_{|(1,1,1)}\\ &= (2,-2,2)\times(2,1,1) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -2 & 2 \\ 2 & 1 & 1 \end{vmatrix} = (-4,2,6).\end{aligned}$$

A vector normal to the surface at $(1,1,1)$ is

$$\mathbf{n} = \nabla(xyz-x^2-6y+6)_{|(1,1,1)} = (yz-2x, xz-6, xy)_{|(1,1,1)} = (-1,-5,1).$$

Since $\mathbf{T}\cdot\mathbf{n} = 4-10+6 = 0$, the vectors are perpendicular, and the curve is tangent to the surface at $(1,1,1)$.

29. Since a vector normal to the surface at (x_0, y_0, z_0) is

$$\nabla[z-f(x,y)]_{|(x_0,y_0,z_0)} = (f_x(x_0,y_0), f_y(x_0,y_0), -1),$$

the equation of the tangent plane at the point is

$$0 = (f_x(x_0,y_0), f_y(x_0,y_0), -1)\cdot(x-x_0, y-y_0, z-z_0) \implies z-z_0 = f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0).$$

30. Since a vector along the curve is $\mathbf{T} = \nabla(x+y+z-4) \times \nabla(x-y+z-2) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = (2,0,-2)$, we find that at $(3,1,0)$,

$$D_{\hat{\mathbf{T}}}f = \nabla f_{|(3,1,0)} \cdot \hat{\mathbf{T}} = (4x, 2yz^2, 2y^2z)_{|(3,1,0)} \cdot \frac{(1,0,-1)}{\sqrt{2}} = (12,0,0) \cdot \frac{(1,0,-1)}{\sqrt{2}} = \frac{12}{\sqrt{2}} = 6\sqrt{2}.$$

31. Since a vector perpendicular to the surface at the point $(1,-2,5)$ is

$$\mathbf{n} = \nabla(x^2+y^2-z)_{|(1,-2,5)} = (2x, 2y, -1)_{|(1,-2,5)} = (2,-4,-1),$$

the required derivative is

$$\begin{aligned}\pm D_{\hat{\mathbf{n}}}f &= \pm\nabla f_{|(1,-2,5)} \cdot \hat{\mathbf{n}} = \pm(yz+y+z, xz+x+z, xy+x+y)_{|(1,-2,5)} \cdot \frac{(2,-4,-1)}{\sqrt{21}} \\ &= \pm(-7,11,-3) \cdot \frac{(2,-4,-1)}{\sqrt{21}} = \frac{\pm 55}{\sqrt{21}}.\end{aligned}$$

32. Since $f(x,y,z) = 0$ everywhere on the curve, its directional derivative must also be zero.

33. Since the slope of the tangent line to the curve is given by $\dfrac{dy}{dx} = -\dfrac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(y)}} = -\dfrac{F_x}{F_y}$, a vector along the tangent line is $(F_y, -F_x)$. Since the scalar product of this vector with $\nabla F = (F_x, F_y)$ is zero, the gradient ∇F is perpendicular to the curve.

34. A vector normal to the surface at (x_0,y_0,z_0) is $\nabla\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}-1\right)_{|(x_0,y_0,z_0)} = \left(\dfrac{2x_0}{a^2}, \dfrac{2y_0}{b^2}, \dfrac{2z_0}{c^2}\right)$. Hence, the equation of the tangent plane is

$$0 = \left(\frac{x_0}{a^2}, \frac{y_0}{b^2}, \frac{z_0}{c^2}\right) \cdot (x-x_0, y-y_0, z-z_0) = \frac{x_0x}{a^2} + \frac{y_0y}{b^2} + \frac{z_0z}{c^2} - \left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2}\right).$$

Since (x_0,y_0,z_0) is on the ellipsoid, $x_0^2/a^2 + y_0^2/b^2 + z_0^2/c^2 = 1$, and the equation of the plane reduces to $x_0x/a^2 + y_0y/b^2 + z_0z/c^2 = 1$.

35. A normal vector to the surface at any point is $\nabla(9x^2-4y^2-36z) = (18x, -8y, -36)$. The tangent plane to the surface is parallel to the plane $x+y+z=4$ if this vector is a multiple of the normal $(1,1,1)$ to the plane,

$$(18x, -8y, -36) = \lambda(1,1,1) \quad \Longrightarrow \quad 18x = \lambda,\ -8y = \lambda,\ -36 = \lambda.$$

Thus, $x = -2$ and $y = 9/2$, and the only point is $(-2, 9/2, -5/4)$.

36. A normal vector to the surface is $\nabla(4x^2+4y^2-z^2) = (8x, 8y, -2z)$, as is $(4x, 4y, -z)$. The tangent plane is parallel to $x-y+2z=3$, which has normal $(1,-1,2)$, if, and where, $(4x,4y,-z) = \lambda(1,-1,2)$ for some λ. This requires $4x = \lambda$, $4y = -\lambda$, and $-z = 2\lambda$. Substitution of $x = \lambda/4$, $y = -\lambda/4$, and $z = -2\lambda$ into the equation of the surface gives $4\lambda^2 = 4\left(\dfrac{\lambda^2}{16}\right) + 4\left(\dfrac{\lambda^2}{16}\right)$. The only solution of this equation is $\lambda = 0$. But this implies that $x = y = z = 0$, and this is unacceptable since there is no tangent plane to the surface at $(0,0,0)$.

37. A tangent vector to the curve is $(dx/dt, dy/dt, dz/dt)$. But,

$$\frac{dx}{dt} = -\frac{\dfrac{\partial(F,G,H)}{\partial(t,y,z)}}{\dfrac{\partial(F,G,H)}{\partial(x,y,z)}}, \quad \frac{dy}{dt} = -\frac{\dfrac{\partial(F,G,H)}{\partial(x,t,z)}}{\dfrac{\partial(F,G,H)}{\partial(x,y,z)}}, \quad \frac{dz}{dt} = -\frac{\dfrac{\partial(F,G,H)}{\partial(x,y,t)}}{\dfrac{\partial(F,G,H)}{\partial(x,y,z)}}.$$

Hence a tangent vector is $\left(\frac{\partial(F,G,H)}{\partial(t,y,z)}, \frac{\partial(F,G,H)}{\partial(x,t,z)}, \frac{\partial(F,G,H)}{\partial(x,y,t)}\right)$. Symmetric equations for the tangent line at P are

$$\frac{x-x_0}{\left.\frac{\partial(F,G,H)}{\partial(t,y,z)}\right|_P} = \frac{y-y_0}{\left.\frac{\partial(F,G,H)}{\partial(x,t,z)}\right|_P} = \frac{z-z_0}{\left.\frac{\partial(F,G,H)}{\partial(x,y,t)}\right|_P}.$$

38. A normal vector to the paraboloid at any point $P(x,y,z)$ is

$$\mathbf{n} = \nabla(x^2+y^2-1-z) = (2x, 2y, -1).$$

This vector coincides with **OP** if $(x,y,z) = \lambda(2x, 2y, -1)$, or, $x = 2\lambda x$, $y = 2\lambda y$, $z = -\lambda$. When these are combined with $z = x^2+y^2-1$, the points obtained are $(0,0,-1)$ and $x^2+y^2 = 1/2$, $z = -1/2$.

39. Since a normal vector to the surface is $\nabla(\sqrt{x}+\sqrt{y}+\sqrt{z}-\sqrt{a}) = \left(\frac{1}{2\sqrt{x}}, \frac{1}{2\sqrt{y}}, \frac{1}{2\sqrt{z}}\right)$, the equation of the tangent plane at any point (x_0, y_0, z_0) is

$$0 = \left(\frac{1}{\sqrt{x_0}}, \frac{1}{\sqrt{y_0}}, \frac{1}{\sqrt{z_0}}\right)\cdot(x-x_0, y-y_0, z-z_0) = \frac{1}{\sqrt{x_0}}(x-x_0) + \frac{1}{\sqrt{y_0}}(y-y_0) + \frac{1}{\sqrt{z_0}}(z-z_0).$$

The x-intercept of this plane is given by

$$0 = \frac{1}{\sqrt{x_0}}(x-x_0) - \sqrt{y_0} - \sqrt{z_0} \quad\Longrightarrow\quad x = x_0 + \sqrt{x_0}(\sqrt{y_0}+\sqrt{z_0}) = \sqrt{x_0}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}).$$

With similar expressions for the y- and z-intercepts, their sum is

$$\begin{aligned}\sqrt{x_0}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}) + \sqrt{y_0}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}) + \sqrt{z_0}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})\\ = (\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})^2 = a.\end{aligned}$$

EXERCISES 12.10

1. For critical points we solve $0 = \frac{\partial f}{\partial x} = 2x+2y$, $0 = \frac{\partial f}{\partial y} = 2x+4y-6$. The only solution is $(-3,3)$.

$$\frac{\partial^2 f}{\partial x^2} = 2, \qquad \frac{\partial^2 f}{\partial x\partial y} = 2, \qquad \frac{\partial^2 f}{\partial y^2} = 4$$

Since $B^2 - AC = 4-8 = -4$ and $A = 2$, there is a relative minimum at $(-3,3)$.

2. For critical points we solve $0 = \frac{\partial f}{\partial x} = 3y-3x^2$, $0 = \frac{\partial f}{\partial y} = 3x-3y^2$. Solutions are $(0,0)$ and $(1,1)$.

$$\frac{\partial^2 f}{\partial x^2} = -6x, \qquad \frac{\partial^2 f}{\partial x\partial y} = 3, \qquad \frac{\partial^2 f}{\partial y^2} = -6y$$

At $(0,0)$, $B^2-AC = 9-0$, and therefore $(0,0)$ yields a saddle point. At $(1,1)$, $B^2-AC = 9-(-6)(-6) = -27$, and $A = -6$, and therefore $(1,1)$ gives a relative maximum.

3. For critical points we solve $0 = \frac{\partial f}{\partial x} = 3x^2-3$, $0 = \frac{\partial f}{\partial y} = 2y+2$. Solutions are $(\pm 1, -1)$.

$$\frac{\partial^2 f}{\partial x^2} = 6x, \qquad \frac{\partial^2 f}{\partial x\partial y} = 0, \qquad \frac{\partial^2 f}{\partial y^2} = 2$$

At $(-1,-1)$, $B^2-AC = 0+12$, and therefore $(-1,-1)$ yields a saddle point. At $(1,-1)$, $B^2-AC = 0-12$ and $A = 6$. Thus, $(1,-1)$ gives a relative minimum.

4. For critical points we solve $0 = \frac{\partial f}{\partial x} = 2xy^2+3$, $0 = \frac{\partial f}{\partial y} = 2x^2y$. Because there are no solutions of these equations, there are no critical points.

5. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = y - 2x$, $0 = \dfrac{\partial f}{\partial y} = x + 2y$. The only solution is $(0, 0)$.

$$\frac{\partial^2 f}{\partial x^2} = -2, \qquad \frac{\partial^2 f}{\partial x \partial y} = 1, \qquad \frac{\partial^2 f}{\partial y^2} = 2$$

Since $B^2 - AC = 1 + 4$, $(0, 0)$ yields a saddle point.

6. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = \sin y$, $0 = \dfrac{\partial f}{\partial y} = x \cos y$. Solutions are $(0, n\pi)$, where n is an integer.

$$\frac{\partial^2 f}{\partial x^2} = 0, \qquad \frac{\partial^2 f}{\partial x \partial y} = \cos y, \qquad \frac{\partial^2 f}{\partial y^2} = -x \sin y$$

At $(0, n\pi)$, $B^2 - AC = [\cos(n\pi)]^2 - 0 > 0$, and therefore all critical points yield saddle points.

7. For critical points we solve

$$0 = \frac{\partial f}{\partial x} = ye^{-(x^2+y^2)} - 2x^2ye^{-(x^2+y^2)} = y(1 - 2x^2)e^{-(x^2+y^2)},$$
$$0 = \frac{\partial f}{\partial y} = xe^{-(x^2+y^2)} - 2xy^2e^{-(x^2+y^2)} = x(1 - 2y^2)e^{-(x^2+y^2)}.$$

Solutions are $(0, 0)$, $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$ and $(\pm 1/\sqrt{2}, \mp 1/\sqrt{2})$.

$$\frac{\partial^2 f}{\partial x^2} = y(-4x)e^{-(x^2+y^2)} - 2xy(1 - 2x^2)e^{-(x^2+y^2)} = 2xy(2x^2 - 3)e^{-(x^2+y^2)},$$
$$\frac{\partial^2 f}{\partial x \partial y} = (1 - 2x^2)e^{-(x^2+y^2)} - 2y^2(1 - 2x^2)e^{-(x^2+y^2)} = (1 - 2x^2)(1 - 2y^2)e^{-(x^2+y^2)},$$
$$\frac{\partial^2 f}{\partial y^2} = x(-4y)e^{-(x^2+y^2)} - 2xy(1 - 2y^2)e^{-(x^2+y^2)} = 2xy(2y^2 - 3)e^{-(x^2+y^2)},$$
$${f_{xy}}^2 - f_{xx}f_{yy} = [(1 - 2x^2)^2(1 - 2y^2)^2 - 4x^2y^2(2x^2 - 3)(2y^2 - 3)]e^{-2(x^2+y^2)}.$$

At $(0, 0)$, $B^2 - AC = 1$, and therefore $(0, 0)$ yields a saddle point.
At $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$, $B^2 - AC = -4(1/2)(1/2)(-2)(-2)e^{-2} < 0$, and $A = 2(1/2)(-2)e^{-1} < 0$. These critical points give relative maxima.
At $(\pm 1/\sqrt{2}, \mp 1/\sqrt{2})$, $B^2 - AC = -4(1/2)(1/2)(-2)(-2)e^{-2} < 0$, and $A = 2(-1/2)(-2)e^{-1} > 0$. These critical points give relative minima.

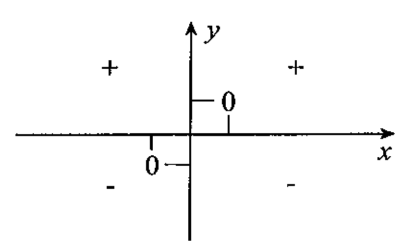
8. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = 2x - 2y$, $0 = \dfrac{\partial f}{\partial y} = -2x + 2y$. All points on the line $y = x$ are critical. Because $f(x, y) = (x - y)^2$, it is clear that each of these points gives a relative minimum.

9. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = \dfrac{4x}{3(x^2 + y^2)^{1/3}}$, $0 = \dfrac{\partial f}{\partial y} = \dfrac{4y}{3(x^2 + y^2)^{1/3}}$. There are no solutions to these equations. Because the derivatives are undefined at $(0, 0)$, but $f(0, 0) = 0$, there is a critical point at $(0, 0)$. Since $f(x, y) > 0$ for all $(x, y) \neq (0, 0)$, there is a relative minimum at $(0, 0)$.

10. For critical points we solve

$$0 = \frac{\partial f}{\partial x} = 4x^3y^3, \qquad 0 = \frac{\partial f}{\partial y} = 3x^4y^2.$$

Every point on the x- and y-axes is critical, and at each of these points $f(x, y) = 0$. The diagram to the right showing the sign of $f(x, y)$ in the four quadrants indicates that the points $(0, y)$ for $y > 0$ yield relative minima; $(0, y)$ for $y < 0$ yield relative maxima; and $(x, 0)$ yield saddle points.

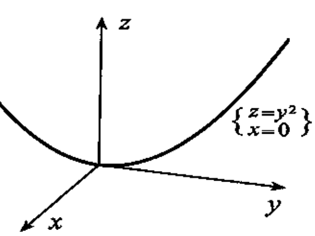


11. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = 2y^2 + 3y + 2xy^3$, $0 = \dfrac{\partial f}{\partial y} = 4xy + 3x + 3x^2y^2$. Solutions are $(0,0)$, $(0,-3/2)$ and $(-4/3, 3/2)$.

$$\frac{\partial^2 f}{\partial x^2} = 2y^3, \qquad \frac{\partial^2 f}{\partial x \partial y} = 4y + 3 + 6xy^2, \qquad \frac{\partial^2 f}{\partial y^2} = 4x + 6x^2 y$$

At $(0,0)$, $B^2 - AC = 9 > 0$, so that $(0,0)$ yields a saddle point.
At $(0,-3/2)$, $B^2 - AC = (-3)^2 > 0$, so that $(0,-3/2)$ yields a saddle point.
At $(-4/3, 3/2)$, $B^2 - AC = (9)^2 - 2(27/8)(32/3) > 0$, so that $(-4/3, 3/2)$ also yields a saddle point.

12. If $x > 0$, then $\partial f/\partial x = 1$, and if $x < 0$, then $\partial f/\partial x = -1$. Consequently, there are no critical points at which $0 = \partial f/\partial x = \partial f/\partial y$. However $\partial f/\partial x$ does not exist when $x = 0$, and therefore every point on the y-axis is critical. They cannot give saddle points because $\partial f/\partial x$ does not exist at these points. Since the cross-section of the surface $z = |x| + y^2$ with the plane $x = 0$ is the parabola $z = y^2$, $x = 0$ (see figure to the right), no critical point except possible $(0,0)$ can yield a relative maximum or minimum. Finally, because $f(0,0) = 0$, and $f(x,y) > 0$ for $(x,y) \neq (0,0)$, it follows that $(0,0)$ gives a relative minimum.



13. For critical points we solve

$$0 = \frac{\partial f}{\partial x} = -(1-y)(x+y-1) + (1-x)(1-y) = (1-y)(2-2x-y),$$
$$0 = \frac{\partial f}{\partial y} = -(1-x)(x+y-1) + (1-y)(1-x) = (1-x)(2-2y-x).$$

Solutions are $(1,1)$, $(0,1)$, $(1,0)$ and $(2/3, 2/3)$.

$$\frac{\partial^2 f}{\partial x^2} = 2(y-1), \qquad \frac{\partial^2 f}{\partial x \partial y} = -(2-2x-y) - (1-y) = -3 + 2x + 2y, \qquad \frac{\partial^2 f}{\partial y^2} = 2(x-1)$$

At $(1,1)$, $(0,1)$ and $(1,0)$, $B^2 - AC = 1$, so that each gives a saddle point. At $(2/3, 2/3)$, $B^2 - AC = (-1/3)^2 - 4(-1/3)(-1/3) = -1/3$ and $A = -2/3$. This critical point therefore yields a relative maximum.

14. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = 4x^3 - 2x = 2x(2x^2 - 1)$, $0 = \dfrac{\partial f}{\partial y} = 4y^3 - 2y = 2y(2y^2 - 1)$. These give $x = 0, \pm 1/\sqrt{2}$ and $y = 0, \pm 1/\sqrt{2}$, and therefore critical points are $(0,0)$, $(0, \pm 1/\sqrt{2})$, $(\pm 1/\sqrt{2}, 0)$, $(1/\sqrt{2}, \pm 1/\sqrt{2})$, $(-1/\sqrt{2}, \pm 1/\sqrt{2})$. With $f_{xx} = 12x^2 - 2$, $f_{xy} = 0$, and $f_{yy} = 12y^2 - 2$, we construct the following table.

	A	B	C	$B^2 - AC$	Classification
$(0,0)$	-2	0	-2	-4	relative maximum
$(0, \pm 1/\sqrt{2})$	-2	0	4	8	saddle points
$(\pm 1/\sqrt{2}, 0)$	4	0	-2	8	saddle points
$(1/\sqrt{2}, \pm 1/\sqrt{2})$	4	0	4	-16	relative minima
$(-1/\sqrt{2}, \pm 1/\sqrt{2})$	4	0	4	-16	relative minima

15. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = 2x + 3$, $0 = \dfrac{\partial f}{\partial y} = 2y - 2$, $0 = \dfrac{\partial f}{\partial z} = -2z$. The only solution is $(-3/2, 1, 0)$.

16. For critical points we solve

$$0 = \frac{\partial f}{\partial x} = 2xy^2z^2 + 2xt^2 + 3, \quad 0 = \frac{\partial f}{\partial y} = 2x^2yz^2, \quad 0 = \frac{\partial f}{\partial z} = 2x^2y^2z, \quad 0 = \frac{\partial f}{\partial t} = 2x^2t.$$

There are no solutions to these equations.

17. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = yz + 2xyz$, $0 = \dfrac{\partial f}{\partial y} = xz + x^2z - 1$, $0 = \dfrac{\partial f}{\partial z} = xy + x^2y$. Solutions of this equation are $\left(x, 0, \dfrac{1}{x^2+x}\right)$, where x is any real number except 0 and -1.

18. For critical points we solve

$$0 = \frac{\partial f}{\partial x} = yze^{x^2+y^2+z^2} + 2x^2yze^{x^2+y^2+z^2} = yz(1+2x^2)e^{x^2+y^2+z^2},$$

$$0 = \frac{\partial f}{\partial y} = xz(1+2y^2)e^{x^2+y^2+z^2}, \qquad 0 = \frac{\partial f}{\partial z} = xy(1+2z^2)e^{x^2+y^2+z^2}.$$

All points on the coordinate axes satisfy these equations.

19. A function $f(x,y,z)$ has a relative maximum at (x_0, y_0, z_0) if there exists a sphere centred at (x_0, y_0, z_0) such that for all points inside this sphere, $f(x,y,z) \le f(x_0,y_0,z_0)$. A function $f(x,y,z)$ has a relative minimum at (x_0,y_0,z_0) if there exists a sphere centred at (x_0,y_0,z_0) such that for all points inside this sphere, $f(x,y,z) \ge f(x_0,y_0,z_0)$.

20. If $f(x,y)$ is harmonic in D, it has continuous second partial derivatives in D and $\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 = 0$ therein. For $f(x,y)$ to have a relative maximum or minimum at a point (x,y) in D, its first partial derivatives must vanish there. In addition,

$$0 > B^2 - AC = \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) = \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 f}{\partial x^2}\right)\left(-\frac{\partial^2 f}{\partial x^2}\right) = \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 f}{\partial x^2}\right)^2,$$

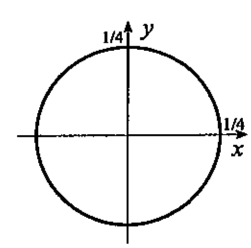
an impossibility if f_{xx} or f_{xy} does not vanish.

21. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = -8xy + 12x^3$, $0 = \dfrac{\partial f}{\partial y} = 2y - 4x^2$.
The only solution is $(0,0)$. Since $B^2 - AC = (-8x)^2 - (-8y + 36x^2)(2) = 0$ at $(0,0)$, the second derivative test fails. On the parabola $y = ax^2$, where a is a nonzero constant, function values are $f(x, ax^2) = (ax^2)^2 - 4x^2(ax^2) + 3x^4 = (a^2 - 4a + 3)x^4 = (a-1)(a-3)x^4$. Since this function takes on positive and negative values in every circle centred at the origin, it follows that the function must have a saddle point at $(0,0)$.

22. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = 4x^3 + 3y^2$, $0 = \dfrac{\partial f}{\partial y} = 6xy + 2y$. The only solutions are $(0,0)$ and $(-1/3, \pm 2/9)$.

$$\frac{\partial^2 f}{\partial x^2} = 12x^2, \qquad \frac{\partial^2 f}{\partial x \partial y} = 6y, \qquad \frac{\partial^2 f}{\partial y^2} = 6x + 2$$

At $(-1/3, \pm 2/9)$, $B^2 - AC = 36(4/81) - (12/9)(0) > 0$, and therefore $(-1/3, \pm 2/9)$ yield saddle points. At $(0,0)$, $B^2 - AC = 0$ and the test fails. Now $f(0,0) = 0$, and in the circle shown to the right, $f(x,y) = x^4 + y^2(1+3x) > 0$ (except at $(0,0)$). This implies that $(0,0)$ gives a relative minimum.



23. If we set $F(x,y,z) = 2x^2 + 3y^2 + z^2 - 12xy + 4xz - 35$, then

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(z)}} = -\frac{F_x}{F_z} = -\frac{4x - 12y + 4z}{2z + 4x} \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial(F)}{\partial(y)}}{\dfrac{\partial(F)}{\partial(z)}} = -\frac{F_y}{F_z} = -\frac{6y - 12x}{2z + 4x}$$

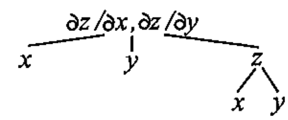
$$= \frac{6y - 2x - 2z}{z + 2x}, \qquad\qquad = \frac{6x - 3y}{z + 2x}.$$

For critical points we solve $0 = 2x - 6y + 2z$ and $0 = 3y - 6x$. Certainly $x = 1$ and $y = 2$ satisfy the second of these. When these values are substituted into the first, $z = 5$ is obtained. Since $x = 1$, $y = 2$, and $z = 5$ also satisfy the original equation, it follows that $(1, 2)$ is indeed a critical point. Consider now finding the second derivatives at the critical point. From the schematic,

$$\frac{\partial^2 z}{\partial x^2}_{|(1,2,5)} = \left[\frac{\partial}{\partial z}\left(\frac{\partial z}{\partial x}\right)\frac{\partial z}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\right]_{|(1,2,5)},$$

$$\frac{\partial^2 z}{\partial x \partial y}_{|(1,2,5)} = \left[\frac{\partial}{\partial z}\left(\frac{\partial z}{\partial x}\right)\frac{\partial z}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\right]_{|(1,2,5)},$$

$$\frac{\partial^2 z}{\partial y^2}_{|(1,2,5)} = \left[\frac{\partial}{\partial z}\left(\frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right]_{|(1,2,5)}.$$



But at the critical point $(1, 2, 5)$, $\partial z/\partial x$ and $\partial z/\partial y$ are both zero. Hence,

$$\frac{\partial^2 z}{\partial x^2}_{|(1,2,5)} = \left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\right]_{(1,2,5)} = \left[\frac{(z+2x)(-2) - (6y - 2x - 2z)(2)}{(z+2x)^2}\right]_{|(1,2,5)} = -\frac{2}{7},$$

$$\frac{\partial^2 z}{\partial x \partial y}_{|(1,2,5)} = \left[\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\right]_{|(1,2,5)} = \left[\frac{6}{z + 2x}\right]_{|(1,2,5)} = \frac{6}{7},$$

$$\frac{\partial^2 z}{\partial y^2}_{|(1,2,5)} = \left[\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right]_{|(1,2,5)} = \left[\frac{-3}{z + 2x}\right]_{|(1,2,5)} = -\frac{3}{7}.$$

Consequently, at $(1, 2)$, $B^2 - AC = (6/7)^2 - (-2/7)(-3/7) > 0$, and the critical point yields a saddle point.

24. (a) For critical points we solve $0 = \dfrac{\partial f}{\partial x} = 2x - yz$, $0 = \dfrac{\partial f}{\partial y} = 2y - xz$, $0 = \dfrac{\partial f}{\partial z} = 2z - xy$. Solutions are $(0,0,0)$, $(2, \pm 2, \pm 2)$, $(-2, \pm 2, \mp 2)$, $(2, \mp 2, \mp 2)$, and $(-2, \mp 2, \pm 2)$.

(b) The value of the function at $(0,0,0)$ is $f(0,0,0) = 0$. Since $(x-y)^2 \geq 0$, it follows that $x^2 + y^2 \geq 2xy$. Similarly, $y^2 + z^2 \geq 2yz$ and $x^2 + z^2 \geq 2xz$. Addition of these gives

$$2x^2 + 2y^2 + 2z^2 \geq 2(xy + yz + xz), \quad \text{or,} \quad x^2 + y^2 + z^2 \geq xy + yz + xz.$$

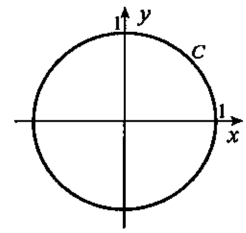
If $|z| < 1$, then $xy > xyz/3$. Similarly, if $|x| < 1$ and $|y| < 1$, then $yz > xyz/3$ and $xz > xyz/3$. Hence, for $|x| < 1$, $|y| < 1$, and $|z| < 1$,

$$x^2 + y^2 + z^2 \geq xy + yz + xz \geq \frac{xyz}{3} + \frac{xyz}{3} + \frac{xyz}{3} = xyz.$$

In other words, for all points inside the sphere $x^2 + y^2 + z^2 = 1$, we can say that $x^2 + y^2 + z^2 - xyz \geq 0$. Thus, $f(x,y,z)$ must have a relative minimum at $(0,0,0)$.

EXERCISES 12.11

1. For critical points of $f(x,y)$, we solve



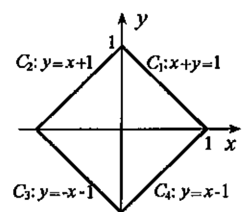
$$0 = \frac{\partial f}{\partial x} = 2x, \quad 0 = \frac{\partial f}{\partial y} = 3y^2.$$

The only solution is $(0,0)$ at which $f(0,0) = \boxed{0}$. On C, $z = f(x,y)$ can be expressed in the form

$$z = F(y) = 1 - y^2 + y^3, \; -1 \le y \le 1.$$

For critical points of this function, we solve $0 = F'(y) = -2y + 3y^2$. Critical points are $y = 0$ and $y = 2/3$. Since $F(-1) = \boxed{-1}$, $F(0) = \boxed{1}$, $F(2/3) = \boxed{23/27}$, and $F(1) = \boxed{1}$, maximum and minimum values of $f(x,y)$ are 1 and -1.

2. For critical points of $f(x,y)$, we solve



$$0 = \frac{\partial f}{\partial x} = 2x + 1, \quad 0 = \frac{\partial f}{\partial y} = 6y + 1.$$

The only solution is $(-1/2, -1/6)$ at which $f(-1/2,-1/6) = \boxed{-1/3}$.

On C_1, $z = f(x,y)$ becomes

$$z = F(x) = x^2 + x + 3(1-x)^2 + (1-x)$$
$$= 4x^2 - 6x + 4, \; 0 \le x \le 1.$$

For critical points of this function, we solve $0 = F'(x) = 8x - 6$. The only critical point is $x = 3/4$ at which $F(3/4) = \boxed{7/4}$.

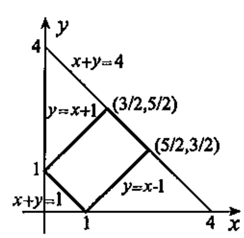
On C_2, $z = f(x,y)$ is $z = F(x) = x^2 + x + 3(x+1)^2 + (x+1) = 4x^2 + 8x + 4$, $-1 \le x \le 0$. For critical points of this function, we solve $0 = F'(x) = 8x + 8$. At the critical point $x = -1$, $F(-1) = \boxed{0}$.

On C_3, $z = f(x,y)$ is $z = F(x) = x^2 + x + 3(-x-1)^2 + (-x-1) = 4x^2 + 6x + 2$, $-1 \le x \le 0$. For critical points of this function, we solve $0 = F'(x) = 8x + 6$. At the critical point $x = -3/4$, $F(-3/4) = \boxed{-1/4}$.

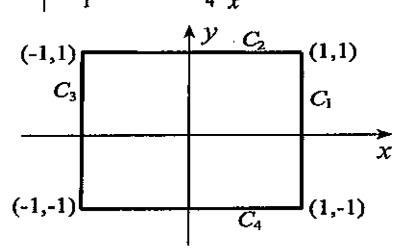
On C_4, $z = f(x,y)$ is $z = F(x) = x^2 + x + 3(x-1)^2 + (x-1) = 4x^2 - 4x + 2$, $0 \le x \le 1$. For critical points we solve $0 = F'(x) = 8x - 4$. At the critical point $x = 1/2$, $F(1/2) = \boxed{1}$.

Finally, at the remaining three corners of R, $f(1,0) = \boxed{2}$, $f(0,1) = \boxed{4}$, $f(0,-1) = \boxed{2}$. Maximum and minimum values of $f(x,y)$ on R are therefore 4 and $-1/3$.

3. The function has no critical points inside C. When $f(x,y)$ is expressed in terms of one variable on each part of C, the resulting function is linear, and therefore has no critical points. It follows that maximum and minimum of the function must occur at the vertices of the rectangle. Since $f(1,0) = \boxed{3}$, $f(0,1) = \boxed{4}$, $f(3/2,5/2) = \boxed{29/2}$, and $f(5/2,3/2) = \boxed{27/2}$, maximum and minimum values are 29/2 and 3.



4. For critical points of $f(x,y)$, we solve



$$0 = \frac{\partial f}{\partial x} = 2xy + y^2, \quad 0 = \frac{\partial f}{\partial y} = x^2 + 2xy + 1.$$

Solutions are $(\pm 1/\sqrt{3}, \mp 2/\sqrt{3})$ at which

$$f(1/\sqrt{3}, -2/\sqrt{3}) = \boxed{-4\sqrt{3}/9}, \quad f(-1/\sqrt{3}, 2/\sqrt{3}) = \boxed{4\sqrt{3}/9}.$$

On C_1, $z = f(x,y)$ becomes

$$z = F(y) = y + y^2 + y = 2y + y^2, \quad -1 \le y \le 1.$$

For critical points of this function, we solve $0 = F'(y) = 2 + 2y$. The only critical point is $y = -1$ at which $F(-1) = \boxed{-1}$.

On C_2, $z = f(x,y)$ is $z = F(x) = x^2 + x + 1$, $-1 \le x \le 1$. For critical points of this function, we solve $0 = F'(x) = 2x + 1$. At the critical point $x = -1/2$, $F(-1/2) = \boxed{3/4}$.

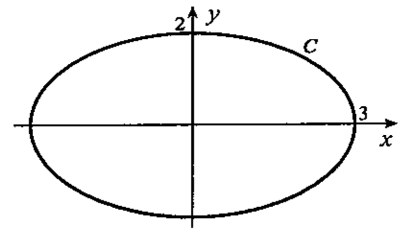
On C_3, $z = f(x,y)$ becomes $z = F(y) = y - y^2 + y = 2y - y^2$, $-1 \le y \le 1$. For critical points, we solve $0 = F'(y) = 2 - 2y$. At the critical point $y = 1$, $F(1) = \boxed{1}$.

On C_4, $z = f(x,y)$ is $z = F(x) = -x^2 + x - 1$, $-1 \le x \le 1$. For critical points we solve $0 = F'(x) = -2x + 1$. At the critical point $x = 1/2$, $F(1/2) = \boxed{-3/4}$.

Finally, we evaluate $f(x,y)$ at the remaining two corners, $f(1,1) = \boxed{3}$ and $f(-1,-1) = \boxed{-3}$. Maximum and minimum values of $f(x,y)$ on R are therefore 3 and -3.

5. For critical points of $f(x,y)$, we solve

$$0 = \frac{\partial f}{\partial x} = 6x + 2y, \quad 0 = \frac{\partial f}{\partial y} = 2x - 2y.$$



The only solution is $(0,0)$ at which $f(0,0) = \boxed{5}$.
On the edge of the ellipse we set

$$x = 3\cos t,\ y = 2\sin t,\ 0 \le t \le 2\pi,$$

in which case

$$f(x,y) = F(t) = 27\cos^2 t + 12\sin t\cos t - 4\sin^2 t + 5, \quad 0 \le t \le 2\pi.$$

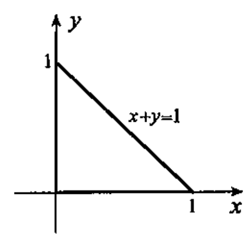
For critical points we solve

$$0 = F'(t) = -54\cos t\sin t + 12(\cos^2 t - \sin^2 t) - 8\sin t\cos t = -31\sin 2t + 12\cos 2t.$$

This implies that $\tan 2t = 12/31$, and the four solutions of this equation in the interval $0 \le t \le 2\pi$ are $(1/2)\text{Tan}^{-1}(12/31)$, $(1/2)\text{Tan}^{-1}(12/31) + \pi/2$, $(1/2)\text{Tan}^{-1}(12/31) + \pi$, and $(1/2)\text{Tan}^{-1}(12/31) + 3\pi/2$. When these are substituted into $F(t)$, two values result, namely $\boxed{33.12}$ and $\boxed{-0.12}$. These are maximum and minimum values of the function on the ellipse.

6. For critical points of $f(x,y)$, we solve

$$0 = \frac{\partial f}{\partial x} = 3x^2 - 3, \quad 0 = \frac{\partial f}{\partial y} = 2y + 2.$$



Solutions are $(\pm 1, -1)$ both of which are outside the region. On the edge $x = 0$,

$$f(0,y) = F(y) = y^2 + 2y, \quad 0 \le y \le 1.$$

It has a critical point when $0 = 2y + 2 \Longrightarrow y = -1$, which we reject.

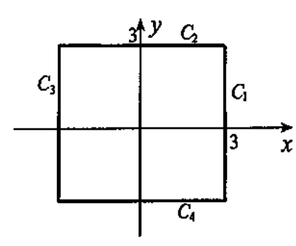
On $y = 0$, $f(x,0) = G(x) = x^3 - 3x$, $0 \le x \le 1$, which has critical points when $0 = 3x^2 - 3 \Longrightarrow x = \pm 1$. The value of $G(x)$ at $x = 1$ is $G(1) = \boxed{-2}$.

On $x + y = 1$, $f(x,y) = H(x) = x^3 - 3x + (1-x)^2 + 2(1-x) = x^3 + x^2 - 7x + 3$, $0 \le x \le 1$. It has critical points when $0 = H'(x) = 3x^2 + 2x - 7$. Neither of the solutions $x = (-1 \pm \sqrt{22})/3$ lie in the interval $0 \le x \le 1$.

We have evaluated $f(x,y)$ at vertex $(1,0)$ of the triangle. Its values at the remaining two vertices are $f(0,1) = \boxed{3}$ and $f(0,0) = \boxed{0}$. Maximum and minimum values are therefore 3 and -2.

7. For critical points of $f(x,y)$, we solve

$$0 = \frac{\partial f}{\partial x} = 3x^2 - 3, \quad 0 = \frac{\partial f}{\partial y} = 3y^2 - 12.$$



Solutions are $(\pm 1, \pm 2)$ and $(\pm 1, \mp 2)$ at which $f(x,y)$ has the values $\boxed{-16}$, $\boxed{20}$, $\boxed{16}$, and $\boxed{-12}$.

On C_1, $z = f(x,y)$ becomes

$$z = F(y) = y^3 - 12y + 20, \quad -3 \le y \le 3.$$

For critical points of this function, we solve $0 = F'(y) = 3y^2 - 12$. Critical points are $y = \pm 2$ at which $F(\pm 2) = \boxed{4}, \boxed{36}$.

On C_2, $z = f(x,y)$ is $z = F(x) = x^3 - 3x - 7$, $-3 \le x \le 3$. For critical points of this function, we solve $0 = F'(x) = 3x^2 - 3$. Critical points are $x = \pm 1$ at which $F(\pm 1) = \boxed{-9}, \boxed{-5}$.

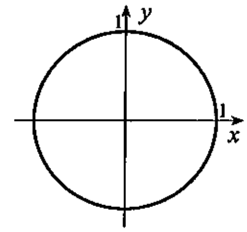
On C_3, $z = f(x,y)$ becomes $z = F(y) = y^3 - 12y - 16$, $-3 \le y \le 3$. For critical points, we solve $0 = F'(y) = 3y^2 - 12$. Critical points are $y = \pm 2$ at which $F(\pm 2) = \boxed{-32}, \boxed{0}$.

On C_4, $z = f(x,y)$ is $z = F(x) = x^3 - 3x + 11$, $-3 \le x \le 3$. For critical points we solve $0 = F'(x) = 3x^2 - 3$. Critical points are $x = \pm 1$ at which $F(\pm 1) = \boxed{9}, \boxed{13}$.

Finally, we evaluate $f(x,y)$ at the corners , $f(3,3) = \boxed{11}$, $f(-3,-3) = \boxed{-7}$, $f(3,-3) = \boxed{29}$, $f(-3,3) = \boxed{-25}$. The maximum and minimum values of $f(x,y)$ on R are therefore 36 and -32.

8. For critical points of $f(x,y)$, we solve

$$0 = \frac{\partial f}{\partial x} = 3x^2 - 3, \quad 0 = \frac{\partial f}{\partial y} = 3y^2 - 3.$$



The solutions $(\pm 1, 1)$ and $(\pm 1, -1)$ are exterior to the circle. On the edge of the circle we set

$$x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi,$$

in which case

$$f(x,y) = F(t) = \cos^3 t + \sin^3 t - 3\cos t - 3\sin t + 2, \quad 0 \le t \le 2\pi.$$

For critical points we solve

$$\begin{aligned} 0 = F'(t) &= -3\cos^2 t\,\sin t + 3\sin^2 t\,\cos t + 3\sin t - 3\cos t \\ &= 3\sin t\,\cos t(\sin t - \cos t) + 3(\sin t - \cos t) \\ &= 3(\sin t\,\cos t + 1)(\sin t - \cos t). \end{aligned}$$

Setting the factor $\sin t\,\cos t + 1 = 0$ leads to $\sin 2t = -2$, an impossibility. The other possibility is to set $\sin t - \cos t = 0$, which leads to $t = \pi/4$, and $t = 5\pi/4$. Since

$$F(0) = F(2\pi) = \boxed{0}, \quad F(\pi/4) = \boxed{\frac{2\sqrt{2}-5}{\sqrt{2}}}, \quad F(5\pi/4) = \boxed{\frac{2\sqrt{2}+5}{\sqrt{2}}},$$

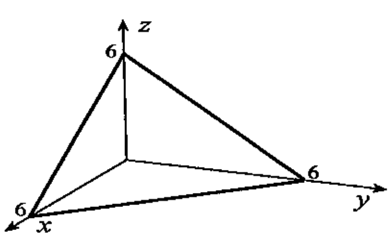
maximum and minimum values are $(2\sqrt{2}+5)/\sqrt{2}$ and $(2\sqrt{2}-5)/\sqrt{2}$.

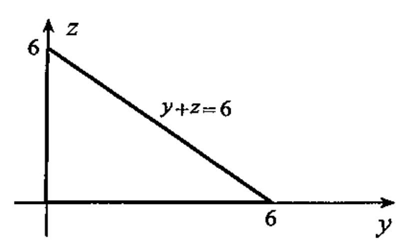
9. (a) If we set $x = 6 - y - z$, then $f(x,y,z) = F(y,z) = (6-y-z)y^2z^3$. For critical points we solve

$$\begin{aligned} 0 = F_y &= -y^2z^3 + 2(6-y-z)yz^3 = yz^3(12 - 3y - 2z), \\ 0 = F_z &= -y^2z^3 + 3(6-y-z)y^2z^2 = y^2z^2(18 - 3y - 4z). \end{aligned}$$

The only solution inside the triangle in the right figure below is $y = 2$ and $z = 3$. Since $F(2,3) = \boxed{108}$, and values of the function become arbitrarily close to 0 as we approach the sides of the triangle, it follows that the function has maximum value 108, but no minimum.

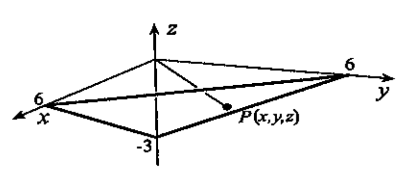
(b) When we include the edges of the plane (left figure below), we must find extreme values of $F(y,z)$ on the triangle in part (a), but now including its edges. Since the value of $F(y,z)$ is identically zero on the edges, it follows that maximum and minimum values now are 108 and 0.





10. The distance D from O to any point $P(x,y,z)$ on the plane is given by

$$\begin{aligned} D^2 &= x^2 + y^2 + z^2 \\ &= (6 - y + 2z)^2 + y^2 + z^2. \end{aligned}$$



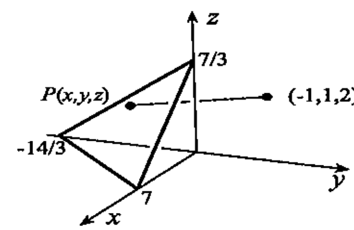
For critical points of D^2 we solve

$$0 = \frac{\partial D^2}{\partial y} = -2(6 - y + 2z) + 2y, \quad 0 = \frac{\partial D^2}{\partial z} = 4(6 - y + 2z) + 2z.$$

The only solution is $y = 1$, $z = -2$. Since y and z can take on all possible values, and D^2 becomes infinite for large values of y or z, the critical point must minimize D^2. The closest point is therefore $(1, 1, -2)$.

11. The distance D from $(-1, 1, 2)$ to any point $P(x, y, z)$ on the plane is given by

$$\begin{aligned} D^2 &= (x+1)^2 + (y-1)^2 + (z-2)^2 \\ &= \left(7 + \frac{3y}{2} - 3z + 1\right)^2 + (y-1)^2 + (z-2)^2 \\ &= \left(8 + \frac{3y}{2} - 3z\right)^2 + (y-1)^2 + (z-2)^2. \end{aligned}$$



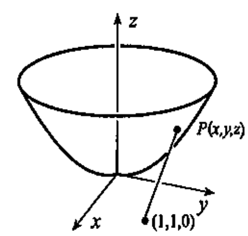
For critical points of D^2 we solve

$$0 = \frac{\partial D^2}{\partial y} = 2\left(8 + \frac{3y}{2} - 3z\right)\left(\frac{3}{2}\right) + 2(y-1), \quad 0 = \frac{\partial D^2}{\partial z} = 2\left(8 + \frac{3y}{2} - 3z\right)(-3) + 2(z-2).$$

The only solution is $y = 4/7$, $z = 20/7$. Since y and z can take on all possible values, and D^2 becomes infinite for large values of y or z, the critical point must minimize D^2. The shortest distance is therefore $D(4/7, 20/7) = 1$.

12. The distance D from $(1, 1, 0)$ to any point $P(x, y, z)$ on the surface is given by

$$\begin{aligned} D^2 &= (x-1)^2 + (y-1)^2 + z^2 \\ &= (x-1)^2 + (y-1)^2 + (x^2 + y^2)^2. \end{aligned}$$



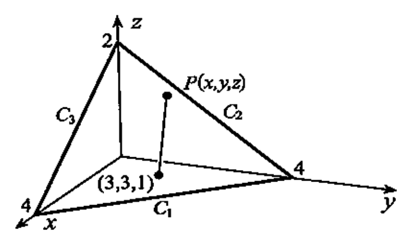
For critical points of D^2, we solve

$$\begin{aligned} 0 &= \frac{\partial D^2}{\partial x} = 2(x-1) + 4x(x^2 + y^2), \\ 0 &= \frac{\partial D^2}{\partial y} = 2(y-1) + 4y(x^2 + y^2). \end{aligned}$$

The only solution is $x = 1/2$, $y = 1/2$. Since x and y can take on all possible values, and D^2 becomes infinite for large values of x and y, the critical point must minimize D^2. The closest point is therefore $(1/2, 1/2, 1/2)$.

13. The distance D from $(3, 3, 1)$ to any point $P(x, y, z)$ on the plane is given by

$$\begin{aligned} D^2 &= (x-3)^2 + (y-3)^2 + (z-1)^2 \\ &= (4 - y - 2z - 3)^2 + (y-3)^2 + (z-1)^2 \\ &= (1 - y - 2z)^2 + (y-3)^2 + (z-1)^2. \end{aligned}$$



For critical points of D^2 we solve

$$0 = \frac{\partial D^2}{\partial y} = -2(1 - y - 2z) + 2(y-3), \quad 0 = \frac{\partial D^2}{\partial z} = -4(1 - y - 2z) + 2(z-1).$$

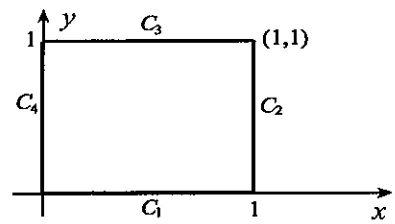
The only solution is $y = 7/3$, $z = -1/3$, which is unacceptable (not being in the first octant). Because of the proximity of $(3, 3, 1)$ to C_1, we can be sure that the closest point lies somewhere along C_1, not on C_2 or C_3. We therefore minimize

$$D^2(x, y, z) = F(y) = (1-y)^2 + (y-3)^2 + 1 = 2y^2 - 8y + 11, \quad 0 \le y \le 4.$$

For critical points we solve $0 = F'(y) = 4y - 8$. Since the solution is $y = 2$, the closest point is $(2, 2, 0)$.

14. For critical points of $V(x,y)$, we solve

$$0 = \frac{\partial V}{\partial x} = 48y - 96x^2, \quad 0 = \frac{\partial V}{\partial y} = 48x - 48y.$$



At the critical points $(0,0)$ and $(1/2,1/2)$,

$V(0,0) = \boxed{0}, \quad V(1/2,1/2) = \boxed{2}.$

On C_1, $V = -32x^3$, $0 \le x \le 1$. This function has a critical point at $x = 0$ corresponding to $(0,0)$.

On C_2, $V = 48y - 32 - 24y^2$, $0 \le y \le 1$. For critical points, $0 = dV/dy = 48 - 48y$. At the critical point $y = 1$, $V = \boxed{-8}$.

On C_3, $V = 48x - 32x^3 - 24$, $0 \le x \le 1$. For critical points, $0 = dV/dx = 48 - 96x^2$. At the critical point $x = 1/\sqrt{2}$, $V = \boxed{8(2\sqrt{2}-3)}$.

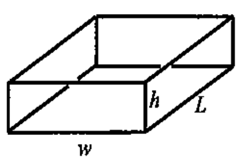
On C_4, $V = -24y^2$, $0 \le y \le 1$. The only critical point of this function is $y = 0$, corresponding to $(0,0)$.

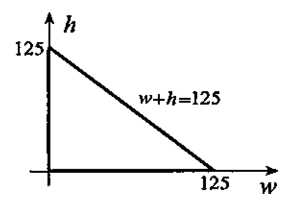
Finally, at the remaining two corners, $V(1,0) = \boxed{-32}$ and $V(0,1) = \boxed{-24}$. Maximum and minimum values of $V(x,y)$ are therefore 2 and -32.

15. The volume of the box is $V = Lwh$. Since $L + 2(w+h) \le 250$, we set $L = 250 - 2w - 2h$, in which case $V = wh(250 - 2w - 2h)$. This function must be maximized on the triangle in the right figure below. For critical points we solve

$$0 = \frac{\partial V}{\partial w} = 250h - 4wh - 2h^2, \qquad 0 = \frac{\partial V}{\partial h} = 250w - 2w^2 - 4wh.$$

The only solution inside the triangle is $w = h = 125/3$. Since $V = 0$ on the edges of the triangle, it follows that this critical point must yield a maximum volume. Dimensions of the box are therefore $w = h = 125/3$ cm and $L = 250/3$ cm.



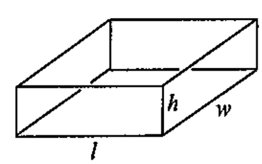


16. If k is the cost per square centimetre for lining the side of the tank, the total cost is

$$C = k(2hw + 2hl) + 3k(wl).$$

Because the tank must hold 1000 L,

$$10^6 = lwh,$$



and therefore

$$C = C(w,h) = k\left[2hw + 2h\left(\frac{10^6}{wh}\right) + 3w\left(\frac{10^6}{wh}\right)\right] = k\left[2hw + \frac{2\times 10^6}{w} + \frac{3\times 10^6}{h}\right].$$

This function must be minimized for all points in the first quadrant of the wh-plane. For critical points of $C(w,h)$,

$$0 = \frac{\partial C}{\partial w} = k\left[2h - \frac{2\times 10^6}{w^2}\right], \qquad 0 = \frac{\partial C}{\partial h} = k\left[2w - \frac{3\times 10^6}{h^2}\right].$$

The only critical point is $w = 100(2/3)^{1/3}$, $h = 100(3/2)^{2/3}$. Since C becomes infinite as $h \to 0$ or $w \to 0$, or h or w become infinite, it follows that the critical point must minimize C. The required dimensions are therefore $w = 100(2/3)^{1/3}$ cm, $h = 100(3/2)^{2/3}$ cm, $l = 100(2/3)^{1/3}$ cm.

17. The distance D from (x_1, y_1, z_1) to any point $P(x, y, z)$ on the plane is given by

$$D^2 = (x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2 = (x-x_1)^2 + (y-y_1)^2 + \left(\frac{-D-Ax-By}{C} - z_1\right)^2$$
$$= (x-x_1)^2 + (y-y_1)^2 + \frac{1}{C^2}(D+Ax+By+Cz_1)^2.$$

For critical points of D^2 we solve

$$0 = \frac{\partial D^2}{\partial x} = 2(x-x_1) + \frac{2A}{C^2}(D+Ax+By+Cz_1), \quad 0 = \frac{\partial D^2}{\partial y} = 2(y-y_1) + \frac{2B}{C^2}(D+Ax+By+Cz_1),$$

getting $x = \dfrac{(B^2+C^2)x_1 - A(By_1+Cz_1+D)}{A^2+B^2+C^2}$, $\quad y = \dfrac{(A^2+C^2)y_1 - B(Ax_1+Cz_1+D)}{A^2+B^2+C^2}$. Since this is the only critical point, and distance becomes infinite as x and y take on large values, it follows that this critical point must minimize D^2. To find the minimum value we substitute these values for x and y into the formula for D^2,

$$D^2 = \left[\frac{(B^2+C^2)x_1 - A(By_1+Cz_1+D)}{A^2+B^2+C^2} - x_1\right]^2 + \left[\frac{(A^2+C^2)y_1 - B(Ax_1+Cz_1+D)}{A^2+B^2+C^2} - y_1\right]^2$$
$$+ \frac{1}{C^2}\left[D + \frac{A(B^2+C^2)x_1 - A^2(By_1+Cz_1+D)}{A^2+B^2+C^2} + \frac{B(A^2+C^2)y_1 - B^2(Ax_1+Cz_1+D)}{A^2+B^2+C^2} + Cz_1\right]^2.$$

This simplifies to $\dfrac{(Ax_1+By_1+Cz_1+D)^2}{A^2+B^2+C^2}$, and its square root gives the desired result.

18. If $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, and $C(x_3, y_3, z_3)$ are vertices of a triangle, and $P(x, y, z)$ is any other point, then the sum of the squares of the distances from P to A, B, and C is

$$\begin{aligned} D &= \|PA\|^2 + \|PB\|^2 + \|PC\|^2 \\ &= (x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2 + (x-x_2)^2 + (y-y_2)^2 + (z-z_2)^2 \\ &\quad + (x-x_3)^2 + (y-y_3)^2 + (z-z_3)^2. \end{aligned}$$

For critical points of this function, we solve

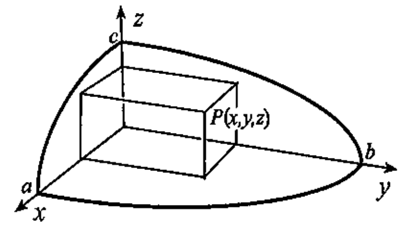
$$\begin{aligned} 0 &= \frac{\partial D}{\partial x} = 2(x-x_1) + 2(x-x_2) + 2(x-x_3), \\ 0 &= \frac{\partial D}{\partial y} = 2(y-y_1) + 2(y-y_2) + 2(y-y_3), \\ 0 &= \frac{\partial D}{\partial z} = 2(z-z_1) + 2(z-z_2) + 2(z-z_3). \end{aligned}$$

The solution is $x = \frac{1}{3}(x_1+x_2+x_3)$, $y = \frac{1}{3}(y_1+y_2+y_3)$, $z = \frac{1}{3}(z_1+z_2+z_3)$, the centroid of the triangle (see Exercise 43 in Section 7.7). Since D becomes infinite as the point (x, y) moves farther and farther away from the triangle, it follows that this one, and only one, critical point must minimize D.

19. The distance D from any point (x, y, z) on the curve to the origin is given by $D^2 = x^2 + y^2 + z^2$. Because every point on the curve satisfies $x^2 + y^2 = 1$, we can write that $D^2 = 1 + z^2$. Furthermore, the equations of the curve imply that $z^2 + xy = 0$, and therefore $D^2 = 1 - xy$. We now set $x = \cos t$ and $y = \sin t$, so that $D^2 = F(t) = 1 - \cos t \sin t$. Since $z^2 + xy = 0$ requires x and y to have opposite signs, we consider this function for values $\pi/2 \le t \le \pi$ and $-\pi/2 \le t \le 0$. Critical points of $F(t)$ are given by $0 = F'(t) = -\cos^2 t + \sin^2 t = -\cos 2t$. Acceptable solution are $t = 3\pi/4$ and $t = -\pi/4$. Since

$$F(-\pi/2) = 1, \quad F(-\pi/4) = 3/2, \quad F(0) = 1, \quad F(\pi/2) = 1, \quad F(3\pi/4) = 3/2, \quad F(\pi) = 1,$$

it follows that the points on the curve closest to the origin are $(\pm 1, 0, 0)$ and $(0, \pm 1, 0)$.



20. The volume obtained from a point $P(x, y, z)$ on that part of the ellipsoid in the first octant is

$$V = 8xyz = 8cxy\sqrt{1 - x^2/a^2 - y^2/b^2}.$$

This function must be maximized in the first quadrant portion R of the ellipse $x^2/a^2 + y^2/b^2 = 1$. For critical points of V, we solve

$$0 = \frac{\partial V}{\partial x} = 8cy\sqrt{1 - x^2/a^2 - y^2/b^2} - \frac{8cx^2y/a^2}{\sqrt{1 - x^2/a^2 - y^2/b^2}},$$

$$0 = \frac{\partial V}{\partial y} = 8cx\sqrt{1 - x^2/a^2 - y^2/b^2} - \frac{8cxy^2/b^2}{\sqrt{1 - x^2/a^2 - y^2/b^2}}.$$

The only solution of these equations inside R is $x = a/\sqrt{3}$, $y = b/\sqrt{3}$. Since $V = 0$ on the three parts of the boundary of R, it follows that V must be maximized at this critical point, and therefore the dimensions of the largest box are $2a/\sqrt{3} \times 2b/\sqrt{3} \times 2c/\sqrt{3}$.

21. On the upper hemisphere $z = \sqrt{1 - x^2 - y^2}$, in which case $f(x, y, z) = F(x, y) = xy\sqrt{1 - x^2 - y^2}$, where $x^2 + y^2 \le 1$. For critical points we solve

$$0 = \frac{\partial F}{\partial x} = y\sqrt{1 - x^2 - y^2} - \frac{x^2y}{\sqrt{1 - x^2 - y^2}}, \quad 0 = \frac{\partial F}{\partial y} = x\sqrt{1 - x^2 - y^2} - \frac{xy^2}{\sqrt{1 - x^2 - y^2}}.$$

Solutions are $x = \pm 1/\sqrt{3}$ and $y = \pm 1/\sqrt{3}$. At the critical points with these coordinates, values of $F(x, y)$ are $\pm\sqrt{3}/9$. On the edge $x^2 + y^2 = 1$ the value of the function is identically zero. Hence, maximum and minimum values on the upper hemisphere are $\pm\sqrt{3}/9$. The same values are obtained for the lower hemisphere.

22. We write $f(x, y) = F(x) = x^2 - (1 - x^2) = 2x^2 - 1$, $-1 \le x \le 1$. For critical points of $F(x)$, we solve $0 = F'(x) = 4x \implies x = 0$. Since $F(-1) = 1$, $F(0) = -1$, and $F(1) = 1$, maximum and minimum values are ± 1.

23. If we set $x = \cos t$, $y = \sin t$, then $f(x, y) = F(t) = |\cos t - \sin t|$, $0 \le t \le 2\pi$. For critical points of $F(t)$ we solve

$$0 = F'(t) = \frac{|\cos t - \sin t|}{\cos t - \sin t}(-\sin t - \cos t).$$

The derivative is zero when $\sin t + \cos t = 0$, and the only angles in $0 \le t \le 2\pi$ satisfying this equation are $t = 3\pi/4$ and $t = 7\pi/4$. The derivative does not exist when $\cos t = \sin t$, and solutions of this are $t = \pi/4$ and $t = 5\pi/4$. Since

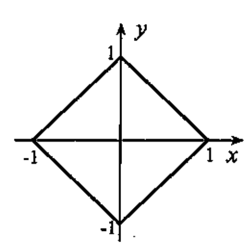
$$F(0) = 1,\ F(\pi/4) = 0,\ F(3\pi/4) = \sqrt{2},\ F(5\pi/4) = 0,\ F(7\pi/4) = \sqrt{2},\ F(2\pi) = 1,$$

maximum and minimum values are $\sqrt{2}$ and 0.

24. Since $y^2 = (1 - |x|)^2$ on the edges of the square, we can express $f(x, y)$ in terms of x alone,

$$f(x, y) = F(x) = x^2 - (1 - |x|)^2 = 2|x| - 1, \quad -1 \le x \le 1.$$

There are no points at which the derivative of this function vanishes, but it does not exist at the critical point $x = 0$. Since $F(-1) = 1$, $F(0) = -1$, and $F(1) = 1$, maximum and minimum values are ± 1.

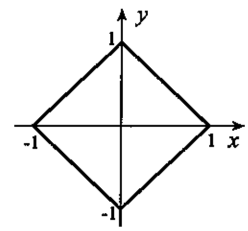


25. On the top half of the square $y = 1 - |x|$, in which case we can write that

$$f(x,y) = F(x) = \big|x - 2 + 2|x|\big|, \ -1 \le x \le 1.$$

For critical points of this function, we solve

$$0 = F'(x) = \frac{\big|x - 2 + 2|x|\big|}{x - 2 + 2|x|}\left(1 + \frac{2|x|}{x}\right).$$



There are no solutions of this equation, but the derivative does not exist at $x = 0$ and $x = 2/3$. We now calculate $F(-1) = 1$, $F(0) = 2$, $F(2/3) = 0$, and $F(1) = 1$. A similar analysis on the bottom half of the square leads to the same values. Hence, maximum and minimum values of $f(x,y)$ are 2 and 0.

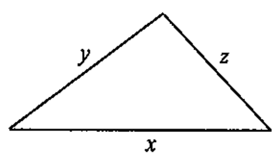
26. We may write

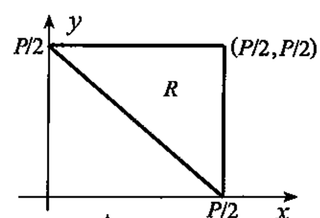
$$A^2 = \frac{P}{2}\left(\frac{P}{2} - x\right)\left(\frac{P}{2} - y\right)\left(\frac{P}{2} + x + y - P\right) = \frac{P}{2}\left(\frac{P}{2} - x\right)\left(\frac{P}{2} - y\right)\left(x + y - \frac{P}{2}\right).$$

This function must be maximized for those points (x, y) in the triangle R shown to the right. For critical points of A^2,

$$0 = \frac{\partial(A^2)}{\partial x} = -\frac{P}{2}\left(\frac{P}{2} - y\right)\left(x + y - \frac{P}{2}\right) + \frac{P}{2}\left(\frac{P}{2} - x\right)\left(\frac{P}{2} - y\right),$$
$$0 = \frac{\partial(A^2)}{\partial y} = -\frac{P}{2}\left(\frac{P}{2} - x\right)\left(x + y - \frac{P}{2}\right) + \frac{P}{2}\left(\frac{P}{2} - x\right)\left(\frac{P}{2} - y\right).$$

The only solution of these equations inside R is $x = y = P/3$. Since $A^2 = 0$ on the boundary of R, it follows that A is maximized when $x = y = P/3$, in which case $z = P/3$ also.

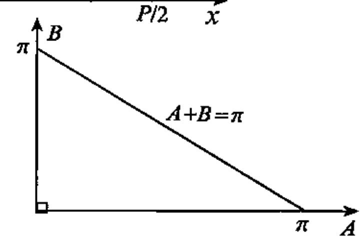




27. Since $A + B + C = \pi$, we may write that

$$f(A, B, C) = F(A, B) = \sin(A/2)\sin(B/2)\sin(\pi/2 - A/2 - B/2).$$

Consider finding the maximum value of this function on the triangle shown. Its minimum is zero (everywhere on the edge). For critical points we solve



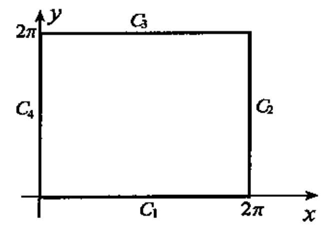
$$0 = \frac{\partial F}{\partial A} = \frac{1}{2}\cos(A/2)\sin(B/2)\sin(\pi/2 - A/2 - B/2) - \frac{1}{2}\sin(A/2)\sin(B/2)\cos(\pi/2 - A/2 - B/2),$$
$$0 = \frac{\partial F}{\partial B} = \frac{1}{2}\sin(A/2)\cos(B/2)\sin(\pi/2 - A/2 - B/2) - \frac{1}{2}\sin(A/2)\sin(B/2)\cos(\pi/2 - A/2 - B/2).$$

From the second,

$$\begin{aligned} 0 &= \sin(A/2)[\sin(\pi/2 - A/2 - B/2)\cos(B/2) - \sin(B/2)\cos(\pi/2 - A/2 - B/2)] \\ &= \sin(A/2)\sin(\pi/2 - A/2 - B). \end{aligned}$$

Thus, $\pi/2 - A/2 - B = n\pi \implies A/2 + B = \pi/2 - n\pi$. Since A and B are angles in a triangle, it follows that $n = 0$, and $A/2 + B = \pi/2$. Similarly, from the first equation, we obtain $A + B/2 = \pi/2$, and together, these equations imply that $A = B = \pi/3$. Since these values yield a maximum value for the function of $F(\pi/3, \pi/3) = 1/8$, it follows that for all other angles $F(A, B) \le 1/8$.

28. Because the function is 2π-periodic in x and y, we need only show that the inequality is valid in the square $R: 0 \le x \le 2\pi$, $0 \le y \le 2\pi$. We shall show that maximum and minimum values of the function $f(x,y) = \cos x + \cos y + \sin x \sin y$ in R are ± 2. Critical points of $f(x,y)$ are given by



$$0 = \frac{\partial f}{\partial x} = -\sin x + \cos x \sin y, \qquad 0 = \frac{\partial f}{\partial y} = -\sin y + \sin x \cos y.$$

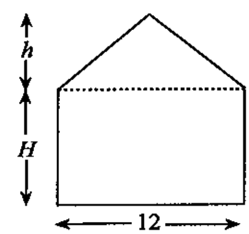
If we solve the first for $\sin x$ and substitute into the second,

$$0 = -\sin y + (\cos x \sin y)\cos y = \sin y(\cos x \cos y - 1).$$

For $\sin y$ to vanish, y must be 0, π, or 2π. The first and third of these are boundaries of R which will be treated later. If $y = \pi$, then $\sin x = 0$, from which x must be 0, π, or 2π. Again, the first and third of these are boundaries of R. In other words, we obtain a critical point of $f(x,y)$ in R to be (π,π) at which $f(\pi,\pi) = -2$. The other possibility for critical points interior to R is to set $\cos x \cos y = 1$. This can be true only if both x and y are equal to 1 or -1. This implies that both x and y must be equal to 0, π, or 2π, and we are led to the same critical point (π,π) inside R. On the boundary C_1, $f(x,y) = 1 + \cos x$, $0 \le x \le 2\pi$. The maximum and minimum values are 2 and 0. The same results are obtained on the remaining three boundaries C_2, C_3, and C_4. Thus, maximum and minimum values of $f(x,y)$ on the square are ± 2, and our proof is complete.

29. The volume of the silo is

$$V = \pi(6)^2 H + \frac{1}{3}\pi(6)^2 h = 12\pi(h + 3H).$$



Since area of the silo is $200 = 2\pi(6)H + \pi(6)\sqrt{36 + h^2}$, it follows that

$$\begin{aligned} V &= 12\pi\left[h + 3\left(\frac{200 - 6\pi\sqrt{36+h^2}}{12\pi}\right)\right] \\ &= 12\pi h + 600 - 18\pi\sqrt{36+h^2}, \quad 0 \le h \le \frac{\sqrt{4\times 10^4 - 36^2\pi^2}}{6\pi}. \end{aligned}$$

For critical points of V, we solve

$$0 = \frac{dV}{dh} = 12\pi - \frac{18\pi h}{\sqrt{36+h^2}}.$$

The only positive solution of this equation is $h = 12/\sqrt{5}$. Since

$$V(0) = 600 - 108\pi = 260.7, \quad V\left(\frac{12}{\sqrt{5}}\right) = 347.1, \quad V\left(\frac{\sqrt{4\times 10^4 - 36^2\pi^2}}{6\pi}\right) = 329.9,$$

V is maximized for $h = 12/\sqrt{5}$ m and $H = (50\sqrt{5} - 27\pi)/(3\sqrt{5}\pi)$ m.

30. If we set $P(x,y) = F(y) = kx^\alpha y^{1-\alpha} = k\left(\frac{C - By}{A}\right)^\alpha y^{1-\alpha}$, $0 \le y \le C/B$, then the minimum value of the function occurs at the end points. The maximum must occur at a critical point. For critical points we solve

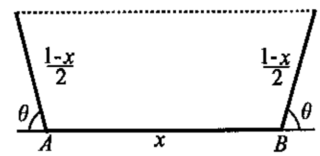
$$\begin{aligned} 0 = F'(y) &= k\left[\left(\frac{C-By}{A}\right)^\alpha (1-\alpha)y^{-\alpha} - \frac{\alpha B}{A}\left(\frac{C-By}{A}\right)^{\alpha-1} y^{1-\alpha}\right] \\ &= k\left(\frac{C-By}{A}\right)^{\alpha-1} y^{-\alpha}\left[\left(\frac{C-By}{A}\right)(1-\alpha) - \frac{\alpha B}{A}y\right]. \end{aligned}$$

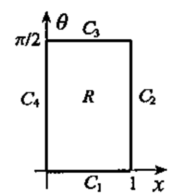
Ths solution is $y = C(1-\alpha)/B = C\beta/B$. The corresponding x-value is $x = C\alpha/A$.

31. If the length of AB is x, and the bends are at angle θ, then the area of the trapezoid is

$$F(x,\theta) = \frac{1}{2}\left(\frac{1-x}{2}\right)\sin\theta\left[2x + 2\left(\frac{1-x}{2}\right)\cos\theta\right] = \frac{1}{4}(1-x)\sin\theta[2x + (1-x)\cos\theta].$$

This function must be maximized for the region R of the $x\theta$-plane shown to the right below.





For critical points, we solve

$$0 = \frac{\partial F}{\partial x} = \frac{1}{4}\sin\theta[2 - 4x - 2(1-x)\cos\theta],$$
$$0 = \frac{\partial F}{\partial \theta} = \frac{1}{4}(1-x)[2x\cos\theta + (1-x)(\cos^2\theta - \sin^2\theta)].$$

Since $\sin\theta = 0$ and $1 - x = 0$ correspond to edges of R, we set

$$1 - 2x - (1-x)\cos\theta = 0, \qquad 2x\cos\theta + (1-x)(\cos^2\theta - \sin^2\theta) = 0.$$

The first equation implies that $\cos\theta = (1-2x)/(1-x)$, and when this is substituted into the second equation,

$$0 = 2x\left(\frac{1-2x}{1-x}\right) + (1-x)\left[2\left(\frac{1-2x}{1-x}\right)^2 - 1\right].$$

This simplifies to $0 = 3x^2 - 4x + 1 = (3x-1)(x-1)$. Thus, $x = 1/3$, and from this $\theta = \pi/3$. The area of the trapezoid so formed is

$$F = \frac{1}{4}\left(\frac{2}{3}\right)\left(\frac{\sqrt{3}}{2}\right)\left[\frac{2}{3} + \frac{2}{3}\left(\frac{1}{2}\right)\right] = \boxed{\frac{\sqrt{3}}{12}}.$$

For values of x and θ along edges C_1 and C_2 of R, the area of the trapezoid is $\boxed{0}$. Along C_3,

$$F = \frac{1}{4}(1-x)2x = \frac{x(1-x)}{2}, \quad 0 \le x \le 1.$$

For critical points, $0 = dF/dx = (1-2x)/2$. At the critical point $x = 1/2$, $F(1/2) = \boxed{1/8}$.

Along C_4,

$$F = \frac{1}{4}\sin\theta\cos\theta = \frac{1}{8}\sin 2\theta, \quad 0 \le \theta \le \pi/2.$$

For critical points, $0 = dF/d\theta = (1/4)\cos 2\theta$. At the critical point $\theta = \pi/4$, $F(\pi/4) = \boxed{1/8}$.

Finally, at the four vertices of the rectangle,

$$F(0,0) = \boxed{0}, \quad F(1,0) = \boxed{0}, \quad F(1,\pi/2) = \boxed{0}, \quad F(0,\pi/2) = \boxed{0}.$$

Thus, area is maximized when $\theta = \pi/3$ and $x = 1/3$ m.

32. For critical points of $f(x,y,z)$ inside the sphere we solve

$$0 = \frac{\partial f}{\partial x} = y + z, \quad 0 = \frac{\partial f}{\partial y} = x, \quad 0 = \frac{\partial f}{\partial z} = x.$$

For the line of critical points $(0,y,-y)$, $f(0,y,-y) = \boxed{0}$. On the boundary $S: x^2+y^2+z^2=1$ of the region,

$$f(x,y,z) = F(y,z) = \pm(y+z)\sqrt{1-y^2-z^2}, \quad y^2+z^2 \le 1.$$

For critical points of $F(y,z)$ inside $y^2+z^2=1$, we solve

$$0 = \frac{\partial F}{\partial y} = \pm\sqrt{1-y^2-z^2} \mp \frac{y(y+z)}{\sqrt{1-y^2-z^2}},$$
$$0 = \frac{\partial F}{\partial z} = \pm\sqrt{1-y^2-z^2} \mp \frac{z(y+z)}{\sqrt{1-y^2-z^2}}.$$

Solutions are $(y,z) = (\pm 1/2, \pm 1/2)$ at which $F(\pm 1/2, \pm 1/2) = \boxed{\pm 1/\sqrt{2}}$. On the boundary $C: y^2+z^2=1$ of S, $f(x,y,z) = \boxed{0}$. Consequently, maximum and minimum values of $f(x,y,z)$ are $\pm 1/\sqrt{2}$.

33. Since values of z are independent of those of x and y in the region, $f(x,y,z)$ is maximized when $z=1$. In other words, we should maximize and minimize $F(x,y) = x^2y$ on the circle $x^2+y^2 \le 1$. For critical points of $F(x,y)$, we solve $0 = \dfrac{\partial F}{\partial x} = 2xy$, $0 = \dfrac{\partial F}{\partial y} = x^2$. At the critical points $(0,y)$, $F(0,y) = \boxed{0}$. On the boundary $C: x^2+y^2=1$, we set $x = \cos t$, $y = \sin t$, in which case $F(x,y) = u(t) = \cos^2 t \sin t$, $0 \le t \le 2\pi$. For critical points of $u(t)$, we solve $0 = u'(t) = -2\cos t \sin^2 t + \cos^3 t = \cos t(-2\sin^2 t + \cos^2 t)$. Either $\cos t = 0$ or $0 = -2\sin^2 t + 1 - \sin^2 t = 1 - 3\sin^2 t \Longrightarrow \sin t = \pm 1/\sqrt{3}$. Thus, $t = \pi/2, 3\pi/2, \mathrm{Sin}^{-1}(\pm 1/\sqrt{3}), \mathrm{Sin}^{-1}(\pm 1/\sqrt{3}) + \pi$. Since

$$u(0) = \boxed{0}, \quad u(\pi/2) = u(3\pi/2) = \boxed{0}, \quad u(\mathrm{Sin}^{-1}(\pm 1/\sqrt{3})) = u(\mathrm{Sin}^{-1}(\mp 1/\sqrt{3}) + \pi) = \boxed{\pm\frac{2\sqrt{3}}{9}}, \quad u(2\pi) = \boxed{0},$$

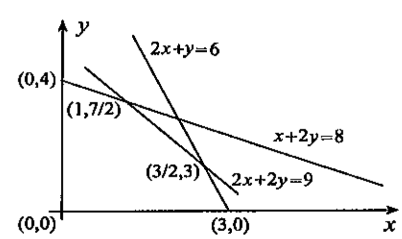
maximum and minimum values of $f(x,y,z)$ are $\pm 2\sqrt{3}/9$.

34. Let x and y be the numbers of X and Y produced per hour. Then the profit per hour is $P(x,y) = 200x + 300y$. But x and y must satisfy the following inequalities:

$$\frac{x}{8} + \frac{y}{4} \le 1, \quad \frac{x}{3} + \frac{y}{6} \le 1, \quad \frac{x}{9/2} + \frac{y}{9/2} \le 1.$$

These can be rewritten

$$x + 2y \le 8, \quad 2x + y \le 6, \quad 2x + 2y \le 9.$$



Points that satisfy these inequalities lie in the polygon shown to the right. Profit $P(x,y)$ does not have any critical points. In addition, when $P(x,y)$ is evaluated along the five edges of the polygon, linear functions are obtained. They do not have critical points. It follows that the maximum value of P must occur at one of the vertices of the polygon. Since $P(0,0) = 0$, $P(0,4) = 1200$, $P(1,7/2) = 1250$, $P(3/2,3) = 1200$, and $P(3,0) = 600$, maximum profit occurs when 2 units of X and 7 units of Y are produced in a two-hour shift.

35. If G and S are the numbers of grams of grain and supplements per day, the cost for feeding the cow per day is

$$C = C(G,S) = \frac{2750}{1000}(11) + \frac{11\,000}{1000}G + \frac{17\,500}{1000}S = 30.25 + 11G + 17.5S.$$

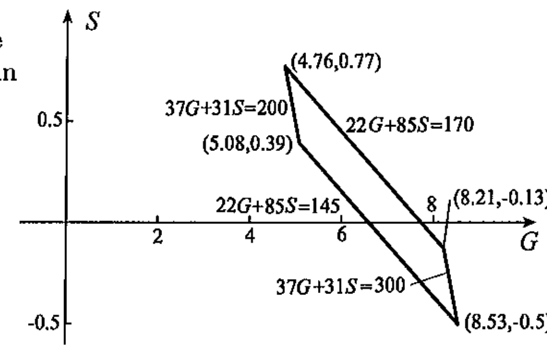
Because the cow's diet must contain between 9.5 and 11.5 kilograms of digestive material and between 1.9 and 2.0 kilograms of protein per day,

$$9.5 \le \frac{11}{2} + \frac{74}{100}G + \frac{62}{100}S \le 11.5, \qquad 1.9 \le \frac{12}{100}(11) + \frac{8.8}{100}G + \frac{34}{100}S \le 2.0,$$

and these inequalitities reduce to

$$200 \le 37G + 31S \le 300, \qquad 145 \le 22G + 85S \le 170.$$

We must therefore minimize $C(G,S)$ for those points (G,S) in the parallelogram R (actually only for those points in the first quadrant). Since $C(G,S)$ is linear in G and S, it has no critical points. On the four parts of the boundary of R, C is also linear, and therefore C has no critical points on these lines. It follows that C is minimized at one of the two corners of the parallelogram in the first quadrant. Since $C(4.76, 0.77) = 96.09$ and $C(5.08, 0.39) = 92.96$, C is minimized for $G = 5.08$ kg and $S = 0.39$ kg.



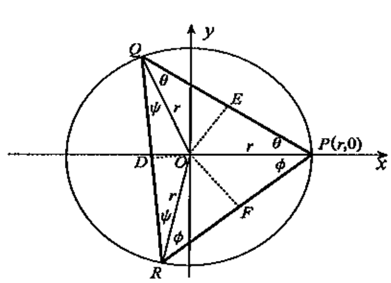
36. We choose one vertex of the triangle at $P(r,0)$. For maximum area, one of the remaining vertices must have a positive y-coordinate and the other a negative y-coordinate. When vertices are denoted by Q and R, the area of ΔPQR is twice the area of ΔPOE plus twice the area of ΔQOD plus twice the area of ΔPOF,

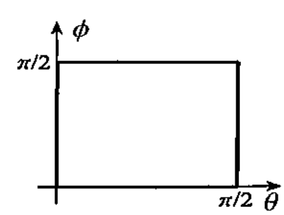
$$\begin{aligned} A &= 2\left(\frac{1}{2}\right)(r\sin\theta)(r\cos\theta) + 2\left(\frac{1}{2}\right)(r\sin\psi)(r\cos\psi) + 2\left(\frac{1}{2}\right)(r\sin\phi)(r\cos\phi) \\ &= \frac{r^2}{2}(\sin 2\theta + \sin 2\psi + \sin 2\phi). \end{aligned}$$

But, $2\theta + 2\phi + 2\psi = \pi$, so that

$$A(\theta,\phi) = \frac{r^2}{2}[\sin 2\theta + \sin 2\phi + \sin(\pi - 2\theta - 2\phi)] = \frac{r^2}{2}[\sin 2\theta + \sin 2\phi + \sin(2\theta + 2\phi)],$$

defined on the square $0 \le \theta \le \pi/2$ and $0 \le \phi \le \pi/2$.





For critical points of this function, we solve

$$0 = \frac{\partial A}{\partial \theta} = \frac{r^2}{2}[2\cos 2\theta + 2\cos(2\theta + 2\phi)], \quad 0 = \frac{\partial A}{\partial \phi} = \frac{r^2}{2}[2\cos 2\phi + 2\cos(2\theta + 2\phi)].$$

These imply that $\cos 2\theta = \cos 2\phi$ and therefore $\theta = \phi$. Then

$$0 = 2\cos 2\theta + 2\cos 4\theta = 2\cos 2\theta + 2(2\cos^2 2\theta - 1) = 2(2\cos 2\theta - 1)(\cos 2\theta + 1).$$

Thus, $\cos 2\theta = 1/2$ or $\cos 2\theta = -1$. The only solution in the interval $0 < \theta < \pi/2$ is $\theta = \pi/6$. The only critical point inside the square is $(\pi/6, \pi/6)$, and for this equilateral triangle

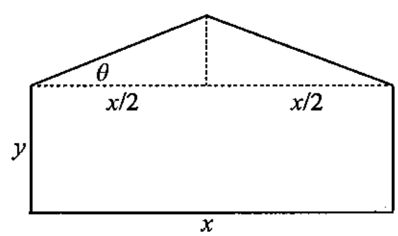
$$A = \frac{r^2}{2}\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}r^2}{4}.$$

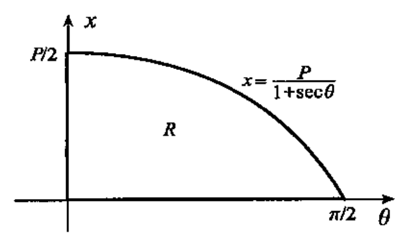
On the edge $\theta = 0$ of the square, area reduces to $r^2 \sin 2\phi$, and this function has maximum value r^2. The same situation occurs for $\phi = 0$. On the edges $\theta = \pi/2$ or $\phi = \pi/2$, area is zero. Hence, maximum area is $3\sqrt{3}r^2/4$ when the triangle is equilateral.

37. The area of the left figure is $A = xy + \frac{x}{2}\left(\frac{x}{2}\tan\theta\right)$. Since $x + 2y + 2\left(\frac{x}{2}\sec\theta\right) = P$, it follows that $y = (P - x - x\sec\theta)/2$, and

$$A(\theta, x) = \frac{x}{2}(P - x - x\sec\theta) + \frac{x^2}{4}\tan\theta = \frac{Px}{2} + \frac{x^2}{4}(\tan\theta - 2 - 2\sec\theta).$$

Now, θ varies from 0 to $\pi/2$. In order to guarantee nonnegativity of y, corresponding values of x must satisfy $P - x - x\sec\theta \geq 0$; that is, $x \leq \dfrac{P}{1+\sec\theta}$. Thus, the region R of the θx-plane over which A must be maximized is shown to the right.





For critical points of $A(\theta, x)$ we solve

$$0 = \frac{\partial A}{\partial x} = \frac{P}{2} + \frac{x}{2}(\tan\theta - 2 - 2\sec\theta), \quad 0 = \frac{\partial A}{\partial \theta} = \frac{x^2}{4}(\sec^2\theta - 2\sec\theta\tan\theta).$$

The second implies that $x = 0$ or $\sec\theta = 2\tan\theta$. The only solution of the latter of these in the interval $0 < \theta < \pi/2$ is $\theta = \pi/6$. The corresponding value of x is $x = (2 - \sqrt{3})P$, and for these values $A(\pi/6, (2-\sqrt{3})P) = \boxed{P^2(2-\sqrt{3})/4}$. On the boundary $x = 0$ of R, A is identically equal to 0. On $\theta = 0$,

$$A(0, x) = F(x) = \frac{Px}{2} + \frac{x^2}{4}(-2-2) = \frac{Px}{2} - x^2, \quad 0 \leq x \leq P/2.$$

For critical points we solve $0 = P/2 - 2x \implies x = P/4$. At this value $F(P/4) = \boxed{P^2/16}$. On the boundary $x = P/(1 + \sec\theta)$,

$$A = G(\theta) = \frac{P^2}{4(1+\sec\theta)^2}\tan\theta, \quad 0 \leq \theta < \pi/2.$$

For critical points we solve $0 = G'(\theta)$. Neglecting the $P^2/4$, we obtain

$$0 = \frac{(1+\sec\theta)^2\sec^2\theta - 2\tan\theta(1+\sec\theta)\sec\theta\tan\theta}{(1+\sec\theta)^2} = \frac{\sec\theta(1+\sec\theta)[\sec\theta(1+\sec\theta) - 2\tan^2\theta]}{(1+\sec\theta)^2}.$$

This implies that

$$0 = \sec\theta + \sec^2\theta - 2(\sec^2\theta - 1) = -\sec^2\theta + \sec\theta + 2 = -(\sec\theta - 2)(\sec\theta + 1).$$

Thus, $\sec\theta = 2 \implies \theta = \pi/3$. For this θ, $G(\pi/3) = \boxed{\sqrt{3}P^2/36}$. Finally, we evaluate $A(0, P/2) = \boxed{0}$. Maximum area is $P^2(2-\sqrt{3})/4$ when $\theta = \pi/6$, $x = (2-\sqrt{3})P$, and $y = (1 - \sqrt{3}/3)P/2$.

38. If x and y are the numbers of computers of models A and B, then the cost of the 100 computers is

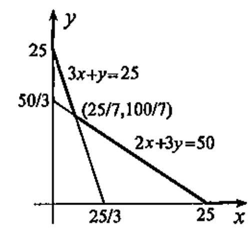
$$C = f(x,y) = 1300x + 1200y + 1000(100 - x - y) = 100\,000 + 300x + 200y.$$

Because the computers must have at least 2000 MB of memory,

$$64x + 32y + 16(100 - x - y) \geq 2000 \quad \Longrightarrow 3x + y \geq 25.$$

Because the computers must have at least 150 GB of disk space,

$$3x + 4y + (100 - x - y) \geq 150 \quad \Longrightarrow \quad 2x + 3y \geq 50.$$



The domain of $f(x,y)$ therefore consists of all non-negative values of x and y satisfying these two inequalities. They are shown to the right. Since $f(x,y)$ is linear, there are no critical points inside the region. The function will also be linear on the boundaries so that no critical points occur there either. The function must be minimized at one of the three corners. We find $f(0,25) = 105\,000$, $f(25,0) = 107\,500$, and $f(25/7, 100/7) = 103\,929$. We need to take either x or y, or both, to the next integer value. For $x = 4$, we require $y \geq 13$ and $3y \geq 42 \Longrightarrow y \geq 14$. For $y = 15$, we require $3x \geq 11$ and $2x \geq 5 \Longrightarrow x \geq 4$. Of these two, we should choose $x = 4$ and $y = 14$. Cost for these along with $z = 87$ is $f(4,13) = 104\,000$.

39. For critical points of $f(x,y)$ inside the polygon, we should solve $0 = f_x = c$ and $0 = f_y = d$. There are no solutions. On any edge of the polygon described by a straight line say $A_i x + B_i y = C_i$, $f(x,y)$ has value

$$F(x) = cx + \frac{d}{B_i}(C_i - A_i x) = \left(c - \frac{A_i d}{B_i}\right)x + \frac{dC_i}{B_i},$$

for some values of x representing the extent of the edge of the polygon. Since $F'(x) = c - A_i d/B_i \neq 0$, there are no critical points on the edge of the polygon. Since this is the case for all edges, it follows that the maximum value of $f(x,y)$ (and there must be one since the function is continuous on a closed polygon) must occur at one of the vertices of the polygon. If $c - A_i d/B_i$ should equal zero along some edge of the polygon, then $F(x)$ would be constant along that edge. If this value turned out to be a maximum for $f(x,y)$, then it would also be taken on at the ends of the edge and therefore at two vertices of the polygon. Once again the maximum is at a vertex of the polygon.

40. Since travel times through the media are distances divided by speeds, it follows that time from source to receiver is

$$t = \frac{2d_1 \sec\theta_1}{v_1} + \frac{2d_2 \sec\theta_2}{v_2} + \frac{s - 2d_1 \tan\theta_1 - 2d_2 \tan\theta_2}{v_3}.$$

For critical points of this function we solve

$$0 = \frac{\partial t}{\partial \theta_1} = \frac{2d_1 \sec\theta_1 \tan\theta_1}{v_1} - \frac{2d_1 \sec^2\theta_1}{v_3}, \quad 0 = \frac{\partial t}{\partial \theta_2} = \frac{2d_2 \sec\theta_2 \tan\theta_2}{v_2} - \frac{2d_2 \sec^2\theta_2}{v_3}.$$

These equation can be solved separately. If we divide the first by $2d_1 \sec\theta_1$,

$$\frac{\tan\theta_1}{v_1} = \frac{\sec\theta_1}{v_3} \quad \Longrightarrow \quad \sin\theta_1 = \frac{v_1}{v_3} \quad \Longrightarrow \quad \theta_1 = \mathrm{Sin}^{-1}\left(\frac{v_1}{v_3}\right).$$

Similarly, $\theta_2 = \mathrm{Sin}^{-1}\left(\dfrac{v_2}{v_3}\right)$.

EXERCISES 12.12

1. The constraint $x^2 + y^2 = 4$ defines a **closed** curve (a circle). We define the Lagrangian

$$L(x, y, \lambda) = x^2 + y + \lambda(x^2 + y^2 - 4).$$

For critical points of L, we solve

$$0 = \frac{\partial L}{\partial x} = 2x + 2\lambda x, \quad 0 = \frac{\partial L}{\partial y} = 1 + 2\lambda y, \quad 0 = \frac{\partial L}{\partial \lambda} = x^2 + y^2 - 4.$$

Critical points (x, y) are $(0, \pm 2)$ and $(\pm\sqrt{15}/2, 1/2)$. Since $f(0, \pm 2) = \pm 2$, and $f(\pm\sqrt{15}/2, 1/2) = 17/4$, maximum and minimum values of $f(x, y)$ are $17/4$ and -2.

2. The constraint $x^2 + 2y^2 + 4z^2 = 9$ defines a **closed** surface (an ellipsoid). We define the Lagrangian

$$L(x, y, z, \lambda) = 5x - 2y + 3z + 4 + \lambda(x^2 + 2y^2 + 4z^2 - 9).$$

For critical points of L, we solve

$$0 = \frac{\partial L}{\partial x} = 5 + 2\lambda x, \quad 0 = \frac{\partial L}{\partial y} = -2 + 4\lambda y, \quad 0 = \frac{\partial L}{\partial z} = 3 + 8\lambda z, \quad 0 = \frac{\partial L}{\partial \lambda} = x^2 + 2y^2 + 4z^2 - 9.$$

Critical points (x, y, z) are $(\pm 10/\sqrt{13}, \mp 2/\sqrt{13}, \pm 3/(2\sqrt{13}))$. Since $f(\pm 10/\sqrt{13}, \mp 2/\sqrt{13}, \pm 3/(2\sqrt{13})) = (8 \pm 9\sqrt{13})/2$, these are the maximum and minimum values of $f(x, y, z)$.

3. The constraint $(x - 1)^2 + y^2 = 1$ defines a **closed** curve (a circle). We define the Lagrangian

$$L(x, y, \lambda) = x + y + \lambda[(x - 1)^2 + y^2 - 1].$$

For critical points of L, we solve

$$0 = \frac{\partial L}{\partial x} = 1 + 2\lambda(x - 1), \quad 0 = \frac{\partial L}{\partial y} = 1 + 2\lambda y, \quad 0 = \frac{\partial L}{\partial \lambda} = (x - 1)^2 + y^2 - 1.$$

Critical points (x, y) are $(1 \pm 1/\sqrt{2}, \pm 1/\sqrt{2})$. Since $f(1 \pm 1/\sqrt{2}, \pm 1/\sqrt{2}) = 1 \pm \sqrt{2}$, these are maximum and minimum values of $f(x, y)$.

4. The constraint $x^2 + y^2 + z^2 = 9$ is a **closed** surface (a sphere). We define the Lagrangian

$$L(x, y, z, \lambda) = x^3 + y^3 + z^3 + \lambda(x^2 + y^2 + z^2 - 9).$$

For critical points of L, we solve

$$0 = \frac{\partial L}{\partial x} = 3x^2 + 2\lambda x, \quad 0 = \frac{\partial L}{\partial y} = 3y^2 + 2\lambda y, \quad 0 = \frac{\partial L}{\partial z} = 3z^2 + 2\lambda z, \quad 0 = \frac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 - 9.$$

Critical points (x, y, z) are $(\pm 3, 0, 0)$, $(0, \pm 3, 0)$, $(0, 0, \pm 3)$, $(0, \pm 3/\sqrt{2}, \pm 3/\sqrt{2})$, $(\pm 3/\sqrt{2}, 0, \pm 3/\sqrt{2})$, $(\pm 3/\sqrt{2}, \pm 3/\sqrt{2}, 0)$, $(\pm\sqrt{3}, \pm\sqrt{3}, \pm\sqrt{3})$. Since $f(x, y, z) = \pm 27$ at the first six critical points, $f(x, y, z) = \pm 27/\sqrt{2}$ at the second set of six critical points, and $f(\pm\sqrt{3}, \pm\sqrt{3}, \pm\sqrt{3}) = \pm 9\sqrt{3}$, maximum and minimum values of $f(x, y, z)$ are ± 27.

5. The constraint $x^2 + 2y^2 + 3z^2 = 12$ defines a **closed** surface (an ellipsoid). We define the Lagrangian

$$L(x, y, z, \lambda) = xyz + \lambda(x^2 + 2y^2 + 3z^2 - 12).$$

For critical points of L, we solve

$$0 = \frac{\partial L}{\partial x} = yz + 2\lambda x, \quad 0 = \frac{\partial L}{\partial y} = xz + 4\lambda y, \quad 0 = \frac{\partial L}{\partial z} = xy + 6\lambda z, \quad 0 = \frac{\partial L}{\partial \lambda} = x^2 + 2y^2 + 3z^2 - 12.$$

Critical points (x, y, z) are $(0, 0, \pm 2)$, $(0, \pm\sqrt{6}, 0)$, $(\pm 2\sqrt{3}, 0, 0)$, $(2, \pm\sqrt{2}, \pm 2/\sqrt{3})$, $(2, \pm\sqrt{2}, \mp 2/\sqrt{3})$, $(-2, \pm\sqrt{2}, \pm 2/\sqrt{3})$, and $(-2, \pm\sqrt{2}, \mp 2/\sqrt{3})$. Since $f(x, y, z)$ has value 0 at the first six critical points and values $\pm 4\sqrt{6}/3$ at the other critical points, maximum and minimum values of $f(x, y, z)$ are $\pm 4\sqrt{6}/3$.

6. The constraints $x^2+y^2=1$, $z=y$ define a **closed** curve. We define the Lagrangian

$$L(x,y,z,\lambda,\mu)=x^2y+z+\lambda(x^2+y^2-1)+\mu(z-y).$$

For critical points of L, we solve

$$0=\frac{\partial L}{\partial x}=2xy+2\lambda x,\quad 0=\frac{\partial L}{\partial y}=x^2+2\lambda y-\mu,\quad 0=\frac{\partial L}{\partial z}=1+\mu,$$

$$0=\frac{\partial L}{\partial \lambda}=x^2+y^2-1,\quad 0=\frac{\partial L}{\partial \mu}=z-y.$$

Solutions (x,y,z) of these equations are $(0,\pm1,\pm1)$, $(1/\sqrt{3},\pm\sqrt{2/3},\pm\sqrt{2/3})$, and $(-1/\sqrt{3},\pm\sqrt{2/3},\pm\sqrt{2/3})$. Since $f(0,\pm1,\pm1)=\pm1$, $f(1/\sqrt{3},\pm\sqrt{2/3},\pm\sqrt{2/3})=\pm\sqrt{32/27}$, and $f(-1/\sqrt{3},\pm\sqrt{2/3},\pm\sqrt{2/3})=\pm\sqrt{32/27}$, maximum and minimum values are $\pm\sqrt{32/27}$.

7. Since all points must satisfy $x^2+y^2+z^2=2z$, we may replace the function $f(x,y,z)=x^2+y^2+z^2$ with $f(x,y,z)=2z$. The constraints define a **closed** curve. We define the Lagrangian

$$L(x,y,z,\lambda,\mu)=2z+\lambda(x^2+y^2+z^2-2z)+\mu(x+y+z-1).$$

For critical points of L, we solve

$$0=\frac{\partial L}{\partial x}=2\lambda x+\mu,\quad 0=\frac{\partial L}{\partial y}=2\lambda y+\mu,\quad 0=\frac{\partial L}{\partial z}=2+\lambda(2z-2)+\mu,$$

$$0=\frac{\partial L}{\partial \lambda}=x^2+y^2+z^2-2z,\quad 0=\frac{\partial L}{\partial \mu}=x+y+z-1.$$

Solutions (x,y,z) of these equations are $(\pm1/\sqrt{6},\pm1/\sqrt{6},1\mp\sqrt{6}/3)$. Since $f(\pm1/\sqrt{6},\pm1/\sqrt{6},1\mp\sqrt{6}/3)=2(1\mp\sqrt{6}/3)$, these are minimum and maximum values of $f(x,y,z)$.

8. The constraints $x^2+y^2=1$, $z=\sqrt{x^2+y^2}$ define the **closed** curve $x^2+y^2=1$, $z=1$, so that we write alternatively, $f(x,y,z)=F(x,y)=xy-x^2$ subject to $x^2+y^2=1$. We define the Lagrangian $L(x,y,\lambda)=xy-x^2+\lambda(x^2+y^2-1)$. For critical points of L, we solve

$$0=\frac{\partial L}{\partial x}=y-2x+2\lambda x,\quad 0=\frac{\partial L}{\partial y}=x+2\lambda y,\quad 0=\frac{\partial L}{\partial \lambda}=x^2+y^2-1.$$

Solutions (x,y) of these equations are $\left(\dfrac{\pm\sqrt{2-\sqrt{2}}}{2},\dfrac{\pm\sqrt{2+\sqrt{2}}}{2}\right)$, $\left(\dfrac{\pm\sqrt{2+\sqrt{2}}}{2},\dfrac{\mp\sqrt{2-\sqrt{2}}}{2}\right)$. Since values of $F(x,y)$ at these points are $(-1\pm\sqrt{2})/2$, maximum and minimum values are $(\sqrt{2}-1)/2$ and $-(\sqrt{2}+1)/2$.

9. The distance D from $(-1,1,2)$ to any point $P(x,y,z)$ is given by $D^2=(x+1)^2+(y-1)^2+(z-2)^2$. To minimize this function subject to the constraint $2x-3y+6z=14$, we define the Lagrangian $L(x,y,z,\lambda)=(x+1)^2+(y-1)^2+(z-2)^2+\lambda(2x-3y+6z-14)$. For critical points of $L(x,y,z,\lambda)$ we solve

$$0=\frac{\partial L}{\partial x}=2(x+1)+2\lambda,\quad 0=\frac{\partial L}{\partial y}=2(y-1)-3\lambda,\quad 0=\frac{\partial L}{\partial z}=2(z-2)+6\lambda,$$

$$0=\frac{\partial L}{\partial \lambda}=2x-3y+6z-14,$$

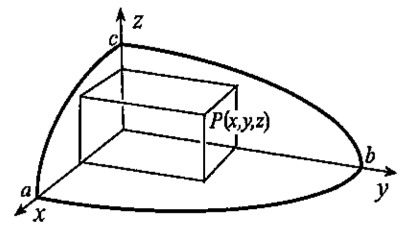
The only solution is $x=-5/7$, $y=4/7$, $z=20/7$. Since x, y and z can take on all possible values, and D^2 becomes infinite for large values of x, y or z, the critical point must minimize D^2. The shortest distance is therefore $D(-5/7,4/7,20/7)=1$.

10. The distance D from $(1,1,0)$ to any point $P(x,y,z)$ is given by $D^2 = (x-1)^2 + (y-1)^2 + z^2$. This function must be minimized subject to the constraint $z = x^2 + y^2$. If we define the Lagrangian $L(x,y,z,\lambda) = (x-1)^2 + (y-1)^2 + z^2 + \lambda(x^2+y^2-z)$, its critical points are given by

$$0 = \frac{\partial L}{\partial x} = 2(x-1) + 2\lambda x, \quad 0 = \frac{\partial L}{\partial y} = 2(y-1) + 2\lambda y, \quad 0 = \frac{\partial L}{\partial z} = 2z - \lambda, \quad 0 = \frac{\partial L}{\partial \lambda} = x^2 + y^2 - z.$$

The solution for (x,y,z) is $(1/2, 1/2, 1/2)$. Since D^2 becomes infinite as x, y, and z become infinite, it follows that this point must minimize D^2 (or D).

11. The volume obtained from a point $P(x,y,z)$ on that part of the ellipsoid in the first octant is $V = 8xyz$. This function must be maximized for (x,y,z) satisfying $x \geq 0$, $y \geq 0$, $z \geq 0$, and $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. We define the Lagrangian



$L(x,y,z,\lambda) = 8xyz + \lambda(x^2/a^2 + y^2/b^2 + z^2/c^2 - 1)$.

For critical points of $L(x,y,z,\lambda)$, we solve

$$0 = \frac{\partial L}{\partial x} = 8yz + \frac{2\lambda x}{a^2}, \quad 0 = \frac{\partial L}{\partial y} = 8xz + \frac{2\lambda y}{b^2}, \quad 0 = \frac{\partial L}{\partial z} = 8xy + \frac{2\lambda z}{c^2}, \quad 0 = \frac{\partial L}{\partial \lambda} = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1.$$

The only solution (x,y,z) of these equations with positive coordinates is $(a/\sqrt{3}, b/\sqrt{3}, c/\sqrt{3})$. Since $V = 0$ on the three edges of the ellipsoid in the first octant, it follows that V must be maximized at this critical point, and therefore the dimensions of the largest box are $2a/\sqrt{3} \times 2b/\sqrt{3} \times 2c/\sqrt{3}$.

12. Since A is maximized when A^2 is maximized, we define the Lagrangian

$$L(x,y,z,\lambda) = \frac{P}{2}\left(\frac{P}{2} - x\right)\left(\frac{P}{2} - y\right)\left(\frac{P}{2} - z\right) + \lambda(x + y + z - P).$$

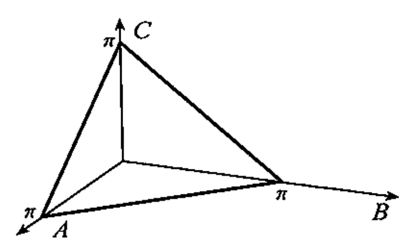
Critical points of L are given by

$$0 = \frac{\partial L}{\partial x} = -\frac{P}{2}\left(\frac{P}{2} - y\right)\left(\frac{P}{2} - z\right) + \lambda, \quad 0 = \frac{\partial L}{\partial y} = -\frac{P}{2}\left(\frac{P}{2} - x\right)\left(\frac{P}{2} - z\right) + \lambda,$$

$$0 = \frac{\partial L}{\partial z} = -\frac{P}{2}\left(\frac{P}{2} - x\right)\left(\frac{P}{2} - y\right) + \lambda, \quad 0 = \frac{\partial L}{\partial \lambda} = x + y + z - P.$$

The only solution of these equations is $x = y = z = P/3$; or, any two of x, y, and z equal to $P/2$ and the third equal to zero. In the latter case, the triangle has degenerated to a straight line with $A = 0$. Since this case represents the bounds for possible values of x, y, and z, it follows that $x = y = z = P/3$ must maximize A.

13. Consider finding the maximum value of the function $f(A,B,C) = \sin(A/2)\sin(B/2)\sin(C/2)$ subject to the constraint $A + B + C = \pi$ (because A, B and C are angles in a triangle). This is a plane in ABC-space, and we consider only that part R of the plane in the first octant. If we define the Lagrangian



$L(A,B,C,\lambda) = f(A,B,C) + \lambda(A + B + C - \pi)$,

its critical points are defined by

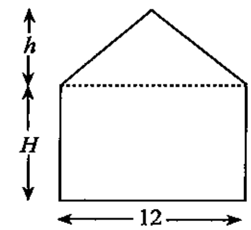
$$0 = \frac{\partial L}{\partial A} = \frac{1}{2}\cos(A/2)\sin(B/2)\sin(C/2) + \lambda, \quad 0 = \frac{\partial L}{\partial B} = \frac{1}{2}\sin(A/2)\cos(B/2)\sin(C/2) + \lambda,$$

$$0 = \frac{\partial L}{\partial C} = \frac{1}{2}\sin(A/2)\sin(B/2)\cos(C/2) + \lambda, \quad 0 = \frac{\partial L}{\partial \lambda} = A + B + C - \pi.$$

The only solution of these equations for which all three angles are between 0 and π is $A = B = C = \pi/3$. On the three edges of R, the value of $f(A,B,C)$ is zero. It follows that $f(A,B,C)$ must have a maximum value at $A = B = C = \pi/3$, and this value is $f(\pi/3,\pi/3,\pi/3) = 1/8$. Hence, $\sin(A/2)\sin(B/2)\sin(C/2 \le 1/8$ for all other values of A, B, and C.

14. The volume of the silo is

$V = \pi(6)^2 H + \frac{1}{3}\pi(6)^2 h = 12\pi(h+3H).$



Since the area of the silo must be 200 m^2,

$200 = 2\pi(6)H + \pi(6)\sqrt{36+h^2}.$

We define the Lagrangian

$L(h,H,\lambda) = 12\pi(h+3H) + \lambda(12\pi H + 6\pi\sqrt{36+h^2} - 200).$

Its critical points are given by

$$0 = \frac{\partial L}{\partial h} = 12\pi + \frac{6\pi\lambda h}{\sqrt{36+h^2}}, \quad 0 = \frac{\partial L}{\partial H} = 36\pi + 12\pi\lambda, \quad 0 = \frac{\partial L}{\partial \lambda} = 12\pi H + 6\pi\sqrt{36+h^2} - 200.$$

The solution of these equations for h and H is $h = 12/\sqrt{5}$ and $H = (50\sqrt{5} - 27\pi)/(3\sqrt{5}\pi)$. Clearly $h \ge 0$ and the constraint requires $h \le \sqrt{4\times 10^4 - 36^2\pi^2}/(6\pi)$. Since

$$V_{|h=0} = 260.7, \quad V_{|h=12/\sqrt{5}} = 347.1, \quad V_{|h=\sqrt{4\times10^4 - 36^2\pi^2}/(6\pi)} = 329.9,$$

it follows that V is maximized for $h = 12/\sqrt{5}$ m and $H = (50\sqrt{5} - 27\pi)/(3\sqrt{5}\pi)$ m.

15. We define the Lagrangian $L(x,y,\lambda) = x^2 - y^2 + \lambda(x^2+y^2-1)$. For critical points of L, we solve

$$0 = \frac{\partial L}{\partial x} = 2x + 2\lambda x, \quad 0 = \frac{\partial L}{\partial y} = -2y + 2\lambda y, \quad 0 = \frac{\partial L}{\partial \lambda} = x^2 + y^2 - 1.$$

Critical points (x,y) are $(\pm 1, 0)$ and $(0,\pm 1)$. Since $f(\pm 1, 0) = 1$, and $f(0,\pm 1) = -1$, maximum and minimum values of $f(x,y)$ are ± 1.

16. We define the Lagrangian $L(x,y,\lambda) = |x-y| + \lambda(x^2+y^2-1)$. For critical points of L, we solve

$$0 = \frac{\partial L}{\partial x} = \frac{|x-y|}{x-y} + 2\lambda x, \quad 0 = \frac{\partial L}{\partial y} = -\frac{|x-y|}{x-y} + 2\lambda y, \quad 0 = \frac{\partial L}{\partial \lambda} = x^2 + y^2 - 1.$$

Critical points (x,y) are $(\pm 1/\sqrt{2}, \mp 1/\sqrt{2})$. The derivatives do not exist when $y = x$ and this leads to the additional critical points $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$. Since $f(\pm 1/\sqrt{2}, \pm 1/\sqrt{2}) = 0$ and $f(\pm 1/\sqrt{2}, \mp 1/\sqrt{2}) = \sqrt{2}$, maximum and minimum values of $f(x,y)$ are $\sqrt{2}$ and 0.

17. We define the Lagrangian $L(x,y,\lambda) = x^2 - y^2 + \lambda(|x| + |y| - 1)$. For critical points of L, we solve

$$0 = \frac{\partial L}{\partial x} = 2x + \frac{\lambda|x|}{x}, \quad 0 = \frac{\partial L}{\partial y} = -2y + \frac{\lambda|y|}{y}, \quad 0 = \frac{\partial L}{\partial \lambda} = |x| + |y| - 1.$$

There are no solutions of these equations. Since the partial derivative with respect to x fails to exist at $x = 0$, and the derivative with respect to y does not exist at $y = 0$, critical points are $(0,\pm 1)$ and $(\pm 1, 0)$. Since $f(\pm 1, 0) = 1$, and $f(0,\pm 1) = -1$, maximum and minimum values of $f(x,y)$ are ± 1.

18. We define the Lagrangian $L(x,y,\lambda) = |x-2y| + \lambda(|x| + |y| - 1)$. For critical points of L, we solve

$$0 = \frac{\partial L}{\partial x} = \frac{|x-2y|}{x-2y} + \frac{\lambda|x|}{x}, \quad 0 = \frac{\partial L}{\partial y} = -2\frac{|x-2y|}{x-2y} + \frac{\lambda|y|}{y}, \quad 0 = \frac{\partial L}{\partial \lambda} = |x| + |y| - 1.$$

There are no solutions of these equations. Since the partial derivative with respect to x fails to exist at $x = 0$ and when $x = 2y$, and the derivative with respect to y does not exist at $y = 0$ or when $x = 2y$, critical points are $(0,\pm 1)$, $(\pm 1, 0)$, and $(\pm 2/3, \pm 1/3)$. Since $f(\pm 1, 0) = 1$, $f(0,\pm 1) = 2$, and $f(\pm 2/3, \pm 1/3) = 0$, maximum and minimum values of $f(x,y)$ are 2 and 0.

19. The distance D from (x_1, y_1, z_1) to any point $P(x, y, z)$ is given by $D^2 = (x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2$. This function must be minimized subject to the constraint $Ax + By + Cz + D = 0$. If we define the Lagrangian $L(x, y, z, \lambda) = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 + \lambda(Ax + By + Cz + D)$, critical points are given by

$$0 = \frac{\partial L}{\partial x} = 2(x - x_1) + \lambda A, \quad 0 = \frac{\partial L}{\partial y} = 2(y - y_1) + \lambda B, \quad 0 = \frac{\partial L}{\partial z} = 2(z - z_1) + \lambda C,$$

$$0 = \frac{\partial L}{\partial \lambda} = Ax + By + Cz + D.$$

The only solution is

$$x = \frac{(B^2 + C^2)x_1 - A(By_1 + Cz_1 + D)}{A^2 + B^2 + C^2}, \quad y = \frac{(A^2 + C^2)y_1 - B(Ax_1 + Cz_1 + D)}{A^2 + B^2 + C^2},$$

$$z = \frac{(A^2 + B^2)z_1 - C(Ax_1 + By_1 + D)}{A^2 + B^2 + C^2}.$$

Since this is the only critical point, and distance becomes infinite as x, y and z take on large values, it follows that this critical point must minimize D^2. To find the minimum value we substitute these values for x, y, and z into the formula for D^2,

$$D^2 = \left[\frac{(B^2 + C^2)x_1 - A(By_1 + Cz_1 + D)}{A^2 + B^2 + C^2} - x_1\right]^2 + \left[\frac{(A^2 + C^2)y_1 - B(Ax_1 + Cz_1 + D)}{A^2 + B^2 + C^2} - y_1\right]^2 + \left[\frac{(A^2 + B^2)z_1 - C(Ax_1 + By_1 + D)}{A^2 + B^2 + C^2} - z_1\right]^2.$$

This simplifies to $\dfrac{(Ax_1 + By_1 + Cz_1 + D)^2}{A^2 + B^2 + C^2}$, and its square root gives the desired result.

20. The distance D between points $P(x, y)$ on $F(x, y) = 0$ and $Q(X, Y)$ on $G(x, y) = 0$ is given by $D^2 = (x - X)^2 + (y - Y)^2$, where $F(x, y) = 0$ and $G(X, Y) = 0$. We define the Lagrangian

$$L(x, y, X, Y, \lambda, \mu) = (x - X)^2 + (y - Y)^2 + \lambda F(x, y) + \mu G(X, Y).$$

For critical points of this function,

$$0 = \frac{\partial L}{\partial x} = 2(x - X) + \lambda F_x, \quad 0 = \frac{\partial L}{\partial y} = 2(y - Y) + \lambda F_y, \quad 0 = \frac{\partial L}{\partial X} = -2(x - X) + \mu G_X,$$

$$0 = \frac{\partial L}{\partial Y} = -2(y - Y) + \mu G_Y, \quad 0 = \frac{\partial L}{\partial \lambda} = F(x, y), \quad 0 = \frac{\partial L}{\partial \mu} = G(X, Y).$$

If $P(x_0, y_0)$ and $Q(X_0, Y_0)$ are the points that minimize D^2, then they must satisfy these equations. In particular, the first four give

$$0 = 2(x_0 - X_0) + \lambda F_x(x_0, y_0), \quad 0 = 2(y_0 - Y_0) + \lambda F_y(x_0, y_0),$$

$$0 = -2(x_0 - X_0) + \mu G_X(X_0, Y_0), \quad 0 = -2(y_0 - Y_0) + \mu G_Y(X_0, Y_0).$$

From the first two equations, we obtain $\dfrac{y_0 - Y_0}{x_0 - X_0} = \dfrac{F_y(x_0, y_0)}{F_x(x_0, y_0)}$. But $(y_0 - Y_0)/(x_0 - X_0)$ is the slope of the line joining $P(x_0, y_0)$ and $Q(X_0, Y_0)$, and the slope of the tangent line to $F(x, y) = 0$ at (x_0, y_0) is $-F_x(x_0, y_0)/F_y(x_0, y_0)$. Hence, these lines are perpendicular. The last two equations indicate that PQ is perpendicular to the tangent line to $G(x, y) = 0$ at Q.

21. Production levels at the four plants are

$$x_1 = \frac{26(500)}{100} = 130, \quad x_2 = \frac{24(500)}{100} = 120, \quad x_3 = \frac{23(500)}{100} = 115, \quad x_4 = \frac{27(500)}{100} = 135.$$

Total cost is $\dfrac{500^2}{100} = 2500$.

22. The volume of a right circular cylinder is $V = \pi r^2 h$, and were there no constraints on r and h, this function would be considered for all points in the first quadrant of the rh-plane. However, r and h must satisfy a constraint that geometrically can be interpreted as a curve in the rh-plane. What we must do then is minimize $V = \pi r^2 h$, considering only those points (r, h) on the curve defined by the constraint. Clearly there is only one independent variable in the problem—either r or h, but not both. If we choose r as the independent variable, then we note from the constraint that as h becomes very large, r approaches $2.4048/\sqrt{k}$. Since there is no upper bound on r, we can state that the values of r to be considered in the minimization of V are $r > 2.4048/\sqrt{k}$.

To find critical points of V we introduce the Lagrangian

$$L(r, h, \lambda) = \pi r^2 h + \lambda\left[\left(\frac{2.4048}{r}\right)^2 + \left(\frac{\pi}{h}\right)^2 - k\right],$$

and first solve the equations

$$0 = \frac{\partial L}{\partial r} = 2\pi r h + \lambda\left[\frac{-2(2.4048)^2}{r^3}\right],$$
$$0 = \frac{\partial L}{\partial h} = \pi r^2 + \lambda\left(\frac{-2\pi^2}{h^3}\right),$$
$$0 = \frac{\partial L}{\partial \lambda} = \left(\frac{2.4048}{r}\right)^2 + \left(\frac{\pi}{h}\right)^2 - k.$$

If we solve each of the first two equations for λ and equate the resulting expressions, we have

$$\frac{\pi r^4 h}{2.4048^2} = \frac{r^2 h^3}{2\pi}.$$

Since neither r nor h can be zero, we divide by $r^2 h$:

$$\frac{\pi r^2}{2.4048^2} = \frac{h^2}{2\pi} \quad \Longrightarrow \quad r = \frac{2.4048h}{\sqrt{2}\pi}.$$

Substitution of this result into the constraint equation gives

$$\left(\frac{\sqrt{2}\pi}{h}\right)^2 + \left(\frac{\pi}{h}\right)^2 = k,$$

and this equation can be solved for $h = \pi\sqrt{3/k}$. This gives

$$r = \frac{2.4048}{\sqrt{2}\pi}\frac{\pi\sqrt{3}}{\sqrt{k}} = 2.4048\sqrt{3/(2k)}.$$

We have obtained therefore only one critical point (r, h) at which the derivatives of L vanish. The only values of r and h at which the derivatives of L do not exist are $r = 0$ and $h = 0$, but these must be rejected since the constraint requires both r and h to be positive.

To finish the problem we note that

$$\lim_{r\to\infty} V = \infty, \qquad \lim_{r\to 2.4048/\sqrt{k}^+} V = \lim_{h\to\infty} V = \infty.$$

It follows, therefore, that the single critical point at which $r = 2.4048\sqrt{3/(2k)}$ and $h = \pi\sqrt{3/k}$ must give the absolute minimum value of $V(r, h)$.

23. We find maximum and minimum values of $D^2 = x^2 + y^2$ subject to the constraint $x^2 + xy + y^2 = 1$. Critical points of the Lagrangian $L(x, y, \lambda) = x^2 + y^2 + \lambda(x^2 + xy + y^2 - 1)$ are given by

$$0 = \frac{\partial L}{\partial x} = 2x + \lambda(2x + y), \quad 0 = \frac{\partial L}{\partial y} = 2y + \lambda(x + 2y), \quad 0 = \frac{\partial L}{\partial \lambda} = x^2 + xy + y^2 - 1.$$

Solutions (x, y) are $(\pm 1, \mp 1)$ and $(\pm 1/\sqrt{3}, \pm 1/\sqrt{3})$. Since $D^2(\pm 1, \mp 1) = 2$ and $D^2(\pm 1/\sqrt{3}, \pm 1/\sqrt{3}) = 2/3$, the closest points are $(\pm 1/\sqrt{3}, \pm 1/\sqrt{3})$ and the farthest points are $(\pm 1, \mp 1)$.

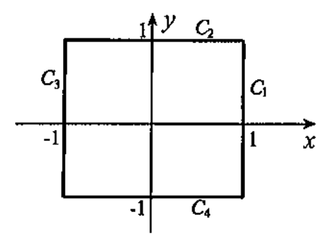
24. For critical points of $f(x, y)$, we solve $0 = \dfrac{\partial f}{\partial x} = 6x + 2y$, $0 = \dfrac{\partial f}{\partial y} = 2x - 2y$. At the only solution $(0, 0)$, $f(0, 0) = 5$. On the **closed** boundary $4x^2 + 9y^2 = 36$ of the region, we define the Lagrangian $L(x, y, \lambda) = 3x^2 + 2xy - y^2 + 5 + \lambda(4x^2 + 9y^2 - 36)$. For its critical points,

$$0 = \frac{\partial L}{\partial x} = 6x + 2y + 8\lambda x, \quad 0 = \frac{\partial L}{\partial y} = 2x - 2y + 18\lambda y, \quad 0 = \frac{\partial L}{\partial \lambda} = 4x^2 + 9y^2 - 36.$$

The solutions (x, y) of these equations are $(\pm 0.55086, \mp 1.96600)$ and $(\pm 2.94899, \pm 0.36724)$. Since $f(\pm 0.55086, \mp 1.96600) = -0.12$ and $f(\pm 2.94899, \pm 0.36724) = 33.12$, maximum and minimum values of $f(x, y)$ are 33.12 and -0.12.

25. For critical points of $f(x, y)$ we solve

$$0 = \frac{\partial f}{\partial x} = 2xy + y^2, \quad 0 = \frac{\partial f}{\partial y} = x^2 + 2xy + 1.$$



Solutions are $(\pm 1/\sqrt{3}, \mp 2/\sqrt{3})$ at which $f(\pm 1/\sqrt{3}, \mp 2/\sqrt{3}) = \boxed{\mp 4\sqrt{3}/9}$.

On C_1, we define the Lagrangian $L_1(x, y, \lambda) = x^2 y + xy^2 + y + \lambda(x - 1)$. Critical points are given by

$$0 = \frac{\partial L_1}{\partial x} = 2xy + y^2 + \lambda, \quad 0 = \frac{\partial L_1}{\partial y} = x^2 + 2xy + 1, \quad 0 = \frac{\partial L_1}{\partial \lambda} = x - 1.$$

The only solution is $(1, -1)$ at which $f(1, -1) = \boxed{-1}$.

On C_2, we define the Lagrangian $L_2(x, y, \lambda) = x^2 y + xy^2 + y + \lambda(y - 1)$. Critical points are given by

$$0 = \frac{\partial L_2}{\partial x} = 2xy + y^2, \quad 0 = \frac{\partial L_2}{\partial y} = x^2 + 2xy + 1 + \lambda, \quad 0 = \frac{\partial L_2}{\partial \lambda} = y - 1.$$

The only solution is $(-1/2, 1)$ at which $f(-1/2, 1) = \boxed{3/4}$.

On C_3, we define the Lagrangian $L_3(x, y, \lambda) = x^2 y + xy^2 + y + \lambda(x + 1)$. Critical points are given by

$$0 = \frac{\partial L_3}{\partial x} = 2xy + y^2 + \lambda, \quad 0 = \frac{\partial L_3}{\partial y} = x^2 + 2xy + 1, \quad 0 = \frac{\partial L_3}{\partial \lambda} = x + 1.$$

The only solution is $(-1, 1)$ at which $f(-1, 1) = \boxed{1}$.

On C_4, we define the Lagrangian $L_4(x, y, \lambda) = x^2 y + xy^2 + y + \lambda(y + 1)$. Critical points are given by

$$0 = \frac{\partial L_4}{\partial x} = 2xy + y^2, \quad 0 = \frac{\partial L_4}{\partial y} = x^2 + 2xy + 1 + \lambda, \quad 0 = \frac{\partial L_4}{\partial \lambda} = y + 1.$$

The only solution is $(1/2, -1)$ at which $f(1/2, -1) = \boxed{-3/4}$. The function must also be evaluated at the two remaining corners $f(1, 1) = \boxed{3}$ and $f(-1, -1) = \boxed{-3}$. Maximum and minimum values are ± 3.

26. For critical points of $f(x,y,z)$ we solve $0 = \dfrac{\partial f}{\partial x} = y + z$, $0 = \dfrac{\partial f}{\partial y} = x$, $0 = \dfrac{\partial f}{\partial z} = x$. For the line of critical points $(0, y, -y)$, we evaluate $f(0, y, -y) = \boxed{0}$. On the **closed** boundary $x^2 + y^2 + z^2 = 1$, we define the Lagrangian $L(x,y,z,\lambda) = xy + xz + \lambda(x^2 + y^2 + z^2 - 1)$. For critical points of L,

$$0 = \frac{\partial L}{\partial x} = y + z + 2\lambda x, \quad 0 = \frac{\partial L}{\partial y} = x + 2\lambda y, \quad 0 = \frac{\partial L}{\partial z} = x + 2\lambda z, \quad 0 = \frac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 - 1.$$

The solutions of these equations for (x,y,z) are $(0, \pm 1/\sqrt{2}, \mp 1/\sqrt{2})$, $(1/\sqrt{2}, \pm 1/2, \pm 1/2)$, and $(-1/\sqrt{2}, \pm 1/2, \pm 1/2)$. Since $f(0, \pm 1/\sqrt{2}, \mp 1/\sqrt{2}) = \boxed{0}$, $f(1/\sqrt{2}, \pm 1/2, \pm 1/2) = \boxed{\pm 1/\sqrt{2}}$, and $f(-1/\sqrt{2}, \pm 1/2, \pm 1/2) = \boxed{\mp 1/\sqrt{2}}$, maximum and minimum values of $f(x,y,z)$ are $\pm 1/\sqrt{2}$.

27. Since z is always equal to 1 on the curve of intersection of the surfaces, we maximize the function $F(x,y) = x^2y - xy^2$ subject to the constraint $x^2 + y^2 = 1$. Critical points of the Lagrangian $L(x,y,\lambda) = x^2y - xy^2 + \lambda(x^2 + y^2 - 1)$ are given by

$$0 = \frac{\partial L}{\partial x} = 2xy - y^2 + 2\lambda x, \quad 0 = \frac{\partial L}{\partial y} = x^2 - 2xy + 2\lambda y, \quad 0 = \frac{\partial L}{\partial \lambda} = x^2 + y^2 - 1.$$

Solutions for (x,y) are $(\pm 1/\sqrt{2}, \mp 1/\sqrt{2})$, $\left(\sqrt{(3 \pm \sqrt{5})/6}, 1/[3\sqrt{3 \pm \sqrt{5})/6}]\right)$, and $\left(-\sqrt{(3 \pm \sqrt{5})/6}, -1/[3\sqrt{3 \pm \sqrt{5})/6}]\right)$. When these are substituted into $F(x,y)$, the largest value is $1/\sqrt{2}$.

28. The distance D from the origin to any point (x,y) on the ellipse is given by $D^2 = x^2 + y^2$, subject to $3x^2 + 4xy + 6y^2 = 140$. Ends of the major and minor axes maximize and minimize this function. To find these points we define the Lagrangian $L(x,y,\lambda) = x^2 + y^2 + \lambda(3x^2 + 4xy + 6y^2 - 140)$. Critical points are given by

$$0 = \frac{\partial L}{\partial x} = 2x + \lambda(6x + 4y), \quad 0 = \frac{\partial L}{\partial y} = 2y + \lambda(4x + 12y), \quad 0 = \frac{\partial L}{\partial \lambda} = 3x^2 + 4xy + 6y^2 - 140.$$

When the first two are solved for λ and the expressions equated, $\dfrac{-x}{3x + 2y} = \dfrac{-y}{2x + 6y}$, which simplifies to $0 = 2x^2 + 3xy - 2y^2 = (2x - y)(x + 2y)$. Thus, $y = 2x$ or $x = -2y$. These lead to the four points $(\pm 2, \pm 4)$ and $(\pm 2\sqrt{14}, \mp\sqrt{14})$. Since $D^2(\pm 2, \pm 4) = 20$ and $D^2(\pm 2\sqrt{14} \mp \sqrt{14}) = 70$, the ends of the major axis are $(\pm 2\sqrt{14}, \mp\sqrt{14})$, and the ends of the minor axis are $(\pm 2, \pm 4)$.

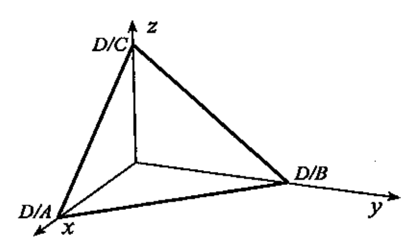
29. Critical points of the Lagrangian

$L(x,y,z,\lambda) = x^p y^q z^r + \lambda(Ax + By + Cz - D)$

are given by

$$0 = \frac{\partial L}{\partial x} = px^{p-1}y^q z^r + \lambda A, \ 0 = \frac{\partial L}{\partial y} = qx^p y^{q-1} z^r + \lambda B,$$

$$0 = \frac{\partial L}{\partial z} = rx^p y^q z^{r-1} + \lambda C, \ 0 = \frac{\partial L}{\partial \lambda} = Ax + By + Cz - D.$$



The only critical point has coordinates

$$x = \frac{pD}{A(p+q+r)}, \ y = \frac{qD}{B(p+q+r)}, \ z = \frac{rD}{C(p+q+r)}.$$

Since $f(x,y,z) = x^p y^q z^r$ vanishes along the edges of that part of the plane in the first octant, and is positive otherwise, it follows that this critical point must maximize the function.

30. (a) The distance D from the origin to any point (x, y) on the folium is given by

$$D^2(t) = x^2 + y^2 = \frac{9a^2t^2}{(1+t^3)^2} + \frac{9a^2t^4}{(1+t^3)^2} = \frac{9a^2(t^2+t^4)}{(1+t^3)^2}, \quad 0 \le t < \infty.$$

For critical points we solve

$$\begin{aligned} 0 = \frac{dD^2}{dt} &= 9a^2 \left[\frac{(1+t^3)^2(2t+4t^3) - (t^2+t^4)2(1+t^3)(3t^2)}{(1+t^3)^4}\right] \\ &= \frac{9a^2}{(1+t^3)^3}(2t + 4t^3 + 2t^4 + 4t^6 - 6t^4 - 6t^6) \\ &= \frac{-18a^2t(t-1)(t^4+t^3+3t^2+t+1)}{(1+t^3)^3}. \end{aligned}$$

Since the quartic polynomial has no positive solutions, the only critical points are $t = 0$ and $t = 1$. Since

$$D^2(0) = 0, \quad D^2(1) = \frac{9a^2}{2}, \quad \lim_{t\to\infty} D^2 = 0,$$

it follows that distance is maximized at the point $(3a/2, 3a/2)$.
(b) An implicit definition of the curve is

$$\begin{aligned} x^3 + y^3 &= \frac{27a^3t^3}{(1+t^3)^3} + \frac{27a^3t^6}{(1+t^3)^3} = \frac{27a^3t^3(1+t^3)}{(1+t^3)^3} \\ &= \frac{27a^3t^3}{(1+t^3)^2} = 3a\left(\frac{3at}{1+t^3}\right)\left(\frac{3at^2}{1+t^3}\right) = 3axy. \end{aligned}$$

To maximize $D^2 = x^2 + y^2$ subject to $x^3 + y^3 = 3axy$, we define the Lagrangian

$$L(x, y, \lambda) = x^2 + y^2 + \lambda(x^3 + y^3 - 3axy).$$

For critical points of L, we solve

$$0 = \frac{\partial L}{\partial x} = 2x + \lambda(3x^2 - 3ay), \quad 0 = \frac{\partial L}{\partial y} = 2y + \lambda(3y^2 - 3ax), \quad 0 = \frac{\partial L}{\partial \lambda} = x^3 + y^3 - 3axy.$$

When x times the second equation is subtracted from y times the first,

$$\begin{aligned} 0 &= \lambda(3x^2y - 3xy^2 - 3ay^2 + 3ax^2) = 3\lambda[xy(x-y) + a(x-y)(x+y)] \\ &= 3\lambda(x-y)[xy + a(x+y)]. \end{aligned}$$

Thus, $y = x$ or $xy + a(x + y) = 0$. The second equation cannot be satisfied for positive x and y. The only critical point is obtained from $x^3 + x^3 - 3ax^2 = 0$. The solution is $x = 3a/2$. Since the curve is **closed**, the maximum value of D^2 must be at $(3a/2, 3a/2)$.

31. (a) We define the Lagrangian $L(x, y, z, \lambda, \mu) = x^2 + y^2 + z^2 + \lambda(x^2 - xy + y^2 - z^2 - 1) + \mu(x^2 + y^2 - 1)$. Its critical points are given by

$$0 = \frac{\partial L}{\partial x} = 2x + \lambda(2x - y) + 2\mu x, \quad 0 = \frac{\partial L}{\partial y} = 2y + \lambda(-x + 2y) + 2\mu y,$$

$$0 = \frac{\partial L}{\partial z} = 2z - 2\lambda z, \quad 0 = \frac{\partial L}{\partial \lambda} = x^2 - xy + y^2 - z^2 - 1, \quad 0 = \frac{\partial L}{\partial \mu} = x^2 + y^2 - 1.$$

Solutions of these equations are $(0, \pm 1, 0)$, $(\pm 1, 0, 0)$, $(1/\sqrt{2}, -1/\sqrt{2}, \pm 1/\sqrt{2})$, and $(-1/\sqrt{2}, 1/\sqrt{2}, \pm 1/\sqrt{2})$. Because the curve is closed (actually two closed curves), we evaluate

$$f(0, \pm 1, 0) = 1, \; f(\pm 1, 0, 0) = 1, \; f(1/\sqrt{2}, -1/\sqrt{2}, \pm 1/\sqrt{2}) = 3/2, \; f(-1/\sqrt{2}, 1/\sqrt{2}, \pm 1/\sqrt{2}) = 3/2,$$

and conclude that $(\pm 1, 0, 0)$ and $(0, \pm 1, 0)$ are all equally close to the origin.
(b) By subtracting one constraint from the other, we obtain $z^2 = -xy$, so that we can write

$$f(x, y, z) = u(x, y) = x^2 + y^2 - xy = 1 - xy,$$

subject to $x^2 + y^2 = 1$. Because x and y must have opposite signs ($z^2 = -xy$), we consider only those points on $x^2 + y^2 = 1$ in the second and fourth quadrants. Critical points of the Lagrangian $L(x, y, \lambda) = 1 - xy + \lambda(x^2 + y^2 - 1)$ are given by

$$0 = \frac{\partial L}{\partial x} = -y + 2\lambda x, \quad 0 = \frac{\partial L}{\partial y} = -x + 2\lambda y, \quad 0 = \frac{\partial L}{\partial \lambda} = x^2 + y^2 - 1.$$

The only acceptable solutions (x, y) of these equations are $(\pm 1/\sqrt{2}, \mp 1/\sqrt{2})$. We now evaluate $u(x, y)$ at these points and the ends of the two arcs of the circle:

$$u(\pm 1/\sqrt{2}, \mp 1/\sqrt{2}) = 3/2, \quad u(\pm 1, 0) = u(0, \pm 1) = 1,$$

and arrive at the same conclusion as in part (a).
(c) Since $y = \begin{cases} -\sqrt{1 - x^2}, & x > 0 \\ \sqrt{1 - x^2}, & x < 0 \end{cases}$, we may write

$$f(x, y, z) = u(x) = 1 - x(\mp\sqrt{1 - x^2}) = 1 \pm x\sqrt{1 - x^2},$$

(the positive being chosen when $0 < x \le 1$ and the negative when $-1 \le x < 0$). For critical points of these functions we solve

$$0 = u'(x) = \pm\sqrt{1 - x^2} \mp \frac{x^2}{\sqrt{1 - x^2}} = \frac{\pm(1 - 2x^2)}{\sqrt{1 - x^2}}.$$

The critical points are $x = \pm 1/\sqrt{2}$. When we evaluate $u(x)$ at these points and the ends of the two arcs of the circle,

$$u(1/\sqrt{2}) = 3/2 = u(-1/\sqrt{2}), \quad u(-1) = u(0) = u(1) = 1,$$

and the same conclusion is in parts (a) and (b) is obtained.
(d) If we write $x = \cos t$ and $y = \sin t$, then

$$f(x, y, z) = u(t) = 1 - \cos t \sin t = 1 - \frac{1}{2}\sin 2t,$$

and this function must be minimized on $-\pi/2 \le t \le 0$, $\pi/2 \le t \le \pi$. For critical points we solve $0 = u'(t) = -\cos 2t$, and obtain $t = -\pi/4$ and $t = 3\pi/4$. When we evaluate $u(t)$ at these points and the ends of the two intervals, we find

$$u(-\pi/4) = 3/2 = u(3\pi/4), \quad u(-\pi/2) = u(0) = u(\pi/2) = u(\pi) = 1.$$

Once again the same points $(0, \pm 1, 0)$ and $\pm 1, 0, 0)$ are obtained.

32. We must maximize and minimize the function $D^2 = x^2 + y^2 + z^2$ subject to the constraints $x^2 + y^2/4 + z^2/9 = 1$ and $x + y + z = 0$. We define the the Lagrangian

$$L(x, y, z, \lambda, \mu) = x^2 + y^2 + z^2 + \lambda\left(x^2 + \frac{y^2}{4} + \frac{z^2}{9} - 1\right) + \mu(x + y + z).$$

For critical points, we solve

$$0 = \frac{\partial L}{\partial x} = 2x + 2\lambda x + \mu, \quad 0 = \frac{\partial L}{\partial y} = 2y + \frac{\lambda y}{2} + \mu, \quad 0 = \frac{\partial L}{\partial z} = 2z + \frac{2\lambda z}{9} + \mu,$$

$$0 = \frac{\partial L}{\partial \lambda} = x^2 + \frac{y^2}{4} + \frac{z^2}{9} - 1, \quad 0 = \frac{\partial L}{\partial \mu} = x + y + z.$$

The second equation subtracted from the first implies that $y = 4x(\lambda + 1)/(4 + \lambda)$. The third subtracted from the first gives $z = 9x(\lambda + 1)/(9 + \lambda)$. When these are substituted into the last equation

$$\begin{aligned} 0 &= x + \frac{4x(\lambda + 1)}{4 + \lambda} + \frac{9x(\lambda + 1)}{9 + \lambda} \\ &= x\left[\frac{(4 + \lambda)(9 + \lambda) + 4(\lambda + 1)(9 + \lambda) + 9(\lambda + 1)(4 + \lambda)}{(4 + \lambda)(9 + \lambda)}\right]. \end{aligned}$$

Since $x \neq 0$ (else $y = z = 0$), we set $0 = 14\lambda^2 + 98\lambda + 108 = 2(7\lambda^2 + 49\lambda + 54)$. Solutions are $\lambda = (-49 \pm \sqrt{889})/14$. Substitution of these results into the ellipsoid constraint gives

$$\begin{aligned} 1 &= x^2 + \frac{1}{4}\left[\frac{4x(\lambda + 1)}{4 + \lambda}\right]^2 + \frac{1}{9}\left[\frac{9x(\lambda + 1)}{9 + \lambda}\right]^2 \\ &= x^2 + \frac{4x^2(\lambda + 1)^2}{(4 + \lambda)^2} + \frac{9x^2(\lambda + 1)^2}{(9 + \lambda)^2}. \end{aligned}$$

Thus,

$$\begin{aligned} x^2 &= \frac{(4 + \lambda)^2(9 + \lambda)^2}{(4 + \lambda)^2(9 + \lambda)^2 + 4(\lambda + 1)^2(9 + \lambda)^2 + 9(\lambda + 1)^2(4 + \lambda)^2}, \\ y^2 &= \frac{16(1 + \lambda)^2(9 + \lambda)^2}{(4 + \lambda)^2(9 + \lambda)^2 + 4(\lambda + 1)^2(9 + \lambda)^2 + 9(\lambda + 1)^2(4 + \lambda)^2}, \\ z^2 &= \frac{81(1 + \lambda)^2(4 + \lambda)^2}{(4 + \lambda)^2(9 + \lambda)^2 + 4(\lambda + 1)^2(9 + \lambda)^2 + 9(\lambda + 1)^2(4 + \lambda)^2}. \end{aligned}$$

For $\lambda = (-49 + \sqrt{889})/14$, we obtain $D = \sqrt{x^2 + y^2 + z^2} = 1.171$, and for $\lambda = (-49 - \sqrt{889})/14$, $D = 2.373$.

33. Geometrically, the constraint represents a cylinder in the z-direction, and therfore z is arbitrary. Because z appears in only the denominator, we should choose $z = 0$ in order to maximize the function. In other words, the maximum value of $f(x, y, z)$ is the maximum value of $F(x, y) = xy + x^2$ subject $x^2(4 - x^2) = y^2$. Critical points of the Lagrangian $L(x, y, \lambda) = xy + x^2 + \lambda(4x^2 - x^4 - y^2)$ are given by

$$0 = \frac{\partial L}{\partial x} = y + 2x + \lambda(8x - 4x^3), \quad 0 = \frac{\partial L}{\partial y} = x - 2\lambda y, \quad 0 = \frac{\partial L}{\partial \lambda} = x^2(4 - x^2) - y^2.$$

The solutions of these equations are $(0, 0)$, $(\pm 1.80158, \pm 1.56470)$ and $(\pm 1.28188, \mp 1.96972)$. Since the curve is closed, we evaluate

$$F(0, 0) = 0, \; F(\pm 1.80158, \pm 1.56470) = 6.06, \; F(\pm 1.28188, \mp 1.96972) = -0.88,$$

and conclude that the maximum value of $f(x, y, z)$ is 6.06.

34. For the solution without Lagrange multipliers we express $f(x,y,z)$ as a function of one variable. We solve the constraint equations for $x^2 = 1-3y^2$ and $y^2 = (1-z)/2$, and substitute into $f(x,y,z)$:

$$\begin{aligned} f(x,y,z) &= 2(1-3y^2)y^2 + 2y^2z^2 + 3z = 2\left(1-\frac{3}{2}+\frac{3z}{2}\right)\left(\frac{1}{2}-\frac{z}{2}\right) + 2z^2\left(\frac{1}{2}-\frac{z}{2}\right) + 3z \\ &= \frac{1}{2}(-2z^3 - z^2 + 10z - 1). \end{aligned}$$

Since $z = x^2+y^2 = (1-3y^2)+y^2 = 1-2y^2$, and y must be restricted to $|y| \le 1/\sqrt{3}$, it follows that the only possible values for z are $1/3 \le z \le 1$. Thus, finding maximimum and minimum values of $f(x,y,z)$ subject to the two constraints is equivalent to finding maximimum and minimum values of

$$F(z) = \frac{1}{2}(-2z^3 - z^2 + 10z - 1), \quad \frac{1}{3} \le z \le 1.$$

For critical points of $F(z)$, we solve $0 = F'(z) = (1/2)(-6z^2 - 2z + 10) \implies z = (-1 \pm \sqrt{61})/6$. These must be rejected as not lying in the required interval, and maximum and minimum values of $F(z)$ must therefore occur at $z = 1/3$ and $z = 1$. Since $F(1/3) = 29/27$ and $F(1) = 3$, these are minimum and maximum values of $F(z)$.

To use Lagrange multipliers, we define the Lagrangian

$$L(x,y,z,\lambda,\mu) = 2x^2y^2 + 2y^2z^2 + 3z + \lambda(x^2+y^2-z) + \mu(x^2+3y^2-1).$$

For critical points we solve

$$0 = \frac{\partial L}{\partial x} = 4xy^2 + 2\lambda x + 2\mu x, \quad 0 = \frac{\partial L}{\partial y} = 4x^2y + 4yz^2 + 2\lambda y + 6\mu y,$$

$$0 = \frac{\partial L}{\partial z} = 4y^2z + 3 - \lambda, \quad 0 = \frac{\partial L}{\partial \lambda} = x^2 + y^2 - z, \quad 0 = \frac{\partial L}{\partial \mu} = x^2 + 3y^2 - 1.$$

If we choose $x = 0$ to satisfy the first equation, then the remaining equations imply that $y = \pm 1/\sqrt{3}$ and $z = 1/3$. If we choose $y = 0$ to satisfy the second equation, the remaining equations require $x = \pm 1$ and $z = 1$. The only other way to satisfy the first two equations is to set

$$2y^2 + \lambda + \mu = 0, \quad 2x^2 + 2z^2 + \lambda + 3\mu = 0.$$

If we multiply the first by three and subtract the second, we obtain

$$\begin{aligned} 0 &= 6y^2 - 2x^2 - 2z^2 + 2\lambda = 6y^2 - 2x^2 - 2(x^2+y^2)^2 + 2\lambda \\ &= 6y^2 - 2x^2 - 2(x^4 + 2x^2y^2 + y^4) + 2\lambda \\ &= 6y^2 - 2(1-3y^2) - 2(1-3y^2)^2 - 4y^2(1-3y^2) - 2y^4 + 2\lambda \\ &= 2(\lambda - 2 + 10y^2 - 4y^4), \end{aligned}$$

But from the equation for $\partial L/\partial z$, we can also write

$$0 = 3 - \lambda + 4y^2(x^2+y^2) = 3 - \lambda + 4y^2(1-3y^2+y^2) = 3 - \lambda + 4y^2 - 8y^4.$$

These two equations in y and λ imply that

$$2 - 10y^2 + 4y^4 = 3 + 4y^2 - 8y^4 \quad \Longrightarrow \quad 12y^4 - 14y^2 - 1 = 0 \quad \Longrightarrow \quad y^2 = \frac{7 \pm \sqrt{61}}{12}.$$

We reject the negative solution. But substitution of this result into the constraint $x^2 + 3y^2 = 1$ requires x^2 to be negative. Consequently, only four critical points (x,y,z) are obtained: $(0, \pm 1/\sqrt{3}, 1/3)$ and $(\pm 1, 0, 1)$. Since the curve defined by the constraints is closed, we evaluate

$$f(0, \pm 1/\sqrt{3}, 1/3) = 29/27, \quad f(\pm 1, 0, 1) = 3.$$

These are minimum and maximum values.

EXERCISES 12.13

1. Least squares estimates for parameters a and b in a linear function $S = aM + b$ must satisfy equations similar to 12.71. For the tabular values these equations are

$$160\,080a + 1260b = 109\,624, \qquad 1260a + 10b = 856.$$

The solution is $a = 1.3394$ and $b = -83.164$.

2. Least squares estimates for parameters a and b in a linear function $P = aA + b$ must satisfy equations similar to 12.71. For the tabular values these equations are

$$1482a + 130b = 13\,663, \qquad 130a + 13b = 1315.$$

The solution is $a = 2.8187$ and $b = 72.967$.

3. Least squares estimates for a and b must satisfy the equations

$$\left(\sum_{i=1}^{11} t_i^2\right) a + \left(\sum_{i=1}^{11} t_i\right) b = \sum_{i=1}^{11} t_i \overline{S}_i, \qquad \left(\sum_{i=1}^{11} t_i\right) a + 11b = \sum_{i=1}^{11} \overline{S}_i,$$

where $\overline{S}_i$ are the values in the table.
(a) Using $t = 0$ in year zero, we obtain

$$41\,870\,521a + 21\,461b = 1\,843\,532.5, \qquad 21\,461a + 11b = 944.7,$$

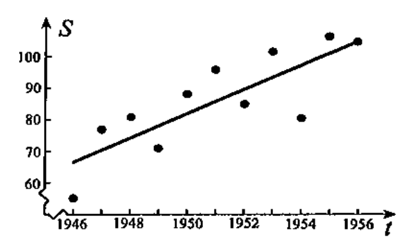
the solution of which is $a = 3.8436$ and $b = -7413.05$. The least-squares line is therefore $S = 3.8436t - 7413.05$.
(b) Using $t = 0$ in year 1946, the equations defining a and b are

$$385a + 55b = 5146.3, \quad 55a + 11b = 944.7,$$

the solution of which is $a = 3.8436$ and $b = 66.664$. The least squares line is therefore $S = 3.8436t + 66.664$. If we replace t by $t - 1946$, where t is now the actual year,

$$S(t) = 3.8436(t - 1946) + 66.664 = 3.8436t - 7412.98.$$

A plot of curve and data points is shown to the right.

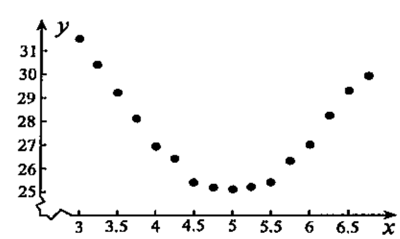


4. (a) The plot is to the right.
(b) For critical points of $S(a, b, c)$ we solve

$$0 = \frac{\partial S}{\partial a} = \sum_{i=1}^{16} 2(a{x_i}^2 + bx_i + c - \overline{y_i})({x_i}^2),$$

$$0 = \frac{\partial S}{\partial b} = \sum_{i=1}^{16} 2(a{x_i}^2 + bx_i + c - \overline{y_i})(x_i),$$

$$0 = \frac{\partial S}{\partial c} = \sum_{i=1}^{16} 2(a{x_i}^2 + bx_i + c - \overline{y_i}).$$



These can be rewritten in the form

$$\left(\sum_{i=1}^{16} {x_i}^4\right) a + \left(\sum_{i=1}^{16} {x_i}^3\right) b + \left(\sum_{i=1}^{16} {x_i}^2\right) c = \sum_{i=1}^{16} {x_i}^2 \overline{y_i},$$

$$\left(\sum_{i=1}^{16} {x_i}^3\right) a + \left(\sum_{i=1}^{16} {x_i}^2\right) b + \left(\sum_{i=1}^{16} x_i\right) c = \sum_{i=1}^{16} x_i \overline{y_i},$$

$$\left(\sum_{i=1}^{16} {x_i}^2\right) a + \left(\sum_{i=1}^{16} x_i\right) b + 16c = \sum_{i=1}^{16} \overline{y_i}.$$

(c) From the tabular values,

$$12\,117.5a + 2164.5b + 401.5c = 10\,979.4,\; 2164.5a + 401.5b + 78c = 2133.47,\; 401.5a + 78b + 16c = 439.4.$$

The solution is $a = 1.6653$, $b = -16.642$, and $c = 66.802$.

5. (a) Using the equations in Exercise 4, least squares estimates for parameters a, b, and c of the quadratic function $y = ax^2 + bx + c$ must satisfy

$$1442.9a + 410.688b + 121.04c = 5613.13, \quad 410.688a + 121.04b + 37.2c = 1559.86,$$

$$121.04a + 37.2b + 12c = 445.69.$$

The solution is $a = 5.9226$, $b = -5.5627$, and $c = -5.3543$.
(b) The value of $S(5.9226, -5.5627, -5.3543)$ is 1.5275.

6. The sum of the squares of the differences between observed and predicted values is

$$S = S(a,b) = \sum_{i=1}^{8} (a + bQ_i^2 - \overline{H}_i)^2,$$

where $(Q_i, \overline{H}_i)$ are the points in the table. For critical points of S, we solve

$$0 = \frac{\partial S}{\partial b} = \sum_{i=1}^{8} 2(a + bQ_i^2 - \overline{H}_i)(Q_i^2), \qquad 0 = \frac{\partial S}{\partial a} = \sum_{i=1}^{8} 2(a + bQ_i^2 - \overline{H}_i).$$

These can rewritten in the form

$$\left(\sum_{i=1}^{8} Q_i^4\right) b + \left(\sum_{i=1}^{8} Q_i^2\right) a = \sum_{i=1}^{n} Q_i^2 \overline{H}_i, \qquad \left(\sum_{i=1}^{8} Q_i^2\right) b + 8a = \sum_{i=1}^{8} \overline{H}_i.$$

From the table, these become

$$2.19185 \times 10^8 b + 34728.8a = 632647, \qquad 34728.8b + 8a = 197.2.$$

The solution is $a = 38.82$ and $b = 0.003265$; that is, the least squares quadratic is $H = 38.82 - 0.003265Q^2$.

7. By analogy with Exercise 4, least squares estimates for parameters a, b, c, and d must satisfy

$$\left(\sum_{i=1}^{12} {x_i}^6\right) a + \left(\sum_{i=1}^{12} {x_i}^5\right) b + \left(\sum_{i=1}^{12} {x_i}^4\right) c + \left(\sum_{i=1}^{12} {x_i}^3\right) d = \sum_{i=1}^{12} {x_i}^3 \overline{y_i},$$

$$\left(\sum_{i=1}^{12} {x_i}^5\right) a + \left(\sum_{i=1}^{12} {x_i}^4\right) b + \left(\sum_{i=1}^{12} {x_i}^3\right) c + \left(\sum_{i=1}^{12} {x_i}^2\right) d = \sum_{i=1}^{12} {x_i}^2 \overline{y_i},$$

$$\left(\sum_{i=1}^{12} {x_i}^4\right) a + \left(\sum_{i=1}^{12} {x_i}^3\right) b + \left(\sum_{i=1}^{12} {x_i}^2\right) c + \left(\sum_{i=1}^{12} x_i\right) d = \sum_{i=1}^{12} x_i \overline{y_i},$$

$$\left(\sum_{i=1}^{12} {x_i}^3\right) a + \left(\sum_{i=1}^{12} {x_i}^2\right) b + \left(\sum_{i=1}^{12} x_i\right) c + 12d = \sum_{i=1}^{12} \overline{y_i}.$$

From the tabular values,

$$19\,279.6a + 5214.9b + 1442.9c + 410.688d = 20\,660.5,\; 5214.9a + 1442.9b + 410.688c + 121.04d = 5613.13,$$

$$1442.9a + 410.688b + 121.04c + 37.2d = 1559.86,\; 410.688a + 121.04b + 37.2c + 12d = 445.69.$$

The solution is $a = -0.007\,693\,4$, $b = 5.9943$, $c = -5.7787$, and $d = -5.1442$.
(b) The value of $S(-0.007\,693\,4, 5.9943, -5.7787, -5.1442)$ is 1.5289. The fact that this is essentially the same as that in Exercise 5 indicates that the parabola fits the data as well as the cubic.

8. (a) When we take logarithms of y-values in the given table, we obtain

x	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5
y	4.94	5.19	5.44	5.67	5.90	6.12	6.34	6.51	6.67	6.91	7.11

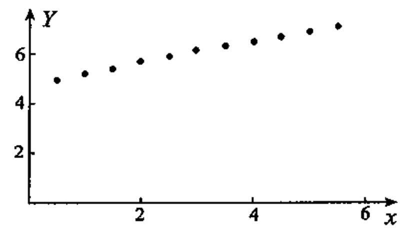
The plot of these to the right indicates that a straight line fit is indeed reasonable.
(b) Equations for a and B corresponding to 12.71 are

$$\left(\sum_{i=1}^{11} x_i^2\right) a + \left(\sum_{i=1}^{11} x_i\right) B = \sum_{i=1}^{11} x_i \overline{Y}_i,$$

$$\left(\sum_{i=1}^{11} x_i\right) a + (11)b = \sum_{i=1}^{11} \overline{Y}_i,$$

from which

$$126.5a + 33B = 212.149, \quad 33a + 11B = 66.795.$$



The solution of these is $a = 0.43$ and $B = 4.79$. Consequently, the least-squares estimates give

$$\ln y = Y = 4.79 + 0.43x.$$

When we take exponentials,

$$y = e^{4.79+0.43x} = 120.3e^{0.43x}.$$

9. The points in the plot of $\ln W$ against $\ln F$ in the left figure below are reasonably collinear so that

$$W = aF^b \quad \Longrightarrow \quad \ln W = \ln a + b \ln F$$

is an acceptable functional representation for $W(F)$. If we set $w = \ln W$, $A = \ln a$, and $f = \ln F$, then $w = A + bf$. Least squares estimates for b and A are defined by

$$\left(\sum_{i=1}^{10} f_i^2\right) b + \left(\sum_{i=1}^{10} f_i\right) A = \sum_{i=1}^{10} f_i \overline{w}_i, \qquad \left(\sum_{i=1}^{10} f_i\right) b + 10A = \sum_{i=1}^{10} \overline{w}_i.$$

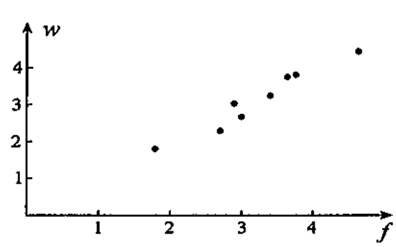
These become

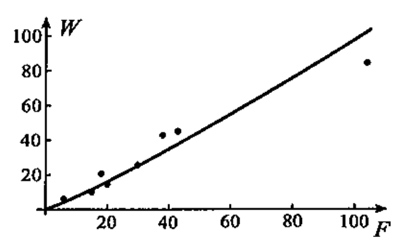
$$99.9544b + 30.5124A = 94.1115, \quad 30.1524b + 10A = 28.3150,$$

the solution of which is $b = 1.126$ and $A = -0.6034$. Thus,

$$w = -0.6034 + 1.126f \quad \Longrightarrow \quad \ln W = -0.6034 + 1.126 \ln F \quad \Longrightarrow \quad W = 0.547F^{1.126}.$$

A plot is shown in the right figure along with the original data points.





10. To find $y = f(v) = a - bv$, the number of kilometres per litre for a truck travelling at speed v, we use least squares for the 18 data points in the table. Equations for a and b are

$$-\left(\sum_{i=1}^{18} v_i^2\right) b + \left(\sum_{i=1}^{18} v_i\right) a = \sum_{i=1}^{18} v_i \overline{y}_i, \qquad -\left(\sum_{i=1}^{18} v_i\right) b + 18a = \sum_{i=1}^{18} \overline{y}_i.$$

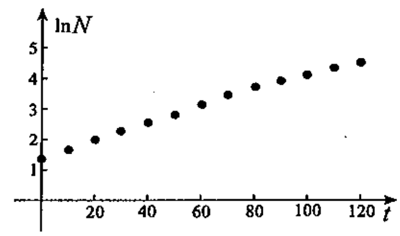
These become

$$-147\,000b + 1620a = 3379, \quad -1620b + 18a = 37.71.$$

The solution is $a = 3.2125$ and $b = 0.0124167$. We can now substitute these into the formula $v = a/(b + \sqrt{bp/w})$ in Exercise 59 of Section 4.7,

$$v = \frac{3.2125}{0.0124167 + \sqrt{\dfrac{0.0124167(0.6)}{20}}} = 101.3 \text{ kilometres per hour.}$$

11. If N denotes the population (in millions) and t is the year (taking $t = 0$ in 1790), then $N(t)$ can be approximated by an exponential if $\ln N$ can be aproximated by a straight line. The plot indicates that this is indeed the case, and we therefore set $N(t) = be^{at} \implies \ln N = at + b$. If we set $Y = \ln N$, then $Y = at + b$. Least squares estimates of a and b are then defined by equations similar to 12.71 with x replaced by t. They are



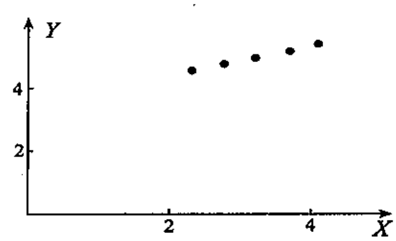
$$65\,000a + 780b = 2878.4809, \quad 780a + 13b = 39.845\,743.$$

The solution is $a = 0.026\,798\,699$ and $b = 1.457\,135\,2$. Consequently,

$$Y = \ln N = 0.026\,798\,699t + 1.457\,135\,2 \implies N = 4.293\,641\,5\, e^{0.026\,798\,699t}.$$

For $t = 0$ in year zero, we set $N = 4.293\,641\,5\, e^{0.026\,798\,699(t-1790)} = 6.308 \times 10^{-21} e^{0.0268t}$.

12. (a) The data points of Y against X are reasonably collinear.
(b) If a line $Y = aX + B$ is to fit the points in the plot, then a and B must satisfy equations similar to 12.71 where B replaces b. They are

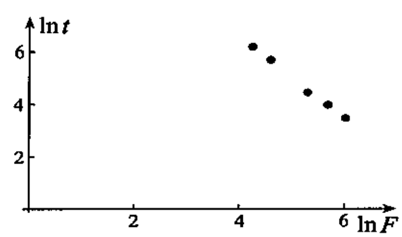


$$53.721\,796a + 16.077\,273B = 81.047\,068,$$
$$16.077\,273a + 5B = 24.904\,540.$$

The solution is $a = 0.477\,603\,27$ and $B = 3.445\,196\,4$.

Thus, $Y = \ln y = 0.477\,603\,27X + 3.445\,196\,4 = 0.477\,603\,27 \ln x + 3.445\,196\,4$, or, $y = 31.35x^{0.4776}$.

13. The points in the plot of $\ln F$ against $\ln t$ are reasonably collinear so that $t = aF^b \implies \ln t = \ln a + b \ln F$ is indeed an acceptable functional representation.
If we set $T = \ln t$, $A = \ln a$, and $f = \ln F$, then $T = bf + A$. Least squares estimates for b and A are defined by



$$\left(\sum_{i=1}^{5} f_i^2\right) b + \left(\sum_{i=1}^{5} f_i\right) A = \sum_{i=1}^{5} f_i \overline{T}_i,$$
$$\left(\sum_{i=1}^{5} f_i\right) b + 5A = \sum_{i=1}^{5} \overline{T}_i.$$

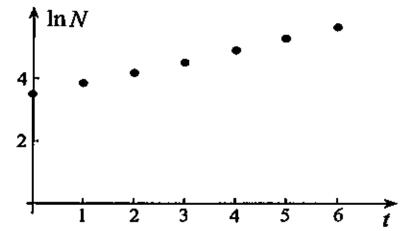
These become

$$135.760b + 25.847A = 118.9965, \qquad 25.847b + 5A = 23.6635,$$

the solution of which is $b = -1.551$ and $A = 12.75$. Thus, $T = -1.551f + 12.75$, and when we substitute $T = \ln t$ and $f = \ln F$,

$$\ln t = -1.551 \ln F + 12.75 \implies t = \frac{e^{12.75}}{F^{1.551}} = \frac{3.45 \times 10^5}{F^{1.551}}.$$

14. (a) A plot is shown to the right.



(b) If we take logarithms of $N = be^{at}$, $\ln N = at + \ln b$, and define $n = \ln N$ and $B = \ln b$, then, $n = at + B$. Least-squares estimates for a and B are given by

$$\left(\sum_{i=1}^{7} t_i^2\right) a + \left(\sum_{i=1}^{7} t_i\right) B = \sum_{i=1}^{7} t_i \overline{n}_i,$$

$$\left(\sum_{i=1}^{7} t_i\right) a + 7B = \sum_{i=1}^{7} \overline{n}_i.$$

These become

$$91a + 21B = 105.2312, \qquad 21a + 7B = 31.7587,$$

the solution of which is $a = 0.35554$ and $B = 3.47034$. Therefore,

$$n = 0.35554t + 3.47034 \Longrightarrow \ln N = 0.35554t + 3.47034 \Longrightarrow N = e^{0.355554t+3.47034} = 32.1476e^{0.35554t}.$$

15. If we take logarithms of $PV^a = b$, we obtain

$$\ln P + a \ln V = \ln b \quad \Longrightarrow \quad \ln P = -a \ln V + \ln b.$$

If we set $p = \ln P$, $v = \ln V$, $A = -a$, and $B = \ln b$, then $p = Av + B$. Least-squares estimates of A and B must satisfy the equations

$$\left(\sum_{i=1}^{6} v_i^2\right) A + \left(\sum_{i=1}^{6} v_i\right) B = \sum_{i=1}^{6} v_i \overline{p}_i, \qquad \left(\sum_{i=1}^{6} v_i\right) A + 6B = \sum_{i=1}^{6} \overline{p}_i.$$

These become

$$121.9758A + 26.9295B = 89.3605, \qquad 26.9295A + 6B = 20.2570,$$

the solution of which is $A = -1.4043$ and $B = 9.6788$. Consequently,

$$p = -1.4043v + 9.6788 \Longrightarrow \ln P = -1.4043 \ln V + 9.6788 \Longrightarrow P = e^{-1.4043 \ln V + 9.6788} = 15975V^{-1.4043}.$$

16. (a) To find a least-squares estimate for b, we form the sum of squares $S(b) = \sum_{i=1}^{n} \left(\overline{y_i} - \dfrac{b}{{x_i}^2}\right)^2$. For critical points of $S(b)$, we solve

$$0 = \frac{dS}{db} = \sum_{i=1}^{n} 2\left(\overline{y_i} - \frac{b}{{x_i}^2}\right)\left(-\frac{1}{{x_i}^2}\right) = \sum_{i=1}^{n} \frac{b}{{x_i}^4} - \sum_{i=1}^{n} \frac{\overline{y_i}}{{x_i}^2}.$$

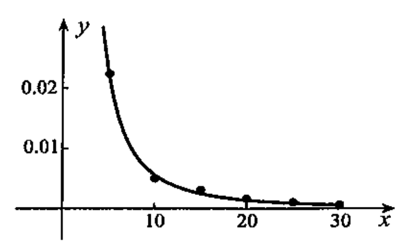
Thus, $b = \left(\sum_{i=1}^{n} \dfrac{\overline{y_i}}{{x_i}^2}\right) \Big/ \left(\sum_{i=1}^{n} \dfrac{1}{{x_i}^4}\right)$.

(b) For the data in the table,

$$b = \frac{9.669\,874 \times 10^{-4}}{1.729\,797\,7 \times 10^{-3}} = 0.559\,017\,65.$$

The least squares approximation for the curve is therefore $y = \dfrac{0.5590}{x^2}$.

(c) The curve and data are plotted to the right.



17. (a) If we set $Y = e^y = ax + b$, then least-squares estimates for a and b are defined by equations similar to 12.71,

$$139a + 27b = 352.831\,52, \quad 27a + 6b = 70.874\,602.$$

The solution is $a = 1.936\,903\,5$ and $b = 3.096\,368\,0$. Thus,

$$Y = e^y = 1.936\,903\,5x + 3.096\,368\,0 \quad \Longrightarrow \quad y = \ln(3.0964 + 1.9369x).$$

(b) If we attempt to do so, we define $S(a,b) = \sum_{i=1}^{n}[\overline{y_i} - \ln(b + ax_i)]^2$. For critical points of S, we then solve

$$0 = \frac{\partial S}{\partial a} = \sum_{i=1}^{n} 2[\overline{y_i} - \ln(b+ax_i)]\left(\frac{-x_i}{b+ax_i}\right), \quad 0 = \frac{\partial S}{\partial b} = \sum_{i=1}^{n} 2[\overline{y_i} - \ln(b+ax_i)]\left(\frac{-1}{b+ax_i}\right).$$

Unfortunately, we cannot solve these equations for a and b.

18. If we set $Y = 1/y = ax + b$, then least-squares estimates for a and b are given by equations similar to 12.71,

$$255a + 35b = 28.285\,127, \quad 35a + 5b = 3.979\,676\,5.$$

The solution is $a = 0.042\,739\,2$ and $b = 0.496\,761\,3$, and therefore

$$Y = \frac{1}{y} = 0.042\,739\,2x + 0.496\,761\,3 \quad \Longrightarrow \quad y = \frac{1}{0.04274x + 0.4968}.$$

19. If we take logarithms of $P/y = k(x/y)^q$, we obtain $\ln(P/y) = \ln k + q\ln(x/y)$. We now set $R = \ln(P/y)$, $K = \ln k$, and $Z = \ln(x/y)$, then $R = K + qZ$. Least-squares estimates for K and q must satisfy

$$\left(\sum_{i=1}^{7} Z_i^2\right)q + \left(\sum_{i=1}^{7} Z_i\right)K = \sum_{i=1}^{7} Z_i\overline{R}_i, \qquad \left(\sum_{i=1}^{7} Z_i\right)q + 7K = \sum_{i=1}^{7}\overline{R}_i.$$

These become

$$147.95q - 32.169K = 81.148, \qquad -32.169q + 7K = -17.643,$$

with solution $q = 0.59388$ and $K = 0.20878$. Consequently, $R = 0.20878 + 0.59388Z$, from which

$$\ln\left(\frac{P}{y}\right) = 0.20878 + 0.59388\ln\left(\frac{x}{y}\right) \Longrightarrow \frac{P}{y} = e^{0.20878}\left(\frac{x}{y}\right)^{0.59388} \Longrightarrow P = 1.232x^{0.59388}y^{0.40612}.$$

EXERCISES 12.14

1. If we set $z = f(x,y)$, then $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 2xy\,dx + (x^2 - \cos y)\,dy$.

2. If we set $z = f(x,y)$, then $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{y}{1+x^2y^2}\,dx + \frac{x}{1+x^2y^2}\,dy = \frac{1}{1+x^2y^2}(y\,dx + x\,dy)$.

3. If we set $u = f(x,y,z)$, then $du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = (yz - 3x^2e^z)\,dx + xz\,dy + (xy - x^3e^z)\,dz$.

4. If we set $u = f(x,y,z)$, then

$$du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = [yz\cos(xyz) - 2xy^2z^2]\,dx + [xz\cos(xyz) - 2x^2yz^2]\,dy + [xy\cos(xyz) - 2x^2y^2z]\,dz.$$

5. If we set $u = f(x, y, z)$, then
$$\begin{aligned} du &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \\ &= \frac{2x}{x^2+y^2+z^2}dx + \frac{2y}{x^2+y^2+z^2}dy + \frac{2z}{x^2+y^2+z^2}dz \\ &= \frac{2}{x^2+y^2+z^2}(x\,dx + y\,dy + z\,dz). \end{aligned}$$

6. If we set $z = f(x, y)$, then
$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{y}{\sqrt{1-x^2y^2}}\,dx + \frac{x}{\sqrt{1-x^2y^2}}\,dy = \frac{1}{\sqrt{1-x^2y^2}}(y\,dx + x\,dy).$$

7. If we set $z = f(x, y)$, then
$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \left[\frac{1}{\sqrt{1-(x+y)^2}} - \frac{1}{\sqrt{1-(x+y)^2}}\right]dx + \left[\frac{1}{\sqrt{1-(x+y)^2}} - \frac{1}{\sqrt{1-(x+y)^2}}\right]dy = 0.$$

8. If we set $u = f(x, y, z, t)$, then
$$du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial t}dt = (y+t)\,dx + (x+z)\,dy + (y+t)\,dz + (z+x)\,dt.$$

9. If we set $u = f(x, y, z, w)$, then
$$\begin{aligned} du &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial w}dw \\ &= y\tan(zw)\,dx + x\tan(zw)\,dy + xyw\sec^2(zw)\,dz + xyz\sec^2(zw)\,dw. \end{aligned}$$

10. If we set $u = f(x, y, z, t)$, then
$$\begin{aligned} du &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial t}dt \\ &= 2xe^{x^2+y^2+z^2-t^2}dx + 2ye^{x^2+y^2+z^2-t^2}dy + 2ze^{x^2+y^2+z^2-t^2}dz - 2te^{x^2+y^2+z^2-t^2}dt \\ &= 2e^{x^2+y^2+z^2-t^2}(x\,dx + y\,dy + z\,dz - t\,dt) \end{aligned}$$

11. If $V(r, h) = (1/3)\pi r^2 h$, then $dV = (2/3)\pi rh\,dr + (1/3)\pi r^2\,dh$. When $r = 10$, $h = 20$, $dr = 0.1$, and $dh = -0.3$, then
$$dV = \left(\frac{2}{3}\right)\pi(10)(20)(0.1) + \left(\frac{1}{3}\right)\pi(100)(-0.3) = 10.47\text{ cm}^3.$$
The actual change is $V(10.1, 19.7) - V(10, 20) = (1/3)\pi(10.1)^2(19.7) - (1/3)\pi(10)^2(20) = 10.05\text{ cm}^3$.

12. $dV = \dfrac{\partial V}{\partial a}da + \dfrac{\partial V}{\partial b}db = \dfrac{4}{3}\pi b^2\,da + \dfrac{8}{3}\pi ab\,db = \dfrac{4}{3}\pi b(b\,da + 2a\,db)$

Since the percentage changes in a and b are 1%, it follows that $100\dfrac{da}{a} = 100\dfrac{db}{b} = 1$, and therefore the approximate percentage change in V is
$$100\frac{dV}{V} = 100\frac{(4/3)\pi b(b\,da + 2a\,db)}{4\pi ab^2/3} = 100\left(\frac{da}{a} + 2\frac{db}{b}\right) = 1 + 2 = 3.$$

EXERCISES 12.15

1. By multiplying the Maclaurin series of $F(x)$ and $G(y)$,
$$\begin{aligned} F(x)G(y) &= (a_0 + a_1x + a_2x^2 + \cdots)(b_0 + b_1y + b_2y^2 + \cdots) \\ &= a_0b_0 + (a_1b_0x + a_0b_1y) + (a_2b_0x^2 + a_1b_1xy + a_0b_2y^2) + \cdots\ , \end{aligned}$$
we get a series of form 12.76 with $c = d = 0$. It must therefore be the Taylor series about $(0, 0)$.

2. By setting $v_x = x - c$ and $v_y = y - d$ in the expression for $F'''(0)$,

$$F'''(0) = f_{xxx}(c,d)(x-c)^3 + 3f_{xxy}(c,d)(x-c)^2(y-d) + 3f_{xyy}(c,d)(x-c)(y-d)^2 + f_{yyy}(c,d)(y-d)^3.$$

When this is substituted into $F'''(0)t^3/3!$ and t is set equal to 1, the cubic terms are

$$\frac{1}{3!}[f_{xxx}(c,d)(x-c)^3 + 3f_{xxy}(c,d)(x-c)^2(y-d) + 3f_{xyy}(c,d)(x-c)(y-d)^2 + f_{yyy}(c,d)(y-d)^3].$$

3. Since $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$, it follows that $\cos(xy) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} y^{2n}$.

4.
$$e^{2x-3y} = e^5 e^{2(x-1)-3(y+1)} = e^5 \sum_{n=0}^{\infty} \frac{1}{n!}[2(x-1) - 3(y+1)]^n$$
$$= \sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{e^5}{n!}\binom{n}{r} 2^r (x-1)^r (-3)^{n-r}(y+1)^{n-r} = \sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{e^5(-1)^{n-r}2^r 3^{n-r}}{(n-r)!r!}(x-1)^r(y+1)^{n-r}$$

5. If we expand $\sqrt{1+x}$ with the binomial expansion 10.33b,

$$\begin{aligned} x^2y\sqrt{1+x} &= x^2y\left[1 + \frac{x}{2} + \frac{(1/2)(-1/2)}{2!}x^2 + \frac{(1/2)(-1/2)(-3/2)}{3!}x^3 + \cdots\right] \\ &= y\left[x^2 + \frac{x^3}{2} - \frac{1}{2^2 2!}x^4 + \frac{1\cdot 3}{2^3 3!}x^5 - \frac{1\cdot 3\cdot 5}{2^4 4!}x^6 + \cdots\right] \\ &= y\left[x^2 + \frac{x^3}{2} + \sum_{n=4}^{\infty} \frac{(-1)^{n+1}[1\cdot 3\cdot 5\cdots(2n-7)]}{2^{n-2}(n-2)!}x^n\right], \end{aligned}$$

valid for all y and $-1 \le x \le 1$.

6. Since $\ln(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \cdots$,

$$\begin{aligned} \ln(1+x^2+y^2) &= x^2 + y^2 - \frac{(x^2+y^2)^2}{2} + \frac{(x^2+y^2)^3}{3} - \cdots \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x^2+y^2)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{r=0}^{n}\binom{n}{r} x^{2r}y^{2(n-r)} \\ &= \sum_{n=1}^{\infty}\sum_{r=0}^{n} \frac{(-1)^{n+1}(n-1)!}{(n-r)!\,r!}x^{2r}y^{2(n-r)} \end{aligned}$$

7. Since the function is undefined at $(3,-4)$, there is no Taylor series for the function about this point.

8.
$$\frac{xy^2}{1+y^2} = [(x+1)-1]\frac{y^2}{1+y^2} = [(x+1)-1]\sum_{n=0}^{\infty}(-1)^n y^{2n+2}$$

9. With $f(-1,1) = -\frac{1}{2}$,

$$\begin{aligned} f_x(-1,1) &= \frac{(x^2+y^2)(y) - xy(2x)}{(x^2+y^2)^2}_{|(-1,1)} = \frac{y^3 - x^2y}{(x^2+y^2)^2}_{|(-1,1)} = 0, \\ f_y(-1,1) &= \frac{(x^2+y^2)(x) - xy(2y)}{(x^2+y^2)^2}_{|(-1,1)} = \frac{x^3 - xy^2}{(x^2+y^2)^2}_{|(-1,1)} = 0, \\ f_{xx}(-1,1) &= \frac{(x^2+y^2)^2(-2xy) - (y^3 - x^2y)(4x)(x^2+y^2)}{(x^2+y^2)^4}_{|(-1,1)} = \frac{1}{2}, \\ f_{xy}(-1,1) &= \frac{(x^2+y^2)^2(3y^2 - x^2) - (y^3 - x^2y)(4y)(x^2+y^2)}{(x^2+y^2)^4}_{|(-1,1)} = \frac{1}{2}, \\ f_{yy}(-1,1) &= \frac{(x^2+y^2)^2(-2xy) - (x^3 - xy^2)(4y)(x^2+y^2)}{(x^2+y^2)^4}_{|(-1,1)} = \frac{1}{2}, \end{aligned}$$

$$\frac{xy}{x^2+y^2} = -\frac{1}{2} + \frac{1}{2!}\left[\frac{1}{2}(x+1)^2 + (x+1)(y-1) + \frac{1}{2}(y-1)^2\right] + \cdots.$$

10. With $f(2,1)=\sqrt{1+2}=\sqrt{3}$,

$$f_x(2,1)=\frac{y}{2\sqrt{1+xy}}\bigg|_{(2,1)}=\frac{1}{2\sqrt{3}},$$

$$f_y(2,1)=\frac{x}{2\sqrt{1+xy}}\bigg|_{(2,1)}=\frac{1}{\sqrt{3}},\qquad f_{xx}(2,1)=\frac{-y^2}{4(1+xy)^{3/2}}\bigg|_{(2,1)}=-\frac{1}{12\sqrt{3}},$$

$$f_{xy}(2,1)=\left[\frac{1}{2\sqrt{1+xy}}-\frac{xy}{4(1+xy)^{3/2}}\right]_{|(2,1)}=\frac{1}{3\sqrt{3}},\qquad f_{yy}(2,1)=\frac{-x^2}{4(1+xy)^{3/2}}\bigg|_{(2,1)}=-\frac{1}{3\sqrt{3}},$$

$$\begin{aligned}\sqrt{1+xy}&=\sqrt{3}+\frac{1}{2\sqrt{3}}(x-2)+\frac{1}{\sqrt{3}}(y-1)\\&\quad+\frac{1}{2!}\left[-\frac{1}{12\sqrt{3}}(x-2)^2+\frac{2}{3\sqrt{3}}(x-2)(y-1)-\frac{1}{3\sqrt{3}}(y-1)^2\right]+\cdots\\&=\frac{1}{24\sqrt{3}}\left[72+12(x-2)+24(y-1)-(x-2)^2+8(x-2)(y-1)-4(y-1)^2\right]+\cdots .\end{aligned}$$

11. With $f(-1,0)=-e^{-1}\sin 3$,

$$\begin{aligned}f_x(-1,0)&=[e^x\sin(3x-y)+3e^x\cos(3x-y)]_{|(-1,0)}=e^{-1}(3\cos 3-\sin 3),\\f_y(-1,0)&=[-e^x\cos(3x-y)]_{|(-1,0)}=-e^{-1}\cos 3,\\f_{xx}(-1,0)&=[6e^x\cos(3x-y)-8e^x\sin(3x-y)]_{|(-1,0)}=e^{-1}(8\sin 3+6\cos 3),\\f_{xy}(-1,0)&=[3e^x\sin(3x-y)-3e^x\cos(3x-y)]_{|(-1,0)}=-3e^{-1}(\cos 3+\sin 3),\\f_{yy}(-1,0)&=[-e^x\sin(3x-y)]_{|(-1,0)}=e^{-1}\sin 3,\end{aligned}$$

$$\begin{aligned}e^x\sin(3x-y)&=-\frac{\sin 3}{e}+\frac{(3\cos 3-\sin 3)}{e}(x+1)-\frac{y\cos 3}{e}\\&\quad+\frac{1}{2!}\left[\frac{8\sin 3+6\cos 3}{e}(x+1)^2-\frac{6(\cos 3+\sin 3)}{e}(x+1)y+\frac{\sin 3}{e}y^2\right]+\cdots .\end{aligned}$$

12. With $f(0,1)=0$,

$$\begin{aligned}f_x(0,1)&=[2(x+y)\ln(x+y)+x+y]_{|(0,1)}=1,\\f_y(0,1)&=[2(x+y)\ln(x+y)+x+y]_{|(0,1)}=1,\\f_{xx}(0,1)&=[2\ln(x+y)+2+1]_{|(0,1)}=3,\\f_{xy}(0,1)&=[2\ln(x+y)+2+1]_{|(0,1)}=3,\\f_{yy}(0,1)&=[2\ln(x+y)+2+1]_{|(0,1)}=3,\end{aligned}$$

$$\begin{aligned}(x+y)^2\ln(x+y)&=x+(y-1)+\frac{1}{2!}[3x^2+6x(y-1)+3(y-1)^2]+\cdots\\&=\frac{1}{2}[2x+2(y-1)+3x^2+6x(y-1)+3(y-1)^2]+\cdots .\end{aligned}$$

13. With $f(1,-1)=\dfrac{\pi}{4}$,

$$\begin{aligned}f_x(1,-1)&=\frac{3}{1+(3x+2y)^2}\bigg|_{(1,-1)}=\frac{3}{2},\\f_y(1,-1)&=\frac{2}{1+(3x+2y)^2}\bigg|_{(1,-1)}=1,\\f_{xx}(1,-1)&=\frac{-18(3x+2y)}{[1+(3x+2y)^2]^2}\bigg|_{(1,-1)}=-\frac{9}{2},\\f_{xy}(1,-1)&=\frac{-12(3x+2y)}{[1+(3x+2y)^2]^2}\bigg|_{(1,-1)}=-3,\end{aligned}$$

$$f_{yy}(1,-1) = \frac{-8(3x+2y)}{[1+(3x+2y)^2]^2}_{|(1,-1)} = -2,$$

$$\operatorname{Tan}^{-1}(3x+2y) = \frac{\pi}{4} + \frac{3}{2}(x-1) + (y+1) + \frac{1}{2!}\left[-\frac{9}{2}(x-1)^2 - 6(x-1)(y+1) - 2(y+1)^2\right] + \cdots .$$

14. The first six terms vanish since the function is its own Taylor series about $(0,0)$.

15. $f(x,y,z) = f(c,d,e) + [f_x(c,d,e)(x-c) + f_y(c,d,e)(y-d) + f_z(c,d,e)(z-e)]$

$$+\frac{1}{2!}[f_{xx}(c,d,e)(x-c)^2 + f_{yy}(c,d,e)(y-d)^2 + f_{zz}(c,d,e)(z-e)^2$$
$$+2f_{xy}(c,d,e)(x-c)(y-d) + 2f_{xz}(c,d,e)(x-c)(z-e) + 2f_{yz}(c,d,e)(y-d)(z-e)] + \cdots .$$

16. If the operator is defined as

$$\left[(x-c)\frac{\partial}{\partial x} + (y-d)\frac{\partial}{\partial y}\right]^n f(c,d) = \sum_{r=0}^{n}\binom{n}{r}(x-c)^{n-r}(y-d)^r \frac{\partial^n f(x,y)}{\partial x^{n-r}\partial y^r}_{|(c,d)},$$

then equation 12.76 becomes

$$f(x,y) = \sum_{n=0}^{\infty}\frac{1}{n!}\left[(x-c)\frac{\partial}{\partial x} + (y-d)\frac{\partial}{\partial y}\right]^n f(c,d).$$

REVIEW EXERCISES

1. $\dfrac{\partial f}{\partial x} = \dfrac{2x}{y^3} - \dfrac{y}{\sqrt{1-x^2y^2}}$

2. From $\dfrac{\partial f}{\partial y} = \dfrac{2y}{x^2+y^2+z^2}$, we obtain $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{(x^2+y^2+z^2)(2) - 2y(2y)}{(x^2+y^2+z^2)^2} = \dfrac{2(x^2-y^2+z^2)}{(x^2+y^2+z^2)^2}$.

3. From $\partial f/\partial y = x^3e^y - \cos(x+y+z+t)$, we obtain $\partial^2 f/\partial x\partial y = 3x^2e^y + \sin(x+y+z+t)$, and $\partial^3 f/\partial x^2\partial y = 6xe^y + \cos(x+y+z+t)$.

4. If we set $F(x,y,z) = z^2x + \operatorname{Tan}^{-1}z + y - 3x$, then

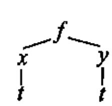
$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(z)}} = -\frac{F_x}{F_z} = -\frac{z^2-3}{2xz + \dfrac{1}{1+z^2}} = \frac{(3-z^2)(1+z^2)}{1+2xz+2xz^3}.$$

5. If we set $F(x,y,z,u) = u\cos y + y\cos(xu) + z^2 - 5x$, then

$$\frac{\partial u}{\partial y} = -\frac{\dfrac{\partial(F)}{\partial(y)}}{\dfrac{\partial(F)}{\partial(u)}} = -\frac{F_y}{F_u} = -\frac{-u\sin y + \cos(xu)}{\cos y - xy\sin(xu)} = \frac{u\sin y - \cos(xu)}{\cos y - xy\sin(xu)}.$$

6. $\dfrac{df}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$

$= (2x - ye^{xy})(3t^2+3) + (2y - xe^{xy})(\ln t + 1)$



7. If we set $F(x,y) = x - y^3 - 3y^2 + 2y - 4$, then

$$\frac{dy}{dx} = -\frac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(y)}} = -\frac{F_x}{F_y} = -\frac{1}{-3y^2-6y+2} = \frac{1}{3y^2+6y-2}.$$

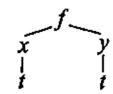
8. If we set $F(x,y,u,v)=u^2+v^2-xy-5$ and $G(x,u,v)=3u-2v+x^2u-2v^3$, then

$$\frac{\partial u}{\partial x}=-\frac{\dfrac{\partial(F,G)}{\partial(x,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}}=-\frac{\begin{vmatrix}F_x & F_v\\ G_x & G_v\end{vmatrix}}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}=-\frac{\begin{vmatrix}-y & 2v\\ 2xu & -2-6v^2\end{vmatrix}}{\begin{vmatrix}2u & 2v\\ 3+x^2 & -2-6v^2\end{vmatrix}}$$

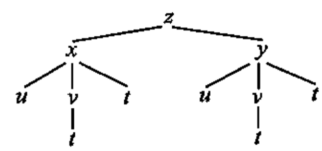
$$=-\frac{y(2+6v^2)-4xuv}{-2u(2+6v^2)-2v(3+x^2)}=\frac{y(1+3v^2)-2xuv}{u(2+6v^2)+v(3+x^2)}.$$

9. From $\dfrac{\partial f}{\partial v}=-\dfrac{u^2}{2v^{3/2}}-\dfrac{1}{\sqrt{u}}$, we obtain $\dfrac{\partial^2 f}{\partial u\partial v}=-\dfrac{u}{v^{3/2}}+\dfrac{1}{2u^{3/2}}$.

10.
$$\frac{df}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$$
$$=(y-2x)(te^t+e^t)+(x-2y)(-te^{-t}+e^{-t})$$
$$=e^t(t+1)(y-2x)+e^{-t}(1-t)(x-2y)$$



11.
$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}\frac{dv}{dt}+\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\frac{dv}{dt}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$
$$=2x(-6v+3ut)(2t-2)+2x(3uv)+(-2y)[-ut\sin(vt)](2t-2)$$
$$+(-2y)[-uv\sin(vt)]$$
$$=6x[uv+2(t-1)(ut-2v)]+2uy\sin(vt)[2t(t-1)+v]$$



12. If we set $F(x,r,\theta)=r\cos\theta-x$ and $G(y,r,\theta)=r\sin\theta-y$, then

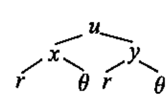
$$\frac{\partial r}{\partial x}=-\frac{\dfrac{\partial(F,G)}{\partial(x,\theta)}}{\dfrac{\partial(F,G)}{\partial(r,\theta)}}=-\frac{\begin{vmatrix}F_x & F_\theta\\ G_x & G_\theta\end{vmatrix}}{\begin{vmatrix}F_r & F_\theta\\ G_r & G_\theta\end{vmatrix}}=-\frac{\begin{vmatrix}-1 & -r\sin\theta\\ 0 & r\cos\theta\end{vmatrix}}{\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}}=\frac{r\cos\theta}{r}=\cos\theta.$$

13. If we set $F(x,r,\phi,\theta)=r\sin\phi\cos\theta-x$, $G(y,r,\phi,\theta)=r\sin\phi\sin\theta-y$ and $H(z,r,\phi)=r\cos\phi-z$, then

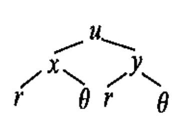
$$\frac{\partial\theta}{\partial x}=-\frac{\dfrac{\partial(F,G,H)}{\partial(r,\phi,x)}}{\dfrac{\partial(F,G,H)}{\partial(r,\phi,\theta)}}=-\frac{\begin{vmatrix}F_r & F_\phi & F_x\\ G_r & G_\phi & G_x\\ H_r & H_\phi & H_x\end{vmatrix}}{\begin{vmatrix}F_r & F_\phi & F_\theta\\ G_r & G_\phi & G_\theta\\ H_r & H_\phi & H_\theta\end{vmatrix}}=-\frac{\begin{vmatrix}\sin\phi\cos\theta & r\cos\phi\cos\theta & -1\\ \sin\phi\sin\theta & r\cos\phi\sin\theta & 0\\ \cos\phi & -r\sin\phi & 0\end{vmatrix}}{\begin{vmatrix}\sin\phi\cos\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta\\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta\\ \cos\phi & -r\sin\phi & 0\end{vmatrix}}$$

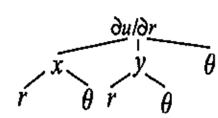
$$=-\frac{r\sin\theta}{\cos\phi(r^2\sin\phi\cos\phi)+r\sin\phi(r\sin^2\phi)}=-\frac{r\sin\theta}{r^2\sin\phi}=-\frac{\sin\theta}{r\sin\phi}.$$

14.
$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$
$$=(2x-3x^2y^2)\cos\theta+(-2yx^3)\sin\theta$$



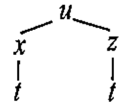
15.
$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$
$$=(2x-3x^2y^2)\cos\theta+(-2x^3y)\sin\theta$$

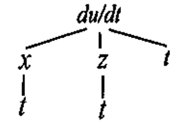




$$\frac{\partial^2 u}{\partial r^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right)\frac{\partial x}{\partial r}+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial r}\right)\frac{\partial y}{\partial r}$$
$$=[(2-6xy^2)\cos\theta-6x^2y\sin\theta]\cos\theta+(-6x^2y\cos\theta-2x^3\sin\theta)\sin\theta$$
$$=2(1-3xy^2)\cos^2\theta-12x^2y\sin\theta\cos\theta-2x^3\sin^2\theta$$

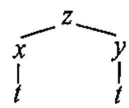
16. $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial z}\dfrac{dz}{dt} = \left(\dfrac{1}{z^2} + \dfrac{2z}{x^3}\right)(3t^2) + \left(-\dfrac{2x}{z^3} - \dfrac{1}{x^2}\right)\left(-\dfrac{3}{t^4}\right)$





$$\frac{d^2u}{dt^2} = \frac{\partial}{\partial x}\left(\frac{du}{dt}\right)\frac{dx}{dt} + \frac{\partial}{\partial z}\left(\frac{du}{dt}\right)\frac{dz}{dt} + \frac{\partial}{\partial t}\left(\frac{du}{dt}\right)$$
$$= \left(-\frac{18zt^2}{x^4} + \frac{6}{z^3t^4} - \frac{6}{x^3t^4}\right)(3t^2) + \left(-\frac{6t^2}{z^3} + \frac{6t^2}{x^3} - \frac{18x}{z^4t^4}\right)\left(-\frac{3}{t^4}\right)$$
$$+ \left(\frac{1}{z^2} + \frac{2z}{x^3}\right)(6t) + \left(\frac{2x}{z^3} + \frac{1}{x^2}\right)\left(-\frac{12}{t^5}\right)$$
$$= \frac{6}{x^4t^8z^4}(-9z^5t^{12} + 6zt^6x^4 - 6xt^6z^4 + 9x^5 + t^9z^2x^4 + 2z^5xt^9 - 4x^5zt^3 - 2x^2t^3z^4)$$

17. $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$



$$= (1 - y^2)\frac{dx}{dt} + (1 - 2xy)\frac{dy}{dt}$$

If we set $F(x,y,t) = x^2 - y^2 + xt - 2t$ and $G(x,y,t) = xy - 4t^2$, then

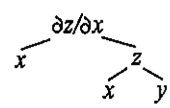
$$\frac{dx}{dt} = -\frac{\dfrac{\partial(F,G)}{\partial(t,y)}}{\dfrac{\partial(F,G)}{\partial(x,y)}} = -\frac{\begin{vmatrix} F_t & F_y \\ G_t & G_y \end{vmatrix}}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}} = -\frac{\begin{vmatrix} x-2 & -2y \\ -8t & x \end{vmatrix}}{\begin{vmatrix} 2x+t & -2y \\ y & x \end{vmatrix}} = \frac{2x + 16ty - x^2}{2x^2 + xt + 2y^2},$$

$$\frac{dy}{dt} = -\frac{\dfrac{\partial(F,G)}{\partial(x,t)}}{\dfrac{\partial(F,G)}{\partial(x,y)}} = -\frac{\begin{vmatrix} F_x & F_t \\ G_x & G_t \end{vmatrix}}{2x^2 + xt + 2y^2} = -\frac{\begin{vmatrix} 2x+t & x-2 \\ y & -8t \end{vmatrix}}{2x^2 + xt + 2y^2} = \frac{16xt + 8t^2 + xy - 2y}{2x^2 + xt + 2y^2},$$

Thus, $\dfrac{dz}{dt} = (1 - y^2)\left(\dfrac{2x + 16ty - x^2}{2x^2 + xt + 2y^2}\right) + (1 - 2xy)\left(\dfrac{16xt + 8t^2 + xy - 2y}{2x^2 + xt + 2y^2}\right)$.

18. If we set $F(x,y,z) = xz - x^2z^3 + y^2 - 3$, then $\dfrac{\partial z}{\partial x} = -\dfrac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(z)}} = -\dfrac{F_x}{F_z} = -\dfrac{z - 2xz^3}{x - 3x^2z^2} = \dfrac{2xz^3 - z}{x - 3x^2z^2}$.

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial z}{\partial x}\right)\frac{\partial z}{\partial x}$$
$$= \left[\frac{(x - 3x^2z^2)(2z^3) - (2xz^3 - z)(1 - 6xz^2)}{(x - 3x^2z^2)^2}\right]$$

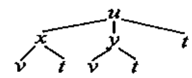


$$+ \left[\frac{(x - 3x^2z^2)(6xz^2 - 1) - (2xz^3 - z)(-6x^2z)}{(x - 3x^2z^2)^2}\right]\left(\frac{2xz^3 - z}{x - 3x^2z^2}\right)$$
$$= \frac{1}{(x - 3x^2z^2)^3}[x(1 - 3xz^2)(2xz^3 - 6x^2z^5 - 2xz^3 + z + 12x^2z^5 - 6xz^3)$$
$$+ z(2xz^2 - 1)(6x^2z^2 - x - 18x^3z^4 + 3x^2z^2 + 12x^3z^4 - 6x^2z^2)]$$
$$= \frac{xz[(1 - 3xz^2)(1 + 6x^2z^4 - 6xz^2) + (1 - 2xz^2)(1 + 6x^2z^4 - 3xz^2)]}{(x - 3x^2z^2)^3}$$

19. If we set $F(x,y,z) = yx - x^2z^2 + 5x - 3$ and $G(x,y,z) = 2xz - 3x^2y^2 - 4z^4$, then

$$\frac{dy}{dx} = -\frac{\dfrac{\partial(F,G)}{\partial(x,z)}}{\dfrac{\partial(F,G)}{\partial(y,z)}} = -\frac{\begin{vmatrix} F_x & F_z \\ G_x & G_z \end{vmatrix}}{\begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}} = -\frac{\begin{vmatrix} y - 2xz^2 + 5 & -2x^2z \\ 2z - 6xy^2 & 2x - 16z^3 \end{vmatrix}}{\begin{vmatrix} x & -2x^2z \\ -6x^2y & 2x - 16z^3 \end{vmatrix}}$$
$$= -\frac{(2x - 16z^3)(y - 2xz^2 + 5) + 2x^2z(2z - 6xy^2)}{x(2x - 16z^3) - 12x^4yz} = \frac{6x^3y^2z - xy - 5x + 8yz^3 - 16xz^5 + 40z^3}{x^2 - 8xz^3 - 6x^4yz}.$$

20. $$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial t}\Big)_{x,y}$$



$$= \left(yt^2 - \frac{3y}{\sqrt{1-x^2y^2}}\right)(2v^2t-2)$$
$$+ \left(xt^2 - \frac{3x}{\sqrt{1-x^2y^2}}\right)(v\sec^2 t) + 2xyt$$
$$= 2y(v^2t-1)\left(t^2 - \frac{3}{\sqrt{1-x^2y^2}}\right) + xv\sec^2 t\left(t^2 - \frac{3}{\sqrt{1-x^2y^2}}\right) + 2xyt$$

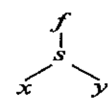
21. This follows from Theorem 12.3 since the function is homogeneous of degree 2.

22. With $\dfrac{\partial u}{\partial x} = 4x + y$, $\dfrac{\partial^2 u}{\partial x^2} = 4$, $\dfrac{\partial^2 u}{\partial x\partial y} = 1$, $\dfrac{\partial u}{\partial y} = -6y + x$, $\dfrac{\partial^2 u}{\partial y^2} = -6$,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 4x^2 + 2xy - 6y^2 = 2u.$$

23. If we set $s = 3x - 2y$, then

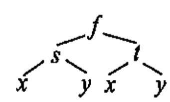
$$\frac{\partial f}{\partial x} = \frac{df}{ds}\frac{\partial s}{\partial x} = 3f'(s), \quad \frac{\partial f}{\partial y} = \frac{df}{ds}\frac{\partial s}{\partial y} = -2f'(s).$$



Thus, $2\dfrac{\partial f}{dx} + 3\dfrac{\partial f}{\partial y} = 6f'(s) - 6f'(s) = 0.$

24. If we set $s = x^2 - y^2$ and $t = y^2 - x^2$, then

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial x} = \frac{\partial f}{\partial s}(2x) + \frac{\partial f}{\partial t}(-2x)$$
$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial y} = \frac{\partial f}{\partial s}(-2y) + \frac{\partial f}{\partial t}(2y).$$



Consequently,

$$y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = y\left(2x\frac{\partial f}{\partial s} - 2x\frac{\partial f}{\partial t}\right) + x\left(-2y\frac{\partial f}{\partial s} + 2y\frac{\partial f}{\partial t}\right) = 0.$$

25. $D_{\mathbf{v}}f = \nabla f_{|(3,-1)} \cdot \hat{\mathbf{v}} = (2x\sin y, x^2\cos y)_{|3,-1)} \cdot \dfrac{(2,4)}{2\sqrt{5}} = (-6\sin 1, 9\cos 1)\cdot\dfrac{(1,2)}{\sqrt{5}} = \dfrac{18\cos 1 - 6\sin 1}{\sqrt{5}}.$

26. With $\nabla f_{|(1,0,1)} = (2x, 2y, 2z)_{|(1,0,1)} = (2,0,2)$, and a unit vector $\hat{\mathbf{v}} = (1,-1,2)/\sqrt{1+1+4}$ in the direction from $(1,0,1)$ to $(2,-1,3)$, $D_{\hat{\mathbf{v}}}f = (2,0,2)\cdot\dfrac{(1,-1,2)}{\sqrt{6}} = \sqrt{6}.$

27. Since $\nabla f_{|(-1,2,5)} = \left(\dfrac{z}{1+(x+y)^2}, \dfrac{z}{1+(x+y)^2}, \mathrm{Tan}^{-1}(x+y)\right)_{|(-1,2,5)} = \left(\dfrac{5}{2}, \dfrac{5}{2}, \dfrac{\pi}{4}\right)$, and a vector perpendicular to the surface is $\mathbf{n} = \nabla(z - x^2 - y^2)_{|(-1,2,5)} = (-2x, -2y, 1)_{|(-1,2,5)} = (2,-4,1)$, the directional derivative is

$$D_{\mathbf{n}}f = \left(\frac{5}{2}, \frac{5}{2}, \frac{\pi}{4}\right)\cdot\hat{\mathbf{n}} = \left(\frac{5}{2}, \frac{5}{2}, \frac{\pi}{4}\right)\cdot\frac{(2,-4,1)}{\sqrt{21}} = \frac{\pi/4 - 5}{\sqrt{21}}.$$

28. With parametric equations, $x = 2t - 1$, $y = -t$, $z = 5 - 3t$ for the line, a vector along the line is $\mathbf{T} = (2,-1,-3)$. Since $\nabla f_{|(1,-1,2)} = (2x, 1, -2)_{|(1,-1,2)} = (2,1,-2)$,

$$D_{\mathbf{T}}f = (2,1,-2)\cdot\frac{(2,-1,-3)}{\sqrt{4+1+9}} = \frac{9}{\sqrt{14}}.$$

29. Since the slope of the curve at $(3,10)$ is 6, a tangent vector to the curve at the point is $\mathbf{T} = (-1,-6)$. The directional derivative is

$$D_{\mathbf{T}}f = \nabla f_{|(3,10)} \cdot \hat{\mathbf{T}} = \left(\frac{1}{x+y}, \frac{1}{x+y}\right)_{|(3,10)} \cdot \frac{(-1,-6)}{\sqrt{37}} = \left(\frac{1}{13}, \frac{1}{13}\right) \cdot \frac{(-1,-6)}{\sqrt{37}} = -\frac{7}{13\sqrt{37}}.$$

30. A vector tangent to the curve at $(0,1,1)$ is

$$\mathbf{T} = \nabla(x^2+y^2+z^2-2)_{|(0,1,1)} \times \nabla(z-y)_{|(0,1,1)} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 2 & 2 \\ 0 & -1 & 1 \end{vmatrix} = (4,0,0).$$

Since $\nabla f_{|(0,1,1)} = (2yz-2x, 2xz, 2xy-2z)_{|(0,1,1)} = (2,0,-2)$, $D_{\mathbf{T}}f = (2,0,-2)\cdot(1,0,0) = 2$.

31. Since a vector normal to the plane is $\nabla(x^2+y^2-z)_{|(1,3,10)} = (2x,2y,-1)_{|(1,3,10)} = (2,6,-1)$, its equation is $0 = (2,6,-1)\cdot(x-1,y-3,z-10) = 2x+6y-z-10$.

32. Since a vector normal to the plane is $\nabla(x^2-y^2+z^3)_{|(-1,3,2)} = (2x,-2y,3z^2)_{|(-1,3,2)} = (-2,-6,12)$, its equation is $0 = (1,3,-6)\cdot(x+1,y-3,z-2) = x+3y-6z+4$.

33. Since a vector normal to the plane is $\nabla(x^2+y^2-z^2-1)_{|(1,0,0)} = (2x,2y,-2z)_{|(1,0,0)} = (2,0,0)$, its equation is $0 = (1,0,0)\cdot(x-1,y-0,z-0) = x-1$.

34. Since a tangent vector to the curve at $(2,0,6)$ is $\mathbf{T} = \dfrac{d\mathbf{r}}{dt}_{|t=1} = (2t,2t,3t^2+5)_{|t=1} = (2,2,8)$, parametric equations for the tangent line are $x = 2+u$, $y = u$, $z = 6+4u$.

35. Since the curve is a straight line, the tangent line to the curve is the curve itself.

36. A tangent vector to the curve at $(1,1,1)$ is

$$\begin{aligned}&\nabla(xy-z)_{|(1,1,1)} \times \nabla(x^2+y^2-2)_{|(1,1,1)} \\ &\quad = (y,x,-1)_{|(1,1,1)} \times (2x,2y,0)_{|(1,1,1)} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & -1 \\ 2 & 2 & 0 \end{vmatrix} = (2,-2,0).\end{aligned}$$

Parametric equations for the tangent line are $x = 1+t$, $y = 1-t$, $z = 1$.

37. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = 3x^2-6$, $0 = \dfrac{\partial f}{\partial y} = 6y$. Solutions are $(\pm\sqrt{2},0)$. Since $f_{xx} = 6x$, $f_{xy} = 0$, and $f_{yy} = 6$, we obtain $(f_{xy})^2 - f_{xx}f_{yy} = -36x$. At $(\sqrt{2},0)$, $B^2-AC = -36\sqrt{2}$ and $A = 6\sqrt{2}$, so that this critical point gives a relative minimum. At $(-\sqrt{2},0)$, $B^2-AC = 36\sqrt{2}$, so that this critical point yields a saddle point.

38. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = ye^x$, $0 = \dfrac{\partial f}{\partial y} = e^x$. There are no solutions to these equations.

39. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = 2x-y+1$, $0 = \dfrac{\partial f}{\partial y} = -x+2y-4$. The only solution is $(2/3,7/3)$. Since $f_{xx} = 2$, $f_{xy} = -1$, and $f_{yy} = 2$, it follows that $B^2-AC = 1-4$. Since $A = 2$, the critical point gives a relative minimum.

40. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = 4x(x^2+y^2-1)$, $0 = \dfrac{\partial f}{\partial y} = 4y(x^2+y^2-1)$. The solutions are $(0,0)$ and every point on the circle $x^2+y^2=1$. Since $f(x,y) = 0$ for every point on $x^2+y^2=1$, but is otherwise positive, each of these critical points yields a relative minimum.

$$\frac{\partial^2 f}{\partial x^2} = 4(x^2+y^2-1)+8x^2, \qquad \frac{\partial^2 f}{\partial x\partial y} = 8xy, \qquad \frac{\partial^2 f}{\partial y^2} = 4(x^2+y^2-1)+8y^2.$$

At $(0,0)$, $B^2-AC = 0-(-4)(-4) < 0$, and $A = -4$, and therefore $(0,0)$ gives a relative maximum.

41. Since $f(x,y) = (x^2+y^2)\left(\dfrac{x^3}{y} - \dfrac{y^3}{x}\right) = \dfrac{x^5}{y} - xy^3 + x^3y - \dfrac{y^5}{x}$,

$$\frac{\partial f}{\partial x} = \frac{5x^4}{y} - y^3 + 3x^2y + \frac{y^5}{x^2}, \qquad \frac{\partial f}{\partial y} = -\frac{x^5}{y^2} - 3xy^2 + x^3 - \frac{5y^4}{x},$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{20x^3}{y} + 6xy - \frac{2y^5}{x^3}, \qquad \frac{\partial^2 f}{\partial y^2} = \frac{2x^5}{y^3} - 6xy - \frac{20y^3}{x}.$$

Thus, $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = \dfrac{20x^3}{y} - \dfrac{2y^5}{x^3} + \dfrac{2x^5}{y^3} - \dfrac{20y^3}{x}$. On the other hand,

$$\frac{\partial F}{\partial x} = \frac{3x^2}{y} + \frac{y^3}{x^2}, \quad \frac{\partial F}{\partial y} = -\frac{x^3}{y^2} - \frac{3y^2}{x}, \quad \frac{\partial^2 F}{\partial x^2} = \frac{6x}{y} - \frac{2y^3}{x^3}, \quad \frac{\partial^2 F}{\partial y^2} = \frac{2x^3}{y^3} - \frac{6y}{x}.$$

Hence,

$$\begin{aligned}(x^2+y^2)\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right) + 12F(x,y) &= (x^2+y^2)\left(\frac{6x}{y} - \frac{2y^3}{x^3} + \frac{2x^3}{y^3} - \frac{6y}{x}\right) + 12\left(\frac{x^3}{y} - \frac{y^3}{x}\right)\\ &= \frac{20x^3}{y} - \frac{2y^5}{x^3} + \frac{2x^5}{y^3} - \frac{20y^3}{x}.\end{aligned}$$

42. For critical points of $f(x,y)$ we solve $0 = \dfrac{\partial f}{\partial x} = y,\ 0 = \dfrac{\partial f}{\partial y} = x$. At the critical point $(0,0)$, $f(0,0) = \boxed{0}$. On the boundary of the region we set $x = \cos t$, $y = \sin t$, in which case $f(x,y)$ becomes

$$z(t) = \cos t \sin t = \frac{1}{2}\sin 2t, \quad 0 \le t \le 2\pi.$$

For critical points of $z(t)$, we set $0 = dz/dt = \cos 2t$. At the critical points $t = \pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$,

$$z(\pi/4) = z(5\pi/4) = \boxed{1/2}, \qquad z(3\pi/4) = z(7\pi/4) = \boxed{-1/2}.$$

Finally, $z(0) = z(2\pi) = \boxed{0}$. Maximum and minimum values are therefore $\pm 1/2$.

43. The function has no critical points inside the sphere so that its maximum and minimum values must occur on the surface of the sphere. To find them we define the Lagrangian $F(x,y,z,\lambda) = 2x + 3y - 4z + \lambda(x^2+y^2+z^2-2)$. For critical points, we solve

$$0 = \frac{\partial F}{\partial x} = 2 + 2\lambda x, \ 0 = \frac{\partial F}{\partial y} = 3 + 2\lambda y, \ 0 = \frac{\partial F}{\partial z} = -4 + 2\lambda z, \ 0 = \frac{\partial F}{\partial \lambda} = x^2+y^2+z^2-2.$$

Solutions are $(\pm\sqrt{8/29}, \pm\sqrt{18/29}, \mp\sqrt{32/29})$. Since the sphere is a closed surface, we need only evaluate $f(x,y,z)$ at these critical points, $f(\pm\sqrt{8/29}, \pm\sqrt{18/29}, \mp\sqrt{32/29}) = \pm\sqrt{58}$. These are maximum and minimum values of $f(x,y,z)$ for the sphere.

44. The distance D from the origin to any point $P(x,y)$ on the curve is given by

$$D^2 = x^2 + y^2 = x^2 + (1 - x^2 - x^4) = 1 - x^4.$$

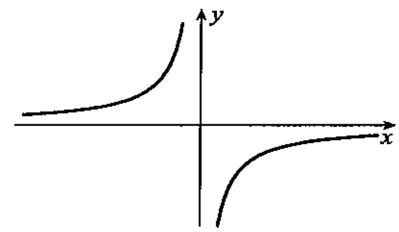
This function must be maximized and minimized on the interval $|x| \le \sqrt{(\sqrt{5}-1)/2}$. Since the only critical point of D^2 is $x = 0$, we evaluate

$$D^2(0) = 1, \qquad D^2\left(\pm\sqrt{(\sqrt{5}-1)/2}\right) = (\sqrt{5}-1)/2.$$

The closest and farthest points are therefore $(\pm\sqrt{(\sqrt{5}-1)/2}, 0)$ and $(0, \pm 1)$ respectively.

45. The distance d from the origin to a point (x, y, z) on the surface is given by $d^2 = x^2 + y^2 + z^2 = x^2 + y^2 + 1 + xy$. Since z^2 must be positive, we consider this function for points between the branches of the hyperbola $y = -1/x$. For critical points of d^2, we solve

$$0 = 2x + y, \qquad 0 = 2y + x.$$



The only solution is $(0, 0)$ at which $d^2 = 1$. Along the right branch of the hyperbola, we can write that

$$d^2 = f(x) = x^2 + \frac{1}{x^2} + 1 - 1 = x^2 + \frac{1}{x^2}, \quad 0 < x < \infty.$$

Critical points of this function are defined by $0 = f'(x) = 2x - 2/x^3$ with solution $x = 1$, at which $f(1) = 2$. A similar result is found on the left branch of the hyperbola. Points on the surface closest to the origin are therefore $(0, 0, \pm 1)$.

46. If the farmer plants x hectares of corn, y hectares of potatoes, and z hectares of sunflowers, his losses are

$$L = pax^2 + qby^2 + rcz^2.$$

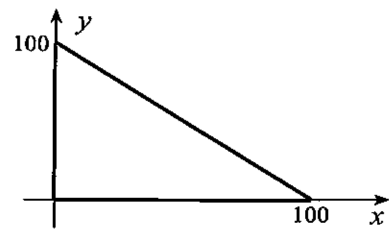
Since $x + y + z = 100$,

$$L = pax^2 + qby^2 + rc(100 - x - y)^2.$$

This function must be minimized for (x, y) in the triangle shown. For critical points of L, we solve

$$0 = \frac{\partial L}{\partial x} = 2apx - 2rc(100 - x - y),$$

$$0 = \frac{\partial L}{\partial y} = 2bqy - 2rc(100 - x - y).$$



The solution of these equations is $x_0 = \dfrac{100bcqr}{acpr + abpq + bcqr}$, $y_0 = \dfrac{100acpr}{acpr + abpq + bcqr}$.

When $x = 0$,

$$L = qby^2 + rc(100 - y)^2, \quad 0 \le y \le 100.$$

For critical points of this function, $0 = \dfrac{dL}{dy} = 2qby - 2rc(100 - y)$. The solution is $y_1 = 100rc/(cr + bq)$.

Similarly, when $y = 0$, we obtain the critical point $x_1 = 100rc/(cr + ap)$ of

$$L = pax^2 + rc(100 - x)^2, \quad 0 \le x \le 100.$$

When $x + y = 100$,

$$L = pax^2 + qb(100 - x)^2, \quad 0 \le x \le 100.$$

The critical point of this function is $x_2 = 100bq/(bq + ap)$.

We now evaluate L at each of these critical points and the corners of the triangle:

$$L(x_0, y_0) = \frac{10^4 abcpqr}{acpr + abpq + bcqr}, \quad L(y_1) = \frac{10^4 bcqr}{cr + bq}, \quad L(x_1) = \frac{10^4 acpr}{cr + ap}, \quad L(x_2) = \frac{10^4 abpq}{ap + bq},$$

$$L(0, 100) = 10^4 bq, \quad L(100, 0) = 10^4 ap, \quad L(0, 0) = 10^4 cr.$$

Notice now that:

$$\frac{1}{L(x_0, y_0)} = 10^{-4}\left(\frac{1}{bq} + \frac{1}{cr} + \frac{1}{ap}\right),$$

$$\frac{1}{L(x_1)} = 10^{-4}\left(\frac{1}{ap} + \frac{1}{cr}\right), \ \frac{1}{L(y_1)} = 10^{-4}\left(\frac{1}{bq} + \frac{1}{cr}\right), \ \frac{1}{L(x_2)} = 10^{-4}\left(\frac{1}{bq} + \frac{1}{ap}\right),$$

$$\frac{1}{L(0,100)} = \frac{10^{-4}}{bq}, \quad \frac{1}{L(100,0)} = \frac{10^{-4}}{ap}, \quad \frac{1}{L(0,0)} = \frac{10^{-4}}{cr}.$$

It follows that $1/L(x_0, y_0)$ is the largest of these numbers, and therefore $L(x_0, y_0)$ is the smallest value for L. Thus L is minimized when $x = x_0$, $y = y_0$, and

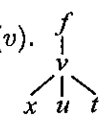
$$z = 100 - x_0 - y_0 = \frac{100abpq}{acpr + abpq + bcqr}.$$

47. If we set $F(x, t, u) = u - f(x - ut)$, then

$$\frac{\partial u}{\partial t} = -\frac{\dfrac{\partial(F)}{\partial(t)}}{\dfrac{\partial(F)}{\partial(u)}} = -\frac{\left.\dfrac{\partial F}{\partial t}\right)_{u,x}}{\left.\dfrac{\partial F}{\partial u}\right)_{x,t}} = -\frac{\left.\dfrac{\partial f}{\partial t}\right)_{u,x}}{1 - \left.\dfrac{\partial f}{\partial u}\right)_{x,t}}, \quad \frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(u)}} = -\frac{\left.\dfrac{\partial F}{\partial x}\right)_{u,t}}{\left.\dfrac{\partial F}{\partial u}\right)_{x,t}} = -\frac{\left.\dfrac{\partial f}{\partial x}\right)_{u,t}}{1 - \left.\dfrac{\partial f}{\partial u}\right)_{x,t}}.$$

If we set $v = x - ut$, then

$$\frac{\partial f}{\partial t} = \frac{df}{dv}\frac{\partial v}{\partial t} = -uf'(v), \quad \frac{\partial f}{\partial x} = \frac{df}{dv}\frac{\partial v}{\partial x} = f'(v), \quad \frac{\partial f}{\partial u} = \frac{df}{dv}\frac{\partial v}{\partial u} = -tf'(v).$$



Consequently, $\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = -\dfrac{-uf'(v)}{1 + tf'(v)} + u\dfrac{-f'(v)}{1 + tf'(v)} = 0.$

48. With $f(1, \pi/4) = 1/\sqrt{2}$,

$$\begin{aligned}
f_x(1, \pi/4) &= [3x^2 \sin(x^2y) + 2x^4y\cos(x^2y)]_{|(1,\pi/4)} = \frac{3}{\sqrt{2}} + \frac{2\pi}{4}\frac{1}{\sqrt{2}} = \frac{3\sqrt{2}}{2} + \frac{\sqrt{2}\pi}{4}, \\
f_y(1, \pi/4) &= x^5\cos(x^2y)_{|(1,\pi/4)} = \frac{1}{\sqrt{2}}, \\
f_{xx}(1, \pi/4) &= [6x\sin(x^2y) + 14x^3y\cos(x^2y) - 4x^5y^2\sin(x^2y)]_{|(1,\pi/4)} \\
&= \frac{6}{\sqrt{2}} + 14\left(\frac{\pi}{4}\right)\frac{1}{\sqrt{2}} - 4\left(\frac{\pi^2}{16}\right)\frac{1}{\sqrt{2}} = \frac{6\sqrt{2}}{2} + \frac{7\sqrt{2}\pi}{4} - \frac{\sqrt{2}\pi^2}{8}, \\
f_{xy}(1, \pi/4) &= [5x^4\cos(x^2y) - 2x^6y\sin(x^2y)]_{|(1,\pi/4)} \\
&= \frac{5}{\sqrt{2}} - 2\left(\frac{\pi}{4}\right)\frac{1}{\sqrt{2}} = \frac{5}{\sqrt{2}} - \frac{\sqrt{2}\pi}{4}, \\
f_{yy}(1, \pi/4) &= -x^7\sin(x^2y)_{|(1,\pi/4)} = -\frac{1}{\sqrt{2}},
\end{aligned}$$

$$\begin{aligned}
x^3\sin(x^2y) = \frac{1}{\sqrt{2}} &+ \left(\frac{3\sqrt{2}}{2} + \frac{\sqrt{2}\pi}{4}\right)(x-1) + \frac{1}{\sqrt{2}}(y - \pi/4) \\
&+ \frac{1}{2}\left[\left(\frac{6\sqrt{2}}{2} + \frac{7\sqrt{2}\pi}{4} - \frac{\sqrt{2}\pi^2}{8}\right)(x-1)^2 + 2\left(\frac{5\sqrt{2}}{2} - \frac{\sqrt{2}\pi}{4}\right)(x-1)(y-\pi/4)\right. \\
&\left. - \frac{1}{\sqrt{2}}(y - \pi/4)^2\right] + \cdots.
\end{aligned}$$

49. Least squares estimates for parameters a and b in a linear function $y = ax + b$ must satisfy equations similar to 12.71. For the tabular values these equations are

$$204a + 36b = 656.8, \qquad 36a + 8b = 11.6.$$

Solutions are $a = 3.6810$ and $b = -2.6143$.