CHAPTER 5

EXERCISES 5.1

1. $\int (x^3 - 2x)\,dx = \frac{x^4}{4} - x^2 + C$

2. $\int (x^4 + 3x^2 + 5x)\,dx = \frac{x^5}{5} + x^3 + \frac{5x^2}{2} + C$

3. $\int (2x^3 - 3x^2 + 6x + 6)\,dx = \frac{x^4}{2} - x^3 + 3x^2 + 6x + C$

4. $\int \sin x\,dx = -\cos x + C$

5. $\int 3\cos x\,dx = 3\sin x + C$

6. $\int \sqrt{x}\,dx = \frac{2}{3}x^{3/2} + C$

7. $\int \left(x^{10} - \frac{1}{x^3}\right) dx = \frac{x^{11}}{11} + \frac{1}{2x^2} + C$

8. $\int \left(\frac{1}{x^2} - \frac{2}{x^4}\right) dx = -\frac{1}{x} + \frac{2}{3x^3} + C$

9. $\int (x^{3/2} - x^{2/7})\,dx = \frac{2}{5}x^{5/2} - \frac{7}{9}x^{9/7} + C$

10. $\int \left(\frac{1}{x^2} + \frac{1}{2\sqrt{x}}\right) dx = -\frac{1}{x} + \sqrt{x} + C$

11. $\int \left(\frac{4}{x^{3/2}} + 2x^{1/3}\right) dx = -\frac{8}{\sqrt{x}} + \frac{3}{2}x^{4/3} + C$

12. $\int \left(-\frac{1}{2x^2} + 3x^3\right) dx = \frac{1}{2x} + \frac{3x^4}{4} + C$

13. $\int \frac{1}{x^\pi} dx = \frac{1}{(1-\pi)x^{\pi-1}} + C$

14. $\int (2\sqrt{x} + 3x^{3/2} - 5x^{5/2})\,dx = \frac{4}{3}x^{3/2} + \frac{6}{5}x^{5/2} - \frac{10}{7}x^{7/2} + C$

15. $\int x^2(x^2 - 3)\,dx = \int (x^4 - 3x^2)\,dx = \frac{x^5}{5} - x^3 + C$

16. $\int \sqrt{x}(x+1)\,dx = \int (x^{3/2} + \sqrt{x})\,dx = \frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + C$

17. $\int \left(\frac{x-2}{x^3}\right) dx = \int \left(\frac{1}{x^2} - \frac{2}{x^3}\right) dx = -\frac{1}{x} + \frac{1}{x^2} + C$

18. $\int x^2(1+x^2)^2\,dx = \int (x^2 + 2x^4 + x^6)\,dx = \frac{x^3}{3} + \frac{2x^5}{5} + \frac{x^7}{7} + C$

19. $\int (x^2+1)^3\,dx = \int (x^6 + 3x^4 + 3x^2 + 1)\,dx = \frac{x^7}{7} + \frac{3x^5}{5} + x^3 + x + C$

20. $\int \frac{(x-1)^2}{\sqrt{x}} dx = \int \left(x^{3/2} - 2\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx = \frac{2}{5}x^{5/2} - \frac{4}{3}x^{3/2} + 2\sqrt{x} + C$

21. If we take the indefinite integral of $f'(x) = x^2 - 3x + 2$ with respect to x, we obtain $f(x) = x^3/3 - 3x^2/2 + 2x + C$. Since $f(2) = 1$, it follows that $1 = 8/3 - 6 + 4 + C \implies C = 1/3$, and therefore $y = x^3/3 - 3x^2/2 + 2x + 1/3$.

22. If we take the indefinite integral of $f'(x) = 2x^3 + 4x$ with respect to x, we obtain $f(x) = x^4/2 + 2x^2 + C$. Since $f(0) = 5$, it follows that $5 = C$, and therefore $y = x^4/2 + 2x^2 + 5$.

23. If we take the indefinite integral of $f'(x) = -2x^4 + 3x^2 + 6$ with respect to x, we obtain $f(x) = -2x^5/5 + x^3 + 6x + C$. Since $f(1) = 0$, it follows that $0 = -2/5 + 1 + 6 + C \implies C = -33/5$, and therefore $y = -2x^5/5 + x^3 + 6x - 33/5$.

24. If we take the indefinite integral of $f'(x) = 2 - 4x + 8x^7$ with respect to x, we obtain $f(x) = 2x - 2x^2 + x^8 + C$. Since $f(1) = 1$, it follows that $1 = 2 - 2 + 1 + C$. Thus, $C = 0$, and $y = 2x - 2x^2 + x^8$.

25. If the equation of the curve is $y = f(x)$, then $f''(x) = 6x^2$. Integration gives $f(x) = x^4/2 + Cx + D$. Since $f(0) = 2$ and $f(-1) = 3$, we have $2 = D$ and $3 = 1/2 - C + D$. Thus, $y = f(x) = x^4/2 - x/2 + 2$.

26. Integration of $f''(x) = -5x$ with respect to x gives $f'(x) = -5x^2/2 + C$. Because $f(2) = 3$ is a relative maximum, it follows that $f'(2) = 0$. Thus, $0 = -5(2) + C$, from which $C = 10$, and $f'(x) = -5x^2/2 + 10$. Another integration now gives $f(x) = -5x^3/6 + 10x + D$. Since $f(2) = 3$, we find that $3 = -20/3 + 20 + D$. Thus, $D = -31/3$, and $f(x) = -5x^3/6 + 10x - 31/3$.

27. No For a relative minimum to occur at $x = 2$, the second derivative there should by positive. But this is impossible if the second derivative is equal to $-5x$.

In Exercises 28–67 we use the following tabular setup to summarize calculations. The first column is an initial proposal. The second column is the derivative of this proposal. The last column is the final answer.

	Initial proposal	Derivative of proposal	Final answer
28.	$(x+2)^{3/2}$	$\frac{3}{2}(x+2)^{1/2}$	$\frac{2}{3}(x+2)^{3/2}+C$
29.	$(x+5)^{5/2}$	$\frac{5}{2}(x+5)^{3/2}$	$\frac{2}{5}(x+5)^{5/2}+C$
30.	$(2-x)^{3/2}$	$\frac{3}{2}(2-x)^{1/2}(-1)$	$-\frac{2}{3}(2-x)^{3/2}+C$
31.	$\sqrt{4x+3}$	$\frac{1}{2}(4x+3)^{-1/2}(4)$	$\frac{1}{2}\sqrt{4x+3}+C$
32.	$(2x-3)^{5/2}$	$\frac{5}{2}(2x-3)^{3/2}(2)$	$\frac{1}{5}(2x-3)^{5/2}+C$
33.	$(3x+1)^6$	$6(3x+1)^5(3)$	$\frac{1}{18}(3x+1)^6+C$
34.	$(1-2x)^8$	$8(1-2x)^7(-2)$	$-\frac{1}{16}(1-2x)^8+C$
35.	$\frac{1}{x+4}$	$\frac{-1}{(x+4)^2}$	$\frac{-1}{x+4}+C$
36.	$\frac{1}{(1+3x)^5}$	$\frac{-5}{(1+3x)^6}(3)$	$\frac{-1}{15(1+3x)^5}+C$
37.	$(x^2+1)^4$	$4(x^2+1)^3(2x)$	$\frac{1}{8}(x^2+1)^4+C$
38.	$(2+3x^3)^8$	$8(2+3x^3)^7(9x^2)$	$\frac{1}{72}(2+3x^3)^8+C$
39.	$\frac{1}{2+x^2}$	$\frac{-2x}{(2+x^2)^2}$	$\frac{-1}{2(2+x^2)}+C$
40.	$\sin 2x$	$2\cos 2x$	$\frac{1}{2}\sin 2x+C$
41.	$\cos^3 x$	$3\cos^2 x(-\sin x)$	$-\frac{1}{3}\cos^3 x+C$
42.	$\sin^2 2x$	$2\sin 2x\cos 2x(2)$	$\frac{3}{4}\sin^2 2x+C$
43.	$\sec 12x$	$\sec 12x\tan 12x\,(12)$	$\frac{1}{12}\sec 12x+C$
44.	$\cot 4x$	$-4\csc^2 4x$	$-\frac{1}{4}\cot 4x+C$
45.	e^{4x}	$4e^{4x}$	$\frac{1}{4}e^{4x}+C$
46.	e^{-x^2}	$e^{-x^2}(-2x)$	$-\frac{1}{2}e^{-x^2}+C$
47.	$e^{3/x}$	$e^{3/x}\left(\frac{-3}{x^2}\right)$	$-\frac{1}{3}e^{3/x}+C$
48.	e^{4x-3}	$e^{4x-3}(4)$	$\frac{1}{4}e^{4x-3}+C$
49.	$\ln\lvert 3x+2\rvert$	$\frac{3}{3x+2}$	$\frac{1}{3}\ln\lvert 3x+2\rvert+C$
50.	$\ln\lvert 7-5x\rvert$	$\frac{1}{7-5x}(-5)$	$-\frac{2}{5}\ln\lvert 7-5x\rvert+C$

51.	$\ln\lvert 1-x^2\rvert$	$\dfrac{-2x}{1-x^2}$	$-\dfrac{1}{2}\ln\lvert 1-x^2\rvert + C$
52.	$\ln\lvert 1-4x^3\rvert$	$\dfrac{1}{1-4x^3}(-12x^2)$	$-\dfrac{1}{4}\ln\lvert 1-4x^3\rvert + C$
53.	2^x	$2^x(\ln 2)$	$\dfrac{2^x}{\ln 2}+C = (\log_2 e)\,2^x + C$
54.	3^{2x}	$3^{2x}(2)\ln 3$	$\dfrac{1}{2\ln 3}3^{2x} = \dfrac{1}{2}(\log_3 e)3^{2x} + C$
55.	$\ln\lvert e^x+1\rvert$	$\dfrac{e^x}{e^x+1}$	$\ln(e^x+1)+C$
56.	$(1+\cos x)^5$	$5(1+\cos x)^4(-\sin x)$	$-\dfrac{1}{5}(1+\cos x)^5 + C$
57.	$\dfrac{1}{\sin^2 x}$	$\dfrac{-2\,\cos x}{\sin^3 x}$	$\dfrac{-1}{2\,\sin^2 x}+C$
58.	$(1+e^{2x})^4$	$4(1+e^{2x})^3(2e^{2x})$	$\dfrac{1}{8}(1+e^{2x})^4 + C$
59.	$\dfrac{1}{\tan x}$	$\dfrac{-\sec^2 x}{\tan^2 x}$	$\dfrac{-1}{\tan x}+C = -\cot x + C$
60.	$\mathrm{Sin}^{-1}2x$	$\dfrac{2}{\sqrt{1-4x^2}}$	$\dfrac{1}{2}\mathrm{Sin}^{-1}2x + C$
61.	$\mathrm{Tan}^{-1}3x$	$\dfrac{3}{1+9x^2}$	$\dfrac{1}{3}\mathrm{Tan}^{-1}3x + C$
62.	$\mathrm{Sec}^{-1}\sqrt{3}x$	$\dfrac{\sqrt{3}}{\sqrt{3}x\sqrt{3x^2-1}}$	$\mathrm{Sec}^{-1}\sqrt{3}x + C$
63.	$\ln(1+5x^2)$	$\dfrac{10x}{1+5x^2}$	$\dfrac{3}{10}\ln(1+5x^2)+C$
64.	$\sinh 4x$	$4\cosh 4x$	$\dfrac{1}{4}\sinh 4x + C$
65.	$\cosh 3x^2$	$6x\,\sinh 3x^2$	$\dfrac{1}{6}\cosh 3x^2 + C$
66.	$\mathrm{sech} 2x$	$-2\mathrm{sech}2x\tanh 2x$	$-\dfrac{1}{2}\mathrm{sech}2x + C$
67.	$\coth 4x^3$	$-12x^2\mathrm{csch}^2 4x^3$	$-\dfrac{1}{12}\coth 4x^3 + C$

68. Integration with respect to x gives $y = \dfrac{x^4}{4} + \dfrac{1}{x} + C$.

69. Since $\dfrac{d}{dx}(3-4x)^{3/2} = (3/2)\sqrt{3-4x}(-4)$, it follows that $y = -(1/6)(3-4x)^{3/2} + C$.

70. Since $\dfrac{d}{dx}\dfrac{1}{(3x+5)^{1/2}} = \dfrac{-1/2}{(3x+5)^{3/2}}(3)$, it follows that $y = \dfrac{-2}{3(3x+5)^{1/2}} + C$.

71. Since $\dfrac{d}{dx}(2x^3+4)^5 = 5(2x^3+4)^4(6x^2)$, it follows that $y = (1/30)(2x^3+4)^5 + C$.

72. Since $\dfrac{d}{dx}\dfrac{1}{2+3x^4} = \dfrac{-1}{(2+3x^4)^2}(12x^3)$, it follows that $y = \dfrac{-1}{12(2+3x^4)} + C$.

73. Since $\dfrac{d}{dx}\cos^3 x = 3\,\cos^2 x\,(-\sin x)$, it follows that $y = -\cos x - (1/3)\cos^3 x + C$.

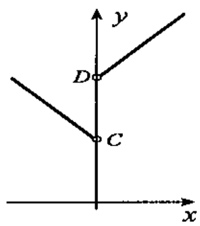
74. Indefinite integrals on the intervals $x<0$ and $x>0$ give $y = f(x) = \begin{cases} -\dfrac{1}{x} + C, & x<0 \\ -\dfrac{1}{x} + D, & x>0. \end{cases}$

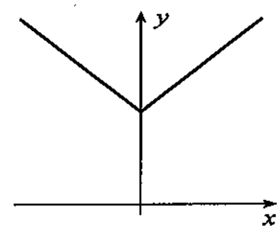
The conditions $f(-1) = -2$ and $f(1) = 1$ require $C = -3$ and $D = 2$.

75. (a) Indefinite integrals on the intervals $x < 0$ and $x > 0$ give

$$\int \operatorname{sgn}(x)\,dx = \begin{cases} -x + C, & x < 0 \\ x + D, & x > 0. \end{cases}$$

(b) Graphs of the indefinite integrals of $\operatorname{sgn}(x)$ (as determined in part (a)) are shown in the left figure below. If $\operatorname{sgn}(x)$ has an antiderivative $F(x)$ on an interval containing $x = 0$, then $F(x)$ has a derivative on this interval, including at $x = 0$. By Theorem 3.6, $F(x)$ is continuous at $x = 0$, in which case its graph must be as shown in the right figure. But this function does not have a derivative at $x = 0$.





EXERCISES 5.2

1. Integration of $dv/dt = a(t) = t + 2$ gives $v(t) = t^2/2 + 2t + C$. Since $v(0) = 0$, it follows that $C = 0$, and $v(t) = dx/dt = t^2/2 + 2t$. Integration now gives $x(t) = t^3/6 + t^2 + D$. The condition $x(0) = 0$ requires $D = 0$, and therefore $x(t) = t^3/6 + t^2$.

2. Integration of $dv/dt = a(t) = 6 - 2t$ gives $v(t) = 6t - t^2 + C$. Since $v(0) = 5$, it follows that $C = 5$, and $v(t) = dx/dt = 6t - t^2 + 5$. Integration now gives $x(t) = 3t^2 - t^3/3 + 5t + D$. The condition $x(0) = 0$ requires $D = 0$, and therefore $x(t) = 3t^2 - t^3/3 + 5t$.

3. Integration of $dv/dt = a(t) = 6 - 2t$ gives $v(t) = 6t - t^2 + C$. Since $v(0) = 5$, it follows that $C = 5$, and $v(t) = dx/dt = 6t - t^2 + 5$. Integration now gives $x(t) = 3t^2 - t^3/3 + 5t + D$. The condition $x(0) = 0$ requires $D = 0$, and therefore $x(t) = 3t^2 - t^3/3 + 5t$.

4. Integration of $dv/dt = a(t) = 120t - 12t^2$ gives $v(t) = 60t^2 - 4t^3 + C$. Since $v(0) = 0$, it follows that $C = 0$, and $v(t) = dx/dt = 60t^2 - 4t^3$. Integration now gives $x(t) = 20t^3 - t^4 + D$. The condition $x(0) = 4$ requires $D = 4$, and therefore $x(t) = 20t^3 - t^4 + 4$.

5. Integration of $dv/dt = a(t) = t^2 + 1$ gives $v(t) = t^3/3 + t + C$. Since $v(0) = -1$, it follows that $C = -1$, and $v(t) = dx/dt = t^3/3 + t - 1$. Integration now gives $x(t) = t^4/12 + t^2/2 - t + D$. The condition $x(0) = 1$ requires $D = 1$, and therefore $x(t) = t^4/12 + t^2/2 - t + 1$.

6. Integration of $dv/dt = a(t) = t^2+5t+4$ gives $v(t) = t^3/3+5t^2/2+4t+C$. Since $v(0) = -2$, it follows that $C = -2$, and $v(t) = dx/dt = t^3/3+5t^2/2+4t-2$. Integration now gives $x(t) = t^4/12+5t^3/6+2t^2-2t+D$. The condition $x(0) = -3$ requires $D = -3$, and therefore $x(t) = t^4/12 + 5t^3/6 + 2t^2 - 2t - 3$.

7. Integration of $dv/dt = a(t) = \cos t$ gives $v(t) = \sin t + C$. Since $v(0) = 0$, it follows that $C = 0$, and $v(t) = dx/dt = \sin t$. Integration now gives $x(t) = -\cos t + D$. The condition $x(0) = 0$ requires $D = 1$, and therefore $x(t) = 1 - \cos t$.

8. Integration of $dv/dt = 3\sin t$ gives $v(t) = -3\cos t + C$. Since $v(0) = 1$, it follows that $1 = -3 + C$. Thus, $C = 4$ and $v(t) = dx/dt = 4 - 3\cos t$. Integration now gives $x(t) = 4t - 3\sin t + D$. The condition $x(0) = 4$ requires $D = 4$, and therefore $x(t) = 4t + 4 - 3\sin t$.

9. (a) Since $a(t) = 6t - 9$, we obtain $a(5) = 21$ m/s^2.
(b) Integration gives $x(t) = t^3 - 9t^2/2 + 6t + C$. Since $x(0) = 1$, it follows that $C = 1$ and $x(t) = t^3 - 9t^2/2 + 6t + 1$. Thus, $x(2) = 3$ m.
(c) Since $v(5/4) = -9/16$ and $a(5/4) = -3/2$, the object is speeding up.
(d) For critical points of $x(t)$ we solve $0 = v(t) = 3(t-1)(t-2)$ for $t = 1, 2$. Since $x(0) = 1$, $x(1) = 7/2$, $x(2) = 3$, and $\lim_{t\to\infty} x = \infty$, the closest the object is to the origin is 1 m.

10. (a) Integration of $dv/dt = 6t - 2$ gives $v(t) = 3t^2 - 2t + C$. Since $v(0) = -3$, we find that $C = -3$ and $v(t) = 3t^2 - 2t - 3$. Integration now gives $x(t) = t^3 - t^2 - 3t + D$. The condition $x(0) = 1$ requires $D = 1$, and therefore $x(t) = t^3 - t^2 - 3t + 1$.
(b) The velocity is zero when $3t^2 - 2t - 3 = 0$, a quadratic equation with solutions $t = (2 \pm \sqrt{4+36})/6 = (1 \pm \sqrt{10})/3$. Only the solution $t = (1+\sqrt{10})/3$ is positive.

11. (a) Integration of $dv/dt = 6t - 15$ gives $v(t) = 3t^2 - 15t + C$. Since $v(2) = 6$, it follows that $6 = 12 - 30 + C \implies C = 24$. Thus, $v(1) = 3 - 15 + 24 = 12$ m/s.
(b) Integration of $dx/dt = 3t^2 - 15t + 24$ gives $x(t) = t^3 - 15t^2/2 + 24t + D$. Since $x(0) = 10$, we obtain $D = 10$, and $x(t) = t^3 - 15t^2/2 + 24t + 10$.
(c) For critical points of $x(t)$, we solve $0 = v(t) = 3(t^2 - 5t + 8)$. Since there are no solutions of this equation, we evaluate $x(0) = 10$ and $\lim_{t\to\infty} x = \infty$. The closest distance is 10 m.

12. (a) Integration of $dv/dt = 3 - t/5$ gives $v(t) = 3t - t^2/10 + C$. Since $v(0) = 0$, it follows that $C = 0$, and $v(t) = 3t - t^2/10$. Integration now gives $x(t) = 3t^2/2 - t^3/30 + D$. If we choose a positive x-axis in the direction of motion of the car with $x = 0$ at $t = 0$, then $x(0) = 0$. This condition requires $D = 0$, and therefore $x(t) = 3t^2/2 - t^3/30$. The position of the car after 10 s is $x(10) = 3(100)/2 - (1000)/30 = 350/3$ m.
(b) For $t > 10$, the acceleration is $a(t) = -2$. Integration of this yields $v(t) = -2t + E$. Because $v(10) = 3(10) - 100/10 = 20$, it follows that $20 = -2(10) + E$. Hence, $E = 40$, and $v(t) = 40 - 2t$. Integration now gives $x(t) = 40t - t^2 + F$. Because $x(10) = 350/3$, it follows that $350/3 = 40(10) - 100 + F$. Thus, $F = -550/3$, and $x(t) = 40t - t^2 - 550/3$. The car comes to a stop when $0 = v(t) = 40 - 2t$, and this implies that $t = 20$. The position of the car at this time is $x(20) = 40(20) - (20)^2 - 550/3 = 650/3$ m.

13. Integration of $v(t) = 180 - 18t$ gives $x(t) = 180t - 9t^2 + C$. If we choose $x = 0$ at the position at which the plane touches the ground, then $C = 0$ and $x(t) = 180t - 9t^2$. Since the speed of the plane is zero in 10 seconds, the distance that it moves after touching the ground is $x(10) = 180(10) - 9(10^2) = 900$ m.

14. We choose y as positive upward with $y = 0$ and $t = 0$ at the point and instant of projection. The acceleration of the stone is $a = -9.81$. Integration gives $v(t) = -9.81t + C$. Since $v(0) = 10$, it follows that $C = 10$, and $v(t) = -9.81t + 10$. Integration now gives $y(t) = -4.905t^2 + 10t + D$. The condition $y(0) = 0$ requires $D = 0$, and therefore $y(t) = -4.905t^2 + 10t$. At the peak height of the stone, $0 = v(t) = -9.81t + 10$, and this occurs when $t = 10/9.81$. The height of the stone at this time is $y(10/9.81) = -4.905(10/9.81)^2 + 10(10/9.81) = 5.1$ m.

15. We choose y as positive downward with $y = 0$ and $t = 0$ at the point and instant the stone is dropped. The acceleration of the stone is $a = 9.81$. Integration gives $v(t) = 9.81t + C$. Since $v(0) = 0$, it follows that $C = 0$, and $v(t) = 9.81t$. Integration now gives $y(t) = 4.905t^2 + D$. The condition $y(0) = 0$ requires $D = 0$, and therefore $y(t) = 4.905t^2$. The time that it takes the stone to drop 25 m is given by $25 = 4.905t^2 \implies t = \sqrt{25/4.905}$. Thus, the stone should be dropped $5/\sqrt{4.905}$ s before the wood reaches the appropriate spot.

16. We choose y as positive upward with $y = 0$ and $t = 0$ at the point and instant the ball is thrown. The acceleration of the ball is $a = -9.81$. Integration gives $v(t) = -9.81t + C$. If v_0 is the initial speed of the ball, then $C = v_0$, and $v(t) = v_0 - 9.81t$. Integration gives $y(t) = v_0 t - 4.905t^2 + D$. The condition $y(0) = 0$ requires $D = 0$, and therefore $y(t) = v_0 t - 4.905t^2$. For the ball just to reach your friend, we must have $v = 0$ when $y = 20$:

$$0 = v_0 - 9.81t, \qquad 20 = v_0 t - 4.905t^2.$$

The first implies that $t = v_0/9.81$, and this can be substituted into the second,

$$20 = v_0\left(\frac{v_0}{9.81}\right) - 4.905\left(\frac{v_0}{9.81}\right)^2.$$

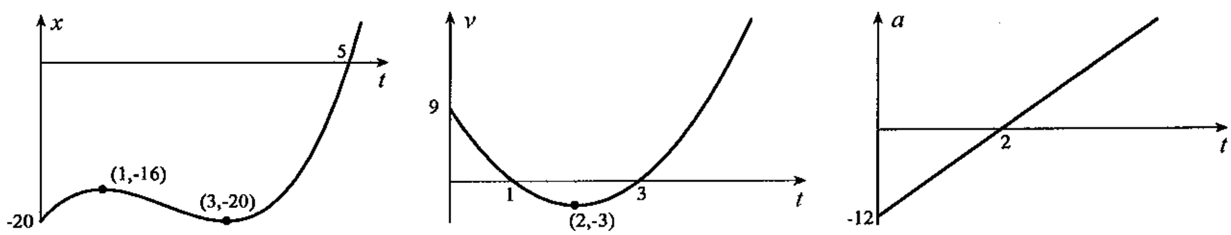
The positive solution of this equation is $v_0 = 19.8$ m/s.

17. Suppose the car is travelling to the right and we choose x positive to the right with $x = 0$ and $t = 0$ at the point and instant the brakes are applied. Let a represent the constant acceleration that will stop the car just as it touches the tree. Integration of $dv/dt = a$ gives $v(t) = at + C$. Since $v(0) = 20$, we have $C = 20$ and $v(t) = at + 20$. A second integration gives $x(t) = at^2/2 + 20t + D$. Since $x(0) = 0$, it follows that $D = 0$ and $x(t) = at^2/2 + 20t$. Since the velocity of the car is zero at the tree when $x = 50$, we can say that at the tree,

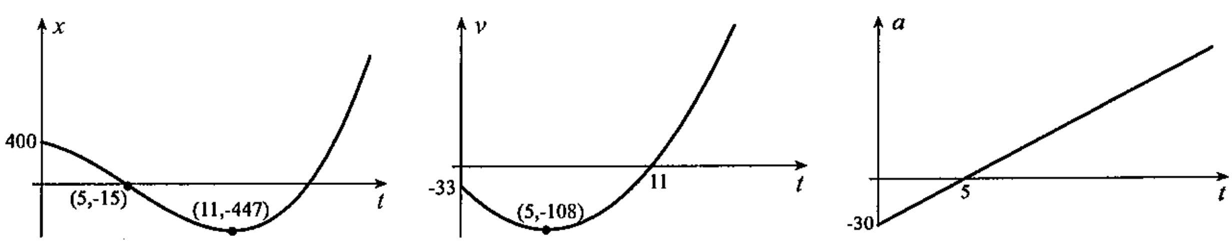
$$50 = \frac{1}{2}at^2 + 20t, \qquad 0 = at + 20.$$

When these are solved for a and t, the result is $a = -4$. Consequently, if the acceleration is less than -4 m/s^2, the car stops before striking the tree.

18. The velocity and acceleration are $v(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3)$ and $a(t) = 6t - 12 = 6(t-2)$. The graph of $x(t)$ has critical points at $t = 1$ and $t = 3$, the first giving a relative maximum of $x(1) = -16$, and the second a relative minimum of $x(3) = -20$. The graph has a point of inflection at $(2, -18)$. The graph of $v(t)$ has zeros at $t = 1$ and $t = 3$, the critical points of $x(t)$. The graph of $a(t)$ has a zero at $t = 2$, the point of inflection for $x(t)$.



19. Integration of $dv/dt = 6t - 30$ gives $v(t) = 3t^2 - 30t + C$. Since $v(0) = -33$, it follows that $C = -33$ and $v(t) = 3t^2 - 30t - 33 = 3(t-11)(t+1)$. A second integration gives $x(t) = t^3 - 15t^2 - 33t + D$. Since $x(0) = 400$, we find that $D = 400$, and $x(t) = t^3 - 15t^2 - 33t + 400$. The graph of $x(t)$ has a critical point at $t = 11$, giving a relative minimum of $x(11) = -447$. There is a point of inflection at $(5, -15)$. The zero $t = 11$ of $v(t)$ is the critical point of $x(t)$. The zero $t = 5$ of $a(t)$ gives the point of inflection for $x(t)$.



20. We choose $x = 0$ and $t = 0$ at the point and instant the brakes are applied. If we assume that the acceleration of the car is -9.81 m/s^2, and determine the initial speed v_0 which produces a skid mark of 9 m, then v_0 is the maximum possible speed. In other words, we are testifying for the defence. Integration of $a = -9.81$ gives $v(t) = -9.81t + C$. The condition $v(0) = v_0$ requires $C = v_0$, and therefore $v(t) = -9.81t + v_0$. Another integration yields $x(t) = -4.905t^2 + v_0 t + D$. Since $x(0) = 0$, it follows that $D = 0$. Because $x = 9$ when $v = 0$,

$$0 = -9.81t + v_0, \qquad 9 = -4.905t^2 + v_0 t.$$

The first requires $t = v_0/9.81$, and when this is substituted into the second,

$$9 = -4.905\left(\frac{v_0}{9.81}\right)^2 + v_0\left(\frac{v_0}{9.81}\right).$$

The positive solution of this equation is $v_0 = 13.3$ m/s or 47.8 km/hr.

21. (a) Suppose we let x measure distance in the direction of motion of the car, taking $x = 0$ and $t = 0$ at the point and instant the brakes are applied. Integration of $dv/dt = -5$ gives $v(t) = -5t + C$. Since $v(0) = 250/9$, we find $C = 250/9$, and $v(t) = -5t+250/9$. A second integration gives $x(t) = -5t^2/2+250t/9+D$. With $x(0) = 0$, we obtain $D = 0$, and $x(t) = -5t^2/2 + 250t/9$. The car comes to a stop when $0 = v = -5t + 250/9 \Longrightarrow t = 50/9$, and at this instant $x(50/9) = -(5/2)(50/9)^2 + (250/9)(50/9) = 6250/81$ m.
(b) With $v(0) = 125/9$, the velocity is $v(t) = -5t + 125/9$, and distance travelled is $x(t) = -5t^2/2 + 125t/9$. The car stops when $t = 25/9$, at which time $x(25/9) = -(5/2)(25/9)^2 + (125/9)(25/9) = 3125/162$ m.
(c) The ratio of these distances is 4.
(d) Stopping times from the instant the driver takes his foot from the accelerator are

$$\frac{6250}{81} + \left(\frac{3}{4}\right)\left(\frac{250}{9}\right) = \frac{15875}{162} \quad \text{and} \quad \frac{3125}{162} + \left(\frac{3}{4}\right)\left(\frac{125}{9}\right) = \frac{9625}{324}.$$

The ratio of these distances is 3.3.

22. We choose y as positive downward with $y = 0$ and $t = 0$ at the point and instant the stone is dropped. The acceleration of the stone is $a = 9.81$. Integration gives $v(t) = 9.81t + C$. Since $v(0) = 0$, it follows that $C = 0$, and $v(t) = 9.81t$. Integration gives $y(t) = 4.905t^2 + D$. The condition $y(0) = 0$ requires $D = 0$, and therefore $y(t) = 4.905t^2$. If d is the distance from the top of the well to the surface of the water and T is the time it takes the stone to fall this distance, then $d = 4.905T^2 \Longrightarrow T = \sqrt{d/4.905}$. Since the time taken for the sound to travel the distance d is $d/340$, it follows that

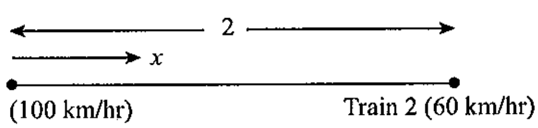
$$\sqrt{\frac{d}{4.905}} + \frac{d}{340} = 3.1 \quad \Longrightarrow \quad \sqrt{\frac{d}{4.905}} = 3.1 - \frac{d}{340}.$$

Squaring gives

$$\frac{d}{4.905} = (3.1)^2 - \frac{3.1d}{170} + \frac{d^2}{340^2} \quad \Longrightarrow \quad \frac{d^2}{340^2} - \left(\frac{3.1}{170} + \frac{1}{4.905}\right)d + (3.1)^2 = 0.$$

Solutions of this quadratic are $d = \dfrac{(3.1/170 + 1/4.905) \pm \sqrt{(3.1/170 + 1/4.905)^2 - 4(3.1)^2/340^2}}{2/340^2}$. Only the solution $d = 43.3$ satisfies the original equation. Hence the depth of the well is 43.3 m.

23. (a)



The acceleration of Train 1 is $a_1 = -1/4$. Thus, $v_1 = -t/4+C$. Since $v_1(0) = 250/9$, we have $C = 250/9$, and $v_1 = -t/4 + 250/9$. Hence, $x_1 = -t^2/8 + 250t/9 + D$. Because $x_1(0) = 0$, it follows that $D = 0$, and $x_1(t) = -t^2/8 + 250t/9$. A similar calculation for Train 2 with $a_2 = 1/4$, $v_2(0) = -50/3$, and $x_2(0) = 2000$ gives $v_2 = t/4 - 50/3$, and $x_2(t) = t^2/8 - 50t/3 + 2000$. These expressions for x_1 and x_2 are valid until each train stops or a collision takes place. Train 1 stops when $t = 1000/9$ and therefore at position

$$x_1 = -\frac{1}{8}\left(\frac{1000}{9}\right)^2 + \frac{250}{9}\left(\frac{1000}{9}\right) = 1543.2 \text{ m.}$$

Train 2 stops when $t = 200/3$, and therefore at position

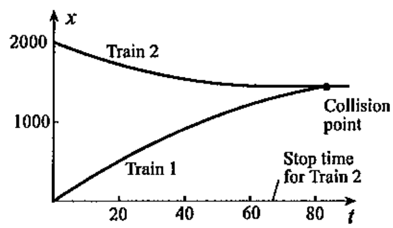
$$x_2 = \frac{1}{8}\left(\frac{200}{3}\right)^2 - \frac{50}{3}\left(\frac{200}{3}\right) + 2000 = 1444.4 \text{ m.}$$

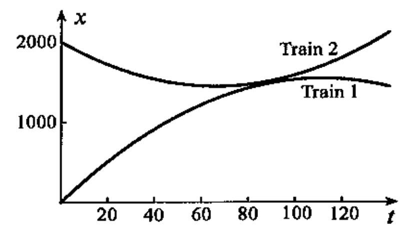
Since Train 2 would stop to the left of Train 1, a collision occurs.

(b) In this case the expressions for x_1 and x_2 are for all $t \geq 0$ since the trains reverse directions after stopping (unless a collision occurs). A collision occurs if and only if $x_1 = x_2$; that is,

$$-\frac{t^2}{8} + \frac{250t}{9} = \frac{t^2}{8} - \frac{50t}{3} + 2000 \implies \frac{t^2}{4} - \frac{400t}{9} + 2000 = 0.$$

Since this equation has no real solutions, a collision does not occur.
(c)





24. We take y as positive downward from the top of the building with $t = 0$ at the instant the bearing is dropped. Let h be the distance from the top of the building to the top of the window and H be the distance from the bottom of the window to the sidewalk. Integration of $dv/dt = 9.81$ gives $v(t) = 9.81t + C$. Since $v(0) = 0$, we find that $C = 0$ and $v(t) = 9.81t$. A second integration gives $y(t) = 4.905t^2 + D$. With $y(0) = 0$, we obtain $D = 0$, and $y(t) = 4.905t^2$. If T is the time taken to reach the top of the window on the way down, then

$$h = 4.905T^2, \qquad h + 1 = 4.905(T + 1/8)^2, \qquad h + 1 + H = 4.905(T + 9/8)^2.$$

When the first of these is subtracted from the second, and the resulting equation is solved for T, we obtain $T = 0.753$. Substitution of this into the third gives $h + 1 + H = 17.3$. Thus, the building is 17.3 metres high.

25. (a) We take y as positive upward with $y = 0$ on the ground, and $t = 0$ at the instant the ball is thrown upward. The acceleration of the ball is $a_b = -9.81$. Integration gives $v_b(t) = -9.81t + C$. Since $v_b(0) = 30$, we have $C = 30$, and $v_b(t) = -9.81t + 30$. Integration gives $y_b(t) = -4.905t^2 + 30t + D$. The condition $y_b(0) = 30$ requires $D = 30$, and therefore $y_b(t) = -4.905t^2 + 30t + 30$. The ball reaches peak height when $v_b = 0 \implies t = 30/9.81$. The height of the ball at this time is $y_b = -4.905(30/9.81)^2 + 30(30/9.81) + 30 = 75.9$ m.
(b) Since the height of the elevator floor at time t is $y_e = 10t + 28$, the elevator catches the ball when

$$y_b = y_e \implies -4.905t^2 + 30t + 30 = 10t + 28 \implies -4.905t^2 + 20t + 2 = 0.$$

The positive solution is $t = 4.18$ s.

26. Let us choose a different coordinate system to that of Example 5.6 by taking y as positive upward with $y = 0$ and $t = 0$ at the point and instant of projection of the first stone. The acceleration of stone 1 is

$$a_1 = -9.81 = \frac{dv_1}{dt},$$

from which $v_1(t) = -9.81t + C$. Since $v_1(0) = 25$, it follows that $25 = C$, and $v_1(t) = -9.81t + 25$, $t \geq 0$. Thus,

$$\frac{dy_1}{dt} = -9.81t + 25,$$

from which we have $y_1(t) = -4.905t^2 + 25t + D$. Since $y_1(0) = 0$, we find that $0 = D$, and $y_1(t) = -4.905t^2 + 25t$, $t \geq 0$.
The acceleration of stone 2 is also -9.81; hence we have

$$a_2 = -9.81 = \frac{dv_2}{dt},$$

from which $v_2(t) = -9.81t + E$. Because $v_2(1) = 20$, we must have $20 = -9.81(1) + E$, or $E = 29.81$. Thus, $v_2(t) = -9.81t + 29.81$, $t \geq 1$. Consequently,

$$\frac{dy_2}{dt} = -9.81t + 29.81,$$

from which $y_2(t) = -4.905t^2 + 29.81t + F$. Since $y_2(1) = 0$, it follows that $0 = -4.905(1)^2 + 29.81(1) + F$, or $F = -24.905$. Thus, $y_2(t) = -4.905t^2 + 29.81t - 24.905$, $t \geq 1$. The stones will pass each other if y_1 and y_2 are ever equal for the same time t; that is, if

$$-4.905t^2 + 25t = -4.905t^2 + 29.81t - 24.905.$$

The solution of this equation is

$$t = \frac{24.905}{4.81} = 5.2.$$

Because the first stone does not strike the base of the cliff for 7.7 s (Example 5.6), it follows that the stones do indeed pass each other 5.2 s after the first stone is projected.

27. $$F = m_0 \left\{ \frac{\sqrt{1-(v^2/c^2)}(dv/dt) - v(1/2)[1-(v^2/c^2)]^{-1/2}(-2v/c^2)(dv/dt)}{1-(v^2/c^2)} \right\}$$
$$= m_0 \frac{1-(v^2/c^2)+(v^2/c^2)}{[1-(v^2/c^2)]^{3/2}}\frac{dv}{dt} = \frac{m_0 a}{[1-(v^2/c^2)]^{3/2}}.$$

Newton's second law $F = ma = m_0 a$ indicates that the force necessary to impart an acceleration a to a mass is independent of the velocity of m. The relativistic formula states that F depends on v as well as a. In addition, notice that as v approaches c, the speed of light, forces necessary to give accelerations become extremely large.

28. Flow rate when distances are a fraction of the safe distance is

$$r(v) = \frac{v}{l + k\left(vT - \dfrac{v^2}{2a}\right)}.$$

The critical point of this function is defined by

$$0 = \frac{\left[l + k\left(vT - \dfrac{v^2}{2a}\right)\right](1) - v\left[k\left(T - \dfrac{v}{a}\right)\right]}{\left[l + k\left(vT - \dfrac{v^2}{2a}\right)\right]^2} \qquad \Longrightarrow \qquad 0 = \frac{1}{2a}(2al + kv^2).$$

Thus, $v = \sqrt{-2al/k}$.

29. (a) We choose y positive upward with $y = 0$ and $t = 0$ at the point and instant the first stone is released. For stone 1, $a_1 = -g$, from which $v_1 = -gt + C$. Since $v_1(0) = v_0'$, it follows that $C = v_0'$, and $v_1 = -gt + v_0'$. A second integration gives $x_1 = -gt^2/2 + v_0't + D$. Since $x_1(0) = 0$, we find that $D = 0$, and $x_1 = -gt^2/2 + v_0't$, $t \geq 0$. For stone 2, $a_2 = -g$, from which $v_2 = -gt + E$. With $v_2(t_0) = v_0''$, we obtain $v_0'' = -gt_0 + E$, and therefore $v_2 = -g(t - t_0) + v_0''$. Integration now gives $x_2 = -g(t-t_0)^2/2 + v_0''t + F$. The initial condition $x_2(t_0) = 0$ gives $0 = v_0''t_0 + F$, and therefore $x_2 = -g(t-t_0)^2/2 + v_0''(t - t_0)$, $t \geq t_0$. The stones pass each other for $t \geq t_0$ if

$$-\frac{gt^2}{2} + v_0't = -\frac{g(t-t_0)^2}{2} + v_0''(t - t_0),$$

and the solution of this equation for t is

$$t = \frac{(gt_0 + 2v_0'')t_0}{2(gt_0 + v_0'' - v_0')}.$$

Since the numerator is positive, it follows that t will be positive if $gt_0 > v_0' - v_0''$. In addition, they will pass one another during the motion if and only if this value of t is greater than t_0; that is,

$$\frac{(gt_0 + 2v_0'')t_0}{2(gt_0 + v_0'' - v_0')} > t_0,$$

and this condition reduces to $v_0' > gt_0/2$.
(b) The times required for the stones to commence their downward trajectories are

$$t_1 = \frac{v_0'}{g} \qquad \text{and} \qquad t_2 = \frac{v_0''}{g} + t_0.$$

Stone 1 will commence downward first therefore if

$$\frac{v_0'}{g} < \frac{v_0''}{g} + t_0,$$

and this is equivalent to $gt_0 > v_0' - v_0''$.
(c) Stone 1 reaches its projection point again at $t = 2v_0'/g$, and this will occur after Stone 2 is released if and only if $2v_0'/g > t_0$, or $v_0' > gt_0/2$.

30. During the acceleration stage, $a = dv/dt = 3$ so that $v = 3t + C$. If we choose time $t = 0$ as the vehicle leaves a speed bump with velocity 2.5 m/3, then $C = 2.5$ and $v = 3t + 2.5$. If x measures displacement, then $x = 3t^2/2 + 2.5t + D$. If we take $x = 0$ at the speed bump, then $D = 0$ and $x = 3t^2/2 + 2.5t$. Since speed is to be 10 m/s after the acceleration stage, which we suppose takes T seconds, $10 = 3T + 2.5 \Longrightarrow T = 2.5$ s. The position of the vehicle at this time is $x = 3(2.5)^2/2 + 2.5(2.5) = 15.625$ m. During the deceleration stage, $a = dv/dt = -7$ so that $v = -7t + E$. Since speed is 10 m/s when $t = 2.5$ s, $10 = -7(2.5) + E \Longrightarrow E = 27.5$, and $v = -7t + 27.5$. Displacement during this stage is $x = -7t^2/2 + 27.5t + F$. Since $x = 15.625$ when $t = 2.5$, $15.625 = -7(2.5)^2/2 + 27.5(2.5) + F \Longrightarrow F = -31.25$. Since speed at the end of this stage is to be 2.5 m/s at the second bump, we can find when this occurs by solving $2.5 = -7t + 27.5 \Longrightarrow t = 25/7$. The displacement of the vehicle at this time is $x = -7(25/7)^2/2 + 27.5(25/7) - 31.25 = 22.3$. This is the distance in metres between speed bumps.

31. We choose x as positive to the right (the direction of motion) with $x = 0$ and $t = 0$ at position and instant motion commences. It t_1 is the time at which acceleration ends, then $T - t_1$ is the time at which deceleration begins. If a is the acceleration of the car during the time interval $0 < t < t_1$, then its velocity and position during this time interval are $v = at$ and $x = at^2/2$. Velocity and position of the car at t_1 are $V = at_1$ and $x_1 = at_1^2/2$. During the time interal $t_1 < t < T - t_1$, position of the car is $x = at_1^2/2 + V(t - t_1)$. Position at time $T - t_1$ is $at_1^2/2 + V(T - 2t_1)$. Since acceleration of the car during the time interval $T - t_1 < t < T$ is $-a$, its velocity is $v = -at + C$. Because $v(T - t_1) = V$, it follows that $V = -a(T - t_1) + C \Longrightarrow C = V + a(T - t_1)$ and $v = -at + V + a(T - t_1)$. Position of the car is $x = -at^2/2 + Vt + a(T - t_1)t + E$. Since $x(T - t_1) = at_1^2/2 + V(T - 2t_1)$, it follows that

$$\frac{at_1^2}{2} + V(T - 2t_1) = -\frac{a}{2}(T - t_1)^2 + V(T - t_1) + a(T - t_1)(T - t_1) + E \Longrightarrow E = \frac{at_1^2}{2} - Vt_1 - \frac{a}{2}(T - t_1)^2.$$

The position of the car at time T is

$$D = x(T) = -\frac{aT^2}{2} + VT + a(T - t_1)T + \frac{at_1^2}{2} - Vt_1 - \frac{a}{2}(T - t_1)^2 = V(T - t_1) \Longrightarrow t_1 = T - \frac{D}{V}.$$

Consequently, the length of time at speed V is $T - 2t_1 = T - 2\left(T - \dfrac{D}{V}\right) = \dfrac{2D}{V} - T$.

EXERCISES 5.3

In Exercises 1–7, it is not necessary to use a substitution; these integrations can be done by adjusting constants.

1. $\displaystyle\int (5x + 14)^9\, dx = \frac{1}{50}(5x + 14)^{10} + C$

2. $\displaystyle\int \sqrt{1 - 2x}\, dx = -\frac{1}{3}(1 - 2x)^{3/2} + C$

3. $\displaystyle\int \frac{1}{(3y-12)^{1/4}}dy = \frac{4}{9}(3y-12)^{3/4} + C$

4. $\displaystyle\int \frac{5}{(5-42x)^{1/4}}dx = \frac{-(5)(4)}{3(42)}(5-42x)^{3/4} + C = -\frac{10}{63}(5-42x)^{3/4} + C$

5. $\displaystyle\int x^2(3x^3+10)^4\,dx = \frac{1}{45}(3x^3+10)^5 + C$

6. $\displaystyle\int \frac{x}{(x^2+4)^2}dx = \frac{-1}{2(x^2+4)} + C$

7. $\displaystyle\int \sin^4 x\,\cos x\,dx = \frac{1}{5}\sin^5 x + C$

8. If we set $u = x-2$, then $du = dx$, and

$$\begin{aligned}\int \frac{x^2}{(x-2)^4}dx &= \int \frac{(u+2)^2}{u^4}du = \int\left(\frac{1}{u^2}+\frac{4}{u^3}+\frac{4}{u^4}\right)du\\ &= -\frac{1}{u}-\frac{2}{u^2}-\frac{4}{3u^3}+C = \frac{-1}{x-2}-\frac{2}{(x-2)^2}-\frac{4}{3(x-2)^3}+C.\end{aligned}$$

9. If we set $u = 1-3z$, then $du = -3\,dz$, and

$$\begin{aligned}\int z\sqrt{1-3z}\,dz &= \int\left(\frac{1-u}{3}\right)\sqrt{u}\left(\frac{du}{-3}\right) = \frac{1}{9}\int(u^{3/2}-\sqrt{u})\,du\\ &= \frac{1}{9}\left(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right)+C = \frac{2}{45}(1-3z)^{5/2}-\frac{2}{27}(1-3z)^{3/2}+C.\end{aligned}$$

10. If we set $u = 2x+3$, then $du = 2\,dx$, and

$$\begin{aligned}\int \frac{x}{\sqrt{2x+3}}dx &= \int \frac{(u-3)/2}{\sqrt{u}}\left(\frac{du}{2}\right) = \frac{1}{4}\int\left(\sqrt{u}-\frac{3}{\sqrt{u}}\right)du\\ &= \frac{1}{4}\left(\frac{2}{3}u^{3/2}-6\sqrt{u}\right)+C = \frac{1}{6}(2x+3)^{3/2}-\frac{3}{2}\sqrt{2x+3}+C.\end{aligned}$$

11. $\displaystyle\int \frac{1+\sqrt{x}}{\sqrt{x}}dx = \int\left(\frac{1}{\sqrt{x}}+1\right)dx = 2\sqrt{x}+x+C$

12. If we set $u = s^2+5$, then $du = 2s\,ds$, and

$$\begin{aligned}\int s^3\sqrt{s^2+5}\,ds &= \int s^2\sqrt{s^2+5}\,s\,ds = \int(u-5)\sqrt{u}\left(\frac{du}{2}\right) = \frac{1}{2}\int(u^{3/2}-5\sqrt{u})\,du\\ &= \frac{1}{2}\left(\frac{2}{5}u^{5/2}-\frac{10}{3}u^{3/2}\right)+C = \frac{1}{5}(s^2+5)^{5/2}-\frac{5}{3}(s^2+5)^{3/2}+C.\end{aligned}$$

13. If we set $u = \sin x$, then $du = \cos x\,dx$, and

$$\begin{aligned}\int \sin^2 x\,\cos^3 x\,dx &= \int \sin^2 x(1-\sin^2 x)\cos x\,dx = \int u^2(1-u^2)\,du\\ &= \frac{u^3}{3}-\frac{u^5}{5}+C = \frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x+C.\end{aligned}$$

14. If we set $u = 1-\cos x$, then $du = \sin x\,dx$, and

$$\int\sqrt{1-\cos x}\,\sin x\,dx = \int\sqrt{u}\,du = \frac{2}{3}u^{3/2}+C = \frac{2}{3}(1-\cos x)^{3/2}+C.$$

15. If we set $u = 3-x^2$, then $du = -2x\,dx$, and

$$\begin{aligned}\int \frac{x^3}{(3-x^2)^3}dx &= \int \frac{x^2}{(3-x^2)^3}x\,dx = \int \frac{3-u}{u^3}\left(\frac{du}{-2}\right) = \frac{1}{2}\int\left(\frac{1}{u^2}-\frac{3}{u^3}\right)du\\ &= \frac{1}{2}\left(-\frac{1}{u}+\frac{3}{2u^2}\right)+C = \frac{-1}{2(3-x^2)}+\frac{3}{4(3-x^2)^2}+C.\end{aligned}$$

16. If we set $u = y - 4$, then $du = dy$, and

$$\int y^2\sqrt{y-4}\,dy = \int (u+4)^2\sqrt{u}\,du = \int (u^{5/2} + 8u^{3/2} + 16\sqrt{u})\,du = \frac{2}{7}u^{7/2} + \frac{16}{5}u^{5/2} + \frac{32}{3}u^{3/2} + C$$
$$= \frac{2}{7}(y-4)^{7/2} + \frac{16}{5}(y-4)^{5/2} + \frac{32}{3}(y-4)^{3/2} + C.$$

17. If we set $y = \sqrt{u}$, then $dy = \dfrac{1}{2\sqrt{u}}du$, and

$$\int \frac{(1+\sqrt{u})^{1/2}}{\sqrt{u}}du = \int (1+y)^{1/2}\,2\,dy = \frac{4}{3}(1+y)^{3/2} + C = \frac{4}{3}(1+\sqrt{u})^{3/2} + C.$$

18. If we set $u = 3x^3 - 5$, then $du = 9x^2\,dx$, and

$$\int x^8(3x^3-5)^6\,dx = \int (x^3)^2(3x^3-5)^6x^2\,dx = \int \left(\frac{u+5}{3}\right)^2 u^6\left(\frac{du}{9}\right) = \frac{1}{81}\int (u^8 + 10u^7 + 25u^6)\,du$$
$$= \frac{1}{81}\left(\frac{u^9}{9} + \frac{5u^8}{4} + \frac{25u^7}{7}\right) + C = \frac{1}{729}(3x^3-5)^9 + \frac{5}{324}(3x^3-5)^8 + \frac{25}{567}(3x^3-5)^7 + C.$$

19. $\displaystyle\int \frac{1+z^{1/4}}{\sqrt{z}}dz = \int\left(\frac{1}{\sqrt{z}} + \frac{1}{z^{1/4}}\right)dz = 2\sqrt{z} + \frac{4z^{3/4}}{3} + C$

20. $\displaystyle\int \frac{x+1}{(x^2+2x+2)^{1/3}}dx = \frac{3}{4}(x^2+2x+2)^{2/3} + C$

21. $\displaystyle\int \frac{(x-1)(x+2)}{\sqrt{x}}dx = \int\left(x^{3/2} + \sqrt{x} - \frac{2}{\sqrt{x}}\right)dx = \frac{2x^{5/2}}{5} + \frac{2x^{3/2}}{3} - 4\sqrt{x} + C$

22. If we set $u = 3 - 4\sin x$, then $du = -4\cos x\,dx$, and

$$\int \frac{\cos^3 x}{(3-4\sin x)^4}dx = \int \frac{(1-\sin^2 x)\cos x}{(3-4\sin x)^4}dx = \int \frac{1-[(3-u)/4]^2}{u^4}\left(-\frac{du}{4}\right) = \frac{1}{64}\int \frac{-7-6u+u^2}{u^4}du$$
$$= \frac{1}{64}\int\left(\frac{1}{u^2} - \frac{6}{u^3} - \frac{7}{u^4}\right)du = \frac{1}{64}\left(-\frac{1}{u} + \frac{3}{u^2} + \frac{7}{3u^3}\right) + C$$
$$= \frac{-1}{64(3-4\sin x)} + \frac{3}{64(3-4\sin x)^2} + \frac{7}{192(3-4\sin x)^3} + C.$$

23. If we set $u = 1 + \sin 4t$, then $du = 4\,\cos 4t\,dt$, and

$$\int \sqrt{1+\sin 4t}\,\cos^3 4t\,dt = \int \sqrt{1+\sin 4t}\,(1-\sin^2 4t)\cos 4t\,dt$$
$$= \int \sqrt{u}[1-(u-1)^2]\left(\frac{du}{4}\right) = \frac{1}{4}\int (2u^{3/2} - u^{5/2})\,du$$
$$= \frac{1}{4}\left(\frac{4u^{5/2}}{5} - \frac{2u^{7/2}}{7}\right) + C = \frac{1}{5}(1+\sin 4t)^{5/2} - \frac{1}{14}(1+\sin 4t)^{7/2} + C.$$

24. If we set $u = 1 + \sqrt{x}$, then $du = \dfrac{1}{2\sqrt{x}}dx$, and

$$\int \sqrt{1+\sqrt{x}}\,dx = \int \sqrt{u}\,2(u-1)\,du = 2\int (u^{3/2} - \sqrt{u})\,du$$
$$= 2\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C = \frac{4}{5}(1+\sqrt{x})^{5/2} - \frac{4}{3}(1+\sqrt{x})^{3/2} + C.$$

25. $\displaystyle\int \tan^2 x\,\sec^2 x\,dx = \frac{1}{3}\tan^3 x + C$

26. $\displaystyle\int \tan x\,\sec^2 x\,dx = \frac{1}{2}\tan^2 x + C$

27. $\displaystyle\int \frac{e^{2x}}{e^{2x}+1}dx = \frac{1}{2}\ln(e^{2x}+1)+C$

28. If we set $u=\ln x$, then $du=\dfrac{1}{x}dx$, and $\displaystyle\int \frac{\ln x}{x}dx = \int u\,du = \frac{u^2}{2}+C = \frac{1}{2}(\ln x)^2+C.$

29. $\displaystyle\int \frac{1}{x\ln x}dx = \ln|\ln x|+C$

30. If we set $u=\ln(x^2+1)$, then $du=\dfrac{2x}{x^2+1}dx$, and

$$\int \frac{x}{(x^2+1)[\ln(x^2+1)]^2}dx = \int \frac{1}{u^2}\left(\frac{du}{2}\right) = -\frac{1}{2u}+C = \frac{-1}{2\ln(x^2+1)}+C.$$

31. (a) If we set $u=\sin x$, then $du=\cos x\,dx$, and

$$\begin{aligned}\int \sin^3 x\cos^3 x\,dx &= \int \sin^3 x(1-\sin^2 x)\cos x\,dx = \int u^3(1-u^2)\,du\\ &= \frac{u^4}{4}-\frac{u^6}{6}+C = \frac{1}{4}\sin^4 x-\frac{1}{6}\sin^6 x+C.\end{aligned}$$

(b) If we set $u=\cos x$, then $du=-\sin x\,dx$, and

$$\begin{aligned}\int \sin^3 x\cos^3 x\,dx &= \int (1-\cos^2 x)\cos^3 x\sin x\,dx = \int (1-u^2)u^3\,(-du)\\ &= -\frac{u^4}{4}+\frac{u^6}{6}+C = -\frac{1}{4}\cos^4 x+\frac{1}{6}\cos^6 x+C.\end{aligned}$$

(c) $$\begin{aligned}\frac{1}{4}\sin^4 x-\frac{1}{6}\sin^6 x+C &= \frac{1}{4}(1-\cos^2 x)^2-\frac{1}{6}(1-\cos^2 x)^3+C\\ &= \left(\frac{1}{4}-\frac{1}{6}\right)+\left(-\frac{1}{2}+\frac{1}{2}\right)\cos^2 x+\left(\frac{1}{4}-\frac{1}{2}\right)\cos^4 x+\frac{1}{6}\cos^6 x+C\\ &= -\frac{1}{4}\cos^4 x+\frac{1}{6}\cos^6 x+D\end{aligned}$$

32. If $x\ge 0$, then $\displaystyle\int\sqrt{\frac{x^2}{1+x}}dx = \int \frac{x}{\sqrt{1+x}}dx$. If we set $u=1+x$, then $du=dx$, and

$$\int\sqrt{\frac{x^2}{1+x}}dx = \int \frac{u-1}{\sqrt{u}}du = \int\left(\sqrt{u}-\frac{1}{\sqrt{u}}\right)du = \frac{2}{3}u^{3/2}-2\sqrt{u}+C = \frac{2}{3}(1+x)^{3/2}-2\sqrt{1+x}+C.$$

If $-1<x<0$, then $\displaystyle\int\sqrt{\frac{x^2}{1+x}}dx = \int\frac{-x}{\sqrt{1+x}}dx = -\frac{2}{3}(1+x)^{3/2}+2\sqrt{1+x}+C.$

33. If we set $u=2/x$, then $du=-(2/x^2)dx$. For $4x-x^2=x(4-x)$ to be nonnegative, x must be in the interval $0\le x\le 4$. It follows that u is positive, and

$$\int\frac{\sqrt{4x-x^2}}{x^3}dx = \int\frac{\sqrt{\dfrac{8}{u}-\dfrac{4}{u^2}}}{2/u}\left(\frac{du}{-2}\right) = -\frac{1}{2}\int\sqrt{2u-1}\,du = -\frac{1}{6}(2u-1)^{3/2}+C = -\frac{1}{6}\left(\frac{4}{x}-1\right)^{3/2}+C.$$

34. If we set $u=1/x$, then $du=-\dfrac{1}{x^2}dx$. For $x-x^2=x(1-x)$ to be nonnegative, x must be in the interval $0\le x\le 1$. It follows that u is positive, and

$$\int\frac{\sqrt{x-x^2}}{x^4}dx = \int\frac{\sqrt{\dfrac{1}{u}-\dfrac{1}{u^2}}}{\dfrac{1}{u^2}}(-du) = -\int u\sqrt{u-1}\,du.$$

We now set $v=u-1$, in which case $dv=du$, and

$$\int \frac{\sqrt{x-x^2}}{x^4}\,dx = -\int (v+1)\sqrt{v}\,dv = -\int (v^{3/2}+\sqrt{v})\,dv = -\left(\frac{2}{5}v^{5/2}+\frac{2}{3}v^{3/2}\right)+C$$
$$= -\frac{2}{5}(u-1)^{5/2} - \frac{2}{3}(u-1)^{3/2} + C = -\frac{2}{5}\left(\frac{1}{x}-1\right)^{5/2} - \frac{2}{3}\left(\frac{1}{x}-1\right)^{3/2} + C.$$

35. Since $u^2 = \dfrac{5+x}{1-x} \implies u^2(1-x) = 5+x \implies x = \dfrac{u^2-5}{u^2+1}$, we obtain $dx = \dfrac{(u^2+1)(2u)-(u^2-5)(2u)}{(u^2+1)^2}du = \dfrac{12u}{(u^2+1)^2}du$. Since

$$5-4x-x^2 = 5-4\left(\frac{u^2-5}{u^2+1}\right) - \left(\frac{u^2-5}{u^2+1}\right)^2 = \frac{5(u^2+1)^2-4(u^2+1)(u^2-5)-(u^2-5)^2}{(u^2+1)^2} = \frac{36u^2}{(u^2+1)^2},$$

$$\int \frac{x}{(5-4x-x^2)^{3/2}}dx = \int \frac{\dfrac{u^2-5}{u^2+1}}{\left[\dfrac{36u^2}{(u^2+1)^2}\right]^{3/2}}\frac{12u}{(u^2+1)^2}du = \int \frac{12u(u^2-5)(u^2+1)^3}{216u^3(u^2+1)^3}du$$
$$= \frac{1}{18}\int \frac{u^2-5}{u^2}du = \frac{1}{18}\left(u+\frac{5}{u}\right)+C = \frac{1}{18}\left(\frac{u^2+5}{u}\right)+C = \frac{1}{18}\left(\frac{\dfrac{5+x}{1-x}+5}{\sqrt{\dfrac{5+x}{1-x}}}\right)+C$$
$$= \frac{1}{18}\left[\frac{5+x+5-5x}{(1-x)\sqrt{\dfrac{5+x}{1-x}}}\right]+C = \frac{5-2x}{9\sqrt{5-4x-x^2}}+C.$$

36. Since $u^2 = \dfrac{1-x}{1+x} \implies u^2(1+x) = 1-x \implies x = \dfrac{1-u^2}{1+u^2}$, we obtain $dx = \dfrac{(1+u^2)(-2u)-(1-u^2)(2u)}{(1+u^2)^2}du = \dfrac{-4u}{(1+u^2)^2}du$, and

$$\int \frac{1}{3(1-x^2)-(5+4x)\sqrt{1-x^2}}dx = \int \frac{1}{3(1-x)(1+x)-(5+4x)\sqrt{(1-x)(1+x)}}dx$$
$$= \int \frac{-4u/(1+u^2)^2}{3\left(1-\dfrac{1-u^2}{1+u^2}\right)\left(1+\dfrac{1-u^2}{1+u^2}\right) - \left(5+\dfrac{4-4u^2}{1+u^2}\right)\sqrt{\left(1-\dfrac{1-u^2}{1+u^2}\right)\left(1+\dfrac{1-u^2}{1+u^2}\right)}}du$$
$$= \int \frac{-4u}{\dfrac{12u^2}{(1+u^2)^2} - \dfrac{9+u^2}{1+u^2}\sqrt{\dfrac{4u^2}{(1+u^2)^2}}}\frac{1}{(1+u^2)^2}du$$
$$= \int \frac{2}{(u-3)^2}du = -\frac{2}{u-3}+C = \frac{-2}{\sqrt{\dfrac{1-x}{1+x}}-3}+C.$$

37. If $u-x = \sqrt{x^2+x+4}$, then $u^2-2ux+x^2 = x^2+x+4 \implies x = \dfrac{u^2-4}{1+2u}$, from which $dx = \dfrac{(1+2u)(2u)-(u^2-4)(2)}{(1+2u)^2}du = \dfrac{2u^2+2u+8}{(1+2u)^2}du$. Then,

$$\int \sqrt{x^2+x+4}\,dx = \int (u-x)\,dx = \int \left(u-\frac{u^2-4}{1+2u}\right)\frac{2u^2+2u+8}{(1+2u)^2}du = \int \frac{2(u^2+u+4)^2}{(1+2u)^3}du,$$

and the integrand is a rational function of u.

38. If $(x+1)u = \sqrt{4+3x-x^2}$, then $(x+1)^2u^2 = 4+3x-x^2 = (4-x)(1+x)$, or, $(x+1)u^2 = 4-x$. When this equation is solved for x, the result is $x = \dfrac{4-u^2}{1+u^2}$, and therefore

$$dx = \frac{(1+u^2)(-2u)-(4-u^2)(2u)}{(1+u^2)^2}du = \frac{-10u}{(1+u^2)^2}du.$$

Since $\sqrt{4+3x-x^2} = (x+1)u = u\left(\dfrac{4-u^2}{1+u^2}+1\right) = \dfrac{5u}{1+u^2}$,

$$\int \frac{1}{\sqrt{4+3x-x^2}}dx = \int \frac{1+u^2}{5u}\frac{-10u}{(1+u^2)^2}du = -2\int \frac{1}{1+u^2}du,$$

and the integrand is a rational function of u.

EXERCISES 5.4

1. Deflections must satisfy equation 5.8 with $F(x) = -9.81m/L$ subject to conditions $y(0) = y''(0) = 0$ and $y(L) = y''(L) = 0$. Integration of the differential equation gives $y(x) = [-9.81mx^4/(24L) + Ax^3 + Bx^2 + Cx + D]/(EI)$. The boundary conditions require

$$0 = EI\,y(0) = D, \quad 0 = EI\,y''(0) = 2B,$$

$$0 = EI\,y(L) = \frac{-9.81mL^3}{24} + AL^3 + BL^2 + CL + D, \quad 0 = EI\,y''(L) = \frac{-9.81mL}{2} + 6AL + 2B.$$

Solutions are $A = 9.81m/12$, $B = 0$, $C = -9.81mL^2/24$, and $D = 0$, so that

$$y(x) = \frac{1}{EI}\left(-\frac{9.81mx^4}{24L} + \frac{9.81mx^3}{12} - \frac{9.81mL^2x}{24}\right) = -\frac{9.81m}{24EIL}(x^4 - 2Lx^3 + L^3x).$$

2. Deflections must satisfy equation 5.8 with $F(x) = -9.81m/L$ subject to conditions $y(0) = y'(0) = 0$ and $y(L) = y'(L) = 0$. Integration of the differential equation gives $y(x) = [-9.81mx^4/(24L) + Ax^3 + Bx^2 + Cx + D]/(EI)$. The boundary conditions require

$$0 = EI\,y(0) = D, \quad 0 = EI\,y'(0) = C,$$

$$0 = EI\,y(L) = \frac{-9.81mL^3}{24} + AL^3 + BL^2 + CL + D, \quad 0 = EI\,y'(L) = \frac{-9.81mL^2}{6} + 3AL^2 + 2BL + C.$$

Solutions are $A = 9.81m/12$, $B = -9.81mL/24$, $C = 0$, and $D = 0$, so that

$$y(x) = \frac{1}{EI}\left(-\frac{9.81mx^4}{24L} + \frac{9.81mx^3}{12} - \frac{9.81mLx^2}{24}\right) = -\frac{9.81m}{24EIL}(x^4 - 2Lx^3 + L^2x^2).$$

3. Deflections must satisfy equation 5.8 with $F(x) = -9.81m/L$ subject to conditions $y(0) = y'(0) = 0$ and $y''(L) = y'''(L) = 0$. Integration of the differential equation gives $y(x) = [-9.81mx^4/(24L) + Ax^3 + Bx^2 + Cx + D]/(EI)$. The boundary conditions require

$$0 = EI\,y(0) = D, \quad 0 = EI\,y'(0) = C,$$

$$0 = EI\,y''(L) = \frac{-9.81mL}{2} + 6AL + 2B, \quad 0 = EI\,y'''(L) = -9.81m + 6A.$$

Solutions are $A = 9.81m/6$, $B = -9.81mL/4$, $C = 0$, and $D = 0$, so that

$$y(x) = \frac{1}{EI}\left(-\frac{9.81mx^4}{24L} + \frac{9.81mx^3}{6} - \frac{9.81mLx^2}{4}\right) = -\frac{9.81m}{24EIL}(x^4 - 4Lx^3 + 6L^2x^2).$$

4. Deflections must satisfy equation 5.8 with $F(x) = -9.81m/L$ subject to conditions $y(0) = y'(0) = 0$ and $y(L) = y''(L) = 0$. Integration of the differential equation gives $y(x) = [-9.81mx^4/(24L) + Ax^3 + Bx^2 + Cx + D]/(EI)$. The boundary conditions require

$$0 = EI\,y(0) = D, \quad 0 = EI\,y'(0) = C,$$

$$0 = EI\,y(L) = \frac{-9.81mL^3}{24} + AL^3 + BL^2 + CL + D, \quad 0 = EI\,y''(L) = \frac{-9.81mL}{2} + 6AL + 2B.$$

Solutions are $A = 5(9.81)m/48$, $B = -9.81mL/16$, $C = 0$, and $D = 0$, so that

$$y(x) = \frac{1}{EI}\left(-\frac{9.81mx^4}{24L} + \frac{5(9.81)mx^3}{48} - \frac{9.81mLx^2}{16}\right) = -\frac{9.81m}{48EIL}(2x^4 - 5Lx^3 + 3L^2x^2).$$

5. Deflections must satisfy the differential equation $\dfrac{d^4y}{dx^4} = -\dfrac{F}{EI}\delta(x - L/2)$ subject to the boundary conditions $y(0) = y'(0) = 0 = y(L) = y'(L)$. Four integrations of the differential equation give

$$y(x) = \frac{1}{EI}\left[-\frac{F}{6}(x - L/2)^3h(x - L/2) + Ax^3 + Bx^2 + Cx + D\right].$$

The boundary conditions require

$$0 = EIy(0) = D, \qquad 0 = EIy'(0) = C,$$

$$0 = EIy(L) = -\frac{F}{6}\left(\frac{L}{2}\right)^3 + AL^3 + BL^2 + CL + D, \quad 0 = EIy'(L) = -\frac{F}{2}\left(\frac{L}{2}\right)^2 + 3AL^2 + 2BL + C.$$

These give $A = \dfrac{F}{12}$, $B = \dfrac{-FL}{16}$, and hence

$$y(x) = \frac{1}{EI}\left[-\frac{F}{6}(x - L/2)^3h(x - L/2) + \frac{Fx^3}{12} - \frac{FLx^2}{16}\right] = \frac{-F}{48EI}\left[8(x - L/2)^3h(x - L/2) - 4x^3 + 3Lx^2\right].$$

6. Deflections must satisfy the differential equation $\dfrac{d^4y}{dx^4} = -\dfrac{F}{EI}\delta(x - L/2)$ subject to the boundary conditions $y(0) = y'(0) = 0 = y''(L) = y'''(L)$. Four integrations of the differential equation give

$$y(x) = \frac{1}{EI}\left[-\frac{F}{6}(x - L/2)^3h(x - L/2) + Ax^3 + Bx^2 + Cx + D\right].$$

The boundary conditions require

$$0 = EIy(0) = D,\ 0 = EIy'(0) = C,\ 0 = EIy''(L) = -F\left(\frac{L}{2}\right) + 6AL + 2B,\ 0 = EIy'''(L) = -F + 6A.$$

These give $A = \dfrac{F}{6}$, $B = \dfrac{-FL}{4}$, and hence

$$y(x) = \frac{1}{EI}\left[-\frac{F}{6}(x - L/2)^3h(x - L/2) + \frac{Fx^3}{6} - \frac{FLx^2}{4}\right] = \frac{-F}{12EI}\left[2F(x - L/2)^3h(x - L/2) - 2x^3 + 3Lx^2\right].$$

For $x > L/2$,

$$y = \frac{-F}{12EI}\left[2F(x - L/2)^3 - 2x^3 + 3Lx^2\right] = \frac{FL^2}{48EI}(L - 6x),$$

the equation of a straight line.

7. Deflections must satisfy the differential equation $\dfrac{d^4y}{dx^4} = -\dfrac{F}{EI}\delta(x - L/2)$ subject to the boundary conditions $y(0) = y''(0) = 0 = y(L) = y''(L)$. Four integrations of the differential equation give

$$y(x) = \frac{1}{EI}\left[-\frac{F}{6}(x - L/2)^3 h(x - L/2) + Ax^3 + Bx^2 + Cx + D\right].$$

The boundary conditions require

$$0 = EIy(0) = D, \qquad 0 = EIy''(0) = 2B,$$

$$0 = EIy(L) = -\frac{F}{6}\left(\frac{L}{2}\right)^3 + AL^3 + BL^2 + CL + D, \qquad 0 = EIy''(L) = -F\left(\frac{L}{2}\right) + 6AL + 2B.$$

These give $A = \dfrac{F}{12}$, $C = \dfrac{-FL^2}{16}$, and hence

$$y(x) = \frac{1}{EI}\left[-\frac{F}{6}(x - L/2)^3 h(x - L/2) + \frac{Fx^3}{12} - \frac{FL^2x}{16}\right] = \frac{-F}{48EI}\left[8F(x - L/2)^3 h(x - L/2) - 4x^3 + 3L^2x\right].$$

8. Deflections must satisfy the differential equation $\dfrac{d^4y}{dx^4} = \dfrac{1}{EI}\left[-F\delta(x - L/2) - \dfrac{mg}{L}\right]$ subject to the boundary conditions $y(0) = y'(0) = 0 = y(L) = y'(L)$. Four integrations of the differential equation give

$$y(x) = \frac{1}{EI}\left[-\frac{F}{6}(x - L/2)^3 h(x - L/2) - \frac{mgx^4}{24L} + Ax^3 + Bx^2 + Cx + D\right].$$

The boundary conditions require

$$0 = EIy(0) = D, \qquad 0 = EIy'(0) = C,$$

$$0 = EIy(L) = -\frac{F}{6}\left(\frac{L}{2}\right)^3 - \frac{mgL^3}{24} + AL^3 + BL^2 + CL + D,$$

$$0 = EIy'(L) = -\frac{F}{2}\left(\frac{L}{2}\right)^2 - \frac{mgL^2}{6} + 3AL^2 + 2BL + C.$$

These give $A = (F + mg)/12$, $B = -L(3F + 2mg)/48$, and therefore

$$y(x) = \frac{1}{EI}\left[-\frac{F}{6}(x - L/2)^3 h(x - L/2) - \frac{mgx^4}{24L} + \frac{(F + mg)x^3}{12} - \frac{L(3F + 2mg)x^2}{48}\right].$$

9. Deflections must satisfy the differential equation $\dfrac{d^4y}{dx^4} = \dfrac{1}{EI}\left[-F\delta(x - L/2) - \dfrac{mg}{L}\right]$ subject to the boundary conditions $y(0) = y'(0) = 0 = y''(L) = y'''(L)$. Four integrations of the differential equation give

$$y(x) = \frac{1}{EI}\left[-\frac{F}{6}(x - L/2)^3 h(x - L/2) - \frac{mgx^4}{24L} + Ax^3 + Bx^2 + Cx + D\right].$$

The boundary conditions require

$$0 = EIy(0) = D, \qquad 0 = EIy'(0) = C,$$

$$0 = EIy''(L) = -F\left(\frac{L}{2}\right) - \frac{mgL}{2} + 6AL + 2B, \; 0 = EIy'''(L) = -F - mg + 6A.$$

These give $A = (F + mg)/6$ and $B = -L(F + mg)/4$, and hence

$$y(x) = \frac{1}{EI}\left[-\frac{F}{6}(x - L/2)^3 h(x - L/2) - \frac{mgx^4}{24L} + \frac{(F + mg)x^3}{6} - \frac{L(F + mg)x^2}{4}\right].$$

10. Deflections must satisfy the differential equation $\dfrac{d^4y}{dx^4} = \dfrac{1}{EI}\left[-F\delta(x-L/2) - \dfrac{mg}{L}\right]$ subject to the boundary conditions $y(0) = y''(0) = 0 = y(L) = y''(L)$. Four integrations of the differential equation give

$$y(x) = \frac{1}{EI}\left[-\frac{F}{6}(x-L/2)^3h(x-L/2) - \frac{mgx^4}{24L} + Ax^3 + Bx^2 + Cx + D\right].$$

The boundary conditions require

$$0 = EIy(0) = D, \qquad 0 = EIy''(0) = 2B,$$

$$0 = EIy(L) = -\frac{F}{6}\left(\frac{L}{2}\right)^3 - \frac{mgL^3}{24} + AL^3 + BL^2 + CL + D,$$

$$0 = EIy''(L) = -F\left(\frac{L}{2}\right) - \frac{mgL}{2} + 6AL + 2B.$$

These give $A = (F+mg)/12$, and $C = -L^2(3F+2mg)/48$, and hence

$$y(x) = \frac{1}{EI}\left[-\frac{F}{6}(x-L/2)^3h(x-L/2) - \frac{mgx^4}{24L} + \frac{(F+mg)x^3}{12} - \frac{L^2(3F+2mg)x}{48}\right].$$

11. (a) According to Exercise 10,

$$\begin{aligned}y(x) &= \frac{1}{10^6}\left[-\frac{1500}{6}(x-2)^3h(x-2) - \frac{1000x^4}{24(4)} + \frac{(1500+1000)x^3}{12} - \frac{16(4500+2000)x}{48}\right]\\ &= \frac{1}{10^6}\left[-250(x-2)^3h(x-2) - \frac{125x^4}{12} + \frac{625x^3}{3} - \frac{6500x}{3}\right].\end{aligned}$$

(b) Maximum deflection occurs at the centre of the beam,

$$y(2) = \frac{1}{10^6}\left[-\frac{125(2)^4}{12} + \frac{625(2)^3}{3} - \frac{6500(2)}{3}\right] = -0.00283.$$

Since maximum deflection cannot exceed $4/360 = 0.011$ m, the beam is acceptable.

12. Deflections must satisfy the differential equation $\dfrac{d^4y}{dx^4} = \dfrac{1}{EI}\{-98.1 - 98.1[h(x-5) - h(x-10)]\}$ $= \dfrac{-98.1}{EI}[1 + h(x-5)]$ since $h(x-10) = 0$ if $0 < x < 10$, subject to the boundary conditions $y(0) = y'(0) = 0 = y''(10) = y'''(10)$. Integration of the differential equation four times using equation 5.11 gives

$$y(x) = \frac{-4.0875}{EI}[x^4 + (x-5)^4h(x-5) + Ax^3 + Bx^2 + Cx + D].$$

The boundary conditions require

$$0 = EIy(0) = -4.0875D, \qquad 0 = EIy'(0) = -4.0875C,$$

$$0 = EIy''(10) = -4.0875[12(10)^2 + 12(5)^2 + 6A(10) + 2B],$$

$$0 = EIy'''(10) = -4.0875[24(10) + 24(5) + 6A].$$

These give $A = -60$ and $B = 1050$, and therefore

$$y(x) = -\frac{4.0875}{EI}[x^4 + (x-5)^4h(x-5) - 60x^3 + 1050x^2].$$

Deflection at $x = 10$ is greater in this case.

13. Deflections must satisfy the differential equation $\dfrac{d^4y}{dx^4} = \dfrac{1}{EI}\{-98.1 - 98.1[h(x-5/2) - h(x-15/2)]\}$, subject to the boundary conditions $y(0) = y'(0) = 0 = y''(10) = y'''(10)$. Integration of the differential equation four times using equation 5.11 gives

$$y(x) = \frac{-4.0875}{EI}[x^4 + (x-5/2)^4h(x-5/2) - (x-15/2)^4h(x-15/2) + Ax^3 + Bx^2 + Cx + D].$$

The boundary conditions require

$$0 = EIy(0) = -4.0875D \qquad 0 = EIy'(0) = -4.0875C,$$
$$0 = EIy''(10) = -4.0875[12(10)^2 + 12(15/2)^2 - 12(5/2)^2 + 6A(10) + 2B],$$
$$0 = EIy'''(10) = -4.0875[24(10) + 24(15/2) - 24(5/2) + 6A].$$

These gives $A = -60$ and $B = 900$, and therefore

$$y(x) = \frac{-4.0875}{EI}[x^4 + (x-5/2)^4h(x-5/2) - (x-15/2)^4h(x-15/2) - 60x^4 + 900x^2].$$

Deflection at $x = 10$ in this case is greater than in Figure 5.9 but less than that in Exercise 12.

14. When the concentrated force is placed at x_0 just to the left of the end of the beam, the differential equation for displacements is $\dfrac{d^4y}{dx^4} = -\dfrac{F}{EI}\delta(x-x_0)$, subject to the boundary conditions $y(0) = y'(0) = 0 = y''(L) = y'''(L)$. Four integrations of the differential equation give

$$y(x) = \frac{1}{EI}\left[-\frac{F}{6}(x-x_0)^3h(x-x_0) + Ax^3 + Bx^2 + Cx + D\right].$$

The boundary conditions require

$$0 = EIy(0) = D,\ 0 = EIy'(0) = C,\ 0 = EIy''(L) = -F(L-x_0) + 6AL + 2B,\ 0 = EIy'''(L) = -F + 6A.$$

These give $A = \dfrac{F}{6}$, $B = \dfrac{-Fx_0}{2}$, and hence $y(x) = \dfrac{1}{EI}\left[-\dfrac{F}{6}(x-x_0)^3h(x-x_0) + \dfrac{Fx^3}{6} - \dfrac{Fx_0x^2}{2}\right]$. If we now take the limit as $x_0 \to L^-$, we obtain

$$y(x) = \frac{1}{EI}\left[-\frac{F}{6}(x-L)^3h(x-L) + \frac{Fx^3}{6} - \frac{FLx^2}{2}\right] = \frac{Fx^2(x-3L)}{6EI}.$$

EXERCISES 5.5

1. If $N(t)$ is the number of bacteria at time t, then the fact that they increase at a rate proportional to N can be expressed as $dN/dt = kN$, where k is a constant. This is a separable differential equation $\dfrac{dN}{N} = k\,dt$. Solutions are defined implicitly by

$$\ln|N| = kt + C \quad\Longrightarrow\quad |N| = e^{kt+C} \quad\Longrightarrow\quad N = De^{kt}, \quad (D = e^C).$$

If N_0 is the number of bacteria at time $t = 0$, then $N_0 = D$, and $N(t) = N_0e^{kt}$. Since $N(2) = (5/4)N_0$, it follows that $(5/4)N_0 = N_0e^{2k}$. Hence, $k = (1/2)\ln(5/4)$. If T is the time when the number of bacteria doubles, then $2N_0 = N_0e^{kT}$. This equation can be solved for $T = k^{-1}\ln 2 = \dfrac{2}{\ln(5/4)}\ln 2 = 6.21$ hours.

2. If $N(t)$ is the number of bacteria at time t, then the fact that they increase at a rate proportional to N can be expressed as $dN/dt = kN$, where k is a constant. This is a separable differential equation $\dfrac{dN}{N} = k\,dt$. Solutions are defined implicitly by

$$\ln|N| = kt + C \quad\Longrightarrow\quad |N| = e^{kt+C} \quad\Longrightarrow\quad N = De^{kt}, \quad (D = e^C).$$

If N_0 is the number of bacteria at time $t = 0$, then $N_0 = D$, and $N(t) = N_0 e^{kt}$. Since $N(3) = 2N_0$, it follows that $2N_0 = N_0 e^{3k}$. Hence, $k = (\ln 2)/3$. If T is the time when the number of bacteria triples, then $3N_0 = N_0 e^{kT}$. This equation can be solved for $T = k^{-1}\ln 3 = \dfrac{3}{\ln 2}\ln 3 = 4.75$ hours.

3. The amount of radioactive material in a sample at any time t is $A = A_0 e^{kt}$ where A_0 is the amount at time $t = 0$. If $A(15) = A_0/2$, then $A_0/2 = A_0 e^{15k}$, from which $k = -(1/15)\ln 2$. If T is the time for 90% of the sample to decay, then, $A_0/10 = A_0 e^{kT}$. This can be solved for $T = -k^{-1}\ln 10 = \dfrac{15}{\ln 2}\ln 10 = 49.83$ days.

4. The amount of radioactive material in a sample at any time t is $A = A_0 e^{kt}$ where A_0 is the amount at time $t = 0$. If $A(3) = 0.9A_0$, then $0.9A_0 = A_0 e^{3k}$, from which $k = \ln(0.9)/3$. The half life T of the material is the time at which $A = A_0/2$; that is, $A_0/2 = A_0 e^{kT}$. This can be solved for $T = -k^{-1}\ln 2 = -\dfrac{3}{\ln(0.9)}\ln 2 = 19.74$ seconds.

5. Since half the amount decreases during each half-life, only 1/16 or 6.25% remains after 4 half-lives.

6. If $A(t)$ is the amount of drug in the body as a function of time t, then $\dfrac{dA}{dt} = kA$, $k < 0$ a constant. This differential equation is separable, $\dfrac{dA}{A} = k\,dt$. Solutions are defined implicitly by $\ln A = kt + C$. Exponentiation yields $A = e^{kt+C} = De^{kt}$, where $D = e^C$. If we choose time $t = 0$ when the original amount is A_0, then $A_0 = D$, and therefore $A = A_0 e^{kt}$. Since $A(1) = 0.95A_0$, we have $0.95A_0 = A_0 e^k$. This equation implies that $k = \ln(0.95)$. The amount of drug in the body will be $A_0/2$ when $A_0/2 = A_0 e^{kt}$, and the solution of this equation for t is $t = -\dfrac{1}{k}\ln 2 = -\dfrac{\ln 2}{\ln 0.95} = 13.51$ hours.

7. (a) Let V be the volume of sugar remaining at any given time t, and x be the length of the side of the cube at this time. Since dissolving occurs at a rate proportional to the surface area of the remaining cube,

$$\frac{dV}{dt} = k(6x^2), \quad k < 0 \text{ a constant.}$$

Since $V = x^3$, it follows that $6kx^2 = \dfrac{d}{dt}(x^3) = 3x^2\dfrac{dx}{dt}$. Thus, $\dfrac{dx}{dt} = 2k$, and integration of this equation gives $x = 2kt + C$. Since $x(0) = 1$, constant C must be 1, and $x = 2kt + 1$. The cube has completely dissolved when $0 = x = 2kt + 1$, and therefore when $t = -1/(2k)$.
(b) When dissolving is proportional to the amount of sugar remaining,

$$\frac{dV}{dt} = kV, \quad k < 0 \text{ a constant.}$$

This is differential equation 5.19 with solution $V = Ce^{kt}$. Since $V(0) = 1$ cubic centimetre, it follows that $C = 1$, and $V = e^{kt}$. The cube completely dissolves when $0 = V = e^{kt}$, but this occurs only after an infinitely long time.

8. (a) Since sugar particles dissolve independently, the time taken for the sugar to dissolve is the time taken for each spherical particle to dissolve. Since dissolving occurs at a rate proportional to the surface area of the particle,

$$\frac{dV}{dt} = k(4\pi r^2), \quad k < 0 \text{ a constant.}$$

Since $V = (4/3)\pi r^3$, it also follows that $4k\pi r^2 = \dfrac{d}{dt}\left(\dfrac{4\pi r^3}{3}\right) = 4\pi r^2\dfrac{dr}{dt}$. Thus, $\dfrac{dr}{dt} = k$, and integration gives $r = kt + C$. The initial condition $r(0) = r_0$ implies that $C = r_0$, and therefore $r = kt + r_0$. The sugar is completely dissolved when $0 = r = kt + r_0 \Longrightarrow t = -r_0/k$.
(b) In this case, $dV/dt = kV$. The solution of this differential equation is $V = Ce^{kt}$. The initial condition $V(0) = (4/3)\pi r_0^3 = V_0$ implies that $C = V_0$, and $V = V_0 e^{kt}$. The sugar is completely dissolved when $V = 0$, but this does not occur in finite time.

9. According to the discussion on carbon dating, the amount of C^{14} in the fossil at time t after the creature's death is $A = A_0 e^{kt}$ where A_0 is the amount present at death and $k = -\ln 2/5550$. If T is the time at which 1.51% of A_0 remains, then $0.0151A_0 = A_0 e^{kT}$. The solution of this equation is $T = \ln(0.0151)/k = 33\,574$ years.

10. According to the discussion on carbon dating, the amount of C^{14} in the fossil at time t after the creature's death is $A = A_0 e^{kt}$ where A_0 is the amount present at death and $k = -\ln 2/5550$. When $t = 100\,000$, the percentage of C^{14} present is

$$\frac{100A}{A_0} = \frac{100A_0}{A_0}e^{100\,000k} = 100e^{100\,000(-\ln 2)/5550} = 3.8 \times 10^{-4}\%.$$

11. If $A(t)$ represents the amount of drug in the body at time t (in hours), then $\dfrac{dA}{dt} = kA$, where $k < 0$ is a constant. Separation of variables gives $\dfrac{1}{A}\,dA = k\,dt$, and therefore $\ln|A| = kt + C$, or, $A = De^{kt}$. If A_0 is the size of the original dose injected at time $t = 0$, then $A_0 = D$, and $A = A_0 e^{kt}$. Since $A(1) = 0.95A_0$, it follows that $0.95A_0 = A_0 e^{k}$. Thus, $k = \ln(0.95)$, and $A = A_0 e^{t\ln(0.95)}$. The dose decreases to $A_0/2$ when $A_0/2 = A_0 e^{t\ln(0.95)}$, the solution of which is $t = -\ln 2/\ln(0.95) = 13.51$ h.

12. The rate of change dC/dt of the amount of glucose in the blood is equal to the rate at which it is added less the rate at which it is used up,

$$\frac{dC}{dt} = R - kC, \quad k > 0 \text{ a constant.}$$

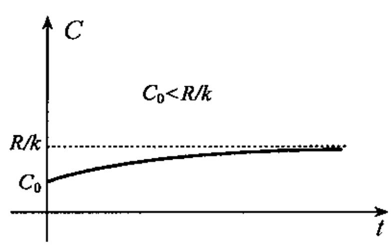
This is separable, $\dfrac{dC}{R - kC} = dt$, and solutions are therefore defined implicitly by $-\dfrac{1}{k}\ln|R - kC| = t + D$. Thus, $|R - kC| = e^{-k(t+D)}$, from which $R - kC = Ee^{-kt}$, where $E = \pm e^{-kD}$ is a constant. We can now solve for

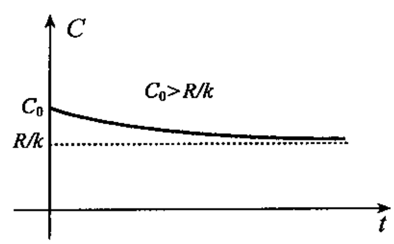
$$C(t) = \frac{1}{k}(R - Ee^{-kt}).$$

Since $C(0) = C_0$, it follows that $C_0 = (R - E)/k$, and this implies that $E = R - kC_0$. Thus,

$$C(t) = \frac{1}{k}\left[R - (R - kC_0)e^{-kt}\right] = \frac{R}{k}(1 - e^{-kt}) + C_0 e^{-kt}.$$

Graphs in the cases $C_0 < R/k$ and $C_0 > R/k$ are shown below.





13. If a quantity decreases at a rate proportional to its present amount, then the amount A present at any given time is given by $A = A_0 e^{kt}$, where A_0 is the amount at time $t = 0$ and k is a negative constant. The percentage decrease in a time interval of length h beginning at time t is

$$100\left[\frac{A(t) - A(t+h)}{A(t)}\right] = 100\left[\frac{A_0 e^{kt} - A_0 e^{k(t+h)}}{A_0 e^{kt}}\right] = 100(1 - e^{kh}).$$

Since this quantity is independent of t, the percentage decrease is the same at any time.

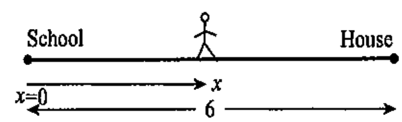
14. Separation of variables leads to $\dfrac{1}{T-20}\,dT = k\,dt$, and therefore $\ln|T-20| = kt + C$. The absolute values may be dropped since $T \geq 20$. Exponentiation then gives $T = 20 + De^{kt}$. Since $T(0) = 90$, we find that $D = 70$, and therefore $T = 20 + 70e^{kt}$. Because $T(40) = 60$, it follows that $60 - 20 = 70e^{40k}$, and $k = (1/40)\ln(4/7)$. Hence,

$$T = 20 + 70e^{(1/40)\ln(4/7)t} = 20 + 70e^{-0.01399t}.$$

15. If $T(t)$ represents the temperature of the mercury in the thermometer as a function of time t, then according to Newton's law of cooling, $\dfrac{dT}{dt} = k(T+20)$, where $k < 0$ is a constant. Separation of variables leads to $\dfrac{1}{T+20}\,dT = k\,dt$, and therefore $\ln|T+20| = kt + C$. When we solve for T, the result is $T(t) = -20 + De^{kt}$. If we choose time $t = 0$ when $T = 25$, then $25 = -20 + D$. Thus, $D = 45$, and $T(t) = -20 + 45e^{kt}$. Because $T(4) = 0$, it follows that $0 = -20 + 45e^{4k}$, from which $k = (1/4)\ln(20/45)$. The temperature is $-19°$ C when $-19 = -20 + 45e^{kt} \implies t = (1/k)\ln(1/45) = 18.8$ minutes.

16. When the boy is x km from school, his velocity is

$$\frac{dx}{dt} = kx^2, \qquad k = \text{a constant}.$$



We separate variables, $\dfrac{dx}{x^2} = k\,dt$, in which case solutions are defined implicitly by

$$-\frac{1}{x} = kt + C \quad\Longrightarrow\quad x = \frac{-1}{kt+C}.$$

If we choose time $t = 0$ when $x = 6$, then $6 = -1/C$. Thus, $x = -1/(kt - 1/6) = 6/(1-6kt)$. Since $x(1) = 3$, it follows that $3 = 6/(1-6k) \implies k = -1/6$, and $x(t) = 6/(1+t)$ km. The boy reaches school when $x = 0$, but this does not happen in finite time.

17. If $S(t)$ represents the number of grams of sugar in the tank at any given time, then dS/dt is equal to the rate at which sugar is added to the tank less the rate at which it leaves the tank. It is being added at (1/5)(10)=2 grams per minute. Since the amount of solution in the tank is always 100 litres, it follows that the rate at which sugar leaves the tank is $(1/5)S/100 = S/500$. Consequently,

$$\frac{dS}{dt} = 2 - \frac{S}{500} \qquad\Longrightarrow\qquad \frac{1}{1000-S}dS = \frac{1}{500}dt,$$

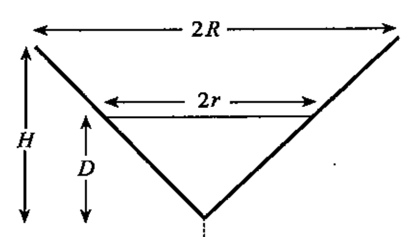
a separated differential equation. Solutions are defined implicitly by

$$-\ln|1000-S| = \frac{t}{500} + C \quad\Longrightarrow\quad 1000 - S = \pm e^{-t/500-C} \quad\Longrightarrow\quad S = 1000 + De^{-t/500}, \quad (D = \pm e^{-C}).$$

Since $S(0) = 4000$, it follows that $4000 = 1000 + D \implies D = 3000$. Hence, the number of grams of sugar in the tank is $S = 1000 + 3000e^{-t/500}$.

18. The volume of water in the tank is

$$V = \frac{1}{3}\pi r^2 D.$$



Because $r/D = R/H$, it follows that

$$V = \frac{1}{3}\pi\left(\frac{RD}{H}\right)^2 D = \frac{\pi R^2}{3H^2}D^3.$$

Thus, $\dfrac{dV}{dt} = \dfrac{\pi R^2 D^2}{H^2}\dfrac{dD}{dt}$.

But the rate at which water exits through the hole is Av. In other words,

$$\frac{\pi R^2 D^2}{H^2}\frac{dD}{dt} = -Av = -Ac\sqrt{2gD}.$$

We separate variables, $D^{3/2}\,dD = -\dfrac{\sqrt{2g}AcH^2}{\pi R^2}\,dt$, in which case solutions are defined implicitly by

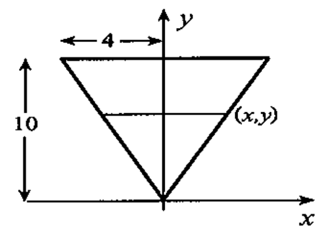
$\frac{2}{5}D^{5/2} = -\frac{\sqrt{2g}AcH^2}{\pi R^2}t + C$. If we choose time $t = 0$ when the tank is full ($D = H$), then $C = (2/5)H^{5/2}$, and

$$\frac{2}{5}D^{5/2} = -\frac{\sqrt{2g}AcH^2}{\pi R^2}t + \frac{2}{5}H^{5/2}.$$

The tank empties when $D = 0$, and this occurs when $t = \frac{2}{5}H^{5/2}\frac{\pi R^2}{\sqrt{2g}AcH^2} = \frac{\pi R^2}{5cA}\sqrt{\frac{2H}{g}}$.

19. When the depth of water is y, the volume is $V = (\pi/3)x^2y$. By similar triangles, $y/x = 10/4$, so that $V = (\pi/3)x^2(5x/2) = (5\pi/6)x^3$. Differentiation with respect to time t gives

$$\frac{dV}{dt} = \frac{5\pi x^2}{2}\frac{dx}{dt}.$$

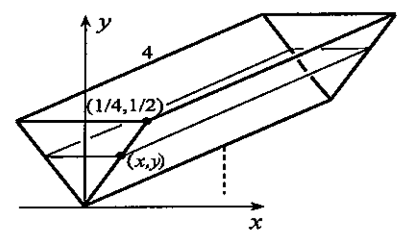


Because water evaporates at a rate proportional to the surface area, we can say that $\frac{dV}{dt} = k(\pi x^2)$, where k is a constant. Consequently,

$$\frac{5\pi x^2}{2}\frac{dx}{dt} = k\pi x^2 \quad \Longrightarrow \quad \frac{dx}{dt} = \frac{2k}{5}.$$

Integration gives $x(t) = 2kt/5 + C$. If we take $t = 0$ when the container is full, then $x(0) = 4$, and this implies that $4 = C$. Thus, $x(t) = 2kt/5 + 4$. Since $x(5) = 18/5$, it follows that $18/5 = 2k(5)/5 + 4 \Longrightarrow k = -1/5$. Thus, $x(t) = 4 - 2t/25$. The water has all evaporated when $x = 0$, and this occurs when $4 - 2t/25 = 0 \Longrightarrow t = 50$ days.

20. When the depth of water in the trough is y, the volume of water is $V = 4(xy) = 4xy$. Similar triangles require $y/x = (1/2)/(1/4) \Longrightarrow y = 2x$. Thus, $V = 4y(y/2) = 2y^2$. Differentiation of this equation with respect to time t gives $dV/dt = 4y(dy/dt)$. Because water exits through a hole of area 10^{-4} m^2 with speed $\sqrt{gy/2}$, it follows that $dV/dt = -10^{-4}\sqrt{gy/2}$.



Hence, $4y\frac{dy}{dt} = -\frac{\sqrt{gy/2}}{10^4} \Longrightarrow \sqrt{y}\,dy = -\frac{\sqrt{g}}{10^4(4\sqrt{2})}\,dt$. Solutions of this separated equation are defined implicitly by $\frac{2}{3}y^{3/2} = -\frac{\sqrt{g}\,t}{10^4(4\sqrt{2})} + C$. If we choose $t = 0$ when the trough is full, then $y(0) = 1/2$, and this implies that $(2/3)(1/2)^{3/2} = C$. Thus, $\frac{2}{3}y^{3/2} = \frac{-\sqrt{g}\,t}{10^4(4\sqrt{2})} + \frac{1}{3\sqrt{2}}$. The tank empties when $y = 0$, and the time at which this occurs is $t = \frac{4 \times 10^4}{3\sqrt{g}} = 4257$ seconds, or 70.95 minutes.

21. If we replace d^2x/dt^2 by $v(dv/dx)$, the differential equation becomes $v\frac{dv}{dx} = -\frac{k}{M}x$, and this can be separated, $v\,dv = -\frac{k}{M}x\,dx$. Solutions are defined implicitly by $v^2/2 = -kx^2/(2M) + C$. Since $v = v_0$ when $x = 0$, it follows that $v_0^2/2 = C$, and

$$\frac{v^2}{2} = -\frac{kx^2}{2M} + \frac{v_0^2}{2} \quad \Longrightarrow \quad v = \pm\sqrt{v_0^2 - kx^2/M}.$$

22. If we set (see Exercise 21) $\frac{d^2r}{dt^2} = \frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = v\frac{dv}{dr}$, then $mv\frac{dv}{dr} = -\frac{GmM}{r^2}$, or $v\,dv = -\frac{GM}{r^2}\,dr$. Solutions of this separated differential equation are defined implicitly by $\frac{v^2}{2} = \frac{GM}{r} + C$. Since $v = 0$ when $r = R + h$, where R is the radius of the earth, it follows that $0 = GM/(R + h) + C$, and therefore

$$\frac{v^2}{2} = \frac{GM}{r} - \frac{GM}{R+h}.$$

The velocity of m when it strikes the earth ($r = R$) is defined by

$$\frac{v^2}{2} = \frac{GM}{R} - \frac{GM}{R+h} = \frac{GMh}{R(R+h)} \implies v = -\sqrt{\frac{2GMh}{R(R+h)}}.$$

Maximum attainable speed occurs when h becomes infinite; that is,

$$|v_{\max}| = \lim_{h\to\infty} \sqrt{\frac{2GMh}{R(R+h)}} = \sqrt{\frac{2GM}{R}}.$$

23. If V and A are the volume and area of the disk of ice, then $\frac{dV}{dt} = kA$ where $k < 0$ is a constant. Since the ratio of the radius R to the thickness T of the disk remains constant, and is 2 at time $t = 0$ when the disk begins to melt, we can say that $R = 2T$ for all t. Since $V = \pi R^2 T/2 = \pi(2T)^2 T/2 = 2\pi T^3$ and $A = \pi R^2 + 2RT + \pi RT = \pi(2T)^2 + 2(2T)T + \pi(2T)T = 2(3\pi + 2)T^2$, it follows that

$$\frac{d}{dt}(2\pi T^3) = 2k(3\pi + 2)T^2 \implies 6\pi T^2 \frac{dT}{dt} = 2k(3\pi + 2)T^2 \implies \frac{dT}{dt} = \frac{k}{3\pi}(3\pi + 2).$$

This diferential equation must be solved subject to the initial condition $T(0) = 1$. Integration gives $T(t) = k(3\pi + 2)t/(3\pi) + C$. The initial condition requires $1 = C$. Because $T(10) = 1/4$, it follows that

$$\frac{1}{4} = \frac{k(3\pi + 2)(10)}{3\pi} + 1 \implies k = -\frac{9\pi}{40(3\pi + 2)}.$$

Hence, $T(t) = -3t/40 + 1$. The disk is totally melted when $0 = T(t) = -3t/40 + 1 \implies t = 40/3$ minutes.

24. If V and A represent volume and area of the mothball at any time, then the assumption that evaporation is proportional to surface area is represented by the equation

$$\frac{dV}{dt} = kA,$$

where k is a constant. We have four variables in the problem: t, r, A, and V. By substituting $V = 4\pi r^3/3$ and $A = 4\pi r^2$, we eliminate V and A:

$$\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2 k \implies 4\pi r^2 \frac{dr}{dt} = 4\pi r^2 k \implies \frac{dr}{dt} = k.$$

Solutions of this differential equation are $r = kt + C$. Using the conditions $r(0) = R$ and $r(1) = R/2$, we find $C = R$ and $k = -R/2$. Consequently,

$$r(t) = R - \frac{Rt}{2} = R\left(1 - \frac{t}{2}\right).$$

The mothball completely disappears when $r = 0$, and this occurs when $t = 2$ years.

25. If we let r be the radius of the raindrop, m its mass, and y the distance that it has fallen, then $\frac{dm}{dt} = k(4\pi r^2)\left(\frac{dy}{dt}\right)$. Since $m = 4\pi r^3 \rho/3$, where ρ is the density of water,

$$\frac{d}{dt}\left(\frac{4}{3}\pi r^3 \rho\right) = 4\pi k r^2 \frac{dy}{dt} \implies 4\pi r^2 \rho \frac{dr}{dt} = 4\pi k r^2 \frac{dy}{dt} \implies \frac{dr}{dy} = \frac{k}{\rho}.$$

Integration of this differential equation gives $r(y) = ky/\rho + C$. If we choose $y = 0$ at the position when the raindrop is initially formed ($r = 0$), then $r(0) = 0$, and this implies that $C = 0$. Thus, $r = ky/\rho$.

26. The modified Torricelli law in equation 5.22 implies that water exits through the hole with horizontal speed $v_x = c\sqrt{2g(H-h)}$, where $0 < c < 1$ is a constant, and $g = 9.81$ is the acceleration due to gravity. Suppose we follow a droplet of water on its journey to the ground if it exits at time $t = 0$. Because the only force acting on it is gravity, its vertical acceleration is $a_y = -g$, from which $v_y = -gt + C$. Since the initial velocity of the droplet is horizontal, $v_y(0) = 0$, and this implies that $C = 0$. Integration of $dy/dt = -gt$ gives $y = -gt^2/2 + D$. Since $y(0) = h$, it follows that $h = D$, and $y = -gt^2/2 + h$. The horizontal acceleration of the droplet is zero so that its horizontal velocity must always be $v_x = c\sqrt{2g(H-h)}$. Since this is dx/dt, we integrate to get $x = ct\sqrt{2g(H-h)} + E$. Because $x(0) = 0$, we obtain $E = 0$, and $x = ct\sqrt{2g(H-h)}$. The droplet hits the ground when

$$0 = y = -\frac{gt^2}{2} + h \quad \Longrightarrow \quad t = \sqrt{\frac{2h}{g}}.$$

The x-coordinate of the point at which the droplet hits the ground is therefore

$$x = c\sqrt{\frac{2h}{g}}\sqrt{2g(H-h)} = 2c\sqrt{h(H-h)}, \quad 0 \le h \le H.$$

We must find the value of h that maximizes this function. For critical points we solve

$$0 = \frac{dx}{dh} = \frac{c}{\sqrt{h(H-h)}}(H - 2h) \quad \Longrightarrow \quad h = \frac{H}{2}.$$

Since $x(0) = x(H) = 0$, it follows that x is maximized when $h = H/2$.

27. (a) Two integrations of the differential equation give $(EI)y = \dfrac{A}{6}x^3 - \dfrac{mg}{24}x^4 + Cx + D$. Because $f(0) = f(L/2) = 0$,

$$0 = D, \qquad 0 = \frac{A}{6}\left(\frac{L}{2}\right)^3 - \frac{mg}{24}\left(\frac{L}{2}\right)^4 + C\left(\frac{L}{2}\right) + D.$$

These imply that $C = (L^2/192)(mgL - 8A)$, and therefore

$$y = \frac{1}{EI}\left[\frac{A}{6}x^3 - \frac{mg}{24}x^4 + \frac{L^2}{192}(mgL - 8A)x\right].$$

(b) The diagram makes it clear that dy/dx must be 0 at $x = L/2$; that is,

$$0 = \frac{A}{2}\left(\frac{L}{2}\right)^2 - \frac{mg}{6}\left(\frac{L}{2}\right)^3 + \frac{L^2}{192}(mgL - 8A) \quad \Longrightarrow \quad A = \frac{3mgL}{16}.$$

28. If we multiply the differential equation by r^2, it can be written in the form

$$0 = r^2\frac{d^2T}{dr^2} + 2r\frac{dT}{dr} = \frac{d}{dr}\left(r^2\frac{dT}{dr}\right).$$

Integration gives $r^2\dfrac{dT}{dr} = C$, where C is a constant, from which $dT/dr = C/r^2$. A second integration now gives $T = -C/r + D$. Since $T(1) = 10$ and $T(2) = 20$, it follows that $10 = -C + D$ and $20 = -C/2 + D$. These imply that $C = 20$ and $D = 30$, so that $T(r) = 30 - 20/r$.

29. Let $C(t)$ be the volume of CO_2 in the room at any time t. Then dC/dt is the rate of change of the volume of CO_2 in the room. It is equal to the rate at which CO_2 enters less the rate at which it exits. It enters at a rate of 0.0025 m^3/min. Since the concentration of CO_2 in the room at any time is $C(t)/100$, the exit rate is $5C(t)/100 = 0.05C(t)$. Consequently,

$$\frac{dC}{dt} = 0.0025 - 0.05C = \frac{1}{400} - \frac{C}{20} = \frac{1 - 20C}{400}.$$

It follows that $\dfrac{dC}{1-20C} = \dfrac{dt}{400}$, a separated equation with solutions defined implicitly by

$$-\frac{1}{20}\ln|1-20C| = \frac{t}{400} + D \quad\Longrightarrow\quad \ln|1-20C| = -\frac{t}{20} + E,$$

where $E = -20D$. Exponentiation gives

$$|1-20C| = e^E e^{-t/20} \quad\Longrightarrow\quad 1-20C = \pm e^E e^{-t/20} \quad\Longrightarrow\quad C = \frac{1}{20}(1 \pm e^E e^{-t/20}) = \frac{1}{20}(1 + Fe^{-t/20}),$$

where $F = \pm e^E$. The fact that $C(0) = 1/10$ requires

$$\frac{1}{10} = \frac{1}{20}(1+F) \quad\Longrightarrow\quad F = 1 \quad\Longrightarrow\quad C(t) = \frac{1}{20}(1 + e^{-t/20}).$$

The limit of this function as $t \to \infty$ is $1/20$ m^3.

30. (a) We can separate the differential equation $\dfrac{1}{\rho^{2-\delta}}d\rho = -\dfrac{1}{k\delta}dh$. Solutions are defined implicitly by

$$\frac{1}{(\delta-1)\rho^{1-\delta}} = -\frac{h}{k\delta} + C.$$

Because $\rho = \rho_0$ when $h = 0$, $\dfrac{1}{(\delta-1)\rho_0^{1-\delta}} = C$, and therefore

$$\frac{1}{(\delta-1)\rho^{1-\delta}} = -\frac{h}{k\delta} + \frac{1}{(\delta-1)\rho_0^{1-\delta}}.$$

This can be written $\rho^{\delta-1} = -\dfrac{h}{k}\left(\dfrac{\delta-1}{\delta}\right) + \rho_0^{\delta-1}$.

(b) If $P = k\rho^\delta$, then $\rho^{\delta-1} = \left(\rho^\delta\right)^{(\delta-1)/\delta} = \left(\dfrac{P}{k}\right)^{(\delta-1)/\delta}$. Because $P = P_0$ when $\rho = \rho_0$, it follows that $\rho_0^{\delta-1} = \left(\dfrac{P_0}{k}\right)^{(\delta-1)/\delta}$. When these are substituted into the result of part (a),

$$\left(\frac{P}{k}\right)^{1-1/\delta} = -\frac{h}{k}\left(\frac{\delta-1}{\delta}\right) + \left(\frac{P_0}{k}\right)^{1-1/\delta},$$

or,

$$P^{1-1/\delta} = P_0^{1-1/\delta} - \frac{h}{k}\left(1-\frac{1}{\delta}\right)k^{1-1/\delta} = P_0^{1-1/\delta} - h\left(1-\frac{1}{\delta}\right)k^{-1/\delta}.$$

But $\rho_0^{\delta-1} = (P_0/k)^{(\delta-1)/\delta}$ implies that $k^{-1/\delta} = \rho_0 P_0^{-1/\delta}$, and therefore

$$P^{1-1/\delta} = P_0^{1-1/\delta} - h\left(1-\frac{1}{\delta}\right)\rho_0 P_0^{-1/\delta}.$$

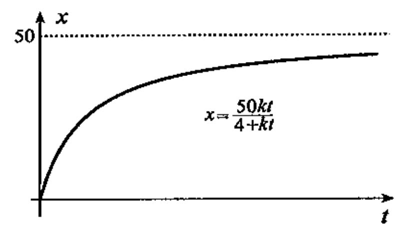
(c) If we define the effective height of the atmosphere when $P = 0$, then this occurs for

$$0 = P_0^{1-1/\delta} - h\left(1-\frac{1}{\delta}\right)\rho_0 P_0^{-1/\delta} \quad\Longrightarrow\quad h = \frac{\delta P_0}{(\delta-1)\rho_0}.$$

31. If $x(t)$ represents the number of grams of dissolved chemical at time t, then

$$\frac{dx}{dt} = k(50-x)\left(\frac{25}{100} - \frac{x}{200}\right) = \frac{k}{200}(50-x)^2,$$

where k is a constant. This equation can be separated, $\frac{1}{(50-x)^2}\,dx = \frac{k}{200}\,dt$, and solutions are defined implicitly by $\frac{1}{50-x} = \frac{kt}{200} + C$. Since $x(0) = 0$, it follows that $1/50 = C$, and $\frac{1}{50-x} = \frac{kt}{200} + \frac{1}{50}$. When we solve this equation for x, we obtain $x(t) = \frac{50kt}{4+kt}$ g.



32. (a) If $x(t)$ represents the amount of C in the mixture at time t, then

$$\frac{dx}{dt} = k\left(20 - \frac{2x}{3}\right)\left(10 - \frac{x}{3}\right) = \frac{2k}{9}(30-x)^2.$$

We can separate this equation, $\frac{dx}{(30-x)^2} = \frac{2k\,dt}{9} \implies \frac{1}{30-x} = \frac{2kt}{9} + C$. For $x(0) = 0$, we must have $1/30 = C$, and therefore $\frac{1}{30-x} = \frac{2kt}{9} + \frac{1}{30} \implies x(t) = \frac{600kt}{20kt+3}$.

REVIEW EXERCISES

1. $\int (3x^3 - 4x^2 + 5)\,dx = \frac{3x^4}{4} - \frac{4x^3}{3} + 5x + C$

2. $\int \left(\frac{1}{x^5} + 2x - \frac{1}{x^3}\right) dx = -\frac{1}{4x^4} + x^2 + \frac{1}{2x^2} + C$

3. $\int (2x^2 - 3x + 7x^6)\,dx = \frac{2x^3}{3} - \frac{3x^2}{2} + x^7 + C$

4. $\int \left(\frac{1}{x^2} - 2\sqrt{x}\right) dx = -\frac{1}{x} - \frac{4}{3}x^{3/2} + C$

5. $\int \sqrt{x-2}\,dx = \frac{2}{3}(x-2)^{3/2} + C$

6. $\int x(1+3x^2)^4\,dx = \frac{1}{30}(1+3x^2)^5 + C$

7. $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx = \frac{2}{3}x^{3/2} - 2\sqrt{x} + C$

8. $\int \left(\frac{x^2+5}{\sqrt{x}}\right) dx = \int \left(x^{3/2} + \frac{5}{\sqrt{x}}\right) dx = \frac{2}{5}x^{5/2} + 10\sqrt{x} + C$

9. $\int \frac{1}{(x+5)^4}dx = \frac{-1}{3(x+5)^3} + C$

10. $\int \left(\frac{\sqrt{x}}{x^2} - \frac{15}{\sqrt{x}}\right) dx = \int \left(\frac{1}{x^{3/2}} - \frac{15}{\sqrt{x}}\right) dx = -\frac{2}{\sqrt{x}} - 30\sqrt{x} + C$

11. $\int \sin 3x\,dx = -\frac{1}{3}\cos 3x + C$

12. $\int x\sqrt{1-x^2}\,dx = -\frac{1}{3}(1-x^2)^{3/2} + C$

13. $\int x\cos x^2\,dx = \frac{1}{2}\sin x^2 + C$

14. $\int x^2(1-2x^2)^2\,dx = \int (x^2 - 4x^4 + 4x^6)\,dx = \frac{x^3}{3} - \frac{4x^5}{5} + \frac{4x^7}{7} + C$

15. If we set $u = 1 + x$, then $du = dx$, and

$$\int x\sqrt{1+x}\,dx = \int (u-1)\sqrt{u}\,du = \int (u^{3/2} - \sqrt{u})\,du$$
$$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C = \frac{2}{5}(1+x)^{5/2} - \frac{2}{3}(1+x)^{3/2} + C.$$

16. If we set $u = 2 - x$, then $du = -dx$, and

$$\int \frac{x}{\sqrt{2-x}}dx = \int \frac{2-u}{\sqrt{u}}(-du) = \int \left(\sqrt{u} - \frac{2}{\sqrt{u}}\right) du = \frac{2}{3}u^{3/2} - 4\sqrt{u} + C = \frac{2}{3}(2-x)^{3/2} - 4\sqrt{2-x} + C.$$

17. $\displaystyle\int \frac{1}{(1+x)^2}dx = \frac{-1}{1+x} + C$

18. $\displaystyle\int (2+\sqrt{x})^2\,dx = \int (4 + 4\sqrt{x} + x)\,dx = 4x + \frac{8}{3}x^{3/2} + \frac{x^2}{2} + C$

19. If we set $u = \sqrt{x}$, then $du = \dfrac{1}{2\sqrt{x}}dx$, and

$$\int \frac{1}{\sqrt{x}(2+\sqrt{x})^2}dx = \int \frac{1}{(2+u)^2}(2\,du) = 2\left(\frac{-1}{2+u}\right) + C = \frac{-2}{2+\sqrt{x}} + C.$$

20. $\displaystyle\int \sin^4 x\,\cos x\,dx = \frac{1}{5}\sin^5 x + C$

21. $\displaystyle\int e^{3-5x}\,dx = -\frac{1}{5}e^{3-5x} + C$

22. $\displaystyle\int xe^{-4x^2}\,dx = -\frac{1}{8}e^{-4x^2} + C$

23. $\displaystyle\int \frac{e^x - 1}{e^{2x}}dx = \int (e^{-x} - e^{-2x})\,dx = -e^{-x} + \frac{1}{2}e^{-2x} + C$

24. If we set $u = \ln x$, then $du = \dfrac{1}{x}dx$, and $\displaystyle\int \frac{1}{5x\ln x}dx = \frac{1}{5}\int \frac{1}{u}du = \frac{1}{5}\ln|u| + C = \frac{1}{5}\ln|\ln x| + C.$

25. If we set $u = 2x^2$ and $du = 4x\,dx$, then

$$\int \frac{x}{\sqrt{1-4x^4}}dx = \int \frac{1}{\sqrt{1-u^2}}\left(\frac{du}{4}\right) = \frac{1}{4}\mathrm{Sin}^{-1}u + C = \frac{1}{4}\mathrm{Sin}^{-1}(2x^2) + C.$$

26. If we set $u = \sqrt{7}x$, then $du = \sqrt{7}\,dx$, and

$$\int \frac{3}{1+7x^2}dx = \int \frac{3}{1+u^2}\left(\frac{du}{\sqrt{7}}\right) = \frac{3}{\sqrt{7}}\mathrm{Tan}^{-1}u + C = \frac{3}{\sqrt{7}}\mathrm{Tan}^{-1}\sqrt{7}x + C.$$

27. $\displaystyle\int x\,\cosh 5x^2\,dx = \frac{1}{10}\sinh 5x^2 + C$

28. $\displaystyle\int \mathrm{sech}^2 5x\,dx = \frac{1}{5}\tanh 5x + C$

29. If $N(t)$ is the number of bacteria in the culture, then N increases at a rate proportional to N (see Exercise 1 in Section 5.5);

$$\frac{dN}{dt} = kN \qquad \Longrightarrow \qquad \frac{dN}{N} = k\,dt.$$

This is a separated differential equation with solutions defined implicitly by

$$\ln N = kt + C \qquad \Longrightarrow \qquad N = e^{kt+C} = De^{kt}, \quad \text{where } D = e^C.$$

If N_0 is the original number of bacteria when $t = 0$, say, then $N_0 = D$. Thus, $N(t) = N_0e^{kt}$. Since the number of bacteria triples in 3 days, $3N_0 = N_0e^{3k}$, and this implies that $k = (1/3)\ln 3$. The number of bacteria quadruples when

$$4N_0 = N_0e^{kt} \qquad \Longrightarrow \qquad t = \frac{1}{k}\ln 4 = \frac{\ln 4}{(1/3)\ln 3} = 3.8 \text{ days.}$$

30. If $T(t)$ represents the temperature of the water as a function of time t, then according to Newton's law of cooling,

$$\frac{dT}{dt} = k(T+20) \qquad \Longrightarrow \qquad \frac{dT}{T+20} = k\,dt, \quad \text{where } k < 0 \text{ is a constant.}$$

This is a separated differential equation with solutions defined implicitly by

$$\ln|T+20| = kt + C \qquad \Longrightarrow \qquad T + 20 = e^{kt+C} \qquad \Longrightarrow \qquad T = -20 + De^{kt}$$

where $D = e^C$. If we choose time $t = 0$ when $T = 70$, then $70 = -20 + D$. Thus, $D = 90$, and $T(t) = -20+90e^{kt}$. Because $T(10) = 50$, it follows that $50 = -20+90e^{4k}$, from which $k = (1/4)\ln(7/9)$. This solution would only be valid until the time at which the water reaches temperature zéro and freezes.

31. If $f''(x) = x^2+1$, then $f'(x) = x^3/3+x+C$. Since $f'(1) = 4$, it follows that $4 = 1/3+1+C \Longrightarrow C = 8/3$, and $f'(x) = x^3/3 + x + 8/3$. Integration gives $f(x) = x^4/12 + x^2/2 + 8x/3 + D$. Since $f(1) = 1$, we obtain $1 = 1/12 + 1/2 + 8/3 + D \Longrightarrow D = -9/4$, and $y = f(x) = x^4/12 + x^2/2 + 8x/3 - 9/4$.

32. If we integrate $f''(x) = 12x^2$, the result is $f'(x) = 4x^3+C$. A second integration gives $f(x) = x^4+Cx+D$. Since $f(1) = 4$ and $f(-1) = -3$, it follows that

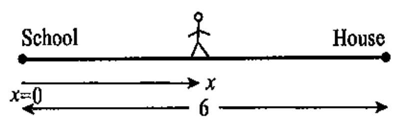
$$4 = (1)^4 + C(1) + D \qquad \text{and} \qquad -3 = (-1)^4 + C(-1) + D.$$

Solutions of these equations are $C = 7/2$ and $D = -1/2$. The equation of the required curve is therefore $y = x^4 + 7x/2 - 1/2$.

33. Integration of $f''(x) = 24x^2 + 6x$ gives $f'(x) = 8x^3 + 3x^2 + C$. Since $f'(1) = 4$, it follows that $4 = 8+3+C \Longrightarrow C = -7$. A second integration of $f'(x) = 8x^3+3x^2-7$ gives $f(x) = 2x^4+x^3-7x+D$. Since $f(1) = 8$, we must have $8 = 2+1-7+D \Longrightarrow D = 12$. Thus, $f(x) = 2x^4 + x^3 - 7x + 12$.

34. When the boy is x km from school, his velocity is

$$\frac{dx}{dt} = k\sqrt{x}, \qquad k = \text{a constant.}$$



If we separate $\dfrac{dx}{\sqrt{x}} = k\,dt$, then solutions are defined implicitly by

$$2\sqrt{x} = kt + C.$$

If we choose time $t = 0$ when $x = 6$, then $2\sqrt{6} = C$. Thus, $2\sqrt{x} = kt + 2\sqrt{6}$. Since $x(1) = 3$, it follows that $2\sqrt{3} = k + 2\sqrt{6}$, from which $k = 2(\sqrt{3} - \sqrt{6})$, and $2\sqrt{x} = 2(\sqrt{3} - \sqrt{6})t + 2\sqrt{6}$. Thus, $x = [(\sqrt{3}-\sqrt{6})t+\sqrt{6}]^2$ km. The boy reaches school when $x = 0$, and this occurs when $t = \sqrt{6}/(\sqrt{6}-\sqrt{3})$ hours.

35. If we take y as positive upward with $y = 0$ and $t = 0$ at the point and instant the ball is released, then the acceleration of the ball is $dv/dt = -9.81$. Integration gives $v = -9.81t + C$. Since $v(0) = 30$, we obtain $C = 30$, and $v = -9.81t+30$. A second integration gives $y = -4.905t^2 + 30t + D$. Since $y(0) = 0$, we find $D = 0$, and $y = -4.905t^2 + 30t$. The ball reaches its peak height when $0 = v = -9.81t + 30 \Longrightarrow t = 30/9.81$. The height of the ball at this time is $-4.905(30/9.81)^2 + 30(30/9.81) = 45.9$ m.

36. We choose y as positive downward with $y = 0$ and $t = 0$ at the instant the stone is released. The acceleration of the stone is $dv/dt = 9.81$. Inegration gives $v(t) = 9.81t+C$. If we denote by v_0 the initial speed of the stone, then $v(0) = v_0$, and this implies that $C = v_0$. Thus, $dy/dt = 9.81t + v_0$. Integration now yields $y(t) = 4.905t^2 + v_0t + D$. The condition $y(0) = 0$ requires $D = 0$. Because $y(2.2) = 50$, it follows that $50 = 4.905(2.2)^2 + v_0(2.2)$, and this implies that $v_0 = 11.9$ m/s.

37. If we set $u = 1 + \sqrt{x}$, then $du = \dfrac{1}{2\sqrt{x}}dx$, and

$$\begin{aligned}\int \frac{1}{\sqrt{1+\sqrt{x}}}dx &= \int \frac{1}{\sqrt{u}}2(u-1)\,du = 2\int\left(\sqrt{u} - \frac{1}{\sqrt{u}}\right)du \\ &= 2\left(\frac{2u^{3/2}}{3} - 2\sqrt{u}\right) + C = \frac{4}{3}(1+\sqrt{x})^{3/2} - 4\sqrt{1+\sqrt{x}} + C.\end{aligned}$$

38. If we set $u = \sqrt{1+x}$, then $du = \dfrac{1}{2\sqrt{1+x}}dx$, and

$$\begin{aligned}\int \frac{x}{\sqrt{1+x}+1}dx &= \int \frac{u^2-1}{u+1}2u\,du = 2\int (u^2 - u)\,du = 2\left(\frac{u^3}{3} - \frac{u^2}{2}\right) + C \\ &= \frac{2}{3}(1+x)^{3/2} - (1+x) + C = \frac{2}{3}(1+x)^{3/2} - x + D.\end{aligned}$$

39. By adjusting constants, $\displaystyle\int \frac{\sin x}{\sqrt{4+3\cos x}}\,dx = -\frac{2}{3}\sqrt{4+3\cos x} + C.$

40. If we set $u = 3-2x^3$, then $du = -6x^2\,dx$, and

$$\int x^8(3-2x^3)^6\,dx = \int (x^3)^2(3-2x^3)^6x^2\,dx = \int \left(\frac{3-u}{2}\right)^2 u^6\left(-\frac{du}{6}\right) = \frac{1}{24}\int(-9u^6+6u^7-u^8)\,du$$
$$= \frac{1}{24}\left(-\frac{9u^7}{7}+\frac{3u^8}{4}-\frac{u^9}{9}\right)+C = \frac{-3}{56}(3-2x^3)^7+\frac{1}{32}(3-2x^3)^8-\frac{1}{216}(3-2x^3)^9+C.$$

41. $\displaystyle\int \frac{(2+x)^4}{x^6}dx = \int\left(\frac{16}{x^6}+\frac{32}{x^5}+\frac{24}{x^4}+\frac{8}{x^3}+\frac{1}{x^2}\right)dx = \frac{-16}{5x^5}-\frac{8}{x^4}-\frac{8}{x^3}-\frac{4}{x^2}-\frac{1}{x}+C$

42. If we set $u=\sin x$, then $du = \cos x\,dx$, and

$$\int \sin^3 x\cos^3 x\,dx = \int \sin^3 x(1-\sin^2 x)\cos x\,dx = \int u^3(1-u^2)\,du$$
$$= \int (u^3-u^5)\,du = \frac{u^4}{4}-\frac{u^6}{6}+C = \frac{1}{4}\sin^4 x-\frac{1}{6}\sin^6 x+C.$$

43. If we set $u=\ln x$, then $du=(1/x)\,dx$, and

$$\int \frac{1}{x\sqrt{1+3\ln x}}dx = \int\frac{1}{\sqrt{1+3u}}du = \frac{2}{3}\sqrt{1+3u}+C = \frac{2}{3}\sqrt{1+3\ln x}+C.$$

44. If we set $u=\cos x$, then $du = -\sin x\,dx$, and

$$\int \tan x\,dx = \int\frac{\sin x}{\cos x}dx = \int\frac{1}{u}(-du) = -\ln|u|+C = -\ln|\cos x|+C.$$

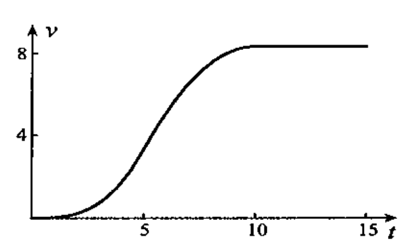
45. Integration of $a(t) = \begin{cases} 2t^2/25, & 0\le t\le 5\\ 4-2t/5, & 5<t\le 10\\ 0, & 10<t\le 15\end{cases}$ gives $v(t) = \begin{cases} 2t^3/75+C, & 0\le t\le 5\\ 4t-t^2/5+D, & 5<t\le 10\\ E, & 10<t\le 15.\end{cases}$
Since $v(0)=0$, it follows that $C=0$. To obtain D and E we demand that velocity be continuous at $t=5$ and $t=10$. This requires

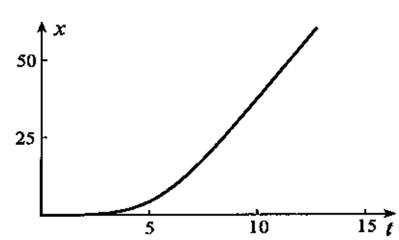
$$\frac{2(5)^3}{75} = 4(5)-\frac{25}{5}+D, \qquad 4(10)-\frac{100}{5}+D = E.$$

These give $D=-35/3$ and $E=25/3$, so that $v(t) = \begin{cases} 2t^3/75, & 0\le t\le 5\\ 4t-t^2/5-35/3, & 5<t\le 10\\ 25/3, & 10<t\le 15.\end{cases}$ A second integration gives $x(t) = \begin{cases} t^4/150+F, & 0\le t\le 5\\ 2t^2-t^3/15-35t/3+G, & 5<t\le 10\\ 25t/3+H, & 10<t\le 15.\end{cases}$ Since $x(0)=0$, we find that $F=0$.
To obtain G and H we demand that displacement be continuous at $t=5$ and $t=10$. This requires

$$\frac{(5)^4}{150} = 2(5)^2-\frac{5^3}{15}-\frac{35(5)}{3}+G, \qquad 2(10)^2-\frac{10^3}{15}-\frac{35(10)}{3}+G = \frac{25(10)}{3}+H.$$

These give $G=125/6$ and $H=-275/6$. These functions are graphed below.





46. Since the slope of $y = f(x)$ at any point is dy/dx, it follows that $\dfrac{dy}{dx} = \dfrac{1}{2}\left(\dfrac{y}{x}\right)^2$. If we write $\dfrac{dy}{y^2} = \dfrac{dx}{2x^2}$, then solutions are defined implicitly by $-\dfrac{1}{y} = -\dfrac{1}{2x} + C$. Since the curve is to pass through $(1,1)$, it follows that $-1/1 = -1/2 + C$. Thus $C = -1/2$, and $-\dfrac{1}{y} = -\dfrac{1}{2x} - \dfrac{1}{2} \Longrightarrow y = 2x/(1+x)$.

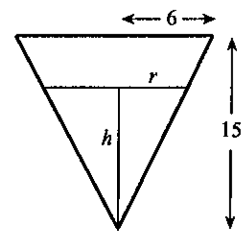
47. The slope of each of these curves is $dy/dx = 3x^2$. For a curve $y = f(x)$ to intersect each of the cubics at right angles, its slope at each point must be $f'(x) = -1/(3x^2)$. Integration gives $f(x) = 1/(3x) + D$. For that curve through the point $(1,1)$, we must have $1 = 1/3 + D$. Hence, the required curve is $y = 1/(3x) + 2/3$.

48. If let $V(t)$ be the volume of water in the cone (figure to the right), then

$$\frac{dV}{dt} = k(\pi r^2).$$

But $V = (1/3)\pi r^2 h$, and from similar triangles, $r/h = 6/15$. Hence,

$$V = \frac{1}{3}\pi r^2\left(\frac{5r}{2}\right) = \frac{5\pi r^3}{6}.$$



If we substitute this into the differential equation, we obtain

$$\frac{5\pi}{6}(3r^2)\frac{dr}{dt} = k\pi r^2 \qquad \Longrightarrow \qquad \frac{dr}{dt} = \frac{2k}{5}.$$

Integration gives $r = 2kt/5 + C$. If we take $t = 0$ when the cone is full, $6 = C$. Thus, $r = 2kt/5 + 6$. Since the water level drops 1 cm in 6 days, and $h = 5r/2$, it follows that

$$14 = \frac{5}{2}\left[\frac{2k(6)}{5} + 6\right] \qquad \Longrightarrow \qquad k = -\frac{1}{6}.$$

The radius of the surface of the water is therefore $r(t) = -t/15 + 6$. Half the water has evaporated when

$$\frac{1}{3}\pi r^2 h = \frac{1}{2}\left[\frac{1}{3}\pi(6)^2(15)\right].$$

Since $h = 5r/2$, this equation becomes $r^2(5r/2) = 270 \Longrightarrow r = 108^{1/3}$. We can now find out how long this takes,

$$108^{1/3} = -\frac{t}{15} + 6 \qquad \Longrightarrow \qquad t = 15(6 - 108^{1/3}) = 18.6 \text{ days}.$$

49. Deflections of the beam must satisfy the differential equation

$$\frac{d^4y}{dx^4} = -\frac{9.81M}{5EI}[h(x) - h(x-5)] = k[1 - h(x-5)], \quad \text{where } k = -9.81M/(5EI),$$

(since $h(x) = 1$ for $0 < x < 10$), subject to the boundary conditions $y(0) = y'(0) = 0 = y''(10) = y'''(10)$. Integration of the differential equation four times gives
$y(x) = k\left[\dfrac{x^4}{24} - \dfrac{(x-5)^4}{24}h(x-5) + Ax^3 + Bx^2 + Cx + D\right]$. The boundary conditions require

$$0 = y(0) = k(D), \qquad 0 = y'(0) = k(C),$$

$$0 = y''(10) = k\left[\frac{10^2}{2} - \frac{5^2}{2} + 60A + 2B\right], \qquad 0 = y'''(10) = k(10 - 5 + 6A).$$

These give $A = -5/6$ and $B = 25/4$, and therefore the function describing deflections of the beam is

$$y(x) = -\frac{9.81M}{5EI}\left[\frac{x^4}{24} - \frac{(x-5)^4}{24}h(x-5) - \frac{5x^3}{6} + \frac{25x^2}{4}\right].$$

For $x > 5$, $y(x) = -\frac{9.81M}{5EI}\left[\frac{x^4}{24} - \frac{(x-5)^4}{24} - \frac{5x^3}{6} + \frac{25x^2}{4}\right] = -\frac{9.81M}{5EI}\left(\frac{125x}{6} - \frac{625}{24}\right)$. Since the equation is linear for $x > 5$, the beam is straight on this interval. This is to be expected since there is no load on the beam for $x > 5$.

50. Deflections of the beam must satisfy the differential equation

$$\frac{d^4y}{dx^4} = -\frac{9.81M}{5EI}[h(x-5) - h(x-10)] = k\,h(x-5), \quad \text{where } k = -9.81M/(5EI),$$

(since $h(x-10) = 0$ for $0 < x < 10$), subject to the boundary conditions $y(0) = y'(0) = 0 = y''(10) = y'''(10)$. Integration of the differential equation four times gives
$y(x) = k\left[\frac{(x-5)^4}{24}h(x-5) + Ax^3 + Bx^2 + Cx + D\right]$. The boundary conditions require

$$0 = y(0) = k(D), \qquad 0 = y'(0) = k(C),$$

$$0 = y''(10) = k\left[\frac{5^2}{2} + 6A(10) + 2B\right], \qquad 0 = y'''(10) = k(5 + 6A).$$

These gives $A = -5/6$ and $B = 75/4$, and therefore the function describing deflections of the beam is

$$y(x) = -\frac{9.81M}{5EI}\left[\frac{(x-5)^4}{24}h(x-5) - \frac{5x^3}{6} + \frac{75x^2}{4}\right].$$

For $x < 5$, $y(x) = -\frac{9.81M}{5EI}\left[-\frac{5x^3}{6} + \frac{75x^2}{4}\right]$. Since this equation is not linear, the beam is not straight on this interval, nor should it be.

51. Deflections of the beam must satisfy the differential equation

$$\frac{d^4y}{dx^4} = -\frac{9.81M}{5EI}[h(x-5/2) - h(x-15/2)] = k[h(x-5/2) - h(x-15/2)],$$

where $k = -9.81M/(5EI)$, subject to the boundary conditions $y(0) = y'(0) = 0 = y''(10) = y'''(10)$. Integration of the differential equation four times gives
$y(x) = k\left[\frac{(x-5/2)^4}{24}h(x-5/2) - \frac{(x-15/2)^4}{24}h(x-15/2) + Ax^3 + Bx^2 + Cx + D\right]$. The boundary conditions require

$$0 = y(0) = k(D), \qquad 0 = y'(0) = k(C),$$

$$0 = y''(10) = k\left[\frac{(15/2)^2}{2} - \frac{(5/2)^2}{2} + 6A(10) + 2B\right], \qquad 0 = y'''(10) = k(15/2 - 5/2 + 6A).$$

These give $A = -5/6$ and $B = 25/2$, and therefore the function describing deflections of the beam is

$$y(x) = -\frac{9.81M}{5EI}\left[\frac{(x-5/2)^4}{24}h(x-5/2) - \frac{(x-15/2)^4}{24}h(x-15/2) - \frac{5x^3}{6} + \frac{25x^2}{2}\right].$$

For $x < 5/2$, $y(x) = -\frac{9.81M}{5EI}\left[-\frac{5x^3}{6} + \frac{25x^2}{2}\right]$, and for $x > 15/2$,

$$y(x) = -\frac{9.81M}{5EI}\left[\frac{(x-5/2)^4}{24} - \frac{(x-15/2)^4}{24} - \frac{5x^3}{6} + \frac{25x^2}{2}\right] = -\frac{9.81M}{5EI}\left[\frac{1625x}{24} - \frac{3125}{24}\right].$$

Since the equation is linear for $x > 15/2$, the beam is straight on this interval. This is to be expected since there is no load on the beam for $x > 15/2$. It is not straight for $x < 5/2$.