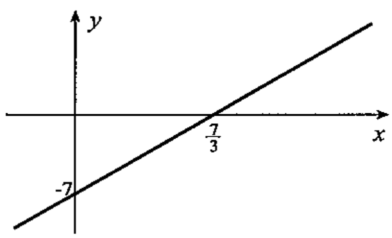
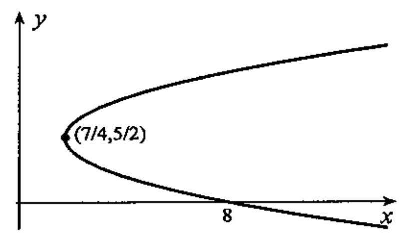
CHAPTER 9

EXERCISES 9.1

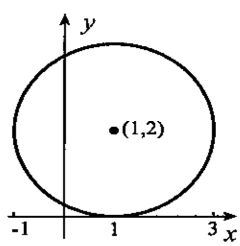
1. This is the straight line
$y = 3(x-2) - 1 = 3x - 7$.



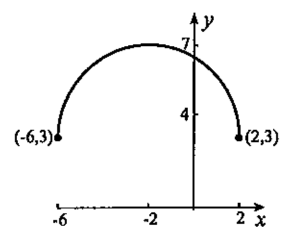
2. This is the parabola
$x = (1-y)^2 + 3(1-y) + 4 = y^2 - 5y + 8$.



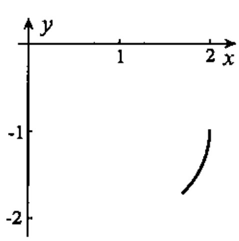
3. Since $(x-1)^2 + (y-2)^2 = 4\cos^2 t + 4\sin^2 t = 4$, this is a circle with centre $(1, 2)$ and radius 2.



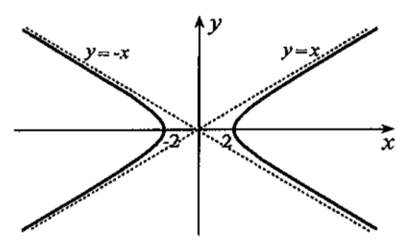
4. Since $(x+2)^2 + (y-3)^2 = 16\cos^2 t + 16\sin^2 t = 16$, points lie on a circle with centre $(-2, 3)$ and radius 4. Values of t in the interval $0 \le t \le \pi$ yield only the upper semicircle.



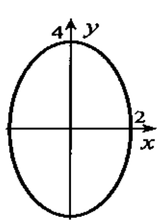
5. Since $(x-1)^2 + (y+1)^2 = \cos^2 t + \sin^2 t = 1$, points lie on a circle with centre $(1, -1)$ and radius 1. Values $0 \le t \le \pi/4$ give only one eighth of the circle.



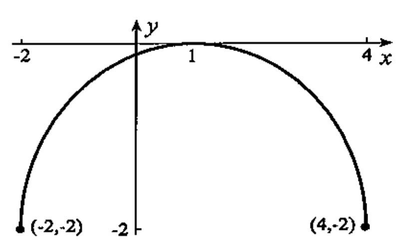
6. Since $x + y = 2t$ and $x - y = 2/t$, multiplication gives $x^2 - y^2 = 4$, a hyperbola.



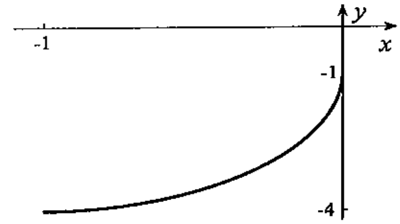
7. Since $\dfrac{x^2}{4} + \dfrac{y^2}{16} = \cos^2 t + \sin^2 t = 1$, this is an ellipse.



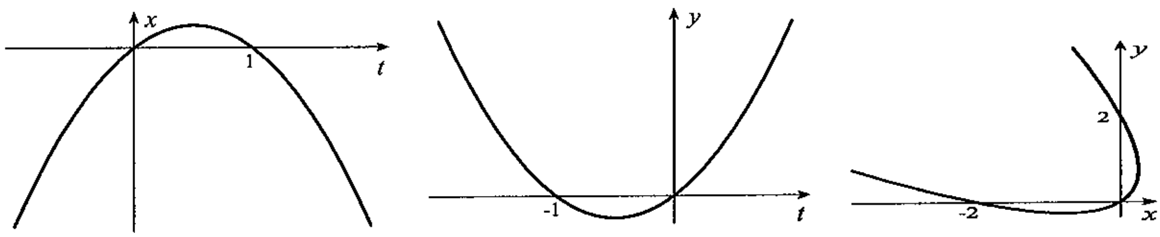
8. Since $\dfrac{(x-1)^2}{9} + \dfrac{(y+2)^2}{4} = \cos^2 t + \sin^2 t = 1$, points lie on an ellipse with centre $(1, -2)$. Values $0 \le t \le \pi$ yield only the top half of the ellipse.



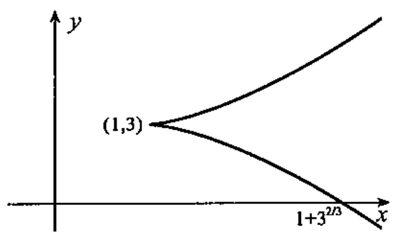
9. Since $(x+1)^2 + \dfrac{(y+1)^2}{9} = \sin^2 t + \cos^2 t = 1$, points lie on an ellipse with centre $(-1,-1)$. Values $0 \le t \le \pi/2$ yield only one quarter of the ellipse.



10. Sketches of x and y against t in the left and middle figures below yield the curve to the right.

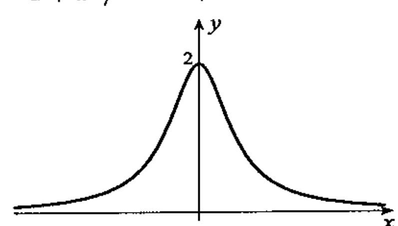


11. When we eliminate the parameter, $x = (y-3)^{2/3} + 1$.



12. When we eliminate the parameter,

$$y = 2\sin^2\theta = \frac{2}{\csc^2\theta} = \frac{2}{1+\cot^2\theta} = \frac{2}{1+x^2/4} = \frac{8}{4+x^2}.$$



13. $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2+3}$$

14. $$\frac{dy}{dx} = \frac{dy/du}{dx/du} = \frac{\dfrac{(u^2-1)(2u) - u^2(2u)}{(u^2-1)^2}}{\dfrac{(u-1)(1) - u(1)}{(u-1)^2}} = \frac{-2u}{(u^2-1)^2}\,\frac{(u-1)^2}{-1} = \frac{2u}{(u+1)^2}$$

15. $$\frac{dy}{dx} = \frac{dy/dv}{dx/dv} = \frac{6v^2}{3v^2\sqrt{v-1} + \dfrac{v^3+2}{2\sqrt{v-1}}} = \frac{12v^2\sqrt{v-1}}{7v^3 - 6v^2 + 2}$$

16. Since $y = 1/x$, it follows that $dy/dx = -1/x^2$.

17. $$\frac{dy}{dx} = \frac{dy/ds}{dx/ds} = \frac{2s+2}{(3/2)\sqrt{s} - (2/3)s^{-1/3}} = \frac{12s^{1/3}(s+1)}{9s^{5/6} - 4}$$

18. $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dfrac{(t+6)(1) - t(1)}{(t+6)^2}}{4(2t+3)^3(2)} = \frac{6}{(t+6)^2 8(2t+3)^3} = \frac{3}{4(t+6)^2(2t+3)^3}$$

19. Since $\dfrac{1+u}{1-u} = x^3$, it follows that $y = 1/x^{12}$. Hence $dy/dx = -12/x^{13}$.

20. $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dfrac{-(2t+2)}{(t^2+2t-5)^2}}{\dfrac{-2t+3}{2\sqrt{-t^2+3t+5}}} = \frac{4(t+1)\sqrt{-t^2+3t+5}}{(2t-3)(t^2+2t-5)^2}$$

21. Since $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1+1/t^2}{1-1/t^2} = \dfrac{t^2+1}{t^2-1}$, the slope of the tangent line at the point $(17/4, 15/4)$, corresponding to $t = 4$ is $17/15$. Equations for the tangent and normal lines are $y - 15/4 = (17/15)(x - 17/4)$ and $y - 15/4 = -(15/17)(x - 17/4)$. These simplify to $17x - 15y = 16$ and $30x + 34y = 255$.

22. The slope will be one when $1 = \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t+1}{t^2-3}$, or, $t^2 - 3 = 3t + 1$. Thus, $0 = t^2 - 3t - 4 = (t-4)(t+1) \Longrightarrow t = -1, 4$. The required points are $(8/3, 1/2)$ and $(28/3, 28)$.

23. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t+1/t^2}{2t-1/t^2} = \dfrac{2t^3+1}{2t^3-1}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{2t^3+1}{2t^3-1}\right) = \frac{d}{dt}\left(\frac{2t^3+1}{2t^3-1}\right)\frac{dt}{dx} = \frac{\dfrac{d}{dt}\left(\dfrac{2t^3+1}{2t^3-1}\right)}{dx/dt} = \frac{\dfrac{(2t^3-1)(6t^2)-(2t^3+1)(6t^2)}{(2t^3-1)^2}}{2t-1/t^2} = \frac{-12t^4}{(2t^3-1)^3}.$$

24. Since $x^2 = t - 1$ and $y^2 = t + 1$, it follows that $y^2 - x^2 = 2$. Thus, $2y\dfrac{dy}{dx} - 2x = 0 \Longrightarrow \dfrac{dy}{dx} = \dfrac{x}{y}$.
Consequently, $\dfrac{d^2y}{dx^2} = \dfrac{y(1) - x(dy/dx)}{y^2} = \dfrac{y - x(x/y)}{y^2} = \dfrac{y^2 - x^2}{y^3} = \dfrac{2}{y^3}$.

25. Since $y = 7 - 14(x-5)/2 = 42 - 7x$, it follows that $dy/dx = -7$ and $d^2y/dx^2 = 0$.

26. Since $\dfrac{dy}{dx} = \dfrac{dy/dv}{dx/dv} = \dfrac{2}{2v+2} = \dfrac{1}{v+1}$, we obtain

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{v+1}\right) = \frac{d}{dv}\left(\frac{1}{v+1}\right)\frac{dv}{dx} = \frac{\dfrac{d}{dv}\left(\dfrac{1}{v+1}\right)}{dx/dv} = \frac{\dfrac{-1}{(v+1)^2}}{2v+2} = \frac{-1}{2(v+1)^3}.$$

27. Yes The parabola $y = 2x^2 - 1$ is defined for all x, whereas the parametric equations define only those points on the parabola with x-coordinates in the interval $-1 \le x \le 1$.

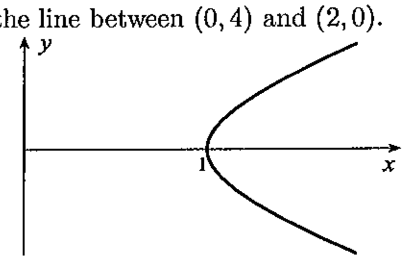
28. Since $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = \cos^2\theta + \sin^2\theta = 1$, the equations describe an ellipse. Values of the parameter yield all points on the ellipse exactly once.

29. Based on the equation $(x-h)^2 + (y-k)^2 = r^2$, parametric equations are $x = h + r\cos\theta$, $y = k + r\sin\theta$. Values $0 \le \theta < 2\pi$ traverse the circle once.

30. By solving each equation for t and equating results $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} \Longrightarrow y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$. This is the equation of the line through P_1 and P_2. By permitting t to take on all possible values, all points on the line are obtained.

31. When we add the second equation to twice the first, $2x + y = 4\sin^2 t + 4\cos^2 t = 4$, thus indicating that points lie on the straight line. Values of t only yield points on the line between $(0, 4)$ and $(2, 0)$.

32. (a) Since $x^2 - y^2 = \sec^2\theta - \tan^2\theta = 1$, points lie on the hyperbola. Values $-\pi/2 < \theta < \pi/2$ yield only the right half of the hyperbola.
(b) Since $x^2 - y^2 = \cosh^2\phi - \sinh^2\phi = 1$, points are again on the hyperbola. Because $\cosh\phi$ is always positive, only the right half of the hyperbola is defined by the parametric equations.



(c) Since $x^2 - y^2 = \dfrac{1}{4}\left(t + \dfrac{1}{t}\right)^2 - \dfrac{1}{4}\left(t - \dfrac{1}{t}\right)^2 = 1$, points are once again on the hyperbola. With $t > 0$, so also is x, and therefore only the right half of the hyperbola is defined by the parametric equations.

33. If we set $x = t$, then $y = \dfrac{t+1}{t-2}$.

34. If we set $y = t$, then $x = \dfrac{5y^2 - y^3}{1+y} = \dfrac{5t^2 - t^3}{1+t}$.

35. Written in the form $(x+1)^2+(y-2)^2=5$, the curve is a circle with centre $(-1,2)$ and radius $\sqrt{5}$. According to Exercise 29, parametric equations are $x=-1+\sqrt{5}\cos t$, $y=2+\sqrt{5}\sin t$, $0\le t<2\pi$.

36. Using Exercise 32, parametric equation for the hyperbola $\dfrac{x^2}{4}-\dfrac{y^2}{2}=1$ are $x=2\sec\theta$, $y=\sqrt{2}\tan\theta$. To get both halves of the hyperbola, we use $-\pi\le\theta\le\pi$, but do not consider $\theta=\pm\pi/2$.

37. The distance D between the particles at any time is given by

$$D^2=(1-t-4t+5)^2+(t-2t+1)^2=(6-5t)^2+(1-t)^2=26t^2-62t+37,\quad t\ge 0.$$

For critical points of D^2, we solve $0=\dfrac{dD^2}{dt}=52t-62$, which implies $t=31/26$. Since

$$D^2(0)=37,\qquad D^2(31/26)=0.04,\qquad \lim_{t\to\infty}D^2=\infty,$$

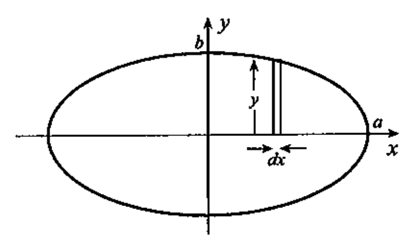
the particles are closest together at $t=31/26$.

38. If we differentiate $\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}$ with respect to x, we obtain

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy/dt}{dx/dt}\right)=\frac{d}{dt}\left(\frac{dy/dt}{dx/dt}\right)\frac{dt}{dx}=\frac{\dfrac{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2}-\dfrac{dy}{dt}\dfrac{d^2x}{dt^2}}{(dx/dt)^2}}{dx/dt}=\frac{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2}-\dfrac{dy}{dt}\dfrac{d^2x}{dt^2}}{(dx/dt)^3}.$$

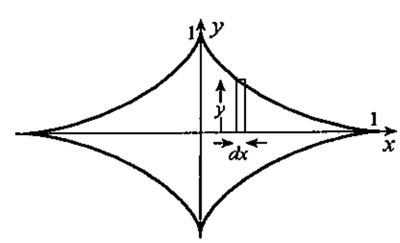
39. The area is four times that in the first quadrant (figure to the right),

$$\begin{aligned}A&=4\int_0^a y\,dx=4\int_{\pi/2}^0 b\sin t(-a\sin t\,dt)\\&=4ab\int_0^{\pi/2}\sin^2 t\,dt=4ab\int_0^{\pi/2}\left(\frac{1-\cos 2t}{2}\right)dt\\&=2ab\left\{t-\frac{1}{2}\sin 2t\right\}_0^{\pi/2}=\pi ab.\end{aligned}$$

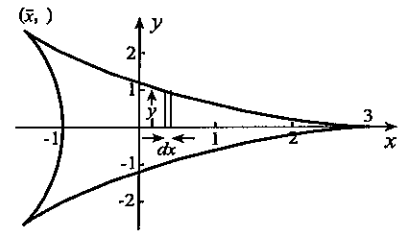


40. The area is four times that in the first quadrant (figure to the right),

$$\begin{aligned}A&=4\int_0^1 y\,dx=4\int_{\pi/2}^0 \sin^3 t(-3\cos^2 t\sin t\,dt)\\&=12\int_0^{\pi/2}\sin^2 t(\sin t\cos t)^2\,dt\\&=12\int_0^{\pi/2}\left(\frac{1-\cos 2t}{2}\right)\left(\frac{\sin 2t}{2}\right)^2dt\\&=\frac{3}{2}\int_0^{\pi/2}(\sin^2 2t-\sin^2 2t\cos 2t)\,dt=\frac{3}{2}\int_0^{\pi/2}\left(\frac{1-\cos 4t}{2}-\sin^2 2t\cos 2t\right)dt\\&=\frac{3}{2}\left\{\frac{t}{2}-\frac{1}{8}\sin 4t-\frac{1}{6}\sin^3 2t\right\}_0^{\pi/2}=\frac{3\pi}{8}.\end{aligned}$$

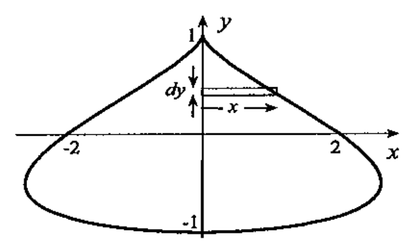


41. The area is twice that above the x-axis (figure to the right). We set up two integrals for this area. If $\overline{t}$ is the value of t giving the point with x-coordinate $\overline{x}$ shown, then



$$\begin{aligned}
A &= 2\int_{\overline{x}}^{3} y\,dx - 2\int_{\overline{x}}^{-1} y\,dx \\
&= 2\int_{\overline{x}}^{3} y\,dx + 2\int_{-1}^{\overline{x}} y\,dx \\
&= 2\int_{\overline{t}}^{0} (2\sin t - \sin 2t)(-2\sin t - 2\sin 2t)\,dt + 2\int_{\pi}^{\overline{t}} (2\sin t - \sin 2t)(-2\sin t - 2\sin 2t)\,dt \\
&= 2\int_{\pi}^{0} (2\sin t - \sin 2t)(-2\sin t - 2\sin 2t)\,dt \\
&= 4\int_{0}^{\pi} (2\sin^2 t - \sin^2 2t + \sin t \sin 2t)\,dt = 4\int_0^{\pi}\left(1 - \cos 2t - \frac{1-\cos 4t}{2} + 2\sin^2 t\cos t\right) dt \\
&= 4\left\{\frac{t}{2} - \frac{1}{2}\sin 2t + \frac{1}{8}\sin 4t + \frac{2}{3}\sin^3 t\right\}_0^{\pi} = 2\pi.
\end{aligned}$$

42. The area is twice that to the right of the y-axis (figure to the right),



$$\begin{aligned}
A &= 2\int_{-1}^{1} x\,dy = 2\int_{-\pi/2}^{\pi/2} (2\cos t - \sin 2t)(\cos t\,dt) \\
&= 2\int_{-\pi/2}^{\pi/2} (1 + \cos 2t - 2\cos^2 t \sin t)\,dt \\
&= 2\left\{t + \frac{1}{2}\sin 2t + \frac{2}{3}\cos^3 t\right\}_{-\pi/2}^{\pi/2} = 2\pi.
\end{aligned}$$

43. We revolve that part of the area in the first quadrant and double the result,

$$\begin{aligned}
V &= 2\int_0^a \pi y^2\,dx = 2\pi\int_{\pi/2}^{0} b^2\sin^2 2t(-a\sin t\,dt) = 2\pi ab^2\int_0^{\pi/2} (2\sin t\cos t)^2 \sin t\,dt \\
&= 8\pi ab^2\int_0^{\pi/2} \sin^3 t\cos^2 t\,dt = 8\pi ab^2\int_0^{\pi/2} (1-\cos^2 t)\cos^2 t \sin t\,dt \\
&= 8\pi ab^2\left\{-\frac{1}{3}\cos^3 t + \frac{1}{5}\cos^5 t\right\}_0^{\pi/2} = \frac{16\pi ab^2}{15}.
\end{aligned}$$

44. The volume is

$$\begin{aligned}
V &= \int_0^{2\pi R} \pi y^2\,dx = \pi\int_0^{2\pi} R^2(1-\cos\theta)^2(R)(1-\cos\theta)\,d\theta \\
&= \pi R^3\int_0^{2\pi} (1 - 3\cos\theta + 3\cos^2\theta - \cos^3\theta)\,d\theta \\
&= \pi R^3\int_0^{2\pi}\left[1 - 3\cos\theta + \frac{3}{2}(1+\cos 2\theta) - \cos\theta(1-\sin^2\theta)\right] d\theta \\
&= \pi R^3\left\{\frac{5\theta}{2} - 4\sin\theta + \frac{3}{4}\sin 2\theta + \frac{1}{3}\sin^3\theta\right\}_0^{2\pi} = 5\pi^2 R^3.
\end{aligned}$$

45. Since these equations define the circle $(x-3)^2 + (y+2)^2 = 16$, its length is $2\pi(4) = 8\pi$.

46. Since small lengths along the curve are given by

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\sqrt{(e^{-t}\cos t-e^{-t}\sin t)^2+(-e^{-t}\sin t-e^{-t}\cos t)^2}=\sqrt{2}\,e^{-t}\,dt,$$

the length of the curve is $\displaystyle\int_0^1\sqrt{2}\,e^{-t}\,dt=\sqrt{2}\left\{-e^{-t}\right\}_0^1=\sqrt{2}(1-e^{-1}).$

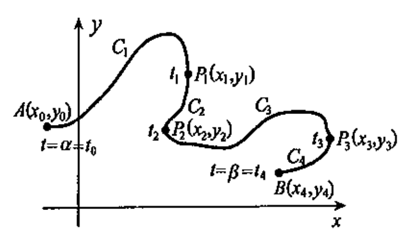
47. By equation 9.3, $L=\displaystyle\int_1^2\sqrt{(1+1/t)^2+(1-1/t)^2}\,dt=\sqrt{2}\int_1^2\frac{\sqrt{1+t^2}}{t}dt.$ Suppose we set $t=\tan\theta$ and $dt=\sec^2\theta\,d\theta$. With $\tilde{\theta}=\mathrm{Tan}^{-1}2$,

$$\begin{aligned}L&=\sqrt{2}\int_{\pi/4}^{\tilde{\theta}}\frac{\sec\theta}{\tan\theta}\sec^2\theta\,d\theta=\sqrt{2}\int_{\pi/4}^{\tilde{\theta}}\frac{\sec\theta}{\tan\theta}(1+\tan^2\theta)\,d\theta=\sqrt{2}\int_{\pi/4}^{\tilde{\theta}}(\csc\theta+\sec\theta\,\tan\theta)\,d\theta\\&=\sqrt{2}\left\{\ln|\csc\theta-\cot\theta|+\sec\theta\right\}_{\pi/4}^{\tilde{\theta}}=1.73.\end{aligned}$$

48. Quadrupling the first quadrant length, we get

$$4\int_0^{\pi/2}\sqrt{(-a\,\sin\theta)^2+(b\,\cos\theta)^2}\,d\theta=4\int_0^{\pi/2}\sqrt{a^2\,\sin^2\theta+b^2\,\cos^2\theta}\,d\theta.$$

49. To verify 9.3 we first note that any curve defined parametrically by 9.1 can be divided into subcurves in such a way that each subcurve represents a function $y=f(x)$. Suppose we denote these subcurves by $C_1,C_2,\cdots,C_n$ and let P_i be the point joining C_i and C_{i+1}. According to equation 7.15, the length of C_i is given by



$$L_i=\int_{x_{i-1}}^{x_i}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

when $x_i>x_{i-1}$, and by

$$L_i=\int_{x_i}^{x_{i-1}}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

when $x_{i-1}>x_i$. Let t_i be the value of t yielding P_i. If we set $x=x(t)$, $y=y(t)$ in these integrals and use 10.2, the first integral becomes

$$L_i=\int_{t_{i-1}}^{t_i}\sqrt{1+\left(\frac{dy/dt}{dx/dt}\right)^2}\,dx=\int_{t_{i-1}}^{t_i}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,\frac{dt}{dx}dx=\int_{t_{i-1}}^{t_i}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt.$$

The second integral for L_i leads to the same integral in t when $x_{i-1}>x_i$. The total length of the curve is therefore

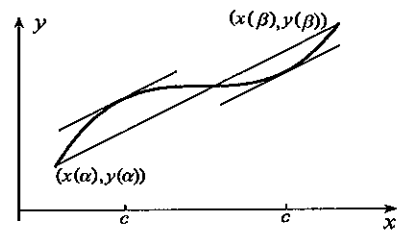
$$L=\sum_{i=1}^n L_i=\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_\alpha^\beta\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt.$$

50. (a) If we set $u=t-1$, then $x=(u+1)^2+2(u+1)-1=u^2+4u+2$, and $y=(u+1)+5=u+6$, define the same curve where $0\le u\le 3$.

(b) If we set $v=u/3$, then $x=(3v)^2+4(3v)+2=9v^2+12v+2$, and $y=3v+6$ define the same curve where $0\le v\le 1$.

51. Theorem 3.18 with $f(x)$ replaced by $y(t)$ and $g(x)$ by $x(t)$ on $\alpha \le t \le \beta$ states that

$$\frac{y(\beta)-y(\alpha)}{x(\beta)-x(\alpha)} = \frac{y'(c)}{x'(c)}.$$

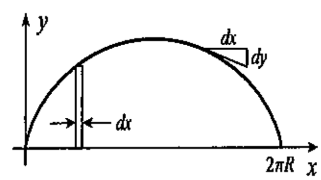


If we interpret $x = x(t)$, $y = y(t)$, $\alpha \le t \le \beta$ as parametric equations for a curve, then $[y(\beta)-y(\alpha)]/[x(\beta)-x(\alpha)]$ is the slope of the line joining the end points of the curve. The ratio $y'(c)/x'(c)$ is the slope of the tangent line to the curve at the point corresponding to $t = c$. Thus, the theorem states that there is at least one point on the curve at which the tangent line is parallel to the line joining the end points of the curve.

52. Parametric equations for the circle are $x = 2\cos\theta$, $y = 2\sin\theta$. If the particle makes two revolutions each sescond, then $\theta = 4\pi t$, where $t \ge 0$. Consequently, parametric equations for the position are $x = 2\cos 4\pi t$, $y = 2\sin 4\pi t$, $t \ge 0$.

53. (a) If we use vertical rectangles, the area is

$$\begin{aligned} A &= \int_0^{2\pi R} y\,dx = \int_0^{2\pi} R(1-\cos\theta)R(1-\cos\theta)\,d\theta \\ &= R^2\int_0^{2\pi}\left[1 - 2\cos\theta + \left(\frac{1+\cos 2\theta}{2}\right)\right]d\theta \\ &= R^2\left\{\frac{3\theta}{2} - 2\sin\theta + \frac{1}{4}\sin 2\theta\right\}_0^{2\pi} = 3\pi R^2. \end{aligned}$$



(b) Small lengths along the cycloid are approximated by

$$\sqrt{(dx)^2+(dy)^2} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta = \sqrt{(R - R\cos\theta)^2 + (R\sin\theta)^2}\,d\theta = R\sqrt{2-2\cos\theta}\,d\theta.$$

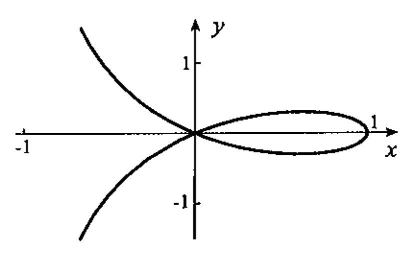
The length of one arch is therefore

$$\begin{aligned} L &= \int_0^{2\pi} R\sqrt{2-2\cos\theta}\,d\theta = \sqrt{2}R\int_0^{2\pi}\sqrt{1-[1-2\sin^2(\theta/2)]}\,d\theta \\ &= \sqrt{2}R\int_0^{2\pi}\sqrt{2}\sin\left(\frac{\theta}{2}\right)d\theta = 2R\left\{-2\cos\left(\frac{\theta}{2}\right)\right\}_0^{2\pi} = 8R. \end{aligned}$$

It represents the distance travelled by the stone as the tire makes one revolution.

54. The curve is shown to the right. Points at which the tangent line is horizontal can be found by solving

$$\begin{aligned} 0 = \frac{dy}{dx} &= \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} \\ &= \frac{\dfrac{(1+t^2)(1-3t^2)-(t-t^3)(2t)}{(1+t^2)^2}}{\dfrac{(1+t^2)(-2t)-(1-t^2)(2t)}{(1+t^2)^2}} \\ &= \frac{1-4t^2-t^4}{-4t}. \end{aligned}$$



But this implies that $t^4 + 4t^2 - 1 = 0$, a quadratic equation in t^2,

$$t^2 = \frac{-4 \pm \sqrt{16+4}}{2} = -2 \pm \sqrt{5}.$$

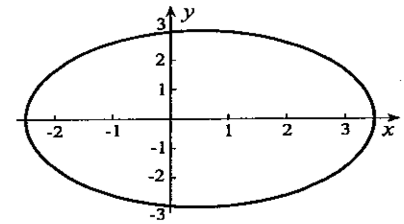
Since t^2 must be nonnegative, it follows that $t^2 = \sqrt{5}-2$, and therefore $t = \pm\sqrt{\sqrt{5}-2}$. For these values of t, $x = 0.62$ and $y = \pm 0.30$. Points with a horizontal tangent line are therefore $(0.62, \pm 0.30)$.

55. When $l_1 = 1$, $l_2 = 3$, and $d = 1/2$, parametric equations are

$$x_c = \frac{1}{2} - \frac{3(1/2 - \cos\theta)}{\sqrt{1 + 1/4 - 2(1/2)\cos\theta}} = \frac{1}{2} - \frac{3(1 - 2\cos\theta)}{\sqrt{5 - 4\cos\theta}},$$

$$y_c = \frac{1(3)\sin\theta}{\sqrt{1 + 1/4 - 2(1/2)\cos\theta}} = \frac{6\sin\theta}{\sqrt{5 - 4\cos\theta}}.$$

The plot is shown to the right.



56. (a) Since the length of CE is l_3,

$$(x_E - x_c)^2 + (y_E - y_c)^2 = l_3^2.$$

Substituting for x_c, and y_c gives

$$\left[x_E - d + \frac{l_2(d - l_1\cos\theta)}{\sqrt{d^2 + l_1^2 - 2dl_1\cos\theta}}\right]^2 + \left[y_E - \frac{l_1 l_2 \sin\theta}{\sqrt{d^2 + l_1^2 - 2dl_1\cos\theta}}\right]^2 = l_3^2.$$

Hence,

$$x_E = d - \frac{l_2(d - l_1\cos\theta)}{\sqrt{d^2 + l_1^2 - 2dl_1\cos\theta}} \pm \sqrt{l_3^2 - \left[y_E - \frac{l_1 l_2 \sin\theta}{\sqrt{d^2 + l_1^2 - 2dl_1\cos\theta}}\right]^2}.$$

Since x_E must be greater than x_c, we choose

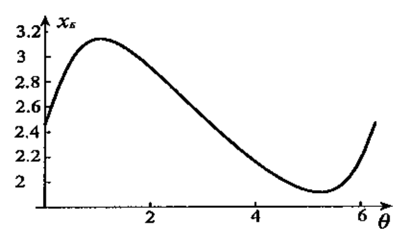
$$x_E = d - \frac{l_2(d - l_1\cos\theta)}{\sqrt{d^2 + l_1^2 - 2dl_1\cos\theta}} + \sqrt{l_3^2 - \left[y_E - \frac{l_1 l_2 \sin\theta}{\sqrt{d^2 + l_1^2 - 2dl_1\cos\theta}}\right]^2}.$$

(b) When $l_1 = 1/2$, $l_2 = 2$, $l_3 = 4$, $d = 1$, and $y_E = 2$,

$$x_E = 1 - \frac{2[1 - (1/2)\cos\theta]}{\sqrt{1 + 1/4 - 2(1/2)\cos\theta}} + \sqrt{16 - \left[2 - \frac{(1/2)(2)\sin\theta}{\sqrt{1 + 1/4 - 2(1/2)\cos\theta}}\right]^2}$$

$$= 1 - \frac{2(2 - \cos\theta)}{\sqrt{5 - 4\cos\theta}} + \sqrt{16 - \left[2 - \frac{2\sin\theta}{\sqrt{5 - 4\cos\theta}}\right]^2}.$$

A plot is shown below. The estimated stroke is 1.2 m.



57. (a) Since A rotates at 60 rpm, or $\dfrac{60(2\pi)}{60} = 2\pi$ radians per second, coordinates of A are $(\cos 2\pi t, \sin 2\pi t)$. The x-coordinate of B when A starts in the first quadrant (left figure below) is

$$x = \|DC\| + \|CB\| = \cos 2\pi t + \sqrt{16 - (3 + \sin 2\pi t)^2}, \quad 0 \le t \le 1/4.$$

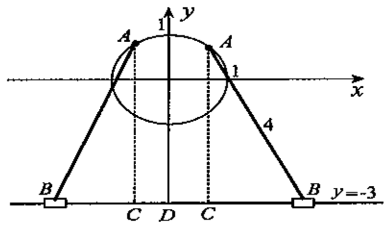
When A now moves through a complete revolution, B will be in the third quadrant, and its x-coordinate is given by

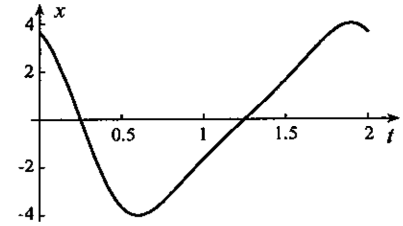
$$x = -\|DC\| - \|CB\| = \cos 2\pi t - \sqrt{16 - (3 + \sin 2\pi t)^2}, \quad 1/4 < t \le 5/4.$$

During the next three-quarters of a revolution of A, B passes into and remains in the fourth quadrant. Its x-coordinate is once again given by

$$x = \cos 2\pi t + \sqrt{16 - (3 + \sin 2\pi t)^2}, \quad 5/4 < t \leq 2.$$

A plot is shown in the right figure below.





(b) The graph suggest that maximum and minimum values for x are $x = \pm 4$. To confirm the positive value, we determine when velocity is zero for that part of the function in the interval $5/4 < t < 2$,

$$0 = \frac{dx}{dt} = -2\pi \sin 2\pi t - \frac{2(2\pi)\cos 2\pi t(3 + \sin 2\pi t)}{2\sqrt{16 - (3 + \sin 2\pi t)^2}}.$$

This implies that

$$\sqrt{16 - (3 + \sin 2\pi t)^2} = -\cot 2\pi t\,(3 + \sin 2\pi t).$$

Squaring gives

$$16 - (3 + \sin 2\pi t)^2 = \cot^2 2\pi t\,(3 + \sin 2\pi t)^2 \quad \Longrightarrow \quad 16 = (3 + \sin 2\pi t)^2 \csc^2 2\pi t.$$

Square roots now yield

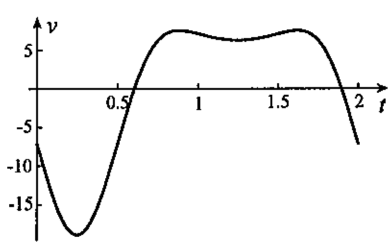
$$\pm 4 = \csc 2\pi t\,(3 + \sin 2\pi t) = 3\csc 2\pi t + 1.$$

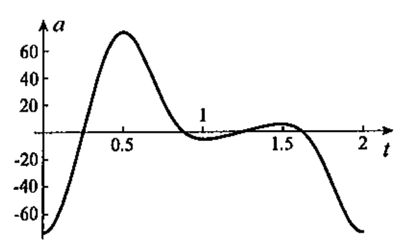
Thus,

$$3\csc 2\pi t = 3,\ -5 \quad \Longrightarrow \quad \sin 2\pi t = 1,\ -\frac{3}{5}.$$

Since dx/dt is undefined when $\sin 2\pi t = 1$, the required critical point satisfies $\sin 2\pi t = -3/5$. Instead of solving this for t, notice that the graph makes it clear that $1.8 < t < 2 \Longrightarrow 1.8(2\pi) < 2\pi t < 2(2\pi) \Longrightarrow 7\pi/2 < 2\pi t < 4\pi$. It follows that $2\pi t$ is an angle in the fourth quadrant, and therefore $\cos 2\pi t = 4/5$. Consequently, the value of x, when the velocity is zero is $x = 4/5 + \sqrt{16 - (3 - 3/5)^2} = 4$.

(c) We analyze the velocity function at $t = 1/4$ as typical. Because $v(t)$ is undefined at this value of t, velocity is not continuous at $t = 1/4$. The plot of $v(t)$ in the left figure below does not show the discontinuity, but it does indicate that the limit of $v(t)$ exists as $t \to 1/4$, and therefore the discontinuity is removable.



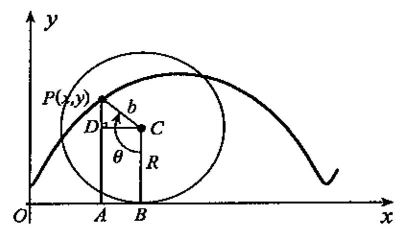


(d) The graph of the velocity function suggests that maximum speed is approximately 19 cm/s.

(e) The graph of the acceleration function in the right figure above suggests that the maximum value of $|a(t)|$ is approximately 73 cm/s^2.

58. From the figure to the right,

$$\begin{aligned} x &= \|OB\| - \|AB\| = R\theta - \|CD\| \\ &= R\theta - b\cos(\theta - \pi/2) = R\theta - b\sin\theta, \\ y &= \|AD\| + \|DP\| = R + b\sin(\theta - \pi/2) \\ &= R - b\cos\theta. \end{aligned}$$

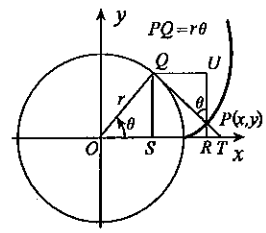


59. From the figure,

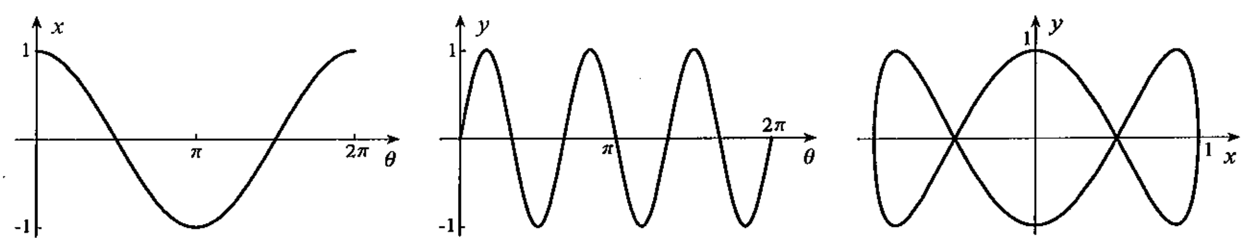
$$\begin{aligned} x &= \|OS\| + \|SR\| \\ &= r\cos\theta + \|QU\| \\ &= r\cos\theta + r\theta\sin\theta, \end{aligned}$$

and

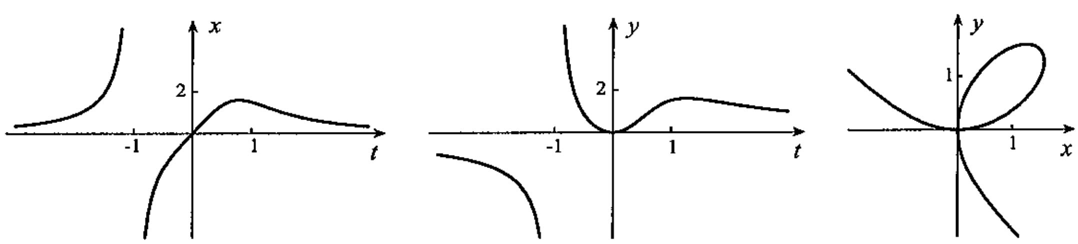
$$\begin{aligned} y &= \|QS\| - \|PU\| \\ &= r\sin\theta - r\theta\cos\theta. \end{aligned}$$



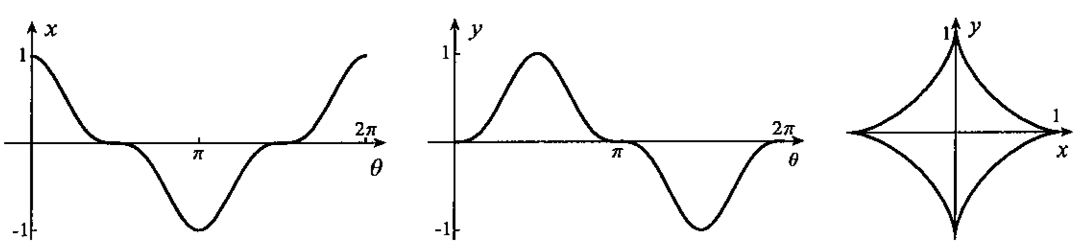
60. The graphs of x and y as functions of θ in the left and middle figures lead to the curve in the right figure.



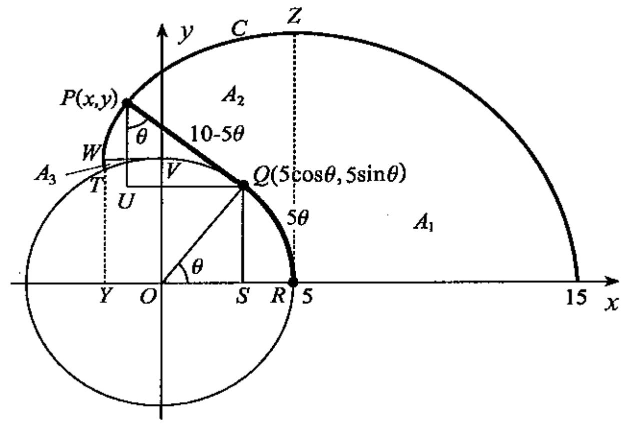
61. The graphs of x and y as functions of t in the left and middle figures lead to the curve in the right figure.



62. The graphs of x and y as functions of θ in the left and middle figures lead to the curve in the right figure.



63. We double the grazing area above the x-axis. We divide this area into three parts: A_1 to the right of the line $x = 5$; A_2 under the curve C, to the left of $x = 5$, and above the horizontal straight line segment WV and also above the quarter circle VR ; and A_3 to the right of the curve C, under the straight line segment WV and above that part $TV$of the circle (figure below).



To find A_2 and A_3, we require the equation for the curve C followed by the end of the rope; we will find parametric equations for C in terms of the angle θ between the x-axis and the final point of contact of the rope when the end of the rope is at position $P(x, y)$. The diagram shows that $x = \|OS\| - \|UQ\| = 5\cos\theta - (10 - 5\theta)\sin\theta$, and $y = \|SQ\| + \|UP\| = 5\sin\theta + (10 - 5\theta)\cos\theta$. These are parametric equations for the end of the rope. The initial value of θ is 0 and the final value occurs when $5\theta - 10 = 0 \Longrightarrow \theta = 2$. Clearly, A_1 is the area of one-quarter of a circle with radius 10, $A_1 = (1/4)\pi(10)^2 = 25\pi$ m^2. We calculate area A_2 by finding the area under C and above the x-axis from W to R and subtracting from this the sum of the areas of quarter circle ORV and rectangle $OVWY$.

$$\begin{aligned}
A_2 &= \int_{\theta=\pi/2}^{\theta=0} y\,dx - \frac{1}{4}\pi(5)^2 - 5\left(10 - \frac{5\pi}{2}\right)\\
&= \int_{\pi/2}^{0} [5\sin\theta + (10 - 5\theta)\cos\theta][-5\sin\theta + 5\sin\theta - (10 - 5\theta)\cos\theta]\,d\theta + \frac{25\pi}{4} - 50\\
&= 25\int_0^{\pi/2} [(2-\theta)\sin\theta\cos\theta + (2-\theta)^2\cos^2\theta]\,d\theta + \frac{25\pi}{4} - 50\\
&= \frac{25}{2}\int_0^{\pi/2} [(2-\theta)\sin 2\theta + (2-\theta)^2(1 + \cos 2\theta)]\,d\theta + \frac{25\pi}{4} - 50.
\end{aligned}$$

Because we need the antiderivative of this integrand again later, we digress to develop it. Integration by parts on the second term with $u = (2-\theta)^2$, $dv = \cos 2\theta\,d\theta$, $du = -2(2-\theta)\,d\theta$, and $v = (1/2)\sin 2\theta$, gives

$$\begin{aligned}
I &= \int [(2-\theta)\sin 2\theta + (2-\theta)^2(1+\cos 2\theta)]\,d\theta\\
&= -\frac{1}{3}(2-\theta)^3 + \frac{1}{2}(2-\theta)^2\sin 2\theta + \int [(2-\theta)\sin 2\theta + (2-\theta)\sin 2\theta]\,d\theta\\
&= -\frac{1}{3}(2-\theta)^3 + \frac{1}{2}(2-\theta)^2\sin 2\theta + 2\int (2-\theta)\sin 2\theta\,d\theta.
\end{aligned}$$

Integration by parts with $u = 2 - \theta$, $dv = \sin 2\theta\,d\theta$, $du = -d\theta$, and $v = -(1/2)\cos 2\theta$, now gives

$$\begin{aligned}
I &= -\frac{1}{3}(2-\theta)^3 + \frac{1}{2}(2-\theta)^2\sin 2\theta - (2-\theta)\cos 2\theta - \int \cos 2\theta\,d\theta\\
&= -\frac{1}{3}(2-\theta)^3 + \frac{1}{2}(2-\theta)^2\sin 2\theta - (2-\theta)\cos 2\theta - \frac{1}{2}\sin 2\theta + C.
\end{aligned}$$

Using this antiderivative,

$$A_2 = \frac{25}{2}\left\{-\frac{1}{3}(2-\theta)^3 + \frac{1}{2}(2-\theta)^2\sin 2\theta - (2-\theta)\cos 2\theta - \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} + \frac{25\pi}{4} - 50 = 33.0350 \text{ m}^2.$$

Area A_3 must be calculated with two integrals:

$$\begin{aligned}A_3 &= \int_{\theta=\pi/2}^{\theta=2}(5-y)\,dx + \int_{5\cos 2}^{0}(5-y)\,dx\\ &= \int_{\pi/2}^{2}[5-5\sin\theta-(10-5\theta)\cos\theta][-5\sin\theta+5\sin\theta-(10-5\theta)\cos\theta]\,d\theta + \int_{5\cos 2}^{0}(5-\sqrt{25-x^2})\,dx\\ &= 25\int_{\pi/2}^{2}[-(2-\theta)\sin\theta\cos\theta-(2-\theta)^2\cos^2\theta+(2-\theta)\cos\theta]\,d\theta + \int_{5\cos 2}^{0}(5-\sqrt{25-x^2})\,dx\\ &= \frac{25}{2}\int_{\pi/2}^{2}[-(2-\theta)\sin 2\theta-(2-\theta)^2(1+\cos 2\theta)+2(2-\theta)\cos\theta]\,d\theta + \int_{5\cos 2}^{0}(5-\sqrt{25-x^2})\,dx.\end{aligned}$$

The first two terms in the first integral were integrated above. For the integral of $(2-\theta)\cos\theta$, we use integration by parts with $u = 2-\theta$, $dv = \cos\theta\,d\theta$, $du = -d\theta$, and$v = \sin\theta$,

$$\int(2-\theta)\cos\theta\,d\theta = (2-\theta)\sin\theta - \int -\sin\theta\,d\theta = (2-\theta)\sin\theta - \cos\theta + C.$$

For the antiderivative of $\sqrt{25-x^2}$, we use the trigonometric substitution $x = 5\sin\phi$ and $dx = 5\cos\phi\,d\phi$,

$$\begin{aligned}\int\sqrt{25-x^2}\,dx &= \int 5\cos\phi\,5\cos\phi\,d\phi = \frac{25}{2}\int(1+\cos 2\phi)\,d\phi\\ &= \frac{25}{2}\left(\phi+\frac{1}{2}\sin 2\phi\right)+C = \frac{25}{2}(\phi+\sin\phi\cos\phi)+C\\ &= \frac{25}{2}\text{Sin}^{-1}(x/5) + \frac{25}{2}\left(\frac{x}{5}\right)\sqrt{1-x^2/25}+C = \frac{25}{2}\text{Sin}^{-1}(x/5)+\frac{x}{2}\sqrt{25-x^2}.\end{aligned}$$

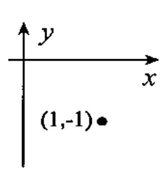
When we put these all together,

$$\begin{aligned}A_3 = &= \frac{25}{2}\left\{\frac{1}{3}(2-\theta)^3 - \frac{1}{2}(2-\theta)^2\sin 2\theta + (2-\theta)\cos 2\theta + \frac{1}{2}\sin 2\theta + 2(2-\theta)\sin\theta - 2\cos\theta\right\}_{\pi/2}^{2}\\ &\quad + \left\{5x - \frac{25}{2}\text{Sin}^{-1}(x/5) - \frac{x}{2}\sqrt{25-x^2}\right\}_{5\cos 2}^{0}\\ &= 0.2878 \text{ m}^2.\end{aligned}$$

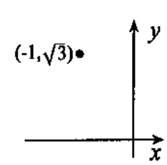
Finally then the grazing area is $2(A_1+A_2+A_3) = 2(25\pi+33.0350+0.2878) = 223.7 \text{ m}^2$.

EXERCISES 9.2

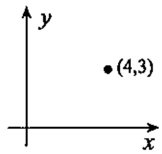
1. $r = \sqrt{1^2+(-1)^2} = \sqrt{2}$. Angles satisfying $\tan\theta = -1$ are $-\pi/4+n\pi$. Since the point is in the fourth quadrant, polar coordinates are $(\sqrt{2}, 2n\pi-\pi/4)$.



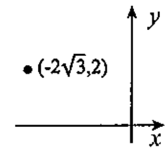
2. $r = \sqrt{(-1)^2+(\sqrt{3})^2} = 2$. Angles satisfying $\tan\theta = -\sqrt{3}$ are $-\pi/3+n\pi$. Since the point is in the second quadrant, polar coordinates are $(2, 2\pi/3+2n\pi)$.



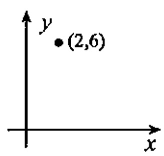
3. $r = \sqrt{4^2 + 3^2} = 5$. Angles satisfying $\tan\theta = 3/4$ are $0.644 + n\pi$. Since the point is in the first quadrant, polar coordinates are $(5, 0.644 + 2n\pi)$.



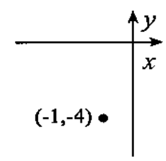
4. $r = \sqrt{(-2\sqrt{3})^2 + 2^2} = 4$. Angles satisfying $\tan\theta = -1/\sqrt{3}$ are $-\pi/6 + n\pi$. Since the point is in the second quadrant, polar coordinates are $(4, 5\pi/6 + 2n\pi)$.



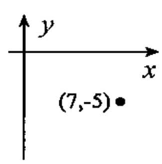
5. $r = \sqrt{2^2 + 6^2} = 2\sqrt{10}$. Angles satisfying $\tan\theta = 3$ are $1.25 + n\pi$. Since the point is in the first quadrant, polar coordinates are $(2\sqrt{10}, 1.25 + 2n\pi)$.



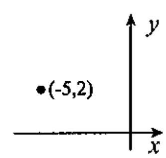
6. $r = \sqrt{(-1)^2 + (-4)^2} = \sqrt{17}$. Angles satisfying $\tan\theta = 4$ are $1.33 + n\pi$. Since the point is in the third quadrant, polar coordinates are $(\sqrt{17}, -1.82 + 2n\pi)$.



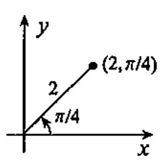
7. $r = \sqrt{7^2 + (-5)^2} = \sqrt{74}$. Angles satisfying $\tan\theta = -5/7$ are $-0.620 + n\pi$. Since the point is in the fourth quadrant, polar polar coordinates are $(\sqrt{74}, -0.620 + 2n\pi)$.



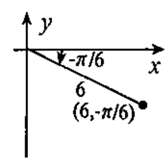
8. $r = \sqrt{(-5)^2 + 2^2} = \sqrt{29}$. Angles satisfying $\tan\theta = -2/5$ are $-0.38 + n\pi$. Since the point is in the second quadrant, polar coordinates are $(\sqrt{29}, 2.76 + 2n\pi)$.



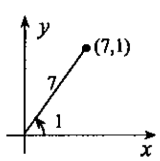
9. $(x, y) = (2\cos(\pi/4), 2\sin(\pi/4)) = (\sqrt{2}, \sqrt{2})$



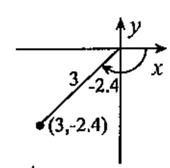
10. $(x, y) = (6\cos(-\pi/6), 6\sin(-\pi/6))$
$= (3\sqrt{3}, -3)$



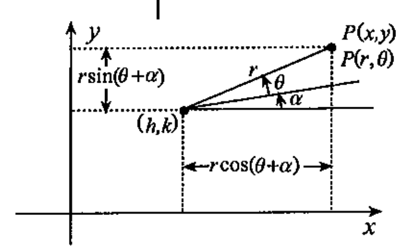
11. $(x, y) = (7\cos 1, 7\sin 1) = (3.78, 5.89)$



12. $(x, y) = (3\cos(-2.4), 3\sin(-2.4))$
$= (-2.21, -2.03)$

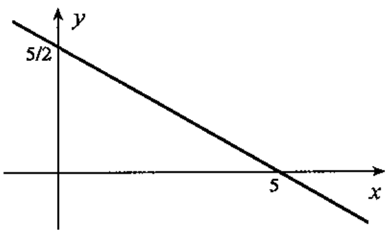


13. The diagram indicates that equations 9.9 are valid.

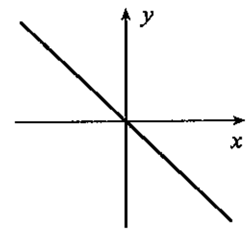


EXERCISES 9.3

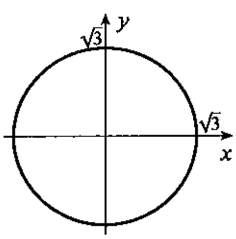
1. If we set $x = r\cos\theta$ and $y = r\sin\theta$ in $x + 2y = 5$, we obtain $r\cos\theta + 2r\sin\theta = 5$.



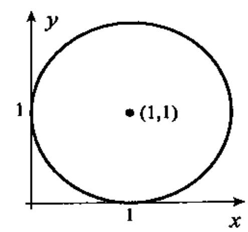
2. If we set $x = r\cos\theta$ and $y = r\sin\theta$ in $y = -x$, we obtain $r\sin\theta = -r\cos\theta$. Either $r = 0$ or $\sin\theta = -\cos\theta$. The first describes the origin (or pole), and the second can be expressed in the form $\tan\theta = -1$. This is equivalent to the two half lines $\theta = 3\pi/4$ and $\theta = -\pi/4$, both of which contain $r = 0$.



3. If we set $x = r\cos\theta$ and $y = r\sin\theta$ in $x^2 + y^2 = 3$, we obtain $r = \sqrt{3}$.

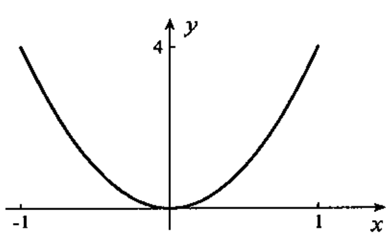


4. In polar coordinates, $r^2\cos^2\theta - 2r\cos\theta + r^2\sin^2\theta - 2r\sin\theta + 1 = 0$, or, $r^2 - 2r(\cos\theta + \sin\theta) + 1 = 0$. The curve is more easily sketched in Cartesian coordinates by writing its equation in the form $(x-1)^2 + (y-1)^2 = 1$, a circle.



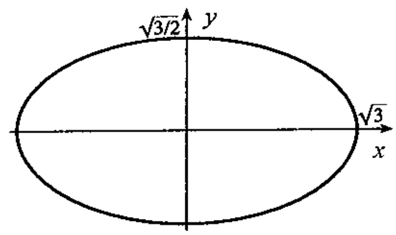
5. In polar coordinates, the equation of the parabola is $r\sin\theta = 4r^2\cos^2\theta$,

or, $4r = \sec\theta\tan\theta$.

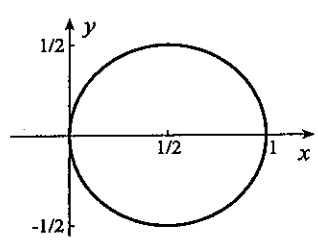


6. The equation of this ellipse in polar coordinates is $r^2\cos^2\theta + 2r^2\sin^2\theta = 3$, or,

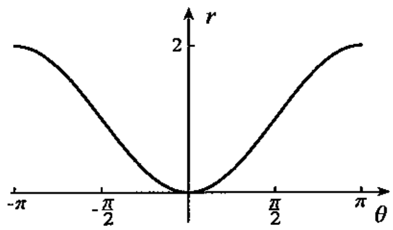
$$r^2 = \frac{3}{\cos^2\theta + 2\sin^2\theta} = \frac{3}{1+\sin^2\theta}.$$

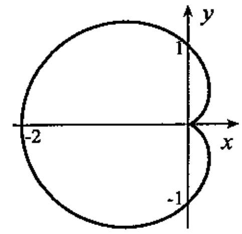


7. In polar coordinates, $r^2 = r\cos\theta$. Either $r = 0$ or $r = \cos\theta$. Since $r = 0$ also satisfies $r = \cos\theta$ (for $\theta = \pi/2$), we need only write $r = \cos\theta$. It is most esily drawn by completing the square, $(x - 1/2)^2 + y^2 = 1/4$, a circle.

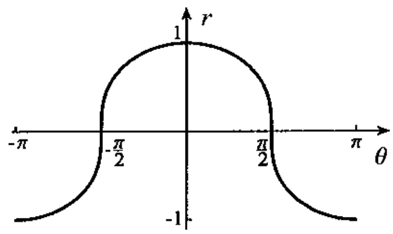


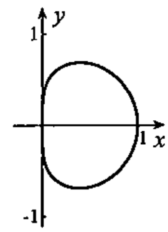
8. In polar coordinates, $r^2 = r - r\cos\theta$. Either $r = 0$ or $r = 1 - \cos\theta$. Since $r = 0$ also satisfies $r = 1 - \cos\theta$ (for $\theta = 0$), we need only write $r = 1 - \cos\theta$. The graph on the left leads to the curve on the right.



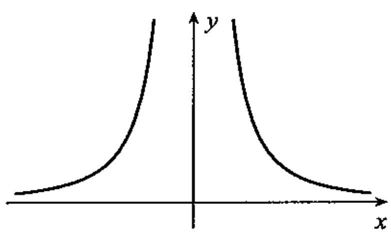


9. In polar coordinates, $r^4 = r\cos\theta$. Either $r = 0$ or $r^3 = \cos\theta$. Since $r = 0$ also satisfies $r^3 = \cos\theta$ (for $\theta = \pi/2$), we need only write $r^3 = \cos\theta$. The graph on the left leads to the curve on the right.

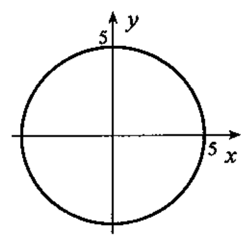




10. In polar coordinates, $r\sin\theta = \dfrac{1}{r^2\cos^2\theta}$, or, $r^3 = \sec^2\theta\,\csc\theta$. It is easily drawn in Cartesian coordinates.

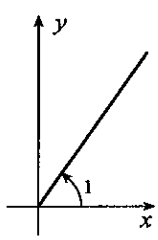


11. In Cartesian coordinates, $\sqrt{x^2+y^2} = 5$, or, $x^2 + y^2 = 25$, a circle.

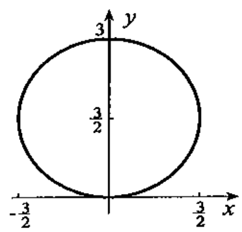


12. This equation describes a half line. Its equation in Cartesian coordinates is

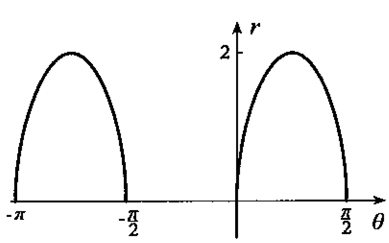
$y = (\tan 1)x$, $x \ge 0$.

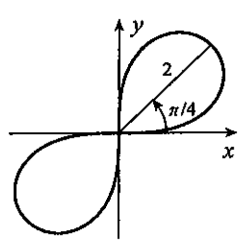


13. In Cartesian coordinates, $\sqrt{x^2+y^2} = \dfrac{3y}{\sqrt{x^2+y^2}}$, or, $x^2 + y^2 = 3y$. This is the circle $x^2 + (y - 3/2)^2 = 9/4$.

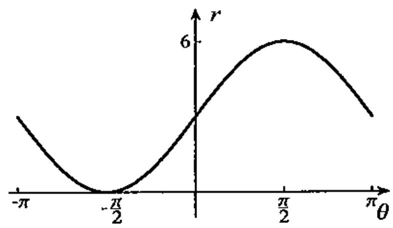


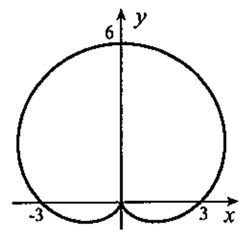
14. In Cartesian coordinates, $x^2 + y^2 = 8\sin\theta\,\cos\theta = 8\dfrac{y}{\sqrt{x^2+y^2}}\dfrac{x}{\sqrt{x^2+y^2}} \implies (x^2+y^2)^2 = 8xy$. The graph on the left leads to the curve on the right.



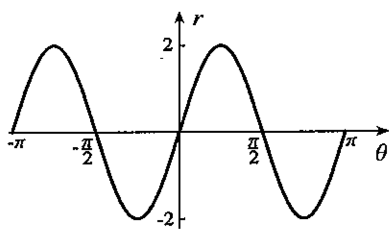


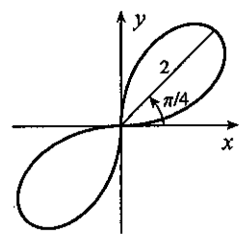
15. In Cartesian coordinates, $\sqrt{x^2+y^2} = 3 + \dfrac{3y}{\sqrt{x^2+y^2}} \Longrightarrow x^2+y^2 = 3(\sqrt{x^2+y^2}+y)$. The graph on the left leads to the curve on the right.



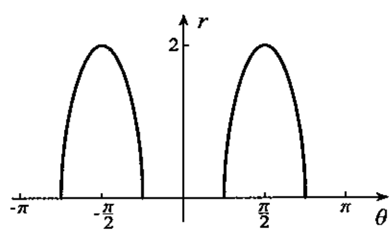


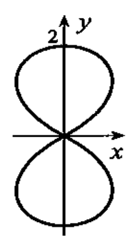
16. In Cartesian coordinates, $\sqrt{x^2+y^2} = 4\sin\theta\cos\theta = 4\dfrac{y}{\sqrt{x^2+y^2}}\dfrac{x}{\sqrt{x^2+y^2}} \Longrightarrow (x^2+y^2)^{3/2} = 4xy$. The graph on the left leads to the curve on the right.



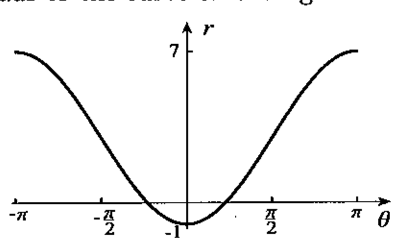


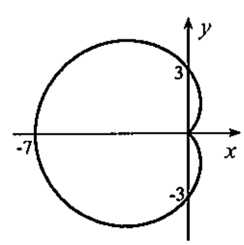
17. Since $r^2 = -4(\cos^2\theta - \sin^2\theta)$, we find that in Cartesian coordinates, $x^2+y^2 = -4\left(\dfrac{x^2}{x^2+y^2} - \dfrac{y^2}{x^2+y^2}\right)$, or, $(x^2+y^2)^2 = 4(y^2-x^2)$. The graph on the left leads to the curve on the right.



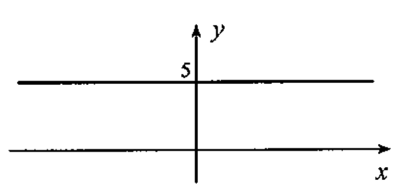


18. In Cartesian coordinates, $\sqrt{x^2+y^2} = 3 - \dfrac{4x}{\sqrt{x^2+y^2}} \Longrightarrow x^2+y^2 = 3\sqrt{x^2+y^2} - 4x$. The graph on the left leads to the curve on the right.

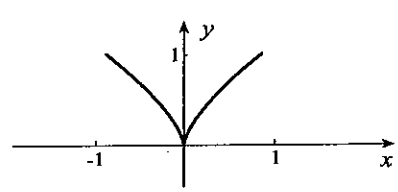




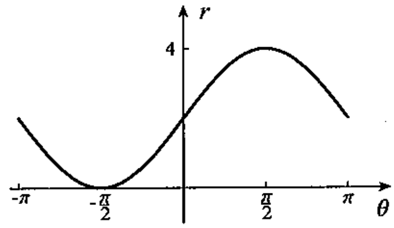
19. In Cartesian coordinates, $\sqrt{x^2+y^2} = \dfrac{5\sqrt{x^2+y^2}}{y}$, or $y = 5$.

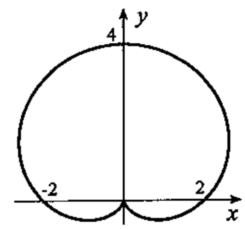


20. In Cartesian coordinates, $\sqrt{x^2+y^2} = \dfrac{\cos^2\theta}{\sin^3\theta} = \dfrac{x^2}{x^2+y^2}\dfrac{(x^2+y^2)^{3/2}}{y^3}$, or, $y^3 = x^2$.

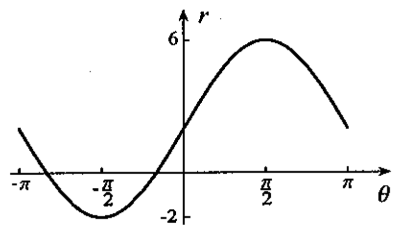


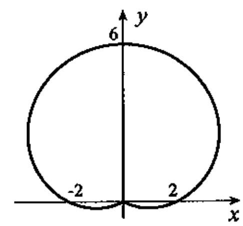
21. (a) The graph on the left leads to the curve on the right.



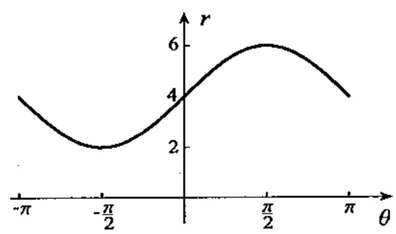


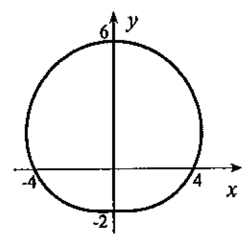
(b) The graph on the left leads to the curve on the right.



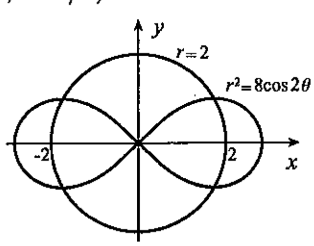


(c) The graph on the left below leads to the curve on the right.

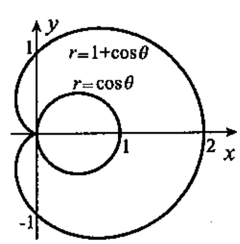




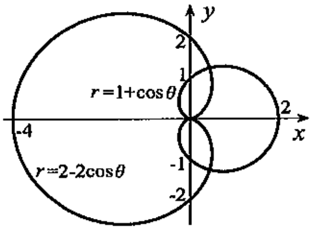
22. When we set $4 = 8\cos 2\theta$, we obtain $\cos 2\theta = 1/2 \Longrightarrow 2\theta = \pm\pi/3 + 2n\pi$. Thus, $\theta = \pm\pi/6 + n\pi$. The figure indicates four points of intersection, $(2, \pm\pi/6)$ and $(2, \pm 5\pi/6)$.



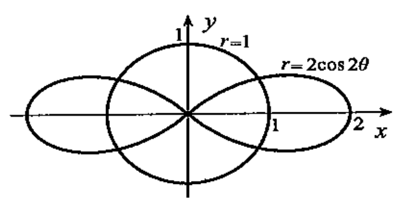
23. When we set $\cos\theta = 1 + \cos\theta$, we obtain no solution. The figure indicates that the curves intersect at the pole.



24. When we set $1 + \cos\theta = 2 - 2\cos\theta$, we obtain $\cos\theta = 1/3$. Thus, $\theta = \pm\text{Cos}^{-1}(1/3) + 2n\pi$. The figure indicates three points of intersection, the pole $(0, \theta)$, and $(4/3, \pm\text{Cos}^{-1}(1/3)) = (4/3, \pm 1.23)$.



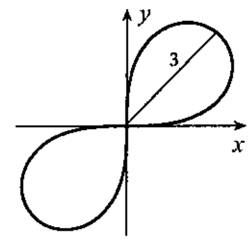
25. When we set $1 = 2\cos 2\theta$, we obtain $2\theta = \pm\pi/3 + 2n\pi$, from which $\theta = \pm\pi/6 + n\pi$. The figure indicates four points of intersection $(1, \pm\pi/6)$ and $(1, \pm 5\pi/6)$.



26. According to equation 9.13, the slope of the tangent line to $r = 9\cos 2\theta$ at $\theta = \pi/6$ is

$$\frac{dy}{dx}_{|\theta=\pi/6} = \frac{-18\sin 2\theta \sin\theta + 9\cos 2\theta \cos\theta}{-18\sin 2\theta \cos\theta - 9\cos 2\theta \sin\theta}_{|\theta=\pi/6} = \frac{\sqrt{3}}{7}.$$

27. According to equation 9.13, the slope of the tangent line to $r = 3\sqrt{\sin 2\theta}$ at $\theta = -5\pi/6$ is



$$\frac{dy}{dx}_{|\theta=-5\pi/6} = \frac{\dfrac{6\cos 2\theta}{2\sqrt{\sin 2\theta}}\sin\theta + 3\sqrt{\sin 2\theta}\cos\theta}{\dfrac{6\cos 2\theta}{2\sqrt{\sin 2\theta}}\cos\theta - 3\sqrt{\sin 2\theta}\sin\theta}_{|\theta=-5\pi/6}$$

$$= \frac{\dfrac{3\cos(-5\pi/3)}{\sqrt{\sin(-5\pi/3)}}\sin(-5\pi/6) + 3\sqrt{\sin(-5\pi/3)}\cos(-5\pi/6)}{\dfrac{3\cos(-5\pi/3)}{\sqrt{\sin(-5\pi/3)}}\cos(-5\pi/6) - 3\sqrt{\sin(-5\pi/3)}\sin(-5\pi/6)}.$$

Since the denominator is equal to 0, either the curve does not have a tangent line at the point corresponding to $\theta = -5\pi/6$, or the tangent line is vertical. The graph of the curve to the right shows that the tangent line is vertical.

28. According to equation 9.13, the slope of the tangent line to $r = 3 - 5\cos\theta$ at $\theta = 3\pi/4$ is

$$\frac{dy}{dx}_{|\theta=3\pi/4} = \frac{5\sin\theta\sin\theta + (3 - 5\cos\theta)\cos\theta}{5\sin\theta\cos\theta - (3 - 5\cos\theta)\sin\theta}_{|\theta=3\pi/4} = \frac{3}{5\sqrt{2}+3}.$$

29. According to equation 9.13, the slope of the tangent line to $r = 2\cos(\theta/2)$ at $\theta = \pi/2$ is

$$\frac{dy}{dx}_{|\theta=\pi/2} = \frac{-\sin(\theta/2)\sin\theta + 2\cos(\theta/2)\cos\theta}{-\sin(\theta/2)\cos\theta - 2\cos(\theta/2)\sin\theta}_{|\theta=\pi/2} = \frac{-1/\sqrt{2}}{-2(1/\sqrt{2})} = \frac{1}{2}.$$

30. (a) According to 9.13,

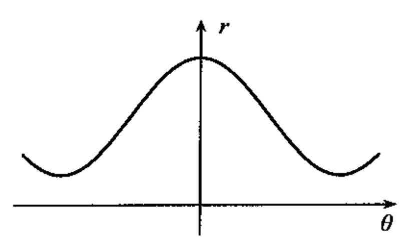
$$\frac{dy}{dx}_{|\theta=\pi/6} = \frac{\dfrac{3\cos\theta\sin\theta}{(1-\sin\theta)^2} + \dfrac{3\cos\theta}{1-\sin\theta}}{\dfrac{3\cos\theta\cos\theta}{(1-\sin\theta)^2} - \dfrac{3\sin\theta}{1-\sin\theta}}_{|\theta=\pi/6} = \sqrt{3}.$$

(b) The equation of the curve in Cartesian coordinates is

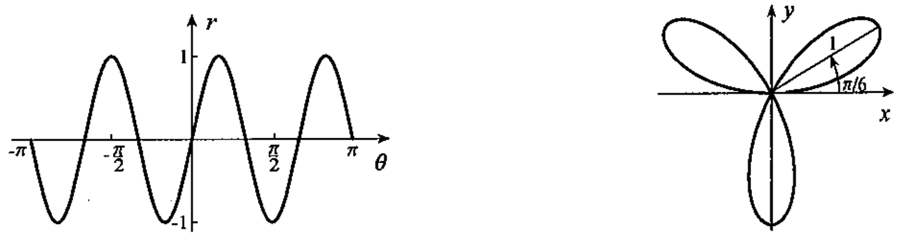
$$\sqrt{x^2+y^2} = \frac{3}{1 - \dfrac{y}{\sqrt{x^2+y^2}}} = \frac{3\sqrt{x^2+y^2}}{\sqrt{x^2+y^2} - y} \quad\Longrightarrow\quad \sqrt{x^2+y^2} - y = 3.$$

When we transpose the the y, square and simplify, the result is $y = (x^2 - 9)/6$. Hence, $dy/dx = x/3$. Since the x-coordinate of the point is $x = 6(\sqrt{3}/2) = 3\sqrt{3}$, the slope of the tangent line at the point is $(3\sqrt{3})/3 = \sqrt{3}$.

31. If $f(\theta)$ is even, then a sketch of the function is symmetric about the r-axis (figure to the right). This sketch indicates that the radial distance r is the same for a positive rotation θ and for a negative rotation $-\theta$. In other words the curve $r = f(\theta)$ is symmetric about the lines $\theta = 0$ and $\theta = \pi$. The curves in Exercise 18 and Example 9.16 are represented by even functions $f(\theta)$.

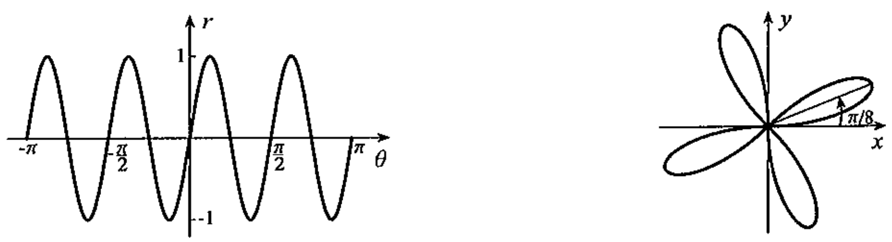


32. The graph on the left leads to the curve on the right.

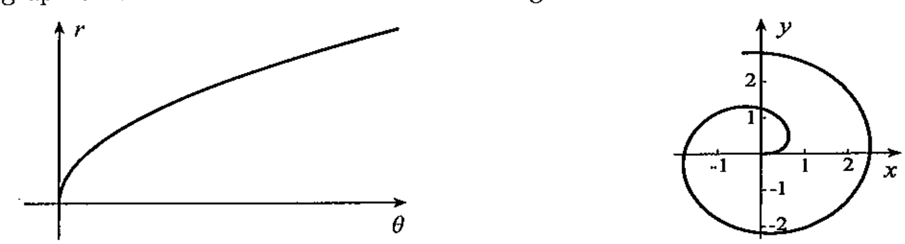


33. The graph on the left leads to the curve on the right.

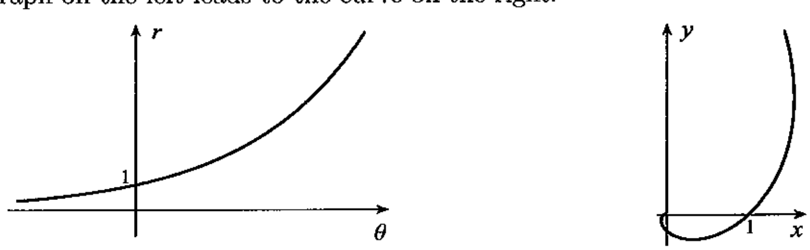
34. The graph on the left leads to the curve on the right.



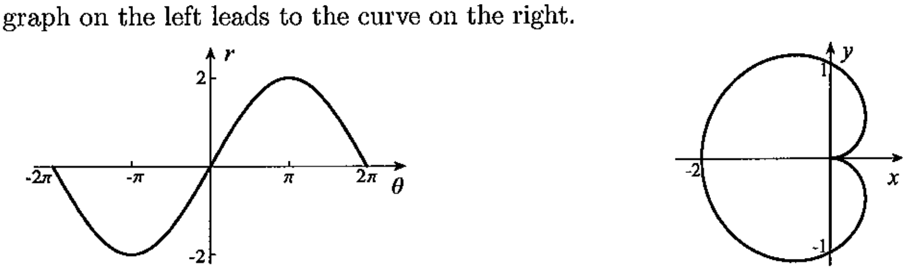
35. The graph on the left leads to the curve on the right.



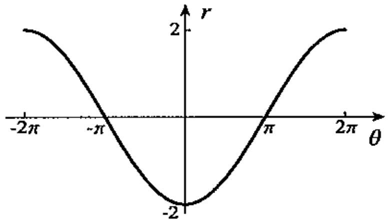
36. The graph on the left leads to the curve on the right.

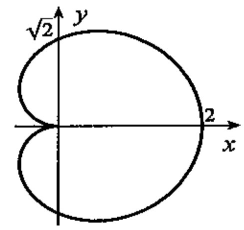


37. The graph on the left leads to the curve on the right.

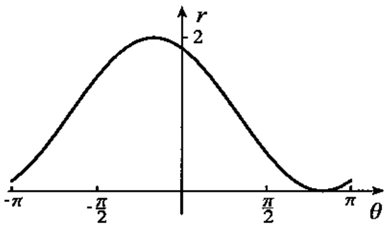


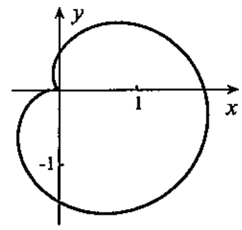
38. The graph on the left leads to the curve on the right.





39. The graph on the left leads to the curve on the right.

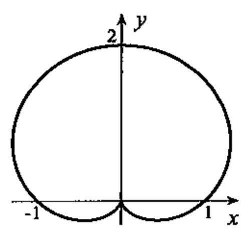




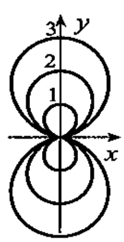
40. Maximum speed occurs when $2\pi\omega t = \pm\dfrac{\pi}{2} + n\pi \Longrightarrow t = \pm\dfrac{1}{4\omega} + \dfrac{n}{2\omega}$, where $n \geq 0$ is an integer. At these times $x = a$. Minimum speed is zero when $2\pi\omega t = n\pi \Longrightarrow t = \dfrac{n}{2\omega}$, where $n \geq 0$ is an integer. At these times, $x = a \pm b$, positions where the follower is furthest from and closest to the axis of rotation.

41. With formula 9.14,

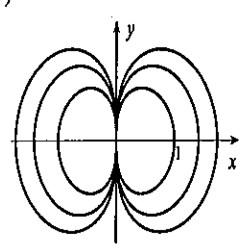
$$\begin{aligned}
L &= 2\int_{-\pi/2}^{\pi/2} \sqrt{a^2(1+\sin\theta)^2 + a^2\cos^2\theta}\, d\theta \\
&= 2a\int_{-\pi/2}^{\pi/2} \sqrt{2+2\sin\theta}\, d\theta \\
&= 2\sqrt{2}a\int_{-\pi/2}^{\pi/2} \sqrt{1+\sin\theta}\,\frac{\sqrt{1-\sin\theta}}{\sqrt{1-\sin\theta}}\, d\theta \\
&= 2\sqrt{2}a\int_{-\pi/2}^{\pi/2} \frac{\sqrt{1-\sin^2\theta}}{\sqrt{1-\sin\theta}}\, d\theta = 2\sqrt{2}a\int_{-\pi/2}^{\pi/2} \frac{\cos\theta}{\sqrt{1-\sin\theta}}\, d\theta = 2\sqrt{2}a\Big\{-2\sqrt{1-\sin\theta}\Big\}_{-\pi/2}^{\pi/2} = 8a.
\end{aligned}$$



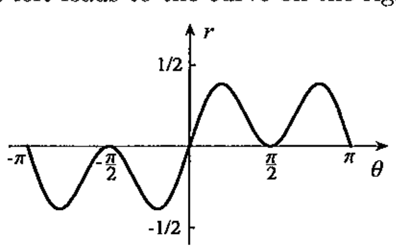
42. (a)

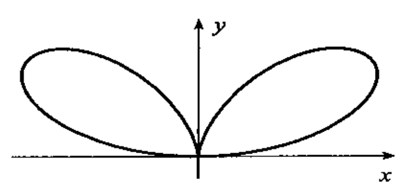


(b)

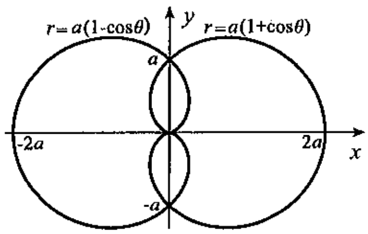


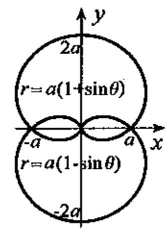
43. The equation of the bifolium in polar coordinates is $r^4 = r^3\cos^2\theta\sin\theta$, or, $r = \cos^2\theta\sin\theta$. The graph on the left leads to the curve on the right.





44. (a) Graphs of the cardioids are shown below.



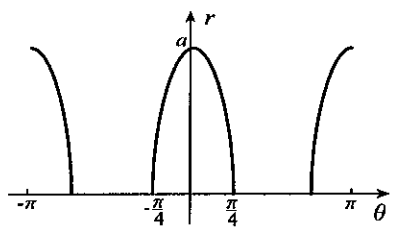


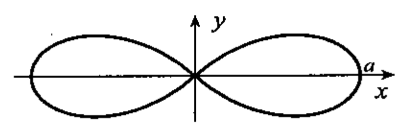
(b) Equations for $r = a(1 \pm \cos\theta)$ in Caresian coordinates are

$$\sqrt{x^2+y^2} = a\left(1 \pm \frac{x}{\sqrt{x^2+y^2}}\right) \implies x^2+y^2 = a(\sqrt{x^2+y^2} \pm x).$$

Similarly, equations for the other cardioids are $x^2+y^2 = a(\sqrt{x^2+y^2} \pm y)$.

45. (a) For $r = a\sqrt{\cos 2\theta}$, the graph on the left leads to the curve on the right.

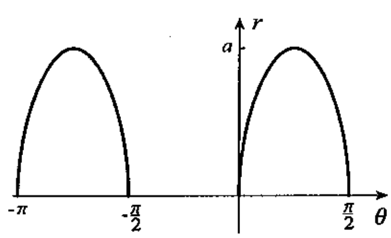


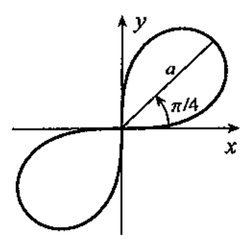


(b) From $r^2 = a^2(2\cos^2\theta - 1)$, we obtain the equation of the curve in Cartesian coordinates as

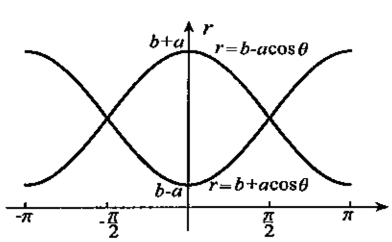
$$x^2+y^2 = a^2\left(\frac{2x^2}{x^2+y^2} - 1\right) \implies (x^2+y^2)^2 = a^2(x^2-y^2).$$

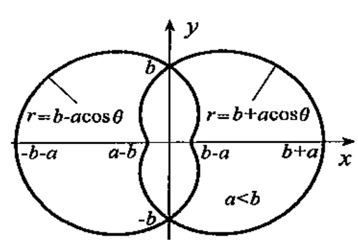
A similar analysis gives the equation $(x^2+y^2)^2 = 2a^2xy$ for $r^2 = a^2\sin 2\theta$ and the curve below.



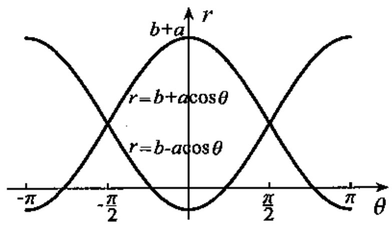


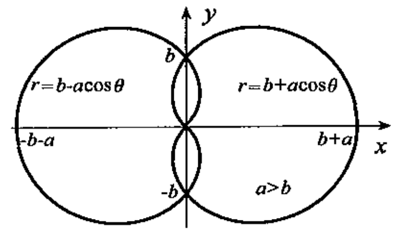
46. (a) We draw the curves $r = b \pm a\,\cos\theta$; curves $r = b \pm a\,\sin\theta$ are rotated $\pi/2$ radians. For the case $b = a$, see the cardioids in Exercise 44. When $a < b$, the graph on the left leads to the curve on the right.





When $a > b$, the graph on the left leads to the curve on the right.





(b) Equations for $r = b \pm a\cos\theta$ in Caresian coordinates are

$$\sqrt{x^2+y^2} = b \pm \frac{ax}{\sqrt{x^2+y^2}} \qquad \Longrightarrow \qquad x^2+y^2 = b\sqrt{x^2+y^2} \pm ax.$$

Similarly, equations for the other curves are $x^2+y^2 = b\sqrt{x^2+y^2} \pm ay$.

47. The function $r = f(\theta) = a\sin n\theta$ (or $a\cos n\theta$) is $2\pi/n$ periodic. This means that the graph of the function has n distinct parts above the θ-axis in the interval $-\pi \le \theta \le \pi$. These yield n loops (or petals) for the curve $r = a\sin n\theta$.

48. The function $r = f(\theta) = a\sin n\theta$ (or $a\cos n\theta$) is $2\pi/n$ periodic. This means that the graph of $r = |a\sin n\theta|$ has $2n$ distinct parts above the θ-axis in the interval $-\pi \le \theta \le \pi$. These yield $2n$ loops (or petals) for the curve $r = |a\sin n\theta|$.

49. (a) When we substitute $x = r\cos\theta$ and $y = r\sin\theta$,

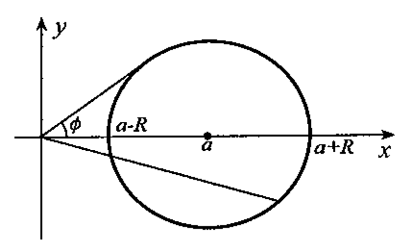
$$(r\cos\theta - a)^2 + r^2\sin^2\theta = R^2 \qquad \Longrightarrow \qquad r^2 - 2ar\cos\theta + (a^2 - R^2) = 0.$$

Solutions of this quadratic equation in r are

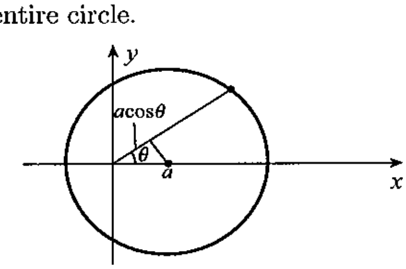
$$r = \frac{2a\cos\theta \pm \sqrt{4a^2\cos^2\theta - 4(a^2-R^2)}}{2} = a\cos\theta \pm \sqrt{R^2 - a^2\sin^2\theta}.$$

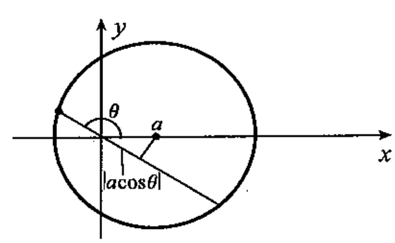
(b) We could use the final result in part (a) to reduce the equation when $a = R$. It is easier however to return to $r^2 - 2ar\cos\theta + (a^2 - R^2) = 0$. When $a = R$, this immediately gives $r = 2a\cos\theta$. It represents the entire circle.

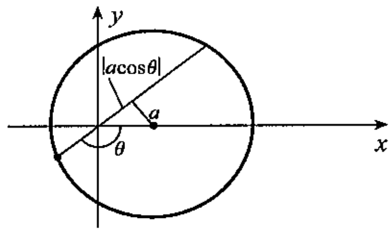
(c) When $a > R$, we require both parts since for each angle in the interval $-\phi < \theta < \phi$, there are two values of r. The positive radical gives point on the right portion of the circle, and the negative radical gives points on the left portion. We can find angle ϕ by setting
$a\cos\phi + \sqrt{R^2 - a^2\sin^2\phi} = a\cos\phi - \sqrt{R^2 - a^2\sin^2\phi}$.
This gives $R^2 = a^2\sin^2\phi$, from which $\phi = \mathrm{Sin}^{-1}(R/a)$.

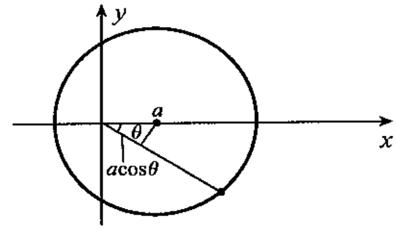


(d) When $a < R$, we have shown $a\cos\theta$ in the following figures for θ in each of the four quadrants. In all cases, r is greater than $a\cos\theta$, with the result that we must choose $r = a\cos\theta + \sqrt{R^2 - a^2\sin^2\theta}$ for the entire circle.





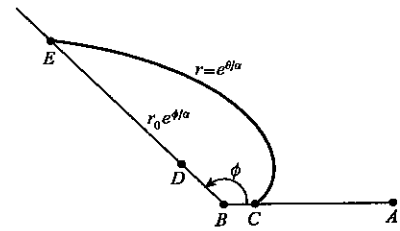




50. (a) If D is the position of the submarine after time $k/(V+v)$, then $\|BD\| = kv/(V+v)$. On the other hand,

$$\|BC\| = k - \|AC\| = k - \frac{kV}{V+v} = \frac{kv}{V+v}.$$

(b) The point of intersection, E, of the paths followed by the two boats is a distance $r_0\, e^{\phi/\alpha}$ from the pole. The submarine reaches E at time $t = (r_0/v)\, e^{\phi/\alpha}$.
To find the time at which the patrol boat reaches E, we first calculate the length of the spiral from C to this point,



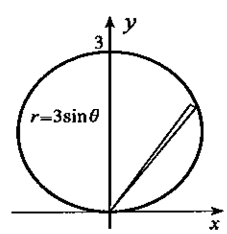
$$\int_0^\phi \sqrt{(r_0\, e^{\theta/\alpha})^2 + \left(\frac{r_0}{\alpha} e^{\theta/\alpha}\right)^2}\, d\theta = \frac{r_0}{\alpha}\sqrt{\alpha^2+1}\int_0^\phi e^{\theta/\alpha}\, d\theta = \frac{r_0}{\alpha}\sqrt{\alpha^2+1}\left\{\alpha\, e^{\theta/\alpha}\right\}_0^\phi = r_0\sqrt{\alpha^2+1}(e^{\phi/\alpha}-1).$$

The time at which the patrol boat reaches E is

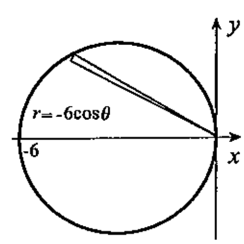
$$\frac{k}{V+v} + \frac{r_0\sqrt{\alpha^2+1}(e^{\phi/\alpha}-1)}{V} = \frac{k}{V+v} + \frac{r_0}{V}\sqrt{\frac{V^2}{v^2} - 1 + 1}(e^{\phi/\alpha}-1) = \frac{k}{V+v} + \frac{r_0}{v}e^{\phi/\alpha} - \frac{r_0}{v}$$
$$= \frac{k}{V+v} + \frac{r_0}{v}e^{\phi/\alpha} - \frac{kv}{v(V+v)} = \frac{r_0}{v}e^{\phi/\alpha}.$$

EXERCISES 9.4

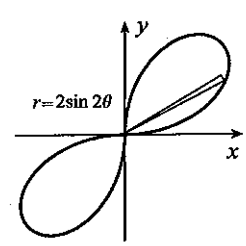
1. $$A = 2\int_0^{\pi/2} \frac{1}{2}(3\sin\theta)^2\, d\theta = 9\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right) d\theta = \frac{9}{2}\left\{\theta - \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = \frac{9\pi}{4}$$



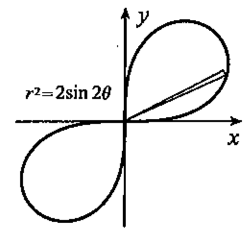
2. $$A = 2\int_{\pi/2}^{\pi} \frac{1}{2}(-6\cos\theta)^2\, d\theta = 36\int_{\pi/2}^{\pi}\left(\frac{1+\cos 2\theta}{2}\right) d\theta = 18\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_{\pi/2}^{\pi} = 9\pi$$



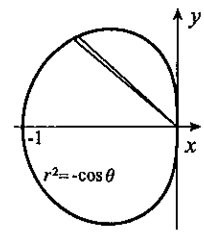
3. $$A = 2\int_0^{\pi/2} \frac{1}{2}(2\sin 2\theta)^2\, d\theta = 4\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right) d\theta = 2\left\{\theta - \frac{1}{4}\sin 4\theta\right\}_0^{\pi/2} = \pi$$



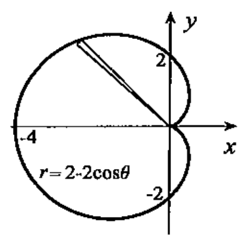
4. $$A = 2\int_0^{\pi/2} \frac{1}{2}(2\sin 2\theta)\,d\theta = \left\{-\cos 2\theta\right\}_0^{\pi/2} = 2$$



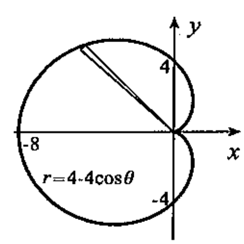
5. $$A = 2\int_{\pi/2}^{\pi} \frac{1}{2}(-\cos\theta)\,d\theta = \left\{-\sin\theta\right\}_{\pi/2}^{\pi} = 1$$



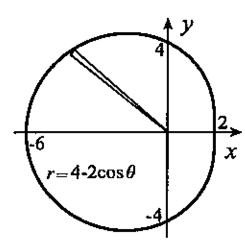
6. $$\begin{aligned} A &= 2\int_0^{\pi} \frac{1}{2}(2 - 2\cos\theta)^2\,d\theta \\ &= 4\int_0^{\pi} (1 - 2\cos\theta + \cos^2\theta)\,d\theta \\ &= 4\int_0^{\pi} \left(1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta \\ &= 4\left\{\frac{3\theta}{2} - 2\sin\theta + \frac{1}{4}\sin 2\theta\right\}_0^{\pi} = 6\pi \end{aligned}$$



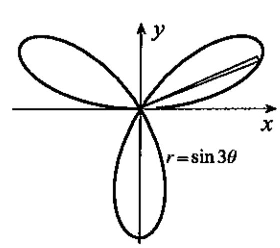
7. $$\begin{aligned} A &= 2\int_0^{\pi} \frac{1}{2}(4 - 4\cos\theta)^2\,d\theta \\ &= 16\int_0^{\pi} (1 - 2\cos\theta + \cos^2\theta)\,d\theta \\ &= 16\int_0^{\pi} \left(1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta \\ &= 16\left\{\frac{3\theta}{2} - 2\sin\theta + \frac{1}{4}\sin 2\theta\right\}_0^{\pi} = 24\pi \end{aligned}$$



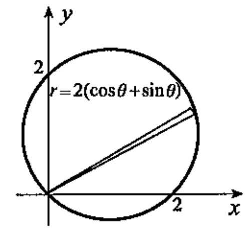
8. $$\begin{aligned} A &= 2\int_0^{\pi} \frac{1}{2}(4 - 2\cos\theta)^2\,d\theta = 4\int_0^{\pi} (4 - 4\cos\theta + \cos^2\theta)\,d\theta \\ &= 4\int_0^{\pi} \left(4 - 4\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta \\ &= 4\left\{\frac{9\theta}{2} - 4\sin\theta + \frac{1}{4}\sin 2\theta\right\}_0^{\pi} = 18\pi \end{aligned}$$



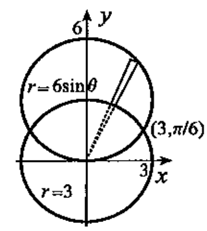
9. $$\begin{aligned} A &= 3\int_0^{\pi/3} \frac{1}{2}(\sin 3\theta)^2\,d\theta \\ &= \frac{3}{2}\int_0^{\pi/3} \left(\frac{1 - \cos 6\theta}{2}\right)d\theta \\ &= \frac{3}{4}\left\{\theta - \frac{1}{6}\sin 6\theta\right\}_0^{\pi/3} = \frac{\pi}{4} \end{aligned}$$



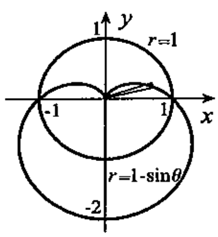
10. $$A = \int_{-\pi/4}^{3\pi/4} \frac{1}{2}[2(\cos\theta + \sin\theta)]^2\,d\theta$$
$$= 2\int_{-\pi/4}^{3\pi/4} (1 + 2\cos\theta\sin\theta)\,d\theta$$
$$= 2\Big\{\theta + \sin^2\theta\Big\}_{-\pi/4}^{3\pi/4} = 2\pi$$



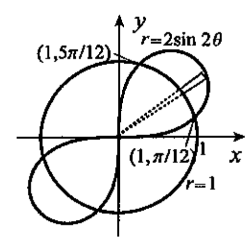
11. $$A = 2\int_{\pi/6}^{\pi/2} \frac{1}{2}[(6\sin\theta)^2 - 9]\,d\theta = 9\int_{\pi/6}^{\pi/2} (4\sin^2\theta - 1)\,d\theta$$
$$= 9\int_{\pi/6}^{\pi/2} [2(1 - \cos 2\theta) - 1]\,d\theta$$
$$= 9\Big\{\theta - \sin 2\theta\Big\}_{\pi/6}^{\pi/2} = 3\pi + \frac{9\sqrt{3}}{2}$$



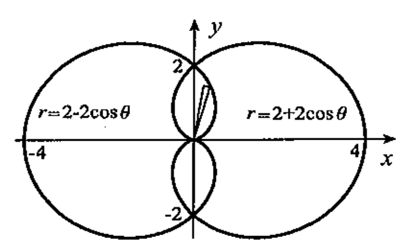
12. $$A = \frac{\pi}{2} + 2\int_0^{\pi/2} \frac{1}{2}(1 - \sin\theta)^2\,d\theta$$
$$= \frac{\pi}{2} + \int_0^{\pi/2} \left(1 - 2\sin\theta + \frac{1 - \cos 2\theta}{2}\right) d\theta$$
$$= \frac{\pi}{2} + \left\{\frac{3\theta}{2} + 2\cos\theta - \frac{1}{4}\sin 2\theta\right\}_0^{\pi/2} = \frac{1}{4}(5\pi - 8)$$



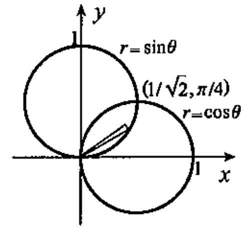
13. $$A = 2\int_{\pi/12}^{5\pi/12} \frac{1}{2}[(2\sin 2\theta)^2 - 1]\,d\theta$$
$$= \int_{\pi/12}^{5\pi/12} [2(1 - \cos 4\theta) - 1]\,d\theta$$
$$= \left\{\theta - \frac{1}{2}\sin 4\theta\right\}_{\pi/12}^{5\pi/12} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$



14. $$A = 4\int_0^{\pi/2} \frac{1}{2}(2 - 2\cos\theta)^2\,d\theta$$
$$= 8\int_0^{\pi/2} \left(1 - 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right) d\theta$$
$$= 8\left\{\frac{3\theta}{2} - 2\sin\theta + \frac{1}{4}\sin 2\theta\right\}_0^{\pi/2} = 6\pi - 16$$



15. $$A = 2\int_0^{\pi/4} \frac{1}{2}(\sin\theta)^2\,d\theta$$
$$= \int_0^{\pi/4} \left(\frac{1 - \cos 2\theta}{2}\right) d\theta$$
$$= \frac{1}{2}\left\{\theta - \frac{1}{2}\sin 2\theta\right\}_0^{\pi/4} = \frac{\pi}{8} - \frac{1}{4}$$



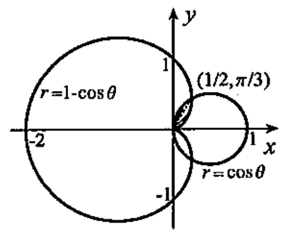
16. $A = 2\int_0^{\pi/3} \frac{1}{2}(1-\cos\theta)^2\,d\theta + 2\int_{\pi/3}^{\pi/2}\frac{1}{2}\cos^2\theta\,d\theta$

$$= \int_0^{\pi/3}\left(1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta$$

$$+ \int_{\pi/3}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \left\{\frac{3\theta}{2} - 2\sin\theta + \frac{1}{4}\sin 2\theta\right\}_0^{\pi/3}$$

$$+ \frac{1}{2}\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_{\pi/3}^{\pi/2} = \frac{1}{12}(7\pi - 12\sqrt{3})$$



17. $A = 2\int_{\tilde\theta}^{\pi}\frac{1}{2}(1-4\cos\theta)^2\,d\theta$

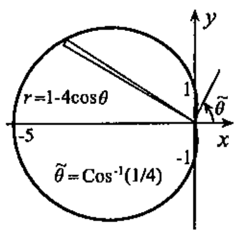
$$= \int_{\tilde\theta}^{\pi}(1 - 8\cos\theta + 16\cos^2\theta)\,d\theta$$

$$= \int_{\tilde\theta}^{\pi}[1 - 8\cos\theta + 8(1+\cos 2\theta)]\,d\theta$$

$$= \left\{9\theta - 8\sin\theta + 4\sin 2\theta\right\}_{\tilde\theta}^{\pi}$$

$$= 9\pi - 9\,\mathrm{Cos}^{-1}(1/4) + 8\sin\tilde\theta - 4\sin 2\tilde\theta = 9\pi - 9\,\mathrm{Cos}^{-1}(1/4) + 8\sin\tilde\theta - 8\sin\tilde\theta\cos\tilde\theta$$

$$= 9\pi - 9\,\mathrm{Cos}^{-1}(1/4) + 8\left(\frac{\sqrt{15}}{4}\right) - 8\left(\frac{\sqrt{15}}{4}\right)\left(\frac{1}{4}\right) = 9\pi - 9\,\mathrm{Cos}^{-1}(1/4) + \frac{3\sqrt{15}}{2}$$



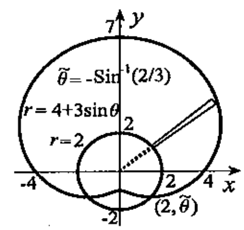
18. $A = 2\int_{\tilde\theta}^{\pi/2}\frac{1}{2}[(4+3\sin\theta)^2 - 4]\,d\theta = \int_{\tilde\theta}^{\pi/2}\left[12 + 24\sin\theta + \frac{9}{2}(1-\cos 2\theta)\right]d\theta$

$$= \left\{\frac{33\theta}{2} - 24\cos\theta - \frac{9}{4}\sin 2\theta\right\}_{\tilde\theta}^{\pi/2}$$

$$= \frac{33\pi}{4} - \frac{33\tilde\theta}{2} + 24\cos\tilde\theta + \frac{9}{4}\sin 2\tilde\theta$$

$$= \frac{33\pi}{4} - \frac{33\tilde\theta}{2} + 24\cos\tilde\theta + \frac{9}{2}\sin\tilde\theta\cos\tilde\theta$$

$$= \frac{33\pi}{4} + \frac{33}{2}\,\mathrm{Sin}^{-1}\!\left(\frac{2}{3}\right) + 24\left(\frac{\sqrt{5}}{3}\right) + \frac{9}{2}\left(-\frac{2}{3}\right)\left(\frac{\sqrt{5}}{3}\right) = \frac{33\pi}{4} + \frac{33}{2}\,\mathrm{Sin}^{-1}\!\left(\frac{2}{3}\right) + 7\sqrt{5}$$

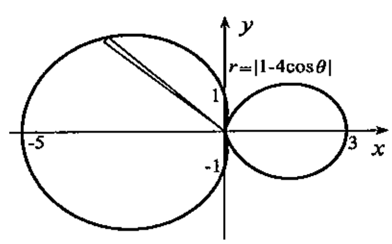


19. $A = 2\int_0^{\pi}\frac{1}{2}(1-4\cos\theta)^2\,d\theta$

$$= \int_0^{\pi}(1 - 8\cos\theta + 16\cos^2\theta)\,d\theta$$

$$= \int_0^{\pi}[1 - 8\cos\theta + 8(1+\cos 2\theta)]\,d\theta$$

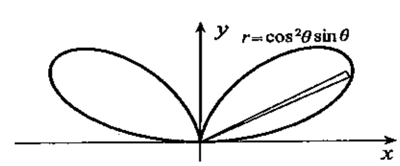
$$= \left\{9\theta - 8\sin\theta + 4\sin 2\theta\right\}_0^{\pi} = 9\pi$$



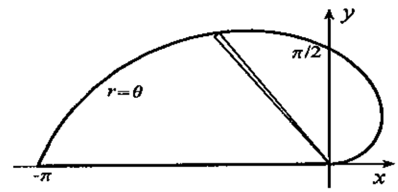
20. $A = 2\int_0^{\pi/2}\frac{1}{2}\cos^4\theta\sin^2\theta\,d\theta = \int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{\sin^2 2\theta}{4}\right)d\theta$

$$= \frac{1}{8}\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2} + \sin^2 2\theta\cos 2\theta\right)d\theta$$

$$= \frac{1}{8}\left\{\frac{\theta}{2} - \frac{1}{8}\sin 4\theta + \frac{1}{6}\sin^3 2\theta\right\}_0^{\pi/2} = \frac{\pi}{32}$$



21. $A = \int_0^{\pi} \frac{1}{2}\theta^2 \, d\theta$
$= \frac{1}{2}\left\{\frac{\theta^3}{3}\right\}_0^{\pi} = \frac{\pi^3}{6}$



22. (a) In polar coordinates,

$$r^2 \sin^2\theta = r^2 \cos^2\theta \left(\frac{a - r\cos\theta}{a + r\cos\theta}\right), \quad \Longrightarrow \quad a\sin^2\theta + r\sin^2\theta\cos\theta = a\cos^2\theta - r\cos^3\theta.$$

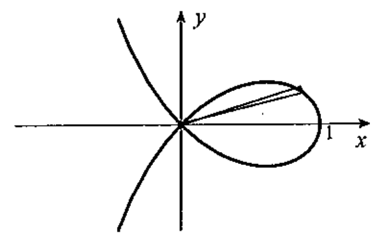
Thus, $r = \frac{a(\cos^2\theta - \sin^2\theta)}{\sin^2\theta\cos\theta + \cos^3\theta} = \frac{a\cos 2\theta}{\cos\theta} = a\cos 2\theta\sec\theta.$

(b) $A = 2\int_0^{\pi/4} \frac{1}{2}a^2\cos^2 2\theta\sec^2\theta \, d\theta = a^2 \int_0^{\pi/4} (2\cos^2\theta - 1)^2 \sec^2\theta \, d\theta$

$= a^2 \int_0^{\pi/4} (4\cos^2\theta - 4 + \sec^2\theta) \, d\theta$

$= a^2 \int_0^{\pi/4} [2(1 + \cos 2\theta) - 4 + \sec^2\theta] \, d\theta$

$= a^2 \left\{-2\theta + \sin 2\theta + \tan\theta\right\}_0^{\pi/4} = \frac{a^2}{2}(4 - \pi)$



EXERCISES 9.5

1. This is a parabola.

2. Completion of squares leads to $(x - 3/2)^2 + (y + 1)^2 = 25 + 9/4 + 1 = 113/4$, a circle.

3. This is a straight line.

4. The presence of y^3 means that the equation describes none of the given curves.

5. This is an ellipse.

6. The equation can be expressed in form 9.26 for a hyperbola, $\frac{y^2}{5/3} - \frac{x^2}{5/2} = 1$.

7. Completion of squares leads to $(x - 1/2)^2 + (y + 3/2)^2 = 33/2$, a circle.

8. In the form $y = x^2/3 + 2x/3 - 4/3$, we have a parabola.

9. Completion of squares leads to $(x - 1)^2 + (y + 3)^2 = -5$. No point can satisfy this equation.

10. No x and y can make $x^2 + 2y^2 + 24$ equal to zero.

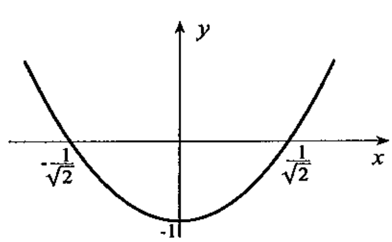
11. This is a hyperbola.

12. This is an ellipse.

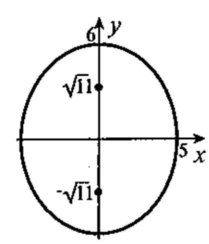
13. The presence of y^3 means that the equation describes none of the given curves.

14. Equation $x + 4y = 3$ represents a straight line.

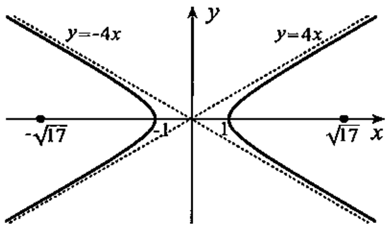
15. This is a parabola.



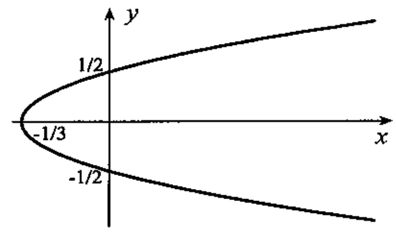
16. Foci for the ellipse are on the y-axis at distances $\pm\sqrt{36 - 25} = \pm\sqrt{11}$ from the origin.



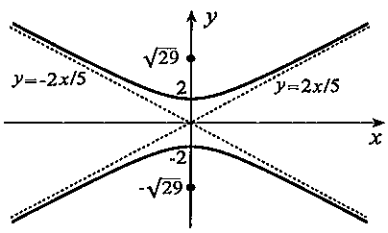
17. Foci for the hyperbola are on the x-axis at distances $\pm\sqrt{1+16} = \pm\sqrt{17}$ from the origin.



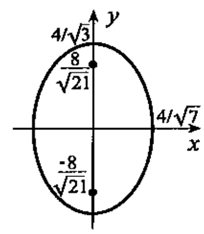
18. This is a parabola.



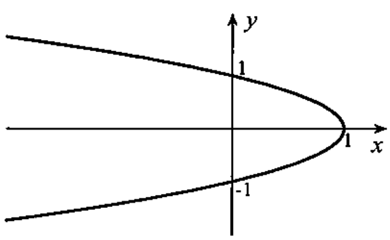
19. Foci for the hyperbola are on the y-axis at distances $\pm\sqrt{4+25} = \pm\sqrt{29}$ from the origin.



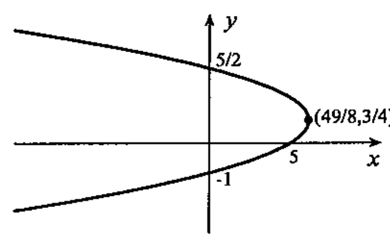
20. Foci for the ellipse $\dfrac{x^2}{16/7} + \dfrac{y^2}{16/3} = 1$ are on the y-axis at distances $\pm\sqrt{16/3 - 16/7} = \pm 8/\sqrt{21}$ from the origin.



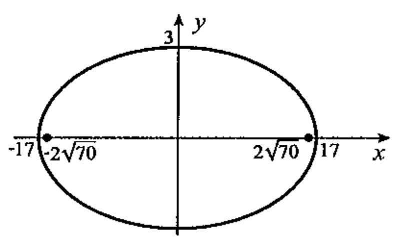
21. This is a parabola.



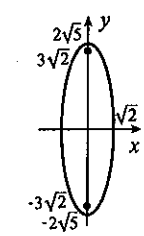
22. This is the parabola $x = -2y^2 + 3y + 5$ $= -(2y-5)(y+1)$.



23. Foci for the ellipse $\dfrac{x^2}{289} + \dfrac{y^2}{9} = 1$ are on the x-axis at distances $\pm\sqrt{289-9} = \pm 2\sqrt{70}$ from the origin.

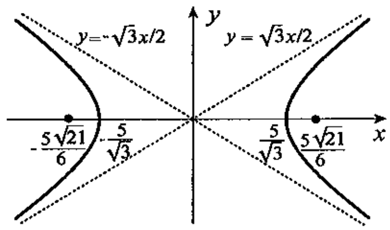


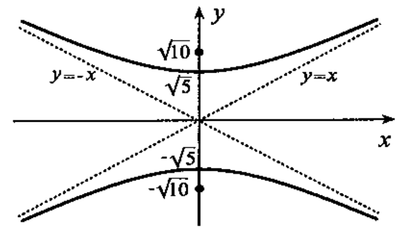
24. Foci for the ellipse $\dfrac{x^2}{2} + \dfrac{y^2}{20} = 1$ are on the y-axis at distances $\pm\sqrt{20-2} = \pm 3\sqrt{2}$ from the origin.



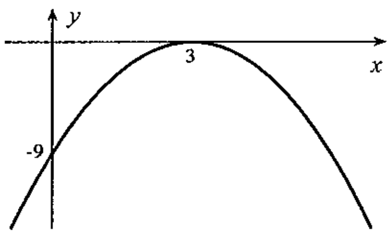
25. Foci for the hyperbola $\dfrac{x^2}{25/3} - \dfrac{y^2}{25/4} = 1$ are on the x-axis at distances $\pm\sqrt{25/3 + 25/4} = \pm 5\sqrt{21}/6$ from the origin.

26. Foci for the hyperbola $\dfrac{y^2}{5} - \dfrac{x^2}{5} = 1$ are on the y-axis at distances $\pm\sqrt{5+5} = \pm\sqrt{10}$ from the origin.

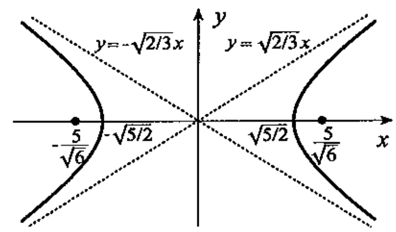




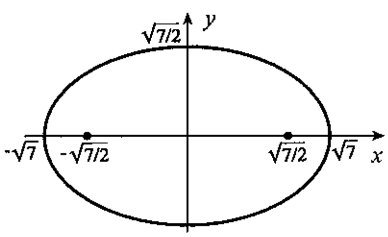
27. This is the parabola $y = -(x-3)^2$.



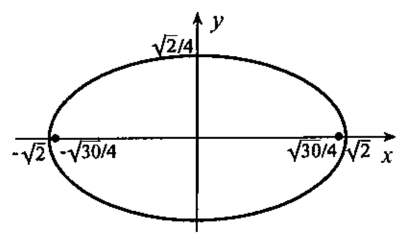
28. Foci for the hyperbola $\dfrac{x^2}{5/2} - \dfrac{y^2}{5/3} = 1$ are on the x-axis at distances $\pm\sqrt{5/2+5/3} = \pm 5/\sqrt{6}$ from the origin.



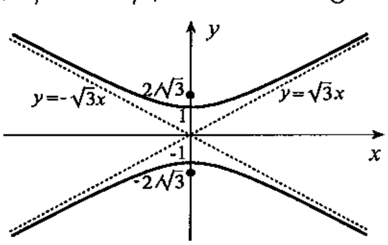
29. Foci for the ellipse $\dfrac{x^2}{7} + \dfrac{y^2}{7/2} = 1$ are on the x-axis at distances $\pm\sqrt{7-7/2} = \pm\sqrt{7/2}$ from the origin.



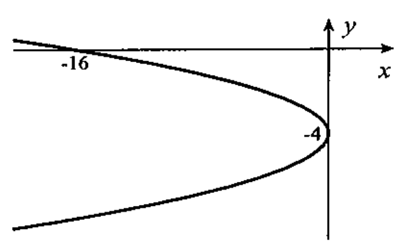
30. Foci for the ellipse $\dfrac{x^2}{2} + \dfrac{y^2}{1/8} = 1$ are on the x-axis at distances $\pm\sqrt{2-1/8} = \pm\sqrt{30}/4$ from the origin.



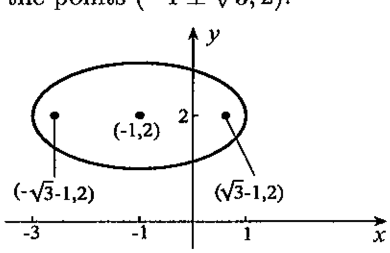
31. Foci for the hyperbola $y^2 - \dfrac{x^2}{1/3} = 1$ are on the y-axis at distances $\pm\sqrt{1+1/3} = \pm 2/\sqrt{3}$ from the origin.



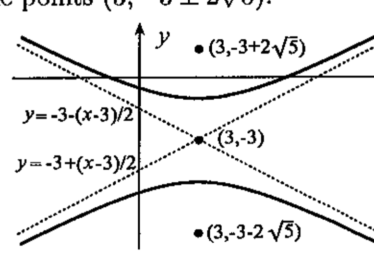
32. This is a parabola.



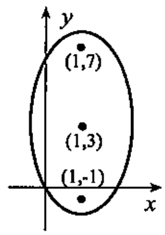
33. Foci for the ellipse $\dfrac{(x+1)^2}{4} + (y-2)^2 = 1$ are at the points $(-1 \pm \sqrt{3}, 2)$.



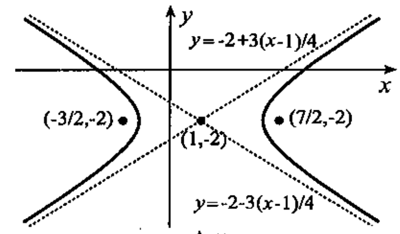
34. Foci for the hyperbola $\dfrac{(y+3)^2}{4} - \dfrac{(x-3)^2}{16} = 1$ are at the points $(3, -3 \pm 2\sqrt{5})$.



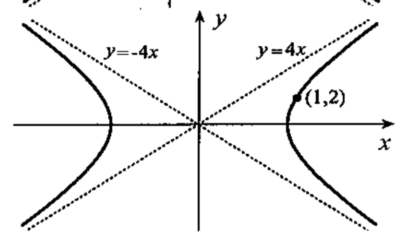
35. Foci for the ellipse $\dfrac{(x-1)^2}{2}+\dfrac{(y-3)^2}{18}=1$ are at the points $(1,-1)$ and $(1,7)$.



36. Foci for the hyperbola $\dfrac{(x-1)^2}{4}-\dfrac{(y+2)^2}{9/4}=1$ are at the points $(7/2,-2)$ and $(-3/2,-2)$.



37. With asymptotes $y=\pm 4x$, and the point $(1,2)$, we take the hyperbola in the form $x^2/a^2-y^2/b^2=1$ with $b/a=4$. Since $(1,2)$ is on the ellipse, $1/a^2-4/b^2=1$. These two equations give $a^2=3/4$ and $b^2=12$. The equation of the hyperbola is therefore $x^2/(3/4)-y^2/12=1$, or $16x^2-y^2=12$.



38. For an ellipse of the form $x^2/a^2+y^2/b^2=1$, a and b must satisfy

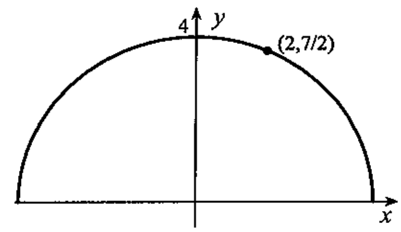
$$\frac{4}{a^2}+\frac{16}{b^2}=1 \qquad \text{and} \qquad \frac{9}{a^2}+\frac{1}{b^2}=1.$$

These imply that $a^2=28/3$ and $b^2=28$. Hence, $3x^2/28+y^2/28=1 \Longrightarrow 3x^2+y^2=28$.

39. With the coordinate system shown, the equation of the ellipse must be of the form $x^2/a^2+y^2/b^2=1$. Since $(0,4)$ and $(2,7/2)$ are points on the ellipse,

$$\frac{16}{b^2}=1, \qquad \frac{4}{a^2}+\frac{49/4}{b^2}=1.$$

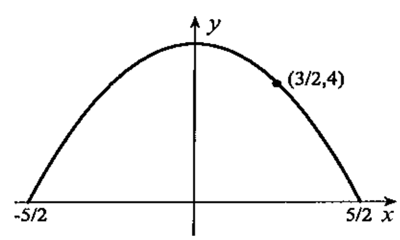
These imply that $a^2=256/15$ and $b^2=16$. The width of the arch is therefore $2a=32/\sqrt{15}$.



40. With the coordinate system shown to the right, the equation of the parabola is $y=ax^2+c$. Since $(5/2,0)$ and $(3/2,4)$ are points thereon,

$$0=\frac{25a}{4}+c, \quad 4=\frac{9a}{4}+c.$$

These imply that $a=-1$ and $c=25/4$. Thus the height of the arch is $25/4$.



41. Suppose the ends of the string are attached to the tacks at fixed positions. If the pencil is placed against the string and moved so that the string is always taut, the curve traced out is an ellipse with the tacks as foci.

42. (a) The foci of the ellipse are $(\pm 4,0)$, so that in equation 9.22, $a^2-b^2=16$. Since the sum of the distances from a point on the ellipse to the foci is $2a$, it follows that $a=5$, and hence $b=3$. The required equation is therefore $x^2/25+y^2/9=1 \Longrightarrow 9x^2+25y^2=225$.

(b) With foci $(\pm 4,0)$ and $2a=10$, equation 9.20 for the ellipse is

$$\sqrt{(x+4)^2+y^2}+\sqrt{(x-4)^2+y^2}=10.$$

If we transpose the second term and square both sides,

$$x^2+8x+16+y^2=100-20\sqrt{(x-4)^2+y^2}+x^2-8x+16+y^2 \implies 100-16x=20\sqrt{(x-4)^2+y^2}.$$

Dividing by 4, and squaring, $625-200x+16x^2=25(x^2-8x+16+y^2) \Longrightarrow 9x^2+25y^2=225$.

43. (a) The foci of the hyperbola are $(0, \pm 3)$, so that in equation 9.26, $a^2 + b^2 = 9$. Since the differences of the distances from a point on the hyperbola to the foci is $2b$, it follows that $b = 1/2$, and hence $a = \sqrt{35}/2$. The required equation is therefore $y^2/(1/4) - x^2/(35/4) = 1 \Longrightarrow 140y^2 - 4x^2 = 35$.
(b) With foci $(0, \pm 3)$ and $2b = 1$, the equation similar to 9.24 for the hyperbola is

$$|\sqrt{x^2 + (y+3)^2} - \sqrt{x^2 + (y-3)^2}| = 1.$$

If we write $\sqrt{x^2 + (y+3)^2} = \pm 1 + \sqrt{x^2 + (y-3)^2}$, and square both sides,

$$x^2 + (y+3)^2 = 1 \pm 2\sqrt{x^2 + (y-3)^2} + x^2 + (y-3)^2 \quad \Longrightarrow \quad 12y - 1 = \pm 2\sqrt{x^2 + (y-3)^2}.$$

Squaring again gives

$$144y^2 - 24y + 1 = 4[x^2 + (y-3)^2] \quad \Longrightarrow \quad 140y^2 - 4x^2 = 35.$$

44. For straight lines we set $A = C = 0$. For circles we set $A = C$. For parabolas of the form $y = ax^2 + bx + c$, we set $C = 0$. For parabolas of the the form $x = ay^2 + by + c$, we set $A = 0$. For ellipse 9.23 we demand that $AC > 0$. For hyperbolas 9.27 we demand that $AC < 0$.

45. If we differentiate $b^2x^2 + a^2y^2 = a^2b^2$ with respect to x, $2b^2x + 2a^2y\dfrac{dy}{dx} = 0 \Longrightarrow \dfrac{dy}{dx} = -\dfrac{b^2x}{a^2y}$. At a point (x_0, y_0), the equation of the tangent line is

$$y - y_0 = -\frac{b^2x_0}{a^2y_0}(x - x_0) \qquad \Longrightarrow \qquad a^2yy_0 - a^2y_0^2 = -b^2xx_0 + b^2x_0^2.$$

Since $b^2x_0^2 + a^2y_0^2 = a^2b^2$, we may also write for the line, $b^2xx_0 + a^2yy_0 = a^2b^2$.

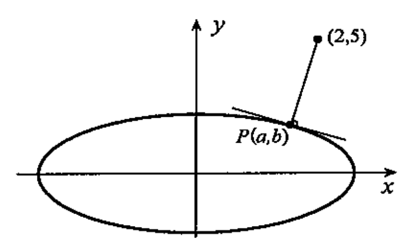
46. If we differentiate $b^2x^2 - a^2y^2 = a^2b^2$ with respect to x, $2b^2x - 2a^2y\dfrac{dy}{dx} = 0 \Longrightarrow \dfrac{dy}{dx} = \dfrac{b^2x}{a^2y}$. At a point (x_0, y_0), the equation of the tangent line is

$$y - y_0 = \frac{b^2x_0}{a^2y_0}(x - x_0) \qquad \Longrightarrow \qquad a^2yy_0 - a^2y_0^2 = b^2xx_0 - b^2x_0^2.$$

Since $b^2x_0^2 - a^2y_0^2 = a^2b^2$, we may also write for the line $b^2xx_0 - a^2yy_0 = a^2b^2$.

47. If we differentiate $2x^2 + 3y^2 = 14$ with respect to x

$$4x + 6y\frac{dy}{dx} = 0 \quad \Longrightarrow \quad \frac{dy}{dx} = -\frac{2x}{3y}.$$



The slope of the tangent line to the ellipse at $P(a, b)$ is therefore $-2a/(3b)$. Since the slope of the line joining P and $(2, 5)$ is $(b-5)/(a-2)$, it follows that

$$-\frac{2a}{3b} = -\frac{a-2}{b-5} \quad \Longrightarrow \quad b = \frac{10a}{6-a}.$$

Since P is on the ellipse, we must also have $2a^2 + 3b^2 = 14$. When we substitute for b,

$$2a^2 + 3\left(\frac{10a}{6-a}\right)^2 = 14,$$

and this equation simplifies to

$$P(a) = a^4 - 12a^3 + 179a^2 + 84a - 252 = 0.$$

We find that $a = 1$ is a solution of this equation. Geometrically, we can see that this is the only solution. To show this algebraically, we factor $a - 1$ from $P(a)$,

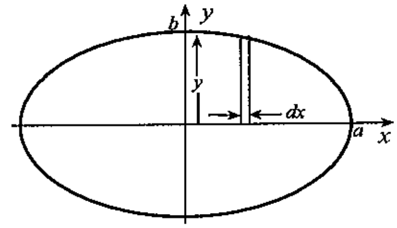
$$P(a) = (a-1)(a^3 - 11a^2 + 168a + 252) = 0.$$

We now examine the cubic polynomial by expressing it in the form

$$Q(a) = a^3 - 11a^2 + 168a + 252 = a(a^2 - 11a + 168) + 252.$$

Since the discriminant of $a^2 - 11a + 168 < 0$, this quantity is always positive. It follows that for $0 < a < \sqrt{7}$, the polynomial $Q(a)$ cannot vanish; that is, $a = 1$ is the only solution of $P(a) = 0$. The required point is therefore $(1, 2)$.

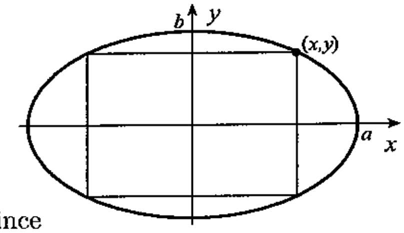
48. $A = 4\int_0^a y\,dx = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$ If we set $x = a\sin\theta$, then $dx = a\cos\theta\,d\theta$, and



$$\begin{aligned}A &= \frac{4b}{a}\int_0^{\pi/2} a\cos\theta\, a\cos\theta\, d\theta \\ &= 4ab\int_0^{\pi/2}\left(\frac{1 + \cos 2\theta}{2}\right) d\theta \\ &= 2ab\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = \pi ab.\end{aligned}$$

49. The area of the rectangle is $A = 4xy$. When we solve the equation of the ellipse for the positive value of y, the result is $y = (b/a)\sqrt{a^2 - x^2}$. The area of the rectangle can therefore be expressed in the form

$$A = A(x) = \frac{4bx}{a}\sqrt{a^2 - x^2}, \quad 0 \le x \le a.$$



For critical points of $A(x)$ we solve

$$0 = A'(x) = \frac{4b}{a}\left(\sqrt{a^2 - x^2} - \frac{x^2}{\sqrt{a^2 - x^2}}\right).$$

This equation can be expressed in the form

$$\sqrt{a^2 - x^2} = \frac{x^2}{\sqrt{a^2 - x^2}},$$

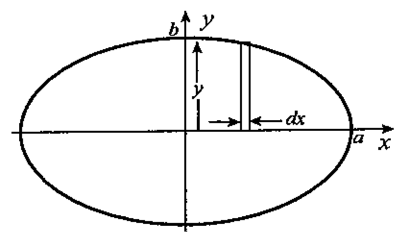
from which $a^2 - x^2 = x^2$. The positive solution is $x = a/\sqrt{2}$. Since

$$A(0) = 0, \qquad A\left(\frac{a}{\sqrt{2}}\right) > 0, \qquad A(a) = 0,$$

area is maximized when the length of the rectangle in the x-direction is $\sqrt{2}a$ and that in the y-direction is $\sqrt{2}b$.

50. For the prolate spheroid,

$$\begin{aligned}V &= 2\int_0^a \pi y^2\,dx = 2\pi\int_0^a \frac{b^2}{a^2}(a^2 - x^2)\,dx \\ &= \frac{2\pi b^2}{a^2}\left\{a^2 x - \frac{x^3}{3}\right\}_0^a = \frac{4}{3}\pi ab^2.\end{aligned}$$



For the oblate spheroid,

$$\begin{aligned}V &= 2\int_0^a 2\pi xy\,dx = 4\pi\int_0^a x\frac{b}{a}\sqrt{a^2 - x^2}\,dx \\ &= \frac{4\pi b}{a}\left\{-\frac{1}{3}(a^2 - x^2)^{3/2}\right\}_0^a = \frac{4}{3}\pi a^2 b.\end{aligned}$$

51. This is due to the fact that the sum of the distances from the foci to any point on the ellipse is always the same.

52. If we equate coefficients in $y = ax^2 + bx + c$ and those in equation 9.18,

$$a = \frac{1}{2(q - r)}, \qquad b = \frac{-p}{q - r}, \qquad c = \frac{p^2 + q^2 - r^2}{2(q - r)}.$$

The second divided by the first gives $b/a = -2p$, or, $p = -b/(2a)$. When the third is divided by the first and p set equal to $-b/(2a)$, the result is $q^2 - r^2 = (4ac - b^2)/(4a^2)$, or,

$$\frac{4ac - b^2}{4a^2} = (q + r)(q - r) = (q + r)\frac{1}{2a}.$$

Thus,

$$q + r = \frac{4ac - b^2}{2a} \qquad \text{and} \qquad q - r = \frac{1}{2a}.$$

Addition and subtraction of these results give the expressions for q and r.

53. Simply interchange the formulas for p and q.

54. Exercise 15: $p = 0$; $q = (1/8)(1-8) = -7/8$; $r = (1/8)(-1-8) = -9/8$. Thus, the focus is $(0, -7/8)$, and the directrix is $y = -9/8$.

Exercise 18: $p = 1/(16/3)[1 + 4(4/3)(-1/3)] = -7/48$; $q = 0$; $r = 1/(16/3)[-1 + 4(4/3)(-1/3)] = -25/48$. Thus, the focus is $(-7/48, 0)$, and the directrix is $x = -25/48$.

Exercise 21: $p = 1/(-4)[1 + 4(-1)(1)] = 3/4$; $q = 0$; $r = 1/(-4)[-1 + 4(-1)(1)] = 5/4$. Thus, the focus is $(3/4, 0)$, and the directrix is $x = 5/4$.

Exercise 22: $p = 1/(-8)[1 + 4(-2)(5) - 9] = 6$; $q = -3/(-4)$; $r = 1/(-8)[-1 + 4(-2)(5) - 9] = 25/4$. Thus, the focus is $(6, 3/4)$, and the directrix is $x = 25/4$.

Exercise 27: $p = -6/(-2) = 3$; $q = 1/(-4)[1 + 4(-1)(-9) - 36] = -1/4$; $r = 1/(-4)[-1 + 4(-1)(-9) - 36] = 1/4$. Thus, the focus is $(3, -1/4)$, and the directrix is $y = 1/4$.

Exercise 32: With $x = -y^2 - 8y - 16$, $p = [1/(-4)][1 + 4(-1)(-16) - 64] = -1/4$; $q = 8/(-2) = -4$; $r = [1/(-4)][-1 + 4(-1)(-16) - 64] = 1/4$. Thus, the focus is $(-1/4, -4)$ and the directrix $x = 1/4$.

55. When the centre of the ellipse is (h, k), and foci lie on $y = k$, the foci are $(h \pm c, k)$. If the sum of the distances from a point(x, y) on the ellipse to the foci is $2a$, then the equation of the ellipse is

$$\sqrt{(x-h-c)^2 + (y-k)^2} + \sqrt{(x-h+c)^2 + (y-k)^2} = 2a.$$

When we transpose the second square root and square both sides

$$(x-h-c)^2 + (y-k)^2 = (x-h+c)^2 + (y-k)^2 - 4a\sqrt{(x-h+c)^2 + (y-k)^2} + 4a^2,$$

and this equation simplifies to $a\sqrt{(x-h+c)^2 + (y-k)^2} = a^2 + c(x-h)$. Squaring again leads to

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2 - c^2} = 1.$$

If we set $b^2 = a^2 - c^2$, then 9.23 is obtained. A similar derivation can be given when foci are on the line $x = h$.

56. When the centre of the hyperbola is (h, k), and foci lie on $y = k$, the foci are $(h \pm c, k)$. If the difference of the distances from a point(x, y) on the hyperbola to the foci is $2a$, then the equation of the hyperbola is

$$\left|\sqrt{(x-h-c)^2 + (y-k)^2} - \sqrt{(x-h+c)^2 + (y-k)^2}\right| = 2a.$$

When we write $\sqrt{(x-h-c)^2 + (y-k)^2} = \sqrt{(x-h+c)^2 + (y-k)^2} \pm 2a$, and square both sides

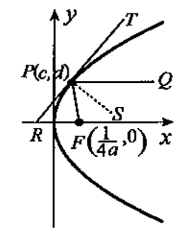
$$(x-h-c)^2 + (y-k)^2 = (x-h+c)^2 + (y-k)^2 \pm 4a\sqrt{(x-h+c)^2 + (y-k)^2} + 4a^2,$$

and this equation simplifies to $\pm a\sqrt{(x-h+c)^2 + (y-k)^2} = a^2 + c(x-h)$. Squaring again leads to

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{c^2 - a^2} = 1.$$

If we set $b^2 = c^2 - a^2$, then the first of equations 9.27 is obtained. The second equation in 9.27 is obtained in a similar fashion when foci are on the line $x = h$.

57. According to Exercise 53, the x-coordinate of the focus of the parabola is $1/(4a)$. If we differentiate the equation $x = ay^2$ with respect to x, then $1 = 2ay(dy/dx)$, from which $dy/dx = 1/(2ay)$. The slope of the tangent line at P is therefore $1/(2ad)$, and the equation of the tangent line is $y - d = [1/(2ad)](x - c)$. The x-intercept of this line is given by



$$-d = \frac{1}{2ad}(x - c) \quad \Longrightarrow \quad x = c - 2ad^2.$$

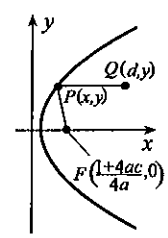
Taking into account the fact that $c = ad^2$, squares of the lengths of PF and RF are

$$\|PF\|^2 = \left(c - \frac{1}{4a}\right)^2 + d^2 = \left(ad^2 - \frac{1}{4a}\right)^2 + d^2 = a^2d^4 + \frac{d^2}{2} + \frac{1}{16a^2},$$

$$\|RF\|^2 = \left(\frac{1}{4a} - c + 2ad^2\right)^2 = \left(\frac{1}{4a} + ad^2\right)^2 = \frac{1}{16a^2} + \frac{d^2}{2} + a^2d^4.$$

Thus, $\|PF\| = \|RF\|$. It now follows that angles FPR and FRP are equal, and both are equal to angle TPQ. Since angles FPR and SPF add to $\pi/2$ as do angles TPQ and QPS, it follows that angles SPF and QPS are equal.

58. What we must verify is that the sum of distances $\|PQ\| + \|PF\|$ is independent of the coordinates of P.

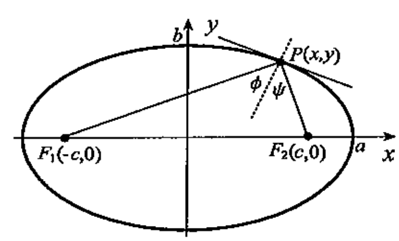


$$\begin{aligned}
\|PQ\| + \|PF\| &= (d - x) + \sqrt{\left(x - \frac{1+4ac}{4a}\right)^2 + y^2} \\
&= d - (ay^2 + c) + \sqrt{\left(ay^2 + c - \frac{1+4ac}{4a}\right)^2 + y^2} \\
&= d - ay^2 - c + \sqrt{\left(\frac{4a^2y^2 - 1}{4a}\right)^2 + y^2} \\
&= d - ay^2 - c + \sqrt{\frac{16a^4y^4 - 8a^2y^2 + 1 + 16a^2y^2}{16a^2}} \\
&= d - ay^2 - c + \sqrt{\frac{(4a^2y^2+1)^2}{16a^2}} = d - ay^2 - c + \frac{4a^2y^2 + 1}{4a} = d - c + \frac{1}{4a},
\end{aligned}$$

and this is indeed independent of x and y.

59. The slope of the tangent line to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at P is given by

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \quad \Longrightarrow \quad \frac{dy}{dx} = -\frac{b^2x}{a^2y}.$$

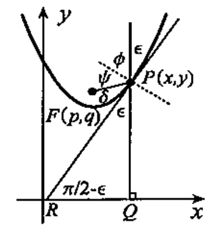


The slope of the normal line to the ellipse at P is $a^2y/(b^2x)$. Slopes of PF_2 and PF_1 are $y/(x-c)$ and $y/(x+c)$, respectively. Using formula 1.60, angles ψ and ϕ are given by

$$\begin{aligned}
\tan\psi &= \left|\frac{\dfrac{y}{x-c} - \dfrac{a^2y}{b^2x}}{1 + \dfrac{y}{x-c}\dfrac{a^2y}{b^2x}}\right| = \left|\frac{b^2xy - a^2y(x-c)}{b^2x(x-c)}\,\frac{b^2x(x-c)}{b^2x(x-c) + a^2y^2}\right| = \left|\frac{b^2xy - a^2y(x-c)}{a^2y^2 + b^2x^2 - b^2cx}\right| \\
&= \left|\frac{y[-x(a^2-b^2) + a^2c]}{a^2b^2 - b^2cx}\right| = \left|\frac{y(-xc^2 + a^2c)}{b^2(a^2 - cx)}\right| = \left|\frac{yc(-cx + a^2)}{b^2(a^2 - cx)}\right| = \left|\frac{cy}{b^2}\right|, \\
\tan\phi &= \left|\frac{\dfrac{a^2y}{b^2x} - \dfrac{y}{x+c}}{1 + \dfrac{a^2y}{b^2x}\dfrac{y}{x+c}}\right| = \left|\frac{a^2y(x+c) - b^2xy}{b^2x(x+c)}\,\frac{b^2x(x+c)}{b^2x(x+c) + a^2y^2}\right| = \left|\frac{a^2y(x+c) - b^2xy}{b^2x^2 + a^2y^2 + b^2cx}\right| \\
&= \left|\frac{y[x(-b^2 + a^2) + a^2c]}{a^2b^2 + b^2cx}\right| = \left|\frac{y(xc^2 + a^2c)}{b^2(a^2 + cx)}\right| = \left|\frac{yc(xc + a^2)}{b^2(a^2 + cx)}\right| = \left|\frac{cy}{b^2}\right|.
\end{aligned}$$

Since $\tan\psi = \tan\phi$ and both angles are betwen 0 and π, we conclude that $\psi = \phi$. In other words, the normal bisects the angle between the focal radii. A similar proof can be constructed for the hyperbola.

60. The figure indicates that $\psi = \phi$ if $\delta = \epsilon$. When we differentiate $y = ax^2 + bx + c$, we obtain $y' = 2ax + b$. The slope of PF is $(y-q)/(x-p)$ and hence, using equation 1.60,



$$\tan\delta = \frac{2ax + b - \dfrac{y-q}{x-p}}{1 + (2ax+b)\left(\dfrac{y-q}{x-p}\right)}$$

or,

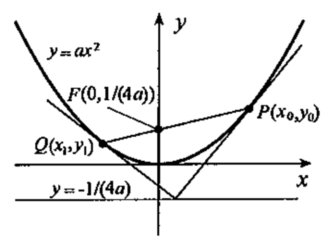
$$\cot\delta = \frac{x - p + (y-q)(2ax+b)}{(2ax+b)(x-p) - y + q}.$$

If we substitute $y = ax^2 + bx + c$, $p = -b/(2a)$, $q = (1 + 4ac - b^2)/(4a)$ (see Exercise 52),

$$\cot\delta = \frac{\left(x + \dfrac{b}{2a}\right) + (2ax+b)\left[ax^2 + bx + c - \dfrac{1}{4a}(1 + 4ac - b^2)\right]}{(2ax+b)\left(x + \dfrac{b}{2a}\right) - \left[ax^2 + bx + c - \dfrac{1}{4a}(1 + 4ac - b^2)\right]}$$

and this quantity simplifies to $\cot\delta = 2ax + b$. But $\cot\epsilon = \tan(\pi/2 - \epsilon) = f'(x) = 2ax + b$. Hence, $\cot\delta = \cot\epsilon$, and $\delta = \epsilon$.

61. Since $dy/dx = 2ax$, slopes of the tangent lines at P and Q are $2ax_0$ and $2ax_1$. The equation of PQ is $y - y_0 = \dfrac{y_1 - y_0}{x_1 - x_0}(x - x_0)$, and since F is on this line



$$\frac{1}{4a} - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(-x_0).$$

Furthermore, $y_0 = ax_0^2$ and $y_1 = ax_1^2$, so that

$$\frac{1}{4a} - ax_0^2 = \frac{ax_1^2 - ax_0^2}{x_1 - x_0}(-x_0).$$

When this equation is multiplied by $4a$,

$$1 - 4a^2x_0^2 = -4a^2x_0(x_1 + x_0) \quad\Longrightarrow\quad 1 = -4a^2x_0x_1 = -(2ax_0)(2ax_1).$$

In other words, the tangent lines at P and Q are perpendicular. Equations of the tangent lines at P and Q are $y - y_0 = 2ax_0(x - x_0)$ and $y - y_1 = 2ax_1(x - x_1)$. The point of intersection of these lines has a y-coordinate given by

$$\frac{y - y_0}{2ax_0} + x_0 = \frac{y - y_1}{2ax_1} + x_1 \quad\Longrightarrow\quad x_1(y - y_0) + 2ax_0^2x_1 = x_0(y - y_1) + 2ax_0x_1^2.$$

Thus, $y = \dfrac{2ax_0x_1^2 - 2ax_0^2x_1 - x_0y_1 + x_1y_0}{x_1 - x_0}$. When we substitute $y_0 = ax_0^2$ and $y_1 = ax_1^2$,

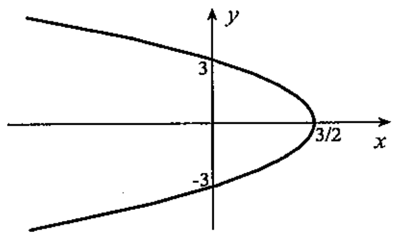
$$y = \frac{2ax_0x_1(x_1 - x_0) - x_0(ax_1^2) + x_1(ax_0^2)}{x_1 - x_0} = \frac{2ax_0x_1(x_1 - x_0) - ax_0x_1(x_1 - x_0)}{x_1 - x_0} = ax_0x_1.$$

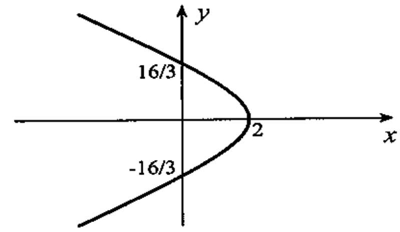
But $4a^2x_0x_1 = -1$, and hence $y = -1/(4a)$; that is, the point of intersection lies on the directrix.

EXERCISES 9.6

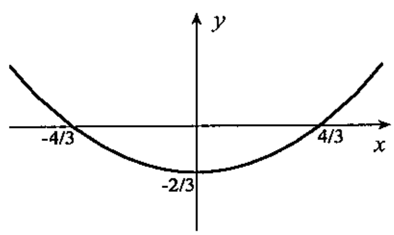
1. We have a parabola with focus at the origin that opens to the left. It crosses the x-axis at $r = 3/2$ when $\theta = 0$, and the y-axis at $r = 3$ when $\theta = \pm\pi/2$.

2. Since $r = \dfrac{16/3}{1 + (5/3)\cos\theta}$, we have a hyperbola with foci on the x-axis. It crosses the x-axis at $r = 2$ when $\theta = 0$, and the y-axis at $r = 16/3$ when $\theta = \pm\pi/2$. Only the left half of the hyperbola is obtained.

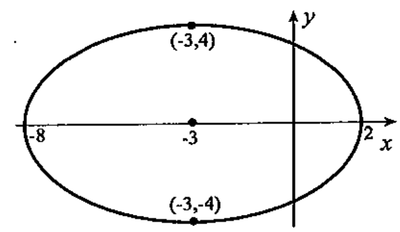




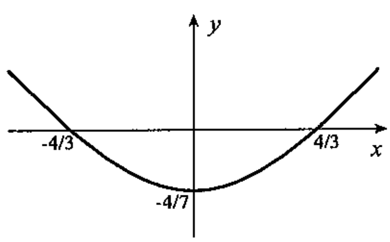
3. Since $r = \dfrac{4/3}{1 - \sin\theta}$, we have a parabola with focus at the origin that opens upward. It crosses the y-axis at $r = 2/3$ when $\theta = -\pi/2$, and the x-axis at $r = 4/3$ when $\theta = 0$ and $\theta = \pi$.



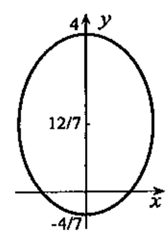
4. Since $r = \dfrac{16/5}{1 + (3/5)\cos\theta}$, we have an ellipse with foci on the x-axis. It crosses the x-axis at $r = 2$ when $\theta = 0$, and at $r = 8$ when $\theta = \pi$. The centre of the ellipse is $(-3, 0)$, and its maximum y-value is $b = \sqrt{25 - 9} = 4$.



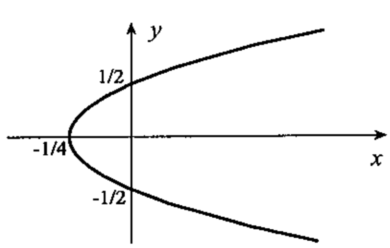
5. Since $r = \dfrac{4/3}{1 - (4/3)\sin\theta}$, we have a hyperbola with foci on the y-axis. It crosses the y-axis at $r = 4/7$ when $\theta = -\pi/2$, and the x-axis at $r = 4/3$ when $\theta = 0$ and $\theta = \pi$. Only the top half of the hyperbola is obtained.



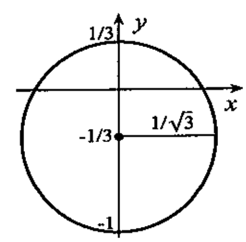
6. Since $r = \dfrac{1}{1 - (3/4)\sin\theta}$, we have an ellipse with foci on the y-axis. It crosses the y-axis at $r = 4$ when $\theta = \pi/2$, and at $r = 4/7$ when $\theta = -\pi/2$. The centre of the ellipse is $(0, 12/7)$ and its maximum x-value is $a = \sqrt{(16/7)^2 - (12/7)^2} = 4/\sqrt{7}$.



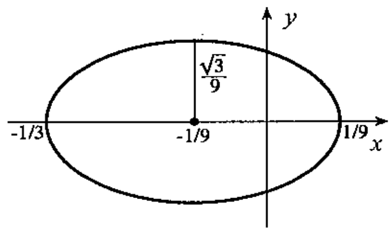
7. Since $r = \dfrac{1/2}{1 - \cos\theta}$, we have a parabola that opens to the right. It crosses the x-axis at $r = 1/4$ when $\theta = \pi$, and the y-axis at $r = 1/2$ when $\theta = \pm\pi/2$.



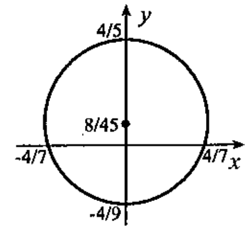
8. Since $r = \dfrac{1/2}{1 + (1/2)\sin\theta}$, we have an ellipse with foci on the y-axis. It crosses the y-axis at $r = 1/3$ when $\theta = \pi/2$, and at $r = 1$ when $\theta = -\pi/2$. The centre of the ellipse is $(0, -1/3)$ and its maximum x-value is $a = \sqrt{(2/3)^2 - (1/3)^2} = 1/\sqrt{3}$.



9. Since $r = \dfrac{1}{3\cos\theta + 6} = \dfrac{1/6}{1 + (1/2)\cos\theta}$, we have an ellipse with foci on the x-axis. It crosses the x-axis at $r = 1/9$ when $\theta = 0$, and at $r = 1/3$ when $\theta = \pi$. Its centre is $(-1/9, 0)$ and its maximum y-value is $\sqrt{(2/9)^2 - (1/9)^2} = \sqrt{3}/9$.



10. Since $r = \dfrac{4}{7 - 2\sin\theta} = \dfrac{4/7}{1 - (2/7)\sin\theta}$, we have an ellipse with foci on the y-axis. It crosses the y-axis at $r = 4/5$ when $\theta = \pi/2$, and at $r = 4/9$ when $\theta = -\pi/2$. The centre of the ellipse is $(0, 8/45)$ and its maximum x-value is $a = \sqrt{(28/45)^2 - (8/45)^2} = 4\sqrt{5}/15$.



11. We set $r = \sqrt{x^2+y^2}$ and $\sin\theta = y/\sqrt{x^2+y^2}$, $\quad \sqrt{x^2+y^2} = \dfrac{3}{1 - \dfrac{y}{\sqrt{x^2+y^2}}} = \dfrac{3\sqrt{x^2+y^2}}{\sqrt{x^2+y^2} - y}$. Thus,

$\sqrt{x^2+y^2} - y = 3 \Longrightarrow x^2 + y^2 = (y+3)^2$. This simplifies to $x^2 = 6y + 9$.

12. We set $r = \sqrt{x^2+y^2}$ and $\cos\theta = x/\sqrt{x^2+y^2}$, $\quad \sqrt{x^2+y^2} = \dfrac{1}{3 + \dfrac{x}{\sqrt{x^2+y^2}}} = \dfrac{\sqrt{x^2+y^2}}{3\sqrt{x^2+y^2} + x}$. Thus,

$3\sqrt{x^2+y^2} + x = 1 \Longrightarrow 9(x^2+y^2) = (1-x)^2$. This simplifies to $8x^2 + 9y^2 + 2x = 1$.

13. We set $r = \sqrt{x^2+y^2}$ and $\cos\theta = x/\sqrt{x^2+y^2}$, $\quad \sqrt{x^2+y^2} = \dfrac{1}{1 + \dfrac{2x}{\sqrt{x^2+y^2}}} = \dfrac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2} + 2x}$. Thus,

$\sqrt{x^2+y^2} + 2x = 1 \Longrightarrow x^2 + y^2 = (1-2x)^2$. This simplifies to $3x^2 - y^2 - 4x + 1 = 0$.

14. We set $r = \sqrt{x^2+y^2}$ and $\cos\theta = x/\sqrt{x^2+y^2}$, $\quad \sqrt{x^2+y^2} = \dfrac{2}{1 - \dfrac{3x}{\sqrt{x^2+y^2}}} = \dfrac{2\sqrt{x^2+y^2}}{\sqrt{x^2+y^2} - 3x}$. Thus,

$\sqrt{x^2+y^2} - 3x = 2 \Longrightarrow x^2 + y^2 = (3x+2)^2$. This simplifies to $8x^2 - y^2 + 12x + 4 = 0$.

15. We set $r = \sqrt{x^2+y^2}$ and $\sin\theta = y/\sqrt{x^2+y^2}$, $\quad \sqrt{x^2+y^2} = \dfrac{4}{6 - \dfrac{3y}{\sqrt{x^2+y^2}}} = \dfrac{4\sqrt{x^2+y^2}}{6\sqrt{x^2+y^2} - 3y}$. Thus,

$6\sqrt{x^2+y^2} - 3y = 4 \Longrightarrow 36(x^2+y^2) = (3y+4)^2$. This simplifies to $36x^2 + 27y^2 - 24y = 16$.

16. We set $r = \sqrt{x^2+y^2}$ and $\cos\theta = x/\sqrt{x^2+y^2}$, $\quad \sqrt{x^2+y^2} = \dfrac{4}{5 + \dfrac{5x}{\sqrt{x^2+y^2}}} = \dfrac{4\sqrt{x^2+y^2}}{5\sqrt{x^2+y^2} + 5x}$. Thus,

$5\sqrt{x^2+y^2} + 5x = 4 \Longrightarrow 25(x^2+y^2) = (4-5x)^2$. This simplifies to $25y^2 = 16 - 40x$.

17. For this hyperbola, $c = \sqrt{16+1} = \sqrt{17}$. If we choose the pole at the focus $(\sqrt{17}, 0)$ and the polar axis parallel to the positive x-axis, then

$$x = \sqrt{17} + r\cos\theta, \quad y = r\sin\theta.$$

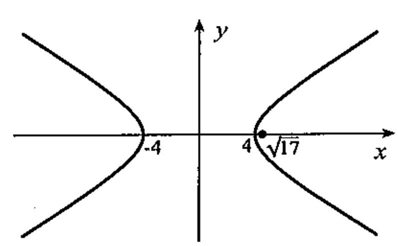
Substitution of these into the equation of the hyperbola gives

$$17 + 2\sqrt{17}r\cos\theta + r^2\cos^2\theta - 16r^2\sin^2\theta = 16,$$

or,

$$r^2(\cos^2\theta - 16\sin^2\theta) + r(2\sqrt{17}\cos\theta) + 1 = 0.$$

Solutions for r are



$$r = \frac{-2\sqrt{17}\cos\theta \pm \sqrt{68\cos^2\theta - 4(\cos^2\theta - 16\sin^2\theta)}}{2(\cos^2\theta - 16\sin^2\theta)} = \frac{-2\sqrt{17}\cos\theta \pm 8}{2(17\cos^2\theta - 16)} = \frac{-\sqrt{17}\cos\theta \pm 4}{17\cos^2\theta - 16}.$$

If we choose -4, then

$$r = \frac{-\sqrt{17}\cos\theta - 4}{(\sqrt{17}\cos\theta + 4)(\sqrt{17}\cos\theta - 4)} = \frac{1}{4 - \sqrt{17}\cos\theta}.$$

This gives the right half of the hyperbola. The other half is obtained when we set

$$r = \frac{-\sqrt{17}\cos\theta + 4}{17\cos^2\theta - 16} = \frac{-1}{4 + \sqrt{17}\cos\theta}.$$

18. For this ellipse, $c = \sqrt{9-4} = \sqrt{5}$. If we choose the pole at the focus $(\sqrt{5}, 0)$ and the polar axis along the positive x-axis, then

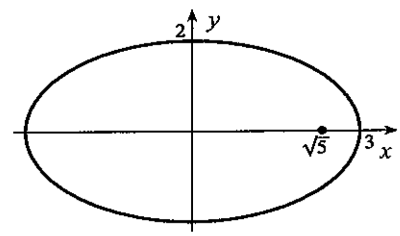
$$x = \sqrt{5} + r\cos\theta, \quad y = r\sin\theta.$$

Substitution of these into the equation of the ellipse gives

$$36 = 4(5 + 2\sqrt{5}r\cos\theta + r^2\cos^2\theta) + 9r^2\sin^2\theta,$$

or,

$$r^2(4\cos^2\theta + 9\sin^2\theta) + r(8\sqrt{5}\cos\theta) - 16 = 0.$$



Solutions for r are

$$r = \frac{-8\sqrt{5}\cos\theta \pm \sqrt{320\cos^2\theta - 4(-16)(4\cos^2\theta + 9\sin^2\theta)}}{2(4\cos^2\theta + 9\sin^2\theta)} = \frac{-8\sqrt{5}\cos\theta \pm 24}{2(9 - 5\cos^2\theta)} = \frac{4(\pm 3 - \sqrt{5}\cos\theta)}{9 - 5\cos^2\theta}.$$

If we choose $+3$ (to correspond to positive r-values),

$$r = \frac{4(3 - \sqrt{5}\cos\theta)}{(3 - \sqrt{5}\cos\theta)(3 + \sqrt{5}\cos\theta)} = \frac{4}{3 + \sqrt{5}\cos\theta}.$$

19. For this ellipse, $c = \sqrt{4-1} = \sqrt{3}$. If we choose the pole at the focus $(1, \sqrt{3} - 1)$ and the polar axis parallel to the positive x-axis, then

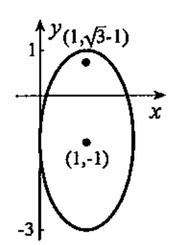
$$x = 1 + r\cos\theta, \quad y = (\sqrt{3} - 1) + r\sin\theta.$$

Substitution of these into the equation of the ellipse gives

$$\begin{aligned} 4 &= 4r^2\cos^2\theta + (\sqrt{3} + r\sin\theta)^2 \\ &= 4r^2\cos^2\theta + r^2\sin^2\theta + 2\sqrt{3}r\sin\theta + 3, \end{aligned}$$

or,

$$r^2(4\cos^2\theta + \sin^2\theta) + r(2\sqrt{3}\sin\theta) - 1 = 0.$$



Solutions for r are

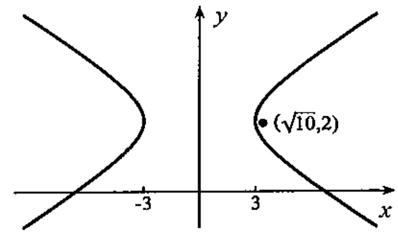
$$r = \frac{-2\sqrt{3}\sin\theta \pm \sqrt{12\sin^2\theta + 4(4\cos^2\theta + \sin^2\theta)}}{2(4\cos^2\theta + \sin^2\theta)} = \frac{-2\sqrt{3}\sin\theta \pm 4}{2(4 - 3\sin^2\theta)} = \frac{\pm 2 - \sqrt{3}\sin\theta}{4 - 3\sin^2\theta}.$$

If we choose $+2$ (to correspond to positive r-values),

$$r = \frac{2 - \sqrt{3}\sin\theta}{(2 - \sqrt{3}\sin\theta)(2 + \sqrt{3}\sin\theta)} = \frac{1}{2 + \sqrt{3}\sin\theta}.$$

20. For this hyperbola, $c = \sqrt{9+1} = \sqrt{10}$. If we choose the pole at the focus $(\sqrt{10}, 2)$ and the polar axis parallel to the positive x-axis, then

$$x = \sqrt{10} + r\cos\theta, \quad y = 2 + r\sin\theta.$$



Substitution of these into the equation of the hyperbola gives

$$10 + 2\sqrt{10}r\cos\theta + r^2\cos^2\theta - 9r^2\sin^2\theta = 9,$$

or,

$$r^2(\cos^2\theta - 9\sin^2\theta) + r(2\sqrt{10}\cos\theta) + 1 = 0.$$

Solutions for r are

$$r = \frac{-2\sqrt{10}\cos\theta \pm \sqrt{40\cos^2\theta - 4(\cos^2\theta - 9\sin^2\theta)}}{2(\cos^2\theta - 9\sin^2\theta)} = \frac{-2\sqrt{10}\cos\theta \pm 6}{2(10\cos^2\theta - 9)} = \frac{-\sqrt{10}\cos\theta \pm 3}{10\cos^2\theta - 9}.$$

If we choose -3, then

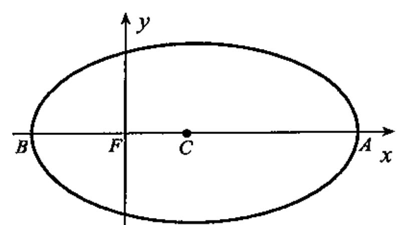
$$r = \frac{-\sqrt{10}\cos\theta - 3}{(\sqrt{10}\cos\theta + 3)(\sqrt{10}\cos\theta - 3)} = \frac{1}{3 - \sqrt{10}\cos\theta}.$$

This gives the right half of the hyperbola. The other half is obtained when we set

$$r = \frac{-\sqrt{10}\cos\theta + 3}{10\cos^2\theta - 9} = \frac{-1}{3 + \sqrt{10}\cos\theta}.$$

21. (a) We prove the case for the ellipse. Equation 9.29 gives the polar coordinate r of A as $\epsilon d/(1-\epsilon)$ (from $\theta = 0$) and that for B as $r = \epsilon d/(1+\epsilon)$ (from $\theta = \pi$). The r-coordinate of the centre of the ellipse is therefore

$$\frac{1}{2}\left(\frac{\epsilon d}{1-\epsilon} - \frac{\epsilon d}{1+\epsilon}\right),$$



and this is the length of CF. The length of AB is

$$\frac{\epsilon d}{1-\epsilon} + \frac{\epsilon d}{1+\epsilon}.$$

Consequently,

$$\frac{\|CF\|}{\|AB\|/2} = \frac{\frac{1}{2}\left(\frac{\epsilon d}{1-\epsilon} - \frac{\epsilon d}{1+\epsilon}\right)}{\frac{1}{2}\left(\frac{\epsilon d}{1-\epsilon} + \frac{\epsilon d}{1+\epsilon}\right)},$$

and this simplifies to ϵ.

(b) If we express equation 9.32 in the form $(x-h)^2/a^2 + y^2/b^2 = 1$, then

$$a^2 = \left(\frac{\epsilon d}{1-\epsilon^2}\right)^2, \qquad b^2 = (1-\epsilon^2)\left(\frac{\epsilon d}{1-\epsilon^2}\right)^2.$$

Consequently, $b^2/a^2 = 1 - \epsilon^2$, and this ratio determines the elongation of the ellipse. For small ϵ, $b \approx a$ and the ellipse is almost circular as in Figure 9.45a. For ϵ close to 1, b is very much less than a and the ellipse is very elongated as in Figure 9.45b.

22. According to equation 9.32, the distance between the foci of the ellipse determined by equation 9.29 is $2d\epsilon^2/(1-\epsilon^2)$. For this distance to approach zero, ϵ must approach zero.

23. (a) When $\epsilon = 0$, corresponding to a circular orbit, $I = \theta + C$.

(b) When $0 < \epsilon < 1$, the orbit is elliptic. We use the Weierstrass substitution $x = \tan(\theta/2)$ of Exercise 35 in Section 8.6. The integral becomes

$$I = \int \frac{1}{\left[1 + \frac{\epsilon(1-x^2)}{1+x^2}\right]^2} \frac{2}{1+x^2}\,dx = 2\int \frac{1+x^2}{[1+x^2+\epsilon-\epsilon x^2]^2}\,dx = 2\int \frac{1+x^2}{[(1+\epsilon)+(1-\epsilon)x^2]^2}\,dx.$$

We now set $x = \sqrt{\frac{1+\epsilon}{1-\epsilon}}\tan\phi$ and $dx = \sqrt{\frac{1+\epsilon}{1-\epsilon}}\sec^2\phi\,d\phi$:

$$\begin{aligned}
I &= 2\int \frac{1+\frac{1+\epsilon}{1-\epsilon}\tan^2\phi}{[1+\epsilon+(1+\epsilon)\tan^2\phi]^2}\sqrt{\frac{1+\epsilon}{1-\epsilon}}\sec^2\phi\,d\phi = \frac{2}{(1-\epsilon^2)^{3/2}}\int \frac{(1-\epsilon)+(1+\epsilon)\tan^2\phi}{\sec^2\phi}\,d\phi \\
&= \frac{2}{(1-\epsilon^2)^{3/2}}\int [(1-\epsilon)\cos^2\phi + (1+\epsilon)\sin^2\phi]\,d\phi \\
&= \frac{1}{(1-\epsilon^2)^{3/2}}\int [(1-\epsilon)(1+\cos 2\phi) + (1+\epsilon)(1-\cos 2\phi)]\,d\phi \\
&= \frac{1}{(1-\epsilon^2)^{3/2}}[2\phi - \epsilon\sin 2\phi] + C = \frac{1}{(1-\epsilon^2)^{3/2}}\left[2\mathrm{Tan}^{-1}\sqrt{\frac{1-\epsilon}{1+\epsilon}}x - \frac{2\epsilon\sqrt{\frac{1+\epsilon}{1-\epsilon}}x}{x^2+\frac{1+\epsilon}{1-\epsilon}}\right] + C \\
&= \frac{1}{(1-\epsilon^2)^{3/2}}\left[2\mathrm{Tan}^{-1}\sqrt{\frac{1-\epsilon}{1+\epsilon}}x - \frac{2\epsilon\sqrt{1-\epsilon^2}x}{(1-\epsilon)x^2+(1+\epsilon)}\right] + C \\
&= \frac{1}{(1-\epsilon^2)^{3/2}}\left[2\mathrm{Tan}^{-1}\left(\sqrt{\frac{1-\epsilon}{1+\epsilon}}\tan(\theta/2)\right) - \frac{2\epsilon\sqrt{1-\epsilon^2}\tan(\theta/2)}{(1-\epsilon)\tan^2(\theta/2)+(1+\epsilon)}\right] + C \\
&= \frac{1}{(1-\epsilon^2)^{3/2}}\left[2\mathrm{Tan}^{-1}\left(\sqrt{\frac{1-\epsilon}{1+\epsilon}}\tan(\theta/2)\right) - \frac{2\epsilon\sqrt{1-\epsilon^2}\sin(\theta/2)}{\cos(\theta/2)\left[(1-\epsilon)\frac{\sin^2(\theta/2)}{\cos^2(\theta/2)}+(1+\epsilon)\right]}\right] + C \\
&= \frac{1}{(1-\epsilon^2)^{3/2}}\left[2\mathrm{Tan}^{-1}\left(\sqrt{\frac{1-\epsilon}{1+\epsilon}}\tan(\theta/2)\right) - \frac{2\epsilon\sqrt{1-\epsilon^2}\sin(\theta/2)\cos(\theta/2)}{(1-\epsilon)\sin^2(\theta/2)+(1+\epsilon)\cos^2(\theta/2)}\right] + C \\
&= \frac{1}{(1-\epsilon^2)^{3/2}}\left[2\mathrm{Tan}^{-1}\left(\sqrt{\frac{1-\epsilon}{1+\epsilon}}\tan(\theta/2)\right) - \frac{\epsilon\sqrt{1-\epsilon^2}\sin\theta}{(1-\epsilon)(1-\cos\theta)/2+(1+\epsilon)(1+\cos\theta)/2}\right] + C \\
&= \frac{1}{(1-\epsilon^2)^{3/2}}\left[2\mathrm{Tan}^{-1}\left(\sqrt{\frac{1-\epsilon}{1+\epsilon}}\tan(\theta/2)\right) - \frac{\epsilon\sqrt{1-\epsilon^2}\sin\theta}{1+\epsilon\cos\theta}\right] + C.
\end{aligned}$$

(c) When $\epsilon = 1$, the orbit is parabolic. Once again we use the Weierstrass substitution,

$$\begin{aligned}
I &= \int \frac{1}{(1+\cos\theta)^2}\,d\theta = \int \frac{1}{\left(1+\frac{1-x^2}{1+x^2}\right)^2}\frac{2}{1+x^2}\,dx \\
&= 2\int \frac{1+x^2}{4}\,dx = \frac{1}{2}\left(x + \frac{x^3}{3}\right) + C = \frac{1}{2}\tan(\theta/2) + \frac{1}{6}\tan^3(\theta/2) + C.
\end{aligned}$$

(d) When $\epsilon > 1$, the orbit is a hyperbola. With the Weierstrass substitution we arrive, as in the elliptic case, at

$$I = 2\int \frac{1+x^2}{[(1+\epsilon)+(1-\epsilon)x^2]^2}\,dx.$$

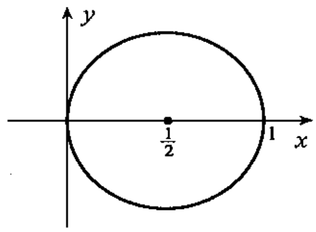
This time we set $x = \sqrt{\frac{1+\epsilon}{\epsilon-1}}\sin\phi$ and $dx = \sqrt{\frac{1+\epsilon}{\epsilon-1}}\cos\phi\,d\phi$,

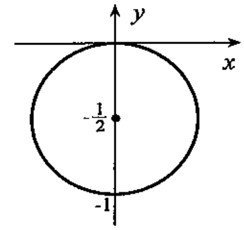
$$
\begin{aligned}
I &= 2\int \frac{1+\dfrac{1+\epsilon}{\epsilon-1}\sin^2\phi}{[(1+\epsilon)-(1+\epsilon)\sin^2\phi]^2}\sqrt{\frac{1+\epsilon}{\epsilon-1}}\cos\phi\,d\phi \\
&= \frac{2}{(\epsilon^2-1)^{3/2}}\int \frac{(\epsilon-1)+(1+\epsilon)\sin^2\phi}{\cos^4\phi}\cos\phi\,d\phi \\
&= \frac{2}{(\epsilon^2-1)^{3/2}}\int [(\epsilon-1)\sec^3\phi+(1+\epsilon)(\sec^3\phi-\sec\phi)]\,d\phi \\
&= \frac{2}{(\epsilon^2-1)^{3/2}}\int [2\epsilon\sec^3\phi-(1+\epsilon)\sec\phi]\,d\phi \\
&= \frac{2}{(\epsilon^2-1)^{3/2}}[\epsilon(\sec\phi\tan\phi+\ln|\sec\phi+\tan\phi|)-(1+\epsilon)\ln|\sec\phi+\tan\phi|]+C \\
&= \frac{2}{(\epsilon^2-1)^{3/2}}[\epsilon\sec\phi\tan\phi-\ln|\sec\phi+\tan\phi|]+C \\
&= \frac{2}{(\epsilon^2-1)^{3/2}}\left[\epsilon\frac{\sqrt{\dfrac{1+\epsilon}{\epsilon-1}}x}{\dfrac{1+\epsilon}{\epsilon-1}-x^2}-\ln\left|\frac{\sqrt{\dfrac{1+\epsilon}{\epsilon-1}}}{\sqrt{\dfrac{1+\epsilon}{\epsilon-1}-x^2}}+\frac{x}{\sqrt{\dfrac{1+\epsilon}{\epsilon-1}-x^2}}\right|\right]+C \\
&= \frac{2}{(\epsilon^2-1)^{3/2}}\left[\frac{\epsilon\sqrt{\epsilon^2-1}\,x}{(1+\epsilon)-(\epsilon-1)x^2}-\ln\left|\frac{\sqrt{\dfrac{1+\epsilon}{\epsilon-1}}+x}{\sqrt{\sqrt{\dfrac{1+\epsilon}{\epsilon-1}}-x}\sqrt{\sqrt{\dfrac{1+\epsilon}{\epsilon-1}}+x}}\right|\right]+C \\
&= \frac{2}{(\epsilon^2-1)^{3/2}}\left[\frac{\epsilon\sqrt{\epsilon^2-1}\,x}{(1+\epsilon)-(\epsilon-1)x^2}-\frac{1}{2}\ln\left|\frac{\sqrt{\dfrac{1+\epsilon}{\epsilon-1}}+x}{\sqrt{\dfrac{1+\epsilon}{\epsilon-1}}-x}\right|\right]+C \\
&= \frac{2}{(\epsilon^2-1)^{3/2}}\left[\frac{\epsilon\sqrt{\epsilon^2-1}\tan(\theta/2)}{(1+\epsilon)-(\epsilon-1)\tan^2(\theta/2)}-\frac{1}{2}\ln\left|\frac{\sqrt{\dfrac{1+\epsilon}{\epsilon-1}}+\tan(\theta/2)}{\sqrt{\dfrac{1+\epsilon}{\epsilon-1}}-\tan(\theta/2)}\right|\right]+C \\
&= \frac{1}{(\epsilon^2-1)^{3/2}}\left[\frac{2\epsilon\sqrt{\epsilon^2-1}\dfrac{\sin(\theta/2)}{\cos(\theta/2)}}{(1+\epsilon)-(\epsilon-1)\dfrac{\sin^2(\theta/2)}{\cos^2(\theta/2)}}-\ln\left|\frac{\sqrt{1+\epsilon}+\sqrt{\epsilon-1}\tan(\theta/2)}{\sqrt{1+\epsilon}-\sqrt{\epsilon-1}\tan(\theta/2)}\right|\right]+C \\
&= \frac{1}{(\epsilon^2-1)^{3/2}}\left[\frac{2\epsilon\sqrt{\epsilon^2-1}\sin(\theta/2)\cos(\theta/2)}{(1+\epsilon)\cos^2(\theta/2)-(\epsilon-1)\sin^2(\theta/2)}-\ln\left|\frac{\sqrt{1+\epsilon}+\sqrt{\epsilon-1}\tan(\theta/2)}{\sqrt{1+\epsilon}-\sqrt{\epsilon-1}\tan(\theta/2)}\right|\right]+C \\
&= \frac{1}{(\epsilon^2-1)^{3/2}}\left[\frac{\epsilon\sqrt{\epsilon^2-1}\sin\theta}{1+\epsilon\cos\theta}-\ln\left|\frac{\sqrt{1+\epsilon}+\sqrt{\epsilon-1}\tan(\theta/2)}{\sqrt{1+\epsilon}-\sqrt{\epsilon-1}\tan(\theta/2)}\right|\right]+C
\end{aligned}
$$

REVIEW EXERCISES

1. In Cartesian coordinates, $\sqrt{x^2+y^2}=\dfrac{x}{\sqrt{x^2+y^2}}\implies(x-1/2)^2+y^2=1/4.$
This is a circle.

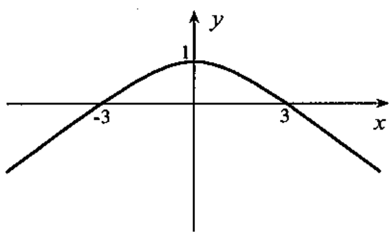
2. In Cartesian coordinates, $\sqrt{x^2+y^2}=\dfrac{-y}{\sqrt{x^2+y^2}}\implies x^2+(y+1/2)^2=1/4.$
This is a circle.



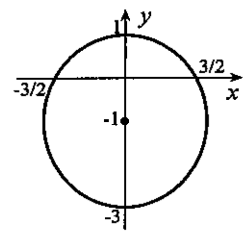


3. This is a hyperbola with foci on the y-axis.

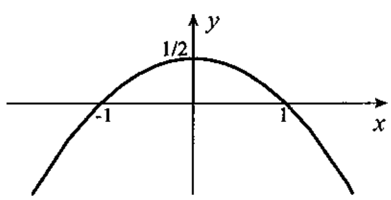
It crosses the y-axis at $r = 1$ when $\theta = \pi/2$ and the x-axis at $r = 3$ when $\theta = 0$ and $\theta = \pi$. Only the bottom half of the hyperbola is obtained.



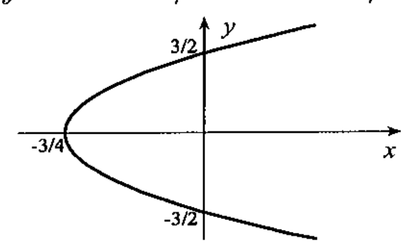
4. Since $r = \dfrac{3/2}{1 + (1/2)\sin\theta}$, the curve is an ellipse with foci on the y-axis. It crosses the y-axis at $r = 1$ when $\theta = \pi/2$, and at $r = 3$ when $\theta = -\pi/2$. The centre of the ellipse is $(0, -1)$, and its maximum x-value is $a = \sqrt{4 - 1} = \sqrt{3}$.



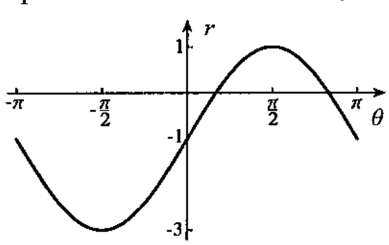
5. This is a parabola that opens downward. It crosses the y-axis at $r = 1/2$ when $\theta = \pi/2$ and the x-axis at $r = 1$ when $\theta = 0$ and $\theta = \pi$.

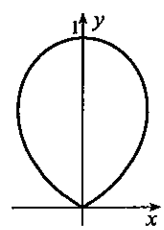


6. This is a parabola that opens to the right. It crosses the x-axis at $r = 3/4$ when $\theta = \pi$ and the y-axis at $r = 3/2$ when $\theta = \pm\pi/2$.

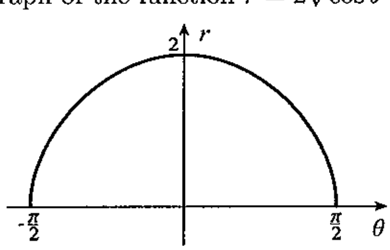


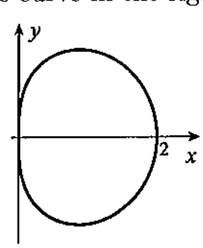
7. The graph of the function $r = -1 + 2\sin\theta$ in the left figure gives the curve in the right figure.



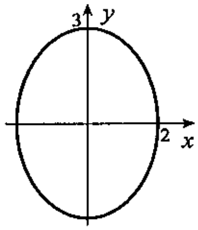


8. The graph of the function $r = 2\sqrt{\cos\theta}$ in the left figure gives the curve in the right figure.

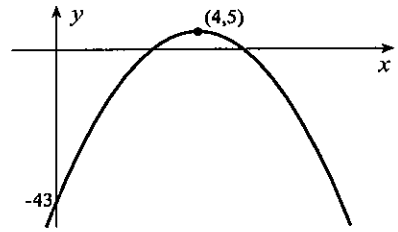




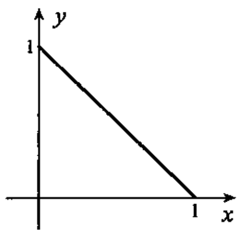
9. This is the ellipse $x^2/4 + y^2/9 = 1$.



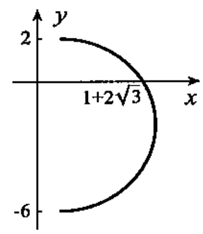
10. This is the parabola $y = 5 - 3(x-4)^2$.



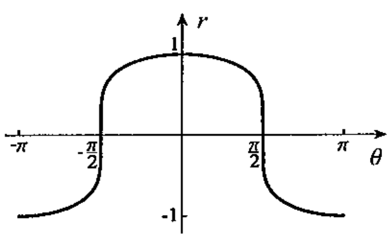
11. This is the line segment $x + y = 1$, $0 \le x \le 1$, traced four times.

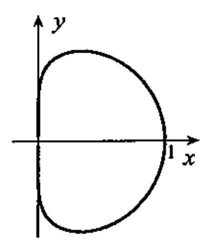


12. These equations describe the right half of of the circle $(x-1)^2 + (y+2)^2 = 16$.

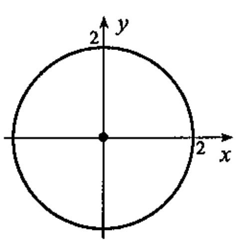


13. In polar coordinates $r^6 = r\cos\theta \implies r^5 = \cos\theta$. The graph of the function $r = (\cos\theta)^{1/5}$ in the left figure gives the curve in the right figure.

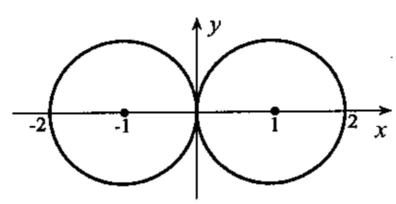




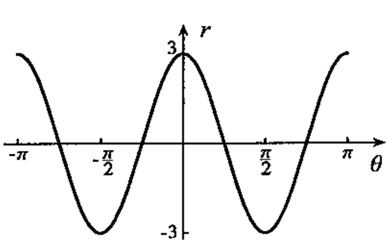
14. In polar coordinates, $r^2 = 2r$, and this describes the circle $r = 2$ and the origin $r = 0$.

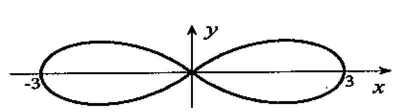


15. In Cartesian coordinates, $x^2 + y^2 = \dfrac{4x^2}{x^2+y^2} \implies x^2 + y^2 = \pm 2x$. From these $(x \pm 1)^2 + y^2 = 1$, two circles.



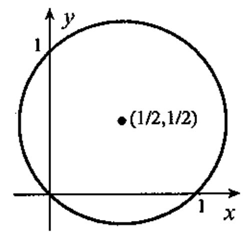
16. The graph of the function $r = 3\cos 2\theta$ in the left figure gives the curve in the right figure.



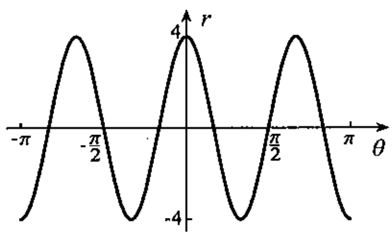


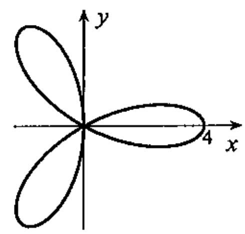
17. If we square the equation, $x^2 + 2xy + x^2 = x^2 + y^2$ and this implies that $x = 0$ or $y = 0$. Since x and y must be nonnegative, the equation defines the nonnegative x- and y-axes.

18. When squares are completed on x and y terms, $(x-1/2)^2+(y-1/2)^2=1/2$, a circle.

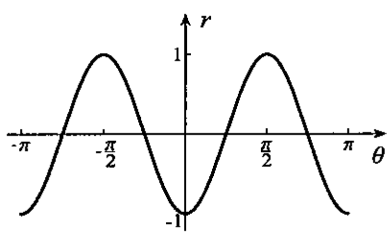


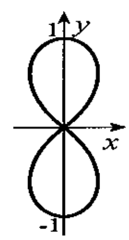
19. The graph of the function $r=4\cos 3\theta$ in the left figure gives the curve in the right figure.



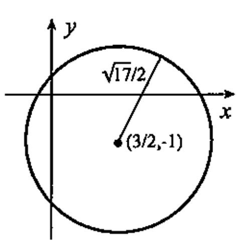


20. The graph of the function $r=\sin^2\theta-\cos^2\theta=-\cos 2\theta$ in the left figure gives the curve in the right figure.

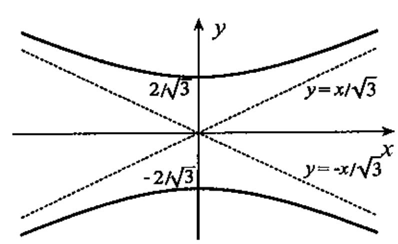




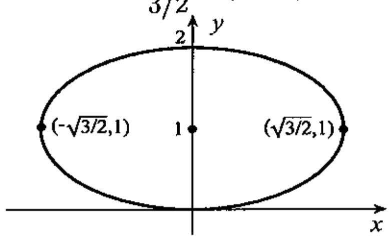
21. When we complete squares on x- and y-terms, $(x-3/2)^2+(y+1)^2=17/4$, a circle.



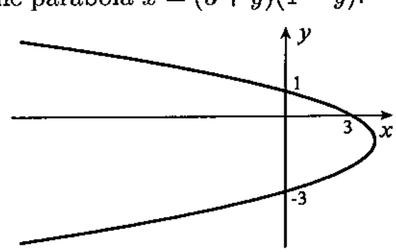
22. This is a hyperbola.



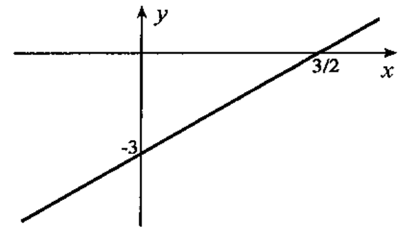
23. This is the ellipse $\dfrac{x^2}{3/2}+(y-1)^2=1$.



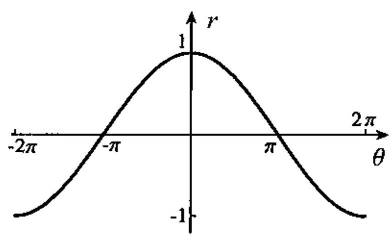
24. This is the parabola $x=(3+y)(1-y)$.

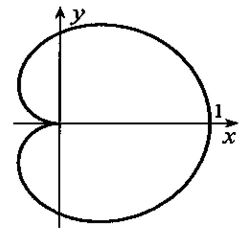


25. This is the line $2x - y = 3$.

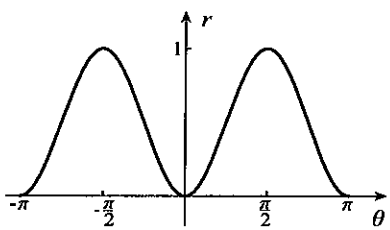


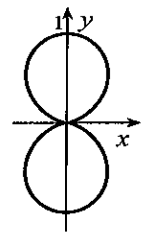
26. The graph of the function $r = \cos(\theta/2)$ in the left figure gives the curve in the right figure.





27. The graph of the function $r = \sin^2\theta$ in the left figure gives the curve in the right figure.

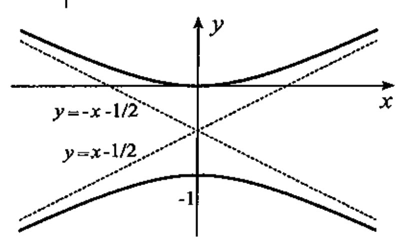




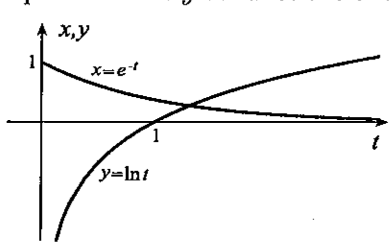
28. Completion of the square on the y term leads to

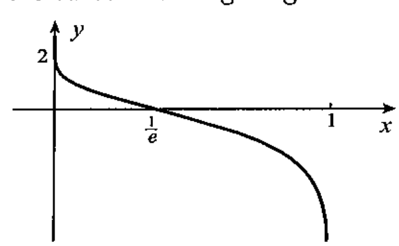
$$\frac{(y+1/2)^2}{1/4} - \frac{x^2}{1/4} = 1.$$

This is a hyperbola.

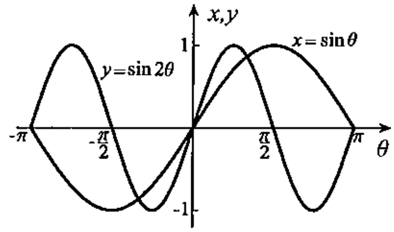


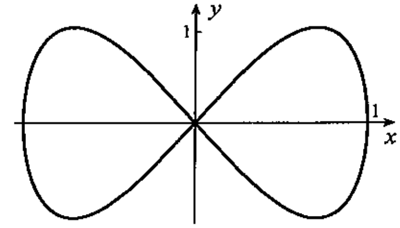
29. The graphs of x and y as functions of t in the left figure lead to the curve in the right figure.



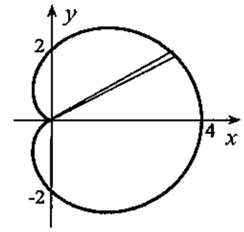


30. The graphs of x and y as functions of θ in the left figure lead to the curve in the right figure.

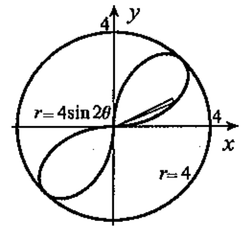




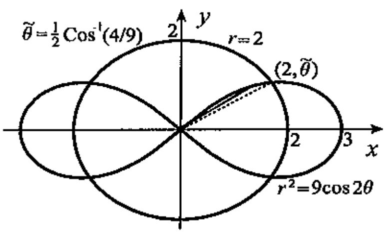
31. $$A = 2\int_0^{\pi} \frac{1}{2}(2+2\cos\theta)^2\,d\theta$$
$$= 4\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta$$
$$= 4\int_0^{\pi}\left(1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta$$
$$= 4\left\{\frac{3\theta}{2}+2\sin\theta+\frac{1}{4}\sin 2\theta\right\}_0^{\pi} = 6\pi$$



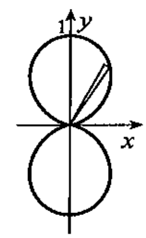
32. $$A = 16\pi - 2\int_0^{\pi/2}\frac{1}{2}(4\sin 2\theta)^2\,d\theta$$
$$= 16\pi - 16\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right)d\theta$$
$$= 16\pi - 8\left\{\theta - \frac{1}{4}\sin 4\theta\right\}_0^{\pi/2} = 12\pi$$



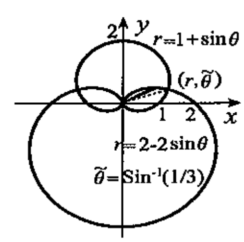
33. $$A = 4\left[4\pi\left(\frac{\tilde{\theta}}{2\pi}\right) + \int_{\tilde{\theta}}^{\pi/4}\frac{1}{2}(9\cos 2\theta)\,d\theta\right]$$
$$= 8\tilde{\theta} + 18\left\{\frac{1}{2}\sin 2\theta\right\}_{\tilde{\theta}}^{\pi/4} = 8\tilde{\theta} + 9(1-\sin 2\tilde{\theta})$$
$$= 8\tilde{\theta} + 9 - 9\sin 2\tilde{\theta} = 9 + 4\,\mathrm{Cos}^{-1}(4/9) - 9\sqrt{1-\frac{16}{81}}$$
$$= 9 + 4\,\mathrm{Cos}^{-1}(4/9) - \sqrt{65}$$



34. $$A = 4\int_0^{\pi/2}\frac{1}{2}(\sin^2\theta)^2\,d\theta$$
$$= 2\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta$$
$$= \frac{1}{2}\int_0^{\pi/2}\left(1-2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)d\theta$$
$$= \frac{1}{2}\left\{\frac{3\theta}{2}-\sin 2\theta+\frac{1}{8}\sin 4\theta\right\}_0^{\pi/2} = \frac{3\pi}{8}$$



35. $$A = 2\int_{-\pi/2}^{\tilde{\theta}}\frac{1}{2}(1+\sin\theta)^2\,d\theta + 2\int_{\tilde{\theta}}^{\pi/2}\frac{1}{2}(2-2\sin\theta)^2\,d\theta$$
$$= \int_{-\pi/2}^{\tilde{\theta}}\left(1+2\sin\theta+\frac{1-\cos 2\theta}{2}\right)d\theta$$
$$+4\int_{\tilde{\theta}}^{\pi/2}\left(1-2\sin\theta+\frac{1-\cos 2\theta}{2}\right)d\theta$$
$$= \left\{\frac{3\theta}{2}-2\cos\theta-\frac{1}{4}\sin 2\theta\right\}_{-\pi/2}^{\tilde{\theta}} + 4\left\{\frac{3\theta}{2}+2\cos\theta-\frac{1}{4}\sin 2\theta\right\}_{\tilde{\theta}}^{\pi/2}$$
$$= \frac{3\tilde{\theta}}{2} - 2\cos\tilde{\theta} - \frac{1}{4}\sin 2\tilde{\theta} + \frac{3\pi}{4} + 3\pi - 6\tilde{\theta} - 8\cos\tilde{\theta} + \sin 2\tilde{\theta} = \frac{15\pi}{4} - \frac{9\tilde{\theta}}{2} - 10\cos\tilde{\theta} + \frac{3}{2}\sin\tilde{\theta}\cos\tilde{\theta}$$
$$= \frac{15\pi}{4} - \frac{9}{2}\,\mathrm{Sin}^{-1}(1/3) - 10\left(\frac{2\sqrt{2}}{3}\right) + \frac{3}{2}\left(\frac{1}{3}\right)\left(\frac{2\sqrt{2}}{3}\right) = \frac{15\pi}{4} - \frac{9}{2}\,\mathrm{Sin}^{-1}(1/3) - \frac{19\sqrt{2}}{3}$$



36. Since $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3-3t^2}{3t^2+2}$,

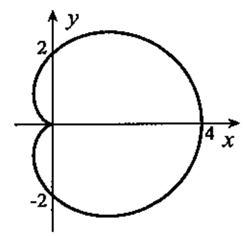
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{3-3t^2}{3t^2+2}\right) = \frac{d}{dt}\left(\frac{3-3t^2}{3t^2+2}\right)\frac{dt}{dx} = \frac{\dfrac{d}{dt}\left(\dfrac{3-3t^2}{3t^2+2}\right)}{dx/dt} = \frac{\dfrac{(3t^2+2)(-6t)-(3-3t^2)(6t)}{(3t^2+2)^2}}{3t^2+2} = \frac{-30t}{(3t^2+2)^3}.$$

37. Since $\dfrac{dy}{dx} = \dfrac{dy/du}{dx/du} = \dfrac{-3\sin u}{2\cos u} = -\dfrac{3}{2}\tan u$,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{3}{2}\tan u\right) = -\frac{3}{2}\frac{d}{du}(\tan u)\frac{du}{dx} = -\frac{3}{2}\frac{\dfrac{d}{du}(\tan u)}{dx/du} = -\frac{3}{2}\left(\frac{\sec^2 u}{2\cos u}\right) = -\frac{3}{4}\sec^3 u.$$

38. By equation 9.14,

$$\begin{aligned} L &= 2\int_0^{\pi}\sqrt{(2+2\cos\theta)^2+(-2\sin\theta)^2}\,d\theta \\ &= 4\sqrt{2}\int_0^{\pi}\sqrt{1+\cos\theta}\,d\theta \\ &= 4\sqrt{2}\int_0^{\pi}\sqrt{1+[2\cos^2(\theta/2)-1]}\,d\theta \\ &= 8\int_0^{\pi}\cos(\theta/2)\,d\theta = 8\Big\{2\sin(\theta/2)\Big\}_0^{\pi} = 16. \end{aligned}$$



39. According to equation 9.3, the length is

$$\int_0^1\sqrt{(2t)^2+(3t^2)^2}\,dt = \int_0^1 t\sqrt{4+9t^2}\,dt = \left\{\frac{1}{27}(4+9t^2)^{3/2}\right\}_0^1 = \frac{1}{27}(13\sqrt{13}-8).$$

40. By equation 9.3, the length is

$$\int_0^{\pi/2}\sqrt{(e^t\cos t - e^t\sin t)^2+(e^t\sin t+e^t\cos t)^2}\,dt = \int_0^{\pi/2}\sqrt{2}e^t\,dt = \sqrt{2}\Big\{e^t\Big\}_0^{\pi/2} = \sqrt{2}(e^{\pi/2}-1).$$

41. The slope of the tangent line is $\dfrac{dy}{dx}_{|\theta=\pi/6} = \dfrac{(-2\cos\theta)\sin\theta+(2-2\sin\theta)\cos\theta}{(-2\cos\theta)\cos\theta-(2-2\sin\theta)\sin\theta}_{|\theta=\pi/6} = 0$. Since Cartesian coordinates of the point are $(\sqrt{3}/2, 1/2)$, the equation of the tangent line is $y = 1/2$.