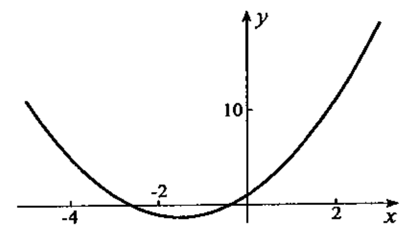
# CHAPTER 4

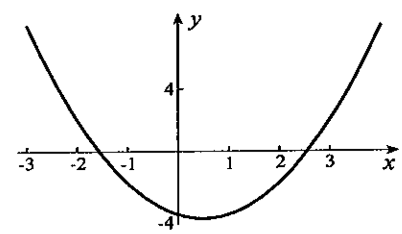
## EXERCISES 4.1

**1.** Newton's iterative procedure defines
$$x_{n+1} = x_n - \frac{x_n^2 + 3x_n + 1}{2x_n + 3}.$$
For the root between $-3$ and $-2$ we use an initial approximation $x_1 = -2.5$. The resulting iterations give
$$x_2 = -2.625, \qquad x_3 = -2.618\,0,$$
$$x_4 = -2.618\,034\,0, \quad x_5 = -2.618\,034\,0.$$



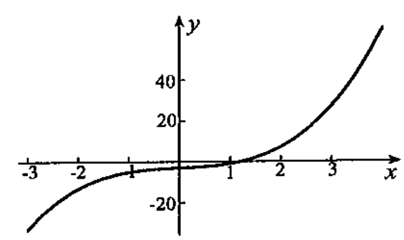

With $f(x) = x^2 + 3x + 1$, we calculate that $f(-2.618\,034\,5) = 1.1 \times 10^{-6}$ and $f(-2.618\,033\,5) = -1.1 \times 10^{-6}$. Hence, to six decimals, a root is $x = -2.618\,034$. A similar procedure for the root between $-1$ and $0$ gives $-0.381\,966$.

**2.** Newton's iterative procedure defines
$$x_{n+1} = x_n - \frac{x_n^2 - x_n - 4}{2x_n - 1}.$$
For the root between 2 and 3 we use an initial approximation $x_1 = 2.5$. The resulting iterations give
$$x_2 = 2.562\,5, \qquad x_3 = 2.561\,553\,0,$$
$$x_4 = 2.561\,552\,8, \quad x_5 = 2.561\,552\,8.$$



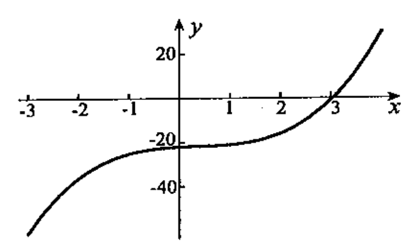

With $f(x) = x^2 - x - 4$, we calculate that $f(2.561\,552\,5) = -1.3 \times 10^{-6}$ and $f(2.561\,553\,5) = 2.8 \times 10^{-6}$. Hence, to six decimals, a root is $x = 2.561\,553$. A similar procedure for the root between $-2$ and $-1$ gives $x = -1.561\,553$.

**3.** Newton's iterative procedure defines
$$x_{n+1} = x_n - \frac{x_n^3 + x_n - 3}{3x_n^2 + 1}.$$
The only root is between 1 and 2, and we use an initial approximation $x_1 = 1.5$. The resulting iterations give
$$x_2 = 1.258, \qquad x_3 = 1.214\,7,$$
$$x_4 = 1.213\,413, \qquad x_5 = 1.213\,411\,6.$$
$$x_6 = 1.213\,411\,6.$$




With $f(x) = x^3 + x - 3$, we calculate that $f(1.213\,411\,5) = -8.8 \times 10^{-7}$ and $f(1.213\,412\,5) = 4.5 \times 10^{-6}$. Hence, to six decimals, a root is $x = 1.213\,412$.

**4.** Newton's iterative procedure defines
$$x_{n+1} = x_n - \frac{x_n^3 - x_n^2 + x_n - 22}{3x_n^2 - 2x_n + 1}.$$
The only root of the equation is slightly larger than 3. To find it we use $x_1 = 3.1$. Iteration gives
$$x_2 = 3.045\,789\,3, \quad x_3 = 3.044\,7236,$$
$$x_4 = 3.044\,723\,1, \quad x_5 = 3.044\,723\,1.$$



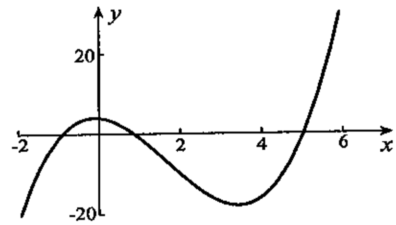

With $f(x) = x^3 - x^2 + x - 22$, we calculate that $f(3.044\,722\,5) = -1.5 \times 10^{-5}$ and $f(3.044\,723\,5) = 8.0 \times 10^{-6}$. Thus, to six decimals, the root is $x = 3.044\,723$.

**5.** Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{x_n^3 - 5x_n^2 - x_n + 4}{3x_n^2 - 10x_n - 1}.$$

For the root slightly larger than 5, we use an initial approximation $x_1 = 5$. The resulting iterations give

$$x_2 = 5.042, \qquad x_3 = 5.040\,965,$$
$$x_4 = 5.040\,964\,6, \quad x_5 = 5.040\,964\,6.$$



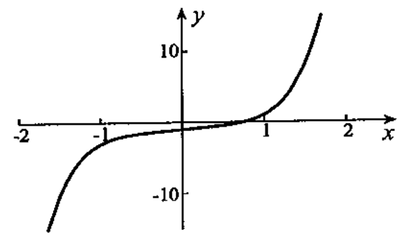

With $f(x) = x^3 - 5x^2 - x + 4$, we calculate that $f(5.040\,964\,5) = -2.4 \times 10^{-6}$ and $f(5.040\,965\,5) = 2.2 \times 10^{-5}$. Hence, to six decimals, a root is $x = 5.040\,965$. A similar procedure for the other roots gives $-0.911\,503$ and $0.870\,539$.

**6.** Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{x_n^5 + x_n - 1}{5x_n^4 + 1}.$$

The only root of the equation is between 0 and 1. To find it we use $x_1 = 0.5$. Iteration gives

$$x_2 = 0.857\,142\,9, \quad x_3 = 0.770\,682\,2,$$
$$x_4 = 0.755\,283\,0, \quad x_5 = 0.754\,877\,9,$$
$$x_6 = 0.754\,877\,7, \quad x_7 = 0.754\,877\,7.$$



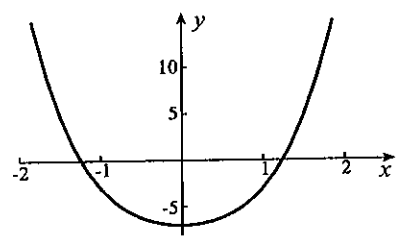

With $f(x) = x^5 + x - 1$, we calculate that $f(0.754\,877\,5) = -4.4 \times 10^{-7}$ and $f(0.754\,878\,5) = 2.2 \times 10^{-6}$. Thus, to six decimals, the root is $x = 0.754\,878$.

**7.** Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{x_n^4 + 3x_n^2 - 7}{4x_n^3 + 6x_n}.$$

For the root between 1 and 2 we use an initial approximation $x_1 = 1.2$. The resulting iterations give

$$x_2 = 1.243, \qquad x_3 = 1.241\,152\,6,$$
$$x_4 = 1.241\,523\,8, \quad x_5 = 1.241\,523\,8.$$



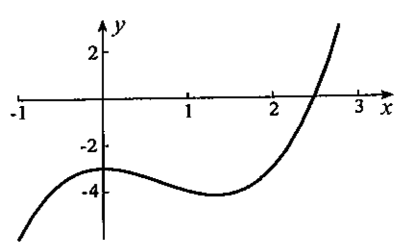

With $f(x) = x^4 + 3x^2 - 7$, we calculate that $f(1.241\,523\,5) = -4.0 \times 10^{-6}$ and $f(1.241\,524\,5) = 1.1 \times 10^{-5}$. Hence, to six decimals, a root is $x = 1.241\,524$. The other root is $-1.241\,524$.

**8.** The equation can be rearranged into the form $x^3 - 2x^2 - 3 = 0$. Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n^2 - 3}{3x_n^2 - 4x_n}.$$

The only root of the equation is between 2 and 3. To find it we use $x_1 = 2.5$. Iteration gives

$$x_2 = 2.485\,714\,3, \quad x_3 = 2.485\,584\,0,$$
$$x_4 = 2.485\,584\,0.$$



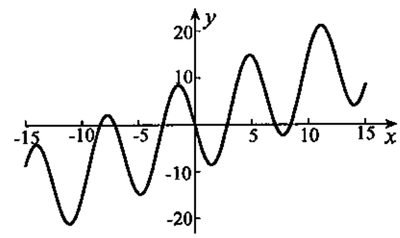

With $f(x) = x^3 - 2x^2 - 3$, we calculate that $f(2.485\,583\,5) = -4.3 \times 10^{-6}$ and $f(2.485\,584\,5) = 4.3 \times 10^{-6}$. Thus, to six decimals, the root is $x = 2.485\,584$.

**9.** Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{x_n - 10\,\sin x_n}{1 - 10\,\cos x_n}.$$

For the root between 2 and 3 we use an initial approximation $x_1 = 2.8$. The resulting iterations give

$$x_2 = 2.852\,76, \qquad x_3 = 2.852\,342,$$
$$x_4 = 2.852\,341\,9, \quad x_5 = 2.852\,341\,9.$$



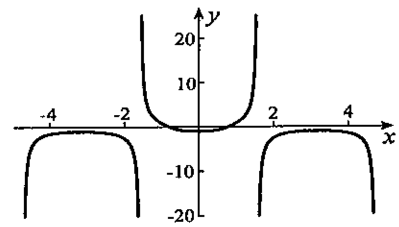
With $f(x) = x - 10\,\sin x$, we calculate that $f(2.852\,341\,5) = -4.2 \times 10^{-6}$ and $f(2.852\,342\,5) = 6.4 \times 10^{-6}$. Hence, to six decimals, a root is $x = 2.852\,342$. A similar procedure yields the additional roots 7.068 174 and 8.423 204. Negatives of these three roots also satisfy the equation. So also does $x = 0$.

**10.** If we rearrange the equation into the form $f(x) = (1 + x^4)\sec x - 2 = 0$, Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{(1 + x_n^4)\sec x_n - 2}{4x_n^3 \sec x_n + (1 + x_n^4)\sec x_n\,\tan x_n}.$$

For the root near 1 we use $x_1 = 1$. Iteration gives

$$x_2 = 0.870\,777\,3, \quad x_3 = 0.807\,267\,5,$$
$$x_4 = 0.795\,650\,2, \quad x_5 = 0.795\,324\,2,$$
$$x_6 = 0.795\,323\,9, \quad x_7 = 0.795\,323\,9.$$



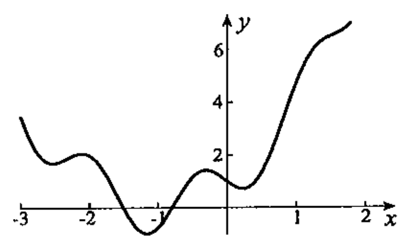
We now calculate that $f(0.795\,323\,5) = -2.0 \times 10^{-6}$ and $f(0.795\,324\,5) = 2.9 \times 10^{-6}$. Thus, to six decimals, the root is $x = 0.795\,324$. Because of the symmetry of the graph, the other root is $-0.795\,324$.

**11.** Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{(x_n + 1)^2 - \sin 4x_n}{2(x_n + 1) - 4\,\cos 4x_n}.$$

For the root between $-2$ and $-1$ we use an initial approximation $x_1 = -1.5$. The resulting iterations give

$$x_2 = -1.506, \qquad x_3 = -1.506\,05,$$
$$x_4 = -1.506\,052\,7, \quad x_5 = -1.506\,052\,7.$$



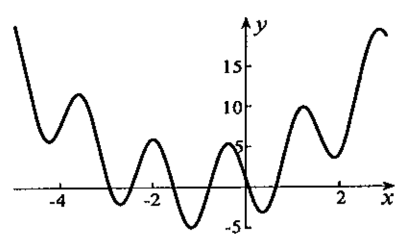
With $f(x) = (x + 1)^2 - \sin 4x$, we calculate that $f(-1.506\,053\,5) = 3.8 \times 10^{-6}$ and $f(-1.506\,052\,5) = -1.0 \times 10^{-6}$. Hence, to six decimals, a root is $x = -1.506\,053$. A similar procedure gives the other root $-0.795\,823$.

**12.** The graph suggests that there are 6 roots of the equation. Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{(x_n + 1)^2 - 5\sin 4x_n}{2(x_n + 1) - 20\cos 4x_n}.$$

For the smallest positive root we use $x_1 = 0$. Iteration gives

$$x_2 = 0.055\,555\,6, \quad x_3 = 0.056\,257\,3,$$
$$x_4 = 0.056\,257\,6, \quad x_5 = 0.056\,257\,6.$$



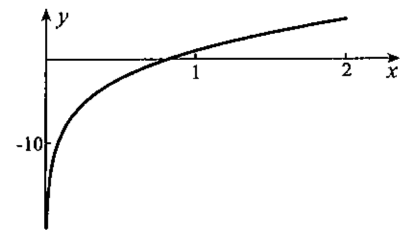
With $f(x) = (x + 1)^2 - 5\sin 4x$, we calculate that $f(0.056\,257\,5) = 1.9 \times 10^{-6}$ and $f(0.056\,258\,5) = -1.6 \times 10^{-5}$. Thus, to six decimals, the root is $x = 0.056\,258$. Similar procedures lead to the other 5 roots $-2.931\,137$, $-2.467\,518$, $-1.555\,365$, $-0.787\,653$, $0.642\,851$.

**13.** Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{x_n + 4\ln x_n}{1 + 4/x_n}.$$

With $x_1 = 1$ as initial approximation, resulting iterations give

$$x_2 = 0.8, \qquad x_3 = 0.815\,4,$$
$$x_4 = 0.815\,553\,4, \quad x_5 = 0.815\,553\,4.$$



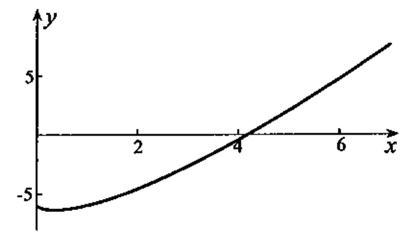

With $f(x) = x + 4\ln x$, we calculate that $f(0.815\,552\,5) = -5.4 \times 10^{-6}$ and $f(0.815\,553\,5) = 4.8 \times 10^{-7}$. Hence, to six decimals, the root is $x = 0.815\,553$.

**14.** Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{x_n \ln x_n - 6}{\ln x_n + 1}.$$

The only root of the equation is between 4 and 5. To find it we use $x_1 = 4.3$. Iteration gives

$$x_2 = 4.189\,350\,5, \quad x_3 = 4.188\,760\,1,$$
$$x_4 = 4.188\,760\,1.$$



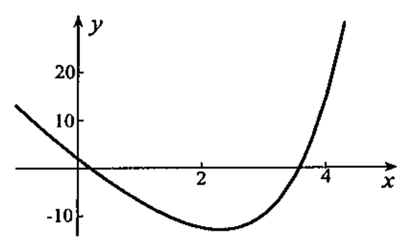

With $f(x) = x\ln x - 6$, we calculate that $f(4.188\,759\,5) = -1.5 \times 10^{-6}$ and $f(4.188\,760\,5) = 9.3 \times 10^{-7}$. Thus, to six decimals, the root is $x = 4.188\,760$.

**15.** Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{e^{x_n} + e^{-x_n} - 10x_n}{e^{x_n} - e^{-x_n} - 10}.$$

For the root between 0 and 1 we use an initial approximation $x_1 = 0.2$. The resulting iterations give

$$x_2 = 0.204\,2, \qquad x_3 = 0.204\,183\,6,$$
$$x_4 = 0.204\,183\,6.$$



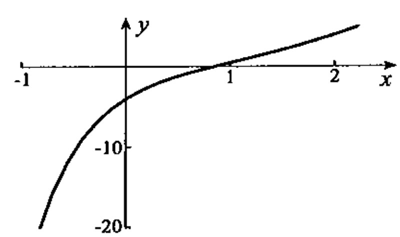

With $f(x) = e^x + e^{-x} - 10x$, we calculate that $f(0.204\,183\,5) = 9.5 \times 10^{-7}$ and $f(0.204\,184\,5) = -8.6 \times 10^{-6}$. Hence, to six decimals, a root is $x = 0.204\,184$. A similar procedure yields the other root 3.576 065.

**16.** Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{x_n^2 - 4e^{-2x_n}}{2x_n + 8e^{-2x_n}}.$$

With $x_1 = 0.8$, we obtain

$$x_2 = 0.852\,123\,5, \quad x_3 = 0.852\,605\,5,$$
$$x_4 = 0.852\,605\,5.$$



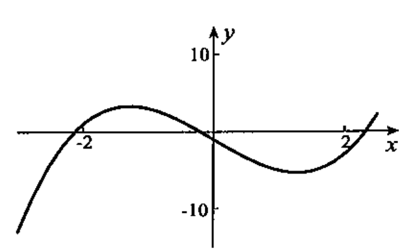

With $f(x) = x^2 - 4e^{-2x}$, we calculate that $f(0.852\,605\,5) = -6.4 \times 10^{-9}$ and $f(0.852\,606\,5) = 3.2 \times 10^{-6}$. Hence, to six decimals, the root is $x = 0.852\,606$.

**17.** Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{x_n^3 - 5x_n - 1}{3x_n^2 - 5}.$$

For the root between 2 and 3 we use an initial approximation $x_1 = 2.3$. The resulting iterations give

$$x_2 = 2.3306, \quad x_3 = 2.3301,$$
$$x_4 = 2.3301.$$




With $f(x) = x^3 - 5x - 1$, we calculate that $f(2.329) = -0.01$ and $f(2.331) = 0.01$. Hence, a root with error less than $10^{-3}$ is $x = 2.330$. A similar procedure yields the roots $-2.128$ and $-0.202$.

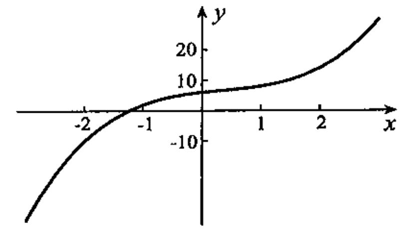
**18.** Obviously $x = 0$ is a solution of the equation. The remaining solutions must satisfy $f(x) = x^3 - x^2 + 2x + 6 = 0$. The graph indicates only one root between $-2$ and $-1$. To find it we use, $x_1 = -1.2$, and

$$x_{n+1} = x_n - \frac{x_n^3 - x_n^2 + 2x_n + 6}{3x_n^2 - 2x_n + 2}.$$

Iteration gives

$x_2 = -1.249\,541$, $x_3 = -1.248\,299$,

$x_4 = -1.248\,298$.




Since $f(-1.248\,4) = -9.4 \times 10^{-4}$ and $f(-1.248\,2) = 8.9 \times 10^{-4}$, we can say that the root is $x = -1.2483$.

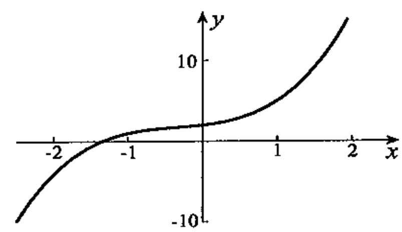
**19.** We rewrite the equation in the form $x^3 + x^2 + x + 2 = 0$. Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{x_n^3 + x_n^2 + x_n + 2}{3x_n^2 + 2x_n + 1}.$$

With an initial approximation $x_1 = -1.5$.
The resulting iterations give

$x_2 = -1.368$, $x_3 = -1.353\,38$,

$x_4 = -1.353\,210$ $x_5 = -1.353\,210$.



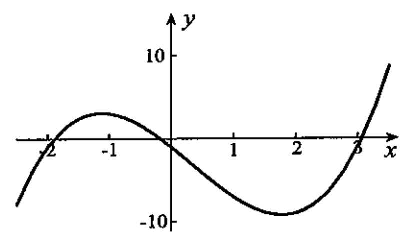

With $f(x) = x^3 + x^2 + x + 2$, we calculate that $f(-1.353\,22) = -3.8 \times 10^{-5}$ and $f(-1.353\,20) = 3.8 \times 10^{-5}$. Hence, the root is $x = -1.353\,21$.

**20.** We rearrange the equation into the form $f(x) = x^3 - x^2 - 6x - 1 = 0$. The graph suggests three solutions. To find the positive one, we use $x_1 = 3$ and

$$x_{n+1} = x_n - \frac{x_n^3 - x_n^2 - 6x_n - 1}{3x_n^2 - 2x_n - 6}.$$

Iteration gives

$x_2 = 3.066\,67$, $x_3 = 3.064\,44$, $x_4 = 3.064\,43$.




Since $f(3.063) = -2.3 \times 10^{-2}$ and $f(3.065) = 9.1 \times 10^{-3}$, the root is $x = 3.064$. A similar procedure leads to the other two roots $-1.892$ and $-0.172$.

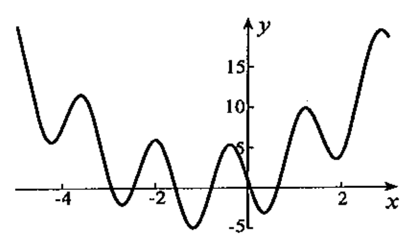
**21.** Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{(x_n + 1)^2 - 5\sin 4x_n}{2(x_n + 1) - 20\cos 4x_n}.$$

For the root nearest $-3$, we use an initial approximation $x_1 = -3$. The resulting iterations give

$x_2 = -2.036\,9$, $x_3 = -2.93\,2$,

$x_4 = -2.931\,1$, $x_5 = -2.931\,1$.



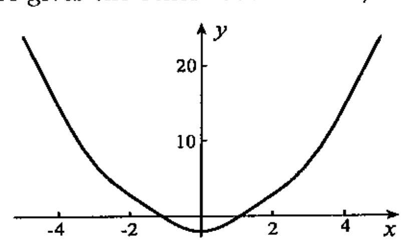

With $f(x) = (x + 1)^2 - 5\sin 4x$, we calculate that $f(-2.932) = 0.01$ and $f(-2.930) = -0.02$. Hence, a root with error less than $10^{-3}$ is $x = -2.931$. A similar procedure gives the other roots $-2.468$, $-1.555$, $-0.788$, $0.056$ and $0.643$.

**22.** The graph indicates two roots. To find the positive one, we use $x_1 = 1$ and

$$x_{n+1} = x_n - \frac{x_n^2 - 1 - \cos^2 x_n}{2x_n + 2\cos x_n \sin x_n}.$$

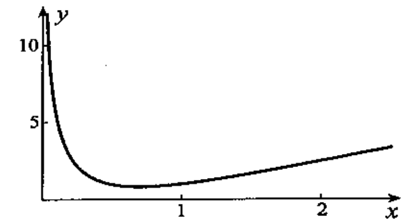
Iteration gives

$x_2 = 1.100\,343$, $x_3 = 1.098\,587$, $x_4 = 1.098\,587$.




With $f(x) = x^2 - 1 - \cos^2 x$, we calculate that $f(1.098\,5) = -2.6 \times 10^{-4}$ and $f(1.098\,7) = 3.4 \times 10^{-4}$. Thus, the root is $x = 1.098\,6$. Symmetry gives the other root as $-1.098\,6$.

**23.** The graph indicates that this equation has no solutions.

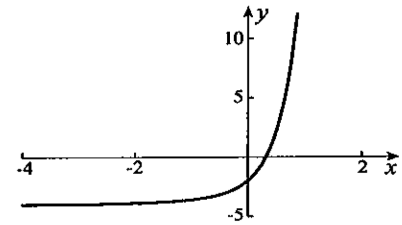



**24.** The graph indicates a solution of this equation between 0 and 1. To find it we use $x_1 = 0.3$ and

$$x_{n+1} = x_n - \frac{e^{3x_n} + e^{x_n} - 4}{3e^{3x_n} + e^{x_n}}.$$

Iteration gives

$$x_2 = 0.321\,83, \quad x_3 = 0.321\,21, \quad x_4 = 0.321\,21.$$



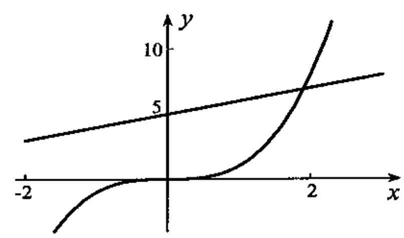

With $f(x) = e^{3x} + e^x - 4$, we calculate that $f(0.3211) = -1.0 \times 10^{-3}$ and $f(0.3213) = 8.2 \times 10^{-4}$. Thus, the root is $x = 0.3212$.

**25.** The $x$-coordinate of the point of intersection must satisfy the equation $f(x) = x^3 - x - 5 = 0$. To find it we use $x_1 = 2$ and

$$x_{n+1} = x_n - \frac{x_n^3 - x_n - 5}{3x_n^2 - 1}.$$

Iteration gives

$$x_2 = 1.909, \qquad x_3 = 1.904\,17,$$
$$x_4 = 1.904\,161, \quad x_5 = 1.904\,161.$$



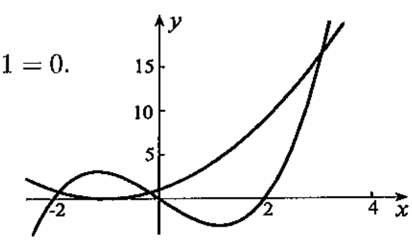

Since $f(1.904\,160\,5) = -3.5 \times 10^{-6}$ and $f(1.904\,161\,5) = 6.3 \times 10^{-6}$, we can say that to six decimals $x = 1.904\,161$. In either of the original equations, $x = 1.904\,161$ yields the same four decimals, $y = 6.904\,2$. The point of intersection is therefore $(1.904\,2, 6.904\,2)$.

**26.** To find $x$-coordinates of the points of intersection, we set $(x+1)^2 = x^3 - 4x$, and this reduces to $f(x) = x^3 - x^2 - 6x - 1 = 0$. With $x_1 = 3$, and

$$x_{n+1} = x_n - \frac{x_n^3 - x_n^2 - 6x_n - 1}{3x_n^2 - 2x_n - 6},$$

iteration gives

$$x_2 = 3.066\,7, \quad x_3 = 3.064\,435, \quad x_4 = 3.064\,435.$$



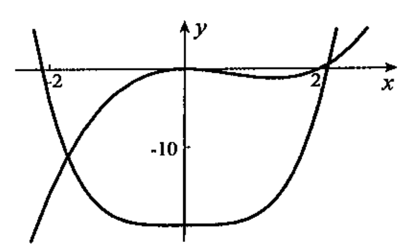

Since $f(3.064\,434\,5) = -5.4 \times 10^{-7}$ and $f(3.064\,435\,5) = 1.6 \times 10^{-5}$, the root is $x = 3.064\,435$. In either of the original equations, $x = 3.064\,435$ yields the same four decimals, $y = 16.519\,6$. A point of intersection is therefore $(3.064\,4, 16.519\,6)$. Similar procedures lead to the other points of intersection $(-1.892\,0, 0.795\,6)$ and $(-0.172\,5, 0.684\,8)$.

**27.** The $x$-coordinates of the points of intersection must satisfy the equation $f(x) = x^4 - x^3 + 2x^2 - 20 = 0$. To find one of them, we use $x_1 = 2$ and

$$x_{n+1} = x_n - \frac{x_n^4 - x_n^3 + 2x_n^2 - 20}{4x_n^3 - 3x_n^2 + 4x_n}.$$

Iteration gives

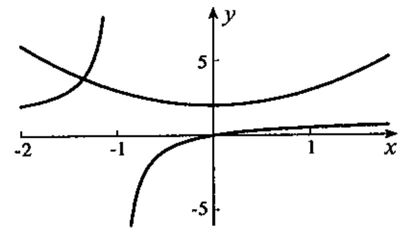
$$x_2 = 2.142\,9, \qquad x_3 = 2.130\,298,$$
$$x_4 = 2.130\,189, \quad x_5 = 2.130\,189.$$




Since $f(2.130\,188\,5) = -7.8 \times 10^{-6}$ and $f(2.130\,189\,5) = 2.6 \times 10^{-5}$, we can say that $x = 2.130\,189$. In either of the original equations, $x = 2.130\,189$ yields the same four decimals, $y = 0.590\,8$. A point of intersection is therefore $(2.130\,2, 0.590\,8)$. A similar procedure leads to the other point of intersection $(-1.7267, -11.1109)$.

**28.** When we equate the two expressions for $y$, and rearrange the equation, $x$ must satisfy $f(x) = x^3 + x^2 + x + 2 = 0$. The graph indicates that there is only one solution, and to find it we use $x_1 = -1.3$ and

$$x_{n+1} = x_n - \frac{x_n^3 + x_n^2 + x_n + 2}{3x_n^2 + 2x_n + 1}.$$




Iteration gives

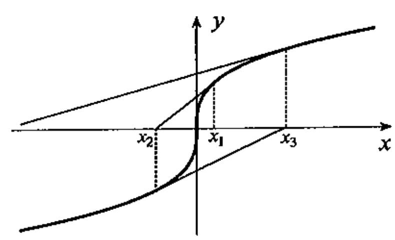
$$x_2 = -1.368, \qquad x_3 = -1.353\,38,$$
$$x_4 = -1.353\,210, \quad x_5 = -1.353\,210.$$

Since $f(-1.353\,210\,5) = -2.0 \times 10^{-6}$ and $f(-1.353\,209\,5) = 1.8 \times 10^{-6}$, we can say that $x = -1.353\,210$. In either of the original equations, $x = -1.353\,210$ yields the same four decimals, $y = 3.831\,2$. The point of intersection is therefore $(-1.353\,2, 3.831\,2)$.

**29.** If $x_1$ is the initial approximation to the solution of $f(x) = x^{1/3} = 0$, then the second approximation as defined by Newton's iterative procedure is

$$x_2 = x_1 - \frac{x_1^{1/3}}{(1/3)x_1^{-2/3}} = x_1 - 3x_1 = -2x_1.$$

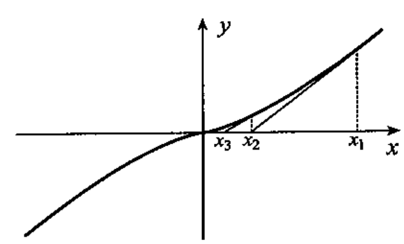
What this implies is that every approximation is $-2$ times the previous one. Approximations therefore do not approach the root $x = 0$. This is illustrated in the graph to the right.




**30.** If $x_1$ is the initial approximation to the solution of $f(x) = x^{7/5} = 0$, then the second approximation as defined by Newton's iterative procedure is

$$x_2 = x_1 - \frac{x_1^{7/5}}{(7/5)x_1^{2/5}} = x_1 - \frac{5x_1}{7} = \frac{2}{5}x_1.$$

What this implies is that every approximation is $2/5$ times the previous one. Approximations therefore must approach zero, the root of the equation.
This is illustrated in the graph to the right.




**31.** (a) For $P = 100\,000$, $i = 5$, $n = 25$, and $m = 12$, $\quad M = 100\,000\left[\dfrac{5/1200}{1 - (1 + 5/1200)^{-300}}\right] = 584.59.$

(b) For $P = 100\,000$, $n = 25$, $M = 500$, and $m = 12$, $i$ must satisfy $500 = 100\,000\left[\dfrac{i/1200}{1 - (1 + i/1200)^{-300}}\right]$.

This simplifies to

$$f(i) = \left(1 + \frac{i}{1200}\right)^{300} - \frac{6}{6 - i} = 0.$$

Newton's iterative procedure with an initial approximation of $i_1 = 4$, defines the sequence

$$i_1 = 4, \qquad i_{n+1} = i_n - \frac{(1 + i_n/1200)^{300} - 6/(6 - i_n)}{(1/4)(1 + i_n/1200)^{299} - 6/(6 - i_n)^2}.$$

Iteration gives

$$i_2 = 3.65, \quad i_3 = 3.51, \quad i_4 = 3.49, \quad i_5 = 3.49.$$

Thus, the interest rate is 3.49%.

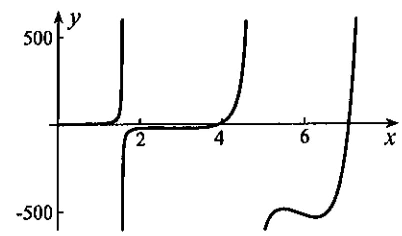
**32.** If we write the equation in the form $f(x) = e^x(-1+\tan x) + e^{-x}(1+\tan x) = 0$, Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{e^{x_n}(-1+\tan x_n) + e^{-x_n}(1+\tan x_n)}{e^{x_n}(-1+\tan x_n + \sec^2 x_n) + e^{-x_n}(-1-\tan x_n + \sec^2 x_n)}.$$

For the root just larger than 7, we use an initial approximation $x_1 = 7$. The resulting iterations give

$$x_2 = 7.068\,8, \qquad x_3 = 7.068\,789,$$
$$x_4 = 7.068\,583, \quad x_5 = 7.068\,583.$$

Since $f(7.068\,582\,5) = -5.8 \times 10^{-4}$ and $f(7.068\,583\,5) = 1.8 \times 10^{-3}$, we can say




that $x = 7.068\,583$. When this is divided by $20\pi$, the result to four decimals is 0.1125. Similarly, the smallest frequency is 0.0625.

**33.** (a) Since $y(3) = 11.8$ and $y(4) = -3.0$, the solution is between 3 and 4. To find it more accurately we use

$$t_1 = 3.8, \qquad t_{n+1} = t_n - \frac{1181(1-e^{-t_n/10}) - 98.1t_n}{118.1e^{-t_n/10} - 98.1}.$$

Iteration gives $t_2 = 3.833\,4$ and $t_3 = 3.833\,2$. Since $y(3.825) = 0.14$ and $y(3.835) = -0.03$, it follows that to 2 decimals $t = 3.83$ s.

(b) When we set $0 = y = 20t - 4.905t^2$, the positive solution is 4.08 s.

**34.** To simplify calculations, we set $z = c/\lambda$. Then, $z$ must satisfy the equation $f(z) = (5-z)e^z - 5 = 0$. Since $f(4) = e^4 - 5$ and $f(5) = -5$, the solution for $z$ is slightly less than 5. To find it more accurately, we use

$$z_1 = 4.9, \qquad z_{n+1} = z_n - \frac{(5-z_n)e^{z_n} - 5}{-e^{z_n} + (5-z_n)e^{z_n}} = z_n - \frac{(5-z_n)e^{z_n} - 5}{(4-z_n)e^{z_n}}.$$

Iteration gives

$$z_2 = 4.969\,741\,205, \quad z_3 = 4.965\,135\,924, \quad z_4 = 4.965\,114\,232, \qquad z_5 = 4.965\,114\,232.$$

With this approximation for $z$, we obtain $\lambda = c/z_5 = 0.000\,028\,974$. For a seven decimal answer, we use $g(\lambda) = (5\lambda - c)e^{c/\lambda} - 5\lambda$ to calculate $g(0.000\,028\,95) = -1.7 \times 10^{-5}$ and $g(0.000\,029\,05) = 5.1 \times 10^{-5}$. Thus, to 7 decimals, $\lambda = 0.000\,029\,0$.

**35.** (i) To find the delay time, we must solve the equation $0.5 = 1 + e^{-2.5\sqrt{11}t}\sin(20t - \pi/2)$, or,

$$g(t) = e^{-2.5\sqrt{11}t}\sin(20t - \pi/2) + 0.5 = 0.$$

Newton's iterative procedure with initial approximation 0.05 defines further approximations by

$$t_1 = 0.05, \quad t_{n+1} = t_n - \frac{e^{-2.5\sqrt{11}t_n}\sin(20t_n - \pi/2) + 0.5}{-2.5\sqrt{11}e^{-2.5\sqrt{11}t_n}\sin(20t_n - \pi/2) + 20e^{-2.5\sqrt{11}t_n}\cos(20t_n - \pi/2)}.$$

Iteration gives $t_2 = 0.0398$, $t_3 = 0.0400$, and $t_4 = 0.0400$. To verify that $t = 0.04$ is the delay time correct to two decimal places we evaluate

$$g(0.035) = -0.07 \qquad \text{and} \qquad g(0.045) = 0.07.$$

(ii) To find the rise time, we must subtract the times when $f(t) = 0.9$ and $f(t) = 0.1$. Following the procedure in (i), we find these times to be 0.0696 and 0.0102 . Rise time is therefore $t = 0.06$.

(iii) Using the above procedures, we find two times at which $f(t) = 1.05$, namely, 0.08 and 0.22. There are three times at which $f(t) = 0.95$, namely, 0.07, 0.26 and 0.34. It follows that $0.95 \le f(t) \le 1.05$ when $t \ge 0.34$; that is, settling time is 0.34.

**36.** Let $P(x)$ be a cubic polynomial with roots $a$, $b$, and $c$. Then, $P(x) = k(x-a)(x-b)(x-c)$, where $k$ is a constant. Suppose we use Newton's iterative procedure with $x_1 = (a+b)/2$. We require

$$P(x_1) = k\left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - b\right)\left(\frac{a+b}{2} - c\right) = \frac{k}{8}(b-a)(a-b)(a+b-2c) = -\frac{k}{8}(b-a)^2(a+b-2c),$$

and

$$\begin{aligned} P'(x_1) &= k\left[\left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - b\right) + \left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - c\right) + \left(\frac{a+b}{2} - b\right)\left(\frac{a+b}{2} - c\right)\right] \\ &= \frac{k}{4}\left[(b-a)(a-b) + (b-a)(a+b-2c) + (a-b)(a+b-2c)\right] \\ &= -\frac{k}{4}(b-a)^2. \end{aligned}$$

Therefore,

$$x_2 = \frac{a+b}{2} - \frac{-(k/8)(b-a)^2(a+b-2c)}{-(k/4)(b-a)^2} = \frac{a+b}{2} - \frac{1}{2}(a+b-2c) = c.$$

## EXERCISES 4.2

**1.** Since $f'(x) = 2$, the function is increasing for all $x$.

**2.** Since $f'(x) = -5$, the function is decreasing for all $x$.

**3.** Since $f'(x) = 2x - 3$, it follows that $f'(x) \le 0$ for $x \le 3/2$ and $f'(x) \ge 0$ for $x \ge 3/2$. The function is therefore decreasing for $x \le 3/2$ and increasing for $x \ge 3/2$.

**4.** Since $f'(x) = -4x + 5$, it follows that $f'(x) \le 0$ for $x \ge 5/4$ and $f'(x) \ge 0$ for $x \le 5/4$. The function is therefore decreasing for $x \ge 5/4$ and increasing for $x \le 5/4$.

**5.** Since $f'(x) = 6x + 6$, it follows that $f'(x) \le 0$ for $x \le -1$ and $f'(x) \ge 0$ for $x \ge -1$. The function is therefore decreasing for $x \le -1$ and increasing for $x \ge -1$.

**6.** Since $f'(x) = 2 - 8x$, it follows that $f'(x) \le 0$ for $x \ge 1/4$ and $f'(x) \ge 0$ for $x \le 1/4$. The function is therefore decreasing for $x \ge 1/4$ and increasing for $x \le 1/4$.

**7.** Since $f'(x) = 6x^2 - 36x + 48 = 6(x-2)(x-4)$, it follows that $f'(x) \le 0$ for $2 \le x \le 4$, and $f'(x) \ge 0$ for $x \le 2$ and $x \ge 4$. The function is therefore decreasing for $2 \le x \le 4$, and increasing for $x \le 2$ and $x \ge 4$.

**8.** Since $f'(x) = 3x^2 + 12x + 12 = 3(x+2)^2 \ge 0$ for all $x$, the function is always increasing.

**9.** Since $f'(x) = 12x^2 - 36x = 12x(x-3)$, it follows that $f'(x) \le 0$ for $0 \le x \le 3$, and $f'(x) \ge 0$ for $x \le 0$ and $x \ge 3$. The function is therefore decreasing for $0 \le x \le 3$, and increasing for $x \le 0$ and $x \ge 3$.

**10.** Since $f'(x) = -18 - 18x - 6x^2 = -6(3 + 3x + x^2) < 0$ for all $x$, the function is always decreasing.

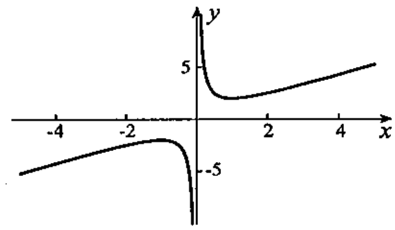
**11.** Since $f'(x) = 12x^3 + 12x^2 - 24 = 12(x-1)(x^2+2x+2)$, it follows that $f'(x) \le 0$ for $x \le 1$ and $f'(x) \ge 0$ for $x \ge 1$. The function is therefore decreasing for $x \le 1$ and increasing for $x \ge 1$.

**12.** Since $f'(x) = 12x^3 - 12x^2 + 48x - 48 = 12(x-1)(x^2+4)$, it follows that $f'(x) \le 0$ for $x \le 1$ and $f'(x) \ge 0$ for $x \ge 1$. The function is therefore decreasing for $x \le 1$ and increasing for $x \ge 1$.

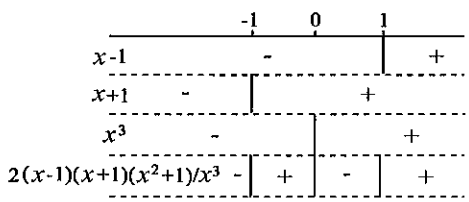
**13.** Since $f'(x) = 4x^3 - 12x^2 - 16x + 48 = 4(x+2)(x-2)(x-3)$, it follows that $f'(x) \le 0$ for $x \le -2$ and $2 \le x \le 3$, and $f'(x) \ge 0$ for $-2 \le x \le 2$ and $x \ge 3$. The function is therefore decreasing for $x \le -2$ and $2 \le x \le 3$, and increasing for $-2 \le x \le 2$ and $x \ge 3$.

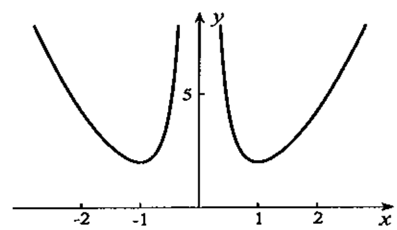
**14.** Since $f'(x) = 5x^4 - 5 = 5(x-1)(x+1)(x^2+1)$, we find that $f'(x) \le 0$ when $-1 \le x \le 1$, and $f'(x) \ge 0$ when $x \le -1$ and $x \ge 1$. The function is therefore decreasing for $-1 \le x \le 1$, and it is increasing on the intervals $x \le -1$ and $x \ge 1$.

**15.** Since $f'(x) = 1 - 1/x^2 = (x^2-1)/x^2 = (x-1)(x+1)/x^2$, it follows that $f'(x) \le 0$ for $-1 \le x < 0$ and $0 < x \le 1$, and $f'(x) \ge 0$ for $x \le -1$ and $x \ge 1$. The function is therefore decreasing for $-1 \le x < 0$ and $0 < x \le 1$, and increasing for $x \le -1$ and $x \ge 1$. The graph corroborates this.

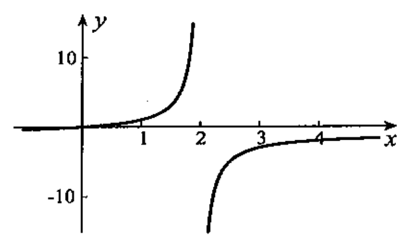



**16.** The sign diagram below evaluates the sign of $f'(x) = 2x - 2/x^3 = 2(x-1)(x+1)(x^2+1)/x^3$. The function is decreasing on the intervals $x \le -1$ and $0 < x \le 1$, and increasing on the intervals $-1 \le x < 0$ and $x \ge 1$. The graph corroborates this.

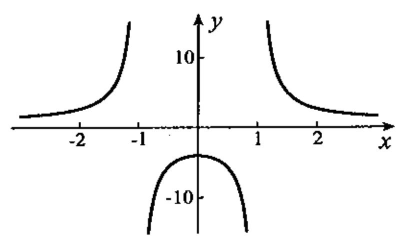


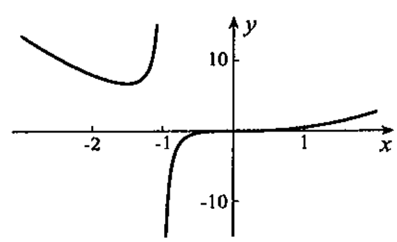

**17.** Since $f'(x) = \dfrac{(2-x)(1) - x(-1)}{(2-x)^2} = \dfrac{2}{(2-x)^2}$, it follows that $f'(x) \ge 0$ for all $x \ne 2$. The function is therefore increasing for $x < 2$ and $x > 2$. The graph shows that it is not increasing for all $x$.



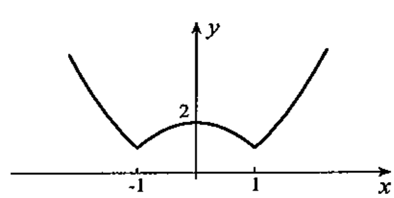

**18.** Since $f'(x) = \dfrac{(x^2-1)(2x) - (x^2+4)(2x)}{(x^2-1)^2} = \dfrac{-10x}{(x^2-1)^2}$, it follows that $f'(x) \le 0$ when $0 \le x < 1$ and $x > 1$, and $f'(x) \ge 0$ when $x < -1$ and $-1 < x \le 0$. The function is decreasing on the intervals $0 \le x < 1$ and $x > 1$, and it is increasing on $x < -1$ and $-1 < x \le 0$. The graph corroborates this.




**19.** Since $f'(x) = \dfrac{(x+1)(3x^2) - x^3(1)}{(x+1)^2} = \dfrac{x^2(2x+3)}{(x+1)^2}$, it follows that $f'(x) \le 0$ when $x \le -3/2$, and $f'(x) \ge 0$ when $-3/2 \le x < -1$ and $x > -1$. The function is decreasing on the interval $x \le -3/2$, and it is increasing on $-3/2 \le x < -1$ and $x > -1$. The graph corroborates this.




**20.** The graph of $f(x)$ indicates that $f(x)$ is decreasing on the intervals $x \le -1$ and $0 \le x \le 1$, and it is increasing on $-1 \le x \le 0$ and $x \ge 1$.




**21.** Since $f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$, we find that $f'(x) \le 0$ when $x \ge 1$, and $f'(x) \ge 0$ when $x \le 1$. The function is therefore decreasing for $x \ge 1$, and it is increasing on the interval $x \le 1$.

**22.** Since $f'(x) = 2xe^{-x} - x^2e^{-x} = x(2-x)e^{-x}$, it follows that $f'(x) \leq 0$ for $x \leq 0$ and $x \geq 2$, and $f'(x) \geq 0$ for $0 \leq x \leq 2$. The function is decreasing on the intervals $x \leq 0$ and $x \geq 2$, and it is increasing for $0 \leq x \leq 2$.

**23.** Since $f'(x) = 2x/(x^2+5)$, we find that $f'(x) \leq 0$ when $x \leq 0$, and $f'(x) \geq 0$ when $x \geq 0$. The function is therefore decreasing for $x \leq 0$, and it is increasing for $x \geq 0$.

**24.** Since $f'(x) = \ln x + 1$, we find that $f'(x) \leq 0$ when $0 < x \leq 1/e$, and $f'(x) \geq 0$ when $x \geq 1/e$. The function is therefore decreasing for $0 < x \leq 1/e$ and increasing for $x \geq 1/e$.

**25.** Since $f(x) = \dfrac{(x+3)(x-3)}{x-3} = x+3$, except for $x = 3$, where $f(x)$ is undefined, the function is increasing for $x < 3$ and $x > 3$.

**26.** Since $f(x) = \dfrac{(x-1)(x+1)(x+2)}{(2+x)(1-x)} = -x-1$, except for $x = -2$ and $x = 1$, where $f(x)$ is undefined, the function is decreasing for $x < -2$, $-2 < x < 1$, and $x > 1$.

**27.** If we differentiate implicitly with respect to $x$, we find

$$1 = -6ar\frac{dr}{dx} + 3r^2\frac{dr}{dx} \quad \Longrightarrow \quad \frac{dr}{dx} = \frac{1}{3r^2 - 6ar} = \frac{1}{3r(r-2a)}.$$

Since this is negative for $0 < r < 2a$, the function is decreasing on this interval.

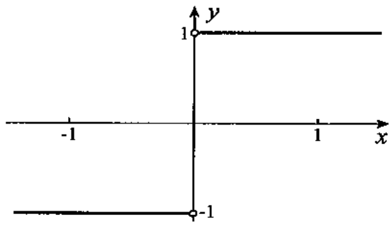
**28.**

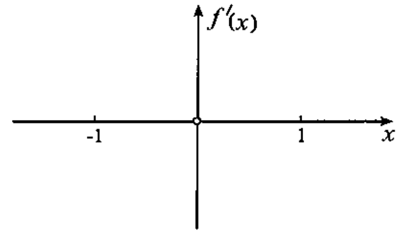

**29.**


**30.**

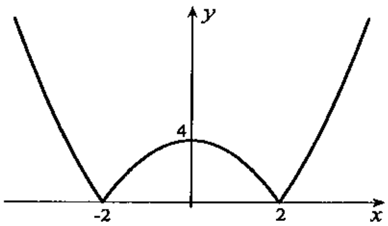

**31.**

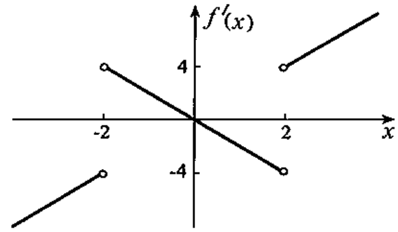
**32.**



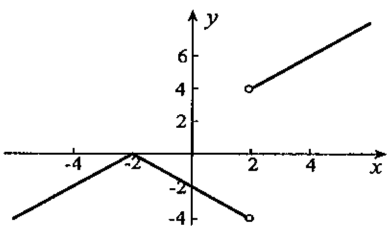



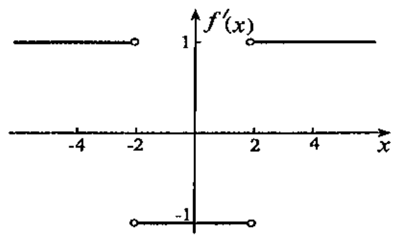

**33.**



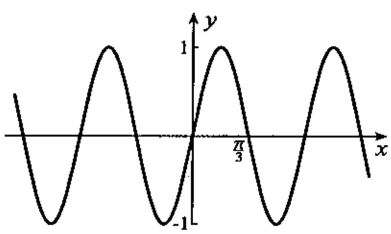



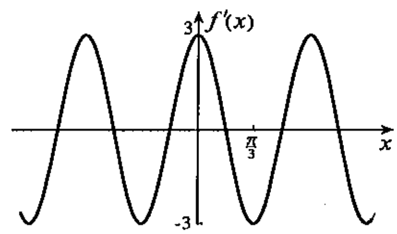

**34.**



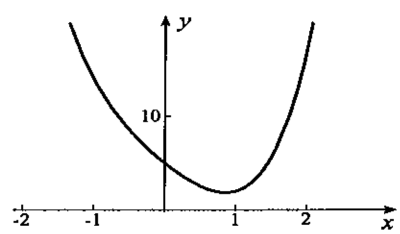

**35.**




**36.** We first consider points where $0 = f'(x) = 4x^3 + 4x - 6$. The plot of $f(x)$ indicates that $f'(x)$ has a zero between 0 and 1. To find it more accurately, we use Newton's iterative procedure with $x_1 = 0.7$ and

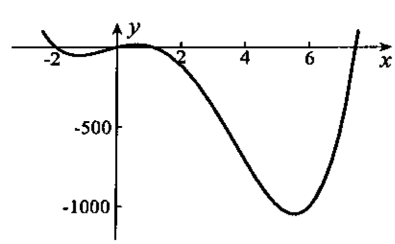
$$x_{n+1} = x_n - \frac{4x_n^3 + 4x_n - 6}{12x_n^2 + 4}.$$




Iteration gives $x_2 = 0.885\,02$, $x_3 = 0.861\,67$, $x_4 = 0.861\,22$, $x_5 = 0.861\,22$. Since $f'(0.861\,15) = -9.6 \times 10^{-4}$ and $f'(0.861\,25) = 3.3 \times 10^{-4}$, it follows that $x = 0.861\,2$ is the solution of $f'(x) = 0$ to four decimals. The graph makes it clear that $f(x)$ is decreasing for $x \leq 0.861\,2$ and increasing for $x \geq 0.861\,2$.

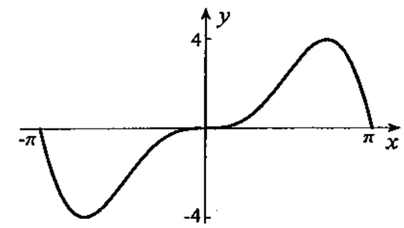
**37.** We first consider points where $0 = f'(x) = 12x^3 - 60x^2 - 48x + 48 = 12(x^3 - 5x^2 - 4x + 4)$. The plot of $f(x)$ indicates that $f'(x)$ has three zeros. To find the zero between 0 and 1, we use Newton's iterative procedure with $x_1 = 0.6$ and

$$x_{n+1} = x_n - \frac{x_n^3 - 5x_n^2 - 4x_n + 4}{3x_n^2 - 10x_n - 4}.$$




Iteration gives $x_2 = 0.601\,79$ and $x_3 = 0.601\,79$. Since $f'(0.601\,75) = 4.6 \times 10^{-3}$ and $f'(0.601\,85) = -6.1 \times 10^{-3}$, it follows that $x = 0.601\,8$ is the solution of $f'(x) = 0$ to four decimals. The other two solutions are $-1.189\,5$ and $5.587\,7$. The graph makes it clear that $f(x)$ is decreasing for $x \leq -1.189\,5$ and $0.601\,8 \leq x \leq 5.587\,7$, and increasing for $-1.189\,5 \leq x \leq 0.601\,8$ and $x \geq 5.587\,7$.

**38.** We first consider points where $0 = f'(x) = 2x\sin x + x^2\cos x = x(2\sin x + x\cos x)$. This equation implies that $x = 0$ or $2\sin x + x\cos x = 0$. The graph of $f(x)$ indicates that the second of these has a solution between $x = 2$ and $x = 3$. To find it we use Newton's iterative procedure with

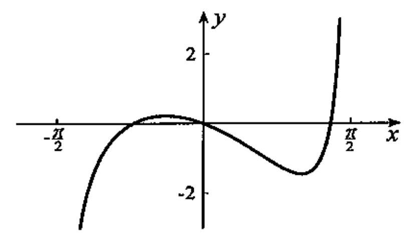



$$x_1 = 2, \qquad x_{n+1} = x_n - \frac{2\sin x_n + x_n\cos x_n}{3\cos x_n - x_n\sin x_n}.$$

Iteration gives $x_2 = 2.321\,58$, $x_3 = 2.289\,13$, $x_4 = 2.288\,93$, and $x_5 = 2.288\,93$. Since $f'(2.288\,85) = 6.7\times 10^{-4}$ and $f'(2.288\,95) = -1.7\times 10^{-4}$, we can say that a solution of $f'(x) = 0$ to four decimals is $x = 2.288\,9$. The graph makes it clear that so also is $x = -2.288\,9$. We conclude that $f(x)$ is decreasing on the intervals $-\pi \le x \le -2.288\,9$ and $2.288\,9 \le x \le \pi$, and it is increasing on $-2.288\,9 \le x \le 2.288\,9$.

**39.** We first consider points where $0 = f'(x) = \sec^2 x - 2x - 2$. The graph of $f(x)$ indicates that there are two solutions. To find the solution near $x = 1$, we use Newton's iterative procedure with

$$x_1 = 1, \qquad x_{n+1} = x_n - \frac{\sec^2 x_n - 2x_n - 2}{2\sec^2 x_n\,\tan x_n - 2}.$$




Iteration gives $x_2 = 1.066$, $x_3 = 1.055\,36$, $x_4 = 1.054\,95$, and $x_5 = 1.054\,95$. Since $f'(1.054\,95) = -2.7\times 10^{-5}$ and $f'(1.055\,05) = 1.2\times 10^{-3}$, we can say that a solution of $f'(x) = 0$ to four decimals is $x = 1.055\,0$. The other solution is $x = -0.407\,1$. We conclude that $f(x)$ is decreasing on the interval $-0.407\,1 \le x \le 1{,}055\,0$, and it is increasing on the intervals $-\pi/2 < x \le -0.407\,1$ and $1.055\,0 \le x < \pi/2$.

**40.** For $f(x) = x^{23} + 3x^{15} + 4x + 1$, we find that $f(-1) = -7$ and $f(0) = 1$. By the zero intermediate value theorem, there exists at least one solution of $f(x) = 0$ between $x = -1$ and $x = 0$. Since $f'(x) = 23x^{22} + 45x^{14} + 4 > 0$, the function is increasing for all $x$. Hence, there can be only one solution of the equation.

**41.** Suppose that $a > 0$ and $b > 0$ (a similar proof holds when $a$ and $b$ are both negative). Since $f(x) = ax^5 + bx^3 + c$ is continuous, and $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$, it follows that the graph of $f(x)$ must cross the $x$-axis at least once. Since $f'(x) = 5ax^4 + 3bx^2 > 0$, the function is increasing for all $x$. Hence, there can be only one solution of the equation.

**42.** Since $f(x) = x^n + ax - 1$ is continuous, and $f(0) = -1$ and $\lim_{x\to\infty} f(x) = \infty$, it follows that the graph of $f(x)$ must cross the $x$-axis at least once for positive $x$. Since $f'(x) = nx^{n-1} + a > 0$ for $x > 0$, the function is increasing for $0 \le x < \infty$. Hence, the graph of $f(x)$ can cross the $x$-axis only once for $x > 0$.

**43.** Since $f(x) = x^n + x^{n-1} - a$ is continuous, and $f(0) = -a$ and $\lim_{x\to\infty} f(x) = \infty$, it follows that the graph of $f(x)$ must cross the $x$-axis at least once for positive $x$. Since $f'(x) = nx^{n-1} + (n-1)x^{n-2} > 0$ for $x > 0$, the function is increasing for $0 \le x < \infty$. Hence, the graph of $f(x)$ can cross the $x$-axis only once for $x > 0$.

**44.** If $f(x) = x - \sin x$, then $f(0) = 0$. Since $f'(x) = 1 - \cos x \ge 0$, it follows that the function $f(x)$ is increasing for all $x > 0$. This means that $f(x) > 0$ for all $x > 0$, and therefore $x > \sin x$.

**45.** The function $f(x) = \cos x - 1 + x^2/2$ has value $f(0) = 0$. Since $f'(x) = -\sin x + x \ge 0$ for $x > 0$ (Exercise 44), it follows that $f(x)$ is increasing for all $x > 0$. This means that $f(x) > 0$ for all $x > 0$, and therefore $\cos x > 1 - x^2/2$.

**46.** If $f(x) = \sin x - x + x^3/6$, then $f(0) = 0$. Since $f'(x) = \cos x - 1 + x^2/2$, Exercise 45 implies that $f'(x) \ge 0$ for $x > 0$. Consequently, $f(x)$ is increasing for $x > 0$, and $\sin x > x - x^3/6$.

**47.** The function $f(x) = 1 - x^2/2 + x^4/24 - \cos x$ has value $f(0) = 0$. Since $f'(x) = -x + x^3/6 + \sin x$, Exercise 46 implies that $f'(x) \ge 0$ for $x > 0$. Consequently, $f(x)$ is increasing for all $x > 0$, and $\cos x < 1 - x^2/2 + x^4/24$.

**48.** If we define a function $f(x) = \dfrac{1}{\sqrt{1+3x}} - 1 + \dfrac{3x}{2}$, then $f(0) = 0$, and

$$f'(x) = \frac{-3}{2(1+3x)^{3/2}} + \frac{3}{2} = \frac{3}{2}\left[1 - \frac{1}{(1+3x)^{3/2}}\right].$$

This derivative is clearly positive for $x > 0$, and therefore the function $f(x)$ is increasing for $x > 0$. Thus, $f(x) > 0$ for $x > 0$, and the required result follows.

**49.** Not necessarily. The derivative of $f(x)g(x)$ is $f'(x)g(x) + f(x)g'(x)$. Nonnegativity of $f'(x)$ and $g'(x)$ does not guarantee nonnegativity of this derivative.

**50.** Yes. Since $f(x) > 0$, $f'(x) \geq 0$, $g(x) > 0$, and $g'(x) \geq 0$ on $I$, it follows that $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x) \geq 0$ on $I$ also; that is, $f(x)g(x)$ is increasing on $I$.

**51.** We prove part (i); the proof of part (ii) is similar. Suppose then that $f'(x) \geq 0$ on an interval $I: a \leq x \leq b$, and let $d_1 < d_2 < \ldots < d_{n-1}$ be the finite number of points in $I$ at which $f'(x) = 0$. Let $d_0 = a$ and $d_n = b$. We prove that $f(x)$ is increasing on each subinterval $I^*: d_{i-1} \leq x \leq d_i$, for $i = 1, \ldots, n$, and therefore it is increasing on $I$. If $x_1$ and $x_2$, where $x_1 > x_2$, are any two points in $I^*$, then the mean value theorem of Section 3.14 implies the existence of at least one point $c$ in the open interval between $d_{i-1}$ and $d_i$ at which

$$f'(c) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} \quad \Longrightarrow \quad f(x_1) = f(x_2) + (x_1 - x_2)f'(c).$$

Since $x_1 - x_2 > 0$ and $f'(c) > 0$, it follows that $f(x_1) > f(x_2)$. Hence, $f(x)$ is increasing on $I^*$.

**52.** Consider the function $g(x) = x - f(x)$. Since $g'(x) = 1 - f'(x)$, and $f'(x) < 1$ for all $x$, it follows that $g(x)$ is a decreasing function for all $x$. Consequently, it can have value 0 at most once; that is, there can exist at most one point $x_0$ at which $g(x_0) = 0$, and therefore at which $f(x_0) = x_0$.

**53.** We can express the inequality in the form $\dfrac{\tan b}{b} > \dfrac{\tan a}{a}$. If we define the function $f(x) = \dfrac{\tan x}{x}$, then we must show that $f(x)$ is increasing on the interval $I: 0 < x < \pi/2$. The plot in the left figure below seems to indicate that this is the case, but it is not conclusive. To verify this algebraically, we show that $f'(x) \geq 0$ on $I$,

$$\begin{aligned} f'(x) &= -\frac{1}{x^2}\tan x + \frac{1}{x}\sec^2 x = \frac{1}{x^2}(x\sec^2 x - \tan x) \\ &= \frac{1}{x^2}\left(\frac{x}{\cos^2 x} - \frac{\sin x}{\cos x}\right) = \frac{1}{x^2\cos^2 x}(x - \sin x\cos x). \end{aligned}$$

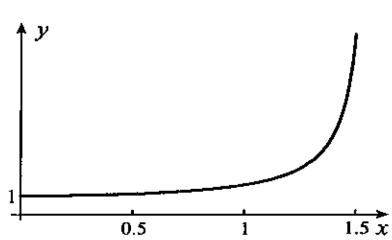
Since $x^2\cos^2 x$ is positive on $I$, the sign of $f'(x)$ is determined by that of $x - \sin x\cos x$. Therefore, we must show that

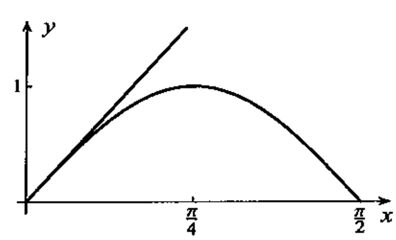
$$0 \leq x - \sin x \cos x.$$

This inequality can be simplified if it is multiplied by 2,

$$0 \leq 2x - 2\sin x\cos x = 2x - \sin 2x \qquad \text{or} \qquad 2x \geq \sin 2x.$$

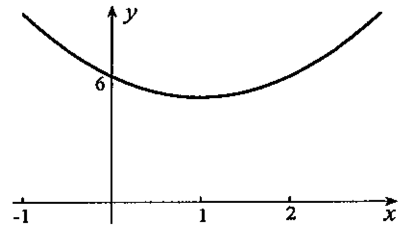
Graphs of the functions $2x$ and $\sin 2x$ are shown in the right figure. Important to this picture are the facts that the line $y = 2x$ and the curve $y = \sin 2x$ both have slope 2 at $x = 0$, but the slope of $y = \sin 2x$ is less than 2 for $x > 0$. Clearly, then, $2x \geq \sin 2x$, and our proof is complete.



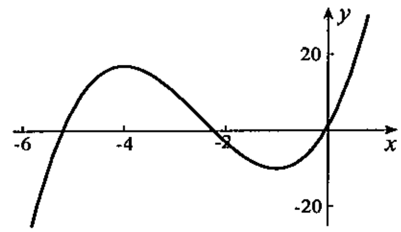



## EXERCISES 4.3

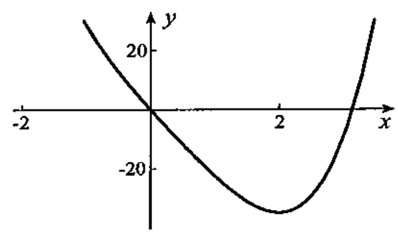
**1.** Since $f'(x) = 2x - 2$, the only critical point is $x = 1$. Because $f'(x)$ changes from negative to positive as $x$ increases through 1, $x = 1$ gives a relative minimum.



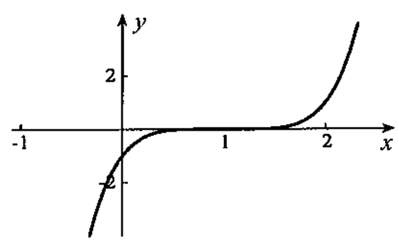

**2.** Since $f'(x) = 6x^2 + 30x + 24 = 6(x+1)(x+4)$, critical points are $x = -1$ and $x = -4$. Because $f'(x)$ changes from positive to negative as $x$ increases through $-4$, $x = -4$ yields a relative maximum. Since $f'(x)$ changes from negative to positive as $x$ increases through $-1$, $x = -1$ gives a relative minimum.



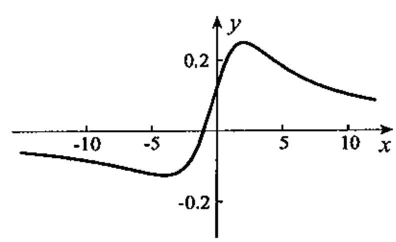

**3.** Since $f'(x) = 4x^3 - 4x^2 + 4x - 24 = 4(x-2)(x^2 + x + 3)$, the only critical point is $x = 2$. Because $f'(x)$ changes from negative to positive as $x$ increases through 2, $x = 2$ gives a relative minimum.



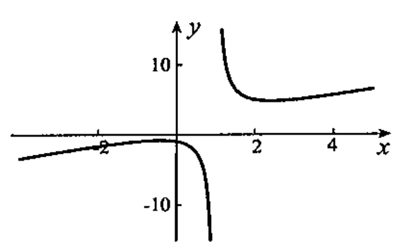

**4.** Since $f'(x) = 5(x-1)^4$, the only critical point is $x = 1$. Because $f'(x) \geq 0$ for all $x$, the function is always increasing, and $x = 1$ does not give a relative maximum or minimum.



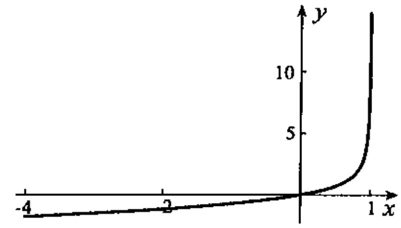

**5.** Since $f'(x) = \dfrac{(x^2+8)(1) - (x+1)(2x)}{(x^2+8)^2} = \dfrac{-(x+4)(x-2)}{(x^2+8)^2}$, critical points are $x = -4$ and $x = 2$. Since $f'(x)$ changes from negative to positive as $x$ increases through $-4$, this critical point gives a relative minimum. Because $f'(x)$ changes from positive to negative as $x$ increases through 2, there is a relative maximum at this value.



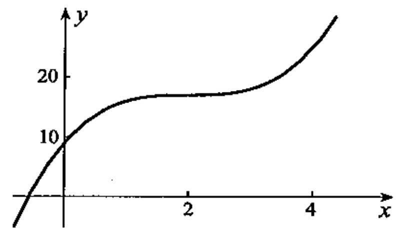

**6.** Since $f'(x) = \dfrac{(x-1)(2x) - (x^2+1)(1)}{(x-1)^2} = \dfrac{x^2 - 2x - 1}{(x-1)^2}$, critical points are $x = (2 \pm \sqrt{4+4})/2 = 1 \pm \sqrt{2}$. The derivative does not exist at $x = 1$, but this is not a critical point because the function is not defined at $x = 1$. Since $f'(x)$ changes from positive to negative as $x$ increases through $1 - \sqrt{2}$, this critical point gives a relative maximum. Because $f'(x)$ changes from negative to positive as $x$ increases through $1 + \sqrt{2}$, there is a relative minimum at this point.



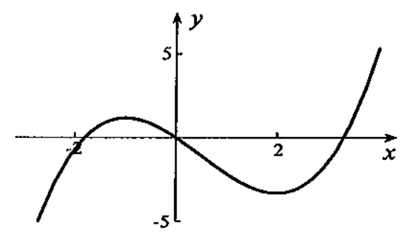
**7.** Since $f'(x) = \dfrac{\sqrt{1-x} - x(1/2)(1-x)^{-1/2}(-1)}{1-x} = \dfrac{2-x}{2(1-x)^{3/2}}$,
There are no critical points at which $f'(x) = 0$.
The derivative does not exist at $x = 1$, but because $f(1)$ is undefined, $x = 1$ is not critical.



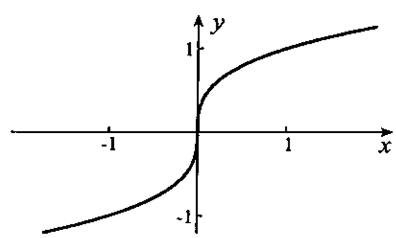

**8.** Since $f'(x) = 3x^2 - 12x + 12 = 3(x-2)^2$, the only critical point is $x = 2$. Since $f'(x) \geq 0$ for all $x$, it follows that $f(x)$ is increasing for all $x$, and there cannot be a relative extremum at $x = 2$.



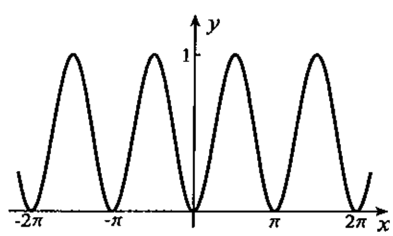

**9.** Since $f'(x) = x^2 - x - 2 = (x-2)(x+1)$, there are two critical points $x = -1$ and $x = 2$. Since $f'(x)$ changes from positive to negative as $x$ increases through $-1$, this critical point gives a relative maximum. Because $f'(x)$ changes from negative to positive as $x$ increases through 2, there is a relative minimum at this point.



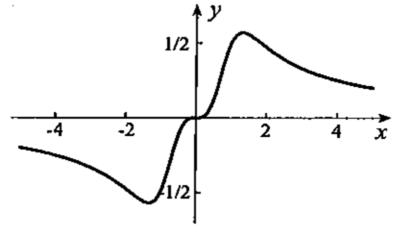

**10.** Since $f'(x) = (1/3)x^{-2/3}$, and this derivative does not exist at $x = 0$, this is a critical point. Because $f'(x)$ is positive for all $x \neq 0$, it follows that the function is always increasing, and $x = 0$ cannot therefore yield a relative extremum.



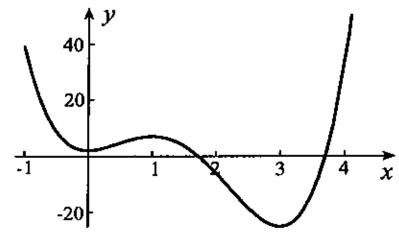

**11.** Since $f'(x) = 2\sin x\cos x = \sin 2x$, critical points are $x = n\pi/2$, where $n$ is an integer. Because $f'(x)$ changes from negative to positive as $x$ increases through $n\pi$, these critical points gives relative minima. Because $f'(x)$ changes from positive to negative as $x$ increases through $(2n+1)\pi/2$, these critical points give relative maxima.



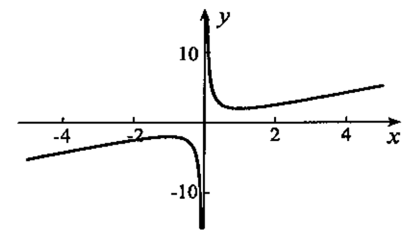

**12.** Since $f'(x) = \dfrac{(x^4+1)(3x^2) - x^3(4x^3)}{(x^4+1)^2} = \dfrac{x^2(3^{1/4}-x)(3^{1/4}+x)(\sqrt{3}+x^2)}{(x^4+1)^2}$,
critical points occur at $x = 0$ and $x = \pm 3^{1/4}$. Because $f'(x)$ changes from negative to positive as $x$ increases through $-3^{1/4}$, this critical point gives a relative minimum. Since $f'(x)$ changes from positive to negative as $x$ increases through $3^{1/4}$, this critical point yields a relative maximum. The derivative remains positive as $x$ increases through 0. Consequently $f(x)$ is increasing in an interval around $x = 0$, and this point cannot yield a relative extremum.



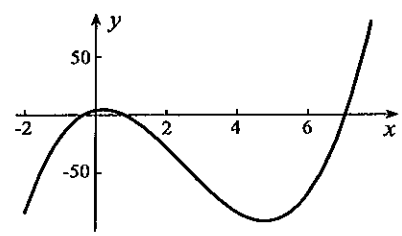
**13.** Since $f'(x) = 12x^3 - 48x^2 + 36x = 12x(x-1)(x-3)$, critical points are $x = 0, 1, 3$. Because $f'(x)$ changes from negative to positive as $x$ increases through 0 and 3, these critical points give relative minima. Since $f'(x)$ changes from positive to negative as $x$ increases through 1, this critical point yields a relative maximum.



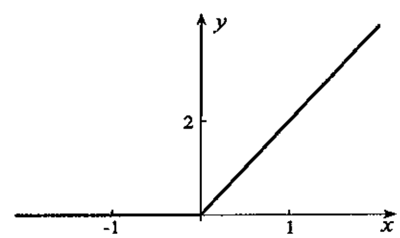

**14.** Since $f'(x) = 1 - 1/x^2$, the derivative vanishes at $x = \pm 1$. It does not exist at $x = 0$, but this point cannot be critical because $f(0)$ is not defined. Since $f'(x)$ changes from positive to negative as $x$ increases through $-1$, $x = -1$ gives a relative maximum. The critical point $x = 1$ gives a relative minimum since $f'(x)$ changes from negative to positive as $x$ increases through this point.



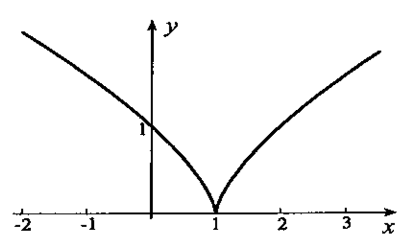

**15.** Since $f'(x) = 6x^2 - 30x + 6 = 6(x^2 - 5x + 1)$, critical points are $x = (5 \pm \sqrt{25-4})/2 = (5 \pm \sqrt{21})/2$. Because $f'(x)$ changes from positive to negative as $x$ increases through $(5 - \sqrt{21})/2$, this critical point gives a relative maximum. Since $f'(x)$ changes from negative to positive as $x$ increases through $(5 + \sqrt{21})/2$, this critical point yields a relative minimum.



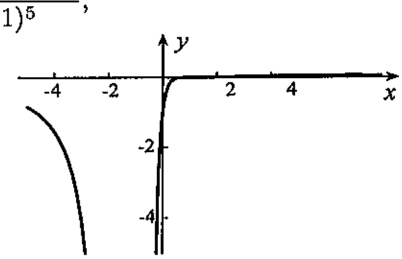

**16.** The point $x = 0$ is critical since $f'(0)$ does not exist, and it yields a relative minimum. Every point on the negative $x$-axis is also critical, and each such point yields a relative maximum and a relative minimum (see Definitions 4.3 and 4.4).



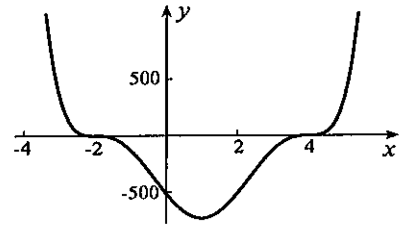

**17.** Since $f'(x) = (2/3)(x-1)^{-1/3}$, the derivative is never 0, but $x = 1$ is critical since $f'(1)$ does not exist. Because $f'(x)$ changes from negative to positive as $x$ increases through $x = 1$, this critical point gives a relative minimum.



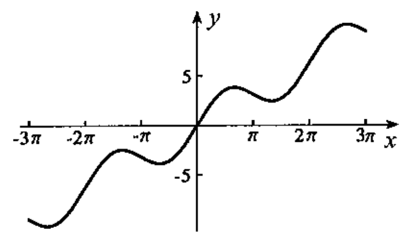

**18.** Since $f'(x) = \dfrac{(x+1)^4 3(x-1)^2 - (x-1)^3 4(x+1)^3}{(x+1)^8} = \dfrac{(x-1)^2(7-x)}{(x+1)^5}$, critical points are $x = 1$ and $x = 7$. Although $f'(x)$ does not exist at $x = -1$, this point is not critical since $f(-1)$ is not defined. Since $f'(x)$ changes from positive to negative as $x$ increases through 7, $x = 7$ gives a relative maximum. Because $f'(x)$ remains positive as $x$ increases through 1, this point does not give a relative extremum.



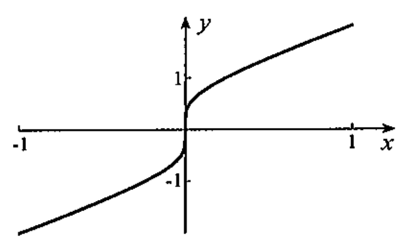
**19.** Since $f'(x) = 3(x+2)^2(x-4)^3 + 3(x+2)^3(x-4)^2 = 6(x+2)^2(x-4)^2(x-1)$, critical points are $x = -2, 1, 4$. Because $f'(x)$ changes from negative to positive as $x$ increases through 1, this critical point yields a relative minimum. Since $f'(x)$ does not change sign as $x$ passes through $x = -2$ and $x = 4$, the function does not have relative extrema at these points.




**20.** Since $f'(x) = 1 + 2\cos x$, the derivative vanishes for all values of $x$ satisfying $\cos x = -1/2$. Solutions of this equation are $x = 2\pi/3 + 2n\pi$ and $x = 4\pi/3 + 2n\pi$, where $n$ is an integer. Because $f'(x)$ changes from positive to negative as $x$ increases through the values $2(3n+1)\pi/3$, these critical points give relative maxima. On the other hand, $f'(x)$ changes from negative to positive as $x$ increases through the remaining critical points $x = 2(3n+2)\pi/3$, which therefore give relative minima.



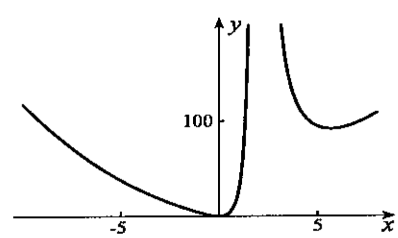

**21.** Since $f'(x) = (1/5)x^{-4/5} + 1 = \dfrac{5x^{4/5}+1}{5x^{4/5}}$, there is no point at which $f'(x) = 0$. Because $f'(0)$ does not exist, $x = 0$ is critical. Since $f'(x) > 0$ for all $x$, the function is always increasing, and $x = 0$ cannot give a relative extremum.




**22.** For critical points we first solve

$$0 = f'(x) = 2x + \frac{(x-2)^2(50x) - 25x^2(2)(x-2)}{(x-2)^4} = 2x + \frac{50x(x-2-x)}{(x-2)^3} = 2x\left[\frac{(x-2)^3-50}{(x-2)^3}\right].$$

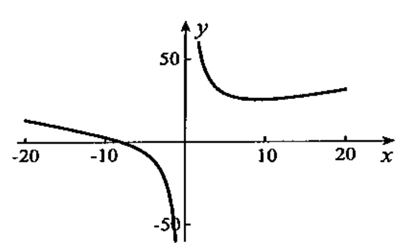
Thus, either $x = 0$, or $(x-2)^3 = 50$, and the latter equation requires $x = 2 + 50^{1/3}$. The derivative does not exist at $x = 2$, but neither does $f(2)$, and therefore $x = 2$ is not critical. Since $f'(x)$ changes from negative to positive as $x$ increases through 0, $x = 0$ gives a relative minimum. Because $f'(x)$ changes from negative to positive as $x$ increases through $2 + 50^{1/3}$, this critical point also yields a relative minimum.



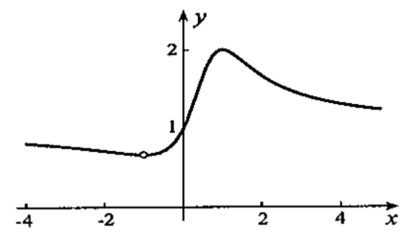

**23.** For critical points we first solve

$$0 = f'(x) = \left(\frac{-8}{x^2}\right)\sqrt{x^2+100} + \left(x + \frac{8}{x}\right)\frac{x}{\sqrt{x^2+100}} =$$

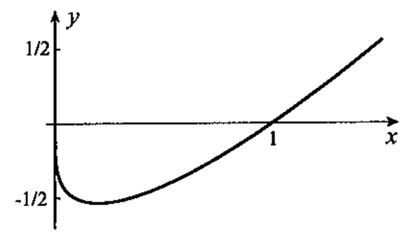
$\dfrac{x^3 - 800}{x^2\sqrt{x^2+100}}$. The only solution is $x = 800^{1/3}$. Since $f'(x)$ changes from negative to positive as $x$ increases through this critical point, it yields a relative minimum.



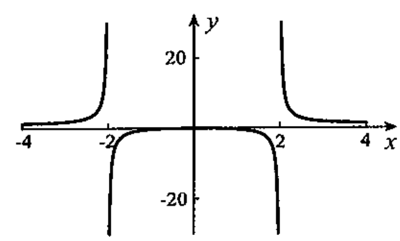
**24.** Since $f'(x) = \dfrac{(1+x^3)(1+2x+3x^2)-(1+x+x^2+x^3)(3x^2)}{(1+x^3)^2} =$ $\dfrac{(1-x)(1+x)^3}{(1+x^3)^2}$, it follows that $x = 1$ is critical. The derivative does not exist at $x = -1$, but this point is not critical since $f(-1)$ is not defined. Because $f'(x)$ changes from positive to negative as $x$ increases through 1, this critical point gives a relative maximum.



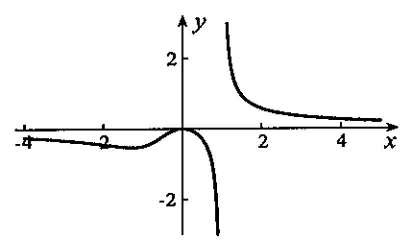

**25.** Since $f'(x) = \dfrac{5}{4}x^{1/4} - \dfrac{1}{4}x^{-3/4} = \dfrac{5x-1}{4x^{3/4}}$, $x = 1/5$ is a critical point at which $f'(x) = 0$. The right-hand derivative does not exist at $x = 0$, but $f(0) = 0$, so that $x = 0$ is also critical. Because $f'(x)$ changes from negative to positive as $x$ increases through $1/5$, this critical point gives a relative minimum. According to Definition 4.3, $x = 0$ cannot yield a relative maximum.




**26.** Since $f'(x) = \dfrac{(x^2-4)(2x) - x^2(2x)}{(x^2-4)^2} = \dfrac{-8x}{(x^2-4)^2}$, the only critical point is $x = 0$. The derivative does not exist at $x = \pm 2$, but these are not critical points because $f(\pm 2)$ do not exist. Since $f'(x)$ changes from positive to negative as $x$ increases through $x = 0$, this critical point yields a relative maximum.



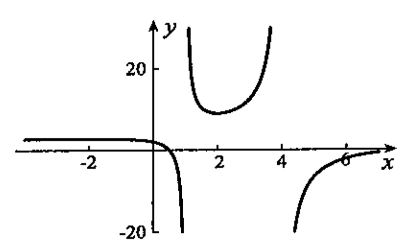

**27.** Since $f'(x) = \dfrac{(x^3-1)(2x) - x^2(3x^2)}{(x^3-1)^3} = \dfrac{-x(x^3+2)}{(x^3-1)^3}$, $x = 0$ is a critical point, as is $x = -2^{1/3}$. The derivative does not exist at $x = 1$, but this is not a critical point because $f(1)$ does not exist. Because $f'(x)$ changes from positive to negative as $x$ increases through 0, this critical point yields a relative maximum. Since $f'(x)$ changes from negative to positive as $x$ increases through $-2^{1/3}$, this critical point gives a relative minimum.



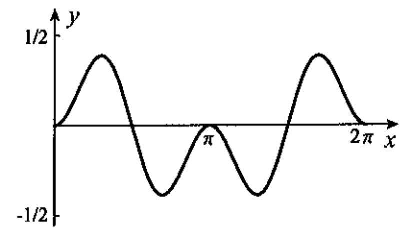

**28.** With the function written in the form $f(x) = \dfrac{2x^2 - 17x + 8}{x^2 - 5x + 4}$, we set

$$0 = f'(x) = \frac{(x^2-5x+4)(4x-17) - (2x^2-17x+8)(2x-5)}{(x^2-5x+4)^2} =$$

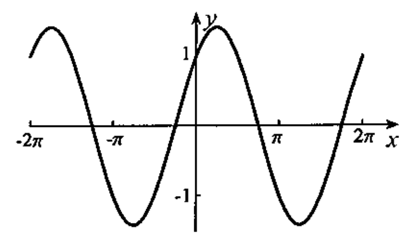
$\dfrac{7(x^2-4)}{(x-1)^2(x-4)^2}$. Critical points are $x = \pm 2$. Since $f'(x)$ changes from positive to negative as $x$ increases through $-2$, this critical point gives a relative maximum. The critical point $x = 2$ gives a relative minimum as $f'(x)$ changes from negative to positive through this point.



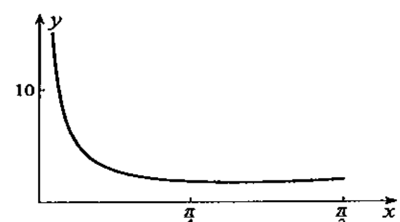
**29.** Since $f'(x) = 2\sin x\cos^2 x - \sin^3 x = \sin x(2\cos^2 x - \sin^2 x) = \sin x(3\cos^2 x - 1)$, critical points are $x = 0, \pi, 2\pi$ and values of $x$ satisfying $\cos x = \pm 1/\sqrt{3}$. From $\cos x = 1/\sqrt{3}$, we obtain $x = 0.955, 2\pi - 0.955$, and from $\cos x = -1/\sqrt{3}$, we get $x = \pi \pm 0.955$. Since $f'(x)$ changes from positive to negative as $x$ increases through $0.955$, $\pi$, and $2\pi - 0.955$, these critical points give relative maxima. Since $f'(x)$ changes from negative to positive as $x$ increases through $\pi \pm 0.955$, these critical points give relative minima. End points $x = 0, 2\pi$ cannot give relative minima (see Definition 4.4).



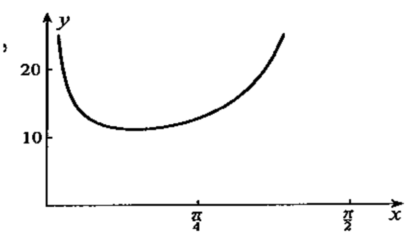

**30.** Since $f'(x) = \cos x - \sin x$, critical points are $x = \pi/4 + n\pi$, where $n$ is an integer. Since $f'(x)$ changes from positive to negative as $x$ increases through the critical points $\pi/4 + 2n\pi$, these points yield relative maxima. The remaining critical points $5\pi/4 + 2n\pi$ give relative minima as $f'(x)$ changes from negative to positive through these points.




**31.** Since $f'(x) = -2\csc x\cot x + \csc^2 x = \dfrac{1 - 2\cos x}{\sin^2 x}$, the only critical point is $x = \pi/3$. Because $f'(x)$ changes from negative to positive as $x$ increases through $\pi/3$, the critical point gives a relative minimum.



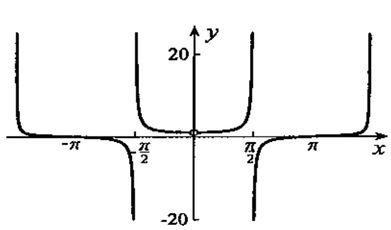

**32.** Since $f'(x) = -\csc x\cot x + 8\sec x\tan x = \dfrac{\cos x}{\sin^2 x}(8\tan^3 x - 1)$, critical points satisfy $\tan x = 1/2$. The only solution of this equation between 0 and $\pi/2$ is 0.464. Because $f'(x)$ changes from negative to positive as $x$ increases through this critical point, it yields a relative minimum.

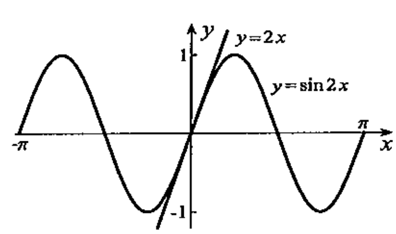



**33.** The left figure below suggests that the function has no critical points. We can verify this by considering

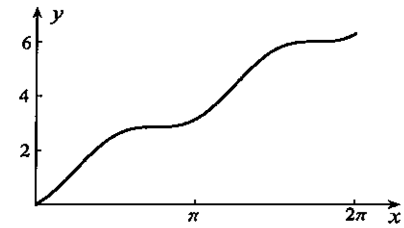
$$0 = f'(x) = \frac{x\sec^2 x - \tan x}{x^2} = \frac{x - \sin x\cos x}{x^2\cos^2 x} = \frac{2x - \sin 2x}{2x^2\cos^2 x}.$$

Graphs of $y = 2x$ and $y = \sin 2x$ in the right figure show $x = 0$ as the only solution of $2x = \sin 2x$. But this cannot be critical because $f(0)$ is not defined.

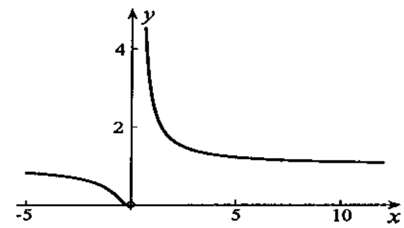


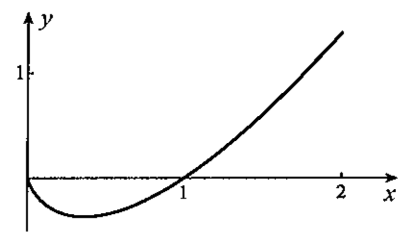
**34.** Since $f'(x) = 1 + 2\sin x\,\cos x = 1 + \sin 2x$, critical points occur when $\sin 2x = -1$. The only solutions of this equation in the interval $0 < x < 2\pi$ are $x = 3\pi/4$ and $x = 7\pi/4$. Because $f'(x)$ is always nonnegative, the function is increasing, and the critical points do not give relative extrema.



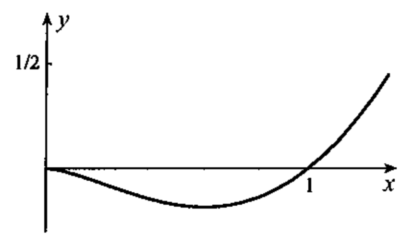

**35.** The graph indicates that the function has no critical points. We can also see this algebraically since $0 = f'(x) = e^{1/x}(-1/x^2)$ has no solutions.



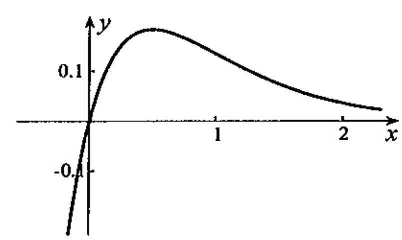

**36.** The derivative $f'(x) = \ln x + 1$ vanishes when $x = 1/e$. This critical point yields a relative minimum because $f'(x)$ changes from negative to positive as $x$ increases through $1/e$.



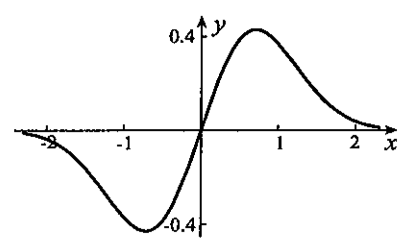

**37.** The derivative $f'(x) = 2x\ln x + x = x(2\ln x + 1)$ vanishes when $x = 1/\sqrt{e}$. This critical point yields a relative minimum because $f'(x)$ changes from negative to positive as $x$ increases through $1/\sqrt{e}$.



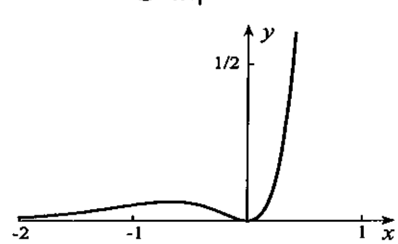

**38.** Since $f'(x) = e^{-2x} - 2xe^{-2x} = (1 - 2x)e^{-2x}$, the only critical point is $x = 1/2$. Since $f'(x)$ changes from positive to negative as $x$ increases through $1/2$, the critical point gives a relative maximum.

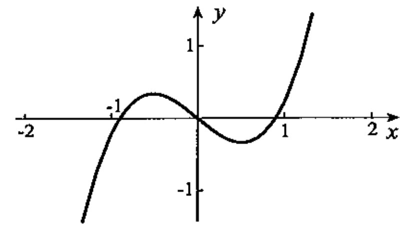



**39.** Since $f'(x) = e^{-x^2} - 2x^2\,e^{-x^2} = (1 - 2x^2)e^{-x^2}$, critical points are $x = \pm 1/\sqrt{2}$. Since $f'(x)$ changes from positive to negative as $x$ increases through $1/\sqrt{2}$, this critical point gives a relative maximum. Since $f'(x)$ changes from negative to positive as $x$ increases through $-1/\sqrt{2}$, this critical point gives a relative minimum.




**40.** Since $f'(x) = 2xe^{3x} + 3x^2e^{3x} = x(3x + 2)e^{3x}$, we have two critical points, $x = 0$ and $x = -2/3$. Because $f'(x)$ changes from negative to positive as $x$ increases through 0, this critical point gives a relative minimum. The critical point $x = -2/3$ gives a relative maximum since $f'(x)$ changes from positive to negative as $x$ increases through this point.





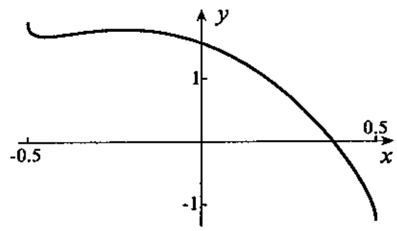

**41.** For critical points we solve

$$0 = f'(x) = 3x^2 - \frac{1}{1+x^2} = \frac{3x^4 + 3x^2 - 1}{1+x^2}.$$

When we set $3x^4 + 3x^2 - 1 = 0$, we obtain

$$x^2 = \frac{-3 \pm \sqrt{9+12}}{6} = \frac{-3 \pm \sqrt{21}}{6}.$$

Thus, $x = \pm\sqrt{\dfrac{3+\sqrt{21}}{6}}$. Since $f'(x)$ changes from positive to negative as $x$ increases through the negative value, $x = -\sqrt{(3+\sqrt{21})/6}$ gives a relative maximum. The positive critical point gives a relative minimum since $f'(x)$ changes from negative to positive as increases through it.




**42.** For critical points we solve

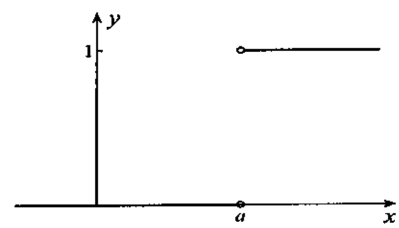
$$0 = f'(x) = \frac{-2}{\sqrt{1-4x^2}} - 10x = \frac{-2(1+5x\sqrt{1-4x^2})}{\sqrt{1-4x^2}}.$$

When we set $1 + 5x\sqrt{1-4x^2} = 0$, we obtain

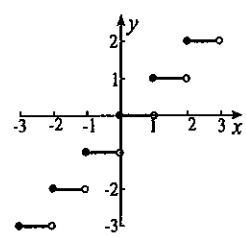
$$25x^2(1-4x^2) = 1$$
$$100x^4 - 25x^2 + 1 = 0$$
$$(20x^2 - 1)(5x^2 - 1) = 0$$

We can only accept the negative solutions of this equation, namely, $x = -1/\sqrt{5}$ and $x = -\sqrt{5}/10$. Since $f'(x)$ changes from negative to positive as $x$ increases through $-1/\sqrt{5}$, this critical point gives a relative minimum. Since $f'(x)$ changes from positive to negative as $x$ increases through $-\sqrt{5}/10$, this critical point gives a relative maximum.

**43.** Every point except $x = a$ is critical. Each gives both a relative maximum and a relative minimum.




**44.** Every point is critical. Every integer gives a relative maximum. Every other value of $x$ gives both a relative maximum and a relative minimum.




**45.** Implicit differentiation gives

$$4x^3 + 3y^2\frac{dy}{dx} + 5y^4\frac{dy}{dx} = 0 \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{-4x^3}{3y^2 + 5y^4}.$$

The only critical point at which the derivative vanishes is $x = 0$. Since the denominator $3y^2 + 5y^4$ is never negative, the derivative changes from positive to negative as $x$ increases through 0. Hence, $x = 0$ gives a relative maximum.

**46.** Implicit differentiation gives

$$2x + 3y^2\frac{dy}{dx} + \frac{dy}{dx} = 0 \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{-2x}{3y^2 + 1}.$$

The only critical point at which the derivative vanishes is $x = 0$. Since the denominator $3y^2 + 1$ is always positive, the derivative changes from positive to negative as $x$ increases through 0. Hence, $x = 0$ gives a relative maximum.

**47.** Implicit differentiation gives

$$3x^2y + x^3\frac{dy}{dx} + y^3 + 3xy^2\frac{dy}{dx} = 0 \qquad \Longrightarrow \qquad \frac{dy}{dx} = -\frac{3x^2y + y^3}{x^3 + 3xy^2} = -\frac{y(3x^2 + y^2)}{x(x^2 + 3y^2)}.$$

For this to vanish we set $y = 0$, but this is impossible in the original equation $x^3y + xy^3 = 2$.

**48.** Implicit differentiation gives

$$4y^3\frac{dy}{dx} + y^3 + 3xy^2\frac{dy}{dx} = 0 \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{-y^3}{3xy^2 + 4y^3}.$$

For this to vanish we set $y = 0$, but this is not permitted by the original equation $y^4 + xy^3 = 1$.

**49.** Implicit differentiation gives

$$4x^3y + x^4\frac{dy}{dx} + 5y^4\frac{dy}{dx} = 0 \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{-4x^3y}{x^4 + 5y^4}.$$

For this to vanish we set either $x = 0$ or $y = 0$. The original equation $x^4y + y^5 = 32$ does not permit $y = 0$. When $x = 0$, the value of $y$ is $y = 2$. Since the denominator $x^4 + 5y^4$ is always positive, it follows that the derivative changes from positive to negative as $x$ increases through 0. Hence, $x = 0$ gives a relative maximum.

**50.** Implicit differentiation gives

$$2xy^4 + 4x^2y^3\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0 \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{-2xy^4}{4x^2y^3 + 3y^2}.$$

For this to vanish we set either $x = 0$ or $y = 0$. The original equation $x^2y^4 + y^3 = 1$ does not permit $y = 0$. Since the denominator $4x^2y^3 + 3y^2$ is always positive, the derivative changes from positive to negative as $x$ increases through 0. Hence, $x = 0$ gives a relative maximum.

**51.** Implicit differentiation gives

$$2x + 2y + 2x\frac{dy}{dx} + 6y\frac{dy}{dx} = 0 \qquad \Longrightarrow \qquad \frac{dy}{dx} = -\frac{x + y}{x + 3y}.$$

For this to vanish we set $y = -x$. When this is substituted into the original equation

$$x^2 - 2x^2 + 3x^2 = 2 \qquad \Longrightarrow \qquad 2x^2 = 2 \qquad \Longrightarrow \qquad x = \pm 1.$$

**52.** Implicit differentiation gives

$$4x^3y + x^4\frac{dy}{dx} + 5y^4\frac{dy}{dx} = 4 \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{4 - 4x^3y}{x^4 + 5y^4}.$$

For this to vanish we set $4 - 4x^3y = 0$, from which $y = 1/x^3$. When this is substituted into the original equation

$$\frac{x^4}{x^3} + \frac{1}{x^{15}} = 4x \qquad \Longrightarrow \qquad 3x^{16} = 1 \qquad \Longrightarrow \qquad x = \pm\frac{1}{3^{1/16}}.$$

**53.** Implicit differentiation gives

$$4x - 3y^2\frac{dy}{dx} + y + x\frac{dy}{dx} = 0 \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{4x + y}{3y^2 - x}.$$

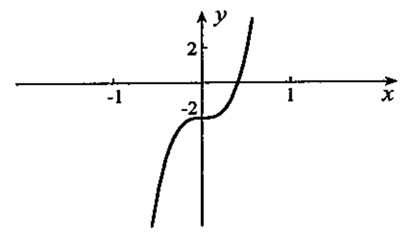
For this to vanish we set $4x + y = 0$, from which $y = -4x$. When this is substituted into the original equation

$$2x^2 - (-4x)^3 + x(-4x) = 4 \qquad \Longrightarrow \qquad 32x^3 - x^2 - 2 = 0.$$

A plot of the function $f(x) = 32x^3 - x^2 - 2$ to the right shows that the only solution of this equation is near 0.5. Newton's iterative procedure can be used to approximate it to four decimals. Iteration of

$$x_1 = 0.5, \quad x_{n+1} = x_n - \frac{32x_n^3 - x_n^2 - 2}{96x_n^2 - 2x_n}$$

leads to the iterates




$$x_2 = 0.42391, \quad x_3 = 0.40819, \quad x_4 = 0.407546, \quad x_5 = 0.407545.$$

To confirm 0.4075 as a four-decimal approximation we evaluate

$$f(0.40745) = -1.4 \times 10^{-3} \qquad \text{and} \qquad f(0.40755) = 7.3 \times 10^{-5}.$$

**54.** (a) Implicit differentiation gives

$$\frac{dy}{dx} = 2x\sqrt{1-y^2} + x^2\left(\frac{-y}{\sqrt{1-y^2}}\right)\frac{dy}{dx} \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{2x\sqrt{1-y^2}}{1 + \dfrac{x^2y}{\sqrt{1-y^2}}} = \frac{2x(1-y^2)}{x^2y + \sqrt{1-y^2}}.$$

Since $y$ cannot be equal to $\pm 1$, the only critical point at which the derivative vanishes is $x = 0$.
(b) If we square the equation, we obtain

$$y^2 = x^4(1-y^2) \qquad \Longrightarrow \qquad y^2(1+x^4) = x^4 \qquad \Longrightarrow \qquad y = \pm\sqrt{\frac{x^4}{1+x^4}}.$$

Since $y$ must be positive, the explicit definition of the function is $y = \dfrac{x^2}{\sqrt{1+x^4}}$. For critical points we solve

$$0 = \frac{dy}{dx} = \frac{\sqrt{1+x^4}(2x) - x^2(1/2)(1+x^4)^{-1/2}(4x^3)}{1+x^4} = \frac{2x(1+x^4) - 2x^5}{(1+x^4)^{3/2}} = \frac{2x}{(1+x^4)^{3/2}}.$$

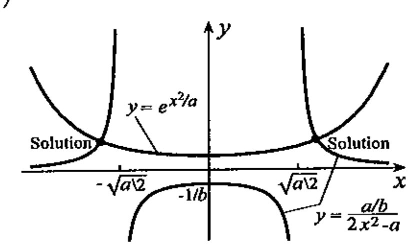
The only solution is $x = 0$.

**55.** For critical points we solve $\quad 0 = f'(x) = \dfrac{(1+be^{x^2/a})(1) - x(be^{x^2/a})(2x/a)}{(1+be^{x^2/a})^2}$. This implies that

$$0 = 1 + be^{x^2/a} - \frac{2b}{a}x^2e^{x^2/a} = 1 + \frac{b}{a}\left(a - 2x^2\right)e^{x^2/a}$$

$$\Longrightarrow \quad e^{x^2/a} = \frac{a/b}{2x^2 - a}.$$

Graphs of these functions to the right show two solutions, one the negative of the other.




**56.** (a) Since $f'_+(0) = 0$, it follows that $x = 0$ is critical. The point $x = 1$ is not critical since $f'_-(1) = 2$.
(b) No. End points of intervals cannot be relative extrema (see Definitions 4.3 and 4.4).

**57.** True The function must be defined at a critical point and derivatives do not exist at points of discontinuity.

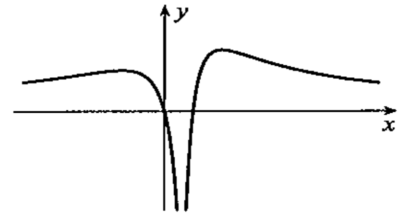
**58.** False According to Definitions 4.3 and 4.4 they can be neither relative maxima nor relative minima.

**59.** False It is true that if a function is discontinuous at a point, then it cannot have a derivative there. But, it can be continuous at points where the derivative does not exist. Take, for example, the function $|x|$ at $x = 0$.

**60.** True All values of $x$ give relative maxima and minima for the function $f(x) = 1$.

**61.** True This is the first derivative test.

**62.** False The discontinuous function to the right has two relative maxima but no relative minima.

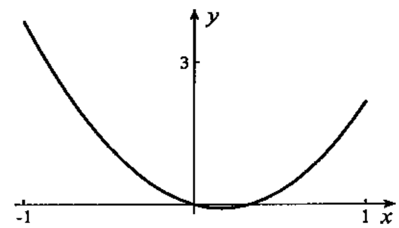



**63.** True Every $x$ is critical for $f(x) = 1$.

**64.** False The function in Figure 2.9(b) has an infinite number of such critical points in the interval $0 < x < 1$.

**65.** Critical points are defined by the equation $0 = f'(x) = 3x^2 - \sin x$. Clearly, $x = 0$ satisfies this equation. The graph of $f'(x)$ shows a second zero near 0.3. To find it, we use Newton's iterative procedure with

$$x_1 = 0.3, \quad x_{n+1} = x_n - \frac{3x_n^2 - \sin x_n}{6x_n - \cos x_n}.$$




Iteration gives $x_2 = 0.330$, $x_3 = 0.3274$, $x_4 = 0.3274$. Since $f'(0.3265) = -9.2 \times 10^{-4}$ and $f(0.3275) = 9.2 \times 10^{-5}$, the critical point is $x = 0.327$.

**66.** (a) Critical points of $P(V) = \dfrac{RTe^{-a/(RTV)}}{V - b}$ are defined by

$$0 = P'(V) = RT\left[\frac{(V-b)e^{-a/(RTV)}[a/(RTV^2)] - e^{-a/(RTV)}}{(V-b)^2}\right].$$

This implies that

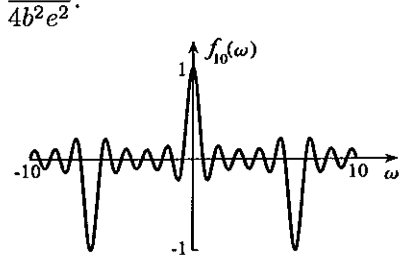
$$0 = \frac{a(V-b)}{RTV^2} - 1 \qquad \Longrightarrow \qquad (RT)V^2 - aV + ab = 0,$$

a quadratic equation with solutions $V = \dfrac{a \pm \sqrt{a^2 - 4abRT}}{2RT}$. These exist provided $a^2 - 4abRT > 0 \Longrightarrow T < a/(4bR)$.

(b) When $T = T_c = a/(4bR)$, there is one critical point, $V = \dfrac{a}{2RT_c} = \dfrac{a}{2R[a/(4bR)]} = 2b$, at which

$$P = \frac{RT_c e^{-\frac{a}{RT_c(2b)}}}{2b - b} = \frac{R\left(\frac{a}{4bR}\right)e^{-2}}{b} = \frac{a}{4b^2e^2}.$$

**67.** (a) The plot is shown to the right.
(b) $f_{10}(0)$ is not defined even though the plot suggests otherwise. The plot suggests a limit of 1.
(c) The function is even.
(d) The plot would indicate a period of about 12, if there is one. This suggests perhaps a period of $4\pi$. This is confirmed by the calculation




$$f_{10}(\omega + 4\pi) = \frac{\sin[5(\omega + 4\pi)]}{10\sin[(\omega + 4\pi)/2]} = \frac{\sin(5\omega + 20\pi)}{10\sin(\omega/2 + 2\pi)} = \frac{\sin(5\omega)}{10\sin(\omega/2)} = f_{10}(\omega).$$

(e) The smallest positive value of $\omega$ at which $f_{10}(\omega)$ has its smallest positive relative maximum is near $x = 2.8$. To find it more accurately, we must solve

$$0 = f_{10}'(\omega) = \frac{\sin(\omega/2)[5\cos(5\omega)] - \sin(5\omega)[(1/2)\cos(\omega/2)]}{10\sin^2(\omega/2)}.$$

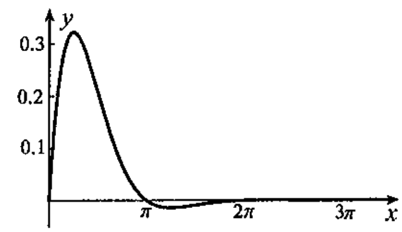
This requires $10\sin(\omega/2)\cos(5\omega) = \sin(5\omega)\cos(\omega/2) \implies g(\omega) = 10\cot(5\omega) - \cot(\omega/2) = 0$. Newton's iterative procedure with initial approximation $\omega_1 = 2.8$ defines the sequence

$$\omega_1 = 2.8, \quad \omega_{n+1} = \omega_n - \frac{10\cot(5\omega_n) - \cot(\omega_n/2)}{-50\csc^2(5\omega_n) + (1/2)\csc^2(\omega_n/2)}.$$

Iteration gives $\omega_2 = 2.824$ and $\omega_3 = 2.824$. Thus, the required value is $\omega \approx 2.824$.

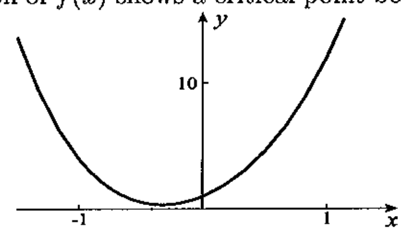
(f) For $f_L(\omega) = 0$, we must have $\sin(\omega L/2) = 0 \implies \omega L/2 = n\pi \implies \omega = 2n\pi/L$, where $n$ is an integer.

**68.** (a) A plot is shown to the right.




(b) Since $f'(x) = e^{-x}\cos x - e^{-x}\sin x = e^{-x}(\cos x - \sin x)$, critical points are $x = \pi/4 + n\pi$, where $n \geq 0$ is an integer. The graph indicates (as would the first derivative test) that relative maxima occur at $x = \pi/4 + 2n\pi$ and relative minima occur at $x = \pi/4 + (2n+1)\pi$.

**69.** Critical points are given by $0 = f'(x) = 4x^3 + 12x + 4$. The graph of $f(x)$ shows a critical point between $-1$ and 0. To find it, we use Newton's iterative procedure with




$$x_1 = 0, \quad x_{n+1} = x_n - \frac{x_n^3 + 3x_n + 1}{6x_n^2 + 3}.$$

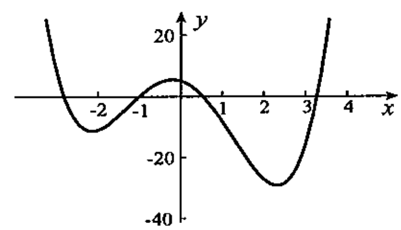
Iteration gives

$x_2 = -0.33333, \quad x_3 = -0.322\,22,$

$x_4 = -0.322\,19, \quad x_5 = -0.322\,19.$

Since $f'(-0.322\,25) = -8.6 \times 10^{-4}$ and $f(-0.322\,15) = 4.7 \times 10^{-4}$, the critical point is $x = -0.322\,2$. Since $f'(x)$ changes from negative to positive as $x$ increases through $-0.322\,2$, the critical point gives a relative minimum.

**70.** Critical points are defined by $0 = f'(x) = 4x^3 - 20x - 4$. The graph of $f(x)$ indicates that there are three critical points. To find the positive one, we use Newton's iterative procedure with




$$x_1 = 2.2, \quad x_{n+1} = x_n - \frac{x_n^3 - 5x_n - 1}{3x_n^2 - 5}.$$

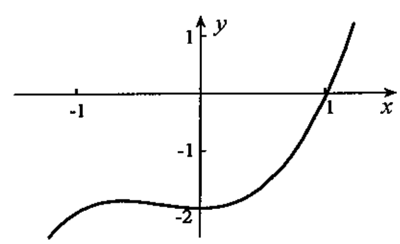
Iteration gives

$x_2 = 2.342\,02, \quad x_3 = 2.330\,15,$

$x_4 = 2.330\,06, \quad x_5 = 2.330\,06.$

Since $f'(2.330\,05) = -3.9 \times 10^{-4}$ and $f'(2.330\,15) = 4.1 \times 10^{-3}$, it follows that $2.330\,1$ is a critical point. Since $f'(x)$ changes from negative to positive as $x$ increases through this value, there is a relative minimum at this critical point. Similar procedures give a relative maximum at $-0.201\,6$ and a relative minimum at $-2.128\,4$.

**71.** Critical points are defined by $0 = f'(x) = 3x^2 + 2\sin x$. The graph of $f(x)$ indicates that there are two critical points $x = 0$ and one to the left of $x = 0$. There is a relative minimum at $x = 0$. To find the negative critical point, we use Newton's iterative procedure with




$$x_1 = -0.6, \quad x_{n+1} = x_n - \frac{3x_n^2 + 2\sin x_n}{6x_n + 2\cos x_n}.$$

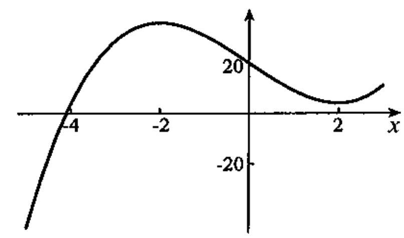
Iteration gives $x_2 = -0.625\,3$, $x_3 = -0.624\,21$, $x_4 = -0.624\,21$. Since $f'(-0.624\,25) = 8.6 \times 10^{-5}$ and $f'(-0.624\,15) = -1.3 \times 10^{-4}$, it follows that $-0.624\,2$ is a critical point. Since $f'(x)$ changes from positive to negative, there is a relative maximum at this critical point.

**72.** For critical points we solve

$$0 = f'(x) = \frac{(x^2-5x+4)^2(2x)-(x^2-4)(2)(x^2-5x+4)(2x-5)}{(x^2-5x+4)^4} = \frac{-2(x^3-12x+20)}{(x-1)^3(x-4)^3}.$$

The graph of $x^3 - 12x + 20$ to the right indicates that the only critical point is slightly less than $-4$. To find it we use Newton's iterative procedure with

$$x_1 = -4, \quad x_{n+1} = x_n - \frac{x_n^3 - 12x_n + 20}{3x_n^2 - 12}.$$



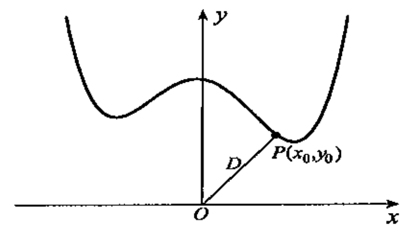

Iteration gives

$$x_2 = -4.111, \qquad x_3 = -4.107\,25,$$
$$x_4 = -4.107\,24, \quad x_5 = -4.107\,25.$$

Since $f'(-4.107\,25) = 7.4 \times 10^{-9}$ and $f'(-4.107\,15) = -1.0 \times 10^{-7}$, the critical point is $-4.107\,2$. It yields a relative maximum.

**73.** We write that $D = \sqrt{x^2 + [f(x)]^2}$. Relative extrema occur at critical points of $D$, obtained by solving

$$0 = \frac{dD}{dx} = \frac{2x + 2f(x)f'(x)}{2\sqrt{x^2+[f(x)]^2}}.$$




If $x_0$ is a critical point, then

$$2x_0 + 2f(x_0)f'(x_0) = 0 \implies f'(x_0) = \frac{-x_0}{f(x_0)} = -\frac{x_0}{y_0}.$$

But $y_0/x_0$ is the slope of $OP$, and therefore slopes of the tangent line at $P$ and the line $OP$ are negative reciprocals; that is, the lines are perpendicular.

**74.** If we differentiate the equation of the cardioid with respect to $x$, we obtain $2(x^2+y^2+x)\left(2x + 2y\dfrac{dy}{dx} + 1\right) = 2x + 2y\dfrac{dy}{dx}$. When we solve this for $dy/dx$ and set it equal to 0, we obtain $0 = \dfrac{dy}{dx} = \dfrac{(2x+1)(x^2+y^2+x) - x}{y - 2y(x^2+y^2+x)}$. We now set the numerator equal to 0, and this implies that $x^2 + y^2 + x = x/(2x+1)$. Substitution of this into the equation of the cardioid gives

$$\frac{x^2}{(2x+1)^2} = x^2 + y^2 \qquad \implies \qquad \frac{x^2}{(2x+1)^2} = \frac{x}{2x+1} - x.$$

Since $x = 0$ does not lead to a maximum for $y$, we set $x/(2x+1)^2 = 1/(2x+1) - 1$, and this equation simplifies to $4x^2 + 3x = 0$, with solution $x = -3/4$. The y-coordinate of the point on the cardioid in the second quadrant corresponding to this value of $x$ is $3\sqrt{3}/4$.

**75.** To find the points on the curve farthest from the origin, we find relative maxima of the function $D = \sqrt{x^2+y^2}$. We do this by finding its critical points,

$$0 = \frac{dD}{dx} = \frac{1}{2\sqrt{x^2+y^2}}\left(2x + 2y\frac{dy}{dx}\right) \quad \implies \quad \frac{dy}{dx} = -\frac{x}{y}.$$

By differentiating the equation of the bifolium with respect to $x$, we obtain $2(x^2+y^2)\left(2x + 2y\dfrac{dy}{dx}\right) = 2xy + x^2\dfrac{dy}{dx}$. Substitution of $dy/dx = -x/y$ into this equation gives

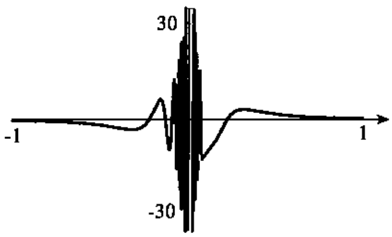
$$4(x^2+y^2)\left[x + y\left(-\frac{x}{y}\right)\right] = 2xy + x^2\left(-\frac{x}{y}\right) \quad \implies \quad \frac{x}{y}(2y^2 - x^2) = 0 \quad \implies \quad x^2 = 2y^2,$$

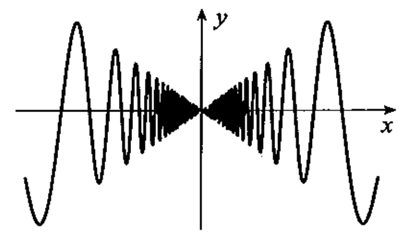
since $x = 0$ gives a relative minimum for $D$. Substitution of this into the equation of the bifolium now gives $(2y^2+y^2)^2 = (2y^2)y \implies 9y^4 = 2y^3 \implies y = 2/9$, since $y = 0$ gives minimum $D$. Thus, the points farthest from the origin are $(\pm 2\sqrt{2}/9, 2/9)$.

**76.** (a) By equation 3.2, $f'(0) = \lim_{h\to0} \dfrac{f(h)-f(0)}{h} = \lim_{h\to0} \dfrac{h\sin(1/h)}{h} = \lim_{h\to0} \sin(1/h)$, and this limit does not exist (see Example 2.9). Since $f'(0)$ does not exist, but $f(0)$ does, $x=0$ is a critical point.

(b) The plot of $f'(x) = \sin(1/x) - (1/x)\cos(1/x)$ in the left figure below clearly shows that the derivative does not change sign as $x$ increases through 0; the sign oscillates more and more rapidly the closer $x$ is to zero.

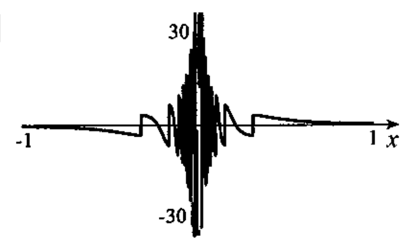
(c) No In every interval around $x=0$, the function takes on both positive and negative values. (right graph below).




**77.** (a) By equation 3.2, $f'(0) = \lim_{h\to0} \dfrac{f(h)-f(0)}{h} = \lim_{h\to0} \dfrac{|h\sin(1/h)|}{h}$, and this limit does not exist. Since the derivative does not exist at $x=0$, but the function has a value there, $x=0$ is a critical point.

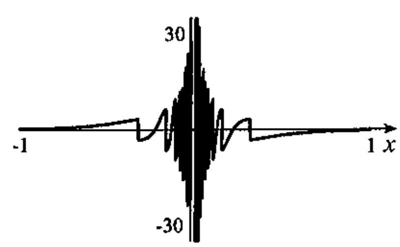
(b) The plot of $f'(x) = \dfrac{|x\sin(1/x)|}{x\sin(1/x)}[\sin(1/x) - (1/x)\cos(1/x)]$ clearly shows that the derivative does not change sign as $x$ increases through 0; the sign oscillates more and more rapidly the closer $x$ is to zero. In addition, there is an increasing number of values of $x$ at which the derivative does not exist, namely $x = 1/(n\pi)$, where $n$ is an integer.




(c) Since function values are always positive for $x \neq 0$, and $f(0)=0$, it follows that a relative minimum occurs at $x=0$.

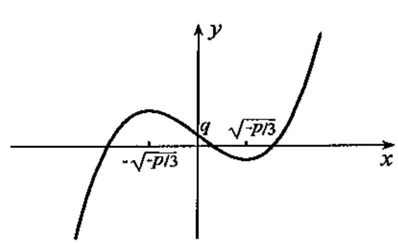
**78.** (a) By definition 3.2, $f'(0) = \lim_{h\to0} \dfrac{f(h)-f(0)}{h} = \lim_{h\to0} \dfrac{-|h\sin(1/h)|}{h}$, and this limit does not exist. Since the derivative does not exist at $x=0$, but the function has a value there, $x=0$ is a critical point.

(b) The plot of $f'(x) = -\dfrac{|x\sin(1/x)|}{x\sin(1/x)}[\sin(1/x) - (1/x)\cos(1/x)]$ clearly shows that the derivative does not change from positive to negative nor from negative to positive as $x$ increases through 0; its sign oscillates more and more rapidly the closer $x$ is to zero. In addition, there is an increasing number of values of $x$ at which the derivative does not exist, namely $x = 1/(n\pi)$, where $n$ is an integer.




(c) Since function values are always negative for $x \neq 0$, and $f(0)=0$, it follows that a relative maximum occurs at $x=0$.

**79.** If the cubic polynomial $f(x) = x^3 + px + q$ has three distinct zeros, it has a relative maximum and a relative minimum; that is, it has two critical points defined by $0 = f'(x) = 3x^2 + p$. Consequently, $p$ must be negative, and the critical points are $x = \pm\sqrt{-p/3}$. The graph of $f(x)$ must be one of the two situations shown depending on whether $q$ is positive or negative. In either case, $f(\sqrt{-p/3})$ must be negative; that is,

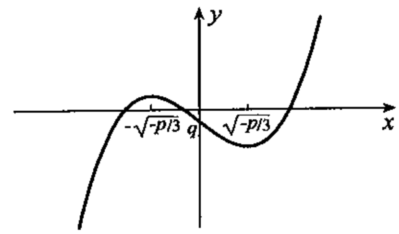


$$0 > \left(\sqrt{\frac{-p}{3}}\right)^3 + p\sqrt{\frac{-p}{3}} + q = \frac{2p}{3}\sqrt{\frac{-p}{3}} + q.$$

Thus, $q < -\frac{2p}{3}\sqrt{\frac{-p}{3}}$. When $q > 0$ the square of this gives $q^2 < \frac{4p^2}{9}\left(\frac{-p}{3}\right) \Longrightarrow 4p^3 + 27q^2 < 0$.

A similar proof can be given when $q < 0$.

Conversely, if $4p^3 + 27q^2 < 0$, then $p$ must be negative. As a result, $f(x) = x^3 + px + q$ has two critical points $x = \pm\sqrt{-p/3}$. If $q < 0$, the value of $f(x)$ at $x = \sqrt{-p/3}$ must be negative, and that at $x = -\sqrt{-p/3}$ is




$$f(-\sqrt{-p/3}) = \left(-\sqrt{\frac{-p}{3}}\right)^3 + p\left(-\sqrt{\frac{-p}{3}}\right) + q = \sqrt{\frac{-p}{3}}\left(-p + \frac{p}{3}\right) + q = -\frac{2p}{3}\sqrt{\frac{-p}{3}} + q.$$

Because $4p^3 + 27q^2 < 0$, it follows that

$$q^2 < -\frac{4p^3}{27} \Longrightarrow -q < \sqrt{\frac{-4p^3}{27}} = \sqrt{-\frac{p}{3}\left(\frac{-2p}{3}\right)^2} = -\frac{2p}{3}\sqrt{\frac{-p}{3}} \Longrightarrow q - \frac{2p}{3}\sqrt{\frac{-p}{3}} > 0.$$

Thus, $f(-\sqrt{-p/3}) > 0$. With a positive relative maximum at $x = -\sqrt{-p/3}$ and a negative relative minimum at $x = \sqrt{-p/3}$, the graph of $y = f(x)$ must cross the $x$-axis at three distinct points. A similar proof works when $q > 0$.

## EXERCISES 4.4

1. Since $f''(x) = 6x - 6 = 6(x-1)$, it follows that $f''(x) \leq 0$ for $x \leq 1$, and $f''(x) \geq 0$ for $x \geq 1$. Consequently, the graph of the function is concave downward on the interval $x \leq 1$, and concave upward on the interval $x \geq 1$. The point $(1, 0)$ that separates these intervals is a point of inflection.

2. Since $f''(x) = 36x^2 + 24x = 12x(3x+2)$, it follows that $f''(x) \leq 0$ for $-2/3 \leq x \leq 0$, and $f''(x) \geq 0$ for $x \leq -2/3$ and $x \geq 0$. Consequently, the graph of the function is concave downward on the interval $-2/3 \leq x \leq 0$, and concave upward on the intervals $x \leq -2/3$ and $x \geq 0$. The points $(-2/3, 470/27)$ and $(0, 2)$ that separate these intervals are points of inflection.

3. Since $f''(x) = 2 + 6/x^4$, the second derivative is always positive except at $x = 0$ where it does not exist. The graph is therefore concave upward for $x < 0$ and $x > 0$.

4. Since $f'(x) = \frac{(x^2-1)(2x) - (x^2+4)(2x)}{(x^2-1)^2} = \frac{-10x}{(x^2-1)^2}$, the second derivative vanishes when

$$0 = f''(x) = -10\left[\frac{(x^2-1)^2(1) - x(2)(x^2-1)(2x)}{(x^2-1)^4}\right] = \frac{10(3x^2+1)}{(x^2-1)^3}.$$

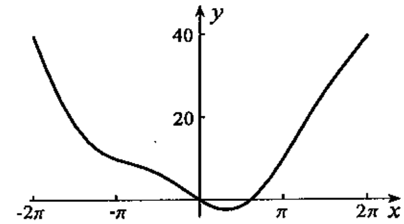
Since $f''(x) > 0$ for $x < -1$ and for $x > 1$, and $f''(x) < 0$ for $-1 < x < 1$, the graph is concave upward on the intervals $x < -1$ and $x > 1$, and concave downward on $-1 < x < 1$. Since $f(1)$ and $f(-1)$ are not defined, there are no points of inflection.

5. Since $f''(x) = -\cos x$, and the cosine function is positive in quadrants one and four, and negative in quadrants two and three, we can say that the graph is concave downward on the intervals $-2\pi < x \leq -3\pi/2$, $-\pi/2 \leq x \leq \pi/2$, and $3\pi/2 \leq x < 2\pi$. It is concave upward on the intervals $-3\pi/2 \leq x \leq -\pi/2$ and $\pi/2 \leq x \leq 3\pi/2$. Points $(-3\pi/2, -3\pi/2)$, $(-\pi/2, -\pi/2)$, $(\pi/2, \pi/2)$ and $(3\pi/2, 3\pi/2)$ that separate these intervals are points of inflection.

6. Since $f''(x) = 2 + \sin x$, it follows that $f''(x) > 0$ for all $x$. Hence the graph is always concave upward.

7. Since $f''(x) = 2 + 2\sin x$, it follows that $f''(x) \geq 0$ for all $x$. Hence the graph is always concave upward.

**8.** Since $f''(x) = 2 + 4\sin x$, the second derivative vanishes when $\sin x = -1/2$. The angles in the interval $|x| < 2\pi$ that satisfy this equation are $x = -5\pi/6$, $-\pi/6$, $7\pi/6$, and $11\pi/6$. The graph of the function indicates that the graph is concave upward on the intervals $-2\pi < x \le -5\pi/6$, $-\pi/6 \le x \le 7\pi/6$, and $11\pi/6 \le x < 2\pi$; it is concave downward on $-5\pi/6 \le x \le -\pi/6$ and $7\pi/6 \le x \le 11\pi/6$. The points which separate these intervals are points of inflection, namely, $(-5\pi/6, 2 + 25\pi^2/36)$, $(-\pi/6, 2 + \pi^2/36)$, $(7\pi/6, 2 + 49\pi^2/36)$, and $(11\pi/6, 2 + 121\pi^2/36)$.




**9.** Since $f''(x) = d/dx(\ln x + 1) = 1/x$, it follows that $f''(x) > 0$ for all $x > 0$. Hence the graph is concave upward for $x > 0$.

**10.** Since $f'(x) = 2x\ln x + x$, we obtain $f''(x) = 2\ln x + 3$. It follows that $f''(x) = 0$ when $x = e^{-3/2}$. Since $f''(x) \ge 0$ for $x \ge e^{-3/2}$, and $f''(x) \le 0$ for $0 < x \le e^{-3/2}$, it follows that the graph is concave downward on the interval $0 < x \le e^{-3/2}$, and it is concave upward for $x \ge e^{-3/2}$. The point $(e^{-3/2}, -3/(2e^3))$ which separates these intervals is a point of inflection.

**11.** Since $f'(x) = e^{1/x}(-1/x^2)$, we obtain $f''(x) = e^{1/x}(1/x^4 + 2/x^3) = e^{1/x}(1 + 2x)/x^4$. It follows that $f''(x) = 0$ when $x = -1/2$. Since $f''(x) \ge 0$ for $-1/2 \le x < 0$ and $x > 0$, and $f''(x) \le 0$ for $x \le -1/2$, it follows that the graph is concave upward on the intervals $-1/2 \le x < 0$ and $x > 0$, and it is concave downward for $x \le -1/2$. The point $(-1/2, 1/e^2)$ is a point of inflection.

**12.** Since $f'(x) = e^{-2x} - 2x\,e^{-2x}$, we find that $f''(x) = -4\,e^{-2x} + 4x\,e^{-2x} = 4(x-1)\,e^{-2x}$. Because $f''(x) \le 0$ for $x \le 1$, and $f''(x) \ge 0$ for $x \ge 1$, the graph is concave downward for $x \le 1$, and concave upward for $x \ge 1$. The point of inflection is $(1, e^{-2})$.

**13.** Since $f'(x) = 2x\,e^{3x} + 3x^2\,e^{3x}$, we find that $f''(x) = 2\,e^{3x} + 6x\,e^{3x} + 6x\,e^{3x} + 9x^2\,e^{3x} = (2 + 12x + 9x^2)\,e^{3x}$. Because $f''(x) = 0$ for $x = (-12 \pm \sqrt{144 - 72})/18 = (-2 \pm \sqrt{2})/3$, it follows that $f''(x) \ge 0$ for $x \le (-2-\sqrt{2})/3$ and $x \ge (-2+\sqrt{2})/3$, and $f''(x) \le 0$ for $(-2-\sqrt{2})/3 \le x \le (-2+\sqrt{2})/3$. Consequently, the graph is concave upward for $x \le (-2 - \sqrt{2})/3$ and $x \ge (-2 + \sqrt{2})/3$, and concave downward for $(-2 - \sqrt{2})/3 \le x \le (-2 + \sqrt{2})/3$. The points $\big((-2 - \sqrt{2})/3, 0.0426\big)$ and $\big((-2 + \sqrt{2})/3, 0.0212\big)$ that separate these parts of the graph are points of inflection.

**14.** Since $f'(x) = 2x + e^{-x}$, we obtain $f''(x) = 2 - e^{-x}$. It follows that $f''(x) = 0$ when $x = -\ln 2$. Because $f''(x) \le 0$ when $x \le -\ln 2$, and $f''(x) \ge 0$ when $x \ge -\ln 2$, the graph is concave downward on the interval $x \le -\ln 2$ and concave upward on $x \ge -\ln 2$. The point $(-\ln 2, (\ln 2)^2 - 2)$ separating these intervals is a point of inflection.

**15.** For critical points we solve $0 = f'(x) = 3x^2 - 6x - 3 = 3(x^2 - 2x - 1)$ for $x = (2 \pm \sqrt{4 + 4})/2 = 1 \pm \sqrt{2}$. Since $f''(x) = 6x - 6 = 6(x - 1)$, we find that $f''(1 + \sqrt{2}) = 6\sqrt{2}$ and $f''(1 - \sqrt{2}) = -6\sqrt{2}$. Consequently, $x = 1 + \sqrt{2}$ gives a relative minimum and $x = 1 - \sqrt{2}$ gives a relative maximum.

**16.** Since $f'(x) = 1 - 1/x^2$, critical points occur at $x = \pm 1$. With $f''(x) = 2/x^3$, we calculate that $f''(-1) = -2$ and $f''(1) = 2$. Hence, $x = -1$ yields a relative maximum and $x = 1$ a relative minimum.

**17.** Since $f'(x) = 12x^3 - 48x^2 + 36x = 12x(x - 1)(x - 3)$, critical points are $x = 0, 1, 3$. With $f''(x) = 36x^2 - 96x + 36 = 12(3x^2 - 8x + 3)$, we find that $f''(0) = 36$, $f''(1) = -24$, and $f''(3) = 72$. Consequently, $x = 0, 3$ yield relative minima and $x = 1$ gives a relative maximum.

**18.** Since $f'(x) = (5/4)x^{1/4} - 1/4x^{-3/4} = \dfrac{5x - 1}{4x^{3/4}}$, the only critical point at which $f'(x) = 0$ is $x = 1/5$. Because $f''(x) = (5/16)x^{-3/4} + (3/16)x^{-7/4} = \dfrac{5x + 3}{16x^{7/4}}$, it follows that $f''(1/5) > 0$, and therefore $x = 1/5$ yields a relative minimum.

**19.** Since $f'(x) = \ln x + 1$, the only critical point at which $f'(x) = 0$ is $x = 1/e$. With $f''(x) = 1/x$, we find that $f''(1/e) = e$, and $x = 1/e$ gives a relative minimum.

**20.** Since $f'(x) = 2x\ln x + x = x(2\ln x + 1)$, the only critical point at which $f'(x) = 0$ is $x = 1/\sqrt{e}$. Because $f''(x) = 2\ln x + 3$, it follows that $f''(1/\sqrt{e}) = 2(-1/2) + 3 = 2 > 0$, and the critical point yields a relative minimum.

**21.** Since $f'(x) = e^{2x} + 2x\,e^{2x} = (1+2x)\,e^{2x}$, the only critical point is $x = -1/2$. With $f''(x) = 2\,e^{2x} + 2(1+2x)\,e^{2x} = 4(1+x)\,e^{2x}$, we find that $f''(-1/2) = 2e^{-1}$, and therefore $x = -1/2$ gives a relative minimum.

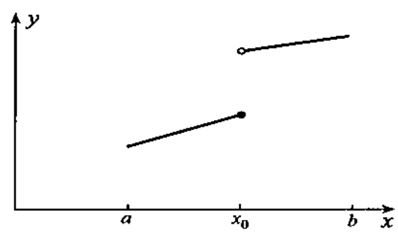
**22.** Since $f'(x) = 2xe^{-2x} - 2x^2e^{-x} = 2x(1-x)e^{-2x}$, critical points at which $f'(x) = 0$ are $x = 0$ and $x = 1$. Because $f''(x) = (2-4x)e^{-2x} - 2(2x - 2x^2)e^{-2x} = (2 - 8x + 4x^2)e^{-2x}$, it follows that $f''(0) = 2$ and $f''(1) = -2e^{-2}$. Hence, $x = 0$ gives a relative minimum and $x = 1$ gives a relative maximum.

**23.** If $f''(x) = 0$ on an interval, then $f(x) = Ax + B$ on that interval. The graph is a straight line and the slope is constant. Hence the graph is neither concave upward nor concave downward.

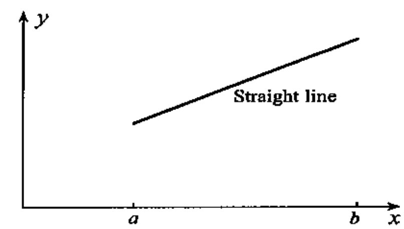
**24.** For points of inflection we solve $0 = \dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(\sin x + x\cos x) = 2\cos x - x\sin x$. If $(x_0, y_0)$ is a point of inflection, then $x_0$ satisfies $0 = 2\cos x_0 - x_0 \sin x_0$, and $y_0$ is given by $y_0 = x_0 \sin x_0$. It follows that $0 = 2\cos x_0 - y_0 \Longrightarrow y_0 = 2\cos x_0$. In other words, $(x_0, y_0)$ is on the curve $y = 2\cos x$.

**25.** The second derivative of $f(x) = ax^3 + bx^2 + cx + d$ is $f''(x) = 6ax + 2b$, a linear equation with one solution $x = -b/(3a)$. Since $f''(x)$ will change sign as $x$ increases through this value, it will yield a point of inflection.

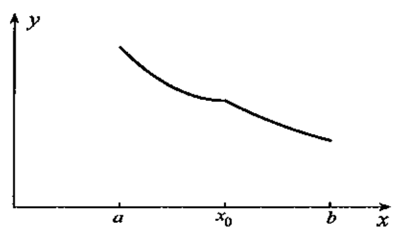
**26.**



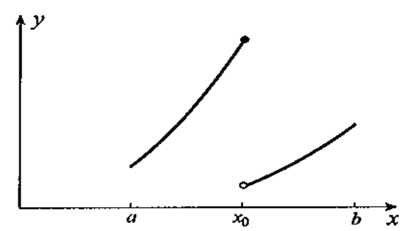
**27.**



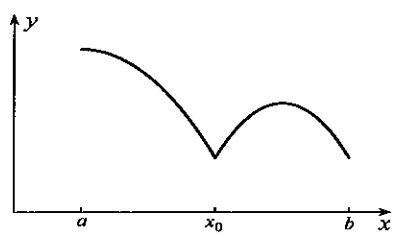

**28.**



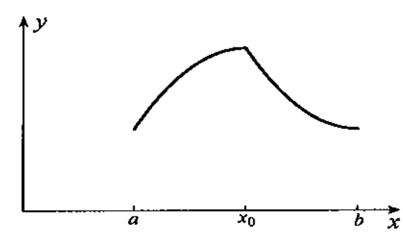
**29.**



**30.**



**31.**



**32.** According to Example 3.19 in Section 3.5, $[f(x)g(x)]'' = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)$. Since the graphs of $f(x)$ and $g(x)$ are concave upward on $I$, we can state that $f''(x) \ge 0$ and $g''(x) \ge 0$ on $I$. This does not, however, guarantee that $[f(x)g(x)]'' \ge 0$ on $I$.

**33.** According to Example 3.19 in Section 3.5, $[f(x)g(x)]'' = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)$. Since the graphs of $f(x)$ and $g(x)$ are concave upward on $I$, we can state that $f''(x) \ge 0$ and $g''(x) \ge 0$ on $I$. Since $f(x)$ and $g(x)$ are increasing on $I$, we can also say that $f'(x) \ge 0$ and $g'(x) \ge 0$ on $I$. These do not, however, guarantee that $[f(x)g(x)]'' \ge 0$ on $I$.

**34.** For a horizontal point of inflection we set first and second derivatives equal to 0,

$$
\begin{aligned}
0 = P'(V) &= \frac{-RT}{(V-b)^2}e^{-a/(RT^{3/2}V)} + \frac{RT}{V-b}e^{-a/(RT^{3/2}V)}\left(\frac{a}{RT^{3/2}V^2}\right) \\
&= RTe^{-a/(RT^{3/2}V)}\left[\frac{-1}{(V-b)^2} + \frac{a}{RT^{3/2}V^2(V-b)}\right] \\
&= \frac{RTe^{-a/(RT^{3/2}V)}}{RT^{3/2}V^2(V-b)^2}[a(V-b) - RT^{3/2}V^2],
\end{aligned}
$$

$$0 = P''(V) = RTe^{-a/(RT^{3/2}V)}\left(\frac{a}{RT^{3/2}V^2}\right)\left[\frac{-1}{(V-b)^2} + \frac{a}{RT^{3/2}V^2(V-b)}\right]$$
$$+ RTe^{-a/(RT^{3/2}V)}\left[\frac{2}{(V-b)^3} - \frac{2a}{RT^{3/2}V^3(V-b)} - \frac{a}{RT^{3/2}V^2(V-b)^2}\right].$$

Because $P'(V) = 0$, the term in the first set of brackets of $P''(V)$ vanishes, and we therefore set

$$0 = \frac{RTe^{-a/(RT^{3/2}V)}}{RT^{3/2}V^3(V-b)^3}[2RT^{3/2}V^3 - 2a(V-b)^2 - aV(V-b)].$$

From $0 = P'(V)$, we obtain $\quad V - b = \dfrac{RT^{3/2}V^2}{a},\quad$ which we substitute into $0 = P''(V)$,

$$0 = 2RT^{3/2}V^3 - 2a\left(\frac{R^2T^3V^4}{a^2}\right) - aV\left(\frac{RT^{3/2}V^2}{a}\right) = \frac{RT^{3/2}V^3}{a}(2a - 2RT^{3/2}V - a)$$
$$= \frac{RT^{3/2}V^3}{a}(a - 2RT^{3/2}V).$$

Thus, $a = 2RT^{3/2}V$. It now follows that

$$b = V - \frac{RT^{3/2}V^2}{2RT^{3/2}V} = \frac{V}{2}.$$

We now have $a$ and $b$ in terms of $T$ and $V$. To replace $V$ with $P$, we use Dieterici's equation to write

$$P = \frac{RT}{V-b}e^{-a/(RT^{3/2}V)}.$$

When we substitute the expressions for $a$ and $b$ into this equation we obtain

$$P = \frac{RT}{V - V/2}e^{-2RT^{3/2}V/(RT^{3/2}V)} = \frac{2RT}{e^2V} \quad\Longrightarrow\quad V = \frac{2RT}{e^2P}.$$

This then gives

$$a = 2RT^{3/2}\left(\frac{2RT}{e^2P}\right) = \frac{4R^2T^{5/2}}{e^2P}, \qquad b = \frac{RT}{e^2P};$$

that is, expressions for $a$ and $b$ in terms of critical temperature and pressure are

$$a = \frac{4R^2T_c^{5/2}}{e^2P_c}, \qquad b = \frac{RT_c}{e^2P_c}.$$

**35.** (a) Since $f'(x) = 6x(x^2-1)^2$, critical points are $x = 0$ and $x = \pm 1$. We now calculate $f''(x) = 6(x^2-1)^2 + 24x^2(x^2-1)$ and $f'''(x) = 24x(x^2-1) + 96x^3 - 48x$. Because $f''(0) = 6$, it follows that $x = 0$ yields a relative minimum. Since $f''(\pm 1) = 0$ and $f'''(\pm 1) \neq 0$, it follows that these critical points give horizontal points of inflection.

(b) Since $f'(x) = 2x\sqrt{1-x} - \dfrac{x^2}{2\sqrt{1-x}} = \dfrac{x(4-5x)}{2\sqrt{1-x}}$, critical points at which $f'(x) = 0$ are $x = 0$ and $x = 4/5$. We now calculate

$$f''(x) = \frac{2\sqrt{1-x}(4-10x) - (4x-5x^2)(1-x)^{-1/2}(-1)}{4(1-x)} = \frac{8 - 24x + 15x^2}{4(1-x)^{3/2}}.$$

Because $f''(0) = 2$, and $f''(4/5) = \dfrac{8 - 24(4/5) + 15(4/5)^2}{4(1-4/5)^{3/2}} < 0$, we conclude that $x = 0$ gives a relative minimum and $x = 4/5$ gives a relative maximum.

**36.** $f'(x) = \dfrac{(x^2+k^2)(-1)-(k-x)(2x)}{(x^2+k^2)^2} = \dfrac{x^2-2kx-k^2}{(x^2+k^2)^2}$ For points of inflection we solve

$$0 = f''(x) = \frac{(x^2+k^2)^2(2x-2k)-(x^2-2kx-k^2)(2)(x^2+k^2)(2x)}{(x^2+k^2)^4},$$

and this simplifies to $0 = \dfrac{-2(x+k)(x^2-4kx+k^2)}{(x^2+k^2)^3}$. Thus, $x=-k$ and $x=(4k\pm\sqrt{16k^2-4k^2})/2 = (2\pm\sqrt{3})k$. Since $f''(x)$ changes sign as $x$ passes through each of these values, each gives a point of inflection, and these points are

$$P\left(-k, \frac{1}{k}\right), \quad Q\left((2+\sqrt{3})k, \frac{1-\sqrt{3}}{4k}\right), \quad R\left((2-\sqrt{3})k, \frac{1+\sqrt{3}}{4k}\right).$$

These points are collinear if the slopes of line segments $PR$ and $PQ$ are equal. This is indeed the case since

$$\text{Slope of } PR = \frac{\dfrac{1+\sqrt{3}}{4k}-\dfrac{1}{k}}{(2-\sqrt{3})k+k} = \frac{1+\sqrt{3}-4}{4k^2(2-\sqrt{3}+1)} = -\frac{1}{4k^2},$$
$$\text{Slope of } PQ = \frac{\dfrac{1-\sqrt{3}}{4k}-\dfrac{1}{k}}{(2+\sqrt{3})k+k} = \frac{1-\sqrt{3}-4}{4k^2(2+\sqrt{3}+1)} = -\frac{1}{4k^2}.$$

**37.** If the cubic polynomial is $y=f(x)=ax^3+bx^2+cx+d$, then critical points are defined by $0=f'(x)=3ax^2+2bx+c$. Solutions of this equation are

$$x = \frac{-2b\pm\sqrt{4b^2-12ac}}{6a} = \frac{-b\pm\sqrt{b^2-3ac}}{3a},$$

and $b^2-3ac$ must be positive since these values must yield the relative extrema. If we set $x_1 = (-b+\sqrt{b^2-3ac})/(3a)$ and $x_2=(-b-\sqrt{b^2-3ac})/(3a)$, the relative extrema are at $P\big(x_1, f(x_1)\big)$ and $Q\big(x_2, f(x_2)\big)$. For the point of inflection we solve $0=f''(x)=6ax+2b$, and obtain $x=-b/(3a)$. The $y$-coordinate of the point of inflection is

$$y = a\left(\frac{-b}{3a}\right)^3 + b\left(\frac{-b}{3a}\right)^2 + c\left(\frac{-b}{3a}\right) + d = \frac{2b^3-9abc+27a^2d}{27a^2}.$$

We now show that the midpoint R of line segment $PQ$ has the same coordinates. The $x$-coordinate of $R$ is

$$\frac{1}{2}(x_1+x_2) = \frac{1}{2}\left[\frac{-b+\sqrt{b^2-3ac}}{3a}+\frac{-b-\sqrt{b^2-3ac}}{3a}\right] = -\frac{b}{3a}.$$

The $y$-coordinate of $R$ is

$$\begin{aligned}
\frac{1}{2}[f(x_1)+f(x_2)] &= \frac{1}{2}\left[a\left(\frac{-b+\sqrt{b^2-3ac}}{3a}\right)^3 + b\left(\frac{-b+\sqrt{b^2-3ac}}{3a}\right)^2 + c\left(\frac{-b+\sqrt{b^2-3ac}}{3a}\right)+d\right.\\
&\qquad \left. + a\left(\frac{-b-\sqrt{b^2-3ac}}{3a}\right)^3 + b\left(\frac{-b-\sqrt{b^2-3ac}}{3a}\right)^2 + c\left(\frac{-b-\sqrt{b^2-3ac}}{3a}\right)+d\right]\\
&= \frac{1}{54a^2}\left[-b^3+3b^2\sqrt{b^2-3ac}-3b(b^2-3ac)+(b^2-3ac)^{3/2}\right.\\
&\qquad \left. -b^3-3b^2\sqrt{b^2-3ac}-3b(b^2-3ac)-(b^2-3ac)^{3/2}\right]\\
&\quad + \frac{b}{18a^2}\left[b^2-2b\sqrt{b^2-3ac}+b^2-3ac+b^2+2b\sqrt{b^2-3ac}+b^2-3ac\right]
\end{aligned}$$

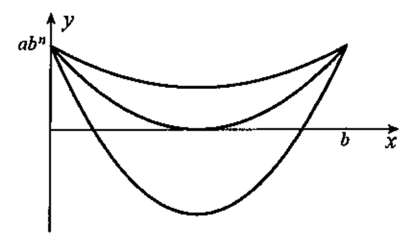
$$+\frac{c}{6a}\left[-b+\sqrt{b^2-3ac}-b-\sqrt{b^2-3ac}\right]+d$$
$$=\frac{1}{54a^2}\left[-8b^3+18abc\right]+\frac{b}{18a^2}\left[4b^2-6ac\right]-\frac{bc}{3a}+d$$
$$=\frac{-4b^3+9abc+6b^3-9abc-9abc+27a^2d}{27a^2}$$
$$=\frac{2b^3-9abc+27a^2d}{27a^2}.$$

**38.** For critical points of $f(x)$, we solve $0=f'(x)=(1-\cos x)\cos(x-\sin x)$. If we set $1-\cos x=0$, we obtain $x=2n\pi$, where $n$ is an integer. (Other solutions can be obtained by setting $\cos(x-\sin x)=0$, but we can answer the question without looking at this equation.) The second derivative is $f''(x)=-(1-\cos x)^2\sin(x-\sin x)+\sin x\cos(x-\sin x)$, and $f''(2n\pi)=0$. The third derivative is

$$f'''(x)=-(1-\cos x)^3\cos(x-\sin x)-2(1-\cos x)\sin x\sin(x-\sin x)$$
$$+\cos x\cos(x-\sin x)-\sin x(1-\cos x)\sin(x-\sin x).$$

Since $f'''(2n\pi)=1$, Exercise 35 implies that there are horizontal points of inflection when $x=2n\pi$.

**39.** First we note that $f(0)=f(b)=ab^n>0$. Next, $f'(x)=(n+1)x^n-b^n$ and $f''(x)=(n+1)nx^{n-1}$. Since $f''(x)>0$ for all $x>0$, the graph of $f(x)$ must be concave upward for all $x>0$. This means that it must be one of the three situations shown to the right. There are exactly two solutions of $f(x)=0$ if and only if the relative minimum is negative. The critical point is given by




$$0=f'(x)=(n+1)x^n-b^n \implies x=\frac{b}{(n+1)^{1/n}}.$$

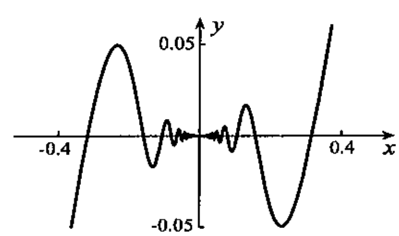
Thus, we require

$$0>f\left(\frac{b}{(n+1)^{1/n}}\right)=\left[\frac{b}{(n+1)^{1/n}}\right]^{n+1}-b^n\left[\frac{b}{(n+1)^{1/n}}\right]+ab^n=b^n\left[\frac{b}{(n+1)^{(n+1)/n}}-\frac{b}{(n+1)^{1/n}}+a\right].$$

This is equivalent to

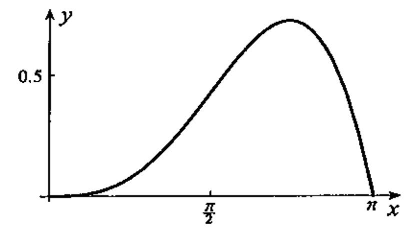
$$a<\frac{b}{(n+1)^{1/n}}-\frac{b}{(n+1)^{(n+1)/n}}=\frac{b}{(n+1)^{(n+1)/n}}(n+1-1)=\frac{bn}{(n+1)^{(n+1)/n}}.$$

**40.** (a) The graph is shown to the right.
(b) Since $f'(x)$ is not continuous at $x=0$ (Exercise 47 in Section 3.9), it follows that $f''(0)$ cannot exist (see the corollary to Theorem 3.6 in Section 3.3).
(c) Since $f(x)$ takes on positive and negative values in every interval around $x=0$, there cannot be a relative maximum or minimum at $x=0$. Since $f'(x)$ takes on negative and and positive values in every interval around $x=0$, the graph cannot change concavity at $x=0$, and therefore $(0,0)$ cannot be a point of inflection.



**41.** A graph of the function $f(\theta) = 2\sin\theta - \theta(1+\cos\theta)$ certainly suggests that this is the case. Consider the following verification. The first two derivatives of $f(\theta)$ are

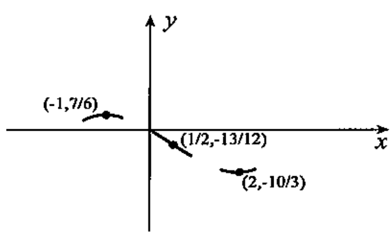
$$f'(\theta) = \theta\sin\theta + \cos\theta - 1,$$
$$f''(\theta) = \theta\cos\theta.$$

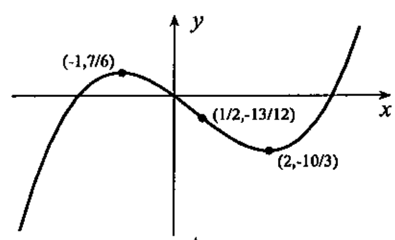



The tangent line at $\theta = 0$ is horizontal. Since $f''(\theta) > 0$ for $0 < \theta < \pi/2$, the graph is concave upward on this interval, and this implies that $f(\theta)$ cannot be equal to zero in this interval. On $\pi/2 < \theta < \pi$, we see that $f''(\theta) < 0$ so that the graph is concave downward on this interval. Consequently, $f(\theta)$ could have at most one zero in this interval. Since $f(\pi) = 0$, this is impossible. Hence, $f(\theta)$ cannot be equal to zero on the interval $0 < \theta < \pi$.

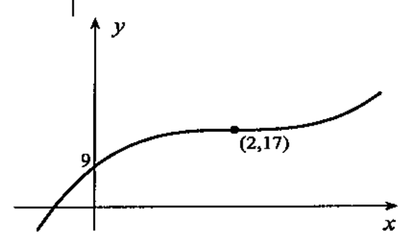
## EXERCISES 4.5

**1.** Since $f'(x) = x^2 - x - 2 = (x-2)(x+1)$, the critical points are $x = -1, 2$. With $f''(x) = 2x - 1$, we find that $f''(-1) = -3$ and $f''(2) = 3$. Consequently, $x = -1$ gives a relative maximum of $f(-1) = 7/6$ and $x = 2$ gives a relative minimum of $f(2) = -10/3$. Since $f''(1/2) = 0$ and $f''(x)$ changes sign as $x$ passes through $1/2$, there is a point of inflection at $(1/2, -13/12)$. This information is shown in the left figure below. We complete the graph of this cubic polynomial as shown in the right figure.

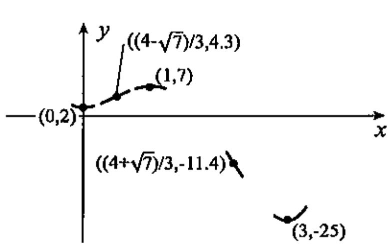


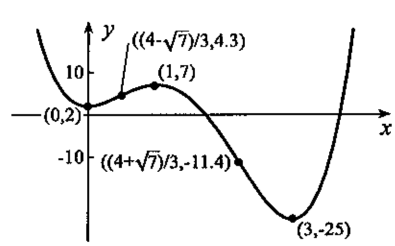




**2.** Since $f'(x) = 3x^2 - 12x + 12 = 3(x-2)^2$, the only critical point is $x = 2$. Because $f'(x)$ remains positive as $x$ passes through 2, the critical point gives a horizontal point of inflection at $(2, 17)$. Since $f''(x) = 6(x-2)$, there are no other points of inflection. The graph is shown to the right.

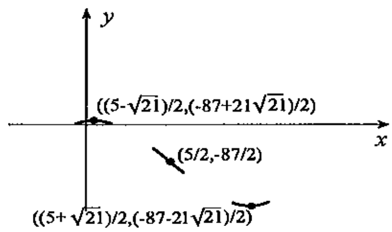


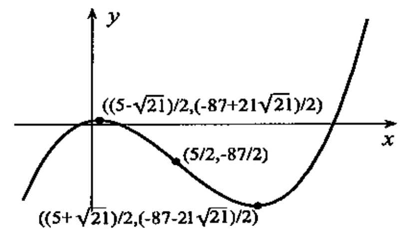

**3.** Since $f'(x) = 12x^3 - 48x^2 + 36x = 12x(x-1)(x-3)$, the critical points are $x = 0, 1, 3$. With $f''(x) = 36x^2 - 96x + 36 = 12(3x^2 - 8x + 3)$, we find that $f''(0) = 36$, $f''(1) = -24$, and $f''(3) = 72$. Consequently, $x = 0$ and $x = 3$ give relative minima of $f(0) = 2$ and $f(3) = -25$, and $x = 1$ gives a relative maximum of $f(1) = 7$. Since $f''(x) = 0$ when $x = (8 \pm \sqrt{64-36})/6 = (4 \pm \sqrt{7})/3$, and $f''(x)$ changes sign as $x$ passes through each of these, there are points of inflection at $\big((4-\sqrt{7})/3, 4.3\big)$ and $\big((4+\sqrt{7})/3, -11.4\big)$. This information is shown in the left figure below. We complete the graph of this quartic polynomial as shown in the right figure.



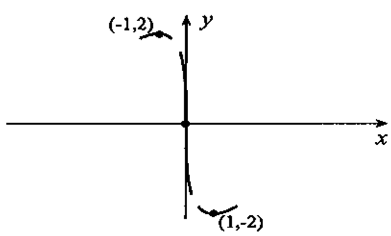



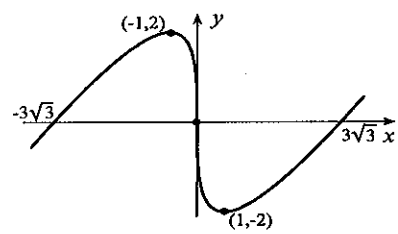
**4.** Since $f'(x) = 6x^2 - 30x + 6 = 6(x^2 - 5x + 1)$, the critical points are $x = (5 \pm \sqrt{25-4})/2 = (5 \pm \sqrt{21})/2$. Since $f'(x)$ changes from positive to negative as $x$ increases through $(5 - \sqrt{21})/2$, there is a relative maximum at $\left((5 - \sqrt{21})/2, (-87 + 21\sqrt{21})/2\right)$. Similarly, there is a relative minimum at $\left((5 + \sqrt{21})/2, (-87 - 21\sqrt{21})/2\right)$. Since $0 = f''(x) = 12x - 30$ at $x = 5/2$, and $f''(x)$ changes sign as $x$ passes through $5/2$, there is a point of inflection at $(5/2, -87/2)$. This information is shown in the left figure below. We complete the graph of this cubic polynomial as shown in the right figure.




**5.** For critical points we solve $0 = f'(x) = 1 - x^{-2/3} = \dfrac{x^{2/3} - 1}{x^{2/3}}$. Solutions are $x = \pm 1$. Since $f'(0)$ is undefined, but $f(0) = 0$, $x = 0$ is also a critical point. With $f''(x) = (2/3)x^{-5/3}$, we find that $f''(-1) = -2/3$ and $f''(1) = 2/3$. Thus, $x = -1$ yields a relative maximum of $f(-1) = 2$ and $x = 1$ gives a relative minimum of $f(1) = -2$. Because $f''(x) < 0$ for $x < 0$, and $f''(x) > 0$ for $x > 0$, it follows that $(0, 0)$ must be a point of inflection. Since $\lim_{x \to 0} f'(x) = -\infty$, $(0, 0)$ is a vertical point of inflection. This information is shown in the left figure below. The final graph is shown to the right.

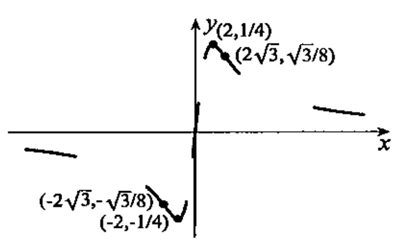


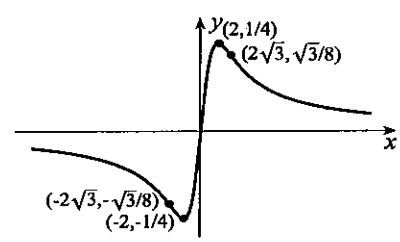




**6.** For critical points we solve $0 = f'(x) = \dfrac{(x^2+4)(1) - x(2x)}{(x^2+4)^2} = \dfrac{4 - x^2}{(x^2+4)^2}$. Solutions are $x = \pm 2$. We now calculate $f''(x) = \dfrac{(x^2+4)^2(-2x) - (4-x^2)(2)(x^2+4)(2x)}{(x^2+4)^4} = \dfrac{2x(x^2-12)}{(x^2+4)^3}$. Since $f''(-2) = 1/16$ and $f''(2) = -1/16$, there is a relative minimum at $x = -2$ equal to $f(-2) = -1/4$, and a relative maximum at $x = 2$ of $f(2) = 1/4$.

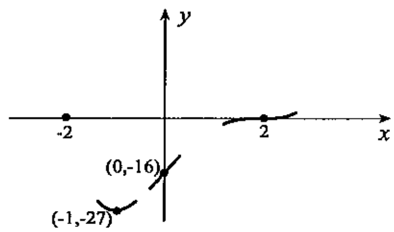
Because $f''(x) = 0$ at $x = 0, \pm 2\sqrt{3}$, and $f''(x)$ changes sign as $x$ passes through each of these, points of inflection occur at $(0, 0)$, $(2\sqrt{3}, \sqrt{3}/8)$, and $(-2\sqrt{3}, -\sqrt{3}/8)$.

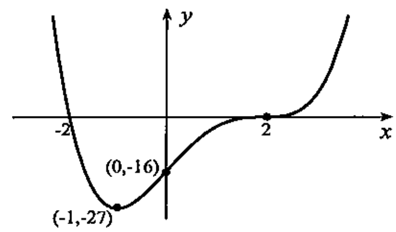
This information is shown in the left diagram below, together with the limits $\lim_{x \to -\infty} f(x) = 0^-$ and $\lim_{x \to \infty} f(x) = 0^+$. The final graph is shown to the right. We could have shortened the analysis by considering only the right half of the graph and using the fact that the function is odd.



**7.** Since $f'(x) = 3(x-2)^2(x+2) + (x-2)^3 = 4(x-2)^2(x+1)$, the critical points are $x = -1, 2$. Since $f'(x)$ changes from negative to positive as $x$ increases through $-1$, there is a relative minimum at $(-1, -27)$. The derivative does not change sign at $x = 2$. We now calculate $f''(x) = 8(x-2)(x+1) + 4(x-2)^2 = 12x(x-2)$. Since $f''(x) = 0$ when $x = 0, 2$, and $f''(x)$ changes sign as $x$ passes through these values, there is a point of inflection at $(0, -16)$ and a horizontal point of inflection at $(2, 0)$. This information is shown in the left figure below. The final graph is shown to the right.



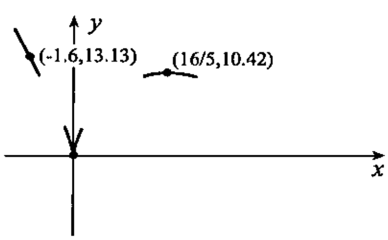

**8.** For critical points we solve $0 = f'(x) = (2/3)x^{-1/3}(8-x) - x^{2/3} = \dfrac{16-5x}{3x^{1/3}}$. Clearly, $x = 16/5$ is critical, but so also is $x = 0$ because $f'(0)$ does not exist and $f(0) = 0$. We now calculate

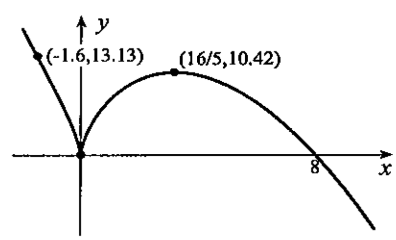
$$f''(x) = \frac{3x^{1/3}(-5) - (16-5x)x^{-2/3}}{9x^{2/3}} = \frac{-2(5x+8)}{9x^{4/3}}.$$

Since $f''(16/5) < 0$, there is a relative maximum of $f(16/5) = 10.42$. Because $f'(x)$ changes from negative to positive as $x$ increases through 0, this critical point gives a relative mimumum of $f(0) = 0$.

Since $f''(-8/5) = 0$, and $f''(x)$ changes sign as $x$ passes through $-8/5$, there is a point of inflection at $(-1.6, 13.13)$.

We have shown this information in the left figure below along with the additional facts that $\lim_{x\to 0^-} f'(x) = -\infty$ and $\lim_{x\to 0^+} f'(x) = \infty$. The final graph is shown to the right.



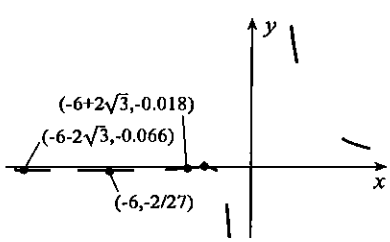

**9.** For critical points we solve $0 = f'(x) = \dfrac{x^3(2)(x+2) - (x+2)^2(3x^2)}{x^6} = -\dfrac{(x+2)(x+6)}{x^4}$. Solutions are $x = -6, -2$. We now calculate

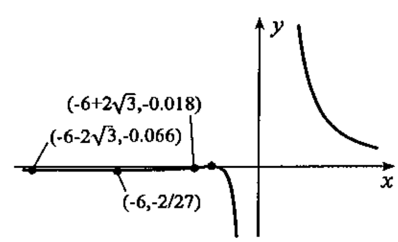
$$f''(x) = -\frac{x^4(2x+8) - (x+2)(x+6)(4x^3)}{x^8} = \frac{2(x^2+12x+24)}{x^5}.$$

Since $f''(-6) > 0$ and $f''(-2) < 0$, there is a relative minimum at $x = -6$ equal to $f(-6) = -2/27$, and a relative maximum at $x = -2$ of $f(-2) = 0$.

Because $f''(x) = 0$ at $x = (-12 \pm \sqrt{144-96})/2 = -6 \pm 2\sqrt{3}$, and $f''(x)$ changes sign as $x$ passes through each of these, points of inflection occur at $(-6-2\sqrt{3}, -0.066)$, $(-6+2\sqrt{3}, -0.018)$.

This information is shown in the left figure below, together with the limits $\lim_{x\to-\infty} f(x) = 0^-$, $\lim_{x\to\infty} f(x) = 0^+$, $\lim_{x\to 0^-} f(x) = -\infty$, and $\lim_{x\to 0^+} f(x) = \infty$. The final graph is shown to the right.



**10.** For critical points we solve

$$0 = f'(x) = 3x^{1/2} - 9 + 6x^{-1/2} = \frac{3(x - 3\sqrt{x} + 2)}{\sqrt{x}} = \frac{3(\sqrt{x} - 1)(\sqrt{x} - 2)}{\sqrt{x}}.$$

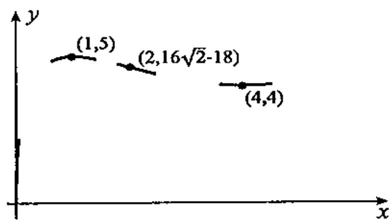
Solutions are $x = 1$ and $x = 4$. The point $x = 0$ is also critical since $f'(0)$ is not defined but $f(0) = 0$. To classify these critical points we calculate

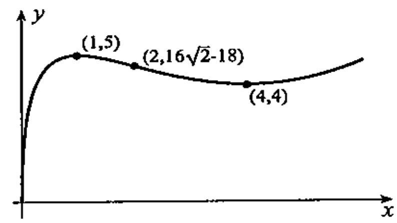
$$f''(x) = \frac{3}{2\sqrt{x}} - \frac{3}{x^{3/2}} = \frac{3(x-2)}{2x^{3/2}}.$$

Since $f''(1) = -3/2$ and $f''(4) = 3/8$, $f(1) = 5$ is a relative maximum and $f(4) = 4$ is a relative minimum. Because $f(x)$ is not defined for $x < 0$, the critical point $x = 0$ does not give a relative extrema. For graphical purposes we note that $\lim_{x\to 0^+} f'(x) = \infty$.

Since $f''(2) = 0$, and $f''(x)$ changes sign as $x$ passes through 2, there is a point of inflection at $(2, 16\sqrt{2} - 18)$.

This information is shown in the left figure below. The final graph is shown to the right.



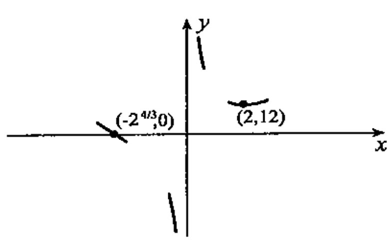

**11.** For critical points we solve

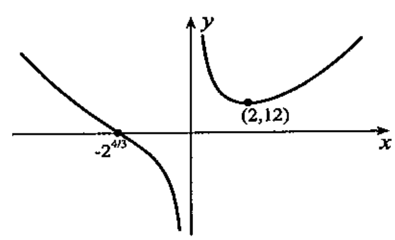
$$0 = f'(x) = \frac{x(3x^2) - (x^3 + 16)(1)}{x^2} = \frac{2(x^3 - 8)}{x^2}.$$

Clearly $x = 2$ is a critical point. To classify this critical point we calculate

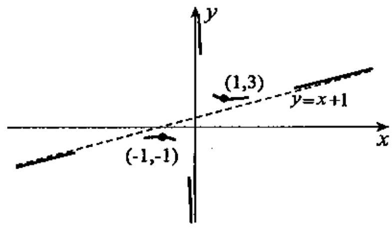
$$f''(x) = 2\left[\frac{x^2(3x^2) - (x^3 - 8)(2x)}{x^4}\right] = \frac{2(x^3 + 16)}{x^3}.$$

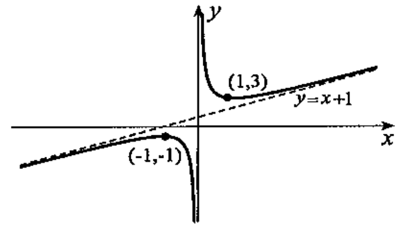
Since $f''(2) = 6$, $f(2) = 12$ is a relative minimum. Since $f''(x) = 0$ when $x = -2^{4/3}$ and $f''(x)$ changes sign as $x$ passes through $-2^{4/3}$, the only point of inflection is $(-2^{4/3}, 0)$. This information is shown in the left figure below. The final graph is shown to the right.



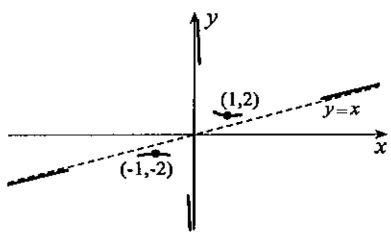



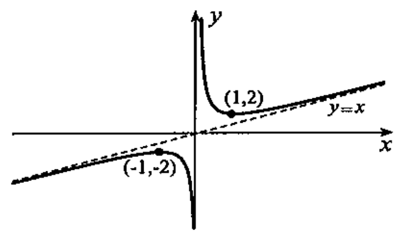

**12.** The function is discontinuous at $x = 0$. Left- and right-hand limits of the function as $x \to 0$ are $\lim_{x\to 0^-} f(x) = -\infty$ and $\lim_{x\to 0^+} f(x) = \infty$. For critical points we express the function in the form $f(x) = x + 1 + \dfrac{1}{x}$, and solve $0 = f'(x) = 1 - \dfrac{1}{x^2}$. Clearly, $x = \pm 1$. The second derivative is $f''(x) = 2/x^3$. Since $f''(1) = 2$, there is a relative minimum at $x = 1$ of $f(1) = 3$. With $f''(-1) = -2$, there is a relative maximum of $f(-2) = -1$. Since $f''(x)$ is never zero, there can be no points of inflection. Finally, we note that $y = x + 1$ is an oblique asymptote. This information is shown in the left figure below. The final graph is shown to the right.




**13.** The function is discontinuous at $x = 0$. Left- and right-hand limits of the function as $x \to 0$ are $\lim_{x\to 0^-} f(x) = -\infty$ and $\lim_{x\to 0^+} f(x) = \infty$. Clearly, $y = x$ is an oblique asymptote. For critical points we solve $0 = f'(x) = 1 - \dfrac{1}{x^2}$. Clearly, $x = \pm 1$. The second derivative is $f''(x) = 2/x^3$. Since $f''(1) = 2$, there is a relative minimum at $x = 1$ of $f(1) = 2$. With $f''(-1) = -2$, there is a relative maximum of $f(-2) = -2$. Since $f''(x)$ is never zero, there can be no points of inflection. This information is shown in the left figure below. The final graph is shown to the right.




**14.** The function is discontinuous at $x = \pm 2$. We take left-and right-hand limits of $f(x) = \dfrac{x^3}{(x+2)(x-2)}$ as $x \to \pm 2$

$$\lim_{x\to -2^-} f(x) = -\infty, \quad \lim_{x\to -2^+} f(x) = \infty, \quad \lim_{x\to 2^-} f(x) = -\infty, \quad \lim_{x\to 2^+} f(x) = \infty.$$

For critical points we solve

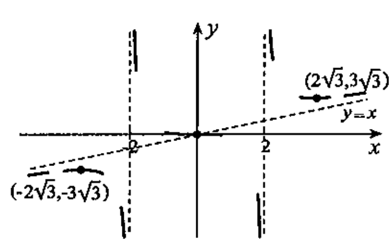
$$0 = f'(x) = \frac{(x^2-4)(3x^2) - x^3(2x)}{(x^2-4)^2} = \frac{x^2(x^2-12)}{(x^2-4)^2}.$$

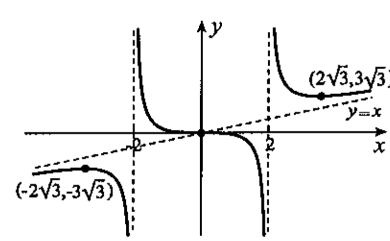
Solutions are $x = 0,\ \pm 2\sqrt{3}$. The second derivative is

$$f''(x) = \frac{(x^2-4)^2(4x^3-24x) - (x^4-12x^2)(2)(x^2-4)(2x)}{(x^2-4)^4} = \frac{8x(x^2+12)}{(x^2-4)^3}.$$

Since $f''(2\sqrt{3}) > 0$, we have a relative minimum $f(2\sqrt{3}) = 3\sqrt{3}$. Similarly, $f''(-2\sqrt{3}) < 0$ indicates that $f(-2\sqrt{3}) = -3\sqrt{3}$ is a relative maximum. Since $f''(x) = 0$ only at $x = 0$, and $f''(x)$ changes sign as x passes through 0, there is a horizontal point of inflection at $(0, 0)$.

By writing $f(x)$ in the form $f(x) = x + \dfrac{4x}{x^2-4}$, we see that the graph is asymptotic to the line $y = x$. This information is shown in the left figure below. The final graph is shown to the right.



**15.** With $f(x) = \dfrac{2x^2}{(x-2)(x-6)}$, we see that the function is discontinuous at $x = 2$ and $x = 6$. Left- and right-hand limits at these values of $x$ are

$$\lim_{x\to 2^-} f(x) = \infty, \quad \lim_{x\to 2^+} f(x) = -\infty, \quad \lim_{x\to 6^-} f(x) = -\infty, \quad \lim_{x\to 6^+} f(x) = \infty.$$

For critical points, we solve

$$0 = f'(x) = \frac{(x^2-8x+12)(4x) - 2x^2(2x-8)}{(x^2-8x+12)^2} = \frac{16x(3-x)}{(x^2-8x+12)^2}.$$

Solutions are $x = 0, 3$. The second derivative is

$$f''(x) = \frac{(x^2-8x+12)^2(48-32x) - (48x-16x^2)(2)(x^2-8x+12)(2x-8)}{(x^2-8x+12)^4} = \frac{16(2x^3-9x^2+36)}{(x^2-8x+12)^3}.$$

Since $f''(0) > 0$, there is a relative minimum at $x = 0$ of $f(0) = 0$. With $f''(3) < 0$, there is a relative maximum of $f(3) = -6$. For points of inflection, we would solve $2x^3 - 9x^2 + 36 = 0$. With Newton's iterative procedure, and an inital value $x_1 = -2$, iteration of $x_{n+1} = x_n - \dfrac{2x_n^3 - 9x_n^2 + 36}{6x_n^2 - 18x_n}$ gives

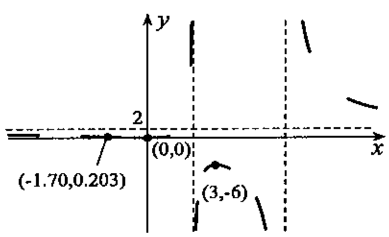
$$x_2 = -1.733, \quad x_3 = -1.704, \quad x_4 = -1.703, \quad x_5 = -1.703.$$

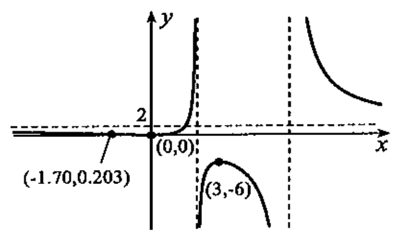
Since $f''(x)$ changes sign as $x$ passes through this solution, there is a point of inflection at $(-1.70, 0.203)$.

By writing $f(x)$ in the form $f(x) = 2 + \dfrac{16x-24}{x^2-8x+12}$, we see that $y = 2$ is a horizontal asymptote, and

$$\lim_{x\to -\infty} f(x) = 2^-, \qquad \lim_{x\to\infty} f(x) = 2^+.$$

This information is shown in the left figure below. The final graph is shown to the right.




**16.** With $f(x) = \dfrac{x^2+1}{(x-1)(x+1)}$, we see that the function is discontinuous at $x = \pm 1$. Left-and right-hand limits at these values of $x$ are

$$\lim_{x\to -1^-} f(x) = \infty, \quad \lim_{x\to -1^+} f(x) = -\infty, \quad \lim_{x\to 1^-} f(x) = -\infty, \quad \lim_{x\to -1^+} f(x) = \infty.$$

For critical points, we solve

$$0 = f'(x) = \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}.$$

The only solution is $x = 0$. Since $f'(x)$ changes from positive to negative as $x$ increases through 0, there is a relative maximum of $f(0) = -1$. For points of inflection, we solve

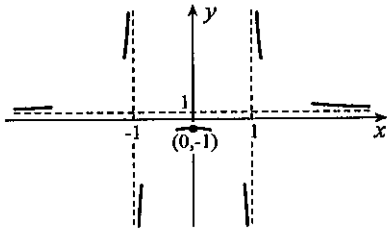
$$0 = f''(x) = \frac{(x^2-1)^2(-4) - (-4x)(2)(x^2-1)(2x)}{(x^2-1)^4} = \frac{12x^2+4}{(x^2-1)^3}.$$

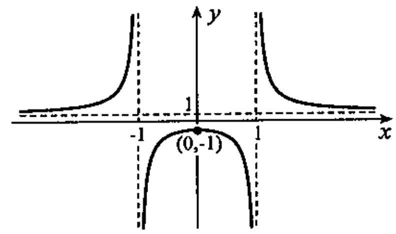
There are no solutions indicating no points of inflection.

By writing $f(x)$ in the form $f(x) = 1 + \dfrac{1}{x^2-1}$, we see that $y = 1$ is a horizontal asymptote, and

$$\lim_{x\to-\infty} f(x) = 1^+, \qquad \lim_{x\to\infty} f(x) = 1^+.$$

This information is shown in the left figure below. The final graph is shown to the right.




**17.** To analyze the graph near the discontinuity $x = -1$, we calculate

$$\lim_{x\to-1^-} f(x) = -\infty, \qquad \lim_{x\to-1^+} f(x) = -\infty.$$

The $x$-axis is a horizontal asymptote, and

$$\lim_{x\to-\infty} f(x) = 0^-, \qquad \lim_{x\to\infty} f(x) = 0^+.$$

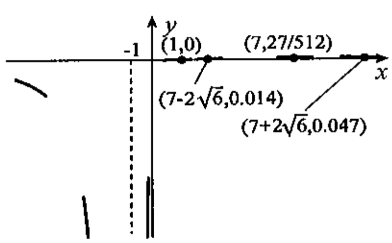
For critical points we solve

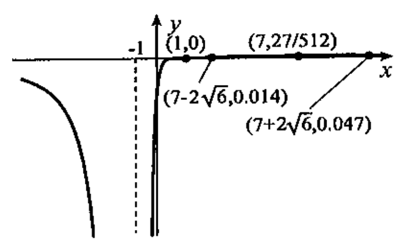
$$0 = f'(x) = \frac{(x+1)^4(3)(x-1)^2 - (x-1)^3(4)(x+1)^3}{(x+1)^8} = \frac{(x-1)^2(7-x)}{(x+1)^5}.$$

Solutions are $x = 1,\ 7$. Since $f'(x)$ changes from positive to negative as $x$ increases through 7, there is a relative maximum at $(7, 27/512)$. The derivative does not change sign at $x = 1$. For points of inflection, we solve

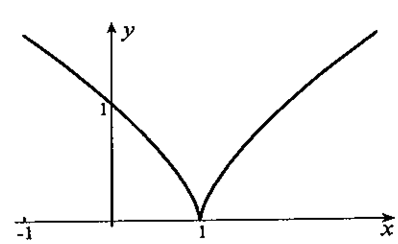
$$0 = f''(x) = \frac{(x+1)^5[2(x-1)(7-x) - (x-1)^2] - (x-1)^2(7-x)(5)(x+1)^4}{(x+1)^{10}} = \frac{2(x-1)(x^2-14x+25)}{(x+1)^6}.$$

Solutions are $x = 1,\ 7 \pm 2\sqrt{6}$. The second derivative changes sign as x passes through each of these. Thus, we have points of inflection at $(1, 0)$ (a horizontal one), and $(7-2\sqrt{6}, 0.014)$ and $(7+2\sqrt{6}, 0.047)$. This information is shown in the left figure below. The final graph is shown to the right.




**18.** For critical points we solve $f'(x) = (2/3)(x-1)^{-1/3} = 0$. There is no solution, but $x = 1$ is critical since $f(1) = 0$ and $f'(1)$ does not exist. Because $f(x) > 0$ for all $x \neq 1$, it follows that $f(1)$ is a relative minimum. Since $f''(x) = -(2/9)(x-1)^{-4/3}$ never vanishes, there are no points of inflection. The graph is shown to the right.



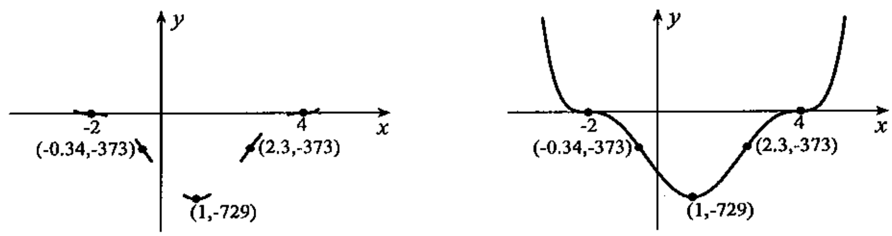
**19.** For critical points we solve

$$0 = f'(x) = 3(x+2)^2(x-4)^3 + 3(x+2)^3(x-4)^2 = 6(x+2)^2(x-4)^2(x-1).$$

Solutions are $x = -2, 1, 4$. The derivative changes from negative to positive as $x$ increases through 1 indicating a relative minimum there of $f(1) = -729$. The derivative does not change sign at $x = -2$ and $x = 4$. For points of inflection we solve

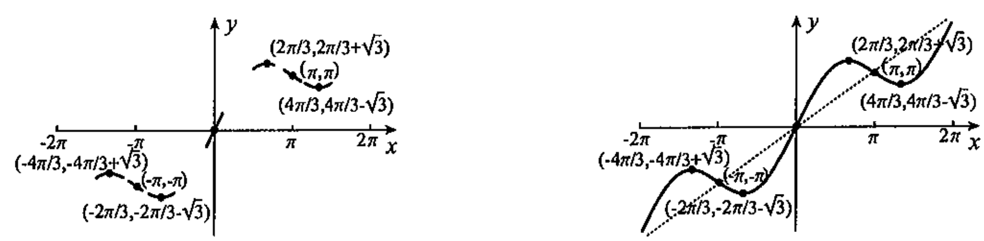
$$\begin{aligned} 0 = f''(x) &= 6[2(x+2)(x-2)^2(x-1) + (x+2)^2(2)(x-4)(x-1) + (x+2)^2(x-4)^2] \\ &= 6(x+2)(x-4)(5x^2 - 10x - 4). \end{aligned}$$

Solutions are $x = -2$, 4, $(5 \pm 3\sqrt{5})/5$. Since $f''(x)$ changes sign as $x$ passes through each of these, there are four points of inflection $(-2, 0)$, $(4, 0)$ (both of which are horizontal), and $((5 + 3\sqrt{5})/5, -373)$ and $((5 - 3\sqrt{5})/5, -373)$. This information is shown in the left figure below. The final graph is shown to the right.




**20.** For critical points we solve $0 = f'(x) = 1 + 2\cos x$. Solutions are $x = \pm 2\pi/3 + 2n\pi$, where $n$ is an integer. To classify them we calculate $f''(x) = -2\sin x$. Since $f''(2\pi/3 + 2n\pi) < 0$, the critical points $x = 2\pi/3 + 2n\pi$ give relative maxima $f(2\pi/3 + 2n\pi) = \sqrt{3} + 2\pi/3 + 2n\pi$. Similarly, there are relative minima at $(-2\pi/3 + 2n\pi, -\sqrt{3} - 2\pi/3 + 2n\pi)$.

For points of inflection, we solve $0 = f''(x) = -2\sin x$. Solutions are $x = n\pi$, where $n$ is an integer. Since $f''(x)$ changes sign at each of these, they all give points of inflection. This information is shown in the left figure below. The final graph is shown to the right. It oscillates about the line $y = x$.



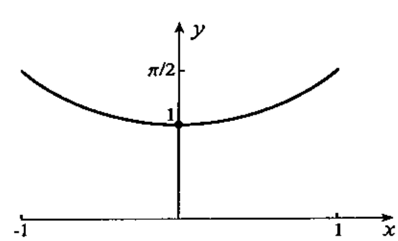

**21.** For critical points we consider $f'(x) = \dfrac{1}{\sqrt{1-x^2}} - \dfrac{1}{\sqrt{1-x^2}} = 0$. Since this is true for all $x$ in the domain $-1 \le x \le 1$ of the function, the function must be equal to a constant, say $\mathrm{Sin}^{-1}x + \mathrm{Cos}^{-1}x = C$. When we set $x = 0$, we obtain $0 + \pi/2 = C$. Thus, the graph is a horizontal line segment joining the points $(-1, \pi/2)$ and $(1, \pi, 2)$. Every point in the interval $-1 < x < 1$ is critical, and each gives a relative maximum and a relative minimum. There are no points of inflection.

**22.** For critical points we solve

$$\begin{aligned} 0 = f'(x) &= \mathrm{Sin}^{-1}x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} \\ &= \mathrm{Sin}^{-1}x. \end{aligned}$$

The only solution is $x = 0$. For points of inflection we solve $0 = f''(x) = \dfrac{1}{\sqrt{1-x^2}}$.

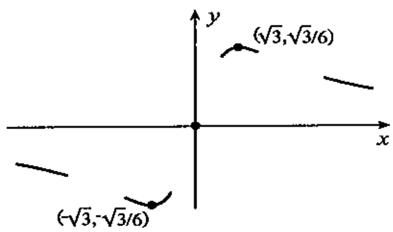
There are no solutions. Because $f''(0) = 1$, $f(x)$ has a relative minimum of $f(0) = 1$.

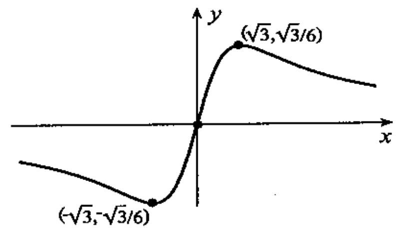


**23.** For critical points we solve

$$0 = f'(x) = \frac{(x^2+3)(1) - x(2x)}{(x^2+3)^2} = \frac{3-x^2}{(x^2+3)^2}.$$

Solutions are $x = \pm\sqrt{3}$. Since $f'(x)$ changes from negative to positive as $x$ increases through $-\sqrt{3}$, there is a relative minimum at this critical point of $f(-\sqrt{3}) = -\sqrt{3}/6$. Since $f'(x)$ changes from positive to negative as $x$ increases through $\sqrt{3}$, there is a relative maximum at this critical point of $f(\sqrt{3}) = \sqrt{3}/6$. The graph is asymptotic to the $x$-axis. This information is shown in the left figure below. The final graph is shown to the right.

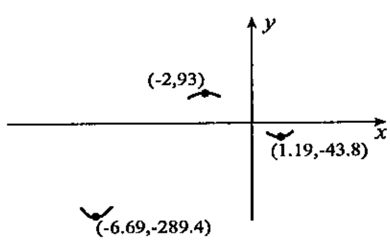


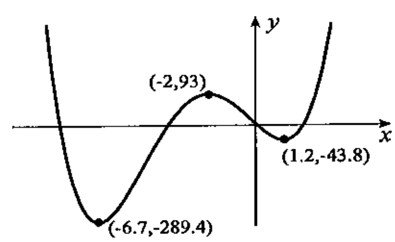




**24.** For critical points we solve

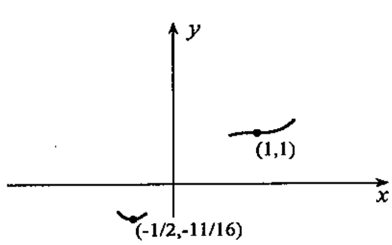
$$0 = f'(x) = 4x^3 + 30x^2 + 12x - 64 = 2(x+2)(2x^2 + 11x - 16).$$

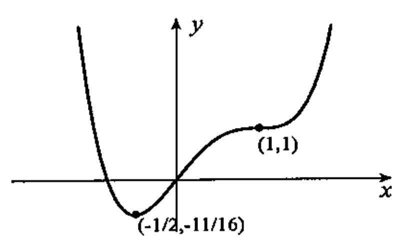
Solutions are $x = -2$ and $x = (-11 \pm \sqrt{121+128})/4 = (-11 \pm \sqrt{249})/4$. Since $f'(x)$ changes from positive to negative as $x$ increases through $-2$, there is a relative maximum at this critical point of $f(-2) = 93$. Since $f'(x)$ changes from negative to positive as $x$ increases through the remaining critical points, they give relative minima of $f\big((-11-\sqrt{249})/4\big) = f(-6.69) = -289.4$ and $f\big((-11+\sqrt{249})/4\big) = f(1.19) = -43.8$. This information is shown in the left figure below. The final graph is shown to the right.




**25.** For critical points we solve $0 = f'(x) = 4x^3 - 6x^2 + 2 = 2(x-1)^2(2x+1)$. Solutions of this equation are $x = -1/2, 1$. Since $f'(x)$ changes from negative to positive as $x$ increases through $-1/2$, there is a relative minimum at this critical point of $f(-1/2) = -11/16$. Since $f'(x)$ does not change sign as $x$ passes through 1, there is a horizontal point of inflection at $(1,1)$. This information is shown in the left figure below. The final graph is shown to the right.



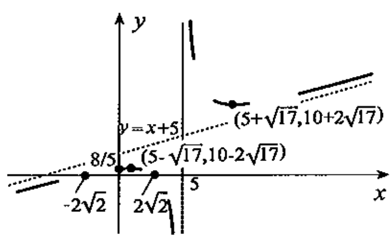
**26.** The function is discontinuous at $x = 5$, and therefore we calculate

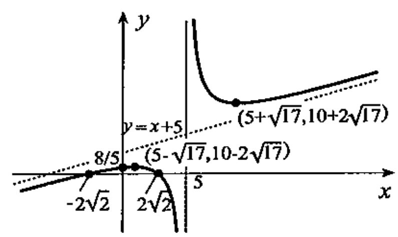
$$\lim_{x\to 5^-} f(x) = -\infty \qquad \text{and} \qquad \lim_{x\to 5^+} f(x) = \infty.$$

For critical points we solve

$$0 = f'(x) = \frac{(x-5)(2x) - (x^2-8)(1)}{(x-5)^2} = \frac{x^2 - 10x + 8}{(x-5)^2}.$$

Solutions are $x = \dfrac{10 \pm \sqrt{100-32}}{2} = 5 \pm \sqrt{17}$. Since $f'(x)$ changes from positive to negative as $x$ increases through $5 - \sqrt{17}$, there is a relative maximum at this critical point of $f(5-\sqrt{17}) = 10 - 2\sqrt{17}$. Since $f'(x)$ changes from negative to positive as $x$ increases through $5+\sqrt{17}$, there is a relative minimum at this critical point of $f(5+\sqrt{17}) = 10 + 2\sqrt{17}$. Since $f(x) = x + 5 + 17/(x-5)$, the graph has oblique asymptote $y = x + 5$. This information, along with $x$- and $y$-intercepts, is shown in the left figure below. The final graph is shown to the right.



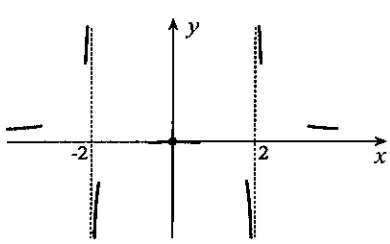

**27.** The function is discontinuous at $x = \pm 2$, and therefore we calculate

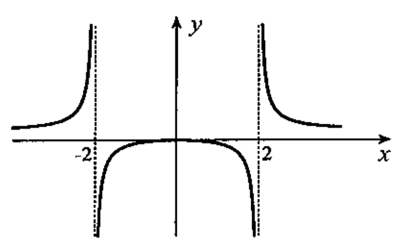
$$\lim_{x\to -2^-} f(x) = \infty, \quad \lim_{x\to -2^+} f(x) = -\infty, \quad \lim_{x\to 2^-} f(x) = -\infty, \quad \lim_{x\to 2^+} f(x) = \infty.$$

For critical points we solve

$$0 = f'(x) = \frac{(x^2-4)(2x) - x^2(2x)}{(x^2-4)^2} = \frac{-8x}{(x^2-4)^2}.$$

The only solution is $x = 0$. Since $f'(x)$ changes from positive to negative as $x$ increases through the critical point, there is a relative maximum of $f(0) = 0$. The $x$-axis is a horizontal asymptote; it approaches it from above for $x \to \pm\infty$. This information is shown in the left figure below. The final graph is shown to the right.

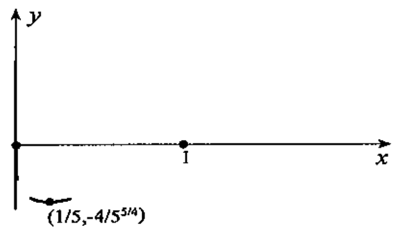


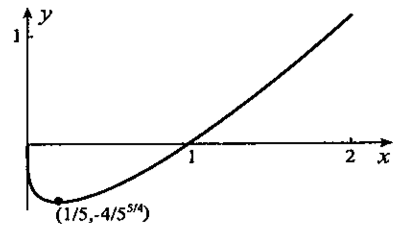




**28.** For critical points we solve

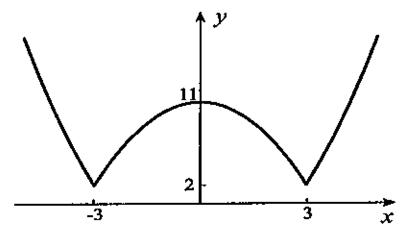
$$0 = f'(x) = \frac{5}{4}x^{1/4} - \frac{1}{4}x^{-3/4} = \frac{5x-1}{4x^{3/4}}.$$

The solution is $x = 1/5$. Since $f'(x)$ changes from negative to positive as $x$ increases through the critical point, there is a relative minimum of $f(1/5) = -4/5^{5/4}$. There is also critical point at $x = 0$, but because the function is not defined for $x < 0$, it cannot yield a relative maximum. We note that $f(0) = 0$ and $\lim_{x\to 0^+} f'(x) = -\infty$. This information, along with the $x$-intercept $x = 1$, is shown in the left figure below. The final graph is shown to the right.




**29.** The graph is most easily drawn by taking absolute values of the parabola $y = x^2 - 9$ and shifting it upward 2 units. The derivative is undefined at $x = \pm 3$ where the graph has sharp corners. There are relative minima equal to $f(\pm 3) = 2$ at these points.




**30.** The function is discontinuous at $x = 1$ and $x = 4$, and we therefore calculate

$$\lim_{x\to 1^-} f(x) = -\infty, \quad \lim_{x\to 1^+} f(x) = \infty, \quad \lim_{x\to 4^-} f(x) = \infty, \quad \lim_{x\to 4^+} f(x) = -\infty.$$

To find critical points we solve

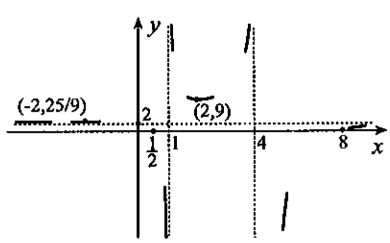
$$0 = f'(x) = \frac{(x^2 - 5x + 4)(4x - 17) - (2x^2 - 17x + 8)(2x - 5)}{(x^2 - 5x + 4)^2} = \frac{7(x-2)(x+2)}{(x^2 - 5x + 4)^2}.$$

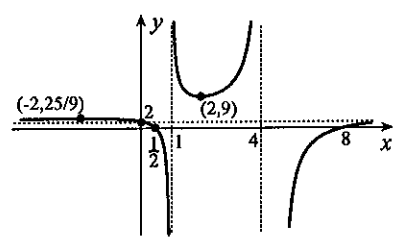
Solutions are $x = -2, 2$. Since $f'(x)$ changes from positive to negative as $x$ increases through $-2$, there is a relative maximum of $f(-2) = 25/9$. Since $f'(x)$ changes from negative to positive as $x$ increases through 2, there is a relative minimum of $f(2) = 9$.

The graph has horizontal asymptote $y = 2$, and to determine how the asymptote is approached, we express $f(x)$ in the form

$$f(x) = \frac{2x^2 - 17x + 8}{x^2 - 5x + 4} = 2 - \frac{7x}{x^2 - 5x + 4}.$$

This shows that $y = 2$ is approached from below as $x \to \infty$ and from above as $x \to -\infty$. This information is shown in the left figure below. The final graph is shown to the right.

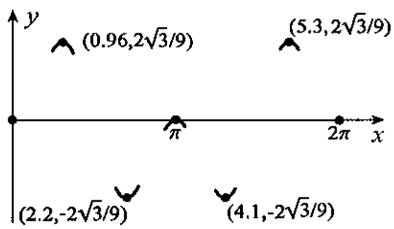


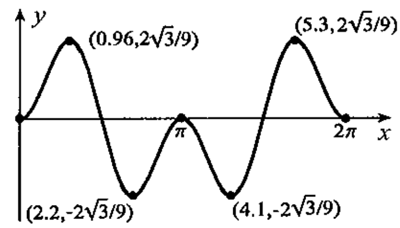




**31.** For critical points we solve

$$0 = f'(x) = 2\sin x\cos^2 x - \sin^3 x = \sin x(2\cos^2 x - \sin^2 x) = \sin x(3\cos^2 x - 1).$$

Solutions are $x = 0, \pi, 2\pi$ and values of $x$ satisfying $\cos x = \pm 1/\sqrt{3}$. From $\cos x = 1/\sqrt{3}$, we obtain $x = \mathrm{Cos}^{-1}(1/\sqrt{3}) = 0.96$ and $x = 2\pi - 0.96 = 5.3$. From $\cos x = -1/\sqrt{3}$, $x = \mathrm{Cos}^{-1}(-1/\sqrt{3}) = 2.2$ and $x = 2\pi - 2.2 = 4.1$. Being end points, $x = 0, 2\pi$ cannot yield relative extrema. Since $f'(x)$ changes from positive to negative as $x$ increases through $\pi$, 0.96 and 5.3, there are a relative maxima of $f(\pi) = 0$, $f(0.96) = f(5.3) = 2\sqrt{3}/9$. Since $f'(x)$ changes from negative to positive as $x$ increases through 2.2 and 4.1, there are relative minima of $f(2.2) = f(4.1) = -2\sqrt{3}/9$. This information is shown in the left figure below. The final graph is shown to the right.



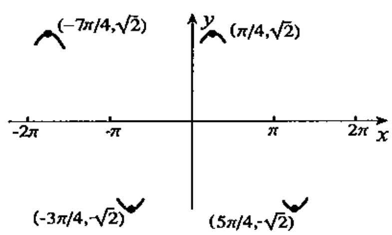



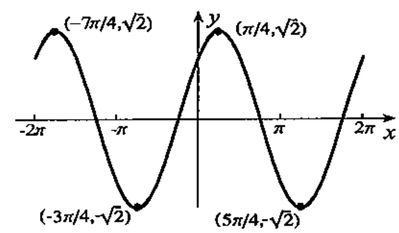

**32.** For critical points we solve $0 = f'(x) = \cos x - \sin x$. Solutions are $x = \pi/4 + 2n\pi$, where $n$ is an integer. Since $f'(x)$ changes from positive to negative as $x$ increases through $\pi/4 + 2n\pi$, there are a relative maxima of $f(\pi/4 + 2n\pi) = \sqrt{2}$. Since $f'(x)$ changes from negative to positive as $x$ increases through $\pi/4 + (2n+1)\pi$, there are a relative minima of $f(\pi/4 + (2n+1)\pi) = -\sqrt{2}$. This information is shown in the left figure below. The final graph is shown to the right.

The function could also be graphed by expressing it as a general sine function $A\sin(x + \phi)$ (see Example 1.45 in Section 1.8),

$$\sin x + \cos x = A\sin(x+\phi) = A(\sin x\,\cos\phi + \cos x\,\sin\phi).$$

To satisfy this equation we set $A\cos\phi = 1$ and $A\sin\phi = 1$. These are satisfied if we choose $A = \sqrt{2}$ and $\phi = \pi/4$. Thus, $f(x)$ can be expressed in the form $f(x) = \sqrt{2}\sin(x + \pi/4)$.

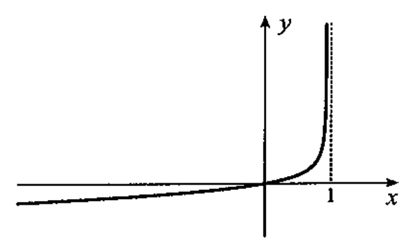



**33.** The function is only defined for $x < 1$.
For critical points we solve

$$0 = f'(x) = \frac{\sqrt{1-x} - x(1/2)(1-x)^{-1/2}(-1)}{1-x} = \frac{2-x}{2(1-x)^{3/2}}.$$

The derivative never vanishes for $x < 1$.
To draw the graph we note that $\lim_{x\to 1^-} f(x) = \infty$, the $x$- and $y$-intercepts are both 0, and the function is negative when $x$ is negative.



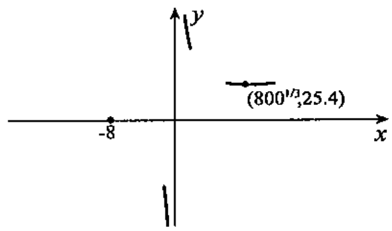

**34.** The function is discontinuous at $x = 0$, and we therefore calculate

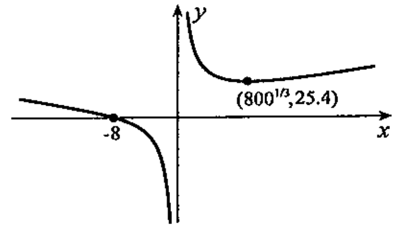
$$\lim_{x\to 0^-} f(x) = -\infty, \qquad \lim_{x\to 0^+} f(x) = \infty.$$

To identify critical points we solve

$$0 = f'(x) = -\frac{8}{x^2}\sqrt{x^2+100} + \left(1 + \frac{8}{x}\right)\frac{x}{\sqrt{x^2+100}} = \frac{x^3 - 800}{x^2\sqrt{x^2+100}}.$$

The only critical point is $800^{1/3}$. Since $f'(x)$ changes from negative to positive as $x$ increases through this critical point, there is a relative minimum of $f(800^{1/3}) = \left(\frac{8 + 800^{1/3}}{800^{1/3}}\right)\sqrt{100 + 800^{2/3}} \approx 25.4$. This information is shown in the left figure below. The final graph is shown to the right.



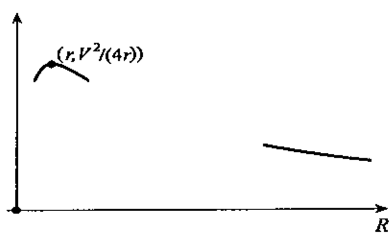

**35.** (a) $P_R(R) = i^2R = \dfrac{RV^2}{(r+R)^2}$ The graph of this function begins at the origin, is in the first quadrant, and is asymptotic to the $R$-axis since $\lim_{R\to\infty} P_R(R) = 0^+$. For critical points we solve

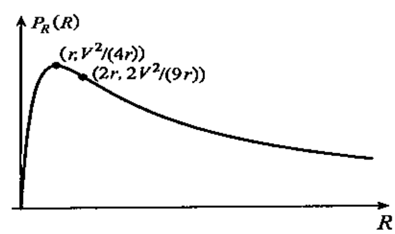
$$0 = P_R'(R) = V^2\left[\frac{(r+R)^2(1) - R(2)(r+R)}{(r+R)^4}\right] = \frac{V^2(r-R)}{(r+R)^3}.$$

Since $P_R'(R)$ changes from positive to negative as $R$ increases through $r$, there is a relative maximum at $\big(r, V^2/(4r)\big)$. These facts are shown in the left figure below. To finish the graph we solve

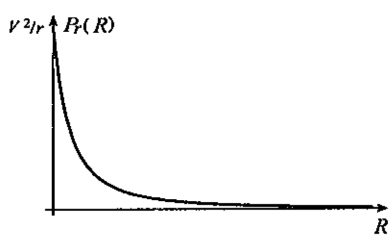
$$0 = P_R''(R) = V^2\left[\frac{(r+R)^3(-1) - (r-R)(3)(r+R)^2}{(r+R)^6}\right] = \frac{2V^2(R-2r)}{(r+R)^4}.$$

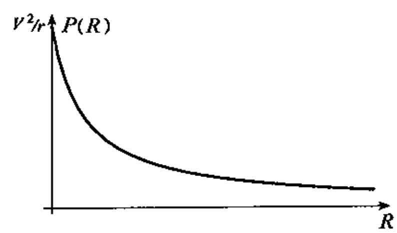
Because $P_R''(R)$ changes sign as $R$ passes through $2r$, there is a point of inflection at $\big(2r, 2V^2/(9r)\big)$. The final graph is to the right.




The function $P_r(R) = V^2r/(r+R)^2$ is decreasing for $R > 0$ and $\lim_{R\to\infty} P_r(R) = 0^+$. Its graph is shown to the left below.



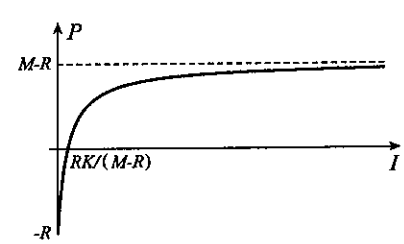




(b) The function $P(R) = V^2/(r+R)$ has the same shape as $P_r(R)$ (right figure above).

**36.** There are no critical points since

$$\frac{dP}{dI} = \frac{(I+K)M - MI(1)}{(I+K)^2} = \frac{KM}{(I+K)^2}$$

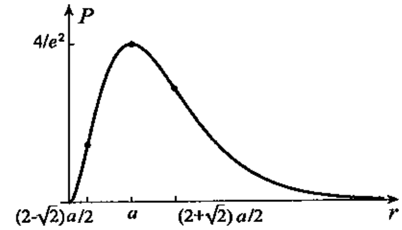
never vanishes. Since $\dfrac{d^2P}{dI^2} = \dfrac{-2KM}{(I+K)^3}$ is always negative, the graph is concave downward.
It crosses the $I$-axis when $0 = \dfrac{MI}{I+K} - R$, and the solution of this equation is $I = RK/(M-R)$.
Finally, we have a horizontal asymptote since $P \to (M-R)^-$ as $I \to \infty$.



**37.** For critical points we solve $0 = P'(r) = \left(\frac{8r}{a^2}\right)e^{-2r/a} + \left(\frac{4r^2}{a^2}\right)e^{-2r/a}\left(\frac{-2}{a}\right) = \frac{8r}{a^3}(a-r)e^{-2r/a}$. Solutions are $r = 0, a$. Since $P'(r)$ changes from positive to negative as $r$ increases through $a$, there is a relative maximum at $P(a) = 4/e^2$. For points of inflection we solve

$$0 = P''(r) = \frac{8}{a^3}[(a-2r)e^{-2r/a} + (ar-r^2)e^{-2r/a}(-2/a)] = \frac{8}{a^4}(a^2 - 4ar + 2r^2)e^{-2r/a}.$$

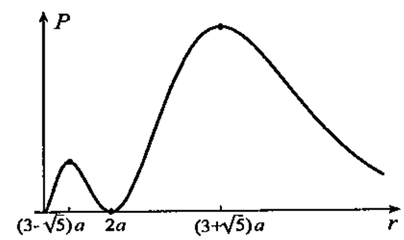
Solutions are $r = (4a \pm \sqrt{16a^2 - 8a^2})/4 = (2 \pm \sqrt{2})a/2$. Since $P''(r)$ changes sign as $r$ passes through these values, there are points of inflection at $((2 \pm \sqrt{2})a/2, (6 \pm 4\sqrt{2})e^{-2\mp\sqrt{2}})$. When combined with the fact that $\lim_{r\to\infty} P(r) = 0^+$, the graph to the right is obtained.




**38.** For critical points we solve

$$0 = P'(r) = \frac{1}{8a^3}\left[2r\left(2-\frac{r}{a}\right)^2 e^{-r/a} + 2r^2\left(2-\frac{r}{a}\right)\left(\frac{-1}{a}\right)e^{-r/a} + r^2\left(2-\frac{r}{a}\right)^2 e^{-r/a}\left(\frac{-1}{a}\right)\right]$$
$$= \frac{r(2a-r)e^{-r/a}}{8a^6}(r^2 - 6ar + 4a^2).$$

Solutions are $r = 0$, $r = 2a$, and $r = (6a \pm \sqrt{36a^2 - 16a^2})/2 = (3 \pm \sqrt{5})a$. Since $P'(r)$ changes from negative to positive as $r$ increases through $2a$, there is a relative minimum of $f(2a) = 0$ at $r = 2a$. Since $P'(r)$ changes from positive to negative as $r$ increases through both $(3 \pm \sqrt{5})a$, they give relative maxima of $P((3 \pm \sqrt{5})a)$, and these simplify to $(2/a)(9 \pm 4\sqrt{5})e^{-3\mp\sqrt{5}}$. When combined with the fact that $\lim_{r\to\infty} P(r) = 0^+$, the graph to the right is obtained.

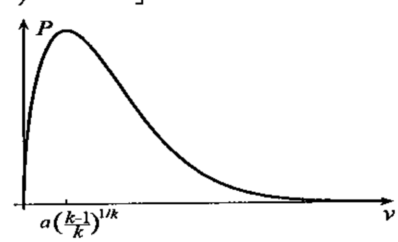



**39.** For critical points we solve

$$0 = P'(v) = \frac{k}{a}\left[\frac{k-1}{a}\left(\frac{v}{a}\right)^{k-2} e^{-(v/a)^k} + \left(\frac{v}{a}\right)^{k-1} e^{-(v/a)^k}\left(\frac{-k}{a}\right)\left(\frac{v}{a}\right)^{k-1}\right]$$
$$= \frac{k}{a^k}v^{k-2}e^{-(v/a)^k}\left[(k-1) - k\left(\frac{v}{a}\right)^k\right].$$

Thus, $v = a\left(\frac{k-1}{k}\right)^{1/k}$. Since $P'(v)$ changes from a positive quantity to a negative quantity as $v$ increases through this value, there is a relative maximum of

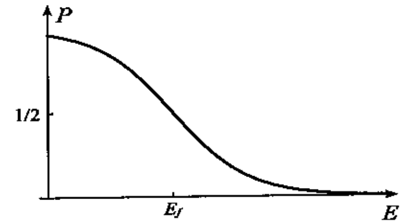
$$\frac{k}{a}\left(\frac{k-1}{k}\right)^{(k-1)/k} e^{-(k-1)/k}.$$




Since $1 < k < 2$, it follows that $\lim_{v\to 0^+} P'(v) = \infty$. Furthermore, with $\lim_{v\to\infty} P(v) = 0$ the graph is shown to the right.

**40.** For critical points we solve $0 = P'(E) = \frac{-e^{(E-E_f)/(kT)}}{kT[e^{(E-E_f)/(kT)} + 1]^2}$. There are no solutions to this equation. For points of inflection we solve

$$0 = P''(E) = \frac{-1}{kT}\left\{\frac{e^{(E-E_f)/(kT)}}{[e^{(E-E_f)/(kT)}+1]^2}\left(\frac{1}{kT}\right) - \frac{2e^{(E-E_f)/(kT)}}{[e^{(E-E_f)/(kT)}+1]^3}e^{(E-E_f)/(kT)}\left(\frac{1}{kT}\right)\right\}$$
$$= \frac{e^{(E-E_f)/(kT)}[e^{(E-E_f)/(kT)} - 1]}{k^2T^2[e^{(E-E_f)/(kT)}+1]^3}.$$




The only solution is $E = E_f$. Since $P''(E)$ changes sign as $E$ passes through $E_f$, there is a point of inflection at $(E_f, 1/2)$. The graph is asymptotic to the $E$-axis.

**41.** (a) A plot of $f(x) = (4 + e^{10/x})^{-1}$ is shown in the left figure below. Since $f'(x) = \dfrac{-e^{10/x}(-10/x^2)}{(4+e^{10/x})^2} = \dfrac{10\,e^{10/x}}{x^2(4+e^{10/x})^2}$, it follows that

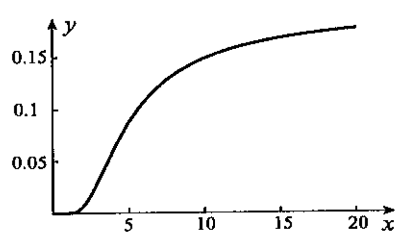
$$\begin{aligned} f''(x) &= \frac{10\,e^{10/x}(-10/x^2)}{x^2(4+e^{10/x})^2} - \frac{20\,e^{10/x}}{x^3(4+e^{10/x})^2} - \frac{20\,e^{10/x}e^{10/x}(-10/x^2)}{x^2(4+e^{10/x})^3} \\ &= \frac{10e^{10/x}[-40 - 10\,e^{10/x} - 2x(4+e^{10/x}) + 20\,e^{10/x}]}{x^4(4+e^{10/x})^3} \\ &= \frac{20e^{10/x}[(5-x)\,e^{10/x} - 4x - 20]}{x^4(4+e^{10/x})^3}. \end{aligned}$$

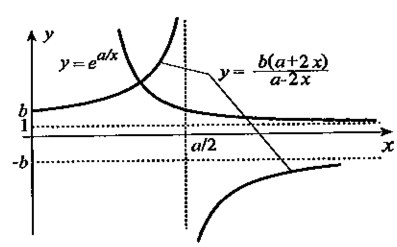
For the point of inflection we solve $0 = (5-x)\,e^{10/x} - 4x - 20$. Using Newton's iterative procedure, we obtain $x = 3.34$. The point of inflection is therefore $(3.34, 0.042)$.

(b) Since $f'(x) = \dfrac{-e^{a/x}(-a/x^2)}{(b+e^{a/x})^2} = \dfrac{a\,e^{a/x}}{x^2(b+e^{a/x})^2}$, there are no critical points of the function. For points of inflection we solve

$$\begin{aligned} 0 = f''(x) &= \frac{a\,e^{a/x}(-a/x^2)}{x^2(b+e^{a/x})^2} - \frac{2a\,e^{a/x}}{x^3(b+e^{a/x})^2} - \frac{2a\,e^{a/x}e^{a/x}(-a/x^2)}{x^2(b+e^{a/x})^3} \\ &= \frac{a\,e^{a/x}}{x^4(b+e^{a/x})^3}[-a(b+e^{a/x}) - 2x(b+e^{a/x}) + 2a\,e^{a/x}] \\ &= \frac{a\,e^{a/x}}{x^4(b+e^{a/x})^3}[(a-2x)\,e^{a/x} - 2bx - ab]. \end{aligned}$$

To show that $(a-2x)\,e^{a/x} - 2bx - ab = 0 \iff e^{a/x} = \dfrac{b(a+2x)}{a-2x}$ has exactly one solution, we draw graphs of $e^{a/x}$ and $b(a+2x)/(a-2x)$ as shown in the right figure below. There is one point of intersection of the curves.

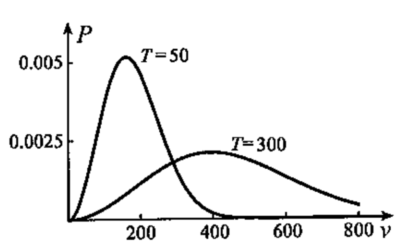



**42.** (a) The plots are shown to the right.
(b) For critical points we solve

$$\begin{aligned} 0 &= 2v\,e^{-Mv^2/(2RT)} + v^2 e^{-Mv^2/(2RT)}\left(\frac{-2Mv}{2RT}\right) \\ &= v\left(2 - \frac{Mv^2}{RT}\right)e^{-Mv^2/(2RT)}. \end{aligned}$$

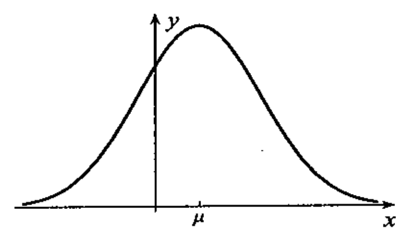
The positive solution is $v = \sqrt{2RT/M}$.
For points of inflection we solve



$$\begin{aligned}0 &= \left(2 - \frac{Mv^2}{RT}\right) e^{-Mv^2/(2RT)} + v\left(\frac{-2Mv}{RT}\right) e^{-Mv^2/(2RT)} + v\left(2 - \frac{Mv^2}{RT}\right) e^{-Mv^2/(2RT)} \left(\frac{-Mv}{RT}\right) \\ &= e^{-Mv^2/(2RT)} \left[2 - \frac{Mv^2}{RT} - \frac{2Mv^2}{RT} - \frac{Mv^2}{RT}\left(2 - \frac{Mv^2}{RT}\right)\right] \\ &= e^{-Mv^2/(2RT)} \left(\frac{M^2v^4}{R^2T^2} - \frac{5Mv^2}{RT} + 2\right).\end{aligned}$$

Solutions of this quadratic in $Mv^2/(RT)$ are $\dfrac{Mv^2}{RT} = \dfrac{5 \pm \sqrt{25-8}}{2} \implies v = \sqrt{\dfrac{5 \pm \sqrt{17}}{2}}\sqrt{\dfrac{RT}{M}}$. Since the second derivative changes sign as $v$ passes through these values, they yield points of inflection.

**43.** In Example 1.53 of Section 1.9 we sketched the function $f(x) = e^{-ax^2}$. The graph in this exercise has the same shape; it is shifted $\mu$ units to the right, and scales are modified.




**44.** The function can be expressed in the form

$$f(x) = \frac{(x+1)(x^2+1)}{(x+1)(x^2-x+1)} = \begin{cases} \dfrac{x^2+1}{x^2-x+1}, & x \neq -1, \\ \text{undefined}, & x = -1. \end{cases}$$

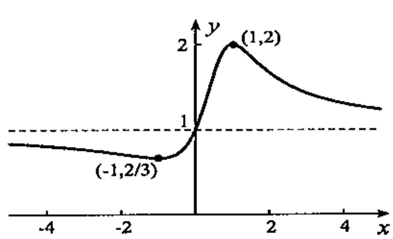
We therefore draw a graph of the function $g(x) = (x^2+1)/(x^2-x+1)$ and then delete the point at $x = -1$. We have a horizontal asymptote for $y = g(x)$ since

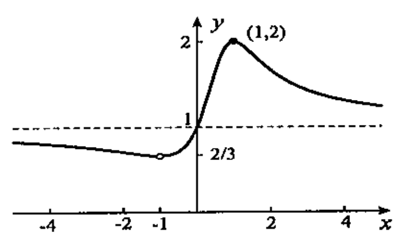
$$\lim_{x\to-\infty} \frac{x^2+1}{x^2-x+1} = 1^- \qquad \text{and} \qquad \lim_{x\to\infty} \frac{x^2+1}{x^2-x+1} = 1^+.$$

For critical points of $g(x)$ we solve

$$0 = g'(x) = \frac{(x^2-x+1)(2x) - (x^2+1)(2x-1)}{(x^2-x+1)^2} = \frac{1-x^2}{(x^2-x+1)^2}.$$

There are two solutions $x = \pm 1$. Since $g'(x)$ changes from negative to positive as $x$ increases through $-1$, $g(x)$ has a relative minimum $g(-1) = 2/3$. There is a relative maximum $g(1) = 2$ because $g'(x)$ changes from positive to negative as $x$ increases through 1. The graph of $g(x)$ is shown in the left diagram below. That of $f(x)$ is to the right.

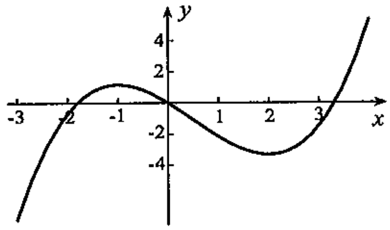


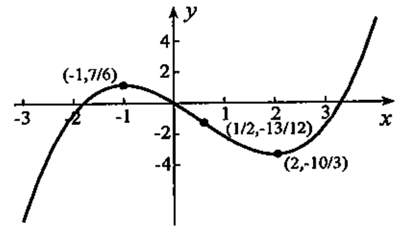




## EXERCISES 4.6

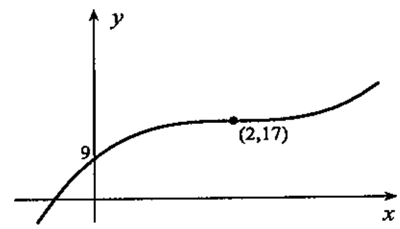
**1.** The plot in the left figure below indicates one relative maximum, one relative minimum, and one point of inflection. To confirm this, we first find critical points. Since $f'(x) = x^2 - x - 2 = (x-2)(x+1)$, the critical points are $x = -1, 2$. With $f''(x) = 2x - 1$, we find that $f''(-1) = -3$ and $f''(2) = 3$.

Consequently, $x = -1$ gives a relative maximum of $f(-1) = 7/6$ and $x = 2$ gives a relative minimum of $f(2) = -10/3$. Since $f''(1/2) = 0$ and $f''(x)$ changes sign as $x$ passes through $1/2$, there is a point of inflection at $(1/2, -13/12)$. These are shown in the right figure.

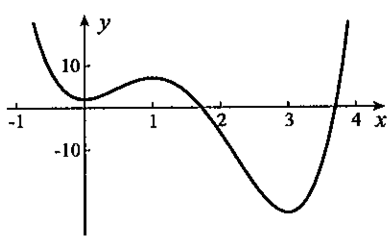


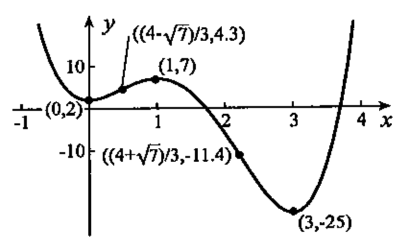



**2.** The plot suggests a horizontal point of inflection. To confirm this we calculate $f'(x) = 3x^2 - 12x + 12 = 3(x-2)^2$. The only critical point is $x = 2$. Because $f'(x)$ remains positive as $x$ passes through 2, the critical point does indeed give a horizontal point of inflection at $(2, 17)$. Since $f''(x) = 6(x-2)$, there are no other points of inflection.

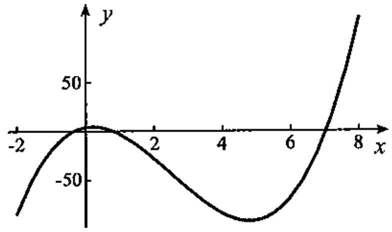


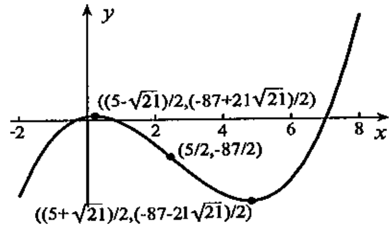

**3.** The plot in the left figure below indicates one relative maximum, two relative minima, and two points of inflection. To confirm this, we calculate $f'(x) = 12x^3 - 48x^2 + 36x = 12x(x-1)(x-3)$. Critical points are $x = 0, 1, 3$. With $f''(x) = 36x^2 - 96x + 36 = 12(3x^2 - 8x + 3)$, we find that $f''(0) = 36$, $f''(1) = -24$, and $f''(3) = 72$. Consequently, $x = 0$ and $x = 3$ give relative minima of $f(0) = 2$ and $f(3) = -25$, and $x = 1$ gives a relative maximum of $f(1) = 7$. Since $f''(x) = 0$ when $x = (8 \pm \sqrt{64 - 36})/6 = (4 \pm \sqrt{7})/3$, and $f''(x)$ changes sign as $x$ passes through each of these, there are points of inflection at $\left((4 - \sqrt{7})/3, 4.3\right)$ and $\left((4 + \sqrt{7})/3, -11.4\right)$. These are shown in the right figure.



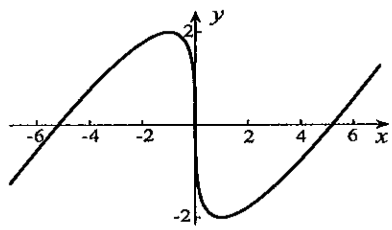


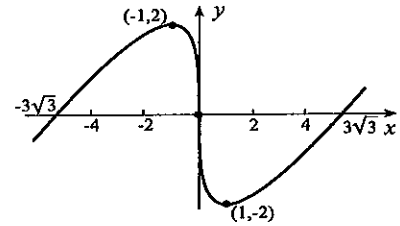

**4.** The plot in the left figure below indicates one relative maximum, one relative minimum, and one point of inflection. To confirm this, we first find critical points. Since $f'(x) = 6x^2 - 30x + 6 = 6(x^2 - 5x + 1)$, the critical points are $x = (5 \pm \sqrt{25 - 4})/2 = (5 \pm \sqrt{21})/2$. Since $f'(x)$ changes from positive to negative as $x$ increases through $(5 - \sqrt{21})/2$, there is a relative maximum at $\left((5 - \sqrt{21})/2, (-87 + 21\sqrt{21})/2\right)$. Similarly, there is a relative minimum at $\left((5 + \sqrt{21})/2, (-87 - 21\sqrt{21})/2\right)$. Since $0 = f''(x) = 12x - 30$ at $x = 5/2$, and $f''(x)$ changes sign as $x$ passes through $5/2$, there is a point of inflection at $(5/2, -87/2)$. These are shown in the right figure.





**5.** The plot in the left figure below indicates one relative maximum, one relative minimum, and a vertical point of inflection at $(0, 0)$. To confirm this, we first find critical points. Solutions of the equation $0 = f'(x) = 1 - x^{-2/3} = \dfrac{x^{2/3} - 1}{x^{2/3}}$ are $x = \pm 1$. Since $f'(0)$ is undefined, but $f(0) = 0$, $x = 0$ is also a critical point. With $f''(x) = (2/3)x^{-5/3}$, we find that $f''(-1) = -2/3$ and $f''(1) = 2/3$. Thus, $x = -1$ yields a relative maximum of $f(-1) = 2$ and $x = 1$ gives a relative minimum of $f(1) = -2$. Because $f''(x) < 0$ for $x < 0$, and $f''(x) > 0$ for $x > 0$, it follows that $(0, 0)$ must be a point of inflection. Since $\lim_{x \to 0} f'(x) = -\infty$, $(0, 0)$ is a vertical point of inflection. These are shown in the right figure.

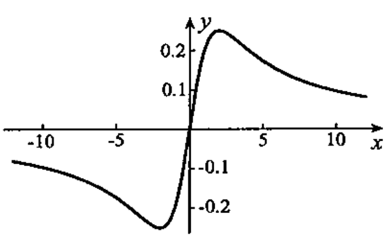


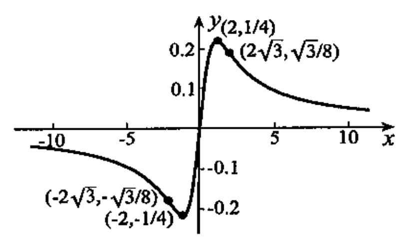




**6.** The plot in the left figure below indicates one relative maximum, one relative minimum, and three points of inflection. To confirm this, we first find critical points. For critical points we solve the equation $0 = f'(x) = \dfrac{(x^2 + 4)(1) - x(2x)}{(x^2 + 4)^2} = \dfrac{4 - x^2}{(x^2 + 4)^2}$. Solutions are $x = \pm 2$. We now calculate the second derivative $f''(x) = \dfrac{(x^2 + 4)^2(-2x) - (4 - x^2)(2)(x^2 + 4)(2x)}{(x^2 + 4)^4} = \dfrac{2x(x^2 - 12)}{(x^2 + 4)^3}$. Since $f''(-2) = 1/16$ and $f''(2) = -1/16$, there is a relative minimum at $x = -2$ equal to $f(-2) = -1/4$, and a relative maximum at $x = 2$ of $f(2) = 1/4$.

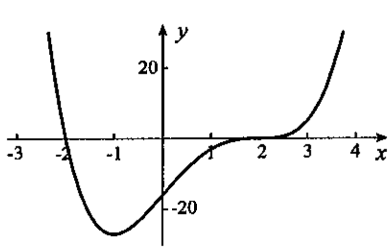
Because $f''(x) = 0$ at $x = 0, \pm 2\sqrt{3}$, and $f''(x)$ changes sign as $x$ passes through each of these, points of inflection occur at $(0, 0)$, $(2\sqrt{3}, \sqrt{3}/8)$, and $(-2\sqrt{3}, -\sqrt{3}/8)$.

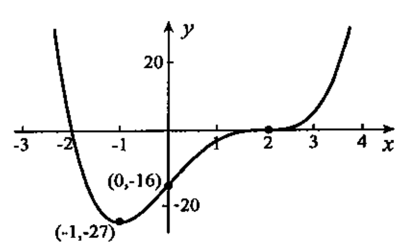
The final graph is shown to the right. We could have shortened the analysis by considering only the right half of the graph and using the fact that the function is odd.




**7.** The plot in the left figure below indicates one relative minimum and two points of inflection one of which is horizontal. To confirm this, we first find critical points. Since $f'(x) = 3(x - 2)^2(x + 2) + (x - 2)^3 = 4(x - 2)^2(x + 1)$, the critical points are $x = -1, 2$. Since $f'(x)$ changes from negative to positive as $x$ increases through $-1$, there is a relative minimum at $(-1, -27)$. The derivative does not change sign at $x = 2$. We now calculate $f''(x) = 8(x-2)(x+1) + 4(x-2)^2 = 12x(x-2)$. Since $f''(x) = 0$ when $x = 0, 2$, and $f''(x)$ changes sign as $x$ passes through these values, there is a point of inflection at $(0, -16)$ and a horizontal point of inflection at $(2, 0)$.



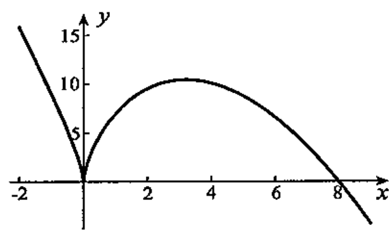
**8.** The plot in the left figure below indicates one relative minimum one relative maximum, perhaps a point of inflection for negative x, and perhaps a point of inflection at $x = 0$. To decide, we first find critical points. Since $0 = f'(x) = (2/3)x^{-1/3}(8 - x) - x^{2/3} = \dfrac{16 - 5x}{3x^{1/3}}$, $x = 16/5$ is critical, but so also is $x = 0$ because $f'(0)$ does not exist and $f(0) = 0$. We now calculate

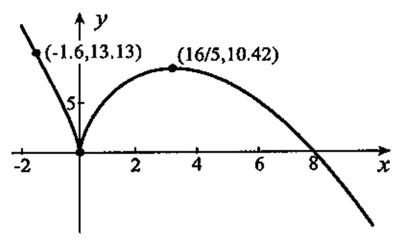
$$f''(x) = \frac{3x^{1/3}(-5) - (16 - 5x)x^{-2/3}}{9x^{2/3}} = \frac{-2(5x + 8)}{9x^{4/3}}.$$

Since $f''(16/5) < 0$, there is a relative maximum of $f(16/5) = 10.42$. Because $f'(x)$ changes from negative to positive as $x$ increases through 0, this critical point gives a relative mimumum of $f(0) = 0$.

Since $f''(-8/5) = 0$, and $f''(x)$ changes sign as $x$ passes through $-8/5$, there is a point of inflection at $(-1.6, 13.13)$. The point $(0, 0)$ is not a point of inflection.

The final graph is shown to the right.

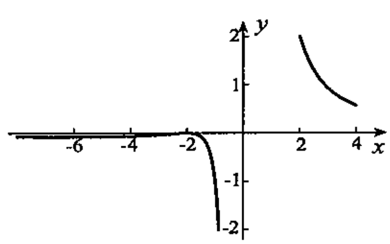


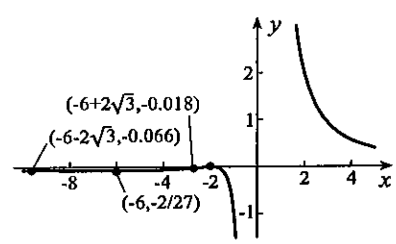




**9.** The plot in the left figure below indicates a relative maximum at $x = -2$, but the situation to the left of $x = -2$ is not at all clear. To confirm the relative maximum, and and assess the $x < -2$ situation, we begin with critical points. We solve $0 = f'(x) = \dfrac{x^3(2)(x + 2) - (x + 2)^2(3x^2)}{x^6} = -\dfrac{(x + 2)(x + 6)}{x^4}$ for $x = -6, -2$. We now calculate $f''(x) = -\dfrac{x^4(2x + 8) - (x + 2)(x + 6)(4x^3)}{x^8} = \dfrac{2(x^2 + 12x + 24)}{x^5}$. Since $f''(-6) > 0$ and $f''(-2) < 0$, there is a relative minimum at $x = -6$ equal to $f(-6) = -2/27$, and a relative maximum at $x = -2$ of $f(-2) = 0$.

Because $f''(x) = 0$ at $x = (-12 \pm \sqrt{144 - 96})/2 = -6 \pm 2\sqrt{3}$, and $f''(x)$ changes sign as $x$ passes through each of these, points of inflection occur at $(-6 - 2\sqrt{3}, -0.066)$ and $(-6 + 2\sqrt{3}, -0.018)$.

The following limits show that the $x$-axis is the horizontal asymptote and the $y$-axis is a vertical asymptote, $\lim_{x\to-\infty} f(x) = 0^-$, $\lim_{x\to\infty} f(x) = 0^+$, $\lim_{x\to 0^-} f(x) = -\infty$, and $\lim_{x\to 0^+} f(x) = \infty$. The final graph is shown to the right.



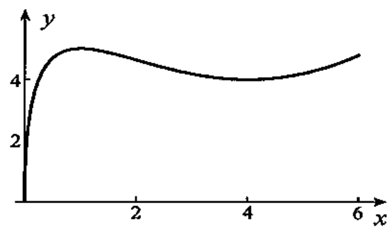



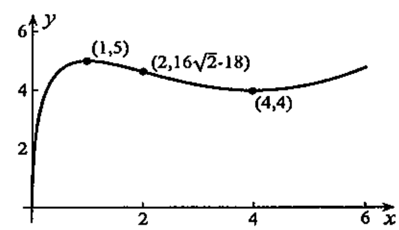

**10.** The plot in the left figure below indicates one relative minimum one relative maximum, and a point of inflection between them. For critical points we solve

$$0 = f'(x) = 3x^{1/2} - 9 + 6x^{-1/2} = \frac{3(x - 3\sqrt{x} + 2)}{\sqrt{x}} = \frac{3(\sqrt{x} - 1)(\sqrt{x} - 2)}{\sqrt{x}}.$$

Solutions are $x = 1$ and $x = 4$. The point $x = 0$ is also critical since $f'(0)$ is not defined but $f(0) = 0$. To classify these critical points we calculate $f''(x) = \dfrac{3}{2\sqrt{x}} - \dfrac{3}{x^{3/2}} = \dfrac{3(x - 2)}{2x^{3/2}}$. Since $f''(1) = -3/2$ and $f''(4) = 3/8$, $f(1) = 5$ is a relative maximum and $f(4) = 4$ is a relative minimum. Because $f(x)$ is not defined for $x < 0$, the critical point $x = 0$ does not give a relative extrema. We note that $\lim_{x\to 0^+} f'(x) = \infty$.

Since $f''(2) = 0$, and $f''(x)$ changes sign as $x$ passes through 2, there is a point of inflection at $(2, 16\sqrt{2} - 18)$. The final graph is shown to the right.



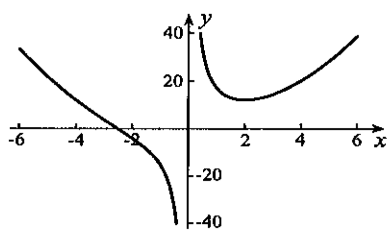

**11.** The plot in the left figure below indicates a relative minimum at or near $x = 2$, and a point of inflection near $x = -2$. For critical points we solve

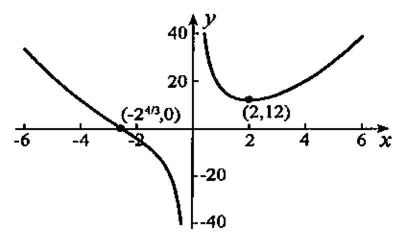
$$0 = f'(x) = \frac{x(3x^2) - (x^3 + 16)(1)}{x^2} = \frac{2(x^3 - 8)}{x^2}.$$

Clearly $x = 2$ is a critical point, and since $f'(x)$ changes from negative to positive as $x$ increases through 2, $f(2) = 12$ is a relative minimum. To determine the point of inflection, we solve

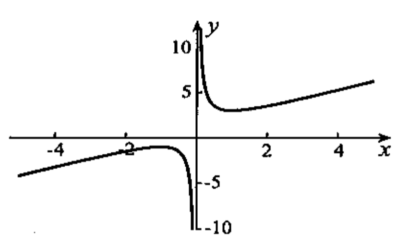
$$0 = f''(x) = 2\left[\frac{x^2(3x^2) - (x^3 - 8)(2x)}{x^4}\right] = \frac{2(x^3 + 16)}{x^3}.$$

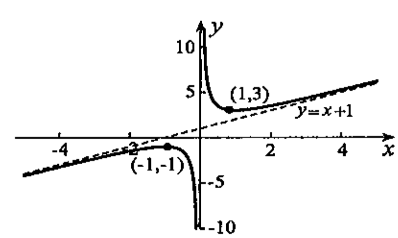
The only solution is $x = -2^{4/3}$. Since $f''(x)$ changes sign as $x$ passes through $-2^{4/3}$, the point of inflection is $(-2^{4/3}, 0)$. The final graph is shown to the right.



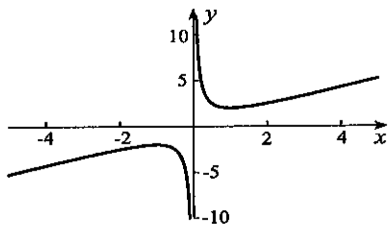



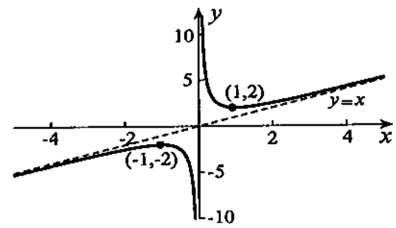

**12.** The plot in the left figure below indicates one relative minimum, one relative maximum, no points of inflection, and an oblique asymptote. By writing $f(x)$ in the form $f(x) = x + 1 + \frac{1}{x}$, we see that the graph is asymptotic to the line $y = x + 1$. Exact locations of the relative extrema can be determined by solving $0 = f'(x) = 1 - \frac{1}{x^2}$. Clearly, $x = \pm 1$. The second derivative is $f''(x) = 2/x^3$. Since $f''(1) = 2$, there is a relative minimum at $x = 1$ of $f(1) = 3$. With $f''(-1) = -2$, there is a relative maximum of $f(-1) = -1$. Since $f''(x)$ is never zero, there can be no points of inflection. The final graph is shown to the right.




**13.** The plot in the left figure below indicates one relative minimum, one relative maximum, no points of inflection, and an oblique asymptote. The graph is asymptotic to the line $y = x$. Exact locations of the relative extrema can be determined by solving $0 = f'(x) = 1 - \frac{1}{x^2}$. Clearly, $x = \pm 1$. The second derivative is $f''(x) = 2/x^3$. Since $f''(1) = 2$, there is a relative minimum at $x = 1$ of $f(1) = 2$. With $f''(-1) = -2$, there is a relative maximum of $f(-1) = -2$. Since $f''(x)$ is never zero, there can be no points of inflection. The final graph is shown to the right.





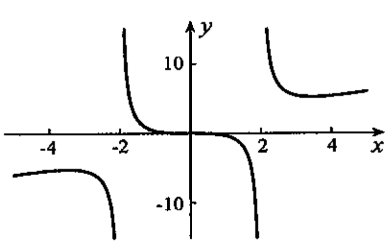

**14.** The plot in the left figure below indicates one relative minimum, one relative maximum, a horizontal point of inflection at $x = 0$, and an oblique asymptote. By writing $f(x)$ in the form $f(x) = x + \dfrac{4x}{x^2-4}$, we see that the graph is asymptotic to the line $y = x$. There are discontinuities at $x = \pm 2$, which yield vertical asymptotes. To identify the relative extrema, we solve

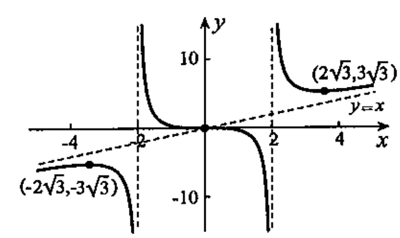
$$0 = f'(x) = \frac{(x^2-4)(3x^2) - x^3(2x)}{(x^2-4)^2} = \frac{x^2(x^2-12)}{(x^2-4)^2}.$$

Solutions are $x = 0,\ \pm 2\sqrt{3}$. The second derivative is

$$f''(x) = \frac{(x^2-4)^2(4x^3-24x) - (x^4-12x^2)(2)(x^2-4)(2x)}{(x^2-4)^4} = \frac{8x(x^2+12)}{(x^2-4)^3}.$$

The fact that $f''(2\sqrt{3}) > 0$ confirms a relative minimum $f(2\sqrt{3}) = 3\sqrt{3}$. Similarly, $f''(-2\sqrt{3}) < 0$ confirms a relative maximum $f(-2\sqrt{3}) = -3\sqrt{3}$. Since $f''(x) = 0$ only at $x = 0$, and $f''(x)$ changes sign as $x$ passes through 0, there is a horizontal point of inflection at $(0, 0)$. The final graph is shown to the right.






**15.** The plot in the left figure below suggests one relative minimum, one relative maximum, one point of inflection, and a horizontal asymptote. The limits $\lim_{x \to \pm\infty} f(x) = 2$, confirm $y = 2$ as a horizontal asymptote. There are discontinuities at $x = 2, 6$, leading to vertical asymptotes. For critical points, we solve

$$0 = f'(x) = \frac{(x^2-8x+12)(4x) - 2x^2(2x-8)}{(x^2-8x+12)^2} = \frac{16x(3-x)}{(x^2-8x+12)^2}.$$

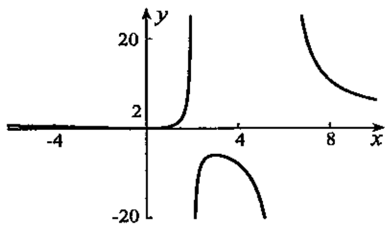
Solutions are $x = 0, 3$. The second derivative is

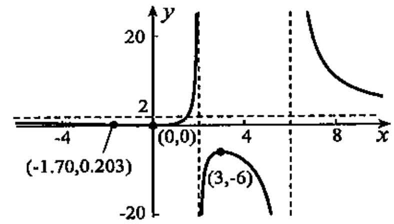
$$f''(x) = \frac{(x^2-8x+12)^2(48-32x) - (48x-16x^2)(2)(x^2-8x+12)(2x-8)}{(x^2-8x+12)^4} = \frac{16(2x^3-9x^2+36)}{(x^2-8x+12)^3}.$$

Since $f''(0) > 0$, there is a relative minimum at $x = 0$ of $f(0) = 0$. With $f''(3) < 0$, there is a relative maximum of $f(3) = -6$. For points of inflection, we would solve $2x^3 - 9x^2 + 36 = 0$. With Newton's iterative procedure, and an inital value $x_1 = -2$, iteration of $x_{n+1} = x_n - \dfrac{2x_n^3 - 9x_n^2 + 36}{6x_n^2 - 18x_n}$ gives

$$x_2 = -1.733, \quad x_3 = -1.704, \quad x_4 = -1.703, \quad x_5 = -1.703.$$

Since $f''(x)$ changes sign as $x$ passes through this solution, there is a point of inflection at $(-1.70, 0.203)$. The final graph is shown to the right.



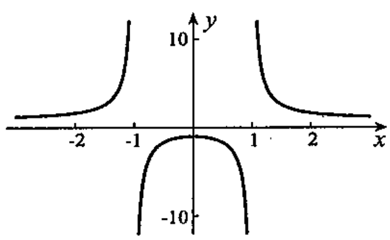

**16.** The plot in the left figure below suggests one relative maximum, no relative minima, no points of inflection, and a horizontal asymptote. The limits $\lim_{x\to\pm\infty} \dfrac{x^2+1}{x^2-1} = 1^+$ confirm $y = 1$ as the horizontal asymptote. There are discontinuities at $x = \pm 1$, leading to vertical asymptotes. For critical points, we solve

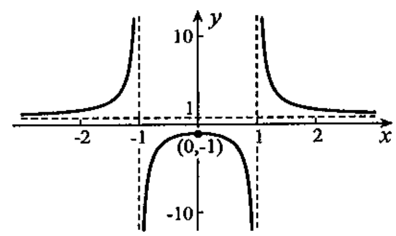
$$0 = f'(x) = \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}.$$

The only solution is $x = 0$. Since $f'(x)$ changes from positive to negative as $x$ increases through 0, there is a relative maximum of $f(0) = -1$. For points of inflection, we solve

$$0 = f''(x) = \frac{(x^2-1)^2(-4) - (-4x)(2)(x^2-1)(2x)}{(x^2-1)^4} = \frac{12x^2+4}{(x^2-1)^3}.$$

There are no solutions confirming no points of inflection. The final graph is shown to the right.



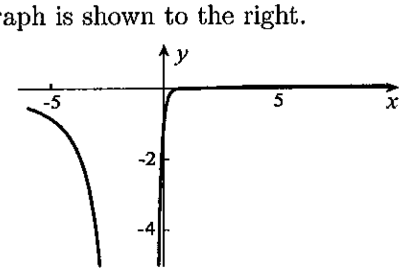

**17.** The plot in the left figure below appears asymptotic to the $x$-axis, but what is happening for $x > 0$ is not clear. The horizontal asymptote is confirmed by the limits $\lim_{x\to\infty} f(x) = 0^+$ and $\lim_{x\to-\infty} f(x) = 0^-$. There is a discontinuity at $x = -1$, leading to a vertical asymptote. For critical points we solve

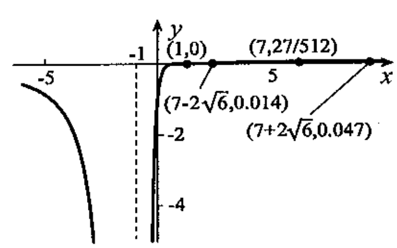
$$0 = f'(x) = \frac{(x+1)^4(3)(x-1)^2 - (x-1)^3(4)(x+1)^3}{(x+1)^8} = \frac{(x-1)^2(7-x)}{(x+1)^5}.$$

Solutions are $x = 1,\ 7$. Since $f'(x)$ changes from positive to negative as $x$ increases through 7, there is a relative maximum at $(7, 27/512)$. The derivative does not change sign at $x = 1$. For points of inflection, we solve

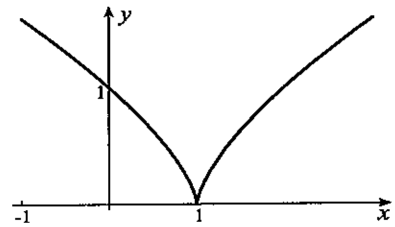
$$0 = f''(x) = \frac{(x+1)^5[2(x-1)(7-x) - (x-1)^2] - (x-1)^2(7-x)(5)(x+1)^4}{(x+1)^{10}} = \frac{2(x-1)(x^2-14x+25)}{(x+1)^6}.$$

Solutions are $x = 1,\ 7 \pm 2\sqrt{6}$. The second derivative changes sign as $x$ passes through each of these. Thus, we have points of inflection at $(1, 0)$ (a horizontal one), and $(7 - 2\sqrt{6}, 0.014)$ and $(7 + 2\sqrt{6}, 0.047)$. The final graph is shown to the right.



**18.** There appears to be a relative minimum at $x = 1$ where the derivative does not exist, and no other significant features to the graph. Because $f(x) > 0$ for all $x \neq 1$, it follows that $f(1)$ is indeed a relative minimum. Since $f'(x) = (2/3)(x-1)^{-1/3}$, we see that $f'(x)$ is not defined at the minimum.



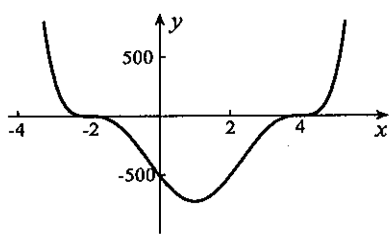

**19.** The plot in the left figure below suggests one relative minimum, two horizontal points of inflection and two others. For critical points we solve

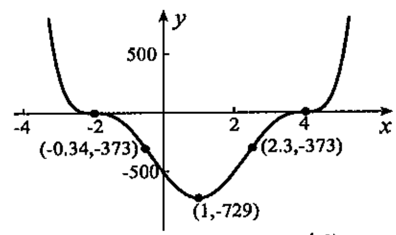
$$0 = f'(x) = 3(x+2)^2(x-4)^3 + 3(x+2)^3(x-4)^2 = 6(x+2)^2(x-4)^2(x-1).$$

Solutions are $x = -2, 1, 4$. The derivative changes from negative to positive as $x$ increases through 1 confirming a relative minimum there of $f(1) = -729$. For points of inflection we solve

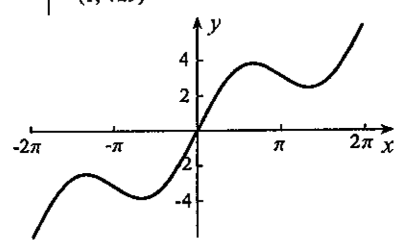
$$\begin{aligned} 0 = f''(x) &= 6[2(x+2)(x-2)^2(x-1) + (x+2)^2(2)(x-4)(x-1) + (x+2)^2(x-4)^2] \\ &= 6(x+2)(x-4)(5x^2-10x-4). \end{aligned}$$

Solutions are $x = -2,\ 4,\ (5 \pm 3\sqrt{5})/5$. Since $f''(x)$ changes sign as $x$ passes through each of these, there are four points of inflection $(-2, 0)$, $(4, 0)$ (both of which are horizontal), and $((5+3\sqrt{5})/5, -373)$ and $((5-3\sqrt{5})/5, -373)$.




**20.** The plot to the right indicates an infinite number of relative extrema and points of inflection. For critical points we solve $0 = f'(x) = 1 + 2\cos x$. Solutions are $x = \pm 2\pi/3 + 2n\pi$, where $n$ is an integer. Relative maxima occur at $(2\pi/3 + 2n\pi, \sqrt{3} + 2\pi/3 + 2n\pi)$, and relative minima at $(-2\pi/3 + 2n\pi, -\sqrt{3} - 2\pi/3 + 2n\pi)$.

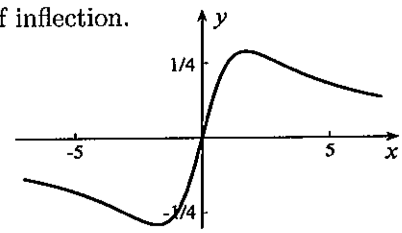



For points of inflection, we solve $0 = f''(x) = -2\sin x$. Solutions are $x = n\pi$, where $n$ is an integer. Since $f''(x)$ changes sign at each of these, they all give points of inflection.

**21.** For critical points we solve

$$0 = f'(x) = \frac{(x^2+3)(1) - x(2x)}{(x^2+3)^2} = \frac{3-x^2}{(x^2+3)^2}.$$

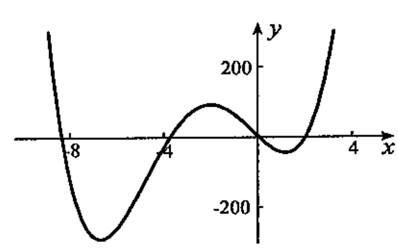
Solutions $x = \pm\sqrt{3}$ give relative extrema at $(\pm\sqrt{3}, \pm\sqrt{3}/6)$.




**22.** For critical points we solve

$$\begin{aligned} 0 = f'(x) &= 4x^3 + 30x^2 + 12x - 64 \\ &= 2(x+2)(2x^2 + 11x - 16). \end{aligned}$$

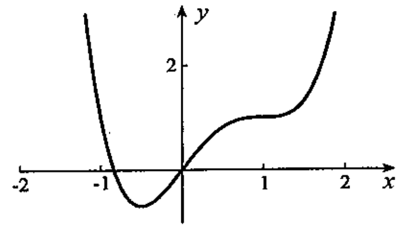
Solutions of this equation are $x = -2$ and $x = (-11 \pm \sqrt{121+128})/4 = (-11 \pm \sqrt{249})/4$. Hence, we have a relative maximum $f(-2) = 93$ and relative minima $f((-11-\sqrt{249})/4) = f(-6.69) = -289.4$ and $f((-11+\sqrt{249})/4) = f(1.19) = -43.8$.



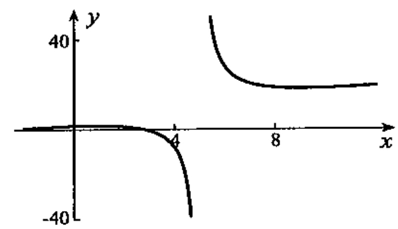
**23.** For critical points we solve

$$0 = f'(x) = 4x^3 - 6x^2 + 2 = 2(x-1)^2(2x+1).$$

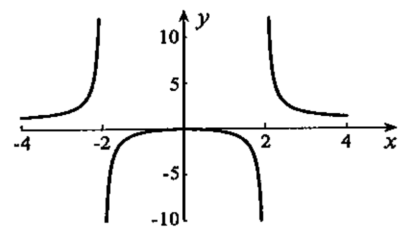
Solutions of this equation are $x = -1/2, 1$. We have a relative minimum $f(-1/2) = -11/16$, but $x = 1$ gives a horizontal point of inflection since $f'(x)$ remains positive as $x$ passes through 1.




**24.** Since $f(x) = x + 5 + 17/(x-5)$, critical points are given by $0 = f'(x) = 1 - 17/(x-5)^2$. Solutions are $x = 5 \pm \sqrt{17}$. These gives relative extrema equal to $10 \pm 2\sqrt{17}$.



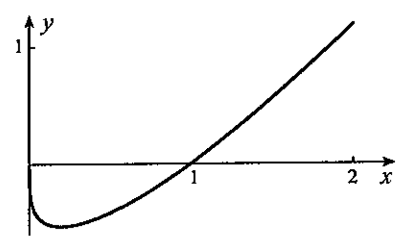

**25.** Since $f'(x) = \dfrac{(x^2-4)(2x) - x^2(2x)}{(x^2-4)^2} = \dfrac{-8x}{(x^2-4)^2}$, the only critical point is $x = 0$ at which there is a relative maximum of $f(0) = 0$.



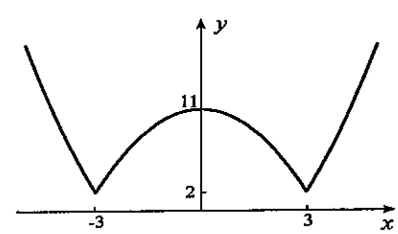

**26.** For critical points we solve

$$0 = f'(x) = \frac{5}{4}x^{1/4} - \frac{1}{4}x^{-3/4} = \frac{5x-1}{4x^{3/4}}.$$

Thus, $x = 1/5$ at which $f(x)$ has a relative minimum of $f(1/5) = -4/5^{5/4}$. There is also critical point at $x = 0$, but because the function is not defined for $x < 0$, it cannot yield a relative maximum.



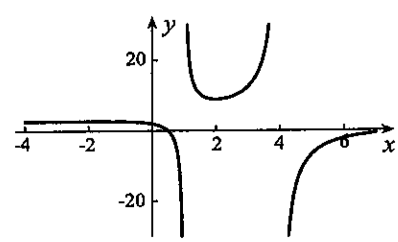

**27.** The derivative is undefined at $x = \pm 3$ where the graph has sharp corners. There are relative minima equal to $f(\pm 3) = 2$ at these points.



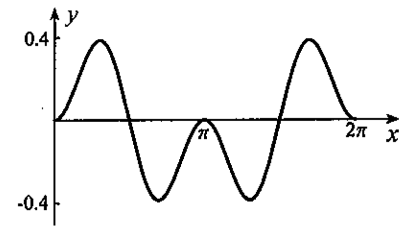

**28.** To find critical points we solve

$$0 = f'(x) = \frac{(x^2-5x+4)(4x-17) - (2x^2-17x+8)(2x-5)}{(x^2-5x+4)^2} = \frac{7(x-2)(x+2)}{(x^2-5x+4)^2}.$$

Solutions are $x = -2, 2$. There is a relative maximum of $f(-2) = 25/9$ and a relative minimum of $f(2) = 9$.



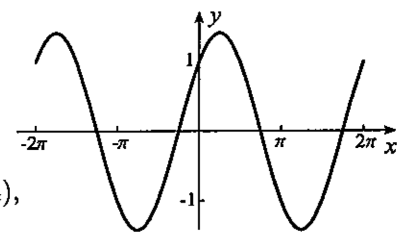
**29.** Since $f'(x) = 2\sin x\cos^2 x - \sin^3 x = \sin x(2\cos^2 x - \sin^2 x) = \sin x(3\cos^2 x - 1)$, critical points are $x = 0, \pi, 2\pi$ and values of $x$ satisfying $\cos x = \pm 1/\sqrt{3}$. Being end points, $x = 0, 2\pi$ cannot yield relative extrema. Relative maxima occur at $\pi$ and the values of $x$ satisfying $\cos x = 1/\sqrt{3}$. Values of these maxima are 0 and $(\pm 2/3)(1/\sqrt{3}) = 2\sqrt{3}/9$. Relative minima occur at the critical points satisfying $\cos x = -1/\sqrt{3}$, and the value of the function at these minima is $(\pm 2/3)(-1/\sqrt{3}) = -2\sqrt{3}/9$.




**30.** This function is most easily analyzed by expressing it as a general sine function $A\sin(x+\phi)$ (see Example 1.45 in Section 1.8),

$$\sin x + \cos x = A\sin(x+\phi) = A(\sin x\cos\phi + \cos x\sin\phi).$$

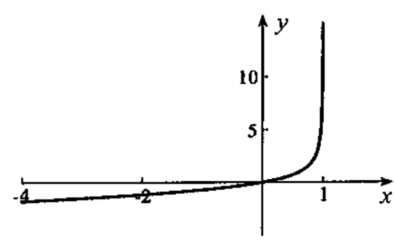
To satisfy this equation we set $A\cos\phi = 1$ and $A\sin\phi = 1$. These are satisfied if we choose $A = \sqrt{2}$ and $\phi = \pi/4$. Thus, $f(x)$ can be expressed in the form $f(x) = \sqrt{2}\sin(x + \pi/4)$, and the relative maxima and minima are $\pm\sqrt{2}$.




**31.** This function has no relative extrema. We can confirm this with

$$f'(x) = \frac{\sqrt{1-x} - x(1/2)(1-x)^{-1/2}(-1)}{1-x} = \frac{2-x}{2(1-x)^{3/2}}.$$

The derivative never vanishes for $x < 1$.

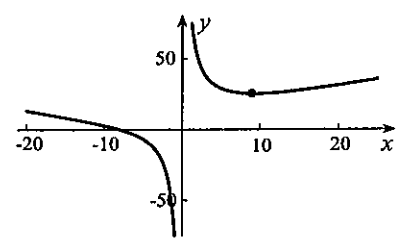



**32.** To identify critical points we solve

$$0 = f'(x) = -\frac{8}{x^2}\sqrt{x^2+100} + \left(1 + \frac{8}{x}\right)\frac{x}{\sqrt{x^2+100}} = \frac{x^3 - 800}{x^2\sqrt{x^2+100}}.$$

The only critical point is $800^{1/3}$ at which there is a relative minimum of

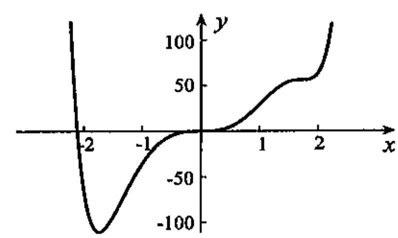
$$f(800^{1/3}) = \left(\frac{8 + 800^{1/3}}{800^{1/3}}\right)\sqrt{100 + 800^{2/3}} \approx 25.4.$$




**33.** (a) A plot of the function is shown to the right.
(b) For critical points we solve

$$0 = f'(x) = 8x^7 - 24x^5 - 40x^4 + 120x^2 = 8x^2(x^2-3)(x^3-5).$$

Solutions are $x = 0, \pm\sqrt{3}, 5^{1/3}$. Since $f'(x)$ changes from a negative quantity to a positive quantity as $x$ increases through $\pm\sqrt{3}$, these critical points give relative minima. Since $f'(x)$ changes from positive to negative as $x$ increases through $5^{1/3}$, there is a relative maximum there. Finally, because $f''(x) = 56x^6 - 120x^4 - 160x^3 + 240x = 8x(7x^5 - 15x^3 - 20x^2 + 30)$, it follows $f''(0) = 0$, and $f''(x)$ changes sign as $x$ increases through 0. Hence, $x = 0$ yields a horizontal point of inflection.

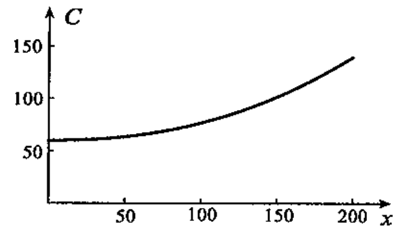


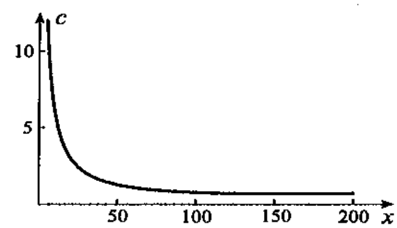
**34.** (a) A plot is shown below in the left figure below. Derivatives of $C(x)$ are

$$C'(x) = \frac{(x+300)(3x^2+200x)-(x^3+100x^2)}{300(x+300)^2} = \frac{x^3+500x^2+30\,000x}{150(x+300)^2},$$

$$C''(x) = \frac{(x+300)^2(3x^2+1000x+30\,000)-(x^3+500x^2+30\,000x)(2x+600)}{150(x+300)^4}.$$

This simplifies to $C''(x) = \dfrac{x^3+900x^2+270\,000x+9\times 10^6}{150(x+300)^3} > 0$. Hence the graph is concave upward.




(b) A plot is shown in the right figure above.

(c)(i) Critical points of $c(x) = \dfrac{x}{300}\left(\dfrac{x+100}{x+300}\right) + \dfrac{60}{x}$ are given by

$$0 = \frac{(x+300)(2x+100)-(x^2+100x)(1)}{300(x+300)^2} - \frac{60}{x^2} = \frac{x^2(x^2+600x+30\,000)-18\,000(x+300)^2}{300x^2(x+300)^2}$$
$$= \frac{x^2(x^2+600x+90\,000)-60\,000x^2-18\,000(x+300)^2}{300x^2(x+300)^2} = \frac{(x+300)^2(x^2-18\,000)-60\,000x^2}{300x^2(x+300)^2}.$$

When we equate the numerator to zero, we obtain the required equation.

(ii) The tangent line to $C(x)$ passes through the origin when the slope of the tangent line is equal to $C(x)/x$,

$$\frac{x}{300}\left(\frac{x+100}{x+300}\right) + \frac{60}{x} = \frac{x^3+500x^2+30\,000x}{150(x+300)^2}.$$

If we multiply by $300x(x+300)^2$,

$$\begin{aligned}
0 &= 2x(x^3+500x^2+30\,000x) - x^2(x+100)(x+300) - 18\,000(x+300)^2\\
&= 2x^2(x^2+500x+30\,000) - x^2(x+300)(x+300) + 200x^2(x+300) - 18\,000(x+300)^2\\
&= 2x^2(x^2+600x+60\,000) - x^2(x+300)^2 - 18\,000(x+300)^2\\
&= 2x^2(x^2+600x+90\,000) - 60\,000x^2 - x^2(x+300)^2 - 18\,000(x+300)^2\\
&= (x+300)^2(x^2-18\,000) - 60\,000x^2.
\end{aligned}$$

## EXERCISES 4.7

**1.** For critical points we solve $0 = f'(x) = 3x^2-2x-5 = (3x-5)(x+1)$. Solutions are $x = -1, 5/3$. Since $f(-2) = 2$, $f(-1) = 7$, $f(5/3) = -67/27$, and $f(3) = 7$, absolute minimum and maximum values are $-67/27$ and 7.

**2.** For critical points we solve $0 = f'(x) = \dfrac{(x+1)(1)-(x-4)(1)}{(x+1)^2} = \dfrac{5}{(x-1)^2}$. Since there are no critical points, we evaluate $f(0) = -4$ and $f(10) = 6/11$. These are the absolute minimum and maximum.

**3.** For critical points we solve $0 = f'(x) = 1 - \dfrac{1}{x^2}$. Solutions are $x = \pm 1$. Since $f(1/2) = 5/2$, $f(1) = 2$, and $f(5) = 26/5$, absolute minimum and maximum values are 2 and 26/5.

**4.** For critical points we solve $0 = f'(x) = 1 - 2\cos x$. Solutions on the given interval are $\pi/3$, $5\pi/3$, $7\pi/3$, and $11\pi/3$. Since

$$f(0) = 0, \quad f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \sqrt{3}, \quad f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sqrt{3},$$

$$f\left(\frac{7\pi}{3}\right) = \frac{7\pi}{3} - \sqrt{3}, \quad f\left(\frac{11\pi}{3}\right) = \frac{11\pi}{3} + \sqrt{3}, \quad f(4\pi) = 4\pi,$$

the absolute minimum is $\pi/3 - \sqrt{3}$ and the absolute maximum is $11\pi/3 + \sqrt{3}$.

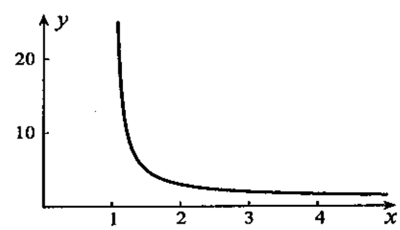
**5.** For critical points we solve $0 = f'(x) = \sqrt{x+1} + \dfrac{x}{2\sqrt{x+1}} = \dfrac{3x+2}{2\sqrt{x+1}}$. The only solution is $x = -2/3$. Since $f(-1) = 0$, $f(-2/3) = -2\sqrt{3}/9$, and $f(1) = \sqrt{2}$, absolute minimum and maximum are $-2\sqrt{3}/9$ and $\sqrt{2}$.

**6.** For critical points we solve $0 = f'(x) = \dfrac{-12(2x+2)}{(x^2+2x+2)^2}$, and obtain $x = -1$. Since

$$\lim_{x\to-\infty} f(x) = 0, \qquad f(-1) = 12, \qquad \lim_{x\to 0^-} f(x) = 6,$$

the absolute maximum is 12 but the function does not have an absolute minimum.

**7.** The graph indicates that the function does not have absolute extrema on this interval.

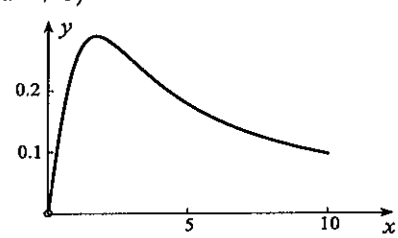



**8.** For critical points we solve $0 = f'(x) = \dfrac{(x^2+3)(1) - x(2x)}{(x^2+3)^2} = \dfrac{3-x^2}{(x^2+3)^2}$.

Only the critical point $x = \sqrt{3}$ is positive. Since

$$\lim_{x\to0^+} f(x) = 0, \quad f(\sqrt{3}) = \frac{\sqrt{3}}{6}, \quad \lim_{x\to\infty} f(x) = 0,$$

the absolute maximum is $\sqrt{3}/6$ but the function does not have an absolute minimum.



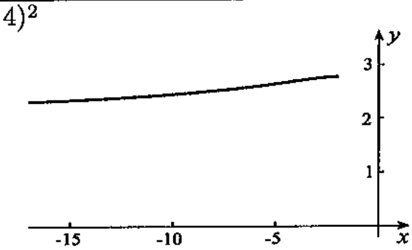

**9.** For critical points, $\quad 0 = f'(x) = \dfrac{(x^2-5x+4)(4x-17) - (2x^2-17x+8)(2x-5)}{(x^2-5x+4)^2}$

$$= \frac{7(x-2)(x+2)}{(x^2-5x+4)^2}.$$

Solutions are $x = -2, 2$. Since

$$f(-2) = \frac{25}{9} \quad \text{and} \quad \lim_{x\to-\infty} f(x) = 2,$$

the absolute maximum is $25/9$, but the function does not have an absolute minimum.




**10.** We separate the proof into three cases:
CASE 1 – $f(x) = k$, a constant. Then $f'(x) = 0$ for all $x$, and $c$ can be chosen as any point in the interval.
CASE 2 – $f(x) > f(a)$ for some $x$ in $a < x < b$. Because $f(x)$ is continuous on $a \le x \le b$, it must have an absolute maximum on this interval, and this maximum must occur at a critical point $c$ interior to the interval. Since $f(x)$ is differentiable at every point in $a < x < b$, it follows that the derivative must vanish at the critical point, $f'(c) = 0$.

CASE 3 – $f(x) < f(a)$ for all $x$ in $a < x < b$. In this case the absolute minimum of $f(x)$ must occur at a critical point $c$ between $a$ and $b$ at which $f'(c) = 0$.

**11.** For critical points of $y(x)$, we solve $0 = 8x^3 - 15Lx^2 + 6L^2x$. Solutions are $x = 0$ and $x = (15L \pm \sqrt{225L^2 - 192L^2})/16 = (15 \pm \sqrt{33})L/16$. Maximum deflection occurs at $x = (15 - \sqrt{33})L/16$.

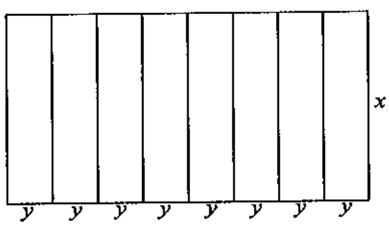
**12.** If $x$ and $y$ represent the length and width of each plot, then the amount of fencing required is $F = 9x+16y$. Since each plot must have area 9000 m$^2$, it follows that $xy = 9000$. Thus, $y = 9000/x$, and

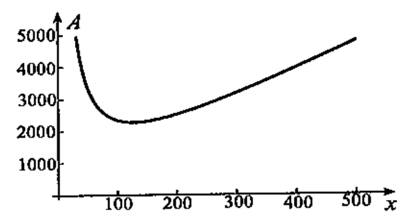
$$F = F(x) = 9x + \frac{144\,000}{x}, \quad x > 0.$$

To find critical points of $F(x)$, we solve $0 = F'(x) = 9 - 144\,000/x^2$. The only positive solution is $x = 40\sqrt{10}$. We now evaluate

$$\lim_{x\to 0^+} F(x) = \infty, \qquad F(40\sqrt{10}) = 720\sqrt{10}, \qquad \lim_{x\to\infty} F(x) = \infty.$$

The minimum amount of fencing is therefore $720\sqrt{10}$ m. The graph of $F(x)$ in the right figure also indicates that $F(x)$ is minimized at its critical point.



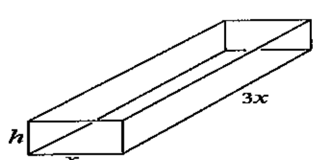

**13.** If $x$ is the width of the box and $h$ is its height (left figure below), the volume of the box is $V = 3x^2h$. Since the surface area must be 30 m$^2$, $x$ and $h$ must satisfy $30 = 2(3x^2) + 2(xh) + 2(3xh) = 6x^2 + 8xh$. Hence, $h = (15 - 3x^2)/(4x)$, and

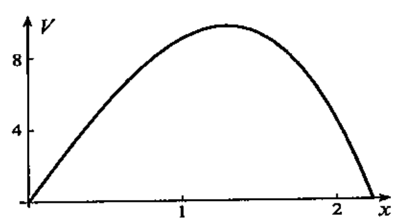
$$V = V(x) = 3x^2\left(\frac{15 - 3x^2}{4x}\right) = \frac{9}{4}(5x - x^3), \quad 0 \le x \le \sqrt{5}.$$

To find critical points of $V(x)$, we solve $0 = V'(x) = (9/4)(5 - 3x^2)$. Solutions are $x = \pm\sqrt{5/3}$. We now evaluate

$$V(0) = 0, \qquad V(\sqrt{15}/3) > 0, \qquad V(\sqrt{5}) = 0.$$

Maximum volume occurs for $x = \sqrt{15}/3$ m, $3x = \sqrt{15}$ m, and $h = \sqrt{15}/2$ m. The graph of $V(x)$ in the right figure below also indicates that $V(x)$ is maximized at its critical point.



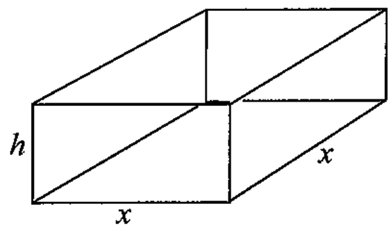

**14.** The surface area of the box is $A = x^2 + 4xh$. Since the volume of the box must be 6000 L or 6 m$^3$, $x$ and $h$ must satisfy $6 = x^2h$. Hence, $h = 6/x^2$, and

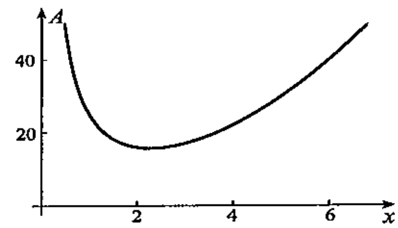
$$A = A(x) = x^2 + 4x\left(\frac{6}{x^2}\right) = x^2 + \frac{24}{x}, \quad x > 0.$$

To find critical points of $A(x)$, we solve $0 = A'(x) = 2x - 24/x^2$. The positive solution is $12^{1/3}$. We now evaluate

$$\lim_{x\to 0^+} A(x) = \infty, \qquad A(12^{1/3}/2) < \infty, \qquad \lim_{x\to\infty} A(x) = \infty.$$

Minimum area therefore occurs when the base of the box is $12^{1/3} \times 12^{1/3}$ m and the height is $6/12^{2/3} = 12^{1/3}/2$ m. The graph of $A(x)$ in the right figure also shows that $A(x)$ is minimized at its critical point.




**15.** If $Y$ represents the yield of the orchard and $x$ the additional number of trees planted, then

$$Y(x) = (255 + x)(25 - x/12), \quad 0 \le x \le 300.$$

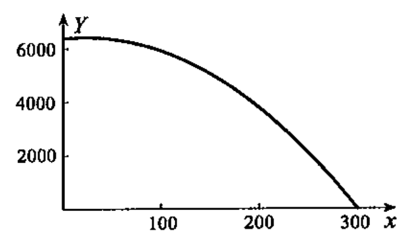
For critical points of this function, we solve

$$0 = Y'(x) = 25 - \frac{255}{12} - \frac{x}{6} \quad \Longrightarrow \quad x = \frac{45}{2}.$$

Since only integer numbers of trees can be planted, we calculate

$$Y(0) = 6375, \qquad Y(22) = 6417.2, \qquad Y(23) = 6417.2, \qquad Y(300) = 0,$$

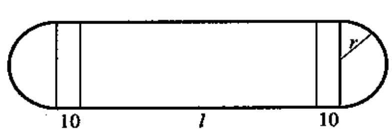
It follows that 22 new trees should be planted (it costing less to plant 22 then 23). The graph of $Y(x)$ to the right also indicates that $Y(x)$ is maximized at its critical point.

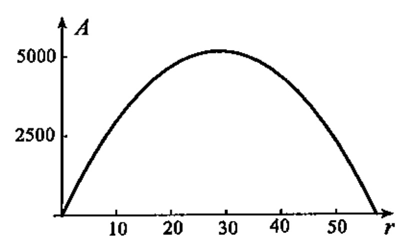



**16.** The area of the playing field (not including end zones) is $A = 2rl$. Since the perimeter must be 400 m, $400 = 2l + 40 + 2\pi r$. Thus, $l = 180 - \pi r$, and $A = A(r) = 2r(180 - \pi r)$. We must choose $r > 0$, and for $l$ to be positive, we must take $r < 180/\pi$. For critical point of $A(r)$, we solve $0 = A'(r) = 360 - 4\pi r$. The solution is $r = 90/\pi$. We now evaluate

$$\lim_{r \to 0^+} A(r) = 0, \qquad A(90/\pi) > 0, \qquad \lim_{r \to 180/\pi^-} A(r) = 0.$$

Thus, $A(r)$ is maximized when the width of the field is $180/\pi$ m and its length is 90 m. The graph of $A(x)$ in the right figure also indicates that $A(x)$ is maximized at its critical point.



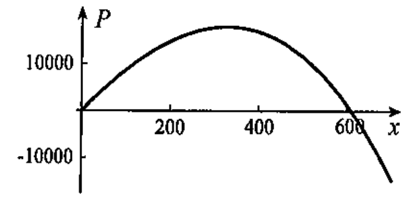

**17.** Profit on the sale of $x$ objects is $P(x) = x\,r(x) - C(x) = 100x - \dfrac{x^3}{10\,000} - \dfrac{x^2}{10} - 2x - 20$, where $0 \le x \le 1000$. For critical points of this function, we solve

$$0 = P'(x) = 100 - \frac{3x^2}{10\,000} - \frac{x}{5} - 2 \qquad \Longrightarrow \qquad 3x^2 + 2000x - 980\,000 = 0.$$

The positive solution of this equation is $x = 328.3$. Since $x$ must be an integer, we evaluate

$$P(0) = -20, \qquad P(328) = 17\,836.84, \qquad P(329) = 17\,836.77, \qquad P(1000) < 0.$$

The company should produce and sell 328 objects. The graph of $P(x)$ in the figure to the right also indicates that $P(x)$ is maximized at the critical point.

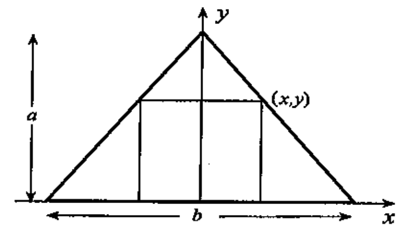



**18.** The area of the rectangle in the diagram is $A = 2xy$. Since the equation of the line containing the point $(x, y)$ is $y = a - 2ax/b$, we can express $A$ in terms of $x$ as follows

$$A(x) = 2x\left(a - \frac{2ax}{b}\right) = 2a\left(x - \frac{2x^2}{b}\right), \quad 0 \le x \le b/2.$$

For critical points of $A(x)$ we solve

$$0 = A'(x) = 2a\left(1 - \frac{4x}{b}\right) \Longrightarrow x = \frac{b}{4}.$$




Since $A(0) = 0$, $A(b/4) = 2a\left(\frac{b}{4} - \frac{b}{8}\right) = \frac{ab}{4}$, and $A(b/2) = 0$, maximum area is $ab/4$.

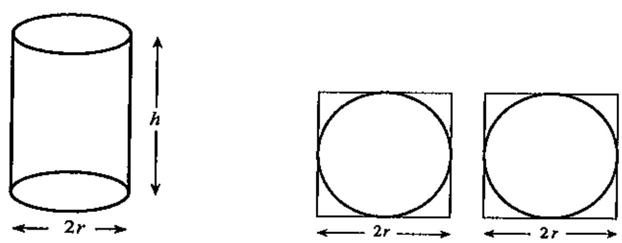
**19.** If $r$ and $h$ are the radius and height of the can (in centimetres), then the area of metal use to make the can is $A = 2\pi rh + 8r^2$. Because the can must hold 1000 cm$^3$, it follows that $1000 = \pi r^2 h \Longrightarrow h = 1000/(\pi r^2)$. Thus,

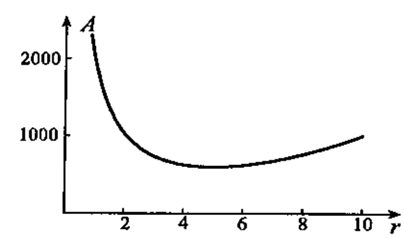
$$A = A(r) = 8r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 8r^2 + \frac{2000}{r}, \quad r > 0.$$

To find critical points of $A(r)$, we solve $0 = A'(r) = 16r - 2000/r^2 \Longrightarrow r = 5$. We now evaluate

$$\lim_{r \to 0+} A(r) = \infty, \qquad A(5) < \infty, \qquad \lim_{r \to \infty} A(r) = \infty.$$

Thus, metal is minimized when $r = 5$ cm and $h = 40/\pi$ cm. The graph of $A(r)$ in the right figure also indicates that $A(r)$ is minimized at its critical point,



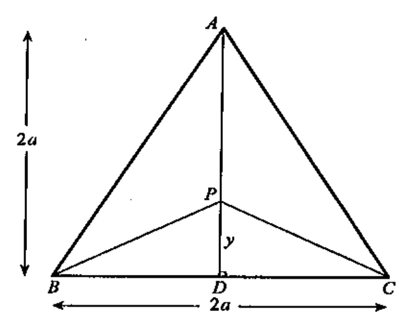

**20.** When $P$ is at height $y$, the sum of the lengths of $AP$, $BP$, and $CP$ is

$$L = L(y) = (2a - y) + 2\sqrt{a^2 + y^2}, \quad 0 \le y \le 2a.$$

For critical points of $L(y)$ we solve

$$0 = \frac{dL}{dy} = -1 + \frac{2y}{\sqrt{a^2 + y^2}}.$$

This equation can be expressed in the form $2y = \sqrt{a^2 + y^2}$, and squaring gives $4y^2 = a^2 + y^2$, the positive solutions of which is $y = a/\sqrt{3}$. Since




$$L(0) = 2a + 2a = 4a,\ L\left(\frac{a}{\sqrt{3}}\right) = 2a - \frac{a}{\sqrt{3}} + 2\sqrt{a^2 + \frac{a^2}{3}} = a(2 + \sqrt{3}),\ L(2a) = 2\sqrt{a^2 + 4a^2} = 2\sqrt{5}a,$$

minimum length is attained when $y = a/\sqrt{3}$.

**21.** When the loop is $x$ m from the longer pole, the length of rope is

$$L(x) = \sqrt{x^2+4} + \sqrt{(3-x)^2+1}, \quad 0 \le x \le 3.$$

To find critical points, we solve

$$0 = \frac{dL}{dx} = \frac{x}{\sqrt{x^2+4}} + \frac{x-3}{\sqrt{(3-x)^2+1}} \quad \Longrightarrow \quad x\sqrt{(3-x)^2+1} = -(x-3)\sqrt{x^2+4}.$$

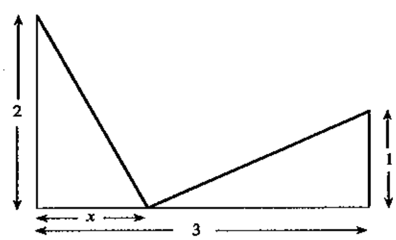
When we square this equation,

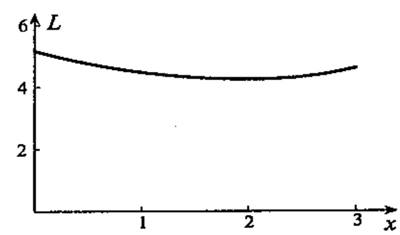
$$x^2(10-6x+x^2) = (x^2-6x+9)(x^2+4) \quad \Longrightarrow \quad 0 = 3x^2-24x+36 = 3(x-2)(x-6).$$

Only the solution $x=2$ is acceptable. We now evaluate

$$L(0) = 2+\sqrt{10}, \qquad L(2) = 3\sqrt{2}, \qquad L(3) = 1+\sqrt{13}.$$

Thus, the loop should be placed 2 m from the taller pole. The graph of $L(x)$ in the right figure also indicates that it is minimized at its critical point.




**22.** If length and width of the page are denoted by $y$ and $x$, then the area of the page is $A = xy$. Since the area of the printed portion of the page must be 150 cm$^2$, we must have $150 = (x-5)(y-7.5)$. This equation can be solved for $x = (10y+225)/(2y-15)$, and therefore

$$A = A(y) = \frac{10y^2+225y}{2y-15}, \quad y > \frac{15}{2}.$$

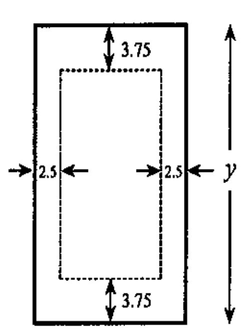
To find critical points, we solve

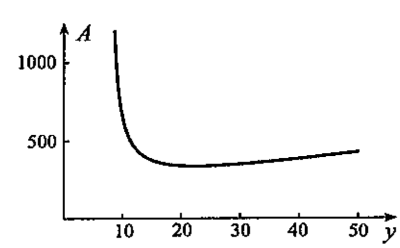
$$0 = A'(y) = \frac{(2y-15)(20y+225)-(10y^2+225y)(2)}{(2y-15)^2}.$$

When we equate the numerator to 0, we obtain $0 = 20y^2-300y-3375 = 5(2y-45)(2y+15) \Longrightarrow y = 45/2$. We now evaluate

$$\lim_{y\to 0^+} A(y) = \infty, \qquad A(45/2) < \infty, \qquad \lim_{y\to\infty} A(y) = \infty.$$

Hence, area is smallest when $y = 45/2$ cm and $x = 15$ cm. The graph of $A(y)$ in the right figure also indicates that $A(y)$ is minimized at its critical point.



**23.** If $Q(x, y)$ is any point on the hyperbola, the length of line $PQ$ is

$$L = L(x) = \sqrt{(x-4)^2 + y^2} = \sqrt{(x-4)^2 + x^2 + 9} = \sqrt{2x^2 - 8x + 25}, \quad 0 \le x < \infty.$$

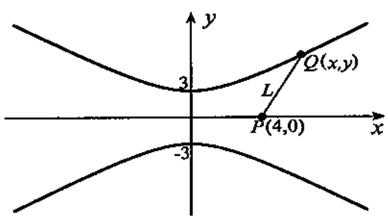
To find critical points, we solve

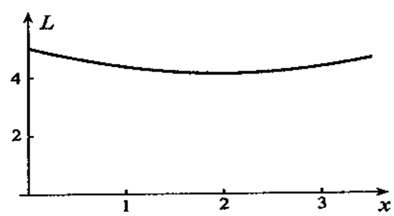
$$0 = \frac{dL}{dx} = \frac{4x - 8}{2\sqrt{2x^2 - 8x + 25}}.$$

The only solution is $x = 2$. We now evaluate

$$L(0) = 5, \qquad L(2) = \sqrt{17}, \qquad \lim_{x \to \infty} L(x) = \infty.$$

Thus, the points on the hyperbola closest to $(4, 0)$ are $(2, \pm\sqrt{13})$. The graph of $L(x)$ to the right also shows that the minimum of $L(x)$ occurs at its critical point.



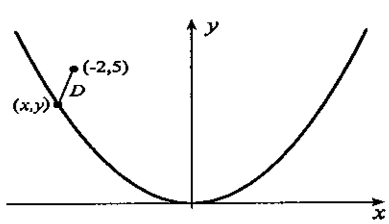

**24.** If $(x, y)$ is any point on the parabola, then the distance $D$ from $(-2, 5)$ to $(x, y)$ is given by $D^2 = (x+2)^2 + (y-5)^2 = (x+2)^2 + (x^2-5)^2$. To minimize $D$ we minimize $D^2$ on the interval $x \le 0$. For critical points, we solve

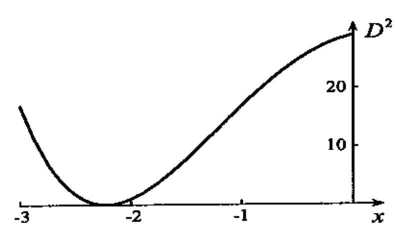
$$0 = \frac{d}{dx}D^2 = 2(x+2) + 2(x^2-5)(2x) = 2(2x^3 - 9x + 2) = 2(x-2)(2x^2 + 4x - 1).$$

The only negative solution is $x = -1 - \sqrt{6}/2$. We now evaluate

$$\lim_{x \to -\infty} D^2(x) = \infty, \qquad D^2(-1 - \sqrt{6}/2) = 0.05, \qquad D^2(0) = 29.$$

The point closest to $(-2, 5)$ is $(-1 - \sqrt{6}/2, 5/2 + \sqrt{6})$. The graph of $D^2$ to the right also shows that $D^2$ is minimized at its critical point.




**25.** The illumination at height $h$ above the table is

$$I = I(h) = \frac{k \cos\theta}{d^2} = \frac{kh}{d^3} = \frac{kh}{(r^2 + h^2)^{3/2}}, \quad h \ge 0,$$

where $k$ is a constant. For critical points we solve

$$0 = \frac{dI}{dh} = k\left[\frac{(r^2+h^2)^{3/2} - h(3/2)\sqrt{r^2+h^2}(2h)}{(r^2+h^2)^3}\right] = \frac{k(r^2 - 2h^2)}{(r^2+h^2)^{5/2}}.$$

The positive solution is $h = r/\sqrt{2}$. Since $\quad I(0) = 0, \quad I(r/\sqrt{2}) = \dfrac{2\sqrt{3}k}{9r^2}, \quad \lim_{h \to \infty} I(h) = 0, \quad$ maximum illumination occurs when $h = r/\sqrt{2}$.

**26.** The area of the triangle is $A = xy/2$. Since slopes of the line segments joining $(0, y)$ to $(x, 0)$ and $(2, 5)$ to $(x, 0)$ must be the same, we have $\dfrac{y-0}{0-x} = \dfrac{0-5}{x-2} \Longrightarrow y = \dfrac{5x}{x-2}$. Thus,

$$A = A(x) = \frac{5x^2}{2(x-2)}, \quad x > 2.$$

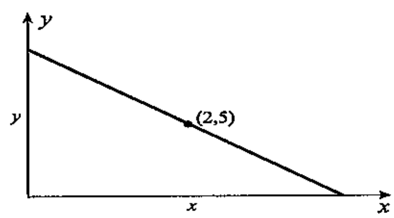
For critical points, we solve

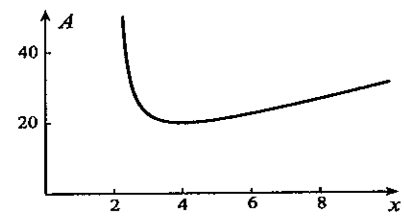
$$0 = A'(x) = \frac{5}{2}\left[\frac{(x-2)(2x) - x^2(1)}{(x-2)^2}\right] = \frac{5x(x-4)}{2(x-2)^2}.$$

We now evaluate

$$\lim_{x\to 2^+} A(x) = \infty, \qquad A(4) < \infty, \qquad \lim_{x\to\infty} A(x) = \infty.$$

Area is minimized when $x = 4$ and $y = 10$. The graph of $A(x)$ in the right figure also indicates that $A(x)$ is minimized at its critical point.




**27.** Maximum flow occurs when area $ABCD$ is a maximum. If we denote the area by $a(\theta)$, then

$$a(\theta) = \|BE\|\left(\frac{\|AD\| + \|BC\|}{2}\right) = \frac{1}{6}\sin\theta(2\|AE\| + 2\|BC\|) = \frac{1}{3}\sin\theta\left(\frac{1}{3}\cos\theta + \frac{1}{3}\right)$$
$$= \frac{1}{9}\sin\theta(1 + \cos\theta), \quad 0 \le \theta \le \pi/2.$$

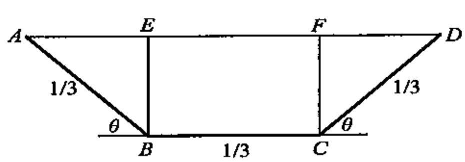
To find critical points, we solve

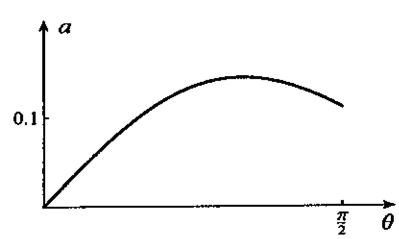
$$0 = \frac{da}{d\theta} = \frac{1}{9}\cos\theta(1+\cos\theta) + \frac{1}{9}\sin\theta(-\sin\theta) = \frac{1}{9}(\cos\theta + \cos^2\theta - \sin^2\theta)$$
$$= \frac{1}{9}(\cos\theta + 2\cos^2\theta - 1) = \frac{1}{9}(2\cos\theta - 1)(\cos\theta + 1).$$

The only solution in the interval $0 \le \theta \le \pi/2$ is $\theta = \pi/3$. We now evaluate

$$a(0) = 0, \qquad a(\pi/3) = \frac{\sqrt{3}}{12}, \qquad a(\pi/2) = \frac{1}{9}.$$

Thus, maximum flow occurs when $\theta = \pi/3$. The graph of $a(\theta)$ in the right figure also indicates that $a(\theta)$ is maximized at its critical point.



**28.** The manufacturer's yearly costs for $x$ employees is $C = 20\,000x + 365y$. When 1000 automobiles are to be produced, we must have $1000 = x^{2/5}y^{3/5} \Longrightarrow y = 10^5/x^{2/3}$, and therefore

$$C(x) = 20\,000x + \frac{365 \times 10^5}{x^{2/3}}, \quad x > 0.$$

For critical points of $C(x)$, we set

$$0 = C'(x) = 20\,000 + 365 \times 10^5 \left(\frac{-2}{3}\right) x^{-5/3} \quad \Longrightarrow \quad x = 70.97.$$

Since $x$ must be an integer, we evalulate

$$\lim_{x \to 0^+} C(x) = \infty, \quad C(70) = 3.5490 \times 10^6, \quad C(71) = 3.5487 \times 10^6, \quad \lim_{x \to \infty} C(x) = \infty.$$

It follows that $C(x)$ is minimized for $x = 71$.

**29.** Since critical points of $Q(p)$ are the same as those of $Q^2(p)$, we solve

$$0 = \frac{dQ^2}{dp} = \frac{2A^2\gamma p_0 \rho_0}{\gamma - 1}\left[\frac{2}{\gamma}\left(\frac{p}{p_0}\right)^{2/\gamma - 1}\left(\frac{1}{p_0}\right) - \left(\frac{\gamma+1}{\gamma}\right)\left(\frac{p}{p_0}\right)^{1/\gamma}\left(\frac{1}{p_0}\right)\right].$$

Consequently,

$$2\left(\frac{p}{p_0}\right)^{2/\gamma - 1} = (\gamma+1)\left(\frac{p}{p_0}\right)^{1/\gamma} \Longrightarrow \left(\frac{p}{p_0}\right)^{1/\gamma - 1} = \frac{\gamma+1}{2} \Longrightarrow p = p_0\left(\frac{\gamma+1}{2}\right)^{\gamma/(1-\gamma)} = p_0\left(\frac{2}{1+\gamma}\right)^{\gamma/(\gamma-1)}.$$

Since $Q(0) = 0 = Q(p_0)$, it follows that this critical point must yield a maximum for $Q$, and is therefore the critical pressure $p_c$.

**30.** The printer will choose the number of set types that will minimize production costs. If $x$ set types are used, then the cost for the set types themselves is $2x$ dollars. With this number of set types, the press prints $1000x$ cards per hour. In order to produce 200 000 cards, the press must therefore run for $200\,000/(1000x)$ hours at a cost of $[10][200\,000/(1000x)]$ dollars. The total cost of producing the cards when $x$ set types are used is therefore given by

$$C(x) = 2x + \frac{2000}{x}, \quad 1 \le x \le 40.$$

To find critical points of $C(x)$, we set

$$0 = C'(x) = 2 - \frac{2000}{x^2}.$$

The only solution of this equation in the interval $1 \le x \le 40$ is $x = 10\sqrt{10} = 31.6$. Since $x$ must be an integer, we evaluate

$$C(1) = 2002, \quad C(31) = 126.52, \quad C(32) = 126.50, \quad C(40) = 130.$$

The printer should therefore use 32 set types.

**31.** The $x$ and $y$-coordinates of $S_2$ and $S_1$, in kilometres, as functions of time are, respectively, $x = 8t$ and $y = 20 - 6t$. The distance between the ships is therefore

$$D(t) = \sqrt{x^2 + y^2} = \sqrt{64t^2 + (20 - 6t)^2} = \sqrt{100t^2 - 240t + 400}, \quad t \ge 0.$$

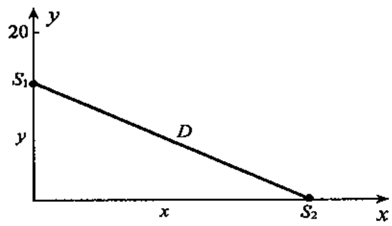
For critical points, we solve

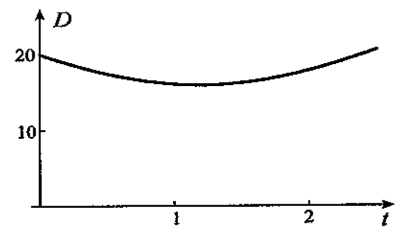
$$0 = \frac{dD}{dt} = \frac{200t - 240}{2\sqrt{100t^2 - 240t + 400}}.$$

The solution is $t = 6/5$. We now evaluate

$$D(0) = 20, \qquad D(6/5) = 16, \qquad \lim_{t \to \infty} D(t) = \infty.$$

The ships are closest together at 1:12 p.m. The graph of $D(t)$ in the right figure also indicates that $D(t)$ is minimized at its critical point.




**32.** (a) If the courier heads to point $R$, his travel time is

$$T(x) = \frac{\sqrt{x^2+36}}{14} + \frac{3-x}{50}, \quad 0 \le x \le 3.$$

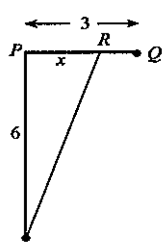
To find critical points, we solve

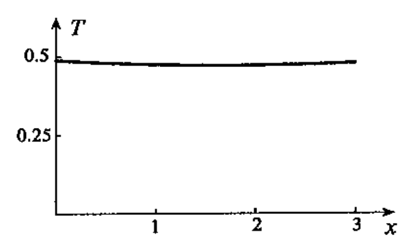
$$0 = T'(x) = \frac{x}{14\sqrt{x^2+36}} - \frac{1}{50}.$$

This equation can be expressed in the form $25x = 7\sqrt{x^2+36}$, and squaring gives $625x^2 = 49(x^2+36)$. The positive solution of this equation is $x = 7/4$.
We now evaluate

$$T(0) = 0.49, \qquad T(7/4) = 0.47, \qquad T(3) = 0.48.$$

Thus, travel time is minimized when the courier heads to the point on the road 7/4 km from $P$. The graph of $T(x)$ function in the right figure also indicates that $T(x)$ is minimized at its critical point.




(b) In this case travel time is given by the formula

$$T(x) = \frac{\sqrt{x^2+36}}{14} + \frac{1-x}{50}, \quad 0 \le x \le 1.$$

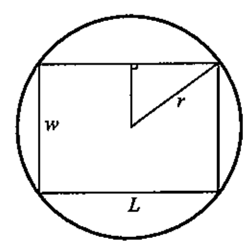
The derivative of this function is identical to that in part (a), but in this case the point $x = 7/4$ must be rejected. Since $T(0) = 6/14 + 1/50 = 0.4486$ and $T(1) = \sqrt{37}/14 = 0.4345$, the courier should head directly to $Q$.

**33.** If the length and width of an inscribed rectangle are $L$ and $w$, then its area is $A = Lw$. Since $r^2 = (L/2)^2 + (w/2)^2$, it follows that
$L = \sqrt{4r^2 - w^2}$, and

$$A(w) = w\sqrt{4r^2-w^2}, \quad 0 \le w \le 2r.$$

For critical points of $A(w)$, we solve

$$0 = \frac{dA}{dw} = \sqrt{4r^2-w^2} + \frac{w(-2w)}{2\sqrt{4r^2-w^2}} = \frac{4r^2-2w^2}{\sqrt{4r^2-w^2}}.$$



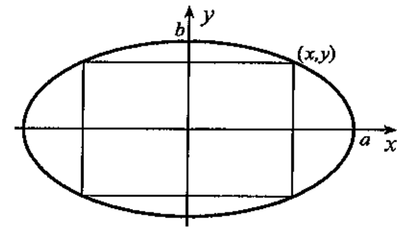

The positive solution is $w = \sqrt{2}r$. Since

$$A(0) = 0, \qquad A(\sqrt{2}r) = \sqrt{2}r\sqrt{4r^2-2r^2} = 2r^2, \qquad A(2r) = 0,$$

the area of the largest rectangle is $2r^2$.

**34.** The area of the rectangle is $A = 4xy$. When we solve the equation of the ellipse for the positive value of $y$, the result is $y = (b/a)\sqrt{a^2 - x^2}$. The area of the rectangle can therefore be expressed in the form

$$A = A(x) = \frac{4bx}{a}\sqrt{a^2 - x^2}, \quad 0 \le x \le a.$$




For critical points of $A(x)$ we solve

$$0 = A'(x) = \frac{4b}{a}\left(\sqrt{a^2 - x^2} - \frac{x^2}{\sqrt{a^2 - x^2}}\right).$$

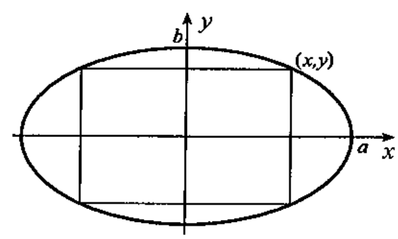
This equation can be expressed in the form $\sqrt{a^2 - x^2} = \dfrac{x^2}{\sqrt{a^2 - x^2}}$, from which $a^2 - x^2 = x^2$. The positive solution is $x = a/\sqrt{2}$. Since

$$A(0) = 0, \qquad A\left(\frac{a}{\sqrt{2}}\right) > 0, \qquad A(a) = 0,$$

area is maximized when the length of the rectangle in the $x$-direction is $\sqrt{2}a$ and that in the $y$-direction is $\sqrt{2}b$.

**35.** The perimeter of the rectangle shown is

$$P(x) = 4x + 4y = 4x + \frac{4b}{a}\sqrt{a^2 - x^2}, \; 0 \le x \le a.$$




For critical points of this function, we solve

$$0 = \frac{dP}{dx} = 4 + \frac{-4bx}{a\sqrt{a^2 - x^2}} = \frac{4a\sqrt{a^2 - x^2} - 4bx}{a\sqrt{a^2 - x^2}}.$$

This implies that $a^2(a^2 - x^2) = b^2x^2$, from which $x = a^2/\sqrt{a^2 + b^2}$. Since

$$P(0) = 4b, \quad P\left(\frac{a^2}{\sqrt{a^2 + b^2}}\right) = \frac{4a^2}{\sqrt{a^2 + b^2}} + \frac{4b}{a}\sqrt{a^2 - \frac{a^4}{a^2 + b^2}} = 4\sqrt{a^2 + b^2}, \quad P(a) = 4a,$$

maximum perimeter occurs for $x = a^2/\sqrt{a^2 + b^2}$.

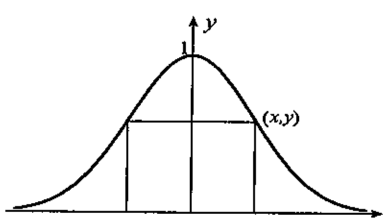
**36.** The area of the rectangle is $A = 2xy$. Since $y = e^{-x^2}$, area can be expressed as $A = A(x) = 2xe^{-x^2}$, $x \ge 0$. To find critical points, we solve

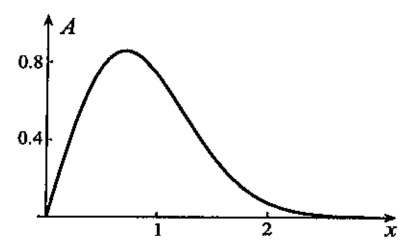
$$0 = A'(x) = 2e^{-x^2} - 4x^2e^{-x^2} = 2(1 - 2x^2)e^{-x^2}.$$

The positive solution of this equation is $x = 1/\sqrt{2}$. We now evaluate

$$A(0) = 0, \qquad T(1/\sqrt{2}) = \sqrt{2/e}, \qquad \lim_{x\to\infty} A(x) = 0.$$

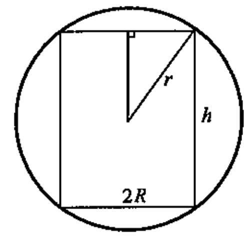
Maximum area is therefore $\sqrt{2/e}$. The graph of $A(x)$ also indicates that it is maximized at its critical point.



**37.** The volume of the cylinder, the cross section of which is shown, is $V = \pi R^2 h$. Since $r^2 = R^2 + h^2/4$,

$$V(h) = \pi\left(r^2 - \frac{h^2}{4}\right)h, \quad 0 \le h \le 2r.$$




For critical points of $V(h)$, we solve

$$0 = \frac{dV}{dh} = \pi\left(r^2 - \frac{3h^2}{4}\right).$$

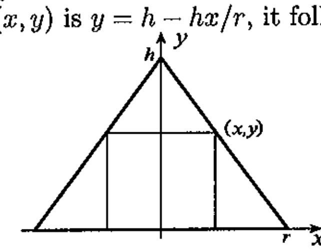
The positive solution is $h = 2r/\sqrt{3}$. Since

$$V(0) = 0, \qquad V(2r/\sqrt{3}) = \pi\left(r^2 - \frac{r^2}{3}\right)\left(\frac{2r}{\sqrt{3}}\right) = \frac{4\pi r^3}{3\sqrt{3}}, \qquad V(2r) = 0,$$

the largest cylinder has volume $4\pi r^3/(3\sqrt{3})$.

**38.** We have shown cross-sections of the cone and an inscribed cylinder. The volume of the cylinder is $V = \pi x^2 y$. Since the equation of the line containing the point $(x, y)$ is $y = h - hx/r$, it follows that

$$V = \pi x^2\left(h - \frac{hx}{r}\right) = \pi h\left(x^2 - \frac{x^3}{r}\right), \quad 0 \le x \le r.$$




For critical points of $V(x)$ we solve

$$0 = \frac{dV}{dx} = \pi h\left(2x - \frac{3x^2}{r}\right).$$

The positive solution is $x = 2r/3$. Since

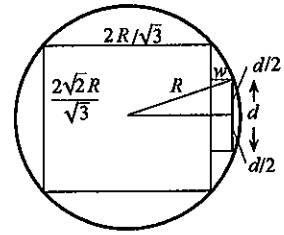
$$V(0) = 0, \qquad V\left(\frac{2r}{3}\right) = \pi h\left(\frac{4r^2}{9} - \frac{8r^3}{27r}\right) = \frac{4\pi h r^2}{27}, \qquad V(r) = 0,$$

maximum volume for the cylinder is $4\pi h r^2/27$.

**39.** The strength of the beams is $S = kwd^2$. Since $\left(\frac{d}{2}\right)^2 + \left(w + \frac{R}{\sqrt{3}}\right)^2 = R^2$,

it follows that

$$d^2 = 4R^2 - 4\left(w + \frac{R}{\sqrt{3}}\right)^2,$$




and

$$S = kw\left[4R^2 - 4\left(w + \frac{R}{\sqrt{3}}\right)^2\right], \; 0 \le w \le R - \frac{R}{\sqrt{3}}.$$

For critical points,

$$0 = \frac{dS}{dw} = k\left[4R^2 - 4\left(w + \frac{R}{\sqrt{3}}\right)^2\right] + kw\left[-8\left(w + \frac{R}{\sqrt{3}}\right)\right] = -\frac{4k}{3}(9w^2 + 4\sqrt{3}Rw - 2R^2).$$

Consequently, $w = (-4\sqrt{3}R \pm \sqrt{48R^2 + 72R^2})/18$, but only $w = (\sqrt{30} - 2\sqrt{3})R/9$ is acceptable. Since $S(0) = S(R - R/\sqrt{3}) = 0$, the strongest beams occur when widths are $w = (\sqrt{30} - 2\sqrt{3})R/9$ cm and depths are

$$d = \sqrt{4R^2 - 4\left[\frac{(\sqrt{30} - 2\sqrt{3})R}{9} + \frac{R}{\sqrt{3}}\right]^2} = \frac{2\sqrt{48 - 6\sqrt{10}}R}{9} \text{ cm}.$$

**40.** For critical points we solve

$$0 = \frac{dR}{d\theta} = \frac{v^2}{g}\left(-\sin^2\theta + \cos^2\theta - \sin\theta\sqrt{\sin^2\theta + \frac{2gh}{v^2}} + \frac{\sin\theta\cos^2\theta}{\sqrt{\sin^2\theta + 2gh/v^2}}\right).$$

From this equation,

$$(\sin^2\theta - \cos^2\theta)\sqrt{\sin^2\theta + \frac{2gh}{v^2}} = \sin\theta\cos^2\theta - \sin\theta\left(\sin^2\theta + \frac{2gh}{v^2}\right)$$
$$= \sin\theta(\cos^2\theta - \sin^2\theta) - \frac{2gh}{v^2}\sin\theta.$$

Hence, $\quad -\cos 2\theta\sqrt{\sin^2\theta + \dfrac{2gh}{v^2}} = \sin\theta\cos 2\theta - \dfrac{2gh}{v^2}\sin\theta$. Squaring this gives

$$\cos^2 2\theta\left(\sin^2\theta + \frac{2gh}{v^2}\right) = \sin^2\theta\cos^2 2\theta - \frac{4gh}{v^2}\sin^2\theta\cos 2\theta + \frac{4g^2h^2}{v^4}\sin^2\theta,$$

from which

$$\cos^2 2\theta = \frac{2gh}{v^2}\sin^2\theta - 2\sin^2\theta\cos 2\theta = \left(\frac{2gh}{v^2} - 2\cos 2\theta\right)\left(\frac{1-\cos 2\theta}{2}\right)$$
$$= \frac{gh}{v^2} - \cos 2\theta - \frac{gh}{v^2}\cos 2\theta + \cos^2 2\theta.$$

Thus, $\cos 2\theta = \dfrac{gh/v^2}{1 + gh/v^2} = \dfrac{gh}{v^2 + gh}$. The only solution of this equation in the interval $0 < \theta < \pi/2$ is $\theta = \dfrac{1}{2}\mathrm{Cos}^{-1}\left(\dfrac{gh}{v^2 + gh}\right)$. It is geometrically clear that there is an angle between $\theta = 0$ and $\theta = \pi/2$ that maximizes $R$, and since only one critical point has been obtained, it must maximize $R$. For $v = 13.7$ m/s and $h = 2.25$ m, $\theta = (1/2)\mathrm{Cos}^{-1}\{9.81(2.25)/[13.7^2 + 9.81(2.25)]\} = 0.733$ radians.

**41.** The longest beam that can be transported around corner $C$ is the shortest of all line segments that touch the walls at $A$ and $B$ and pass through $C$. If $L$ is the length of $AB$, then

$$L(\theta) = \|AC\| + \|BC\| = 6\csc\theta + 3\sec\theta, \quad 0 < \theta < \pi/2.$$

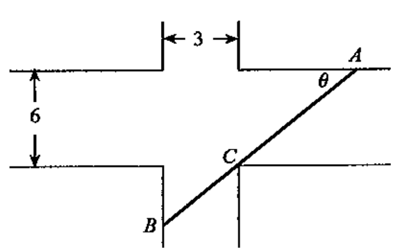
For critical points, we solve

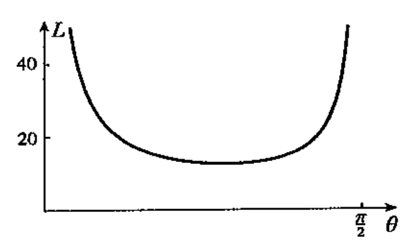
$$0 = L'(\theta) = -6\csc\theta\cot\theta + 3\sec\theta\tan\theta \quad\Longrightarrow\quad \frac{6\cos\theta}{\sin^2\theta} = \frac{3\sin\theta}{\cos^2\theta}.$$

This equation implies that $\tan\theta = 2^{1/3}$. The acute angle satisfying this equation is $\theta = 0.90$ radians. We now evaluate

$$\lim_{\theta\to 0^+} L(\theta) = \infty, \qquad L(0.90) = 12.5, \qquad \lim_{\theta\to\pi/2^-} L(\theta) = \infty.$$

The longest beam that can be transported around the corner is 12.5 m. The graph of $L(\theta)$ also indicates that it is minimized at its critical point.




**42.** Length $L$ of the beam is the sum of the lengths $L_1$ and $L_2$. Now,

$$L_1 = \|FD\|\sec\theta = \sec\theta\left(3 - \frac{1}{3}\sin\theta\right), \quad \text{and} \quad L_2 = \|GE\|\csc\theta = \csc\theta\left(6 - \frac{1}{3}\cos\theta\right).$$

Hence, $\qquad L = L_1 + L_2 = \dfrac{1}{3}\left[\sec\theta(9 - \sin\theta) + \csc\theta(18 - \cos\theta)\right],$

and this function is defined for $0 < \theta < \pi/2$. The longest beam that can be transported around the corner is represented by the minimum value of $L(\theta)$. To find critical points, we solve

$$0 = \frac{dL}{d\theta} = \frac{1}{3}\left[\sec\theta(-\cos\theta) + \sec\theta\tan\theta(9-\sin\theta) + \csc\theta(\sin\theta) - \csc\theta\cot\theta(18-\cos\theta)\right]$$
$$= \frac{1}{3}\left[-1 + \frac{9\sin\theta}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta} + 1 - \frac{18\cos\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta}\right] = \frac{1}{3}\left[\frac{9\sin\theta - \sin^2\theta}{\cos^2\theta} - \frac{18\cos\theta - \cos^2\theta}{\sin^2\theta}\right].$$

Critical points therefore satisfy

$$\begin{aligned} 0 = f(\theta) &= 9\sin^3\theta - \sin^4\theta - 18\cos^3\theta + \cos^4\theta \\ &= 9\sin^3\theta - 18\cos^3\theta + (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta) \\ &= 9\sin^3\theta - 18\cos^3\theta + 2\cos^2\theta - 1. \end{aligned}$$

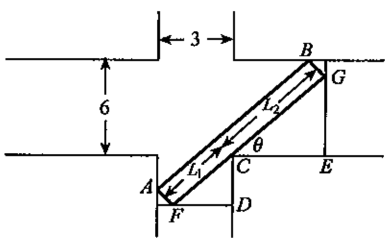
To solve this equation we use Newton's iterative procedure with

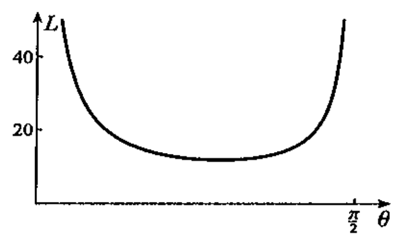
$$\theta_1 = 0.9, \quad \theta_{n+1} = \theta_n - \frac{9\sin^3\theta_n - 18\cos^3\theta_n + 2\cos^2\theta_n - 1}{27\sin^2\theta_n\cos\theta_n + 54\cos^2\theta_n\sin\theta_n - 4\cos\theta_n\sin\theta_n}.$$

Iteration gives $\theta_2 = 0.9091$ and $\theta_3 = 0.9091$. We now evaluate

$$\lim_{\theta\to 0^+} L(\theta) = \infty, \qquad L(0.90) = 11.8, \qquad \lim_{\theta\to\pi/2^-} L(\theta) = \infty.$$

Thus, the length of the longest beam is 11.8 m. The graph of $L(\theta)$ in the right figure also shows that it is minimized at its critical point.



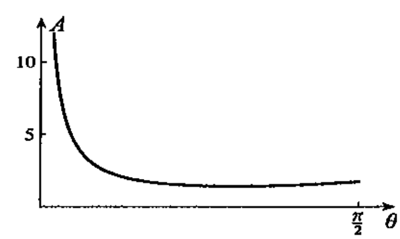




**43.** With $x$ and $y$ fixed, we minimize $f(\theta) = \sqrt{3}\csc\theta - \cot\theta$. Critical points of $f(\theta)$ are given by

$$0 = f'(\theta) = -\sqrt{3}\csc\theta\cot\theta + \csc^2\theta = \frac{1-\sqrt{3}\cos\theta}{\sin^2\theta}.$$

The angle in the first quadrant for which $\cos\theta = 1/\sqrt{3}$ is $\theta = 0.96$ radians. We now evaluate

$$\lim_{\theta\to 0^+} L(\theta) = \infty, \qquad L(0.96) = 1.41, \qquad L(\pi/2) = \sqrt{3}.$$




Thus, $f(\theta)$ is minimized for $\theta = 0.96$ radians. The graph of $f(\theta)$ also indicates that it is minimized at its critical point.

**44.** For critical points we solve $0 = f'(x) = \dfrac{(x^2+c)(1) - x(2x)}{(x^2+c)^2} = \dfrac{c-x^2}{(x^2+c)^2}$. The only solution in $0 \le x \le c$ is $x = \sqrt{c}$. We now calculate

$$f(0) = 0, \qquad f(\sqrt{c}) = \frac{\sqrt{c}}{c+c} = \frac{1}{2\sqrt{c}}, \qquad f(c) = \frac{c}{c^2+c} = \frac{1}{c+1}.$$

Since $c > 0$, we can say that $1/(c+1) \le 1/(2\sqrt{c})$ if and only if

$$c + 1 \ge 2\sqrt{c} \iff c - 2\sqrt{c} + 1 \ge 0 \iff (\sqrt{c}-1)^2 \ge 0,$$

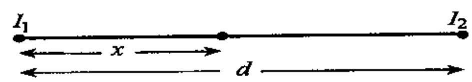
and this is always valid. Hence, the absolute maximum and minimum values of $f(x)$ on $0 \le x \le c$ are $1/(2\sqrt{c})$ and 0.

**45.** (a) If we substitute $n = t^\alpha/\beta$ into the formula for $C(t)$, we obtain $C(t) = \dfrac{p}{t} + \dfrac{r}{\beta}t^{\alpha-1}$, $t > 0$. For critical points of $C(t)$ we solve $0 = -\dfrac{p}{t^2} + \dfrac{r}{\beta}(\alpha-1)t^{\alpha-2} = \dfrac{1}{\beta t^2}[-p\beta + r(\alpha-1)t^\alpha]$. The solution of this equation is $[p\beta/(r\alpha - r)]^{1/\alpha}$. Since $\lim_{t\to 0^+} C(t) = \infty$, $C\left(\left(\dfrac{p\beta}{r\alpha - r}\right)^{1/\alpha}\right) < \infty$, $\lim_{t\to\infty} C(t) = \infty$, the machine should be replaced in $[p\beta/(r\alpha - r)]^{1/\alpha}$ years.

(b) When $\alpha = 1$, then $C(t) = \dfrac{p}{t} + \dfrac{r}{\beta}$. Since this is a decreasing function for $t > 0$, and is asymptotic to the line $C = r/\beta$, it follows that the machine should be kept as long as possible. When $\alpha < 1$, then $C(t) = \dfrac{p}{t} + \dfrac{r}{\beta t^{1-\alpha}}$. Once again this is a decreasing function for $t > 0$, and is asymptotic to the $t$-axis. The machine should therefore be kept as long as possible.

**46.** The illumination $L$ at a point which is a distance $x$ from the source with intensity $I_1$ is

$$L = L(x) = \frac{kI_1}{x^2} + \frac{kI_2}{(d-x)^2}, \quad 0 < x < d,$$




where $k$ is a constant. For critical points of this function we solve

$$0 = \frac{dL}{dx} = -\frac{2kI_1}{x^3} + \frac{2kI_2}{(d-x)^3} \Longrightarrow I_2x^3 = I_1(d-x)^3 \Longrightarrow \left(\frac{d-x}{x}\right)^3 = \frac{I_2}{I_1}.$$

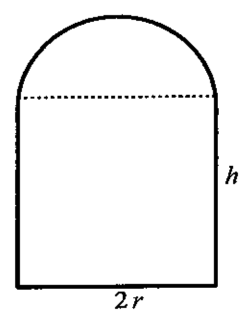
The solution of this equation is $x = dI_1^{1/3}/(I_1^{1/3} + I_2^{1/3})$. Since

$$\lim_{x\to 0^+} L(x) = \infty \qquad \text{and} \qquad \lim_{x\to d^-} L(x) = \infty,$$

illumination is minimized at the critical point $dI_1^{1/3}/(I_1^{1/3} + I_2^{1/3})$ units from the $I_1$ source.

**47.** If $k$ is the amount of light per unit area transmitted by the clear glass, then the total amount of light admitted by the window is

$$A = k(2rh) + \frac{k}{2}\left(\frac{1}{2}\pi r^2\right) = \frac{k(8rh + \pi r^2)}{4}.$$




If $P$ represents the fixed perimeter of the window, then $P = \pi r + 2h + 2r$. Thus, $h = (P - \pi r - 2r)/2$, and

$$A(r) = \frac{k}{4}\left[8r\left(\frac{P - \pi r - 2r}{2}\right) + \pi r^2\right] = \frac{k}{4}(4rP - 3\pi r^2 - 8r^2).$$

We must take $r \geq 0$, and for $h \geq 0$, we must also take $P - \pi r - 2r \geq 0$ or $r \leq P/(\pi + 2)$. For critical points of $A(r)$, we solve

$$0 = \frac{dA}{dr} = \frac{k}{4}(4P - 6\pi r - 16r).$$

The solution is $r = 2P/(3\pi + 8)$. Since

$$A(0) = 0, \quad A\left(\frac{2P}{3\pi + 8}\right) = \frac{kP^2}{3\pi + 8}, \quad A\left(\frac{P}{\pi + 2}\right) = \frac{k\pi P^2}{4(\pi + 2)^2},$$

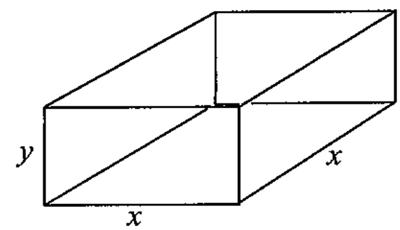
$A$ is maximized for $r = 2P/(3\pi + 8)$. For this $r$, $h = P(\pi + 4)/(6\pi + 16)$, and the ratio of $h$ to $2r$ is

$$\frac{h}{2r} = \left[\frac{P(\pi + 4)}{2(3\pi + 8)}\right]\left(\frac{3\pi + 8}{4P}\right) = \frac{\pi + 4}{8}.$$

**48.** (a) The cost in pennies for the material to construct a tank with base $x$ m and height $y$ m is

$$C = 125(x^2 + 4xy) + 475x^2.$$

Since the tank must hold 100 m$^3$, $x$ and $y$ must satisfy $100 = x^2y$. Thus, $y = 100/x^2$, and




$$C = C(x) = 125x^2 + 500x\left(\frac{100}{x^2}\right) + 475x^2$$
$$= 600x^2 + \frac{50\,000}{x}, \quad x > 0.$$

For critical points of $C(x)$ we solve

$$0 = 1200x - \frac{50\,000}{x^2}.$$

The solution is $x = 5/3^{1/3}$. Since $\lim_{x\to 0^+} C(x) = \infty$ and $\lim_{x\to\infty} C(x) = \infty$, it follows that cost is minimized when $x = 5/3^{1/3}$ m and $y = 4 \cdot 3^{2/3}$ m.

(b) When welding is taken into account the cost is

$$C(x) = 125(x^2 + 4xy) + 475x^2 + 750(8x + 4y) = 600x^2 + \frac{50\,000}{x} + 750\left(8x + \frac{400}{x^2}\right)$$
$$= 600x^2 + \frac{50\,000}{x} + 6000x + \frac{300\,000}{x^2}, \quad x > 0.$$

For critical points of this function we solve

$$0 = 1200x - \frac{50\,000}{x^2} + 6000 - \frac{600\,000}{x^3},$$

and this equation simplifies to $3x^4 + 15x^3 - 125x - 1500 = 0$. The positive solution of this equation, $x = 4.19$, can be found using Newton's iterative procedure (see Example 4.1 in Section 4.1). Since limits of $C(x)$ are once again "infinity" as $x \to 0^+$ and $x \to \infty$, cost is minimized for $x = 4.19$ m and $y = 5.70$ m.

**49.** For critical points of $f(x)$, we solve

$$0 = f'(x) = \frac{n!}{m!\,(n-m)!}[mx^{m-1}(1-x)^{n-m} - (n-m)x^m(1-x)^{n-m-1}]$$
$$= \frac{n!x^{m-1}(1-x)^{n-m-1}(m-nx)}{m!\,(n-m)!}.$$

Solutions are $x = 0, m/n, 1$. Since $f(0) = 0$, $f(m/n) > 0$, and $f(1) = 0$, it follows that $x = m/n$ maximizes $f(x)$.

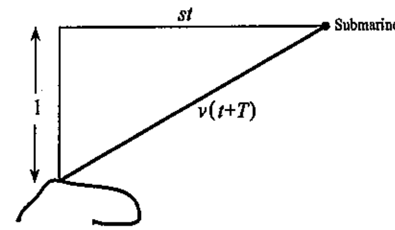
**50.** Total cost for the pipeline is $C(x) = c_1\sqrt{x^2 + y_1^2} + c_2\sqrt{(x_2 - x)^2 + y_2^2}$, $0 \le x \le x_2$. For critical points of this function we solve $0 = C'(x) = \dfrac{c_1 x}{\sqrt{x^2 + y_1^2}} + \dfrac{c_2(x_2 - x)(-1)}{\sqrt{(x_2 - x)^2 + y_2^2}}$. Since $\sin\theta_1 = x/\sqrt{x^2 + y_1^2}$ and $\sin\theta_2 = (x_2 - x)/\sqrt{(x_2 - x)^2 + y_2^2}$, it follows that at the critical point $0 = c_1\sin\theta_1 - c_2\sin\theta_2$. We can show that the critical point yields a minimum by calculating

$$C''(x) = \frac{c_1}{\sqrt{x^2 + y_1^2}} - \frac{c_1x^2}{(x^2 + y_1^2)^{3/2}} + \frac{c_2}{\sqrt{(x_2 - x)^2 + y_2^2}} - \frac{c_2(x_2 - x)^2}{[(x_2 - x)^2 + y_2^2]^{3/2}}$$
$$= \frac{c_1y_1^2}{(x^2 + y_1^2)^{3/2}} + \frac{c_2y_2^2}{[(x_2 - x)^2 + y_2^2]^{3/2}}.$$

Since $C''(x)$ is positive for $0 \le x \le x_2$, the graph of $C(x)$ is concave upward. Consequently, the critical point must minimize $C(x)$.

**51.** Let $T$ be the number of hours before midnight that the raft must leave the island in order to intercept the submarine $t$ hours after midnight. Then, $1 + s^2t^2 = v^2(t+T)^2$, and this can be solved for

$$T(t) = \frac{\sqrt{1+s^2t^2}}{v} - t, \quad t \geq 0.$$




For critical points of this function, we solve

$$0 = \frac{dT}{dt} = \frac{2s^2t}{2v\sqrt{1+s^2t^2}} - 1 = \frac{s^2t - v\sqrt{1+s^2t^2}}{v\sqrt{1+s^2t^2}}.$$

When we square $s^2t = v\sqrt{1+s^2t^2}$, we obtain $s^4t^2 = v^2(1+s^2t^2)$, from which $t = \dfrac{v}{s\sqrt{s^2-v^2}}$. To finish the problem, we calculate that $T(0) = 1/v$,

$$T\left(\frac{v}{s\sqrt{s^2-v^2}}\right) = \frac{\sqrt{1+\dfrac{s^2v^2}{s^2(s^2-v^2)}}}{v} - \frac{v}{s\sqrt{s^2-v^2}} = \frac{s^2}{vs\sqrt{s^2-v^2}} - \frac{v}{s\sqrt{s^2-v^2}} = \frac{\sqrt{s^2-v^2}}{vs},$$

and $\lim_{t\to\infty} T(t) = \lim_{t\to\infty} \dfrac{\sqrt{1+s^2t^2} - vt}{v} = \infty$, (since $s > v$). Since $\sqrt{s^2-v^2}/(vs) < 1/v$, the raft should leave $v/(s\sqrt{s^2-v^2})$ hours before midnight.

**52.** The volume of the box is $V = lwh$. Dimensions on the cardboard make it clear that $2l + 2w = 2$, and therefore $w = 1-l$. The fact that the outer flaps must meet in the centre requires $l = 2(1/2)(1-h) = 1-h$. Hence, $w = 1-(1-h) = h$ and

$$V = V(h) = (1-h)hh = h^2 - h^3, \quad 0 \leq h \leq 1.$$

For critical points we solve $0 = V'(h) = 2h - 3h^2 = h(2-3h)$. Since $V(0) = 0$, $V(2/3) > 0$, and $V(1) = 0$, maximum volume occurs when $h = w = 2/3$ m and $l = 1/3$ m. Since the sum of the spaces between inner flaps on top and bottom is $2w + 2l - 2 = 2/3$, the inner flaps are $1/3$ m apart.

**53.** If $C_f$ is the fuel cost per hour, then $C_f = ks^3$ where $s$ is the speed of the ship and $k$ is a constant. Since $C_f = B$ when $s = b$, we obtain $k = B/b^3$. Thus, $C_f = Bs^3/b^3$, and the total cost per hour for running the ship is $C^* = Bs^3/b^3 + A$. If the length of the trip is $D$ km, then the time to make the trip at speed $s$ is $D/s$, and the total cost of the trip is

$$C = C(s) = (C^*)\left(\frac{D}{s}\right) = \frac{D}{s}\left(\frac{Bs^3}{b^3} + A\right) = D\left(\frac{Bs^2}{b^3} + \frac{A}{s}\right), \quad s > 0.$$

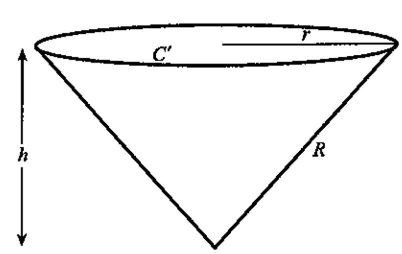
For critical points we solve $0 = C'(s) = D(2Bs/b^3 - A/s^2)$, the solution of which is $s = \left(Ab^3/(2B)\right)^{1/3}$ Since $\lim_{s\to 0^+} C(s) = \infty$ and $\lim_{s\to\infty} C(s) = \infty$, it follows that $C(s)$ is minimized for $s = \left(Ab^3/(2B)\right)^{1/3}$ km/hr.

**54.** The volume of the drinking cup is $V = \pi r^2h/3$. Since $R^2 = r^2 + h^2$, we may write

$$V = V(h) = \frac{1}{3}\pi(R^2 - h^2)h.$$

To obtain the domain of this function, we relate $\theta$ and $h$ by noting that the length of the arc joining $A$ and $B$ is the same as that of $C'$:

$$R(2\pi - \theta) = 2\pi r = 2\pi\sqrt{R^2 - h^2}.$$




This equation implies that $h = 0$ when $\theta = 0$, and $h = R$ when $\theta = 2\pi$. The appropriate domain for $V(h)$ is therefore $0 \leq h \leq R$. For critical points of the function we solve

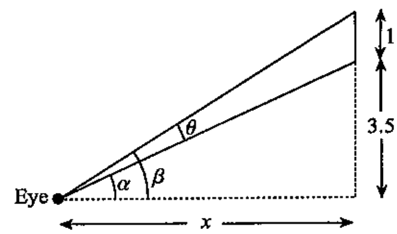
$$0 = V'(h) = \frac{1}{3}\pi(R^2 - 3h^2).$$

The positive solution is $h = R/\sqrt{3}$. Since $V(0) = 0 = V(R)$, it follows that $V$ is maximized for $h = R/\sqrt{3}$. For this $h$, $R(2\pi - \theta) = 2\pi\sqrt{R^2 - R^2/3}$, and the solution of this equation is $\theta = 2\pi(1 - \sqrt{6}/3)$.

**55.** The letters appear tallest when the angle $\theta$ that they subtend at the eye is greatest. Since

$\theta = \beta - \alpha = \mathrm{Tan}^{-1}\left(\dfrac{9/2}{x}\right) - \mathrm{Tan}^{-1}\left(\dfrac{7/2}{x}\right)$,

$0 < x < \infty$, critical points of $\theta(x)$ are given by




$$0 = \frac{d\theta}{dx} = \frac{1}{1 + \left(\frac{9}{2x}\right)^2}\left(-\frac{9}{2x^2}\right) - \frac{1}{1 + \left(\frac{7}{2x}\right)^2}\left(-\frac{7}{2x^2}\right) = \frac{-18}{4x^2 + 81} + \frac{14}{4x^2 + 49}.$$

Equivalently, $18(4x^2 + 49) = 14(4x^2 + 81)$, which has only one positive solution, $x = 3\sqrt{7}/2$. Since $\theta(x)$ approaches zero as $x$ approaches zero and as $x$ becomes very large, it follows that $\theta$ is maximized when the motorist is $3\sqrt{7}/2$ m from the sign.

**56.** The thrust to speed ratio is

$$g(v) = \frac{F}{v} = \frac{1}{2}\rho A v\left(0.000182 + \frac{4w^2}{6.5\pi\rho^2 A^2 v^4}\right) = \frac{1}{2}\rho A\left(0.000182v + \frac{4w^2}{6.5\pi\rho^2 A^2 v^3}\right).$$

For critical points of this function we solve

$$0 = g'(v) = \frac{1}{2}\rho A\left(0.000182 - \frac{12w^2}{6.5\pi\rho^2 A^2 v^4}\right) \qquad \Longrightarrow \qquad v = \left[\frac{12w^2}{0.000182(6.5)\pi\rho^2 A^2}\right]^{1/4}.$$

Since $g(v)$ becomes infinite as $v \to 0$ and $v \to \infty$, it follows that this value of $v$ must minimize $g(v)$.

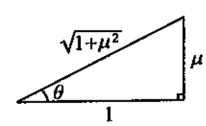
(a) At sea level, when $A = 1600$, $w = 150,000$, and $\rho = 0.0238$, the speed is 323 mph.

(b) At 30,000 feet, when $A = 1600$, $w = 150,000$, and $\rho = (0.375)(0.0238)$, the speed is 527 mph.

**57.** For critical points of $F(\theta)$ we solve

$0 = F'(\theta) = \dfrac{-\mu mg}{(\cos\theta + \mu\sin\theta)^2}(-\sin\theta + \mu\cos\theta).$

Thus, $0 = -\sin\theta + \mu\cos\theta$, or, $\tan\theta = \mu$, and this implies that $\theta = \mathrm{Tan}^{-1}\mu$. Since




$$F(0) = \mu mg, \qquad F(\mathrm{Tan}^{-1}\mu) = \frac{\mu mg}{\dfrac{1}{\sqrt{1+\mu^2}} + \dfrac{\mu^2}{\sqrt{1+\mu^2}}} = \frac{\mu mg}{\sqrt{1+\mu^2}}, \qquad F(\pi/2) = mg,$$

it follows that $F$ is minimized for $\theta = \mathrm{Tan}^{-1}\mu$.

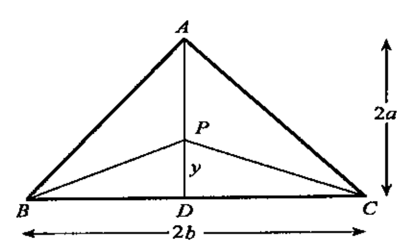
**58.** From the figure to the right, the sum of the distance from $P$ to the vertices is

$L(y) = (2a - y) + 2\sqrt{b^2 + y^2}, \quad 0 \le y \le 2a.$

For critical points of this function, we solve

$0 = L'(y) = -1 + \dfrac{2y}{\sqrt{b^2 + y^2}}.$

The solution is $y = b/\sqrt{3}$. To show that this value minimizes $L(y)$, we evaluate



$$L(0) = 2a + 2b, \qquad L(b/\sqrt{3}) = 2a - \frac{b}{\sqrt{3}} + 2\sqrt{b^2 + \frac{b^2}{3}} = 2a + \sqrt{3}b, \qquad L(2a) = 2\sqrt{b^2 + 4a^2}.$$

Certainly the second of these is less than the first. Since the second and third are both positive, the second is less than the third if, and only if,

$$(2a + \sqrt{3}b)^2 < 4(b^2 + 4a^2) \qquad \iff \qquad 4a^2 + 4\sqrt{3}ab + 3b^2 < 4b^2 + 16a^2.$$

But this is equivalent to

$$0 < 12a^2 - 4\sqrt{3}ab + b^2 = (2\sqrt{3}a - b)^2,$$

which is obviously true. Thus, $y = b/\sqrt{3}$ does indeed minimize $L$.

**59.** Suppose a trip of length $d$ km is to be driven. Since the time for the trip at speed $v$ is $d/v$, the wages earned by the driver are $wd/v$, and the cost for gas is $pd/f(v)$; that is, company costs are

$$C(v) = \frac{wd}{v} + \frac{pd}{a - bv}, \quad 80 \le v \le 100.$$

For critical points,

$$0 = C'(v) = -\frac{wd}{v^2} + \frac{bpd}{(a - bv)^2}.$$

This gives

$$\frac{bp}{(a - bv)^2} = \frac{w}{v^2} \quad \Longrightarrow \quad \left(\frac{a - bv}{v}\right)^2 = \frac{bp}{w} \quad \Longrightarrow \quad \frac{a - bv}{v} = \sqrt{\frac{bp}{w}} \quad \Longrightarrow \quad v = \frac{a}{b + \sqrt{bp/w}}.$$

This must minimize $C(v)$ since it is the only positive critical point and $C(v)$ becomes infinite as $v \to 0$ and $v \to a/b^-$.

**60.** The distance $D$ from $(x_1, y_1)$ to any point $P(x, y)$ on the line $Ax + By + C = 0$ is given by

$$D^2 = (x - x_1)^2 + (y - y_1)^2 = (x - x_1)^2 + \left(-\frac{C}{B} - \frac{Ax}{B} - y_1\right)^2 \qquad \text{(provided } B \neq 0\text{)}$$
$$= (x - x_1)^2 + \frac{(C + Ax + By_1)^2}{B^2}, \quad -\infty < x < \infty.$$

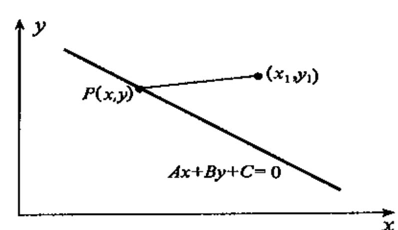
To minimize $D$ we minimize $D^2$. For critical points of $D^2$ we solve

$$0 = 2(x - x_1) + 2\frac{(C + Ax + By_1)}{B^2}A.$$

The solution of this equation is

$$x = \frac{B^2 x_1 - AC - ABy_1}{A^2 + B^2}.$$

Since $D^2$ becomes infinite as $x \to \pm\infty$, it follows that $D^2$ is minimized for this value of $x$, and the minimum distance is

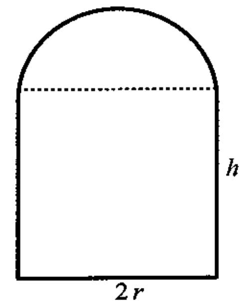



$$\sqrt{\left(\frac{B^2 x_1 - AC - ABy_1}{A^2 + B^2} - x_1\right)^2 + \frac{1}{B^2}\left(C + \frac{AB^2 x_1 - A^2 C - A^2 B y_1}{A^2 + B^2} + By_1\right)^2}$$
$$= \sqrt{A^2\left(\frac{Ax_1 + By_1 + C}{A^2 + B^2}\right)^2 + B^2\left(\frac{Ax_1 + By_1 + C}{A^2 + B^2}\right)^2} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

If $B = 0$, then the line $Ax + C = 0$ is vertical and the minimum distance is $|C/A + x_1|$. But this is predicted by the above formula when $B = 0$. Consequently, the formula is correct for any line whatsoever.

**61.** If $k$ is the amount of light per unit area admitted by the clear glass, then the total amount of light admitted by the entire window is

$$L = k(2rh) + \frac{pk}{2}(\pi r^2) = \frac{k}{2}(4rh + \pi pr^2).$$




The total cost of the window plus frame is

$$A = a(2rh) + b(\pi r^2/2) + c(\pi r)$$

from which $h = \dfrac{A - \pi r^2 b/2 - \pi rc}{2ar}$. Thus,

$$L = L(r) = \frac{k}{2}\left[\frac{2}{a}\left(A - \frac{\pi r^2 b}{2} - \pi rc\right) + \pi pr^2\right].$$

Clearly, $r > 0$, and for $h \geq 0$, we must choose $A - \pi r^2 b/2 - \pi rc \geq 0$. This condition implies that $r \leq r_m = (-c + \sqrt{c^2 + 2Ab/\pi})/b$. For critical points of $L(r)$ we solve

$$0 = L'(r) = \frac{k}{2}\left(-\frac{2\pi br}{a} - \frac{2\pi c}{a} + 2\pi pr\right) \implies r = \frac{c}{ap - b}.$$

We now calculate that

$$\lim_{r \to 0^+} L(r) = \frac{kA}{a}, \quad L\left(\frac{c}{ap-b}\right) = \frac{kA}{a} - \frac{k\pi c^2}{2a(ap-b)}, \quad L(r_m) = \frac{k\pi p}{2b^2}\left(-c + \sqrt{c^2 + \frac{2Ab}{\pi}}\right)^2.$$

If $p < b/a$, then the critical point $c/(ap - b) < 0$, and is therefore inadmissible. If $p > b/a$, then $c/(ap - b) > 0$, but clearly in this case the limit of $L$ as $r \to 0^+$ is greater than $L$ evaluated at $c/(ap - b)$. Thus, in either case, maximum $L(r)$ is achieved at one of the ends of the interval. The difference between the values of $L$ at the end points is

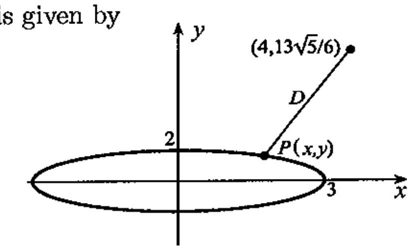
$$\begin{aligned}\lim_{r \to 0^+} L(r) - L(r_m) &= \frac{kA}{a} - \frac{k\pi p}{2b^2}\left(-c + \sqrt{c^2 + \frac{2Ab}{\pi}}\right)^2 \\ &= \frac{k}{b}\left[A\left(\frac{b}{a} - p\right) + \frac{c\pi p}{b}\left(\sqrt{c^2 + \frac{2Ab}{\pi}} - c\right)\right].\end{aligned}$$

If $p < b/a$, then this difference is positive, implying that $r$ should be chosen as small as possible. When $p > b/a$, no conclusion can yet be drawn. This difference could be negative or positive.

**62.** The distance $D$ from $(4, 13\sqrt{5}/6)$ to any point $P(x, y)$ on the ellipse is given by

$D^2 = (x-4)^2 + \left(y - \dfrac{13\sqrt{5}}{6}\right)^2$, where $x$ and $y$ must satisfy $4x^2 + 9y^2 = 36$. For critical points of $D^2$, we solve

$$0 = 2(x-4) + 2\left(y - \frac{13\sqrt{5}}{6}\right)\frac{dy}{dx}.$$

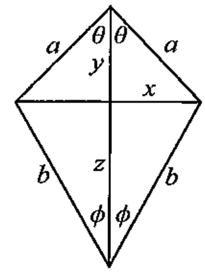



We obtain $dy/dx$ by differentiating the equation of the ellipse, $8x + 18y\,dy/dx = 0$. We solve this for $dy/dx$ and substitute into the equation defining critical points, $0 = x - 4 + \left(y - \dfrac{13\sqrt{5}}{6}\right)\left(-\dfrac{4x}{9y}\right)$. This equation can be solved for $y = \dfrac{26\sqrt{5}x}{3(36-5x)}$. When this is substituted into the equation of the ellipse,

$$36 = 4x^2 + 9\left[\frac{26\sqrt{5}x}{3(36-5x)}\right]^2,$$

and this equation simplifies to $(36-5x)^2(36-4x^2) - 3380x^2 = 0$. The only solution between 0 and 3 is $x = 2$. The corresponding $y$-coordinate of the point is $y = 2\sqrt{5}/3$. Since $D^2(0) = 24.09$, $D^2(2) = 15.25$, and $D^2(3) = 24.47$, the point $(2, 2\sqrt{5}/3)$ is indeed closest to $(4, 13\sqrt{5}/6)$.

**63.** The area of the kite is




$$\begin{aligned} A &= xy + xz \\ &= (a \sin\theta)(a \cos\theta) + (b \sin\phi)(b \cos\phi) \\ &= \frac{1}{2}(a^2 \sin 2\theta + b^2 \sin 2\phi). \end{aligned}$$

Angles $\theta$ and $\phi$ are related by the sine law,

$$\frac{\sin\theta}{b} = \frac{\sin\phi}{a}.$$

For critical points of $A$, we solve

$$0 = \frac{dA}{d\theta} = \frac{1}{2}\left(2a^2 \cos 2\theta + 2b^2 \cos 2\phi \frac{d\phi}{d\theta}\right), \qquad \text{where} \quad a \cos\theta = b \cos\phi \frac{d\phi}{d\theta}.$$

For critical points then

$$0 = a^2 \cos 2\theta + b^2 \cos 2\phi \left(\frac{a \cos\theta}{b \cos\phi}\right) \quad \Longrightarrow \quad \frac{a \cos 2\theta}{\cos\theta} = -\frac{b \cos 2\phi}{\cos\phi}.$$

If we divide this result by $a \sin\theta = b \sin\phi$, we obtain

$$\frac{a \cos 2\theta}{a \cos\theta \sin\theta} = -\frac{b \cos 2\phi}{b \cos\phi \sin\phi} \quad \Longrightarrow \quad \tan 2\theta = -\tan 2\phi.$$

Thus, $2\theta = -2\phi + n\pi$, or, $\theta = -\phi + n\pi/2$, where $n$ is an integer. Since $\theta$ and $\phi$ are both acute angles, $n$ must be 1, and substitution of this into the sine law gives $a \sin\theta = b \sin(\pi/2 - \theta) = b \cos\theta$. Finally, then, the critical point is the acute angle for which $\tan\theta = b/a$. When $\theta = 0$, $\phi$ must also be zero, and in this case $A = 0$. When $\theta = \pi/2$, $\sin\phi = a/b$, and

$$A = b^2\left(\frac{a}{b}\right)\sqrt{1 - a^2/b^2} = a\sqrt{b^2 - a^2}.$$

When $\tan\theta = b/a$, we obtain

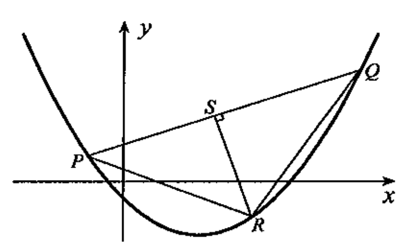
$$\sin\theta = \frac{b}{\sqrt{a^2+b^2}}, \ \cos\theta = \frac{a}{\sqrt{a^2+b^2}}, \ \sin\phi = \frac{a}{\sqrt{a^2+b^2}}, \ \cos\phi = \frac{b}{\sqrt{a^2+b^2}},$$

and

$$A = a^2\left(\frac{ab}{a^2+b^2}\right) + b^2\left(\frac{ab}{a^2+b^2}\right) = ab.$$

Since $ab > \sqrt{a^2 - b^2}$, it follows that $A$ is maximized for $\theta$ defined by $\tan\theta = b/a$.

**64.** The area of triangle PQR is $F = \|RS\|\|PQ\|/2$. If we use formula 1.16 for length $\|RS\|$, then




$$F(x) = \frac{\|PQ\|}{2\sqrt{A^2+B^2}}|Ax + B(ax^2 + bx + c) + C|,$$

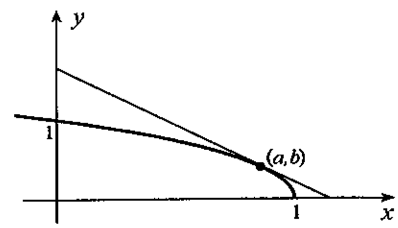
where $\|PQ\|$ is constant. For critical points of this function we solve

$$0 = F'(x) = \frac{\|PQ\|}{2\sqrt{A^2+B^2}}(A + 2Bax + bB)\frac{|Ax + B(ax^2 + bx + c) + C|}{Ax + B(ax^2 + bx + c) + C}$$

(see formula 3.13). The only solution of this equation is $x = -(A + bB)/(2aB)$. Now, there must be a value of $x$ between $P$ and $Q$ which maximizes $F$ since coordinates of these points lead to degenerate triangles with zero area. Hence, this critical point maximizes the area.

**65.** Since the slope of the tangent line to the curve at a point $(a,b)$ is $-(1/2)/\sqrt{1-a}$, the equation of the tangent line at the point is

$$y-b=\frac{-1}{2\sqrt{1-a}}(x-a).$$




The $x$- and $y$-intercepts of this line are

$$x=2b\sqrt{1-a}+a,\quad y=b+\frac{a}{2\sqrt{1-a}}.$$

The area of the triangle formed by the tangent line at $(a,b)$ is therefore

$$A=\frac{1}{2}\left(b+\frac{a}{2\sqrt{1-a}}\right)(2b\sqrt{1-a}+a)=\frac{1}{2}\left(\sqrt{1-a}+\frac{a}{2\sqrt{1-a}}\right)[2(1-a)+a]=\frac{(2-a)^2}{4\sqrt{1-a}},$$

defined for $-\infty<a<1$. For critical points of $A(a)$, we solve

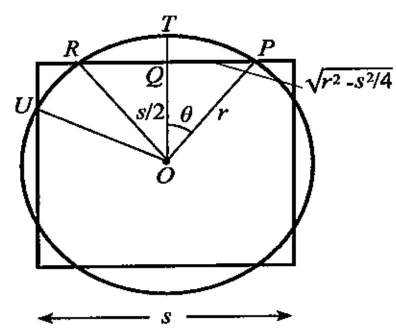
$$0=\frac{dA}{da}=\frac{1}{4}\left[\frac{\sqrt{1-a}(2)(2-a)(-1)-(2-a)^2(1/2)(1-a)^{-1/2}(-1)}{1-a}\right]$$
$$=\frac{(2-a)[-4(1-a)+(2-a)]}{8(1-a)^{3/2}}=\frac{(2-a)(3a-2)}{8(1-a)^{3/2}}.$$

The only acceptable solution is $a=2/3$. Since area becomes infinite as $a\to-\infty$ and $a\to1^-$, it follows that area is minimized when $a=2/3$. The minimum area is $A(2/3)=4\sqrt{3}/9$.

**66.** Since $b>a+c$, irrigating land produces more profit than leaving it unirrigated. It follows that the length $r$ of the arm should be at least $s/2$. When $r>s/2$, however, uncultivated land is being irrigated as well, with no return, and clearly $r$ should be no larger than $s/\sqrt{2}$, the distance to the corner of the field. The profit for an arm of length $r$ is

$$P=b(A_1)+a(A_2)-c(\pi r^2),$$

where $A_1$ is the area of irrigated land and $A_2$ is the area of unirrigated land.




Area $A_1$ is $\pi r^2$, less four times the area of the segment $PRT$ of the circle,

$$A_1=\pi r^2-4\left[\frac{1}{2}r^2(2\theta)-\frac{s}{2}\sqrt{r^2-s^2/4}\right]=\pi r^2-4r^2\mathrm{Cos}^{-1}\left(\frac{s}{2r}\right)+s\sqrt{4r^2-s^2}.$$

Area $A_2$ is the area of the field, less four times the area of sector $ORU$, less eight times the area of triangle $OQR$,

$$A_2=s^2-4\left[\frac{1}{2}r^2\left(\frac{\pi}{2}-2\theta\right)\right]-8\left[\frac{s}{4}\sqrt{r^2-s^2/4}\right]=s^2-2r^2\left[\frac{\pi}{2}-2\mathrm{Cos}^{-1}\left(\frac{s}{2r}\right)\right]-s\sqrt{4r^2-s^2}$$
$$=s^2-\pi r^2+4r^2\mathrm{Cos}^{-1}\left(\frac{s}{2r}\right)-s\sqrt{4r^2-s^2}.$$

Profit when the arm has length $r$ is

$$P(r)=b\left[\pi r^2-4r^2\mathrm{Cos}^{-1}\left(\frac{s}{2r}\right)+s\sqrt{4r^2-s^2}\right]+a\left[s^2-\pi r^2+4r^2\mathrm{Cos}^{-1}\left(\frac{s}{2r}\right)-s\sqrt{4r^2-s^2}\right]-c(\pi r^2)$$
$$=\pi r^2(b-a-c)+as^2+4r^2(a-b)\mathrm{Cos}^{-1}\left(\frac{s}{2r}\right)+(b-a)s\sqrt{4r^2-s^2},$$

defined for $s/2\le r\le s/\sqrt{2}$. For critical points of the function we solve

$$\begin{aligned}0 = \frac{dP}{dr} &= 2\pi r(b-a-c) + 8r(a-b)\mathrm{Cos}^{-1}\left(\frac{s}{2r}\right) - \frac{4r^2(a-b)}{\sqrt{1-s^2/(4r^2)}}\left(\frac{-s}{2r^2}\right) + \frac{(b-a)s(8r)}{2\sqrt{4r^2-s^2}}\\ &= 2\pi r(b-a-c) + 8r(a-b)\mathrm{Cos}^{-1}\left(\frac{s}{2r}\right) + \frac{4rs(a-b)}{\sqrt{4r^2-s^2}} + \frac{4rs(b-a)}{\sqrt{4r^2-s^2}}\\ &= 2\pi r(b-a-c) + 8r(a-b)\mathrm{Cos}^{-1}\left(\frac{s}{2r}\right).\end{aligned}$$

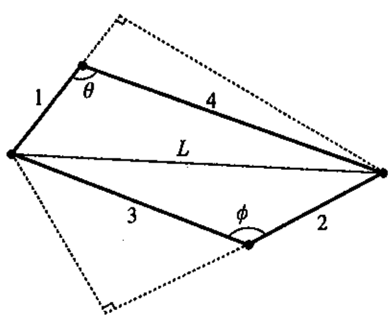
Since $r$ cannot be zero, we set

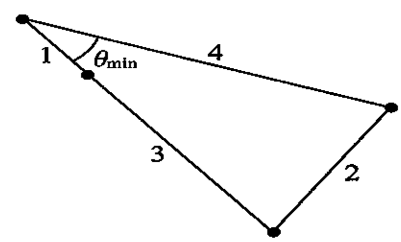
$$0 = \pi(b-a-c) + 4(a-b)\mathrm{Cos}^{-1}\left(\frac{s}{2r}\right) \quad \Longrightarrow \quad r = \frac{s}{2}\sec\left[\frac{\pi(b-a-c)}{4(b-a)}\right].$$

Since profit decreases as $r \to (s/2)^+$ and $r \to (s/\sqrt{2})^-$, it follows that this value of $r$ must maximize profit.

**67.** If we divide the quadrilateral into two triangles as shown in the left figure below, the area of the quadrilateral is

$$A = \frac{1}{2}(1)[4\sin(\pi-\theta)] + \frac{1}{2}(2)[3\sin(\pi-\phi)] = 2\sin\theta + 3\sin\phi.$$




By the cosine law,

$$L^2 = 1^2 + 4^2 - 2(1)(4)\cos\theta = 2^2 + 3^2 - 2(2)(3)\cos\phi \implies 12\cos\phi - 8\cos\theta + 4 = 0 \implies \cos\phi = \frac{2\cos\theta - 1}{3}.$$

We can now express $A$ in terms of $\theta$,

$$A(\theta) = 2\sin\theta + 3\sqrt{1-\cos^2\phi} = 2\sin\theta + 3\sqrt{1-\left(\frac{2\cos\theta-1}{3}\right)^2} = 2\sin\theta + 2\sqrt{2+\cos\theta-\cos^2\theta}.$$

Because $1+4=5=2+3$, the maximum value for $\theta$ is $\pi$. The minimum value of $\theta$ is shown in the configuration in the right figure above. Using the cosine law, we find that $4 = 16+16-2(4)(4)\cos\theta_{\min}$, and this gives $\theta_{\min} = \mathrm{Cos}^{-1}(7/8)$. For critical points of $A(\theta)$, we solve

$$0 = \frac{dA}{d\theta} = 2\cos\theta + \frac{-\sin\theta + 2\sin\theta\cos\theta}{\sqrt{2+\cos\theta-\cos^2\theta}}.$$

When we transpose the first term and square the equation, we obtain

$$4\cos^2\theta(2+\cos\theta-\cos^2\theta) = \sin^2\theta(1-4\cos\theta+4\cos^2\theta).$$

When we replace $\sin^2\theta$ with $1-\cos^2\theta$, the equation simplies to

$$0 = 5\cos^2\theta + 4\cos\theta - 1 = (5\cos\theta-1)(\cos\theta+1).$$

Thus, $\theta = \pi$ or $\theta = \mathrm{Cos}^{-1}(1/5)$. Since

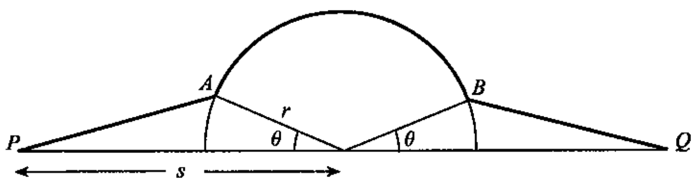
$$A(\mathrm{Cos}^{-1}(7/8)) = 2\sqrt{1-\frac{49}{64}} + 2\sqrt{2+\frac{7}{8}-\frac{49}{64}} = \sqrt{15}, \quad A(\mathrm{Cos}^{-1}(1/5)) = 2\sqrt{1-\frac{1}{25}} + 2\sqrt{2+\frac{1}{5}-\frac{1}{25}} = 2\sqrt{6},$$

and $A(\pi) = 2\sqrt{2-1-1} = 0$, it follows that maximum area is $2\sqrt{6}$.

**68.** If we take speed in the bush as $v$, then it is $2v$ on the beach. The time to walk from $P$ to $Q$ by the path in the figure below is

$$T = \frac{\text{Length of arc}AB}{2v} + \frac{2\|AP\|}{v} = \frac{r(\pi - 2\theta)}{2v} + \frac{2}{v}\sqrt{r^2+s^2-2rs\cos\theta}.$$

Obviously the smallest value of $\theta$ is 0. The largest value occurs when $PA$ is tangent to the circle and this occurs when $\theta = \mathrm{Cos}^{-1}(r/s)$.




For critical values of $T(\theta)$, we solve

$$0 = \frac{dT}{d\theta} = -\frac{r}{v} + \frac{2}{v}\left(\frac{2rs\sin\theta}{2\sqrt{r^2+s^2-2rs\cos\theta}}\right).$$

When we transpose the first term and square both sides of the equation, we obtain

$$r^2+s^2-2rs\cos\theta = 4s^2\sin^2\theta = 4s^2(1-\cos^2\theta) \quad\Longrightarrow\quad 4s^2\cos^2\theta - 2rs\cos\theta + (r^2-3s^2) = 0.$$

Solutions of this quadratic equation are

$$\cos\theta = \frac{2rs \pm \sqrt{4r^2s^2-16s^2(r^2-3s^2)}}{8s^2} = \frac{r\pm\sqrt{12s^2-3r^2}}{4s}.$$

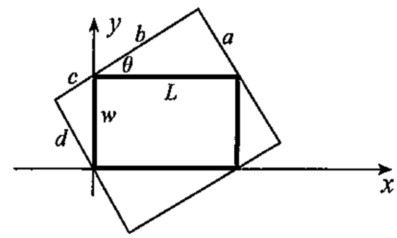
Since $\cos\theta$ must be positive, the only critical point is $\theta = \mathrm{Cos}^{-1}\left(\dfrac{r+\sqrt{12s^2-3r^2}}{4s}\right)$. We could show that this value of $\theta$ minimizes time by evaluating $T(\theta)$ at this value and at $\theta = 0$ and $\theta = \mathrm{Cos}^{-1}(7/8)$. It is difficult to see which is the smallest. As an alternative, we calculate the second derivative of $T(\theta)$,

$$\begin{aligned}\frac{d^2T}{d\theta^2} &= \frac{2rs}{v}\left[\frac{\cos\theta}{\sqrt{r^2+s^2-2rs\cos\theta}} + \frac{(-1/2)\sin\theta(2rs\sin\theta)}{(r^2+s^2-2rs\cos\theta)^{3/2}}\right]\\ &= \frac{2rs}{v(r^2+s^2-2rs\cos\theta)^{3/2}}\left[(r^2+s^2-2rs\cos\theta)\cos\theta - rs\sin^2\theta\right]\\ &= \frac{2rs}{v(r^2+s^2-2rs\cos\theta)^{3/2}}\left[(r^2+s^2)\cos\theta - 2rs\cos^2\theta - rs(1-\cos^2\theta)\right]\\ &= \frac{2rs}{v(r^2+s^2-2rs\cos\theta)^{3/2}}\left[-rs\cos^2\theta + (r^2+s^2)\cos\theta - rs\right]\\ &= \frac{-2r^2s^2}{v(r^2+s^2-2rs\cos\theta)^{3/2}}\left(\cos\theta - \frac{r}{s}\right)\left(\cos\theta - \frac{s}{r}\right).\end{aligned}$$

Since $r < s$, the last factor is always negative. In addition, because $\cos\theta > r/s$, the middle factor is positive. It follows that $d^2T/d\theta^2$ is always positive, and therefore a graph of $T(\theta)$ would be concave upward. The critical point must minimize travel time. You should therefore head to point $P$ that creates the critical angle.

**69.** The area of the rectangle shown to the right is

$$A = Lw + ab + cd.$$



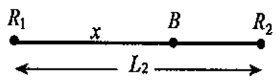

Since $a = L\sin\theta$, $b = L\cos\theta$, $c = w\sin\theta$, and $d = w\cos\theta$, we can express $A$ in terms of $\theta$,

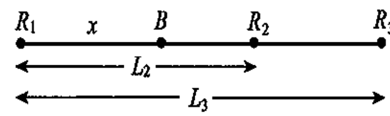
$$\begin{aligned} A &= Lw + L^2\sin\theta\cos\theta + w^2\sin\theta\cos\theta \\ &= Lw + \left(\frac{L^2}{2} + \frac{w^2}{2}\right)\sin 2\theta,\ 0 < \theta < \frac{\pi}{2}. \end{aligned}$$

This function is a maximum when $\sin 2\theta = 1$. Hence, maximum area is

$$Lw + L^2/2 + w^2/2 = (L+w)^2/2.$$

**70.** Consider first the case when $n = 2$ (left figure below). If the blue stake is at a distance $x$ from one of the red stakes, then the sum of the lengths of the two ropes is $L = x + (L_2 - x) = L_2$ (provided the blue stake is placed between the red ones). In this case then, the blue stake can be placed anywhere between the red one and $L$ will always be the same.




Consider now the case when $n = 3$ (right figure above). If the blue stake is at a distance $x$ from the left most red stake, then the total length of the three ropes is

$$L = L(x) = x + |x - L_2| + |x - L_3|, \quad 0 \le x \le L_3.$$

The derivative of this function is

$$L'(x) = 1 + \frac{|x - L_2|}{x - L_2} + \frac{|x - L_3|}{x - L_3}$$

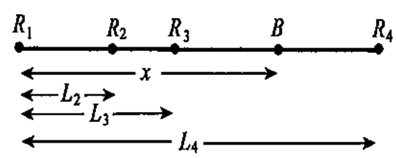
except at $x = L_2$ and $x = L_3$ where the derivative does not exist. Because the only values for the quotient terms are $\pm 1$, it follows that at no point can $L'(x) = 0$. The only critical points are $x = L_2, L_3$. Since

$$L(0) = L_2 + L_3, \quad L(L_2) = L_2 + |L_2 - L_3| = L_3, \quad L(L_3) = L_3 + |L_3 - L_2|,$$

it follows that $L(x)$ is minimized when the blue stake is placed at the position of the intermediate stake. Now consider the case when $n = 4$.

If we proceed as in the $n = 3$ case, the total length of the four ropes is

$$L = L(x) = x + |x - L_2| + |x - L_3| + |x - L_4|, \quad 0 \le x \le L_4.$$




The derivative of this function (except at $x = L_2, L_3, L_4$) is

$$L'(x) = 1 + \frac{|x - L_2|}{x - L_2} + \frac{|x - L_3|}{x - L_3} + \frac{|x - L_4|}{x - L_4}.$$

This derivative is equal to zero for all $x$ in the interval $L_2 < x < L_3$. Since

$$\begin{aligned} L(0) &= L_2 + L_3 + L_4, \\ L(L_2) &= L_2 + |L_2 - L_3| + |L_2 - L_4| = L_4 + L_3 - L_2, \\ L(L_3) &= L_3 + |L_3 - L_2| + |L_3 - L_4| = L_4 + L_3 - L_2, \\ L(L_4) &= L_4 + |L_4 - L_2| + |L_4 - L_3| = L_4 + (2L_4 - L_3) - L_2, \\ L(L_2 < x < L_3) &= x + (x - L_2) + (L_3 - x) + (L_4 - x) = L_4 + L_3 - L_2, \end{aligned}$$

it follows that the minimum value is $L_4 + L_3 - L_2$ and this is achieved at every position between $R_2$ and $R_3$, inclusively.

In general, for $n$ red stakes at positions $L_i$,

$$L(x) = x + \sum_{i=2}^{n} |x - L_i|, \qquad 0 \le x \le L_n.$$

Critical points of this function are defined by

$$0 = L'(x) = 1 + \sum_{i=2}^{n} \frac{|x - L_i|}{x - L_i}.$$

If $n$ is odd, it is impossible for $L'(x)$ to be equal to zero, and therefore $L(x)$ is minimized at one of the positions of the $n$ red stakes. Case $n = 3$ suggests that the optimun stake is the middle one. If $n$ is even, then $L'(x)$ is zero for all values of $x$ in the middle interval. Cases $n = 2$ and $n = 4$ suggest that the value of $L(x)$ is the same for all values of $x$ in the middle interval, and $L(x)$ has a larger value at any other red stake position. The optimum position is therefore anywhere in the middle interval.

**71.** (a) Since $L_1 = \|AE\| - \|DE\| = X - \|CE\| \cot\theta = X - Y\cot\theta$, and $L_2 = \|CE\| \csc\theta = Y\csc\theta$,

$$R = f(\theta) = \frac{k}{r_1^4}(X - Y\cot\theta) + \frac{k}{r_2^4} Y \csc\theta.$$

(b) For critical points of $f(\theta)$ we solve

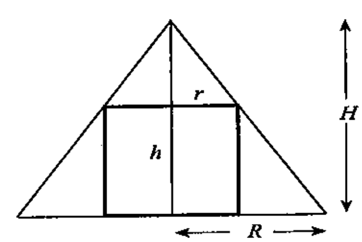
$$0 = f'(\theta) = \frac{k}{r_1^4}(Y\csc^2\theta) - \frac{k}{r_2^4} Y \csc\theta \cot\theta = \frac{kY}{\sin^2\theta}\left(\frac{1}{r_1^4} - \frac{\cos\theta}{r_2^4}\right).$$

Thus, $\cos\theta = r_2^4/r_1^4$, and there is only one solution $\overline{\theta}$ of this equation in the range $0 < \theta < \pi/2$.
(c) Since $f'(\theta)$ changes from negative to positive as $\theta$ increases through $\overline{\theta}$, this critical point gives a relative minimum.
(d) When $f(\theta)$ is evaluated at $\overline{\theta}$,

$$\begin{aligned} f(\overline{\theta}) &= \frac{kX}{r_1^4} + kY\left(\frac{\csc\overline{\theta}}{r_2^4} - \frac{\cot\overline{\theta}}{r_1^4}\right) = \frac{kX}{r_1^4} + kY\left(\frac{r_1^4}{r_2^4\sqrt{r_1^8 - r_2^8}} - \frac{r_2^4}{r_1^4\sqrt{r_1^8 - r_2^8}}\right) \\ &= \frac{kX}{r_1^4} + kY\frac{\sqrt{r_1^8 - r_2^8}}{r_1^4 r_2^4} = \frac{kX}{r_1^4} + \frac{kY}{r_2^4}\sqrt{1 - \left(\frac{r_2}{r_1}\right)^8}. \end{aligned}$$

Since $f(\pi/2) = kX/r_1^4 + kY/r_2^4$, it follows that $f(\overline{\theta}) < f(\pi/2)$.
(e) The smallest value of $\theta$ is defined by $\tan\theta = Y/X$. If we call this angle $\theta_m$, then $\overline{\theta}$ yields an absolute minimum if $\theta_m < \overline{\theta}$. If, however, $\theta_m > \overline{\theta}$, then $f(\theta)$ is minimized for $\theta = \theta_m$.

**72.** If $r$ and $h$ are the radius and height of the cylinder (figure to the right), then its surface area is $A = 2\pi r^2 + 2\pi rh$. Similar triangles give $r/R = (H - h)/H$, from which $h = H(R - r)/R$. Thus,




$$A(r) = 2\pi r^2 + \frac{2\pi r H(R - r)}{R}, \quad 0 \le r \le R.$$

For critical points we solve

$$0 = \frac{dA}{dr} = 4\pi r + \frac{2\pi H}{R}(R - 2r) \qquad \Longrightarrow \qquad r = \frac{HR}{2H - 2R}.$$

This is positive only if $H > R$ so that we consider two cases. Case 1: $H > R$ In this case, we evaluate $A(0) = 0$, $A(R) = 2\pi R^2$, and

$$\begin{aligned} A\left(\frac{HR}{2H - 2R}\right) &= 2\pi\left[\frac{H^2R^2}{4(H - R)^2}\right] + \frac{2\pi H}{R}\left[\frac{HR}{2(H - R)}\right]\left[R - \frac{HR}{2(H - R)}\right] \\ &= \frac{\pi H^2 R^2}{2(H - R)^2} + \frac{\pi H^2[2R(H - R) - HR]}{2(H - R)^2} = \frac{\pi H^2R^2 + \pi H^2(RH - 2R^2)}{2(H - R)^2} \\ &= \frac{\pi R H^3 - \pi H^2 R^2}{2(H - R)^2} = \frac{\pi H^2 R}{2(H - R)}. \end{aligned}$$

This will be maximum surface area if

$$0 < \frac{\pi H^2 R}{2(H-R)} - 2\pi R^2 = \frac{\pi R}{2(H-R)}[H^2 - 4R(H-R)] = \frac{\pi R(H-2R)^2}{2(H-R)}.$$

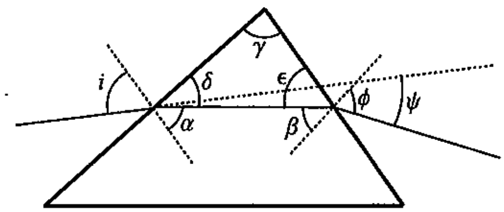
This is obviously true, and therefore surface area is maximized for the critical value of $r$.
Case 2: $H \le R$ In this case there is no critical point. Since $A(0) = 0$ and $A(R) = 2\pi R^2$, surface area is maximized when $r = R$. In this case, the cylinder has no height.

**73.** (a) If we introduce angles $\delta$, $\epsilon$, and $\gamma$ in the figure to the right, then $\gamma + \delta + \epsilon = \pi$, $\alpha + \delta = \pi/2$, and $\beta + \epsilon = \pi/2$. From these,

$$\gamma = \pi - (\delta + \epsilon)$$
$$= \pi - (\pi/2 - \alpha + \pi/2 - \beta) = \alpha + \beta,$$

and

$$\psi = (i - \alpha) + (\phi - \beta)$$
$$= i + \phi - (\alpha + \beta) = i + \phi - \gamma.$$




Angle $\gamma$ is known. It remains to find $\phi$ explicitly in terms of $i$. First,

$$\sin\phi = n\sin\beta = n\sin(\gamma - \alpha) = n(\sin\gamma\cos\alpha - \cos\gamma\sin\alpha).$$

Now, $\sin\alpha = (1/n)\sin i$, and because $\alpha$ is an acute angle,

$$\cos\alpha = \sqrt{1 - \sin^2\alpha} = \sqrt{1 - \frac{\sin^2 i}{n^2}}.$$

Thus,

$$\sin\phi = n\left(\sin\gamma\sqrt{1 - \frac{\sin^2 i}{n^2}} - \cos\gamma\frac{\sin i}{n}\right) = \sin\gamma\sqrt{n^2 - \sin^2 i} - \cos\gamma\sin i.$$

Since $\phi$ is between 0 and $\pi/2$, we may write that

$$\psi(i) = i - \gamma + \phi = i - \gamma + \mathrm{Sin}^{-1}\left(\sin\gamma\sqrt{n^2 - \sin^2 i} - \cos\gamma\sin i\right).$$

(b) Since $\psi = i + \phi - \gamma$, where $\sin\phi = \sin\gamma\sqrt{n^2 - \sin^2 i} - \cos\gamma\sin i$, differentiation gives $d\psi/di = 1 + d\phi/di$, where

$$\cos\phi\frac{d\phi}{di} = \frac{\sin\gamma}{2\sqrt{n^2 - \sin^2 i}}(-2\sin i\cos i) - \cos\gamma\cos i.$$

Thus,

$$\frac{d\psi}{di} = 1 + \frac{-\sin i\cos i\sin\gamma - \cos\gamma\cos i\sqrt{n^2 - \sin^2 i}}{\cos\phi\sqrt{n^2 - \sin^2 i}},$$

and for critical points of $\psi(i)$, we solve

$$0 = \cos\phi\sqrt{n^2 - \sin^2 i} - \sin i\cos i\sin\gamma - \cos\gamma\cos i\sqrt{n^2 - \sin^2 i}$$
$$= (\cos\phi - \cos\gamma\cos i)\frac{\sin\phi + \cos\gamma\sin i}{\sin\gamma} - \sin i\cos i\sin\gamma.$$

From this equation,

$$0 = \cos\phi\sin\phi - \cos\gamma\cos i\sin\phi + \cos\phi\cos\gamma\sin i - \cos^2\gamma\cos i\sin i - \sin i\cos i\sin^2\gamma$$
$$= \frac{1}{2}\sin 2\phi + \cos\gamma(\sin i\cos\phi - \cos i\sin\phi) - \frac{1}{2}\sin 2i$$

$$= \frac{1}{2}(\sin 2\phi - \sin 2i) + \cos\gamma \sin(i-\phi)$$
$$= \frac{1}{2}[2\cos(\phi+i)\sin(\phi-i)] + \cos\gamma\sin(i-\phi)$$
$$= \sin(\phi - i)[\cos(\phi+i) - \cos\gamma]$$
$$= \sin(\phi-i)\{-2\sin[(\phi+i+\gamma)/2]\sin[(\phi+i-\gamma)/2]\}.$$

To satisfy this equation, we must have one of the following situations:

$$i = \phi + n\pi; \qquad \phi + i + \gamma = 2n\pi; \qquad \phi + i - \gamma = 2n\pi.$$

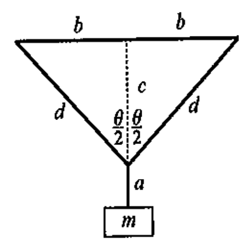
Since the last two are impossible, we conclude that the minimum value $\psi_m$ for $\psi$ occurs when $i = \phi$. For this value, $\psi_m = 2i - \gamma$, and

$$\sin i = \sin\gamma\sqrt{n^2 - \sin^2 i} - \cos\gamma \sin i.$$

When we solve this equation for $n$, the result is

$$n = \sqrt{\frac{2\sin^2 i(1+\cos\gamma)}{\sin^2\gamma}} = \sqrt{\frac{2\sin^2 i[1 + 2\cos^2(\gamma/2) - 1]}{[2\sin(\gamma/2)\cos(\gamma/2)]^2}}$$
$$= \frac{\sin i}{\sin(\gamma/2)} = \frac{\sin[(\psi_m+\gamma)/2]}{\sin(\gamma/2)}.$$

**74.** The distance from $m$ to the line through the rings is $L = a + c$. But, $c = b\cot(\theta/2)$. Furthermore, if $k$ is the length of the rope, then $k = a + 2d + 2b$, or, $a = k - 2b - 2d = k - 2b - 2b\csc(\theta/2)$. Thus, $L = f(\theta) = k - 2b - 2b\csc(\theta/2) + b\cot(\theta/2)$. The minimum value for $\theta$ is $2\,\mathrm{Sin}^{-1}[2b/(k-2b)]$ occurring when $a = 0$, and its maximum value is $\pi$. For critical points of $f(\theta)$, we solve




$$0 = f'(\theta) = b\csc(\theta/2)\cot(\theta/2) - \frac{b}{2}\csc^2(\theta/2) = \frac{b}{2}\csc(\theta/2)[2\cot(\theta/2) - \csc(\theta/2)].$$

This equation imples that $2\cos(\theta/2) = 1$, and therefore $\theta = 2\pi/3$. We now evaluate

$$f(2\pi/3) = k - 2b - 2b(2/\sqrt{3}) + b(1/\sqrt{3}) = k - (2+\sqrt{3})b,$$
$$f(\pi) = k - 2b - 2b = k - 4b,$$
$$f\left[2\,\mathrm{Sin}^{-1}\left(\frac{2b}{k-2b}\right)\right] = k - 2b - 2b\left(\frac{k-2b}{2b}\right) + b\left(\frac{\sqrt{k^2-4kb}}{2b}\right) = \frac{\sqrt{k^2-4kb}}{2}.$$

Certainly $f(2\pi/3) > f(\pi)$. Furthermore, $f(2\pi/3)$ is greater than $f[2\,\mathrm{Sin}^{-1}[2b/(k-2b)]]$ if and only if $[k - (2+\sqrt{3})b]^2 > (1/4)(k^2 - 4kb)$, or,

$$0 < 4[k - (2+\sqrt{3})b]^2 - (k^2 - 4kb) = 4[k^2 - 2(2+\sqrt{3})bk + (7+4\sqrt{3})b^2] - k^2 + 4kb$$
$$= 3k^2 - 4(3+2\sqrt{3})bk + 4(7+4\sqrt{3})b^2 = [\sqrt{3}k - 2(2+\sqrt{3})b]^2,$$

which is obviously valid. Thus, $\theta = 2\pi/3$ maximizes $f(\theta)$. Note, however, that this is the case provided the rope is sufficiently long that $\theta = 2\pi/3$ is indeed a possible configuration. This is true if $a$ is greater than zero when $\theta = 2\pi/3$; that is, if

$$0 < k - 2b - 2b(2/\sqrt{3}) = k - 2b(1 + 2/\sqrt{3}) \quad \Longrightarrow \quad k > \frac{2}{\sqrt{3}}(\sqrt{3}+2)b.$$

If $k$ is less than this value, $f(\theta)$ is largest for the largest value of $\theta$.

**75.** If $O$ is the centre of the circular arc, then the area of the pasture is the area of sector $OBCD$ less the area of triangle $OBD$ (left figure below),

$$A = \frac{1}{2}r^2(\theta) - [r\sin(\theta/2)][r\cos(\theta/2)] = \frac{r^2}{2}(\theta - \sin\theta).$$

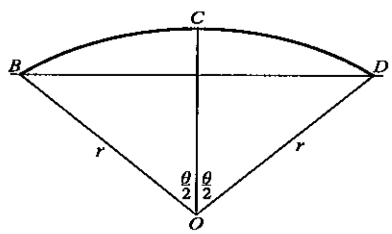
Since $1000 = r\theta$, it follows that $r = 1000/\theta$, and

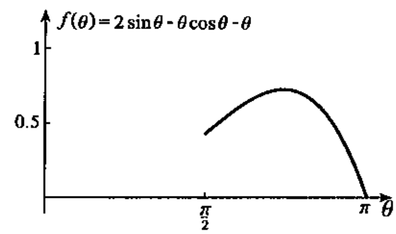
$$A(\theta) = \left(\frac{1000}{\theta}\right)^2\left(\frac{1}{2}\right)(\theta - \sin\theta), \quad \pi/2 \le \theta \le \pi.$$

For critical points of $A(\theta)$ we solve

$$0 = \frac{\theta^2(1-\cos\theta) - (\theta - \sin\theta)(2\theta)}{\theta^4} = \frac{-\theta - \theta\cos\theta + 2\sin\theta}{\theta^3}.$$

There is no solution to the equation obtained by setting the numerator equal to zero in the interval $\pi/2 < \theta < \pi$ (right figure below). Since $A(\pi/2) \approx 115\,668$ and $A(\pi) = 500\,000/\pi > 115\,668$, maximum area is $500\,000/\pi$ m$^2$ when it is a semicircle.




**76.** Triangle $ABC$ in the left diagram below is right-angled at $C$ and therefore $y^2 = x^2 + c^2$. Since triangle $ADC$ is also right-angled, $c^2 = a^2 + (c-z)^2$, and this equation can be solved for $c = (a^2+z^2)/(2z)$. Finally, from triangle $BCE$, $x^2 = z^2 + (a-x)^2$, and this equation can be solved for $x = (a^2+z^2)/(2a)$. We may write therefore that

$$y^2 = \left(\frac{a^2+z^2}{2a}\right)^2 + \left(\frac{a^2+z^2}{2z}\right)^2 = (a^2+z^2)^2\left(\frac{1}{4a^2} + \frac{1}{4z^2}\right) = \frac{(a^2+z^2)^3}{4a^2z^2}.$$

Since $a$ is constant, we have expressed $y$ in terms of $z$. The smallest value for $z$ occurs when the fold begins in the lower left corner (see the middle diagram below). In this case $b^2 = a^2 + (b-z)^2$, and this equation can be solved for $z = b - \sqrt{b^2 - a^2}$. The largest value for $z$ is $a$ shown in the right diagram. We therefore minimize the function

$$y^2 = f(z) = \frac{(z^2+a^2)^3}{4a^2z^2}, \qquad b - \sqrt{b^2-a^2} \le z \le a.$$

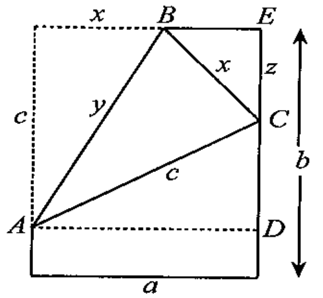
For critical points we solve

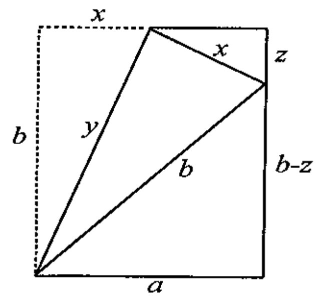
$$0 = f'(z) = \frac{1}{4a^2}\left[\frac{z^2(3)(z^2+a^2)^2(2z) - (z^2+a^2)^3(2z)}{z^4}\right] = \frac{(z^2+a^2)^2(2z^2-a^2)}{2a^2z^3}.$$

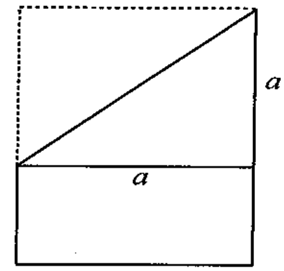
The only critical point is $z = a/\sqrt{2}$. We now calculate

$$f(a) = \frac{(a^2+a^2)^3}{4a^2a^2} = 2a^2 \quad \text{and} \quad f\left(\frac{a}{\sqrt{2}}\right) = \frac{\left(\frac{a^2}{2} + a^2\right)^3}{4a^2\left(\frac{a^2}{2}\right)} = \frac{27a^2}{16}.$$

We should also calculate $f(z)$ at $z = b - \sqrt{b^2-a^2}$, but the middle diagram makes it clear that $f(z)$ cannot have a minimum value in this situation. Hence the minimum length of the fold is $3\sqrt{3}a/4$ when $z = a/\sqrt{2}$ and $x = 3a/4$.

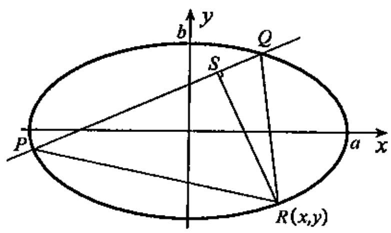


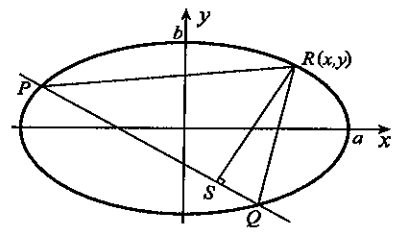




**77.** The area of triangle $PQR$ is $\|PQ\|\|RS\|/2$ where $\|RS\| = \dfrac{|y - mx - c|}{\sqrt{1+m^2}}$. Thus,

$$A = \frac{\|PQ\|}{2\sqrt{1+m^2}}|y - mx - c|.$$




It is not necessary to divide the discussion into two parts depending on whether $R$ is above or below the line and find intervals on which the functions are defined. It is clear that the maximum occurs at a critical point. To find critical points we solve

$$0 = \frac{dA}{dx} = \frac{\|PQ\|}{2\sqrt{1+m^2}}\frac{|y-mx-c|}{y-mx-c}\left(\frac{dy}{dx} - m\right).$$

Thus, we must have $dy/dx = m$ at critical points. Differentiation of the equation of the ellipse gives

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \quad \Longrightarrow \quad \frac{dy}{dx} = -\frac{b^2x}{a^2y}.$$

When we combine this with $dy/dx = m$, we obtain $-b^2x/(a^2y) = m \Longrightarrow y = -b^2x/(a^2m)$. Substitution of this into the equation of the ellipse gives

$$\frac{x^2}{a^2} + \frac{1}{b^2}\left(\frac{b^4x^2}{a^4m^2}\right) = 1 \quad \Longrightarrow \quad x = \frac{\pm a^2m}{\sqrt{a^2m^2+b^2}} \quad \text{and} \quad y = \frac{\pm b^2}{\sqrt{a^2m^2+b^2}}.$$

The equation $y = -b^2x/(a^2m)$ requires $x$ and $y$ to have opposite signs when $m > 0$ and the same signs when $m < 0$. In either case, critical points are $\pm\left(\dfrac{a^2m}{\sqrt{a^2m^2+b^2}}, \dfrac{-b^2}{\sqrt{a^2m^2+b^2}}\right)$. The diagrams makes it clear that when $m > 0$, the sign of $x$ should be chosen opposite to that of $c$, and when $m < 0$, the sign of $x$ should be the same as that of $c$. If $c = 0$, either choice gives the same area.

**78.** This is essentially the same problem as Exercise 75.

## EXERCISES 4.8

**1.** The velocity and acceleration are

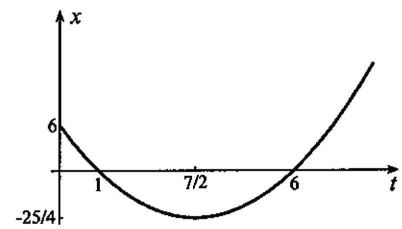
$$v(t) = 2 \text{ m/s},$$
$$a(t) = 0 \text{ m/s}^2.$$

The object begins 15 m to the right of the origin, and moves to the right with constant velocity 2 m/s forever.

**2.** The velocity and acceleration are

$$v(t) = 2t - 7 \text{ m/s},$$
$$a(t) = 2 \text{ m/s}^2.$$

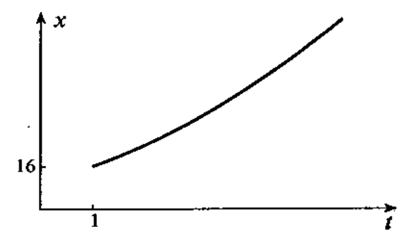
At time $t = 0$, the object is at $x = 6$ m and moving to the left with speed 7 m/s. It continues to move to the left until $t = 7/2$ s when it stops at $x = -25/4$ m. It then moves to the right with ever increasing speed.




**3.** The velocity and acceleration are

$$v(t) = 2t + 5 \text{ m/s},$$
$$a(t) = 2 \text{ m/s}^2.$$

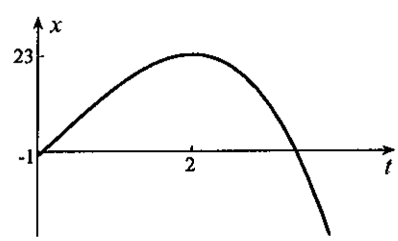
At time $t = 1$, the object is at $x = 16$ m and moving to the right with speed 7 m/s. It continues to move to the right with ever increasing speed.




**4.** The velocity and acceleration are

$$v(t) = -6t^2 + 4t + 16 = -2(3t + 4)(t - 2) \text{ m/s},$$
$$a(t) = -12t + 4 = -4(3t - 1) \text{ m/s}^2.$$

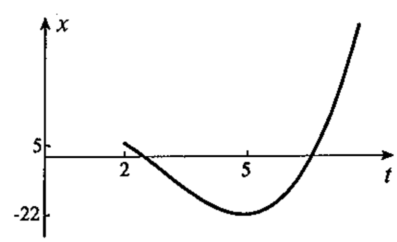
The object begins at time $t = 0$ at position $x = -1$ m with speed 16 m/s to the right. Its acceleration at this time is 4 m/s$^2$ so that it is picking up speed. At time $t = 1/3$ s, acceleration is zero and thereafter acceleration is negative. This means that its velocity decreases for $t \geq 1/3$ s. At time $t = 2$ s (and $x = 23$ m), the object's velocity is zero. Thereafter it moves to the left with increasing speed.




**5.** The velocity and acceleration are

$$v(t) = 3t^2 - 18t + 15 = 3(t - 1)(t - 5) \text{ m/s},$$
$$a(t) = 6t - 18 = 6(t - 3) \text{ m/s}^2.$$

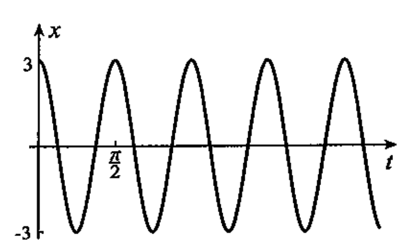
The object begins at time $t = 2$ at position $x = 5$ m with speed 9 m/s to the left. Its acceleration at this time is $-6$ m/s$^2$ so that it is speeding up. At time $t = 3$ s, acceleration is zero and thereafter acceleration is positive. This means that its velocity increases for $t \geq 3$ s. At time $t = 5$ s (and $x = -22$ m), the object's velocity is zero. Thereafter it moves to the right with increasing speed.




**6.** The velocity and acceleration are

$$v(t) = -12 \sin 4t \text{ m/s},$$
$$a(t) = -48 \cos 4t \text{ m/s}^2.$$

The object moves back and forth along the $x$-axis between $x = \pm 3$ m. Its velocity is zero in the turns, and its acceleration is equal to zero each time it passes through $x = 0$. The period of each oscillation is $\pi/2$ s.

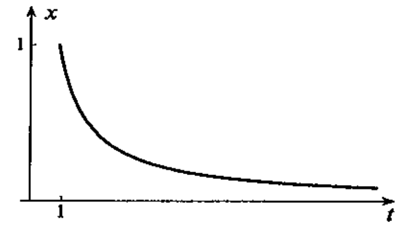


7. The velocity and acceleration are

$v(t) = -1/t^2$ m/s,

$a(t) = 2/t^3$ m/s$^2$.

The object is at position $x = 1$ m and moving to the left with speed 1 m/s at time $t = 1$. It continues to move to the left forever with ever decreasing speed gradually approaching the origin.

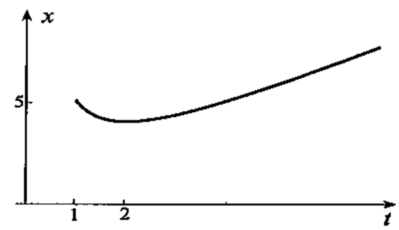



8. The velocity and acceleration are

$v(t) = 1 - 4/t^2$ m/s,

$a(t) = 8/t^3$ m/s$^2$.

At time $t = 1$, the object is at position $x = 5$ m and moving to the left with speed 3 m/s. Its acceleration is always to the right resulting in an instantaneous stop at $t = 2$ s at $x = 4$ m. The object then moves to the right thereafter with increasing speed. For large $t$, the velocity of the object approaches 1 m/s, and its acceleration approaches 0.

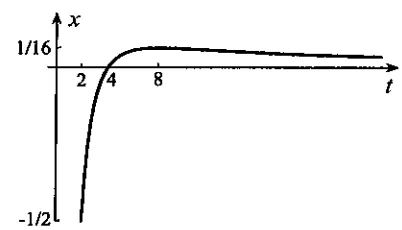



9. The velocity and acceleration are

$v(t) = -1/t^2 + 8/t^3 = (8 - t)/t^3$ m/s,

$a(t) = 2/t^3 - 24/t^4 = 2(t - 12)/t^4$ m/s$^2$.

At time $t = 2$, the object is at position $x = -1/2$ m and moving to the right with speed 3/4 m/s. Its acceleration is negative for $2 \le t \le 12$ resulting in an instantaneous stop at $t = 8$ s at $x = 1/16$ m. The object then moves to the left thereafter picking up speed until $t = 12$ s. For large $t > 12$, its speed decreases as it gradually approaches the origin.




10. The velocity and acceleration are

$$v(t) = \frac{5}{2}t^{3/2} - 3\sqrt{t} + \frac{1}{2\sqrt{t}} = \frac{(5t-1)(t-1)}{2\sqrt{t}} \text{ m/s},$$

$$a(t) = \frac{15}{4}\sqrt{t} - \frac{3}{2\sqrt{t}} - \frac{1}{4t^{3/2}} = \frac{15t^2 - 6t - 1}{4t^{3/2}} \text{ m/s}^2.$$

The object starts at the origin with zero velocity, but with positive acceleration. It therefore moves to the right, continuing to do so forever with ever increasing speed.


11. A plot of the displacement function is shown to the right. We need its velocity and acceleration:

$v(t) = 3t^2 - 18t + 24 = 3(t - 2)(t - 4)$ m/s,

$a(t) = 6t - 18 = 6(t - 3)$ m/s$^2$.

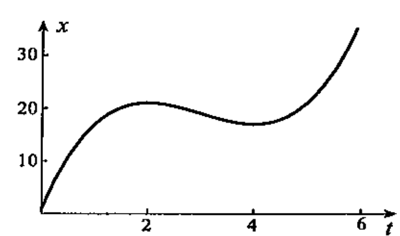
(a) Since $v(1) = 9$ and $a(1) = -12$, the object is slowing down, its speed is decreasing.

(b) Critical points of $v(t)$ occur when acceleration vanishes, namely, at $t = 3$. Since $v(0) = 24$, $v(3) = -3$, and $v(6) = 24$, maximum and minimum velocities are 24 m/s and $-3$ m/s.

(c) Maximum speed is 24 m/s and minimum speed is 0.

(d) Since acceleration is linear in $t$, maximum and minimum accelerations occur at $t = 0$ and $t = 6$. They are $\pm 18$ m/s$^2$.

(e) Since $a'(t) = 6 > 0$, the acceleration is always increasing.

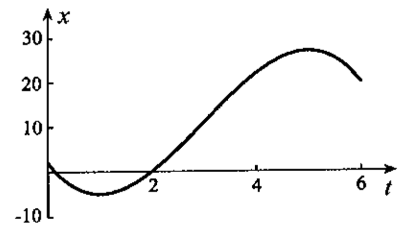


**12.** A plot of the displacement function is shown to the right. We need its velocity and acceleration:

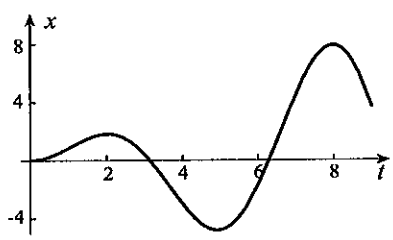
$v(t) = -15 + 18t - 3t^2 = -3(t-1)(t-5)$ m/s,

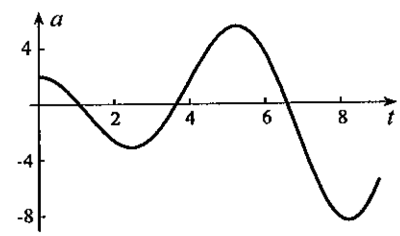
$a(t) = 18 - 6t = -6(t-3)$ m/s$^2$.

(a) Since $v(1) = 0$, the object is stopped at $t = 1$, it's speed is neither increasing nor decreasing.
(b) Critical points of $v(t)$ occur when acceleration vanishes, namely, at $t = 3$. Since $v(0) = -15$, $v(3) = 12$, and $v(6) = -15$, maximum and minimum velocities are 12 m/s and $-15$ m/s.
(c) Maximum speed is 15 m/s and minimum speed is 0.
(d) Since acceleration is linear in $t$, maximum and minimum acceleration occur at $t = 0$ and $t = 6$. They are $\pm 18$ m/s$^2$.
(e) Since $a'(t) = -6 < 0$, the acceleration is never increasing.




**13.** (a) A plot is shown in the left figure below.




(b) Velocity vanishes when $0 = v(t) = \sin t + t\cos t$. This occurs for $t = 0$ and three additional times. To find the next time, we use Newton's iterative procedure with $t_1 = 2$ and

$$t_{n+1} = t_n - \frac{\sin t_n + t_n \cos t_n}{2\cos t_n - t_n \sin t_n}.$$

Iteration gives $t_2 = 2.029$, $t_3 = 2.0288$, and $t_4 = 2.0288$. Since $v(2.0285) = 7.0 \times 10^{-4}$ and $v(2.0295) = -2.0 \times 10^{-3}$, to three decimals, the velocity is zero at $t = 2.029$ s. Similar procedures give the remaining times $t = 4.913$ s and $t = 7.979$ s.
(c) Acceleration is equal to 1 when $1 = a(t) = 2\cos t - t\sin t$. The right plot above for $a(t)$ indicates that this occurs three times. To find the first time, we use Newton's iterative procedure with $t_1 = 0.7$ and

$$t_{n+1} = t_n - \frac{2\cos t_n - t_n \sin t_n - 1}{-3\sin t_n - t_n \cos t_n}.$$

Iteration gives $t_2 = 0.7314$ and $t_3 = 0.7314$. Since $a(0.7305) - 1 = 2.3 \times 10^{-3}$ and $a(0.7315) - 1 = -2.9 \times 10^{-4}$, we can say that acceleration is 1 at $t = 0.731$ (accurate to three decimals). Similar procedures give the additional times $t = 3.853$ and $t = 6.436$.

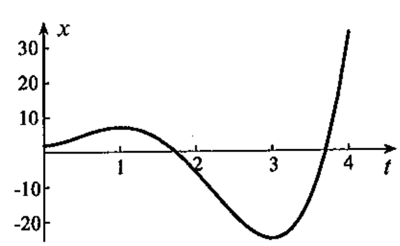
**14.** A plot of the displacement function is shown to the right. We need the velocity and acceleration functions:

$v(t) = 12t^3 - 48t^2 + 36t = 12t(t-1)(t-3)$ m/s,

$a(t) = 36t^2 - 96t + 36 = 12(3t^2 - 8t + 3)$ m/s$^2$.

(a) Acceleration is zero for

$$t = \frac{8 \pm \sqrt{64-36}}{6} = \frac{4 \pm \sqrt{7}}{3}.$$




Since $a(t) \geq 0$ for $0 \leq t \leq (4-\sqrt{7})/3$ and $(4+\sqrt{7})/3 \leq t \leq 4$, velocity is increasing on these intervals, and it is decreasing for $(4-\sqrt{7})/3 \leq t \leq (4+\sqrt{7})/3$.
(b) Speed is increasing on the intervals $0 \leq t \leq (4-\sqrt{7})/3$, $1 \leq t \leq (4+\sqrt{7})/3$, and $3 \leq t \leq 4$. It is decreasing for $(4-\sqrt{7})/3 \leq t \leq 1$ and $(4+\sqrt{7})/3 \leq t \leq 3$.

(c) Critical points of velocity occur where acceleration is zero, namely $t = (4 \pm \sqrt{7})/3$. Since

$$v(0) = 0, \quad v((4-\sqrt{7})/3) = \frac{8(-10+7\sqrt{7})}{9}, \quad v((4+\sqrt{7})/3) = \frac{-8(10+7\sqrt{7})}{9}, \quad v(4) = 144,$$

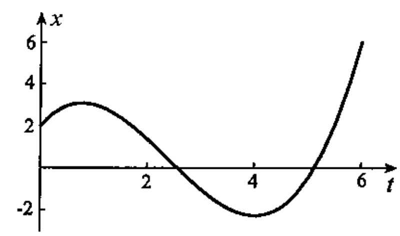
velocity is a maximum at $t = 4$ and a minimum at $t = (4+\sqrt{7})/3$.
(d) Speed is a maximum at $t = 4$ and a minimum at $t = 0, 1, 3$ where it is zero.
(e) The graph makes it clear that maximum distance from the origin is $x(4) = 34$.
(f) To answer this we should maximize $x(t) - 5$. Critical points of this function are $t = 1, 3$. Since

$$x(0) - 5 = -3, \quad x(1) - 5 = 2, \quad x(3) - 5 = -30, \quad x(4) - 5 = 29,$$

maximum distance from $x = 5$ is 30 at $t = 3$.

**15.** A plot of the displacement function is shown to the right. We need the velocity and acceleration functions:

$$v(t) = 14t^2/15 - 202t/45 + 132/45$$
$$= 2(21t^2 - 101t + 66)/45 \text{ m/s},$$
$$a(t) = 28t/15 - 202/45 \text{ m/s}^2.$$



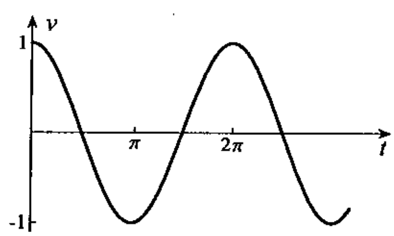

(a) Acceleration is zero for $t = 101/42$, and velocity is decreasing for $0 \le t \le 101/42$, and increasing for $101/42 \le t \le 6$.
(b) Velocity is equal to zero when
$$t = \frac{101 \pm \sqrt{101^2 - 84(66)}}{42} = \frac{101 \pm \sqrt{4657}}{42}.$$
Speed is therefore increasing on the intervals $(101 - \sqrt{4657})/42 \le t \le 101/42$ and $(101 + \sqrt{4657})/42 \le t \le 6$. It is decreasing for $0 \le t \le (101 - \sqrt{4657})/42$ and $101/42 \le t \le (101 + \sqrt{4657})/42$.
(c) Critical points of velocity occur where acceleration is zero, namely $t = 101/42$. Since

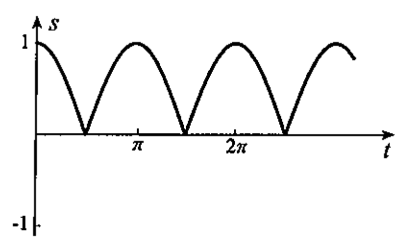
$$v(0) = \frac{132}{45}, \quad v(101/42) = -\frac{4657}{1890}, \quad v(6) = \frac{48}{5},$$

velocity is a maximum at $t = 6$ and a minimum at $t = 101/42$.
(d) Speed is a maximum at $t = 6$ and a minimum at $t = (101 \pm \sqrt{4657})/42$ when it is zero.
(e) The graph makes it clear that maximum distance from the origin is $x(6) = 6$.
(f) The graph makes it clear that maximum distance from $x = 5$ is $5 - x\big((101 + \sqrt{4657})/42\big) = 7.27$.

**16.** A horizontal point of inflection.

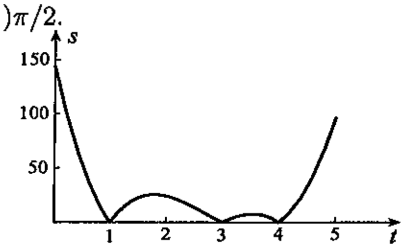
**17.** Not always. Suppose the displacement function is $x(t) = \sin t$, $t \ge 0$. Graphs of velocity and speed functions are shown below.



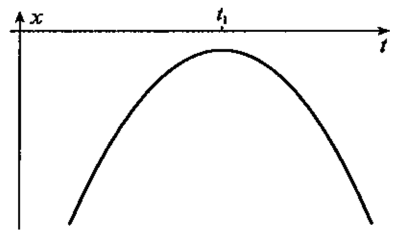

Critical points of the velocity graph are $t = n\pi$, where $n \ge 0$ is an integer. On the other hand, the speed graph has critical points at these points and also at $t = (2n+1)\pi/2$.

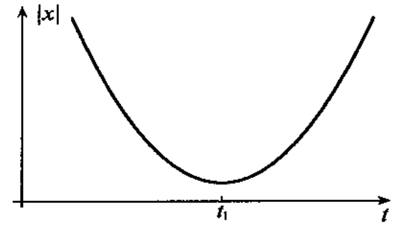
**18.** The speed graph to the right indicates maximum speed at $t = 0$ of $|v(0)| = 144$ m/s.



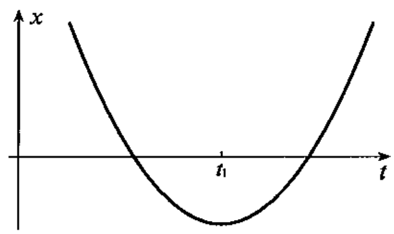
**19.** If we regard velocity as a function of position, and apply the chain rule to the situation, $v = v(x)$, $x = x(t)$, then $a = \dfrac{dv}{dt} = \dfrac{dv}{dx}\dfrac{dx}{dt} = \dfrac{dv}{dx}v$.

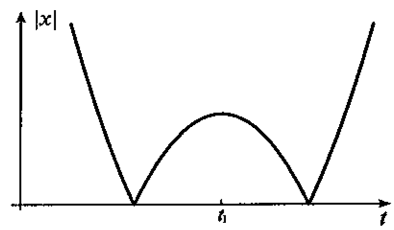
**20.** This is not always true. Consider the position function shown to the left below together with its absolute value to the right. Position has relative maximum at $t_1$, whereas its absolute value has relative maximum at $t_2$.



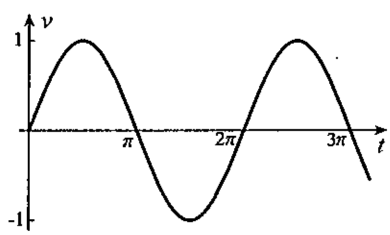



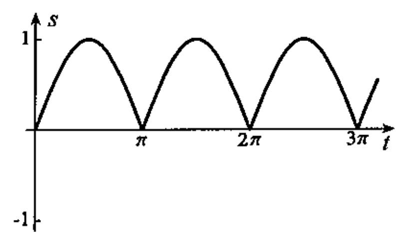

**21.** This is not always true. Consider the position function shown to the left below together with its absolute value to the right. Position has relative minimum at $t_1$, whereas its absolute value has relative maximum at $t_1$.



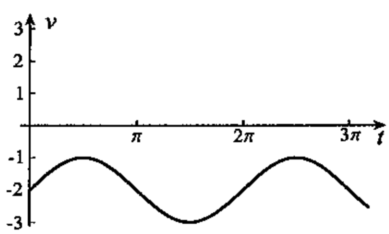



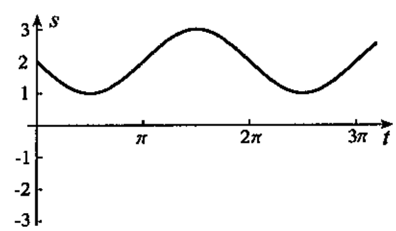

**22.** This is not always true. Consider the velocity function $v(t) = \sin t$, $t \geq 0$. Graphs of the velocity and speed functions are shown below. Velocity has relative minima at $t = 3\pi/2 + 2n\pi$, where $n$ is a nonnegative integer, whereas speed has relative maxima at these times.




**23.** This is not always true. Consider the velocity function $v(t) = -2 + \sin t$, $t \geq 0$. Graphs of the velocity and speed functions are shown below. Velocity has relative maxima at $t = \pi/2 + 2n\pi$, where $n$ is a nonnegative integer, whereas speed has relative minima at these times.




**24.** (a) Since $\phi$ lies in the interval $-\pi/2 < \phi < \pi/2$, we can solve $e = L\sin\phi - r\sin\theta$ for

$$\sin\phi = \frac{e + r\sin\theta}{L} \qquad \Longrightarrow \qquad \cos\phi = \sqrt{1 - \left(\frac{e + r\sin\theta}{L}\right)^2}.$$

Hence,

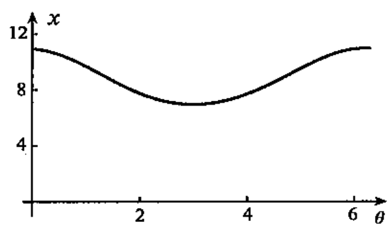
$$x = r\cos\theta + L\sqrt{1 - \left(\frac{e + r\sin\theta}{L}\right)^2} = r\cos\theta + \sqrt{L^2 - (e + r\sin\theta)^2}.$$

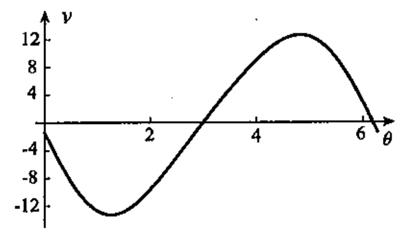
(b) When $L = 9$, $r = 2$, and $e = 1$,

$$x = 2\cos\theta + \sqrt{81 - (1 + 2\sin\theta)^2},$$

a plot of which is shown in the left figure below.
(c) Since the length of the stroke is the difference between maximum and minimum values of $x$, the graph suggests a stroke length of about 4 cm. The formula in Example 4.28 gives the stroke length

$$s = \sqrt{(9+2)^2 - 1^2} - \sqrt{(9-2)^2 - 1^2} = 2\sqrt{30} - 4\sqrt{3} = 4.03.$$




(d) Differentiation of $x$ with respect to $t$ gives

$$v = \frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt} = \omega\left[-r\sin\theta + \frac{-2(e + r\sin\theta)r\cos\theta}{2\sqrt{L^2 - (e + r\sin\theta)^2}}\right]$$
$$= -\omega r\left[\sin\theta + \frac{\cos\theta(e + r\sin\theta)}{\sqrt{L^2 - (e + r\sin\theta)^2}}\right].$$

If we substitute for $\sin\phi$ and $\cos\phi$ in the velocity formula of Example 4.31,

$$v = \frac{-\omega r(\sin\theta\cos\phi + \cos\theta\sin\phi)}{\cos\phi} = -\omega r\left[\sin\theta + \frac{\cos\theta\left(\dfrac{e + r\sin\theta}{L}\right)}{\sqrt{1 - \left(\dfrac{e + r\sin\theta}{L}\right)^2}}\right]$$
$$= -\omega r\left[\sin\theta + \frac{\cos\theta(e + r\sin\theta)}{\sqrt{L^2 - (e + r\sin\theta)^2}}\right],$$

the above result.
(e) With $\omega = 2\pi$, $L = 9$, $r = 2$, and $e = 1$,

$$v = -4\pi\left[\sin\theta + \frac{\cos\theta(1 + 2\sin\theta)}{\sqrt{81 - (1 + 2\sin\theta)^2}}\right],$$

a plot of which is shown in the right figure above. The velocity would appear to be zero when the displacement graph is at its highest and lowest points. These are the inner and outer dead positions in Example 4.28. Maximum and minimum velocities are approximately $\pm 13$ cm/s.

**25.** If we let $y = f(x) = ax^3 + bx^2 + cx + d$ denote the cubic polynomial for the landing curve (left figure below), then $d$ must be zero for the curve to pass through $(0, 0)$. For a smooth touchdown, we also demand that

$$0 = f'(0) = (3ax^2 + 2bx + c)_{|x=0} = c.$$

Suppose we denote horizontal distance from $x = 0$ to the point at which descent commences by $L$. Then descent begins at the point $(L, h)$, and

$$h = f(L) = aL^3 + bL^2 \qquad \text{and} \qquad 0 = f'(L) = 3aL^2 + 2bL.$$

These equations can be solved for $a = -2h/L^3$ and $b = 3h/L^2$, so that the glide path has equation

$$y = f(x) = -\frac{2hx^3}{L^3} + \frac{3hx^2}{L^2}.$$

Since a constant horizontal speed $U$ must be maintained, we can say that $dx/dt = -U$ (the negative sign is necessary because $dx/dt < 0$). It follows that

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \left(-\frac{6hx^2}{L^3} + \frac{6hx}{L^2}\right)(-U).$$

Vertical acceleration is therefore

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dx}\left(\frac{dy}{dt}\right)\frac{dx}{dt} = \left(\frac{-12hx}{L^3} + \frac{6h}{L^2}\right)(-U)^2 = \frac{6hU^2}{L^2}\left(1 - \frac{2x}{L}\right).$$

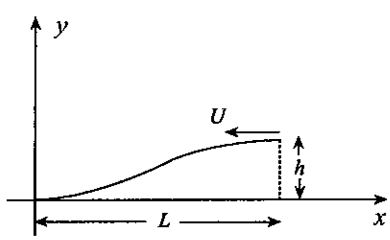
The right figure below shows that acceleration is positive for $0 < x < L/2$ and negative for $L/2 < x < L$. In addition, in magnitude, it is largest at $x = 0$ and $x = L$, where

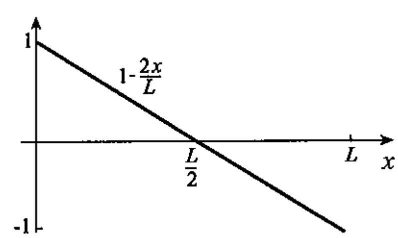
$$\left|\frac{d^2y}{dt^2}\right| = \frac{6hU^2}{L^2}.$$

Consequently, we must have

$$\frac{6hU^2}{L^2} \le k \qquad \Longrightarrow \qquad L \ge \sqrt{\frac{6hU^2}{k}};$$

that is, the plane must begin descent when it is at least $\sqrt{6hU^2/k}$ metres from touchdown.•



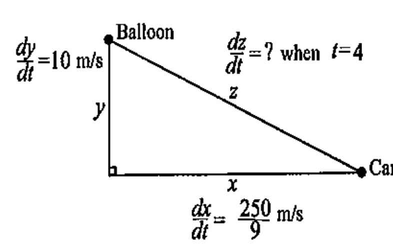




## EXERCISES 4.9

**1.** The distance $z$ between the balloon and the car is given by $z^2 = x^2 + y^2$. Differentiation of this equation with respect to time $t$ gives

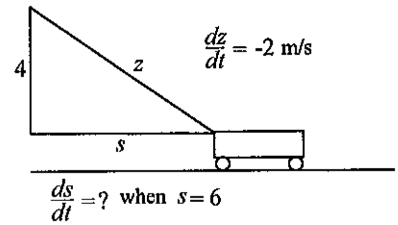
$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}.$$

When $t = 4$, we know that $x = 1000/9$, $y = 40$, and $z = \sqrt{(1000/9)^2 + (40)^2} = (40/9)\sqrt{706}$. Consequently, at this time




$$\frac{40}{9}\sqrt{706}\frac{dz}{dt} = \frac{1000}{9}\left(\frac{250}{9}\right) + 40(10) \qquad \Longrightarrow \qquad \frac{dz}{dt} = \frac{9}{40\sqrt{706}}\left(\frac{250\,000}{81} + 400\right) = 29.5.$$

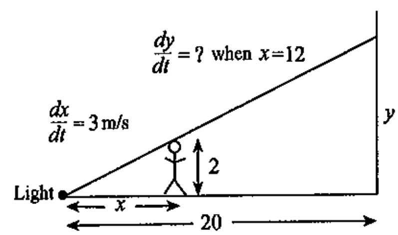
The balloon and child are separating at 29.5 m/s.




**2.** For the right-angled triangle, we may write $z^2 = s^2 + 16$. Differentiation with respect to time gives $2z\frac{dz}{dt} = 2s\frac{ds}{dt}$. When $s = 6$, we find that $z = 2\sqrt{13}$, and

$$(2\sqrt{13})(-2) = (6)\frac{ds}{dt}.$$

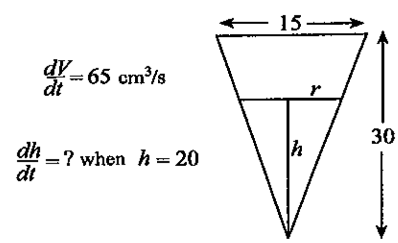
Thus, $ds/dt = -2\sqrt{13}/3$. The cart is therefore moving to the left at $2\sqrt{13}/3$ m/s.




**3.** By similar triangles, $y/20 = 2/x \implies y = 40/x$. If we differentiate this with respect to time $t$, $\frac{dy}{dt} = -\frac{40}{x^2}\frac{dx}{dt}$. When $x = 12$, we obtain

$$\frac{dy}{dt} = -\frac{40}{144}(3) = -\frac{5}{6}.$$

The length of the shadow is therefore decreasing at the rate of 5/6 m/s.




**4.** When the depth of liquid is $h$, the volume of liquid in the funnel is $V = \frac{1}{3}\pi r^2 h$. Similar triangles give $\frac{r}{h} = \frac{15/2}{30}$. Thus, $r = h/4$, and

$$V = \frac{1}{3}\pi\left(\frac{h}{4}\right)^2 h = \frac{\pi}{48}h^3.$$

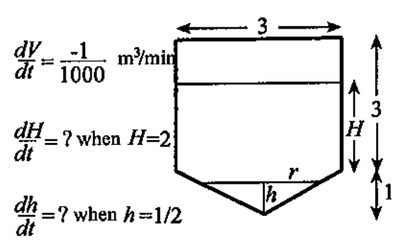
Differentiation with respect to time gives

$$\frac{dV}{dt} = \frac{\pi}{16}h^2\frac{dh}{dt}.$$

Since the net rate of change of the volume of liquid in the funnel is 65 $\text{cm}^3$/s, we can say that when $h = 20$,

$$65 = \frac{\pi}{16}(20)^2\frac{dh}{dt} \implies \frac{dh}{dt} = \frac{65 \cdot 16}{400\pi} = \frac{13}{5\pi}.$$

The liquid level is therefore rising at $13/(5\pi)$ cm/s.



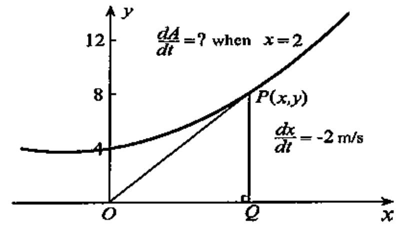

**5.** (a) When the water level is in the cylinder, the volume of water in the tank is $V = V_{\text{cone}} + \pi(3/2)^2 H$ where $V_{\text{cone}}$, the volume of water in the cone is constant. Differentiation with respect to time gives $\frac{dV}{dt} = \frac{9\pi}{4}\frac{dH}{dt}$. Consequently,
$\frac{dH}{dt} = \frac{4}{9\pi}\left(\frac{-1}{1000}\right) = \frac{-1}{2250\pi}$. The water level is dropping at $1/(2250\pi)$ metres per minute.

(b) When the water level is in the cone, the volume of water in the tank is $V = (1/3)\pi r^2 h$. By similar triangles, $r/h = (3/2)/1 \implies r = 3h/2$. Hence, $V = (1/3)\pi(3h/2)^2 h = 3\pi h^3/4$. When we differentiate this equation with respect to time, $dV/dt = (9\pi h^2/4)dh/dt$, and when $h = 1/2$, we obtain

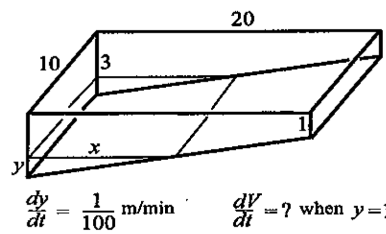
$$-\frac{1}{1000} = \frac{9\pi}{4}\left(\frac{1}{2}\right)^2\frac{dh}{dt} \implies \frac{dh}{dt} = \frac{16}{9\pi}\left(\frac{-1}{1000}\right) = \frac{-2}{1125\pi}.$$

The water level is dropping at $2/(1125\pi)$ metres per minute.

**6.** The area of the triangle is
$$A = \frac{1}{2}xy = \frac{1}{2}x(x^2 + x + 4).$$
If we differentiate with respect to time,
$$\frac{dA}{dt} = \frac{1}{2}\left(3x^2\frac{dx}{dt} + 2x\frac{dx}{dt} + 4\frac{dx}{dt}\right) = \frac{1}{2}(3x^2 + 2x + 4)\frac{dx}{dt}.$$
When $x = 2$, we obtain
$$\frac{dA}{dt} = \frac{1}{2}[3(2)^2 + 2(2) + 4](-2) = -20.$$
The area is therefore decreasing at 20 $\text{m}^2/\text{s}$.

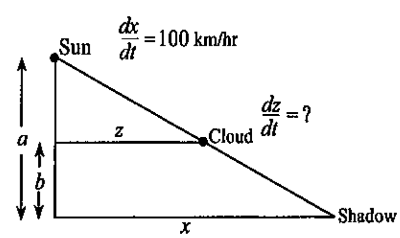



**7.** The volume of water in the pool is given by $V = (1/2)xy(10) = 5xy$. Similar triangles gives $y/x = 2/20 \implies x = 10y$. Hence, $V = 5y(10y) = 50y^2$. Differentiation with respect to time gives $dV/dt = (100y)dy/dt$. Consequently, when $y = 1$,
$$\frac{dV}{dt} = 100(1)\left(\frac{1}{100}\right) = 1.$$
The pool is being filled at 1 cubic metre per minute.

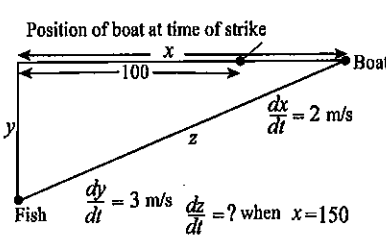



**8.** If $P$ and $V$ are pressure and volume of the gas, Boyle's law states that $P = k/V$, where $k$ is a constant. Differentiation with respect to time gives
$$\frac{dP}{dt} = -\frac{k}{V^2}\frac{dV}{dt}.$$
Since $V = 1/100$ $\text{m}^3$ when $P = 50$ $\text{N/m}^2$, it follows that $50 = k(100)$, which implies that $k = 1/2$. At this instant,
$$\frac{dP}{dt} = -\frac{1/2}{(1/100)^2}\left(\frac{1}{2000}\right) = -2.5.$$
Pressure is decreasing at 2.5 $\text{N/m}^2/\text{s}$.

**9.** If $b$ is the height of the clouds and $a$ is the height of the sun, then similar triangles give $\dfrac{z}{a-b} = \dfrac{x}{a}$, from which $z = \left(\dfrac{a-b}{a}\right)x$. Differentiation with respect to time gives $\dfrac{dz}{dt} = \left(\dfrac{a-b}{a}\right)\dfrac{dx}{dt}$. Since $b$ is so much smaller than $a$, we can say that $a - b \approx a$, and therefore $dz/dt \approx dx/dt = 100$ km/hr.




**10.** The amount of line between reel and fish is $z^2 = x^2 + y^2$. Differentiation of this equation with respect to time gives $2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt}$. It takes 25 s for the boat to travel the 50 m from the instant the fish struck, and during this time the fish dives 75 m. The amount of line line between fish and reel at this instant is $\sqrt{150^2 + 75^2} = 75\sqrt{5}$. Consequently,
$$75\sqrt{5}\frac{dz}{dt} = (150)(2) + 75(3) \implies \frac{dz}{dt} = \frac{525}{75\sqrt{5}} = \frac{7}{\sqrt{5}}.$$
The line is therefore being played out at $7/\sqrt{5}$ m/s.



**11.** Differentiation of $PV^{7/5} = k$ with respect to time gives

$$V^{7/5}\frac{dP}{dt} + \frac{7}{5}PV^{2/5}\frac{dV}{dt} = 0.$$

At the instant in question, $V = 1/10$ m$^3$, $P = 4 \times 10^5$ N/m$^2$, and $dV/dt = -1/1000$ m$^3$/s, so that

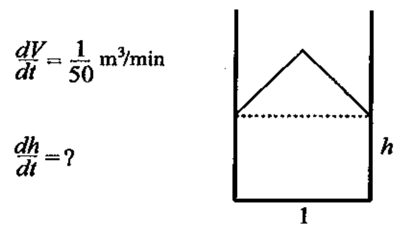
$$\left(\frac{1}{10}\right)^{7/5}\frac{dP}{dt} + \frac{7}{5}(4\times 10^5)\left(\frac{1}{10}\right)^{2/5}\left(\frac{-1}{1000}\right) = 0 \quad\Longrightarrow\quad \frac{dP}{dt} = 5600.$$

The pressure is increasing at 5600 N/m$^2$/s.

**12.** (a) When sand completely covers the bottom of the cylinder, the volume of sand is

$$V = C + \pi\left(\frac{1}{2}\right)^2 h = C + \frac{\pi h}{4},$$

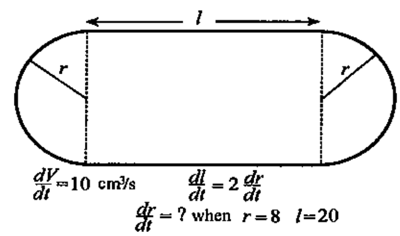
where $C$ is the volume of sand in the cone. Since $C$ is constant, differentiation of this equation with respect to time gives $\dfrac{dV}{dt} = \dfrac{\pi}{4}\dfrac{dh}{dt}$.




Since $dV/dt = 1/50$, it follows that $dh/dt = 2/(25\pi)$, and the top of the pile is rising at $2/(25\pi)$ m/min.
(b) Since the height of the cone is constant, sand also travels along the side of the cylinder at $2/(25\pi)$ m/min.

**13.** The volume of the balloon is $V = \dfrac{4}{3}\pi r^3 + \pi r^2 l$.
Differentiation with respect to time gives

$$\begin{aligned}\frac{dV}{dt} &= 4\pi r^2\frac{dr}{dt} + 2\pi r l\frac{dr}{dt} + \pi r^2\frac{dl}{dt}\\ &= 4\pi r^2\frac{dr}{dt} + 2\pi r l\frac{dr}{dt} + 2\pi r^2\frac{dr}{dt}\\ &= (6\pi r^2 + 2\pi r l)\frac{dr}{dt}.\end{aligned}$$




When $r = 8$ and $l = 20$, $\quad 10 = [6\pi(64) + 2\pi(8)(20)]\dfrac{dr}{dt} \implies \dfrac{dr}{dt} = \dfrac{5}{352\pi}$. The radius is increasing at $5/(352\pi)$ cm/s.

**14.** (a) When Car 2 is on that part of the racetrack between $D$ and $B$, its $x$- and $y$-coordinates must satisfy the equation $(x-50)^2 + y^2 = 2500$, the equation of a circle of radius 50 centred at the point $(50, 0)$. If we differentiate this equation with respect to time,

$$2(x-50)\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \quad\Longrightarrow\quad \frac{dy}{dt} = -\frac{x-50}{y}\frac{dx}{dt}.$$

Since the rate of change of the $x$-coordinate of Car 2 is the same as that of Car 1, namely 10 m/s, it follows that $dx/dt = 10$, and

$$\frac{dy}{dt} = \frac{10(50-x)}{y} \text{ m/s}.$$

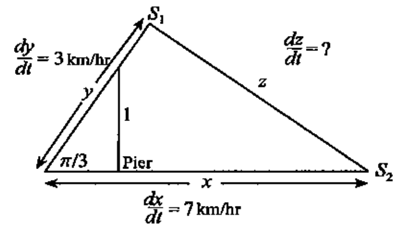
(b) When the car is at $E$, its $x$-coordinate is 75 and its $y$-coordinate is $-\sqrt{2500-(75-50)^2} = -25\sqrt{3}$. At this point,

$$\frac{dy}{dt} = \frac{10(50-75)}{-25\sqrt{3}} = \frac{10}{\sqrt{3}} \text{ m/s}.$$

(c) As Car 2 approaches $B$, its $x$-coordinate approaches 100 and its $y$-coordinate approaches 0. This means that the rate of change of its $y$-coordinate is $\displaystyle\lim_{x\to 100}\frac{10(50-x)}{y} = \infty$. Car 1 therefore suffers the most damage.

**15.** When the ships are at the positions $S_1$ and $S_2$, the distance $z$ between them is given by the cosine law,

$$z^2 = x^2 + y^2 - 2xy\cos(\pi/3) \\ = x^2 + y^2 - xy.$$




Differentiation with respect to time gives

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} - y\frac{dx}{dt} - x\frac{dy}{dt}.$$

At the instant in question, $x = 3/4 + \cot(\pi/3) = 3/4 + 1/\sqrt{3}$, $y = \csc(\pi/3) = 2/\sqrt{3}$, and $z = \sqrt{(3/4 + 1/\sqrt{3})^2 + (2/\sqrt{3})^2 - (3/4 + 1/\sqrt{3})(2/\sqrt{3})} = 5/4.$
At this instant,

$$2\left(\frac{5}{4}\right)\frac{dz}{dt} = 2\left(\frac{3}{4} + \frac{1}{\sqrt{3}}\right)(7) + 2\left(\frac{2}{\sqrt{3}}\right)(3) - \left(\frac{2}{\sqrt{3}}\right)(7) - \left(\frac{3}{4} + \frac{1}{\sqrt{3}}\right)(3) \quad \Longrightarrow \quad \frac{dz}{dt} = \frac{3(11 + 4\sqrt{3})}{10}.$$

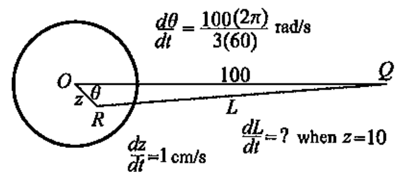
The ships are separating at $3(11 + 4\sqrt{3})/10$ km/hr.

**16.** The cosine law applied to triangle $ORQ$ gives $L^2 = z^2 + 100^2 - 200z\cos\theta$. Differentiation of this equation with respect to time gives

$$2L\frac{dL}{dt} = 2z\frac{dz}{dt} - 200\cos\theta\frac{dz}{dt} + 200z\sin\theta\frac{d\theta}{dt}.$$

When $z = 10$ and $\theta = \pi/4$, we obtain

$$L = \sqrt{100 + 100^2 - 200(10)(1/\sqrt{2})} = 93.197567.$$

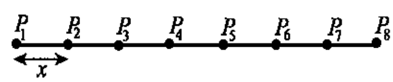



Substitution of these values into the equation involving $dL/dt$ gives

$$2(93.197567)\frac{dL}{dt} = 2(10)(1) - 200\left(\frac{1}{\sqrt{2}}\right)(1) + 200(10)\left(\frac{1}{\sqrt{2}}\right)\left(\frac{10\pi}{9}\right),$$

and this equation can be solved for $dL/dt = 25.83$. The distance is therefore increasing at 25.83 cm/s.

**17.** We assume that the first person $P_1$ on the whip stays on the same spot and the distance $x$ between skaters is constant. The fourth person $P_4$ travels on a circle of radius $3x$, and therefore distance travelled by this skater along the arc of the circle is given by $3x\theta$, where $\theta$ is the angle through which the skater has turned. Similarly, distance travelled by the seventh skater $P_7$ is given $6x\theta$. Since the rate of change of $\theta$ is the same for the skaters, it follows that the seventh skater travels twice as fast the the fourth skater.

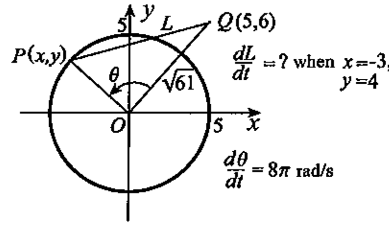



**18.** The cosine law on triangle $OPQ$ gives

$$L^2 = 61 + 25 - 2(5)\sqrt{61}\cos\theta = 86 - 10\sqrt{61}\cos\theta.$$

Differentiation of this equation with respect to time gives

$$2L\frac{dL}{dt} = 10\sqrt{61}\sin\theta\frac{d\theta}{dt}.$$

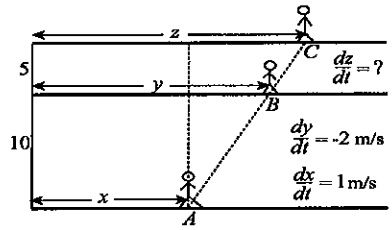



When the particle is at $P(-3,4)$, the length of $PQ$ is $L = \sqrt{(5+3)^2 + (6-4)^2} = 2\sqrt{17}$, and the equation $68 = 86 - 10\sqrt{61}\cos\theta$ gives $\cos\theta = 9/(5\sqrt{61})$. It follows that $\sin\theta = \sqrt{1 - 81/(25\cdot 61)} = 38/(5\sqrt{61})$. Substitution of these into the equation for $dL/dt$ gives

$$2(2\sqrt{17})\frac{dL}{dt} = 10\sqrt{61}\left(\frac{38}{5\sqrt{61}}\right)(8\pi).$$

When this is solved for $dL/dt$, the result is 115.8 cm/s.

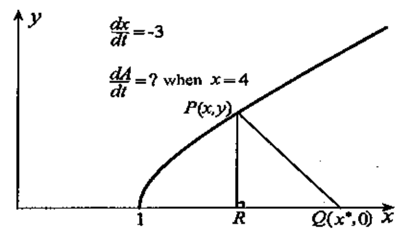
**19.** If $x$, $y$, and $z$ are distances from some fixed vertical line to $A$, $B$, and $C$, respectively, similar triangles give $\dfrac{y-x}{10} = \dfrac{z-x}{15} \implies z = \dfrac{3y-x}{2}$. Differentiation with respect to time gives $dz/dt = (1/2)(3dy/dt - dx/dt) = (1/2)(-6-1) = -7/2$. Thus, $C$ walks to the left at $7/2$ m/s.




**20.** (a) The slope of the tangent line to the hyperbola can be obtained by differentiating $x^2 - y^2 = 1$ with respect to $x$, $2x - 2y(dy/dx) = 0 \implies dy/dx = x/y$. The slope of the normal line at $P$ is therefore $-y/x$. Since the slope of this line is also $(y-0)/(x-x^*)$, it follows that

$$-\frac{y}{x} = \frac{y}{x - x^*}.$$

When we cross multiply and solve for $x^*$, we obtain the required result that $x^* = 2x$.




(b) The area of triangle $PQR$ is

$$A = \frac{1}{2}\|PR\|\|RQ\| = \frac{1}{2}y(x^* - x) = \frac{1}{2}\sqrt{x^2-1}(2x - x) = \frac{1}{2}x\sqrt{x^2-1}.$$

Differentiation with respect to time gives

$$\frac{dA}{dt} = \frac{1}{2}\left[\sqrt{x^2-1}\frac{dx}{dt} + x\left(\frac{x}{\sqrt{x^2-1}}\frac{dx}{dt}\right)\right] = \frac{2x^2-1}{2\sqrt{x^2-1}}\frac{dx}{dt}.$$

When $x = 4$, we obtain $\dfrac{dA}{dt} = \dfrac{2(4)^2-1}{2\sqrt{16-1}}(-3) = -\dfrac{31\sqrt{15}}{10}$. The area is therefore decreasing at $31\sqrt{15}/10$ units of area per unit of time.

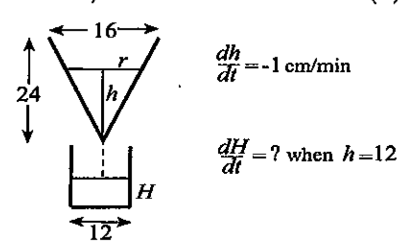
**21.** The total volume $V$ of solution in the filter and cylinder is constant, and therefore $V = \pi(6)^2H + (1/3)\pi r^2 h$. Similar triangles give $r/h = 8/24$, so that

$$V = 36\pi H + \frac{\pi}{3}\left(\frac{h}{3}\right)^2 h = 36\pi H + \frac{\pi h^3}{27}.$$

Differentiation with respect to time gives

$$0 = 36\pi\frac{dH}{dt} + \frac{\pi h^2}{9}\frac{dh}{dt}.$$

When $h = 12$,



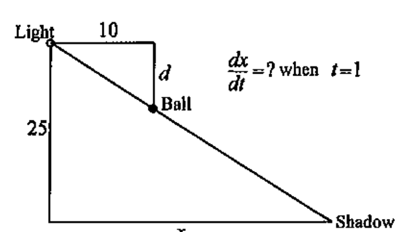

$$0 = 36\pi\frac{dH}{dt} + \frac{\pi(12)^2}{9}(-1) \implies \frac{dH}{dt} = \frac{4}{9}.$$

The water level is rising at $4/9$ centimetres per minute in the cylinder.

**22.** Similar triangles give $\dfrac{x}{25} = \dfrac{10}{d} \implies x = \dfrac{250}{d} = \dfrac{250}{4.905t^2}$.

Differentiation of this gives $\dfrac{dx}{dt} = -\dfrac{500}{4.905t^3}$.

At $t = 1$ s, we obtain $dx/dt = -500/4.905 = -102$. The shadow is therefore moving at 102 m/s.



**23.** The distance $D$ from $(1,2)$ to any point $(x,y)$ on the parabola is given by

$$D^2 = (x-1)^2 + (y-2)^2$$
$$= (x-1)^2 + (x^2-3x-2)^2.$$

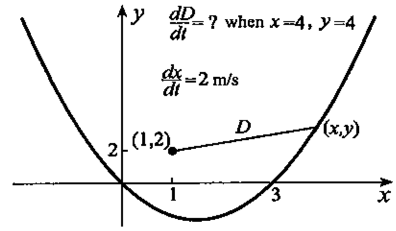
Differentiation with respect to time gives

$$2D\frac{dD}{dt} = 2(x-1)\frac{dx}{dt} + 2(x^2-3x-2)(2x-3)\frac{dx}{dt}.$$

At the point $(4,4)$, $D = \sqrt{(4-1)^2+(4-2)^2} = \sqrt{13}$, and

$$2\sqrt{13}\frac{dD}{dt} = 2(3)(2) + 2(2)(5)(2) \implies \frac{dD}{dt} = 2\sqrt{13}.$$

The distance is increasing at $2\sqrt{13}$ m/s.




**24.** When we solve the equation of the curve for $y$ in terms of $x$, we obtain $y = -x \pm 4\sqrt{x}$. Since we are concerned with that part of the curve which contains the point $(4,4)$, we must choose $y = 4\sqrt{x} - x$. Distance $D$ from any point $(x,y)$ on the curve to $(1,2)$ is given by $D^2 = (x-1)^2 + (y-2)^2 = (x-1)^2 + (4\sqrt{x}-x-2)^2$. Differentiation with respect to time gives

$$2D\frac{dD}{dt} = 2(x-1)\frac{dx}{dt} + 2(4\sqrt{x}-x-2)\left(\frac{2}{\sqrt{x}}-1\right)\frac{dx}{dt}.$$

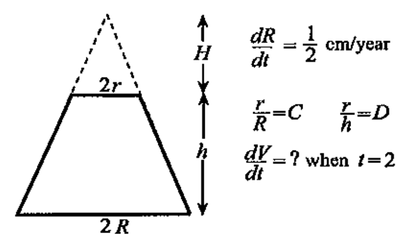
When $(x,y)=(4,4)$, we find $D = \sqrt{(4-1)^2+(4-2)^2} = \sqrt{13}$, and

$$2\sqrt{13}\frac{dD}{dt} = 2(3)(2) + 2(8-4-2)(1-1)(2) \quad\implies\quad \frac{dD}{dt} = \frac{6}{\sqrt{13}}.$$

The distance is changing at $6/\sqrt{13}$ m/s.

**25.** (a) By similar triangles, $H/r = (H+h)/R \implies H = rh/(R-r)$.
The volume of the trunk is the difference in the volumes of two cones,

$$\begin{aligned} V &= \frac{1}{3}\pi R^2(H+h) - \frac{1}{3}\pi r^2 H = \frac{1}{3}\pi(R^2-r^2)H + \frac{1}{3}\pi R^2 h \\ &= \frac{1}{3}\pi(R^2-r^2)\left(\frac{rh}{R-r}\right) + \frac{1}{3}\pi R^2 h \\ &= \frac{\pi h}{3(R-r)}[r(R^2-r^2) + R^2(R-r)] \\ &= \frac{\pi h}{3}(R^2 + rR + r^2). \end{aligned}$$




(b) If we differentiate this equation with respect to time,

$$\frac{dV}{dt} = \frac{\pi}{3}(R^2+rR+r^2)\frac{dh}{dt} + \frac{\pi h}{3}\left(2R\frac{dR}{dt} + r\frac{dR}{dt} + R\frac{dr}{dt} + 2r\frac{dr}{dt}\right).$$

But $r/R = C$ and $r/h = D$, where $C$ and $D$ are constants. At the present time $r = 10$, $R = 50$, and $h = 3000$, and therefore $C = 10/50 = 1/5$, and $D = 10/3000 = 1/300$. Differentiation of $r = R/5$ and $h = 300r$ with respect to $t$ gives

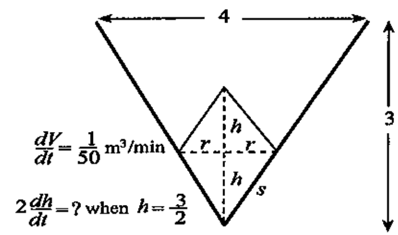
$$\frac{dr}{dt} = \frac{1}{5}\frac{dR}{dt} = \frac{1}{10}, \qquad \frac{dh}{dt} = 300\frac{dr}{dt} = 30.$$

In two years, $r = 10 + 2/10 = 51/5$, $h = 3000 + 60 = 3060$, $R = 50 + 2/2 = 51$, and

$$\begin{aligned} \frac{dV}{dt} &= \frac{\pi}{3}\left[51^2 + \left(\frac{51}{5}\right)(51) + \left(\frac{51}{5}\right)^2\right](30) \\ &\quad + \frac{\pi}{3}(3060)\left[2(51)\left(\frac{1}{2}\right) + \left(\frac{51}{5}\right)\left(\frac{1}{2}\right) + 51\left(\frac{1}{10}\right) + 2\left(\frac{51}{5}\right)\left(\frac{1}{10}\right)\right] = 3.0\times 10^5. \end{aligned}$$

The volume is increasing at 0.30 cubic metres per year.

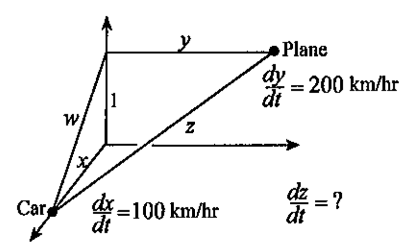
**26.** (a) When the height of each cone is $h$, the volume of sand in the container is $V = 2\pi r^2 h/3$. Similar triangles require $r/h = 2/3$, and therefore
$$V = \frac{2}{3}\pi\left(\frac{2h}{3}\right)^2 h = \frac{8}{27}\pi h^3.$$
Differentiation with respect to time gives
$$\frac{dV}{dt} = \frac{8}{9}\pi h^2 \frac{dh}{dt}.$$
When $h = 3/2$, this result yields
$$\frac{1}{50} = \frac{8}{9}\pi\left(\frac{3}{2}\right)^2 \frac{dh}{dt}.$$



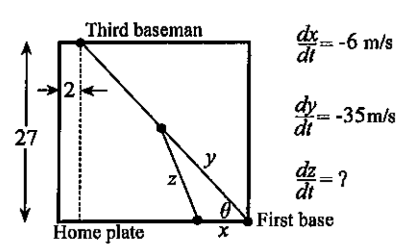

Thus, $dh/dt = 1/(100\pi)$, and the top of the pile is rising at $1/(50\pi)$ metres per minute.
(b) Since $s^2 = h^2 + r^2 = h^2 + (2h/3)^2 = 13h^2/9$, differentiation gives $2s(ds/dt) = (26/9)h(dh/dt)$. When $h = 3/2$, $s = \sqrt{13(3/2)^2/9} = \sqrt{13}/2$, and
$$2\left(\frac{\sqrt{13}}{2}\right)\frac{ds}{dt} = \frac{26}{9}\left(\frac{3}{2}\right)\left(\frac{1}{100\pi}\right) \quad\Longrightarrow\quad \frac{ds}{dt} = \frac{\sqrt{13}}{300\pi}.$$
The sand is rising along the side of the container at $\sqrt{13}/(300\pi)$ metres per minute.

**27.** If we denote the length of rod $AC$ by $q$, then $q^2 = y^2 + (k-x)^2$. Differentiation of this equation with respect to time gives
$$0 = 2y\frac{dy}{dt} - 2(k-x)\frac{dx}{dt} \qquad\Longrightarrow\qquad \frac{dy}{dt} = \frac{k-x}{y}\frac{dx}{dt}.$$
But from Consulting Project 6, $dx/dt = 4\pi x l \sin\theta/(l\cos\theta - x)$, and therefore
$$\frac{dy}{dt} = \frac{k-x}{y}\left(\frac{4\pi x l \sin\theta}{l\cos\theta - x}\right) \text{ m/s}.$$

**28.** If we differentiate $z^2 = y^2 + w^2 = y^2 + (1 + x^2)$, with respect to time $t$,
$$2z\frac{dz}{dt} = 2y\frac{dy}{dt} + 2x\frac{dx}{dt}.$$
At the instant in question, $x = 2 + 100/60 = 11/3$ km, $y = 200/60 = 10/3$ km, and $z^2 = 100/9 + 1 + 121/9 = 230/9$ km. Hence,
$$\frac{\sqrt{230}}{3}\frac{dz}{dt} = \frac{10}{3}(200) + \frac{11}{3}(100),$$
from which $dz/dt = 3100/\sqrt{230}$ km/hr.




**29.** By the cosine law, $z^2 = x^2 + y^2 - 2xy\cos\theta$. If we differentiate this equation with respect to time $t$,
$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} - 2\cos\theta\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right).$$
At the instant in question, $x = 27/4$, $y = \sqrt{25^2 + 27^2}/2 = \sqrt{1354}/2$, $\cos\theta = 25/\sqrt{25^2 + 27^2} = 25/\sqrt{1354}$, and



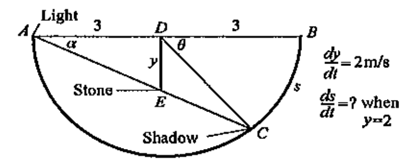

$$z^2 = \left(\frac{27}{4}\right)^2 + \frac{1354}{4} - 2\left(\frac{27}{4}\right)\left(\frac{\sqrt{1354}}{2}\right)\left(\frac{25}{\sqrt{1354}}\right) = \frac{3445}{16}.$$
Hence,
$$\frac{\sqrt{3445}}{4}\frac{dz}{dt} = \frac{27}{4}(-6) + \frac{\sqrt{1354}}{2}(-35) - \frac{25}{\sqrt{1354}}\left[\frac{27}{4}(-35) + \frac{\sqrt{1354}}{2}(-6)\right] \Longrightarrow \frac{dz}{dt} = -30.6 \text{ m/s}.$$

**30.** As a point on the chain travels around the front sprocket, the distance $S$ that it travels is $S = R\theta$, where $\theta$ is the angle through which it turns. Similarly, a point on the rear sprocket travels through a distance $s = r\phi$ where $\phi$ is the angle through which it turns. Since the chain is inextensible, it follows that $S = s \Longrightarrow R\theta = r\phi$. Since the stone rotates through the same angle $\phi$ as the point on the chain on the rear sprocket, the distance the stone travels as it rotates through angle $\phi$ is $\overline{S} = \overline{R}\phi$. Consequently, $\overline{S} = \overline{R}R\theta/r$, and differentiation of this equation with respect to time gives

$$\frac{d\overline{S}}{dt} = \frac{\overline{R}R}{r}\frac{d\theta}{dt} = \frac{\overline{R}R}{r}(2\pi).$$

The stone therefore travels at $2\pi\overline{R}R/r$ m/s when it leaves the tire.

**31.** We must find an equation relating $y$, distance the stone has fallen, and $s$, the distance the shadow of the stone has moved along the side of the pool. Since arc length along a circle is the product of the radius of the circle and the angle $\theta$ that it subtends at the centre of the circle, we know that $s = 3\theta$.




We now relate $y$ and $\theta$. Since triangle $ADC$ is isosceles, angles $DAC$ and $DCA$ are equal. Hence, $s = 3(2\alpha) = 6\alpha$. From triangle $ADE$, we may write $y = 3\tan\alpha$. Combine this with $s = 6\alpha$, and we finally relate $s$ and $y$:

$$y = 3\tan\left(\frac{s}{6}\right).$$

Differentiation with respect to $t$ gives

$$\frac{dy}{dt} = 3\sec^2\left(\frac{s}{6}\right)\left(\frac{1}{6}\frac{ds}{dt}\right).$$

When the stone is 1 m from the bottom of the tank, $y = 2$ m, and therefore $2 = 3\tan\left(\frac{s}{6}\right)$. It follows that at this instant

$$\sec^2\left(\frac{s}{6}\right) = 1 + \tan^2\left(\frac{s}{6}\right) = 1 + \left(\frac{2}{3}\right)^2 = \frac{13}{9}.$$

Substitution of this into the equation that relates $dy/dt$ and $ds/dt$, along with $dy/dt = 2$, yields

$$2 = 3\left(\frac{13}{9}\right)\left(\frac{1}{6}\frac{ds}{dt}\right).$$

Hence, $ds/dt = 36/13$, and the shadow is moving at $36/13$ m/s.

**32.** If we differentiate $A^2 = s(s-a)(s-b)(s-c)$ with respect to time,

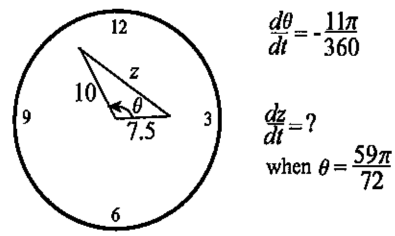
$$2A\frac{dA}{dt} = (s-a)(s-b)(s-c)\frac{ds}{dt} + s(s-b)(s-c)\left(\frac{ds}{dt} - \frac{da}{dt}\right)$$
$$+ s(s-a)(s-c)\left(\frac{ds}{dt} - \frac{db}{dt}\right) + s(s-a)(s-b)\left(\frac{ds}{dt} - \frac{dc}{dt}\right).$$

Furthermore, when we differentiate $s = (a+b+c)/2$, $\quad \dfrac{ds}{dt} = \dfrac{1}{2}\left(\dfrac{da}{dt} + \dfrac{db}{dt} + \dfrac{dc}{dt}\right)$.
Since $da/dt = db/dt = dc/dt = 1$, it follows that $ds/dt = 3/2$. When $a = 3$, $b = 4$, and $c = 5$, we obtain $s = 6$, and $A = \sqrt{6(6-3)(6-4)(6-5)} = 6$. For these values,

$$2(6)\frac{dA}{dt} = (3)(2)(1)\left(\frac{3}{2}\right) + 6(2)(1)\left(\frac{3}{2} - 1\right) + 6(3)(1)\left(\frac{3}{2} - 1\right) + 6(3)(2)\left(\frac{3}{2} - 1\right).$$

This implies that $dA/dt = 7/2$ square centimetres per minute.

**33.** (a) Since the minute hand sweeps out $2\pi$ radians each hour, or $\pi/30$ radians every minute, and the hour hand sweeps out $\pi/360$ radians each minute, the rate of change of the angle between them is $\pm(\pi/30 - \pi/360) = \pm 11\pi/360$ radians per minute, the plus or minus depending on the whether the hands are approaching each other or separating from each other.




(b) When the cosine law is applied to the triangle to the right,

$$z^2 = 100 + \frac{225}{4} - 150\cos\theta.$$

Differentiation with respect to time gives

$$2z\frac{dz}{dt} = 150\sin\theta\frac{d\theta}{dt}.$$

In this case, $d\theta/dt = -11\pi/360$. At 3:00, $\cos\theta = 0$, and therefore $z = \sqrt{100 + 225/4} = 25/2$. At this time then

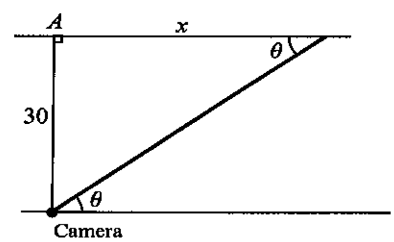
$$2\left(\frac{25}{2}\right)\frac{dz}{dt} = 150(1)\left(\frac{-11\pi}{360}\right) \implies \frac{dz}{dt} = -11\pi/60 \text{ cm/min}.$$

(c) At 8:05 the hands are separating at $d\theta/dt = 11\pi/360$. At this time, $\theta = 59\pi/72$, and $z = \sqrt{100 + 225/4 - 150\cos(59\pi/72)} = 16.81543$. When these are substituted into the equation for $dz/dt$ in part (b),

$$2(16.81543)\frac{dz}{dt} = 150\sin\left(\frac{59\pi}{72}\right)\left(\frac{11\pi}{360}\right) \implies \frac{dz}{dt} = 0.23 \text{ cm/min}.$$

**34.** (a) As the runner approaches $A$, $x = 30\cot\theta$. Differentiation with respect to time $t$ gives

$$\frac{dx}{dt} = -30\csc^2\theta\,\frac{d\theta}{dt}.$$

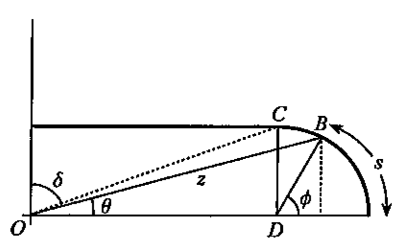



When the runner is at $A$, $\theta = \pi/2$, and

$$-4 = -30(1)\frac{d\theta}{dt}.$$

Hence, $d\theta/dt = 2/15$ radians per second.

(b) At point $C$ where the straightaway begins, $\delta = \mathrm{Tan}^{-1}(50/30) = 1.03 < \pi/3$; that is, $B$ is on the curved portion of the track. Since length $s$ along the curved portion is given by $s = 30\phi$,




$$\frac{ds}{dt} = 30\frac{d\phi}{dt}.$$

Thus, $d\phi/dt = 2/15$ rad/s. By the cosine law,

$$\begin{aligned} z^2 &= \|OD\|^2 + \|BD\|^2 - 2\|OD\|\|BD\|\cos(\pi - \phi) \\ &= \|OD\|^2 + \|BD\|^2 + 2\|OD\|\|BD\|\cos\phi, \end{aligned}$$

and therefore

$$2z\frac{dz}{dt} = -2(50)(30)\sin\phi\,\frac{d\phi}{dt}.$$

This gives $\dfrac{dz}{dt} = \dfrac{-1500\sin\phi}{z}\left(\dfrac{2}{15}\right) = \dfrac{-200\sin\phi}{z}$. By the sine law, $\dfrac{\sin\theta}{\|BD\|} = \dfrac{\sin(\pi-\phi)}{z} = \dfrac{\sin\phi}{z}$, or, $z\sin\theta = 30\sin\phi$. Differentiation with respect to $t$ gives $\sin\theta\,\dfrac{dz}{dt} + z\cos\theta\,\dfrac{d\theta}{dt} = 30\cos\phi\,\dfrac{d\phi}{dt}$, and therefore at $B$,

$$\frac{d\theta}{dt} = \frac{1}{z\cos\theta}\left[30\cos\phi\left(\frac{2}{15}\right) + \frac{200\sin\phi\sin\theta}{z}\right] = \frac{2}{\sqrt{3}z}\left(4\cos\phi + \frac{100\sin\phi}{z}\right) = \frac{8}{\sqrt{3}z^2}(z\cos\phi + 25\sin\phi).$$

When $\theta = \pi/6$, we have $z = 60\sin\phi$ and $z^2 = 2500 + 900 + 3000\cos\phi$. These give $3600\sin^2\phi = 3400 + 3000\sqrt{1-\sin^2\phi}$, from which $324\sin^4\phi - 387\sin^2\phi + 64 = 0$. The solution of this for $\phi$ is $\phi = 1.5087$, and this gives $z = 60\sin(1.5087) = 59.884$. Thus,

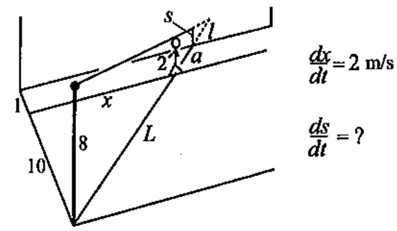
$$\frac{d\theta}{dt} = \frac{8}{\sqrt{3}(59.884)^2}[59.884\cos 1.5087 + 25\sin 1.5087] = 0.0369 \text{ rad/s}.$$

**35.** From similar vertical triangles,

$$\frac{s}{l} = \frac{8}{L+l+a} \quad \text{and} \quad \frac{2}{l+a} = \frac{s}{l}.$$

The second implies that

$$2l = s(l+a) \Longrightarrow l = \frac{as}{2-s}.$$




Substitution into the first gives

$$8\left(\frac{as}{2-s}\right) = s\left(L + \frac{as}{2-s} + a\right) \Longrightarrow 8as = s(L+a)(2-s) + as^2 \Longrightarrow a = \frac{sL(2-s)}{8s - s^2 - s(2-s)} = \frac{L(2-s)}{6}.$$

From horizontal right-angled triangles, we have $(L+a)^2 = (x + \sqrt{a^2-1})^2 + 121$ and $L^2 = x^2 + 100$. If we replace $a$ in the first of these

$$\left[L + \frac{L(2-s)}{6}\right]^2 = \left[x + \sqrt{\frac{L^2(2-s)^2}{36} - 1}\right]^2 + 121 \Longrightarrow \frac{L^2}{36}(8-s)^2 = \left[x + \frac{\sqrt{L^2(2-s)^2 - 36}}{6}\right]^2 + 121.$$

If we now replace $L^2$ with $x^2 + 100$, and multiply by 36,

$$\begin{aligned}(x^2+100)(8-s)^2 &= [6x + \sqrt{(x^2+100)(2-s)^2 - 36}]^2 + 121(36)\\ &= 36x^2 + 12x\sqrt{(x^2+100)(2-s)^2-36} + (x^2+100)(2-s)^2 - 36 + 121(36).\end{aligned}$$

Consequently,

$$\begin{aligned}36x^2 + 12x\sqrt{(x^2+100)(2-s)^2-36} + 120(36) &= (x^2+100)(64 - 16s + s^2 - 4 + 4s - s^2)\\ &= (x^2+100)(60-12s).\end{aligned}$$

Division by 12 gives

$$3x^2 + x\sqrt{(x^2+100)(2-s)^2-36} + 360 = (x^2+100)(5-s) = 5x^2 + 500 - x^2s - 100s,$$

from which

$$x\sqrt{(x^2+100)(2-s)^2-36} = 2x^2 + 140 - x^2s - 100s = x^2(2-s) + 20(7-5s).$$

If we square both sides,

$$x^2(x^2+100)(2-s)^2 - 36x^2 = x^4(2-s)^2 + 40x^2(2-s)(7-5s) + 400(7-5s)^2,$$

from which

$$\begin{aligned}0 &= x^2(560 - 680s + 200s^2 - 400 + 400s - 100s^2 + 36) + 400(7-5s)^2\\ &= x^2(196 - 280s + 100s^2) + 400(7-5s)^2 = 4(x^2+100)(7-5s)^2.\end{aligned}$$

This implies that $s = 7/5$ for all $x$. In other words, the height of the shadow on the wall never changes, and therefore $ds/dt = 0$.

**EXERCISES 4.10**

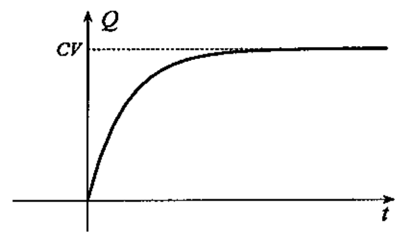
**1.** (a) When we substitute the function into the left side of the equation,

$$R\frac{dQ}{dt}+\frac{Q}{C}=R\left[D\,e^{-t/(RC)}\left(\frac{-1}{RC}\right)\right]+\frac{1}{C}\left[D\,e^{-t/(RC)}+CV\right]=V.$$

(b) With $Q(0)=Q_0$, we find that
$Q_0=D+CV \implies D=Q_0-CV$,
and therefore

$$\begin{aligned}Q(t)&=(Q_0-CV)e^{-t/(RC)}+CV\\&=CV[1-e^{-t/(RC)}]+Q_0\,e^{-t/(RC)}.\end{aligned}$$

(c) The graph when $Q_0=0$ is shown to the right.




**2.** If we substitute for $i$ into the left side of the equation,

$$\begin{aligned}R\frac{di}{dt}+\frac{i}{C}&=R\left[Ae^{-t/(RC)}\left(\frac{-1}{RC}\right)+\frac{\omega V_0}{Z}\cos(\omega t-\phi)\right]+\frac{A}{C}e^{-t/(RC)}+\frac{V_0}{CZ}\sin(\omega t-\phi)\\&=\frac{\omega RV_0}{Z}(\cos\omega t\,\cos\phi+\sin\omega t\,\sin\phi)+\frac{V_0}{CZ}(\sin\omega t\,\cos\phi-\cos\omega t\,\sin\phi).\end{aligned}$$

Because $\tan\phi=-1/(\omega CR)$, it follows that

$$\begin{aligned}\sin\phi&=\frac{-1}{\sqrt{1+\omega^2C^2R^2}}=\frac{-1}{\omega C\sqrt{R^2+1/(\omega^2C^2)}}=\frac{-1}{\omega CZ},\\\cos\phi&=\frac{\omega CR}{\sqrt{1+\omega^2C^2R^2}}=\frac{\omega CR}{\omega CZ}=\frac{R}{Z},\end{aligned}$$

provided we choose $\phi$ in the fourth quadrant. With these,

$$\begin{aligned}R\frac{di}{dt}+\frac{i}{C}&=\frac{\omega RV_0}{Z}\left[\frac{R}{Z}\cos\omega t-\frac{1}{\omega CZ}\sin\omega t\right]+\frac{V_0}{CZ}\left[\frac{R}{Z}\sin\omega t+\frac{1}{\omega CZ}\cos\omega t\right]\\&=\frac{V_0}{Z^2}\left(\omega R^2+\frac{1}{\omega C^2}\right)\cos\omega t=\frac{V_0\omega}{Z^2}\left(R^2+\frac{1}{\omega^2C^2}\right)\cos\omega t=V_0\omega\cos\omega t=\frac{dV}{dt}.\end{aligned}$$

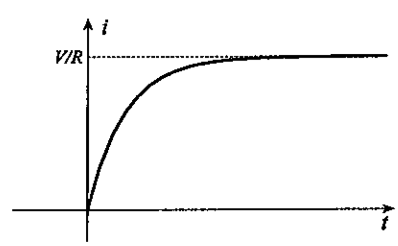
**3.** (a) When we substitute the function into the left side of the equation,

$$L\frac{di}{dt}+Ri=L\left[D\,e^{-Rt/L}\left(\frac{-R}{L}\right)\right]+R\left(D\,e^{-Rt/L}+\frac{V}{R}\right)=V.$$

(b) With $i(0)=0$, we find that
$0=D+V/R \implies D=-V/R$, and therefore

$$i(t)=\left(\frac{-V}{R}\right)e^{-Rt/L}+\frac{V}{R}=\frac{V}{R}(1-e^{-Rt/L}).$$

The graph is shown to the right.




**4.** If we substitute for $i$ into the left side of the equation,

$$\begin{aligned}L\frac{di}{dt}+Ri&=L\left[Ae^{-Rt/L}(-R/L)+\frac{\omega V_0}{Z}\cos(\omega t-\phi)\right]+RAe^{-Rt/L}+\frac{RV_0}{Z}\sin(\omega t-\phi)\\&=\frac{\omega LV_0}{Z}(\cos\omega t\,\cos\phi+\sin\omega t\,\sin\phi)+\frac{RV_0}{Z}(\sin\omega t\,\cos\phi-\cos\omega t\,\sin\phi).\end{aligned}$$

Because $\tan\phi = \omega L/R$, it follows that

$$\sin\phi = \frac{\omega L}{\sqrt{R^2+\omega^2L^2}} = \frac{\omega L}{Z}, \qquad \cos\phi = \frac{R}{\sqrt{R^2+\omega^2L^2}} = \frac{R}{Z},$$

provided we choose $\phi$ in the first quadrant. With these,

$$\begin{aligned} L\frac{di}{dt} + Ri &= \frac{\omega L V_0}{Z}\left(\frac{R}{Z}\cos\omega t + \frac{\omega L}{Z}\sin\omega t\right) + \frac{RV_0}{Z}\left(\frac{R}{Z}\sin\omega t - \frac{\omega L}{Z}\cos\omega t\right) \\ &= \frac{V_0}{Z^2}(\omega^2L^2+R^2)\sin\omega t = V_0\sin\omega t. \end{aligned}$$

**5.** (a) When we substitute the function into the left side of the equation,

$$\begin{aligned} L\frac{d^2Q}{dt^2} + \frac{Q}{C} &= L\left(-\frac{D}{LC}\cos\frac{t}{\sqrt{LC}} - \frac{E}{LC}\sin\frac{t}{\sqrt{LC}} + \frac{A\omega}{\omega L - \frac{1}{\omega C}}\sin\omega t\right) \\ &\quad + \frac{1}{C}\left(D\cos\frac{t}{\sqrt{LC}} + E\sin\frac{t}{\sqrt{LC}} - \frac{A/\omega}{\omega L - \frac{1}{\omega C}}\sin\omega t\right) \\ &= \left(\frac{\omega L - \frac{1}{\omega C}}{\omega L - \frac{1}{\omega C}}\right) A\sin\omega t = A\sin\omega t = V. \end{aligned}$$

(b) Using the facts that $Q(0) = Q_0$ and $i(0) = 0$, we obtain

$$Q_0 = D, \qquad 0 = \frac{E}{\sqrt{LC}} - \frac{A}{\omega L - \frac{1}{\omega C}}.$$

Thus, $E = \dfrac{A\sqrt{LC}}{\omega L - \frac{1}{\omega C}}$, and this gives the required function.

**6.** Since $\dfrac{di}{dt} = -\dfrac{A}{\sqrt{LC}}\sin\left(\dfrac{t}{\sqrt{LC}}\right) + \dfrac{B}{\sqrt{LC}}\cos\left(\dfrac{t}{\sqrt{LC}}\right)$, it follows that

$$L\frac{d^2i}{dt^2} + \frac{i}{C} = L\left[-\frac{A}{LC}\cos\left(\frac{t}{\sqrt{LC}}\right) - \frac{B}{LC}\sin\left(\frac{t}{\sqrt{LC}}\right)\right] + \frac{1}{C}\left[A\cos\left(\frac{t}{\sqrt{LC}}\right) + B\sin\left(\frac{t}{\sqrt{LC}}\right)\right] = 0.$$

**7.** (a) When we substitute the function into the left side of the equation,

$$\begin{aligned} R\frac{dQ}{dt} + \frac{Q}{C} &= R\left\{\frac{CA\omega}{1+\omega^2R^2C^2}\cos\omega t + \frac{\omega RC^2A}{1+\omega^2R^2C^2}\left[e^{-t/(RC)}\left(\frac{-1}{RC}\right) + \omega\sin\omega t\right]\right\} \\ &\quad + \frac{1}{C}\left\{\frac{CA}{1+\omega^2R^2C^2}\sin\omega t + \frac{\omega RC^2A}{1+\omega^2R^2C^2}[e^{-t/(RC)} - \cos\omega t]\right\} \\ &= \left(\frac{\omega^2R^2C^2A + A}{1+\omega^2R^2C^2}\right)\sin\omega t = A\sin\omega t. \end{aligned}$$

(b) We express $Q(t)$ in the form

$$Q(t) = \frac{\omega RC^2A}{1+\omega^2R^2C^2}e^{-t/(RC)} + \frac{CA}{1+\omega^2R^2C^2}(\sin\omega t - \omega RC\cos\omega t),$$

and set

$$\sin\omega t - \omega RC\cos\omega t = B\cos(\omega t - \phi) = B(\cos\omega t\cos\phi + \sin\omega t\sin\phi).$$

This equation is satisfied if $B$ and $\phi$ are chosen to satisfy

$$1 = B\sin\phi, \qquad -\omega RC = B\cos\phi.$$

These imply that $B^2 = 1 + \omega^2R^2C^2$ and $\tan\phi = -1/(\omega CR)$, and therefore

$$
\begin{aligned}
Q(t) &= \frac{\omega RC^2 A}{1+\omega^2R^2C^2}e^{-t/(RC)} + \frac{CA}{1+\omega^2R^2C^2}\sqrt{1+\omega^2R^2C^2}\cos(\omega t-\phi)\\
&= \frac{\omega RC^2 A}{1+\omega^2R^2C^2}e^{-t/(RC)} + \frac{A/\omega}{\sqrt{R^2+1/(\omega^2C^2)}}\cos(\omega t-\phi).
\end{aligned}
$$

**8.** (a) When we substitute the function into the left side of the equation,

$$
\begin{aligned}
L\frac{di}{dt}+Ri &= L\left\{\frac{\omega LA}{R^2+\omega^2L^2}\left[e^{-Rt/L}\left(\frac{-R}{L}\right)+\omega\sin\omega t\right]+\frac{RA\omega}{R^2+\omega^2L^2}\cos\omega t\right\}\\
&\quad + R\left[\frac{\omega LA}{R^2+\omega^2L^2}(e^{-Rt/L}-\cos\omega t)+\frac{RA}{R^2+\omega^2L^2}\sin\omega t\right]\\
&= \left(\frac{\omega^2L^2A+R^2A}{R^2+\omega^2L^2}\right)\sin\omega t = A\sin\omega t = V.
\end{aligned}
$$

(b) We express $i(t)$ in the form

$$
i(t) = \frac{\omega LA}{R^2+\omega^2L^2}e^{-Rt/L}+\frac{A}{R^2+\omega^2L^2}(R\sin\omega t-\omega L\cos\omega t),
$$

and set

$$
R\sin\omega t-\omega L\cos\omega t = B\sin(\omega t-\phi) = B(\sin\omega t\cos\phi-\cos\omega t\sin\phi).
$$

This equation is satisfied if $B$ and $\phi$ are chosen to satisfy

$$
R = B\cos\phi, \qquad \omega L = B\sin\phi.
$$

These imply that $B^2=R^2+\omega^2L^2$ and $\tan\phi=\omega L/R$, and therefore

$$
\begin{aligned}
i(t) &= \frac{\omega LA}{R^2+\omega^2L^2}e^{-Rt/L}+\frac{A}{R^2+\omega^2L^2}\sqrt{R^2+\omega^2L^2}\sin(\omega t-\phi)\\
&= \frac{\omega LA}{R^2+\omega^2L^2}e^{-Rt/L}+\frac{A}{\sqrt{R^2+\omega^2L^2}}\sin(\omega t-\phi)
\end{aligned}
$$

**9.** If we substitute for $i$ into the left side of the equation,

$$
\begin{aligned}
L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{i}{C} &= \frac{1}{C}\left\{e^{-Rt/(2L)}[D\cos\nu t+E\sin\nu t]+\frac{A}{Z}\sin(\omega t-\phi)\right\}\\
&\quad + R\left\{-\frac{R}{2L}e^{-Rt/(2L)}[D\cos\nu t+E\sin\nu t]+e^{-Rt/(2L)}[-D\nu\sin\nu t+E\nu\cos\nu t]+\frac{\omega A}{Z}\cos(\omega t-\phi)\right\}\\
&\quad + L\left\{\frac{R^2}{4L^2}e^{-Rt/(2L)}[D\cos\nu t+E\sin\nu t]-\frac{R}{L}e^{-Rt/(2L)}[-D\nu\sin\nu t+E\nu\cos\nu t]\right.\\
&\quad \left. + e^{-Rt/(2L)}[-D\nu^2\cos\nu t-E\nu^2\sin\nu t]-\frac{\omega^2A}{Z}\sin(\omega t-\phi)\right\}\\
&= e^{-Rt/(2L)}\cos\nu t\left[\frac{D}{C}-\frac{DR^2}{2L}+E\nu R+\frac{DR^2}{4L}-E\nu R-DL\nu^2\right]\\
&\quad + e^{-Rt/(2L)}\sin\nu t\left[\frac{E}{C}-\frac{ER^2}{2L}-D\nu R+\frac{ER^2}{4L}+D\nu R-EL\nu^2\right]\\
&\quad + A\left(\frac{1}{CZ}-\frac{\omega^2L}{Z}\right)\sin(\omega t-\phi)+\frac{\omega AR}{Z}\cos(\omega t-\phi)\\
&= D\,e^{-Rt/(2L)}\cos\nu t\left[\frac{1}{C}-\frac{R^2}{4L}-L\left(\frac{1}{LC}-\frac{R^2}{4L^2}\right)\right]\\
&\quad + E\,e^{-Rt/(2L)}\sin\nu t\left[\frac{1}{C}-\frac{R^2}{4L}-L\left(\frac{1}{LC}-\frac{R^2}{4L^2}\right)\right]\\
&\quad + \frac{\omega A}{Z}\left(\frac{1}{\omega C}-\omega L\right)[\sin\omega t\cos\phi-\cos\omega t\sin\phi]+\frac{\omega AR}{Z}[\cos\omega t\cos\phi+\sin\omega t\sin\phi].
\end{aligned}
$$

Because $\tan\phi = \dfrac{\omega L - 1/(\omega C)}{R}$, it follows that

$$\sin\phi = \frac{\omega L - 1/(\omega C)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{\omega L - 1/(\omega C)}{Z}, \qquad \cos\phi = \frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{R}{Z}.$$

With these,

$$\begin{aligned} L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} &= \frac{\omega A}{Z}\left[\frac{1}{\omega C} - \omega L\right]\left[\frac{R}{Z}\sin\omega t - \frac{\omega L - 1/(\omega C)}{Z}\cos\omega t\right] \\ &\quad + \frac{\omega A R}{Z}\left[\frac{R}{Z}\cos\omega t + \frac{\omega L - 1/(\omega C)}{Z}\sin\omega t\right] \\ &= \frac{\omega A}{Z^2}\left[\left(\frac{1}{\omega C} - \omega L\right)^2 + R^2\right]\cos\omega t = \omega A\cos\omega t = \frac{dV}{dt}. \end{aligned}$$

**10.** (a) If we set

$$R\,\cos\omega t + \left(\omega L - \frac{1}{\omega C}\right)\sin\omega t = A\cos(\omega t - \phi) = A(\cos\omega t\,\cos\phi + \sin\omega t\,\sin\phi),$$

then, $R = A\,\cos\phi$ and $\omega L - \dfrac{1}{\omega C} = A\,\sin\phi$. These imply that $A = \sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}$, and therefore

$$\begin{aligned} i(t) &= \frac{V_0}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\cos(\omega t - \phi) \\ &= \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\cos(\omega t + \phi). \end{aligned}$$

To maximize the amplitude, we minimize the denominator. This occurs when $\omega L - \dfrac{1}{\omega C} = 0$, and this implies that $\omega = 1/\sqrt{LC}$.

(b) Critical points of $i(t)$ are given by

$$0 = \frac{V_0}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\left[-R\omega\,\sin\omega t + \omega\left(\omega L - \frac{1}{\omega C}\right)\cos\omega t\right].$$

If $\bar{t}$ denotes a solution of this equation, then $\tan\omega\bar{t} = \dfrac{\omega L - \frac{1}{\omega C}}{R}$. This implies that

$$\sin\omega\bar{t} = \pm\frac{\omega L - 1/(\omega C)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}, \qquad \cos\omega\bar{t} = \pm\frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}.$$

Consequently,

$$i(\bar{t}) = \frac{V_0}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\left[\frac{\pm R^2}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \pm \frac{\left(\omega L - 1/(\omega C)\right)^2}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\right] = \frac{\pm V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}.$$

The value of $\omega$ that makes this a maximum is $1/\sqrt{LC}$.

## EXERCISES 4.11

Although many of the limits in these exercises can be done without L'Hôpital's rule, we shall demonstrate use of this rule whenever it is applicable.

1. $\lim_{x\to 0} \dfrac{x^2+3x}{x^3+5x^2} = \lim_{x\to 0} \dfrac{2x+3}{3x^2+10x} = \pm\infty$, depending on whether $x \to 0^+$ or $x \to 0^-$

2. $\lim_{x\to 3} \dfrac{x^2-9}{x-3} = \lim_{x\to 3} \dfrac{2x}{1} = 6$

3. $\lim_{x\to -\infty} \dfrac{x^3+3x-2}{x^2+5x+1} = \lim_{x\to -\infty} \dfrac{3x^2+3}{2x+5} = \lim_{x\to -\infty} \dfrac{6x}{2} = -\infty$

4. $\lim_{x\to\infty} \dfrac{2x^2+3x}{5x^3+4} = \lim_{x\to\infty} \dfrac{4x+3}{15x^2} = \lim_{x\to\infty} \dfrac{4}{30x} = 0$ **5.** $\lim_{x\to 5} \dfrac{x^2-10x+25}{x^3-125} = \lim_{x\to 5} \dfrac{2x-10}{3x^2} = 0$

6. L'Hôpital's rule is not appliable. $\lim_{x\to 1} \dfrac{1}{(x-1)^2} = \infty$

7. L'Hôpital's rule does not work on this limit. Instead we divide numerator and denominator by $x$,

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{2x+5} = \lim_{x\to\infty} \frac{\sqrt{1+\frac{1}{x^2}}}{2+\frac{5}{x}} = \frac{1}{2}.$$

8. L'Hôpital's rule is not applicable. $\lim_{x\to -\infty} \dfrac{\sin x}{x} = 0$

9. $\lim_{x\to\infty} \dfrac{\sin(2/x)}{\sin(1/x)} = \lim_{x\to\infty} \dfrac{(-2/x^2)\cos(2/x)}{(-1/x^2)\cos(1/x)} = \lim_{x\to\infty} \dfrac{2\cos(2/x)}{\cos(1/x)} = 2$

10. $\lim_{x\to\pi/2} \dfrac{\cos x}{(x-\pi/2)^2} = \lim_{x\to\pi/2} \dfrac{-\sin x}{2(x-\pi/2)} = \pm\infty$ depending on whether $x$ approaches $\pi/2$ from left or right

11. $\lim_{x\to 1^+} \dfrac{(1-1/x)^3}{\sqrt{x-1}} = \lim_{x\to 1^+} \dfrac{3(1-1/x)^2(1/x^2)}{\dfrac{1}{2\sqrt{x-1}}} = \lim_{x\to 1^+} \dfrac{6\sqrt{x-1}}{x^2}\left(1-\dfrac{1}{x}\right)^2 = 0$

12. $\lim_{x\to\infty} \dfrac{\sin(1/x)}{1/x^2} = \lim_{x\to\infty} \dfrac{-(1/x^2)\cos(1/x)}{-2/x^3} = \lim_{x\to\infty} \dfrac{x}{2}\cos\left(\dfrac{1}{x}\right) = \infty$

13. $\lim_{x\to 9^-} \dfrac{\sqrt{x}-3}{\sqrt{9-x}} = \lim_{x\to 9^-} \dfrac{1/(2\sqrt{x})}{-1/(2\sqrt{9-x})} = \lim_{x\to 9^-} \dfrac{\sqrt{9-x}}{-\sqrt{x}} = 0$

14. $\lim_{x\to 0} \dfrac{\sqrt{5+x}-\sqrt{5-x}}{x} = \lim_{x\to 0} \dfrac{\dfrac{1}{2\sqrt{5+x}} + \dfrac{1}{2\sqrt{5-x}}}{1} = \dfrac{1}{\sqrt{5}}$

15. $\lim_{x\to 0} \dfrac{x-\sin x}{x^3} = \lim_{x\to 0} \dfrac{1-\cos x}{3x^2} = \lim_{x\to 0} \dfrac{\sin x}{6x} = \lim_{x\to 0} \dfrac{\cos x}{6} = \dfrac{1}{6}$

16. $\lim_{x\to a} \dfrac{x^n-a^n}{x-a} = \lim_{x\to a} \dfrac{nx^{n-1}}{1} = na^{n-1}$

17. $\lim_{x\to 0} \dfrac{(1-\cos x)^2}{3x^2} = \lim_{x\to 0} \dfrac{2\sin x(1-\cos x)}{6x} = \lim_{x\to 0} \dfrac{2\cos x - 2\cos^2 x + 2\sin^2 x}{6} = 0$

18. $\lim_{x\to 0} \dfrac{\tan x}{x} = \lim_{x\to 0} \dfrac{\sec^2 x}{1} = 1$

19. $\lim_{x\to 0} \dfrac{\sin 3x}{\tan 2x} = \lim_{x\to 0} \dfrac{3\cos 3x}{2\sec^2 2x} = \dfrac{3}{2}$

20. $\lim_{x\to 1} \dfrac{\left(1-\sqrt{2-x}\right)^{3/2}}{x-1} = \lim_{x\to 1} \dfrac{\dfrac{3}{2}\left(1-\sqrt{2-x}\right)^{1/2}\left(\dfrac{1}{2\sqrt{2-x}}\right)}{1} = 0$

**21.** $\lim_{x\to 0} \dfrac{\sqrt{x+1}-\sqrt{2x+1}}{\sqrt{3x+4}-\sqrt{2x+4}} = \lim_{x\to 0} \dfrac{\dfrac{1}{2\sqrt{x+1}} - \dfrac{1}{\sqrt{2x+1}}}{\dfrac{3}{2\sqrt{3x+4}} - \dfrac{1}{\sqrt{2x+4}}} = -2$

**22.** $\lim_{x\to 0} \dfrac{(1-\cos x)^2}{3x^4} = \lim_{x\to 0} \dfrac{2(1-\cos x)\sin x}{12x^3} = \lim_{x\to 0} \dfrac{2\sin x - \sin 2x}{12x^3} = \lim_{x\to 0} \dfrac{2\cos x - 2\cos 2x}{36x^2}$

$$= \lim_{x\to 0} \frac{-2\sin x + 4\sin 2x}{72x} = \lim_{x\to 0} \frac{-2\cos x + 8\cos 2x}{72} = \frac{1}{12}$$

**23.** $\lim_{x\to\infty} x \sin\left(\dfrac{1}{x}\right) = \lim_{x\to\infty} \dfrac{\sin(1/x)}{1/x} = \lim_{x\to\infty} \dfrac{(-1/x^2)\cos(1/x)}{-1/x^2} = 1$

**24.** It is easier to factor the numerator in this limit than to apply L'Hôpital's rule,

$$\lim_{x\to 2} \frac{(x-2)^{10}}{(\sqrt{x}-\sqrt{2})^{10}} = \lim_{x\to 2} \frac{(\sqrt{x}+\sqrt{2})^{10}(\sqrt{x}-\sqrt{2})^{10}}{(\sqrt{x}-\sqrt{2})^{10}} = \lim_{x\to 2} (\sqrt{x}+\sqrt{2})^{10} = 2^{15}.$$

**25.** $\lim_{x\to 0} \left[\dfrac{4}{x^2} - \dfrac{2}{1-\cos x}\right] = \lim_{x\to 0} \left[\dfrac{4-4\cos x - 2x^2}{x^2(1-\cos x)}\right] = \lim_{x\to 0} \dfrac{4\sin x - 4x}{2x(1-\cos x) + x^2 \sin x}$

$$= \lim_{x\to 0} \frac{4\cos x - 4}{2(1-\cos x) + 4x\sin x + x^2\cos x} = \lim_{x\to 0} \frac{-4\sin x}{6\sin x + 6x\cos x - x^2\sin x}$$

$$= \lim_{x\to 0} \frac{-4\cos x}{12\cos x - 8x\sin x - x^2\cos x} = -\frac{1}{3}$$

**26.** L'Hôpital's rule is not applicable. $\lim_{x\to\infty} xe^x = \infty$

**27.** $\lim_{x\to\infty} x^2 e^{-4x} = \lim_{x\to\infty} \dfrac{x^2}{e^{4x}} = \lim_{x\to\infty} \dfrac{2x}{4e^{4x}} = \lim_{x\to\infty} \dfrac{2}{16e^{4x}} = 0$

**28.** $\lim_{x\to-\infty} x\sin\left(\dfrac{4}{x}\right) = \lim_{x\to-\infty} \dfrac{\sin(4/x)}{1/x} = \lim_{x\to-\infty} \dfrac{(-4/x^2)\cos(4/x)}{-1/x^2} = \lim_{x\to-\infty} 4\cos\left(\dfrac{4}{x}\right) = 4$

**29.** $\lim_{x\to 0} x\cot x = \lim_{x\to 0} \dfrac{x}{\tan x} = \lim_{x\to 0} \dfrac{1}{\sec^2 x} = 1$ **30.** $\lim_{x\to 0} \csc x(1-\cos x) = \lim_{x\to 0} \dfrac{1-\cos x}{\sin x} = \lim_{x\to 0} \dfrac{\sin x}{\cos x} = 0$

**31.** If we set $L = \lim_{x\to 0^+} (\sin x)^x$, and take natural logarithms,

$$\ln L = \ln\left[\lim_{x\to 0^+} (\sin x)^x\right] = \lim_{x\to 0^+} [\ln(\sin x)^x] = \lim_{x\to 0^+} [x\ln(\sin x)] = \lim_{x\to 0^+} \frac{\ln(\sin x)}{1/x}$$

$$= \lim_{x\to 0^+} \frac{\cos x/\sin x}{-1/x^2} = -\lim_{x\to 0^+} \frac{x^2\cos x}{\sin x} = -\lim_{x\to 0^+} \frac{2x\cos x - x^2\sin x}{\cos x} = 0.$$

Thus, $L = \lim_{x\to 0^+} (\sin x)^x = e^0 = 1$.

**32.** If we set $L = \lim_{x\to 0^+} x^{\sin x}$, and take natural logarithms,

$$\ln L = \ln\left[\lim_{x\to 0^+} x^{\sin x}\right] = \lim_{x\to 0^+} \left[\ln(x^{\sin x})\right] = \lim_{x\to 0^+} [\sin x \ln x] = \lim_{x\to 0^+} \frac{\ln x}{\csc x}$$

$$= \lim_{x\to 0^+} \frac{1/x}{-\csc x\cot x} = \lim_{x\to 0^+} \frac{-\sin x\tan x}{x} = \lim_{x\to 0^+} \frac{-(\sin x\sec^2 x + \cos x\tan x)}{1} = 0.$$

Thus, $L = \lim_{x\to 0^+} x^{\sin x} = e^0 = 1$.

**33.** If we set $L = \lim_{x\to\infty}\left(\frac{x+5}{x+3}\right)^x$, and take natural logarithms,

$$\ln L = \ln\left[\lim_{x\to\infty}\left(\frac{x+5}{x+3}\right)^x\right] = \lim_{x\to\infty}\left[\ln\left(\frac{x+5}{x+3}\right)^x\right] = \lim_{x\to\infty}\left[x\ln\left(\frac{x+5}{x+3}\right)\right] = \lim_{x\to\infty}\frac{\ln\left(\frac{x+5}{x+3}\right)}{1/x}$$

$$= \lim_{x\to\infty}\frac{\left(\frac{x+3}{x+5}\right)\left[\frac{(x+3)-(x+5)}{(x+3)^2}\right]}{-1/x^2} = \lim_{x\to\infty}\frac{2x^2}{(x+3)(x+5)} = \lim_{x\to\infty}\frac{4x}{2x+8} = \lim_{x\to\infty}\frac{4}{2} = 2.$$

Thus, $L = \lim_{x\to\infty}\left(\frac{x+5}{x+3}\right)^x = e^2$.

**34.** If we set $L = \lim_{x\to 0}(1+x)^{\cot x}$, and take natural logarithms,

$$\ln L = \ln\left[\lim_{x\to 0}(1+x)^{\cot x}\right] = \lim_{x\to 0}[\cot x\,\ln(1+x)] = \lim_{x\to 0}\frac{\ln(1+x)}{\tan x} = \lim_{x\to 0}\frac{\frac{1}{1+x}}{\sec^2 x} = 1.$$

Thus, $L = \lim_{x\to 0}(1+x)^{\cot x} = e$.

**35.** If we set $L = \lim_{x\to\infty} x^{1/x}$, and take natural logarithms,

$$\ln L = \ln\left[\lim_{x\to\infty} x^{1/x}\right] = \lim_{x\to\infty}\left[\ln(x^{1/x})\right] = \lim_{x\to\infty}\left(\frac{1}{x}\ln x\right) = \lim_{x\to\infty}\frac{1/x}{1} = 0.$$

Thus, $L = \lim_{x\to\infty} x^{1/x} = e^0 = 1$.

**36.** If we set $L = \lim_{x\to 0^+}|\ln x|^{\sin x}$, and take natural logarithms,

$$\ln L = \ln\left[\lim_{x\to 0^+}|\ln x|^{\sin x}\right] = \lim_{x\to 0^+}\ln\left[|\ln x|^{\sin x}\right] = \lim_{x\to 0^+}(\sin x\,\ln|\ln x|) = \lim_{x\to 0^+}\frac{\ln|\ln x|}{\csc x}$$

$$= \lim_{x\to 0^+}\frac{\frac{1}{|\ln x|}\frac{1}{x}\frac{|\ln x|}{\ln x}}{-\csc x\,\cot x} = \lim_{x\to 0^+}\frac{-\sin x\,\tan x}{x\ln x}.$$

To verify that we can use L'Hôpital's rule again, we note that

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x} = \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0.$$

Thus, $\ln L = \lim_{x\to 0^+}\frac{-(\sin x\,\sec^2 x + \cos x\,\tan x)}{x/x + \ln x} = 0$, and $L = \lim_{x\to 0^+}|\ln x|^{\sin x} = e^0 = 1$.

**37.** $\lim_{x\to 0^+} x\,e^{1/x} = \lim_{x\to 0^+}\frac{e^{1/x}}{1/x} = \lim_{x\to 0^+}\frac{(-1/x^2)e^{1/x}}{-1/x^2} = \infty$

**38.** This limit does not exist since $\lim_{x\to 0^+}(\tan x - \csc x) = -\infty$ and $\lim_{x\to 0^-}(\tan x - \csc x) = \infty$.

**39.** $\lim_{x\to 0}(\csc x - \cot x) = \lim_{x\to 0}\left(\frac{1-\cos x}{\sin x}\right) = \lim_{x\to 0}\frac{\sin x}{\cos x} = 0$

**40.** $\lim_{x\to 1}\left(\frac{x}{\ln x} - \frac{1}{x\ln x}\right) = \lim_{x\to 1}\frac{x^2-1}{x\ln x} = \lim_{x\to 1}\frac{2x}{\ln x + 1} = 2$

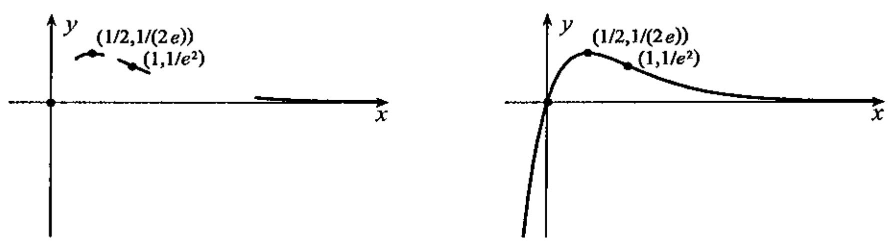
**41.** $\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to 1}\frac{x\ln x - x + 1}{(x-1)\ln x} = \lim_{x\to 1}\frac{\ln x + 1 - 1}{\ln x + (x-1)/x} = \lim_{x\to 1}\frac{1/x}{1/x + 1/x^2} = \frac{1}{2}$

**42.** $$\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)=\lim_{x\to 0}\frac{\sin^2 x - x^2}{x^2\sin^2 x}=\lim_{x\to 0}\frac{(1-\cos 2x)/2-x^2}{x^2(1-\cos 2x)/2}$$
$$=\lim_{x\to 0}\frac{1-\cos 2x - 2x^2}{x^2-x^2\cos 2x}=\lim_{x\to 0}\frac{2\sin 2x - 4x}{2x+2x^2\sin 2x - 2x\cos 2x}$$
$$=\lim_{x\to 0}\frac{4\cos 2x - 4}{2+8x\sin 2x+4x^2\cos 2x - 2\cos 2x}$$
$$=\lim_{x\to 0}\frac{-8\sin 2x}{12\sin 2x + 24x\cos 2x - 8x^2\sin 2x}$$
$$=\lim_{x\to 0}\frac{-16\cos 2x}{48\cos 2x - 64x\sin 2x - 16x^2\cos 2x}=-\frac{1}{3}$$

**43.** For critical points, we solve $0=f'(x)=e^{-2x}-2x\,e^{-2x}=(1-2x)e^{-2x}$. The only solution is $x=1/2$. Since $f'(x)$ changes from positive to negative as $x$ increases through $1/2$, this critical point gives a relative maximum of $f(1/2)=1/(2e)$. For points of inflection, we solve $0=f''(x)=-2e^{-2x}-2(1-2x)e^{-2x}=4(x-1)e^{-2x}$. The solution is $x=1$. Since $f''(x)$ changes sign as $x$ passses through 1, there is a point of inflection at $(1,1/e^2)$. Since

$$\lim_{x\to\infty} x\,e^{-2x}=\lim_{x\to\infty}\frac{x}{e^{2x}}=\lim_{x\to\infty}\frac{1}{2e^{2x}}=0^+,$$

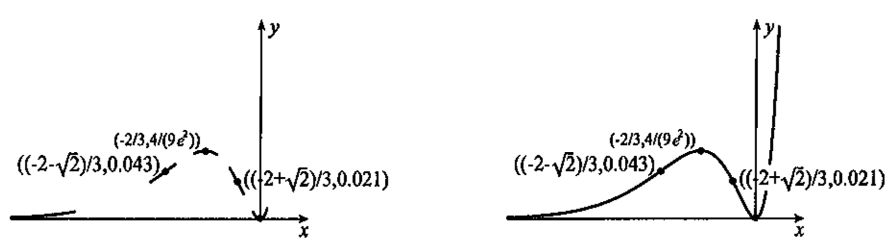
the $x$-axis is a horizontal asymptote. This information, shown in the left figure below, leads to the final graph in the right figure.




**44.** For critical points, we solve $0=f'(x)=2x\,e^{3x}+3x^2\,e^{3x}=x(2+3x)e^{3x}$. Solutions are $x=0$ and $x=-2/3$. Since $f'(x)$ changes from positive to negative as $x$ increases through $-2/3$, this critical point gives a relative maximum of $f(-2/3)=4/(9e^2)$. There is a relative minimum of $f(0)=0$ at $x=0$ because $f'(x)$ changes from negative to positive as $x$ increases through this value. For points of inflection, we solve $0=f''(x)=(2+6x)e^{3x}+3x(2+3x)e^{3x}=(9x^2+12x+2)e^{3x}$. Since $f''(x)$ changes sign as $x$ passes through the solutions $x=(-2\pm\sqrt{2})/3$, there are points of inflection at $((-2-\sqrt{2})/3, 0.043)$ and $((-2+\sqrt{2})/3, 0.021)$. Since

$$\lim_{x\to-\infty} x^2\,e^{3x}=\lim_{x\to-\infty}\frac{x^2}{e^{-3x}}=\lim_{x\to-\infty}\frac{2x}{-3e^{-3x}}=\lim_{x\to-\infty}\frac{2}{9e^{-3x}}=0^+,$$

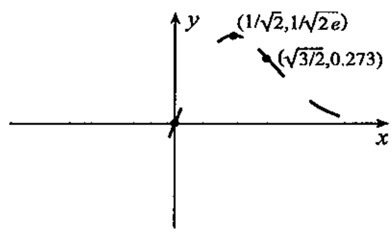
the $x$-axis is a horizontal asymptote. This information, shown in the left figure below, leads to the final graph in the right figure.

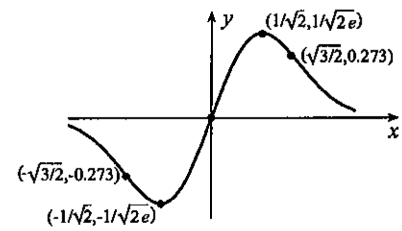


**45.** The function is odd so that we need only draw the graph for $x \geq 0$. For critical points, we solve $0 = f'(x) = e^{-x^2} - 2x^2 e^{-x^2} = (1 - 2x^2)e^{-x^2}$. The positive solution is $x = 1/\sqrt{2}$. Since $f'(x)$ changes from positive to negative as $x$ increases through $1/\sqrt{2}$, this critical point gives a relative maximum of $f(1/\sqrt{2}) = 1/\sqrt{2e}$. For points of inflection, we solve $0 = f''(x) = -4xe^{-x^2} - 2x(1 - 2x^2)e^{-x^2} = 2x(2x^2 - 3)e^{-x^2}$. Since $f''(x)$ changes sign as $x$ passes through the solution $x = 0$ and the positive solution $x = \sqrt{3/2}$, they give points of inflection $(0, 0)$ and $(\sqrt{3/2}, 0.273)$. Since

$$\lim_{x\to\infty} x\,e^{-x^2} = \lim_{x\to\infty} \frac{x}{e^{x^2}} = \lim_{x\to\infty} \frac{1}{2xe^{x^2}} = 0^+,$$

the $x$-axis is a horizontal asymptote. This information, shown in the left figure below, leads to the final graph in the right figure.



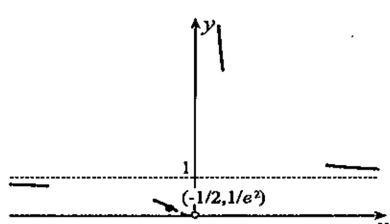

**46.** The following limits show that $y = 1$ is a horizontal asymptote, and the $y$-axis is a vertical asymptote,

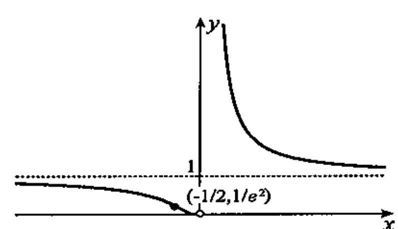
$$\lim_{x\to-\infty} e^{1/x} = 1^-, \quad \lim_{x\to\infty} e^{1/x} = 1^+, \quad \lim_{x\to0^+} e^{1/x} = \infty.$$

Since $f'(x) = -(1/x^2)e^{1/x}$, the function has no critical points, but we notice that

$$\lim_{x\to0^-} f'(x) = \lim_{x\to0^-} \frac{-1/x^2}{e^{-1/x}} = \lim_{x\to0^-} \frac{2/x^3}{(1/x^2)e^{-1/x}} = \lim_{x\to0^-} \frac{2/x}{e^{-1/x}} = \lim_{x\to0^-} \frac{-2/x^2}{(1/x^2)e^{-1/x}} = \lim_{x\to0^-} (-2e^{1/x}) = 0^-.$$

We can locate points of inflection by solving $0 = f''(x) = \dfrac{2}{x^3}e^{1/x} + \dfrac{1}{x^4}e^{1/x} = \dfrac{1}{x^4}(2x+1)e^{1/x}$. Since the only solution is $x = -1/2$, and $f''(x)$ changes sign as $x$ passes through this value, a point of inflection is $(-1/2, 1/e^2)$. This information, shown in the left figure below, leads to the final graph in the right figure.

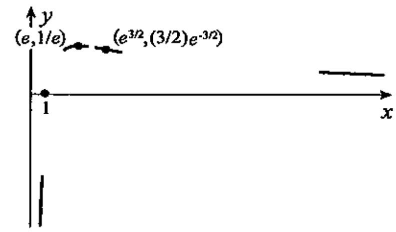


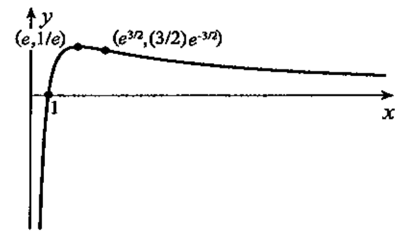




**47.** The following limits show that $y = 0$ is a horizontal asymptote, and the $y$-axis is a vertical asymptote,

$$\lim_{x\to0^+} \frac{\ln x}{x} = -\infty, \qquad \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0^+.$$

For critical points we solve $0 = f'(x) = \dfrac{x(1/x) - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2}$. The only solution is $x = e$. Since $f'(x)$ changes from positive to negative as $x$ increases through $e$, this critical point gives a relative maximum of $f(e) = 1/e$. For points of inflection, we solve $0 = f''(x) = \dfrac{x^2(-1/x) - (1 - \ln x)(2x)}{x^4} = \dfrac{-3 + 2\ln x}{x^3}$. Since $f''(x)$ changes sign as $x$ passes through the solution $x = e^{3/2}$, there is a point of inflection at $(e^{3/2}, (3/2)/e^{3/2})$. This information, shown in the left figure below, leads to the final graph in the right figure.



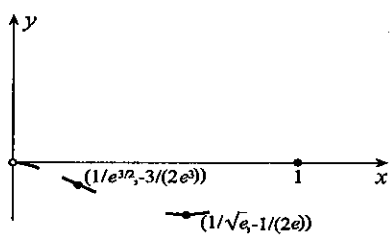

**48.** For critical points, we solve $0 = f'(x) = 2x\ln x + x^2/x = x(2\ln x + 1)$. The only solution is $x = 1/\sqrt{e}$. Since $f''(x) = 2\ln x + 2x/x + 1 = 2\ln x + 3$, it follows that $f''(1/\sqrt{e}) = 2$. The critical point therefore gives a relative minimum of $f(1/\sqrt{e}) = -1/(2e)$. Since $f''(e^{-3/2}) = 0$, and $f''(x)$ changes sign as $x$ passes through $e^{-3/2}$, there is a point of inflection at $\left(e^{-3/2}, -3/(2e^3)\right)$. We use L'Hôpital's rule to show that the graph approaches the origin,

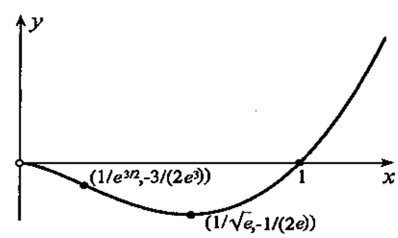
$$\lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x^2} = \lim_{x\to 0^+} \frac{1/x}{-2/x^3} = \lim_{x\to 0^+} \left(-\frac{x^2}{2}\right) = 0^-.$$

The slope of the graph also approaches zero as $x \to 0^+$,

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} [x(2\ln x + 1)] = \lim_{x\to 0^+} \frac{2\ln x + 1}{1/x} = \lim_{x\to 0^+} \frac{2/x}{-1/x^2} = \lim_{x\to 0^+} (-2x) = 0^-.$$

This information, shown in the left figure below, leads to the final graph in the right figure.




**49.** We evaluate the following limits to examine the graph near the discontinuity at $x = 0$:

$$\lim_{x\to 0^+} x\,e^{1/x} = \lim_{x\to 0^+} \frac{e^{1/x}}{1/x} = \lim_{x\to 0^+} \frac{e^{1/x}(-1/x^2)}{-1/x^2} = \infty, \qquad \lim_{x\to 0^-} x\,e^{1/x} = 0^-.$$

For critical points we solve $0 = f'(x) = e^{1/x} + x\,e^{1/x}(-1/x^2) = [(x-1)/x]e^{1/x}$. The only solution is $x = 1$. Since $f'(x)$ changes from negative to positive as $x$ increases through $x = 1$, this critical point gives a relative minimum of $f(1) = e$. To determine the slope of the graph as $x \to 0^-$, we consider

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-}\left(\frac{x-1}{x}\right)e^{1/x} = \lim_{x\to 0^-} \frac{x-1}{x\,e^{-1/x}}.$$

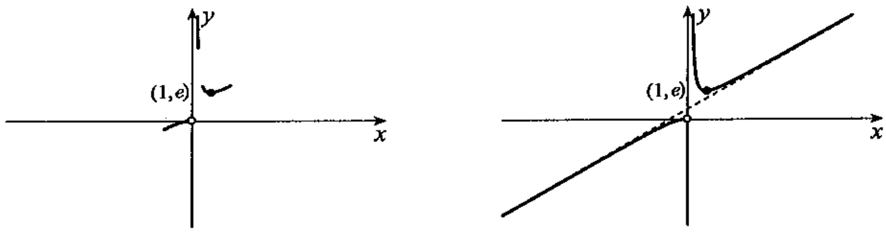
Since

$$\lim_{x\to 0^-} x\,e^{-1/x} = \lim_{x\to 0^-} \frac{e^{-1/x}}{1/x} = \lim_{x\to 0^-} \frac{e^{-1/x}(1/x^2)}{-1/x^2} = -\infty,$$

it follows that $\lim_{x\to 0^-} f'(x) = 0^+$. For points of inflection, we solve

$$0 = f''(x) = \frac{1}{x^2}e^{1/x} + \left(\frac{x-1}{x}\right)e^{1/x}\left(\frac{-1}{x^2}\right) = \frac{1}{x^3}e^{1/x}.$$

There are no points of inflection. This information, shown in the left figure below, leads to the final graph in the right figure. The line $y = x + 1$ is an oblique asymptote, but we do not yet have the tools to show this.




**50.** The limits

$$\lim_{x\to 0^+} \frac{x^2}{\ln x} = 0^-, \qquad \lim_{x\to 1^-} \frac{x^2}{\ln x} = -\infty, \qquad \lim_{x\to 1^+} \frac{x^2}{\ln x} = \infty,$$

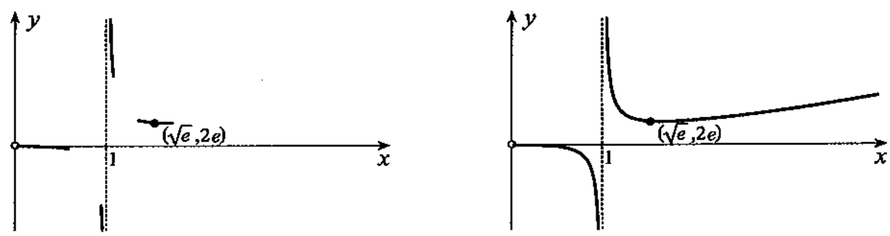
show that the curve approaches the origin as $x \to 0^+$, and that the line $x = 1$ is a vertical asymptote. For critical points, we solve

$$0 = f'(x) = \frac{2x}{\ln x} - \frac{x^2}{(\ln x)^2}\left(\frac{1}{x}\right) = \frac{x}{(\ln x)^2}(2\ln x - 1).$$

Thus, $x = \sqrt{e}$, and this gives a relative minimum of $f(\sqrt{e}) = 2e$ ($f'(x)$ changing from negative to positive as $x$ increases through $\sqrt{e}$). The following limit shows that the slope of the graph approaches zero as $x \to 0^+$,

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} \left[\frac{2x}{\ln x} - \frac{x}{(\ln x)^2}\right] = 0^-.$$

This information, shown in the left figure below, leads to the final graph in the right figure.



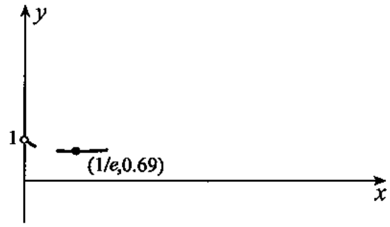

**51.** To find critical points of the function, we set $y = x^x$ and take logarithms, $\ln y = x\ln x$. Implicit differentiation gives

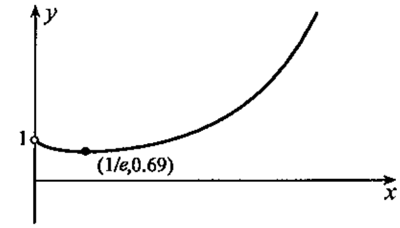
$$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x \quad \Longrightarrow \quad \frac{dy}{dx} = y(1 + \ln x) = x^x(1 + \ln x).$$

Since $f'(x) = 0$ when $x = 1/e$ and $f'(x)$ changes from negative to positive as $x$ increases through $1/e$, there is a relative minimum of $f(1/e) \approx 0.69$. The function is undefined for $x = 0$, and we therefore set $L = \lim_{x\to 0^+} x^x$ and take natural logarithms,

$$\ln L = \ln \lim_{x\to 0^+} x^x = \lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0.$$

Thus, $L = e^0 = 1$, so that the graph approaches 1 as $x \to 0^+$. To find the slope of the graph as $x \to 0^+$ we note that $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} x^x(1 + \ln x) = -\infty$. This information, shown in the left figure below, leads to the final graph in the right figure.

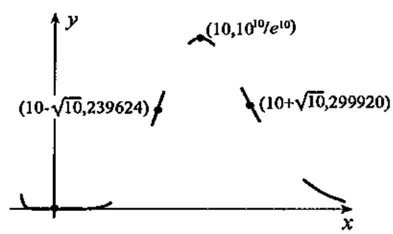


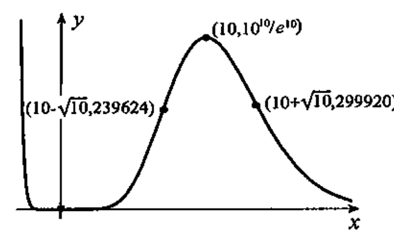




**52.** For critical points we solve $0 = f'(x) = 10x^9e^{-x} - x^{10}e^{-x} = x^9(10-x)e^{-x}$. The solutions are $x = 0$ and $x = 10$. Since $f'(x)$ changes from positive to negative as $x$ increases through 10, this critical point gives a relative maximum of $f(10) = 10^{10}/e^{10}$. We have a relative minimum at $f(0) = 0$ since $f'(x)$ changes from negative to positive as $x$ increases through 0. For points of inflection, we solve

$$0 = f''(x) = (90x^8 - 10x^9)e^{-x} - (10x^9 - x^{10})e^{-x} = x^8(x^2 - 20x^9 + 90)e^{-x}.$$

Solutions are $x = \dfrac{20 \pm \sqrt{400-360}}{2} = 10 \pm \sqrt{10}$. Since $f''(x)$ changes sign as $x$ passes through these values, there are points of inflection at $(10-\sqrt{10}, 239\,624)$ and $(10+\sqrt{10}, 299\,920)$. Repeated applications of L'Hôpital's rule show that $\lim_{x\to\infty} x^{10}e^{-x} = 0^+$. This information, shown in the left figure below, leads to the final graph in the right figure.

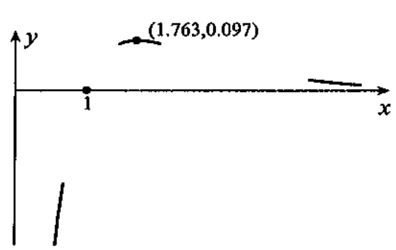


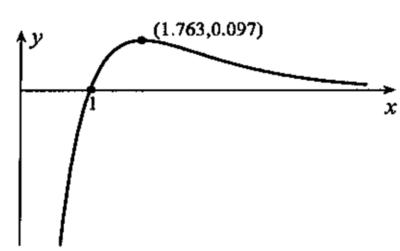




**53.** The limit $\lim_{x\to 0^+} e^{-x}\ln x = -\infty$ shows that the $y$-axis is a vertical asymptote. For critical points we solve $0 = f'(x) = -e^{-x}\ln x + e^{-x}(1/x) = (1/x)e^{-x}(1 - x\ln x)$. To find the only solution of this equation, we use Newton's iterative procedure with $x_1 = 2$, $\quad x_{n+1} = x_n - \dfrac{1 - x_n\ln x_n}{-1 - \ln x_n}$. Iteration gives $x_2 = 1.77$, $x_3 = 1.763$, and $x_4 = 1.763$. It is straightforward to verify that this is the root to 3 decimals. Since $f'(x)$ changes from positive to negative as $x$ increases through 1.763, there is a relative maximum of $f(1.763) = 0.097$. L'Hôpital's rule shows that the $x$-axis is a horizontal asymptote,

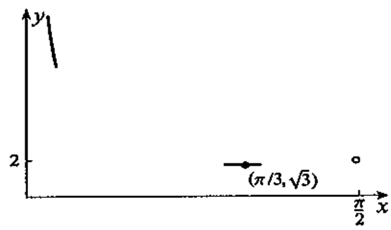
$$\lim_{x\to\infty} e^{-x}\ln x = \lim_{x\to\infty} \frac{\ln x}{e^x} = \lim_{x\to\infty} \frac{1/x}{e^x} = 0^+.$$

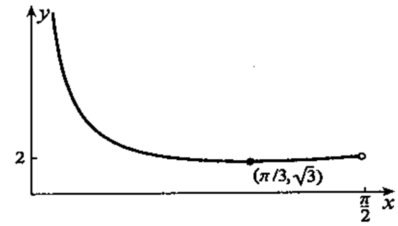
This information, shown in the left figure below, leads to the final graph in the right figure.




**54.** For critical points, we solve $0 = f'(x) = -2\csc x\cot x + \csc^2 x = \csc^2 x(1 - 2\cos x)$. The only critical point is $x = \pi/3$. Since $f'(x)$ changes from negative to positive as $x$ increases through $\pi/3$, there is a relative minimum of $f(\pi/3) = \sqrt{3}$ at this critical point. The limit $\lim_{x\to 0^+}(2\csc x - \cot x) = \lim_{x\to 0^+}\frac{2-\cos x}{\sin x} = \infty$ indicates that the $y$-axis is a vertical asymptote. This information, shown in the left figure below, leads to the final graph in the right figure.




**55.** If we set $L = \lim_{x\to\infty}\left(\frac{x+a}{x+b}\right)^{cx}$, and take natural logarithms, then

$$\ln L = \ln\left[\lim_{x\to\infty}\left(\frac{x+a}{x+b}\right)^{cx}\right] = \lim_{x\to\infty}\left[cx\ln\left(\frac{x+a}{x+b}\right)\right] = c\lim_{x\to\infty}\left[\frac{\ln\left(\frac{x+a}{x+b}\right)}{1/x}\right]$$

$$= c\lim_{x\to\infty}\left\{\frac{\frac{x+b}{x+a}\left[\frac{(x+b)-(x+a)}{(x+b)^2}\right]}{-1/x^2}\right\} = c\lim_{x\to\infty}\frac{(a-b)x^2}{(x+a)(x+b)}$$

$$= c\lim_{x\to\infty}\frac{2(a-b)x}{2x+a+b} = c\lim_{x\to\infty}\frac{2(a-b)}{2} = c(a-b),$$ and this implies that $L = e^{c(a-b)}$.

**56.** If we set $L = \lim_{x\to\infty}(x - \ln x)$, and take exponentials on both sides of the equation, $e^L = e^{\lim_{x\to\infty}(x-\ln x)}$. If we interchange the limit and exponentiation operations,

$$e^L = \lim_{x\to\infty} e^{x-\ln x} = \lim_{x\to\infty}\frac{e^x}{x} = \lim_{x\to\infty}\frac{e^x}{1} = \infty.$$

It follows therefore that $L = \lim_{x\to\infty}(x - \ln x) = \infty$ also.

**57.** $$\lim_{E\to 0^+}\left(\frac{e^E+e^{-E}}{e^E-e^{-E}} - \frac{1}{E}\right) = \lim_{E\to 0^+}\frac{E(e^E+e^{-E}) - e^E + e^{-E}}{E(e^E-e^{-E})} = \lim_{E\to 0^+}\frac{e^E+e^{-E}+E(e^E-e^{-E}) - e^E - e^{-E}}{e^E-e^{-E}+E(e^E+e^{-E})}$$

$$= \lim_{E\to 0^+}\frac{E(e^E-e^{-E})}{e^E-e^{-E}+E(e^E+e^{-E})} = \lim_{E\to 0^+}\frac{e^E-e^{-E}+E(e^E+e^{-E})}{e^E+e^{-E}+e^E+e^{-E}+E(e^E-e^{-E})}$$

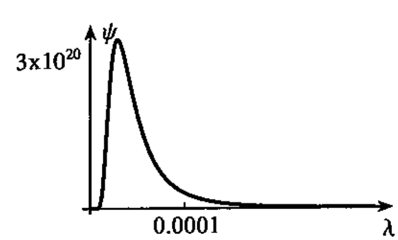
$$= 0$$

**58.** (a) $$\lim_{\lambda\to 0^+}\psi(\lambda) = \lim_{\lambda\to 0^+}\frac{-5k\lambda^{-6}}{e^{c/\lambda}(-c/\lambda^2)} = \lim_{\lambda\to 0^+}\frac{5k\lambda^{-4}}{ce^{c/\lambda}} = \lim_{\lambda\to 0^+}\frac{5k(-4)\lambda^{-5}}{ce^{c/\lambda}(-c/\lambda^2)}$$

$$= \lim_{\lambda\to 0^+}\frac{20k\lambda^{-3}}{c^2e^{c/\lambda}} = \cdots = \lim_{\lambda\to 0^+}\frac{120k}{c^5e^{c/\lambda}} = 0$$

$$\lim_{\lambda\to\infty}\psi(\lambda) = \lim_{\lambda\to\infty}\frac{-5k\lambda^{-6}}{e^{c/\lambda}(-c/\lambda^2)} = \lim_{\lambda\to\infty}\frac{5k}{c\lambda^4e^{c/\lambda}} = 0$$

(b) For critical points of $\psi(\lambda)$ we solve $0 = \psi'(\lambda) = \frac{-k}{[\lambda^5(e^{c/\lambda}-1)]^2}\left[5\lambda^4(e^{c/\lambda}-1) + \lambda^5e^{c/\lambda}(-c/\lambda^2)\right]$,
and therefore $0 = \lambda^3[5\lambda(e^{c/\lambda}-1) - ce^{c/\lambda}]$.
Since $\lambda \neq 0$, we must set $(5\lambda - c)e^{c/\lambda} = 5\lambda$.
This equation was solved using Newton's iterative procedure in Exercise 34 of Section 4.1. The critical point is $\lambda = 0.000\,029\,0$.
(c) The limits in (a) together with the fact that $\psi(\lambda)$ is always positive for $\lambda > 0$ implies that the critical point must give a relative maximum. The graph of the function is shown to the right.



**59.** $f(\theta) = \sin\theta\left[\dfrac{\sin\left(\frac{\pi}{2}\cos\theta - \frac{\pi}{2}\right)}{\cos\theta - 1} + \dfrac{\sin\left(\frac{\pi}{2}\cos\theta + \frac{\pi}{2}\right)}{\cos\theta + 1}\right] = \sin\theta\left[\dfrac{-\cos\left(\frac{\pi}{2}\cos\theta\right)}{\cos\theta - 1} + \dfrac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\cos\theta + 1}\right]$

$$= \sin\theta\,\cos\left(\frac{\pi}{2}\cos\theta\right)\left(\frac{-\cos\theta - 1 + \cos\theta - 1}{\cos^2\theta - 1}\right) = \sin\theta\,\cos\left(\frac{\pi}{2}\cos\theta\right)\left(\frac{-2}{-\sin^2\theta}\right) = \frac{2\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

With L'Hôpital's rule, $\displaystyle\lim_{\theta\to 0} f(\theta) = \lim_{\theta\to 0}\frac{-2\sin\left(\frac{\pi}{2}\cos\theta\right)\left(-\frac{\pi}{2}\sin\theta\right)}{\cos\theta} = 0.$

**60.** By L'Hôpital's rule,

$$5 = \lim_{x\to 0}\frac{ae^{ax} - b + (1 + 2cx)\sin(x + cx^2)}{6x^2 + 10x}.$$

Since the limit of the numerator is $a - b$ and that of the denominator is 0, the only way this limit can be 5 is for $a = b$. In this case,

$$\begin{aligned} 5 &= \lim_{x\to 0}\frac{ae^{ax} - a + (1 + 2cx)\sin(x + cx^2)}{6x^2 + 10x} \\ &= \lim_{x\to 0}\frac{a^2e^{ax} + 2c\sin(x + cx^2) + (1 + 2cx)^2\cos(x + cx^2)}{12x + 10} = \frac{a^2 + 1}{10}. \end{aligned}$$

Thus, $a = \pm 7 = b$, and $c$ is arbitrary.

**61.** (a) A plot on the interval $0 \le x \le 20$ is shown in the left figure below. We added the hole at $x = 1$ since the function is undefined there. Suppose we let

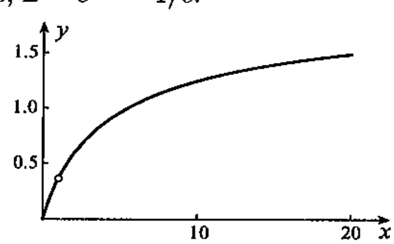
$$L = \lim_{x\to 1} x\left(\frac{2}{x+1}\right)^{(x+1)/(x-1)} = \lim_{x\to 1}\left(\frac{2}{x+1}\right)^{(x+1)/(x-1)},$$

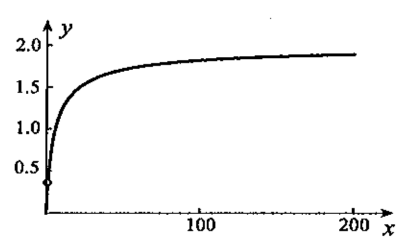
provided the latter limit exists. To evaluate it we take natural logarithms and use L/Hôpital's rule,

$$\ln L = \ln\left[\lim_{x\to 1}\left(\frac{2}{x+1}\right)^{(x+1)/(x-1)}\right] = \lim_{x\to 1}\left[\left(\frac{x+1}{x-1}\right)\ln\left(\frac{2}{x+1}\right)\right] = \lim_{x\to 1}\frac{\ln\left(\dfrac{2}{x+1}\right)}{\dfrac{x-1}{x+1}}$$

$$= \lim_{x\to 1}\left[\frac{\left(\dfrac{x+1}{2}\right)\left(\dfrac{-2}{(x+1)^2}\right)}{\dfrac{(x+1)-(x-1)}{(x+1)^2}}\right] = \lim_{x\to 1}\left[-\frac{x+1}{2}\right] = -1.$$

Hence, $L = e^{-1} = 1/e$.




(b) A plot on the interval $0 \le x \le 200$ is shown in the right figure above. To evaluate the limit of $f(x)$ as $x \to \infty$, we write

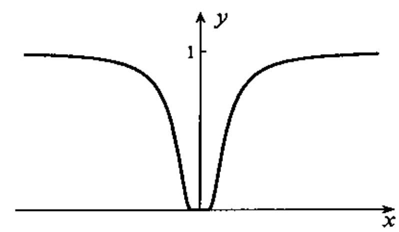
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x\left(\frac{2}{x+1}\right)^{1+2/(x-1)} = \lim_{x\to\infty}\left(\frac{2x}{x+1}\right)\left(\frac{2}{x+1}\right)^{2/(x-1)} = 2\lim_{x\to\infty}\left(\frac{2}{x+1}\right)^{2/(x-1)},$$

provided this limit exists. If we set it equal to $L$ and take natural logarithms,

$$\ln L = \ln\left[\lim_{x\to\infty}\left(\frac{2}{x+1}\right)^{2/(x-1)}\right] = \lim_{x\to\infty}\left[\frac{2}{x-1}\ln\left(\frac{2}{x+1}\right)\right] = 2\lim_{x\to\infty}\left[\frac{\left(\frac{x+1}{2}\right)\left(\frac{-2}{(x+1)^2}\right)}{1}\right] = 0.$$

Hence $L = e^0 = 1$, and it follows that $\lim_{x\to\infty} f(x) = 2$.

**62.** (a) Limits as $x \to 0^+$ and $x \to \infty$, together with symmetry about the $y$-axis, give the graph to the right.




(b) If we can show that the right-hand limit is zero, then the left-hand limit must also be zero. Suppose we set $L = \lim_{x\to 0^+} x^{-n}e^{-1/x^2}$ and take natural logarithms,

$$\ln L = \ln\left(\lim_{x\to 0^+}\frac{e^{-1/x^2}}{x^n}\right) = -\lim_{x\to 0^+}\left(\frac{1}{x^2} + n\ln x\right) = -\lim_{x\to 0^+}\left(\frac{1+nx^2\ln x}{x^2}\right).$$

Since

$$\lim_{x\to 0^+} x^2\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x^2} = \lim_{x\to 0^+}\frac{1/x}{-2/x^3} = \lim_{x\to 0^+}(-x^2/2) = 0,$$

it follows that $\ln L = -\lim_{x\to 0^+}\left(\frac{1+nx^2\ln x}{x^2}\right) = -\infty$. Consequently, $L = 0$.

(c) $f'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{e^{-1/h^2}}{h} = 0$ (by part (b))

Suppose that $k$ is some integer for which $f^{(k)}(0) = 0$. Then

$$f^{(k+1)}(0) = \lim_{h\to 0}\frac{f^{(k)}(h) - f^{(k)}(0)}{h} = \lim_{h\to 0}\frac{f^{(k)}(h)}{h}.$$

Now, any number of differentiations of $f(x) = e^{-1/x^2}$ gives rise to terms of the form $(A/x^n)\,e^{-1/x^2}$, where $n$ is a positive integer and $A$ is a constant. It follows then that $f^{(k)}(h)/h$ must consist of terms of the form $(A/h^n)e^{-1/h^2}$ which have limit zero as $h \to 0$. Thus, $f^{(k+1)}(0) = 0$, and by mathematical induction, $f^{(n)}(0) = 0$ for all $n \geq 1$.

## EXERCISES 4.12

**1.** $dy = f'(x)\,dx = (2x+3)\,dx$

**2.** $dy = f'(x)\,dx = \left[\frac{(x-1)(1)-(x+1)(1)}{(x-1)^2}\right]dx = \frac{-2}{(x-1)^2}dx$

**3.** $dy = f'(x)\,dx = \frac{2x-2}{2\sqrt{x^2-2x}}dx = \frac{x-1}{\sqrt{x^2-2x}}dx$

**4.** $dy = f'(x)\,dx = [2x\cos(x^2+2) + \sin x]\,dx$

**5.** $dy = f'(x)\,dx = [(1/3)x^{-2/3} - (5/3)x^{2/3}]dx$

**6.** $dy = f'(x)\,dx = [3x^2\sqrt{3-4x^2} + (1/2)x^3(3-4x^2)^{-1/2}(-8x)]\,dx$

$$= \frac{3x^2(3-4x^2)-4x^4}{\sqrt{3-4x^2}}dx = \frac{x^2(9-16x^2)}{\sqrt{3-4x^2}}dx$$

**7.** $dy = f'(x)\,dx = (2x\sin x + x^2\cos x)dx$

**8.** Since $f(x) = \frac{(x-1)^3+6}{(x-1)^2} = x - 1 + \frac{6}{(x-1)^2}$, $dy = f'(x)dx = \left[1 - \frac{12}{(x-1)^3}\right]dx$.

**9.** $dy = f'(x)\,dx = [(1/2)(1+\sqrt{1-x})^{-1/2}(1/2)(1-x)^{-1/2}(-1)]dx = \frac{-1}{4\sqrt{1-x}\sqrt{1+\sqrt{1-x}}}dx$

**10.** $dy = f'(x)\,dx = \left[\dfrac{(x^3+5x)(3x^2-4x)-(x^3-2x^2)(3x^2+5)}{(x^3+5x)^2}\right]dx$

$$= \frac{2x^4+10x^3-10x^2}{(x^3+5x)^2}dx = \frac{2(x^2+5x-5)}{(x^2+5)^2}dx$$

**11.** Approximate percentage changes are

$$\frac{100\,dM}{M} = \frac{100(m\,dv)}{mv} = \frac{100\,dv}{v} = 1, \qquad \frac{100\,dK}{K} = \frac{100(mv\,dv)}{mv^2/2} = 2\left(\frac{100\,dv}{v}\right) = 2.$$

**12.** The approximate percentage change in $F$ is

$$100\frac{dF}{F} = \frac{100}{F}\left(-\frac{2GmM}{r^3}dr\right) = -\frac{200GmM}{r^3}dr\left(\frac{r^2}{GmM}\right) = -2\left(100\frac{dr}{r}\right) = -2(2) = -4.$$

**13.** The approximate percentage change in $R$ is

$$100\frac{dR}{R} = \frac{100}{R}\left(\frac{2v^2\cos 2\theta}{9.81}d\theta\right) = \frac{200v^2\cos 2\theta}{9.81}d\theta\left(\frac{9.81}{v^2\sin 2\theta}\right)$$
$$= 200\cot 2\theta\,d\theta = 2\theta\cot 2\theta\left(100\frac{d\theta}{\theta}\right) = 2\theta\cot 2\theta,$$

since the change in $\theta$ is 1%. When $\theta = \pi/3$, this becomes $100\dfrac{dR}{R} = 2\left(\dfrac{\pi}{3}\right)\left(-\dfrac{1}{\sqrt{3}}\right) = -\dfrac{2\sqrt{3}\pi}{9}$.

**14.** The approximate percentage change in $H$ is

$$100\frac{dH}{H} = \frac{100}{H}\left(\frac{2v^2\sin\theta\cos\theta}{19.62}d\theta\right) = \frac{200v^2\sin\theta\cos\theta}{19.62}d\theta\left(\frac{19.62}{v^2\sin^2\theta}\right)$$
$$= 200\cot\theta\,d\theta = 2\theta\cot\theta\left(100\frac{d\theta}{\theta}\right) = 4\theta\cot\theta,$$

since the change in $\theta$ is 2%. When $\theta = \pi/3$, this becomes

$$100\frac{dH}{H} = 4\left(\frac{\pi}{3}\right)\left(\frac{1}{\sqrt{3}}\right) = \frac{4\sqrt{3}\pi}{9}.$$

**15.** Since $V = kP^{-5/7}$ for some constant $k$, the approximate percentage change in $V$ is

$$\frac{100\,dV}{V} = \frac{100(-5/7)kP^{-12/7}\,dP}{kP^{-5/7}} = -\frac{5}{7}\left(\frac{100\,dP}{P}\right) = -\frac{10}{7}.$$

**16.** The differential of $F$ is $dF = -\dfrac{2GmM}{r^3}dr$. If we set $dF = -0.01m$ for a decrease from $9.81m$ to $9.80m$, then

$$-0.01m = -\frac{2GmM}{r^3}dr.$$

But when $r = 6.37\times 10^6$, we know that $F = 9.81m$ so that

$$9.81m = \frac{GmM}{(6.37\times 10^6)^2}.$$

Consequently,

$$-0.01m = \frac{-2}{(6.37\times 10^6)^3}(9.81m)(6.37\times 10^6)^2\,dr,$$

and this equation implies that $dr = 3.25\times 10^3$. Thus, at a height of 3.25 km, the gravitational force of attraction decreases to $9.80m$ N.

**17.** The approximate percentage change in $F$ is

$$\begin{aligned}
100\frac{dF}{F} &= \frac{100}{F}\left[\frac{-9.81\mu m(-\sin\theta+\mu\cos\theta)}{(\cos\theta+\mu\sin\theta)^2}d\theta\right] \\
&= \left[\frac{9.81(100)\mu m(\sin\theta-\mu\cos\theta)}{(\cos\theta+\mu\sin\theta)^2}\right]d\theta\left(\frac{\cos\theta+\mu\sin\theta}{9.81\mu m}\right) \\
&= \frac{100(\sin\theta-\mu\cos\theta)}{\cos\theta+\mu\sin\theta}d\theta = \frac{\theta(\sin\theta-\mu\cos\theta)}{\cos\theta+\mu\sin\theta}\left(100\frac{d\theta}{\theta}\right) \\
&= \frac{2\theta(\sin\theta-\mu\cos\theta)}{\cos\theta+\mu\sin\theta},
\end{aligned}$$

since the change in $\theta$ is 2%. When $\theta=\pi/4$, this becomes

$$100\frac{dF}{F} = \frac{2(\pi/4)(1/\sqrt{2}-\mu/\sqrt{2})}{1/\sqrt{2}+\mu/\sqrt{2}} = \frac{\pi(1-\mu)}{2(\mu+1)}.$$

**18.** (a) The approximate percentage error in $V$ due to an $a\%$ error in $r$ is

$$100\frac{dV}{V} = \frac{100}{V}(2\pi rh\,dr) = \frac{200\pi rh}{\pi r^2h}dr = 2\left(100\frac{dr}{r}\right) = 2a.$$

(b) The approximate percentage error in $V$ due to a $b\%$ error in $h$ is

$$100\frac{dV}{V} = \frac{100}{V}(\pi r^2\,dh) = \frac{100\pi r^2}{\pi r^2h}dh = 100\frac{dh}{h} = b.$$

(c) The maximum approximate percentage error in $V$ due to errors $a\%$ in $r$ and $b\%$ in $h$ is

$$100\frac{\text{Maximum change in V}}{V} = \frac{100}{\pi r^2h}(2\pi rh\,dr+\pi r^2\,dh) = 2\left(100\frac{dr}{r}\right)+\left(100\frac{dh}{h}\right) = 2a+b.$$

**19.** The approximate percentage error in $y$ is $\quad 100\dfrac{dy}{y} = \dfrac{100}{x^n}nx^{n-1}\,dx = n\left(100\dfrac{dx}{x}\right) = na.$

**20.** (a) The approximate percentage error in $z$ due to an $a\%$ error in $x$ is

$$100\frac{dz}{z} = \frac{100}{x^ny^m}(nx^{n-1}y^m\,dx) = n\left(100\frac{dx}{x}\right) = na.$$

(b) The approximate percentage error in $z$ due to a $b\%$ error in $y$ is

$$100\frac{dz}{z} = \frac{100}{x^ny^m}(my^{m-1}x^n\,dx) = m\left(100\frac{dy}{y}\right) = mb.$$

(c) The maximum approximate percentage error in $z$ due to errors $a\%$ in $x$ and $b\%$ in $y$ is

$$100\frac{\text{Maximum change in z}}{z} = \frac{100}{x^ny^m}(nx^{n-1}y^m\,dx+my^{m-1}x^n\,dy) = n\left(100\frac{dx}{x}\right)+m\left(100\frac{dy}{y}\right) = na+mb.$$

**21.** (a) The approximate percentage error in $z$ due to an $a\%$ error in $x$ is

$$100\frac{dz}{z} = \frac{100}{x^n/y^m}\left(\frac{nx^{n-1}}{y^m}dx\right) = n\left(100\frac{dx}{x}\right) = na.$$

(b) The approximate percentage error in $z$ due to a $b\%$ error in $y$ is

$$100\frac{dz}{z} = \frac{100}{x^n/y^m}\left(\frac{-mx^n}{y^{m+1}}dx\right) = -m\left(100\frac{dy}{y}\right) = -mb.$$

(c) The maximum approximate percentage error in $z$ due to errors $a\%$ in $x$ and $b\%$ in $y$ is

$$100\frac{\text{Maximum change in z}}{z} = \frac{100}{x^n/y^m}\left(\frac{nx^{n-1}}{y^m}dx - \frac{mx^n}{y^{m+1}}dy\right)$$
$$= n\left(100\frac{dx}{x}\right) - m\left(100\frac{dy}{y}\right).$$

For maximum error we take $dx$ as positive and $dy$ as negative in which case

$$100\frac{\text{Maximum change in z}}{z} = n\left(100\frac{dx}{x}\right) + m\left(100\frac{-dy}{y}\right) = na + mb.$$

**22.** The approximate percentage change in $n$ is

$$100\frac{dn}{n} = \frac{100}{n}\frac{(1/2)\cos[(\psi_m+\gamma)/2]}{\sin(\gamma/2)}d\psi_m$$
$$= \frac{50\cos[(\psi_m+\gamma)/2]}{\sin(\gamma/2)}d\psi_m\left\{\frac{\sin(\gamma/2)}{\sin[(\psi_m+\gamma)/2]}\right\} = \frac{\psi_m}{2}\cot[(\psi_m+\gamma)/2]\left(100\frac{d\psi_m}{\psi_m}\right).$$

Since the error in $\psi_m$ is 1% when it is $\pi/6$ and $\gamma = \pi/3$,

$$100\frac{dn}{n} = \frac{\pi/6}{2}\cot[(\pi/6+\pi/3)/2](1) = \frac{\pi}{12}.$$

**23.** The approximate percentage change in $n$ is

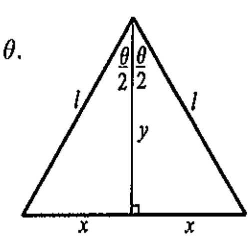
$$100\frac{dn}{n} = \frac{100}{n}\left\{\frac{\sin(\gamma/2)(1/2)\cos[(\psi_m+\gamma)/2] - \sin[(\psi_m+\gamma)/2](1/2)\cos(\gamma/2)}{\sin^2(\gamma/2)}\right\}d\gamma$$
$$= 50\left[\frac{-\sin(\psi_m/2)}{\sin^2(\gamma/2)}\right]d\gamma\left\{\frac{\sin(\gamma/2)}{\sin[(\psi_m+\gamma)/2]}\right\} = \frac{-\gamma\sin(\psi_m/2)}{2\sin(\gamma/2)\sin[(\psi_m+\gamma)/2]}\left(100\frac{d\gamma}{\gamma}\right).$$

Since the error in $\gamma$ is 1% when $\psi_m = \pi/6$ and $\gamma = \pi/3$,

$$100\frac{dn}{n} = -\frac{(\pi/3)\sin(\pi/12)}{2\sin(\pi/6)\sin[(\pi/6+\pi/3)/2]}(1) = -\frac{\sqrt{2}\pi}{3}\sin(\pi/12).$$

## REVIEW EXERCISES

**1.** (a) The area of the triangle is $A = 2\left(\frac{1}{2}xy\right) = \left(l\sin\frac{\theta}{2}\right)\left(l\cos\frac{\theta}{2}\right) = \frac{l^2}{2}\sin\theta.$



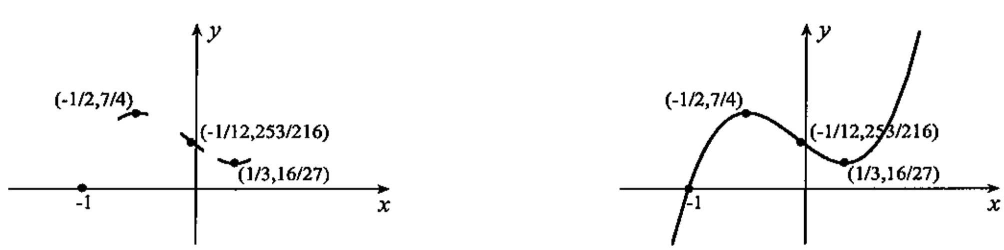

(b) If we differentiate with respect to $t$,

$$\frac{dA}{dt} = \frac{l^2}{2}\cos\theta\frac{d\theta}{dt} = \frac{l^2}{4}\cos\theta.$$

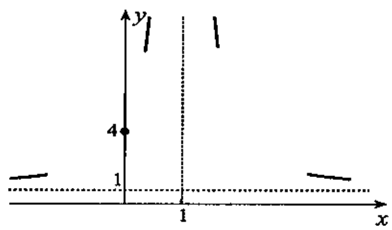
When $\theta = 0, \pi/2, \pi$, this rate has values, respectively, $l^2/4$, 0, $-l^2/4$.

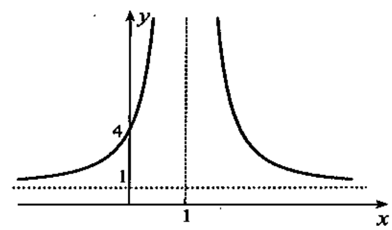
(c) $|dA/dt|$ is largest when $\theta = 0, \pi$ and smallest when $\theta = \pi/2$.

**2.** (a) For critical points we solve $0 = f'(x) = 12x^2 + 2x - 2 = 2(3x-1)(2x+1)$. Solutions are $x = 1/3$ and $x = -1/2$. Since $f''(x) = 24x + 2$, it follows that $f''(1/3) = 10$ and $f''(-1/2) = -10$. Consequently, $x = 1/3$ gives a relative minimum of $f(1/3) = 16/27$, and $x = -1/2$ gives a relative maximum of $f(-1/2) = 7/4$. Since $f''(-1/12) = 0$ and $f''(x)$ changes sign as $x$ passes through $-1/12$, there is a point of inflection at $(-1/12, 253/216)$. This information, shown in the left figure below, leads to the final graph in the right figure.



(b) With the function written as $f(x) = \dfrac{(x^2 - 2x + 1) + 3}{(x-1)^2} = 1 + \dfrac{3}{(x-1)^2}$, critical points are given by $0 = f'(x) = -6/(x-1)^3$. There are no solutions and hence no relative extrema. Since the second derivative $f''(x) = 18/(x-1)^4$ never vanishes, there are no points of inflection. Limits as $x \to \pm\infty$ and right- and left-limits at $x = 1$ give the information in the left figure below. The final graph is to the right.




**3.** If $x$ and $y$ are any two positive numbers with sum $x + y = c$, their product is $P = P(x) = xy = x(c - x)$, $0 < x < c$. For critical points of $P(x)$, we solve $0 = P'(x) = c - 2x$. The solution is $x = c/2$. Since

$$\lim_{x\to 0^+} P(x) = 0, \qquad P(c/2) = \frac{c^2}{4}, \qquad \lim_{x\to c^-} P(x) = 0,$$

the maximum value of $P(x)$ is $c^2/4$, occurring when $x = y = c/2$.

**4.** If $x$ and $y$ are any two positive numbers with sum $x + y = c$, their product is $P = P(x) = xy = x(c - x)$, $0 < x < c$. For critical points of $P(x)$, we solve $0 = P'(x) = c - 2x$. The solution is $x = c/2$. Since

$$\lim_{x\to 0^+} P(x) = 0, \qquad P(c/2) = \frac{c^2}{4}, \qquad \lim_{x\to c^-} P(x) = 0,$$

it follows that $P(x)$ has no absolute minimum on the interval $0 < x < c$. The product can be made arbitrarily close to 0 by choosing $x$ or $y$ sufficiently close to 0.

**5.** If $x$ and $y$ are any two positive numbers with product $xy = c$, their sum is $S = S(x) = x + y = x + (c/x)$, $x > 0$. For critical points of $S(x)$ we solve $0 = S'(x) = 1 - c/x^2$. The positive solution is $x = \sqrt{c}$. Since

$$\lim_{x\to 0^+} S(x) = \infty, \qquad S(\sqrt{c}) = 2\sqrt{c}, \qquad \lim_{x\to\infty} S(x) = \infty,$$

the minimum value of $S(x)$ is $2\sqrt{c}$, occurring when $x = y = \sqrt{c}$.

**6.** If $x$ and $y$ are any two positive numbers with product $xy = c$, their sum is $S = S(x) = x + y = x + (c/x)$, $x > 0$. For critical points of $S(x)$ we solve $0 = S'(x) = 1 - c/x^2$. The positive solution is $x = \sqrt{c}$. Since

$$\lim_{x\to 0^+} S(x) = \infty, \qquad S(\sqrt{c}) = 2\sqrt{c}, \qquad \lim_{x\to\infty} S(x) = \infty,$$

it follows that $S(x)$ has no absolute maximum on the interval $x > 0$. The sum can be made arbitrarily large by choosing $x$ or $y$ sufficiently close to 0.

**7.** If we differentiate the cosine law $\ell^2 = 16 + 9 - 2(4)(3)\cos\theta = 25 - 24\cos\theta$ with respect to time $t$, we obtain $2\ell\dfrac{d\ell}{dt} = 24\sin\theta\dfrac{d\theta}{dt}$. When $\ell = 4$, we find that $16 = 25 - 24\cos\theta \implies \cos\theta = 3/8$. It follows that $\sin\theta = \sqrt{1 - 9/64} = \sqrt{55}/8$, and at this instant,

$$2(4)(-1) = 24\left(\frac{\sqrt{55}}{8}\right)\frac{d\theta}{dt} \quad\implies\quad \frac{d\theta}{dt} = \frac{-8}{3\sqrt{55}}.$$

The angle is therefore decreasing at $8/(3\sqrt{55})$ radians per minute.

**8.** $\displaystyle\lim_{x\to 0}\frac{3x^2 + 2x^3}{3x^3 - 2x^2} = \lim_{x\to 0}\frac{6x + 6x^2}{9x^2 - 4x} = \lim_{x\to 0}\frac{6 + 12x}{18x - 4} = -\frac{3}{2}$

**9.** $\displaystyle\lim_{x\to\infty}\frac{\sin 3x}{2x} = 0$

**10.** $\displaystyle\lim_{x\to 4}\frac{x^2 - 16}{x - 4} = \lim_{x\to 4}\frac{2x}{1} = 8$

**11.** $\lim_{x\to 0} \frac{\sin 3x}{2x} = \lim_{x\to 0} \frac{3\cos 3x}{2} = \frac{3}{2}$ **12.** $\lim_{x\to -\infty} \frac{\sin x^2}{2x} = 0$

**13.** $\lim_{x\to 2^+} \frac{\sqrt{x-2}}{\sqrt{x}-\sqrt{2}} = \lim_{x\to 2^+} \frac{1/(2\sqrt{x-2})}{1/(2\sqrt{x})} = \lim_{x\to 2^+} \frac{\sqrt{x}}{\sqrt{x-2}} = \infty$

**14.** $\lim_{x\to\infty} x^2 e^{-3x} = \lim_{x\to\infty} \frac{x^2}{e^{3x}} = \lim_{x\to\infty} \frac{2x}{3e^{3x}} = \lim_{x\to\infty} \frac{2}{9e^{3x}} = 0$

**15.** If we set $L = \lim_{x\to 0^+} x^{2x}$ and take natural logarithms, then

$$\ln L = \ln\left(\lim_{x\to 0^+} x^{2x}\right) = \lim_{x\to 0^+} (2x\ln x) = 2\lim_{x\to 0^+} \frac{\ln x}{1/x} = 2\lim_{x\to 0^+} \frac{1/x}{-1/x^2} = 0.$$

Hence, $L = e^0 = 1$.

**16.** $\lim_{x\to 0^+} x^4 \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x^4} = \lim_{x\to 0^+} \frac{1/x}{-4/x^5} = \lim_{x\to 0^+} \left(\frac{-x^4}{4}\right) = 0$

**17.** $\lim_{x\to 0} \frac{\sin 2x}{\tan 3x} = \lim_{x\to 0} \frac{2\cos 2x}{3\sec^2 3x} = \frac{2}{3}$

**18.** If we set $L = \lim_{x\to\infty} \left(\frac{x+1}{x-1}\right)^x$ and take natural logarithms,

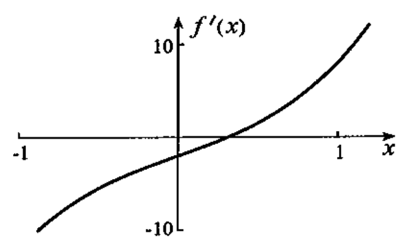
$$\ln L = \lim_{x\to\infty} x\ln\left(\frac{x+1}{x-1}\right) = \lim_{x\to\infty} \frac{\ln\left(\frac{x+1}{x-1}\right)}{1/x} = \lim_{x\to\infty} \frac{\left(\frac{x-1}{x+1}\right)\left[\frac{(x-1)(1)-(x+1)(1)}{(x-1)^2}\right]}{-1/x^2}$$
$$= \lim_{x\to\infty} \frac{2x^2}{x^2-1} = \lim_{x\to\infty} \frac{4x}{2x} = 2.$$

Thus, $L = e^2$.

**19.** $\lim_{x\to -\infty} x e^x = \lim_{x\to -\infty} \frac{x}{e^{-x}} = \lim_{x\to -\infty} \frac{1}{-e^{-x}} = 0$

**20.** (a) For critical points we solve
$0 = f'(x) = 4x^3 + 6x - 2 = 2(2x^3 + 3x - 1)$.
The graph of $f'(x)$ to the right indicates that there is a critical point between $x = 0$ and $x = 1$. To find it with Newton's iterative procdure, we use $x_1 = 1/3$ and

$$x_{n+1} = x_n - \frac{2x_n^3 + 3x_n - 1}{6x_n^2 + 3}.$$



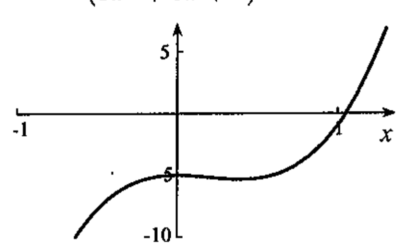

Iteration gives $x_2 = 0.313\,131\,3$, $x_3 = 0.312\,908\,4$, $x_4 = 0.312\,908\,4$. Since $f'(0.312\,907\,5) = -6.5\times 10^{-6}$ and $f'(0.312\,908\,5) = 6.4\times 10^{-7}$, the critical point is $0.312\,908$.

(b) For critical points we solve

$$0 = f'(x) = \frac{(3x^3+5x+1)(3x^2) - (x^3+1)(9x^2+5)}{(3x^3+5x+1)^2} = \frac{10x^3 - 6x^2 - 5}{(3x^3+5x+1)^2}.$$

Thus, critical points are defined by the equation $10x^3 - 6x^2 - 5 = 0$. The graph of the function $10x^3 - 6x^2 - 5$ to the right indicates only one critical point just larger than 1.
To find it we use $x_1 = 1$ and

$$x_{n+1} = x_n - \frac{10x_n^3 - 6x_n^2 - 5}{30x_n^2 - 12x_n}.$$

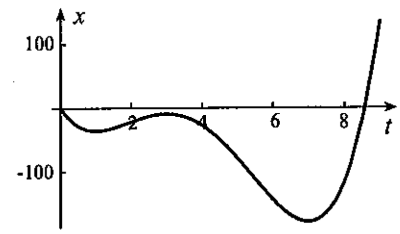



Iteration gives $x_2 = 1.055\,555\,6$, $x_3 = 1.051\,904\,7$, $x_4 = 1.051\,888\,1$, $x_5 = 1.051\,888\,1$. Since $f'(1.051\,887\,5) = -1.3\times 10^{-7}$ and $f'(1.051\,888\,5) = 8.5\times 10^{-8}$, the critical point is $1.051\,888$.

**21.** A plot of the displacement function is shown to the right. The velocity and acceleartion are

$v(t) = 4t^3 - 44t^2 + 124t - 84,$

$a(t) = 12t^2 - 88t + 124.$




The velocity is zero when

$0 = 4(t-1)(t-3)(t-7) \Longrightarrow t = 1, 3, 7.$

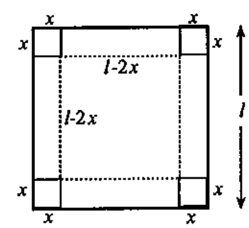
The acceleration is zero when $0 = 4(3t^2 - 22t + 31)$ and solutions of this equation are $t = (11 \pm 2\sqrt{7})/3$.

**22.** The velocity at $t_0$ abruptly changes from a positive quantity to a negative quantity. This could be caused by a collision with a large object.

**23.** No. If the acceleration is constant, then the second derivative can never be zero (unless it is always zero in which case the graph is a straight line).

**24.** If squares of side length $x$ are cut from the corners, the resulting box has volume

$V = x(l-2x)^2 = 4x^3 - 4lx^2 + l^2x,\ 0 \le x \le l/2.$




Critical points of $V$ are given by

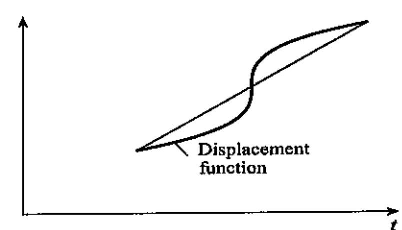
$0 = V'(x) = 12x^2 - 8lx + l^2 = (2x-l)(6x-l).$

Thus, $x = l/2$ or $x = l/6$. Since

$$V(0) = 0, \quad V(l/6) = \frac{2l^3}{27}, \quad V(l/2) = 0,$$

maximum volume is $2l^3/27$.

**25.** (a) Yes The average velocity is the slope of the line joining the end points of the graph of the displacement function. Mean Value Theorem 3.19 guarantees at least one value of $t$ for which the slope of the tangent line is equal to the slope of the line joining the end points. The slope of the tangent line is the instantaneous velocity.




(b) No

**26.** (a) We take the following limits at the discontinuities $x = \pm 1$:

$$\lim_{x\to -1^-} f(x) = -\infty, \quad \lim_{x\to -1^+} f(x) = \infty, \quad \lim_{x\to 1^-} f(x) = -\infty, \quad \lim_{x\to 1^+} f(x) = \infty.$$

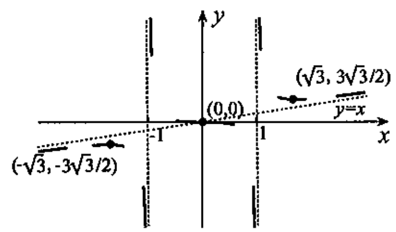
Critical points are given by

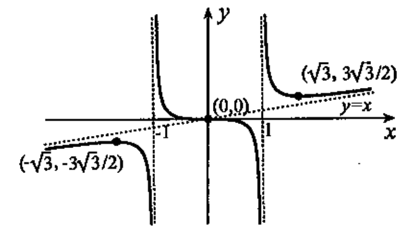
$$0 = f'(x) = \frac{(x^2-1)(3x^2) - x^3(2x)}{(x^2-1)^2} = \frac{x^2(x^2-3)}{(x^2-1)^2}.$$

Solutions are $x = 0, \pm\sqrt{3}$. Since $f'(x)$ changes from a positive quantity to a negative quantity as $x$ increases through $-\sqrt{3}$, there is a relative maximum at $x = -\sqrt{3}$ of $-3\sqrt{3}/2$. Since $f'(x)$ changes from a negative quantity to a positive quantity as $x$ increases through $\sqrt{3}$, there is a relative minimum at $x = \sqrt{3}$ of $3\sqrt{3}/2$. Because $f'(x)$ does not change sign at $x = 0$, it gives a horizontal point of inflection $(0,0)$. To verify that no other points of inflection occur, we calculate

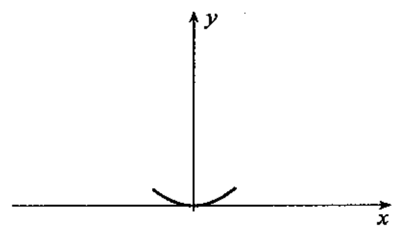
$$f''(x) = \frac{(x^2-1)^2(4x^3-6x) - (x^4-3x^2)(2)(x^2-1)(2x)}{(x^2-1)^4} = \frac{x(2x^2+6)}{(x^2-1)^3}.$$

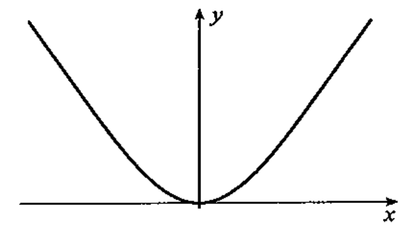
Since $f''(x)$ is 0 only at $x = 0$, there are no other points of inflection. By writing $f(x)$ in the form $f(x) = x + x/(x^2-1)$, we see that $y = x$ is an oblique asymptote for the graph. This information, shown in the left figure below, leads to the final graph in the right figure.




(b) For critical points we consider $0 = f'(x) = 2x + 2\sin x \cos x = 2x + \sin 2x$. The only solution of this equation is $x = 0$. For points of inflection we solve $0 = f''(x) = 2 + 2\cos 2x$. Solutions of this equation are $x = (2n+1)\pi/2$, where $n$ is an integer. Since $f''(x)$ does not change sign as $x$ passes through these points, the graph has no points of inflection. In addition, since $f''(0) = 4$, there is a relative minimum at $x = 0$ of $f(0) = 0$. This information, little as it is, shown in the left figure below, leads to the final graph in the right figure.




**27.** Not necessarily It depends on what side $c$ is doing.

**28.** If the price is raised $x$ dollars per ticket, then the expected numbers of $10 + x$, $9 + x$, and $8 + x$ dollar tickets the team expects to sell are respectively

$$10\,000(1 - x/10), \qquad 20\,000(1 - x/10), \qquad 30\,000(1 - x/10).$$

Total revenue at the new prices is therefore

$$\begin{aligned} R(x) &= 10\,000(1 - x/10)(10 + x) + 20\,000(1 - x/10)(9 + x) + 30\,000(1 - x/10)(8 + x) \\ &= 10\,000(1 - x/10)(6x + 52) = 2000(10 - x)(3x + 26). \end{aligned}$$

We must take $x \geq 0$ and $x$ cannot be greater than 10, else no tickets will be sold. For critical points of $R(x)$ we solve $0 = R'(x) = 2000(-6x + 4)$. Thus, $x = 2/3$. Since $R''(x) = -12\,000$, the graph of the function $R(x)$ is always concave downward. This means that $x = 2/3$ must yield an absolute maximum. The price increase should be 67 cents.

**29.** If the price is raised $x$ dollars per ticket, then the expected numbers of $10 + x$, $9 + x$, and $8 + x$ dollar tickets the team expects to sell are respectively

$$10\,000(1 - x/10), \qquad 20\,000(1 - x/10), \qquad 30\,000(1 - x/10).$$

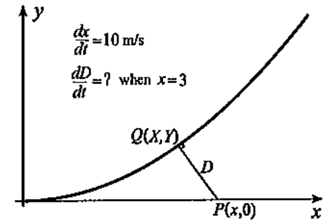
Total revenue at the new prices is therefore

$$\begin{aligned} R(x) &= 10\,000(1 - x/10)(10.5 + x) + 20\,000(1 - x/10)(9.5 + x) + 30\,000(1 - x/10)(8.5 + x) \\ &= 1000(10 - x)(6x + 55). \end{aligned}$$

We must take $x \geq 0$ and $x$ cannot be greater than 10, else no tickets will be sold. For critical points, we solve $0 = R'(x) = 1000(-12x + 5)$. Thus, $x = 5/12$. Since $R''(x) = -12\,000$, the graph of the function $R(x)$ is always concave downward. This means that $x = 5/12$ must yield an absolute maximum. The price increase should be 42 cents.

**30.** When $\|PQ\|$ is the shortest distance from $P$ to the parabola $y = x^2$, line $PQ$ is perpendicular to the tangent line to $y = x^2$ at $Q(X, Y)$. It follows therefore that

$$2X = -\frac{1}{\frac{Y-0}{X-x}} = \frac{x-X}{Y}.$$




We combine this with $Y = X^2$ to obtain

$$2X(X^2) = x - X,$$

or $2X^3 + X - x = 0$. This equation defines the $x$-coordinate $X$ of $Q$ in terms of the $x$-coordinate $x$ of $P$. The distance $D$ from $P$ to $y = x^2$ is then given by

$$D^2 = (x-X)^2 + Y^2 = (x-X)^2 + X^4 = (2X^3)^2 + X^4 = 4X^6 + X^4.$$

Differentiation of this equation with respect to time gives

$$2D\frac{dD}{dt} = 24X^5\frac{dX}{dt} + 4X^3\frac{dX}{dt}.$$

But differentiation of $2X^3 + X - x = 0$ gives

$$6X^2\frac{dX}{dt} + \frac{dX}{dt} - \frac{dx}{dt} = 0.$$

When $x = 3$, $X$ is defined by $2X^3 + X - 3 = 0$, and the only solution of this equation is $X = 1$. At this instant then

$$6(1)^2\frac{dX}{dt} + \frac{dX}{dt} - 10 = 0,$$

from which $dX/dt = 10/7$. Since $D = \sqrt{(3-1)^2 + 1^2} = \sqrt{5}$ at this instant,

$$2\sqrt{5}\frac{dD}{dt} = 24(1)^5\left(\frac{10}{7}\right) + 4(1)^3\left(\frac{10}{7}\right),$$

and therefore $dD/dt = 4\sqrt{5}$. The distance is therefore increasing at $4\sqrt{5}$ m/s.

**31.** The travel time for the cow is

$$t(x) = \frac{\|AB\| + \|BC\|}{2} + \frac{1}{30} = \frac{\sqrt{x^2 + 9/16} + \sqrt{(1-x)^2 + 1}}{2} + \frac{1}{30}, \quad 0 \le x \le 1.$$

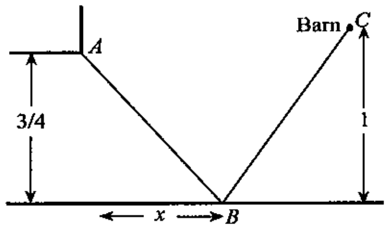
For critical points we solve

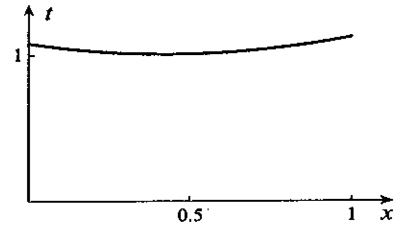
$$0 = \frac{dt}{dx} = \frac{1}{2}\left(\frac{x}{\sqrt{x^2 + 9/16}} + \frac{x-1}{\sqrt{(1-x)^2 + 1}}\right).$$

If we tanspose one of the terms, and square, $\dfrac{x^2}{x^2 + 9/16} = \dfrac{(x-1)^2}{(1-x)^2 + 1}$. When we cross multiply, the equation simplifies to $7x^2 + 18x - 9 = 0$ with solutions $x = 3/7$ and $x = -3$. We now evaluate

$$t(0) = 1.115, \qquad t(3/7) = 1.041, \qquad t(1) = 1.158.$$

Minimum time is therefore $t(3/7) = 62.5$ minutes. The graph of $t(x)$ in the right figure also indicates that it is minimized at its critical point.




When the cow walks twice as fast, travel time is $T(x) = \sqrt{x^2 + 9/16} + \sqrt{(1-x)^2 + 1} + 1/30$. This function has the same critical points as $t(x)$, and therefore minimum travel time again occurs for $x = 3/7$. Minimum time is $T(3/7) \approx 32.2$ minutes.

**32.** The farmer's losses when $x$ hectares of corn and $y$ hectares of potatoes are planted are $L = pax^2 + qby^2$. Since $x + y = 100$,

$$L = L(x) = pax^2 + qb(100 - x)^2, \quad 0 \le x \le 100.$$

For critical points of $L(x)$ we solve $0 = L'(x) = 2pax - 2qb(100 - x)$. The solution is $x = x_c = 100qb/(pa + qb)$. Now

$$L(0) = 10\,000qb, \qquad L(x_c) = \frac{10\,000abpq}{pa + qb}, \qquad L(100) = 10\,000pa.$$

If we write

$$\frac{1}{L(0)} = \frac{10^{-4}}{qb}, \qquad \frac{1}{L(x_c)} = 10^{-4}\left(\frac{1}{qb} + \frac{1}{pa}\right), \qquad \frac{1}{L(100)} = \frac{10^{-4}}{pa},$$

it is clear that $1/L(x_c)$ is greater than $1/L(0)$ and $1/L(100)$. Consequently, $L(x_c)$ is less than $L(0)$ and $L(100)$, and $L(x)$ is minimized for $x = 100qb/(pa + qb)$ hectares and $y = 100pa/(pa + qb)$ hectares.

Notice that if $a$ increases while $p$, $q$, and $b$ remain constant, our results suggest that more and more potatoes should be planted. Conversely large $b$ implies planting more corn. On the other hand, if $a$, $b$, and $q$ remain constant but $p$ increases, the farmer should plant more potatoes. The reason is that with a large area in corn, he will suffer a substantial loss of money.