CHAPTER 13

EXERCISES 13.1

1. $$\int_{-1}^{2}\int_{y}^{y+2}(x^2-xy)\,dx\,dy = \int_{-1}^{2}\left\{\frac{x^3}{3}-\frac{x^2y}{2}\right\}_{y}^{y+2}dy = \frac{1}{6}\int_{-1}^{2}[2(y+2)^3-3y(y+2)^2-2y^3+3y^3]\,dy$$
$$= \frac{1}{6}\int_{-1}^{2}[2(y+2)^3-2y^3-12y^2-12y]\,dy$$
$$= \frac{1}{6}\left\{\frac{1}{2}(y+2)^4-\frac{y^4}{2}-4y^3-6y^2\right\}_{-1}^{2} = 11$$

2. $$\int_{-3}^{3}\int_{-\sqrt{18-2y^2}}^{\sqrt{18-2y^2}}x\,dx\,dy = \int_{-3}^{3}\left\{\frac{x^2}{2}\right\}_{-\sqrt{18-2y^2}}^{\sqrt{18-2y^2}}dy = \int_{-3}^{3}0\,dy = 0$$

3. $$\int_{0}^{1}\int_{x^2}^{x}(2xy+3y^2)\,dy\,dx = \int_{0}^{1}\left\{xy^2+y^3\right\}_{x^2}^{x}dx = \int_{0}^{1}(x^3+x^3-x^5-x^6)\,dx = \left\{\frac{x^4}{2}-\frac{x^6}{6}-\frac{x^7}{7}\right\}_{0}^{1} = \frac{4}{21}$$

4. $$\int_{-1}^{0}\int_{y}^{2}(1+y)^2\,dx\,dy = \int_{-1}^{0}\left\{x(1+y)^2\right\}_{y}^{2}dy = \int_{-1}^{0}(2+3y-y^3)\,dy = \left\{2y+\frac{3y^2}{2}-\frac{y^4}{4}\right\}_{-1}^{0} = \frac{3}{4}$$

5. $$\int_{3}^{4}\int_{0}^{\pi/2}x\,\sin y\,dy\,dx = \int_{3}^{4}\left\{-x\,\cos y\right\}_{0}^{\pi/2}dx = \int_{3}^{4}x\,dx = \left\{\frac{x^2}{2}\right\}_{3}^{4} = \frac{7}{2}$$

6. $$\int_{1}^{2}\int_{1}^{y}e^{x+y}\,dx\,dy = \int_{1}^{2}\left\{e^{x+y}\right\}_{1}^{y}dy = \int_{1}^{2}(e^{2y}-e^{y+1})\,dy = \left\{\frac{e^{2y}}{2}-e^{y+1}\right\}_{1}^{2} = \frac{e^2(1-e)^2}{2}$$

7. $$\int_{-1}^{1}\int_{-x}^{5}(x^2+y^2)\,dy\,dx = \int_{-1}^{1}\left\{x^2y+\frac{y^3}{3}\right\}_{-x}^{5}dx = \int_{-1}^{1}\left(5x^2+\frac{125}{3}+x^3+\frac{x^3}{3}\right)dx$$
$$= \left\{\frac{5x^3}{3}+\frac{125x}{3}+\frac{x^4}{3}\right\}_{-1}^{1} = \frac{260}{3}$$

8. $$\int_{-1}^{1}\int_{x}^{2x}(xy+x^3y^3)\,dy\,dx = \int_{-1}^{1}\left\{\frac{xy^2}{2}+\frac{x^3y^4}{4}\right\}_{x}^{2x}dx = \frac{1}{4}\int_{-1}^{1}(6x^3+15x^7)\,dx = \frac{1}{4}\left\{\frac{3x^4}{2}+\frac{15x^8}{8}\right\}_{-1}^{1} = 0$$

9. $$\int_{0}^{1}\int_{x}^{1}(x+y)^4\,dy\,dx = \int_{0}^{1}\left\{\frac{1}{5}(x+y)^5\right\}_{x}^{1}dx = \frac{1}{5}\int_{0}^{1}[(x+1)^5-(2x)^5]\,dx$$
$$= \frac{1}{5}\left\{\frac{1}{6}(x+1)^6-\frac{16x^6}{3}\right\}_{0}^{1} = \frac{31}{30}$$

10. $$\int_{1}^{2}\int_{x}^{2x}\frac{1}{(x+y)^3}\,dy\,dx = \int_{1}^{2}\left\{\frac{-1}{2(x+y)^2}\right\}_{x}^{2x}dx = \frac{5}{72}\int_{1}^{2}\frac{1}{x^2}dx = \frac{5}{72}\left\{-\frac{1}{x}\right\}_{1}^{2} = \frac{5}{144}$$

11. $$\int_{0}^{1}\int_{0}^{3x}\sqrt{x+y}\,dy\,dx = \int_{0}^{1}\left\{\frac{2}{3}(x+y)^{3/2}\right\}_{0}^{3x}dx = \frac{2}{3}\int_{0}^{1}(8x^{3/2}-x^{3/2})\,dx = \frac{14}{3}\left\{\frac{2x^{5/2}}{5}\right\}_{0}^{1} = \frac{28}{15}$$

12. $$\int_{-1}^{1}\int_{1}^{e}\frac{y}{x}\,dx\,dy = \int_{-1}^{1}\left\{y\ln|x|\right\}_{1}^{e}dy = \int_{-1}^{1}y\,dy = \left\{\frac{y^2}{2}\right\}_{-1}^{1} = 0$$

13. $$\int_{1}^{4}\int_{\sqrt{x}}^{x^2}(x^2+2xy-3y^2)\,dy\,dx = \int_{1}^{4}\left\{x^2y+xy^2-y^3\right\}_{\sqrt{x}}^{x^2}dx = \int_{1}^{4}(x^4+x^5-x^6-x^{5/2}-x^2+x^{3/2})\,dx$$
$$= \left\{\frac{x^5}{5}+\frac{x^6}{6}-\frac{x^7}{7}-\frac{2x^{7/2}}{7}-\frac{x^3}{3}+\frac{2x^{5/2}}{5}\right\}_{1}^{4} = -\frac{20\,975}{14}$$

14. $\displaystyle\int_0^2\int_{x^2}^{2x^2} x\cos y\,dy\,dx = \int_0^2 \Big\{x\sin y\Big\}_{x^2}^{2x^2} dx = \int_0^2 x[\sin 2x^2 - \sin x^2]\,dx$

$$= \left\{-\frac{1}{4}\cos 2x^2 + \frac{1}{2}\cos x^2\right\}_0^2 = -0.54$$

15. $\displaystyle\int_0^1\int_1^{\tan x} \frac{1}{1+y^2}\,dy\,dx = \int_0^1 \Big\{\mathrm{Tan}^{-1}y\Big\}_1^{\tan x} dx = \int_0^1 (x-\pi/4)\,dx = \left\{\frac{(x-\pi/4)^2}{2}\right\}_0^1 = \frac{2-\pi}{4}$

16. $\displaystyle\int_0^1\int_0^{y^3} \frac{1}{1+y^2}\,dx\,dy = \int_0^1 \left\{\frac{x}{1+y^2}\right\}_0^{y^3} dy = \int_0^1 \frac{y^3}{1+y^2}\,dy = \int_0^1 \left(y - \frac{y}{1+y^2}\right) dy$

$$= \left\{\frac{y^2}{2} - \frac{1}{2}\ln(y^2+1)\right\}_0^1 = \frac{1-\ln 2}{2}$$

17. $\displaystyle\int_2^3\int_0^1 \frac{x}{\sqrt{1-y^2}}\,dy\,dx = \int_2^3 \Big\{x\,\mathrm{Sin}^{-1}y\Big\}_0^1 dx = \int_2^3 \frac{\pi x}{2}\,dx = \frac{\pi}{2}\left\{\frac{x^2}{2}\right\}_2^3 = \frac{5\pi}{4}$

18. $\displaystyle\int_0^2\int_{-x}^{x} (8-2x^2)^{3/2}\,dy\,dx = \int_0^2 \Big\{y(8-2x^2)^{3/2}\Big\}_{-x}^{x} dx = 2\int_0^2 x(8-2x^2)^{3/2}\,dx$

$$= 2\left\{-\frac{(8-2x^2)^{5/2}}{10}\right\}_0^2 = \frac{128\sqrt{2}}{5}$$

19. $\displaystyle\int_0^1\int_0^x \frac{1}{\sqrt{1-y^2}}\,dy\,dx = \int_0^1 \Big\{\mathrm{Sin}^{-1}y\Big\}_0^x dx = \int_0^1 \mathrm{Sin}^{-1}x\,dx$

If we set $u = \mathrm{Sin}^{-1}x$, $dv = dx$, $du = \dfrac{1}{\sqrt{1-x^2}}dx$, $v = x$, and use integration by parts,

$$\int_0^1\int_0^x \frac{1}{\sqrt{1-y^2}}\,dy\,dx = \Big\{x\,\mathrm{Sin}^{-1}x\Big\}_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2} + \Big\{\sqrt{1-x^2}\Big\}_0^1 = \frac{\pi}{2} - 1.$$

20. $\displaystyle\int_{-9}^0\int_0^{x^2\sqrt{9+x}} dy\,dx = \int_{-9}^0 \Big\{y\Big\}_0^{x^2\sqrt{9+x}} dx = \int_{-9}^0 x^2\sqrt{9+x}\,dx$ If we set $u = 9+x$, then $du = dx$, and

$$\int_{-9}^0\int_0^{x^2\sqrt{9+x}} dy\,dx = \int_0^9 (u-9)^2\sqrt{u}\,du = \int_0^9 (81\sqrt{u} - 18u^{3/2} + u^{5/2})\,du$$

$$= \left\{\frac{162u^{3/2}}{3} - \frac{36u^{5/2}}{5} + \frac{2u^{7/2}}{7}\right\}_0^9 = \frac{11\,664}{35}.$$

21. $\displaystyle\int_0^2\int_{\sqrt{4-x^2}}^2 y^2\,dy\,dx = \int_0^2 \left\{\frac{y^3}{3}\right\}_{\sqrt{4-x^2}}^2 dx = \frac{1}{3}\int_0^2 8 - (4-x^2)^{3/2}]\,dx$

If we set $x = 2\sin\theta$, then $dx = 2\cos\theta\,d\theta$, and

$$\int_0^2\int_{\sqrt{4-x^2}}^2 y^2\,dy\,dx = \frac{1}{3}\Big\{8x\Big\}_0^2 - \frac{1}{3}\int_0^{\pi/2} 8\cos^3\theta(2\cos\theta\,d\theta) = \frac{16}{3} - \frac{16}{3}\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \frac{16}{3} - \frac{4}{3}\int_0^{\pi/2}\left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) d\theta$$

$$= \frac{16}{3} - \frac{4}{3}\left\{\frac{3\theta}{2} + \sin 2\theta + \frac{1}{8}\sin 4\theta\right\}_0^{\pi/2} = \frac{16}{3} - \pi.$$

22. $\displaystyle\int_{-1}^0\int_y^0 x\sqrt{x^2+y^2}\,dx\,dy = \int_{-1}^0 \left\{\frac{1}{3}(x^2+y^2)^{3/2}\right\}_y^0 dy$

$$= \frac{1}{3}\int_{-1}^0 (2\sqrt{2}-1)y^3\,dy = \frac{2\sqrt{2}-1}{3}\left\{\frac{y^4}{4}\right\}_{-1}^0 = \frac{1-2\sqrt{2}}{12}$$

23. $$\int_2^3\int_1^{2x}\frac{1}{(xy+x^2)^2}\,dy\,dx=\int_2^3\left\{\frac{-1}{x(xy+x^2)}\right\}_1^{2x}dx=\int_2^3\left[\frac{-1}{x(2x^2+x^2)}+\frac{1}{x(x+x^2)}\right]dx$$
$$=\int_2^3\left[\frac{-1}{3x^3}+\frac{1}{x^2(1+x)}\right]dx=\int_2^3\left(\frac{-1}{3x^3}-\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x+1}\right)dx$$
$$=\left\{\frac{1}{6x^2}-\ln|x|-\frac{1}{x}+\ln|x+1|\right\}_2^3=0.0257$$

24. $$\int_0^1\int_0^{\mathrm{Cos}^{-1}x}x\cos y\,dy\,dx=\int_0^1\left\{x\sin y\right\}_0^{\mathrm{Cos}^{-1}x}dx=\int_0^1 x\sqrt{1-x^2}\,dx=\left\{-\frac{1}{3}(1-x^2)^{3/2}\right\}_0^1=\frac{1}{3}$$

25. If we set $u=x^2-y^2$, then $du=2x\,dx$, and
$$\int_0^1\int_{\sqrt{y^2+y}}^{\sqrt{2}}x^3\sqrt{x^2-y^2}\,dx\,dy=\int_0^1\int_y^{y^2}(u+y^2)\sqrt{u}\frac{du}{2}dy=\frac{1}{2}\int_0^1\int_y^{y^2}(u^{3/2}+y^2\sqrt{u})\,du\,dy$$
$$=\frac{1}{2}\int_0^1\left\{\frac{2u^{5/2}}{5}+\frac{2y^2u^{3/2}}{3}\right\}_y^{y^2}dy=\frac{1}{15}\int_0^1(-3y^{5/2}-5y^{7/2}+8y^5)\,dy$$
$$=\frac{1}{15}\left\{-\frac{6y^{7/2}}{7}-\frac{10y^{9/2}}{9}+\frac{4y^6}{3}\right\}_0^1=-\frac{8}{189}.$$

26. If we set $u=x^2-y^2$, then $du=2x\,dx$, and
$$\int_0^1\int_{\sqrt{2}y}^{\sqrt{y^2+y}}x^3\sqrt{x^2-y^2}\,dx\,dy=\int_0^1\int_{y^2}^{y}(u+y^2)\sqrt{u}\frac{du}{2}dy=\frac{1}{2}\int_0^1\int_{y^2}^{y}(u^{3/2}+y^2\sqrt{u})\,du\,dy$$
$$=\frac{1}{2}\int_0^1\left\{\frac{2u^{5/2}}{5}+\frac{2y^2u^{3/2}}{3}\right\}_{y^2}^{y}dy=\frac{1}{15}\int_0^1(3y^{5/2}+5y^{7/2}-8y^5)\,dy$$
$$=\frac{1}{15}\left\{\frac{6y^{7/2}}{7}+\frac{10y^{9/2}}{9}-\frac{4y^6}{3}\right\}_0^1=\frac{8}{189}.$$

27. $$\int_{-2}^0\int_{x^4}^{4x^2}\sqrt{y-x^4}\,dy\,dx=\int_{-2}^0\left\{\frac{2}{3}(y-x^4)^{3/2}\right\}_{x^4}^{4x^2}dx=\frac{2}{3}\int_{-2}^0(4x^2-x^4)^{3/2}\,dx=\frac{2}{3}\int_{-2}^0-x^3(4-x^2)^{3/2}\,dx$$
If we set $u=\sqrt{4-x^2}$, then $du=\dfrac{-x}{\sqrt{4-x^2}}dx$, and
$$\int_{-2}^0\int_{x^4}^{4x^2}\sqrt{y-x^4}\,dy\,dx=-\frac{2}{3}\int_0^2(4-u^2)u^3(-u\,du)=\frac{2}{3}\left\{\frac{4u^5}{5}-\frac{u^7}{7}\right\}_0^2=\frac{512}{105}.$$

28. $$\int_{-2}^0\int_y^0\frac{x}{\sqrt{x^2+y^2}}\,dx\,dy=\int_{-2}^0\left\{\sqrt{x^2+y^2}\right\}_y^0dy=\int_{-2}^0(\sqrt{2}-1)y\,dy=(\sqrt{2}-1)\left\{\frac{y^2}{2}\right\}_{-2}^0=2(1-\sqrt{2})$$

29. $$\int_{-1}^2\int_{-1}^{y^3}\sqrt{1+y}\,dx\,dy=\int_{-1}^2\left\{x\sqrt{1+y}\right\}_{-1}^{y^3}dy=\int_{-1}^2(y^3\sqrt{1+y}+\sqrt{1+y})\,dy$$
If we set $u=\sqrt{1+y}$, then $du=\dfrac{1}{2\sqrt{1+y}}dy$, and
$$\int_{-1}^2\int_{-1}^{y^3}\sqrt{1+y}\,dx\,dy=\int_0^{\sqrt{3}}[(u^2-1)^3u+u](2u\,du)=2\int_0^{\sqrt{3}}(u^8-3u^6+3u^4)\,du$$
$$=2\left\{\frac{u^9}{9}-\frac{3u^7}{7}+\frac{3u^5}{5}\right\}_0^{\sqrt{3}}=\frac{198\sqrt{3}}{35}.$$

30. If we set $y = x\tan\theta$, then $dy = x\sec^2\theta\,d\theta$, and

$$\begin{aligned}\int_0^1\int_0^x \sqrt{x^2+y^2}\,dy\,dx &= \int_0^1\int_0^{\pi/4} x\sec\theta\, x\sec^2\theta\,d\theta\,dx = \int_0^1\int_0^{\pi/4} x^2\sec^3\theta\,d\theta\,dx \\ &= \int_0^1 \frac{x^2}{2}\Big\{\sec\theta\,\tan\theta + \ln|\sec\theta+\tan\theta|\Big\}_0^{\pi/4} dx \quad \text{(see Example 8.9)} \\ &= \frac{\sqrt{2}+\ln(\sqrt{2}+1)}{2}\int_0^1 x^2\,dx = \frac{\sqrt{2}+\ln(\sqrt{2}+1)}{6}.\end{aligned}$$

31. From the continuity equation,

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -k.$$

Integration gives $v(x,y) = -ky + f(x)$, where $f(x)$ is any differentiable function of x.

32. From the continuity equation,

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -2x.$$

Integration gives $v(x,y) = -2xy + f(x)$, where $f(x)$ is any differentiable function of x.

33. From the continuity equation,

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{1}{1+y^2/x^2}\left(\frac{-y}{x^2}\right) = \frac{y}{x^2+y^2}.$$

Integration gives $v(x,y) = (1/2)\ln(x^2+y^2) + f(x)$, where $f(x)$ is any differentiable function of x.

34. From the continuity equation,

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} = \frac{-xy}{\sqrt{x^2+y^2}}.$$

Integration gives $u(x,y) = -y\sqrt{x^2+y^2} + f(y)$, where $f(y)$ is any differentiable function of y.

35. From the continuity equation,

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} = \sin x\sin y.$$

Integration gives $u(x,y) = -\cos x\sin y + f(y)$, where $f(y)$ is any differentiable function of y.

36. Stream functions must satisfy

$$\frac{\partial\psi}{\partial x} = y, \qquad \frac{\partial\psi}{\partial y} = x.$$

Integration of the first gives $\psi(x,y) = xy + f(y)$, where $f(y)$ is any differentiable function of y. Substitution of this into the second equation requires $x + f'(y) = x \implies f(y) = C$, where C is a constant. Thus, $\psi(x,y) = xy + C$.

37. Stream functions must satisfy

$$\frac{\partial\psi}{\partial x} = 2xy, \qquad \frac{\partial\psi}{\partial y} = x^2+y^2.$$

Integration of the first gives $\psi(x,y) = x^2y + f(y)$, where $f(y)$ is any differentiable function of y. Substitution of this into the second equation requires $x^2 + f'(y) = x^2 + y^2 \implies f(y) = y^3/3 + C$, where C is a constant. Thus, $\psi(x,y) = x^2y + y^3/3 + C$.

38. Stream functions must satisfy

$$\frac{\partial\psi}{\partial x} = -x\sqrt{x^2+y^2}, \qquad \frac{\partial\psi}{\partial y} = -y\sqrt{x^2+y^2}.$$

Integration of the first gives $\psi(x,y) = -\frac{1}{3}(x^2+y^2)^{3/2} + f(y)$, where $f(y)$ is any differentiable function of y. Substitution of this into the second equation requires

$$-y\sqrt{x^2+y^2} + f'(y) = -y\sqrt{x^2+y^2} \quad\Longrightarrow\quad f(y) = C,$$

where C is a constant. Thus, $\psi(x,y) = -(1/3)(x^2+y^2)^{3/2} + C$.

39. Stream functions must satisfy

$$\frac{\partial\psi}{\partial x} = -\sin x\cos y - x, \qquad \frac{\partial\psi}{\partial y} = -\cos x\sin y.$$

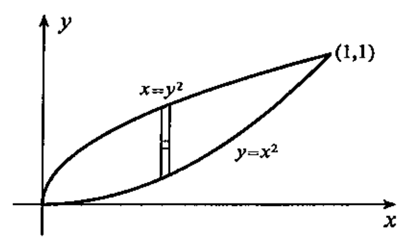
Integration of the second gives $\psi(x,y) = \cos x\cos y + f(x)$, where $f(x)$ is any differentiable function of x. Substitution of this into the first equation requires

$$-\sin x\cos y + f'(x) = -\sin x\cos y - x \quad\Longrightarrow\quad f(x) = -\frac{x^2}{2} + C,$$

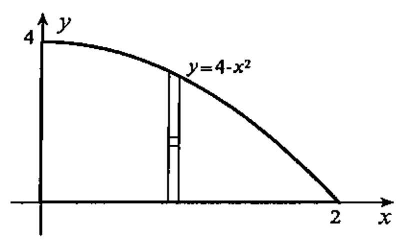
where C is a constant. Thus, $\psi(x,y) = \cos x\cos y - x^2/2 + C$.

EXERCISES 13.2

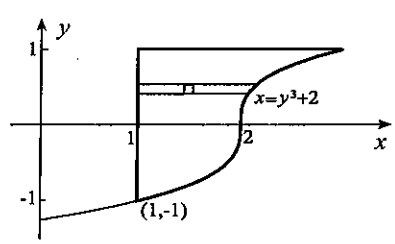
1.
$$\begin{aligned}\iint_R (x^2+y^2)\,dA &= \int_0^1\int_{x^2}^{\sqrt{x}} (x^2+y^2)\,dy\,dx \\ &= \int_0^1 \left\{x^2y + \frac{y^3}{3}\right\}_{x^2}^{\sqrt{x}} dx \\ &= \frac{1}{3}\int_0^1 (3x^{5/2} + x^{3/2} - 3x^4 - x^6)\,dx \\ &= \frac{1}{3}\left\{\frac{6x^{7/2}}{7} + \frac{2x^{5/2}}{5} - \frac{3x^5}{5} - \frac{x^7}{7}\right\}_0^1 = \frac{6}{35}\end{aligned}$$



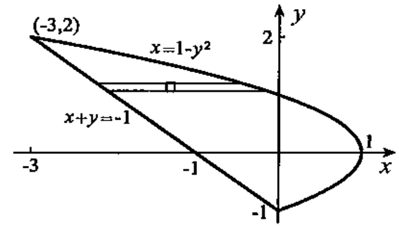
2.
$$\begin{aligned}\iint_R (4-x^2-y)\,dA &= \int_0^2\int_0^{4-x^2} (4-x^2-y)\,dy\,dx \\ &= \int_0^2 \left\{4y - x^2y - \frac{y^2}{2}\right\}_0^{4-x^2} dx \\ &= \frac{1}{2}\int_0^2 (16 - 8x^2 + x^4)\,dx \\ &= \frac{1}{2}\left\{16x - \frac{8x^3}{3} + \frac{x^5}{5}\right\}_0^2 = \frac{128}{15}\end{aligned}$$



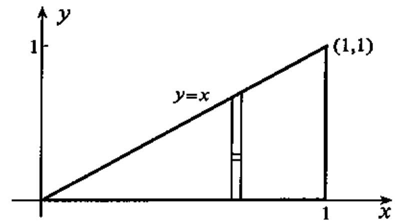
3.
$$\begin{aligned}\iint_R (x+y)\,dA &= \int_{-1}^1\int_1^{y^3+2} (x+y)\,dx\,dy \\ &= \int_{-1}^1 \left\{\frac{x^2}{2} + xy\right\}_1^{y^3+2} dy \\ &= \frac{1}{2}\int_{-1}^1 (y^6 + 2y^4 + 4y^3 + 2y + 3)\,dy \\ &= \frac{1}{2}\left\{\frac{y^7}{7} + \frac{2y^5}{5} + y^4 + y^2 + 3y\right\}_{-1}^1 = \frac{124}{35}\end{aligned}$$



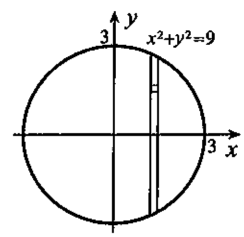
4. $$\iint_R xy^2\,dA = \int_{-1}^{2}\int_{-1-y}^{1-y^2} xy^2\,dx\,dy = \int_{-1}^{2}\left\{\frac{x^2y^2}{2}\right\}_{-1-y}^{1-y^2} dy$$
$$= \frac{1}{2}\int_{-1}^{2}(y^6 - 3y^4 - 2y^3)\,dy$$
$$= \frac{1}{2}\left\{\frac{y^7}{7} - \frac{3y^5}{5} - \frac{y^4}{2}\right\}_{-1}^{2} = -\frac{621}{140}$$



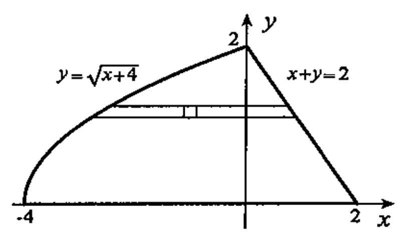
5. $$\iint_R xe^y\,dA = \int_0^1\int_0^x xe^y\,dy\,dx = \int_0^1\left\{xe^y\right\}_0^x dx$$
$$= \int_0^1 (xe^x - x)\,dx$$
$$= \left\{xe^x - e^x - \frac{x^2}{2}\right\}_0^1 = \frac{1}{2}$$



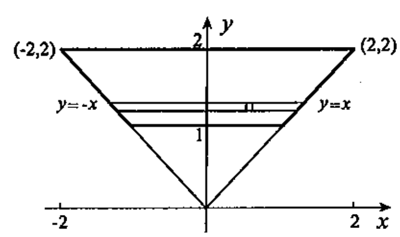
6. $$\iint_R (x+y)\,dA = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x+y)\,dy\,dx$$
$$= \int_{-3}^{3}\left\{xy + \frac{y^2}{2}\right\}_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx$$
$$= 2\int_{-3}^{3} x\sqrt{9-x^2}\,dx = 2\left\{-\frac{1}{3}(9-x^2)^{3/2}\right\}_{-3}^{3} = 0$$



7. $$\iint_R x^2y\,dA = \int_0^2\int_{y^2-4}^{2-y} x^2y\,dx\,dy = \int_0^2\left\{\frac{x^3y}{3}\right\}_{y^2-4}^{2-y} dy$$
$$= \frac{1}{3}\int_0^2 (-y^7 + 12y^5 - y^4 - 42y^3 - 12y^2 + 72y)\,dy$$
$$= \frac{1}{3}\left\{-\frac{y^8}{8} + 2y^6 - \frac{y^5}{5} - \frac{21y^4}{2} - 4y^3 + 36y^2\right\}_0^2 = \frac{56}{5}$$

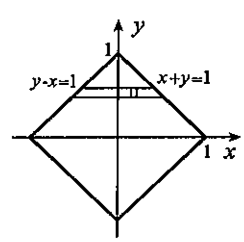


8. $$\iint_R (xy + y^2 - 3x^2)\,dA = \int_1^2\int_{-y}^{y} (xy + y^2 - 3x^2)\,dx\,dy$$
$$= \int_1^2\left\{\frac{x^2y}{2} + xy^2 - x^3\right\}_{-y}^{y} dy$$
$$= \int_1^2 (0)\,dy = 0$$

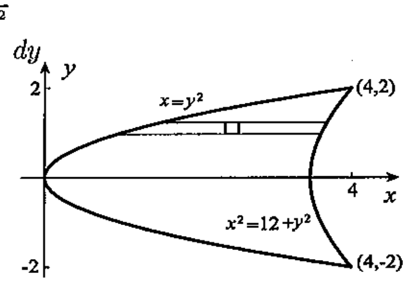


9. Integration over the top half of the square is equal to that over the bottom half. Hence,

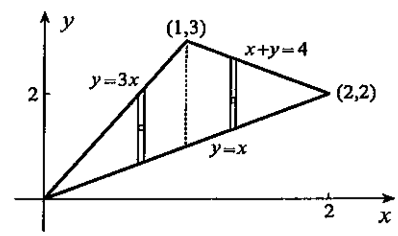
$$\iint_R (1-x)^2\,dA = 2\int_0^1\int_{y-1}^{1-y} (1-x)^2\,dx\,dy$$
$$= 2\int_0^1\left\{-\frac{(1-x)^3}{3}\right\}_{y-1}^{1-y} dy$$
$$= -\frac{2}{3}\int_0^1 [y^3 - (2-y)^3]\,dy$$
$$= -\frac{2}{3}\left\{\frac{y^4}{4} + \frac{(2-y)^4}{4}\right\}_0^1 = \frac{7}{3}$$



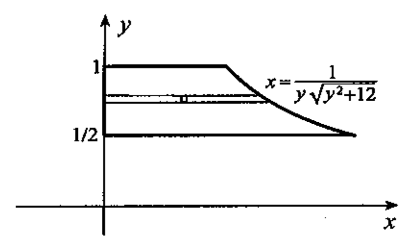
10. $$\iint_R (x+y)\,dA = \int_{-2}^{2}\int_{y^2}^{\sqrt{12+y^2}} (x+y)\,dx\,dy = \int_{-2}^{2}\left\{\frac{x^2}{2}+xy\right\}_{y^2}^{\sqrt{12+y^2}} dy$$
$$= \frac{1}{2}\int_{-2}^{2}\left(12+y^2-2y^3-y^4+2y\sqrt{12+y^2}\right)dy$$
$$= \frac{1}{2}\left\{12y+\frac{y^3}{3}-\frac{y^4}{2}-\frac{y^5}{5}+\frac{2}{3}(12+y^2)^{3/2}\right\}_{-2}^{2}$$
$$= 304/15$$



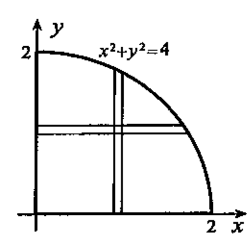
11. $$\iint_R x\,dA = \int_0^1\int_x^{3x} x\,dy\,dx + \int_1^2\int_x^{4-x} x\,dy\,dx$$
$$= \int_0^1 \left\{xy\right\}_x^{3x} dx + \int_1^2 \left\{xy\right\}_x^{4-x} dx$$
$$= \int_0^1 2x^2\,dx + \int_1^2 (4x-2x^2)\,dx$$
$$= \left\{\frac{2x^3}{3}\right\}_0^1 + \left\{2x^2-\frac{2x^3}{3}\right\}_1^2 = 2$$



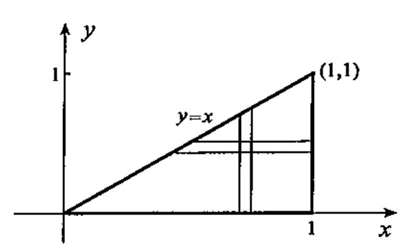
12. $$\iint_R y^2\,dA = \int_{1/2}^{1}\int_0^{1/(y\sqrt{y^2+12})} y^2\,dx\,dy$$
$$= \int_{1/2}^{1}\left\{xy^2\right\}_0^{1/(y\sqrt{y^2+12})} dy$$
$$= \int_{1/2}^{1}\frac{y}{\sqrt{y^2+12}}dy = \left\{\sqrt{y^2+12}\right\}_{1/2}^{1} = \sqrt{13}-\frac{7}{2}$$



13. $$\int_0^2\int_0^{\sqrt{4-x^2}} (4-y^2)^{3/2}\,dy\,dx = \int_0^2\int_0^{\sqrt{4-y^2}} (4-y^2)^{3/2}\,dx\,dy$$
$$= \int_0^2 \left\{x(4-y^2)^{3/2}\right\}_0^{\sqrt{4-y^2}} dy$$
$$= \int_0^2 (16-8y^2+y^4)\,dy$$
$$= \left\{16y-\frac{8y^3}{3}+\frac{y^5}{5}\right\}_0^2 = \frac{256}{15}$$



14. $$\int_0^1\int_y^1 \sin x^2\,dx\,dy = \int_0^1\int_0^x \sin x^2\,dy\,dx = \int_0^1 \left\{y\sin x^2\right\}_0^x dx$$
$$= \int_0^1 x\sin x^2\,dx$$
$$= \left\{-\frac{\cos x^2}{2}\right\}_0^1 = \frac{1-\cos 1}{2}$$

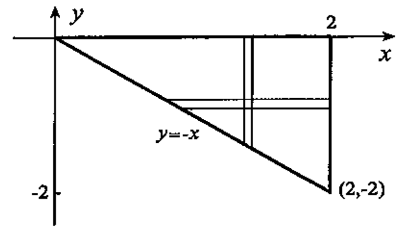


15. $\displaystyle\int_{-2}^{0}\int_{-y}^{2} y(x^2+y^2)^8\,dx\,dy = \int_0^2\int_{-x}^{0} y(x^2+y^2)^8\,dy\,dx$

$$= \int_0^2 \left\{\frac{1}{18}(x^2+y^2)^9\right\}_{-x}^{0} dx$$

$$= \frac{1}{18}\int_0^2 (x^{18} - 512x^{18})\,dx$$

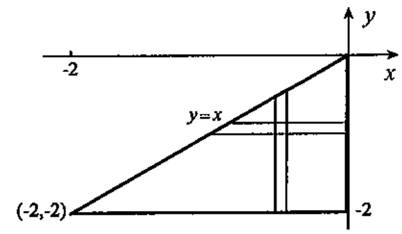
$$= \frac{1}{18}\left\{-\frac{511x^{19}}{19}\right\}_0^2 = \frac{-511(2^{18})}{171}$$



16. $\displaystyle\int_{-2}^{0}\int_{-2}^{x} \frac{x}{\sqrt{x^2+y^2}}dy\,dx = \int_{-2}^{0}\int_{y}^{0} \frac{x}{\sqrt{x^2+y^2}}dx\,dy$

$$= \int_{-2}^{0}\left\{\sqrt{x^2+y^2}\right\}_y^0 dy = (\sqrt{2}-1)\int_{-2}^{0} y\,dy$$

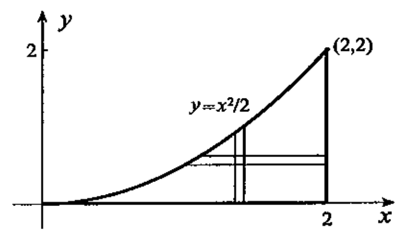
$$= (\sqrt{2}-1)\left\{\frac{y^2}{2}\right\}_{-2}^{0} = 2(1-\sqrt{2})$$



17. $\displaystyle\int_0^2\int_0^{x^2/2} \frac{x}{\sqrt{1+x^2+y^2}}dy\,dx = \int_0^2\int_{\sqrt{2y}}^{2} \frac{x}{\sqrt{1+x^2+y^2}}dx\,dy$

$$= \int_0^2 \left\{\sqrt{1+x^2+y^2}\right\}_{\sqrt{2y}}^{2} dy$$

$$= \int_0^2 [\sqrt{5+y^2} - (1+y)]\,dy$$



If we set $y = \sqrt{5}\tan\theta$, then $dy = \sqrt{5}\sec^2\theta\,d\theta$, and

$$\int_0^2\int_0^{x^2/2} \frac{x}{\sqrt{1+x^2+y^2}}dy\,dx = \int_0^{\mathrm{Tan}^{-1}(2/\sqrt{5})} \sqrt{5}\sec\theta\,\sqrt{5}\sec^2\theta\,d\theta - \left\{y + \frac{y^2}{2}\right\}_0^2$$

$$= 5\int_0^{\mathrm{Tan}^{-1}(2/\sqrt{5})} \sec^3\theta\,d\theta - 4$$

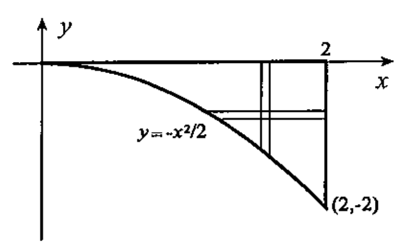
$$= \frac{5}{2}\left\{\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right\}_0^{\mathrm{Tan}^{-1}(2/\sqrt{5})} - 4 \quad \text{(see Example 8.9)}$$

$$= \frac{5}{4}\ln 5 - 1$$

18. $\displaystyle\int_0^2\int_{-x^2/2}^{0} \frac{x}{\sqrt{1+x^2+y^2}}dy\,dx = \int_{-2}^{0}\int_{\sqrt{-2y}}^{2} \frac{x}{\sqrt{1+x^2+y^2}}dx\,dy$

$$= \int_{-2}^{0}\left\{\sqrt{1+x^2+y^2}\right\}_{\sqrt{-2y}}^{2} dy$$

$$= \int_{-2}^{0}(\sqrt{5+y^2} - \sqrt{1-2y+y^2})\,dy$$



In the first term we set $y = \sqrt{5}\tan\theta$ and $dy = \sqrt{5}\sec^2\theta\,d\theta$,

$$\int_0^2\int_{-x^2/2}^{0} \frac{x}{\sqrt{1+x^2+y^2}}dy\,dx = \int_{-\mathrm{Tan}^{-1}(2/\sqrt{5})}^{0} \sqrt{5}\sec\theta\,\sqrt{5}\sec^2\theta\,d\theta - \int_{-2}^{0}|y-1|\,dy$$

$$= \frac{5}{2}\left\{\sec\theta\,\tan\theta + \ln|\sec\theta+\tan\theta|\right\}_{-\mathrm{Tan}^{-1}(2/\sqrt{5})}^{0}$$

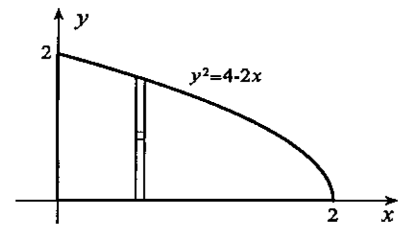
$$- \int_{-2}^{0}(1-y)\,dy \quad \text{(see Example 8.9)}$$

$$= -\frac{5}{2}\left[\frac{3}{\sqrt{5}}\left(\frac{-2}{\sqrt{5}}\right) + \ln\left|\frac{3}{\sqrt{5}} - \frac{2}{\sqrt{5}}\right|\right] - \left\{y - \frac{y^2}{2}\right\}_{-2}^{0}$$
$$= -\frac{5}{2}\left(-\frac{6}{5} - \ln\sqrt{5}\right) + (-2-2) = \frac{5}{4}\ln 5 - 1$$

19. We verify the right inequality; the left is similar. Using equation 13.3,

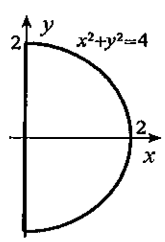
$$\iint_R f(x,y)\,dA = \lim_{\|\Delta A_i\|\to 0}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta A_i \le \lim_{\|\Delta A_i\|\to 0}\sum_{i=1}^{n} M\,\Delta A_i = M \lim_{\|\Delta A_i\|\to 0}\sum_{i=1}^{n}\Delta A_i = M(\text{Area of } R).$$

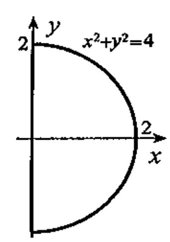
20.
$$\iint_R \frac{1}{\sqrt{2x-x^2}}dA = \int_0^2\int_0^{\sqrt{4-2x}} \frac{1}{\sqrt{2x-x^2}}dy\,dx$$
$$= \int_0^2 \frac{\sqrt{4-2x}}{\sqrt{2x-x^2}}dx$$
$$= \int_0^2 \frac{\sqrt{2}\sqrt{2-x}}{\sqrt{x}\sqrt{2-x}}dx$$
$$= \sqrt{2}\left\{2\sqrt{x}\right\}_0^2 = 4$$



21. Since x^2y^3 is an odd function of y, and the region is symmetric about the x-axis, the value of the integral is zero.

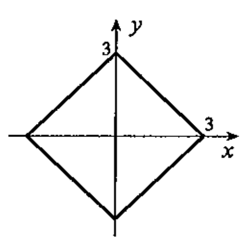
22. Since x^2y^2 is an even function of y, and the region is symmetric about the x-axis, we may double the value of the integral over that part of the region above the x-axis.

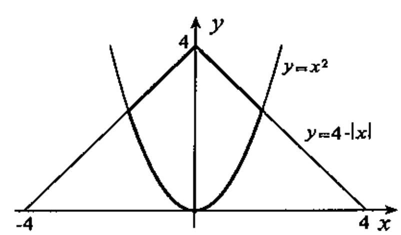




23. Since x is an odd function of x, and the area is symmetric about the y-axis, the double integal of x is equal to zero. Since y is an odd function of y, and the area is symmetric about the x-axis, the double integral of y is equal to zero also.

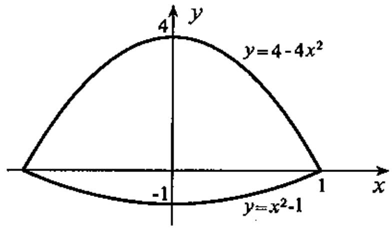
24. Since the integrand is an odd function of x, and the region is symmetric about the y-axis, the value of the integral is zero.

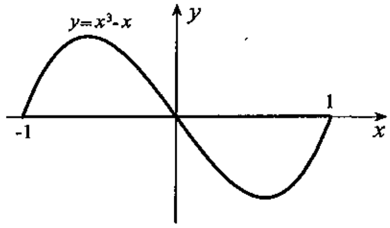




25. Since $e^{x^2+y^2}$ is an even function of x, and the region is symmetric about the y-axis, we could integrate over the right half, and double the result.

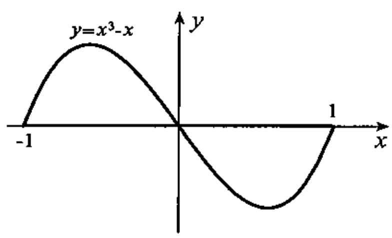
26. Since the integrand is an even function of x and y, and the region is symmetric about the origin, we may double the value of the integral over that part of the region to the right of the y-axis.

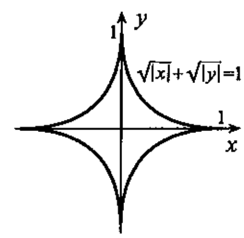




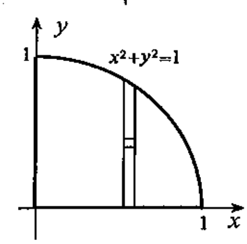
27. Since $\sin(x^2y)$ is an even function of x, and an odd function of y, and the region is symmetric about the origin, the value of the double integral is zero.

28. The first term of the integrand is an odd function of y and the second term is an odd function of x. Since the region is symmetric about the x- and y-axes, the value of the integral is zero.

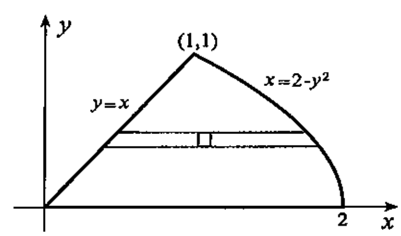




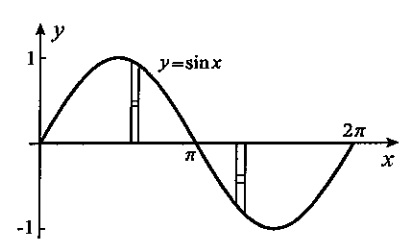
29. $$\overline{f} = \frac{1}{\pi/4}\int_0^1\int_0^{\sqrt{1-x^2}} xy\,dy\,dx = \frac{4}{\pi}\int_0^1 \left\{\frac{xy^2}{2}\right\}_0^{\sqrt{1-x^2}} dx$$
$$= \frac{2}{\pi}\int_0^1 x(1-x^2)\,dx$$
$$= \frac{2}{\pi}\left\{\frac{x^2}{2} - \frac{x^4}{4}\right\}_0^1 = \frac{1}{2\pi}$$



30. Since $\text{Area}(R) = \int_0^1 (2-y^2-y)\,dy = \left\{2y - \frac{y^3}{3} - \frac{y^2}{2}\right\}_0^1 = \frac{7}{6}$,
$$\overline{f} = \frac{6}{7}\int_0^1\int_y^{2-y^2} (x+y)\,dx\,dy = \frac{6}{7}\int_0^1 \left\{\frac{x^2}{2} + xy\right\}_y^{2-y^2} dy$$
$$= \frac{3}{7}\int_0^1 (4+4y-7y^2-2y^3+y^4)\,dy$$
$$= \frac{3}{7}\left\{4y + 2y^2 - \frac{7y^3}{3} - \frac{y^4}{2} + \frac{y^5}{5}\right\}_0^1 = \frac{101}{70}.$$



31. Since $\text{Area}(R) = 2\int_0^\pi \sin x\,dx = 2\left\{-\cos x\right\}_0^\pi = 4$,
$$\overline{f} = \frac{1}{4}\int_0^\pi\int_0^{\sin x} x\,dy\,dx + \frac{1}{4}\int_\pi^{2\pi}\int_{\sin x}^0 x\,dy\,dx$$
$$= \frac{1}{4}\int_0^\pi \left\{xy\right\}_0^{\sin x} dx + \frac{1}{4}\int_\pi^{2\pi}\left\{xy\right\}_{\sin x}^0 dx$$
$$= \frac{1}{4}\int_0^\pi x\sin x\,dx + \frac{1}{4}\int_\pi^{2\pi} -x\sin x\,dx$$
$$= \frac{1}{4}\left\{-x\cos x + \sin x\right\}_0^\pi - \frac{1}{4}\left\{-x\cos x + \sin x\right\}_\pi^{2\pi} = \pi$$



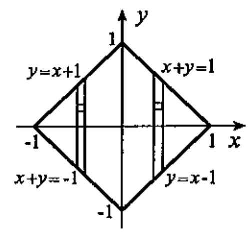
32. $\overline{f} = \frac{1}{2}\iint_R e^{x+y}\,dA$

$$= \frac{1}{2}\int_{-1}^{0}\int_{-1-x}^{x+1} e^{x+y}\,dy\,dx + \frac{1}{2}\int_{0}^{1}\int_{x-1}^{1-x} e^{x+y}\,dy\,dx$$

$$= \frac{1}{2}\int_{-1}^{0}\left\{e^{x+y}\right\}_{-1-x}^{x+1} dx + \frac{1}{2}\int_{0}^{1}\left\{e^{x+y}\right\}_{x-1}^{1-x} dx$$

$$= \frac{1}{2}\int_{-1}^{0}(e^{2x+1} - e^{-1})\,dx + \frac{1}{2}\int_{0}^{1}(e - e^{2x-1})\,dx$$

$$= \frac{1}{2}\left\{\frac{1}{2}e^{2x+1} - \frac{x}{e}\right\}_{-1}^{0} + \frac{1}{2}\left\{ex - \frac{1}{2}e^{2x-1}\right\}_{0}^{1} = \frac{e^2-1}{2e}$$



33. Average $= \frac{1}{(4)(10)}\int_{45}^{55}\int_{8}^{12} 10\,000x^{0.3}y^{0.7}\,dy\,dx = 250\int_{45}^{55}\left\{\frac{x^{0.3}y^{1.7}}{1.7}\right\}_{8}^{12} dx$

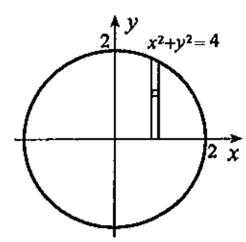
$$= \frac{250(12^{1.7} - 8^{1.7})}{1.7}\left\{\frac{x^{1.3}}{1.3}\right\}_{45}^{55} = 161\,781$$

34. Average $= \frac{1}{(4)(10)}\int_{45}^{55}\int_{8}^{12} 10\,000x^{0.7}y^{0.3}\,dy\,dx = 250\int_{45}^{55}\left\{\frac{x^{0.7}y^{1.3}}{1.3}\right\}_{8}^{12} dx$

$$= \frac{250(12^{1.3} - 8^{1.3})}{1.3}\left\{\frac{x^{1.7}}{1.7}\right\}_{45}^{55} = 307\,973$$

35. $\iint_R x^2\,dA = 4\int_0^2\int_0^{\sqrt{4-x^2}} x^2\,dy\,dx = 4\int_0^2\left\{x^2y\right\}_0^{\sqrt{4-x^2}} dx$

$$= 4\int_0^2 x^2\sqrt{4-x^2}\,dx$$



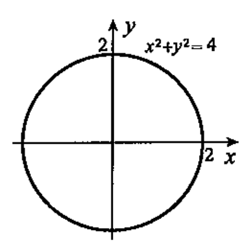
If we set $x = 2\sin\theta$, then $dx = 2\cos\theta\,d\theta$, and

$$\iint_R x^2\,dA = 4\int_0^{\pi/2} 4\sin^2\theta(2\cos\theta)2\cos\theta\,d\theta = 64\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta$$

$$= 64\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^2 d\theta = 16\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right) d\theta = 8\left\{\theta - \frac{1}{4}\sin 4\theta\right\}_0^{\pi/2} = 4\pi.$$

36. Since $f(x,y) = x$ is an odd function of x, and R is symmetric about the y-axis, the integral of the second term vanishes. Similarly, the integral of the third term vanishes. Thus,

$$\iint_R (6 - x - 2y)\,dA = 6\iint_R dA$$
$$= 6(\text{area of } R) = 6\pi(2)^2 = 24\pi.$$



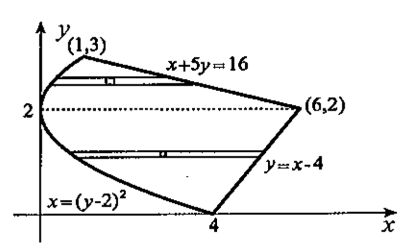
37. $\iint_R 6x^5\,dA = \int_0^2\int_{(y-2)^2}^{y+4} 6x^5\,dx\,dy + \int_2^3\int_{(y-2)^2}^{16-5y} 6x^5\,dx\,dy = \int_0^2\left\{x^6\right\}_{(y-2)^2}^{y+4} dy + \int_2^3\left\{x^6\right\}_{(y-2)^2}^{16-5y} dy$

$$= \int_0^2 [(y+4)^6 - (y-2)^{12}]\,dy$$

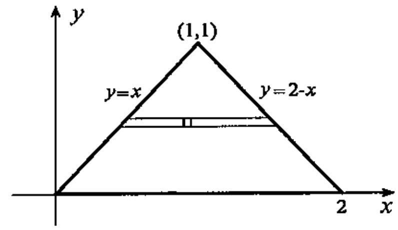
$$+ \int_2^3 [(16-5y)^6 - (y-2)^{12}]\,dy$$

$$= \left\{\frac{(y+4)^7}{7} - \frac{(y-2)^{13}}{13}\right\}_0^2$$

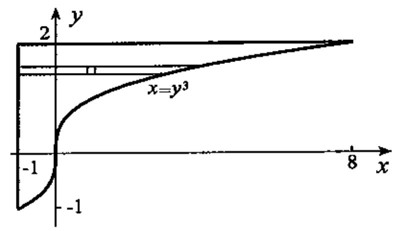
$$+ \left\{-\frac{(16-5y)^7}{35} - \frac{(y-2)^{13}}{13}\right\}_2^3 = 4.50\times 10^4$$



38. $$\iint_R ye^x\,dA = \int_0^1\int_y^{2-y} ye^x\,dx\,dy = \int_0^1 \left\{ye^x\right\}_y^{2-y} dy$$
$$= \int_0^1 (ye^{2-y} - ye^y)dy$$
$$= \left\{-ye^{2-y} - e^{2-y} - ye^y + e^y\right\}_0^1$$
$$= e^2 - 2e - 1$$



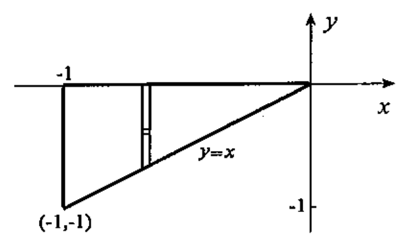
39. $$\iint_R \sqrt{1+y}\,dA = \int_{-1}^2\int_{-1}^{y^3} \sqrt{1+y}\,dx\,dy$$
$$= \int_{-1}^2 \left\{x\sqrt{1+y}\right\}_{-1}^{y^3} dy$$
$$= \int_{-1}^2 (y^3+1)\sqrt{y+1}\,dy$$



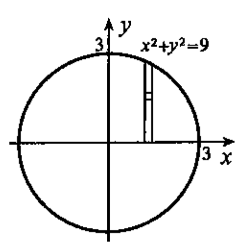
If we set $u = y+1$, then $du = dy$, and

$$\iint_R \sqrt{1+y}\,dA = \int_0^3 (u-1)^3\sqrt{u}\,du + \left\{\frac{2(y+1)^{3/2}}{3}\right\}_{-1}^2 = \int_0^3 (u^{7/2} - 3u^{5/2} + 3u^{3/2} - u^{1/2})\,du + 2\sqrt{3}$$
$$= \left\{\frac{2u^{9/2}}{9} - \frac{6u^{7/2}}{7} + \frac{6u^{5/2}}{5} - \frac{2u^{3/2}}{3}\right\}_0^3 + 2\sqrt{3} = \frac{198\sqrt{3}}{35}.$$

40. $$\iint_R y\sqrt{x^2+y^2}\,dA = \int_{-1}^0\int_x^0 y\sqrt{x^2+y^2}\,dy\,dx$$
$$= \int_{-1}^0 \left\{\frac{1}{3}(x^2+y^2)^{3/2}\right\}_x^0 dx$$
$$= \frac{2\sqrt{2}-1}{3}\int_{-1}^0 x^3\,dx$$
$$= \frac{2\sqrt{2}-1}{3}\left\{\frac{x^4}{4}\right\}_{-1}^0 = \frac{1-2\sqrt{2}}{12}$$



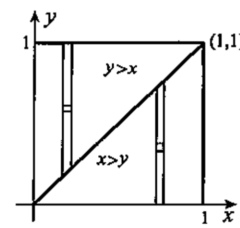
41. $$\iint_R (x^2+y^2)\,dA = 4\int_0^3\int_0^{\sqrt{9-x^2}} (x^2+y^2)\,dy\,dx$$
$$= 4\int_0^3 \left\{x^2y + \frac{y^3}{3}\right\}_0^{\sqrt{9-x^2}} dx$$
$$= \frac{4}{3}\int_0^3 [3x^2\sqrt{9-x^2} + (9-x^2)^{3/2}]\,dx$$
$$= \frac{4}{3}\int_0^3 (2x^2+9)\sqrt{9-x^2}\,dx$$



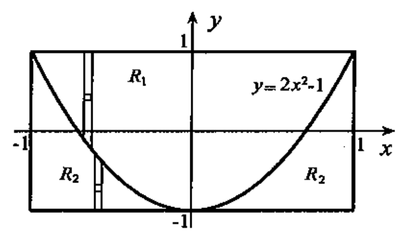
If we set $x = 3\sin\theta$, then $dx = 3\cos\theta\,d\theta$, and

$$\iint_R (x^2+y^2)\,dA = \frac{4}{3}\int_0^{\pi/2} (18\sin^2\theta + 9)(3\cos\theta)3\cos\theta\,d\theta = 108\int_0^{\pi/2} (2\sin^2\theta\cos^2\theta + \cos^2\theta)\,d\theta$$
$$= 108\int_0^{\pi/2}\left[2\left(\frac{\sin 2\theta}{2}\right)^2 + \frac{1+\cos 2\theta}{2}\right]d\theta = 54\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2} + 1 + \cos 2\theta\right)d\theta$$
$$= 54\left\{\frac{3\theta}{2} - \frac{1}{8}\sin 4\theta + \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = \frac{81\pi}{2}.$$

42. $$\int_0^1\int_0^1 |x-y|\,dy\,dx = \int_0^1\int_0^x (x-y)\,dy\,dx + \int_0^1\int_x^1 (y-x)\,dy\,dx$$
$$= \int_0^1 \left\{-\frac{1}{2}(x-y)^2\right\}_0^x dx + \int_0^1 \left\{\frac{1}{2}(y-x)^2\right\}_x^1 dx$$
$$= \frac{1}{2}\int_0^1 x^2\,dx + \frac{1}{2}\int_0^1 (1-x)^2\,dx$$
$$= \frac{1}{2}\left\{\frac{x^3}{3}\right\}_0^1 + \frac{1}{2}\left\{-\frac{(1-x)^3}{3}\right\}_0^1 = \frac{1}{3}$$



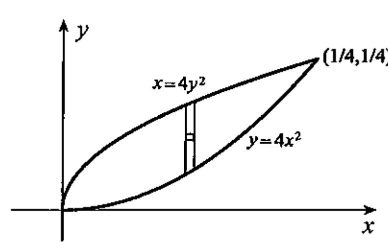
43. $$\iint_R |y-2x^2+1|\,dA = \iint_{R_1} (y-2x^2+1)\,dA + \iint_{R_2} (-y+2x^2-1)\,dA$$
$$= \int_{-1}^1\int_{2x^2-1}^1 (y-2x^2+1)\,dy\,dx + \int_{-1}^1\int_{-1}^{2x^2-1} (-y+2x^2-1)\,dy\,dx$$
$$= \int_{-1}^1 \left\{\frac{y^2}{2} - 2x^2y + y\right\}_{2x^2-1}^1 dx$$
$$+ \int_{-1}^1 \left\{-\frac{y^2}{2} + 2x^2y - y\right\}_{-1}^{2x^2-1} dx$$
$$= 2\int_{-1}^1 (x^4-2x^2+1)\,dx + 2\int_{-1}^1 x^4\,dx$$
$$= 2\left\{\frac{x^5}{5} - \frac{2x^3}{3} + x\right\}_{-1}^1 + 2\left\{\frac{x^5}{5}\right\}_{-1}^1 = \frac{44}{15}$$



44. If we set $b = ay/c$, then
$$n = \frac{2n_cL}{\pi}\int_0^{2d}(1-b^2)\int_0^\infty \frac{x^2}{(1+x^2)(x^2+b^2)}\,dx\,dy,$$
and partial fractions gives
$$n = \frac{2n_cL}{\pi}\int_0^{2d}(1-b^2)\int_0^\infty \left[\frac{1/(1-b^2)}{1+x^2} - \frac{b^2/(1-b^2)}{b^2+x^2}\right]dx\,dy$$
$$= \frac{2n_cL}{\pi}\int_0^{2d}\left\{\mathrm{Tan}^{-1}x - b\,\mathrm{Tan}^{-1}\left(\frac{x}{b}\right)\right\}_0^\infty dy$$
$$= \frac{2n_cL}{\pi}\int_0^{2d}\left(\frac{\pi}{2} - \frac{b\pi}{2}\right)dy$$
$$= n_cL\int_0^{2d}\left(1-\frac{ay}{c}\right)dy = n_cL\left\{y - \frac{ay^2}{2c}\right\}_0^{2d}$$
$$= n_cL\left(2d - \frac{2ad^2}{c}\right) = 2n_cdL\left(1-\frac{ad}{c}\right).$$

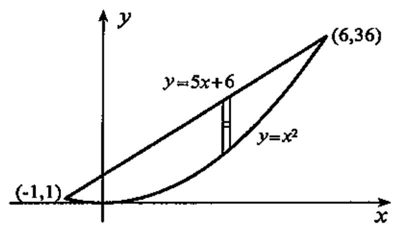
EXERCISES 13.3

1. $$A = \int_0^{1/4}\int_{4x^2}^{\sqrt{x}/2} dy\,dx = \int_0^{1/4}\left(\frac{\sqrt{x}}{2} - 4x^2\right)dx$$
$$= \left\{\frac{x^{3/2}}{3} - \frac{4x^3}{3}\right\}_0^{1/4} = \frac{1}{48}$$



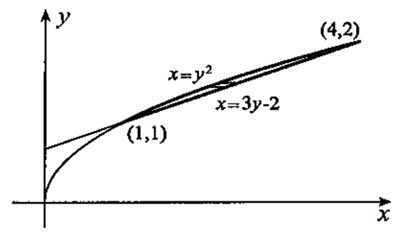
2. $A = \int_{-1}^{6}\int_{x^2}^{5x+6} dy\,dx = \int_{-1}^{6}(5x+6-x^2)\,dx$

$$= \left\{\frac{5x^2}{2} + 6x - \frac{x^3}{3}\right\}_{-1}^{6} = \frac{343}{6}$$



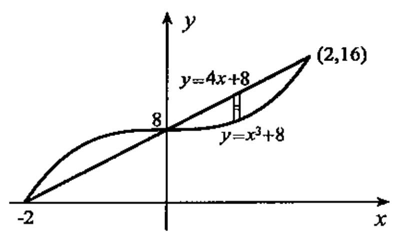
3. $A = \int_{1}^{2}\int_{y^2}^{3y-2} dx\,dy = \int_{1}^{2}(3y-2-y^2)\,dy$

$$= \left\{\frac{3y^2}{2} - 2y - \frac{y^3}{3}\right\}_{1}^{2} = \frac{1}{6}$$



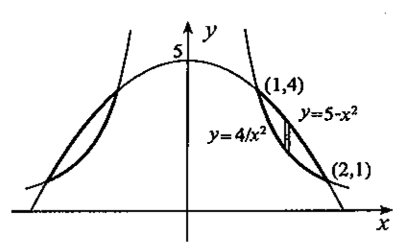
4. $A = 2\int_{0}^{2}\int_{x^3+8}^{4x+8} dy\,dx = 2\int_{0}^{2}(4x-x^3)\,dx$

$$= 2\left\{2x^2 - \frac{x^4}{4}\right\}_{0}^{2} = 8$$



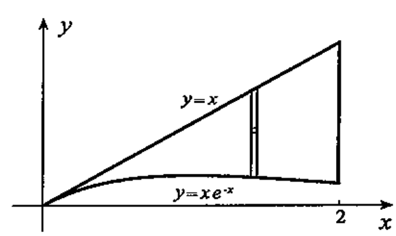
5. $A = 2\int_{1}^{2}\int_{4/x^2}^{5-x^2} dy\,dx = 2\int_{1}^{2}\left(5 - x^2 - \frac{4}{x^2}\right)dx$

$$= 2\left\{5x - \frac{x^3}{3} + \frac{4}{x}\right\}_{1}^{2} = \frac{4}{3}$$



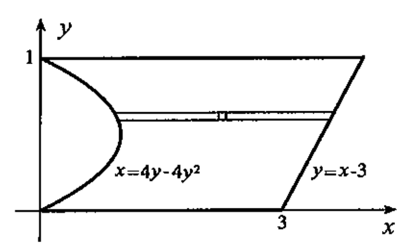
6. $A = \int_{0}^{2}\int_{xe^{-x}}^{x} dy\,dx = \int_{0}^{2}(x - xe^{-x})\,dx$

$$= \left\{\frac{x^2}{2} + xe^{-x} + e^{-x}\right\}_{0}^{2} = 1 + 3e^{-2}$$



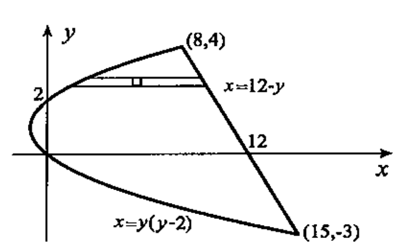
7. $A = \int_{0}^{1}\int_{4y-4y^2}^{y+3} dx\,dy = \int_{0}^{1}(3 - 3y + 4y^2)\,dy$

$$= \left\{3y - \frac{3y^2}{2} + \frac{4y^3}{3}\right\}_{0}^{1} = \frac{17}{6}$$

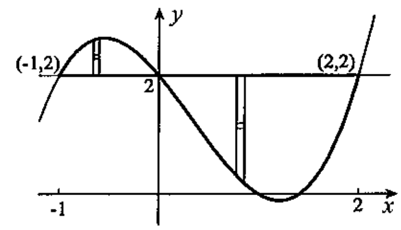


8. $A = \int_{-3}^{4}\int_{y(y-2)}^{12-y} dx\,dy = \int_{-3}^{4}(12 - y^2 + y)\,dy$

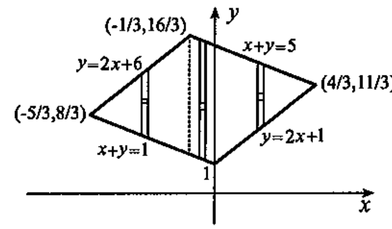
$$= \left\{12y - \frac{y^3}{3} + \frac{y^2}{2}\right\}_{-3}^{4} = \frac{343}{6}$$



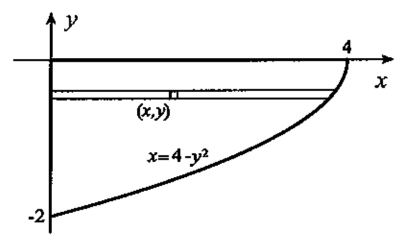
9. $$A = \int_{-1}^{0}\int_{2}^{x^3-x^2-2x+2} dy\,dx + \int_{0}^{2}\int_{x^3-x^2-2x+2}^{2} dy\,dx$$
$$= \int_{-1}^{0}(x^3 - x^2 - 2x)\,dx + \int_{0}^{2}(-x^3 + x^2 + 2x)\,dx$$
$$= \left\{\frac{x^4}{4} - \frac{x^3}{3} - x^2\right\}_{-1}^{0} + \left\{-\frac{x^4}{4} + \frac{x^3}{3} + x^2\right\}_{0}^{2} = \frac{37}{12}$$



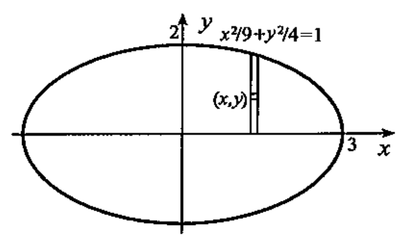
10. $$A = \int_{-5/3}^{-1/3}\int_{1-x}^{2x+6} dy\,dx + \int_{-1/3}^{0}\int_{1-x}^{5-x} dy\,dx + \int_{0}^{4/3}\int_{2x+1}^{5-x} dy\,dx$$
$$= \int_{-5/3}^{-1/3}(3x+5)\,dx + \int_{-1/3}^{0} 4\,dx + \int_{0}^{4/3}(4-3x)\,dx$$
$$= \left\{\frac{3x^2}{2} + 5x\right\}_{-5/3}^{-1/3} + \left\{4x\right\}_{-1/3}^{0} + \left\{4x - \frac{3x^2}{2}\right\}_{0}^{4/3} = \frac{20}{3}$$



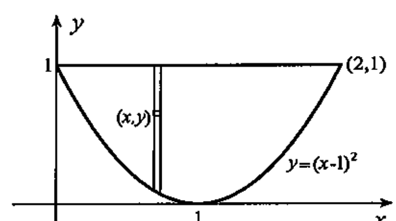
11. $$V = \int_{-2}^{0}\int_{0}^{4-y^2} 2\pi(-y)\,dx\,dy = -2\pi\int_{-2}^{0}\left\{xy\right\}_{0}^{4-y^2} dy$$
$$= -2\pi\int_{-2}^{0} y(4-y^2)\,dy = -2\pi\left\{2y^2 - \frac{y^4}{4}\right\}_{-2}^{0} = 8\pi$$



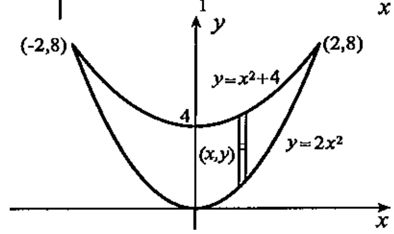
12. $$V = 2\int_{0}^{3}\int_{0}^{2\sqrt{9-x^2}/3} 2\pi y\,dy\,dx = 2\pi\int_{0}^{3}\left\{y^2\right\}_{0}^{2\sqrt{9-x^2}/3} dx$$
$$= \frac{8\pi}{9}\int_{0}^{3}(9-x^2)\,dx = \frac{8\pi}{9}\left\{9x - \frac{x^3}{3}\right\}_{0}^{3} = 16\pi$$



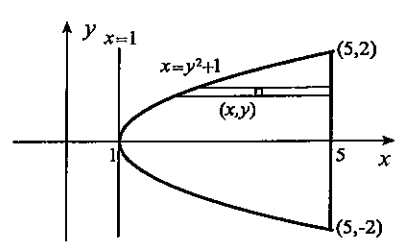
13. $$V = \int_{0}^{2}\int_{(x-1)^2}^{1} 2\pi x\,dy\,dx = 2\pi\int_{0}^{2}\left\{xy\right\}_{(x-1)^2}^{1} dx$$
$$= 2\pi\int_{0}^{2}(-x^3 + 2x^2)\,dx = 2\pi\left\{-\frac{x^4}{4} + \frac{2x^3}{3}\right\}_{0}^{2} = \frac{8\pi}{3}$$



14. $$V = 2\int_{0}^{2}\int_{2x^2}^{x^2+4} 2\pi y\,dy\,dx = 2\pi\int_{0}^{2}\left\{y^2\right\}_{2x^2}^{x^2+4} dx$$
$$= 2\pi\int_{0}^{2}(16 + 8x^2 - 3x^4)\,dx$$
$$= 2\pi\left\{16x + \frac{8x^3}{3} - \frac{3x^5}{5}\right\}_{0}^{2} = \frac{1024\pi}{15}$$



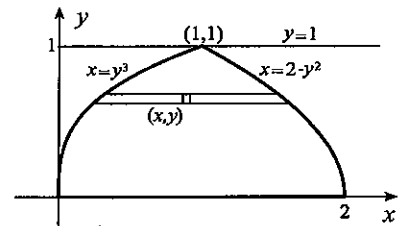
15. $$V = 2\int_{0}^{2}\int_{y^2+1}^{5} 2\pi(x-1)\,dx\,dy = 4\pi\int_{0}^{2}\left\{\frac{x^2}{2} - x\right\}_{y^2+1}^{5} dy$$
$$= 2\pi\int_{0}^{2}(16 - y^4)\,dy = 2\pi\left\{16y - \frac{y^5}{5}\right\}_{0}^{2} = \frac{256\pi}{5}$$



16. $V = \int_0^1 \int_{y^3}^{2-y^2} 2\pi(1-y)\,dx\,dy = 2\pi \int_0^1 \left\{x(1-y)\right\}_{y^3}^{2-y^2} dy$

$= 2\pi \int_0^1 (2 - 2y - y^2 + y^4)\,dy$

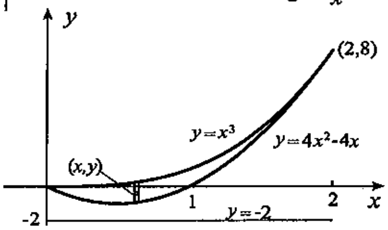
$= 2\pi \left\{2y - y^2 - \frac{y^3}{3} + \frac{y^5}{5}\right\}_0^1 = \frac{26\pi}{15}$



17. $V = \int_0^2 \int_{4x^2-4x}^{x^3} 2\pi(y+2)\,dy\,dx = 2\pi \int_0^2 \left\{\frac{y^2}{2} + 2y\right\}_{4x^2-4x}^{x^3} dx$

$= \pi \int_0^2 (x^6 - 16x^4 + 36x^3 - 32x^2 + 16x)\,dx$

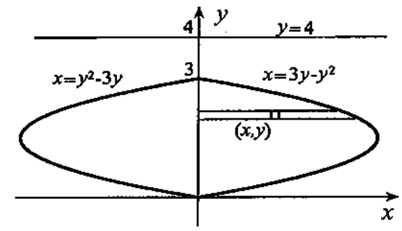
$= \pi \left\{\frac{x^7}{7} - \frac{16x^5}{5} + 9x^4 - \frac{32x^3}{3} + 8x^2\right\}_0^2 = \frac{668\pi}{105}$



18. $V = 2\int_0^3 \int_0^{3y-y^2} 2\pi(4-y)\,dx\,dy = 4\pi \int_0^3 \left\{x(4-y)\right\}_0^{3y-y^2} dy$

$= 4\pi \int_0^3 (12y - 7y^2 + y^3)\,dy$

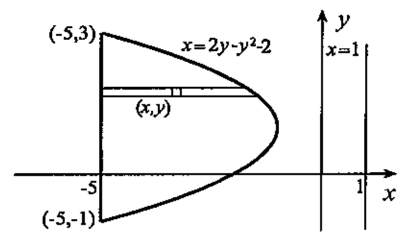
$= 4\pi \left\{6y^2 - \frac{7y^3}{3} + \frac{y^4}{4}\right\}_0^3 = 45\pi$



19. $V = \int_{-1}^3 \int_{-5}^{2y-y^2-2} 2\pi(1-x)\,dx\,dy = 2\pi \int_{-1}^3 \left\{x - \frac{x^2}{2}\right\}_{-5}^{2y-y^2-2} dy$

$= \pi \int_{-1}^3 (27 + 12y - 10y^2 + 4y^3 - y^4)\,dy$

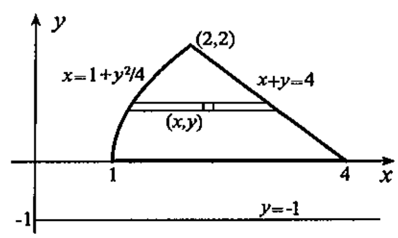
$= \pi \left\{27y + 6y^2 - \frac{10y^3}{3} + y^4 - \frac{y^5}{5}\right\}_{-1}^3 = \frac{1408\pi}{15}$



20. $V = \int_0^2 \int_{y^2/4+1}^{4-y} 2\pi(y+1)\,dx\,dy = 2\pi \int_0^2 \left\{x(y+1)\right\}_{y^2/4+1}^{4-y} dy$

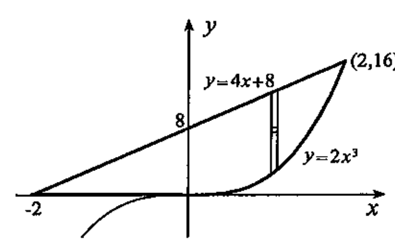
$= \frac{\pi}{2} \int_0^2 (12 + 8y - 5y^2 - y^3)\,dy$

$= \frac{\pi}{2} \left\{12y + 4y^2 - \frac{5y^3}{3} - \frac{y^4}{4}\right\}_0^2 = \frac{34\pi}{3}$



21. $A = \frac{1}{2}(2)(8) + \int_0^2 \int_{2x^3}^{4x+8} dy\,dx = 8 + \int_0^2 (8 + 4x - 2x^3)\,dx$

$= 8 + \left\{8x + 2x^2 - \frac{x^4}{2}\right\}_0^2 = 24$

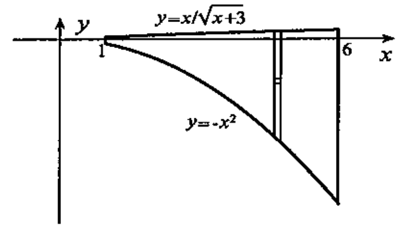


22. $A = \int_1^6 \int_{-x^2}^{x/\sqrt{x+3}} dy\,dx = \int_1^6 \left(\frac{x}{\sqrt{x+3}} + x^2 \right) dx$

If we set $u = x + 3$ and $du = dx$,

$$A = \int_4^9 \left(\frac{u-3}{\sqrt{u}} \right) du + \left\{ \frac{x^3}{3} \right\}_1^6 = \int_4^9 \left(\sqrt{u} - \frac{3}{\sqrt{u}} \right) du + \frac{215}{3}$$

$$= \left\{ \frac{2}{3} u^{3/2} - 6\sqrt{u} \right\}_4^9 + \frac{215}{3} = \frac{235}{3}$$

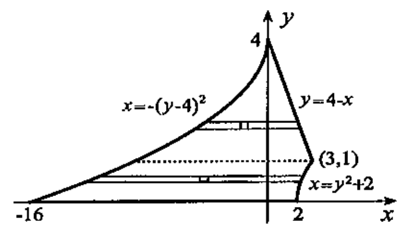


23. $A = \int_0^1 \int_{-(y-4)^2}^{y^2+2} dx\,dy + \int_1^4 \int_{-(y-4)^2}^{4-y} dx\,dy$

$$= \int_0^1 [y^2 + 2 + (y-4)^2]\,dy + \int_1^4 [4 - y + (y-4)^2]\,dy$$

$$= \left\{ \frac{y^3}{3} + 2y + \frac{(y-4)^3}{3} \right\}_0^1 + \left\{ 4y - \frac{y^2}{2} + \frac{(y-4)^3}{3} \right\}_1^4$$

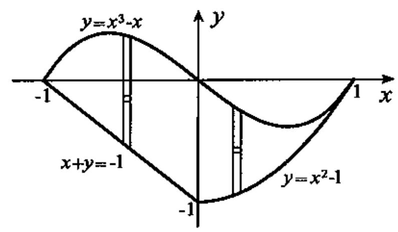
$$= \frac{169}{6}$$



24. $A = \int_{-1}^0 \int_{-1-x}^{x^3-x} dy\,dx + \int_0^1 \int_{x^2-1}^{x^3-x} dy\,dx$

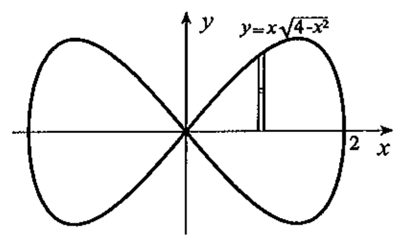
$$= \int_{-1}^0 (x^3 + 1)\,dx + \int_0^1 (x^3 - x - x^2 + 1)\,dx$$

$$= \left\{ \frac{x^4}{4} + x \right\}_{-1}^0 + \left\{ \frac{x^4}{4} - \frac{x^2}{2} - \frac{x^3}{3} + x \right\}_0^1 = \frac{7}{6}$$



25. $A = 4 \int_0^2 \int_0^{x\sqrt{4-x^2}} dy\,dx = 4 \int_0^2 x\sqrt{4-x^2}\,dx$

$$= 4 \left\{ -\frac{1}{3}(4-x^2)^{3/2} \right\}_0^2 = \frac{32}{3}$$

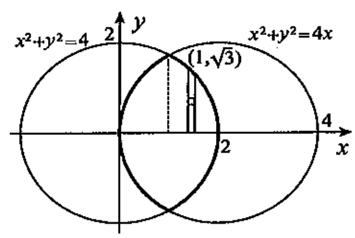


26. $A = 4 \int_1^2 \int_0^{\sqrt{4-x^2}} dy\,dx = 4 \int_1^2 \sqrt{4-x^2}\,dx$

If we set $x = 2\sin\theta$ and $dx = 2\cos\theta\,d\theta$,

$$A = 4 \int_{\pi/6}^{\pi/2} 2\cos\theta\, 2\cos\theta\,d\theta = 16 \int_{\pi/6}^{\pi/2} \left(\frac{1 + \cos 2\theta}{2} \right) d\theta$$

$$= 8 \left\{ \theta + \frac{1}{2}\sin 2\theta \right\}_{\pi/6}^{\pi/2} = \frac{8\pi}{3} - 2\sqrt{3}$$

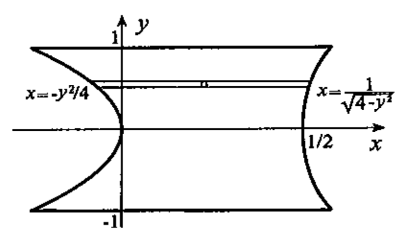


27. $A = 2 \int_0^1 \int_{-y^2/4}^{1/\sqrt{4-y^2}} dx\,dy = 2 \int_0^1 \left(\frac{1}{\sqrt{4-y^2}} + \frac{y^2}{4} \right) dy$

If we set $y = 2\sin\theta$ and $dy = 2\cos\theta\,d\theta$, then

$$A = 2 \int_0^{\pi/6} \frac{1}{2\cos\theta} 2\cos\theta\,d\theta + 2 \left\{ \frac{y^3}{12} \right\}_0^1$$

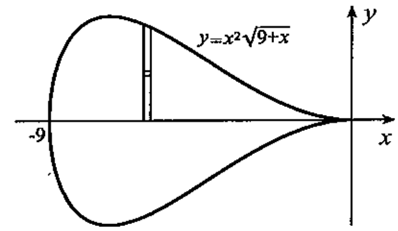
$$= 2\Big\{ \theta \Big\}_0^{\pi/6} + \frac{1}{6} = \frac{\pi}{3} + \frac{1}{6}$$



28. $A = 2\int_{-9}^{0}\int_{0}^{x^2\sqrt{9+x}} dy\,dx = 2\int_{-9}^{0} x^2\sqrt{9+x}\,dx$

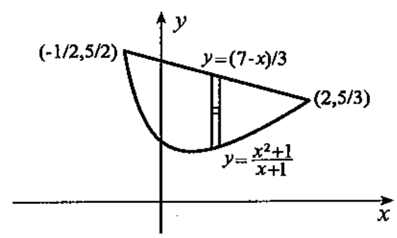
If we set $u = 9 + x$ and $du = dx$,

$$A = 2\int_0^9 (u-9)^2\sqrt{u}\,du$$
$$= 2\int_0^9 (u^{5/2} - 18u^{3/2} + 81u^{1/2})\,du$$
$$= 2\left\{\frac{2u^{7/2}}{7} - \frac{36u^{5/2}}{5} + \frac{162u^{3/2}}{3}\right\}_0^9 = \frac{23\,328}{35}$$



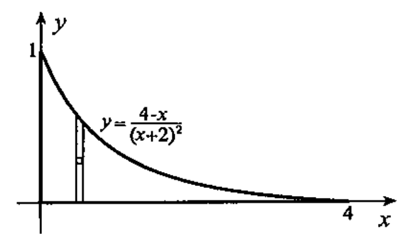
29. $A = \int_{-1/2}^{2}\int_{(x^2+1)/(x+1)}^{(7-x)/3} dy\,dx$

$$= \int_{-1/2}^{2}\left(\frac{7-x}{3} - \frac{x^2+1}{x+1}\right)dx$$
$$= \int_{-1/2}^{2}\left(\frac{10}{3} - \frac{4x}{3} - \frac{2}{x+1}\right)dx$$
$$= \left\{\frac{10x}{3} - \frac{2x^2}{3} - 2\ln|x+1|\right\}_{-1/2}^{2} = \frac{35}{6} - 2\ln 6$$



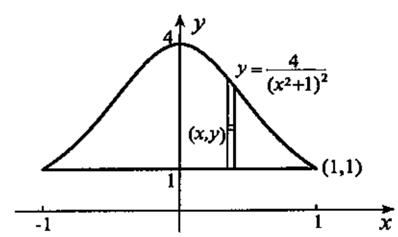
30. $A = \int_0^4\int_0^{(4-x)/(x+2)^2} dy\,dx = \int_0^4 \frac{4-x}{(x+2)^2}dx$

$$= \int_0^4\left[\frac{6}{(x+2)^2} - \frac{1}{x+2}\right]dx$$
$$= \left\{-\frac{6}{x+2} - \ln|x+2|\right\}_0^4 = 2 - \ln 3$$



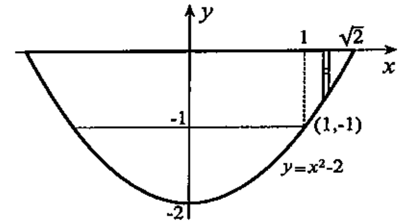
31. $V = \int_0^1\int_1^{4/(x^2+1)^2} 2\pi x\,dy\,dx = 2\pi\int_0^1\left\{xy\right\}_1^{4/(x^2+1)^2}dx$

$$= 2\pi\int_0^1\left[\frac{4x}{(x^2+1)^2} - x\right]dx = 2\pi\left\{\frac{-2}{x^2+1} - \frac{x^2}{2}\right\}_0^1 = \pi$$

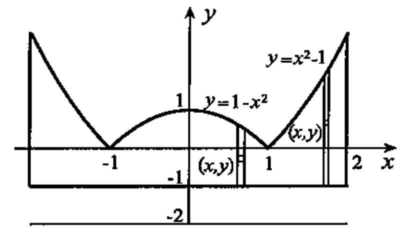


32. We reject the area below $y = -1$ to obtain

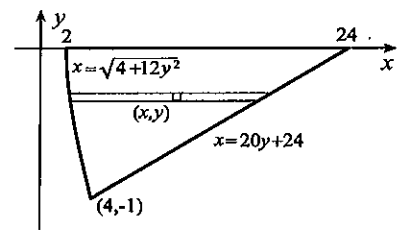
$$V = 2\pi(1)^2(1) + 2\int_1^{\sqrt{2}}\int_{x^2-2}^{0} 2\pi(y+1)\,dy\,dx$$
$$= 2\pi + 4\pi\int_1^{\sqrt{2}}\left\{\frac{1}{2}(y+1)^2\right\}_{x^2-2}^{0}dx$$
$$= 2\pi + 2\pi\int_1^{\sqrt{2}}(-x^4 + 2x^2)\,dx$$
$$= 2\pi + 2\pi\left\{-\frac{x^5}{5} + \frac{2x^3}{3}\right\}_1^{\sqrt{2}} = \frac{16\pi(\sqrt{2}+1)}{15}$$



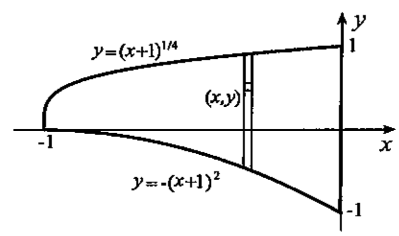
33. $$V = 2\int_0^1 \int_{-1}^{1-x^2} 2\pi(y+2)\,dy\,dx + 2\int_1^2 \int_{-1}^{x^2-1} 2\pi(y+2)\,dy\,dx$$
$$= 4\pi \int_0^1 \left\{ \frac{(y+2)^2}{2} \right\}_{-1}^{1-x^2} dx + 4\pi \int_1^2 \left\{ \frac{(y+2)^2}{2} \right\}_{-1}^{x^2-1} dx$$
$$= 2\pi \int_0^1 (8 - 6x^2 + x^4)\,dx + 2\pi \int_1^2 (2x^2 + x^4)\,dx$$
$$= 2\pi \left\{ 8x - 2x^3 + \frac{x^5}{5} \right\}_0^1 + 2\pi \left\{ \frac{2x^3}{3} + \frac{x^5}{5} \right\}_1^2 = \frac{512\pi}{15}$$



34. $$V = \int_{-1}^0 \int_{\sqrt{4+12y^2}}^{20y+24} 2\pi(-y)\,dx\,dy = 2\pi \int_{-1}^0 \left\{ -xy \right\}_{\sqrt{4+12y^2}}^{20y+24} dy$$
$$= 2\pi \int_{-1}^0 (y\sqrt{4+12y^2} - 20y^2 - 24y)\,dy$$
$$= 2\pi \left\{ \frac{1}{36}(4+12y^2)^{3/2} - \frac{20y^3}{3} - 12y^2 \right\}_{-1}^0 = \frac{68\pi}{9}$$

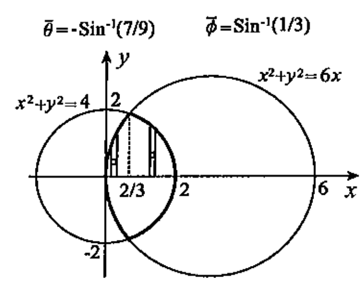


35. $$V = \int_{-1}^0 \int_{-(x+1)^2}^{(x+1)^{1/4}} 2\pi(-x)\,dy\,dx = -2\pi \int_{-1}^0 \left\{ xy \right\}_{-(x+1)^2}^{(x+1)^{1/4}} dx$$
$$= -2\pi \int_{-1}^0 [x(x+1)^{1/4} + x(x+1)^2]\,dx$$
If we set $u = x+1$ and $du = dx$ in the first term,



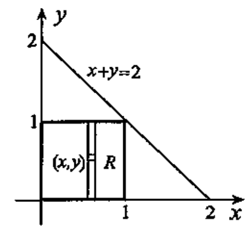
$$V = -2\pi \int_0^1 (u-1)u^{1/4}\,du - 2\pi \int_{-1}^0 (x^3 + 2x^2 + x)\,dx$$
$$= -2\pi \left\{ \frac{4u^{9/4}}{9} - \frac{4u^{5/4}}{5} \right\}_0^1 - 2\pi \left\{ \frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} \right\}_{-1}^0 = \frac{79\pi}{90}$$

36. $$A = 2\int_0^{2/3} \int_0^{\sqrt{6x-x^2}} dy\,dx + 2\int_{2/3}^2 \int_0^{\sqrt{4-x^2}} dy\,dx$$
$$= 2\int_0^{2/3} \sqrt{6x - x^2}\,dx + 2\int_{2/3}^2 \sqrt{4-x^2}\,dx$$
$$= 2\int_0^{2/3} \sqrt{9 - (x-3)^2}\,dx + 2\int_{2/3}^2 \sqrt{4-x^2}\,dx$$
If we set $x - 3 = 3\sin\theta$ in the first integral,
and $x = 2\sin\phi$ in the second,

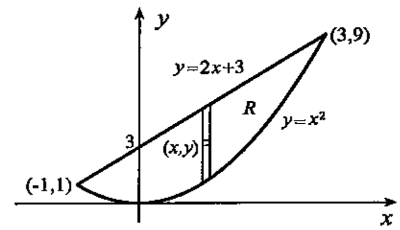


$$A = 2\int_{-\pi/2}^{\overline{\theta}} 3\cos\theta\, 3\cos\theta\,d\theta + 2\int_{\overline{\phi}}^{\pi/2} 2\cos\phi\, 2\cos\phi\,d\phi$$
$$= 18\int_{-\pi/2}^{\overline{\theta}} \left(\frac{1+\cos 2\theta}{2} \right) d\theta + 8\int_{\overline{\phi}}^{\pi/2} \left(\frac{1+\cos 2\phi}{2} \right) d\phi$$
$$= 9\left\{ \theta + \frac{1}{2}\sin 2\theta \right\}_{-\pi/2}^{\overline{\theta}} + 4\left\{ \phi + \frac{1}{2}\sin 2\phi \right\}_{\overline{\phi}}^{\pi/2}$$
$$= 9\left(\overline{\theta} + \sin\overline{\theta}\,\cos\overline{\theta} + \frac{\pi}{2} \right) + 4\left(\frac{\pi}{2} - \overline{\phi} - \sin\overline{\phi}\,\cos\overline{\phi} \right) = 5.38.$$

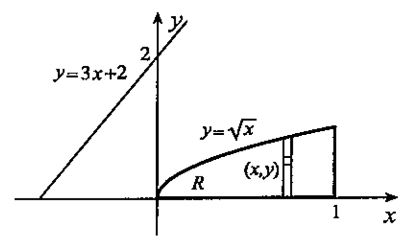
37. $$V = \iint_R 2\pi \frac{|x+y-2|}{\sqrt{2}} dA = \sqrt{2}\pi \int_0^1 \int_0^1 (2-x-y)\, dy\, dx$$
$$= \sqrt{2}\pi \int_0^1 \left\{ 2y - xy - \frac{y^2}{2} \right\}_0^1 dx$$
$$= \sqrt{2}\pi \int_0^1 \left(2 - x - \frac{1}{2}\right) dx = \sqrt{2}\pi \left\{ \frac{3x}{2} - \frac{x^2}{2} \right\}_0^1 = \sqrt{2}\pi$$



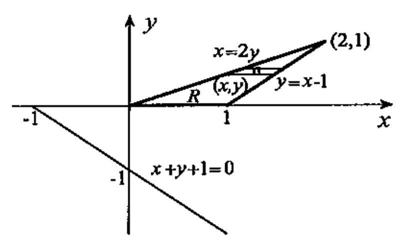
38. $$V = \iint_R 2\pi \frac{|2x-y+3|}{\sqrt{5}} dA = \frac{2\pi}{\sqrt{5}} \int_{-1}^3 \int_{x^2}^{2x+3} (2x-y+3)\, dy\, dx$$
$$= \frac{2\pi}{\sqrt{5}} \int_{-1}^3 \left\{ (2x+3)y - \frac{y^2}{2} \right\}_{x^2}^{2x+3} dx$$
$$= \frac{2\pi}{\sqrt{5}} \int_{-1}^3 \left[\frac{1}{2}(2x+3)^2 + \frac{x^4}{2} - 2x^3 - 3x^2 \right] dx$$
$$= \frac{2\pi}{\sqrt{5}} \left\{ \frac{1}{12}(2x+3)^3 + \frac{x^5}{10} - \frac{x^4}{2} - x^3 \right\}_{-1}^3 = \frac{512\pi}{15\sqrt{5}}$$



39. $$V = \iint_R 2\pi \frac{|y-3x-2|}{\sqrt{10}} dA = \frac{2\pi}{\sqrt{10}} \int_0^1 \int_0^{\sqrt{x}} (3x+2-y)\, dy\, dx$$
$$= \frac{2\pi}{\sqrt{10}} \int_0^1 \left\{ (3x+2)y - \frac{y^2}{2} \right\}_0^{\sqrt{x}} dx$$
$$= \frac{\pi}{\sqrt{10}} \int_0^1 (6x^{3/2} + 4\sqrt{x} - x)\, dx$$
$$= \frac{\pi}{\sqrt{10}} \left\{ \frac{12x^{5/2}}{5} + \frac{8x^{3/2}}{3} - \frac{x^2}{2} \right\}_0^1 = \frac{137\pi}{30\sqrt{10}}$$

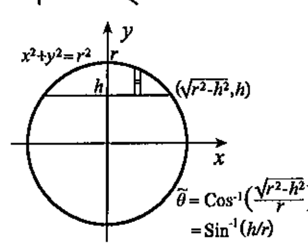


40. $$V = \iint_R 2\pi \frac{|x+y+1|}{\sqrt{2}} dA = \sqrt{2}\pi \int_0^1 \int_{2y}^{y+1} (x+y+1)\, dx\, dy$$
$$= \sqrt{2}\pi \int_0^1 \left\{ \frac{1}{2}(x+y+1)^2 \right\}_{2y}^{y+1} dy$$
$$= \frac{\pi}{\sqrt{2}} \int_0^1 [(2y+2)^2 - (3y+1)^2]\, dy$$
$$= \frac{\pi}{\sqrt{2}} \left\{ \frac{1}{6}(2y+2)^3 - \frac{1}{9}(3y+1)^3 \right\}_0^1 = \frac{7\sqrt{2}\pi}{6}$$



41. $$A = 2\int_0^{\sqrt{r^2-h^2}} \int_h^{\sqrt{r^2-x^2}} dy\, dx = 2\int_0^{\sqrt{r^2-h^2}} (\sqrt{r^2-x^2} - h)\, dx$$

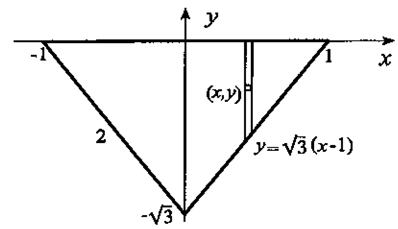
If we set $x = r\cos\theta$ and $dx = -r\sin\theta\, d\theta$,



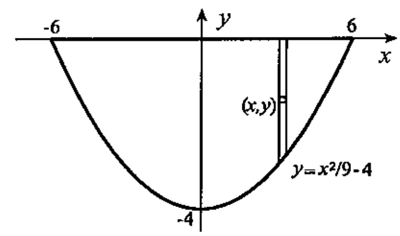
$$A = 2\int_{\pi/2}^{\tilde{\theta}} r\sin\theta(-r\sin\theta\, d\theta) - 2h\Big\{x\Big\}_0^{\sqrt{r^2-h^2}}$$
$$= 2r^2 \int_{\pi/2}^{\tilde{\theta}} \left(\frac{\cos 2\theta - 1}{2} \right) d\theta - 2h\sqrt{r^2-h^2}$$
$$= r^2 \left\{ \frac{1}{2}\sin 2\theta - \theta \right\}_{\pi/2}^{\tilde{\theta}} - 2h\sqrt{r^2-h^2} = r^2(\sin\tilde{\theta}\cos\tilde{\theta} - \tilde{\theta} + \pi/2) - 2h\sqrt{r^2-h^2}$$
$$= \frac{\pi r^2}{2} + r^2\left[\frac{h}{r}\frac{\sqrt{r^2-h^2}}{r} - \mathrm{Sin}^{-1}(h/r) \right] - 2h\sqrt{r^2-h^2} = \frac{\pi r^2}{2} - h\sqrt{r^2-h^2} - r^2\mathrm{Sin}^{-1}(h/r).$$

EXERCISES 13.4

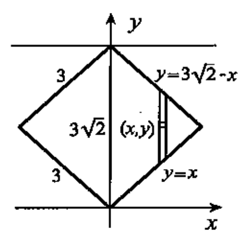
1. $$F = 2\int_0^1 \int_{\sqrt{3}(x-1)}^0 \rho g(-y)\,dy\,dx \quad (g = 9.81)$$
$$= -2\rho g \int_0^1 \left\{\frac{y^2}{2}\right\}_{\sqrt{3}(x-1)}^0 dx$$
$$= \rho g \int_0^1 3(x-1)^2\,dx = \rho g \Big\{(x-1)^3\Big\}_0^1 = \rho g$$



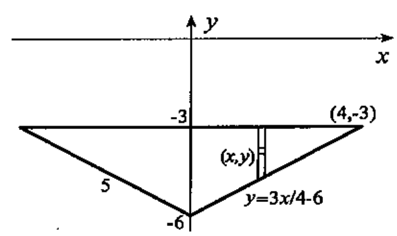
2. $$F = 2\int_0^6 \int_{x^2/9-4}^0 \rho g(-y)\,dy\,dx \quad (g = 9.81)$$
$$= -2\rho g \int_0^6 \left\{\frac{y^2}{2}\right\}_{x^2/9-4}^0 dx = \rho g \int_0^6 \left(\frac{x^2}{9} - 4\right)^2 dx$$
$$= \frac{\rho g}{81}\int_0^6 (x^4 - 72x^2 + 1296)\,dx$$
$$= \frac{\rho g}{81}\left\{\frac{x^5}{5} - 24x^3 + 1296x\right\}_0^6 = \frac{256\rho g}{5}$$



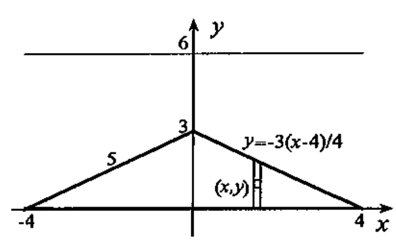
3. $$F = 2\int_0^{3/\sqrt{2}} \int_x^{3\sqrt{2}-x} \rho g(3\sqrt{2} - y)\,dy\,dx \quad (g = 9.81)$$
$$= 2\rho g \int_0^{3/\sqrt{2}} \left\{-\frac{1}{2}(3\sqrt{2} - y)^2\right\}_x^{3\sqrt{2}-x} dx$$
$$= \rho g \int_0^{3/\sqrt{2}} \left[(3\sqrt{2} - x)^2 - x^2\right] dx$$
$$= \rho g \left\{-\frac{1}{3}(3\sqrt{2} - x)^3 - \frac{x^3}{3}\right\}_0^{3/\sqrt{2}} = \frac{27\rho g}{\sqrt{2}}$$



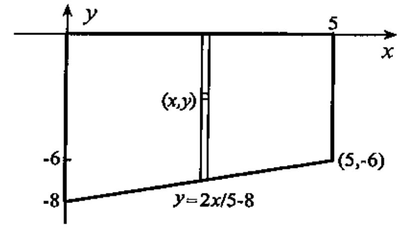
4. $$F = 2\int_0^4 \int_{3x/4-6}^{-3} \rho g(-y)\,dy\,dx \quad (g = 9.81)$$
$$= -2\rho g \int_0^4 \left\{\frac{y^2}{2}\right\}_{3x/4-6}^{-3} dx = \rho g \int_0^4 \left[\left(\frac{3x}{4} - 6\right)^2 - 9\right] dx$$
$$= \rho g \left\{\frac{4}{9}\left(\frac{3x}{4} - 6\right)^3 - 9x\right\}_0^4 = 48\rho g$$



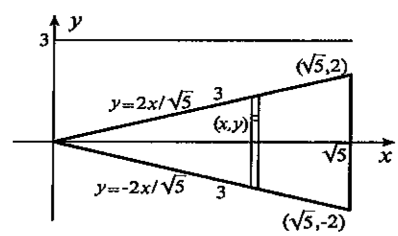
5. $$F = 2\int_0^4 \int_0^{-3(x-4)/4} \rho g(6 - y)\,dy\,dx \quad (g = 9.81)$$
$$= 2\rho g \int_0^4 \left\{-\frac{1}{2}(6 - y)^2\right\}_0^{-3(x-4)/4} dx$$
$$= \rho g \int_0^4 \left\{36 - \left[6 + \frac{3}{4}(x-4)\right]^2\right\} dx$$
$$= \rho g \left\{36x - \frac{4}{9}\left(3 + \frac{3x}{4}\right)^3\right\}_0^4 = 60\rho g$$



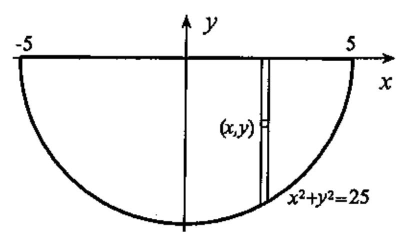
6. $$F = \int_0^5 \int_{2x/5-8}^0 \rho g(-y)\,dy\,dx \quad (g = 9.81)$$
$$= -\rho g \int_0^5 \left\{\frac{y^2}{2}\right\}_{2x/5-8}^0 dx = \frac{\rho g}{2}\int_0^5 \left(\frac{2x}{5} - 8\right)^2 dx$$
$$= \frac{\rho g}{2}\left\{\frac{5}{6}\left(\frac{2x}{5} - 8\right)^3\right\}_0^5 = \frac{370\rho g}{3}$$



7. $$F = \int_0^{\sqrt5}\int_{-2x/\sqrt5}^{2x/\sqrt5} \rho g(3-y)\,dy\,dx \quad (g = 9.81)$$
$$= \rho g \int_0^{\sqrt5}\left\{-\frac{1}{2}(3-y)^2\right\}_{-2x/\sqrt5}^{2x/\sqrt5} dx$$
$$= \frac{\rho g}{2}\int_0^{\sqrt5}\left[\left(3 + \frac{2x}{\sqrt5}\right)^2 - \left(3 - \frac{2x}{\sqrt5}\right)^2\right] dx$$
$$= \frac{\rho g}{2}\left\{\frac{\sqrt5}{6}\left(3 + \frac{2x}{\sqrt5}\right)^3 + \frac{\sqrt5}{6}\left(3 - \frac{2x}{\sqrt5}\right)^3\right\}_0^{\sqrt5} = 6\sqrt5\rho g$$

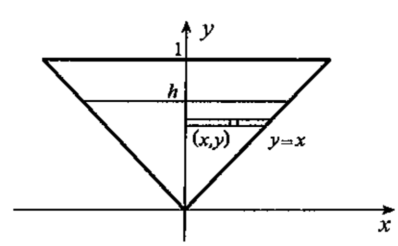


8. $$F = 2\int_0^5 \int_{-\sqrt{25-x^2}}^0 \rho g(-y)\,dy\,dx \quad (g = 9.81)$$
$$= -2\rho g \int_0^5 \left\{\frac{y^2}{2}\right\}_{-\sqrt{25-x^2}}^0 dx = \rho g \int_0^5 (25 - x^2)\,dx$$
$$= \rho g \left\{25x - \frac{x^3}{3}\right\}_0^5 = \frac{250\rho g}{3}$$

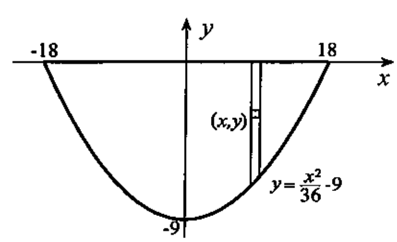


9. The trough is half-full by volume when the vertical end is half covered by area. If h is the depth of water when this happens, $h^2 = 1/2 \implies h = 1/\sqrt2$.

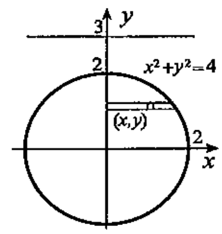
$$F = 2\int_0^{1/\sqrt2}\int_0^y \rho g\left(\frac{1}{\sqrt2} - y\right) dx\,dy \quad (g = 9.81)$$
$$= 2\rho g \int_0^{1/\sqrt2}\left\{x\left(\frac{1}{\sqrt2} - y\right)\right\}_0^y dy$$
$$= 2\rho g \int_0^{1/\sqrt2}\left(\frac{y}{\sqrt2} - y^2\right) dy = 2\rho g\left\{\frac{y^2}{2\sqrt2} - \frac{y^3}{3}\right\}_0^{1/\sqrt2} = \frac{\rho g}{6\sqrt2}$$



10. $$F = 2\int_0^{18}\int_{x^2/36-9}^0 \rho g(-y)\,dy\,dx \quad (g = 9.81)$$
$$= -2\rho g \int_0^{18}\left\{\frac{y^2}{2}\right\}_{x^2/36-9}^0 dx = \rho g \int_0^{18}\left(\frac{x^4}{1296} - \frac{x^2}{2} + 81\right) dx$$
$$= 9810\left\{\frac{x^5}{1296\cdot5} - \frac{x^3}{6} + 81x\right\}_0^{18} = 7.63\times10^6\ \text{N}$$



11. $F = 2\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} \rho g(3-y)\,dx\,dy \quad (g = 9.81)$



$$= 2\rho g\int_{-2}^{2}\Big\{x(3-y)\Big\}_0^{\sqrt{4-y^2}}dy$$

$$= 2\rho g\int_{-2}^{2}(3-y)\sqrt{4-y^2}\,dy$$

If we set $y = 2\sin\theta$ and $dy = 2\cos\theta\,d\theta$,

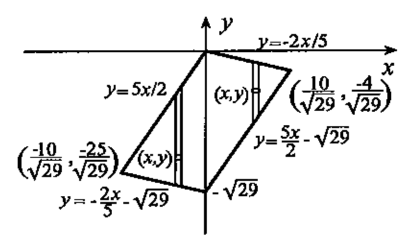
$$F = 6\rho g\int_{-\pi/2}^{\pi/2}2\cos\theta(2\cos\theta\,d\theta) - 2\rho g\left\{-\frac{1}{3}(4-y^2)^{3/2}\right\}_{-2}^{2}$$

$$= 24\rho g\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta = 12\rho g\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_{-\pi/2}^{\pi/2} = 12\rho g\pi$$

12. $F = \int_{-10/\sqrt{29}}^{0}\int_{-2x/5-\sqrt{29}}^{5x/2}\rho g(-y)\,dy\,dx \quad (g = 9.81)$

$$+\int_{0}^{10/\sqrt{29}}\int_{5x/2-\sqrt{29}}^{-2x/5}\rho g(-y)\,dy\,dx$$

$$= -\rho g\int_{-10/\sqrt{29}}^{0}\left\{\frac{y^2}{2}\right\}_{-2x/5-\sqrt{29}}^{5x/2}dx$$



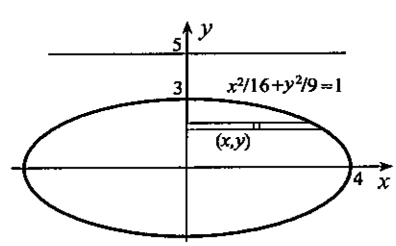
$$-\rho g\int_{0}^{10/\sqrt{29}}\left\{\frac{y^2}{2}\right\}_{5x/2-\sqrt{29}}^{-2x/5}dx$$

$$= \frac{\rho g}{2}\int_{-10/\sqrt{29}}^{0}\left[\left(-\frac{2x}{5}-\sqrt{29}\right)^2 - \frac{25x^2}{4}\right]dx$$

$$+\frac{\rho g}{2}\int_{0}^{10/\sqrt{29}}\left[\left(\frac{5x}{2}-\sqrt{29}\right)^2 - \frac{4x^2}{25}\right]dx$$

$$= \frac{\rho g}{2}\left\{\frac{5}{6}\left(\frac{2x}{5}+\sqrt{29}\right)^3 - \frac{25x^3}{12}\right\}_{-10/\sqrt{29}}^{0} + \frac{\rho g}{2}\left\{\frac{2}{15}\left(\frac{5x}{2}-\sqrt{29}\right)^3 - \frac{4x^3}{75}\right\}_{0}^{10/\sqrt{29}} = 5\sqrt{29}\rho g$$

13. $F = 2\int_{-3}^{3}\int_{0}^{(4/3)\sqrt{9-y^2}}\rho g(5-y)\,dx\,dy \quad (g = 9.81)$



$$= 2\rho g\int_{-3}^{3}\Big\{x(5-y)\Big\}_0^{(4/3)\sqrt{9-y^2}}dy$$

$$= \frac{8\rho g}{3}\int_{-3}^{3}(5-y)\sqrt{9-y^2}\,dy$$

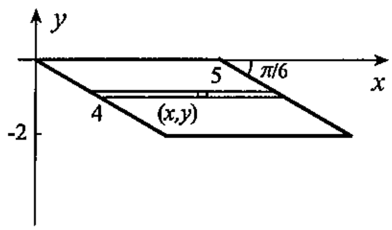
If we set $y = 3\sin\theta$ and $dy = 3\cos\theta\,d\theta$,

$$F = \frac{8\rho g}{3}\int_{-\pi/2}^{\pi/2}5(3\cos\theta)(3\cos\theta\,d\theta) - \frac{8\rho g}{3}\left\{-\frac{1}{3}(9-y^2)^{3/2}\right\}_{-3}^{3} = 120\rho g\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= 60\rho g\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_{-\pi/2}^{\pi/2} = 60\rho g\pi.$$

14. $F = \int_{-2}^{0} \rho g(-y)5\,dy \quad (g = 9.81)$

$$= -5\rho g \left\{ \frac{y^2}{2} \right\}_{-2}^{0} = 10\rho g$$

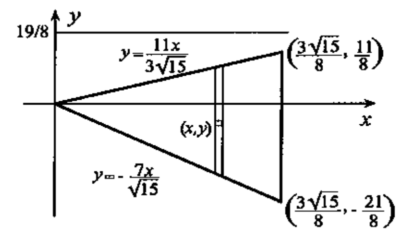


15. $F = \int_0^{3\sqrt{15}/8} \int_{-7x/\sqrt{15}}^{11x/(3\sqrt{15})} \rho g\left(\frac{19}{8} - y\right) dy\,dx \quad (g = 9.81)$

$$= \rho g \int_0^{3\sqrt{15}/8} \left\{ -\frac{1}{2}\left(\frac{19}{8} - y\right)^2 \right\}_{-7x/\sqrt{15}}^{11x/(3\sqrt{15})} dx$$

$$= \frac{\rho g}{2} \int_0^{3\sqrt{15}/8} \left[\left(\frac{19}{8} + \frac{7x}{\sqrt{15}}\right)^2 - \left(\frac{19}{8} - \frac{11x}{3\sqrt{15}}\right)^2 \right] dx$$

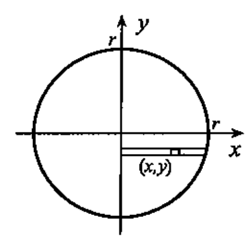
$$= \frac{\rho g}{2} \left\{ \frac{\sqrt{15}}{21}\left(\frac{19}{8} + \frac{7x}{\sqrt{15}}\right)^3 + \frac{\sqrt{15}}{11}\left(\frac{19}{8} - \frac{11x}{3\sqrt{15}}\right)^3 \right\}_0^{3\sqrt{15}/8} = \frac{67\sqrt{15}\rho g}{32}$$



16. $F = 2\int_{-r}^{r} \int_0^{\sqrt{r^2-y^2}} \rho g(r - y)\,dx\,dy \quad (g = 9.81)$

$$= 2\rho g \int_{-r}^{r} \left\{ x(r - y) \right\}_0^{\sqrt{r^2-y^2}} dy$$

$$= 2\rho g \int_{-r}^{r} (r - y)\sqrt{r^2 - y^2}\,dy$$



If we set $y = r\sin\theta$ and $dy = r\cos\theta\,d\theta$ in the first term,

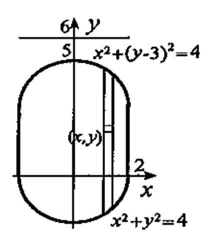
$$F = 2\rho g \int_{-\pi/2}^{\pi/2} r(r\cos\theta)r\cos\theta\,d\theta + 2\rho g \left\{ \frac{1}{3}(r^2 - y^2)^{3/2} \right\}_{-r}^{r}$$

$$= 2\rho g r^3 \int_{-\pi/2}^{\pi/2} \left(\frac{1 + \cos 2\theta}{2} \right) d\theta = \rho g r^3 \left\{ \theta + \frac{\sin 2\theta}{2} \right\}_{-\pi/2}^{\pi/2} = \pi \rho g r^3$$

17. $F = 2\int_0^2 \int_{-\sqrt{4-x^2}}^{3+\sqrt{4-x^2}} \rho g(6 - y)\,dy\,dx \quad (g = 9.81)$

$$= 2\rho g \int_0^2 \left\{ -\frac{1}{2}(6 - y)^2 \right\}_{-\sqrt{4-x^2}}^{3+\sqrt{4-x^2}} dx$$

$$= 9\rho g \int_0^2 (3 + 2\sqrt{4 - x^2})\,dx$$

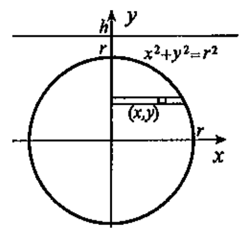


If we set $x = 2\sin\theta$ and $dx = 2\cos\theta\,d\theta$,

$$F = 9\rho g \left\{ 3x \right\}_0^2 + 18\rho g \int_0^{\pi/2} 2\cos\theta 2\cos\theta\,d\theta = 54\rho g + 72\rho g \int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2} \right) d\theta$$

$$= 54\rho g + 36\rho g \left\{ \theta + \frac{1}{2}\sin 2\theta \right\}_0^{\pi/2} = 1.08 \times 10^6 \text{ N}.$$

18. According to Exercise 40 in Section 7.7, the force on the plate is $F = \rho gh(\pi r^2) = \pi\rho ghr^2$. By symmetry, $x_c = 0$. If we integrate over the right-half of the circle and double the result,

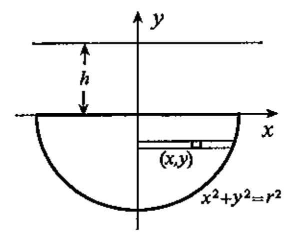


$$\begin{aligned} Fy_c &= 2\int_{-r}^{r}\int_{0}^{\sqrt{r^2-y^2}} y\rho g(h-y)\,dx\,dy \\ &= 2\rho g\int_{-r}^{r}\left\{xy(h-y)\right\}_{0}^{\sqrt{r^2-y^2}} dy \\ &= 2\rho g\int_{-r}^{r}\left(hy\sqrt{r^2-y^2} - y^2\sqrt{r^2-y^2}\right) dy. \end{aligned}$$

If we set $y = r\sin\theta$ and $dy = r\cos\theta\,d\theta$ in the last term,

$$\begin{aligned} y_c &= \frac{2\rho gh}{F}\left\{-\frac{1}{3}(r^2-y^2)^{3/2}\right\}_{-r}^{r} - \frac{2\rho g}{F}\int_{-\pi/2}^{\pi/2} r^2\sin^2\theta\, r\cos\theta\, r\cos\theta\,d\theta \\ &= -\frac{2\rho g r^4}{F}\int_{-\pi/2}^{\pi/2}\left(\frac{1-\cos 4\theta}{8}\right)d\theta = -\frac{\rho g r^4}{4F}\left\{\theta - \frac{\sin 4\theta}{4}\right\}_{-\pi/2}^{\pi/2} = -\frac{\rho g\pi r^4}{4(\pi\rho g h r^2)} = -\frac{r^2}{4h}. \end{aligned}$$

19. The fluid force on the surface is



$$\begin{aligned} F &= 2\int_{-r}^{0}\int_{0}^{\sqrt{r^2-y^2}} \rho g(h-y)\,dx\,dy \\ &= 2\rho g\int_{-r}^{0}\left\{x(h-y)\right\}_{0}^{\sqrt{r^2-y^2}} dy = 2\rho g\int_{-r}^{0}(h-y)\sqrt{r^2-y^2}\,dy \\ &= 2\rho gh\int_{-r}^{0}\sqrt{r^2-y^2}\,dy - 2\rho g\int_{-r}^{0} y\sqrt{r^2-y^2}\,dy \\ &= 2\rho gh\left(\frac{\pi r^2}{4}\right) - 2\rho g\left\{-\frac{(r^2-y^2)^{3/2}}{3}\right\}_{-r}^{0} = \frac{\rho g r^2(3\pi h + 4r)}{6}. \end{aligned}$$

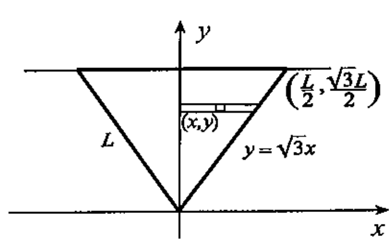
By symmetry, $x_c = 0$, and

$$Fy_c = 2\int_{-r}^{0}\int_{0}^{\sqrt{r^2-y^2}} \rho g y(h-y)\,dx\,dy = 2\rho g\int_{-r}^{0}\left\{xy(h-y)\right\}_{0}^{\sqrt{r^2-y^2}} dy = 2\rho g\int_{-r}^{0} y(h-y)\sqrt{r^2-y^2}\,dy.$$

If we set $y = r\sin\theta$ and $dy = r\cos\theta\,d\theta$ in the second term,

$$\begin{aligned} y_c &= \frac{2\rho gh}{F}\left\{-\frac{(r^2-y^2)^{3/2}}{3}\right\}_{-r}^{0} - \frac{2\rho g}{F}\int_{-\pi/2}^{0} r^2\sin^2\theta\, r\cos\theta\, r\cos\theta\,d\theta \\ &= -\frac{2\rho g h r^3}{3F} - \frac{2\rho g r^4}{F}\int_{-\pi/2}^{0}\left(\frac{1-\cos 4\theta}{8}\right)d\theta = -\frac{2\rho g h r^3}{3F} - \frac{\rho g r^4}{4F}\left\{\theta - \frac{\sin 4\theta}{4}\right\}_{-\pi/2}^{0} \\ &= -\frac{2\rho g h r^3}{3F} - \frac{\rho g r^4\pi}{8F} = -\frac{\rho g r^3(3\pi r + 16h)}{24}\,\frac{6}{\rho g r^2(3\pi h + 4r)} = -\frac{r(3\pi r + 16h)}{4(3\pi h + 4r)}. \end{aligned}$$

20. The fluid force on the triangle is



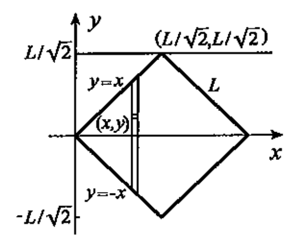
$$\begin{aligned} F &= 2\int_{0}^{\sqrt{3}L/2}\int_{0}^{y/\sqrt{3}} \rho g\left(\frac{\sqrt{3}L}{2} - y\right)dx\,dy \\ &= \rho g\int_{0}^{\sqrt{3}L/2}\left\{x(\sqrt{3}L - 2y)\right\}_{0}^{y/\sqrt{3}} dy \\ &= \frac{\rho g}{\sqrt{3}}\int_{0}^{\sqrt{3}L/2}(\sqrt{3}Ly - 2y^2)\,dy \\ &= \frac{\rho g}{\sqrt{3}}\left\{\frac{\sqrt{3}Ly^2}{2} - \frac{2y^3}{3}\right\}_{0}^{\sqrt{3}L/2} = \frac{\rho g L^3}{8}. \end{aligned}$$

By symmetry, $x_c = 0$, and

$$\begin{aligned}
y_c &= \frac{2}{F}\int_0^{\sqrt{3}L/2}\int_0^{y/\sqrt{3}} \rho g y\left(\frac{\sqrt{3}L}{2}-y\right)dx\,dy = \frac{\rho g}{F}\int_0^{\sqrt{3}L/2}\Big\{xy(\sqrt{3}L-2y)\Big\}_0^{y/\sqrt{3}}dy\\
&= \frac{\rho g}{\sqrt{3}F}\int_0^{\sqrt{3}L/2}(\sqrt{3}Ly^2-2y^3)\,dy = \frac{\rho g}{\sqrt{3}F}\left\{\frac{Ly^3}{\sqrt{3}}-\frac{y^4}{2}\right\}_0^{\sqrt{3}L/2}\\
&= \frac{\sqrt{3}\rho g L^4}{32F} = \frac{\sqrt{3}\rho g L^4}{32}\frac{8}{\rho g L^3} = \frac{\sqrt{3}L}{4}.
\end{aligned}$$

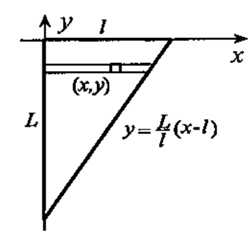
21. According to Exercise 40 in Section 7.7, the force on the square is $F = \rho g(L/\sqrt{2})L^2 = \rho g L^3/\sqrt{2}$. By symmetry, $x_c = L/\sqrt{2}$, and

$$\begin{aligned}
y_c &= \frac{2}{F}\int_0^{L/\sqrt{2}}\int_{-x}^{x}\rho g y\left(\frac{L}{\sqrt{2}}-y\right)dy\,dx\\
&= \frac{2\rho g}{F}\int_0^{L/\sqrt{2}}\left\{\frac{Ly^2}{2\sqrt{2}}-\frac{y^3}{3}\right\}_{-x}^{x}dx\\
&= -\frac{4\rho g}{3F}\int_0^{L/\sqrt{2}}x^3\,dx = -\frac{4\rho g}{3F}\left\{\frac{x^4}{4}\right\}_0^{L/\sqrt{2}}\\
&= -\frac{\rho g L^4}{12F} = -\frac{\rho g L^4}{12}\frac{\sqrt{2}}{\rho g L^3} = -\frac{L}{6\sqrt{2}}.
\end{aligned}$$



22. The force on the triangle is

$$\begin{aligned}
F &= \int_{-L}^{0}\int_0^{ly/L+l}\rho g(-y)\,dx\,dy = -\rho g\int_{-L}^{0}\Big\{xy\Big\}_0^{ly/L+l}dy\\
&= -\rho g\int_{-L}^{0}y\left(\frac{ly}{L}+l\right)dy = -\rho g\left\{\frac{ly^3}{3L}+\frac{ly^2}{2}\right\}_{-L}^{0} = \frac{\rho g l L^2}{6}.
\end{aligned}$$



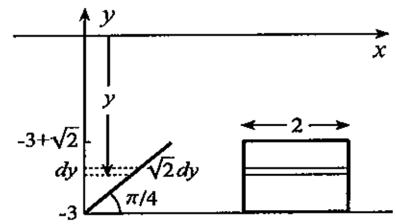
According to equations 13.30,

$$\begin{aligned}
x_c &= \frac{1}{F}\int_{-L}^{0}\int_0^{ly/L+l}\rho g(-y)x\,dx\,dy = -\frac{\rho g}{F}\int_{-L}^{0}\left\{\frac{x^2y}{2}\right\}_0^{ly/L+l}dy\\
&= -\frac{\rho g}{2F}\int_{-L}^{0}y\left(\frac{l^2y^2}{L^2}+\frac{2l^2y}{L}+l^2\right)dy = -\frac{\rho g l^2}{2F}\left\{\frac{y^4}{4L^2}+\frac{2y^3}{3L}+\frac{y^2}{2}\right\}_{-L}^{0}\\
&= \frac{\rho g l^2L^2}{24F} = \frac{\rho g l^2 L^2}{24}\frac{6}{\rho g l L^2} = \frac{l}{4},
\end{aligned}$$

$$\begin{aligned}
y_c &= \frac{1}{F}\int_{-L}^{0}\int_0^{ly/L+l}\rho g(-y)y\,dx\,dy = -\frac{\rho g}{F}\int_{-L}^{0}\Big\{xy^2\Big\}_0^{ly/L+l}dy\\
&= -\frac{\rho g}{F}\int_{-L}^{0}y^2\left(\frac{ly}{L}+l\right)dy = -\frac{\rho g l}{F}\left\{\frac{y^4}{4L}+\frac{y^3}{3}\right\}_{-L}^{0} = -\frac{\rho g l L^3}{12F} = -\frac{\rho g l L^3}{12}\frac{6}{\rho g l L^2} = -\frac{L}{2}.
\end{aligned}$$

23. When the geometric centre of a circle with radius r is at depth $h > r$ below the surface of a fluid, its centre of pressure is at depth $h + r^2/(4h)$ (see Exercise 18). This is not a fixed point in the plate. The centre of pressure is below the geometric centre and approaches the geometric centre as h increases.

24.
$$\begin{aligned}
F &= \int_{-3}^{-3+\sqrt{2}}\rho g(-y)(2)(\sqrt{2}\,dy) \quad (g = 9.81)\\
&= -\sqrt{2}\rho g\Big\{y^2\Big\}_{-3}^{-3+\sqrt{2}}\\
&= 9.00\times 10^4\ \text{N}
\end{aligned}$$

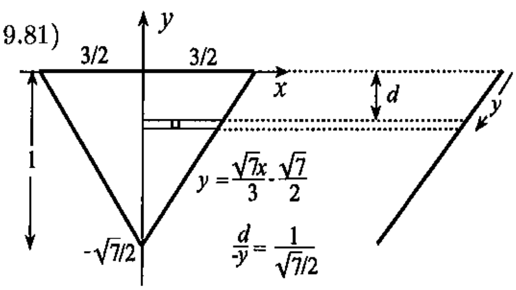


25. $F = 2\int_{-\sqrt{7}/2}^{0}\int_{0}^{3(2y+\sqrt{7})/(2\sqrt{7})} \rho g\left(-\frac{2y}{\sqrt{7}}\right) dx\, dy \quad (g = 9.81)$

$$= -\frac{4\rho g}{\sqrt{7}}\int_{-\sqrt{7}/2}^{0}\left\{xy\right\}_{0}^{3(2y+\sqrt{7})/(2\sqrt{7})} dy$$

$$= -\frac{6\rho g}{7}\int_{-\sqrt{7}/2}^{0}(2y^2+\sqrt{7}y)\, dy$$

$$= -\frac{6\rho g}{7}\left\{\frac{2y^3}{3}+\frac{\sqrt{7}y^2}{2}\right\}_{-\sqrt{7}/2}^{0} = \frac{\sqrt{7}\rho g}{4}$$



26. With the coordinate system in Figure 13.24,

$$y_c = \frac{1}{F}\iint_R yP\, dA = \frac{1}{\rho g(-\overline{y})A}\iint_R y\rho g(-y)\, dA = \frac{I_x}{\overline{y}A},$$

where A is the area of R and I_x is the second moment of area of R about the x-axis. But, according to the parallel axis theorem (see Exercise 20 in Section 7.8), $I_x = I_{CM} + \overline{y}^2 A$, where I_{CM} is the second moment of area of R about the line through the centroid and parallel to the x-axis. Thus,

$$y_c = \frac{I_{CM}+\overline{y}^2A}{\overline{y}A} = \overline{y} + \frac{I_{CM}}{\overline{y}A}.$$

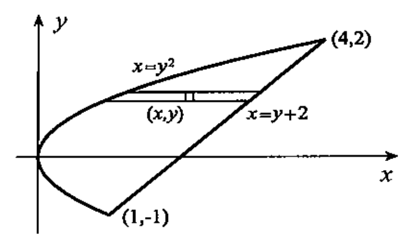
Since $\overline{y} < 0$, it follows that $y_c < \overline{y}$.

EXERCISES 13.5

1. $A = \int_{-1}^{2}\int_{y^2}^{y+2} dx\, dy = \int_{-1}^{2}(y+2-y^2)\, dy = \left\{\frac{y^2}{2}+2y-\frac{y^3}{3}\right\}_{-1}^{2} = \frac{9}{2}$

Since $A\overline{x} = \int_{-1}^{2}\int_{y^2}^{y+2} x\, dx\, dy = \int_{-1}^{2}\left\{\frac{x^2}{2}\right\}_{y^2}^{y+2} dy$

$$= \frac{1}{2}\int_{-1}^{2}[(y+2)^2-y^4]\, dy = \frac{1}{2}\left\{\frac{(y+2)^3}{3}-\frac{y^5}{5}\right\}_{-1}^{2} = \frac{36}{5},$$



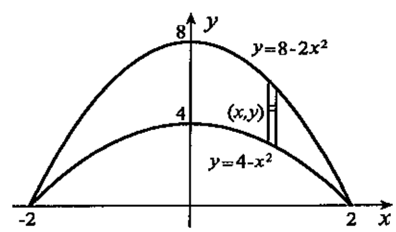
it follows that $\overline{x} = (36/5)(2/9) = 8/5$. Since

$$A\overline{y} = \int_{-1}^{2}\int_{y^2}^{y+2} y\, dx\, dy = \int_{-1}^{2}\left\{xy\right\}_{y^2}^{y+2} dy = \int_{-1}^{2}(y^2+2y-y^3)\, dy = \left\{\frac{y^3}{3}+y^2-\frac{y^4}{4}\right\}_{-1}^{2} = \frac{9}{4},$$

we obtain $\overline{y} = (9/4)(2/9) = 1/2$.

2. $A = 2\int_{0}^{2}\int_{4-x^2}^{8-2x^2} dy\, dx = 2\int_{0}^{2}(4-x^2)\, dx$

$$= 2\left\{4x-\frac{x^3}{3}\right\}_{0}^{2} = \frac{32}{3}$$



By symmetry, $\overline{x} = 0$. Since

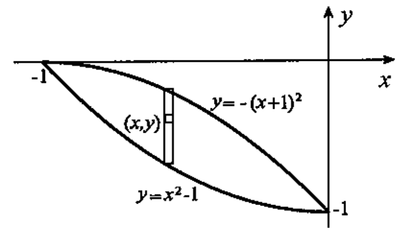
$$A\overline{y} = 2\int_{0}^{2}\int_{4-x^2}^{8-2x^2} y\, dy\, dx = 2\int_{0}^{2}\left\{\frac{y^2}{2}\right\}_{4-x^2}^{8-2x^2} dx$$

$$= 3\int_{0}^{2}(16-8x^2+x^4)\, dx = 3\left\{16x-\frac{8x^3}{3}+\frac{x^5}{5}\right\}_{0}^{2} = \frac{256}{5},$$

we obtain $\overline{y} = \frac{256}{5}\frac{3}{32} = \frac{24}{5}$.

3. $A = \displaystyle\int_{-1}^{0}\int_{x^2-1}^{-(x+1)^2} dy\,dx = \int_{-1}^{0}[-(x+1)^2 - x^2 + 1]\,dx$

$= \left\{-\dfrac{(x+1)^3}{3} - \dfrac{x^3}{3} + x\right\}_{-1}^{0} = \dfrac{1}{3}$



Since

$A\overline{x} = \displaystyle\int_{-1}^{0}\int_{x^2-1}^{-(x+1)^2} x\,dy\,dx = \int_{-1}^{0}\left\{xy\right\}_{x^2-1}^{-(x+1)^2} dx$

$= \displaystyle\int_{-1}^{0}(-2x^3 - 2x^2)\,dx = \left\{-\dfrac{x^4}{2} - \dfrac{2x^3}{3}\right\}_{-1}^{0} = -\dfrac{1}{6}$

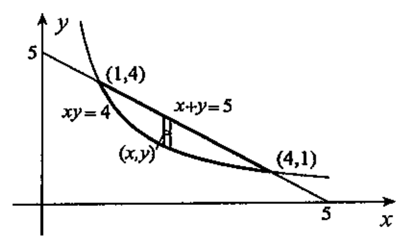
it follows that $\overline{x} = -(1/6)3 = -1/2$. Since

$$A\overline{y} = \int_{-1}^{0}\int_{x^2-1}^{-(x+1)^2} y\,dy\,dx = \int_{-1}^{0}\left\{\frac{y^2}{2}\right\}_{x^2-1}^{-(x+1)^2} dx = \frac{1}{2}\int_{-1}^{0}[(x+1)^4 - x^4 + 2x^2 - 1]\,dx$$
$$= \frac{1}{2}\left\{\frac{(x+1)^5}{5} - \frac{x^5}{5} + \frac{2x^3}{3} - x\right\}_{-1}^{0} = -\frac{1}{6},$$

$\overline{y}$ is also equal to $-1/2$.

4. $A = \displaystyle\int_{1}^{4}\int_{4/x}^{5-x} dy\,dx = \int_{1}^{4}(5 - x - 4/x)\,dx$

$= \left\{5x - \dfrac{x^2}{2} - 4\ln|x|\right\}_{1}^{4} = \dfrac{15}{2} - 4\ln 4$



From

$A\overline{x} = \displaystyle\int_{1}^{4}\int_{4/x}^{5-x} x\,dy\,dx = \int_{1}^{4}\left\{xy\right\}_{4/x}^{5-x} dx$

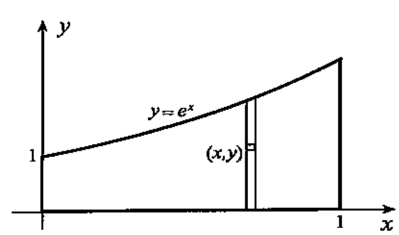
$= \displaystyle\int_{1}^{4}(5x - x^2 - 4)\,dx = \left\{\dfrac{5x^2}{2} - \dfrac{x^3}{3} - 4x\right\}_{1}^{4} = \dfrac{9}{2},$

we obtain $\overline{x} = \dfrac{9}{2}\dfrac{2}{15 - 8\ln 4} = \dfrac{9}{15 - 16\ln 2}$. Since

$$A\overline{y} = \int_{1}^{4}\int_{4/x}^{5-x} y\,dy\,dx = \int_{1}^{4}\left\{\frac{y^2}{2}\right\}_{4/x}^{5-x} dx = \frac{1}{2}\int_{1}^{4}\left[(5-x)^2 - \frac{16}{x^2}\right]dx = \frac{1}{2}\left\{-\frac{1}{3}(5-x)^3 + \frac{16}{x}\right\}_{1}^{4} = \frac{9}{2},$$

$\overline{y}$ is also equal to $9/(15 - 16\ln 2)$.

5. $A = \displaystyle\int_{0}^{1}\int_{0}^{e^x} dy\,dx = \int_{0}^{1} e^x\,dx = \left\{e^x\right\}_{0}^{1} = e - 1$



From

$A\overline{x} = \displaystyle\int_{0}^{1}\int_{0}^{e^x} x\,dy\,dx = \int_{0}^{1}\left\{xy\right\}_{0}^{e^x} dx$

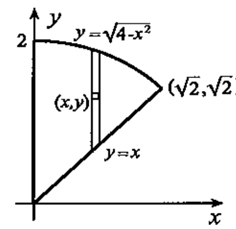
$= \displaystyle\int_{0}^{1} xe^x\,dx = \left\{xe^x - e^x\right\}_{0}^{1} = 1,$

we obtain $\overline{x} = \dfrac{1}{e-1}$. Since

$$A\overline{y} = \int_{0}^{1}\int_{0}^{e^x} y\,dy\,dx = \int_{0}^{1}\left\{\frac{y^2}{2}\right\}_{0}^{e^x} dx = \frac{1}{2}\int_{0}^{1} e^{2x}\,dx = \frac{1}{2}\left\{\frac{e^{2x}}{2}\right\}_{0}^{1} = \frac{e^2 - 1}{4},$$

we find $\overline{y} = \dfrac{e^2 - 1}{4}\dfrac{1}{e-1} = \dfrac{e+1}{4}$.

6. $A = \frac{1}{8}\pi(2)^2 = \frac{\pi}{2}$ Since



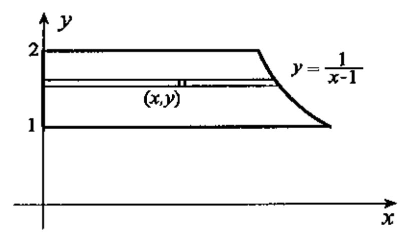
$$A\overline{x} = \int_0^{\sqrt{2}}\int_x^{\sqrt{4-x^2}} x\,dy\,dx$$
$$= \int_0^{\sqrt{2}} \left\{xy\right\}_x^{\sqrt{4-x^2}} dx = \int_0^{\sqrt{2}} (x\sqrt{4-x^2} - x^2)\,dx$$
$$= \left\{-\frac{1}{3}(4-x^2)^{3/2} - \frac{x^3}{3}\right\}_0^{\sqrt{2}} = \frac{4}{3}(2-\sqrt{2}),$$

it follows that $\overline{x} = \frac{4}{3}(2-\sqrt{2})\frac{2}{\pi} = \frac{8(2-\sqrt{2})}{3\pi}$. Since

$$A\overline{y} = \int_0^{\sqrt{2}}\int_x^{\sqrt{4-x^2}} y\,dy\,dx = \int_0^{\sqrt{2}} \left\{\frac{y^2}{2}\right\}_x^{\sqrt{4-x^2}} dx = \int_0^{\sqrt{2}} (2-x^2)\,dx = \left\{2x - \frac{x^3}{3}\right\}_0^{\sqrt{2}} = \frac{4\sqrt{2}}{3},$$

we obtain $\overline{y} = \frac{4\sqrt{2}}{3}\frac{2}{\pi} = \frac{8\sqrt{2}}{3\pi}$.

7. $A = \int_1^2\int_0^{(y+1)/y} dx\,dy = \int_1^2 \left(\frac{y+1}{y}\right) dy = \left\{y + \ln|y|\right\}_1^2 = 1 + \ln 2$



Since

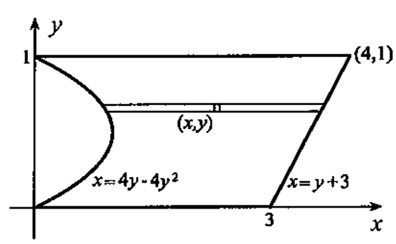
$$A\overline{x} = \int_1^2\int_0^{(y+1)/y} x\,dx\,dy = \int_1^2 \left\{\frac{x^2}{2}\right\}_0^{(y+1)/y} dy$$
$$= \frac{1}{2}\int_1^2 \left(1 + \frac{2}{y} + \frac{1}{y^2}\right) dy = \frac{1}{2}\left\{y + 2\ln|y| - \frac{1}{y}\right\}_1^2 = \frac{3+4\ln 2}{4},$$

it follows that $\overline{x} = \frac{3+4\ln 2}{4}\,\frac{1}{1+\ln 2} = \frac{3+4\ln 2}{4+4\ln 2}$. Since

$$A\overline{y} = \int_1^2\int_0^{(y+1)/y} y\,dx\,dy = \int_1^2 \left\{xy\right\}_0^{(y+1)/y} dy = \int_1^2 (y+1)\,dy = \left\{\frac{y^2}{2} + y\right\}_1^2 = \frac{5}{2},$$

we obtain $\overline{y} = \frac{5}{2}\,\frac{1}{1+\ln 2} = \frac{5}{2+2\ln 2}$.

8. $A = \int_0^1\int_{4y-4y^2}^{y+3} dx\,dy = \int_0^1 (3 - 3y + 4y^2)\,dy$



$$= \left\{3y - \frac{3y^2}{2} + \frac{4y^3}{3}\right\}_0^1 = \frac{17}{6}$$

Since

$$A\overline{x} = \int_0^1\int_{4y-4y^2}^{y+3} x\,dx\,dy = \int_0^1 \left\{\frac{x^2}{2}\right\}_{4y-4y^2}^{y+3} dy$$
$$= \frac{1}{2}\int_0^1 [(3+y)^2 - 16y^2 + 32y^3 - 16y^4]\,dy = \frac{1}{2}\left\{\frac{(3+y)^3}{3} - \frac{16y^3}{3} + 8y^4 - \frac{16y^5}{5}\right\}_0^1 = \frac{59}{10},$$

we find $\overline{x} = \frac{59}{10}\frac{6}{17} = \frac{177}{85}$. Since

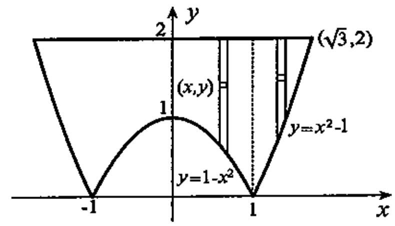
$$A\overline{y} = \int_0^1\int_{4y-4y^2}^{y+3} y\,dx\,dy = \int_0^1 \left\{xy\right\}_{4y-4y^2}^{y+3} dy = \int_0^1 (3y - 3y^2 + 4y^3)\,dy = \left\{\frac{3y^2}{2} - y^3 + y^4\right\}_0^1 = \frac{3}{2},$$

we obtain $\overline{y} = \frac{3}{2}\frac{6}{17} = \frac{9}{17}$.

9. $A = 2\int_0^1 \int_{1-x^2}^2 dy\,dx + 2\int_1^{\sqrt{3}} \int_{x^2-1}^2 dy\,dx$

$= 2\int_0^1 (1+x^2)\,dx + 2\int_1^{\sqrt{3}} (3-x^2)\,dx$

$= 2\left\{x + \frac{x^3}{3}\right\}_0^1 + 2\left\{3x - \frac{x^3}{3}\right\}_1^{\sqrt{3}} = \frac{12\sqrt{3}-8}{3}$



By symmetry, $\overline{x} = 0$. Since

$$A\overline{y} = 2\int_0^1 \int_{1-x^2}^2 y\,dy\,dx + 2\int_1^{\sqrt{3}} \int_{x^2-1}^2 y\,dy\,dx = 2\int_0^1 \left\{\frac{y^2}{2}\right\}_{1-x^2}^2 dx + 2\int_1^{\sqrt{3}} \left\{\frac{y^2}{2}\right\}_{x^2-1}^2 dx$$

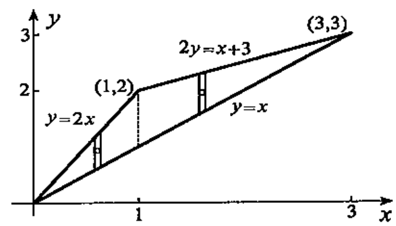
$$= \int_0^1 (3+2x^2-x^4)\,dx + \int_1^{\sqrt{3}} (3+2x^2-x^4)\,dx = \left\{3x + \frac{2x^3}{3} - \frac{x^5}{5}\right\}_0^{\sqrt{3}} = \frac{16\sqrt{3}}{5}$$

it follows that $\overline{y} = \frac{16\sqrt{3}}{5}\,\frac{3}{12\sqrt{3}-8} = \frac{12\sqrt{3}}{15\sqrt{3}-10}$.

10. $A = \int_0^1 \int_x^{2x} dy\,dx + \int_1^3 \int_x^{(x+3)/2} dy\,dx$

$= \int_0^1 x\,dx + \frac{1}{2}\int_1^3 (3-x)\,dx$

$= \left\{\frac{x^2}{2}\right\}_0^1 + \frac{1}{2}\left\{-\frac{1}{2}(3-x)^2\right\}_1^3 = \frac{3}{2}$



Since

$$A\overline{x} = \int_0^1 \int_x^{2x} x\,dy\,dx + \int_1^3 \int_x^{(x+3)/2} x\,dy\,dx$$

$$= \int_0^1 x^2\,dx + \frac{1}{2}\int_1^3 (3x-x^2)\,dx = \left\{\frac{x^3}{3}\right\}_0^1 + \frac{1}{2}\left\{\frac{3x^2}{2} - \frac{x^3}{3}\right\}_1^3 = 2,$$

it follows that $\overline{x} = 2\cdot\frac{2}{3} = \frac{4}{3}$. With

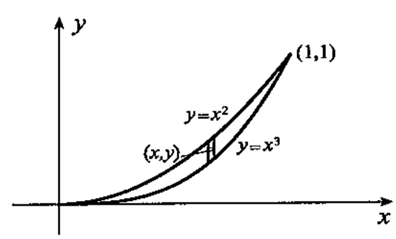
$$A\overline{y} = \int_0^1 \int_x^{2x} y\,dy\,dx + \int_1^3 \int_x^{(x+3)/2} y\,dy\,dx = \int_0^1 \left\{\frac{y^2}{2}\right\}_x^{2x} dx + \int_1^3 \left\{\frac{y^2}{2}\right\}_x^{(x+3)/2} dx$$

$$= \frac{3}{2}\int_0^1 x^2\,dx + \frac{1}{2}\int_1^3 \left[\frac{1}{4}(x+3)^2 - x^2\right] dx = \frac{3}{2}\left\{\frac{x^3}{3}\right\}_0^1 + \frac{1}{2}\left\{\frac{1}{12}(x+3)^3 - \frac{x^3}{3}\right\}_1^3 = \frac{5}{2},$$

we obtain $\overline{y} = (5/2)(2/3) = 5/3$.

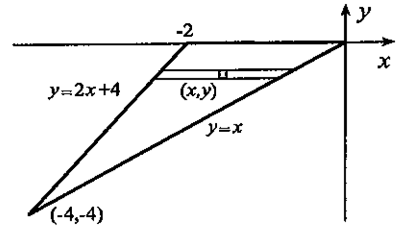
11. $I = \int_0^1 \int_{x^3}^{x^2} x^2\,dy\,dx = \int_0^1 \left\{x^2 y\right\}_{x^3}^{x^2} dx$

$= \int_0^1 (x^4 - x^5)\,dx = \left\{\frac{x^5}{5} - \frac{x^6}{6}\right\}_0^1 = \frac{1}{30}$



12. $I = \int_{-4}^0 \int_{(y-4)/2}^{y} y^2\,dx\,dy = \int_{-4}^0 \left\{xy^2\right\}_{(y-4)/2}^{y} dy$

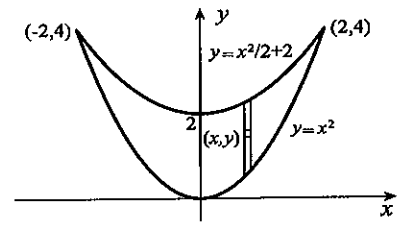
$= \frac{1}{2}\int_{-4}^0 (4y^2 + y^3)\,dy = \frac{1}{2}\left\{\frac{4y^3}{3} + \frac{y^4}{4}\right\}_{-4}^0 = \frac{32}{3}$



13. $I = 2\int_0^2 \int_{x^2}^{2+x^2/2} y^2\,dy\,dx = 2\int_0^2 \left\{\frac{y^3}{3}\right\}_{x^2}^{2+x^2/2} dx$

$$= \frac{2}{3}\int_0^2 \left(8 + 6x^2 + \frac{3x^4}{2} - \frac{7x^6}{8}\right) dx$$

$$= \frac{2}{3}\left\{8x + 2x^3 + \frac{3x^5}{10} - \frac{x^7}{8}\right\}_0^2 = \frac{256}{15}$$

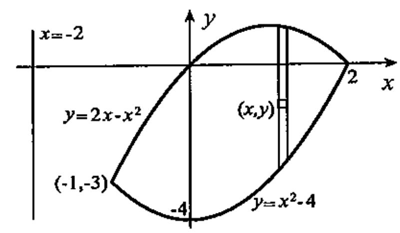


14. $I = \int_{-1}^2 \int_{x^2-4}^{2x-x^2} (x+2)^2\,dy\,dx$

$$= \int_{-1}^2 \left\{y(x+2)^2\right\}_{x^2-4}^{2x-x^2} dx$$

$$= 2\int_{-1}^2 (8 + 12x + 2x^2 - 3x^3 - x^4)\,dx$$

$$= 2\left\{8x + 6x^2 + \frac{2x^3}{3} - \frac{3x^4}{4} - \frac{x^5}{5}\right\}_{-1}^2 = \frac{603}{10}$$

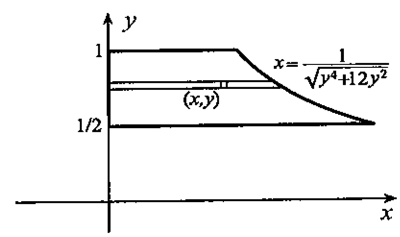


15. $I = \int_{1/2}^1 \int_0^{1/\sqrt{y^4+12y^2}} y^2\,dx\,dy$

$$= \int_{1/2}^1 \left\{xy^2\right\}_0^{1/\sqrt{y^4+12y^2}} dy$$

$$= \int_{1/2}^1 \frac{y^2}{\sqrt{y^4+12y^2}}\,dy = \int_{1/2}^1 \frac{y}{\sqrt{y^2+12}}\,dy$$

$$= \left\{\sqrt{y^2+12}\right\}_{1/2}^1 = \sqrt{13} - 7/2$$

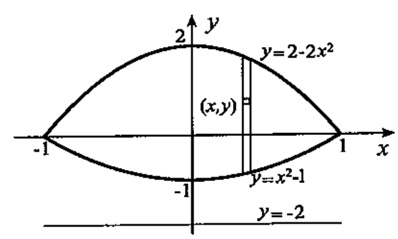


16. Moment $= 2\int_0^1 \int_{x^2-1}^{2-2x^2} \rho(y+2)\,dy\,dx$

$$= 2\rho \int_0^1 \left\{\frac{1}{2}(y+2)^2\right\}_{x^2-1}^{2-2x^2} dx$$

$$= 3\rho \int_0^1 (5 - 6x^2 + x^4)\,dx$$

$$= 3\rho\left\{5x - 2x^3 + \frac{x^5}{5}\right\}_0^1 = \frac{48\rho}{5}$$



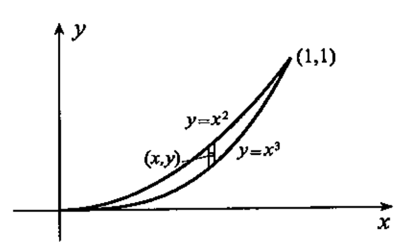
17. Due to the symmetry of the plate, the product moment of inertia about the axes is zero.

18. $I_{xy} = \int_0^1 \int_{x^3}^{x^2} xy\rho\,dy\,dx$

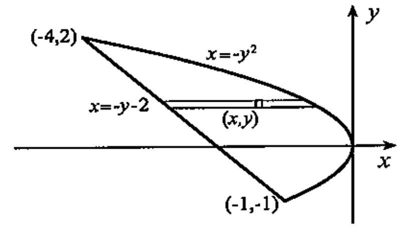
$$= \rho\int_0^1 \left\{\frac{xy^2}{2}\right\}_{x^3}^{x^2} dx$$

$$= \frac{\rho}{2}\int_0^1 (x^5 - x^7)\,dx$$

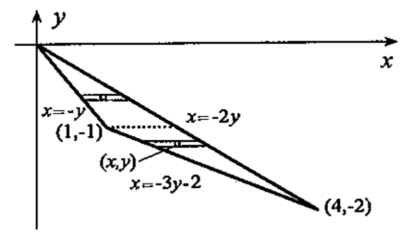
$$= \frac{\rho}{2}\left\{\frac{x^6}{6} - \frac{x^8}{8}\right\}_0^1 = \frac{\rho}{48}$$



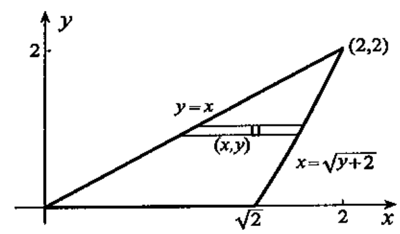
19. $$I_{xy} = \int_{-1}^{2}\int_{-y-2}^{-y^2} xy\rho\,dx\,dy$$
$$= \rho\int_{-1}^{2}\left\{\frac{x^2y}{2}\right\}_{-y-2}^{-y^2} dy$$
$$= \frac{\rho}{2}\int_{-1}^{2}(y^5 - y^3 - 4y^2 - 4y)\,dy$$
$$= \frac{\rho}{2}\left\{\frac{y^6}{6} - \frac{y^4}{4} - \frac{4y^3}{3} - 2y^2\right\}_{-1}^{2} = -\frac{45\rho}{8}$$



20. $$I_{xy} = \int_{-2}^{-1}\int_{-3y-2}^{-2y} xy\rho\,dx\,dy + \int_{-1}^{0}\int_{-y}^{-2y} xy\rho\,dx\,dy$$
$$= \rho\int_{-2}^{-1}\left\{\frac{x^2y}{2}\right\}_{-3y-2}^{-2y} dy + \rho\int_{-1}^{0}\left\{\frac{x^2y}{2}\right\}_{-y}^{-2y} dy$$
$$= \frac{\rho}{2}\int_{-2}^{-1}(-5y^3 - 12y^2 - 4y)\,dy + \frac{3\rho}{2}\int_{-1}^{0} y^3\,dy$$
$$= \frac{\rho}{2}\left\{-\frac{5y^4}{4} - 4y^3 - 2y^2\right\}_{-2}^{-1} + \frac{3\rho}{2}\left\{\frac{y^4}{4}\right\}_{-1}^{0}$$
$$= -2\rho$$



21. $$A = \int_0^2\int_y^{\sqrt{y+2}} dx\,dy = \int_0^2(\sqrt{y+2} - y)\,dy$$
$$= \left\{\frac{2(y+2)^{3/2}}{3} - \frac{y^2}{2}\right\}_0^2 = \frac{10 - 4\sqrt{2}}{3}$$



Since

$$A\overline{x} = \int_0^2\int_y^{\sqrt{y+2}} x\,dx\,dy = \int_0^2\left\{\frac{x^2}{2}\right\}_y^{\sqrt{y+2}} dy$$
$$= \frac{1}{2}\int_0^2(y + 2 - y^2)\,dy = \frac{1}{2}\left\{\frac{y^2}{2} + 2y - \frac{y^3}{3}\right\}_0^2 = \frac{5}{3},$$

it follows that $\overline{x} = \dfrac{5}{3}\dfrac{3}{10 - 4\sqrt{2}} = \dfrac{5}{10 - 4\sqrt{2}}$. We now calculate

$$A\overline{y} = \int_0^2\int_y^{\sqrt{y+2}} y\,dx\,dy = \int_0^2\left\{xy\right\}_y^{\sqrt{y+2}} dy = \int_0^2(y\sqrt{y+2} - y^2)\,dy.$$

If we set $u = y + 2$ and $du = dy$ in the first term,

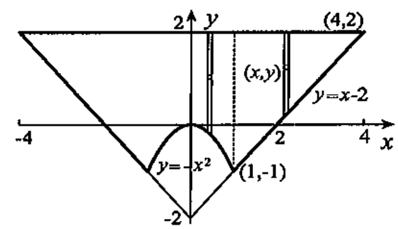
$$A\overline{y} = \int_2^4(u - 2)\sqrt{u}\,du - \left\{\frac{y^3}{3}\right\}_0^2 = \int_2^4(u^{3/2} - 2\sqrt{u})\,du - \frac{8}{3} = \left\{\frac{2u^{5/2}}{5} - \frac{4u^{3/2}}{3}\right\}_2^4 - \frac{8}{3} = \frac{16\sqrt{2} - 8}{15}.$$

Thus, $\overline{y} = \dfrac{16\sqrt{2} - 8}{15}\dfrac{3}{10 - 4\sqrt{2}} = \dfrac{8\sqrt{2} - 4}{25 - 10\sqrt{2}}$.

22. $A = 2\int_0^1 \int_{-x^2}^2 dy\,dx + 2\left(\frac{1}{2}\right)(3)(3)$

$= 2\int_0^1 (2 + x^2)\,dx + 9$

$= 2\left\{2x + \frac{x^3}{3}\right\}_0^1 + 9 = \frac{41}{3}$



By symmetry, $\overline{x} = 0$. Since

$$A\overline{y} = 2\int_0^1 \int_{-x^2}^2 y\,dy\,dx + 2\int_1^4 \int_{x-2}^2 y\,dy\,dx$$

$$= 2\int_0^1 \left\{\frac{y^2}{2}\right\}_{-x^2}^2 dx + 2\int_1^4 \left\{\frac{y^2}{2}\right\}_{x-2}^2 dx$$

$$= \int_0^1 (4 - x^4)\,dx + \int_1^4 [4 - (x-2)^2]\,dx = \left\{4x - \frac{x^5}{5}\right\}_0^1 + \left\{4x - \frac{1}{3}(x-2)^3\right\}_1^4 = \frac{64}{5},$$

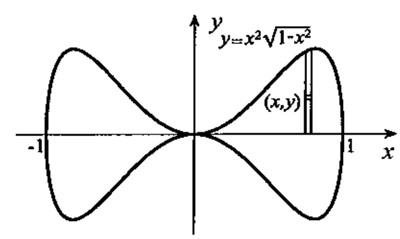
we obtain $\overline{y} = \frac{64}{5}\frac{3}{41} = \frac{192}{205}$.

23. $A = 2\int_0^1 \int_0^{x^2\sqrt{1-x^2}} dy\,dx = 2\int_0^1 x^2\sqrt{1-x^2}\,dx$

If we set $x = \sin\theta$ and $dx = \cos\theta\,d\theta$, then

$A = 2\int_0^{\pi/2} \sin^2\theta(\cos\theta)\cos\theta\,d\theta = 2\int_0^{\pi/2} \left(\frac{\sin 2\theta}{2}\right)^2 d\theta$

$= \frac{1}{2}\int_0^{\pi/2} \left(\frac{1 - \cos 4\theta}{2}\right) d\theta = \frac{1}{4}\left\{\theta - \frac{1}{4}\sin 4\theta\right\}_0^{\pi/2} = \frac{\pi}{8}.$



By symmetry, $\overline{y} = 0$. We now calculate

$$A\overline{x} = 2\int_0^1 \int_0^{x^2\sqrt{1-x^2}} x\,dy\,dx = 2\int_0^1 \left\{xy\right\}_0^{x^2\sqrt{1-x^2}} dx = 2\int_0^1 x^3\sqrt{1-x^2}\,dx.$$

If we set $u = 1 - x^2$ and $du = -2x\,dx$, then

$$A\overline{x} = 2\int_1^0 (1-u)\sqrt{u}\left(\frac{du}{-2}\right) = \int_0^1 (\sqrt{u} - u^{3/2})\,du = \left\{\frac{2u^{3/2}}{3} - \frac{2u^{5/2}}{5}\right\}_0^1 = \frac{4}{15}.$$

Thus, $\overline{x} = (4/15)(8/\pi) = 32/(15\pi)$.

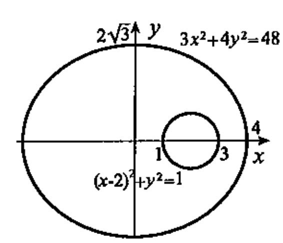
24. Since the area inside an ellipse is π multiplied by the product of half the major and minor axes,

$A = \pi(4)(2\sqrt{3}) - \pi(1)^2 = \pi(8\sqrt{3} - 1).$

By symmetry, $\overline{y} = 0$. Since the first moment of the area about the y-axis is that of the ellipse less that of the circle,

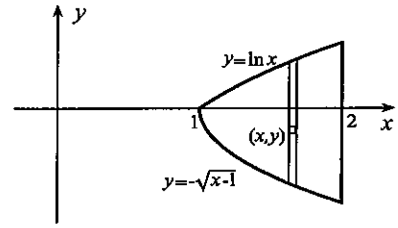
$A\overline{x} = 0 - 2\pi(1)^2 = -2\pi.$

Thus, $\overline{x} = \frac{-2\pi}{\pi(8\sqrt{3} - 1)} = \frac{-2}{8\sqrt{3} - 1}.$



25. $A = \int_1^2 \int_{-\sqrt{x-1}}^{\ln x} dy\,dx = \int_1^2 (\ln x + \sqrt{x-1})\,dx$

$$= \left\{x\ln x - x + \frac{2(x-1)^{3/2}}{3}\right\}_1^2 = 2\ln 2 - 1/3$$



We now calculate that

$$A\overline{x} = \int_1^2 \int_{-\sqrt{x-1}}^{\ln x} x\,dy\,dx = \int_1^2 \left\{xy\right\}_{-\sqrt{x-1}}^{\ln x} dx$$
$$= \int_1^2 (x\ln x + x\sqrt{x-1})\,dx$$

If we use integration by parts on the first term with $u = \ln x$, $dv = x\,dx$, $du = (1/x)dx$, and $v = x^2/2$, and set $u = x-1$ and $du = dx$ in the second,

$$A\overline{x} = \left\{\frac{x^2}{2}\ln x\right\}_1^2 - \int_1^2 \frac{x}{2}dx + \int_0^1 (u+1)\sqrt{u}\,du = 2\ln 2 - \left\{\frac{x^2}{4}\right\}_1^2 + \left\{\frac{2u^{5/2}}{5} + \frac{2u^{3/2}}{3}\right\}_0^1 = \frac{120\ln 2 + 19}{60}.$$

Thus, $\overline{x} = \dfrac{120\ln 2 + 19}{60}\dfrac{3}{6\ln 2 - 1} = \dfrac{120\ln 2 + 19}{120\ln 2 - 20}$. Next, we find

$$A\overline{y} = \int_1^2 \int_{-\sqrt{x-1}}^{\ln x} y\,dy\,dx = \int_1^2 \left\{\frac{y^2}{2}\right\}_{-\sqrt{x-1}}^{\ln x} dx = \frac{1}{2}\int_1^2 [(\ln x)^2 - (x-1)]\,dx.$$

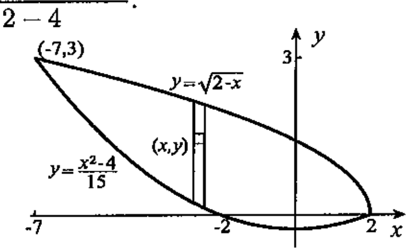
If we set $u = (\ln x)^2$, $dv = dx$, $du = (2/x)\ln x\,dx$ and $v = x$ in the first term,

$$A\overline{y} = \frac{1}{2}\left\{x(\ln x)^2\right\}_1^2 - \frac{1}{2}\int_1^2 2\ln x\,dx - \frac{1}{2}\left\{\frac{x^2}{2} - x\right\}_1^2 = (\ln 2)^2 - \left\{x\ln x - x\right\}_1^2 - \frac{1}{4} = (\ln 2)^2 - 2\ln 2 + \frac{3}{4}.$$

Thus, $\overline{y} = \dfrac{4(\ln 2)^2 - 8\ln 2 + 3}{4}\dfrac{3}{6\ln 2 - 1} = \dfrac{12(\ln 2)^2 - 24\ln 2 + 9}{24\ln 2 - 4}$.

26. $A = \int_{-7}^2 \int_{(x^2-4)/15}^{\sqrt{2-x}} dy\,dx$

$$= \int_{-7}^2 \left[\sqrt{2-x} - \frac{1}{15}(x^2-4)\right] dx$$
$$= \left\{-\frac{2}{3}(2-x)^{3/2} - \frac{x^3}{45} + \frac{4x}{15}\right\}_{-7}^2 = \frac{63}{5}$$



$$A\overline{x} = \int_{-7}^2 \int_{(x^2-4)/15}^{\sqrt{2-x}} x\,dy\,dx = \int_{-7}^2 \left[x\sqrt{2-x} - \frac{1}{15}(x^3-4x)\right] dx$$

If we set $u = 2-x$ and $du = -dx$ in the first term,

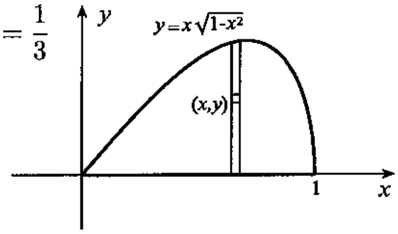
$$A\overline{x} = \int_9^0 (2-u)\sqrt{u}(-du) - \frac{1}{15}\left\{\frac{x^4}{4} - 2x^2\right\}_{-7}^2 = \left\{\frac{4}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right\}_0^9 - \frac{1}{15}\left\{\frac{x^4}{4} - 2x^2\right\}_{-7}^2 = -\frac{549}{20}.$$

Thus, $\overline{x} = -(549/20)(5/63) = -61/28$. Since

$$A\overline{y} = \int_{-7}^2 \int_{(x^2-4)/15}^{\sqrt{2-x}} y\,dy\,dx = \int_{-7}^2 \left\{\frac{y^2}{2}\right\}_{(x^2-4)/15}^{\sqrt{2-x}} dx = \frac{1}{2}\int_{-7}^2 \left[2 - x - \frac{1}{225}(x^4 - 8x^2 + 16)\right] dx$$
$$= \frac{1}{450}\left\{450x - \frac{225x^2}{2} - \frac{x^5}{5} + \frac{8x^3}{3} - 16x\right\}_{-7}^2 = \frac{7263}{500},$$

we find $\overline{y} = (7263/500)(5/63) = 807/700$.

27. $A = \int_0^1 \int_0^{x\sqrt{1-x^2}} dy\,dx = \int_0^1 x\sqrt{1-x^2}\,dx = \left\{-\frac{1}{3}(1-x^2)^{3/2}\right\}_0^1 = \frac{1}{3}$



We now calculate

$$A\overline{x} = \int_0^1 \int_0^{x\sqrt{1-x^2}} x\,dy\,dx = \int_0^1 \left\{xy\right\}_0^{x\sqrt{1-x^2}} dx$$
$$= \int_0^1 x^2\sqrt{1-x^2}\,dx$$

If we set $x = \sin\theta$ and $dx = \cos\theta\,d\theta$,

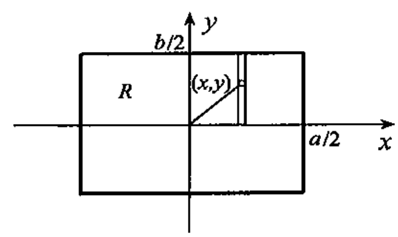
$$A\overline{x} = \int_0^{\pi/2} \sin^2\theta(\cos\theta)\cos\theta\,d\theta = \int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^2 d\theta = \frac{1}{4}\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right)d\theta$$
$$= \frac{1}{8}\left\{\theta - \frac{1}{4}\sin 4\theta\right\}_0^{\pi/2} = \frac{\pi}{16}.$$

Thus, $\overline{x} = (\pi/16)3 = 3\pi/16$. Since

$$A\overline{y} = \int_0^1 \int_0^{x\sqrt{1-x^2}} y\,dy\,dx = \int_0^1 \left\{\frac{y^2}{2}\right\}_0^{x\sqrt{1-x^2}} dx = \frac{1}{2}\int_0^1 x^2(1-x^2)\,dx = \frac{1}{2}\left\{\frac{x^3}{3} - \frac{x^5}{5}\right\}_0^1 = \frac{1}{15},$$

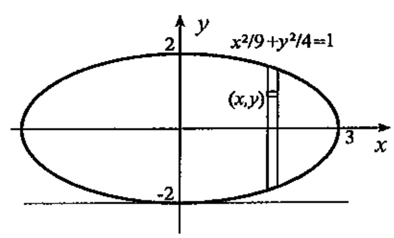
we obtain $\overline{y} = (1/15)3 = 1/5$.

28. $I = \iint_R (x^2+y^2)\rho\,dA = 4\rho\int_0^{a/2}\int_0^{b/2}(x^2+y^2)\,dy\,dx$



$$= 4\rho\int_0^{a/2}\left\{x^2y + \frac{y^3}{3}\right\}_0^{b/2} dx = 4\rho\int_0^{a/2}\left(\frac{bx^2}{2} + \frac{b^3}{24}\right)dx$$
$$= 4\rho\left\{\frac{bx^3}{6} + \frac{b^3x}{24}\right\}_0^{a/2} = \frac{\rho ab}{12}(a^2+b^2)$$

29. $I = 2\int_0^3 \int_{-(2/3)\sqrt{9-x^2}}^{(2/3)\sqrt{9-x^2}} (y+2)^2\,dy\,dx$

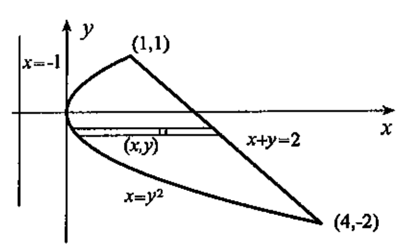


$$= 2\int_0^3 \left\{\frac{(y+2)^3}{3}\right\}_{-(2/3)\sqrt{9-x^2}}^{(2/3)\sqrt{9-x^2}} dx$$
$$= \frac{32}{81}\int_0^3 (36-x^2)\sqrt{9-x^2}\,dx$$

If we set $x = 3\sin\theta$ and $dx = 3\cos\theta\,d\theta$,

$$I = \frac{32}{81}\int_0^{\pi/2}(36 - 9\sin^2\theta)(3\cos\theta)3\cos\theta\,d\theta = 32\int_0^{\pi/2}(4\cos^2\theta - \sin^2\theta\cos^2\theta)\,d\theta$$
$$= 32\int_0^{\pi/2}\left[2 + 2\cos 2\theta - \left(\frac{\sin 2\theta}{2}\right)^2\right]d\theta = 32\int_0^{\pi/2}\left[2 + 2\cos 2\theta - \frac{1}{4}\left(\frac{1-\cos 4\theta}{2}\right)\right]d\theta$$
$$= 32\left\{\frac{15\theta}{8} + \sin 2\theta + \frac{1}{32}\sin 4\theta\right\}_0^{\pi/2} = 30\pi.$$

30. $I = \int_{-2}^1 \int_{y^2}^{2-y} (x+1)^2\,dx\,dy = \int_{-2}^1 \left\{\frac{1}{3}(x+1)^3\right\}_{y^2}^{2-y} dy$

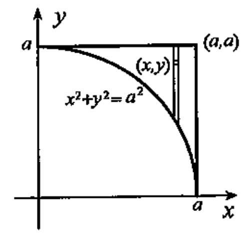


$$= \frac{1}{3}\int_{-2}^1 [(3-y)^3 - y^6 - 3y^4 - 3y^2 - 1]\,dy$$
$$= \frac{1}{3}\left\{-\frac{1}{4}(3-y)^4 - \frac{y^7}{7} - \frac{3y^5}{5} - y^3 - y\right\}_{-2}^1 = \frac{4761}{140}$$

31. $I = \int_0^a \int_{\sqrt{a^2-x^2}}^a y^2\,dy\,dx = \int_0^a \left\{\frac{y^3}{3}\right\}_{\sqrt{a^2-x^2}}^a dx$

$= \frac{1}{3}\int_0^a [a^3 - (a^2-x^2)^{3/2}]\,dx$

If we set $x = a\sin\theta$ and $dx = a\cos\theta\,d\theta$,

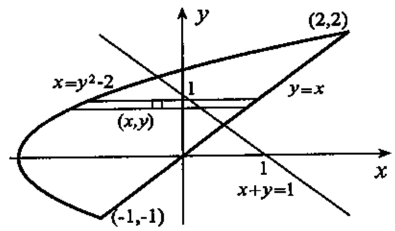


$$I = \frac{1}{3}\left\{a^3x\right\}_0^a - \frac{1}{3}\int_0^{\pi/2} a^3\cos^3\theta(a\cos\theta\,d\theta)$$

$$= \frac{a^4}{3} - \frac{a^4}{3}\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta = \frac{a^4}{3} - \frac{a^4}{12}\int_0^{\pi/2}\left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)d\theta$$

$$= \frac{a^4}{3} - \frac{a^4}{12}\left\{\frac{3\theta}{2} + \sin 2\theta + \frac{1}{8}\sin 4\theta\right\}_0^{\pi/2} = \frac{a^4(16-3\pi)}{48}.$$

32. If we take directed distances to the right of the line $x+y=1$ as positive, then the directed distance from the line to the area $dy\,dx$ at point (x,y) is $(x+y-1)/\sqrt{2}$. The first moment of the plate about the line is



$$M = \int_{-1}^2\int_{y^2-2}^y \frac{x+y-1}{\sqrt{2}}dx\,dy = \frac{1}{\sqrt{2}}\int_{-1}^2\left\{\frac{(x+y-1)^2}{2}\right\}_{y^2-2}^y dy$$

$$= \frac{1}{2\sqrt{2}}\int_{-1}^2 [(2y-1)^2 - y^4 - 2y^3 + 5y^2 + 6y - 9]\,dy$$

$$= \frac{1}{2\sqrt{2}}\left\{\frac{(2y-1)^3}{6} - \frac{y^5}{5} - \frac{y^4}{2} + \frac{5y^3}{3} + 3y^2 - 9y\right\}_{-1}^2 = -\frac{81\sqrt{2}}{40}.$$

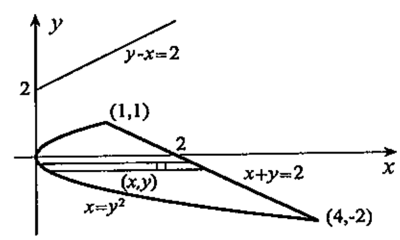
The second moment about the line is

$$I = \int_{-1}^2\int_{y^2-2}^y \frac{(x+y-1)^2}{2}dx\,dy = \frac{1}{2}\int_{-1}^2\left\{\frac{(x+y-1)^3}{3}\right\}_{y^2-2}^y dy$$

$$= \frac{1}{6}\int_{-1}^2 [(2y-1)^3 - y^6 - 3y^5 + 6y^4 + 17y^3 - 18y^2 - 27y + 27]\,dy$$

$$= \frac{1}{6}\left\{\frac{(2y-1)^4}{8} - \frac{y^7}{7} - \frac{y^6}{2} + \frac{6y^5}{5} + \frac{17y^4}{4} - 6y^3 - \frac{27y^2}{2} + 27y\right\}_{-1}^2 = \frac{1863}{280}.$$

33. If we take directed distances to the right of the line $y-x=2$ as positive, then the directed distance from the line to the area $dy\,dx$ at point (x,y) is $(x-y+2)/\sqrt{2}$. The first moment of the plate about the line is



$$M = \int_{-2}^1\int_{y^2}^{2-y} \frac{x-y+2}{\sqrt{2}}dx\,dy = \frac{1}{\sqrt{2}}\int_{-2}^1\left\{\frac{(x-y+2)^2}{2}\right\}_{y^2}^{2-y} dy$$

$$= \frac{1}{2\sqrt{2}}\int_{-2}^1 [(4-2y)^2 - y^4 + 2y^3 - 5y^2 + 4y - 4]\,dy$$

$$= \frac{1}{2\sqrt{2}}\left\{\frac{(4-2y)^3}{-6} - \frac{y^5}{5} + \frac{y^4}{2} - \frac{5y^3}{3} + 2y^2 - 4y\right\}_{-2}^1 = \frac{369\sqrt{2}}{40}.$$

The second moment about the line is

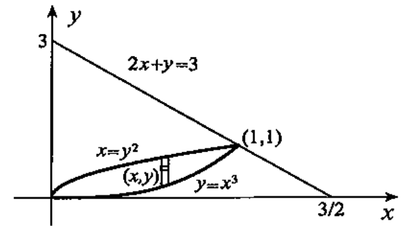
$$I = \int_{-2}^1\int_{y^2}^{2-y} \frac{(x-y+2)^2}{2}dx\,dy = \frac{1}{2}\int_{-2}^1\left\{\frac{(x-y+2)^3}{3}\right\}_{y^2}^{2-y} dy$$

$$= \frac{1}{6}\int_{-2}^1 [(4-2y)^3 - y^6 + 3y^5 - 9y^4 + 13y^3 - 18y^2 + 12y - 8]\,dy$$

$$= \frac{1}{6}\left\{\frac{(4-2y)^4}{-8} - \frac{y^7}{7} + \frac{y^6}{2} - \frac{9y^5}{5} + \frac{13y^4}{4} - 6y^3 + 6y^2 - 8y\right\}_{-2}^1 = \frac{11\,943}{280}.$$

34. If we take directed distances to the right of the line $2x+y=3$ as positive, then the directed distance from the line to the area $dy\,dx$ at point (x,y) is $(2x+y-3)/\sqrt{5}$. The first moment of the plate about the line is

$$M = \int_0^1 \int_{x^3}^{\sqrt{x}} \frac{2x+y-3}{\sqrt{5}}\,dy\,dx$$
$$= \frac{1}{\sqrt{5}} \int_0^1 \left\{ \frac{(2x+y-3)^2}{2} \right\}_{x^3}^{\sqrt{x}} dx$$
$$= \frac{1}{2\sqrt{5}} \int_0^1 (-x^6 - 4x^4 + 6x^3 + 4x^{3/2} + x - 6\sqrt{x})\,dx$$
$$= \frac{1}{2\sqrt{5}} \left\{ -\frac{x^7}{7} - \frac{4x^5}{5} + \frac{3x^4}{2} + \frac{8x^{5/2}}{5} + \frac{x^2}{2} - 4x^{3/2} \right\}_0^1 = -\frac{47\sqrt{5}}{350}.$$

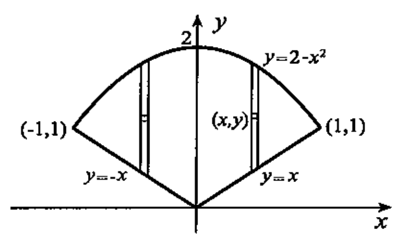


The second moment about the line is

$$I = \int_0^1 \int_{x^3}^{\sqrt{x}} \frac{(2x+y-3)^2}{5}\,dy\,dx = \frac{1}{5} \int_0^1 \left\{ \frac{(2x+y-3)^3}{3} \right\}_{x^3}^{\sqrt{x}} dx$$
$$= \frac{1}{15} \int_0^1 (-x^9 - 6x^7 + 9x^6 - 12x^5 + 36x^4 - 27x^3 + 6x^2 - 9x + 12x^{5/2} - 35x^{3/2} + 27\sqrt{x})\,dx$$
$$= \frac{1}{15} \left\{ -\frac{x^{10}}{10} - \frac{3x^8}{4} + \frac{9x^7}{7} - 2x^6 + \frac{36x^5}{5} - \frac{27x^4}{4} + 2x^3 - \frac{9x^2}{2} + \frac{24x^{7/2}}{7} - 14x^{5/2} + 18x^{3/2} \right\}_0^1 = \frac{89}{350}.$$

35. If we take directed distances to the left of the line $y=x$ as positive, then the directed distance from the line to the area $dy\,dx$ at point (x,y) is $(y-x)/\sqrt{2}$. The first moment of the plate about the line is

$$M = \int_{-1}^0 \int_{-x}^{2-x^2} \frac{y-x}{\sqrt{2}}\,dy\,dx + \int_0^1 \int_x^{2-x^2} \frac{y-x}{\sqrt{2}}\,dy\,dx$$
$$= \frac{1}{\sqrt{2}} \int_{-1}^0 \left\{ \frac{(y-x)^2}{2} \right\}_{-x}^{2-x^2} dx$$
$$+ \frac{1}{\sqrt{2}} \int_0^1 \left\{ \frac{(y-x)^2}{2} \right\}_x^{2-x^2} dx$$

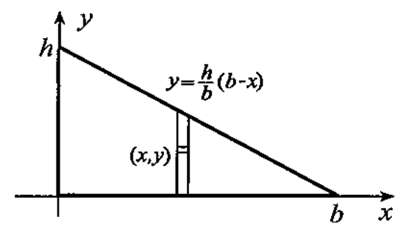


$$= \frac{1}{2\sqrt{2}} \int_{-1}^0 (x^4 + 2x^3 - 7x^2 - 4x + 4)\,dx + \frac{1}{2\sqrt{2}} \int_0^1 (x^4 + 2x^3 - 3x^2 - 4x + 4)\,dx$$
$$= \frac{1}{2\sqrt{2}} \left\{ \frac{x^5}{5} + \frac{x^4}{2} - \frac{7x^3}{3} - 2x^2 + 4x \right\}_{-1}^0 + \frac{1}{2\sqrt{2}} \left\{ \frac{x^5}{5} + \frac{x^4}{4} - x^3 - 2x^2 + 4x \right\}_0^1 = \frac{19\sqrt{2}}{15}.$$

The second moment about the line is

$$I = \int_{-1}^0 \int_{-x}^{2-x^2} \frac{(y-x)^2}{2}\,dy\,dx + \int_0^1 \int_x^{2-x^2} \frac{(y-x)^2}{2}\,dy\,dx$$
$$= \frac{1}{2} \int_{-1}^0 \left\{ \frac{(y-x)^3}{3} \right\}_{-x}^{2-x^2} dx + \frac{1}{2} \int_0^1 \left\{ \frac{(y-x)^3}{3} \right\}_x^{2-x^2} dx$$
$$= \frac{1}{6} \int_{-1}^0 (8 - 12x - 6x^2 + 19x^3 + 3x^4 - 3x^5 - x^6)\,dx$$
$$+ \frac{1}{6} \int_0^1 (8 - 12x - 6x^2 + 11x^3 + 3x^4 - 3x^5 - x^6)\,dx$$
$$= \frac{1}{6} \left\{ 8x - 6x^2 - 2x^3 + \frac{19x^4}{4} + \frac{3x^5}{5} - \frac{x^6}{2} - \frac{x^7}{7} \right\}_{-1}^0 + \frac{1}{6} \left\{ 8x - 6x^2 - 2x^3 + \frac{11x^4}{4} + \frac{3x^5}{5} - \frac{x^6}{2} - \frac{x^7}{7} \right\}_0^1$$
$$= \frac{191}{105}.$$

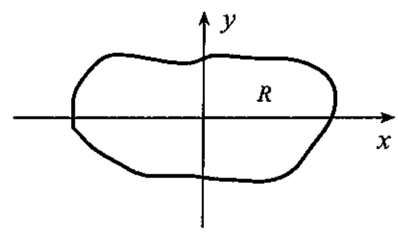
36. (a) $$I_{xy} = \int_0^b \int_0^{h(b-x)/b} xy\rho\,dy\,dx$$
$$= \rho \int_0^b \left\{\frac{xy^2}{2}\right\}_0^{h(b-x)/b} dx$$
$$= \frac{\rho h^2}{2b^2}\int_0^b (b^2x - 2bx^2 + x^3)\,dx$$
$$= \frac{\rho h^2}{2b^2}\left\{\frac{b^2x^2}{2} - \frac{2bx^3}{3} + \frac{x^4}{4}\right\}_0^b = \frac{\rho b^2h^2}{24}$$



(b) The centre of mass is $(b/3, h/3)$. The product moment of inertia about horizontal and vertical lines through this point is

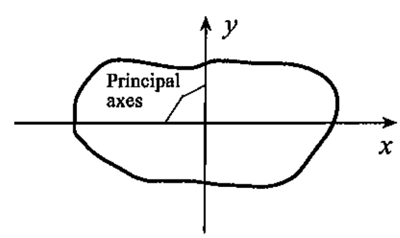
$$I = \int_0^b \int_0^{h(b-x)/b} \left(x - \frac{b}{3}\right)\left(y - \frac{h}{3}\right)\rho\,dy\,dx = \rho\int_0^b \left\{\frac{1}{2}\left(x-\frac{b}{3}\right)\left(y-\frac{h}{3}\right)^2\right\}_0^{h(b-x)/b} dx$$
$$= \frac{\rho h^2}{18b^2}\int_0^b (9x^3 - 15bx^2 + 7b^2x - b^3)\,dx = \frac{\rho h^2}{18b^2}\left\{\frac{9x^4}{4} - 5bx^3 + \frac{7b^2x^2}{2} - b^3x\right\}_0^b = -\frac{\rho b^2h^2}{72}.$$

37. Since $\iint_R (x-y)^2\rho\,dA \ge 0$, it follows that
$$0 \le \iint_R x^2\rho\,dA - 2\iint_R xy\rho\,dA + \iint_R y^2\rho\,dA$$
$$= I_y - 2I_{xy} + I_x,$$
and this implies that $I_{xy} \le (I_x + I_y)/2$. Similarly, by considering the double integral of $\rho(x+y)^2$ over R, we obtain $I_{xy} \ge -(I_x + I_y)/2$. Together these give



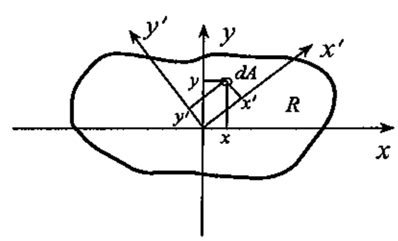
$$-\frac{I_x + I_y}{2} \le I_{xy} \le \frac{I_x + I_y}{2} \quad\Longrightarrow\quad |I_{xy}| \le \frac{I_x + I_y}{2}.$$

38. Suppose we choose the point as the origin and axes along the principal axes of the plate. Then one principal axis has slope zero while the slope of the other is undefined. According to equation 13.39, this occurs when $I_{xy} = 0$. This can also be seen by taking $\theta = 0$ or $\theta = \pi/2$ in Exercise 45.



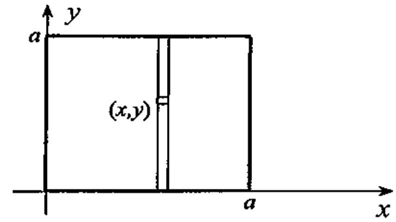
39. $$I_{x'} + I_{y'} = \iint_R (y')^2\rho\,dA + \iint_R (x')^2\rho\,dA$$
$$= \iint_R [(x')^2 + (y')^2]\rho\,dA$$
But $(x')^2 + (y')^2$ is the square of the distance from dA to the origin, and therefore $(x')^2 + (y')^2 = x^2 + y^2$.

Hence,



$$I_{x'} + I_{y'} = \iint_R (x^2 + y^2)\rho\,dA = \iint_R x^2\rho\,dA + \iint_R y^2\rho\,dA = I_x + I_y.$$

40. Since



$$I_x = I_y = \int_0^a \int_0^a y^2 \rho \, dy \, dx = \rho \int_0^a \left\{\frac{y^3}{3}\right\}_0^a dx$$
$$= \frac{\rho a^3}{3}\left\{x\right\}_0^a = \frac{\rho a^4}{3},$$

and

$$I_{xy} = \int_0^a \int_0^a xy\rho \, dy \, dx = \rho \int_0^a \left\{\frac{xy^2}{2}\right\}_0^a dx = \frac{\rho a^2}{2}\left\{\frac{x^2}{2}\right\}_0^a = \frac{\rho a^4}{4},$$

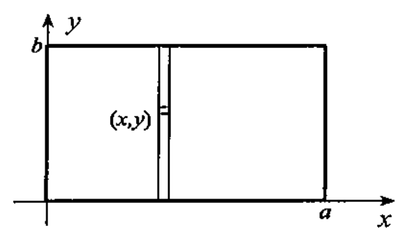
slopes of the principal axes are defined by equation 13.39,

$$m = \frac{I_x - I_y}{2I_{xy}} \pm \sqrt{1 + \left(\frac{I_x - I_y}{2I_{xy}}\right)^2} = \pm 1.$$

Principal axes are therefore the lines $y = \pm x$. According to equation 13.40, principal moments of inertia are

$$\frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + (I_{xy})^2} = I_x \pm I_{xy} = \frac{7\rho a^4}{12}, \frac{\rho a^4}{12}.$$

41. For the rectangle,



$$I_x = \int_0^a \int_0^b y^2 \rho \, dy \, dx = \rho \int_0^a \left\{\frac{y^3}{3}\right\}_0^b dx = \frac{\rho b^3}{3}\left\{x\right\}_0^a = \frac{\rho a b^3}{3},$$

$$I_y = \int_0^a \int_0^b x^2 \rho \, dy \, dx = \rho \int_0^a \left\{x^2 y\right\}_0^b dx = \rho b \left\{\frac{x^3}{3}\right\}_0^a = \frac{\rho a^3 b}{3},$$

$$I_{xy} = \int_0^a \int_0^b xy\rho \, dy \, dx = \rho \int_0^a \left\{\frac{xy^2}{2}\right\}_0^b dx = \frac{\rho b^2}{2}\left\{\frac{x^2}{2}\right\}_0^a = \frac{\rho a^2 b^2}{4}.$$

With $\frac{I_x - I_y}{2I_{xy}} = \left(\frac{\rho a b^3}{3} - \frac{\rho a^3 b}{3}\right)\frac{2}{\rho a^2 b^2} = \frac{2}{3}\left(\frac{b}{a} - \frac{a}{b}\right)$, slopes of the principal axes are defined by equation 13.39,

$$m = \frac{2}{3}\left(\frac{b}{a} - \frac{a}{b}\right) \pm \sqrt{1 + \frac{4}{9}\left(\frac{b}{a} - \frac{a}{b}\right)^2}.$$

Principal moments of inertia in these directions are

$$\frac{1}{2}\left(\frac{\rho a b^3}{3} + \frac{\rho a^3 b}{3}\right) \mp \sqrt{\left(\frac{\rho a b^3}{6} - \frac{\rho a^3 b}{6}\right)^2 + \left(\frac{\rho a^2 b^2}{4}\right)^2} = \frac{\rho a b}{6}(a^2 + b^2) \mp \frac{\rho a b}{12}\sqrt{4(b^2 - a^2)^2 + 9a^2 b^2}.$$

42. Since $\lim_{m \to \pm\infty} I(m) = I_y$, we must show that

$$\frac{I_x + I_y}{2} - \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + (I_{xy})^2} \le I_y \le \frac{I_x + I_y}{2} + \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + (I_{xy})^2}.$$

But this is equivalent to

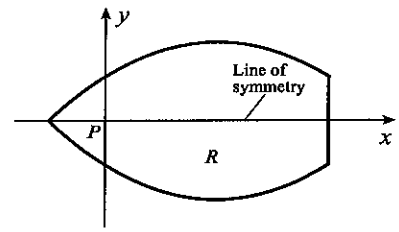
$$-\sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + (I_{xy})^2} \le \frac{I_y - I_x}{2} \le \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + (I_{xy})^2},$$

which is valid for any I_x and I_y.

43. Choose a coordinate system with the x-axis along ℓ and the y-axis through P. The product moment of inertia about ℓ (the x-axis) and the y-axis is

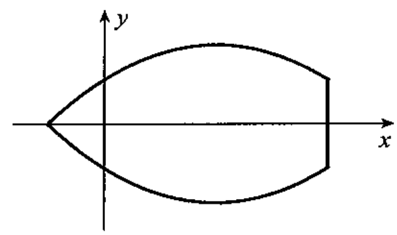
$$I_{xy} = \iint_R xy\,dA.$$

Because xy is an odd function of y and R is symmetric about the x-axis, this integral has value zero.



44. Suppose the x-axis is chosen as the axis of symmetry. Choose the y-axis through any point on the line. According to Exercise 43, the product moment of inertia about the origin for the coordinate axes is zero. Consequently the moment of inertia about any line through the origin with slope m is

$$I(m) = \frac{1}{m^2+1}(I_x + m^2 I_y) = I_y + \frac{I_x - I_y}{m^2+1},$$



(see equation 13.38). If $I_x > I_y$, then this is an even function of m, decreasing from $I(0) = I_x$ to $\lim_{m\to\infty} I(m) = I_y$; that is, principal axes are $x = 0$ and $y = 0$. If $I_x < I_y$, then this even function increases from $I(0) = I_x$ to $\lim_{m\to\infty} I(m) = I_y$, and once again I_x and I_y are principal moments of inertia. If $I_x = I_y$, then $I(m) = I_y$ for all m, in which case all pairs of perpendicular lines through the origin are principal axes.

45. We know that when θ is the angle of inclination of a line, then $\tan\theta = m$, and when the line is a principal axis about the origin, m is defined by equation 13.39. Using the double angle formula for $\tan 2\theta$ gives

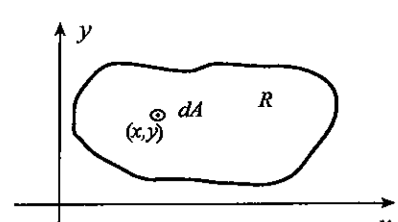
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2m}{1-m^2} = \frac{2\left[\dfrac{I_x - I_y}{2I_{xy}} \pm \sqrt{1 + \left(\dfrac{I_x - I_y}{2I_{xy}}\right)^2}\,\right]}{1 - \left[\dfrac{I_x - I_y}{2I_{xy}} \pm \sqrt{1 + \left(\dfrac{I_x - I_y}{2I_{xy}}\right)^2}\,\right]^2}$$

$$= \frac{\dfrac{I_x - I_y \pm \sqrt{(I_x - I_y)^2 + 4(I_{xy})^2}}{I_{xy}}}{\dfrac{4(I_{xy})^2 - (I_x - I_y)^2 \mp 2(I_x - I_y)\sqrt{(I_x - I_y)^2 + 4(I_{xy})^2} - (I_x - I_y)^2 - 4(I_{xy})^2}{4(I_{xy})^2}}$$

$$= \frac{4I_{xy}\left[I_x - I_y \pm \sqrt{(I_x - I_y)^2 + 4(I_{xy})^2}\right]}{-2(I_x - I_y)^2 \mp 2(I_x - I_y)\sqrt{(I_x - I_y)^2 + 4(I_{xy})^2}} = \frac{2I_{xy}}{-(I_x - I_y)} = \frac{2I_{xy}}{I_y - I_x}.$$

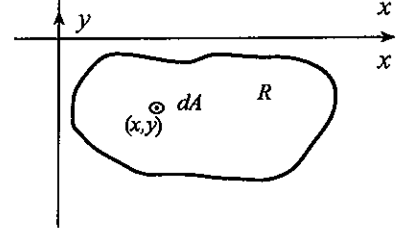
46. If we orient the area so that the axis of rotation is the y-axis, then

$$V = \iint_R 2\pi x\,dA = 2\pi \iint_R x\,dA$$
$$= 2\pi(A\overline{x}) = (2\pi\overline{x})A.$$

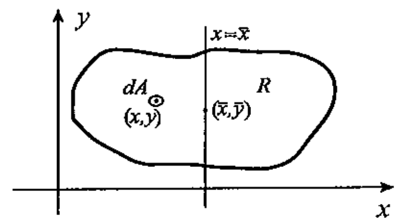


47. The fluid force on each side of the plate R is

$$F = \iint_R \rho g(-y)\,dA \quad (g = 9.81)$$
$$= -\rho g \iint_R y\,dA$$
$$= -\rho g(A\overline{y}) = \rho g(-\overline{y})A.$$



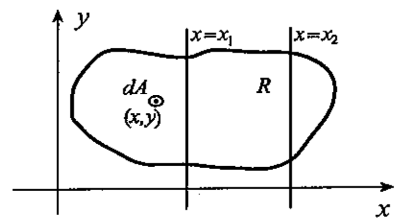
48. If we orient the area so that the line is the y-axis, then



$$\begin{aligned}
I &= \iint_R x^2\rho\,dA = \rho\iint_R [(x-\overline{x})+\overline{x}]^2\,dA\\
&= \rho\iint_R [(x-\overline{x})^2 + 2\overline{x}(x-\overline{x}) + \overline{x}^2]\,dA\\
&= \iint_R \rho(x-\overline{x})^2\,dA + 2\overline{x}\iint_R x\rho\,dA - \overline{x}^2\iint_R \rho\,dA\\
&= \iint_R \rho(x-\overline{x})^2\,dA + 2\overline{x}(M\overline{x}) - \overline{x}^2 M = \iint_R \rho(x-\overline{x})^2\,dA + M\overline{x}^2.
\end{aligned}$$

49. Since $I_{x_2} = \iint_R (x-x_2)^2\rho\,dA$ and $I_{x_1} = \iint_R (x-x_1)^2\rho\,dA$,

$$\begin{aligned}
I_{x_2} - I_{x_1} &= \iint_R (x-x_2)^2\rho\,dA - \iint_R (x-x_1)^2\rho\,dA = \iint_R (x^2 - 2xx_2 + x_2^2 - x^2 + 2xx_1 - x_1^2)\rho\,dA\\
&= (x_2^2 - x_1^2)\iint_R \rho\,dA + 2(x_1-x_2)\iint_R x\rho\,dA\\
&= (x_2^2 - x_1^2)M + 2(x_1-x_2)M\overline{x}.
\end{aligned}$$



Thus,

$$I_{x_2} = I_{x_1} + M[x_2^2 - x_1^2 + 2\overline{x}(x_1 - x_2)].$$

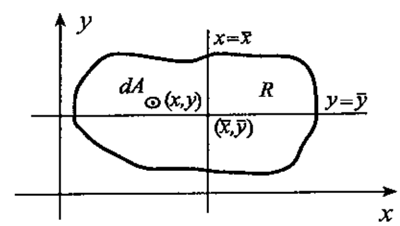
When $x_1 = \overline{x}$, this reduces to

$$I_{x_2} = I_{\overline{x}} + M[x_2^2 - \overline{x}^2 + 2\overline{x}(\overline{x} - x_2)] = I_{\overline{x}} + M(x_2^2 - 2\overline{x}x_2 + \overline{x}^2) = I_{\overline{x}} + M(x_2 - \overline{x})^2,$$

and this is the parallel axis theorem.

50.

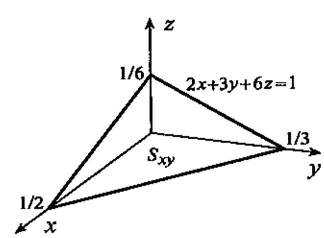
$$\begin{aligned}
I_{\overline{x}\,\overline{y}} &= \iint_R (x-\overline{x})(y-\overline{y})\rho\,dA\\
&= \iint_R xy\rho\,dA - \overline{x}\iint_R y\rho\,dA\\
&\quad -\overline{y}\iint_R x\rho\,dA + \iint_R \overline{x}\,\overline{y}\rho\,dA\\
&= I_{xy} - \overline{x}(M\overline{y}) - \overline{y}(M\overline{x}) + \overline{x}\,\overline{y}(M) = I_{xy} - M\overline{x}\,\overline{y}
\end{aligned}$$



EXERCISES 13.6

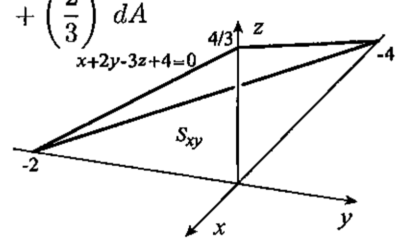
1.

$$\begin{aligned}
\text{Area} &= \iint_{S_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA\\
&= \iint_{S_{xy}} \sqrt{1 + (-1/3)^2 + (-1/2)^2}\,dA\\
&= \frac{7}{6}\iint_{S_{xy}} dA = \frac{7}{6}(\text{Area of } S_{xy}) = \frac{7}{6}\left(\frac{1}{12}\right) = \frac{7}{72}
\end{aligned}$$



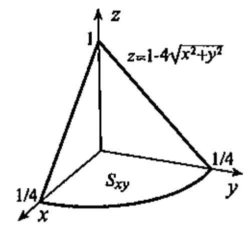
2.

$$\begin{aligned}
\text{Area} &= \iint_{S_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA = \iint_{S_{xy}} \sqrt{1 + \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2}\,dA\\
&= \frac{\sqrt{14}}{3}\iint_{S_{xy}} dA = \frac{\sqrt{14}}{3}(\text{Area of } S_{xy})\\
&= \frac{\sqrt{14}}{3}\frac{1}{2}(2)(4) = \frac{4\sqrt{14}}{3}
\end{aligned}$$

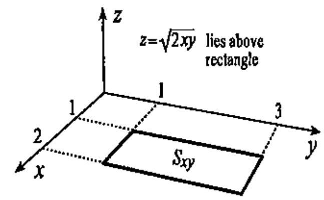


3. We quadruple the area in the first octant.

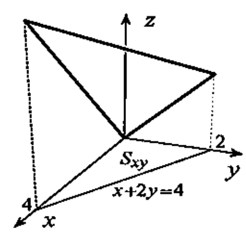
$$\begin{aligned}\text{Area} &= 4\iint_{S_{xy}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA\\ &= 4\iint_{S_{xy}}\sqrt{1+\left(\frac{-4x}{\sqrt{x^2+y^2}}\right)^2+\left(\frac{-4y}{\sqrt{x^2+y^2}}\right)^2}\,dA\\ &= 4\sqrt{17}\iint_{S_{xy}}dA = 4\sqrt{17}(\text{Area of } S_{xy}) = 4\sqrt{17}\frac{\pi}{64} = \frac{\sqrt{17}\pi}{16}\end{aligned}$$



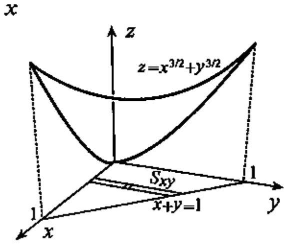
4. $$\begin{aligned}\text{Area} &= \iint_{S_{xy}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA\\ &= \iint_{S_{xy}}\sqrt{1+\left(\frac{y}{\sqrt{2xy}}\right)^2+\left(\frac{x}{\sqrt{2xy}}\right)^2}\,dA\\ &= \iint_{S_{xy}}\left(\frac{x+y}{\sqrt{2xy}}\right)dA = \frac{1}{\sqrt{2}}\int_1^2\int_1^3\left(\frac{\sqrt{x}}{\sqrt{y}}+\frac{\sqrt{y}}{\sqrt{x}}\right)dy\,dx\\ &= \frac{1}{\sqrt{2}}\int_1^2\left\{2\sqrt{xy}+\frac{2y^{3/2}}{3\sqrt{x}}\right\}_1^3dx = \frac{1}{\sqrt{2}}\int_1^2\left[2(\sqrt{3}-1)\sqrt{x}+\frac{6\sqrt{3}-2}{3\sqrt{x}}\right]dx\\ &= \frac{1}{\sqrt{2}}\left\{\frac{4}{3}(\sqrt{3}-1)x^{3/2}+\frac{4(3\sqrt{3}-1)\sqrt{x}}{3}\right\}_1^2 = \frac{4}{3}(5\sqrt{3}-2\sqrt{6}-3+\sqrt{2})\end{aligned}$$



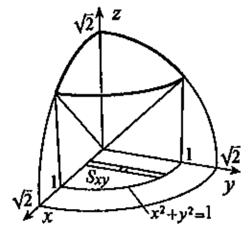
5. $$\begin{aligned}\text{Area} &= \iint_{S_{xy}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA\\ &= \iint_{S_{xy}}\sqrt{1+(1)^2+(1)^2}dA\\ &= \sqrt{3}\iint_{S_{xy}}dA = \sqrt{3}(\text{Area of } S_{xy})\\ &= \sqrt{3}\left(\frac{1}{2}\right)(4)(2) = 4\sqrt{3}\end{aligned}$$



6. $$\begin{aligned}\text{Area} &= \iint_{S_{xy}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA\\ &= \iint_{S_{xy}}\sqrt{1+\left(\frac{3\sqrt{x}}{2}\right)^2+\left(\frac{3\sqrt{y}}{2}\right)^2}\,dA\\ &= \frac{1}{2}\iint_{S_{xy}}\sqrt{4+9x+9y}\,dA\\ &= \frac{1}{2}\int_0^1\int_0^{1-x}\sqrt{4+9x+9y}\,dy\,dx = \frac{1}{2}\int_0^1\left\{\frac{2}{27}(4+9x+9y)^{3/2}\right\}_0^{1-x}dx\\ &= \frac{1}{27}\int_0^1[13\sqrt{13}-(4+9x)^{3/2}]\,dx = \frac{1}{27}\left\{13\sqrt{13}x-\frac{2(4+9x)^{5/2}}{45}\right\}_0^1 = \frac{247\sqrt{13}+64}{1215}\end{aligned}$$



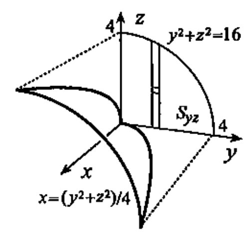
7. We quadruple the first octant area.



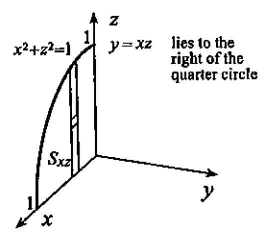
$$\begin{aligned}\text{Area } &= 4\iint_{S_{xy}} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA\\ &= 4\iint_{S_{xy}} \sqrt{1+\left(\frac{-x}{\sqrt{2-x^2-y^2}}\right)^2+\left(\frac{-y}{\sqrt{2-x^2-y^2}}\right)^2}\,dA\\ &= 4\iint_{S_{xy}} \frac{\sqrt{2}}{\sqrt{2-x^2-y^2}}dA = 4\sqrt{2}\int_0^1\int_0^{\sqrt{1-x^2}} \frac{1}{\sqrt{2-x^2-y^2}}dy\,dx\end{aligned}$$

8. We quadruple the area in the first octant.

$$\begin{aligned}\text{Area } &= 4\iint_{S_{yz}} \sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2}\,dA\\ &= 4\iint_{S_{yz}} \sqrt{1+\left(\frac{y}{2}\right)^2+\left(\frac{z}{2}\right)^2}\,dA\\ &= 2\iint_{S_{yz}} \sqrt{4+y^2+z^2}\,dA\\ &= 2\int_0^4\int_0^{\sqrt{16-y^2}} \sqrt{4+y^2+z^2}\,dz\,dy\end{aligned}$$

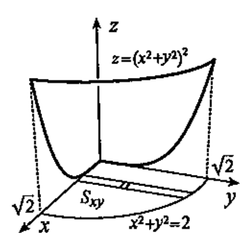


9.
$$\begin{aligned}\text{Area } &= \iint_{S_{xz}} \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dA\\ &= \iint_{S_{xz}} \sqrt{1+(z)^2+(x)^2}dA\\ &= \int_0^1\int_0^{\sqrt{1-x^2}} \sqrt{1+x^2+z^2}\,dz\,dx\end{aligned}$$



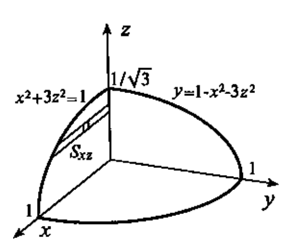
10. We quadruple the area in the first octant.

$$\begin{aligned}\text{Area } &= 4\iint_{S_{xy}} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA\\ &= 4\iint_{S_{xy}} \sqrt{1+[4x(x^2+y^2)]^2+[4y(x^2+y^2)]^2}dA\\ &= 4\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}} \sqrt{1+16(x^2+y^2)^3}\,dy\,dx\end{aligned}$$

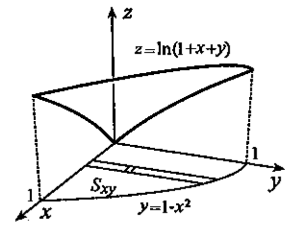


11. We quadruple the area in the first octant.

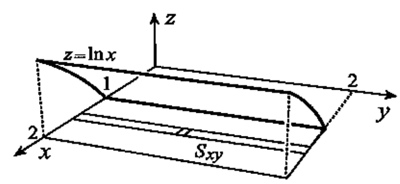
$$\begin{aligned}\text{Area } &= 4\iint_{S_{xz}} \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dA\\ &= 4\iint_{S_{xz}} \sqrt{1+(-2x)^2+(-6z)^2}\,dA\\ &= 4\int_0^{1/\sqrt{3}}\int_0^{\sqrt{1-3z^2}} \sqrt{1+4x^2+36z^2}\,dx\,dz\end{aligned}$$



12.
$$\begin{aligned}\text{Area} &= \iint_{S_{xy}} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA\\ &= \iint_{S_{xy}} \sqrt{1+\left(\frac{1}{1+x+y}\right)^2+\left(\frac{1}{1+x+y}\right)^2}\,dA\\ &= \int_0^1\int_0^{1-x^2}\sqrt{1+\frac{2}{(1+x+y)^2}}\,dy\,dx\end{aligned}$$



13.
$$\begin{aligned}\text{Area} &= \iint_{S_{xy}} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA\\ &= \iint_{S_{xy}} \sqrt{1+(1/x)^2}\,dA = \int_1^2\int_0^2 \frac{\sqrt{x^2+1}}{x}\,dy\,dx\\ &= \int_1^2\left\{\frac{y\sqrt{x^2+1}}{x}\right\}_0^2 dx = 2\int_1^2\frac{\sqrt{x^2+1}}{x}\,dx\end{aligned}$$

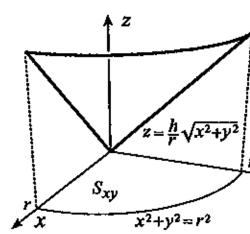


If we set $x=\tan\theta$ and $dx=\sec^2\theta\,d\theta$,

$$\text{Area} = 2\int_{\pi/4}^{\mathrm{Tan}^{-1}2}\frac{\sec\theta}{\tan\theta}\sec^2\theta\,d\theta = 2\int_{\pi/4}^{\mathrm{Tan}^{-1}2}\frac{(1+\tan^2\theta)\sec\theta}{\tan\theta}d\theta = 2\int_{\pi/4}^{\mathrm{Tan}^{-1}2}(\csc\theta+\sec\theta\tan\theta)\,d\theta$$
$$= 2\Big\{\ln|\csc\theta-\cot\theta|+\sec\theta\Big\}_{\pi/4}^{\mathrm{Tan}^{-1}2} = 2[\ln(\sqrt{5}-1)-\ln 2+\sqrt{5}-\ln(\sqrt{2}-1)-\sqrt{2}].$$

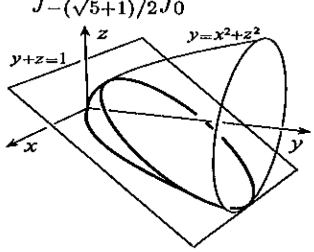
14. We quadruple the area in the first octant.

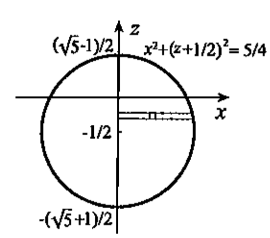
$$\begin{aligned}\text{Area} &= 4\iint_{S_{xy}} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA\\ &= 4\iint_{S_{xy}}\sqrt{1+\left(\frac{hx}{r\sqrt{x^2+y^2}}\right)^2+\left(\frac{hy}{r\sqrt{x^2+y^2}}\right)^2}\,dA\\ &= \frac{4\sqrt{r^2+h^2}}{r}\iint_{S_{xy}}dA = \frac{4\sqrt{r^2+h^2}}{r}(\text{Area of } S_{xy})\\ &= \frac{4\sqrt{r^2+h^2}}{r}\left(\frac{\pi r^2}{4}\right) = \pi r\sqrt{r^2+h^2}\end{aligned}$$



15. The surface projects onto the area in the xz-plane bounded by the circle $1-z=x^2+z^2$, or, $x^2+(z+1/2)^2=5/4$. Thus,

$$\begin{aligned}A &= \iint_{S_{xz}}\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dA = \iint_{S_{xz}}\sqrt{1+(2x)^2+(2z)^2}\,dA\\ &= 2\int_{-(\sqrt{5}+1)/2}^{(\sqrt{5}-1)/2}\int_0^{\sqrt{1-z-z^2}}\sqrt{1+4x^2+4z^2}\,dx\,dz\end{aligned}$$





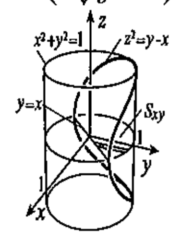
16. We double the area of the upper half.

$$\text{Area} = 2\iint_{S_{xy}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA = 2\iint_{S_{xy}}\sqrt{1+\left(\frac{-1}{2\sqrt{y-x}}\right)^2+\left(\frac{1}{2\sqrt{y-x}}\right)^2}\,dA$$

$$= \sqrt{2}\iint_{S_{xy}}\sqrt{2+\frac{1}{y-x}}\,dA$$

$$= \sqrt{2}\int_{-1/\sqrt{2}}^{1/\sqrt{2}}\int_{x}^{\sqrt{1-x^2}}\sqrt{2+\frac{1}{y-x}}\,dy\,dx$$

$$+\sqrt{2}\int_{-1}^{-1/\sqrt{2}}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\sqrt{2+\frac{1}{y-x}}\,dy\,dx$$

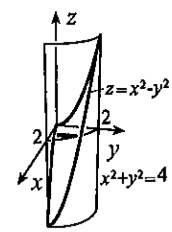


17. We quadruple the area in the first octant.

$$\text{Area} = 4\iint_{S_{xy}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$= 4\iint_{S_{xy}}\sqrt{1+(-2x)^2+(2y)^2}\,dA$$

$$= 4\int_0^2\int_0^{\sqrt{4-x^2}}\sqrt{1+4x^2+4y^2}\,dy\,dx$$

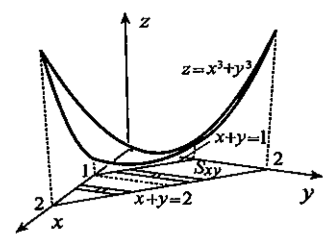


18. $$\text{Area} = \iint_{S_{xy}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$= \iint_{S_{xy}}\sqrt{1+(3x^2)^2+(3y^2)^2}\,dA$$

$$= \int_0^1\int_{1-x}^{2-x}\sqrt{1+9x^4+9y^4}\,dy\,dx$$

$$+\int_1^2\int_0^{2-x}\sqrt{1+9x^4+9y^4}\,dy\,dx$$



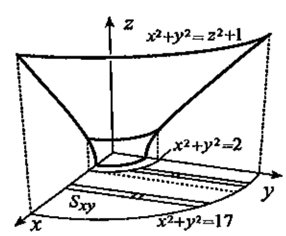
19. We quadruple the area in the first octant.

$$\text{Area} = 4\iint_{S_{xy}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA = 4\iint_{S_{xy}}\sqrt{1+\left(\frac{x}{\sqrt{x^2+y^2-1}}\right)^2+\left(\frac{y}{\sqrt{x^2+y^2-1}}\right)^2}\,dA$$

$$= 4\iint_{S_{xy}}\sqrt{\frac{2x^2+2y^2-1}{x^2+y^2-1}}\,dA$$

$$= 4\int_0^{\sqrt{2}}\int_{\sqrt{2-x^2}}^{\sqrt{17-x^2}}\sqrt{\frac{2x^2+2y^2-1}{x^2+y^2-1}}\,dy\,dx$$

$$+4\int_{\sqrt{2}}^{\sqrt{17}}\int_0^{\sqrt{17-x^2}}\sqrt{\frac{2x^2+2y^2-1}{x^2+y^2-1}}\,dy\,dx$$

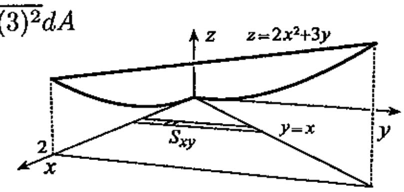


20. If S_{xy} is the region of the xy-plane bounded by the lines $x=2$, $y=0$, and $y=x$, then

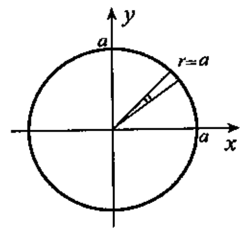
$$\text{Area} = \iint_{S_{xy}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA = \iint_{S_{xy}}\sqrt{1+(4x)^2+(3)^2}\,dA$$

$$= \int_0^2\int_0^x\sqrt{10+16x^2}\,dy\,dx = \int_0^2 x\sqrt{10+16x^2}\,dx$$

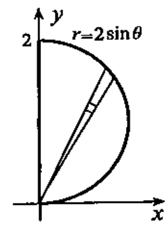
$$= \left\{\frac{1}{48}(10+16x^2)^{3/2}\right\}_0^2 = \frac{1}{24}(37\sqrt{74}-5\sqrt{10}).$$



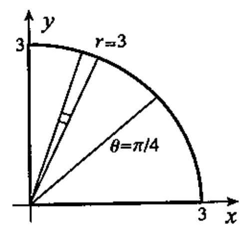
EXERCISES 13.7



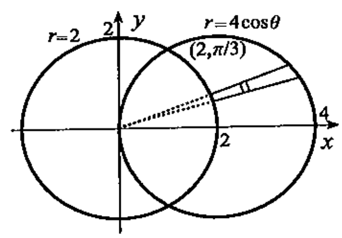
1. $$\iint_R e^{x^2+y^2}\,dA = \int_{-\pi}^{\pi}\int_0^a e^{r^2}\,r\,dr\,d\theta$$
$$= \int_{-\pi}^{\pi}\left\{\frac{1}{2}e^{r^2}\right\}_0^a d\theta$$
$$= \frac{1}{2}(e^{a^2}-1)\left\{\theta\right\}_{-\pi}^{\pi} = \pi(e^{a^2}-1)$$



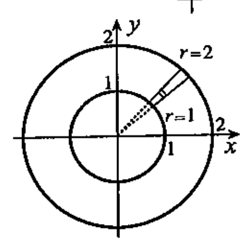
2. $$\iint_R x\,dA = \int_0^{\pi/2}\int_0^{2\sin\theta} r\cos\theta\,r\,dr\,d\theta = \int_0^{\pi/2}\left\{\frac{r^3}{3}\cos\theta\right\}_0^{2\sin\theta} d\theta$$
$$= \frac{8}{3}\int_0^{\pi/2}\sin^3\theta\,\cos\theta\,d\theta$$
$$= \frac{8}{3}\left\{\frac{\sin^4\theta}{4}\right\}_0^{\pi/2} = \frac{2}{3}$$



3. $$\iint_R \sqrt{x^2+y^2}\,dA = \int_{\pi/4}^{\pi/2}\int_0^3 (r)r\,dr\,d\theta$$
$$= \int_{\pi/4}^{\pi/2}\left\{\frac{r^3}{3}\right\}_0^3 d\theta$$
$$= 9\left\{\theta\right\}_{\pi/4}^{\pi/2} = \frac{9\pi}{4}$$

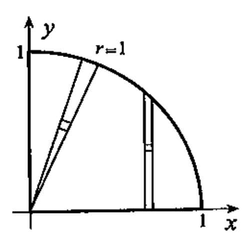


4. $$\iint_R \frac{1}{\sqrt{x^2+y^2}}\,dA = 2\int_0^{\pi/3}\int_2^{4\cos\theta}\frac{1}{r}r\,dr\,d\theta$$
$$= 2\int_0^{\pi/3}(4\cos\theta-2)\,d\theta$$
$$= 4\left\{2\sin\theta-\theta\right\}_0^{\pi/3} = \frac{4}{3}(3\sqrt{3}-\pi)$$



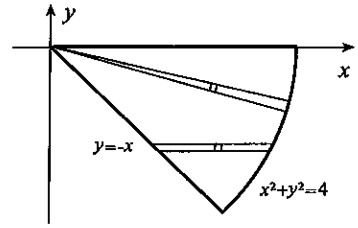
5. $$\iint_R \sqrt{1+2x^2+2y^2}\,dA = \int_{-\pi}^{\pi}\int_1^2\sqrt{1+2r^2}\,r\,dr\,d\theta$$
$$= \int_{-\pi}^{\pi}\left\{\frac{1}{6}(1+2r^2)^{3/2}\right\}_1^2 d\theta$$
$$= \frac{1}{6}(27-3\sqrt{3})\left\{\theta\right\}_{-\pi}^{\pi} = (9-\sqrt{3})\pi$$

6. This double iterated integral represents the double integral of $\sqrt{x^2+y^2}$ over the quarter circle shown. When we change to polar coordinates,



$$\int_0^1\int_0^{\sqrt{1-x^2}}\sqrt{x^2+y^2}\,dy\,dx = \int_0^{\pi/2}\int_0^1 r\,r\,dr\,d\theta$$
$$= \int_0^{\pi/2}\left\{\frac{r^3}{3}\right\}_0^1 d\theta = \frac{1}{3}\left\{\theta\right\}_0^{\pi/2} = \frac{\pi}{6}.$$

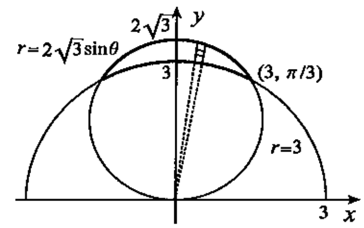
7. Limits define the quarter-circle shown. Changing to polar coordinates gives



$$\int_{-\sqrt{2}}^{0}\int_{-y}^{\sqrt{4-y^2}} x^2\,dx\,dy = \int_{-\pi/4}^{0}\int_{0}^{2} r^2\cos^2\theta\, r\,dr\,d\theta$$

$$= \int_{-\pi/4}^{0}\left\{\frac{r^4}{4}\cos^2\theta\right\}_0^2 d\theta = 4\int_{-\pi/4}^{0}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

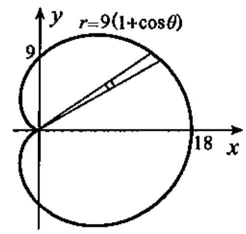
$$= 2\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_{-\pi/4}^{0} = \frac{2+\pi}{2}$$



8. $$A = 2\int_{\pi/3}^{\pi/2}\int_{3}^{2\sqrt{3}\sin\theta} r\,dr\,d\theta = 2\int_{\pi/3}^{\pi/2}\left\{\frac{r^2}{2}\right\}_3^{2\sqrt{3}\sin\theta} d\theta$$

$$= 3\int_{\pi/3}^{\pi/2}(4\sin^2\theta - 3)\,d\theta = 3\int_{\pi/3}^{\pi/2}[2(1-\cos 2\theta) - 3]\,d\theta$$

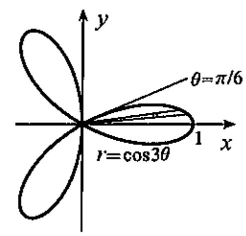
$$= 3\Big\{-\theta - \sin 2\theta\Big\}_{\pi/3}^{\pi/2} = \frac{3\sqrt{3}-\pi}{2}$$



9. $$A = 2\int_0^{\pi}\int_0^{9(1+\cos\theta)} r\,dr\,d\theta = 2\int_0^{\pi}\left\{\frac{r^2}{2}\right\}_0^{9(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi} 81(1+\cos\theta)^2\,d\theta = 81\int_0^{\pi}\left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= 81\left\{\frac{3\theta}{2} + 2\sin\theta + \frac{1}{4}\sin 2\theta\right\}_0^{\pi} = \frac{243\pi}{2}$$

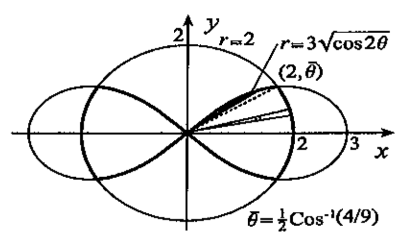


10. $$A = 6\int_0^{\pi/6}\int_0^{\cos 3\theta} r\,dr\,d\theta = 6\int_0^{\pi/6}\left\{\frac{r^2}{2}\right\}_0^{\cos 3\theta} d\theta$$

$$= 3\int_0^{\pi/6}\cos^2 3\theta\,d\theta = 3\int_0^{\pi/6}\left(\frac{1+\cos 6\theta}{2}\right)d\theta$$

$$= \frac{3}{2}\left\{\theta + \frac{\sin 6\theta}{6}\right\}_0^{\pi/6} = \frac{\pi}{4}$$

11. $$A = 4\int_0^{\bar\theta}\int_0^{2} r\,dr\,d\theta + 4\int_{\bar\theta}^{\pi/4}\int_0^{3\sqrt{\cos 2\theta}} r\,dr\,d\theta$$



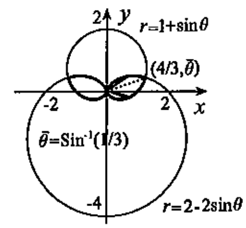
$$= 4\int_0^{\bar\theta}\left\{\frac{r^2}{2}\right\}_0^{2} d\theta + 4\int_{\bar\theta}^{\pi/4}\left\{\frac{r^2}{2}\right\}_0^{3\sqrt{\cos 2\theta}} d\theta$$

$$= 8\Big\{\theta\Big\}_0^{\bar\theta} + 2\int_{\bar\theta}^{\pi/4} 9\cos 2\theta\,d\theta = 8\bar\theta + 18\left\{\frac{1}{2}\sin 2\theta\right\}_{\bar\theta}^{\pi/4}$$

$$= 8\bar\theta + 9(1-\sin 2\bar\theta) = 4\mathrm{Cos}^{-1}(4/9) + 9 - \sqrt{65}$$

12. $$A = 2\int_{-\pi/2}^{\overline{\theta}}\int_0^{1+\sin\theta} r\,dr\,d\theta + 2\int_{\overline{\theta}}^{\pi/2}\int_0^{2-2\sin\theta} r\,dr\,d\theta = 2\int_{-\pi/2}^{\overline{\theta}}\left\{\frac{r^2}{2}\right\}_0^{1+\sin\theta} d\theta + 2\int_{\overline{\theta}}^{\pi/2}\left\{\frac{r^2}{2}\right\}_0^{2-2\sin\theta} d\theta$$

$$= \int_{-\pi/2}^{\overline{\theta}}(1+\sin\theta)^2\,d\theta + 4\int_{\overline{\theta}}^{\pi/2}(1-\sin\theta)^2\,d\theta$$



$$= \int_{-\pi/2}^{\overline{\theta}}\left(1+2\sin\theta+\frac{1-\cos 2\theta}{2}\right)d\theta$$

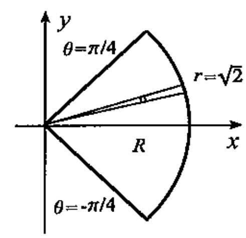
$$+4\int_{\overline{\theta}}^{\pi/2}\left(1-2\sin\theta+\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= \left\{\frac{3\theta}{2} - 2\cos\theta - \frac{\sin 2\theta}{4}\right\}_{-\pi/2}^{\overline{\theta}} + 4\left\{\frac{3\theta}{2} + 2\cos\theta - \frac{\sin 2\theta}{4}\right\}_{\overline{\theta}}^{\pi/2}$$

$$= \frac{15\pi}{4} - \frac{9\overline{\theta}}{2} - 10\cos\overline{\theta} + \frac{3}{4}\sin 2\overline{\theta} = \frac{15\pi}{4} - \frac{9}{2}\mathrm{Sin}^{-1}(1/3) - \frac{19\sqrt{2}}{3}$$

13. By symmetry, $\overline{y} = 0$. The area is $A = (1/4)\pi(2) = \pi/2$. Since

$$A\overline{x} = \iint_R x\,dA = \int_{-\pi/4}^{\pi/4}\int_0^{\sqrt{2}} r\cos\theta\, r\,dr\,d\theta$$



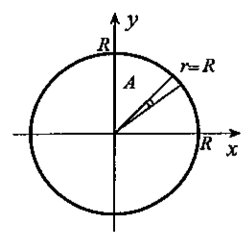
$$= \int_{-\pi/4}^{\pi/4}\left\{\frac{r^3}{3}\cos\theta\right\}_0^{\sqrt{2}} d\theta$$

$$= \frac{2\sqrt{2}}{3}\left\{\sin\theta\right\}_{-\pi/4}^{\pi/4} = \frac{4}{3},$$

it follows that $\overline{x} = (4/3)(2/\pi) = 8/(3\pi)$.

14. If we choose the x-axis as diameter,

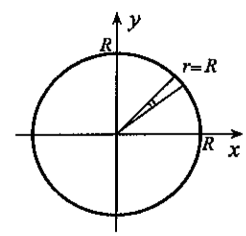
$$I = \iint_A y^2\,dA = \int_{-\pi}^{\pi}\int_0^R r^2\sin^2\theta\, r\,dr\,d\theta$$



$$= \int_{-\pi}^{\pi}\left\{\frac{r^4}{4}\sin^2\theta\right\}_0^R d\theta = \frac{R^4}{4}\int_{-\pi}^{\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= \frac{R^4}{8}\left\{\theta - \frac{\sin 2\theta}{2}\right\}_{-\pi}^{\pi} = \frac{\pi R^4}{4}$$

15. $$F = \int_{-\pi}^{\pi}\int_0^R 1000(9.81)(R - r\sin\theta)r\,dr\,d\theta$$



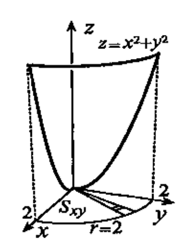
$$= 9810\int_{-\pi}^{\pi}\left\{\frac{Rr^2}{2} - \frac{r^3}{3}\sin\theta\right\}_0^R d\theta$$

$$= \frac{9810}{6}\int_{-\pi}^{\pi}(3R^3 - 2R^3\sin\theta)\,d\theta$$

$$= 1635R^3\left\{3\theta + 2\cos\theta\right\}_{-\pi}^{\pi} = 9810\pi R^3\ \mathrm{N}$$

16. We quadruple the area in the first octant.

$$A = 4\iint_{S_{xy}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA = 4\iint_{S_{xy}}\sqrt{1+(2x)^2+(2y)^2}\,dA$$

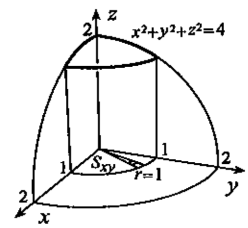


$$= 4\int_0^{\pi/2}\int_0^2\sqrt{1+4r^2}\,r\,dr\,d\theta = 4\int_0^{\pi/2}\left\{\frac{(1+4r^2)^{3/2}}{12}\right\}_0^2 d\theta$$

$$= \frac{17\sqrt{17}-1}{3}\left\{\theta\right\}_0^{\pi/2} = \frac{(17\sqrt{17}-1)\pi}{6}$$

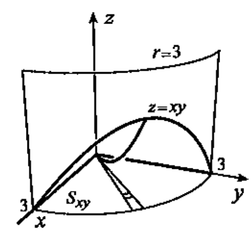
17. We multiply the area in the first octant by 8.

$$A = 8\iint_{S_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$
$$= 8\iint_{S_{xy}} \sqrt{1 + \left(\frac{-x}{\sqrt{4-x^2-y^2}}\right)^2 + \left(\frac{-y}{\sqrt{4-x^2-y^2}}\right)^2}\, dA$$
$$= 8\iint_{S_{xy}} \frac{2}{\sqrt{4-x^2-y^2}} dA = 16\int_0^{\pi/2}\int_0^1 \frac{1}{\sqrt{4-r^2}} r\, dr\, d\theta$$
$$= 16\int_0^{\pi/2} \left\{-\sqrt{4-r^2}\right\}_0^1 d\theta = 16(2-\sqrt{3})\left\{\theta\right\}_0^{\pi/2} = 8\pi(2-\sqrt{3})$$



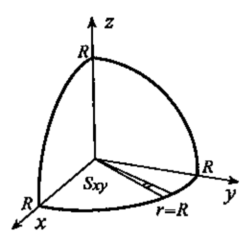
18. We quadruple the area in the first octant.

$$A = 4\iint_{S_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = 4\iint_{S_{xy}} \sqrt{1+(y)^2+(x)^2}\, dA$$
$$= 4\int_0^{\pi/2}\int_0^3 \sqrt{1+r^2}\, r\, dr\, d\theta = 4\int_0^{\pi/2} \left\{\frac{(1+r^2)^{3/2}}{3}\right\}_0^3 d\theta$$
$$= 4\frac{10\sqrt{10}-1}{3}\left\{\theta\right\}_0^{\pi/2} = \frac{2\pi}{3}(10\sqrt{10}-1)$$



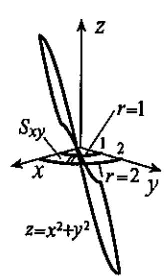
19. We multiply the first octant area by 8.

$$A = 8\iint_{S_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = 8\iint_{S_{xy}} \sqrt{1 + \left(\frac{-x}{\sqrt{R^2-x^2-y^2}}\right)^2 + \left(\frac{-y}{\sqrt{R^2-x^2-y^2}}\right)^2}\, dA$$
$$= 8\iint_{S_{xy}} \frac{R}{\sqrt{R^2-x^2-y^2}} dA$$
$$= 8R\int_0^{\pi/2}\int_0^R \frac{1}{\sqrt{R^2-r^2}} r\, dr\, d\theta$$
$$= 8R\int_0^{\pi/2} \left\{-\sqrt{R^2-r^2}\right\}_0^R d\theta$$
$$= 8R^2\left\{\theta\right\}_0^{\pi/2} = 4\pi R^2$$



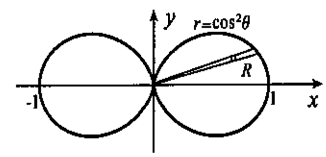
20. We quadruple the area in the first octant.

$$A = 4\iint_{S_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$
$$= 4\iint_{S_{xy}} \sqrt{1+(2x)^2+(-2y)^2}\, dA$$
$$= 4\int_0^{\pi/2}\int_1^2 \sqrt{1+4r^2}\, r\, dr\, d\theta$$
$$= 4\int_0^{\pi/2} \left\{\frac{(1+4r^2)^{3/2}}{12}\right\}_1^2 d\theta$$
$$= \frac{17\sqrt{17}-5\sqrt{5}}{3}\left\{\theta\right\}_0^{\pi/2} = \frac{(17\sqrt{17}-5\sqrt{5})\pi}{6}.$$



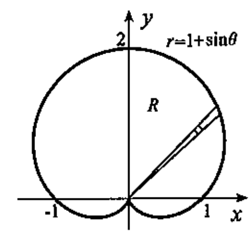
21. If R is the area in the first quadrant.

$$V = 2\iint_R 2\pi y\,dA = 4\pi\int_0^{\pi/2}\int_0^{\cos^2\theta} r\sin\theta\,r\,dr\,d\theta$$
$$= 4\pi\int_0^{\pi/2}\left\{\frac{r^3}{3}\sin\theta\right\}_0^{\cos^2\theta} d\theta = \frac{4\pi}{3}\int_0^{\pi/2}\cos^6\theta\sin\theta\,d\theta$$
$$= \frac{4\pi}{3}\left\{-\frac{1}{7}\cos^7\theta\right\}_0^{\pi/2} = \frac{4\pi}{21}$$



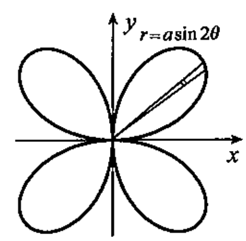
22. If R is that part of the cardioid to the right of the y-axis, then

$$V = \iint_R 2\pi x\,dA = 2\pi\int_{-\pi/2}^{\pi/2}\int_0^{1+\sin\theta} r\cos\theta\,r\,dr\,d\theta$$
$$= 2\pi\int_{-\pi/2}^{\pi/2}\left\{\frac{r^3}{3}\cos\theta\right\}_0^{1+\sin\theta} d\theta$$
$$= \frac{2\pi}{3}\int_{-\pi/2}^{\pi/2}(1+\sin\theta)^3\cos\theta\,d\theta = \frac{2\pi}{3}\left\{\frac{1}{4}(1+\sin\theta)^4\right\}_{-\pi/2}^{\pi/2} = \frac{8\pi}{3}$$



23. The equation of the curve in polar coordinates is $r^6 = 4a^2(r^2\cos^2\theta)(r^2\sin^2\theta) \implies r^2 = a^2\sin^2 2\theta$.

$$A = 4\int_0^{\pi/2}\int_0^{a\sin 2\theta} r\,dr\,d\theta = 4\int_0^{\pi/2}\left\{\frac{r^2}{2}\right\}_0^{a\sin 2\theta} d\theta$$
$$= 2\int_0^{\pi/2} a^2\sin^2 2\theta\,d\theta = 2a^2\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right)d\theta$$
$$= a^2\left\{\theta - \frac{1}{4}\sin 4\theta\right\}_0^{\pi/2} = \frac{\pi a^2}{2}$$



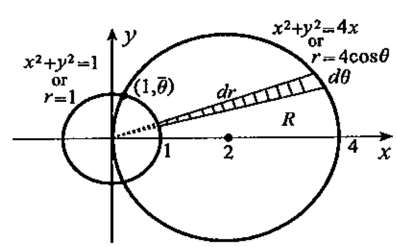
24. The equation of the inner surface of the shell is $r = 700$ in polar coordinates. The equation of the right-half of the outer surface of the shell is $r = 710 - 10\theta/\pi$. The volume of the shell is the product of its length 5000 cm and the cross-sectional area shown,

$$V = 5000(2)\int_0^{\pi/2}\int_{700}^{710-10\theta/\pi} r\,dr\,d\theta = 10\,000\int_0^{\pi/2}\left\{\frac{r^2}{2}\right\}_{700}^{710-10\theta/\pi} d\theta$$
$$= 5000\int_0^{\pi/2}\left[\left(710 - \frac{10\theta}{\pi}\right)^2 - 490\,000\right]d\theta = 5000\left\{-\frac{\pi}{30}\left(710 - \frac{10\theta}{\pi}\right)^3 - 490\,000\theta\right\}_0^{\pi/2}$$
$$= 8.29\times 10^7 \text{ cc.}$$

25. If R is the region bounded by these circles and above the x-axis, then the required area is

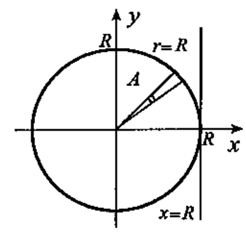
$$2\iint_R dA.$$

Since the curves intersect in the first quadrant at a point where $\theta = \overline{\theta} = \mathrm{Cos}^{-1}(\frac{1}{4})$, then



$$\text{area} = 2\int_0^{\overline{\theta}}\int_1^{4\cos\theta} r\,dr\,d\theta = 2\int_0^{\overline{\theta}}\left\{\frac{r^2}{2}\right\}_1^{4\cos\theta} d\theta = \int_0^{\overline{\theta}}(16\cos^2\theta - 1)d\theta$$
$$= \int_0^{\overline{\theta}}\left[16\left(\frac{1+\cos 2\theta}{2}\right) - 1\right]d\theta = \int_0^{\overline{\theta}}(7 + 8\cos 2\theta)d\theta = \{7\theta + 4\sin 2\theta\}_0^{\overline{\theta}}$$
$$= 7\overline{\theta} + 4\sin 2\overline{\theta} = 7\mathrm{Cos}^{-1}(\tfrac{1}{4}) + 8\cos\overline{\theta}\sin\overline{\theta}$$
$$= 7\,\mathrm{Cos}^{-1}(\tfrac{1}{4}) + 8(\tfrac{1}{4})\sqrt{1 - \tfrac{1}{16}} = 7\,\mathrm{Cos}^{-1}(\tfrac{1}{4}) + \sqrt{15}/2.$$

26. If we rotate $x^2 + y^2 \le R^2$ about $x = R$,



$$V = \iint_A 2\pi(R - x)\,dA = 2\pi \int_{-\pi}^{\pi}\int_0^R (R - r\cos\theta)\,r\,dr\,d\theta$$
$$= 2\pi \int_{-\pi}^{\pi} \left\{\frac{Rr^2}{2} - \frac{r^3}{3}\cos\theta\right\}_0^R d\theta = 2\pi R^3 \int_{-\pi}^{\pi}\left(\frac{1}{2} - \frac{1}{3}\cos\theta\right) d\theta$$
$$= 2\pi R^3 \left\{\frac{\theta}{2} - \frac{1}{3}\sin\theta\right\}_{-\pi}^{\pi} = 2\pi^2 R^3.$$

27. (a) We set $s = \sqrt{r^2 + d^2}$ where (r, θ) are the polar coordinates of dA, and integrate over the plate,

$$V = \int_{-\pi}^{\pi}\int_0^R \frac{\rho}{4\pi\epsilon_0\sqrt{r^2 + d^2}}\,r\,dr\,d\theta = \frac{\rho}{4\pi\epsilon_0}\int_{-\pi}^{\pi}\int_0^R \frac{r}{\sqrt{r^2 + d^2}}\,dr\,d\theta.$$

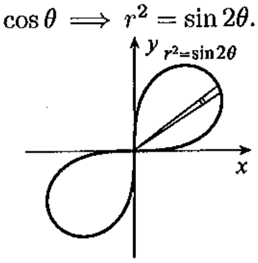
(b) $V = \dfrac{\rho}{4\pi\epsilon_0}\displaystyle\int_{-\pi}^{\pi}\left\{\sqrt{r^2 + d^2}\right\}_0^R d\theta = \dfrac{\rho}{4\pi\epsilon_0}(\sqrt{R^2 + d^2} - d)\left\{\theta\right\}_{-\pi}^{\pi} = \dfrac{\rho}{2\epsilon_0}(\sqrt{R^2 + d^2} - d)$

28. The force on q due to the charge ρdA in dA has magnitude $\dfrac{q\rho dA}{4\pi\epsilon_0 s^2}$. Since x- and y-components of contributions from all parts of the plate cancel, only the z-components survive, and for the contribution from dA, the z-component is $\dfrac{q\rho dA}{4\pi\epsilon_0 s^2}\cos\psi$, where ψ is the angle between the z-axis and the line joining P and dA. The total force therefore has z-component

$$F_z = \iint_A \frac{q\rho\cos\psi}{4\pi\epsilon_0 s^2}\,dA = \frac{q\rho}{4\pi\epsilon_0}\iint_A \frac{d}{s^3}\,dA = \frac{q\rho d}{4\pi\epsilon_0}\int_{-\pi}^{\pi}\int_0^R \frac{1}{(r^2 + d^2)^{3/2}}\,r\,dr\,d\theta$$
$$= \frac{q\rho d}{4\pi\epsilon_0}\int_{-\pi}^{\pi}\left\{\frac{-1}{\sqrt{r^2 + d^2}}\right\}_0^R d\theta = \frac{q\rho d}{4\pi\epsilon_0}\left(\frac{1}{d} - \frac{1}{\sqrt{R^2 + d^2}}\right)(2\pi) = \frac{q\rho}{2\epsilon_0}\left(1 - \frac{d}{\sqrt{R^2 + d^2}}\right).$$

As the radius of the plate becomes very large, $\lim\limits_{R\to\infty} F_z = \dfrac{q\rho}{2\epsilon_0}$.

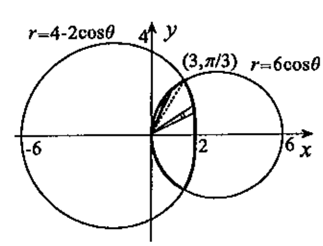
29. The equation of the curve in polar coordinates is $r^4 = 2r^2\sin\theta\cos\theta \implies r^2 = \sin 2\theta$.



$$A = 2\int_0^{\pi/2}\int_0^{\sqrt{\sin 2\theta}} r\,dr\,d\theta$$
$$= 2\int_0^{\pi/2}\left\{\frac{r^2}{2}\right\}_0^{\sqrt{\sin 2\theta}} d\theta$$
$$= \int_0^{\pi/2}\sin 2\theta\,d\theta = \left\{-\frac{1}{2}\cos 2\theta\right\}_0^{\pi/2} = 1$$

30. $A = 2\displaystyle\int_0^{\pi/3}\int_0^{4-2\cos\theta} r\,dr\,d\theta + 2\int_{\pi/3}^{\pi/2}\int_0^{6\cos\theta} r\,dr\,d\theta$

$$= 2\int_0^{\pi/3}\left\{\frac{r^2}{2}\right\}_0^{4-2\cos\theta} d\theta + 2\int_{\pi/3}^{\pi/2}\left\{\frac{r^2}{2}\right\}_0^{6\cos\theta} d\theta = 4\int_0^{\pi/3}(2 - \cos\theta)^2\,d\theta + 36\int_{\pi/3}^{\pi/2}\cos^2\theta\,d\theta$$
$$= 4\int_0^{\pi/3}\left(4 - 4\cos\theta + \frac{1 + \cos 2\theta}{2}\right) d\theta$$
$$+36\int_{\pi/3}^{\pi/2}\left(\frac{1 + \cos 2\theta}{2}\right) d\theta$$



$$= 4\left\{\frac{9\theta}{2} - 4\sin\theta + \frac{\sin 2\theta}{4}\right\}_0^{\pi/3} + 18\left\{\theta + \frac{\sin 2\theta}{2}\right\}_{\pi/3}^{\pi/2}$$
$$= 9\pi - 12\sqrt{3}$$

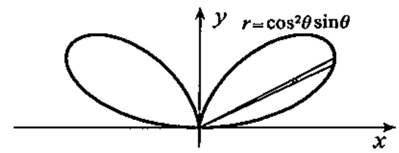
31. $A = 2\int_0^{\pi/2}\int_0^{\cos^2\theta\sin\theta} r\,dr\,d\theta = 2\int_0^{\pi/2}\left\{\frac{r^2}{2}\right\}_0^{\cos^2\theta\sin\theta} d\theta$

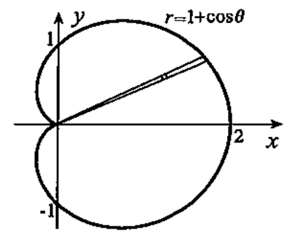
$$= \int_0^{\pi/2}\cos^4\theta\sin^2\theta\,d\theta$$

$$= \int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{\sin 2\theta}{2}\right)^2 d\theta$$

$$= \frac{1}{8}\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2} + \sin^2 2\theta\cos 2\theta\right) d\theta$$

$$= \frac{1}{8}\left\{\frac{\theta}{2} - \frac{1}{8}\sin 4\theta + \frac{1}{6}\sin^3 2\theta\right\}_0^{\pi/2} = \frac{\pi}{32}$$





32. $A = 2\int_0^{\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta = \int_0^{\pi}(1+\cos\theta)^2\,d\theta$

$$= \int_0^{\pi}\left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \left\{\frac{3\theta}{2} + 2\sin\theta + \frac{\sin 2\theta}{4}\right\}_0^{\pi} = \frac{3\pi}{2}$$

By symmetry, $\overline{y} = 0$. Since

$$A\overline{x} = 2\int_0^{\pi}\int_0^{1+\cos\theta} r\cos\theta\, r\,dr\,d\theta = 2\int_0^{\pi}\left\{\frac{r^3}{3}\cos\theta\right\}_0^{1+\cos\theta} d\theta = \frac{2}{3}\int_0^{\pi}(1+\cos\theta)^3\cos\theta\,d\theta$$

$$= \frac{2}{3}\int_0^{\pi}\left[\cos\theta + 3\left(\frac{1+\cos 2\theta}{2}\right) + 3\cos\theta(1-\sin^2\theta) + \frac{1}{4}\left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)\right] d\theta$$

$$= \frac{2}{3}\left\{4\sin\theta + \frac{15\theta}{8} + \sin 2\theta - \sin^3\theta + \frac{\sin 4\theta}{32}\right\}_0^{\pi} = \frac{5\pi}{4},$$

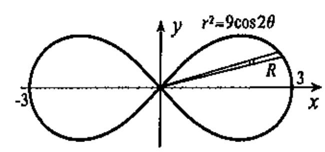
we find $\overline{x} = \dfrac{5\pi}{4}\dfrac{2}{3\pi} = \dfrac{5}{6}$.

33. $I = 4\iint_R y^2\,dA = 4\int_0^{\pi/4}\int_0^{3\sqrt{\cos 2\theta}} r^2\sin^2\theta\, r\,dr\,d\theta = 4\int_0^{\pi/4}\left\{\frac{r^4}{4}\sin^2\theta\right\}_0^{3\sqrt{\cos 2\theta}} d\theta$

$$= 81\int_0^{\pi/4}\cos^2 2\theta\sin^2\theta\,d\theta = 81\int_0^{\pi/4}\cos^2 2\theta\left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{81}{2}\int_0^{\pi/4}\left[\frac{1+\cos 4\theta}{2} - (1-\sin^2 2\theta)\cos 2\theta\right] d\theta$$

$$= \frac{81}{2}\left\{\frac{\theta}{2} + \frac{1}{8}\sin 4\theta - \frac{1}{2}\sin 2\theta + \frac{1}{6}\sin^3 2\theta\right\}_0^{\pi/4} = \frac{27(3\pi-8)}{16}$$



34. $I = \iint_R \sqrt{\dfrac{1-x^2-y^2}{1+x^2+y^2}}\,dA = \int_0^1\int_{-\pi}^{\pi}\sqrt{\dfrac{1-r^2}{1+r^2}}\,r\,d\theta\,dr = 2\pi\int_0^1\sqrt{\dfrac{1-r^2}{1+r^2}}\,r\,dr$

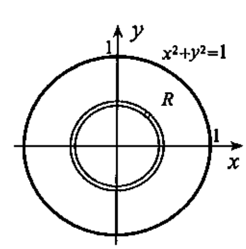
If we set $u = \sqrt{1+r^2}$, then $2u\,du = 2r\,dr$, and

$$I = 2\pi\int_1^{\sqrt{2}}\sqrt{\frac{1-(u^2-1)}{u^2}}\,u\,du = 2\pi\int_1^{\sqrt{2}}\sqrt{2-u^2}\,du.$$

If we now set $u = \sqrt{2}\sin\phi$ and $du = \sqrt{2}\cos\phi\,d\phi$, then

$$I = 2\pi\int_{\pi/4}^{\pi/2}\sqrt{2}\cos\phi\sqrt{2}\cos\phi\,d\phi = 4\pi\int_{\pi/4}^{\pi/2}\left(\frac{1+\cos 2\phi}{2}\right) d\phi$$

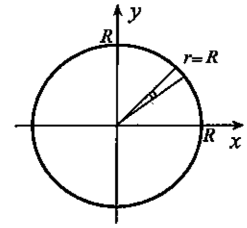
$$= 2\pi\left\{\phi + \frac{\sin 2\phi}{2}\right\}_{\pi/4}^{\pi/2} = \frac{\pi(\pi-2)}{2}.$$



35. $B = \int_{-\pi}^{\pi}\int_0^R v\,r\,dr\,d\theta = \int_{-\pi}^{\pi}\int_0^R \frac{P}{4nL}(R^2 - r^2)r\,dr\,d\theta$

$= \frac{P}{4nL}\int_{-\pi}^{\pi}\left\{\frac{R^2r^2}{2} - \frac{r^4}{4}\right\}_0^R d\theta$

$= \frac{PR^4}{16nL}\left\{\theta\right\}_{-\pi}^{\pi} = \frac{\pi PR^4}{8nL}$



36. The volume of blood flowing through any cross-section of the larger blood vessel per unit time is

$$\int_{-\pi}^{\pi}\int_0^R V_{\max}\left[1 - \left(\frac{r}{R}\right)^2\right] r\,dr\,d\theta = V_{\max}\int_{-\pi}^{\pi}\left\{\frac{r^2}{2} - \frac{r^4}{4R^2}\right\}_0^R d\theta = \frac{R^2V_{\max}}{4}\left\{\theta\right\}_{-\pi}^{\pi} = \frac{\pi R^2 V_{\max}}{2}.$$

Similarly, the volume of blood flowing through any cross-section of the smaller blood vessel is

$$\int_{-\pi}^{\pi}\int_0^{R_1} U_{\max}\left[1 - \left(\frac{r}{R_1}\right)^2\right] r\,dr\,d\theta = \frac{\pi R_1^2 U_{\max}}{2}.$$

When we equate these and set $R_1 = \alpha R$,

$$\frac{\pi R^2 V_{\max}}{2} = \frac{\pi\alpha^2R^2U_{\max}}{2} \quad\Longrightarrow\quad U_{\max} = \frac{V_{\max}}{\alpha^2}.$$

37. The volume of blood flowing through any cross-section of the larger blood vessel per unit time is

$$\int_{-L/2}^{L/2}\int_{-L/2}^{L/2} V_{\max}\left(1 - \frac{4x^2}{L^2}\right)\left(1 - \frac{4y^2}{L^2}\right) dy\,dx = V_{\max}\int_{-L/2}^{L/2}\left(1 - \frac{4x^2}{L^2}\right)\left\{y - \frac{4y^3}{3L^2}\right\}_{-L/2}^{L/2} dx$$

$$= \frac{2LV_{\max}}{3}\left\{x - \frac{4x^3}{3L^2}\right\}_{-L/2}^{L/2} = \frac{4L^2V_{\max}}{9}.$$

A similar calculation for the flow through the smaller pipe gives $4(\alpha^2L^2)U_{\max}/9$, where $U_{\max}$ is the maximum velocity at the centre of the pipe. When we equate flows,

$$\frac{4L^2V_{\max}}{9} = \frac{4\alpha^2L^2U_{\max}}{9} \quad\Longrightarrow\quad U_{\max} = \frac{V_{\max}}{\alpha^2}.$$

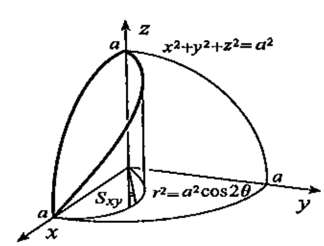
38. We multiply the area in the first octant by 8,

$$A = 8\iint_{S_{xy}}\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA = 8\iint_{S_{xy}}\sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{a^2 - x^2 - y^2}}\right)^2}\,dA$$

$= 8\iint_{S_{xy}}\sqrt{\frac{a^2 - x^2 - y^2 + x^2 + y^2}{a^2 - x^2 - y^2}}\,dA = 8a\iint_{S_{xy}}\frac{1}{\sqrt{a^2 - x^2 - y^2}}dA$

$= 8a\int_0^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}}\frac{1}{\sqrt{a^2 - r^2}}r\,dr\,d\theta$

$= 8a\int_0^{\pi/4}\left\{-\sqrt{a^2 - r^2}\right\}_0^{a\sqrt{\cos 2\theta}} d\theta$



$= 8a\int_0^{\pi/4}(a - \sqrt{a^2 - a^2\cos 2\theta})\,d\theta = 8a^2\int_0^{\pi/4}[1 - \sqrt{1 - (1 - 2\sin^2\theta)}]\,d\theta$

$= 8a^2\int_0^{\pi/4}(1 - \sqrt{2}\sin\theta)\,d\theta = 8a^2\left\{\theta + \sqrt{2}\cos\theta\right\}_0^{\pi/4} = 2a^2(\pi + 4 - 4\sqrt{2})$

39. We multiply the area in the first octant by 8,

$$A = 8\iint_{S_{xy}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA = 8\iint_{S_{xy}}\sqrt{1+\left(\frac{-x}{\sqrt{a^2-x^2}}\right)^2}\,dA = 8a\iint_{S_{xy}}\frac{1}{\sqrt{a^2-x^2}}dA$$

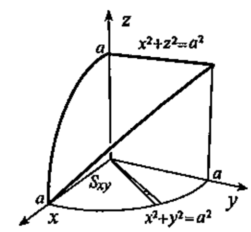
$$= 8a\int_0^{\pi/2}\int_0^a \frac{1}{\sqrt{a^2-r^2\cos^2\theta}}r\,dr\,d\theta = 8a\int_0^{\pi/2}\left\{\frac{\sqrt{a^2-r^2\cos^2\theta}}{-\cos^2\theta}\right\}_0^a d\theta$$

$$= 8a^2\int_0^{\pi/2}\frac{1-\sin\theta}{\cos^2\theta}d\theta = 8a^2\int_0^{\pi/2}(\sec^2\theta-\tan\theta\sec\theta)\,d\theta$$

$$= 8a^2\Big\{\tan\theta-\sec\theta\Big\}_0^{\pi/2} = 8a^2\left[\lim_{\theta\to\pi/2^-}(\tan\theta-\sec\theta)+1\right]$$

$$= 8a^2+8a^2\lim_{\theta\to\pi/2^-}\frac{\sin\theta-1}{\cos\theta} \qquad \text{(and now using L'hôpital's rule)}$$

$$= 8a^2+8a^2\lim_{\theta\to\pi/2^-}\frac{\cos\theta}{-\sin\theta} = 8a^2$$



40. $I^2 = \int_0^\infty e^{-x^2}dx\int_0^\infty e^{-y^2}dy = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dy\,dx$

We now interpret this double iterated integral as the double integral of $e^{-(x^2+y^2)}$ over the first quadrant of the xy-plane, and change to polar coordinates,

$$I^2 = \int_0^{\pi/2}\int_0^\infty e^{-r^2}r\,dr\,d\theta = \int_0^{\pi/2}\left\{-\frac{1}{2}e^{-r^2}\right\}_0^\infty d\theta = \frac{1}{2}\int_0^{\pi/2}d\theta = \frac{\pi}{4}.$$

Thus, $I = \sqrt{\pi}/2$.

41. When $n = 1/2$, $\quad\Gamma(1/2) = \int_0^\infty x^{-1/2}e^{-x}\,dx$. If we set $u = \sqrt{x} \Longrightarrow x = u^2$, and $dx = 2u\,du$,

$$\Gamma(1/2) = \int_0^\infty \frac{1}{u}e^{-u^2}(2u\,du) = 2\int_0^\infty e^{-u^2}\,du = 2\left(\frac{\sqrt{\pi}}{2}\right) = \sqrt{\pi} \quad \text{(from Exercise 40).}$$

EXERCISES 13.8

1. $$\iiint_V (x^2z+ye^x)\,dV = \int_0^1\int_1^2\int_0^1 (x^2z+ye^x)\,dz\,dy\,dx = \int_0^1\int_1^2\left\{\frac{x^2z^2}{2}+zye^x\right\}_0^1 dy\,dx$$

$$= \int_0^1\int_1^2\left(\frac{x^2}{2}+ye^x\right)dy\,dx = \int_0^1\left\{\frac{x^2y}{2}+\frac{y^2e^x}{2}\right\}_1^2 dx$$

$$= \frac{1}{2}\int_0^1(x^2+3e^x)\,dx = \frac{1}{2}\left\{\frac{x^3}{3}+3e^x\right\}_0^1 = \frac{9e-8}{6}$$

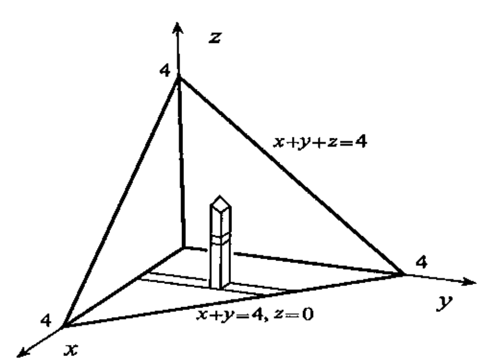
2. $$\iiint_V x\,dV = \int_0^4\int_0^{4-x}\int_0^{4-x-y} x\,dz\,dy\,dx$$

$$= \int_0^4\int_0^{4-x} x(4-x-y)\,dy\,dx$$

$$= \int_0^4\left\{x(4-x)y-\frac{xy^2}{2}\right\}_0^{4-x} dx$$

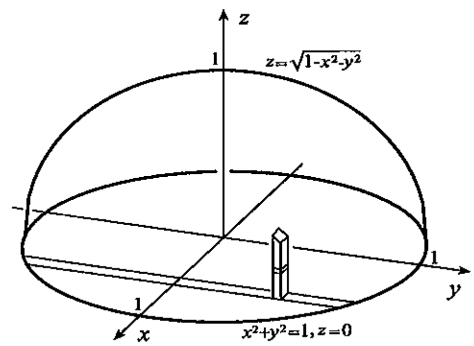
$$= \frac{1}{2}\int_0^4(16x-8x^2+x^3)\,dx$$

$$= \frac{1}{2}\left\{8x^2-\frac{8x^3}{3}+\frac{x^4}{4}\right\}_0^4 = \frac{32}{3}$$



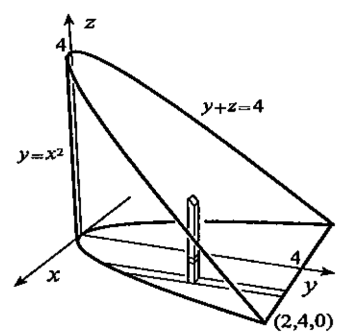
3. $$\iiint_V \sin(y+z)\,dV = \int_0^1\int_0^{2x}\int_0^{x+2y}\sin(y+z)\,dz\,dy\,dx = \int_0^1\int_0^{2x}\left\{-\cos(y+z)\right\}_0^{x+2y}dy\,dx$$
$$= \int_0^1\int_0^{2x}[\cos y - \cos(x+3y)]\,dy\,dx = \int_0^1\left\{\sin y - \frac{1}{3}\sin(x+3y)\right\}_0^{2x}dx$$
$$= \int_0^1\left(\sin 2x - \frac{1}{3}\sin 7x + \frac{1}{3}\sin x\right)dx = \left\{-\frac{1}{2}\cos 2x + \frac{1}{21}\cos 7x - \frac{1}{3}\cos x\right\}_0^1$$
$$= (2\cos 7 - 14\cos 1 - 21\cos 2 + 33)/42$$

4. $$\iiint_V xy\,dV = \int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xy\,dz\,dy\,dx$$
$$= \int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} xy\sqrt{1-x^2-y^2}\,dy\,dx$$
$$= \int_{-1}^1\left\{-\frac{x}{3}(1-x^2-y^2)^{3/2}\right\}_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}dx = 0$$



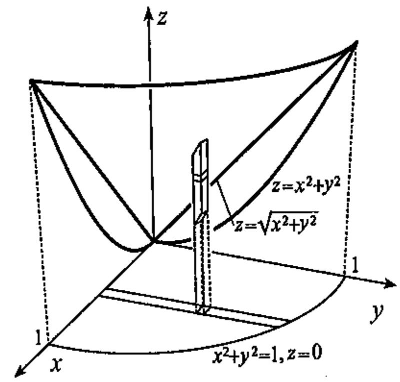
5. $$\iiint_V dV = 8\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}}dz\,dy\,dx = 8\int_0^1\int_0^{\sqrt{1-x^2}}\sqrt{1-x^2}\,dy\,dx$$
$$= 8\int_0^1\left\{y\sqrt{1-x^2}\right\}_0^{\sqrt{1-x^2}}dx = 8\int_0^1(1-x^2)\,dx = 8\left\{x - \frac{x^3}{3}\right\}_0^1 = \frac{16}{3}$$

6. $$\iiint_V (x^2+2z)\,dV = \int_{-2}^2\int_{x^2}^4\int_0^{4-y}(x^2+2z)\,dz\,dy\,dx$$
$$= \int_{-2}^2\int_{x^2}^4\left\{x^2z + z^2\right\}_0^{4-y}dy\,dx$$
$$= \int_{-2}^2\int_{x^2}^4[x^2(4-y)+(4-y)^2]\,dy\,dx$$
$$= \int_{-2}^2\left\{-\frac{x^2}{2}(4-y)^2 - \frac{(4-y)^3}{3}\right\}_{x^2}^4dx$$
$$= \frac{1}{6}\int_{-2}^2(128-48x^2+x^6)\,dx$$
$$= \frac{1}{6}\left\{128x - 16x^3 + \frac{x^7}{7}\right\}_{-2}^2 = \frac{1024}{21}$$



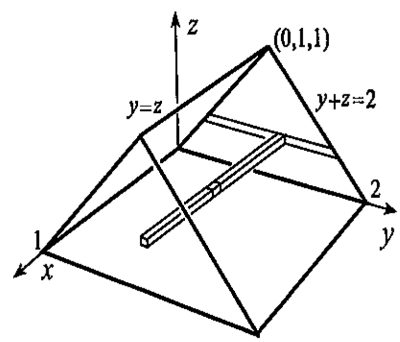
7. $$\iiint_V x^2y^2z^2\,dV = \int_0^1\int_{z-1}^{1-z}\int_0^1 x^2y^2z^2\,dx\,dy\,dz = \int_0^1\int_{z-1}^{1-z}\left\{\frac{x^3y^2z^2}{3}\right\}_0^1dy\,dz = \frac{1}{3}\int_0^1\int_{z-1}^{1-z}y^2z^2\,dy\,dz$$
$$= \frac{1}{3}\int_0^1\left\{\frac{y^3z^2}{3}\right\}_{z-1}^{1-z}dz = \frac{2}{9}\int_0^1(z^2-3z^3+3z^4-z^5)\,dz$$
$$= \frac{2}{9}\left\{\frac{z^3}{3} - \frac{3z^4}{4} + \frac{3z^5}{5} - \frac{z^6}{6}\right\}_0^1 = \frac{1}{270}$$

8. $$\iiint_V xyz\,dV = \int_0^1\int_0^{\sqrt{1-x^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\,dz\,dy\,dx = \int_0^1\int_0^{\sqrt{1-x^2}}\left\{\frac{xyz^2}{2}\right\}_{x^2+y^2}^{\sqrt{x^2+y^2}} dy\,dx$$
$$= \frac{1}{2}\int_0^1\int_0^{\sqrt{1-x^2}}(x^3y + xy^3 - x^5y - 2x^3y^3 - xy^5)\,dy\,dx$$
$$= \frac{1}{2}\int_0^1\left\{\frac{x^3y^2}{2} + \frac{xy^4}{4} - \frac{x^5y^2}{2} - \frac{x^3y^4}{2} - \frac{xy^6}{6}\right\}_0^{\sqrt{1-x^2}} dx$$
$$= \frac{1}{24}\int_0^1 [3x(1-x^2)^2 - 2x(1-x^2)^3]\,dx$$
$$= \frac{1}{24}\left\{-\frac{1}{2}(1-x^2)^3 + \frac{1}{4}(1-x^2)^4\right\}_0^1 = \frac{1}{96}$$



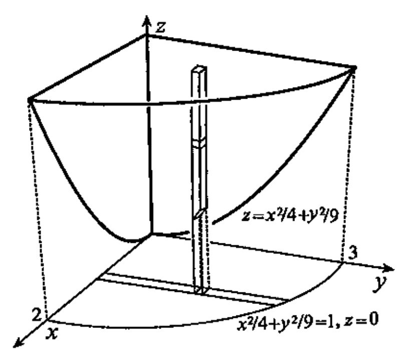
9. We double the integal over the first octant volume.
$$\iiint_V dV = 2\int_0^2\int_{x^2}^4\int_0^{4-z} dy\,dz\,dx = 2\int_0^2\int_{x^2}^4 (4-z)\,dz\,dx = 2\int_0^2\left\{4z - \frac{z^2}{2}\right\}_{x^2}^4 dx$$
$$= 2\int_0^2\left(16 - 8 - 4x^2 + \frac{x^4}{2}\right)dx = 2\left\{8x - \frac{4x^3}{3} + \frac{x^5}{10}\right\}_0^2 = \frac{256}{15}$$

10. $$\iiint_V (x+y+z)\,dV = \int_0^1\int_z^{2-z}\int_0^1 (x+y+z)\,dx\,dy\,dz$$
$$= \int_0^1\int_z^{2-z}\left\{\frac{1}{2}(x+y+z)^2\right\}_0^1 dy\,dz$$
$$= \frac{1}{2}\int_0^1\int_z^{2-z}[(1+y+z)^2 - (y+z)^2]\,dy\,dz$$
$$= \frac{1}{2}\int_0^1\left\{\frac{1}{3}(1+y+z)^3 - \frac{1}{3}(y+z)^3\right\}_z^{2-z} dz$$
$$= \frac{1}{6}\int_0^1 [19 + 8z^3 - (1+2z)^3]\,dz$$
$$= \frac{1}{6}\left\{19z + 2z^4 - \frac{(1+2z)^4}{8}\right\}_0^1 = \frac{11}{6}$$



11. Because of the symmetry of the volume about the z-axis, and the fact that the integrand xyz is an odd function of x and y, the triple integral must have value zero.

12. $$\iiint_V x^2y\,dV = \int_0^2\int_0^{3\sqrt{4-x^2}/2}\int_{x^2/4+y^2/9}^1 x^2y\,dz\,dy\,dx$$
$$= \int_0^2\int_0^{3\sqrt{4-x^2}/2} x^2y\left(1 - \frac{x^2}{4} - \frac{y^2}{9}\right)dy\,dx$$
$$= \int_0^2\left\{x^2\left(1 - \frac{x^2}{4}\right)\frac{y^2}{2} - \frac{x^2y^4}{36}\right\}_0^{3\sqrt{4-x^2}/2} dx$$
$$= \frac{9}{64}\int_0^2 (16x^2 - 8x^4 + x^6)\,dx$$
$$= \frac{9}{64}\left\{\frac{16x^3}{3} - \frac{8x^5}{5} + \frac{x^7}{7}\right\}_0^2 = \frac{48}{35}$$



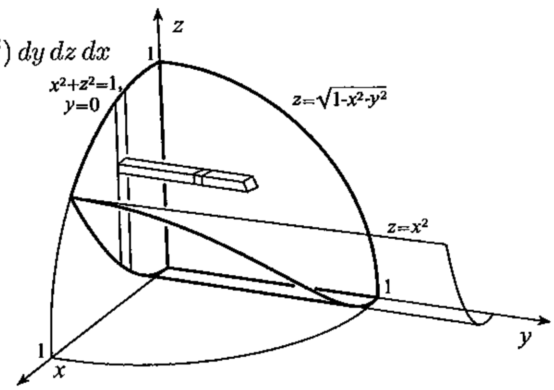
13. The six triple iterated integrals are:

$$\int_{-1}^{1}\int_{0}^{1-x^2}\int_{0}^{y} f(x,y,z)\,dz\,dy\,dx,\quad \int_{0}^{1}\int_{-\sqrt{1-y}}^{\sqrt{1-y}}\int_{0}^{y} f(x,y,z)\,dz\,dx\,dy,\quad \int_{-1}^{1}\int_{0}^{1-x^2}\int_{z}^{1-x^2} f(x,y,z)\,dy\,dz\,dx,$$

$$\int_{0}^{1}\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{z}^{1-x^2} f(x,y,z)\,dy\,dx\,dz,\quad \int_{0}^{1}\int_{0}^{y}\int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y,z)\,dx\,dz\,dy,\quad \int_{0}^{1}\int_{z}^{1}\int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y,z)\,dx\,dy\,dz.$$

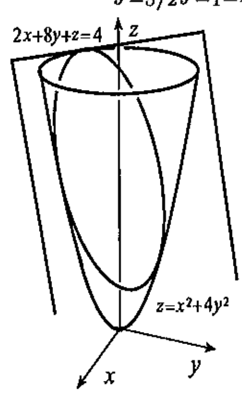
14. $$\iiint_V (x^2+y^2+z^2)\,dV$$

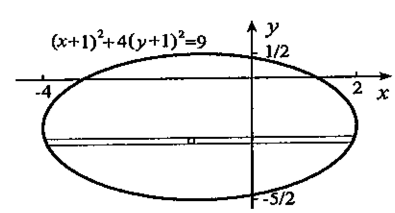
$$= 4\int_{0}^{\sqrt{(\sqrt{5}-1)/2}}\int_{x^2}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-z^2}} (x^2+y^2+z^2)\,dy\,dz\,dx$$



15. $$\iiint_V xz\sin(x+y)\,dV = \int_{-1}^{1}\int_{-(1/2)\sqrt{3-3x^2}}^{(1/2)\sqrt{3-3x^2}}\int_{\sqrt{1+4x^2+4z^2}}^{\sqrt{4+x^2}} xz\sin(x+y)\,dy\,dz\,dx$$

16. $$\iiint_V xyz\,dV = \int_{-5/2}^{1/2}\int_{-1-\sqrt{9-4(y+1)^2}}^{-1+\sqrt{9-4(y+1)^2}}\int_{x^2+4y^2}^{4-2x-8y} xyz\,dz\,dx\,dy$$



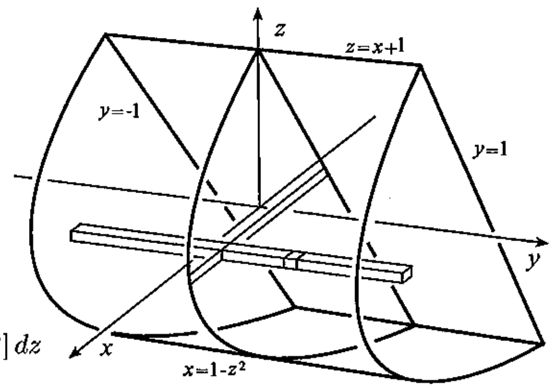


Area onto which vertical columns project

17. The surfaces intersect in a plane parallel to the yz-plane defined by $x+1=x^2$, from which $x=(1\pm\sqrt{1+4})/2=(1\pm\sqrt{5})/2$, only the positive result being acceptable. The equation of the projection of the curve in the yz-plane is $y^2+z^2=(1+\sqrt{5})/2$. Hence,

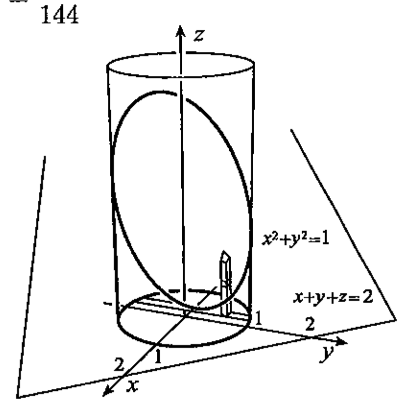
$$\iiint_V x^2y^2z^2\,dV = 4\int_{0}^{\sqrt{(1+\sqrt{5})/2}}\int_{0}^{\sqrt{(1+\sqrt{5})/2-y^2}}\int_{(y^2+z^2)^2-1}^{y^2+z^2} x^2y^2z^2\,dx\,dz\,dy.$$

18. $$\iiint_V (y+x^2)\,dV = \int_{-2}^{1}\int_{z-1}^{1-z^2}\int_{-1}^{1}(y+x^2)\,dy\,dx\,dz$$
$$= \int_{-2}^{1}\int_{z-1}^{1-z^2}\left\{\frac{y^2}{2}+x^2y\right\}_{-1}^{1}dx\,dz$$
$$= 2\int_{-2}^{1}\int_{z-1}^{1-z^2}x^2\,dx\,dz$$
$$= 2\int_{-2}^{1}\left\{\frac{x^3}{3}\right\}_{z-1}^{1-z^2}dz$$
$$= \frac{2}{3}\int_{-2}^{1}[1-3z^2+3z^4-z^6-(z-1)^3]\,dz$$
$$= \frac{2}{3}\left\{z-z^3+\frac{3z^5}{5}-\frac{z^7}{7}-\frac{(z-1)^4}{4}\right\}_{-2}^{1} = \frac{729}{70}$$



19. $$\iiint_V (xy+z)\,dV = \int_0^{1/3}\int_y^{2y}\int_0^3(xy+z)\,dx\,dz\,dy + \int_{1/3}^{1/2}\int_y^{1-y}\int_0^3(xy+z)\,dx\,dz\,dy$$
$$= \int_0^{1/3}\int_y^{2y}\left\{\frac{x^2y}{2}+xz\right\}_0^3 dz\,dy + \int_{1/3}^{1/2}\int_y^{1-y}\left\{\frac{x^2y}{2}+xz\right\}_0^3 dz\,dy$$
$$= \frac{1}{2}\int_0^{1/3}\int_y^{2y}(9y+6z)\,dz\,dy + \frac{1}{2}\int_{1/3}^{1/2}\int_y^{1-y}(9y+6z)\,dz\,dy$$
$$= \frac{1}{2}\int_0^{1/3}\left\{9yz+3z^2\right\}_y^{2y}dy + \frac{1}{2}\int_{1/3}^{1/2}\left\{9yz+3z^2\right\}_y^{1-y}dy$$
$$= \frac{1}{2}\int_0^{1/3}18y^2\,dy + \frac{1}{2}\int_{1/3}^{1/2}[9y-21y^2+3(1-y)^2]\,dy$$
$$= \frac{1}{2}\left\{6y^3\right\}_0^{1/3} + \frac{1}{2}\left\{\frac{9y^2}{2}-7y^3-(1-y)^3\right\}_{1/3}^{1/2} = \frac{29}{144}$$

20. $$\iiint_V dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{2-x-y}dz\,dy\,dx$$
$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(2-x-y)\,dy\,dx$$
$$= \int_{-1}^{1}\left\{(2-x)y-\frac{y^2}{2}\right\}_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}dx$$
$$= 2\int_{-1}^{1}(2-x)\sqrt{1-x^2}\,dx$$



If we set $x=\sin\theta$, then $dx=\cos\theta\,d\theta$, and
$$\iiint_V dV = 2\int_{-\pi/2}^{\pi/2}(2-\sin\theta)\cos\theta\,\cos\theta\,d\theta$$
$$= 2\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta-\cos^2\theta\,\sin\theta)\,d\theta = 2\left\{\theta+\frac{\sin 2\theta}{2}+\frac{\cos^3\theta}{3}\right\}_{-\pi/2}^{\pi/2} = 2\pi.$$

21. We quadruple the integral over the first octant volume.

$$\iiint_V dV = 4\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}\int_{x^2+y^2}^{4-x^2-y^2} dz\,dy\,dx = 4\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}(4-2x^2-2y^2)\,dy\,dx$$

$$= 4\int_0^{\sqrt{2}}\left\{(4-2x^2)y - \frac{2y^3}{3}\right\}_0^{\sqrt{2-x^2}} dx = \frac{16}{3}\int_0^{\sqrt{2}}(2-x^2)^{3/2}dx$$

If we set $x = \sqrt{2}\sin\theta$ and $dx = \sqrt{2}\cos\theta\,d\theta$,

$$\iiint_V dV = \frac{16}{3}\int_0^{\pi/2} 2\sqrt{2}\cos^3\theta(\sqrt{2}\cos\theta\,d\theta) = \frac{64}{3}\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \frac{16}{3}\int_0^{\pi/2}\left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)d\theta = \frac{16}{3}\left\{\frac{3\theta}{2} + \sin 2\theta + \frac{1}{8}\sin 4\theta\right\}_0^{\pi/2} = 4\pi.$$

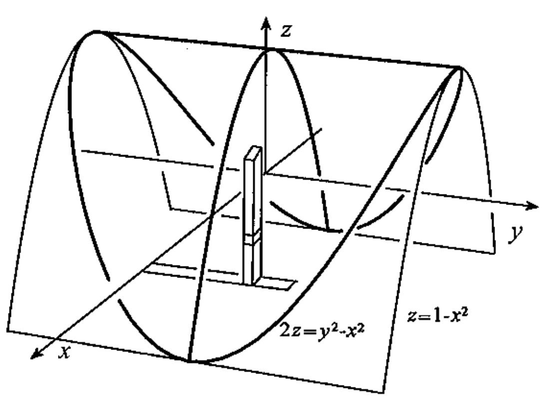
22. Because of the symmetry, integrals of x and y vanish. We multiply the integral of z over the first octant volume by 4,

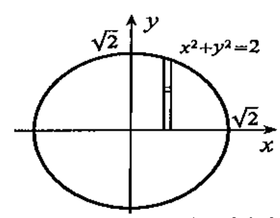
$$\iiint_V (x+y+z)\,dV = \iiint_V z\,dV = 4\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}\int_{y^2/2-x^2/2}^{1-x^2} z\,dz\,dy\,dx$$

$$= 4\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}\left\{\frac{z^2}{2}\right\}_{y^2/2-x^2/2}^{1-x^2} dy\,dx = \frac{1}{2}\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}(4-8x^2+3x^4-y^4+2x^2y^2)\,dy\,dx$$

$$= \frac{1}{2}\int_0^{\sqrt{2}}\left\{4y - 8x^2y + 3x^4y - \frac{y^5}{5} + \frac{2x^2y^3}{3}\right\}_0^{\sqrt{2-x^2}} dx$$

$$= \frac{1}{2}\int_0^{\sqrt{2}}\left[4\sqrt{2-x^2} - 8x^2\sqrt{2-x^2} + 3x^4\sqrt{2-x^2} - \frac{1}{5}(2-x^2)^{5/2} + \frac{2x^2}{3}(2-x^2)^{3/2}\right]dx.$$





Columns project onto interior of circle

If we set $x = \sqrt{2}\sin\theta$, then $dx = \sqrt{2}\cos\theta\,d\theta$, and

$$\iiint_V (x+y+z)\,dV = \frac{1}{2}\int_0^{\pi/2}\left(4\sqrt{2}\cos\theta - 16\sqrt{2}\sin^2\theta\,\cos\theta + 12\sqrt{2}\sin^4\theta\,\cos\theta - \frac{4\sqrt{2}}{5}\cos^5\theta\right.$$

$$\left. + \frac{8\sqrt{2}}{3}\sin^2\theta\,\cos^3\theta\right)\sqrt{2}\cos\theta\,d\theta$$

$$= 4\int_0^{\pi/2}\left[\cos^2\theta - \sin^2 2\theta + 3\left(\frac{\sin^2 2\theta}{4}\right)\left(\frac{1-\cos 2\theta}{2}\right)\right.$$

$$\left. - \frac{1}{5}\left(\frac{1+\cos 2\theta}{2}\right)^3 + \frac{2}{3}\left(\frac{\sin^2 2\theta}{4}\right)\left(\frac{1+\cos 2\theta}{2}\right)\right]d\theta$$

$$= 4\int_0^{\pi/2}\left\{\frac{1+\cos 2\theta}{2} - \left(\frac{1-\cos 4\theta}{2}\right) + \frac{3}{8}\left[\frac{1-\cos 4\theta}{2} - \sin^2 2\theta\,\cos 2\theta\right]\right.$$

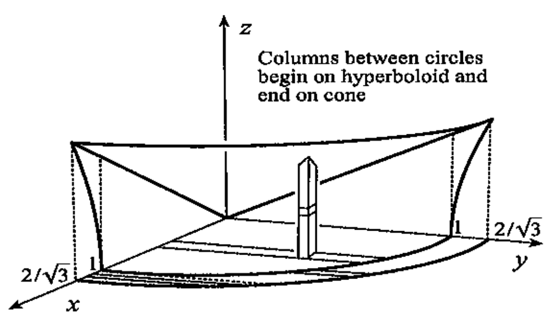
$$
\begin{aligned}
&-\frac{1}{40}\left[1+3\cos 2\theta+\frac{3}{2}(1+\cos 4\theta)+\cos 2\theta(1-\sin^2 2\theta)\right]\\
&\qquad+\frac{1}{12}\left(\frac{1-\cos 4\theta}{2}+\sin^2 2\theta\,\cos 2\theta\right)\Big\}\,d\theta\\
&=4\Big\{\frac{\theta}{2}+\frac{\sin 2\theta}{4}-\frac{\theta}{2}+\frac{\sin 4\theta}{8}+\frac{3\theta}{16}-\frac{3\sin 4\theta}{64}-\frac{\sin^3 2\theta}{16}\\
&\quad-\frac{\theta}{40}-\frac{3\sin 2\theta}{80}-\frac{3\theta}{80}-\frac{3\sin 4\theta}{320}-\frac{\sin 2\theta}{80}\\
&\qquad+\frac{\sin^3 2\theta}{240}+\frac{\theta}{24}-\frac{\sin 4\theta}{96}+\frac{\sin^3 2\theta}{72}\Big\}_0^{\pi/2}\\
&=\pi/3.
\end{aligned}
$$

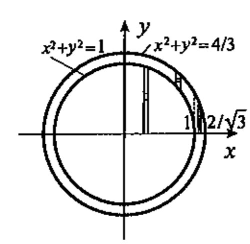
23. We quadruple the integral over the first octant volume.

$$
\begin{aligned}
\iiint_V |yz|\,dV &= 4\int_0^{\sqrt{3/2}}\int_0^{\sqrt{3/2-x^2}}\int_{\sqrt{1+x^2+y^2}}^{\sqrt{4-x^2-y^2}} yz\,dz\,dy\,dx = 4\int_0^{\sqrt{3/2}}\int_0^{\sqrt{3/2-x^2}}\left\{\frac{yz^2}{2}\right\}_{\sqrt{1+x^2+y^2}}^{\sqrt{4-x^2-y^2}} dy\,dx\\
&= 2\int_0^{\sqrt{3/2}}\int_0^{\sqrt{3/2-x^2}}(3y-2x^2y-2y^3)\,dy\,dx = 2\int_0^{\sqrt{3/2}}\left\{\frac{3y^2}{2}-x^2y^2-\frac{y^4}{2}\right\}_0^{\sqrt{3/2-x^2}} dx\\
&= \int_0^{\sqrt{3/2}}[3(3/2-x^2)-2x^2(3/2-x^2)-(3/2-x^2)^2]\,dx = \frac{1}{4}\int_0^{\sqrt{3/2}}(9-12x^2+4x^4)\,dx\\
&= \frac{1}{4}\left\{9x-4x^3+\frac{4x^5}{5}\right\}_0^{\sqrt{3/2}} = \frac{3\sqrt{6}}{5}
\end{aligned}
$$

24. We quadruple the integral over the first octant volume.

$$
\begin{aligned}
\iiint_V (x^2+y^2+z^2)\,dV &= 4\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{x^2+y^2}/2}(x^2+y^2+z^2)\,dz\,dy\,dx\\
&\quad+4\int_0^1\int_{\sqrt{1-x^2}}^{\sqrt{4/3-x^2}}\int_{\sqrt{x^2+y^2-1}}^{\sqrt{x^2+y^2}/2}(x^2+y^2+z^2)\,dz\,dy\,dx\\
&\quad+4\int_1^{2/\sqrt{3}}\int_0^{\sqrt{4/3-x^2}}\int_{\sqrt{x^2+y^2-1}}^{\sqrt{x^2+y^2}/2}(x^2+y^2+z^2)\,dz\,dy\,dx
\end{aligned}
$$



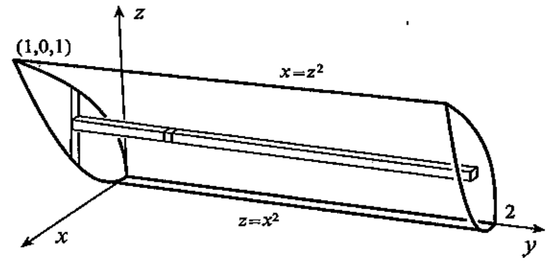


EXERCISES 13.9

1. We double the first octant volume.

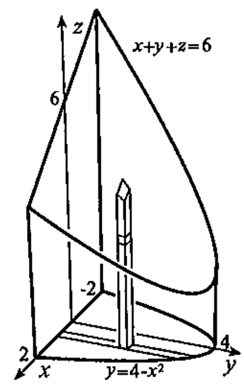
$$
V = 2\int_0^1\int_{x^2}^1\int_0^4 dz\,dy\,dx = 2\int_0^1\int_{x^2}^1 4\,dy\,dx = 8\int_0^1(1-x^2)\,dx = 8\left\{x-\frac{x^3}{3}\right\}_0^1 = \frac{16}{3}
$$

2. $$V=\int_0^1\int_{x^2}^{\sqrt{x}}\int_0^2 dy\,dz\,dx=\int_0^1\int_{x^2}^{\sqrt{x}}2\,dz\,dx$$
$$=2\int_0^1(\sqrt{x}-x^2)\,dx$$
$$=2\left\{\frac{2x^{3/2}}{3}-\frac{x^3}{3}\right\}_0^1=\frac{2}{3}$$



3. $$V=\int_0^2\int_{x/3}^{3x}\int_0^1 dy\,dz\,dx=\int_0^2\int_{x/3}^{3x}dz\,dx=\int_0^2\left(3x-\frac{x}{3}\right)dx=\left\{\frac{4x^2}{3}\right\}_0^2=\frac{16}{3}$$

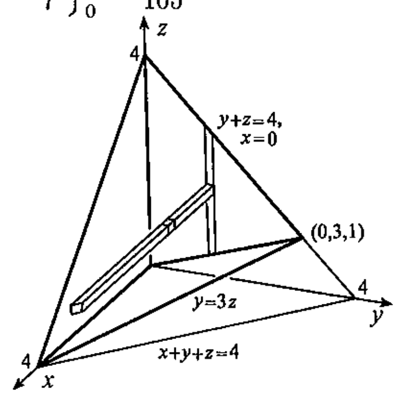
4. $$V=\int_{-2}^2\int_0^{4-x^2}\int_0^{6-x-y}dz\,dy\,dx$$
$$=\int_{-2}^2\int_0^{4-x^2}(6-x-y)\,dy\,dx$$
$$=\int_{-2}^2\left\{(6-x)y-\frac{y^2}{2}\right\}_0^{4-x^2}dx$$
$$=\frac{1}{2}\int_{-2}^2(32-8x-4x^2+2x^3-x^4)\,dx$$
$$=\frac{1}{2}\left\{32x-4x^2-\frac{4x^3}{3}+\frac{x^4}{2}-\frac{x^5}{5}\right\}_{-2}^2=\frac{704}{15}$$



5. We double the first octant volume.

$$V=2\int_0^2\int_{x^2}^4\int_0^{x^2+y^2}dz\,dy\,dx=2\int_0^2\int_{x^2}^4(x^2+y^2)\,dy\,dx=2\int_0^2\left\{x^2y+\frac{y^3}{3}\right\}_{x^2}^4dx$$
$$=\frac{2}{3}\int_0^2(12x^2+64-3x^4-x^6)\,dx=\frac{2}{3}\left\{4x^3+64x-\frac{3x^5}{5}-\frac{x^7}{7}\right\}_0^2=\frac{8576}{105}$$

6. $$V=\int_0^3\int_{y/3}^{4-y}\int_0^{4-y-z}dx\,dz\,dy=\int_0^3\int_{y/3}^{4-y}(4-y-z)\,dz\,dy$$
$$=\int_0^3\left\{(4-y)z-\frac{z^2}{2}\right\}_{y/3}^{4-y}dy$$
$$=\frac{1}{18}\int_0^3[7y^2-24y+9(4-y)^2]\,dy$$
$$=\frac{1}{18}\left\{\frac{7y^3}{3}-12y^2-3(4-y)^3\right\}_0^3=8$$

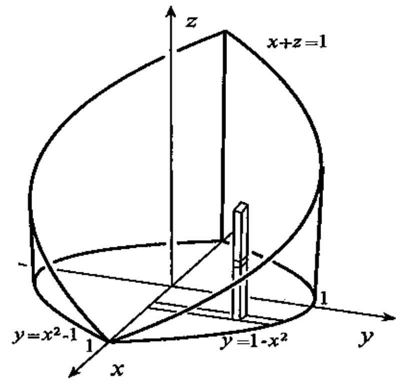


7. We multiply the first octant volume by 8.

$$V=8\int_0^2\int_0^{\sqrt{4-y^2}}\int_0^{\sqrt{4-y^2}}dz\,dx\,dy=8\int_0^2\int_0^{\sqrt{4-y^2}}\sqrt{4-y^2}\,dx\,dy=8\int_0^2\left\{x\sqrt{4-y^2}\right\}_0^{\sqrt{4-y^2}}dy$$
$$=8\int_0^2(4-y^2)\,dy=8\left\{4y-\frac{y^3}{3}\right\}_0^2=\frac{128}{3}$$

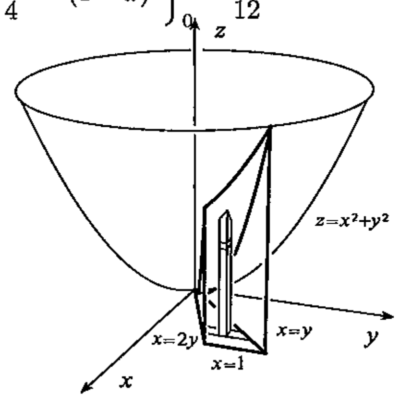
8. We double the volume to the right of the xz-plane.

$$\begin{aligned} V &= 2\int_{-1}^{1}\int_{0}^{1-x^2}\int_{0}^{1-x} dz\,dy\,dx \\ &= 2\int_{-1}^{1}\int_{0}^{1-x^2}(1-x)\,dy\,dx \\ &= 2\int_{-1}^{1}(1-x)(1-x^2)\,dx \\ &= 2\left\{x-\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^4}{4}\right\}_{-1}^{1}=\frac{8}{3} \end{aligned}$$



9. $$V=\int_0^1\int_0^{1-x}\int_{16-x^2-4y^2}^{16} dz\,dy\,dx=\int_0^1\int_0^{1-x}(16-16+x^2+4y^2)\,dy\,dx=\int_0^1\left\{x^2y+\frac{4y^3}{3}\right\}_0^{1-x}dx$$
$$=\frac{1}{3}\int_0^1[3x^2(1-x)+4(1-x)^3]\,dx=\frac{1}{3}\left\{x^3-\frac{3x^4}{4}-(1-x)^4\right\}_0^1=\frac{5}{12}$$

10. $$\begin{aligned} V &= \int_0^1\int_{x/2}^{x}\int_0^{x^2+y^2} dz\,dy\,dx \\ &= \int_0^1\int_{x/2}^{x}(x^2+y^2)\,dy\,dx \\ &= \int_0^1\left\{x^2y+\frac{y^3}{3}\right\}_{x/2}^{x}dx \\ &= \frac{19}{24}\int_0^1 x^3\,dx=\frac{19}{24}\left\{\frac{x^4}{4}\right\}_0^1=\frac{19}{96} \end{aligned}$$



11. We quadruple the first octant volume.

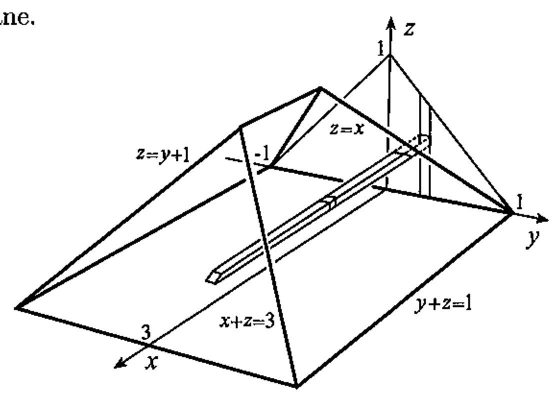
$$V=4\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{1-x^2-y^2}dz\,dy\,dx=4\int_0^1\int_0^{\sqrt{1-x^2}}(1-x^2-y^2)\,dy\,dx=4\int_0^1\left\{y-x^2y-\frac{y^3}{3}\right\}_0^{\sqrt{1-x^2}}dx$$
$$=\frac{8}{3}\int_0^1(1-x^2)^{3/2}\,dx$$

If we set $x=\sin\theta$ and $dx=\cos\theta\,d\theta$,

$$\begin{aligned} V &= \frac{8}{3}\int_0^{\pi/2}\cos^3\theta\cos\theta\,d\theta=\frac{8}{3}\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2d\theta \\ &= \frac{2}{3}\int_0^{\pi/2}\left(1+2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)d\theta=\frac{2}{3}\left\{\frac{3\theta}{2}+\sin 2\theta+\frac{1}{8}\sin 4\theta\right\}_0^{\pi/2}=\frac{\pi}{2}. \end{aligned}$$

12. We double the volume to the right of the xz-plane.

$$\begin{aligned} V &= 2\int_0^1\int_0^{1-y}\int_z^{3-z}dx\,dz\,dy \\ &= 2\int_0^1\int_0^{1-y}(3-2z)\,dz\,dy \\ &= 2\int_0^1\left\{-\frac{1}{4}(3-2z)^2\right\}_0^{1-y}dy \\ &= \frac{1}{2}\int_0^1[9-(2y+1)^2]\,dy \\ &= \frac{1}{2}\left\{9y-\frac{1}{6}(2y+1)^3\right\}_0^1=\frac{7}{3} \end{aligned}$$



13. $V = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{2-x-y} dz\,dy\,dx = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(2-x-y)\,dy\,dx = \int_{-1}^{1}\left\{(2-x)y - \frac{y^2}{2}\right\}_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$

$= 2\int_{-1}^{1}(2-x)\sqrt{1-x^2}\,dx$

If we set $x = \sin\theta$ and $dx = \cos\theta\,d\theta$ in the first term,

$$V = 4\int_{-\pi/2}^{\pi/2}\cos\theta\cos\theta\,d\theta - 2\left\{-\frac{1}{3}(1-x^2)^{3/2}\right\}_{-1}^{1} = 4\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta = 2\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_{-\pi/2}^{\pi/2} = 2\pi.$$

14. $V = \int_{0}^{1/3}\int_{y}^{2y}\int_{0}^{4-y-z} dx\,dz\,dy + \int_{1/3}^{1/2}\int_{y}^{1-y}\int_{0}^{4-y-z} dx\,dz\,dy$

$= \int_{0}^{1/3}\int_{y}^{2y}(4-y-z)\,dz\,dy + \int_{1/3}^{1/2}\int_{y}^{1-y}(4-y-z)\,dz\,dy$

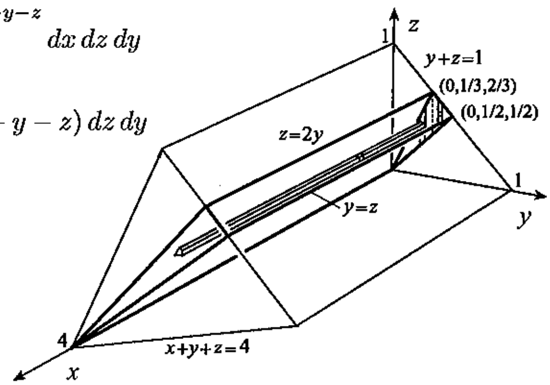
$= \int_{0}^{1/3}\left\{-\frac{1}{2}(4-y-z)^2\right\}_{y}^{2y} dy$

$\quad + \int_{1/3}^{1/2}\left\{-\frac{1}{2}(4-y-z)^2\right\}_{y}^{1-y} dy$

$= \frac{1}{2}\int_{0}^{1/3}(8y-5y^2)\,dy$

$\quad + \frac{1}{2}\int_{1/3}^{1/2}(7-16y+4y^2)\,dy$

$= \frac{1}{2}\left\{4y^2 - \frac{5y^3}{3}\right\}_{0}^{1/3} + \frac{1}{2}\left\{7y - 8y^2 + \frac{4y^3}{3}\right\}_{1/3}^{1/2} = \frac{5}{18}$



15. We quadruple the first octant volume.

$$V = 4\int_{0}^{1}\int_{0}^{2\sqrt{1-y^2}}\int_{x^2+4y^2}^{6-x^2/2-2y^2} dz\,dx\,dy = 4\int_{0}^{1}\int_{0}^{2\sqrt{1-y^2}}\left(6 - \frac{3x^2}{2} - 6y^2\right)dx\,dy$$

$$= 6\int_{0}^{1}\int_{0}^{2\sqrt{1-y^2}}(4-x^2-4y^2)\,dx\,dy = 6\int_{0}^{1}\left\{4x - \frac{x^3}{3} - 4xy^2\right\}_{0}^{2\sqrt{1-y^2}} dy = 32\int_{0}^{1}(1-y^2)^{3/2}\,dy$$

If we set $y = \sin\theta$ and $dy = \cos\theta\,d\theta$,

$$V = 32\int_{0}^{\pi/2}\cos^3\theta\cos\theta\,d\theta = 32\int_{0}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= 8\int_{0}^{\pi/2}\left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)d\theta = 8\left\{\frac{3\theta}{2} + \sin 2\theta + \frac{1}{8}\sin 4\theta\right\}_{0}^{\pi/2} = 6\pi.$$

16. We multiply the volume in the first octant for which $x > y$ by 16.

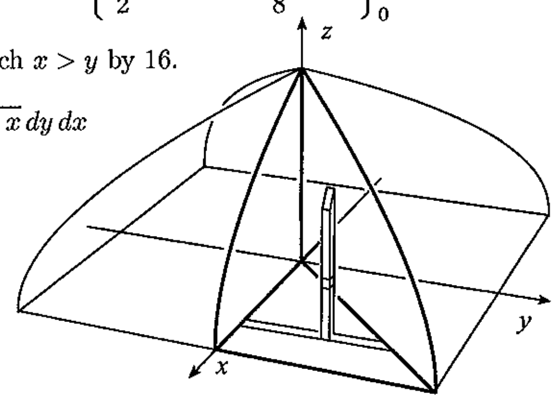
$V = 16\int_{0}^{1}\int_{0}^{x}\int_{0}^{\sqrt{1-x}} dz\,dy\,dx = 16\int_{0}^{1}\int_{0}^{x}\sqrt{1-x}\,dy\,dx$

$= 16\int_{0}^{1} x\sqrt{1-x}\,dx$

If we set $u = 1-x$, then $du = -dx$, and

$V = 16\int_{1}^{0}(1-u)\sqrt{u}(-du)$

$= 16\left\{\frac{2u^{3/2}}{3} - \frac{2u^{5/2}}{5}\right\}_{0}^{1} = \frac{64}{15}.$



17. $V = \int_0^{6/7}\int_x^{2x}\int_0^{3-x/2-3y/2} dz\,dy\,dx + \int_{6/7}^{3/2}\int_x^{2-x/3}\int_0^{3-x/2-3y/2} dz\,dy\,dx$

$= \frac{1}{2}\int_0^{6/7}\int_x^{2x}(6-x-3y)\,dy\,dx + \frac{1}{2}\int_{6/7}^{3/2}\int_x^{2-x/3}(6-x-3y)\,dy\,dx$

$= \frac{1}{2}\int_0^{6/7}\left\{-\frac{1}{6}(6-x-3y)^2\right\}_x^{2x}dx + \frac{1}{2}\int_{6/7}^{3/2}\left\{-\frac{1}{6}(6-x-3y)^2\right\}_x^{2-x/3}dx$

$= -\frac{1}{12}\int_0^{6/7}[(6-7x)^2-(6-4x)^2]\,dx - \frac{1}{12}\int_{6/7}^{3/2}-(6-4x)^2\,dx$

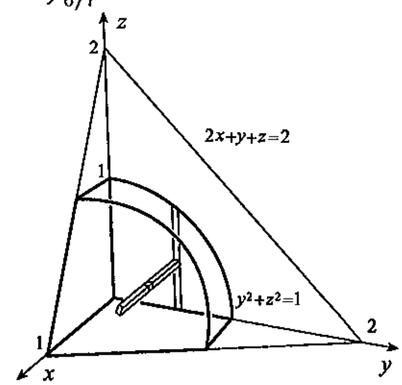
$= -\frac{1}{12}\left\{-\frac{1}{21}(6-7x)^3+\frac{1}{12}(6-4x)^3\right\}_0^{6/7} + \frac{1}{12}\left\{-\frac{1}{12}(6-4x)^3\right\}_{6/7}^{3/2} = \frac{9}{14}$

18. $V = \int_0^1\int_0^{\sqrt{1-y^2}}\int_0^{1-y/2-z/2} dx\,dz\,dy$

$= \int_0^1\int_0^{\sqrt{1-y^2}}\left(1-\frac{y}{2}-\frac{z}{2}\right)dz\,dy$

$= \frac{1}{2}\int_0^1\left\{(2-y)z-\frac{z^2}{2}\right\}_0^{\sqrt{1-y^2}}dy$

$= \frac{1}{4}\int_0^1(4\sqrt{1-y^2}-2y\sqrt{1-y^2}-1+y^2)\,dy$

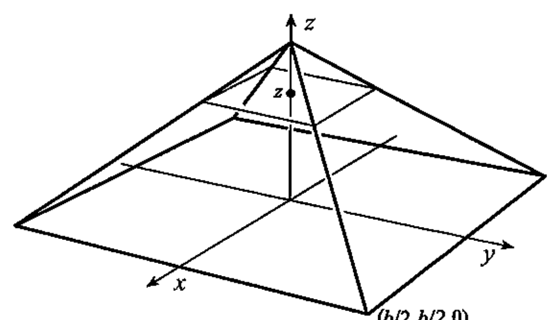


If we set $y=\sin\theta$ and $dy=\cos\theta\,d\theta$ in the first term,

$$V = \int_0^{\pi/2}\cos\theta\,\cos\theta\,d\theta + \frac{1}{4}\left\{\frac{2}{3}(1-y^2)^{3/2}-y+\frac{y^3}{3}\right\}_0^1$$
$$= \int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta - \frac{1}{3} = \frac{1}{2}\left\{\theta+\frac{\sin 2\theta}{2}\right\}_0^{\pi/2} - \frac{1}{3} = \frac{\pi}{4}-\frac{1}{3}$$

19. (a) The square cross section at height z has sides of length $b(h-z)/h$. Consequently, the area of the cross section is $b^2(h-z)^2/h^2$, and the volume of the pyramid is

$V = \int_0^h \frac{b^2}{h^2}(h-z)^2\,dz = \frac{b^2}{h^2}\left\{-\frac{1}{3}(h-z)^3\right\}_0^h = \frac{b^2h}{3}.$



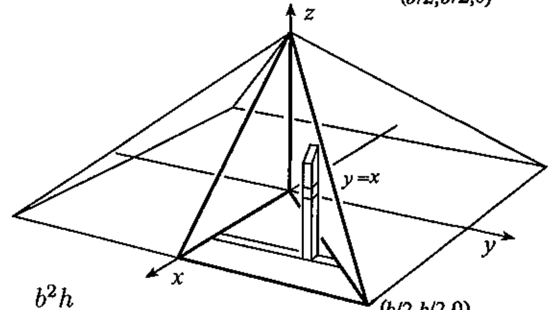
(b) Since the equation of the face of the pyramid containing the point $(b/2,0,0)$ is $2x/b+z/h=1$,

$V = 8\int_0^{b/2}\int_0^x\int_0^{h(1-2x/b)} dz\,dy\,dx$

$= 8\int_0^{b/2}\int_0^x h\left(1-\frac{2x}{b}\right)dy\,dx$

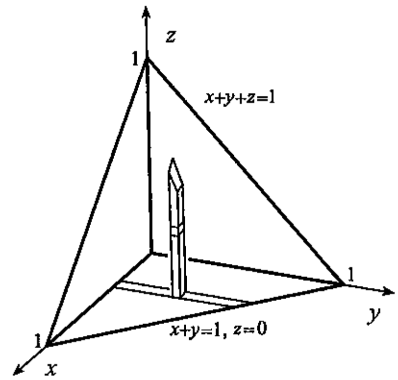
$= \frac{8h}{b}\int_0^{b/2}\left\{(b-2x)y\right\}_0^x dx$

$= \frac{8h}{b}\int_0^{b/2}(bx-2x^2)\,dx = \frac{8h}{b}\left\{\frac{bx^2}{2}-\frac{2x^3}{3}\right\}_0^{b/2} = \frac{b^2h}{3}.$



20. Since Volume

$$= \iiint_V dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\,dy\,dx$$
$$= \int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx$$
$$= \int_0^1 \left\{-\frac{1}{2}(1-x-y)^2\right\}_0^{1-x} dx$$
$$= \frac{1}{2}\int_0^1 (1-x)^2\,dx = \frac{1}{2}\left\{-\frac{1}{3}(1-x)^3\right\}_0^1 = \frac{1}{6},$$



$$\overline{f} = 6\iiint_V xy\,dV = 6\int_0^1 \int_0^{1-x} \int_0^{1-x-y} xy\,dz\,dy\,dx = 6\int_0^1 \int_0^{1-x} xy(1-x-y)\,dy\,dx$$
$$= 6\int_0^1 \left\{x(1-x)\frac{y^2}{2} - \frac{xy^3}{3}\right\}_0^{1-x} dx = \int_0^1 (x - 3x^2 + 3x^3 - x^4)\,dx$$
$$= \left\{\frac{x^2}{2} - x^3 + \frac{3x^4}{4} - \frac{x^5}{5}\right\}_0^1 = \frac{1}{20}$$

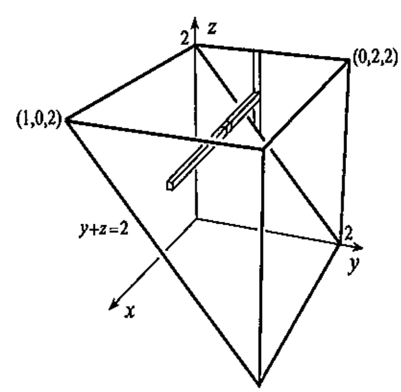
21. Since V

$$= \int_0^1 \int_0^1 \int_0^{9-x^2-y^2} dz\,dy\,dx = \int_0^1 \int_0^1 (9-x^2-y^2)\,dy\,dx = \int_0^1 \left\{9y - x^2y - \frac{y^3}{3}\right\}_0^1 dx$$
$$= \int_0^1 \left(9 - x^2 - \frac{1}{3}\right) dx = \left\{\frac{26x}{3} - \frac{x^3}{3}\right\}_0^1 = \frac{25}{3},$$

$$\overline{f} = \frac{3}{25}\int_0^1 \int_0^1 \int_0^{9-x^2-y^2} (x+y+z)\,dz\,dy\,dx = \frac{3}{25}\int_0^1 \int_0^1 \left\{(x+y)z + \frac{z^2}{2}\right\}_0^{9-x^2-y^2} dy\,dx$$
$$= \frac{3}{50}\int_0^1 \int_0^1 (81 + 18x + 18y - 18x^2 - 18y^2 - 2x^3 - 2y^3 - 2xy^2 - 2x^2y + x^4 + y^4 + 2x^2y^2)\,dy\,dx$$
$$= \frac{3}{50}\int_0^1 \left\{81y + 18xy + 9y^2 - 18x^2y - 6y^3 - 2x^3y - \frac{y^4}{2} - \frac{2xy^3}{3} - x^2y^2 + x^4y + \frac{y^5}{5} + \frac{2x^2y^3}{3}\right\}_0^1 dx$$
$$= \frac{3}{50}\int_0^1 \left(\frac{837}{10} + \frac{52x}{3} - \frac{55x^2}{3} - 2x^3 + x^4\right) dx = \frac{3}{50}\left\{\frac{837x}{10} + \frac{26x^2}{3} - \frac{55x^3}{9} - \frac{x^4}{2} + \frac{x^5}{5}\right\}_0^1 = \frac{1934}{375}.$$

22. Since Volume $= \frac{1}{2}(2)(2)(1) = 2$,

$$\overline{f} = \frac{1}{2}\iiint_V (x^2+y^2+z^2)\,dV$$
$$= \frac{1}{2}\int_0^2 \int_{2-y}^2 \int_0^1 (x^2+y^2+z^2)\,dx\,dz\,dy$$
$$= \frac{1}{2}\int_0^2 \int_{2-y}^2 \left\{\frac{x^3}{3} + x(y^2+z^2)\right\}_0^1 dz\,dy$$
$$= \frac{1}{6}\int_0^2 \int_{2-y}^2 [1 + 3(y^2+z^2)]\,dz\,dy$$
$$= \frac{1}{6}\int_0^2 \left\{z(1+3y^2) + z^3\right\}_{2-y}^2 dy$$
$$= \frac{1}{6}\int_0^2 [8 + y + 3y^3 - (2-y)^3]\,dy = \frac{1}{6}\left\{8y + \frac{y^2}{2} + \frac{3y^4}{4} + \frac{(2-y)^4}{4}\right\}_0^2 = \frac{13}{3}$$



23. The projection in the xy-plane of the curve of intersection of the surfaces has equation $x^2 - y^2 = 4 - 2(x^2 + y^2) \implies 3x^2 + y^2 = 4$. We quadruple the first octant volume.

$$\begin{aligned}V &= 4\int_0^{2/\sqrt{3}}\int_0^{\sqrt{4-3x^2}}\int_{x^2-y^2}^{4-2x^2-2y^2} dz\,dy\,dx = 4\int_0^{2/\sqrt{3}}\int_0^{\sqrt{4-3x^2}}(4-3x^2-y^2)\,dy\,dx \\ &= 4\int_0^{2/\sqrt{3}}\left\{(4-3x^2)y - \frac{y^3}{3}\right\}_0^{\sqrt{4-3x^2}} dx = \frac{4}{3}\int_0^{2/\sqrt{3}}[3(4-3x^2)^{3/2} - (4-3x^2)^{3/2}]\,dx \\ &= \frac{8}{3}\int_0^{2/\sqrt{3}}(4-3x^2)^{3/2}\,dx\end{aligned}$$

If we set $x = (2/\sqrt{3})\sin\theta$ and $dx = (2/\sqrt{3})\cos\theta\,d\theta$,

$$\begin{aligned}V &= \frac{8}{3}\int_0^{\pi/2} 8\cos^3\theta(2/\sqrt{3})\cos\theta\,d\theta = \frac{128}{3\sqrt{3}}\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta \\ &= \frac{32}{3\sqrt{3}}\int_0^{\pi/2}\left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)d\theta = \frac{32}{3\sqrt{3}}\left\{\frac{3\theta}{2} + \sin 2\theta + \frac{1}{8}\sin 4\theta\right\}_0^{\pi/2} = \frac{8\pi}{\sqrt{3}}.\end{aligned}$$

24. If we set $x^2 - y^2 = 4 - x^2 - y^2$, then $2x^2 = 4$ or $x = \pm\sqrt{2}$. This implies that the curve of intersection of the surfaces divides into two parts, two parabolas $z = 2 - y^2$, $x = \pm\sqrt{2}$ in parallel planes. There is no bounded volume.

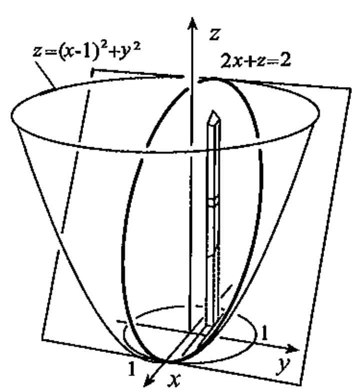
25. The projection in the yz-plane of the curve of intersection of the surfaces has equation $4y^2 = (2-z)^2 \implies z = 2 \pm 2y$. We double the first octant volume.

$$\begin{aligned}V &= 2\int_0^1\int_0^{2-2y}\int_{4y^2/(2-z)}^{2-z} dx\,dz\,dy = 2\int_0^1\int_0^{2-2y}\left(2 - z - \frac{4y^2}{2-z}\right)dz\,dy \\ &= 2\int_0^1\left\{2z - \frac{z^2}{2} + 4y^2\ln|2-z|\right\}_0^{2-2y} dy = \int_0^1(4 - 4y^2 + 8y^2\ln y)\,dy.\end{aligned}$$

If we set $u = \ln y$, $dv = y^2\,dy$, $du = (1/y)\,dy$, and $v = y^3/3$, in the last term,

$$V = \left\{4y - \frac{4y^3}{3}\right\}_0^1 + 8\left\{\frac{y^3}{3}\ln y\right\}_0^1 - 8\int_0^1\frac{y^2}{3}dy = \frac{8}{3} - \frac{8}{3}\left\{\frac{y^3}{3}\right\}_0^1 = \frac{16}{9}.$$

26. We double the volume to the right of the xz-plane.



$$\begin{aligned}V &= 2\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{(x-1)^2+y^2}^{2-2x} dz\,dx\,dy \\ &= 2\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}(1 - x^2 - y^2)\,dx\,dy \\ &= 2\int_0^1\left\{x(1-y^2) - \frac{x^3}{3}\right\}_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dy \\ &= \frac{8}{3}\int_0^1(1-y^2)^{3/2}\,dy\end{aligned}$$

If we set $y = \sin\theta$, then $dy = \cos\theta\,d\theta$, and

$$\begin{aligned}V &= \frac{8}{3}\int_0^{\pi/2}\cos^4\theta\,d\theta = \frac{8}{3}\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta \\ &= \frac{2}{3}\int_0^{\pi/2}\left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)d\theta = \frac{2}{3}\left\{\frac{3\theta}{2} + \sin 2\theta + \frac{1}{8}\sin 4\theta\right\}_0^{\pi/2} = \frac{\pi}{2}.\end{aligned}$$

27. We multiple the first octant volume by eight.

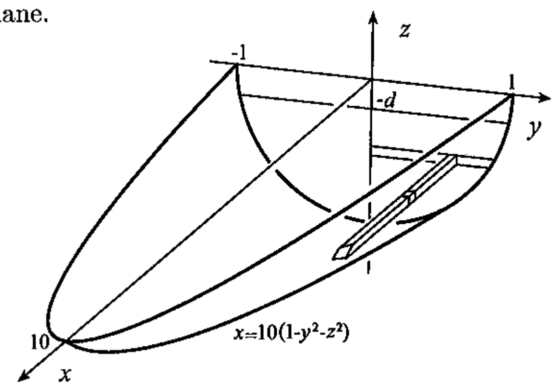
$$V = 8\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} dz\,dy\,dx = 8\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{1-x^2/a^2-y^2/b^2}\,dy\,dx$$

If we set $y = b\sqrt{1-x^2/a^2}\sin\theta$ and $dy = b\sqrt{1-x^2/a^2}\cos\theta\,d\theta$, then

$$V = 8c\int_0^a \int_0^{\pi/2} \sqrt{1-x^2/a^2}\cos\theta\, b\sqrt{1-x^2/a^2}\cos\theta\,d\theta\,dx = 8c\int_0^a b(1-x^2/a^2)\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta\,dx$$

$$= 4bc\int_0^a (1-x^2/a^2)\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} dx = 2\pi bc\int_0^a (1-x^2/a^2)\,dx = 2\pi bc\left\{x - \frac{x^3}{3a^2}\right\}_0^a = \frac{4\pi abc}{3}.$$

28. (a) We double the volume to the right of the xz-plane.



$$V = 2\int_{-1}^{-d}\int_0^{\sqrt{1-z^2}}\int_0^{10(1-y^2-z^2)} dx\,dy\,dz$$

$$= 20\int_{-1}^{-d}\int_0^{\sqrt{1-z^2}} (1-y^2-z^2)\,dy\,dz$$

$$= 20\int_{-1}^{-d}\left\{y(1-z^2) - \frac{y^3}{3}\right\}_0^{\sqrt{1-z^2}} dz$$

$$= \frac{40}{3}\int_{-1}^{-d}(1-z^2)^{3/2}\,dz$$

If we set $z = \sin\theta$, then $dz = \cos\theta\,d\theta$, and

$$V = \frac{40}{3}\int_{-\pi/2}^{\overline{\theta}} \cos^4\theta\,d\theta \qquad (\sin\overline{\theta} = -d)$$

$$= \frac{40}{3}\int_{-\pi/2}^{\overline{\theta}}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \frac{10}{3}\int_{-\pi/2}^{\overline{\theta}}\left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)d\theta = \frac{10}{3}\left\{\frac{3\theta}{2} + \sin 2\theta + \frac{1}{8}\sin 4\theta\right\}_{-\pi/2}^{\overline{\theta}}$$

$$= \frac{5}{3}[3\overline{\theta} + 4\sin\overline{\theta}\,\cos\overline{\theta} + \sin\overline{\theta}\,\cos\overline{\theta}(1-2\sin^2\overline{\theta}) + 3\pi/2] = \frac{5}{3}[3\pi/2 - 3\mathrm{Sin}^{-1}d - d\sqrt{1-d^2}(5-2d^2)]$$

(b) The boat will sink when $d = 0$, at which point $V = 5\pi/2$. The buoyant force when $d = 0$ is $1000gV = 2500\pi g$, and this is the maximum weight.

29. $$V = 16\int_0^{a/\sqrt{2}}\int_0^x\int_0^{\sqrt{a^2-x^2}} dz\,dy\,dx + 16\int_{a/\sqrt{2}}^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2}} dz\,dy\,dx$$

$$= 16\int_0^{a/\sqrt{2}}\int_0^x \sqrt{a^2-x^2}\,dy\,dx + 16\int_{a/\sqrt{2}}^a\int_0^{\sqrt{a^2-x^2}}\sqrt{a^2-x^2}\,dy\,dx$$

$$= 16\int_0^{a/\sqrt{2}} x\sqrt{a^2-x^2}\,dx + 16\int_{a/\sqrt{2}}^a (a^2-x^2)\,dx = 16\left\{-\frac{1}{3}(a^2-x^2)^{3/2}\right\}_0^{a/\sqrt{2}} + 16\left\{a^2x - \frac{x^3}{3}\right\}_{a/\sqrt{2}}^a$$

$$= \frac{16a^3(\sqrt{2}-1)}{\sqrt{2}} = 8(2-\sqrt{2})a^3$$

EXERCISES 13.10

1. $$M = \int_0^1\int_0^1\int_0^{x^2+y^2} \rho\,dz\,dy\,dx = \rho\int_0^1\int_0^1 (x^2+y^2)\,dy\,dx = \rho\int_0^1\left\{x^2y + \frac{y^3}{3}\right\}_0^1 dx$$

$$= \frac{\rho}{3}\int_0^1 (3x^2+1)\,dx = \frac{\rho}{3}\left\{x^3+x\right\}_0^1 = \frac{2\rho}{3}$$

Since $M\overline{x} = \int_0^1\int_0^1\int_0^{x^2+y^2} x\rho\,dz\,dy\,dx = \rho\int_0^1\int_0^1 x(x^2+y^2)\,dy\,dx = \rho\int_0^1 \left\{x^3y + \frac{xy^3}{3}\right\}_0^1 dx$

$$= \frac{\rho}{3}\int_0^1 (3x^3+x)\,dx = \frac{\rho}{3}\left\{\frac{3x^4}{4}+\frac{x^2}{2}\right\}_0^1 = \frac{5\rho}{12},$$

it follows that $\overline{x} = \frac{5\rho}{12}\frac{3}{2\rho} = \frac{5}{8}$. By symmetry, $\overline{y} = 5/8$. Since

$$M\overline{z} = \int_0^1\int_0^1\int_0^{x^2+y^2} z\rho\,dz\,dy\,dx = \rho\int_0^1\int_0^1\left\{\frac{z^2}{2}\right\}_0^{x^2+y^2} dy\,dx = \frac{\rho}{2}\int_0^1\int_0^1 (x^4+2x^2y^2+y^4)\,dy\,dx$$

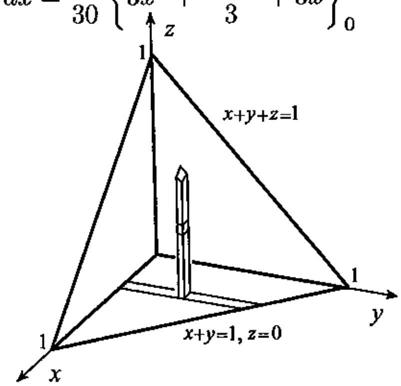
$$= \frac{\rho}{2}\int_0^1\left\{x^4y + \frac{2x^2y^3}{3}+\frac{y^5}{5}\right\}_0^1 dx = \frac{\rho}{30}\int_0^1 (15x^4+10x^2+3)\,dx = \frac{\rho}{30}\left\{3x^5+\frac{10x^3}{3}+3x\right\}_0^1 = \frac{14\rho}{45},$$

we obtain $\overline{z} = \frac{14\rho}{45}\frac{3}{2\rho} = \frac{7}{15}$.

2. $M = \int_0^1\int_0^{1-x}\int_0^{1-x-y} \rho\,dz\,dy\,dx = \rho\int_0^1\int_0^{1-x}(1-x-y)\,dy\,dx$

$$= \rho\int_0^1\left\{-\frac{1}{2}(1-x-y)^2\right\}_0^{1-x} dx$$

$$= \frac{\rho}{2}\int_0^1 (1-x)^2\,dx = \frac{\rho}{2}\left\{-\frac{1}{3}(1-x)^3\right\}_0^1 = \frac{\rho}{6}$$



Since $M\overline{x} = \int_0^1\int_0^{1-x}\int_0^{1-x-y} x\rho\,dz\,dy\,dx = \rho\int_0^1\int_0^{1-x} x(1-x-y)\,dy\,dx = \rho\int_0^1\left\{x(1-x)y - \frac{xy^2}{2}\right\}_0^{1-x} dx$

$$= \frac{\rho}{2}\int_0^1 (x-2x^2+x^3)\,dx = \frac{\rho}{2}\left\{\frac{x^2}{2}-\frac{2x^3}{3}+\frac{x^4}{4}\right\}_0^1 = \frac{\rho}{24},$$

it follows by symmetry that $\overline{x} = \overline{y} = \overline{z} = \frac{\rho}{24}\frac{6}{\rho} = \frac{1}{4}$.

3. $M = 2\int_0^2\int_{x^2}^4\int_0^{4-z} \rho\,dy\,dz\,dx = 2\rho\int_0^2\int_{x^2}^4 (4-z)\,dz\,dx = 2\rho\int_0^2\left\{-\frac{1}{2}(4-z)^2\right\}_{x^2}^4 dx$

$$= \rho\int_0^2 (16-8x^2+x^4)\,dx = \rho\left\{16x - \frac{8x^3}{3}+\frac{x^5}{5}\right\}_0^2 = \frac{256\rho}{15}$$

By symmetry, $\overline{x} = 0$. Since

$$M\overline{y} = 2\int_0^2\int_{x^2}^4\int_0^{4-z} y\rho\,dy\,dz\,dx = 2\rho\int_0^2\int_{x^2}^4\left\{\frac{y^2}{2}\right\}_0^{4-z} dz\,dx = \rho\int_0^2\int_{x^2}^4 (4-z)^2\,dz\,dx$$

$$= \rho\int_0^2\left\{\frac{-(4-z)^3}{3}\right\}_{x^2}^4 dx = \frac{\rho}{3}\int_0^2 (64-48x^2+12x^4-x^6)\,dx = \frac{\rho}{3}\left\{64x-16x^3+\frac{12x^5}{5}-\frac{x^7}{7}\right\}_0^2$$

$$= \frac{2048\rho}{105},$$

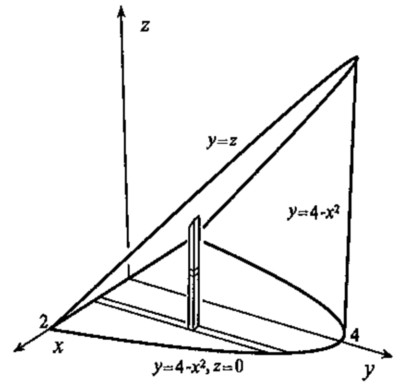
it follows that $\overline{y} = \frac{2048\rho}{105}\frac{15}{256\rho} = \frac{8}{7}$. Since

$$M\overline{z} = 2\int_0^2\int_{x^2}^4\int_0^{4-z} z\rho\,dy\,dz\,dx = 2\rho\int_0^2\int_{x^2}^4 z(4-z)\,dz\,dx = 2\rho\int_0^2\left\{2z^2-\frac{z^3}{3}\right\}_{x^2}^4 dx$$

$$= \frac{2\rho}{3}\int_0^2 (96-64-6x^4+x^6)\,dx = \frac{2\rho}{3}\left\{32x - \frac{6x^5}{5}+\frac{x^7}{7}\right\}_0^2 = \frac{1024\rho}{35},$$

we obtain $\overline{z} = \frac{1024\rho}{35}\frac{15}{256\rho} = \frac{12}{7}$.

4. $$M = 2\int_0^2\int_0^{4-x^2}\int_0^y \rho\,dz\,dy\,dx$$
$$= 2\rho\int_0^2\int_0^{4-x^2} y\,dy\,dx$$
$$= 2\rho\int_0^2 \left\{\frac{y^2}{2}\right\}_0^{4-x^2} dx$$
$$= \rho\int_0^2 (16-8x^2+x^4)\,dx$$
$$= \rho\left\{16x - \frac{8x^3}{3} + \frac{x^5}{5}\right\}_0^2 = \frac{256\rho}{15}$$



Since $$M\overline{y} = 2\int_0^2\int_0^{4-x^2}\int_0^y y\rho\,dz\,dy\,dx = 2\rho\int_0^2\int_0^{4-x^2} y^2\,dy\,dx = 2\rho\int_0^2\left\{\frac{y^3}{3}\right\}_0^{4-x^2} dx$$
$$= \frac{2\rho}{3}\int_0^2 (64-48x^2+12x^4-x^6)\,dx = \frac{2\rho}{3}\left\{64x-16x^3+\frac{12x^5}{5}-\frac{x^7}{7}\right\}_0^2 = \frac{4096\rho}{105},$$
it follows that $\overline{y} = \dfrac{4096\rho}{105}\dfrac{15}{256\rho} = \dfrac{16}{7}$. By symmetry, $\overline{x} = 0$. We find that $\overline{z} = 8/7$ since
$$M\overline{z} = 2\int_0^2\int_0^{4-x^2}\int_0^y z\rho\,dz\,dy\,dx = 2\rho\int_0^2\int_0^{4-x^2}\left\{\frac{z^2}{2}\right\}_0^y dy\,dx = \rho\int_0^2\int_0^{4-x^2} y^2\,dy\,dx = \frac{1}{2}M\overline{y}.$$

5. $$M = \int_0^3\int_{y/3}^{4-y}\int_0^{4-y-z}\rho\,dx\,dz\,dy = \rho\int_0^3\int_{y/3}^{4-y}(4-y-z)\,dz\,dy = \rho\int_0^3\left\{-\frac{1}{2}(4-y-z)^2\right\}_{y/3}^{4-y} dy$$
$$= \frac{\rho}{2}\int_0^3\left(4-\frac{4y}{3}\right)^2 dy = \frac{8\rho}{9}\left\{-\frac{1}{3}(3-y)^3\right\}_0^3 = 8\rho$$

Since $$M\overline{x} = \int_0^3\int_{y/3}^{4-y}\int_0^{4-y-z} x\rho\,dx\,dz\,dy = \rho\int_0^3\int_{y/3}^{4-y}\left\{\frac{x^2}{2}\right\}_0^{4-y-z} dz\,dy = \frac{\rho}{2}\int_0^3\int_{y/3}^{4-y}(4-y-z)^2\,dz\,dy$$
$$= \frac{\rho}{2}\int_0^3\left\{-\frac{1}{3}(4-y-z)^3\right\}_{y/3}^{4-y} dy = \frac{\rho}{6}\int_0^3\left(4-\frac{4y}{3}\right)^3 dy = \frac{32\rho}{81}\left\{-\frac{1}{4}(3-y)^4\right\}_0^3 = 8\rho,$$
we find that $\overline{x} = 1$. Since
$$M\overline{y} = \int_0^3\int_{y/3}^{4-y}\int_0^{4-y-z} y\rho\,dx\,dz\,dy = \rho\int_0^3\int_{y/3}^{4-y} y(4-y-z)\,dz\,dy = \rho\int_0^3\left\{-\frac{y}{2}(4-y-z)^2\right\}_{y/3}^{4-y} dy$$
$$= \frac{\rho}{2}\int_0^3 y\left(4-\frac{4y}{3}\right)^2 dy = \frac{8\rho}{9}\int_0^3 (9y-6y^2+y^3)\,dy = \frac{8\rho}{9}\left\{\frac{9y^2}{2}-2y^3+\frac{y^4}{4}\right\}_0^3 = 6\rho,$$

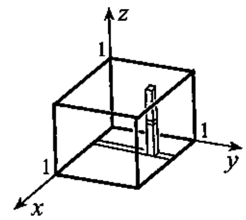
it follows that $\overline{y} = 6\rho/(8\rho) = 3/4$. Since
$$M\overline{z} = \int_0^3\int_{y/3}^{4-y}\int_0^{4-y-z} z\rho\,dx\,dz\,dy = \rho\int_0^3\int_{y/3}^{4-y} z(4-y-z)\,dz\,dy = \rho\int_0^3\left\{2z^2-\frac{yz^2}{2}-\frac{z^3}{3}\right\}_{y/3}^{4-y} dy$$
$$= \frac{\rho}{6}\int_0^3\left[12(4-y)^2-3y(4-y)^2-2(4-y)^3-\frac{4y^2}{3}+\frac{y^3}{3}+\frac{2y^3}{27}\right] dy$$
$$= \frac{\rho}{6}\int_0^3\left[12(4-y)^2-48y+\frac{68y^2}{3}-\frac{70y^3}{27}-2(4-y)^3\right] dy$$
$$= \frac{\rho}{6}\left\{-4(4-y)^3-24y^2+\frac{68y^3}{9}-\frac{35y^4}{54}+\frac{(4-y)^4}{2}\right\}_0^3 = 10\rho,$$

we find that $\overline{z} = 10\rho/(8\rho) = 5/4$.

6. $I = \int_0^1 \int_0^1 \int_0^1 (y^2+z^2)\rho\, dz\, dy\, dx = \rho \int_0^1 \int_0^1 \left\{ y^2 z + \frac{z^3}{3} \right\}_0^1 dy\, dx$

$= \frac{\rho}{3} \int_0^1 \int_0^1 (3y^2+1)\, dy\, dx$

$= \frac{\rho}{3} \int_0^1 \left\{ y^3 + y \right\}_0^1 dx = \frac{2\rho}{3} \left\{ x \right\}_0^1 = \frac{2\rho}{3}$



7. $I = \int_0^3 \int_0^2 \int_0^{2x} (x^2+z^2)\rho\, dz\, dy\, dx = \rho \int_0^3 \int_0^2 \left\{ x^2 z + \frac{z^3}{3} \right\}_0^{2x} dy\, dx = \frac{\rho}{3} \int_0^3 \int_0^2 (6x^3 + 8x^3)\, dy\, dx$

$= \frac{14\rho}{3} \int_0^3 \left\{ x^3 y \right\}_0^2 dx = \frac{28\rho}{3} \left\{ \frac{x^4}{4} \right\}_0^3 = 189\rho$

8. $I = \int_0^1 \int_0^{1-z^2} \int_0^{2-y-z} (y^2+z^2)\rho\, dx\, dy\, dz$

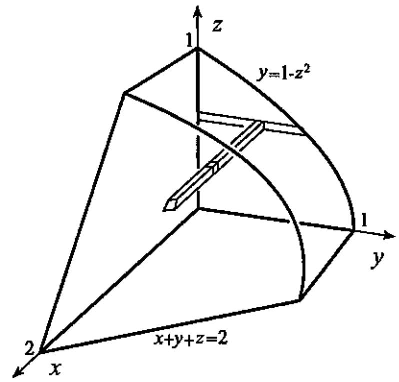
$= \rho \int_0^1 \int_0^{1-z^2} \left\{ (y^2+z^2)x \right\}_0^{2-y-z} dy\, dz$

$= \rho \int_0^1 \int_0^{1-z^2} (2y^2 - y^3 - zy^2 + 2z^2 - yz^2 - z^3)\, dy\, dz$

$= \rho \int_0^1 \left\{ \frac{2y^3}{3} - \frac{y^4}{4} - \frac{zy^3}{3} + 2z^2 y - \frac{y^2 z^2}{2} - yz^3 \right\}_0^{1-z^2} dz$

$= \frac{\rho}{12} \int_0^1 [5 + 6z^2 - 12z^3 - 6z^4 + 12z^5 - 2z^6 - 3z^8 - 4z(1-z^2)^3]\, dz$

$= \frac{\rho}{12} \left\{ 5z + 2z^3 - 3z^4 - \frac{6z^5}{5} + 2z^6 - \frac{2z^7}{7} - \frac{z^9}{3} + \frac{1}{2}(1-z^2)^4 \right\}_0^1 = \frac{773\rho}{2520}$



9. $I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{xy} (x^2+y^2)\rho\, dz\, dy\, dx = \rho \int_0^1 \int_0^{\sqrt{1-x^2}} xy(x^2+y^2)\, dy\, dx$

$= \rho \int_0^1 \left\{ \frac{x^3 y^2}{2} + \frac{xy^4}{4} \right\}_0^{\sqrt{1-x^2}} dx = \frac{\rho}{4} \int_0^1 (x - x^5)\, dx = \frac{\rho}{4} \left\{ \frac{x^2}{2} - \frac{x^6}{6} \right\}_0^1 = \frac{\rho}{12}$

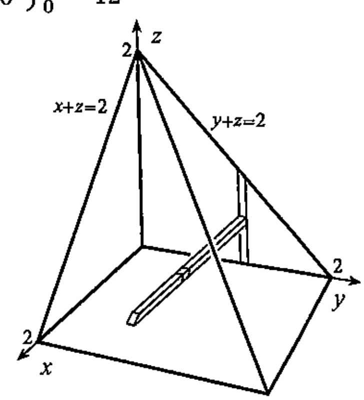
10. $I = \int_0^2 \int_0^{2-y} \int_0^{2-z} (x^2+y^2)\rho\, dx\, dz\, dy$

$= \rho \int_0^2 \int_0^{2-y} \left\{ \frac{x^3}{3} + xy^2 \right\}_0^{2-z} dz\, dy$

$= \frac{\rho}{3} \int_0^2 \int_0^{2-y} [(2-z)^3 + 3y^2(2-z)]\, dz\, dy$

$= \frac{\rho}{3} \int_0^2 \left\{ -\frac{1}{4}(2-z)^4 - \frac{3}{2} y^2 (2-z)^2 \right\}_0^{2-y} dy$

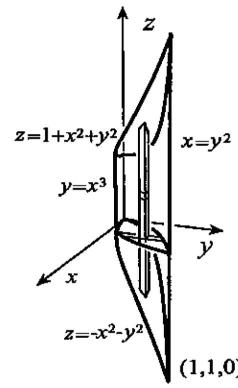
$= \frac{\rho}{12} \int_0^2 (16 + 24y^2 - 7y^4)\, dy = \frac{\rho}{12} \left\{ 16y + 8y^3 - \frac{7y^5}{5} \right\}_0^2 = \frac{64\rho}{15}$



11. Moment $= \int_0^2 \int_{-z}^{z} \int_0^2 z\rho\, dy\, dx\, dz = \rho \int_0^2 \int_{-z}^{z} 2z\, dx\, dz = 2\rho \int_0^2 \left\{ xz \right\}_{-z}^{z} dz$

$= 4\rho \int_0^2 z^2\, dz = 4\rho \left\{ \frac{z^3}{3} \right\}_0^2 = \frac{32\rho}{3}$

12. $$M = \int_0^1 \int_{x^3}^{\sqrt{x}} \int_{-x^2-y^2}^{1+x^2+y^2} \rho\, dz\, dy\, dx = \rho \int_0^1 \int_{x^3}^{\sqrt{x}} (1 + 2x^2 + 2y^2)\, dy\, dx$$
$$= \rho \int_0^1 \left\{ y + 2x^2 y + \frac{2y^3}{3} \right\}_{x^3}^{\sqrt{x}} dx$$
$$= \frac{\rho}{3} \int_0^1 (3\sqrt{x} + 6x^{5/2} + 2x^{3/2} - 3x^3 - 6x^5 - 2x^9)\, dx$$
$$= \frac{\rho}{3} \left\{ 2x^{3/2} + \frac{12x^{7/2}}{7} + \frac{4x^{5/2}}{5} - \frac{3x^4}{4} - x^6 - \frac{x^{10}}{5} \right\}_0^1 = \frac{359\rho}{420}$$



Since
$$M\overline{x} = \int_0^1 \int_{x^3}^{\sqrt{x}} \int_{-x^2-y^2}^{1+x^2+y^2} x\rho\, dz\, dy\, dx = \rho \int_0^1 \int_{x^3}^{\sqrt{x}} x(1 + 2x^2 + 2y^2)\, dy\, dx$$
$$= \rho \int_0^1 \left\{ xy + 2x^3 y + \frac{2xy^3}{3} \right\}_{x^3}^{\sqrt{x}} dx = \frac{\rho}{3} \int_0^1 (3x^{3/2} + 6x^{7/2} + 2x^{5/2} - 3x^4 - 6x^6 - 2x^{10})\, dx$$
$$= \frac{\rho}{3} \left\{ \frac{6x^{5/2}}{5} + \frac{4x^{9/2}}{3} + \frac{4x^{7/2}}{7} - \frac{3x^5}{5} - \frac{6x^7}{7} - \frac{2x^{11}}{11} \right\}_0^1 = \frac{1693\rho}{3465},$$
we obtain $\overline{x} = \dfrac{1693\rho}{3465} \dfrac{420}{359\rho} = \dfrac{6772}{11\,847}$. Since
$$M\overline{y} = \int_0^1 \int_{x^3}^{\sqrt{x}} \int_{-x^2-y^2}^{1+x^2+y^2} y\rho\, dz\, dy\, dx = \rho \int_0^1 \int_{x^3}^{\sqrt{x}} y(1 + 2x^2 + 2y^2)\, dy\, dx$$
$$= \rho \int_0^1 \left\{ \frac{y^2}{2} + x^2 y^2 + \frac{y^4}{2} \right\}_{x^3}^{\sqrt{x}} dx = \frac{\rho}{2} \int_0^1 (x + x^2 + 2x^3 - x^6 - 2x^8 - x^{12})\, dx$$
$$= \frac{\rho}{2} \left\{ \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{2} - \frac{x^7}{7} - \frac{2x^9}{9} - \frac{x^{13}}{13} \right\}_0^1 = \frac{365\rho}{819},$$
we find $\overline{y} = \dfrac{365\rho}{819} \dfrac{420}{359\rho} = \dfrac{7300}{14\,001}$. Since the top and bottom surfaces have exactly the same shape, it follows that $\overline{z} = 1/2$.

13. $$M = \int_{-2}^1 \int_{x^2}^{2-x} \int_0^z \rho\, dy\, dz\, dx = \rho \int_{-2}^1 \int_{x^2}^{2-x} z\, dz\, dx = \rho \int_{-2}^1 \left\{ \frac{z^2}{2} \right\}_{x^2}^{2-x} dx$$
$$= \frac{\rho}{2} \int_{-2}^1 [(2-x)^2 - x^4]\, dx = \frac{\rho}{2} \left\{ -\frac{1}{3}(2-x)^3 - \frac{x^5}{5} \right\}_{-2}^1 = \frac{36\rho}{5}$$
Since
$$M\overline{x} = \int_{-2}^1 \int_{x^2}^{2-x} \int_0^z x\rho\, dy\, dz\, dx = \rho \int_{-2}^1 \int_{x^2}^{2-x} xz\, dz\, dx = \rho \int_{-2}^1 \left\{ \frac{xz^2}{2} \right\}_{x^2}^{2-x} dx$$
$$= \frac{\rho}{2} \int_{-2}^1 (4x - 4x^2 + x^3 - x^5)\, dx = \frac{\rho}{2} \left\{ 2x^2 - \frac{4x^3}{3} + \frac{x^4}{4} - \frac{x^6}{6} \right\}_{-2}^1 = -\frac{45\rho}{8},$$
we obtain $\overline{x} = -\dfrac{45\rho}{8} \dfrac{5}{36\rho} = -\dfrac{25}{32}$. Since
$$M\overline{y} = \int_{-2}^1 \int_{x^2}^{2-x} \int_0^z y\rho\, dy\, dz\, dx = \rho \int_{-2}^1 \int_{x^2}^{2-x} \left\{ \frac{y^2}{2} \right\}_0^z dz\, dx = \frac{\rho}{2} \int_{-2}^1 \int_{x^2}^{2-x} z^2\, dz\, dx$$
$$= \frac{\rho}{2} \int_{-2}^1 \left\{ \frac{z^3}{3} \right\}_{x^2}^{2-x} dx = \frac{\rho}{6} \int_{-2}^1 [(2-x)^3 - x^6]\, dx = \frac{\rho}{6} \left\{ -\frac{1}{4}(2-x)^4 - \frac{x^7}{7} \right\}_{-2}^1 = \frac{423\rho}{56},$$

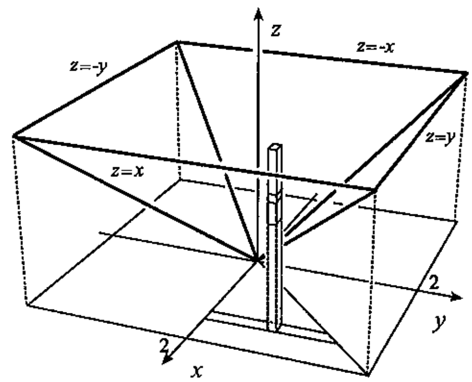
we find that $\overline{y} = \dfrac{423\rho}{56}\dfrac{5}{36\rho} = \dfrac{235}{224}$. Since

$$M\overline{z} = \int_{-2}^{1}\int_{x^2}^{2-x}\int_0^z z\rho\,dy\,dz\,dx = \rho\int_{-2}^{1}\int_{x^2}^{2-x} z^2\,dz\,dx = \frac{423\rho}{28} \quad \text{(see } M\overline{y} \text{ integral)},$$

it follows that $\overline{z} = 235/112$.

14. $M = 8\int_0^2\int_0^x\int_x^2 \rho\,dz\,dy\,dx = 8\rho\int_0^2\int_0^x (2-x)\,dy\,dx$

$$= 8\rho\int_0^2 (2x - x^2)\,dx = 8\rho\left\{x^2 - \frac{x^3}{3}\right\}_0^2 = \frac{32\rho}{3}$$



By symmetry, $\overline{x} = \overline{y} = 0$. Since

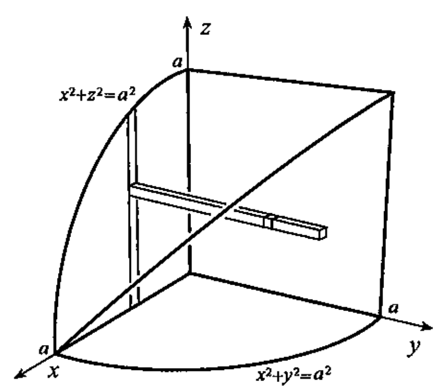
$$\begin{aligned} M\overline{z} &= 8\int_0^2\int_0^x\int_x^2 z\rho\,dz\,dy\,dx \\ &= 8\rho\int_0^2\int_0^x\left\{\frac{z^2}{2}\right\}_x^2 dy\,dx = 4\rho\int_0^2\int_0^x (4-x^2)\,dy\,dx \\ &= 4\rho\int_0^2 (4x - x^3)\,dx = 4\rho\left\{2x^2 - \frac{x^4}{4}\right\}_0^2 = 16\rho, \end{aligned}$$

we find $\overline{z} = 16\rho\dfrac{3}{32\rho} = \dfrac{3}{2}$.

15. $I = \int_{-2}^{3}\int_{-2-x}^{4-x^2}\int_0^2 (x^2+z^2)\rho\,dy\,dz\,dx = \rho\int_{-2}^{3}\int_{-2-x}^{4-x^2} 2(x^2+z^2)\,dz\,dx = 2\rho\int_{-2}^{3}\left\{x^2z + \frac{z^3}{3}\right\}_{-2-x}^{4-x^2} dx$

$$\begin{aligned} &= \frac{2\rho}{3}\int_{-2}^{3}[3x^2(4-x^2) + (4-x^2)^3 - 3x^2(-2-x) - (-2-x)^3]\,dx \\ &= \frac{2\rho}{3}\int_{-2}^{3}(72 + 12x - 24x^2 + 4x^3 + 9x^4 - x^6)\,dx \\ &= \frac{2\rho}{3}\left\{72x + 6x^2 - 8x^3 + x^4 + \frac{9x^5}{5} - \frac{x^7}{7}\right\}_{-2}^{3} = \frac{4750\rho}{21} \end{aligned}$$

16. $I = 8\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2}} (y^2+z^2)\rho\,dy\,dz\,dx$



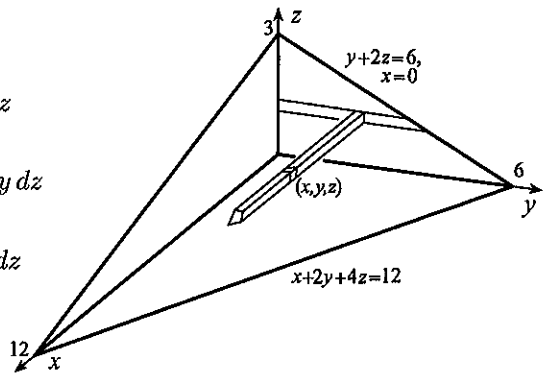
$$\begin{aligned} &= 8\rho\int_0^a\int_0^{\sqrt{a^2-x^2}}\left\{\frac{y^3}{3} + yz^2\right\}_0^{\sqrt{a^2-x^2}} dz\,dx \\ &= \frac{8\rho}{3}\int_0^a\int_0^{\sqrt{a^2-x^2}}[(a^2-x^2)^{3/2} + 3z^2\sqrt{a^2-x^2}]\,dz\,dx \\ &= \frac{8\rho}{3}\int_0^a\left\{(a^2-x^2)^{3/2}z + z^3\sqrt{a^2-x^2}\right\}_0^{\sqrt{a^2-x^2}} dx \\ &= \frac{16\rho}{3}\int_0^a (a^4 - 2a^2x^2 + x^4)\,dx = \frac{16\rho}{3}\left\{a^4x - \frac{2a^2x^3}{3} + \frac{x^5}{5}\right\}_0^a = \frac{128\rho a^5}{45} \end{aligned}$$

17. $I = \int_0^3\int_{x/3}^{2x/3}\int_0^{x+y} (x^2+y^2)\rho\,dz\,dy\,dx = \rho\int_0^3\int_{x/3}^{2x/3}(x^2+y^2)(x+y)\,dy\,dx$

$$= \rho\int_0^3\left\{x^3y + \frac{x^2y^2}{2} + \frac{xy^3}{3} + \frac{y^4}{4}\right\}_{x/3}^{2x/3} dx = \frac{205\rho}{324}\int_0^3 x^4\,dx = \frac{205\rho}{324}\left\{\frac{x^5}{5}\right\}_0^3 = \frac{123\rho}{4}$$

18. The distance from the volume $dz\,dy\,dx$ at point (x,y,z) to the plane $x+y+z=1$ is $|x+y+z-1|/\sqrt{3}$. If we take distances from those points on the origin side of the plane as negative, then the required first moment is

$$\int_0^3\int_0^{6-2z}\int_0^{12-2y-4z}\frac{x+y+z-1}{\sqrt{3}}\rho\,dx\,dy\,dz$$
$$=\frac{\rho}{\sqrt{3}}\int_0^3\int_0^{6-2z}\left\{\frac{(x+y+z-1)^2}{2}\right\}_0^{12-2y-4z}dy\,dz$$
$$=\frac{\rho}{2\sqrt{3}}\int_0^3\int_0^{6-2z}[(11-y-3z)^2-(y+z-1)^2]\,dy\,dz$$
$$=\frac{\rho}{2\sqrt{3}}\int_0^3\left\{\frac{(11-y-3z)^3}{-3}-\frac{(y+z-1)^3}{3}\right\}_0^{6-2z}dz$$
$$=\frac{\rho}{6\sqrt{3}}\int_0^3[-2(5-z)^3+(11-3z)^3+(z-1)^3]\,dz$$
$$=\frac{\rho}{6\sqrt{3}}\left\{\frac{(5-z)^4}{2}-\frac{(11-3z)^4}{12}+\frac{(z-1)^4}{4}\right\}_0^3=51\sqrt{3}\rho.$$



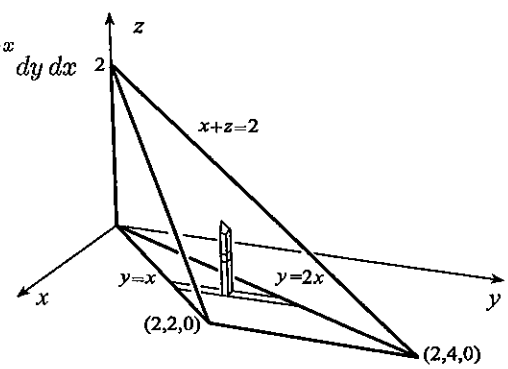
19. The product moment of inertia I_{xy} is

$$I_{xy}=\int_0^{1/a}\int_0^{(1-ax)/b}\int_0^{(1-ax-by)/c}xy\rho\,dz\,dy\,dx=\rho\int_0^{1/a}\int_0^{(1-ax)/b}\left\{xyz\right\}_0^{(1-ax-by)/c}dy\,dx$$
$$=\frac{\rho}{c}\int_0^{1/a}\int_0^{(1-ax)/b}xy(1-ax-by)\,dy\,dx=\frac{\rho}{c}\int_0^{1/a}\left\{\frac{xy^2}{2}-\frac{ax^2y^2}{2}-\frac{bxy^3}{3}\right\}_0^{(1-ax)/b}dx$$
$$=\frac{\rho}{6b^2c}\int_0^{1/a}(x-3ax^2+3a^2x^3-a^3x^4)\,dx=\frac{\rho}{6b^2c}\left\{\frac{x^2}{2}-ax^3+\frac{3a^2x^4}{4}-\frac{a^3x^5}{5}\right\}_0^{1/a}=\frac{\rho}{120a^2b^2c}.$$

Similarly, $I_{yz}=\rho/(120ab^2c^2)$ and $I_{xz}=\rho/(120a^2bc^2)$.

20. The product moment of inertia I_{xy} is

$$I_{xy}=\int_0^2\int_x^{2x}\int_0^{2-x}xy\rho\,dz\,dy\,dx=\rho\int_0^2\int_x^{2x}\left\{xyz\right\}_0^{2-x}dy\,dx$$
$$=\rho\int_0^2\int_x^{2x}xy(2-x)\,dy\,dx$$
$$=\rho\int_0^2\left\{\frac{x(2-x)y^2}{2}\right\}_x^{2x}dx$$
$$=\frac{3\rho}{2}\int_0^2(2x^3-x^4)\,dx=\frac{3\rho}{2}\left\{\frac{x^4}{2}-\frac{x^5}{5}\right\}_0^2=\frac{12\rho}{5}.$$



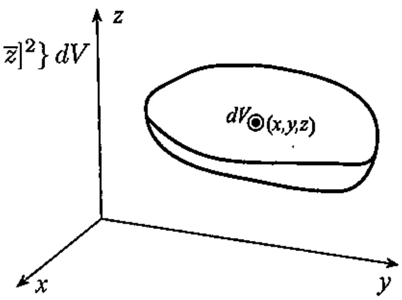
The other two are

$$I_{yz}=\int_0^2\int_x^{2x}\int_0^{2-x}yz\rho\,dz\,dy\,dx=\rho\int_0^2\int_x^{2x}\left\{\frac{yz^2}{2}\right\}_0^{2-x}dy\,dx=\frac{\rho}{2}\int_0^2\int_x^{2x}y(2-x)^2\,dy\,dx$$
$$=\frac{\rho}{2}\int_0^2\left\{\frac{(2-x)^2y^2}{2}\right\}_x^{2x}dx=\frac{3\rho}{4}\int_0^2(4x^2-4x^3+x^4)\,dx=\frac{3\rho}{4}\left\{\frac{4x^3}{3}-x^4+\frac{x^5}{5}\right\}_0^2=\frac{4\rho}{5},$$
$$I_{xz}=\int_0^2\int_x^{2x}\int_0^{2-x}xz\rho\,dz\,dy\,dx=\rho\int_0^2\int_x^{2x}\left\{\frac{xz^2}{2}\right\}_0^{2-x}dy\,dx=\frac{\rho}{2}\int_0^2\int_x^{2x}x(2-x)^2\,dy\,dx$$
$$=\frac{\rho}{2}\int_0^2\left\{x(2-x)^2y\right\}_x^{2x}dx=\frac{\rho}{2}\int_0^2(4x^2-4x^3+x^4)\,dx=\frac{\rho}{2}\left\{\frac{4x^3}{3}-x^4+\frac{x^5}{5}\right\}_0^2=\frac{8\rho}{15}.$$

21. We show that $I_x \le I_y + I_z$.

$$\begin{aligned}I_y + I_z &= \iiint_V (x^2+z^2)\rho\,dV + \iiint_V (x^2+y^2)\rho\,dV = \iiint_V (y^2+z^2)\rho\,dV + 2\iiint_V x^2\rho\,dV\\ &= I_x + 2\iiint_V x^2\rho\,dV \ge I_x, \qquad \text{(since the last integral is positive).}\end{aligned}$$

22. If we orient the volume so that the line is the x-axis, then



$$\begin{aligned}I_x &= \iiint_V (y^2+z^2)\rho\,dV = \iiint_V \{[(y-\overline{y})+\overline{y}]^2 + [(z-\overline{z})+\overline{z}]^2\}\,dV\\ &= \iiint_V [(y-\overline{y})^2 + 2\overline{y}(y-\overline{y}) + \overline{y}^2\\ &\qquad + (z-\overline{z})^2 + 2\overline{z}(z-\overline{z}) + \overline{z}^2]\,dV\\ &= \iiint_V [(y-\overline{y})^2 + (z-\overline{z})^2]\rho\,dV + 2\overline{y}\iiint_V y\rho\,dV\\ &\qquad - \overline{y}^2\iiint_V \rho\,dV + 2\overline{z}\iiint_V z\rho\,dV - \overline{z}^2\iiint_V \rho\,dV\\ &= I_{\overline{x}} + 2\overline{y}(M\overline{y}) - \overline{y}^2(M) + 2\overline{z}(M\overline{z}) - \overline{z}^2(M) = I_{\overline{x}} + M(\overline{y}^2+\overline{z}^2).\end{aligned}$$

23. $M = \displaystyle\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} \rho\,dz\,dy\,dx = \rho\int_0^a\int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dx$

If we set $y = b\sqrt{1-x^2/a^2}\sin\theta$, then $dy = b\sqrt{1-x^2/a^2}\cos\theta\,d\theta$, and

$$\begin{aligned}M &= \rho c\int_0^a\int_0^{\pi/2}\sqrt{\left(1-\frac{x^2}{a^2}\right) - \left(1-\frac{x^2}{a^2}\right)\sin^2\theta}\, b\sqrt{1-\frac{x^2}{a^2}}\cos\theta\,d\theta\,dx\\ &= \rho bc\int_0^a\int_0^{\pi/2}\left(1-\frac{x^2}{a^2}\right)\cos^2\theta\,d\theta\,dx = \frac{\rho bc}{a^2}\int_0^a\int_0^{\pi/2}(a^2-x^2)\left(\frac{1+\cos 2\theta}{2}\right)d\theta\,dx\\ &= \frac{\rho bc}{2a^2}\int_0^a \left\{(a^2-x^2)\left(\theta + \frac{\sin 2\theta}{2}\right)\right\}_0^{\pi/2} dx = \frac{\rho bc}{2a^2}\left(\frac{\pi}{2}\right)\left\{a^2x - \frac{x^3}{3}\right\}_0^a = \frac{\rho\pi abc}{6}.\end{aligned}$$

We now calculate

$$\begin{aligned}M\overline{x} &= \int_0^b\int_0^{a\sqrt{1-y^2/b^2}}\int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} \rho x\,dz\,dx\,dy = \rho\int_0^b\int_0^{a\sqrt{1-y^2/b^2}} cx\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dx\,dy\\ &= \rho c\int_0^b\left\{-\frac{a^2}{3}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{3/2}\right\}_0^{a\sqrt{1-y^2/b^2}} dy = \frac{\rho a^2 c}{3b^3}\int_0^b (b^2-y^2)^{3/2}\,dy.\end{aligned}$$

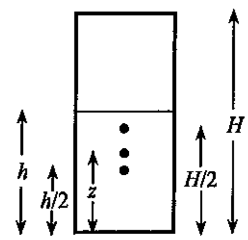
If we set $y = b\sin\theta$, then $dy = b\cos\theta\,d\theta$, and

$$\begin{aligned}M\overline{x} &= \frac{\rho a^2 c}{3b^3}\int_0^{\pi/2} b^3\cos^3\theta\, b\cos\theta\,d\theta = \frac{\rho a^2 bc}{3}\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta\\ &= \frac{\rho a^2 bc}{12}\int_0^{\pi/2}\left[1 + 2\cos 2\theta + \left(\frac{1+\cos 4\theta}{2}\right)\right]d\theta = \frac{\rho a^2 bc}{12}\left\{\frac{3\theta}{2} + \sin 2\theta + \frac{\sin 4\theta}{8}\right\}_0^{\pi/2} = \frac{\rho\pi a^2 bc}{16}.\end{aligned}$$

Thus, $\overline{x} = \dfrac{\rho\pi a^2 bc}{16}\dfrac{6}{\rho\pi abc} = \dfrac{3a}{8}$. Similarly, $\overline{y} = 3b/8$ and $\overline{z} = 3c/8$.

24. Let H be the height of the can and h be the depth of pop. Let m and M be the mass of the pop and can, respectively. Let A be the cross-sectional area of the can and pop and ρ be the density of the pop. If z is the centre of mass of can plus pop, then $(m+M)z = m(h/2) + M(H/2)$. Hence,

$$z = \frac{mh+MH}{2(m+M)} = \frac{(\rho Ah)h+MH}{2(\rho Ah+M)},$$



where $0 < h < H$. For critical points of z as a function of h, we solve

$$0 = \frac{dz}{dh} = \frac{(\rho Ah+M)(2\rho Ah) - (\rho Ah^2+MH)(\rho A)}{2(\rho Ah+M)^2} = \frac{\rho A(\rho Ah^2+2Mh-MH)}{2(\rho Ah+M)^2}.$$

Solutions are $h = \dfrac{-2M \pm \sqrt{4M^2+4\rho AMH}}{2\rho A}$, only the positive root being acceptable. Since $z(0) = z(H) = H/2$, and there is only one critical point, it follows that this critical point must yield a minimum for z. We could substitute the critical value of h into the function $z(h)$ to find the minimum. Instead, notice that if we substitute $\rho Ah = M(H-2h)/h$, then the minimum value is

$$z = \frac{M(H-2h)+MH}{(2M/h)(H-2h)+2M} = \frac{2M(H-h)}{(2M/h)(H-2h+h)} = h;$$

that is, the centre of mass is in the surface of the pop.

25. Since the density of the sphere is half that of water, half the sphere will be above water and half under water. Suppose we take the xy-plane as the surface of the water and $z = -\sqrt{R^2-x^2-y^2}$ as the surface of the hemisphere beneath the surface. The mass of displaced water is $M = (2/3)\pi R^3(1000)$. If $\overline{z}$ is the z-coordinate of the centre of mass of this displaced water, then

$$\begin{aligned} M\overline{z} &= 4\int_0^R\int_0^{\sqrt{R^2-x^2}}\int_{-\sqrt{R^2-x^2-y^2}}^0 z(1000)\,dz\,dy\,dx = 4000\int_0^R\int_0^{\sqrt{R^2-x^2}} \left\{\frac{z^2}{2}\right\}_{-\sqrt{R^2-x^2-y^2}}^0 dy\,dx \\ &= 2000\int_0^R\int_0^{\sqrt{R^2-x^2}} -(R^2-x^2-y^2)\,dy\,dx. \end{aligned}$$

If we transform this double iterated integral to polar coordinates,

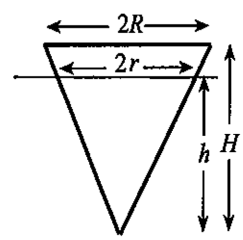
$$M\overline{z} = -2000\int_0^{\pi/2}\int_0^R (R^2-r^2)r\,dr\,d\theta = -2000\int_0^{\pi/2}\left\{\frac{R^2r^2}{2} - \frac{r^4}{4}\right\}_0^R d\theta = -500R^4\Big\{\theta\Big\}_0^{\pi/2} = -250\pi R^4.$$

Hence, $\overline{z} = -250\pi R^4\left(\dfrac{3}{2000\pi R^3}\right) = -\dfrac{3R}{8}$. The centre of buoyancy is $3R/8$ below the surface.

26. Suppose we let r and h be the radius and height of that part of the cone under water. For buoyancy, the weight of the water displaced must be equal to the weight of the cone,

$\frac{1}{3}\pi r^2h(1000)g = \frac{1}{3}\pi R^2H(800)g \implies r^2h = \frac{4}{5}R^2H.$

By similar triangles, $r/h = R/H \implies r = Rh/H$, and therefore

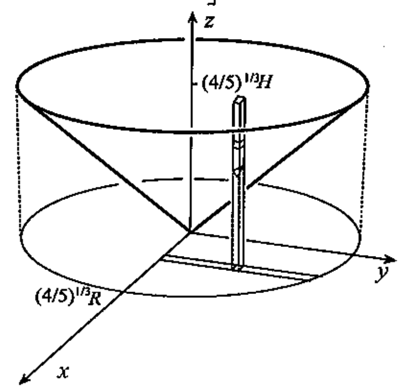


$$\left(\frac{Rh}{H}\right)^2 h = \frac{4}{5}R^2H \implies h = \left(\frac{4}{5}\right)^{1/3} H \quad\text{and}\quad r = \left(\frac{4}{5}\right)^{1/3} R.$$

To find the centre of buoyancy we require the centre of mass of a right-circular cone of water with radius $r = (4/5)^{1/3}R$ and height $h = (4/5)^{1/3}H$. Such a cone with apex at the origin has equation $z = (H/R)\sqrt{x^2+y^2}$. The mass of the displaced water is $M = (1000/3)\pi R^2 H$ kg. If $\overline{z}$ is the z-coordinate of its centre of mass, then

$$\begin{aligned}
M\overline{z} &= 4\int_0^{(4/5)^{1/3}R}\int_0^{\sqrt{(4/5)^{2/3}R^2-x^2}}\int_{(H/R)\sqrt{x^2+y^2}}^{(4/5)^{1/3}H} z(1000)\,dz\,dy\,dx \\
&= 4000\int_0^{(4/5)^{1/3}R}\int_0^{\sqrt{(4/5)^{2/3}R^2-x^2}} \left\{\frac{z^2}{2}\right\}_{(H/R)\sqrt{x^2+y^2}}^{(4/5)^{1/3}H} dy\,dx \\
&= 2000\int_0^{(4/5)^{1/3}R}\int_0^{\sqrt{(4/5)^{2/3}R^2-x^2}} \left[\left(\frac{4}{5}\right)^{2/3}H^2 - \frac{H^2}{R^2}(x^2+y^2)\right] dy\,dx.
\end{aligned}$$

If we transform this double iterated integral to polar coordinates,

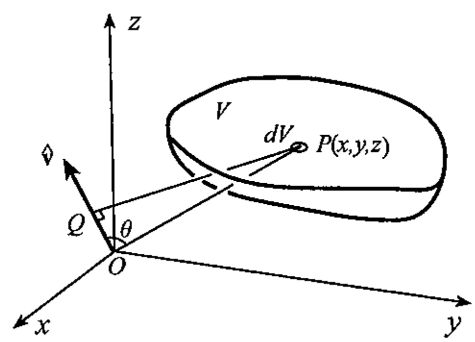


$$\begin{aligned}
M\overline{z} &= 2000\int_0^{\pi/2}\int_0^{(4/5)^{1/3}R}\left[\left(\frac{4}{5}\right)^{2/3}H^2 - \frac{H^2r^2}{R^2}\right] r\,dr\,d\theta \\
&= 2000\int_0^{\pi/2}\left\{\left(\frac{4}{5}\right)^{2/3}\frac{H^2r^2}{2} - \frac{H^2r^4}{4R^2}\right\}_0^{(4/5)^{1/3}R} d\theta \\
&= 500(4/5)^{4/3}H^2R^2\Big\{\theta\Big\}_0^{\pi/2} = 250(4/5)^{4/3}\pi H^2R^2.
\end{aligned}$$

Thus, $\overline{z} = 250(4/5)^{4/3}\pi H^2R^2 \dfrac{3}{1000\pi R^2 H} = \dfrac{3(4/5)^{4/3}H}{4}$. The centre of buoyancy of the floating cone is therefore $\left(\dfrac{4}{5}\right)^{1/3} H - \dfrac{3}{4}\left(\dfrac{4}{5}\right)^{4/3} H = \dfrac{2}{5}\left(\dfrac{4}{5}\right)^{1/3} H$ below the surface.

27. If $\mathbf{PQ}$ is the line from (x,y,z) perpendicular to $\hat{\mathbf{v}}$ at Q, then

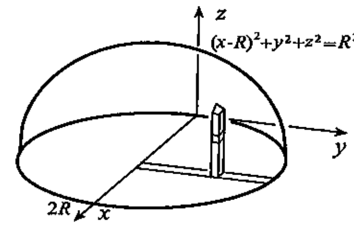
$$\begin{aligned}
|\mathbf{PQ}| &= |\mathbf{OP}|\sin\theta = |\mathbf{OP}||\hat{\mathbf{v}}|\sin\theta = |\mathbf{OP}\times\hat{\mathbf{v}}| \\
&= \left\|\begin{matrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x & y & z \\ v_x & v_y & v_z \end{matrix}\right\| \\
&= \sqrt{(yv_z - zv_y)^2 + (zv_x - xv_z)^2 + (xv_y - yv_x)^2}.
\end{aligned}$$



The moment of inertia of the mass about the line containing $\hat{\mathbf{v}}$ is

$$\begin{aligned}
I &= \iiint_V [yv_z - zv_y)^2 + (zv_x - xv_z)^2 + (xv_y - yv_x)^2]\rho\,dV \\
&= \iiint_V (y^2v_z^2 - 2yzv_yv_z + z^2v_y^2 + z^2v_x^2 - 2xzv_xv_z + x^2v_z^2 + x^2v_y^2 - 2xyv_xv_y + y^2v_x^2)\rho\,dV \\
&= v_x^2\iiint_V (y^2+z^2)\rho\,dV + v_y^2\iiint_V (x^2+z^2)\rho\,dV + v_z^2\iiint_V (x^2+y^2)\rho\,dV \\
&\qquad - 2v_xv_y\iiint_v xy\,\rho\,dV - 2v_yv_z\iiint_V yz\,\rho\,dV - 2v_zv_x\iiint_V xz\,\rho\,dV \\
&= v_x^2I_x + v_y^2I_y + v_z^2I_z - 2v_xy_yI_{xy} - 2v_yv_zI_{yz} - 2v_zv_xI_{xz}.
\end{aligned}$$

28. We choose the sphere $(x-R)^2+y^2+z^2=R^2$ and the z-axis. Then



$$I = 4\int_0^{2R}\int_0^{\sqrt{R^2-(x-R)^2}}\int_0^{\sqrt{R^2-(x-R)^2-y^2}} \rho(x^2+y^2)\,dz\,dy\,dx$$
$$= 4\rho\int_0^{2R}\int_0^{\sqrt{R^2-(x-R)^2}} (x^2+y^2)\sqrt{R^2-(x-R)^2-y^2}\,dy\,dx$$

In the inner integral, we set $a^2 = R^2-(x-R)^2$, in which case

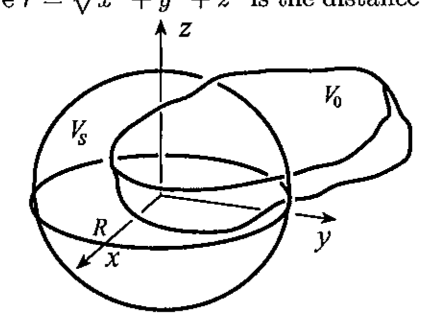
$$\int_0^{\sqrt{R^2-(x-R)^2}} (x^2+y^2)\sqrt{R^2-(x-R)^2-y^2}\,dy = \int_0^a (x^2+y^2)\sqrt{a^2-y^2}\,dy.$$

If we set $y = a\sin\theta$, then $dy = a\cos\theta\,d\theta$,

$$\int_0^{\sqrt{R^2-(x-R)^2}} (x^2+y^2)\sqrt{R^2-(x-R)^2-y^2}\,dy = \int_0^{\pi/2} (x^2+a^2\sin^2\theta)a\cos\theta\, a\cos\theta\,d\theta$$
$$= a^2\int_0^{\pi/2}\left[\frac{x^2(1+\cos 2\theta)}{2}+\frac{a^2}{4}\left(\frac{1-\cos 4\theta}{2}\right)\right]d\theta = a^2\left\{\frac{x^2}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)+\frac{a^2}{8}\theta-\frac{a^2}{32}\sin 4\theta\right\}_0^{\pi/2}$$
$$= \frac{\pi a^2}{16}(a^2+4x^2) = \frac{\pi}{16}[R^2-(x-R)^2][R^2-(x-R)^2+4x^2].$$

$$\text{Thus,}\quad I = \frac{\rho\pi}{4}\int_0^{2R}[2Rx-x^2)(2Rx+3x^2)]\,dx = \frac{\rho\pi}{4}\int_0^{2R}(4R^2x^2+4Rx^3-3x^4)\,dx$$
$$= \frac{\rho\pi}{4}\left\{\frac{4R^2x^3}{3}+Rx^4-\frac{3x^5}{5}\right\}_0^{2R} = \frac{28\rho\pi R^5}{15}.$$

29. $I_x+I_y+I_z = \iiint_V 2(x^2+y^2+z^2)\rho\,dV = 2\iiint_V r^2\rho\,dV$ where $r=\sqrt{x^2+y^2+z^2}$ is the distance from the origin to the element of mass ρdV at point (x,y,z). Let R be the radius of the sphere and $\overline{V}$ the region that it occupies. Let V_0 represent the region occupied by that part of the object outside the sphere and V_S represent the region occupied by that part of the sphere outside the object. The masses M_0 and M_S of these regions must be the same. Since $V=\overline{V}+V_0-V_S$, it follows that



$$\iiint_V r^2\rho\,dV = \iiint_{\overline{V}} r^2\rho\,dV + \iiint_{V_0} r^2\rho\,dV - \iiint_{V_S} r^2\rho\,dV.$$

In V_0, $r>R$, and in V_S, $r<R$, so that

$$\iiint_V r^2\rho\,dV \ge \iiint_{\overline{V}} r^2\rho\,dV + \iiint_{V_0} R^2\rho\,dV - \iiint_{V_S} R^2\rho\,dV$$
$$= \iiint_{\overline{V}} r^2\rho\,dV + R^2M_0 - R^2M_S = \iiint_{\overline{V}} r^2\rho\,dV.$$

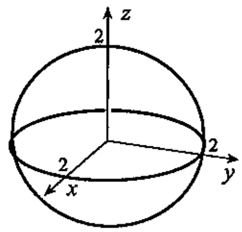
Thus,

$$I_x+I_y+I_z = 2\iiint_V r^2\rho\,dV \ge 2\iiint_{\overline{V}}(x^2+y^2+z^2)\rho\,dV,$$

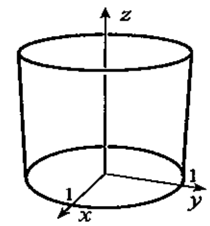
and the right side is the sum of the moments of inertia of the sphere about the axes.

EXERCISES 13.11

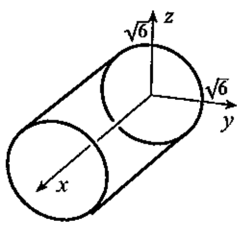
1. The equation is $r^2 + z^2 = 4$.
It is symmetric about the z-axis.



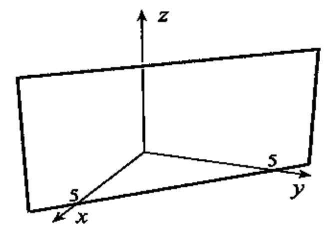
2. The equation is $r = 1$.
It is symmetric about the z-axis.



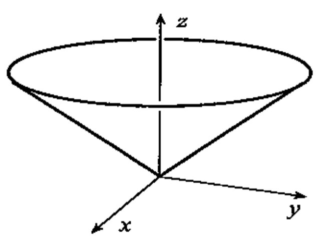
3. The equation is $r^2 \sin^2\theta + z^2 = 6$.
It is not symmetric about the z-axis.



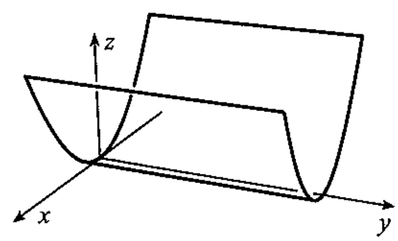
4. The equation is $r\cos\theta + r\sin\theta = 5$.
It is not symmetric about the z-axis.



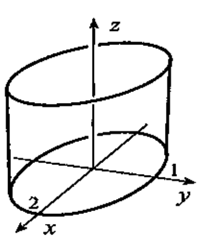
5. The equation is $z = 2r$.
It is symmetric about the z-axis.



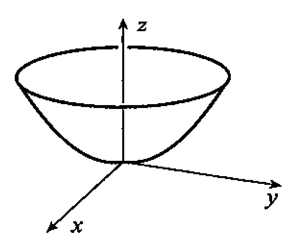
6. The equation is $z = r^2\cos^2\theta$.
It is not symmetric about the z-axis.



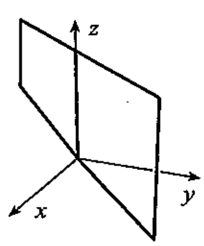
7. The equation is $r^2 = 4/(\cos^2\theta + 4\sin^2\theta)$.
It is not symmetric about the z-axis.



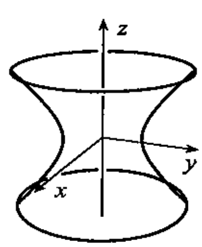
8. The equation is $4z = r^2$.
It is symmetric about the z-axis.



9. The equations are $\theta = \pi/4$ and $\theta = 5\pi/4$.
It is symmetric about the z-axis.



10. The equation is $r^2 = 1 + z^2$.
It is symmetric about the z-axis.



11. $V = 4\int_0^{\pi/2}\int_0^2\int_0^r r\,dz\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^2 \left\{rz\right\}_0^r dr\,d\theta = 4\int_0^{\pi/2}\int_0^2 r^2\,dr\,d\theta$

$= 4\int_0^{\pi/2}\left\{\frac{r^3}{3}\right\}_0^2 d\theta = \frac{32}{3}\left\{\theta\right\}_0^{\pi/2} = \frac{16\pi}{3}$

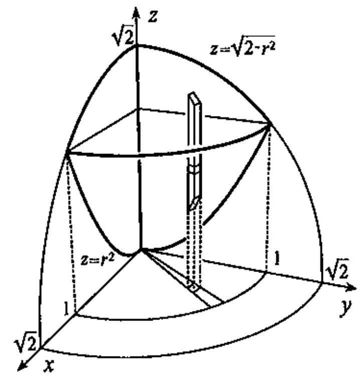
12. We quadruple the volume in the first octant.

$$V = 4\int_0^{\pi/2}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\int_0^1 (r\sqrt{2-r^2} - r^3)\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\left\{-\frac{1}{3}(2-r^2)^{3/2} - \frac{r^4}{4}\right\}_0^1 d\theta$$

$$= \frac{8\sqrt{2}-7}{3}\left\{\theta\right\}_0^{\pi/2} = \frac{(8\sqrt{2}-7)\pi}{6}$$



13. $V = 4\int_0^{\pi/2}\int_0^1\int_0^{r^2\sin\theta\cos\theta} r\,dz\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^1 \left\{rz\right\}_0^{r^2\sin\theta\cos\theta} dr\,d\theta = 4\int_0^{\pi/2}\int_0^1 r^3\sin\theta\cos\theta\,dr\,d\theta$

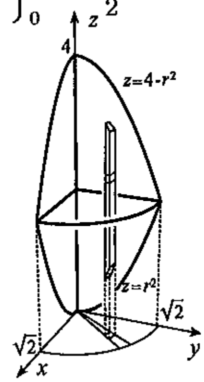
$= 4\int_0^{\pi/2}\left\{\frac{r^4}{4}\sin\theta\cos\theta\right\}_0^1 d\theta = \int_0^{\pi/2}\sin\theta\cos\theta\,d\theta = \left\{\frac{1}{2}\sin^2\theta\right\}_0^{\pi/2} = \frac{1}{2}$

14. We quadruple the volume in the first octant.

$$V = 4\int_0^{\pi/2}\int_0^{\sqrt{2}}\int_{r^2}^{4-r^2} r\,dz\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\int_0^{\sqrt{2}} r(4-2r^2)\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\left\{2r^2 - \frac{r^4}{2}\right\}_0^{\sqrt{2}} d\theta = 8\left\{\theta\right\}_0^{\pi/2} = 4\pi$$



15. $V = \int_{-\pi}^{\pi}\int_0^1\int_0^{2-r\cos\theta-r\sin\theta} r\,dz\,dr\,d\theta = \int_{-\pi}^{\pi}\int_0^1 r(2-r\cos\theta-r\sin\theta)\,dr\,d\theta$

$= \int_{-\pi}^{\pi}\left\{r^2 - \frac{r^3}{3}\cos\theta - \frac{r^3}{3}\sin\theta\right\}_0^1 d\theta = \frac{1}{3}\int_{-\pi}^{\pi}(3-\cos\theta-\sin\theta)\,d\theta$

$= \frac{1}{3}\left\{3\theta - \sin\theta + \cos\theta\right\}_{-\pi}^{\pi} = 2\pi$

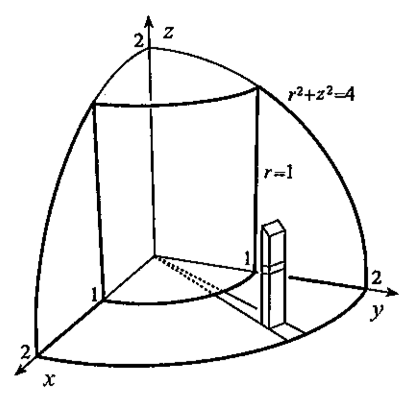
16. We multiply the first octant volume by eight.

$$V = 8\int_0^{\pi/2}\int_1^2\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$$

$$= 8\int_0^{\pi/2}\int_1^2 r\sqrt{4-r^2}\,dr\,d\theta$$

$$= 8\int_0^{\pi/2}\left\{-\frac{1}{3}(4-r^2)^{3/2}\right\}_1^2 d\theta$$

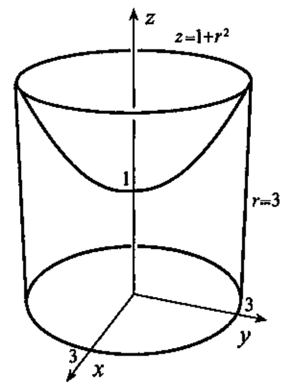
$$= 8\sqrt{3}\left\{\theta\right\}_0^{\pi/2} = 4\sqrt{3}\pi$$



17. By symmetry, $\overline{x} = \overline{y} = 0$ for the hemisphere bounded by $z = \sqrt{R^2 - x^2 - y^2}$ and $z = 0$. Since $M = (2/3)\pi R^3\rho$, where ρ is the density, and

$$M\overline{z} = 4\int_0^{\pi/2}\int_0^R\int_0^{\sqrt{R^2-r^2}} z\rho r\,dz\,dr\,d\theta = 4\rho\int_0^{\pi/2}\int_0^R\left\{\frac{rz^2}{2}\right\}_0^{\sqrt{R^2-r^2}} dr\,d\theta = 2\rho\int_0^{\pi/2}\int_0^R r(R^2-r^2)\,dr\,d\theta$$
$$= 2\rho\int_0^{\pi/2}\left\{\frac{R^2r^2}{2} - \frac{r^4}{4}\right\}_0^R d\theta = \frac{\rho R^4}{2}\left\{\theta\right\}_0^{\pi/2} = \frac{\rho\pi R^4}{4},$$

it follows that $\overline{z} = \dfrac{\rho\pi R^4}{4}\dfrac{3}{2\pi R^3\rho} = \dfrac{3R}{8}$.



18. The six triple iterated integrals are

$$\int_{-\pi}^{\pi}\int_0^3\int_0^{1+r^2} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta,$$

$$\int_0^3\int_{-\pi}^{\pi}\int_0^{1+r^2} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,d\theta\,dr,$$

$$\int_0^3\int_0^{1+r^2}\int_{-\pi}^{\pi} f(r\cos\theta, r\sin\theta, z)\,r\,d\theta\,dz\,dr,$$

$$\int_0^1\int_0^3\int_{-\pi}^{\pi} f(r\cos\theta, r\sin\theta, z)\,r\,d\theta\,dr\,dz + \int_1^{10}\int_{\sqrt{z-1}}^3\int_{-\pi}^{\pi} f(r\cos\theta, r\sin\theta, z)\,r\,d\theta\,dr\,dz,$$

$$\int_{-\pi}^{\pi}\int_0^1\int_0^3 f(r\cos\theta, r\sin\theta, z)\,r\,dr\,dz\,d\theta + \int_{-\pi}^{\pi}\int_1^{10}\int_{\sqrt{z-1}}^3 f(r\cos\theta, r\sin\theta, z)\,r\,dr\,dz\,d\theta,$$

$$\int_0^1\int_{-\pi}^{\pi}\int_0^3 f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz + \int_1^{10}\int_{-\pi}^{\pi}\int_{\sqrt{z-1}}^3 f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz$$

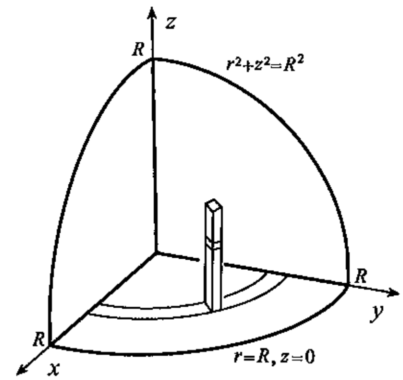
19. For the cylinder $x^2 + y^2 \le R^2$, $0 \le z \le h$,

(a) $$I = 4\int_0^{\pi/2}\int_0^R\int_0^h (x^2+y^2)\rho r\,dz\,dr\,d\theta = 4\rho\int_0^{\pi/2}\int_0^R\left\{r^3z\right\}_0^h dr\,d\theta$$
$$= 4\rho h\int_0^{\pi/2}\left\{\frac{r^4}{4}\right\}_0^R d\theta = \rho hR^4\left\{\theta\right\}_0^{\pi/2} = \frac{\rho\pi R^4 h}{2}$$

(b) The moment of inertia about the x-axis is

$$I = 4\int_0^{\pi/2}\int_0^R\int_0^h (y^2+z^2)\rho r\,dz\,dr\,d\theta = 4\rho\int_0^{\pi/2}\int_0^R\left\{r\left(zr^2\sin^2\theta + \frac{z^3}{3}\right)\right\}_0^h dr\,d\theta$$
$$= \frac{4\rho}{3}\int_0^{\pi/2}\int_0^R (3hr^3\sin^2\theta + h^3r)\,dr\,d\theta = \frac{4\rho}{3}\int_0^{\pi/2}\left\{\frac{3hr^4}{4}\sin^2\theta + \frac{h^3r^2}{2}\right\}_0^R d\theta$$
$$= \frac{\rho}{3}\int_0^{\pi/2}(3hR^4\sin^2\theta + 2h^3R^2)\,d\theta = \frac{\rho}{3}\int_0^{\pi/2}\left[2h^3R^2 + 3hR^4\left(\frac{1-\cos 2\theta}{2}\right)\right]d\theta$$
$$= \frac{\rho}{3}\left\{2h^3R^2\theta + 3hR^4\left(\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right)\right\}_0^{\pi/2} = \frac{\rho\pi hR^2(4h^2+3R^2)}{12}.$$

20. We multiply the moment of inertia about the z-axis of that part in the first octant by eight.



$$\begin{aligned} I_z &= 8\int_0^R\int_0^{\pi/2}\int_0^{\sqrt{R^2-r^2}} r^2\rho\, r\,dz\,d\theta\,dr \\ &= 8\rho\int_0^R\int_0^{\pi/2} r^3\sqrt{R^2-r^2}\,d\theta\,dr \\ &= 4\pi\rho\int_0^R r^3\sqrt{R^2-r^2}\,dr \end{aligned}$$

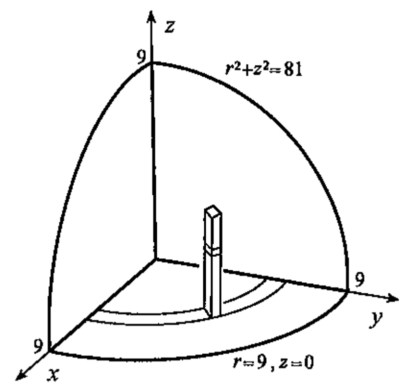
If we set $u = R^2 - r^2$, then $du = -2r\,dr$, and

$$\begin{aligned} I_z &= 4\pi\rho\int_{R^2}^0 (R^2-u)\sqrt{u}\left(-\frac{du}{2}\right) \\ &= 2\pi\rho\left\{\frac{2}{3}R^2u^{3/2} - \frac{2}{5}u^{5/2}\right\}_0^{R^2} = \frac{8\pi\rho R^5}{15}. \end{aligned}$$

21. The limits define the first octant volume under the cone $z = \sqrt{x^2+y^2}$ and inside the cylinder $x^2+y^2 = 9$. The value of the triple integral is therefore

$$\int_0^{\pi/2}\int_0^3\int_0^r r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^3 r^2\,dr\,d\theta = \int_0^{\pi/2}\left\{\frac{r^3}{3}\right\}_0^3 d\theta = 9\left\{\theta\right\}_0^{\pi/2} = \frac{9\pi}{2}.$$

22. The limits define the first octant volume inside the sphere $x^2+y^2+z^2 = 81$. The value of the triple iterated integral is therefore given by



$$\begin{aligned} &\int_0^9\int_0^{\pi/2}\int_0^{\sqrt{81-r^2}} \frac{1}{r}r\,dz\,d\theta\,dr \\ &\quad = \int_0^9\int_0^{\pi/2}\sqrt{81-r^2}\,d\theta\,dr = \frac{\pi}{2}\int_0^9\sqrt{81-r^2}\,dr. \end{aligned}$$

If we set $r = 9\sin\phi$, then $dr = 9\cos\phi\,d\phi$, and

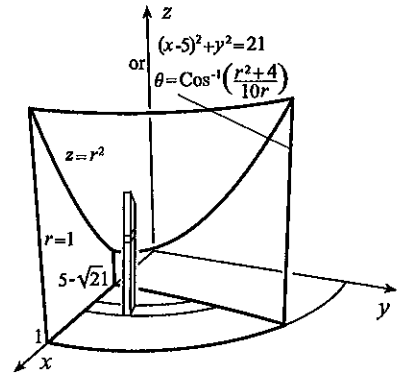
$$\begin{aligned} \int_0^9\int_0^{\pi/2}\int_0^{\sqrt{81-r^2}} \frac{1}{r}r\,dz\,d\theta\,dr &= \frac{\pi}{2}\int_0^{\pi/2} 9\cos\phi\, 9\cos\phi\,d\phi \\ &= \frac{81\pi}{2}\int_0^{\pi/2}\left(\frac{1+\cos 2\phi}{2}\right)d\phi = \frac{81\pi}{4}\left\{\phi + \frac{\sin 2\phi}{2}\right\}_0^{\pi/2} = \frac{81\pi^2}{8}. \end{aligned}$$

23. The limits define the first octant volume under $z = y + x^2$ and inside the cylinder $x^2+y^2 = 4y$. The value of the triple iterated integral is therefore given by

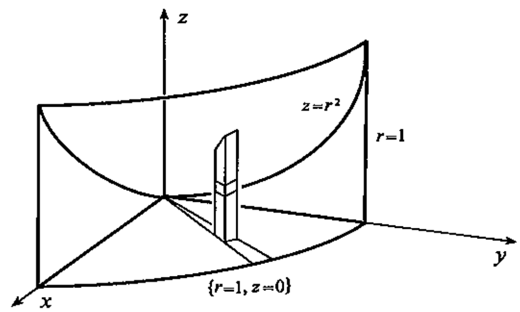
$$\begin{aligned} &\int_0^{\pi/2}\int_0^{4\sin\theta}\int_0^{r\sin\theta + r^2\cos^2\theta} r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^{4\sin\theta} r(r\sin\theta + r^2\cos^2\theta)\,dr\,d\theta \\ &= \int_0^{\pi/2}\left\{\frac{r^3}{3}\sin\theta + \frac{r^4}{4}\cos^2\theta\right\}_0^{4\sin\theta} d\theta = \frac{1}{12}\int_0^{\pi/2}[4\sin\theta(4\sin\theta)^3 + 3\cos^2\theta(4\sin\theta)^4]\,d\theta \\ &= \frac{64}{3}\int_0^{\pi/2}(\sin^4\theta + 3\cos^2\theta\sin^4\theta)\,d\theta = \frac{64}{3}\int_0^{\pi/2}\left[\left(\frac{1-\cos 2\theta}{2}\right)^2 + 3\left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{1-\cos 2\theta}{2}\right)^2\right]d\theta \\ &= \frac{8}{3}\int_0^{\pi/2}[2(1-2\cos 2\theta+\cos^2 2\theta) + 3(1-2\cos 2\theta+\cos^2 2\theta) + 3\cos 2\theta(1-2\cos 2\theta+\cos^2 2\theta)]\,d\theta \\ &= \frac{8}{3}\int_0^{\pi/2}\left[5 - 7\cos 2\theta - \left(\frac{1+\cos 4\theta}{2}\right) + 3\cos 2\theta(1-\sin^2 2\theta)\right]d\theta \\ &= \frac{8}{3}\left\{\frac{9\theta}{2} - 2\sin 2\theta - \frac{1}{8}\sin 4\theta - \sin^3 2\theta\right\}_0^{\pi/2} = 6\pi \end{aligned}$$

24. The limits define the volume under $z = x^2 + y^2$, above $z = 0$, and bounded on the sides by the cylinders $x^2 + y^2 = 1$ and $(x-5)^2 + y^2 = 21$, and the xz-plane. The value of the triple iterated integral is therefore given by

$$\begin{aligned}\int_{5-\sqrt{21}}^{1}\int_{0}^{\theta(r)}\int_{0}^{r^2} r\sin\theta\, r\,dz\,d\theta\,dr &= \int_{5-\sqrt{21}}^{1}\int_{0}^{\theta(r)} r^4\sin\theta\,d\theta\,dr \\ &= \int_{5-\sqrt{21}}^{1}\left\{-r^4\cos\theta\right\}_0^{\theta(r)} dr \\ &= \int_{5-\sqrt{21}}^{1}\left(r^4 - \frac{r^5}{10} - \frac{2r^3}{5}\right)dr \\ &= \left\{\frac{r^5}{5} - \frac{r^6}{60} - \frac{r^4}{10}\right\}_{5-\sqrt{21}}^{1} = 0.084.\end{aligned}$$



25. The limits determine the volume in the first octant bounded by the paraboloid $z = x^2 + y^2$ and the right circular cylinder $x^2 + y^2 = 1$. If we use a triple iterated integral with respect to z, r, and θ, then



$$\begin{aligned}I &= \int_0^{\pi/2}\int_0^1\int_0^{r^2} r^2\sin^2\theta\, r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 r^5\sin^2\theta\,dr\,d\theta = \int_0^{\pi/2}\frac{1}{6}\sin^2\theta\,d\theta \\ &= \frac{1}{6}\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right)d\theta = \frac{1}{12}\left\{\theta - \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = \frac{\pi}{24}.\end{aligned}$$

26. The moment of inertia of the upper leg about a line through its centre of mass G_U is

$$I_{G_U} = (0.137)(73)\left(\frac{0.07^2}{4} + \frac{0.45^2}{12}\right) = 0.181\ \text{kg}\cdot\text{m}^2.$$

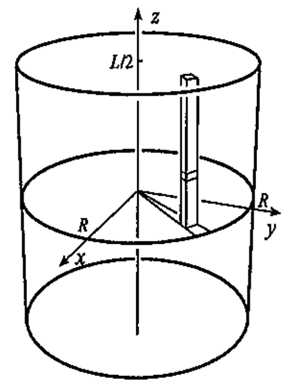
Similarly,

$$I_{G_L} = (0.06)(73)\left(\frac{0.05^2}{4} + \frac{0.5^2}{12}\right) = 0.094\ \text{kg}\cdot\text{m}^2.$$

Since $HG_U = 0.225$ and $HG_L = \sqrt{0.45^2 + 0.25^2 - 2(0.45)(0.25)\cos(\pi/3)} = 0.391$, the moment of inertia of the leg about the hip is

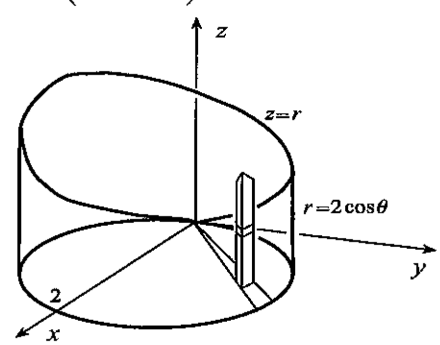
$$[0.181 + (0.137)(73)(0.225)^2] + [0.094 + (0.06)(73)(0.391)^2] = 1.45\ \text{kg}\cdot\text{m}^2.$$

27. The moment of inertia about the x-axis is eight times that in the first octant.



$$I = 8\int_0^{\pi/2}\int_0^R\int_0^{L/2}(y^2+z^2)\rho r\,dz\,dr\,d\theta$$
$$= 8\rho\int_0^{\pi/2}\int_0^R\int_0^{L/2}(r^2\sin^2\theta+z^2)r\,dz\,dr\,d\theta$$
$$= 8\rho\int_0^{\pi/2}\int_0^R\left\{r^3\sin^2\theta\,z+\frac{rz^3}{3}\right\}_0^{L/2}dr\,d\theta$$
$$= \frac{\rho L}{3}\int_0^{\pi/2}\int_0^R(12r^3\sin^2\theta+L^2r)\,dr\,d\theta$$
$$= \frac{\rho L}{3}\int_0^{\pi/2}\left\{3r^4\sin^2\theta+\frac{L^2r^2}{2}\right\}_0^R d\theta$$
$$= \frac{\rho LR^2}{6}\int_0^{\pi/2}(6R^2\sin^2\theta+L^2)\,d\theta = \frac{\rho LR^2}{6}\int_0^{\pi/2}[3R^2(1-\cos 2\theta)+L^2]\,d\theta$$
$$= \frac{\rho LR^2}{6}\left\{3R^2\left(\theta-\frac{\sin 2\theta}{2}\right)+L^2\theta\right\}_0^{\pi/2} = \frac{\rho\pi LR^2(3R^2+L^2)}{12} = m\left(\frac{R^2}{4}+\frac{L^2}{12}\right).$$

28. $$M = 2\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^r \rho r\,dz\,dr\,d\theta = 2\rho\int_0^{\pi/2}\int_0^{2\cos\theta}r^2\,dr\,d\theta$$



$$= 2\rho\int_0^{\pi/2}\left\{\frac{r^3}{3}\right\}_0^{2\cos\theta}d\theta = \frac{16\rho}{3}\int_0^{\pi/2}\cos^3\theta\,d\theta$$
$$= \frac{16\rho}{3}\int_0^{\pi/2}\cos\theta(1-\sin^2\theta)\,d\theta$$
$$= \frac{16\rho}{3}\left\{\sin\theta-\frac{1}{3}\sin^3\theta\right\}_0^{\pi/2} = \frac{32\rho}{9}$$

By symmetry, $\overline{y}=0$. Since

$$M\overline{x} = 2\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^r r\cos\theta\rho r\,dz\,dr\,d\theta = 2\rho\int_0^{\pi/2}\int_0^{2\cos\theta}r^3\cos\theta\,dr\,d\theta$$
$$= 2\rho\int_0^{\pi/2}\left\{\frac{r^4}{4}\cos\theta\right\}_0^{2\cos\theta}d\theta = 8\rho\int_0^{\pi/2}\cos^5\theta\,d\theta = 8\rho\int_0^{\pi/2}\cos\theta(1-\sin^2\theta)^2\,d\theta$$
$$= 8\rho\int_0^{\pi/2}\cos\theta(1-2\sin^2\theta+\sin^4\theta)\,d\theta = 8\rho\left\{\sin\theta-\frac{2}{3}\sin^3\theta+\frac{1}{5}\sin^5\theta\right\}_0^{\pi/2} = \frac{64\rho}{15},$$

it follows that $\overline{x} = \dfrac{64\rho}{15}\dfrac{9}{32\rho} = \dfrac{6}{5}$. Since

$$M\overline{z} = 2\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^r z\rho r\,dz\,dr\,d\theta = 2\rho\int_0^{\pi/2}\int_0^{2\cos\theta}\left\{\frac{rz^2}{2}\right\}_0^r dr\,d\theta$$
$$= \rho\int_0^{\pi/2}\int_0^{2\cos\theta}r^3\,dr\,d\theta = \rho\int_0^{\pi/2}\left\{\frac{r^4}{4}\right\}_0^{2\cos\theta}d\theta = 4\rho\int_0^{\pi/2}\cos^4\theta\,d\theta$$
$$= 4\rho\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta = \rho\int_0^{\pi/2}\left(1+2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)d\theta$$
$$= \rho\left\{\frac{3\theta}{2}+\sin 2\theta+\frac{1}{8}\sin 4\theta\right\}_0^{\pi/2} = \frac{3\pi\rho}{4},$$

we obtain $\overline{z} = \dfrac{3\pi\rho}{4}\dfrac{9}{32\rho} = \dfrac{27\pi}{128}$.

29. First we find the volume interior to the cylinder $x^2+y^2=a^2$ and the sphere $x^2+y^2+z^2=b^2$,

$$V_1 = 8\int_0^{\pi/2}\int_0^a\int_0^{\sqrt{b^2-r^2}} r\,dz\,dr\,d\theta = 8\int_0^{\pi/2}\int_0^a r\sqrt{b^2-r^2}\,dr\,d\theta$$
$$= 8\int_0^{\pi/2}\left\{-\frac{1}{3}(b^2-r^2)^{3/2}\right\}_0^a d\theta = -\frac{8}{3}[(b^2-a^2)^{3/2}-b^3]\left\{\theta\right\}_0^{\pi/2} = \frac{4\pi[b^3-(b^2-a^2)^{3/2}]}{3}.$$

We now find the volume common to both cylinders $x^2+y^2=a^2$ and $y^2+z^2=a^2$,

$$V_2 = 8\int_0^a\int_0^{\sqrt{a^2-y^2}}\int_0^{\sqrt{a^2-y^2}} dz\,dx\,dy = 8\int_0^a\int_0^{\sqrt{a^2-y^2}}\sqrt{a^2-y^2}\,dx\,dy$$
$$= 8\int_0^a\left\{x\sqrt{a^2-y^2}\right\}_0^{\sqrt{a^2-y^2}} dy = 8\int_0^a (a^2-y^2)\,dy = 8\left\{a^2y-\frac{y^3}{3}\right\}_0^a = \frac{16a^3}{3}.$$

It now follows that the volume for the casting is

$$V = (\text{volume of sphere}) - 2V_1 + V_2 = \frac{4}{3}\pi b^3 - \frac{8\pi}{3}[b^3-(b^2-a^2)^{3/2}] + \frac{16a^3}{3} = \frac{16a^3}{3} + \frac{4\pi}{3}[2(b^2-a^2)^{3/2}-b^3].$$

30. The volume bounded by the planes and cylinder is

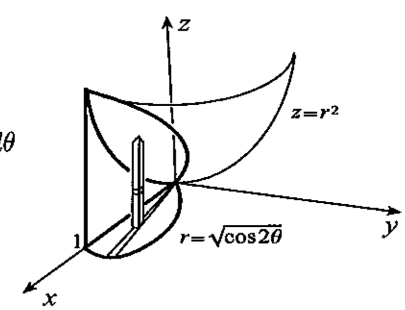
$$V = \int_0^{2\pi}\int_0^R\int_{my}^{my+h} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^R r(h)\,dr\,d\theta = h(\pi R^2).$$

31. We multiply the first octant volume by eight.

$$V = 8\int_0^{\pi/2}\int_0^{1/\sqrt{3}}\int_{2z}^{\sqrt{1+z^2}} r\,dr\,dz\,d\theta = 8\int_0^{\pi/2}\int_0^{1/\sqrt{3}}\left\{\frac{r^2}{2}\right\}_{2z}^{\sqrt{1+z^2}} dz\,d\theta$$
$$= 4\int_0^{\pi/2}\int_0^{1/\sqrt{3}}(1-3z^2)\,dz\,d\theta = 4\int_0^{\pi/2}\left\{z-z^3\right\}_0^{1/\sqrt{3}} d\theta = \frac{8\sqrt{3}}{9}\left\{\theta\right\}_0^{\pi/2} = \frac{4\sqrt{3}\pi}{9}$$

32. We quadruple the first octant volume.

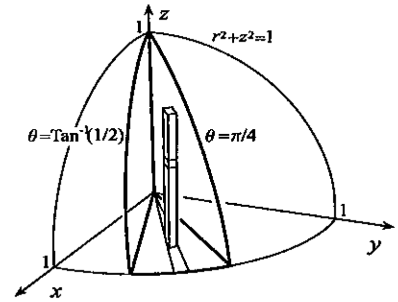
$$\text{V} = 4\int_0^{\pi/4}\int_0^{\sqrt{\cos 2\theta}}\int_0^{r^2} r\,dz\,dr\,d\theta$$
$$= 4\int_0^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} r^3\,dr\,d\theta = 4\int_0^{\pi/4}\left\{\frac{r^4}{4}\right\}_0^{\sqrt{\cos 2\theta}} d\theta$$
$$= \int_0^{\pi/4}\cos^2 2\theta\,d\theta = \int_0^{\pi/4}\left(\frac{1+\cos 4\theta}{2}\right)d\theta$$
$$= \frac{1}{2}\left\{\theta+\frac{1}{4}\sin 4\theta\right\}_0^{\pi/4} = \frac{\pi}{8}$$



33. We quadruple the first octant volume.

$$V = 4\int_0^{\pi/2}\int_0^{\sqrt{2}}\int_{\sqrt{4-r^2}}^{\sqrt{16-r^2}} r\,dz\,dr\,d\theta + 4\int_0^{\pi/2}\int_{\sqrt{2}}^{2\sqrt{2}}\int_r^{\sqrt{16-r^2}} r\,dz\,dr\,d\theta$$
$$= 4\int_0^{\pi/2}\int_0^{\sqrt{2}}(r\sqrt{16-r^2}-r\sqrt{4-r^2})\,dr\,d\theta + 4\int_0^{\pi/2}\int_{\sqrt{2}}^{2\sqrt{2}}(r\sqrt{16-r^2}-r^2)\,dr\,d\theta$$
$$= 4\int_0^{\pi/2}\left\{-\frac{1}{3}(16-r^2)^{3/2}+\frac{1}{3}(4-r^2)^{3/2}\right\}_0^{\sqrt{2}} d\theta + 4\int_0^{\pi/2}\left\{-\frac{1}{3}(16-r^2)^{3/2}-\frac{r^3}{3}\right\}_{\sqrt{2}}^{2\sqrt{2}} d\theta$$
$$= \frac{112}{3}(2-\sqrt{2})\left\{\theta\right\}_0^{\pi/2} = \frac{56(2-\sqrt{2})\pi}{3}$$

34. $$\begin{aligned}V &= \int_{\text{Tan}^{-1}(1/2)}^{\pi/4}\int_0^1\int_0^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta \\ &= \int_{\text{Tan}^{-1}(1/2)}^{\pi/4}\int_0^1 r\sqrt{1-r^2}\,dr\,d\theta \\ &= \int_{\text{Tan}^{-1}(1/2)}^{\pi/4}\left\{-\frac{1}{3}(1-r^2)^{3/2}\right\}_0^1 d\theta \\ &= \frac{1}{3}\left\{\theta\right\}_{\text{Tan}^{-1}(1/2)}^{\pi/4} = \frac{1}{3}\left[\frac{\pi}{4}-\text{Tan}^{-1}(1/2)\right]\end{aligned}$$

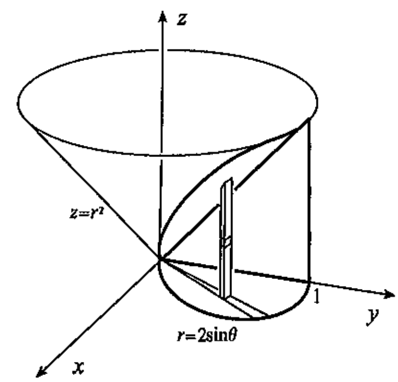


35. We multiply the first octant volume by eight.

$$\begin{aligned}V &= 8\int_0^{\pi/2}\int_0^1\int_0^{\sqrt{8-2r^2}} r\,dz\,dr\,d\theta = 8\int_0^{\pi/2}\int_0^1 r\sqrt{8-2r^2}\,dr\,d\theta \\ &= 8\int_0^{\pi/2}\left\{-\frac{1}{6}(8-2r^2)^{3/2}\right\}_0^1 d\theta = \frac{4}{3}(16\sqrt{2}-6\sqrt{6})\left\{\theta\right\}_0^{\pi/2} = \frac{4(8\sqrt{2}-3\sqrt{6})\pi}{3}\end{aligned}$$

36. We quadruple the first octant volume.

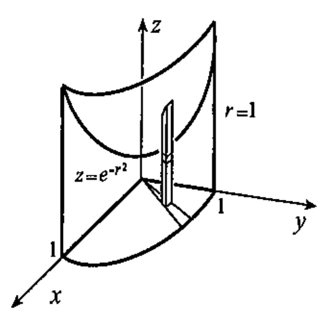
$$\begin{aligned}V &= 4\int_0^{\pi/2}\int_0^{2\sin\theta}\int_0^{r^2} r\,dz\,dr\,d\theta \\ &= 4\int_0^{\pi/2}\int_0^{2\sin\theta} r^3\,dr\,d\theta = 4\int_0^{\pi/2}\left\{\frac{r^4}{4}\right\}_0^{2\sin\theta} d\theta \\ &= \int_0^{\pi/2} 16\sin^4\theta\,d\theta = 16\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta \\ &= 4\int_0^{\pi/2}\left(1-2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)d\theta \\ &= 4\left\{\frac{3\theta}{2}-\sin 2\theta+\frac{1}{8}\sin 4\theta\right\}_0^{\pi/2} = 3\pi\end{aligned}$$



37. $$\begin{aligned}V &= \frac{4}{3}\pi a^3 - 4\int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta = \frac{4}{3}\pi a^3 - 4\int_0^{\pi/2}\int_0^{a\sin\theta} r\sqrt{a^2-r^2}\,dr\,d\theta \\ &= \frac{4}{3}\pi a^3 - 4\int_0^{\pi/2}\left\{-\frac{1}{3}(a^2-r^2)^{3/2}\right\}_0^{a\sin\theta} d\theta = \frac{4}{3}\pi a^3 + \frac{4}{3}\int_0^{\pi/2}(a^3\cos^3\theta - a^3)\,d\theta \\ &= \frac{4}{3}\pi a^3 + \frac{4}{3}a^3\int_0^{\pi/2}[\cos\theta(1-\sin^2\theta)-1]\,d\theta = \frac{4}{3}\pi a^3 + \frac{4}{3}a^3\left\{\sin\theta - \frac{1}{3}\sin^3\theta - \theta\right\}_0^{\pi/2} = \frac{2a^3(3\pi+4)}{9}\end{aligned}$$

38. We quadruple the first octant volume.

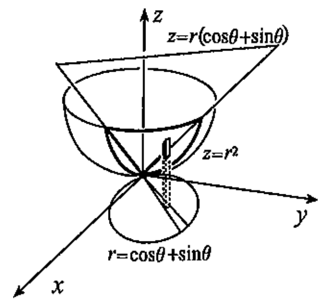
$$\begin{aligned}V &= 4\int_0^{\pi/2}\int_0^1\int_0^{e^{-r^2}} r\,dz\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^1 re^{-r^2}\,dr\,d\theta \\ &= 4\int_0^{\pi/2}\left\{-\frac{1}{2}e^{-r^2}\right\}_0^1 d\theta \\ &= -2(e^{-1}-1)\left\{\theta\right\}_0^{\pi/2} = \pi(1-1/e)\end{aligned}$$



39. We multiply the first octant volume by eight.

$$\begin{aligned}V &= 8\int_0^2\int_0^{\pi/2}\int_{\sqrt{1+z^2}}^{\sqrt{9-z^2}} r\,dr\,d\theta\,dz = 8\int_0^2\int_0^{\pi/2}\left\{\frac{r^2}{2}\right\}_{\sqrt{1+z^2}}^{\sqrt{9-z^2}} d\theta\,dz \\ &= 4\int_0^2\int_0^{\pi/2}(9-z^2-1-z^2)\,d\theta\,dz = 4\int_0^2\left\{(8-2z^2)\theta\right\}_0^{\pi/2} dz = 2\pi\left\{8z-\frac{2z^3}{3}\right\}_0^2 = \frac{64\pi}{3}\end{aligned}$$

40. $$V = \int_{-\pi/4}^{3\pi/4}\int_0^{\cos\theta+\sin\theta}\int_{r^2}^{r\cos\theta+r\sin\theta} r\,dz\,dr\,d\theta$$



$$= \int_{-\pi/4}^{3\pi/4}\int_0^{\cos\theta+\sin\theta}(r^2\cos\theta + r^2\sin\theta - r^3)\,dr\,d\theta$$

$$= \int_{-\pi/4}^{3\pi/4}\left\{\frac{r^3}{3}\cos\theta + \frac{r^3}{3}\sin\theta - \frac{r^4}{4}\right\}_0^{\cos\theta+\sin\theta} d\theta$$

$$= \frac{1}{12}\int_{-\pi/4}^{3\pi/4}[4(\cos\theta+\sin\theta)^4 - 3(\cos\theta+\sin\theta)^4]\,d\theta$$

$$= \frac{1}{12}\int_{-\pi/4}^{3\pi/4}(\cos^4\theta + 4\cos^3\theta\sin\theta + 6\cos^2\theta\sin^2\theta + 4\cos\theta\sin^3\theta + \sin^4\theta)\,d\theta$$

$$= \frac{1}{12}\int_{-\pi/4}^{3\pi/4}[(\cos^2\theta+\sin^2\theta)^2 + 4(\cos^3\theta\sin\theta + \cos^2\theta\sin^2\theta + \cos\theta\sin^3\theta)]\,d\theta$$

$$= \frac{1}{12}\int_{-\pi/4}^{3\pi/4}[1 + 4(\cos^3\theta\sin\theta + \cos\theta\sin^3\theta) + (\sin 2\theta)^2]\,d\theta$$

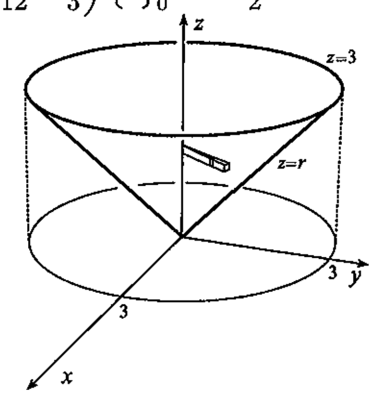
$$= \frac{1}{12}\int_{-\pi/4}^{3\pi/4}\left[1 + 4(\cos^3\theta\sin\theta + \cos\theta\sin^3\theta) + \frac{1-\cos 4\theta}{2}\right]d\theta$$

$$= \frac{1}{12}\left\{\frac{3\theta}{2} - \cos^4\theta + \sin^4\theta - \frac{1}{8}\sin 4\theta\right\}_{-\pi/4}^{3\pi/4} = \frac{\pi}{8}$$

41. $$V = \frac{4}{3}\pi(2)^3 - 4\int_0^{\pi/2}\int_0^{\sqrt{3}}\int_{r^2/3}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta = \frac{32\pi}{3} - 4\int_0^{\pi/2}\int_0^{\sqrt{3}}\left(r\sqrt{4-r^2} - \frac{r^3}{3}\right)dr\,d\theta$$

$$= \frac{32\pi}{3} - 4\int_0^{\pi/2}\left\{-\frac{1}{3}(4-r^2)^{3/2} - \frac{r^4}{12}\right\}_0^{\sqrt{3}} d\theta = \frac{32\pi}{3} + 4\left(\frac{1}{3} + \frac{9}{12} - \frac{8}{3}\right)\left\{\theta\right\}_0^{\pi/2} = \frac{15\pi}{2}$$

42. $$\iiint_V \sqrt{x^2+y^2+z^2}\,dV = 4\int_0^{\pi/2}\int_0^3\int_0^z \sqrt{r^2+z^2}\,r\,dr\,dz\,d\theta$$



$$= 4\int_0^{\pi/2}\int_0^3\left\{\frac{1}{3}(r^2+z^2)^{3/2}\right\}_0^z dz\,d\theta$$

$$= \frac{4}{3}\int_0^{\pi/2}\int_0^3(2\sqrt{2}z^3 - z^3)\,dz\,d\theta$$

$$= \frac{4(2\sqrt{2}-1)}{3}\int_0^{\pi/2}\left\{\frac{z^4}{4}\right\}_0^3 d\theta$$

$$= 27(2\sqrt{2}-1)\left\{\theta\right\}_0^{\pi/2} = \frac{27\pi(2\sqrt{2}-1)}{2}$$

43. We quadruple the integral over the first octant volume.

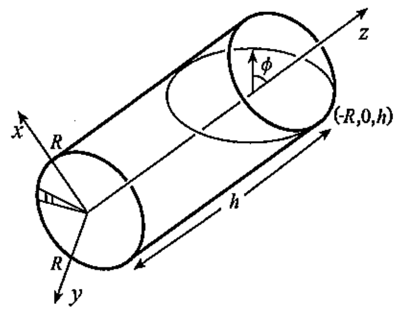
$$\iiint_V |yz|\,dV = 4\int_0^{\pi/2}\int_0^{\sqrt{3/2}}\int_{\sqrt{1+r^2}}^{\sqrt{4-r^2}} r\sin\theta\,z\,r\,dz\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^{\sqrt{3/2}}\left\{\frac{r^2\sin\theta z^2}{2}\right\}_{\sqrt{1+r^2}}^{\sqrt{4-r^2}} dr\,d\theta$$

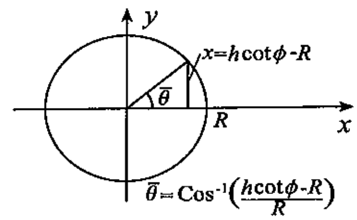
$$= 2\int_0^{\pi/2}\int_0^{\sqrt{3/2}}(3r^2 - 2r^4)\sin\theta\,dr\,d\theta = 2\int_0^{\pi/2}\left\{\left(r^3 - \frac{2r^5}{5}\right)\sin\theta\right\}_0^{\sqrt{3/2}} d\theta$$

$$= \frac{3\sqrt{6}}{5}\left\{-\cos\theta\right\}_0^{\pi/2} = \frac{3\sqrt{6}}{5}$$

44. If we choose a coordinate system as shown, the equation of the surface of the water is

$$0 = (\sin\phi, 0, \cos\phi)\cdot(x+R, 0, z-h) \quad\Longrightarrow\quad x\sin\phi + z\cos\phi = h\cos\phi - R\sin\phi.$$





CASE 1: Water touches only sides of the tumbler $(0 \le \phi \le \mathrm{Tan}^{-1}[h/(2R)])$.
In this case, the volume of water is

$$\begin{aligned}V &= 2\int_0^{\pi}\int_0^{R}\int_0^{h-R\tan\phi - r\tan\phi\cos\theta} r\,dz\,dr\,d\theta = 2\int_0^{\pi}\int_0^{R} r(h - R\tan\phi - r\tan\phi\cos\theta)\,dr\,d\theta\\
&= 2\int_0^{\pi}\left(\frac{R^2h}{2} - \frac{R^3}{2}\tan\phi - \frac{R^3}{3}\tan\phi\cos\theta\right)d\theta = \pi R^2(h - R\tan\phi).\end{aligned}$$

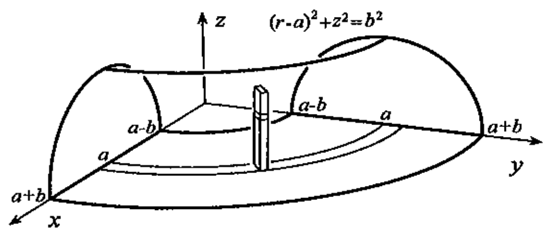
CASE 2: Water touches bottom of tumbler $(\mathrm{Tan}^{-1}[h/(2R)] < \phi < \pi/2)$
In this case, the volume of water is

$$\begin{aligned}V &= 2\int_0^{\bar\theta}\int_0^{(h\cot\phi - R)\sec\theta}\int_0^{h-R\tan\phi - r\tan\phi\cos\theta} r\,dz\,dr\,d\theta\\
&\quad + 2\int_{\bar\theta}^{\pi}\int_0^{R}\int_0^{h-R\tan\phi-r\tan\phi\cos\theta} r\,dz\,dr\,d\theta\\
&= 2\int_0^{\bar\theta}\int_0^{(h\cot\phi-R)\sec\theta} r(h - R\tan\phi - r\tan\phi\cos\theta)\,dr\,d\theta\\
&\quad + 2\int_{\bar\theta}^{\pi}\int_0^{R} r(h - R\tan\phi - r\tan\phi\cos\theta)\,dr\,d\theta\\
&= 2\int_0^{\bar\theta}\left\{\frac{r^2}{2}(h - R\tan\phi) - \frac{r^3}{3}\tan\phi\cos\theta\right\}\Bigg|_0^{(h\cot\phi-R)\sec\theta} d\theta\\
&\quad + 2\int_{\bar\theta}^{\pi}\left\{\frac{r^2}{2}(h - R\tan\phi) - \frac{r^3}{3}\tan\phi\cos\theta\right\}\Bigg|_0^{R} d\theta\\
&= \int_0^{\bar\theta}\left[(h - R\tan\phi)(h\cot\phi - R)^2\sec^2\theta - \frac{2}{3}\tan\phi(h\cot\phi - R)^3\sec^2\theta\right]d\theta\\
&\quad + \int_{\bar\theta}^{\pi}\left[R^2(h - R\tan\phi) - \frac{2R^3}{3}\tan\phi\cos\theta\right]d\theta\\
&= \left[(h - R\tan\phi)(h\cot\phi - R)^2 - \frac{2}{3}\tan\phi(h\cot\phi - R)^3\right]\tan\bar\theta\\
&\quad + R^2(h - R\tan\phi)(\pi - \bar\theta) + \frac{2R^3}{3}\tan\phi\sin\bar\theta\\
&= \frac{1}{3}\tan\phi(h\cot\phi - R)^3\frac{\sqrt{2Rh\cot\phi - h^2\cot^2\phi}}{h\cot\phi - R}\\
&\quad + R^2\tan\phi(h\cot\phi - R)\left[\pi - \mathrm{Cos}^{-1}\left(\frac{h\cot\phi - R}{R}\right)\right]\end{aligned}$$

$$+\frac{2R^3}{3}\tan\phi\frac{\sqrt{2Rh\cot\phi - h^2\cot^2\phi}}{R}$$
$$= \frac{1}{3}\tan\phi\sqrt{2Rh\cot\phi - h^2\cot^2\phi}[(h\cot\phi - R)^2 + 2R^2]$$
$$+ R^2\tan\phi(h\cot\phi - R)\left[\pi - \text{Cos}^{-1}\left(\frac{h\cot\phi - R}{R}\right)\right].$$

45. We multiply the first octant volume by eight.

$$V = 8\int_{a-b}^{a+b}\int_0^{\pi/2}\int_0^{\sqrt{b^2-(r-a)^2}} r\,dz\,d\theta\,dr$$
$$= 8\int_{a-b}^{a+b}\int_0^{\pi/2} r\sqrt{b^2-(r-a)^2}\,d\theta\,dr$$
$$= 4\pi\int_{a-b}^{a+b} r\sqrt{b^2-(r-a)^2}\,dr$$



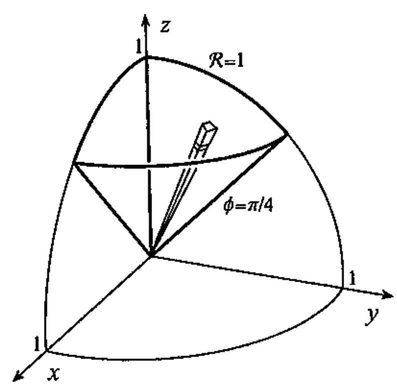
If we set $r - a = b\sin\phi$, then $dr = b\cos\phi\,d\phi$, and

$$V = 4\pi\int_{-\pi/2}^{\pi/2}(a + b\sin\phi)b\cos\phi\,b\cos\phi\,d\phi = 4\pi b^2\int_{-\pi/2}^{\pi/2}\left[a\left(\frac{1+\cos 2\phi}{2}\right) + b\cos^2\phi\,\sin\phi\right]d\phi$$
$$= 4\pi b^2\left\{\frac{a}{2}\left(\phi + \frac{\sin 2\phi}{2}\right) - \frac{b}{3}\cos^3\phi\right\}_{-\pi/2}^{\pi/2} = 2\pi^2ab^2.$$

EXERCISES 13.12

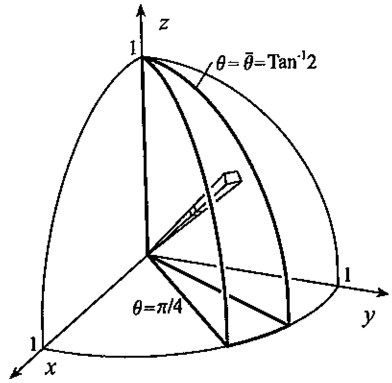
1. The equation is $\Re = 2$. See figure for Exercise 13.11-1.
2. The equation is $\Re\sin\phi = 1$. See figure for Exercise 13.11-2.
3. The equation is $\phi = \text{Tan}^{-1}3$. The figure has the same shape as that in Exercise 13.11-5.
4. The equation is $\Re = 4\csc\phi\cot\phi$. See figure for Exercise 13.11-8.
5. The equations are $\theta = \pi/4$ and $\theta = 5\pi/4$. See figure for Exercise 13.11-9.
6. The equation is $\Re^2 = -\sec 2\phi$. See figure for Exercise 13.11-10.
7. The equation is $\phi = \pi - \text{Tan}^{-1}(1/2)$. Turn the figure in Exercise 13.11-5 upside down.
8. We quadruple the volume in the first octant.

$$V = 4\int_0^{\pi/2}\int_0^{\pi/4}\int_0^1 \Re^2\sin\phi\,d\Re\,d\phi\,d\theta$$
$$= \frac{4}{3}\int_0^{\pi/2}\int_0^{\pi/4}\sin\phi\,d\phi\,d\theta$$
$$= \frac{4}{3}\int_0^{\pi/2}\left\{-\cos\phi\right\}_0^{\pi/4}d\theta$$
$$= \frac{2(2-\sqrt{2})}{3}\left\{\theta\right\}_0^{\pi/2} = \frac{(2-\sqrt{2})\pi}{3}$$



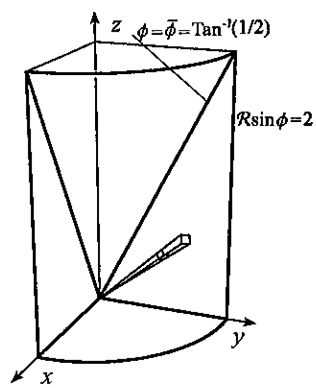
9. $$V = 4\int_0^{\pi/2}\int_0^{\pi/3}\int_{\sec\phi}^{2}\Re^2\sin\phi\,d\Re\,d\phi\,d\theta = 4\int_0^{\pi/2}\int_0^{\pi/3}\left\{\frac{\Re^3}{3}\sin\phi\right\}_{\sec\phi}^{2}d\phi\,d\theta$$
$$= \frac{4}{3}\int_0^{\pi/2}\int_0^{\pi/3}(8\sin\phi - \sec^3\phi\sin\phi)d\phi\,d\theta = \frac{4}{3}\int_0^{\pi/2}\int_0^{\pi/3}(8\sin\phi - \tan\phi\sec^2\phi)d\phi\,d\theta$$
$$= \frac{4}{3}\int_0^{\pi/2}\left\{-8\cos\phi - \frac{1}{2}\tan^2\phi\right\}_0^{\pi/3}d\theta = \frac{10}{3}\left\{\theta\right\}_0^{\pi/2} = \frac{5\pi}{3}$$

10. $$V = \int_{\pi/4}^{\overline{\theta}}\int_0^{\pi/2}\int_0^1 \Re^2 \sin\phi \, d\Re \, d\phi \, d\theta$$
$$= \frac{1}{3}\int_{\pi/4}^{\overline{\theta}}\int_0^{\pi/2} \sin\phi \, d\phi \, d\theta$$
$$= \frac{1}{3}\int_{\pi/4}^{\overline{\theta}} \left\{-\cos\phi\right\}_0^{\pi/2} d\theta$$
$$= \frac{1}{3}\left\{\theta\right\}_{\pi/4}^{\overline{\theta}} = \frac{1}{3}(\mathrm{Tan}^{-1}2 - \pi/4)$$



11. $$V = 8\int_0^{\pi/2}\int_{\pi/4}^{\pi/2}\int_{\csc\phi}^{\sqrt{2}} \Re^2 \sin\phi \, d\Re \, d\phi \, d\theta = 8\int_0^{\pi/2}\int_{\pi/4}^{\pi/2}\left\{\frac{\Re^3}{3}\sin\phi\right\}_{\csc\phi}^{\sqrt{2}} d\phi \, d\theta$$
$$= \frac{8}{3}\int_0^{\pi/2}\int_{\pi/4}^{\pi/2}(2\sqrt{2}\sin\phi - \csc^2\phi)\, d\phi \, d\theta = \frac{8}{3}\int_0^{\pi/2}\left\{-2\sqrt{2}\cos\phi + \cot\phi\right\}_{\pi/4}^{\pi/2} d\theta$$
$$= \frac{8}{3}\int_0^{\pi/2}(2-1)\, d\theta = \frac{8}{3}\left\{\theta\right\}_0^{\pi/2} = \frac{4\pi}{3}$$

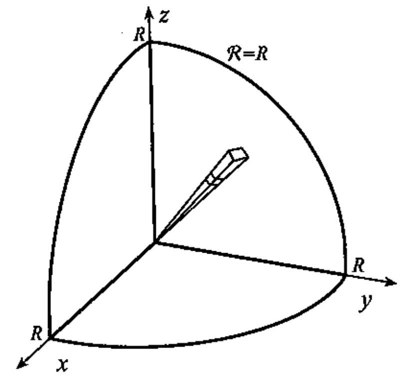
12. We quadruple the volume in the first octant.
$$V = 4\int_0^{\pi/2}\int_{\overline{\phi}}^{\pi/2}\int_0^{2\csc\phi} \Re^2 \sin\phi \, d\Re \, d\phi \, d\theta$$
$$= 4\int_0^{\pi/2}\int_{\overline{\phi}}^{\pi/2}\left\{\frac{\Re^3}{3}\sin\phi\right\}_0^{2\csc\phi} d\phi \, d\theta$$
$$= \frac{32}{3}\int_0^{\pi/2}\int_{\overline{\phi}}^{\pi/2}\csc^2\phi \, d\phi \, d\theta$$
$$= \frac{32}{3}\int_0^{\pi/2}\left\{-\cot\phi\right\}_{\overline{\phi}}^{\pi/2} d\theta$$
$$= \frac{64}{3}\left\{\theta\right\}_0^{\pi/2} = \frac{32\pi}{3}$$



13. For the hemisphere bounded by $z = \sqrt{R^2 - x^2 - y^2}$ and $z = 0$, $\overline{x} = \overline{y} = 0$. Since $M = (2/3)\pi\rho R^3$, and
$$M\overline{z} = 4\int_0^{\pi/2}\int_0^{\pi/2}\int_0^R (\Re\cos\phi)\rho\Re^2\sin\phi \, d\Re \, d\phi \, d\theta = 4\rho\int_0^{\pi/2}\int_0^{\pi/2}\left\{\frac{\Re^4}{4}\cos\phi\sin\phi\right\}_0^R d\phi \, d\theta$$
$$= \rho R^4\int_0^{\pi/2}\left\{\frac{1}{2}\sin^2\phi\right\}_0^{\pi/2} d\theta = \frac{\rho R^4}{2}\left\{\theta\right\}_0^{\pi/2} = \frac{\pi\rho R^4}{4},$$

it follows that $\overline{z} = \dfrac{\pi\rho R^4}{4}\dfrac{3}{2\pi\rho R^3} = \dfrac{3R}{8}$.

14. We multiply the moment of inertia of the first octant portion of the sphere about the z-axis by eight.



$$I_z = 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^R (\Re^2\sin^2\phi)\rho\Re^2\sin\phi\,d\Re\,d\phi\,d\theta$$
$$= 8\rho\int_0^{\pi/2}\int_0^{\pi/2}\left\{\frac{\Re^5}{5}\sin^3\phi\right\}_0^R d\phi\,d\theta$$
$$= \frac{8\rho R^5}{5}\int_0^{\pi/2}\int_0^{\pi/2}(1-\cos^2\phi)\sin\phi\,d\phi\,d\theta$$
$$= \frac{8\rho R^5}{5}\int_0^{\pi/2}\left\{-\cos\phi+\frac{\cos^3\phi}{3}\right\}_0^{\pi/2} d\theta$$
$$= \frac{16\rho R^5}{15}\left\{\theta\right\}_0^{\pi/2} = \frac{8\pi\rho R^5}{15}.$$

15. $$\text{Moment } = \iiint_V x\rho\,dV = \int_0^{\pi/3}\int_0^{\pi/2}\int_2^3 (\Re\sin\phi\cos\theta)\Re^2\sin\phi\,d\Re\,d\phi\,d\theta$$
$$= \int_0^{\pi/3}\int_0^{\pi/2}\left\{\frac{\Re^4}{4}\sin^2\phi\cos\theta\right\}_2^3 d\phi\,d\theta = \frac{65}{4}\int_0^{\pi/3}\int_0^{\pi/2}\cos\theta\left(\frac{1-\cos 2\phi}{2}\right)d\phi\,d\theta$$
$$= \frac{65}{8}\int_0^{\pi/3}\left\{\cos\theta\left(\phi-\frac{1}{2}\sin 2\phi\right)\right\}_0^{\pi/2} d\theta = \frac{65\pi}{16}\int_0^{\pi/3}\cos\theta\,d\theta = \frac{65\pi}{16}\left\{\sin\theta\right\}_0^{\pi/3} = \frac{65\sqrt{3}\pi}{32}$$

16. Using the figure in Exercise 14,

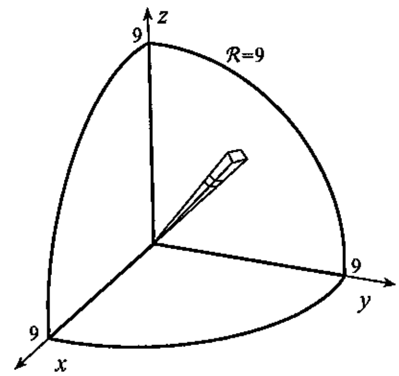
$$Q = \int_0^{2\pi}\int_0^{\pi}\int_0^R k\Re\,\Re^2\sin\phi\,d\Re\,d\phi\,d\theta = \frac{kR^4}{4}\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta$$
$$= \frac{kR^4}{4}\int_0^{2\pi}\left\{-\cos\phi\right\}_0^{\pi} d\theta = \frac{kR^4}{2}\left\{\theta\right\}_0^{2\pi} = k\pi R^4\text{ C.}$$

17. In each of the following integrals f stands for $f(\Re\sin\phi\cos\theta, \Re\sin\phi\sin\theta, \Re\cos\phi)$.

$$\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sqrt{2}} f\Re^2\sin\phi\,d\Re\,d\phi\,d\theta + \int_0^{\pi/2}\int_{\pi/4}^{\pi/2}\int_0^{\csc\phi} f\Re^2\sin\phi\,d\Re\,d\phi\,d\theta,$$
$$\int_0^{\pi/4}\int_0^{\pi/2}\int_0^{\sqrt{2}} f\Re^2\sin\phi\,d\Re\,d\theta\,d\phi + \int_{\pi/4}^{\pi/2}\int_0^{\pi/2}\int_0^{\csc\phi} f\Re^2\sin\phi\,d\Re\,d\theta\,d\phi,$$
$$\int_0^{\pi/4}\int_0^{\sqrt{2}}\int_0^{\pi/2} f\Re^2\sin\phi\,d\theta\,d\Re\,d\phi + \int_{\pi/4}^{\pi/2}\int_0^{\csc\phi}\int_0^{\pi/2} f\Re^2\sin\phi\,d\theta\,d\Re\,d\phi,$$
$$\int_0^1\int_0^{\pi/2}\int_0^{\pi/2} f\Re^2\sin\phi\,d\theta\,d\phi\,d\Re + \int_1^{\sqrt{2}}\int_0^{\mathrm{Csc}^{-1}\Re}\int_0^{\pi/2} f\Re^2\sin\phi\,d\theta\,d\phi\,d\Re,$$
$$\int_0^{\pi/2}\int_1^{\sqrt{2}}\int_0^{\mathrm{Csc}^{-1}\Re} f\Re^2\sin\phi\,d\phi\,d\Re\,d\theta + \int_0^{\pi/2}\int_0^1\int_0^{\pi/2} f\Re^2\sin\phi\,d\phi\,d\Re\,d\theta,$$
$$\int_1^{\sqrt{2}}\int_0^{\pi/2}\int_0^{\mathrm{Csc}^{-1}\Re} f\Re^2\sin\phi\,d\phi\,d\theta\,d\Re + \int_0^1\int_0^{\pi/2}\int_0^{\pi/2} f\Re^2\sin\phi\,d\phi\,d\theta\,d\Re.$$

18. The limits define the first octant volume inside the sphere $x^2 + y^2 + z^2 = 81$. The value of the triple iterated integral is therefore given by

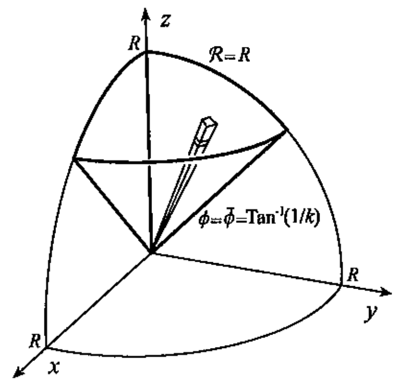
$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^9 \frac{1}{\Re^2}\Re^2 \sin\phi \, d\Re\, d\phi\, d\theta$$
$$= 9\int_0^{\pi/2}\int_0^{\pi/2} \sin\phi\, d\phi\, d\theta$$
$$= 9\int_0^{\pi/2} \Big\{-\cos\phi\Big\}_0^{\pi/2} d\theta$$
$$= 9\Big\{\theta\Big\}_0^{\pi/2} = \frac{9\pi}{2}$$



19. The limits define the first octant volume under the sphere $x^2 + y^2 + z^2 = 2$ and above the cone $z = \sqrt{x^2 + y^2}$. The value of the triple iterated integral is therefore given by

$$\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sqrt{2}} \Re^2 \sin\phi\, d\Re\, d\phi\, d\theta = \int_0^{\pi/2}\int_0^{\pi/4}\left\{\frac{\Re^3}{3}\sin\phi\right\}_0^{\sqrt{2}} d\phi\, d\theta = \frac{2\sqrt{2}}{3}\int_0^{\pi/2}\Big\{-\cos\phi\Big\}_0^{\pi/4} d\theta$$
$$= \frac{2\sqrt{2}}{3}\left(1 - \frac{1}{\sqrt{2}}\right)\Big\{\theta\Big\}_0^{\pi/2} = \frac{(\sqrt{2}-1)\pi}{3}.$$

20.
$$V = 4\int_0^{\pi/2}\int_0^{\overline{\phi}}\int_0^R \Re^2 \sin\phi\, d\Re\, d\phi\, d\theta$$
$$= \frac{4R^3}{3}\int_0^{\pi/2}\int_0^{\overline{\phi}} \sin\phi\, d\phi\, d\theta$$
$$= \frac{4R^3}{3}\int_0^{\pi/2}\Big\{-\cos\phi\Big\}_0^{\overline{\phi}} d\theta$$
$$= \frac{4R^3}{3}(1-\cos\overline{\phi})\Big\{\theta\Big\}_0^{\pi/2}$$
$$= \frac{2\pi R^3}{3}\left(1 - \frac{k}{\sqrt{1+k^2}}\right)$$

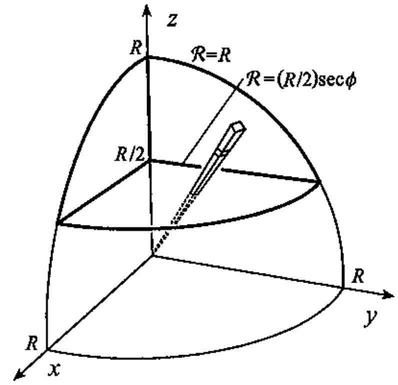


21. In spherical coordinates the equation of the surface is $\Re^4 = \Re\sin\phi\cos\theta \implies \Re^3 = \sin\phi\cos\theta$. As ϕ increases from 0 to π, values of $\sin\phi$ increase from 0 to 1, and then decrease from 1 to 0. Only for θ in the interval $-\pi/2 \le \theta \le \pi/2$ is $\Re > 0$. This leads to

$$V = \int_{-\pi/2}^{\pi/2}\int_0^{\pi}\int_0^{(\sin\phi\cos\theta)^{1/3}} \Re^2\sin\phi\, d\Re\, d\phi\, d\theta = \int_{-\pi/2}^{\pi/2}\int_0^{\pi}\left\{\frac{\Re^3}{3}\sin\phi\right\}_0^{(\sin\phi\cos\theta)^{1/3}} d\phi\, d\theta$$
$$= \frac{1}{3}\int_{-\pi/2}^{\pi/2}\int_0^{\pi}\sin^2\phi\cos\theta\, d\phi\, d\theta = \frac{1}{3}\int_{-\pi/2}^{\pi/2}\int_0^{\pi}\left(\frac{1-\cos 2\phi}{2}\right)\cos\theta\, d\phi\, d\theta$$
$$= \frac{1}{6}\int_{-\pi/2}^{\pi/2}\left\{\left(\phi - \frac{1}{2}\sin 2\phi\right)\cos\theta\right\}_0^{\pi} d\theta = \frac{\pi}{6}\Big\{\sin\theta\Big\}_{-\pi/2}^{\pi/2} = \frac{\pi}{3}.$$

22. (a) Let ρ_b and ρ_w represent the densities of the ball and water. The magnitude of the force of gravity on the ball is $(4/3)\pi R^3\rho_b g$ where R is its radius, and $g > 0$ is the acceleration due to gravity. Since this must be equal to the weight of water displaced by the half-submerged ball, $\frac{4}{3}\pi R^3\rho_b g = \frac{2}{3}\pi R^3\rho_w g$. This equation implies that $\rho_b = \rho_w/2$.

(b) In the diagram, we let the plane $z = R/2$ represent the surface of the water. The volume of ball above water is given by



$$4\int_0^{\pi/2}\int_0^{\pi/3}\int_{(R/2)\sec\phi}^{R} \Re^2 \sin\phi \, d\Re \, d\phi \, d\theta$$
$$= 4\int_0^{\pi/2}\int_0^{\pi/3} \left\{\frac{\Re^3}{3}\sin\phi\right\}_{(R/2)\sec\phi}^{R} d\phi \, d\theta$$
$$= \frac{R^3}{6}\int_0^{\pi/2}\int_0^{\pi/3} (8\sin\phi - \tan\phi \sec^2\phi)\, d\phi \, d\theta$$
$$= \frac{R^3}{6}\int_0^{\pi/2} \left\{-8\cos\phi - \frac{\tan^2\phi}{2}\right\}_0^{\pi/3} d\theta$$
$$= \frac{5R^3}{12}\left\{\theta\right\}_0^{\pi/2} = \frac{5\pi R^3}{24}.$$

The force required to keep the ball at this position is equal to the extra weight of water (above that in (a)) displaced; i.e., $\left(\frac{2}{3}\pi R^3 - \frac{5}{24}\pi R^3\right)\rho_w g = \frac{11}{24}\pi\rho_w g R^3.$

23. The equation of the surface in spherical coordinates is $\Re^4 = 2\Re\cos\phi(\Re^2\sin^2\phi) \implies \Re = 2\sin^2\phi\cos\phi.$

$$V = 4\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2\sin^2\phi\cos\phi} \Re^2\sin\phi\, d\Re\, d\phi\, d\theta = 4\int_0^{\pi/2}\int_0^{\pi/2}\left\{\frac{\Re^3}{3}\sin\phi\right\}_0^{2\sin^2\phi\cos\phi} d\phi\, d\theta$$
$$= \frac{4}{3}\int_0^{\pi/2}\int_0^{\pi/2} 8\sin^7\phi\cos^3\phi\, d\phi\, d\theta = \frac{32}{3}\int_0^{\pi/2}\int_0^{\pi/2}\sin^7\phi(1-\sin^2\phi)\cos\phi\, d\phi\, d\theta$$
$$= \frac{32}{3}\int_0^{\pi/2}\left\{\frac{1}{8}\sin^8\phi - \frac{1}{10}\sin^{10}\phi\right\}_0^{\pi/2} d\theta = \frac{4}{15}\left\{\theta\right\}_0^{\pi/2} = \frac{2\pi}{15}$$

24. (a) Since $s^2 = \Re^2 + d^2 - 2\Re d\cos\phi$,

$$V = \int_{-\pi}^{\pi}\int_0^{\pi}\int_0^R \frac{\rho}{4\pi\epsilon_0 s}\Re^2\sin\phi\, d\Re\, d\phi\, d\theta = \frac{\rho}{4\pi\epsilon_0}\int_{-\pi}^{\pi}\int_0^{\pi}\int_0^R \frac{\Re^2\sin\phi}{\sqrt{\Re^2+d^2-2\Re d\cos\phi}}d\Re\, d\phi\, d\theta.$$

(b) In order to change ϕ to s we first write $V = \frac{\rho}{4\pi\epsilon_0}\int_{-\pi}^{\pi}\int_0^R\int_0^{\pi}\frac{\Re^2\sin\phi}{\sqrt{\Re^2+d^2-2\Re d\cos\phi}}d\phi\, d\Re\, d\theta.$ If $s^2 = \Re^2 + d^2 - 2\Re d\cos\phi$, then $2s\, ds = 2\Re d\sin\phi\, d\phi$, and

$$V = \frac{\rho}{4\pi\epsilon_0}\int_{-\pi}^{\pi}\int_0^R\int_{d-\Re}^{d+\Re}\frac{\Re^2}{s}\left(\frac{s\, ds}{\Re d}\right)d\Re\, d\theta = \frac{\rho}{4\pi\epsilon_0 d}\int_{-\pi}^{\pi}\int_0^R\int_{d-\Re}^{d+\Re}\Re\, ds\, d\Re\, d\theta.$$

(c) $$V = \frac{\rho}{4\pi\epsilon_0 d}\int_{-\pi}^{\pi}\int_0^R\left\{\Re s\right\}_{d-\Re}^{d+\Re}d\Re\, d\theta = \frac{\rho}{2\pi\epsilon_0 d}\int_{-\pi}^{\pi}\int_0^R \Re^2\, d\Re\, d\theta$$
$$= \frac{\rho}{2\pi\epsilon_0 d}\int_{-\pi}^{\pi}\left\{\frac{\Re^3}{3}\right\}_0^R d\theta = \frac{\rho R^3}{6\pi\epsilon_0 d}\left\{\theta\right\}_{-\pi}^{\pi} = \frac{\rho R^3}{3\epsilon_0 d}$$

Since $Q = (4/3)\pi R^3\rho$, $\frac{1}{4\pi\epsilon_0}\frac{Q}{d} = \frac{1}{4\pi\epsilon_0 d}\left(\frac{4}{3}\pi R^3\rho\right) = \frac{\rho R^3}{3\epsilon_0 d}$, and therefore $V = \frac{1}{4\pi\epsilon_0}\frac{Q}{d}$.

25. (a) The cosine law for the triangle joining O, P, and dV gives $\Re^2 = s^2 + d^2 - 2sd\cos\psi$, and therefore

$$F_z = \iiint_V -\frac{Gm\rho}{s^2}\left(\frac{s^2+d^2-\Re^2}{2sd}\right)dV = -\frac{Gm\rho}{2d}\int_{-\pi}^{\pi}\int_0^{\pi}\int_0^R\left(\frac{s^2+d^2-\Re^2}{s^3}\right)\Re^2\sin\phi\,d\Re\,d\phi\,d\theta.$$

(b) In order to replace ϕ with s we first write

$$F_z = -\frac{Gm\rho}{2d}\int_{-\pi}^{\pi}\int_0^R\int_0^{\pi}\left(\frac{s^2+d^2-\Re^2}{s^3}\right)\Re^2\sin\phi\,d\phi\,d\Re\,d\theta.$$

If we set $s = \sqrt{\Re^2+d^2-2d\Re\cos\phi} \implies s^2 = \Re^2+d^2-2d\Re\cos\phi$, from which $2s\,ds = 2d\Re\sin\phi\,d\phi$, then

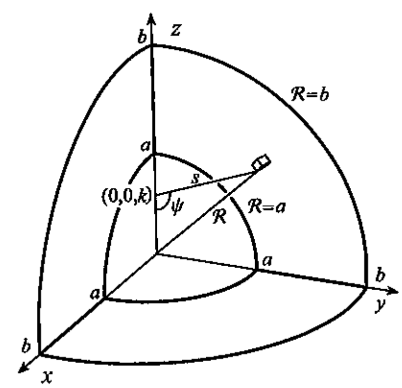
$$\begin{aligned}
F_z &= -\frac{Gm\rho}{2d}\int_{-\pi}^{\pi}\int_0^R\int_{d-\Re}^{d+\Re}\left(\frac{s^2+d^2-\Re^2}{s^3}\right)\Re^2\left(\frac{s\,ds}{d\Re}\right)d\Re\,d\theta\\
&= -\frac{Gm\rho}{2d^2}\int_{-\pi}^{\pi}\int_0^R\int_{d-\Re}^{d+\Re}\Re\left(\frac{s^2+d^2-\Re^2}{s^2}\right)ds\,d\Re\,d\theta\\
&= -\frac{Gm\rho}{2d^2}\int_{-\pi}^{\pi}\int_0^R\left\{\Re\left(s-\frac{d^2-\Re^2}{s}\right)\right\}_{d-\Re}^{d+\Re}d\Re\,d\theta\\
&= -\frac{Gm\rho}{2d^2}\int_{-\pi}^{\pi}\int_0^R\Re\left(d+\Re-\frac{d^2-\Re^2}{d+\Re}-d+\Re+\frac{d^2-\Re^2}{d-\Re}\right)d\Re\,d\theta\\
&= -\frac{2Gm\rho}{d^2}\int_{-\pi}^{\pi}\int_0^R\Re^2\,d\Re\,d\theta = -\frac{2Gm\rho}{d^2}\int_{-\pi}^{\pi}\left\{\frac{\Re^3}{3}\right\}_0^R d\theta\\
&= -\frac{2Gm\rho R^3}{3d^2}\left\{\theta\right\}_{-\pi}^{\pi} = -\frac{4\pi Gm\rho R^3}{3d^2} = -\frac{GmM}{d^2},
\end{aligned}$$

where M is the mass of the sphere.

26. We can always choose a coordinate system so that the point is on the z-axis. Symmetry makes it clear that x- and y-components of the force vanish.

The contribution to the z-component of the force due to the mass in dV is

$$-\frac{Gm\rho dV}{s^2}\cos\psi = -\frac{Gm\rho}{s^2}\left(\frac{k^2+s^2-\Re^2}{2ks}\right)dV.$$



Therefore

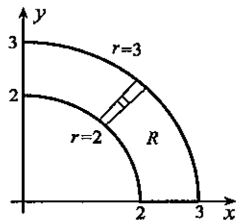
$$\begin{aligned}
F_z &= \int_0^{\pi}\int_a^b\int_{-\pi}^{\pi} -\frac{Gm\rho}{2ks^3}(k^2+s^2-\Re^2)\Re^2\sin\phi\,d\theta\,d\Re\,d\phi\\
&= -\frac{Gm\rho\pi}{k}\int_0^{\pi}\int_a^b\left(\frac{s^2+k^2-\Re^2}{s^3}\right)\Re^2\sin\phi\,d\Re\,d\phi\\
&= -\frac{Gm\rho\pi}{k}\int_a^b\int_0^{\pi}\left(\frac{s^2+k^2-\Re^2}{s^3}\right)\Re^2\sin\phi\,d\phi\,d\Re.
\end{aligned}$$

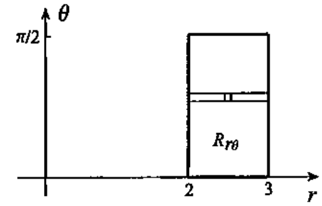
If we set $s = \sqrt{\Re^2+k^2-2k\Re\cos\phi}$ in the inner integral, then $2s\,ds = 2k\Re\sin\phi\,d\phi$, and

$$\begin{aligned}
F_z &= -\frac{Gm\rho\pi}{k}\int_a^b\int_{\Re-k}^{\Re+k}\left(\frac{s^2+k^2-\Re^2}{s^3}\right)\Re^2\sin\phi\left(\frac{s\,ds}{k\Re\sin\phi}\right)d\Re\\
&= -\frac{Gm\rho\pi}{k^2}\int_a^b\int_{\Re-k}^{\Re+k}\Re\left(\frac{s^2+k^2-\Re^2}{s^2}\right)ds\,d\Re = -\frac{Gm\rho\pi}{k^2}\int_a^b\Re\left\{s-\frac{k^2-\Re^2}{s}\right\}_{\Re-k}^{\Re+k}d\Re\\
&= -\frac{Gm\rho\pi}{k^2}\int_a^b\Re\left[\Re+k-\frac{k^2-\Re^2}{\Re+k}-(\Re-k)+\frac{k^2-\Re^2}{\Re-k}\right]d\Re = 0.
\end{aligned}$$

EXERCISES 13.13

1. (a) This is a change to polar coordinates, $\displaystyle\iint_R \sqrt{x^2+y^2}\,dA = \int_0^{\pi/2}\int_2^3 r\,r\,dr\,d\theta = \int_0^{\pi/2}\int_2^3 r^2\,dr\,d\theta.$



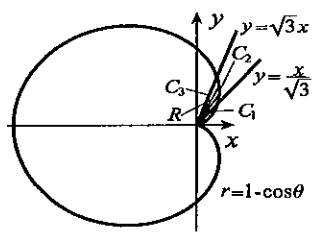


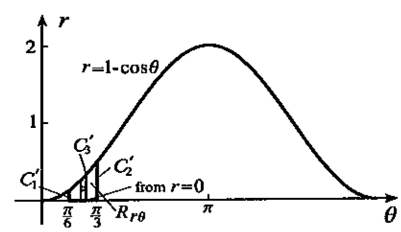
(b) Alternatively, region R in the xy-plane is mapped to the rectangle $R_{r\theta}$ in the $r\theta$-plane shown above. With $\dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$, equation 13.70 gives

$$\iint_R \sqrt{x^2+y^2}\,dA = \iint_{R_{r\theta}} r\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta = \int_0^{\pi/2}\int_2^3 r^2\,dr\,d\theta.$$

2. (a) This is a change to polar coordinates,

$$\iint_R xy\,dA = \int_{\pi/6}^{\pi/3}\int_0^{1-\cos\theta} r\cos\theta\, r\sin\theta\, r\,dr\,d\theta = \int_{\pi/6}^{\pi/3}\int_0^{1-\cos\theta} r^3\cos\theta\sin\theta\,dr\,d\theta.$$

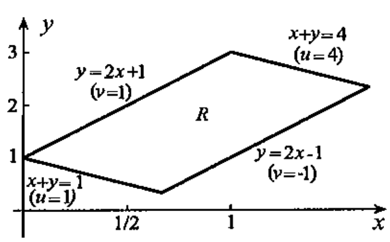


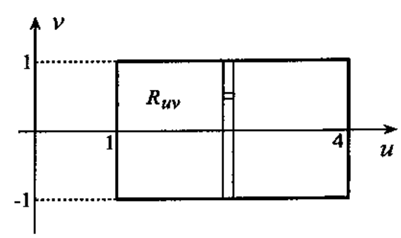


(b) Alternatively, region R in the xy-plane is mapped to the region $R_{r\theta}$ in the $r\theta$-plane shown above. With $\dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$, equation 13.70 gives

$$\iint_R xy\,dA = \iint_{R_{r\theta}} r\cos\theta\, r\sin\theta\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta = \int_{\pi/6}^{\pi/3}\int_0^{1-\cos\theta} r^3\cos\theta\sin\theta\,dr\,d\theta.$$

3. The parallelogram R in the xy-plane is mapped to the rectangle R_{uv} in the uv-plane shown below.

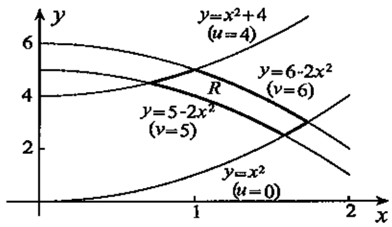


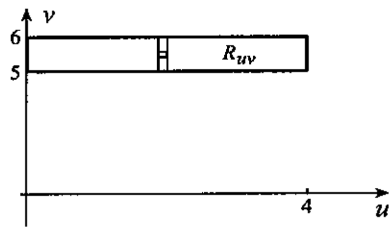


With $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \dfrac{1}{\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}} = \dfrac{1}{3}$, equation 13.70 gives

$$\iint_R x^2\cos y\,dA = \iint_{R_{uv}} \left(\frac{u-v}{3}\right)^2 \cos\left(\frac{2u+v}{3}\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du = \frac{1}{27}\int_1^4\int_{-1}^1 (u-v)^2\cos\left(\frac{2u+v}{3}\right) dv\,du.$$

4. Region R in the xy-plane is mapped to the rectangle R_{uv} in the uv-plane shown below.



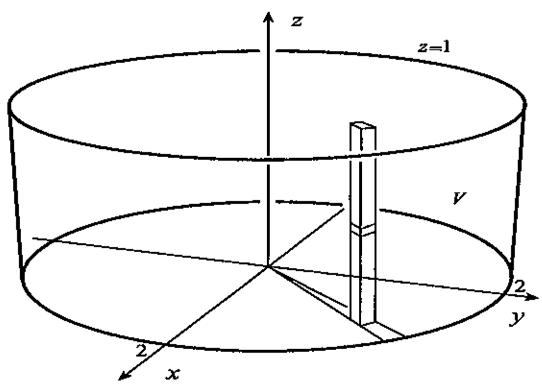


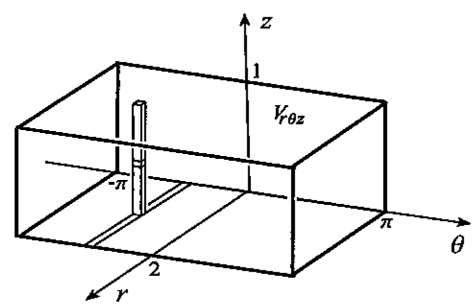
With $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \dfrac{1}{\begin{vmatrix} -2x & 1 \\ 4x & 1 \end{vmatrix}} = -\dfrac{1}{6x}$, equation 13.70 gives

$$\iint_R (x^2+y)\,dA = \iint_{R_{uv}} (x^2+y)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du = \frac{1}{6}\iint_{R_{uv}} \left(\frac{x^2+y}{x}\right) dv\,du$$

$$= \frac{1}{6}\int_0^4\int_5^6 \left(\sqrt{\frac{v-u}{3}} + \frac{(v+2u)/3}{\sqrt{(v-u)/3}}\right) dv\,du = \frac{1}{6\sqrt{3}}\int_0^4\int_5^6 \frac{2v+u}{\sqrt{v-u}}dv\,du.$$

5. (a) This is a change to cylindrical coordinates, $\displaystyle\iiint_V ze^{x^2+y^2}\,dV = \int_{-\pi}^{\pi}\int_0^2\int_0^1 ze^{r^2}\,r\,dz\,dr\,d\theta.$



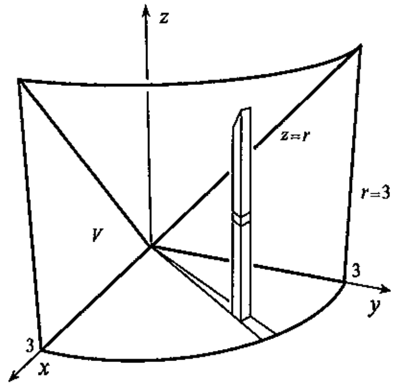


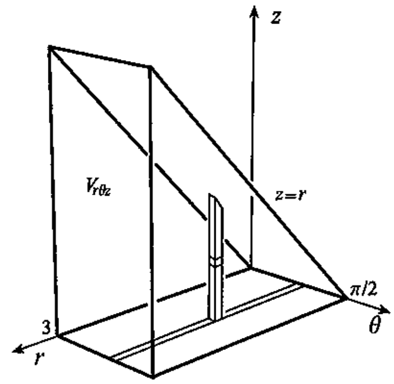
(b) Region V in xyz-space is mapped to the region $V_{r\theta z}$ in $r\theta z$-space shown above. With $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$, equation 13.73 gives

$$\iiint_V ze^{x^2+y^2}dV = \iiint_{V_{r\theta z}} ze^{x^2+y^2}\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| dz\,dr\,d\theta = \int_{-\pi}^{\pi}\int_0^2\int_0^1 ze^{r^2}\,r\,dz\,dr\,d\theta.$$

6. (a) This is a change to cylindrical coordinates,

$$\iiint_V (x^2+y^2)\,dV = \int_0^{\pi/2}\int_0^3\int_0^r r^2\,r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^3\int_0^r r^3\,dz\,dr\,d\theta.$$



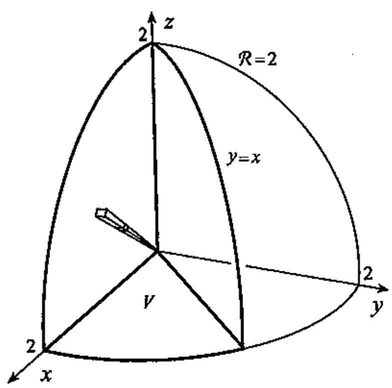


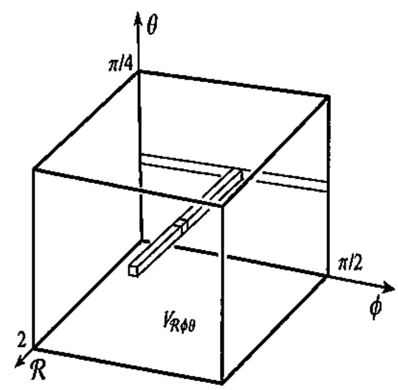
(b) Region V in xyz-space is mapped to the region $V_{r\theta z}$ in $r\theta z$-space shown above. With $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$, equation 13.73 gives

$$\iiint_V (x^2+y^2)\,dV = \iiint_{V_{r\theta z}} r^2 \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^3\int_0^r r^3\,dz\,dr\,d\theta.$$

7. (a) This is a change to spherical coordinates,

$$\iiint_V \frac{1}{x^2+y^2}dV = \int_0^{\pi/4}\int_0^{\pi/2}\int_0^2 \frac{1}{\Re^2\sin\phi}\Re^2\sin\phi\,d\Re\,d\phi\,d\theta = \int_0^{\pi/4}\int_0^{\pi/2}\int_0^2 d\Re\,d\phi\,d\theta.$$





(b) Region V in xyz-space is mapped to the region $V_{\Re\phi\theta}$ in $\Re\phi\theta$-space shown above. With

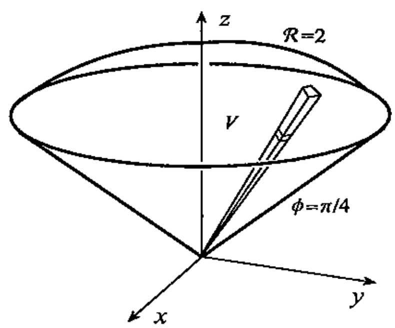
$$\begin{aligned}\frac{\partial(x,y,z)}{\partial(\Re,\phi,\theta)} &= \begin{vmatrix} \sin\phi\cos\theta & \Re\cos\phi\cos\theta & -\Re\sin\phi\sin\theta \\ \sin\phi\sin\theta & \Re\cos\phi\sin\theta & \Re\sin\phi\cos\theta \\ \cos\phi & -\Re\sin\phi & 0 \end{vmatrix} \\ &= \cos\phi(\Re^2\sin\phi\cos\phi\cos^2\theta + \Re^2\sin\phi\cos\phi\sin^2\theta) \\ &\quad + \Re\sin\phi(\Re\sin^2\phi\cos^2\theta + \Re\sin^2\phi\sin^2\theta) \\ &= \Re^2\cos^2\phi\sin\phi + \Re^2\sin^3\phi = \Re^2\sin\phi,\end{aligned}$$

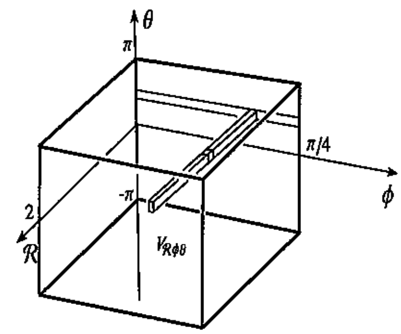
equation 13.73 gives

$$\iiint_V \frac{1}{x^2+y^2}dV = \iiint_{V_{\Re\phi\theta}} \frac{1}{\Re^2\sin\phi}\left|\frac{\partial(x,y,z)}{\partial(\Re,\phi,\theta)}\right| d\Re\,d\phi\,d\theta = \int_0^{\pi/4}\int_0^{\pi/2}\int_0^2 d\Re\,d\phi\,d\theta.$$

8. (a) This is a change to spherical coordinates,

$$\iiint_V x^2y^2z\,dV = \int_{-\pi}^{\pi}\int_0^{\pi/4}\int_0^2 (\Re^2\sin^2\phi\cos^2\theta)(\Re^2\sin^2\phi\sin^2\theta)(\Re\cos\phi)\,\Re^2\sin\phi\,d\Re\,d\phi\,d\theta$$
$$= \int_{-\pi}^{\pi}\int_0^{\pi/4}\int_0^2 \Re^7\sin^5\phi\cos\phi\sin^2\theta\cos^2\theta\,d\Re\,d\phi\,d\theta.$$





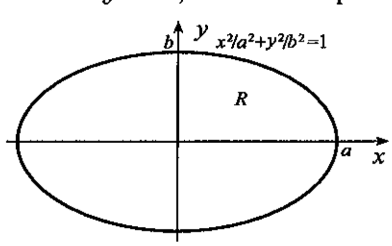
(b) Region V in xyz-space is mapped to the region $V_{\Re\phi\theta}$ in $\Re\phi\theta$-space shown above. With

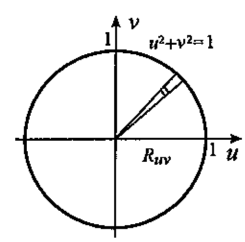
$$\frac{\partial(x,y,z)}{\partial(\Re,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \Re\cos\phi\cos\theta & -\Re\sin\phi\sin\theta \\ \sin\phi\sin\theta & \Re\cos\phi\sin\theta & \Re\sin\phi\cos\theta \\ \cos\phi & -\Re\sin\phi & 0 \end{vmatrix}$$
$$= \cos\phi(\Re^2\sin\phi\cos\phi\cos^2\theta + \Re^2\sin\phi\cos\phi\sin^2\theta)$$
$$+ \Re\sin\phi(\Re\sin^2\phi\cos^2\theta + \Re\sin^2\phi\sin^2\theta)$$
$$= \Re^2\cos^2\phi\sin\phi + \Re^2\sin^3\phi = \Re^2\sin\phi,$$

equation 13.73 gives

$$\iiint_V x^2y^2z\,dV = \iiint_{V_{\Re\phi\theta}} x^2y^2z\left|\frac{\partial(x,y,z)}{\partial(\Re,\phi,\theta)}\right| d\Re\,d\phi\,d\theta$$
$$= \iiint_{V_{\Re\phi\theta}} (\Re^2\sin^2\phi\cos^2\theta)(\Re^2\sin^2\phi\sin^2\theta)(\Re\cos\phi)\Re^2\sin\phi\,d\Re\,d\phi\,d\theta$$
$$= \int_{-\pi}^{\pi}\int_0^{\pi/4}\int_0^2 \Re^7\sin^5\phi\cos\phi\sin^2\theta\cos^2\theta\,d\Re\,d\phi\,d\theta.$$

9. If we let $x = au$ and $y = bv$, then the ellipse $x^2/a^2 + y^2/b^2 = 1$ is mapped to the circle $u^2 + v^2 = 1$.





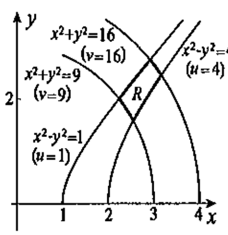
With $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$, equation 13.70 gives

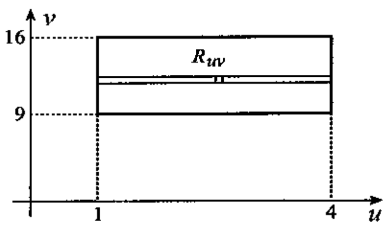
$$\iint_R \sqrt{x^2/a^2+y^2/b^2}\,dA = \iint_{R_{uv}} \sqrt{u^2+v^2}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \iint_{R_{uv}} \sqrt{u^2+v^2}(ab)\,du\,dv.$$

If we now use polar coordinates in the R_{uv}-plane,

$$\iint_R \sqrt{x^2/a^2+y^2/b^2}\,dA = ab\int_{-\pi}^{\pi}\int_0^1 r\,r\,dr\,d\theta = ab\int_{-\pi}^{\pi}\left\{\frac{r^3}{3}\right\}_0^1 d\theta = \frac{ab}{3}\left\{\theta\right\}_{-\pi}^{\pi} = \frac{2\pi ab}{3}.$$

10. If we let $u = x^2 - y^2$ and $v = x^2 + y^2$, then the region R in the xy-plane is mapped to the rectangle R_{uv} in the uv-plane shown below.

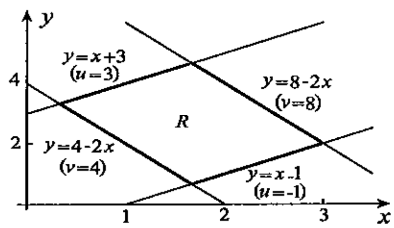


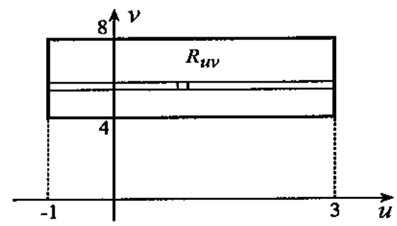


With $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \dfrac{1}{\begin{vmatrix} 2x & -2y \\ 2x & 2y \end{vmatrix}} = \dfrac{1}{8xy}$, equation 13.70 gives

$$\iint_R xy\,dA = \iint_{R_{uv}} xy \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \iint_{R_{uv}} xy \left|\frac{1}{8xy}\right| du\,dv = \frac{1}{8}\iint_{R_{uv}} du\,dv = \frac{1}{8}(\text{Area of } R_{uv}) = \frac{21}{8}.$$

11. If we let $u = y - x$ and $v = y + 2x$, then the region R in the xy-plane is mapped to the rectangle R_{uv} in the uv-plane shown below.



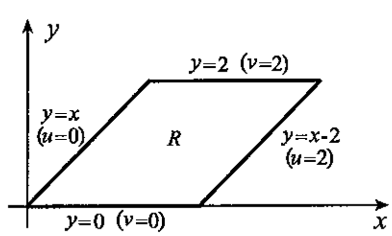


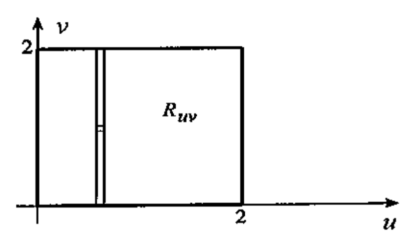
With $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \dfrac{1}{\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix}} = -\dfrac{1}{3}$, and the fact that $2x^2 - xy - y^2 = (2x+y)(x-y) = -uv$, equation 13.70 gives

$$\iint_R (2x^2 - xy - y^2)\,dA = \iint_{R_{uv}} (-uv)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = -\iint_{R_{uv}} uv\left|-\frac{1}{3}\right| du\,dv = -\frac{1}{3}\int_4^8\int_{-1}^3 uv\,du\,dv$$

$$= -\frac{1}{3}\int_4^8 \left\{\frac{u^2 v}{2}\right\}_{-1}^{3} dv = -\frac{4}{3}\left\{\frac{v^2}{2}\right\}_4^8 = -32.$$

12. The transformation $u = x - y$ and $v = y$, maps the parallelogram R in the xy-plane to the square R_{uv} in the uv-plane shown below.



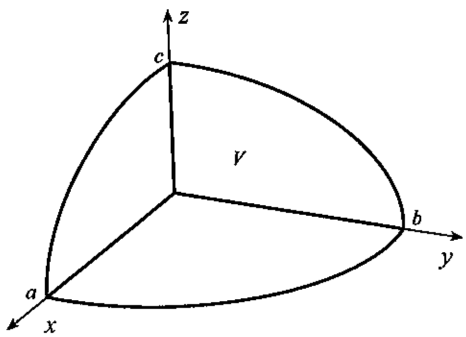


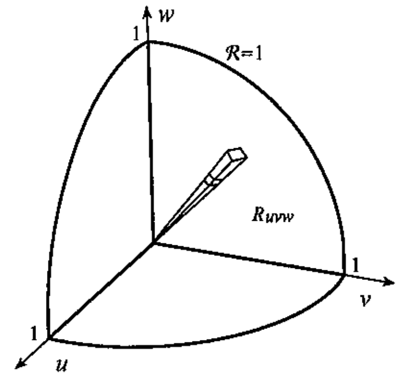
With $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \dfrac{1}{\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}} = 1$, equation 13.70 gives

$$\iint_R (x+y)\,dA = \iint_{R_{uv}} (u+2v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du = \iint_{R_{uv}} (u+2v)\,dv\,du = \int_0^2\int_0^2 (u+2v)\,dv\,du$$

$$= \int_0^2 \left\{uv + v^2\right\}_0^2 du = \int_0^2 (2u+4)\,du = \left\{u^2 + 4u\right\}_0^2 = 12.$$

13. If we let $u = x/a$, $v = y/b$, and $w = z/c$, then the region V in the first octant of xyz-space bounded by the ellipsoid is mapped to the first octant part of the sphere $u^2 + v^2 + w^2 = 1$ in uvw-space.





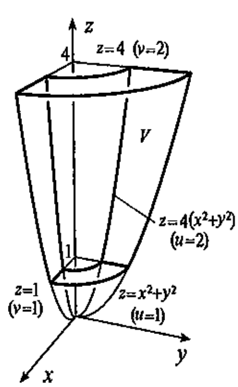
Since $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$, equation 13.73 gives

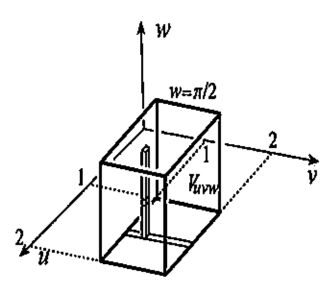
$$8\iiint_V x^2y^2z^2\,dV = 8\iiint_{R_{uvw}} (au)^2(bv)^2(cw)^2 \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw = 8a^3b^3c^3 \iiint_{R_{uvw}} u^2v^2w^2\,du\,dv\,dw.$$

If we now change to spherical coordinates in uvw-space,

$$\begin{aligned}
8\iiint_V x^2y^2z^2\,dV &= 8a^3b^3c^3 \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \Re^2\sin^2\phi\cos^2\theta\,\Re^2\sin^2\phi\sin^2\theta\,\Re^2\cos^2\phi\,\Re^2\sin\phi\,d\Re\,d\phi\,d\theta \\
&= 8a^3b^3c^3 \int_0^{\pi/2}\int_0^{\pi/2} \left\{\frac{\Re^9}{9}\sin^5\phi\cos^2\phi\sin^2\theta\cos^2\theta\right\}_0^1 d\phi\,d\theta \\
&= \frac{8a^3b^3c^3}{9}\int_0^{\pi/2}\int_0^{\pi/2} \cos^2\phi(1 - 2\cos^2\phi + \cos^4\phi)\sin\phi\sin^2\theta\cos^2\theta\,d\phi\,d\theta \\
&= \frac{8a^3b^3c^3}{9}\int_0^{\pi/2} \left\{\left(-\frac{1}{3}\cos^3\phi + \frac{2}{5}\cos^5\phi - \frac{1}{7}\cos^7\phi\right)\left(\frac{1}{4}\right)\sin^2 2\theta\right\}_0^{\pi/2} d\theta \\
&= \frac{16a^3b^3c^3}{945}\int_0^{\pi/2}\left(\frac{1 - \cos 4\theta}{2}\right)d\theta = \frac{8a^3b^3c^3}{945}\left\{\theta - \frac{1}{4}\sin 4\theta\right\}_0^{\pi/2} = \frac{4\pi a^3b^3c^3}{945}.
\end{aligned}$$

14. The transformation maps the first octant volume V bounded by the surfaces to the box V_{uvw} in uvw-space shown below.

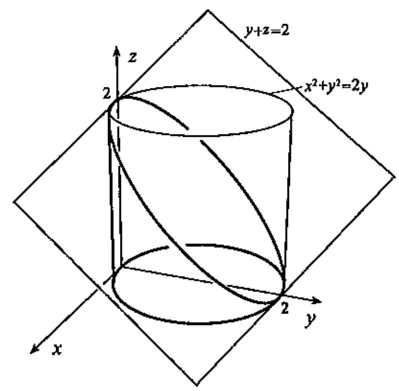




With $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} -(v/u^2)\cos w & (1/u)\cos w & -(v/u)\sin w \\ -(v/u^2)\sin w & (1/u)\sin w & (v/u)\cos w \\ 0 & 2v & 0 \end{vmatrix} = \dfrac{2v^3}{u^3}$, equation 13.73 gives

$$
\begin{aligned}
4\iiint_V (x^2+y^2)\,dV &= 4\iiint_{V_{uvw}} \left(\frac{v^2}{u^2}\cos^2 w + \frac{v^2}{u^2}\sin^2 w\right)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dw\,dv\,du \\
&= 4\iiint_{V_{uvw}} \left(\frac{v^2}{u^2}\right)\left(\frac{2v^3}{u^3}\right) dw\,dv\,du = 8\int_1^2\int_1^2\int_0^{\pi/2} \frac{v^5}{u^5} dw\,dv\,du \\
&= 8\int_1^2\int_1^2 \left\{\frac{v^5 w}{u^5}\right\}_0^{\pi/2} dv\,du = 4\pi\int_1^2 \left\{\frac{v^6}{6u^5}\right\}_1^2 du = 42\pi\left\{-\frac{1}{4u^4}\right\}_1^2 = \frac{315\pi}{32}.
\end{aligned}
$$

15. (a) With the usual cylindrical coordinates,

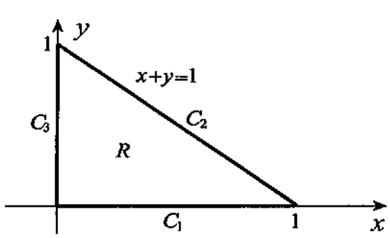


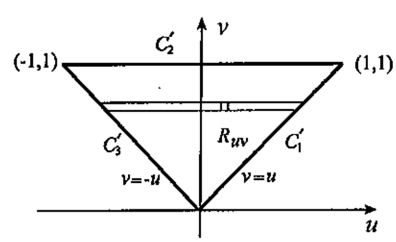
$$
\begin{aligned}
\iiint_V y\,dV &= 2\int_0^{\pi/2}\int_0^{2\sin\theta}\int_0^{2-r\sin\theta} r\sin\theta\, r\,dz\,dr\,d\theta \\
&= 2\int_0^{\pi/2}\int_0^{2\sin\theta} r^2\sin\theta(2-r\sin\theta)\,dr\,d\theta \\
&= 2\int_0^{\pi/2}\left\{\frac{2r^3\sin\theta}{3} - \frac{r^4\sin^2\theta}{4}\right\}_0^{2\sin\theta} d\theta \\
&= \frac{8}{3}\int_0^{\pi/2}(4\sin^4\theta - 3\sin^6\theta)\,d\theta \\
&= \frac{8}{3}\int_0^{\pi/2}\left[4\left(\frac{1-\cos 2\theta}{2}\right)^2 - 3\left(\frac{1-\cos 2\theta}{2}\right)^3\right] d\theta \\
&= \frac{8}{3}\int_0^{\pi/2}\left[1 - 2\cos 2\theta + \cos^2 2\theta - \frac{3}{8}(1 - 3\cos 2\theta + 3\cos^2 2\theta - \cos^3 2\theta)\right] d\theta \\
&= \frac{8}{3}\int_0^{\pi/2}\left[\frac{5}{8} - \frac{7}{8}\cos 2\theta - \frac{1}{16}(1+\cos 4\theta) + \frac{3}{8}\cos 2\theta(1-\sin^2 2\theta)\right] d\theta \\
&= \frac{8}{3}\left\{\frac{9\theta}{16} - \frac{1}{4}\sin 2\theta - \frac{1}{64}\sin 4\theta + \frac{1}{16}\sin^3 2\theta\right\}_0^{\pi/2} = \frac{3\pi}{4}.
\end{aligned}
$$

(b) With cylindrical coordinates based at $(0,1)$, $x = r\cos\theta$, $y = 1 + r\sin\theta$, and

$$
\begin{aligned}
\iiint_V y\,dV &= 2\int_0^{\pi}\int_0^1\int_0^{1-r\sin\theta}(1+r\sin\theta)r\,dz\,dr\,d\theta = 2\int_0^{\pi}\int_0^1 (1-r\sin\theta)(1+r\sin\theta)r\,dr\,d\theta \\
&= 2\int_0^{\pi}\int_0^1 (r - r^3\sin^2\theta)\,dr\,d\theta = 2\int_0^{\pi}\left\{\frac{r^2}{2} - \frac{r^4\sin^2\theta}{4}\right\}_0^1 d\theta \\
&= \int_0^{\pi}\left[1 - \frac{1}{4}(1-\cos 2\theta)\right] d\theta = \left\{\frac{3\theta}{4} + \frac{\sin 2\theta}{8}\right\}_0^{\pi} = \frac{3\pi}{4}.
\end{aligned}
$$

16. The transformation maps the triangle R in the xy-plane to the triangle R_{uv} in the uv-plane shown below.

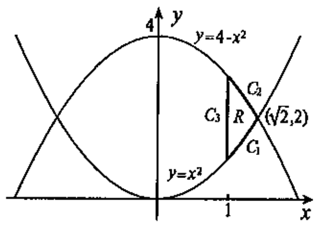


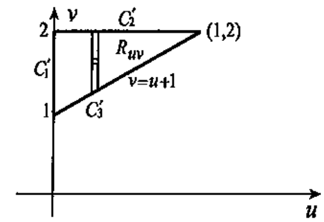


With $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \dfrac{1}{\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}} = \dfrac{1}{2}$, equation 13.70 gives

$$\iint_R \cos\left(\frac{x-y}{x+y}\right) dA = \iint_{R_{uv}} \cos\left(\frac{u}{v}\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \frac{1}{2}\int_0^1\int_{-v}^{v} \cos\left(\frac{u}{v}\right) du\,dv$$
$$= \frac{1}{2}\int_0^1 \left\{ v \sin\left(\frac{u}{v}\right)\right\}_{-v}^{v} dv = \sin 1 \int_0^1 v\,dv = \sin 1 \left\{\frac{v^2}{2}\right\}_0^1 = \frac{\sin 1}{2}.$$

17. The transformation maps the region R in the xy-plane to the triangle R_{uv} in the uv-plane shown below.

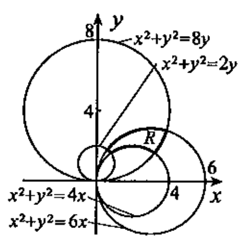


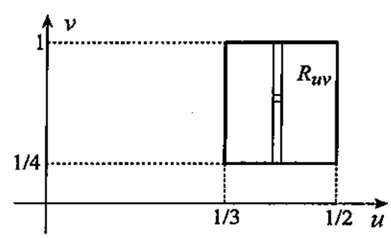


With $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\dfrac{1}{2\sqrt{v-u}} & \dfrac{1}{2\sqrt{v-u}} \\ 1 & 1 \end{vmatrix} = -\dfrac{1}{\sqrt{v-u}}$, equation 13.70 gives

$$\iint_R \frac{x}{x^2+y} dA = \iint_{R_{uv}} \frac{\sqrt{v-u}}{v-u+u+v}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du = \iint_{R_{uv}} \frac{\sqrt{v-u}}{2v}\left(\frac{1}{\sqrt{v-u}}\right) dv\,du$$
$$= \frac{1}{2}\int_0^1\int_{u+1}^{2} \frac{1}{v} dv\,du = \frac{1}{2}\int_0^1 \Big\{\ln|v|\Big\}_{u+1}^{2} du = \frac{1}{2}\int_0^1 [\ln 2 - \ln(u+1)]\,du$$
$$= \frac{1}{2}\Big\{u\ln 2 - (u+1)\ln|u+1| + u\Big\}_0^1 = \frac{1}{2}(1-\ln 2).$$

18. The transformation maps the region R in the xy-plane to the rectangle R_{uv} in the uv-plane shown below.

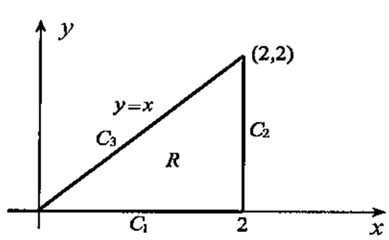


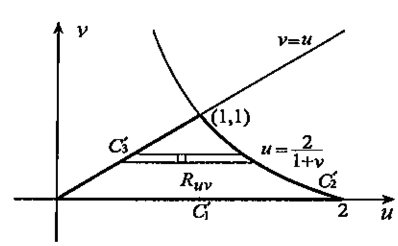


With $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \dfrac{1}{\begin{vmatrix} \dfrac{2(y^2-x^2)}{(x^2+y^2)^2} & \dfrac{-4xy}{(x^2+y^2)^2} \\ \dfrac{-4xy}{(x^2+y^2)^2} & \dfrac{2(x^2-y^2)}{(x^2+y^2)^2}\end{vmatrix}} = -\dfrac{(x^2+y^2)^2}{4}$, equation 13.70 gives

$$\iint_R \frac{1}{(x^2+y^2)^2} dA = \iint_{R_{uv}} \frac{1}{(x^2+y^2)^2}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du = \frac{1}{4}\int_{1/3}^{1/2}\int_{1/4}^{1} dv\,du = \frac{1}{4}\left(\frac{3}{4}\right)\left(\frac{1}{6}\right) = \frac{1}{32}.$$

19. The transformation maps the triangle R in the xy-plane to the region R_{uv} in the uv-plane shown below.





The Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix} = 1+u+v$, and

$$\frac{1}{\sqrt{(x-y)^2+2(x+y)+1}} = \frac{1}{\sqrt{(u-v)^2+2(u+v+2uv)+1}} = \frac{1}{\sqrt{(u+v+1)^2}} = \frac{1}{u+v+1}.$$

Equation 13.70 gives

$$\iint_R \frac{1}{\sqrt{(x-y)^2+2(x+y)+1}}\,dA = \iint_{R_{uv}} \frac{1}{u+v+1}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \iint_{R_{uv}} du\,dv = \int_0^1 \int_v^{2/(1+v)} du\,dv$$

$$= \int_0^1 \left(\frac{2}{1+v} - v\right) dv = \left\{2\ln|1+v| - \frac{v^2}{2}\right\}_0^1 = 2\ln 2 - \frac{1}{2}.$$

20. (a) We require two iterated integrals for direct evaluation,

$$\iiint_V (x+y+z)\,dV = \int_0^1 \int_{1-x}^{1+x} \int_{2x+y}^{2x+y+1} (x+y+z)\,dz\,dy\,dx + \int_1^2 \int_{x-1}^{3-x} \int_{2x+y}^{2x+y+1} (x+y+z)\,dz\,dy\,dx$$

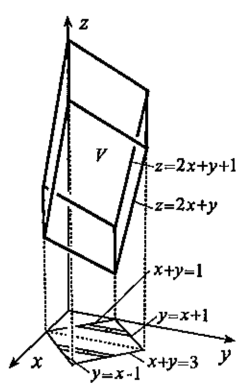
$$= \int_0^1 \int_{1-x}^{1+x} \left\{\frac{(x+y+z)^2}{2}\right\}_{2x+y}^{2x+y+1} dy\,dx + \int_1^2 \int_{x-1}^{3-x} \left\{\frac{(x+y+z)^2}{2}\right\}_{2x+y}^{2x+y+1} dy\,dx$$

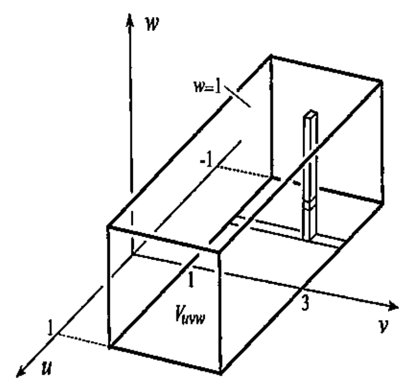
$$= \frac{1}{2}\int_0^1 \int_{1-x}^{1+x} (6x+4y+1)\,dy\,dx + \frac{1}{2}\int_1^2 \int_{x-1}^{3-x} (6x+4y+1)\,dy\,dx$$

$$= \frac{1}{2}\int_0^1 \left\{\frac{(6x+4y+1)^2}{8}\right\}_{1-x}^{1+x} dx + \frac{1}{2}\int_1^2 \left\{\frac{(6x+4y+1)^2}{8}\right\}_{x-1}^{3-x} dx$$

$$= \frac{1}{16}\int_0^1 [(10x+5)^2 - (2x+5)^2]\,dx + \frac{1}{16}\int_1^2 [(2x+13)^2 - (10x-3)^2]\,dx$$

$$= \frac{1}{16}\left\{\frac{(10x+5)^3}{30} - \frac{(2x+5)^3}{6}\right\}_0^1 + \frac{1}{16}\left\{\frac{(2x+13)^3}{6} - \frac{(10x-3)^3}{30}\right\}_1^2 = 11.$$





(b) The transformation maps the region V in xyz-space to the box V_{uvw} in uvw-space shown above. The Jacobian of the transformation is $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \dfrac{1}{\dfrac{\partial(u,v,w)}{\partial(x,y,z)}} = \dfrac{1}{\begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ -2 & -1 & 1 \end{vmatrix}} = \dfrac{1}{2}$. Equation 13.73 gives

$$\iiint_V (x+y+z)\,dV = \iiint_{V_{uvw}} (x+y+z)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dw\,dv\,du$$

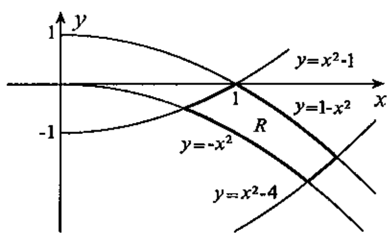
$$= \frac{1}{2}\iiint_{V_{uvw}} \left(\frac{u+v}{2} + \frac{v-u}{2} + w + \frac{u}{2} + \frac{3v}{2}\right) dw\,dv\,du$$

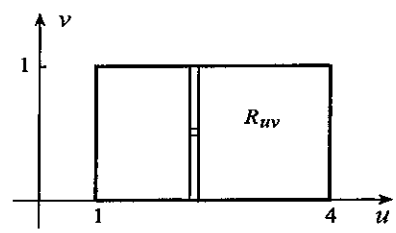
$$= \frac{1}{4}\int_{-1}^1 \int_1^3 \int_0^1 (2w+u+5v)\,dw\,dv\,du = \frac{1}{4}\int_{-1}^1 \int_1^3 \left\{w^2+uw+5vw\right\}_0^1 dv\,du$$

$$= \frac{1}{4}\int_{-1}^{1}\int_{1}^{3}(1+u+5v)\,dv\,du = \frac{1}{4}\int_{-1}^{1}\left\{\frac{(1+u+5v)^2}{10}\right\}_{1}^{3} du$$
$$= \frac{1}{40}\int_{-1}^{1}[(u+16)^2-(u+6)^2]\,du = \frac{1}{40}\left\{\frac{(u+16)^3}{3}-\frac{(u+6)^3}{3}\right\}_{-1}^{1} = 11.$$

21. (a) We require three iterated integrals for direct evaluation,

$$\iint_R (x+y)\,dA = \int_{1/\sqrt{2}}^{1}\int_{-x^2}^{x^2-1}(x+y)\,dy\,dx + \int_{1}^{\sqrt{2}}\int_{-x^2}^{1-x^2}(x+y)\,dy\,dx + \int_{\sqrt{2}}^{\sqrt{5/2}}\int_{x^2-4}^{1-x^2}(x+y)\,dy\,dx$$
$$= \int_{1/\sqrt{2}}^{1}\left\{\frac{(x+y)^2}{2}\right\}_{-x^2}^{x^2-1} dx + \int_{1}^{\sqrt{2}}\left\{\frac{(x+y)^2}{2}\right\}_{-x^2}^{1-x^2} dx + \int_{\sqrt{2}}^{\sqrt{5/2}}\left\{\frac{(x+y)^2}{2}\right\}_{x^2-4}^{1-x^2} dx$$
$$= \frac{1}{2}\int_{1/\sqrt{2}}^{1}(4x^3-2x^2-2x+1)\,dx + \frac{1}{2}\int_{1}^{\sqrt{2}}(1+2x-2x^2)\,dx + \frac{1}{2}\int_{\sqrt{2}}^{\sqrt{5/2}}(-4x^3+6x^2+10x-15)\,dx$$
$$= \frac{1}{2}\left\{x^4-\frac{2x^3}{3}-x^2+x\right\}_{1/\sqrt{2}}^{1} + \frac{1}{2}\left\{x+x^2-\frac{2x^3}{3}\right\}_{1}^{\sqrt{2}} + \frac{1}{2}\left\{-x^4+2x^3+5x^2-15x\right\}_{\sqrt{2}}^{\sqrt{5/2}}$$
$$= \frac{1}{12}(9+62\sqrt{2}-30\sqrt{10}).$$





(b) The transformation maps the region R in the xy-plane to the rectangle R_{uv} in the uv-plane shown above. The Jacobian of the transformation is $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \dfrac{1}{\begin{vmatrix} 2x & -1 \\ 2x & 1 \end{vmatrix}} = \dfrac{1}{4x}$. Equation 13.70 gives

$$\iint_R (x+y)\,dA = \iint_{R_{uv}}(x+y)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du = \iint_{R_{uv}}(x+y)\left(\frac{1}{4x}\right) dv\,du$$
$$= \frac{1}{4}\int_{1}^{4}\int_{0}^{1}\left(\sqrt{\frac{u+v}{2}}+\frac{v-u}{2}\right)\sqrt{\frac{2}{u+v}}\,dv\,du = \frac{1}{4}\int_{1}^{4}\int_{0}^{1}\left(1+\frac{v-u}{\sqrt{2}\sqrt{u+v}}\right) dv\,du$$
$$= \frac{1}{4}\int_{1}^{4}\left\{v\right\}_{0}^{1} du - \frac{\sqrt{2}}{8}\int_{1}^{4}\left\{2u\sqrt{u+v}\right\}_{0}^{1} du + \frac{\sqrt{2}}{8}\int_{1}^{4}\int_{0}^{1}\frac{v}{\sqrt{u+v}}\,dv\,du.$$

We set $z=u+v$ and $dz=dv$ in the last integral,

$$\iint_R (x+y)\,dA = \frac{1}{4}\left\{u\right\}_{1}^{4} - \frac{\sqrt{2}}{4}\int_{1}^{4}(u\sqrt{u+1}-u^{3/2})\,du + \frac{\sqrt{2}}{8}\int_{1}^{4}\int_{u}^{u+1}\frac{z-u}{\sqrt{z}}\,dz\,du$$
$$= \frac{3}{4} - \frac{\sqrt{2}}{4}\int_{1}^{4}u\sqrt{u+1}\,du + \frac{\sqrt{2}}{4}\left\{\frac{2u^{5/2}}{5}\right\}_{1}^{4} + \frac{\sqrt{2}}{8}\int_{1}^{4}\left\{\frac{2z^{3/2}}{3}-2u\sqrt{z}\right\}_{u}^{u+1} du$$
$$= \frac{3}{4} - \frac{\sqrt{2}}{4}\int_{1}^{4}u\sqrt{u+1}\,du + \frac{31\sqrt{2}}{10} + \frac{\sqrt{2}}{8}\int_{1}^{4}\left[\frac{2(u+1)^{3/2}}{3}-2u\sqrt{u+1}+\frac{4u^{3/2}}{3}\right] du.$$

We now combine the first integral and the second term in the last integral and set $z=u+1$ and $dz=du$,

$$\iint_R (x+y)\,dA = \frac{3}{4} - \frac{\sqrt{2}}{2}\int_2^5 (z-1)\sqrt{z}\,dz + \frac{31\sqrt{2}}{10} + \frac{\sqrt{2}}{8}\left\{\frac{4(u+1)^{5/2}}{15} + \frac{8u^{5/2}}{15}\right\}_1^4$$
$$= \frac{3}{4} - \frac{\sqrt{2}}{2}\left\{\frac{2z^{5/2}}{5} - \frac{2z^{3/2}}{3}\right\}_2^5 + \frac{31\sqrt{2}}{10} + \frac{\sqrt{2}}{30}(25\sqrt{5} + 62 - 4\sqrt{2})$$
$$= \frac{1}{12}(9 + 62\sqrt{2} - 30\sqrt{10}).$$

EXERCISES 13.14

1. With Leibnitz's rule, $F'(x) = \int_0^3 (2xy^2 + 3y)\,dy = \left\{\frac{2xy^3}{3} + \frac{3y^2}{2}\right\}_0^3 = 18x + \frac{27}{2}$. If we evaluate the integral, $F(x) = \int_0^3 (x^2y^2 + 3xy)\,dy = \left\{\frac{x^2y^3}{3} + \frac{3xy^2}{2}\right\}_0^3 = 9x^2 + \frac{27x}{2}$, and therefore $F'(x) = 18x + 27/2$.

2. With Leibnitz's rule, $F'(x) = \int_1^x \left(\frac{2x}{y^2}\right) dy + \left(\frac{x^2}{x^2} + e^x\right)(1) = \left\{-\frac{2x}{y}\right\}_1^x + 1 + e^x = e^x + 2x - 1$. If we evaluate the integral, $F(x) = \left\{-\frac{x^2}{y} + e^y\right\}_1^x = -x + e^x + x^2 - e$, in which case $F'(x) = -1 + e^x + 2x$.

3. With Leibnitz's rule,
$$F'(x) = \int_{x-1}^{x^2} 3x^2y\,dy + (x^5 + x^4 + 1)(2x) - [x^3(x-1) + (x-1)^2 + 1](1)$$
$$= \left\{\frac{3x^2y^2}{2}\right\}_{x-1}^{x^2} + 2x^6 + 2x^5 - x^4 + x^3 - x^2 + 4x - 2$$
$$= \frac{3x^6}{2} - \frac{3x^2(x-1)^2}{2} + 2x^6 + 2x^5 - x^4 + x^3 - x^2 + 4x - 2$$
$$= \frac{7x^6}{2} + 2x^5 - \frac{5x^4}{2} + 4x^3 - \frac{5x^2}{2} + 4x - 2.$$

If we evaluate the integral,
$$F(x) = \int_{x-1}^{x^2} (x^3y + y^2 + 1)\,dy = \left\{\frac{x^3y^2}{2} + \frac{y^3}{3} + y\right\}_{x-1}^{x^2}$$
$$= \frac{x^7}{2} + \frac{x^6}{3} + x^2 - \frac{x^3(x-1)^2}{2} - \frac{(x-1)^3}{3} - (x-1),$$
and therefore
$$F'(x) = \frac{7x^6}{2} + 2x^5 + 2x - \frac{3x^2(x-1)^2}{2} - x^3(x-1) - (x-1)^2 - 1$$
$$= \frac{7x^6}{2} + 2x^5 - \frac{5x^4}{2} + 4x^3 - \frac{5x^2}{2} + 4x - 2.$$

4. With Leibnitz's rule,
$$F'(x) = \int_{x^2}^{x^3-1} dy + [x + (x^3-1)\ln(x^3-1)](3x^2) - [x + x^2\ln(x^2)](2x)$$
$$= x^3 - 1 - x^2 + 3x^2[x + (x^3-1)\ln(x^3-1)] - 2x[x + x^2\ln(x^2)]$$
$$= 4x^3 - 3x^2 - 1 + 3x^2(x^3-1)\ln(x^3-1) - 2x^3\ln(x^2).$$

If we evaluate the integral,
$$F(x) = \left\{xy + \frac{y^2}{2}\ln y - \frac{y^2}{4}\right\}_{x^2}^{x^3-1} = x(x^3-1) + \frac{1}{2}(x^3-1)^2\ln(x^3-1) - \frac{1}{4}(x^3-1)^2 - x^3$$
$$- \frac{x^4}{2}\ln(x^2) + \frac{x^4}{4},$$

in which case
$$F'(x) = 4x^3 - 1 + 3x^2(x^3-1)\ln(x^3-1) + \frac{1}{2}(x^3-1)(3x^2) - \frac{1}{2}(x^3-1)(3x^2)$$

$$-3x^2 - 2x^3\ln(x^2) - x^3 + x^3$$
$$= 4x^3 - 3x^2 - 1 + 3x^2(x^3-1)\ln(x^3-1) - 2x^3\ln(x^2).$$

5. With Leibnitz's rule,

$$F'(x) = \int_0^x \left[\frac{(y+x)(-1)-(y-x)(1)}{(y+x)^2}\right]dy = \int_0^x \frac{-2y}{(y+x)^2}dy = -2\int_0^x \left[\frac{1}{y+x} - \frac{x}{(y+x)^2}\right]dy$$
$$= -2\left\{\ln|y+x| + \frac{x}{y+x}\right\}_0^x = -2\left(\ln|2x| + \frac{1}{2} - \ln|x| - 1\right) = 1 - 2\ln 2.$$

If we evaluate the integral, $F(x) = \displaystyle\int_0^x \frac{y-x}{y+x}dy = \int_0^x \left(1 - \frac{2x}{y+x}\right)dy = \Big\{y - 2x\ln|y+x|\Big\}_0^x$
$$= x - 2x\ln|2x| + 2x\ln|x| = x - (2\ln 2)x,$$
and therefore, $F'(x) = 1 - 2\ln 2$.

6. $F(x) = \left\{\dfrac{y^4}{4}\ln y - \dfrac{y^4}{16} + x^3e^y\right\}_x^{2x} = (4\ln 2)x^4 + \dfrac{15x^4}{4}\ln x - \dfrac{15x^4}{16} + x^3(e^{2x} - e^x)$, and therefore
$$F'(x) = (16\ln 2)x^3 + 15x^3\ln x + \frac{15x^3}{4} - \frac{15x^3}{4} + 3x^2(e^{2x}-e^x) + x^3(2e^{2x}-e^x)$$
$$= x^3(16\ln 2 + 15\ln x + 2e^{2x} - e^x) + 3x^2(e^{2x}-e^x).$$

7. Using Example 13.39,
$$\int_0^1 \frac{x^p - x^q}{\ln x}dx = \int_0^1 \frac{x^p-1}{\ln x}dx - \int_0^1 \frac{x^q-1}{\ln x}dx = \ln(p+1) - \ln(q+1) = \ln\left(\frac{p+1}{q+1}\right).$$

8. With Leibnitz's rule,
$$F'(x) = \int_{\sin x}^{e^x} 0\,dy + \sqrt{1+e^{3x}}(e^x) - \sqrt{1+\sin^3 x}(\cos x) = e^x\sqrt{1+e^{3x}} - \cos x\sqrt{1+\sin^3 x}.$$

9. $x\dfrac{dy}{dx} + 2y = x\left[-\dfrac{2}{x^3}\displaystyle\int_0^x t^2 f(t)\,dt + \dfrac{1}{x^2}x^2 f(x)\right] + \dfrac{2}{x^2}\displaystyle\int_0^x tf(t)\,dt = xf(x)$

10. Since $\dfrac{dy}{dx} = \dfrac{1}{2}\displaystyle\int_0^x f(t)(e^{x-t}+e^{t-x})\,dt$, it follows that $\dfrac{d^2y}{dx^2} = \dfrac{1}{2}\displaystyle\int_0^x f(t)(e^{x-t}-e^{t-x})\,dt + \dfrac{1}{2}f(x)(2)(1)$, and therefore
$$\frac{d^2y}{dx^2} - y = \frac{1}{2}\int_0^x f(t)(e^{x-t}-e^{t-x})\,dt + f(x) - \frac{1}{2}\int_0^x f(t)(e^{x-t}-e^{t-x})\,dt = f(x).$$

11. Since $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{2}}\displaystyle\int_0^x \{-2e^{2(t-x)}\sin[\sqrt{2}(x-t)] + \sqrt{2}e^{2(t-x)}\cos[\sqrt{2}(x-t)]\}f(t)\,dt$, and
$$\frac{d^2y}{dx^2} = \frac{1}{\sqrt{2}}\int_0^x \{4e^{2(t-x)}\sin[\sqrt{2}(x-t)] - 4\sqrt{2}e^{2(t-x)}\cos[\sqrt{2}(x-t)] - 2e^{2(t-x)}\sin[\sqrt{2}(x-t)]\}f(t)\,dt + f(x),$$
it follows that
$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 6y = \frac{1}{\sqrt{2}}\int_0^x \{2e^{2(t-x)}\sin[\sqrt{2}(x-t)] - 4\sqrt{2}e^{2(t-x)}\cos[\sqrt{2}(x-t)]\}f(t)\,dt + f(x)$$
$$+ \frac{4}{\sqrt{2}}\int_0^x \{-2e^{2(t-x)}\sin[\sqrt{2}(x-t)] + \sqrt{2}e^{2(t-x)}\cos[\sqrt{2}(x-t)]\}f(t)\,dt$$
$$+ \frac{6}{\sqrt{2}}\int_0^x e^{2(t-x)}\sin[\sqrt{2}(x-t)]f(t)\,dt = f(x).$$

12. Differentiation of $\int_0^b \frac{1}{1+ax}dx = \frac{1}{a}\ln(1+ab)$ with respect to a gives

$$\int_0^b \frac{-x}{(1+ax)^2}dx = -\frac{1}{a^2}\ln(1+ab) + \frac{1}{a}\frac{b}{1+ab} \implies \int_0^b \frac{x}{(1+ax)^2}dx = \frac{1}{a^2}\ln(1+ab) - \frac{b}{a(1+ab)}.$$

13. If we write $\int_0^b \frac{1}{\sqrt{a^2-x^2}}dx = \mathrm{Sin}^{-1}\left(\frac{b}{a}\right)$, then differentiation with respect to a gives

$$\int_0^b \frac{-a}{(a^2-x^2)^{3/2}}dx = \frac{1}{\sqrt{1-b^2/a^2}}\left(\frac{-b}{a^2}\right),$$

and therefore, $\int_0^b \frac{1}{(a^2-x^2)^{3/2}}dx = -\frac{1}{a}\left[\frac{a}{\sqrt{a^2-b^2}}\left(\frac{-b}{a^2}\right)\right] = \frac{b}{a^2\sqrt{a^2-b^2}}.$

Thus, $\int \frac{1}{(a^2-x^2)^{3/2}}dx = \frac{x}{a^2\sqrt{a^2-x^2}} + C.$

14. If we write $\int_0^b \frac{1}{a^2+x^2}dx = \frac{1}{a}\mathrm{Tan}^{-1}\left(\frac{b}{a}\right)$, then differentiation with respect to a gives

$$\int_0^b \frac{-2a}{(a^2+x^2)^2}dx = -\frac{1}{a^2}\mathrm{Tan}^{-1}\left(\frac{b}{a}\right) + \frac{1}{a}\frac{1}{1+b^2/a^2}\left(\frac{-b}{a^2}\right),$$

or,

$$\int_0^b \frac{1}{(a^2+x^2)^2}dx = \frac{1}{2a^3}\mathrm{Tan}^{-1}\left(\frac{b}{a}\right) + \frac{b}{2a^2(a^2+b^2)}.$$

Another differentiation gives

$$\int_0^b \frac{-4a}{(a^2+x^2)^3}dx = -\frac{3}{2a^4}\mathrm{Tan}^{-1}\left(\frac{b}{a}\right) + \frac{1}{2a^3}\frac{1}{1+b^2/a^2}\left(\frac{-b}{a^2}\right) - \frac{b(8a^3+4ab^2)}{4a^4(a^2+b^2)^2},$$

or,

$$\begin{aligned}\int_0^b \frac{1}{(a^2+x^2)^3}dx &= -\frac{1}{4a}\left[-\frac{3}{2a^4}\mathrm{Tan}^{-1}\left(\frac{b}{a}\right) - \frac{b}{2a^3(a^2+b^2)} - \frac{b(2a^2+b^2)}{a^3(a^2+b^2)^2}\right]\\ &= \frac{3}{8a^5}\mathrm{Tan}^{-1}\left(\frac{b}{a}\right) + \frac{b(3b^2+5a^2)}{8a^4(a^2+b^2)^2}.\end{aligned}$$

Thus, $\int \frac{1}{(a^2+x^2)^3}dx = \frac{3}{8a^5}\mathrm{Tan}^{-1}\left(\frac{x}{a}\right) + \frac{x(3x^2+5a^2)}{8a^4(a^2+x^2)^2} + C.$

15. Differentiation of the given formula with respect to a and b gives

$$\int_0^{\pi/2} \frac{-2a\cos^2 x}{(a^2\cos^2 x + b^2\sin^2 x)^2}dx = \frac{-\pi}{2a|ab|}, \qquad \int_0^{\pi/2} \frac{-2b\sin^2 x}{(a^2\cos^2 x + b^2\sin^2 x)^2}dx = \frac{-\pi}{2b|ab|}.$$

If we divide the first by $-2a$, the second by $-2b$, and add, the result is

$$\int_0^{\pi/2} \frac{1}{(a^2\cos^2 x + b^2\sin^2 x)^2}dx = \frac{\pi(a^2+b^2)}{4|ab|^3}.$$

16. If we set $F(a) = \int_0^\pi \frac{\ln(1+a\cos x)}{\cos x}dx$, then $F'(a) = \int_0^\pi \frac{1}{1+a\cos x}dx.$

To evaluate this integral we let $t = \tan\frac{x}{2}$. Then $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2}{1+t^2}dt$ (see Exercise 35 in Section 8.6), and

$$F'(a) = \int_0^\infty \frac{1}{1 + a\left(\dfrac{1-t^2}{1+t^2}\right)} \frac{2}{1+t^2} dt = 2\int_0^\infty \frac{1}{1+t^2+a(1-t^2)} dt = 2\int_0^\infty \frac{1}{(a+1)+(1-a)t^2} dt.$$

We now set $t = \sqrt{(1+a)/(1-a)}\tan\theta$ and $dt = \sqrt{(1+a)/(1-a)}\sec^2\theta\, d\theta$,

$$F'(a) = 2\int_0^{\pi/2} \frac{1}{(a+1)+(a+1)\tan^2\theta}\sqrt{\frac{1+a}{1-a}}\sec^2\theta\, d\theta = 2\left\{\frac{\theta}{\sqrt{1-a^2}}\right\}_0^{\pi/2} = \frac{\pi}{\sqrt{1-a^2}}.$$

Hence, $F(a) = \pi \text{Sin}^{-1} a + C$. Since $F(0) = 0$, it follows that $0 = C$, and $F(a) = \pi\text{Sin}^{-1} a$.

17. If we set $F(a) = \displaystyle\int_0^\infty \frac{\text{Tan}^{-1}(ax)}{x(1+x^2)} dx$, then differentiation with respect to a gives

$$\begin{aligned} F'(a) &= \int_0^\infty \frac{x}{x(1+x^2)(1+a^2x^2)} dx = \int_0^\infty \left(\frac{1/(1-a^2)}{1+x^2} - \frac{a^2/(1-a^2)}{1+a^2x^2}\right) dx \\ &= \left\{\frac{1}{1-a^2}\text{Tan}^{-1}x - \frac{a}{1-a^2}\text{Tan}^{-1}(ax)\right\}_0^\infty = \frac{\pi/2}{1+a}. \end{aligned}$$

Integration gives $F(a) = \dfrac{\pi}{2}\ln(1+a) + C$. Since $F(0) = 0$, it follows that $C = 0$ and therefore $F(a) = (\pi/2)\ln(1+a)$.

18. We calculate:

$$\begin{aligned} \frac{\partial T}{\partial t} &= e^{-(1-x)^2/(4t)}\left(\frac{x-1}{4t^{3/2}}\right) + e^{-(1+x)^2/(4t)}\left(\frac{-x-1}{4t^{3/2}}\right), \\ \frac{\partial T}{\partial x} &= e^{-(1-x)^2/(4t)}\left(\frac{-1}{2\sqrt{t}}\right) + e^{-(1+x)^2/(4t)}\left(\frac{1}{2\sqrt{t}}\right), \\ \frac{\partial^2 T}{\partial x^2} &= e^{-(1-x)^2/(4t)}\left(\frac{x-1}{4t^{3/2}}\right) + e^{-(1+x)^2/(4t)}\left(\frac{-1-x}{4t^{3/2}}\right). \end{aligned}$$

Thus, $\partial T/\partial t = \partial^2 T/\partial x^2$.

19. (a) Since $1 - x^2y^2$ must be positive, it follows that $x^2 < 1/y^2$. Since y ranges from 0 to 9, it follows that x must be restricted to $-1/9 < x < 1/9$. Clearly, $F(0) = \displaystyle\int_0^9 \ln(1)\, dy = 0$.

(b)
$$\begin{aligned} F'(x) &= \int_0^9 \frac{-2xy^2}{1-x^2y^2} dy = \int_0^9 \left(\frac{2}{x} + \frac{1/x}{xy-1} - \frac{1/x}{xy+1}\right) dy \\ &= \left\{\frac{2y}{x} + \frac{1}{x^2}\ln|xy-1| - \frac{1}{x^2}\ln|xy+1|\right\}_0^9 = \frac{18}{x} + \frac{1}{x^2}\ln\left(\frac{1-9x}{1+9x}\right), \quad x \neq 0. \end{aligned}$$

From $F'(x) = \displaystyle\int_0^9 \frac{-2xy^2}{1-x^2y^2} dy$, we obtain $F'(0) = 0$.

(c) If we use Leibnitz's rule,

$$F''(x) = \int_0^9 \left[\frac{(1-x^2y^2)(-2y^2) - (-2xy^2)(-2xy^2)}{(1-x^2y^2)^2}\right] dy = \int_0^9 \frac{-2y^2(1+x^2y^2)}{(1-x^2y^2)^2} dy.$$

Since $F''(x)$ is clearly negative, the graph of $F(x)$ is always concave downward.

20. We calculate that

$$\begin{aligned} \frac{\partial u}{\partial r} &= -\frac{2r}{2\pi}\int_{-\pi}^{\pi} \frac{u(R,\phi)}{R^2+r^2-2rR\cos(\theta-\phi)} d\phi + \frac{R^2-r^2}{2\pi}\int_{-\pi}^{\pi} \frac{-u(R,\phi)[2r-2R\cos(\theta-\phi)]}{[R^2+r^2-2rR\cos(\theta-\phi)]^2} d\phi \\ &= -\frac{r}{\pi}\int_{-\pi}^{\pi} \frac{u(R,\phi)}{R^2+r^2-2rR\cos(\theta-\phi)} d\phi + \frac{R^2-r^2}{\pi}\int_{-\pi}^{\pi} \frac{-u(R,\phi)[r-R\cos(\theta-\phi)]}{[R^2+r^2-2rR\cos(\theta-\phi)]^2} d\phi \\ \frac{\partial u}{\partial \theta} &= \frac{R^2-r^2}{2\pi}\int_{-\pi}^{\pi} \frac{-u(R,\phi)[2rR\sin(\theta-\phi)]}{[R^2+r^2-2rR\cos(\theta-\phi)]^2} d\phi = \frac{R^2-r^2}{\pi}\int_{-\pi}^{\pi} \frac{-u(R,\phi)rR\sin(\theta-\phi)}{[R^2+r^2-2rR\cos(\theta-\phi)]^2} d\phi \end{aligned}$$

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{u(R,\phi)}{R^2+r^2-2rR\cos(\theta-\phi)}d\phi - \frac{r}{\pi}\int_{-\pi}^{\pi}\frac{-u(R,\phi)[2r-2R\cos(\theta-\phi)]}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}d\phi$$
$$-\frac{2r}{\pi}\int_{-\pi}^{\pi}\frac{u(R,\phi)[R\cos(\theta-\phi)-r]}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}d\phi + \frac{R^2-r^2}{\pi}\int_{-\pi}^{\pi}u(R,\phi)\left\{\frac{-1}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}\right.$$
$$\left.+\frac{4[R\cos(\theta-\phi)-r]^2}{[R^2+r^2-2rR\cos(\theta-\phi)]^3}\right\}d\phi$$
$$= -\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{u(R,\phi)}{R^2+r^2-2rR\cos(\theta-\phi)}d\phi - \frac{4r}{\pi}\int_{-\pi}^{\pi}\frac{u(R,\phi)[R\cos(\theta-\phi)-r]}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}d\phi$$
$$+\frac{R^2-r^2}{\pi}\int_{-\pi}^{\pi}u(R,\phi)\left\{\frac{-1}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}+\frac{4[R\cos(\theta-\phi)-r]^2}{[R^2+r^2-2rR\cos(\theta-\phi)]^3}\right\}d\phi$$
$$\frac{\partial^2 u}{\partial \theta^2} = \frac{R^2-r^2}{\pi}\int_{-\pi}^{\pi}-u(R,\phi)\left\{\frac{rR\cos(\theta-\phi)}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}-\frac{4r^2R^2\sin^2(\theta-\phi)}{[R^2+r^2-2rR\cos(\theta-\phi)]^3}\right\}d\phi.$$

With these,

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial r^2} = -\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{u(R,\phi)}{R^2+r^2-2rR\cos(\theta-\phi)}d\phi - \frac{4r}{\pi}\int_{-\pi}^{\pi}\frac{u(R,\phi)[R\cos(\theta-\phi)-r]}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}d\phi$$
$$+\frac{R^2-r^2}{\pi}\int_{-\pi}^{\pi}u(R,\phi)\left\{\frac{-1}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}+\frac{4[R\cos(\theta-\phi)-r]^2}{[R^2+r^2-2rR\cos(\theta-\phi)]^3}\right\}d\phi$$
$$-\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{u(R,\phi)}{R^2+r^2-2rR\cos(\theta-\phi)}d\phi + \frac{R^2-r^2}{\pi r}\int_{-\pi}^{\pi}\frac{-u(R,\phi)[r-R\cos(\theta-\phi)]}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}d\phi$$
$$\frac{R^2-r^2}{\pi r^2}\int_{-\pi}^{\pi}-u(R,\phi)\left\{\frac{rR\cos(\theta-\phi)}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}-\frac{4r^2R^2\sin^2(\theta-\phi)}{[R^2+r^2-2rR\cos(\theta-\phi)]^3}\right\}d\phi$$
$$= -\frac{2}{\pi}\int_{-\pi}^{\pi}\frac{u(R,\phi)}{R^2+r^2-2rR\cos(\theta-\phi)}d\phi + \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{u(R,\phi)\{-4r[R\cos(\theta-\phi)-r]-2R^2+2r^2\}}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}d\phi$$
$$+\frac{R^2-r^2}{\pi}\int_{-\pi}^{\pi}\frac{4u(R,\phi)\{[R\cos(\theta-\phi)-r]^2+R^2\sin^2(\theta-\phi)\}}{[R^2+r^2-2rR\cos(\theta-\phi)]^3}d\phi$$
$$= -\frac{2}{\pi}\int_{-\pi}^{\pi}\frac{u(R,\phi)}{R^2+r^2-2rR\cos(\theta-\phi)}d\phi - \frac{2}{\pi}\int_{-\pi}^{\pi}\frac{u(R,\phi)[R^2-3r^2+2rR\cos(\theta-\phi)]}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}d\phi$$
$$+\frac{R^2-r^2}{\pi}\int_{-\pi}^{\pi}\frac{4u(R,\phi)[R^2+r^2-2rR\cos(\theta-\phi)]}{[R^2+r^2-2rR\cos(\theta-\phi)]^3}d\phi$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{u(R,\phi)\{-2[R^2+r^2-2rR\cos(\theta-\phi)]-2[R^2-3r^2+2rR\cos(\theta-\phi)]+4R^2-4r^2\}}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}d\phi = 0.$$

21. Using Leibnitz's rule:

$$\frac{\partial u}{\partial r} = -\frac{R}{2\pi}\int_{-\pi}^{\pi}\frac{f(u)[2r-2R\cos(\theta-u)]}{R^2+r^2-2rR\cos(\theta-u)}du;$$
$$\frac{\partial u}{\partial \theta} = -\frac{R}{2\pi}\int_{-\pi}^{\pi}\frac{f(u)[2rR\sin(\theta-u)]}{R^2+r^2-2rR\cos(\theta-u)}du;$$
$$\frac{\partial^2 u}{\partial r^2} = -\frac{R}{2\pi}\int_{-\pi}^{\pi}f(u)\left\{\frac{2}{R^2+r^2-2rR\cos(\theta-u)}-\frac{[2r-2R\cos(\theta-u)]^2}{[R^2+r^2-2rR\cos(\theta-u)]^2}\right\}du;$$
$$\frac{\partial^2 u}{\partial \theta^2} = -\frac{R}{2\pi}\int_{-\pi}^{\pi}f(u)\left\{\frac{2rR\cos(\theta-u)}{R^2+r^2-2rR\cos(\theta-u)}-\frac{[2rR\sin(\theta-u)]^2}{[R^2+r^2-2rR\cos(\theta-u)]^2}\right\}du.$$

Thus,

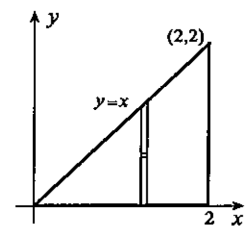
$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{R}{2\pi}\int_{-\pi}^{\pi}f(u)\left\{\frac{2[R^2+r^2-2rR\cos(\theta-u)]-[2r-2R\cos(\theta-u)]^2}{[R^2+r^2-2rR\cos(\theta-u)]^2}du\right\}du$$

$$- \frac{R}{2\pi r}\int_{-\pi}^{\pi}\frac{f(u)[2r-2R\cos(\theta-u)]}{R^2+r^2-2rR\cos(\theta-u)}\,du$$
$$- \frac{R}{2\pi r^2}\int_{-\pi}^{\pi} f(u)\left\{\frac{2rR\cos(\theta-u)[R^2+r^2-2rR\cos(\theta-u)]-[2rR\sin(\theta-u)]^2}{[R^2+r^2-2rR\cos(\theta-u)]^2}\right\}du.$$

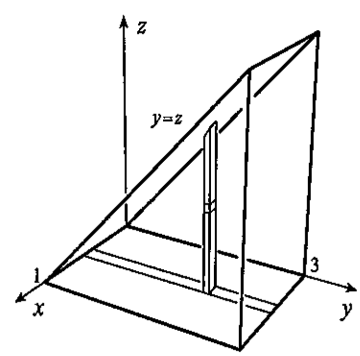
If we bring all three integrals together, and factor $-Rf(u)/\{2\pi r^2[R^2+r^2-2rR\cos(\theta-u)]^2$ from each term, the remaining factor in the integrand is

$$\begin{aligned}
&2r^2[R^2+r^2-2rR\cos(\theta-u)]-r^2[2r-2R\cos(\theta-u)]^2+r[2r-2R\cos(\theta-u)][R^2+r^2-2rR\cos(\theta-u)]\\
&\qquad +2rR\cos(\theta-u)[R^2+r^2-2rR\cos(\theta-u)]-[2rR\sin(\theta-u)]^2\\
&= 2r^2[R^2+r^2-2rR\cos(\theta-u)]-r^2[4r^2-8rR\cos(\theta-u)+4R^2\cos^2(\theta-u)]\\
&\qquad +2r^2[R^2+r^2-2rR\cos(\theta-u)]-2rR\cos(\theta-u)[R^2+r^2-2rR\cos(\theta-u)]\\
&\qquad +2rR\cos(\theta-u)[R^2+r^2-2rR\cos(\theta-u)]-4r^2R^2\sin^2(\theta-u)\\
&= 4r^2R^2-4r^2R^2\cos^2(\theta-u)-4r^2R^2\sin^2(\theta-u)=0.
\end{aligned}$$

REVIEW EXERCISES



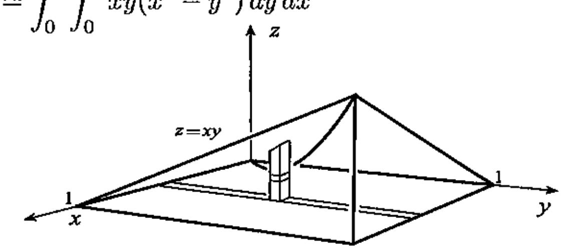
1. $$\iint_R (2x+y)\,dA = \int_0^2\int_0^x (2x+y)\,dy\,dx = \int_0^2 \left\{2xy+\frac{y^2}{2}\right\}_0^x dx$$
$$= \frac{1}{2}\int_0^2 (4x^2+x^2)\,dx$$
$$= \frac{5}{2}\left\{\frac{x^3}{3}\right\}_0^2 = \frac{20}{3}$$



2. $$\iiint_V xyz\,dV = \int_0^1\int_0^3\int_0^y xyz\,dz\,dy\,dx$$
$$= \int_0^1\int_0^3 \left\{\frac{xyz^2}{2}\right\}_0^y dy\,dx$$
$$= \frac{1}{2}\int_0^1\int_0^3 xy^3\,dy\,dx$$
$$= \frac{1}{2}\int_0^1 \left\{\frac{xy^4}{4}\right\}_0^3 dx$$
$$= \frac{81}{8}\int_0^1 x\,dx = \frac{81}{8}\left\{\frac{x^2}{2}\right\}_0^1 = \frac{81}{16}$$

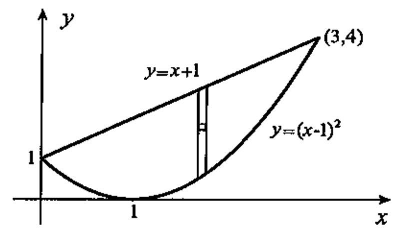
3. Since R is symmetric about the y-axis, and x^3y^2 is an odd function of x, the value of the double integral is 0.

4. $$\iiint_V (x^2-y^3)\,dV = \int_0^1\int_0^1\int_0^{xy}(x^2-y^3)\,dz\,dy\,dx = \int_0^1\int_0^1 xy(x^2-y^3)\,dy\,dx$$
$$= \int_0^1\left\{\frac{x^3y^2}{2}-\frac{xy^5}{5}\right\}_0^1 dx$$
$$= \frac{1}{10}\int_0^1 (5x^3-2x)\,dx$$
$$= \frac{1}{10}\left\{\frac{5x^4}{4}-x^2\right\}_0^1 = \frac{1}{40}$$

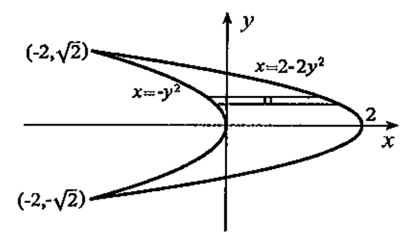


5. $$\iiint_V (x^2-y^2)\,dV = \int_0^1\int_0^1\int_0^{xy}(x^2-y^2)\,dz\,dy\,dx = \int_0^1\int_0^1 (x^3y-xy^3)\,dy\,dx$$
$$= \int_0^1\left\{\frac{x^3y^2}{2}-\frac{xy^4}{4}\right\}_0^1 dx = \frac{1}{4}\int_0^1 (2x^3-x)\,dx = \frac{1}{4}\left\{\frac{x^4}{2}-\frac{x^2}{2}\right\}_0^1 = 0$$

6. $$\iint_R y\,dA = \int_0^3 \int_{(x-1)^2}^{x+1} y\,dy\,dx$$
$$= \int_0^3 \left\{\frac{y^2}{2}\right\}_{(x-1)^2}^{x+1} dx$$
$$= \frac{1}{2}\int_0^3 [(x+1)^2 - (x-1)^4]\,dx$$
$$= \frac{1}{2}\left\{\frac{1}{3}(x+1)^3 - \frac{1}{5}(x-1)^5\right\}_0^3 = \frac{36}{5}$$



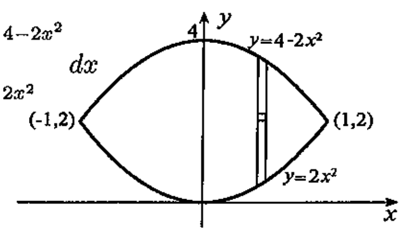
7. $$\iint_R xy^2\,dA = \int_{-\sqrt{2}}^{\sqrt{2}}\int_{-y^2}^{2-2y^2} xy^2\,dx\,dy = \int_{-\sqrt{2}}^{\sqrt{2}} \left\{\frac{x^2y^2}{2}\right\}_{-y^2}^{2-2y^2} dy$$
$$= \frac{1}{2}\int_{-\sqrt{2}}^{\sqrt{2}} [y^2(2-2y^2)^2 - y^2(-y^2)^2]\,dy$$
$$= \frac{1}{2}\int_{-\sqrt{2}}^{\sqrt{2}} (4y^2 - 8y^4 + 3y^6)\,dy$$
$$= \frac{1}{2}\left\{\frac{4y^3}{3} - \frac{8y^5}{5} + \frac{3y^7}{7}\right\}_{-\sqrt{2}}^{\sqrt{2}} = -\frac{32\sqrt{2}}{105}$$



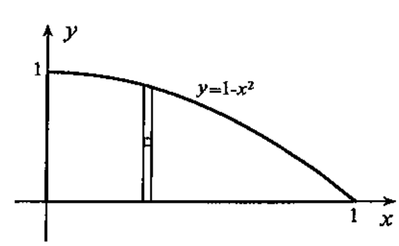
8. Since R is symmetric about the x-axis and x^2y is an odd function of y, $\iint_R x^2y\,dA = 0$.

9. $$\iiint_V (x^2+y^2+z^2)\,dV = \int_0^2\int_{-z}^{z}\int_0^1 (x^2+y^2+z^2)\,dy\,dx\,dz = \int_0^2\int_{-z}^{z}\left\{x^2y + \frac{y^3}{3} + z^2y\right\}_0^1 dx\,dz$$
$$= \frac{1}{3}\int_0^2\int_{-z}^{z} (3x^2 + 1 + 3z^2)\,dx\,dz = \frac{1}{3}\int_0^2 \left\{x^3 + x + 3z^2x\right\}_{-z}^{z} dz$$
$$= \frac{1}{3}\int_0^2 (8z^3 + 2z)\,dz = \frac{1}{3}\left\{2z^4 + z^2\right\}_0^2 = 12$$

10. $$\iint_R (xy - x^2y^2)\,dA = 2\int_0^1\int_{2x^2}^{4-2x^2} -x^2y^2\,dy\,dx = -2\int_0^1\left\{\frac{x^2y^3}{3}\right\}_{2x^2}^{4-2x^2} dx$$
$$= -\frac{32}{3}\int_0^1 (4x^2 - 6x^4 + 3x^6 - x^8)\,dx$$
$$= -\frac{32}{3}\left\{\frac{4x^3}{3} - \frac{6x^5}{5} + \frac{3x^7}{7} - \frac{x^9}{9}\right\}_0^1 = -\frac{4544}{945}$$

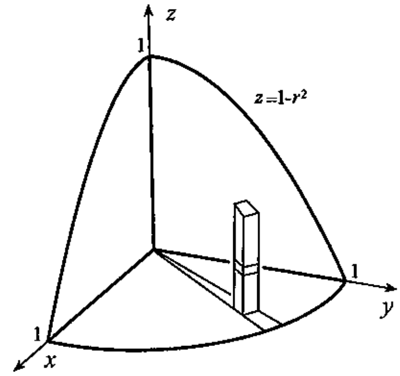


11. $$\iint_R x\sin y\,dA = \int_0^1\int_0^{1-x^2} x\sin y\,dy\,dx$$
$$= \int_0^1 \left\{-x\cos y\right\}_0^{1-x^2} dx$$
$$= \int_0^1 [-x\cos(1-x^2) + x]\,dx$$
$$= \left\{\frac{1}{2}\sin(1-x^2) + \frac{x^2}{2}\right\}_0^1 = \frac{1-\sin 1}{2}$$

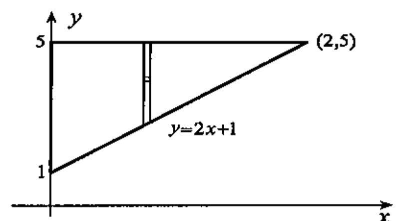


12. Integrals of x and y yield zero. We quadruple the integral of z over the first octant volume.

$$\begin{aligned}\iiint_V (x+y+z)\,dV &= \iiint_V z\,dV\\ &= 4\int_0^{\pi/2}\int_0^1\int_0^{1-r^2} z\,r\,dz\,dr\,d\theta\\ &= 4\int_0^{\pi/2}\int_0^1\left\{\frac{rz^2}{2}\right\}_0^{1-r^2} dr\,d\theta\\ &= 2\int_0^{\pi/2}\int_0^1 r(1-r^2)^2\,dr\,d\theta\\ &= 2\int_0^{\pi/2}\left\{-\frac{1}{6}(1-r^2)^3\right\}_0^1 d\theta\\ &= \frac{1}{3}\Big\{\theta\Big\}_0^{\pi/2} = \frac{\pi}{6}\end{aligned}$$



13. $$\begin{aligned}\iint_R xe^y\,dA &= \int_0^2\int_{2x+1}^5 xe^y\,dy\,dx\\ &= \int_0^2\Big\{xe^y\Big\}_{2x+1}^5 dx\\ &= \int_0^2 (e^5x - xe^{2x+1})\,dx\end{aligned}$$

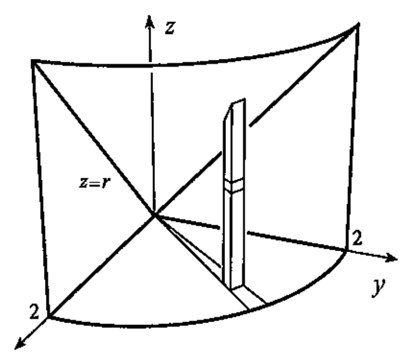


If we set $u = x$, $dv = e^{2x+1}\,dx$, $du = dx$, and $v = (1/2)e^{2x+1}$ in the second term,

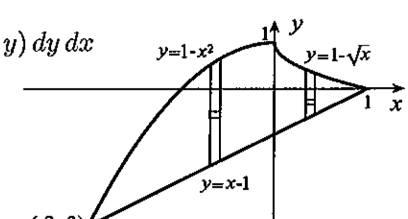
$$\iint_R xe^y\,dA = \left\{\frac{e^5x^2}{2}\right\}_0^2 - \left\{\frac{x}{2}e^{2x+1}\right\}_0^2 + \int_0^2 \frac{1}{2}e^{2x+1}\,dx = e^5 + \frac{1}{2}\left\{\frac{e^{2x+1}}{2}\right\}_0^2 = \frac{5}{4}e^5 - \frac{e}{4}.$$

14. We multiply the first octant volume by 8.

$$\begin{aligned}\iiint_V dV &= 8\int_0^{\pi/2}\int_0^2\int_0^r r\,dz\,dr\,d\theta\\ &= 8\int_0^{\pi/2}\int_0^2 r^2\,dr\,d\theta\\ &= 8\int_0^{\pi/2}\left\{\frac{r^3}{3}\right\}_0^2 d\theta\\ &= \frac{64}{3}\Big\{\theta\Big\}_0^{\pi/2} = \frac{32\pi}{3}\end{aligned}$$

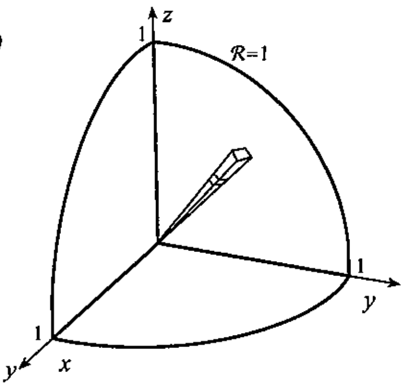


15. $$\begin{aligned}\iint_R (x+y)\,dA &= \int_{-2}^0\int_{x-1}^{1-x^2}(x+y)\,dy\,dx + \int_0^1\int_{x-1}^{1-\sqrt{x}}(x+y)\,dy\,dx\\ &= \int_{-2}^0\left\{\frac{(x+y)^2}{2}\right\}_{x-1}^{1-x^2} dx\\ &\quad + \int_0^1\left\{\frac{(x+y)^2}{2}\right\}_{x-1}^{1-\sqrt{x}} dx\\ &= \frac{1}{2}\int_{-2}^0 (x^4 - 2x^3 - 5x^2 + 6x)\,dx + \frac{1}{2}\int_0^1 (7x - 3x^2 - 2\sqrt{x} - 2x^{3/2})\,dx\\ &= \frac{1}{2}\left\{\frac{x^5}{5} - \frac{x^4}{2} - \frac{5x^3}{3} + 3x^2\right\}_{-2}^0 + \frac{1}{2}\left\{\frac{7x^2}{2} - x^3 - \frac{4x^{3/2}}{3} - \frac{4x^{5/2}}{5}\right\}_0^1 = -\frac{317}{60}\end{aligned}$$

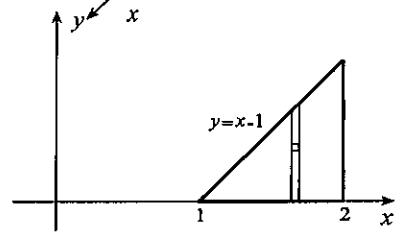


16. We quadruple the integral over the volume in the first octant.

$$\iiint_V (x^2+y^2+z^2)\,dV = 4\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \Re^2\,\Re^2\,\sin\phi\,d\Re\,d\phi\,d\theta$$
$$= 4\int_0^{\pi/2}\int_0^{\pi/2}\left\{\frac{\Re^5}{5}\right\}_0^1 \sin\phi\,d\phi\,d\theta$$
$$= \frac{4}{5}\int_0^{\pi/2}\left\{-\cos\phi\right\}_0^{\pi/2} d\theta$$
$$= \frac{4}{5}\left\{\theta\right\}_0^{\pi/2} = \frac{2\pi}{5}$$



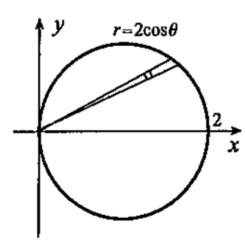
17. $$\iint_R \frac{x}{x+y}dA = \int_1^2\int_0^{x-1}\frac{x}{x+y}dy\,dx$$
$$= \int_1^2\left\{x\ln|x+y|\right\}_0^{x-1}dx$$
$$= \int_1^2 [x\ln(2x-1) - x\ln x]\,dx$$



We use integration by parts on each of these. In the first we set $u = \ln(2x-1)$, $dv = x\,dx$, $du = 2dx/(2x-1)$, $v = x^2/2$; in the second $u = \ln x$, $dv = x\,dx$, $du = (1/x)dx$, $v = x^2/2$;

$$\iint_R \frac{x}{x+y}dA = \left\{\frac{x^2}{2}\ln(2x-1)\right\}_1^2 - \int_1^2\frac{x^2}{2x-1}dx - \left\{\frac{x^2}{2}\ln x\right\}_1^2 + \int_1^2\frac{x}{2}dx$$
$$= 2\ln 3 - \int_1^2\left(\frac{x}{2}+\frac{1}{4}+\frac{1/4}{2x-1}\right)dx - 2\ln 2 + \left\{\frac{x^2}{4}\right\}_1^2$$
$$= 2\ln 3 - 2\ln 2 + \frac{3}{4} - \left\{\frac{x^2}{4}+\frac{x}{4}+\frac{1}{8}\ln(2x-1)\right\}_1^2 = \frac{15}{8}\ln 3 - 2\ln 2 - \frac{1}{4}.$$

18. $$\iint_R (x^2+y^2)\,dA = 2\int_0^{\pi/2}\int_0^{2\cos\theta} r^2\,r\,dr\,d\theta = 2\int_0^{\pi/2}\left\{\frac{r^4}{4}\right\}_0^{2\cos\theta} d\theta$$
$$= 8\int_0^{\pi/2}\cos^4\theta\,d\theta$$
$$= 2\int_0^{\pi/2}\left(1+2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)d\theta$$
$$= 2\left\{\frac{3\theta}{2}+\sin 2\theta+\frac{1}{8}\sin 4\theta\right\}_0^{\pi/2} = \frac{3\pi}{2}$$

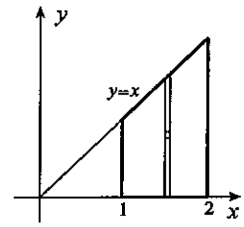


19. $$\iiint_V \frac{x^2}{z^2}dV = 4\int_0^{\pi/2}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}}\frac{r^2\cos^2\theta}{z^2}r\,dz\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^{\sqrt{3}}\left\{\frac{-r^3\cos^2\theta}{z}\right\}_1^{\sqrt{4-r^2}} dr\,d\theta$$
$$= 4\int_0^{\pi/2}\int_0^{\sqrt{3}}\left(r^3 - \frac{r^3}{\sqrt{4-r^2}}\right)\cos^2\theta\,dr\,d\theta$$

If we set $u = 4 - r^2$ and $du = -2r\,dr$ in the second term,

$$\iiint_V \frac{x^2}{z^2}dV = 4\int_0^{\pi/2}\left\{\frac{r^4}{4}\cos^2\theta\right\}_0^{\sqrt{3}} d\theta - 4\int_0^{\pi/2}\int_4^1\left(\frac{4-u}{\sqrt{u}}\right)\cos^2\theta\left(\frac{du}{-2}\right)d\theta$$
$$= 9\int_0^{\pi/2}\cos^2\theta\,d\theta + 2\int_0^{\pi/2}\left\{\left(8\sqrt{u}-\frac{2u^{3/2}}{3}\right)\cos^2\theta\right\}_4^1 d\theta$$
$$= \frac{7}{3}\int_0^{\pi/2}\cos^2\theta\,d\theta = \frac{7}{3}\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta = \frac{7}{6}\left\{\theta+\frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = \frac{7\pi}{12}.$$

20. $\displaystyle\iint_R \frac{1}{x^2+y^2}\,dA = \int_1^2\int_0^x \frac{1}{x^2+y^2}\,dy\,dx$



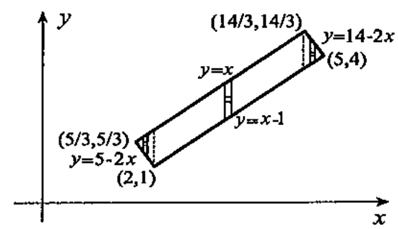
If we set $y = x\tan\theta$, then $dy = x\sec^2\theta\,d\theta$, and

$$\iint_R \frac{1}{x^2+y^2}\,dA = \int_1^2\int_0^{\pi/4} \frac{1}{x^2\sec^2\theta}x\sec^2\theta\,d\theta\,dx$$

$$= \int_1^2 \left\{\frac{\theta}{x}\right\}_0^{\pi/4} dx = \frac{\pi}{4}\int_1^2 \frac{1}{x}dx = \frac{\pi}{4}\Big\{\ln|x|\Big\}_1^2 = \frac{\pi}{4}\ln 2.$$

21. $\displaystyle\iint_R (x^2-y^2)\,dA = \int_{5/3}^2\int_{5-2x}^x (x^2-y^2)\,dy\,dx + \int_2^{14/3}\int_{x-1}^x (x^2-y^2)\,dy\,dx + \int_{14/3}^5\int_{x-1}^{14-2x} (x^2-y^2)\,dy\,dx$

$$= \int_{5/3}^2 \left\{x^2y - \frac{y^3}{3}\right\}_{5-2x}^x dx + \int_2^{14/3}\left\{x^2y - \frac{y^3}{3}\right\}_{x-1}^x dx$$



$$+ \int_{14/3}^5 \left\{x^2y - \frac{y^3}{3}\right\}_{x-1}^{14-2x} dx$$

$$= \frac{1}{3}\int_{5/3}^2 [3x^3 - x^3 - 3x^2(5-2x) + (5-2x)^3]\,dx$$

$$+\frac{1}{3}\int_2^{14/3} [3x^3 - x^3 - 3x^2(x-1) + (x-1)^3]\,dx$$

$$+\frac{1}{3}\int_{14/3}^5 [3x^2(14-2x) - (14-2x)^3 - 3x^2(x-1) + (x-1)^3]\,dx$$

$$= \frac{1}{3}\left\{2x^4 - 5x^3 - \frac{(5-2x)^4}{8}\right\}_{5/3}^2 + \frac{1}{3}\left\{-\frac{x^4}{4} + x^3 + \frac{(x-1)^4}{4}\right\}_2^{14/3}$$

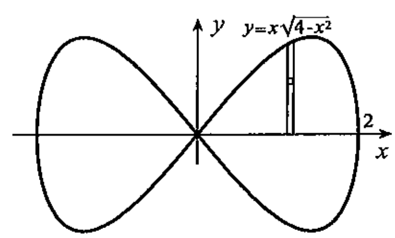
$$+\frac{1}{3}\left\{15x^3 - \frac{9x^4}{4} + \frac{(14-2x)^4}{8} + \frac{(x-1)^4}{4}\right\}_{14/3}^5 = \frac{55}{6}$$

22. See answer in text.

23. $\displaystyle V = 4\int_0^{\pi/2}\int_0^{\sqrt{\ln 2}}\int_{1-2e^{-r^2}}^0 r\,dz\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^{\sqrt{\ln 2}} r(-1+2e^{-r^2})\,dr\,d\theta = 4\int_0^{\pi/2}\left\{-\frac{r^2}{2} - e^{-r^2}\right\}_0^{\sqrt{\ln 2}} d\theta$

$$= 2(-\ln 2 - 2e^{-\ln 2} + 2)\Big\{\theta\Big\}_0^{\pi/2} = \pi(1-\ln 2)$$

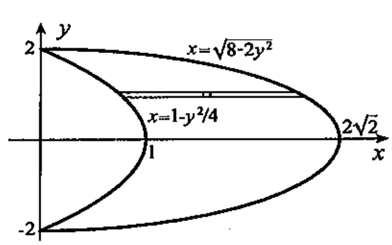
24. $\displaystyle A = 4\int_0^2\int_0^{x\sqrt{4-x^2}} dy\,dx$



$$= 4\int_0^2 x\sqrt{4-x^2}\,dx$$

$$= 4\left\{-\frac{1}{3}(4-x^2)^{3/2}\right\}_0^2 = \frac{32}{3}$$

25. $\displaystyle A = 2\int_0^2\int_{1-y^2/4}^{\sqrt{8-2y^2}} dx\,dy$



$$= 2\int_0^2 \left(\sqrt{8-2y^2} - 1 + \frac{y^2}{4}\right) dy$$

If we set $y = 2\sin\theta$ and $dy = 2\cos\theta\,d\theta$,

$$A = 2\int_0^{\pi/2} 2\sqrt{2}\cos\theta(2\cos\theta\,d\theta) + 2\left\{-y + \frac{y^3}{12}\right\}_0^2 = 8\sqrt{2}\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right) d\theta - \frac{8}{3}$$

$$= 4\sqrt{2}\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} - \frac{8}{3} = \frac{6\sqrt{2}\pi - 8}{3}.$$

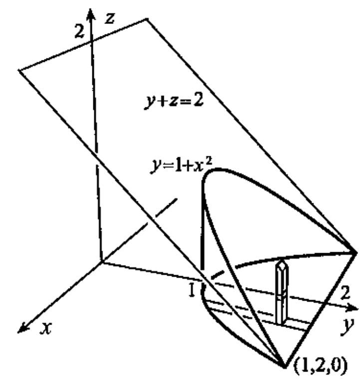
By symmetry, $\overline{y} = 0$. Since

$$A\overline{x} = 2\int_0^2 \int_{1-y^2/4}^{\sqrt{8-2y^2}} x\,dx\,dy = 2\int_0^2 \left\{\frac{x^2}{2}\right\}_{1-y^2/4}^{\sqrt{8-2y^2}} dy = \int_0^2 [(8-2y^2) - (1-y^2/4)^2]\,dy$$

$$= \int_0^2 \left(7 - \frac{3y^2}{2} - \frac{y^4}{16}\right) dy = \left\{7y - \frac{y^3}{2} - \frac{y^5}{80}\right\}_0^2 = \frac{48}{5},$$

it follows that $\overline{x} = \dfrac{48}{5}\dfrac{3}{6\sqrt{2}\pi - 8} = \dfrac{72}{15\sqrt{2}\pi - 20}$.

26. We quadruple the volume in the first octant.



$$V = 4\int_0^1 \int_{1+x^2}^2 \int_0^{2-y} dz\,dy\,dx$$

$$= 4\int_0^1 \int_{1+x^2}^2 (2-y)\,dy\,dx$$

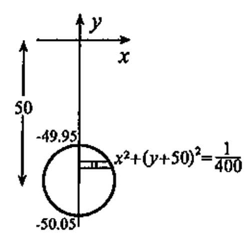
$$= 4\int_0^1 \left\{-\frac{1}{2}(2-y)^2\right\}_{1+x^2}^2 dx$$

$$= 2\int_0^1 (1 - 2x^2 + x^4)\,dx$$

$$= 2\left\{x - \frac{2x^3}{3} + \frac{x^5}{5}\right\}_0^1 = \frac{16}{15}$$

27. $F = 2\displaystyle\int_{-50.05}^{-49.95} \int_0^{\sqrt{1/400-(y+50)^2}} 1000(9.81)(-y)\,dx\,dy = -19\,620 \int_{-50.05}^{-49.95} y\sqrt{\frac{1}{400} - (y+50)^2}\,dy$

If we set $y + 50 = (1/20)\sin\theta$ and $dy = (1/20)\cos\theta\,d\theta$,



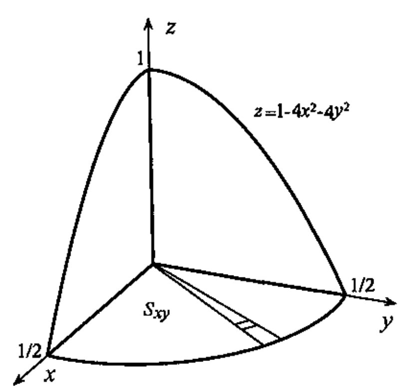
$$F = -19\,620 \int_{-\pi/2}^{\pi/2} \left(-50 + \frac{1}{20}\sin\theta\right)\left(\frac{1}{20}\cos\theta\right)\frac{1}{20}\cos\theta\,d\theta$$

$$= -\frac{981}{400}\int_{-\pi/2}^{\pi/2} (-1000\cos^2\theta + \cos^2\theta\sin\theta)\,d\theta$$

$$= -\frac{981}{400}\int_{-\pi/2}^{\pi/2} [\cos^2\theta\sin\theta - 500(1+\cos 2\theta)]\,d\theta$$

$$= -\frac{981}{400}\left\{-\frac{1}{3}\cos^3\theta - 500\theta - 250\sin 2\theta\right\}_{-\pi/2}^{\pi/2} = \frac{4905\pi}{4}\text{ N.}$$

28. We quadruple the area in the first octant.



$$A = 4\iint_{S_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

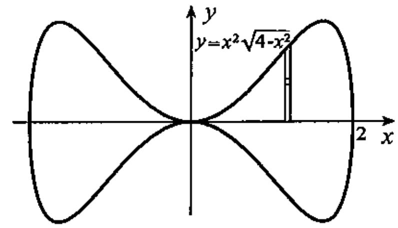
$$= 4\iint_{S_{xy}} \sqrt{1 + (-8x)^2 + (-8y)^2}\,dA$$

$$= 4\int_0^{\pi/2}\int_0^{1/2} \sqrt{1 + 64r^2}\,r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2} \left\{\frac{1}{192}(1+64r^2)^{3/2}\right\}_0^{1/2} d\theta$$

$$= \frac{17\sqrt{17} - 1}{48}\left\{\theta\right\}_0^{\pi/2} = \frac{(17\sqrt{17} - 1)\pi}{96}$$

29. $A = 4\int_0^2 \int_0^{x^2\sqrt{4-x^2}} dy\,dx = 4\int_0^2 x^2\sqrt{4-x^2}\,dx$

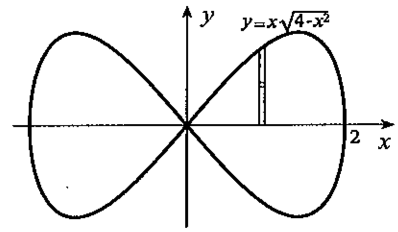


If we set $x = 2\sin\theta$ and $dx = 2\cos\theta\,d\theta$,

$$A = 4\int_0^{\pi/2} 4\sin^2\theta(2\cos\theta)(2\cos\theta\,d\theta) = 64\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^2 d\theta$$

$$= 16\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right) d\theta = 8\left\{\theta - \frac{1}{4}\sin 4\theta\right\}_0^{\pi/2} = 4\pi.$$

30. $V_x = 2\int_0^2 \int_0^{x\sqrt{4-x^2}} 2\pi y\,dy\,dx = 2\pi\int_0^2 \left\{y^2\right\}_0^{x\sqrt{4-x^2}} dx$



$$= 2\pi\int_0^2 x^2(4-x^2)\,dx = 2\pi\left\{\frac{4x^3}{3} - \frac{x^5}{5}\right\}_0^2 = \frac{128\pi}{15}$$

$$V_y = 2\int_0^2 \int_0^{x\sqrt{4-x^2}} 2\pi x\,dy\,dx = 4\pi\int_0^2 x^2\sqrt{4-x^2}\,dx$$

If we set $x = 2\sin\theta$ and $dx = 2\cos\theta\,d\theta$, then

$$V_y = 4\pi\int_0^{\pi/2} 4\sin^2\theta\, 2\cos\theta\, 2\cos\theta\,d\theta = 64\pi\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^2 d\theta$$

$$= 16\pi\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right) d\theta = 8\pi\left\{\theta - \frac{1}{4}\sin 4\theta\right\}_0^{\pi/2} = 4\pi^2$$

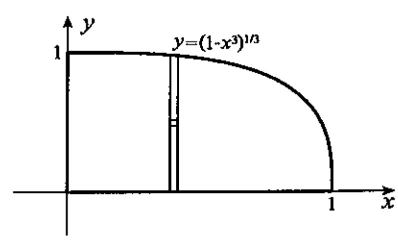
31. $I = 4\int_0^{\pi/2}\int_0^1\int_r^{2-r} \rho r^2\, r\,dz\,dr\,d\theta = 4\rho\int_0^{\pi/2}\int_0^1 \left\{r^3 z\right\}_r^{2-r} dr\,d\theta = 8\rho\int_0^{\pi/2}\int_0^1 (r^3 - r^4)\,dr\,d\theta$

$$= 8\rho\int_0^{\pi/2}\left\{\frac{r^4}{4} - \frac{r^5}{5}\right\}_0^1 d\theta = \frac{2\rho}{5}\left\{\theta\right\}_0^{\pi/2} = \frac{\rho\pi}{5}$$

32. $I = \int_0^1 \int_0^{(1-x^3)^{1/3}} x^2\,dy\,dx$

$$= \int_0^1 x^2(1-x^3)^{1/3}\,dx$$

$$= \left\{-\frac{1}{4}(1-x^3)^{4/3}\right\}_0^1 = \frac{1}{4}$$



33. By symmetry, $\overline{x} = \overline{y} = 0$.

$$M = 4\int_0^{\pi/2}\int_0^1\int_0^{1+r^2} \rho r\,dz\,dr\,d\theta = 4\rho\int_0^{\pi/2}\int_0^1 r(1+r^2)\,dr\,d\theta = 4\rho\int_0^{\pi/2}\left\{\frac{r^2}{2} + \frac{r^4}{4}\right\}_0^1 d\theta = 3\rho\left\{\theta\right\}_0^{\pi/2} = \frac{3\pi\rho}{2}$$

Since $M\overline{z} = 4\int_0^{\pi/2}\int_0^1\int_0^{1+r^2} z\rho r\,dz\,dr\,d\theta = 4\rho\int_0^{\pi/2}\int_0^1 \left\{\frac{rz^2}{2}\right\}_0^{1+r^2} dr\,d\theta$

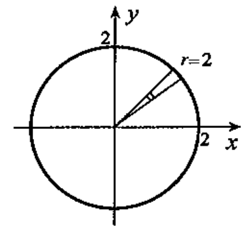
$$= 2\rho\int_0^{\pi/2}\int_0^1 r(1+r^2)^2\,dr\,d\theta = 2\rho\int_0^{\pi/2}\left\{\frac{1}{6}(1+r^2)^3\right\}_0^1 d\theta = \frac{7\rho}{3}\left\{\theta\right\}_0^{\pi/2} = \frac{7\pi\rho}{6},$$

we find that $\overline{z} = \dfrac{7\pi\rho}{6}\dfrac{2}{3\pi\rho} = \dfrac{7}{9}$.

34. $\overline{f} = \dfrac{1}{\pi(2)^2}\displaystyle\int_{-\pi}^{\pi}\int_0^2 r^2\, r\, dr\, d\theta$

$= \dfrac{1}{4\pi}\displaystyle\int_{-\pi}^{\pi}\left\{\frac{r^4}{4}\right\}_0^2 d\theta$

$= \dfrac{1}{\pi}\left\{\theta\right\}_{-\pi}^{\pi} = 2$

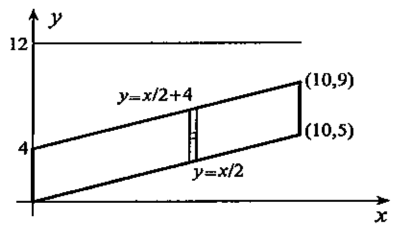


35. $F = \displaystyle\int_0^{10}\int_{x/2}^{4+x/2} 1000(9.81)(12-y)\, dy\, dx$

$= 9810\displaystyle\int_0^{10}\left\{-\frac{1}{2}(12-y)^2\right\}_{x/2}^{4+x/2} dx$

$= -4905\displaystyle\int_0^{10}[(8-x/2)^2-(12-x/2)^2]\, dx$

$= -4905\left\{-\dfrac{2}{3}\left(8-\dfrac{x}{2}\right)^3+\dfrac{2}{3}\left(12-\dfrac{x}{2}\right)^3\right\}_0^{10} = 2943$ kN



36. We quadruple the moment of inertia of the first octant volume.

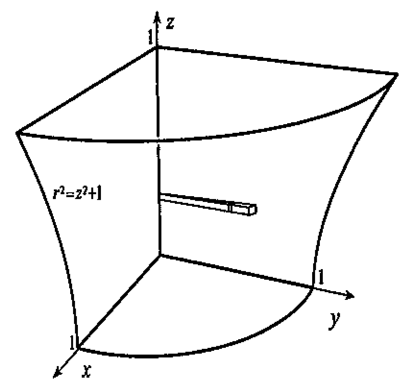
$I = 4\displaystyle\int_0^{\pi/2}\int_0^1\int_0^{\sqrt{1+z^2}} r^2\, \rho\, r\, dr\, dz\, d\theta$

$= 4\rho\displaystyle\int_0^{\pi/2}\int_0^1\left\{\frac{r^4}{4}\right\}_0^{\sqrt{1+z^2}} dz\, d\theta$

$= \rho\displaystyle\int_0^{\pi/2}\int_0^1 (1+2z^2+z^4)\, dz\, d\theta$

$= \rho\displaystyle\int_0^{\pi/2}\left\{z+\frac{2z^3}{3}+\frac{z^5}{5}\right\}_0^1 d\theta$

$= \dfrac{28\rho}{15}\left\{\theta\right\}_0^{\pi/2} = \dfrac{14\pi\rho}{15}$



37. Since $V = \displaystyle\int_0^2\int_0^{2-y}\int_{2y+2z-4}^{2-y-z} dx\, dz\, dy = \int_0^2\int_0^{2-y}(6-3y-3z)\, dz\, dy$

$= 3\displaystyle\int_0^2\left\{-\frac{1}{2}(2-y-z)^2\right\}_0^{2-y} dy = \frac{3}{2}\int_0^2(2-y)^2\, dy$

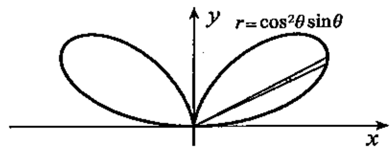
$= \dfrac{3}{2}\left\{-\dfrac{1}{3}(2-y)^3\right\}_0^2 = 4,$

$$\overline{f} = \frac{1}{4}\int_0^2\int_0^{2-y}\int_{2y+2z-4}^{2-y-z}(x+y+z)\, dx\, dz\, dy = \frac{1}{4}\int_0^2\int_0^{2-y}\left\{\frac{1}{2}(x+y+z)^2\right\}_{2y+2z-4}^{2-y-z} dz\, dy$$

$$= \frac{1}{8}\int_0^2\int_0^{2-y}[4-(3y+3z-4)^2]\, dz\, dy = \frac{1}{8}\int_0^2\left\{4z-\frac{1}{9}(3y+3z-4)^3\right\}_0^{2-y} dy$$

$$= \frac{1}{72}\int_0^2[36(2-y)-8+(3y-4)^3]\, dy = \frac{1}{72}\left\{-18(2-y)^2-8y+\frac{1}{12}(3y-4)^4\right\}_0^2 = \frac{1}{2}.$$

38. $$A = 2\int_0^{\pi/2}\int_0^{\sin\theta\cos^2\theta} r\,dr\,d\theta$$
$$= 2\int_0^{\pi/2}\left\{\frac{r^2}{2}\right\}_0^{\sin\theta\cos^2\theta} d\theta$$
$$= \int_0^{\pi/2}\sin^2\theta\cos^4\theta\,d\theta$$
$$= \int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^2\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$
$$= \frac{1}{8}\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}+\sin^2 2\theta\cos 2\theta\right)d\theta = \frac{1}{8}\left\{\frac{\theta}{2}-\frac{1}{8}\sin 4\theta+\frac{1}{6}\sin^3 2\theta\right\}_0^{\pi/2} = \frac{\pi}{32}$$



By symmetry, $\overline{x} = 0$. Since
$$A\overline{y} = 2\int_0^{\pi/2}\int_0^{\sin\theta\cos^2\theta} r\sin\theta\, r\,dr\,d\theta = 2\int_0^{\pi/2}\left\{\frac{r^3}{3}\sin\theta\right\}_0^{\sin\theta\cos^2\theta} d\theta$$
$$= \frac{2}{3}\int_0^{\pi/2}\sin^4\theta\cos^6\theta\,d\theta = \frac{2}{3}\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^4\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$
$$= \frac{1}{48}\int_0^{\pi/2}\left[\left(\frac{1-\cos 2\theta}{2}\right)^2+\sin^4 2\theta\cos 2\theta\right]d\theta$$
$$= \frac{1}{48}\int_0^{\pi/2}\left[\frac{1}{4}\left(1-2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)+\sin^4 2\theta\cos 2\theta\right]d\theta$$
$$= \frac{1}{48}\left\{\frac{3\theta}{8}-\frac{1}{4}\sin 2\theta+\frac{1}{32}\sin 4\theta+\frac{1}{10}\sin^5 2\theta\right\}_0^{\pi/2} = \frac{\pi}{256},$$

it follows that $\overline{y} = \dfrac{\pi}{256}\dfrac{32}{\pi} = \dfrac{1}{8}$.

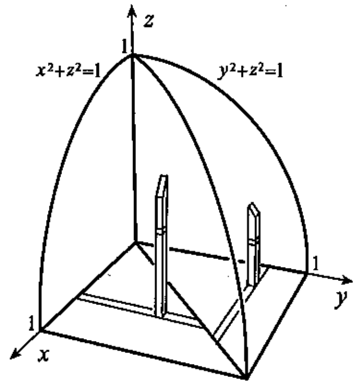
39. By Leibnitz's rule,
$$\frac{dy}{dx} = -3e^{-3x}(C_1\cos x + C_2\sin x) + e^{-3x}(-C_1\sin x + C_2\cos x)$$
$$+\int_0^x f(t)[-3e^{3(t-x)}\sin(x-t) + e^{3(t-x)}\cos(x-t)]\,dt$$
$$= (-3C_1 + C_2)e^{-3x}\cos x + (-3C_2 - C_1)e^{-3x}\sin x$$
$$+\int_0^x f(t)e^{3(t-x)}[\cos(x-t) - 3\sin(x-t)]\,dt,$$

and

$$\frac{d^2y}{dx^2} = (-3C_1 + C_2)(-3e^{-3x}\cos x - e^{-3x}\sin x) + (-3C_2 - C_1)(-3e^{-3x}\sin x + e^{-3x}\cos x)$$
$$+\int_0^x f(t)\{-3e^{3(t-x)}[\cos(x-t) - 3\sin(x-t)] + e^{3(t-x)}[-\sin(x-t) - 3\cos(x-t)]\}dt + f(x).$$

Thus,
$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 10y = (8C_1 - 6C_2)e^{-3x}\cos x + (6C_1 + 8C_2)e^{-3x}\sin x$$
$$+\int_0^x f(t)e^{3(t-x)}[-6\cos(x-t) + 8\sin(x-t)]\,dt + f(x)$$
$$+6(-3C_1 + C_2)e^{-3x}\cos x + 6(-3C_2 - C_1)e^{-3x}\sin x$$
$$+6\int_0^x f(t)e^{3(t-x)}[\cos(x-t) - 3\sin(x-t)]\,dt$$
$$+10e^{-3x}(C_1\cos x + C_2\sin x) + 10\int_0^x f(t)e^{3(t-x)}\sin(x-t)\,dt$$
$$= f(x).$$

40. $M = 2\int_0^1 \int_0^x \int_0^{\sqrt{1-x^2}} \rho \, dz \, dy \, dx$
$= 2\rho \int_0^1 \int_0^x \sqrt{1-x^2} \, dy \, dx$
$= 2\rho \int_0^1 x\sqrt{1-x^2} \, dx$
$= 2\rho \left\{ -\frac{1}{3}(1-x^2)^{3/2} \right\}_0^1 = \frac{2\rho}{3}$



$$M\overline{x} = \int_0^1 \int_0^x \int_0^{\sqrt{1-x^2}} x\rho \, dz \, dy \, dx + \int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} x\rho \, dz \, dx \, dy$$
$$= \rho \int_0^1 \int_0^x x\sqrt{1-x^2} \, dy \, dx + \rho \int_0^1 \int_0^y x\sqrt{1-y^2} \, dx \, dy$$
$$= \rho \int_0^1 x^2\sqrt{1-x^2} \, dx + \frac{\rho}{2} \int_0^1 y^2\sqrt{1-y^2} \, dy = \frac{3\rho}{2} \int_0^1 x^2\sqrt{1-x^2} \, dx$$

If we now set $x = \sin\theta$ and $dx = \cos\theta \, d\theta$, then

$$M\overline{x} = \frac{3\rho}{2} \int_0^{\pi/2} \sin^2\theta \, \cos\theta \, \cos\theta \, d\theta = \frac{3\rho}{2} \int_0^{\pi/2} \left(\frac{\sin 2\theta}{2} \right)^2 d\theta$$
$$= \frac{3\rho}{8} \int_0^{\pi/2} \left(\frac{1-\cos 4\theta}{2} \right) d\theta = \frac{3\rho}{16} \left\{ \theta - \frac{1}{4}\sin 4\theta \right\}_0^{\pi/2} = \frac{3\pi\rho}{32},$$

we obtain $\overline{x} = \frac{3\pi\rho}{32}\frac{3}{2\rho} = \frac{9\pi}{64}$. By symmetry, $\overline{y} = \overline{x} = 9\pi/64$. Since

$$M\overline{z} = 2\int_0^1 \int_0^x \int_0^{\sqrt{1-x^2}} z\rho \, dz \, dy \, dx = 2\rho \int_0^1 \int_0^x \left\{ \frac{z^2}{2} \right\}_0^{\sqrt{1-x^2}} dy \, dx$$
$$= \rho \int_0^1 \int_0^x (1-x^2) \, dy \, dx = \rho \int_0^1 (x - x^3) \, dx = \rho \left\{ \frac{x^2}{2} - \frac{x^4}{4} \right\}_0^1 = \frac{\rho}{4},$$

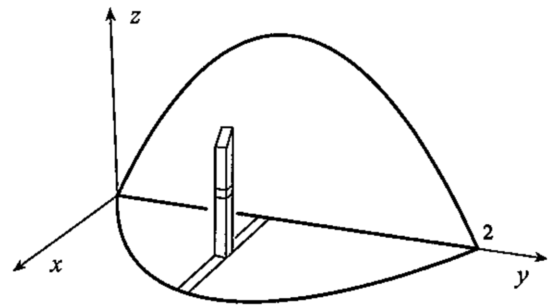
we find $\overline{z} = \frac{\rho}{4}\frac{3}{2\rho} = \frac{3}{8}$.

41. We quadruple the first octant area.

$$A = 4\iint_{S_{xy}} \sqrt{1 + \left(\frac{2x}{x^2+y^2} \right)^2 + \left(\frac{2y}{x^2+y^2} \right)^2} \, dA = 4\iint_{S_{xy}} \sqrt{\frac{(x^2+y^2)^2 + 4(x^2+y^2)}{(x^2+y^2)^2}} \, dA$$
$$= 4\int_0^{\pi/2} \int_1^2 \sqrt{\frac{r^2+4}{r^2}} r \, dr \, d\theta = 4\int_0^{\pi/2} \int_1^2 \sqrt{4+r^2} \, dr \, d\theta = 4\int_1^2 \left\{ \sqrt{4+r^2}\theta \right\}_0^{\pi/2} dr = 2\pi \int_1^2 \sqrt{4+r^2} \, dr$$

If we set $r = 2\tan\theta$ and $dr = 2\sec^2\theta \, d\theta$,

$$A = 2\pi \int_{\mathrm{Tan}^{-1}(1/2)}^{\pi/4} (2\sec\theta) 2\sec^2\theta \, d\theta = 8\pi \int_{\mathrm{Tan}^{-1}(1/2)}^{\pi/4} \sec^3\theta \, d\theta$$
$$= \frac{8\pi}{2} \left\{ \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| \right\}_{\mathrm{Tan}^{-1}(1/2)}^{\pi/4} \quad \text{(see Example 8.9)}$$
$$= 4\pi \left[\sqrt{2} - \frac{\sqrt{5}}{4} + \ln(\sqrt{2}+1) - \ln(\sqrt{5}+1) + \ln 2 \right].$$



42. We quadruple the first octant volume.

$$V = 4\int_0^2 \int_0^{y(2-y)} \int_0^{\sqrt{y^2(2-y)^2-x^2}} dz\,dx\,dy$$
$$= 4\int_0^2 \int_0^{y(2-y)} \sqrt{y^2(2-y)^2 - x^2}\,dx\,dy$$

If we set $x = y(2-y)\sin\theta$ and $dx = y(2-y)\cos\theta\,d\theta$,

$$V = 4\int_0^2 \int_0^{\pi/2} y(2-y)\cos\theta\, y(2-y)\cos\theta\,d\theta\,dy$$
$$= 4\int_0^2 \int_0^{\pi/2} y^2(2-y)^2\left(\frac{1+\cos 2\theta}{2}\right)d\theta\,dy = 2\int_0^2 \left\{y^2(2-y)^2\left(\theta + \frac{\sin 2\theta}{2}\right)\right\}_0^{\pi/2} dy$$
$$= \pi\int_0^2 (4y^2 - 4y^3 + y^4)\,dy = \pi\left\{\frac{4y^3}{3} - y^4 + \frac{y^5}{5}\right\}_0^2 = \frac{16\pi}{15}.$$