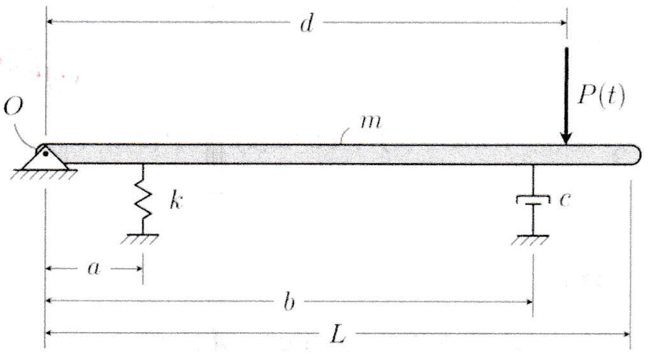
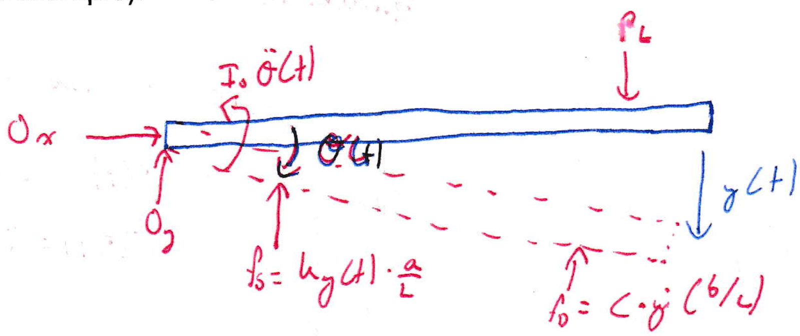
Example 2.3 Write the equation of motion for the following system:



Solution:

Begin by drawing the free body diagram for the mass (self-weight contribution neglected in this example).



Summing moments about O, (if you consider the self-weight, include the term $-mgL/2$)

$$\sum M_o = I_o \ddot{\theta} + k\,y(t)\cdot\frac{a}{L}\cdot a + c\dot{y}\,\frac{b}{L}\cdot b - Pd = 0 \tag{2.32}$$

For small angle θ, $\sin\theta \sim \theta$

$$\theta(t) = \frac{y(t)}{L}, \quad \dot{\theta} = \frac{\dot{y}(t)}{L}, \quad \ddot{\theta} = \frac{\ddot{y}(t)}{L} \tag{2.33}$$

Also, using the parallel-axis theorem,

$$I_o = \frac{mL^2}{12} + m\left(\frac{L}{2}\right)^2 = \frac{mL^2}{3} \tag{2.34}$$

Therefore, we can write

$$\frac{mL^2}{3}\ddot{\theta} + \frac{ka^2}{L}(L\theta) + \frac{cb^2}{L}(L\dot{\theta}) = Pd \tag{2.35}$$

$$\frac{mL^2}{3}\ddot{\theta} + ka^2\theta + cb^2\dot{\theta} = Pd \tag{2.36}$$

If self-weight is considered:

neglected bc not dynamic

$$\frac{mL^2}{3}\ddot{\theta}(t) + ka^2\theta(t) + cb^2\dot{\theta}(t) = P(t)d + mg\frac{L}{2} \tag{2.37}$$

2.3.1 Effect of Gravity Load

So far in our discussion of formulating the equations of motion, we have ignored the effect of gravity loads. We will consider this effect by examining the system shown in Figure 2.9.

$F: \frac{PL}{EA}$)

$u^t(t)$: total displacement from static equilibrium position

Since mg goes away,

$u^t(t) = \dot{u}(t)$,

$\ddot{u}^t(t) = \ddot{u}(t)$

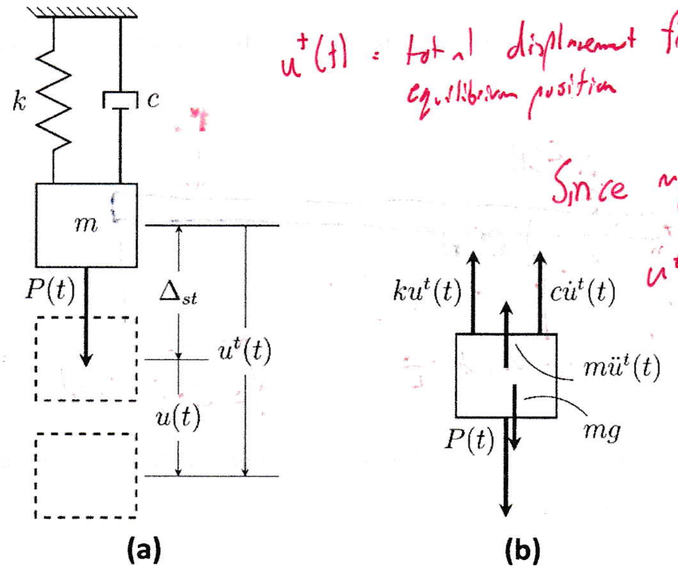


Figure 2.9 Vibrating suspended mass

The position of the mass at rest under gravity load is Δ_{st}.

- The spring is stretched an amount Δ_{st}.
- The amount Δ_{st} is given by

$$\Delta_{st} = \frac{mg}{K} \tag{2.38}$$

The position of the vibrating mass is specified by the distance $u^t(t)$ from the position of the mass when the spring is unstretched. Consider the free body diagram in Figure 2.9 (b).

- Applying the virtual work equation gives the equation of motion for the system.

$$m\ddot{u}^t(t) + c\dot{u}^t(t) + ku^t(t) = P(t) + mg \tag{2.39}$$

- By substituting $u^t(t) = u(t) + \Delta_{st}$, where $u(t)$ is the displacement from the position the mass occupies when it is at rest under the gravity loads, Equation 2.39 becomes

$$m\ddot{u}(t) + c\dot{u}(t) + k[u(t)+\Delta_{st}] = P(t) + mg \tag{2.40}$$

. Since $k\Delta_{st} = mg$, mg cancels out

- Substituting Equation 2.38 into Equation 2.40 gives

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = P(t) \tag{2.41}$$

. To find total displacement add mg back

- Therefore, the effect of the gravity load can be ignored in the formulation of the equation of motion.
- The solution to the equation of motion will produce the relative displacement from the at rest position of the mass under gravity loads.
- If the total displacement is of interest, the displacement of the mass under gravity loads can be added to the relative displacement obtained by the solution of the dynamic problem.

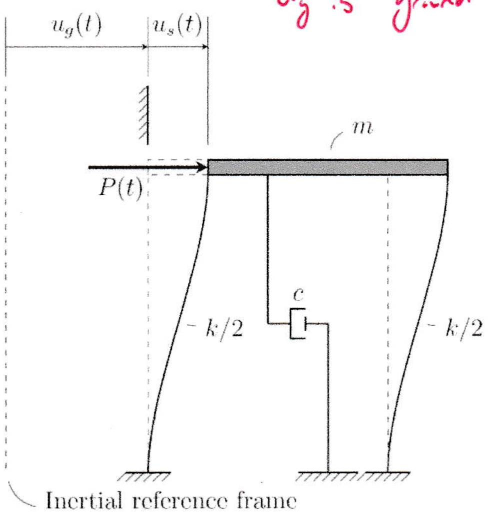
2.3.2 Ground-excited SDOF Structure

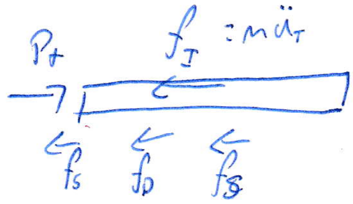
We will now consider how to develop the equation of motion when the force exciting a SDOF is the movement of the ground underneath, such as occurs during earthquake excitation.

u_g is ground displacement

u_s = structure displacement

$f_D = c\dot{u}_s(t)$
$= c(\dot{u}_T - \dot{u}_g)$





$f_S = k(u_T - u_g)$
$= k\,u_s(t)$

Figure 2.10 Ground-excited SDOF structure

The total displacement of the mass, with respect to the inertial reference frame, is

$$u_T(t) = u_g(t) + u_s(t) \tag{2.42}$$

where $u_g(t)$ is the displacement of the ground with respect to the inertial reference frame, and $u_s(t)$ is the displacement of the mass relative to the ground. The equation of motion for the SDOF system subjected to ground excitation is developed as follows:

$$m\ddot{u}_t + c\dot{u}_s(t) + ku_s(t) = P(t)$$
$$m(\ddot{u}_g + \ddot{u}_s) + c\dot{u}_s(t) + ku_s(t) = P(t) \quad (2.43)$$

Equation 2.43 can be rearranged so that the effect of the ground excitation is included as a force acting on the system.

$$m\ddot{u}_s(t) + c\dot{u}_s(t) + ku_s(t) = P(t) - m\ddot{u}_g(t) \quad (2.44)$$

Force on the structure is coming from itself

where $m\ddot{u}_g(t)$ is the inertial force acting on the structure due to the ground-excitation.

2.4 Natural Frequency for a SDOF Oscillator

Recall from Equation 2.20 the equation of motion for the SDOF oscillator is

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = P(t) \tag{2.45}$$

Begin by dividing Equation 2.45 by m.

$$\ddot{x}(t) + \frac{c}{m}\dot{x}(t) + \frac{k}{m}x(t) = \frac{P(t)}{m} \tag{2.46}$$

Now we introduce two important system parameters that characterize the vibration of linear SDOF systems. The first is the *undamped circular natural frequency*, ω_n and is given by

$$\omega_n^2 = \frac{k}{m}, \quad \omega_n = \sqrt{\frac{k}{m}}, \quad \text{units} = \text{rads/s} \tag{2.47}$$

and has units of radians per second (rad/s). The second quantity is the unitless *viscous damping* zeta *factor*, and is given by $\zeta = \frac{c}{2\sqrt{km}}$

(2.48)

The viscous damping factor can also be expressed in terms of c_{cr}, which is the *critical damping coefficient*.

$$c_{cr} = 2\sqrt{km}, \quad \zeta = \frac{c}{c_{cr}}$$
$$= 2m\omega_n \tag{2.49}$$

where

(2.50)

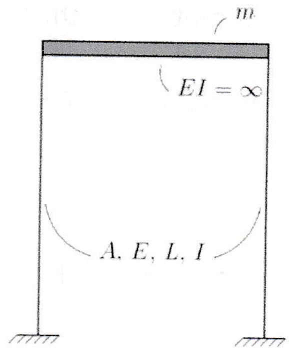
Substituting these quantities into Equation 2.46 gives

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = \omega_n^2\frac{P(t)}{k} \tag{2.48}$$

The circular natural frequency (ω_n) has a very important role to play in the study of structural dynamics.

- It depends on two key system properties, namely the mass (m) and stiffness (k).
- $\omega_n = 2\pi f_n$, where f_n is the natural frequency of the structure in cycles per second or Hertz (Hz).
- $\omega_n = 2\pi/T_n$, where T_n is the natural period of the structure in seconds (s).

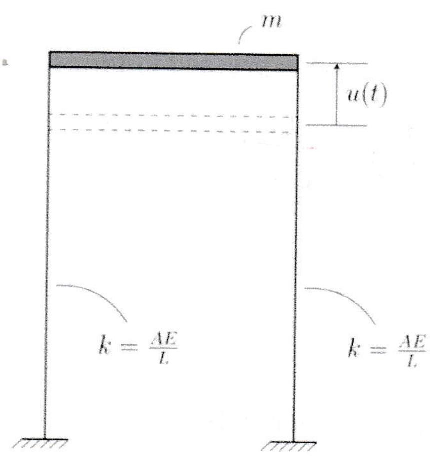
Example 2.4 Consider the following frame:



What is the natural frequency?

Without any knowledge of the nature of the applied external loads, it is hard to determine exactly how the frame will respond. We proceed by considering each mode of vibration separately.

<u>Mode 1: Axial</u>



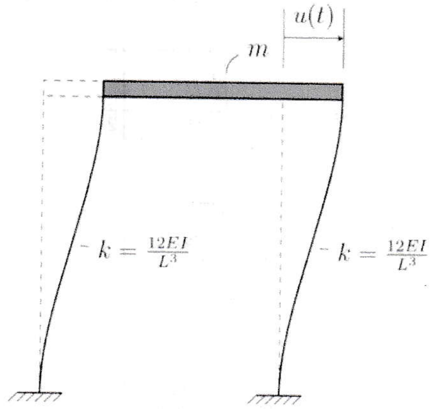
$m\ddot{u}(t) + k_{eff}\, u(t) = p(t)$

$\omega_{n,a} = \sqrt{\frac{k_{eff}}{m}} = \sqrt{\frac{2EA}{Lm}}$ rads/second

a for axial

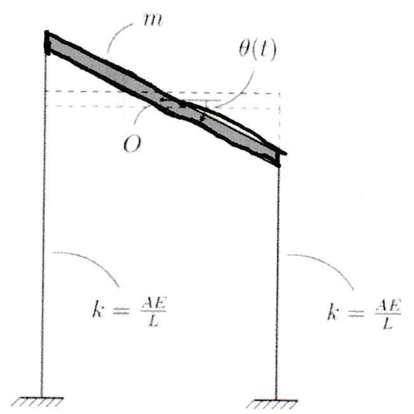
(2.52)

Mode 2: Lateral



$$\omega_{n,x} = \sqrt{\frac{k}{m}} = \sqrt{\frac{24EI}{L^3 m}} \text{ rads/sec} \tag{2.53}$$

Mode 3: Pitching



Assuming the span dimension is b:

$$\begin{aligned}\sum M_O^* &= \frac{mb^2}{12}\ddot{\theta}(t) + 2\left[\frac{kb\theta(t)}{2}\frac{b}{2}\right] \\ &= \underbrace{\frac{mb^2}{12}}_{I_o}\ddot{\theta}(t) + \underbrace{\frac{kb^2}{2}}_{k_{ef}}\theta(t) = 0\end{aligned} \tag{2.54}$$

Recognizing that this result is similar to the equation of motion for an undamped ($c = 0$) SDOF system under free vibration ($P(t) = 0$), then the natural frequency is

$\omega_{n,p} = \sqrt{\frac{k_{eff}}{I_c}}$

$$\omega_{np} = \sqrt{\frac{kb^2}{2}\frac{12}{mb^2}}$$

$$= \sqrt{\frac{6k}{m}} \qquad (2.55)$$

$$= \sqrt{\frac{6AE}{Lm}}$$

Review Appendix 6

2.5 Free Vibration of a SDOF System

= no damping, no external force

A structure is said to undergo free vibration when it is disturbed from its static equilibrium position and then allowed to oscillate without any further external disturbance. Let's investigate the behaviour of such a system.

- Recall the prototype SDOF equation of motion defined earlier in Equation 2.51 in the following form:

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x = \omega_n^2\frac{P(t)}{k} \tag{2.56}$$

- Consider when the system is undamped ($\zeta = 0$) and there is no external excitation applied to the system ($P(t) = 0$). Note that under these conditions, the system will only oscillate due to an initial disturbance.
- Equation 2.56 becomes

$$\ddot{x}(t) + \omega_n^2 x(t) = 0 \tag{2.57}$$

- We will now solve Equation 2.57. Begin by assuming a solution of the form

$$x(t) = Ce^{\lambda x}, \quad \dot{x} = C\lambda e^{\lambda t}, \quad \ddot{x} = C\lambda^2 e^{\lambda t} \tag{2.58}$$

- Substitute Equation 2.58 into Equation 2.57.

$$\lambda^2 Ce^{\lambda t} + \omega_n^2 Ce^{\lambda t} = 0$$
$$C(\lambda^2 + \omega_n^2)e^{\lambda t} = 0 \tag{2.59}$$

- Therefore, the constant λ is

For non-trivial solution, $\lambda = \pm\sqrt{-\omega_n^2} = \pm j\omega_n, \quad j = \sqrt{-1}$ (2.60)

- Hence, the solution to Equation 2.57 is given by

$$x_1(t) = a_1 e^{j\omega_n t}$$
$$x_2(t) = a_2 e^{-j\omega_n t} \tag{2.61}$$

- Since Equation 2.57 is linear, the superposition is also a solution.

$$x(t) = a_1 e^{j\omega_n t} + a_2 e^{-j\omega_n t} \tag{2.62}$$

- Let us try to simplify Equation 2.62 using Euler's Identities

$x(0) = a_1 + a_2$

$\dot{x}(0) = j\omega_n(a_1 + a_2)$

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta \tag{2.63}$$

$$2\cos\theta = e^{j\theta} + e^{-j\theta} \tag{2.64}$$

- The constants of integration a_1 and a_2 in Equation 2.62 are complex quantities. Using Euler's Identities, rewrite each constant in terms of its magnitude and a complex exponential. That is

$$a_1 = \frac{A}{2} e^{-j\phi} \tag{2.65}$$

Since $e^{j\omega_n t}$ and $e^{-j\omega_n t}$ in Equation 2.62 are complex conjugates, then a_1 and a_2 should also be complex conjugates.

$$a_2 = \frac{A}{2} e^{j\phi} \tag{2.66}$$

- Substituting Equations 2.65 and 2.66 into Equation 2.62,

$$x(t) = \frac{A}{2}\left[e^{j(\omega_n t-\phi)} + e^{-j(\omega_n t-\phi)}\right] \tag{2.67}$$

$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$

- Using Euler's Identity in Equation 2.64

Amplitude-phase form

$$x(t) = A\cos(\omega_n t - \phi) \tag{2.68}$$

- In Equation 2.68, A is the *amplitude* and ϕ is known as the *phase*. The phase, ϕ, is the amount by which $\cos(\omega_n t)$ "leads" or "lags" $x(t)$. In Equation 2.68, $x(t)$ lags an amount ϕ/ω_n due to the negative sign in front of ϕ.

- Continue by determining the following initial conditions:

$x(0) = A\cos\phi$ Displacement initial condition

$\dot{x}(0) = \omega_n A \sin\phi$ Velocity initial condition

- Now Equation 2.68 can be written as

initial displacement initial velocity

$$x(t) = A[\cos\omega_n t \cos\phi + \sin\omega_n t \sin\phi] \tag{2.69}$$

Undamped free vibration

$$\boxed{x(t) = x(0)\cos\omega_n t + \frac{\dot{x}(0)}{\omega_n}\sin\omega_n t} \tag{2.70}$$

Equation 2.70 is the solution for free-vibration of an undamped SDOF system.
Let us put Equation 2.70 in a slightly different form.

- We know that $x(0) = A\cos\phi$ and $\dot{x}(0) = \omega_n A\sin\phi$. Using these results, we can solve for A as

$$A = \sqrt{[x(0)]^2 + \left[\frac{\dot{x}(0)}{\omega_n}\right]^2} \tag{2.71}$$

2.72 APF

- It can also be shown that

$$\phi = \tan^{-1}\left[\frac{\dot{x}(0)}{x(0)\omega_n}\right] \tag{2.71}$$

- Then Equation 2.68 becomes

$$x(t) = \sqrt{[x(0)]^2 + \left[\frac{\dot{x}(0)}{\omega_n}\right]^2} \cos\left\{\omega_n t - \tan^{-1}\left[\frac{\dot{x}(0)}{x(0)\omega_n}\right]\right\} \tag{2.72}$$

  which is the same as Equation 2.70.

Consider the following alternate solution for the free-vibration of an undamped SDOF system:

$$\ddot{x}(t) + \omega_n^2 x = 0 \tag{2.74}$$

- Assuming a solution of the form

$$x(t) = Ce^{\lambda t} \tag{2.75}$$

- Substitute Equation 2.75 into Equation 2.74.

$$\begin{aligned} \lambda^2 Ce^{\lambda t} + \omega_n^2 Ce^{\lambda t} &= 0 \\ C(\lambda^2 + \omega_n^2)e^{\lambda t} &= 0 \end{aligned} \tag{2.76}$$

- Therefore, the constant λ is

$$\lambda = \pm j\omega_n \tag{2.77}$$

  where $j = \sqrt{-1}$.

- Hence, the solution is

$$x(t) = a_1 e^{j\omega_n t} + a_2 e^{-j\omega_n t} \tag{2.78}$$

  where a_1 and a_2 are complex quantities that need to be determined using initial conditions. Let

$$a_1 = a_{1R} + j a_{1I}, \quad a_2 = a_{2R} + j a_{2I} \tag{2.79}$$

- Recall the following identities:

$$\begin{aligned} e^{j\theta} &= \cos\theta + j\sin\theta \\ e^{-j\theta} &= \cos\theta - j\sin\theta \end{aligned} \tag{2.80}$$

$$\begin{aligned} e^{j\theta} + e^{-j\theta} &= 2\cos\theta \\ e^{j\theta} - e^{-j\theta} &= 2j\sin\theta \end{aligned} \tag{2.81}$$

- Equation 2.78 becomes

$$x(t) = (a_{1R} + ja_{1I})(\cos\omega_n t + j\sin\omega_n t) + (a_{2R} + ja_{2I})(\cos\omega_n t - j\sin\omega_n t)$$

$$= a_{1R}\cos\omega_n t - a_{1I}\sin\omega_n t + a_{2R}\cos\omega_n t + a_{2I}\sin\omega_n t + j(a_{1R}\sin\omega_n t + a_{1I}\cos\omega_n t + a_{2I}\cos\omega_n t - a_{2R}\sin\omega_n t) \quad (2.82)$$

- We know that $x(t)$ is real; therefore, the imaginary term must vanish. That is

$$x(t) = (a_{1R} - a_{2R})\sin\omega t + (a_{1I} + a_{2I})\cos\omega t = 0 \quad (2.83)$$

- This means

$$a_{1R} = a_{2R}$$
$$a_{2I} = -a_{1I} \quad (2.84)$$

That is, $a_1 = a_{1R} + ja_{1I}$ and $a_2 = a_{1R} - ja_{1I}$ are complex conjugates as well.

- Substituting this result into Equation 2.78

$$x(t) = (a_{1R} + ja_{1I})e^{j\omega t} + (a_{1R} - ja_{1I})e^{-j\omega t} \quad (2.85)$$
$$= 2a_{1R}\cos(\omega t) + 2j^2 a_{1I}\sin(\omega t)$$
$$x = A_1\cos\omega t + A_2\sin\omega t$$

where $A_1 = 2a_{1R}$ and $A_2 = -2a_{1I}$.

- A_1 and A_2 can be determined using initial conditions.

$$x(0) = A_1 \quad (2.86)$$
$$\dot{x}(0) = \omega A_2 \quad (2.87)$$

- Therefore

$$x(t) = x(0)\cos\omega_n t + \frac{\dot{x}(0)}{\omega_n}\sin\omega_n(t) \quad (2.88)$$

which is the same as Equation 2.70. we can also easily express this solution in amplitude-phase form as shown in Equation 2.72. Refer to the appendix.

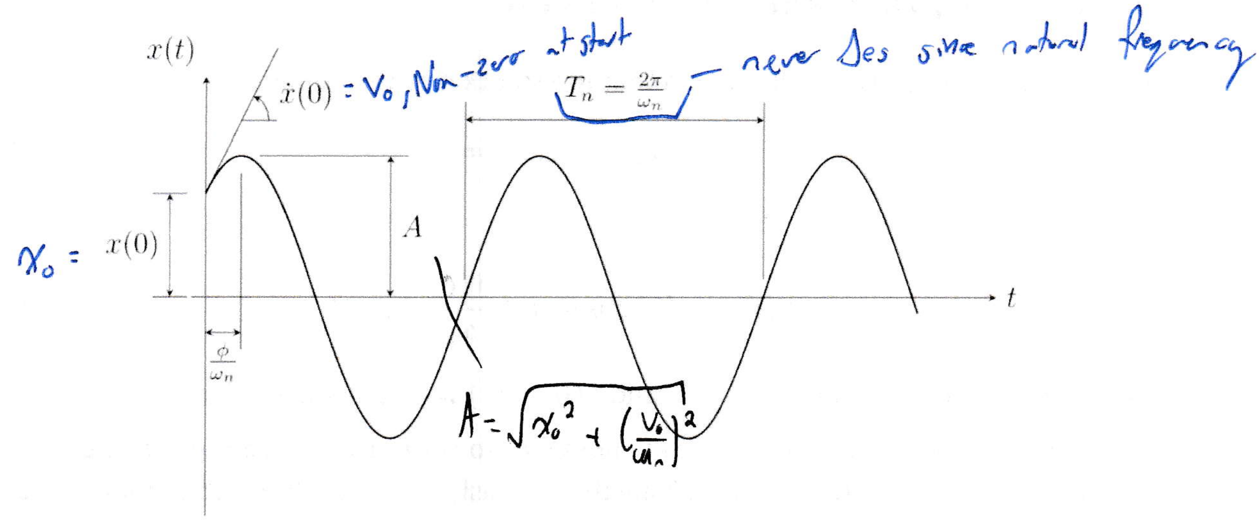


Figure 2.11 Free vibration response for a SDOF undamped system

$$x(t) = A\cos(\omega_n t - \phi)$$

2.6 Viscously Damped Free Vibration

We have seen in our previous discussion that the solution to the equation

$$m\ddot{x}(t) + kx(t) = 0 \tag{2.89}$$

is given by

$$x(t) = x(0)\cos\omega_n t + \frac{\dot{x}(0)}{\omega_n}\sin\omega_n t \tag{2.90}$$

- Given the initial conditions, $x(0)$ and $\dot{x}(0)$, $x(t)$ will oscillate indefinitely.
- We know that this is not physically possible, so we must include the effect of damping. We now turn our attention to solving the viscously damped SDOF system for free vibration.

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0 \tag{2.91}$$

- Begin by assuming the following form of the solution:

$$x(t) = Ce^{\lambda t} \tag{2.92}$$

- Substituting this into Equation 2.91, we get

$$Ce^{\lambda t}(m\lambda^2 + c\lambda + k) = 0 \tag{2.93}$$

- Since $Ce^{\lambda t}$ cannot be zero

$$m\lambda^2 + c\lambda + k = 0 \quad \rightarrow \text{Characteristic Equation} \tag{2.94}$$

- The roots are

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} = \frac{-c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4km} \tag{2.95}$$

- Using the following definitions

$\omega_n = \sqrt{\frac{k}{m}}$ (2.96) $\quad \zeta = \frac{c}{c_{cr}}$ (2.97) $\quad c_{cr} = 2\sqrt{km} = 2m\omega_n$ (2.98)

the roots can be rewritten as

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{2.99}$$

We will continue by considering three levels of damping.

Underdamped ($\zeta < 1$, $c < c_{cr}$), Overdamped ($\zeta > 1$, $c > c_{cr}$), Critically damped ($\zeta = 1$, $c = c_{cr}$)

Typical

Unpractical

2.6.1 Underdamped Motion ($\zeta < 1$)

When $\zeta < 1$, Equation 2.99 can be written as

$$\left.\begin{aligned} \lambda_1 &= -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} \\ \lambda_2 &= -\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2} \end{aligned}\right\} \text{Complex roots} \qquad \begin{aligned} &(2.100) \\ &(2.101) \end{aligned}$$

The overall solution can be written as

$$\begin{aligned} x(t) &= a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} \\ &= a_1 e^{\left(-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}\right)t} + a_2 e^{\left(-\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}\right)t} \\ &= \underbrace{e^{-\zeta\omega_n t}}_{\text{new "decay" term}} \left(a_1 e^{j\omega_n\sqrt{1-\zeta^2}t} + a_2 e^{-j\omega_n\sqrt{1-\zeta^2}t}\right) \end{aligned} \qquad (2.102)$$

The quantity $\omega_n\sqrt{1-\zeta^2}$ is known as the *damped natural frequency* and is denoted by

$$\text{let } \omega_n\sqrt{1-\zeta^2} = \omega_D \qquad . \; T_D = \frac{2\pi}{\omega_D} \; . \; \text{Since } \omega_D \downarrow,\; T_D \uparrow \qquad (2.103)$$

Therefore, we can rewrite Equation 2.102 as

$$x(t) = e^{-\zeta\omega_n t}\left[A\cos(\omega_D t) + B\sin(\omega_D t)\right] \qquad (2.104)$$

Following a procedure similar to the undamped case, we get

Letting $x(0) = x_0$ & $\dot{x}(0) = V_0$, we have

(2.105)

where A and B are integration constants determined from the initial conditions. From the displacement and velocity initial conditions, represented by

$$x(t) = e^{-\zeta\omega_n t}\left[x_0\cos\omega_D t + \frac{V_0 + \zeta\omega_n x_0}{\omega_D}\sin\omega_D t\right] \qquad (2.106)$$

Equation 2.105 becomes

(2.107)

where $\omega_D = \omega_n\sqrt{1-\zeta^2}$ is the damped circular natural frequency. The effect of the damping ratio, ζ, on the circular natural frequency, ω_n, is to lower ω_D compared to ω_n. That means that damping results in lowering the natural frequency or lengthening the natural period.

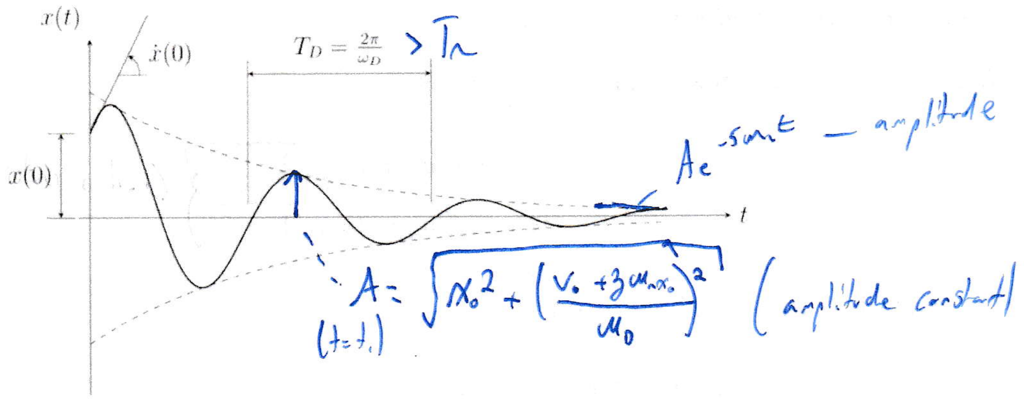


Figure 2.12 Free vibration response for a SDOF underdamped system

2.6.2 Overdamped Motion ($\zeta > 1$)

When $\zeta > 1$, Equation 2.99 can be written as

$$\lambda_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \tag{2.108}$$

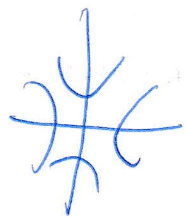
$$\lambda_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \tag{2.109}$$

Let $\omega_n\sqrt{\zeta^2 - 1} = \omega_n^*$. Then

$$\begin{aligned} x(t) &= a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} \\ &= a_1 e^{(-\zeta\omega_n + \omega_n^*)t} + a_2 e^{(-\zeta\omega_n - \omega_n^*)t} \\ &= e^{-\zeta\omega_n t}\left(a_1 e^{\omega_n^* t} + a_2 e^{-\omega_n^* t}\right) \end{aligned} \tag{2.110}$$

Using the following relationship between hyperbolic sine and cosine functions and the exponential function

$$\begin{aligned} e^{x} &= \cosh x + \sinh x \\ e^{-x} &= \cosh x - \sinh x \end{aligned} \tag{2.111}$$



∴ not vibrating back and forth

we get

$$\begin{aligned} x(t) &= e^{-\zeta\omega_n t}[a_1\cosh(\omega_n^* t) + a_1\sinh(\omega_n^* t) + a_2\cosh(\omega_n^* t) - a_2\sinh(\omega_n^* t)] \\ &= e^{-\zeta\omega_n t}[(a_1 + a_2)\cosh(\omega_n^* t) + (a_1 - a_2)\sinh(\omega_n^* t)] \\ &= e^{-\zeta\omega_n t}[A\cosh(\omega_n^* t) + B\sinh(\omega_n^* t)] \end{aligned} \tag{2.112}$$

where A and B are constants of integration which can be determined by initial conditions. From the displacement and velocity initial conditions $x_0 = x(0)$ and $v_0 = \dot{x}(0)$ we get

$$x(t) = e^{-\zeta\omega_n t}\left[x_0\cosh(\omega_n^* t) + \frac{v_0 + \zeta\omega_n x_0}{\omega_n^*}\sinh(\omega_n^* t)\right] \tag{2.113}$$

2.6.3 Critically Damped motion ($\zeta = 1$)

Finally, we will consider the case of critically damped motion, when $\zeta = 1$. Equation 2.99 can be written as

$$x_1(t) = a_1 e^{-\omega_n t} \quad \lambda_1 = \lambda_2 = -\omega_n \text{ . we need 2 roots! } \therefore \text{ find linear combination}$$
$$x_2(t) = f(t)\, x_1(t) \tag{2.114}$$

The solution is

$$x(t) = e^{-\zeta\omega_n t}(A + Bt) \tag{2.115}$$

where A and B are constants of integration determined using initial conditions. From the displacement and velocity initial conditions $x_0 = x(0)$ and $v_0 = \dot{x}(0)$ we get

$$x(t) = e^{-\omega_n t}\left(x_0 + (v_0 + \omega_n x_0)\right) \tag{2.116}$$

The response for the overdamped case (Equation 2.113) and critically damped case (Equation 2.116) are non-oscillatory in nature. Typical response of the two systems are compared in Figure 2.13.

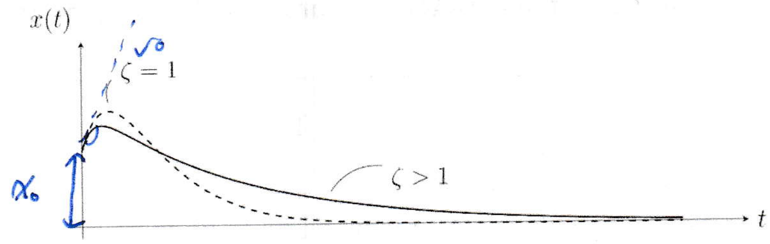


Figure 2.13 Free vibration response for a SDOF over-damped and critically damped system

2.6.4 Damped SDOF Oscillator

We have seen earlier that the damped circular frequency for the underdamped system is

$$\omega_D = \omega_n \sqrt{1 - \zeta^2} \tag{2.117}$$

or in terms of the period, in seconds,

$$T_D = \frac{T_n}{\sqrt{1 - \zeta^2}}, \quad T_D > T_n \tag{2.118}$$

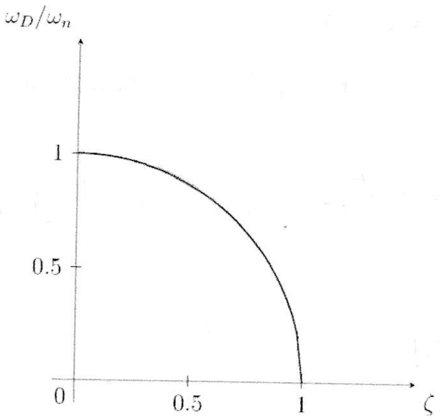


Figure 2.14 Relationship between damping and frequency

- The effect of damping is the lower the frequency or lengthen the period
- Now, let us look at the time response of the underdamped system given by

$$x(t) = e^{-\zeta\omega_n t}\left[x_0 \cos\omega_D t + \left(\frac{V_0+\zeta\omega_n x_0}{\omega_D}\right)\sin(\omega_D t)\right] \tag{2.119}$$

- The time response is generated in MATLAB™ for $\zeta = 0.1$ and $\omega_n = 2\pi$ as follows:

```
omega_n=2*pi; % Natural circular frequency
zeta=0.1; % Damping ratio
omega_D=omega_n*sqrt(1-zeta^2);
x_0=1; % Displacement initial condition
v_0=3; % Velocity initial condition
t=0:0.01:10;
x=exp(-zeta*omega_n*t).*(x_0*cos(omega_D.*t)+ ...
    (v_0+zeta*omega_n*x_0)/omega_D*sin(omega_D.*t));
plot(t,x)
```

- Similarly, for the case of the over-damped system given by

$$x(t) = e^{-\zeta\omega_n t}\left[x_0 \cosh(\omega_n^* t) + \frac{v_0 + \zeta\omega_n x_0}{\omega_n^*}\sinh(\omega_n^* t)\right] \tag{2.120}$$

- The time response is generated in MATLAB™ for $\zeta = 2$ and $\omega_n = 2\pi$ as follows:

```
omega_n=2*pi; % Natural circular frequency
zeta=2; % Damping ratio
omega_n2=omega_n*sqrt(zeta^2-1);
x_0=1; % Displacement initial condition
v_0=3; % Velocity initial condition
t=0:0.01:10;
x=exp(-zeta*omega_n.*t).*(x_0*cosh(omega_n2.*t)+ ...
    (v_0+zeta*omega_n*x_0)/omega_n2*sinh(omega_n2.*t));
plot(t,x)
```

Figure 2.15 compares the underdamped and over-damped response with the undamped system.

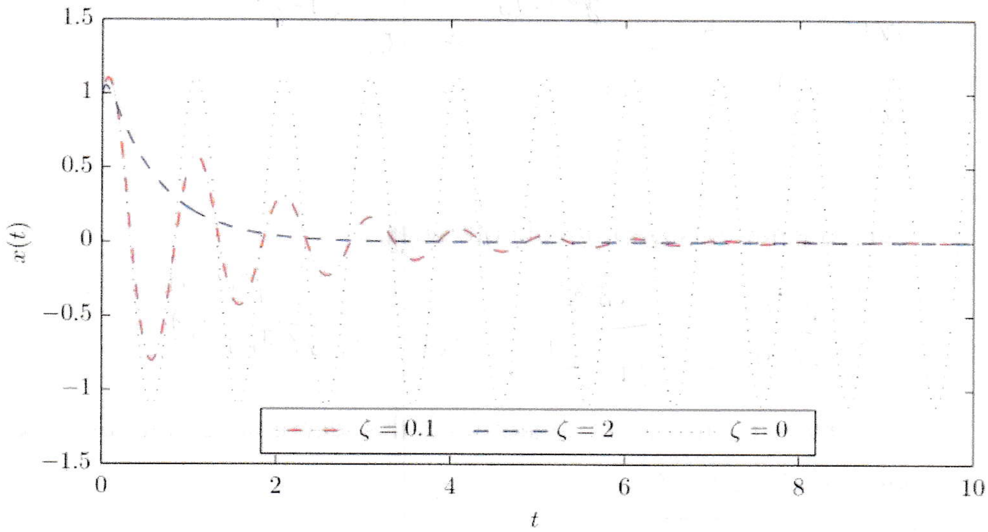


Figure 2.15 Comparison of undamped, underdamped, and over-damped system responses

2.6.5 Decay of Motion for Underdamped Systems

- Most civil structures exhibit underdamped motion.
- If you assumed that ζ is very small (most structures have damping ratios of less than 0.1), then the damped natural frequency, ω_D, and the undamped natural frequency, ω_n, are nearly the same. Hence, given a free-response plot of the displacement, velocity, or acceleration, we can calculated ω_n (or ω_D) or T_n simply by inspection.
- Calculating the damping ratio, ζ, is a bit more involved. Let us investigate how this can be done.

– First, consider the solution to the underdamped free-vibration response in Equation 2.107.

$$x(t) = e^{-\zeta\omega_n t}\left(x_0 \cos\omega_D t + \frac{v_0 + \zeta\omega_n x_0}{\omega_D}\sin\omega_D t\right) \tag{2.121}$$

– Now, if we substitute $t = t + T_D$ where $T_D = 2\pi/\omega_D$, we get – At 1 cycle away

$$x(t+T_D) = e^{-\zeta\omega_n(t+T_D)}\left[x_0 \cos\omega_D(t+T_D) + \left(\frac{v_0+\zeta\omega_n x_0}{\omega_D}\right)\sin\omega_D(t+T_D)\right] \tag{2.122}$$

– Hence,

$$\frac{x(t)}{x(t+T_D)} = e^{-\zeta\omega_n T_D} = e^{\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}} \tag{2.123}$$

– Taking the natural logarithm of both sides,

$$\ln\left[\frac{x(t)}{x(t+T_D)}\right] = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}, \quad \text{let } \ln\left[\frac{x(t)}{x(t+T_D)}\right] = \delta \tag{2.124}$$

In other words, the natural logarithm of the "decrement" between two successive peaks is $2\pi\zeta/\sqrt{1-\zeta^2}$.

– If ζ is small, $\sqrt{1-\zeta^2} \approx 1$. Therefore,

$$\delta \approx 2\pi\zeta \tag{2.125}$$

where x_i and x_{i+1} are the values of the displacement, velocity, or acceleration response for two successive peaks.

– Therefore, if you are able to measure δ, you can calculate ζ.

– Let's investigate for what values of ζ the approximation $\sqrt{1-\zeta^2} \approx 1$ is true.

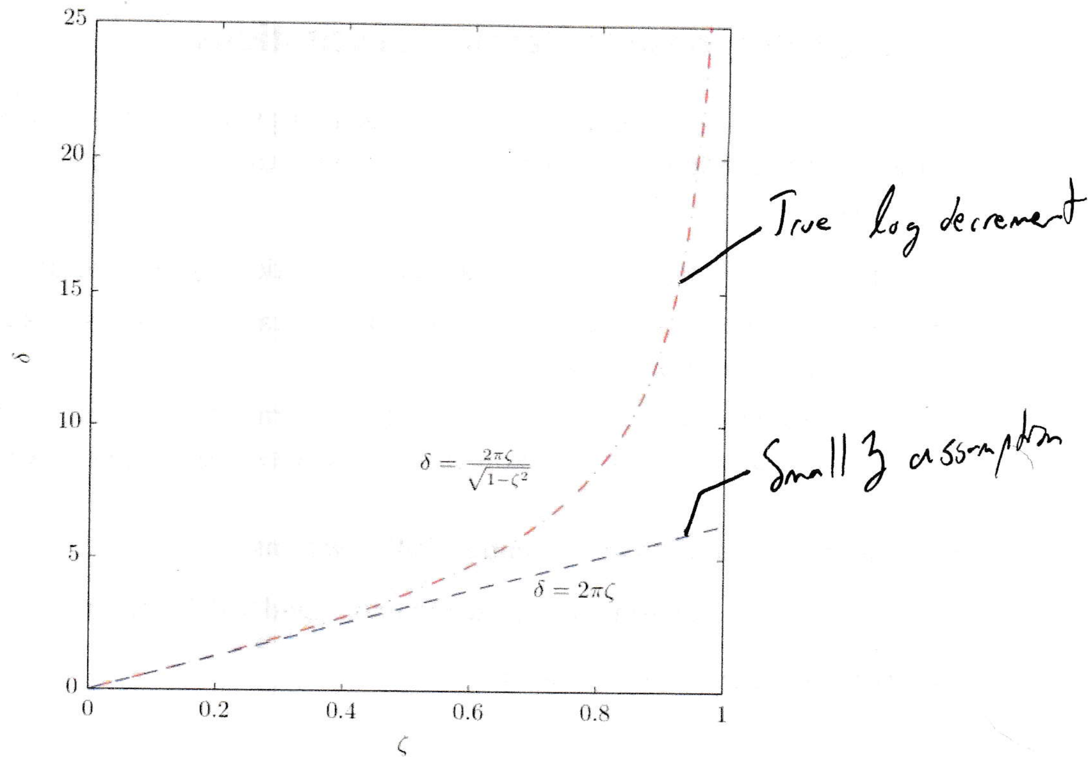


Figure 2.16 Comparison of δ quantities for $\zeta < 1$

- From Figure 2.16 the approximation is valid for $\zeta < 0.2$. The MATLAB™ script used to generate the plot is

```
zeta=0:0.01:1;
Delta1=2*pi.*zeta./(sqrt(1.-zeta.^2));
Delta2=2*pi.*zeta;
plot(zeta,Delta1,zeta,Delta2)
```

- When the amount of damping is small, measuring δ is difficult. You may have to measure the decrement across several cycles. In such a situation

$$\delta = \frac{1}{N}\ln\left(\frac{x_i}{x_{i+N}}\right) = 2\pi\zeta \qquad (2.126)$$

where N is the number of cycles you are measuring δ across.

2.7 SDOF Response to Harmonic Excitations

We will now begin the study of forced motion, beginning with structures subjected to a harmonic (or sinusoidal) excitation. The study of the response to harmonic excitation is important for several reasons.

- Many structures are subject to harmonic excitations, such as rotating machinery.
- Understanding the response to harmonic excitations provides insight into other more complex types of excitations.
- The response due to harmonic excitations forms the basis for experimental structural dynamics and is useful for identifying frequencies and damping in experimental studies.

2.7.1 Harmonic Vibration in Undamped SDOF Systems

We begin by studying a harmonically excited undamped SDOF system.

- Let the excitation be of the form

(2.127)

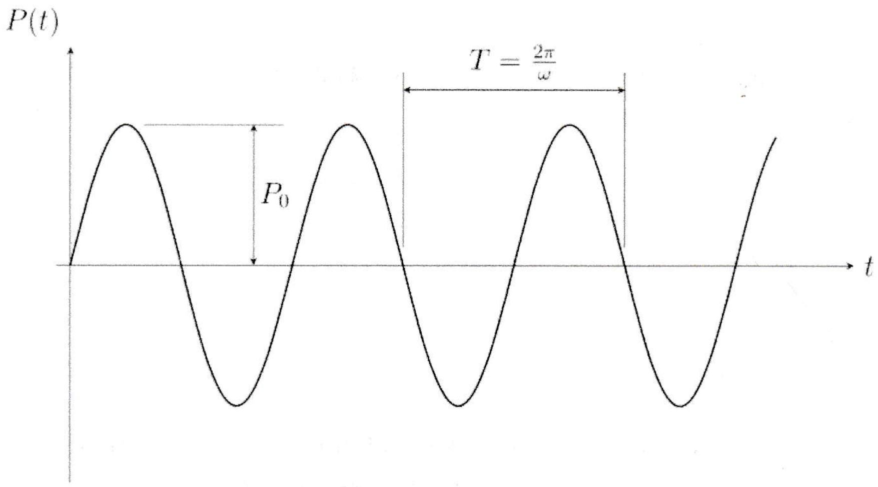


Figure 2.17 Harmonic excitation of the form $P(t) = P_0 \sin \omega t$

- The equation of motion for an undamped SDOF system is

$m\ddot{x} + kx = P(t)$
$= P_0 \sin(\omega t)$ (2.128)

- The solutions to Equation 2.128 is given by the sum of the complementary and particular solutions.

$$x(t) = x_c(t) + x_p(t) \tag{2.129}$$

- The complementary solution is of the free-vibration form seen earlier in Equation 2.85

$$m\ddot{x} + kx = 0 \rightarrow x_c(t) = A_1\cos(\omega_n t) + A_2\sin(\omega_n t) \tag{2.130}$$

- The particular solution depends on the form of the dynamic loading. In the case of harmonic excitation, it is reasonable to assume that the corresponding motion is also harmonic and in phase with the loading. Therefore, assume the particular solution is of the form

$$x_p(t) = C\sin(\omega t) \qquad \dot{x} = C\omega\cos(\omega t) \qquad \ddot{x} = -C\omega^2\sin(\omega t) \tag{2.131}$$

- Substituting Equation 2.131 into Equation 2.128 gives

$$-Cm\omega^2\sin\omega t + Ck\sin(\omega t) = P_0\sin(\omega t)$$
$$C[k - m\omega^2]\sin\omega t = P_0\sin\omega t \tag{2.132}$$

- Therefore,

$$P_0 = C[k - m\omega^2] \tag{2.133}$$

$$C = \frac{P_0}{k - m\omega^2} \tag{2.134}$$

- The particular solution is

$$x_p(t) = \frac{P_0}{k - m\omega^2}\sin(\omega t) \tag{2.135}$$

- Recognizing that $k/m = \omega_n^2$,

$$x_p(t) = \frac{P_0}{k\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]}\sin\omega t \tag{2.136}$$

$x_c(t)$ is known as transient response (free vibration). Depends on initial condition, vibrates @ ω_n, decays w/ damping

$x_p(t)$ is known as steady state response, depends on excitation, vibrates @ ω (forcing frequency)

↳ Continues as long as there is input excitation

- The complete solution is given by

$$x(t) = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) + \frac{P_0}{k\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]} \sin \omega t \tag{2.137}$$

where the constants A_1 and A_2 in Equation 2.137 can be found using the initial conditions.

  – Using the displacement initial condition $x(0) = x_0$, we find that

$$\left[A_1 = x_0\right] \text{ Initial displacement} \tag{2.138}$$

  – Using the velocity initial condition $\dot{x}(0) = v_0$, we find that

$$\left[A_2 = \frac{v_0}{\omega_n} - \frac{P_0\left(\frac{\omega}{\omega_n}\right)}{k\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\right] \text{ Initial velocity} \tag{2.139}$$

↑ resonance when $\omega \approx \omega_n$

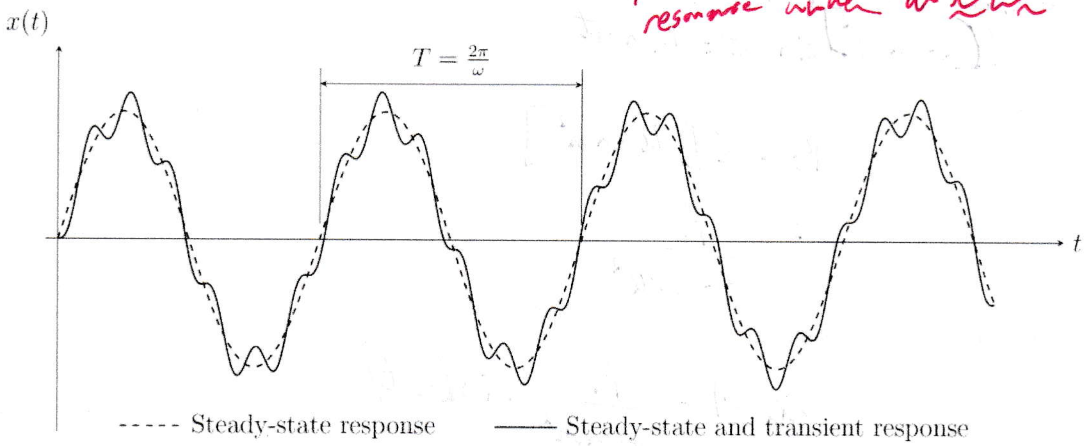


Figure 2.18 Combined transient and steady-state response for a harmonically excited system

2.7.2 Amplification Factor for Undamped SDOF Systems

Let us examine the steady-state response (say, after the transient response decays, which occurs in damped systems).

$$\begin{aligned} x_p(t) &= \frac{P_0}{k\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]} \sin \omega t \\ &= \frac{P_0}{k} \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2} \sin \omega t \end{aligned} \tag{2.140}$$

let $x_{st} = P_0/k$

- Now, P_0/k is the static deflection of a spring with stiffness k due to an applied load P. Let

$$x_{st} = P_0/k \tag{2.141}$$

- Equation 2.140 becomes

$$\frac{x_p(t)}{x_{st}} = \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}\sin(\omega t) \tag{2.142}$$

- Since the maximum value of $\sin\omega t = 1$, we define a new quantity known as the *amplification factor*, AF.

$$\frac{|x_{p,max}|}{x_{st}} = |AF| = \left(\frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right) \text{ free response} \tag{2.143}$$

Figure 2.19 shows a plot of the absolute value of the amplification factor for illustration purposes.

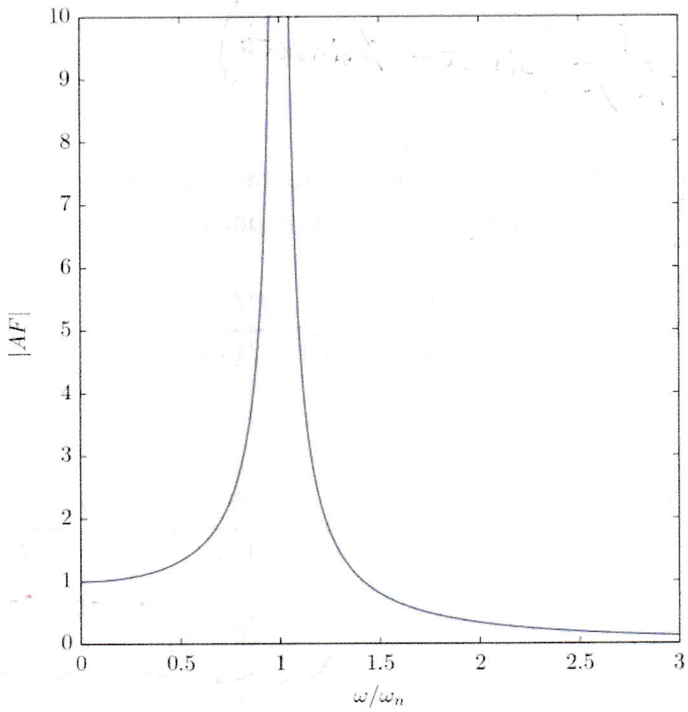


Figure 2.19 Amplification factor for an harmonically excited undamped system

The MATLAB™ script used to generate the above plot is given is as follows:

```
omega_ratio=[(0:0.01:0.99)'; (1.01:0.01:3)']';
AF=abs(1./(1-omega_ratio.^2));
plot(omega_ratio,AF)
ylim([0 10])
```

2.7.3 Resonant Response of an Undamped System

Recall the total response for the undamped SDOF oscillator. transient , steady state

$$x(t) = \left\{ x_0 \cos\omega_n t + \left[\frac{v_0}{\omega_n} - \frac{P_0\left(\frac{\omega}{\omega_n}\right)}{k\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\right]\sin\omega_n t \right\} + \left\{\frac{P_0}{k\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\sin\omega t\right\} \tag{2.144}$$

Under the special case of at rest initial conditions ($x(0) = \dot{x}(0) = 0$), Equation 2.144 becomes:

$$x(t) = \frac{P_0}{k}\cdot\frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}\left(\sin\omega t - \frac{\omega}{\omega_n}\sin\omega_n t\right), \quad \text{let } \frac{\omega}{\omega_n} = \phi \tag{2.145}$$

which can be rewritten as

$$x(t) = \frac{P_0}{k}\cdot\frac{1}{1-\phi^2}\left(\sin\omega t - \phi\sin\omega_n t\right) \tag{2.146}$$

Letting $\phi = 1$ in Equation 2.146 does not allow us to understand the resonant response. Hence, we use L'Hopital's rule to understand the resonant response.

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)} \tag{2.147}$$

Therefore,

$$\begin{aligned}\lim_{\phi\to 1}\frac{P_0}{k}\frac{1}{1-\phi^2}(\sin\omega t - \phi\sin\omega_n t) &= \lim_{\phi\to 1}\frac{P_0}{k}\frac{\omega_n t\cos\phi\omega_n t - \sin\omega_n t}{-2\phi} \\ &\quad \text{dominant as } \phi \to 1 \\ &= \frac{P_0}{2k}(\sin\omega_n t - \omega_n t\cos\omega_n t)\end{aligned} \tag{2.148}$$

Figure 2.20 plots this response for $P_0 = k = 1$ and $\omega_n = 5\pi$.

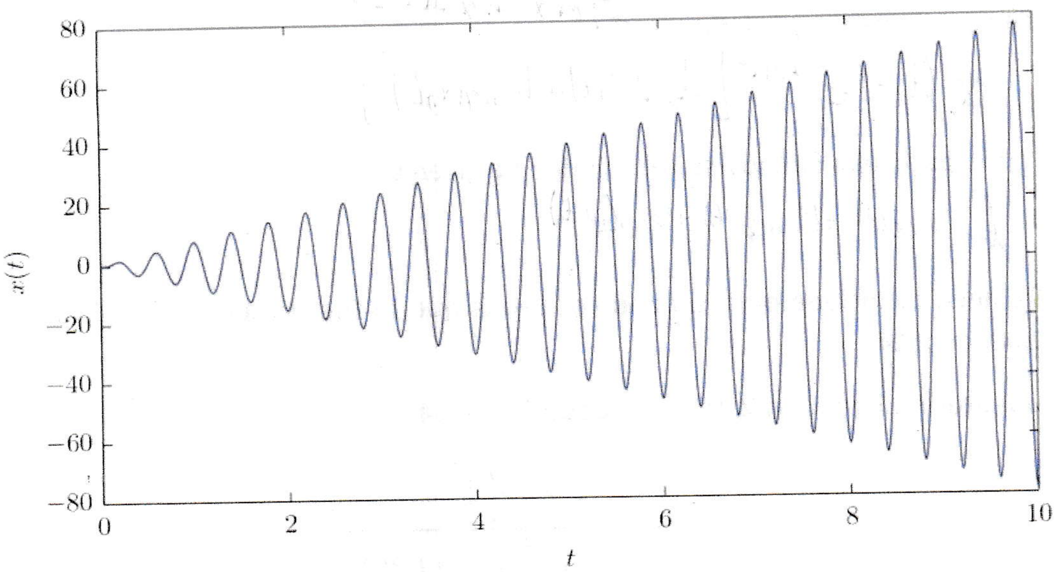


Figure 2.20 Resonant response

The MATLAB™ script used to generate the above plot is as follows:

```
t=0:0.01:10;
omega_n=2.5*2*pi;
P_0=1;
k=1;
x=P_0/(2*k)*(sin(omega_n.*t)-omega_n.*t.*cos(omega_n.*t));
plot(t,x)
```

The condition where $\phi = 1$, or $\omega = \omega_n$ is known as *resonance*, and it is obvious from Figures 2.19 and 2.20 that excitation frequencies near the structure's natural frequency results in a very large response. The interest in resonance stems in large part from the necessity to avoid the resonance condition, where large-amplitude motion is likely to occur.

2.7.4 Harmonic Vibration of Viscously Damped SDOF Systems

We will now consider the solution to the equation of motion for a SDOF system with viscous damping subjected to a harmonic forcing function.

- The equation of motion can be written as

$$m\ddot{x} + c\dot{x} + kx = P_0 \sin \omega t$$

$$\ddot{x} + 2\xi\omega_n \dot{x} + \omega_n^2 x = \frac{P_0}{k}\omega_n^2 \sin \omega t \tag{2.149}$$

- The complementary solution is $\rightarrow m\ddot{x} + c\dot{x} + kx = 0$

$$x_c(t) = e^{-\zeta\omega_n t}\left[A\cos(\omega_D t) + B\sin(\omega_D t)\right] \tag{2.150}$$

- Let the particular (steady-state) solution be given by

$$\text{let } x_p(t) = M\sin(\omega t) + N\cos(\omega t) \tag{2.151}$$

We need to include both sin and cos terms due to the presence of odd derivatives in Equation 2.149.

- Substituting Equation 2.151 into Equation 2.149

$$\begin{aligned} M &= \frac{1-\left(\frac{\omega}{\omega_n}\right)^2}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2} x_{st} \\ &= \frac{1-\phi^2}{(1-\phi^2)^2 + 4\zeta^2\phi^2} x_{st} \end{aligned} \tag{2.152}$$

$$\begin{aligned} N &= \frac{-2\zeta\left(\frac{\omega}{\omega_n}\right)}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2} x_{st} \\ &= \frac{-2\zeta\phi}{(1-\phi^2)^2 + 4\zeta^2\phi^2} x_{st} \end{aligned} \tag{2.153}$$

where $\phi = \omega/\omega_n$ and $x_{st} = P_0/k$.

- The total solution (complementary and particular) is

Transient

Steady-State

$$\begin{aligned} x(t) &= \underbrace{e^{-\zeta\omega_n t}(A\cos\omega_D t + B\sin\omega_D t)}_{\text{Transient}} \\ &\quad + \underbrace{x_{st}\left[\frac{1-\phi^2}{(1-\phi^2)^2 + 4\zeta^2\phi^2}\sin\omega t + \frac{-2\zeta\phi}{(1-\phi^2)^2 + 4\zeta^2\phi^2}\cos\omega t\right]}_{\text{Steady-State}} \end{aligned} \tag{2.154}$$

- If we assume that transient terms decay with time, then the response is given by the steady- state component as

$$\frac{x(t)}{x_{st}} = \frac{1-\phi^2}{(1-\phi^2)^2 + 4\zeta^2\phi^2}\sin\omega t + \frac{-2\zeta\phi}{(1-\phi^2)^2 + 4\zeta^2\phi^2}\cos\omega t \tag{2.155}$$

- We can put Equation 2.155 in amplitude-phase form (see appendix) as

$$\frac{x(t)}{x_{st}} = \underbrace{U}_{\text{Amplitude}}\sin(\omega t - \underbrace{\alpha}_{\text{Phase}}) \tag{2.156}$$

where

$$U = \frac{1}{\sqrt{(1-\phi^2)^2 + 4\zeta^2\phi^2}} \tag{2.157}$$

and

$$\alpha = \tan^{-1}\left(\frac{2\zeta\phi}{1-\phi^2}\right) \tag{2.158}$$

The maximum value of $\sin(\omega t - \alpha) = 1$, therefore, the amplification factor is given by

$$AF = \frac{x_{max}}{x_{st}} = \frac{1}{\sqrt{(1-\phi^2)^2 + 4\zeta^2\phi^2}} \tag{2.159}$$

and is equal to

$$\left(AF = \frac{1}{2\zeta} \right) \text{- max amplitude} \tag{2.160}$$

when $\omega = \omega_n$. The effect of damping on the amplification is clear from this result.

- Damping reduces the response.
- The effect of damping is not uniform across the spectrum (along the frequency, ω/ω_n, axis).
- The effect of damping is greatest near resonance ($\phi \approx 1$).

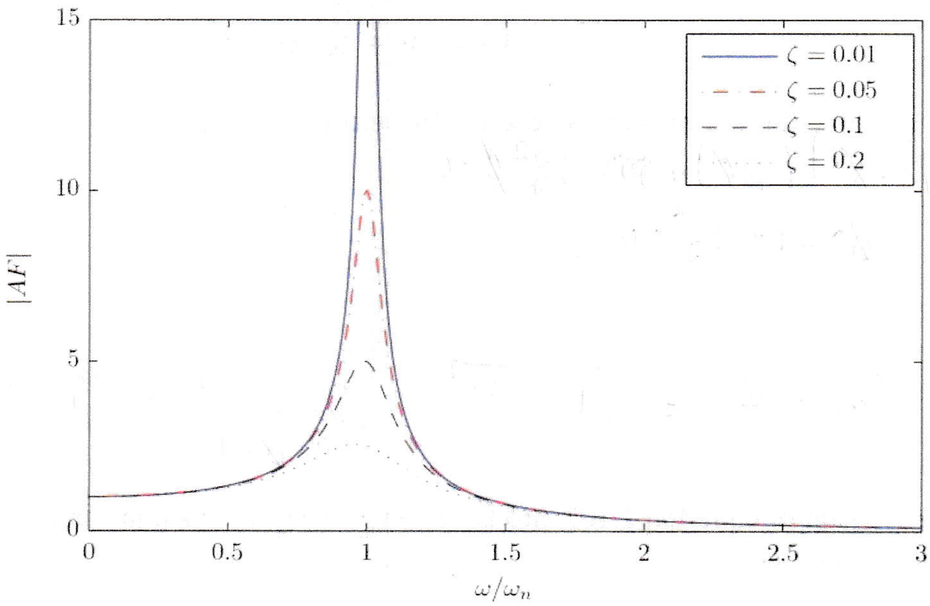
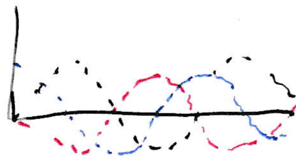
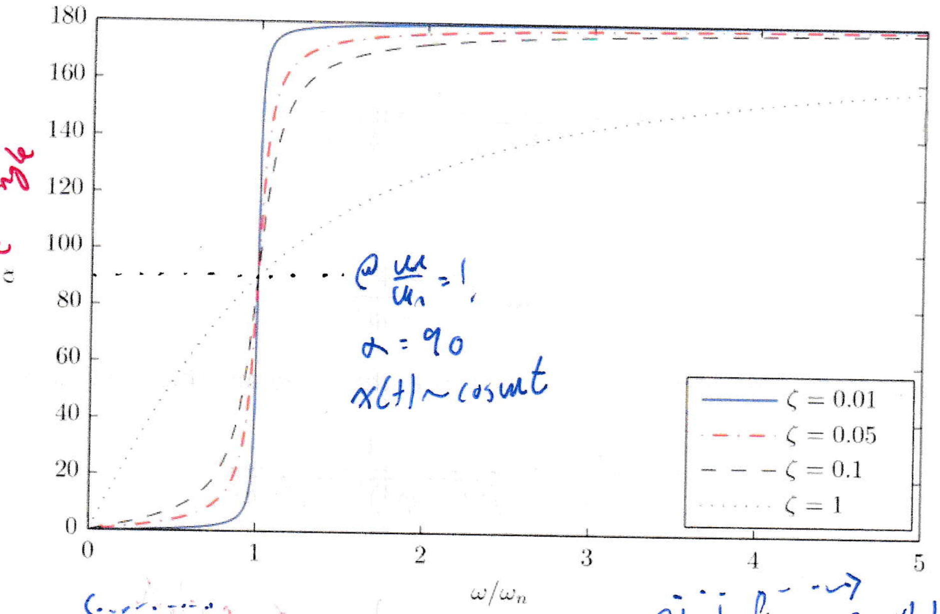


Figure 2.21 Amplification factor versus frequency ratio

m = low excitation
m = high excitation
m = ω/ω_n = 1



phase angle



@ a low excitation, In Phase. $x(t) \sim \sin(\omega t)$

Figure 2.22 Phase versus frequency ratio

@ high frequency excitation, $x(t) \sim (\sin \omega t - 180)$ Out of phase!!!!!

2.7.5 Resonant Frequency for Viscously Damped SDOF Systems

In order to determine $|AF|_{max}$, we need to set the derivative of Equation 2.159 to zero.

$$\frac{d}{d\phi}(AF) = 0 = \frac{d}{d\phi}\left[\frac{1}{\sqrt{(1-\phi^2)^2 + 4\zeta^2\phi^2}}\right]$$
$$= -\frac{1}{2}\frac{2(1-\phi^2)(-2\phi) + 8\zeta^2\phi}{[(1-\phi^2)^2 + 4\zeta^2\phi^2]^{3/2}} \tag{2.161}$$

This is only true if the numerator is equal to zero. Therefore

$$2(1-\phi^2)(-2\phi) + 8\zeta^2\phi = 0$$
$$\phi^2 - 1 + 2\zeta^2 = 0 \tag{2.162}$$

Therefore,

$$\phi_{res} = \frac{\omega_{res}}{\omega_n} = \sqrt{1-2\zeta^2} \tag{2.163}$$

, Does not exactly occur @ $\phi = 1$

Substituting the result of Equation 2.163 into Equation 2.159 gives the peak amplification factor for a viscously damped system.

$$|AF|_{max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \tag{2.164}$$

For very small values of ζ ($\zeta < 0.05$)

$$|AF|_{max} \approx \frac{1}{2\zeta} \tag{2.165}$$

The circular natural frequency, ω_n and damping ratio, ζ can be determined experimentally using harmonic excitation.

- An early and popular method to determine ω_n and ζ is based on the results we have just derived.
- It is a relatively straight-forward process to excite full-scale structures at different frequencies. On the other hand, imparting an initial condition is very difficult.
- To determine ω_n, excite the structure at various frequencies and generate a plot of the maximum response (such as displacement or acceleration) versus the excitation frequency, ω. For small levels of damping, the natural frequency, ω_n, can then be determined by finding the frequency, ω for which the response is maximum. If damping is high, use Equation 2.163 to determine ω_n.
- Determining the damping level, ζ, is more difficult. Two ways include

  Logarithmic decrement method

  Eqn. 2.164

  However, these methods require that you determine the constants of integration based on initial conditions or have free-vibration response data, or know x_{st}, which is rarely available.
- An alternate method is the *Half-Power Bandwidth Method*, which we will discuss next.

2.7.6 Half-Power Bandwidth Method

Previously, we used the logarithmic decrement method to calculate the inherent damping in a structure from the free vibration decay. The principles of forced response that we have just learned can be used to calculate damping as well.

- Recall the amplification factor for the case of the viscously damped SDOF structure subjected to harmonic excitation.

$$AF = \frac{X_{max}}{X_{static}} = \frac{1}{\sqrt{(1-\phi^2)^2 + 4\zeta^2\phi^2}} \tag{2.166}$$

- The resonant frequency at which the maximum AF occurs, denoted by AF_{max}, is found by setting the first derivative of the AF equal to zero. This gave a frequency corresponding to AF_{max} of

$$\phi_{res} = \frac{\omega_{res}}{\omega_n} = \sqrt{1-2\zeta^2} \quad , \quad \omega_{res} = \omega_n\sqrt{1-2\zeta^2} \tag{2.167}$$

- The corresponding value of AF_{max} becomes

$$AF_{max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \tag{2.168}$$

- The basic procedure involve equating Equation 2.166, which is the response amplitude, with $1/\sqrt{2}$ times the maximum value in Equation 2.168.

$$AF = \frac{1}{\sqrt{(1-\phi^2)^2+4\zeta^2\phi^2}} = \frac{1}{\sqrt{2}}\left(\frac{1}{2\zeta\sqrt{1-\zeta^2}}\right. \tag{2.169}$$

- Solve for ϕ, we get two roots

$$\phi^2 = (1-2\zeta^2) \pm 2\zeta\sqrt{1-\zeta^2} \quad , \tag{2.170}$$

- Assuming $\zeta \ll 1$ and neglecting the higher-order terms in ζ, we arrive at the result

$$\phi \sim \sqrt{1\pm 2\zeta} \quad , \quad \phi_1 = \sqrt{1+2\zeta} \quad \phi_2 = \sqrt{1-2\zeta} \tag{2.171}$$

- Using binomial expansion, we get

$$\phi_2 = (1+2\zeta)^{1/2} = 1 + \frac{1}{2}(2\zeta) - 1/8\,(2\zeta)^2 + 1/16\,(2\zeta)^3$$
$$\phi_1 = (1-2\zeta)^{1/2} = 1 - \frac{1}{2}(2\zeta) + 1/8\,(2\zeta)^2 - 1/16\,(2\zeta)^3 \tag{2.172}$$

or, since $\zeta \ll 1$,

$$\phi_1 \sim 1+\zeta \quad , \quad \phi_2 \sim 1-\zeta \tag{2.173}$$

- Therefore,

$$\zeta = \frac{\phi_2-\phi_1}{2} = \frac{1}{2}\left[\frac{\omega_2-\omega_1}{\omega_n}\right] \tag{2.173}$$

The procedure for estimating damping with the half-power bandwidth method is as follows:

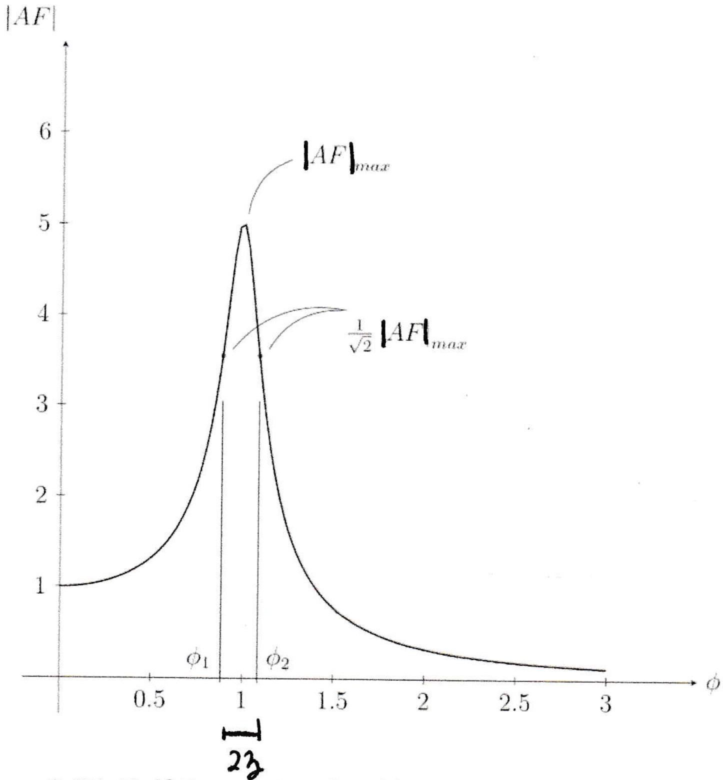


Figure 2.23 Half-Power Bandwidth Method for determining ζ

$$\zeta = \frac{\phi_2 - \phi_1}{2}$$

To do this,

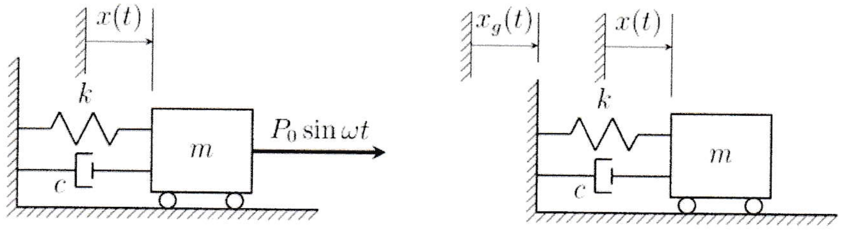
1. Harmonically excite over wide range of ω.
   Generate frequency response plot
2. From $|AF|_{max}$, estimate ω_n
3. Find $\frac{1}{\sqrt{2}}|AF|_{max}$ points
4. Determine ω_2, ω_1, then calculate ζ

2.8 Force Transmissibility and Base Motion

We will now study two important topics related to the frequency response behaviour of a SDOF system.

1. Force transmissibility - force transmitted through spring and damper into fixed base
2. Base Motion – how much ground motion is imparted onto mass

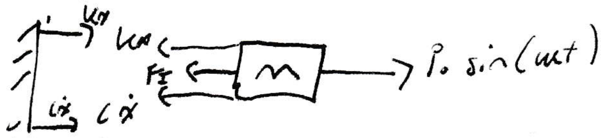
The two topics are important in the study of vibration isolation.



(a) (b)

Figure 2.24 Vibration isolation situations: **(a)** Force transmitted into a stationary base, and **(b)** Excitation by a moving base

2.8.1 Force Transmissibility



We are interested in how much force is transmitted into the support as a result of a harmonic force.

- From Figure 2.24 (a), the total force transmitted to the base, at any time t, can be written as

$$F_{TR}(t) = kx(t) + c\dot{x}(t) = k\left[x(t) + \frac{c}{k}\dot{x}(t)\right] \tag{2.175}$$

- We have seen earlier (Equation 2.156), the solution to the viscously damped SDOF system subjected to harmonic excitation is given in amplitude-phase form as

$$x(t) = \frac{P_0}{k}\frac{1}{\sqrt{(1-\phi^2)^2 + 4\zeta^2\phi^2}}\sin(\omega t - \alpha) \tag{2.176}$$

where

$\dot{x}(t) = \frac{dx}{dt}$

$$\alpha = \tan^{-1}\left(\frac{2\zeta\phi}{1-\phi^2}\right) \tag{2.177}$$

- Substituting Equation 2.176 and its first derivative into Equation 2.175, we get

$$
\begin{aligned}
F_{TR}(t) &= k\frac{P_0}{k}\frac{1}{\sqrt{(1-\phi^2)^2+4\zeta^2\phi^2}}\left[\sin(\omega t-\alpha)+\frac{c}{k}\omega\cos(\omega t-\alpha)\right] \\
&= P_0\frac{1}{\sqrt{(1-\phi^2)^2+4\zeta^2\phi^2}}\left[\overbrace{\sin(\omega t-\alpha)}^{f_s}+\overbrace{2\zeta\phi\cos(\omega t-\alpha)}^{f_D}\right]
\end{aligned}
\qquad (2.178)
$$

$F_{TR}: U\sin(\omega t-\alpha)$ $\sin(u\pm v)$ $\cos(u\pm v)$

- The maximum value of $F_{TR}(t)$ over t can be expressed as

transmitted force

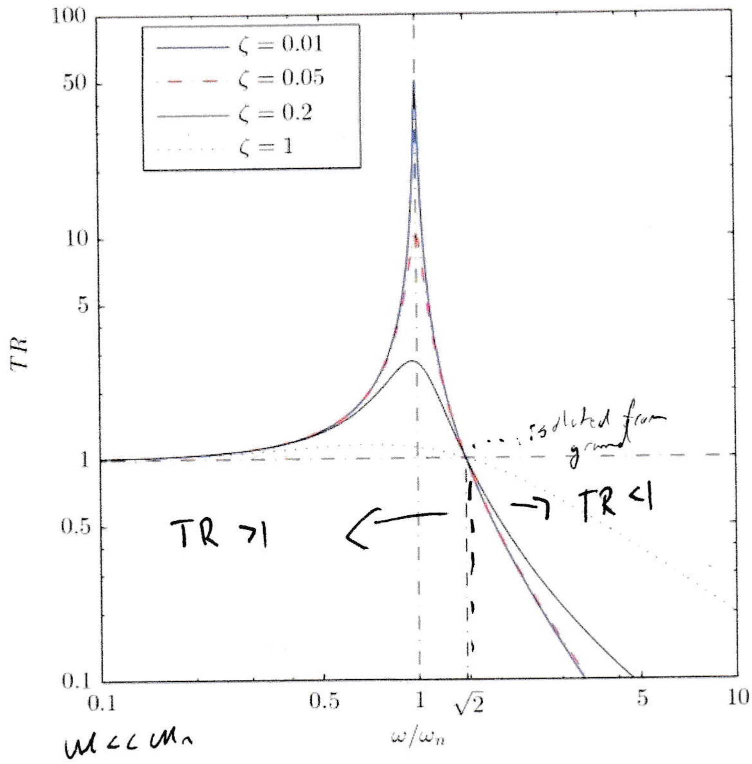
$$|F_{TR_0}| = P_0 \cdot \frac{}{\sqrt{(1-\phi^2)^2+4\zeta^2\phi^2}} \cdot \sqrt{1+4\zeta^2\phi^2} \qquad (2.179)$$

- This quantity is known as the *transmissibility* of the system, and is given by

$$TR = \frac{F_{TR_0}}{P_0} = \sqrt{\frac{1+4\zeta^2\phi^2}{(1-\phi^2)^2+4\zeta^2\phi^2}} \qquad (2.180)$$

Transmissibility is plotted in Figure 2.25 as a function of the frequency ratio, ω/ω_n, for several damping ratios ζ.

- While damping decreases the amplitude of motion for all frequencies (Figure 2.21) damping decreases the transmitted force only for frequency ratios $\omega/\omega_n < \sqrt{2}$.
- For the transmitted force to be less than the applied force, the stiffness of the support or the natural frequency of the system, should be small enough such that $\omega/\omega_n < \sqrt{2}$.
- No damping is desired in the support system, as damping increases the transmitted force in this frequency range.
- In the case of rotating machinery, the frequency will vary during startup and shutdown, and therefore it must have sufficient damping as it moves through resonance, but not so much that it adds significantly to the transmitted force at operating frequencies.



• Ensure enough damping to pass through resonance
• Amplification not too high

Figure 2.25 Transmissibility for a harmonic excitation

The MATLAB™ script used to generate the above plot is as follows:

```
phi=(0.1:0.01:10)';
zeta=[0.01 0.05 0.2 1];
TR=zeros(length(phi),length(zeta));
for j=1:length(zeta)
    for i=1:length(phi)
        TR(i,j)=sqrt((1+(2*zeta(j)*phi(i))^2)/ ...
            ((1-phi(i)^2)^2+(2*zeta(j)*phi(i))^2));
    end
end
loglog(phi,TR)
```

2.8.2 Base Motion

We now consider the case where $P(t) = 0$ and the base undergoes simple harmonic motion,

Assume $x_g(t) = x_{g_0} \sin(\omega t)$ (2.181)

$\therefore \dot{x}_g(t) = x_{g_0} \omega \cos \omega t$ $\ddot{x}_g(t) = -x_{g_0} \omega^2 \sin \omega t$

- The equation of motion is

$$m\ddot{x} + c\dot{x} + kx = -m\ddot{x}_g(t) = m x_{g_0}\omega^2 \sin(\omega t) \quad \text{Similar to } P_0 \sin(\omega t) \tag{2.182}$$

  where $x(t)$ is the relative displacement between the mass and the ground.

- Comparing Equation 2.182 with Equation 2.156, we get

$$x(t) = \frac{m x_{g_0}\omega^2}{k} \frac{1}{\sqrt{(1-\phi^2)^2 + 4\zeta^2\phi^2}} \sin(\omega t - \alpha) \tag{2.183}$$

- The total displacement of the mass is

$$\begin{aligned} x_t(t) &= x_g(t) + x(t) \quad \text{replace } P_0 \text{ w/ } m x_{g_0}\omega^2 \\ &= x_{g_0} \sin\omega t + \frac{m x_{g_0}\omega^2}{k} \frac{1}{\sqrt{(1-\phi^2)^2 + 4\zeta^2\phi^2}} \sin(\omega t - \alpha) \end{aligned} \tag{2.184}$$

  which can also be written as

$$\begin{aligned} x_t(t) &= x_{g_0} \sin\omega t + \frac{x_{g_0}\phi^2}{(1-\phi^2)^2 + 4\zeta^2\phi^2} [(1-\phi^2)\sin\omega t - 2\zeta\phi\cos\omega t] \\ &= \frac{x_{g_0}\phi^2}{(1-\phi^2)^2 + 4\zeta^2\phi^2} [(1-\phi^2 + 4\zeta^2\phi^2)\sin\omega t - 2\zeta\phi^3\cos\omega t] \end{aligned} \tag{2.185}$$

- Converting the solution to amplitude-phase form gives

$$x_t(t) = x_{g_0} \sqrt{\frac{1 + 4\zeta^2\phi^2}{(1-\phi^2)^2 + 4\zeta^2\phi^2}} \sin(\omega t - \gamma) \tag{2.186}$$

  where

$$\gamma = \tan^{-1}\left[\frac{2\zeta\phi^3}{(1-\phi^2)^2 + 4\zeta^2\phi^2}\right] \tag{2.187}$$

- Since the maximum value of $\sin(\omega t - \gamma)$ is 1, then the ratio of the maximum value of $x_t(t)$ over t to x_{g_0} is

$$\frac{(x_t)_{max}}{x_{g_0}} = \sqrt{\frac{1 + 4\zeta^2\phi^2}{(1-\phi^2)^2 + 4\zeta^2\phi^2}} \tag{2.188}$$

- Therefore, the transmissibility is

$$TR = \sqrt{\frac{1 + 4\zeta^2\phi^2}{(1-\phi^2)^2 + 4\zeta^2\phi^2}} \tag{2.189}$$

- It is clear that transmissibility for the ground excitation problem is the same as for the applied force problem.

The study of transmissibility of ground motion lead to some interesting observations using Figure 2.25.

- If the excitation frequency is much smaller than the natural frequency of the system, then the mass moves almost rigidly with the ground, both undergoing the same acceleration.
- If the excitation frequency is much larger than the natural frequency of the system, then the mass stays still while the ground beneath moves. This is the premise for the concept known as *base isolation*, where the structure is isolated from a moving base using a very flexible support system.

On exam, graph observations will be done.

2.9 Equivalent Viscous Damping - how to represent real damping as dashpot

Viscous damping is an idealization of the actual damping mechanisms that are present in a system. Recall from our earlier discussion of damping that it is difficult or impractical to relate damping to measurable physical characteristics on the structure, unlike mass and stiffness. We have already discussed a number of ways to estimate damping based on the response of the structure to a given excitation.

We now introduce the concept of *equivalent viscous damping* in order to quantify other types of damping in a structure, such as hysteretic or Coulomb (friction) damping. Viscous, or velocity proportional damping, is simple to quantify mathematically. We now seek to equate the amount of energy removed by the actual damping mechanism present to an equivalent amount of energy removed by a viscous damper.

Consider the amount of energy dissipated by viscous damping in one cycle of a response to harmonic excitation.

- Recall from Equation 2.156 that the response of a SDOF to a harmonic excitation is given in phase-amplitude for as

$$x(t) = \frac{P_0 U}{k}\sin[\omega t - \alpha], \quad U = \frac{1}{\sqrt{(1-\beta^2)^2 + 4\beta^2\zeta^2}} \tag{2.190}$$

$$\alpha = \tan^{-1}\left(\frac{2\zeta\beta}{1-\beta^2}\right)$$

- The damping force is given by

$$f_D = c\dot{x}(t) = c\cdot\frac{\omega P_0 U}{k}\cos(\omega t - \alpha) \tag{2.191}$$

where

letting $\eta = \frac{P_0 U}{k}$

- The work done in undergoing a small displacement $dx = \omega\eta\cos(\omega t - \alpha)\,dt$ is given by

$$\left.\begin{array}{l} dW = f_D\,dx \\ dx/dt = \dot{x}(t) \end{array}\right\} dW = f_D\,\dot{x}(t)\,dt = c\,\omega^2\eta^2\cos^2(\omega t - \alpha)\,dt \tag{2.192}$$

- Therefore, the energy dissipated in one cycle, $t = 2\pi/\omega$ is given by

$$W_D = \int_0^{2\pi/\omega} c\,\omega^2\eta^2\cos^2(\omega t - \alpha)\,dt = c\pi\omega\eta^2 \tag{2.193}$$

- Observe Equation 2.191. It can be written as

2.191 Plugged in

$$\left(\frac{f_D}{c\omega\eta}\right)^2 + \left[\frac{x(t)}{\eta}\right]^2 = 1 \tag{2.194}$$

which is plotted in Figure 2.26.

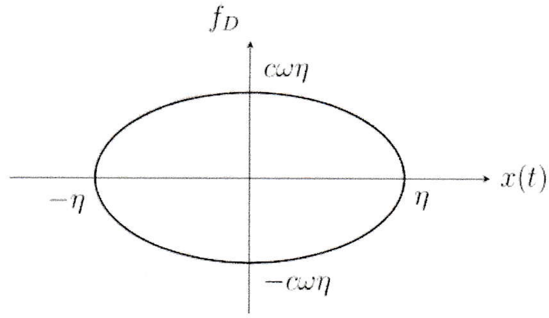


Figure 2.26 Energy dissipated by a viscous damper in one cycle

- The area of the ellipse is

$$A = \pi \eta^2 c \omega \tag{2.195}$$

which is the same as Equation 2.193.

- If we want to find the equivalent viscous damping coefficient, we simply equate the energy dissipated in one cycle of each system.

2.9.1 Coulomb Damping

Damping resistance may be provided by friction against sliding along a dry surface. This friction force is known as the *Coulomb damping force*. The Coulomb damping force is proportional to the force acting normal to the contact surface, and opposite the direction of motion. Consider the simple block and spring system shown in Figure 2.27.

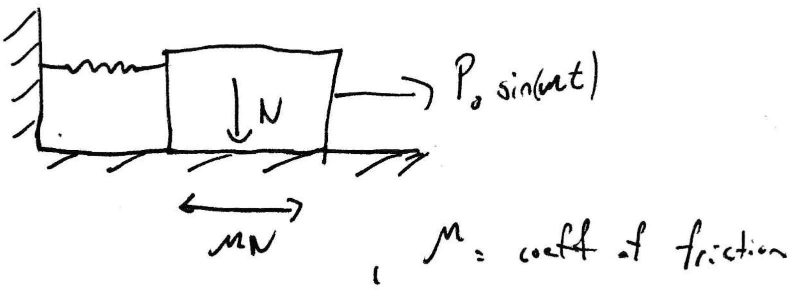


Figure 2.27 Coulomb damping under harmonic excitation

- The equation of harmonic motion is given by

$$m\ddot{x}(t) + kx(t) \pm \mu N = P_0 \sin \omega t \tag{2.196}$$

- The positive sign for the friction force applies when the mass is moving to the right; the negative sign applies when the mass is moving to the left.
- The solution to Equation 2.196 is a nonlinear equation and quite complex.
- We can quite easily obtain the equivalent viscous damping coefficient by equation the energy loss per cycle (shown in Figure 2.28) to the energy loss in a viscous damper.

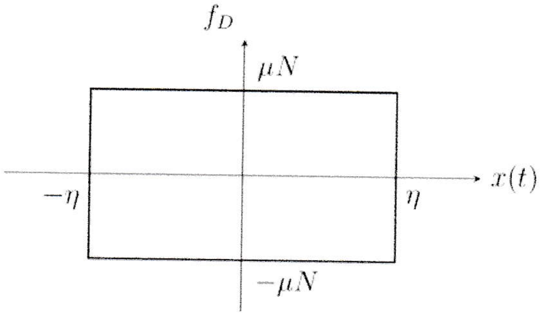


Figure 2.28 Energy dissipated by Coulomb damping in one cycle

- From the friction force versus displacement diagram in Figure 2.28, the work done by the friction force per cycle of motion is

$$W_D = 4\mu N \eta \tag{2.197}$$

- We can now obtain the equivalent viscous damping coefficient by equating the energy loss per cycle in the case of viscous damping to that given in Equation 2.197.

$$c_{eq} \pi \omega \eta^2 = 4 \mu N \eta \tag{2.198}$$

$$c_{eq} = \frac{4\mu N}{\pi \omega \eta} \quad , \quad \text{; idealization}$$

2.10 Response of an SDOF System to Arbitrary Excitations

The analysis of a structure's response to a forcing function that is not harmonic is comparatively more complex. We begin by consider the *step function load.*

2.10.1 Response to a Step Function Load

A *step function load* is a suddenly applied load that remains constant after application. Figure 2.29 shows a step function load applied at time $t = 0$, (or $t =$ another time, t_0)

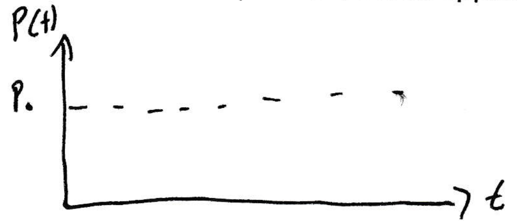


Figure 2.29 Step function load

- The equation of motion for an SDOF system subjected to a step input force of magnitude P_0 is

$$m\ddot{x} + c\dot{x} + kx = P_0 \tag{2.199}$$

- The complementary solution of Equation 2.199 is given by Equation 2.150. - damped free vibration

$$x_c(t) = e^{-\xi\omega_n t}\left[A\cos\omega_D t + B\sin\omega_D t\right] \tag{2.200}$$

- The particular solution to Equation 2.199 is

$$x_p = \frac{P_0}{k} \quad (\text{constant}) \tag{2.201}$$

- Hence, the total solution is

$$x(t) = x_c(t) + x_p(t) \tag{2.202}$$

- The initial conditions are $x(0) = \dot{x}(0) = 0$. After evaluating A and B, using these initial conditions, we get

$$x(t) = x_{st}\left[1 - e^{-\xi\omega_n t}\left(\cos\omega_D t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_D t\right)\right] \tag{2.203}$$

where

$$x_{st} = \frac{P_o}{k} \qquad (2.204)$$

- For the case of an undamped system, Equation 2.203 becomes

$$x(t) = x_{st}\left[1 - \cos\omega_n t\right] \qquad (2.205)$$

- The maximum value of $x(t)/x_{st}$ in Equation 2.205 is 2. This occurs when

$$\frac{x(t)}{x_{st}} = 1 - \cos(\omega_n t) \quad , \cos(\omega_n t) = -1$$

The damped ($\zeta = 0.1$) and undamped dynamic amplification is shown for $\omega_n = 5\pi$ in Figure 2.30.

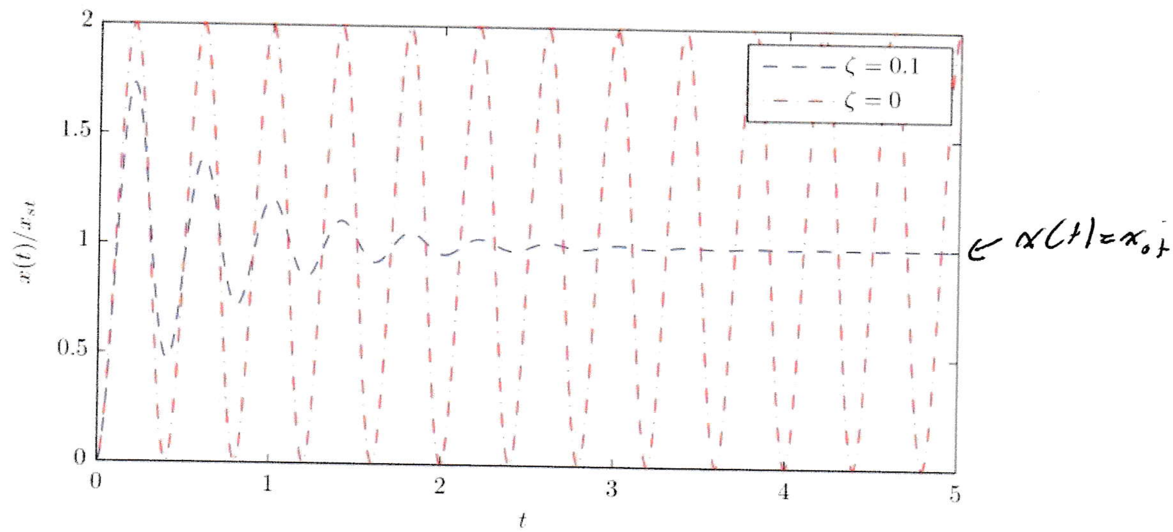


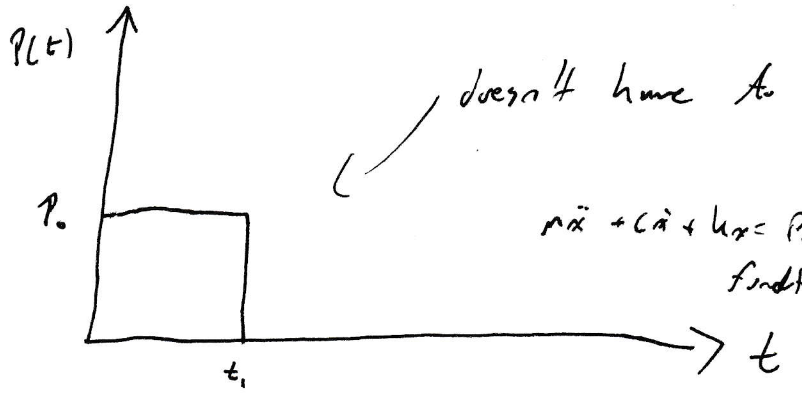
Figure 2.30 Undamped and damped response for a step input

The MATLAB™ script used to generate the above plot is as follows:

```
t=0:0.01:5;
omega_n=5*pi;
% Undamped
ampl_ud=1-cos(omega_n.*t);
% Damped, zeta=0.1
zeta=0.1;
omega_D=omega_n*sqrt(1-zeta^2);
ampl_d=1-exp(-omega_n.*zeta.*t).*(cos(omega_D.*t)+ ...
    zeta/sqrt(1-zeta^2).*sin(omega_D.*t));
plot(t,ampl_ud,t,ampl_d)
```

2.10.2 Undamped SDOF Response to a Rectangular Pulse

A rectangular pulse load of duration t_1 is shown in Figure 2.31.



doesn't have to be applied at $t=0$

$m\ddot{x} + c\dot{x} + kx = P_0$ do apply be not always function of P_0

$\ldots = \begin{Bmatrix} P_0, & 0 \le t_1 \\ 0 & t > t_1 \end{Bmatrix}$

Up to t_1 = forced vibration
Beyond t_1 = free vibration

Figure 2.31 Rectangular pulse load

- The response of the system for $t \leq t_1$ is the same as for the step input. Therefore,

Undamped: $$x(t) = \frac{P_0}{k}\left[1 - \cos\omega_n t\right] \quad , \; t \le t_1 \tag{2.206}$$

- At time $t = t_1$, the displacement and velocity of the system are given by

$$x(t_1) = \frac{P_0}{k}\left[1 - \cos\omega_n t_1\right] \tag{2.207}$$

$$\dot{x}(t_1) = \frac{P_0 \omega_n}{k} \sin\omega_n t_1 \tag{2.208}$$

- In the second phase of the response ($t > t_1$), the response is undergoing free vibration with initial conditions specified in Equations 2.207 and 2.208. Recall the solution for free vibration of a SDOF oscillator from Equation 2.70, replacing $x(0)$ with $x(t_1)$ and $\dot{x}(0)$ with $\dot{x}(t_1)$. $\bar{t}$ means t greater than t_1

$$\begin{aligned} x(\bar{t}) &= x(t_1)\cos\omega_n\bar{t} + \frac{\dot{x}(t_1)}{\omega_n}\sin\omega_n\bar{t} \quad , \; t > \tilde{t}_1 \\ &= \frac{P_0}{k}(1 - \cos\omega_n t_1)\cos\omega_n\bar{t} + \frac{P_0}{k}\sin\omega_n t_1 \sin\omega_n\bar{t} \end{aligned} \tag{2.209}$$

where $\bar{t} = t - t_1$ is the time measured from the beginning of the free vibration phase. To emphasize that it only applies after $t = t_1$ and substituting $x_{st} = P_0/k$,

$$\begin{aligned} x(t - t_1) &= x_{st}(1 - \cos\omega_n t_1)\cos\omega_n(t - t_1) + x_{st}\sin\omega_n t_1 \sin\omega_n(t - t_1) \\ x(t-t_1) &= x_{st}[\cos\omega_n(t - t_1) - \cos\omega_n t] \end{aligned} \tag{2.210}$$

Figure 2.32 shows the response for four different values of t_1/T.

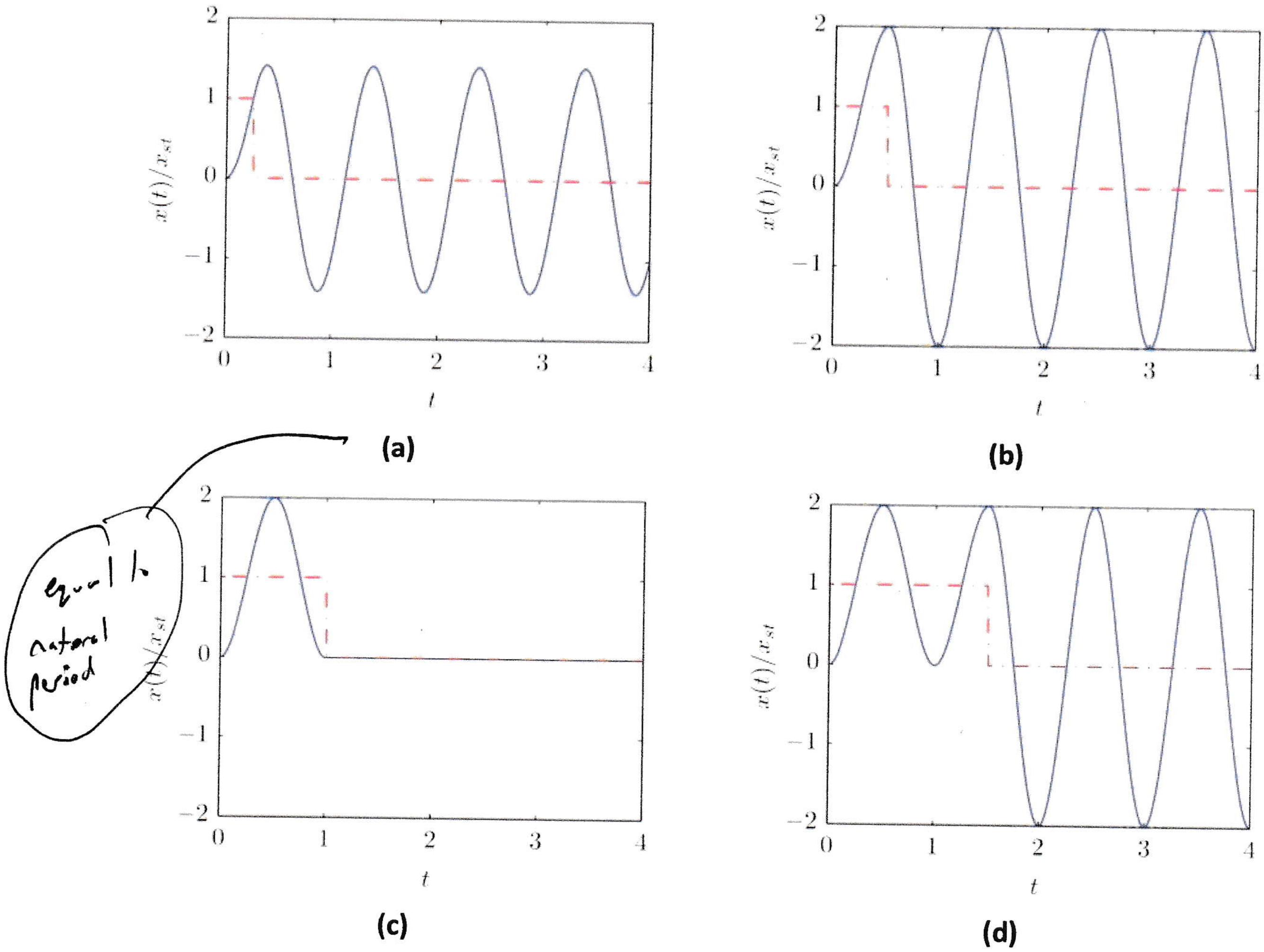


Figure 2.32 Response to a rectangular pulse load for **(a)** $t_1/T = 0.25$ **(b)** $t_1/T = 0.5$ **(c)** $t_1/T = 1$ **(d)** $t_1/T = 1.5$

Several observations can be made from Figure 2.32.

- The maximum value of the response is attained before or after the end of the rectangular pulse, depending on the time length of the pulse, t_1.
- For $t_1/T = 0.25$, the maximum response occurs during the free vibration phase, and is equal to $1.414P_0/k$.
- For $t_1/T = 0.5$, the maximum response occurs at $t = t_1$ and is equal to $2P_0/k$.
- For the special case where $\omega_n t_1 = 2n\pi$, or $t_1/T = n$, the displacement and velocity at this point are both equal to zero, and the motion ceases at this point
- For $t_1/T = 1$ and 1.5, the maximum response occurs during the duration of the pulse and is equal to $2P_0/k$.

• duration Δ... amplitude and when maximum response will be seen $\frac{t_1}{T} \geq 0.5$, X_{max} occurs at $t \leq t_1$

$\frac{t_1}{T} > 0.5$, X_{max} occurs $t \geq t_1$ $X_{max} \leq 2X_{st}$

The MATLAB™ script used to generate the above plot is as follows:

```
t=(0:0.01:4)';
omega_n=2*pi;
T=2*pi/omega_n;
t1_T_ratio=[0.25 0.5 1 1.5];
ampl=zeros(length(t),length(t1_T_ratio));
for j=1:length(t1_T_ratio)
    for i=1:length(t)
        t_1=t1_T_ratio(j)*T;
        if t(i)<=t_1
            ampl(i,j)=1-cos(omega_n*t(i));
        else
            ampl(i,j)=cos(omega_n*(t(i)-t_1))-cos(omega_n*t(i));
        end
    end
end
for i=1:4
    subplot(2,2,i), plot(t,ampl(:,i))
end
```

2.10.3 Response Spectrum

spectrum of natural frequencies

This leads into an important concept known as the *response spectrum*. Consider the solutions to the response for the rectangular pulse input.

- During the duration of the pulse, the response is

$$AF = \frac{x_{max}}{x_{st}} = 1 - \cos(\omega_n t) \tag{2.211}$$

  The maximum value of the AF in this phase will occur corresponding to $t_1/T \geq 0.5$ and the maximum value will be 2.

- If this condition does not hold, the maximum will occur in the free vibration phase of the response.

- After the end of the pulse, during the free vibration response phase, the amplification factor is

$$AF = \frac{x_{max} \text{ @ } t - t_1}{x_{st}} = \cos \omega_n (t - t_1) - \cos \omega_n \tag{2.212}$$

  The maximum value during the free vibration phase is found by setting the differential of Equation 2.212 to zero. Substituting this result back into Equation 2.212 gives

$$|AF| = 2 \sin \left| \frac{\pi t_1}{T} \right| \tag{2.213}$$

The response spectrum for the rectangular pulse is plotted as a function of t_1/T in Figure 2.33.

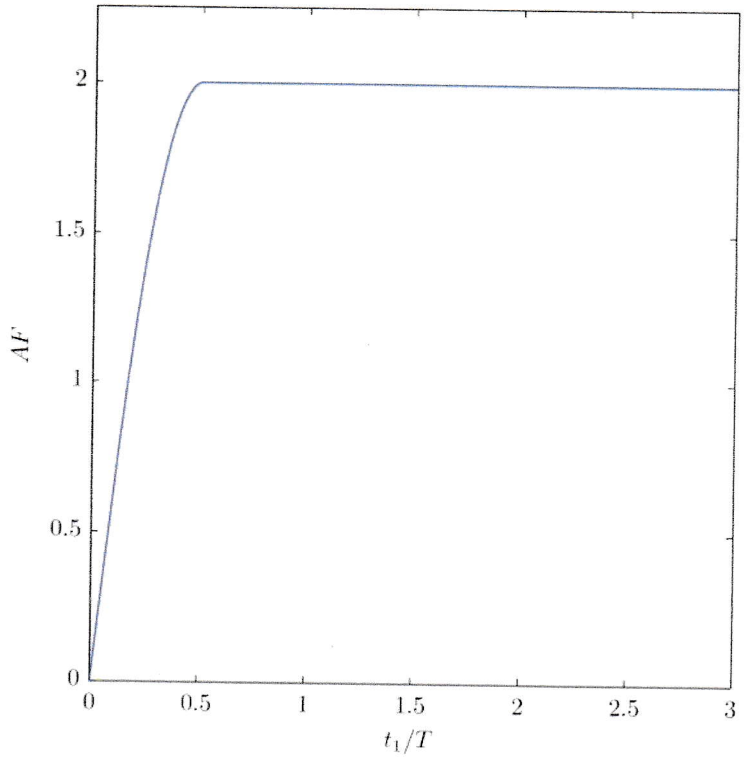


Figure 2.33 Response spectrum for a rectangular pulse load

– only valid for rectangular input

The MATLAB™ script used to generate the above plot is as follows:

```
t1_T_ratio=(0:0.01:3)';
AF=zeros(length(i),1);
for i=1:length(t1_T_ratio)
    if t1_T_ratio(i)>=0.5
        AF(i)=2;
    else
        AF(i)=2*abs(sin(pi*t1_T_ratio(i)));
    end
end
plot(t1_T_ratio,AF)
```

2.10.4 Response to an Impulsive Force

An *impulsive force* is a large force that acts for a very short duration of time. This is an important result which will subsequently be used to develop the convolution theorem (Duhamel's Integral).

- Let us begin by defining the *Dirac delta function*

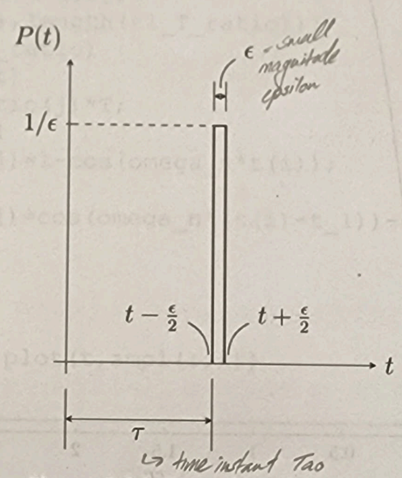


Figure 2.34 Dirac delta function

$$\delta(t-\tau) = \begin{cases} +\infty & t=\tau \\ 0 & t\neq\tau \end{cases} \tag{2.214}$$

- The delta function has the important property, known as the *magnitude of the impulse*. For a unit impulse, it is

$$\int_{-\infty}^{\infty} \delta(t-\tau)\,dt = 1 \tag{2.215}$$

which is also the $\boxed{\text{area under the curve}}$ in Figure 2.34

$F = ma$
$= m\,d\dot{x}/dt$

- Newton's second law of motion states that the action of an impulsive force on a mass results in a change in the velocity of the mass, and hence its momentum. Representing the change in velocity as $\Delta\dot{x}(t)$, the change of momentum is equal to the impulse of the impulsive force.

$$\int_{t_1}^{t_2} F\,dt = m(\dot{x}_2 - \dot{x}_1) = (m)(\Delta\dot{x}) \tag{2.216}$$

$$\int_{-\infty}^{\infty} \delta(t-\tau)\,dt = m\Delta\dot{x} \;\overset{at}{(t=\tau)} = 1 \rightarrow \Delta\dot{x}(t=\tau) = \frac{1}{m} \tag{2.217}$$

- This means that the presence of an impulse function is synonymous with a velocity initial condition at $t = \tau$ of magnitude 1 .

$$\dot{x}(\tau) = \frac{1}{m} \tag{2.218}$$

- The action of the impulsive force will start the system vibrating. The ensuing free vibration response can be obtained by recognizing the system behaves as a free vibration system with zero displacement initial condition ($x(\tau) = x_0 = 0$) and a velocity initial condition of $\dot{x}(\tau) = v_0 = 1/m$.
- For the undamped free vibration response, Equation 2.70 can be written as

$x(t) = \underbrace{\cancel{x_0 \cos \omega_n t}}_{\text{no initial displacement}} + \frac{v_0}{\omega_n} \sin \omega_n t$

$$\hookrightarrow x(t-\tau) = \frac{1}{m\omega_n} \sin \omega_n (t-\tau) \tag{2.219}$$

for $t \geq \tau$.

- For the damped free vibration response, Equation 2.107 can be written as

$$x(t-\tau) = \frac{1}{m\omega_D} e^{-\zeta\omega_n(t-\tau)} \sin \omega_D (t-\tau) \tag{2.220}$$

for $t \geq \tau$.

- Note that Equations 2.219 and 2.220 are valid for systems initially at rest (i.e. $x(0) = \dot{x}(0) = 0$).

2.10.5 Response to General Dynamic Loading

The unit impulse response functions given in Equations 2.219 and 2.220 can be used to obtain the response to any general dynamic loading.

- The system being excited is assumed to be linear; therefore, the principle of superposition applies.
- The applied force can be viewed as a series of impulses, as shown in Figure 2.35.

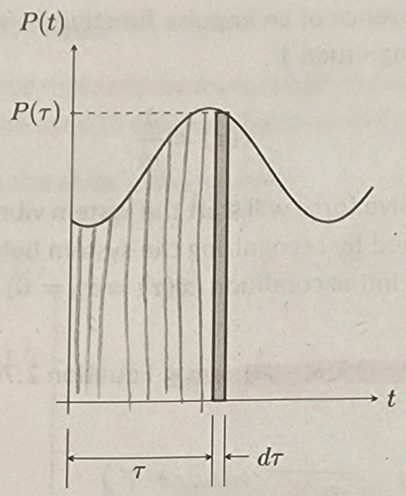


Figure 2.35 Representation of general dynamic loading by a series of impulses

- The shaded impulse has a magnitude of $P(\tau)d\tau$ and is imparted a time $t = \tau$.
- At any time $t \geq \tau$, the incremental response, dx due to the impulse is given by

$$dx(t-\tau) = \frac{P(\tau)d\tau}{m\omega_n}\sin\omega_n(t-\tau) \tag{2.221}$$

for an undamped system.

- The total response at time t is given by superposing the response for all impulses from time $\tau = 0$ to $\tau = t$.

from (2.219)

$$x(t) = \frac{1}{m\omega_n}\int_0^t P(\tau)\sin\omega_n(t-\tau)\,d\tau \tag{2.222}$$

for an undamped system. Similarly, for a damped system,

from (2.220)

$$x(t) = \frac{1}{m\omega_D}\int_0^t P(\tau)e^{-\zeta\omega_n(t-\tau)}\sin\omega_D(t-\tau)\,d\tau \tag{2.223}$$

- These results can be expressed in the general form

$$x(t) = \int_0^t P(\tau)h(t-\tau)\,d\tau \tag{2.224}$$

where the appropriate expression for the unit impulse function, h, is used.

- Equation 2.224 is known as the *convolution integral* or *Duhamel's integral* and provides a general method for determining the response of a linear system to arbitrary loading.

- If $P(t)$ is a simple mathematical function, closed-form analytical evaluation of the integral in Equation 2.224 is possible.
- For other cases, Equation 2.224 may need to be solved numerically in order to evaluate the response. Becomes too complex

- Note that Equations 2.222 and 2.223 are only applicable for at-rest initial conditions. If the body is not at rest initially, then the response is, for the undamped case

$$x(t) = x_0 \cos \omega_n t + \frac{v_0}{\omega_n} \sin \omega_n t + \frac{1}{m\omega_n} \int_0^t P(\tau) \sin \omega_n (t-\tau)\, d\tau \tag{2.225}$$

and for the damped case

$$x(t) = e^{-\xi\omega_n t}\left(x_0 \cos \omega_D t - \frac{v_0 + \xi\omega_n x_0}{\omega_D} \sin \omega_D t\right) + \frac{1}{m\omega_D} \int_0^t P(\tau) e^{-\xi\omega_n (t-\tau)} \sin \omega_D (t-\tau)\, d\tau \tag{2.226}$$

We will now consider several examples of the use of Duhamel's integral to find the response to a general dynamic load.

2.10.6 Response to a Ramp Function Load

A *ramp function load* is a load that increases linearly with time.

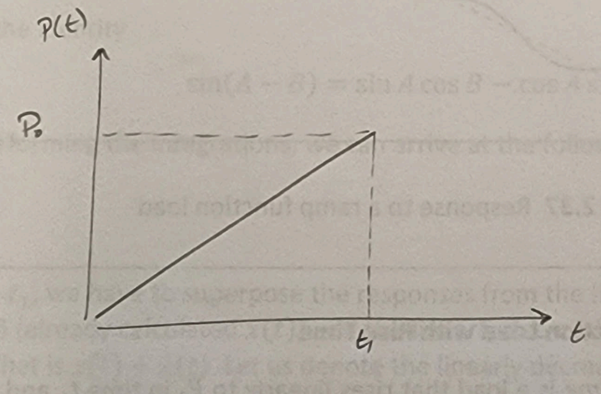


Figure 2.36 Ramp function load

Mathematically, the function can be expressed as

$$P(t) = \frac{P_0 t}{t_1} \tag{2.227}$$

Example 2.5 Determine the response of an undamped SDOF system to the ramp function load shown in Figure 2.36. Assume the system begins from rest.

Will need to do Integration by parts on the midterm/ Quiz

Solution: Substituting the ramp function load for $P(\tau)$ in Duhamel's integral (Equation 2.222) gives

$$x(t) = \frac{1}{m\omega_n} \int_0^t \frac{P_0}{t_1} \tau \sin \omega_n (t-\tau) \, d\tau \tag{2.228}$$

solution to 2.229 : $\int u\,dv = uv - \int v\,du$ → Int by Parts

let $u = \tau$ $dv = \sin \omega_n (t-\tau) d\tau$

$du = d\tau$ $v = \frac{1}{\omega_n} \cos \omega_n (t-\tau)$ → $x(t) = \frac{P_0}{m\omega_n t_1} \left[\frac{\tau}{\omega_n} \cos \omega_n (t-\tau) \Big|_0^t \right] - \int_0^t \frac{1}{\omega_n} \cos \omega_n (t-\tau) \, d\tau$

which can be simplified as

$$x(t) = x_{st} \left(\frac{t}{t_1} - \frac{\sin \omega_n t}{\omega_n t_1} \right) \tag{2.229}$$

$m\omega_n^2 = k$ → $\boxed{x_{st} = \frac{P_0}{k}}$

$= \frac{P_0}{m\omega_n t_1} \left[\frac{t}{\omega_n} + \frac{1}{\omega_n^2} \sin \omega_n (t-\tau) \Big|_0^t \right]$

After simplification

The response is plotted in Figure 2.37

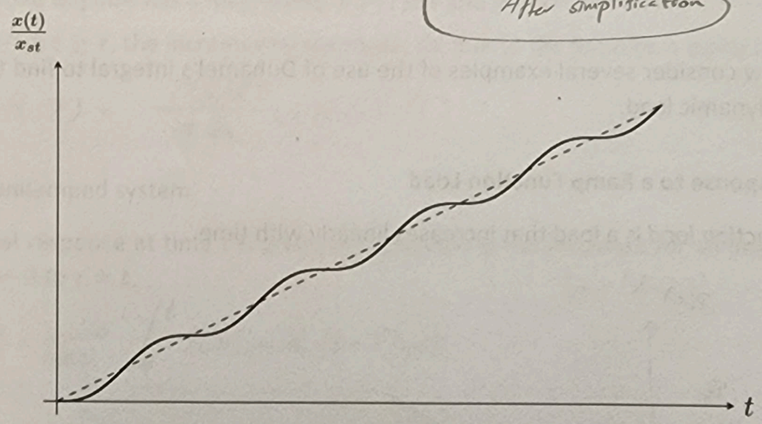


Figure 2.37 Response to a ramp function load

2.10.7 Response to a Step Function Load with Rise Time

A *step function load with rise time* is a load that rises linearly to P_0 in time t_1 and then remains constant, as shown in Figure 2.38. The following example illustrates the concept of superposition of responses.

Will need to do Integration by parts on the midterm/ Quiz

Example 2.5 Determine the response of an undamped SDOF system to the ramp function load shown in Figure 2.36. Assume the system begins from rest.

<u>Solution</u>: Substituting the ramp function load for $P(\tau)$ in Duhamel's integral (Equation 2.222) gives

$$x(t) = \frac{1}{m\omega_n}\int_0^t \frac{P_0}{t_1}\tau \sin\omega_n(t-\tau)\, d\tau \tag{2.228}$$

Solution to 2.229 : $\int u\,dv = uv - \int v\,du$ → Int by Parts

let $u = \tau$ $\quad dv = \sin\omega_n(t-\tau)\,d\tau$

$du = d\tau$ $\quad v = \frac{1}{\omega_n}\cos\omega_n(t-\tau)$ → $x(t) = \frac{P_0}{m\omega_n t_1}\left[\frac{\tau}{\omega_n}\cos\omega_n(t-\tau)\Big|_0^t - \int_0^t \frac{1}{\omega_n}\cos\omega_n(t-\tau)\,d\tau\right]$

which can be simplified as

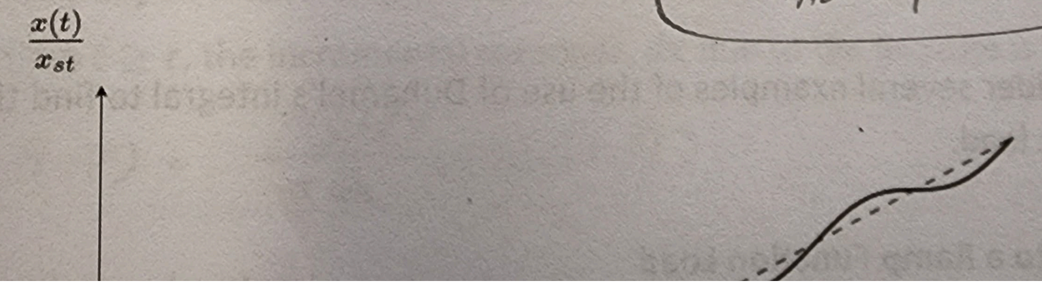
$$x(t) = x_{st}\left(\frac{t}{t_1} - \frac{\sin\omega_n t}{\omega_n t_1}\right) \tag{2.229}$$

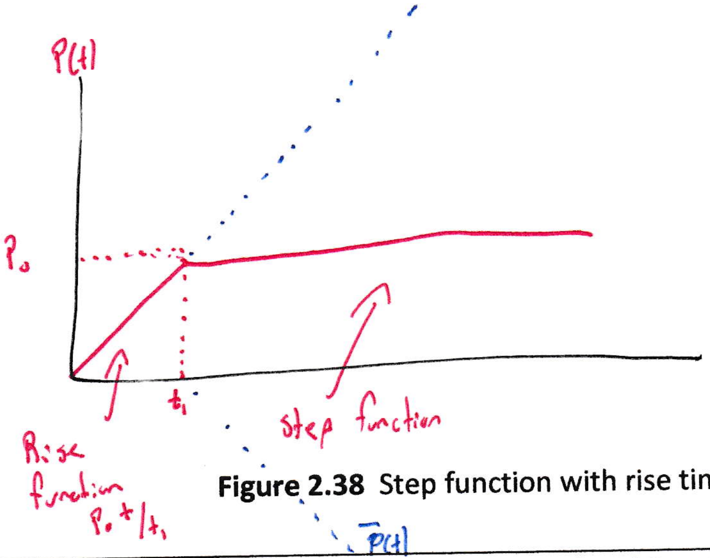
$= \frac{P_0}{m\omega_n t_1}\left[\frac{t}{\omega_n} + \frac{1}{\omega_n^2}\sin\omega_n(t-\tau)\Big|_0^t\right]$

After simplification

$m\omega_n^2 = k$ → $\boxed{x_{st} = \frac{P_0}{k}}$

The response is plotted in Figure 2.37





Since rise function shouldn't exist the $t > t_1$, introduce negative $\bar{P}(t)$, $\bar{P}(t) = \frac{-P_o(t-t_1)}{t_1}$

Figure 2.38 Step function with rise time

Example 2.6 Determine the response of an undamped SDOF system to the step function load with rise time shown in Figure 2.38. Assume the system begins from rest.

Solution: We can solve this in two ways. We can use Duhamel's integral to solve it as follows.

For the initial ramp phase ($t \leq t_1$), the response can be calculated using Duhamel's integral according to:

$$x(t) = \frac{P_o}{m\omega_n} \int_0^{t} \frac{\tau}{t_1} \sin\omega_n (t-\tau)\, d\tau \quad (2.230)$$

Using the identity

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

and performing the integrations, we can arrive at the following solution:

$$x(t) = \frac{P_o}{k}\left[\frac{t}{t_1} - \frac{\sin(\omega_n t)}{t_1 \omega_n}\right], \quad 0 \leq t \leq t_1 \quad (2.231)$$

For $t > t_1$, we have to superpose the responses from the linearly increasing ramp starting at $t = 0$ (already calculated $x(t)$) and the linearly decreasing ramp starting at $t = t_1$ (say, $\bar{x}(t)$). That is $x(t) + \bar{x}(t)$. Let us denote the linearly decreasing ramp by $\bar{P}(t)$. Hence,

$$\bar{x}(t) = \frac{1}{m\omega_n} \int_0^{t} \bar{P}(\tau) \sin\omega_n (t-\tau)\, d\tau \quad (2.232)$$

This can be written as the sum of two parts:

$$\bar{x}(t) = \frac{1}{m\omega_n} \cancel{\int_0^{t_1} \bar{P}(\tau) \sin\omega_n (t-\tau)\, d\tau}^{\,0} + \frac{1}{m\omega_n} \int_{t_1}^{t} \bar{P}(\tau) \sin\omega_n (t-\tau)\, d\tau \quad (2.233)$$

The first term is zero, since $\bar{P}(t)$ is 0 for $t < t_1$. Hence,

$$\bar{x}(t) = \frac{1}{m\omega_n}\int_{t_1}^{t} \bar{P}(\tau)\sin\omega_n(t-\tau)\,d\tau \tag{2.234}$$

$$\bar{x}(t) = \frac{-1}{m\omega_n}\int_{t_1}^{t} P_o\left(\frac{\tau-t_1}{t_1}\right)\sin\omega_n(t-\tau)\,d\tau \tag{2.235}$$

Which, upon simplification results in:

$$\bar{x}(t) = \frac{-P_o}{k}\left(\frac{t-t_1}{t_1} - \frac{\sin\omega_n(t-t_1)}{t_1\omega_n}\right) \tag{2.236}$$

Therefore, the solution for $t > t_1$ is

$$x(t) + \bar{x}(t) = \frac{P_o}{k}\left[1 - \frac{\sin\omega_n t}{t_1\omega_n} + \frac{\sin\omega_n(t-t_1)}{t_1\omega_n}\right], \quad t > t_1 \tag{2.237}$$

Alternatively, we can directly use the solutions as follows:

- The response to a ramp function applied at time $t = 0$ is

$0 \le t \le t_1$ $\qquad I_1$ = ramp function

$$x(t) = x_{st}\left(\frac{t}{t_1} - \frac{\sin\omega_n t}{t_1\omega_n}\right) \quad \leftarrow x_1 \tag{2.238}$$

Let $x_2 = \frac{P_0}{m\omega_n}\int_0^{t_1}\frac{\tau}{t_1}\sin\omega_n(t-\tau)\,d\tau + \frac{P_0}{m\omega_n}\int_{t_1}^{t}\sin\omega_n(t-\tau)\,d\tau$

- The response to a ramp function at time $t = t_1$ is

$t - t_1$

$$\bar{x}(t) = x_{st}\left[\frac{t-t_1}{t_1} - \frac{\sin\omega_n(t-t_1)}{t_1\omega_n}\right] \tag{2.239}$$

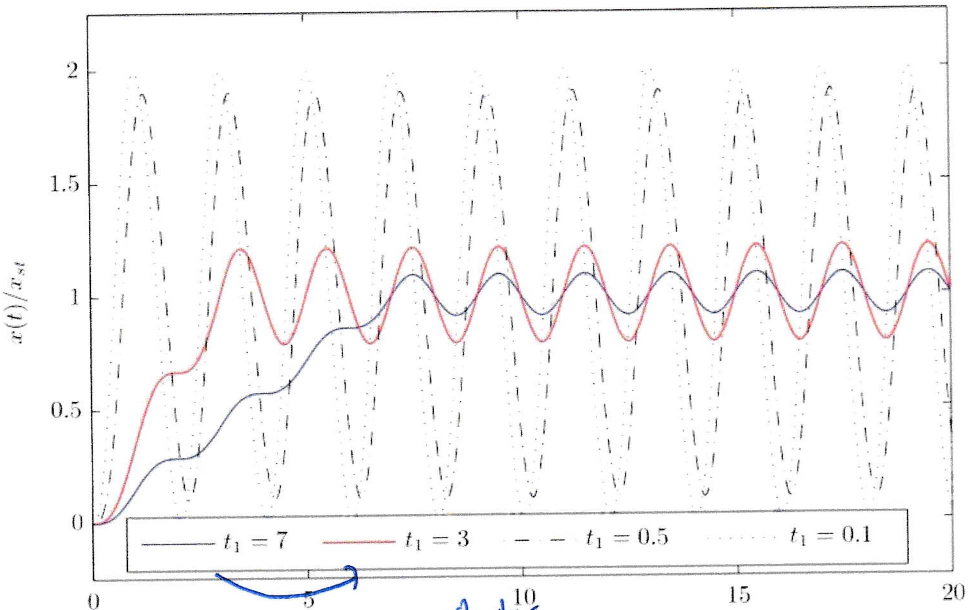
I_2 = step function

- The response to the step function with rise time for $t \le t_1$ is the solution in Equation 2.238. For $t > t_1$, the solution is obtained by subtracting Equation 2.239 from Equation 2.238. ... cont

$t > t_1$

$$x(t) = \begin{cases} x_{st}\left(\frac{t}{t_1} - \frac{\sin\omega_n t}{t_1\omega_n}\right) & t \le t_1 \\ x_{st}\left[1 - \frac{\sin\omega_n t}{t_1\omega_n} + \frac{\sin\omega_n(t-t_1)}{t_1\omega_n}\right] & t > t_1 \end{cases} \tag{2.240}$$

The response of the undamped SDOF system to the step function load with rise time is shown in Figure 2.39 for $\omega_n = \pi$ and $t_1 = 7, 3, 0.5$, and 0.1.



·ramp function gets shorter ~~(t time total)~~

Figure 2.39 Response to a step function with various rise times

... Cont.

$I_1 = \frac{P_0}{m\omega_n^2}\left[\cos\omega_n(t-t_1) + \frac{1}{\omega_n t_1}\left(\sin\omega_n(t-t_1) - \sin\omega_n t\right)\right]$, $\quad I_2 = \frac{P_0}{k}\left[1 - \cos\omega_n(t-t_1)\right]$

The MATLAB™ script used to generate the above plot is as follows:

```
t=(0:0.01:20)';
t_1=[7 3 0.5 0.1];
omega_n=pi;
for j=1:length(t_1)
    for i=1:length(t)
        if t(i)<=t_1(j)
            disp(i,j)=t(i)/t_1(j)-sin(omega_n*t(i))/(omega_n*t_1(j));
        else
            disp(i,j)=1-sin(omega_n*t(i))/(omega_n*t_1(j))+ ...
                sin(omega_n*(t(i)-t_1(j)))/(omega_n*t_1(j));
        end
    end
end
plot(t,disp)
```

- For $t \leq t_1$, the displacement continues to grow with time, oscillating about a static deformation, similar to the ramp function response.
- For $t > t_1$, the system executes a simple harmonic response about the position of equilibrium, much like the step function response.
- The maximum response occurs either at $t = t_1$ or when $t > t_1$. To find the time t at which the maximum response occurs, obtain the first time derivative of Equation 2.240.

$$\frac{d}{dt}\left\{x_{st}\left[1 - \frac{\sin\omega_n t}{t_1\omega_n} + \frac{\sin\omega_n(t-t_1)}{t_1\omega_n}\right]\right\} = 0 \qquad (2.241)$$
$$= x_{st}[-\cos\omega_n t + \cos\omega_n(t-t_1)]$$

- After simplification, we get

$$\tan\omega_n t = \tan\frac{\omega_n t_1}{2} \qquad (2.242)$$

The time corresponding to the maximum response are given by

$$t_p = \frac{n\pi}{\omega_n} + \frac{t_1}{2} \qquad n = 1, 2, \ldots \qquad (2.243)$$

where n must be chosen so that $t_p \geq t_1$.

- To find the maximum response, substitute this result back into Equation 2.240.

$$x_{max} = x_{st}\left[1 + \frac{2\sin(n\pi - \omega_n t_1/2)}{\omega_n t_1}\right] \qquad (2.244)$$

- The true maximum is found by selecting a value for n that makes the second term positive. Therefore,

$$x_{max} = x_{st}\left[1 + \frac{2|\sin(\omega_n t_1/2)|}{\omega_n t_1}\right] \tag{2.245}$$

This result facilitates plotting the response spectrum, or the maximum values of the response parameter versus a frequency (period) parameter. In this case, x_{max}/x_{st} versus t_1/T, where $T = 2\pi/\omega_n$.

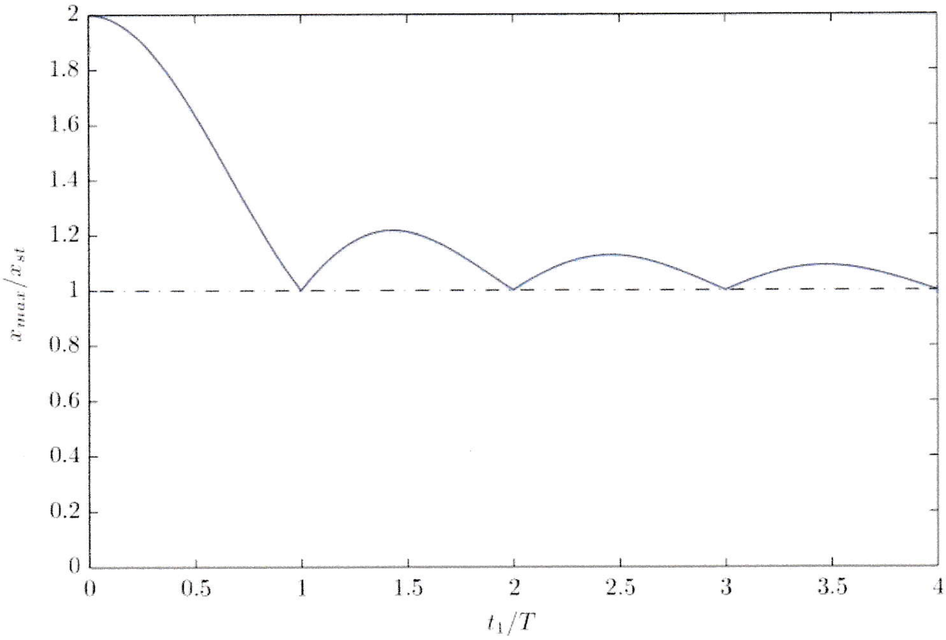


Figure 2.40 Response spectrum for a step function load with rise time

2.10.8 Sinusoidal Pulse

The sinusoidal pulse can be considered by superimposing two sine waves, the second beginning at $t = t_1$, as shown in Figure 2.41.

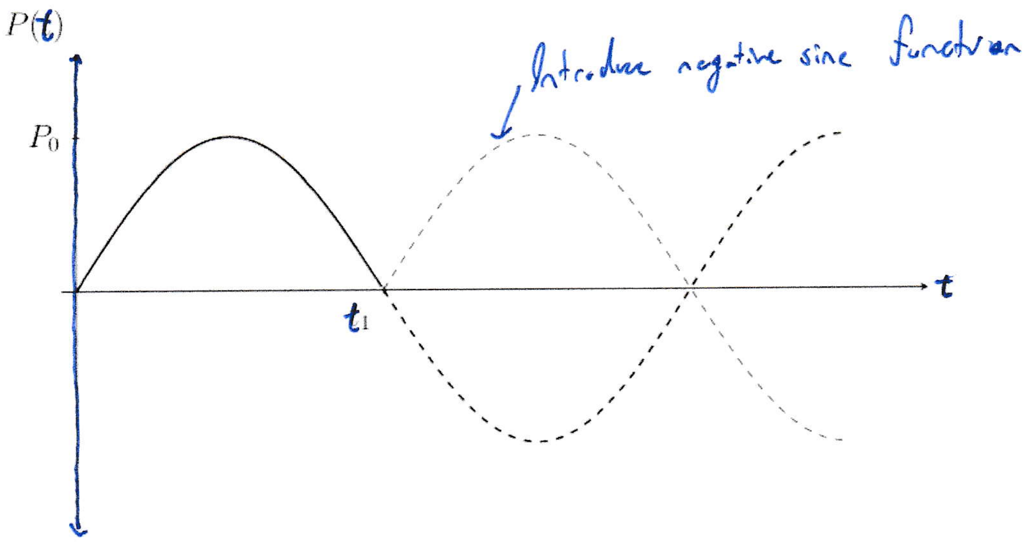


Figure 2.41 Sinusoidal pulse

- The response for $t \le t_1$ is obtained from the response for a harmonically excited undamped system, as in Equation 2.146 assuming the system starts from rest.

$$x(t) = \frac{P_0}{k}\frac{1}{1-\phi^2}(\sin \omega t - \phi \sin \omega_n t) \quad , \phi = \frac{\omega}{\omega_n} \tag{2.246}$$

- The response for $t > t_1$ is similar to Equation 2.246, replacing t with $t - t_1$. The total response for $t > t_1$ is

2nd sine fn. starting @ $t = t_1$

$$x(t-t_1) = \frac{P_0}{k}\frac{1}{1-\phi^2}[\sin \omega t - \phi \sin \omega_n t + \sin \omega (t - t_1) - \phi \sin \omega_n (t - t_1)] \tag{2.247}$$

- We can simplify Equation 2.247 by recognizing the forcing frequency is $\omega = \pi/t_1$. Therefore, substituting $t_1 = \pi/\omega$, we get

$$x(t-t_1) = \frac{P_0}{k}\frac{\phi}{1-\phi^2}[-\sin \omega_n t - \sin \omega_n (t - t_1)] \tag{2.248}$$

- The response obtained in Equations 2.246 and 2.248 are plotted for various values of t_1/T in Figure 2.42, where $T = 2\pi/\omega_n$.

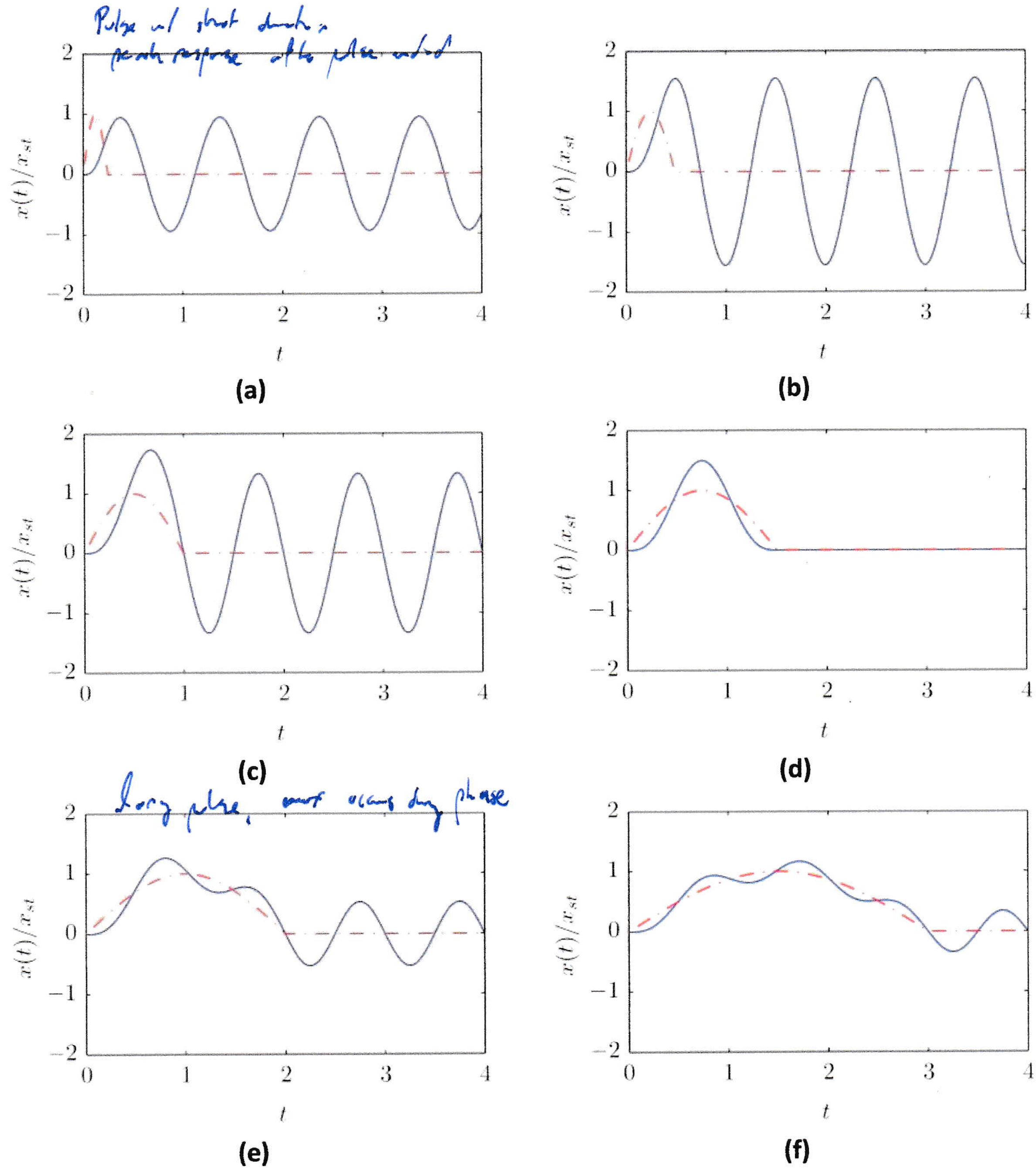


Figure 2.42 Response to a half sine pulse load for **(a)** $t_1/T = 0.25$ **(b)** $t_1/T = 0.49$ **(c)** $t_1/T = 1$ **(d)** $t_1/T = 1.5$ **(e)** $t_1/T = 2$ **(f)** $t_1/T = 3$

- For $t_1/T = 0.5$, which corresponds to $\phi = 1$, Equations 2.246 and 2.248 both become indeterminate. We can determine the limiting values of these equations by invoking L'Hopital's rule. This will enable you to determine the response for $t_1/T = 0.5$.
- In order to generate a response spectrum, we repeat the procedure for plotting the response for several values of t_1/T and pick the absolute maximum value. This result is plotted in Figure 2.43.

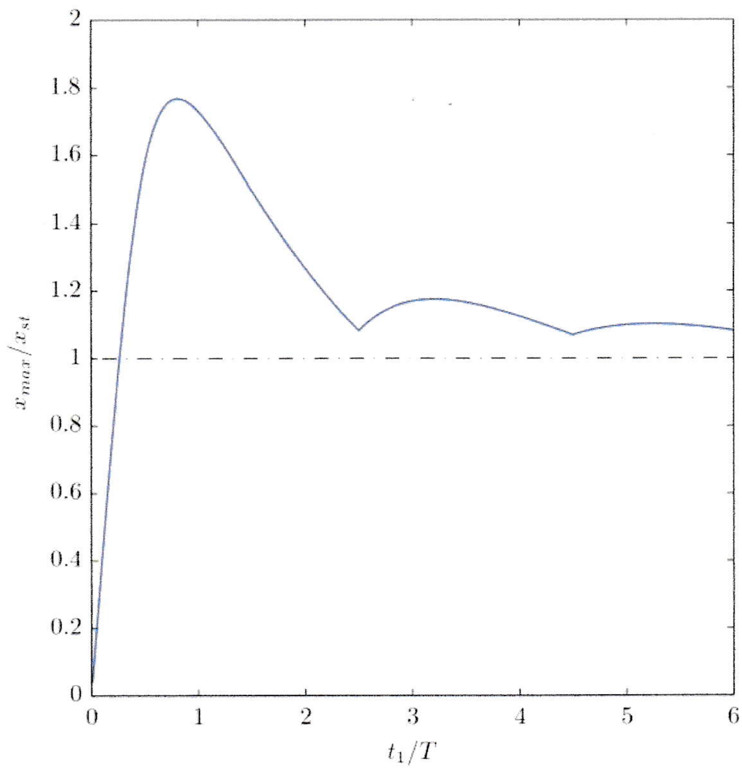


Figure 2.43 Response spectrum for a half sine pulse

For given excitation
maximum excitation!

The MATLAB™ script used to generate the above plot is as follows:

```
t=(0:0.01:4)';
omega_n=2*pi;
T=2*pi/omega_n;
n=[0:0.01:0.49,0.51:0.01:6];
t_1=T.*n;
for j=1:length(t_1)
    omega=pi/t_1(j);
    phi=omega/omega_n;
    for i=1:length(t)
        if t(i)<=t_1(j)
            disp(i,j)=1/(1-phi^2)*(sin(omega*t(i))-phi*sin(omega_n*t(i)));
            input(i,j)=sin(omega*t(i));
        else
            disp(i,j)=phi/(1-phi^2)*(-sin(omega_n*t(i))- ...
                sin(omega_n*(t(i)-t_1(j))));
            input(i,j)=0;
        end
    end
    max_resp(j)=max(abs(disp(:,j)));
end
plot(n,max_resp)
```

2.11 Numerical Evaluation

While Duhamel's integral provides an efficient procedure to evaluate the response of a SDOF system to arbitrary excitations, an analytical evaluation is only possible when the excitation can be described analytically. Where the excitation either cannot be described in this manner (e.g. earthquake forces), or when a discrete approximation can be made, then the response can be evaluated numerically. Such a numerical evaluation of the convolution sum can be undertaken using the procedure described in the Appendix.

Trapezoidal rule