① DE Assignment 6 Benjamin Klassen

$(D^2-1)y = 12x^2e^x + 3e^{2x} + 10\cos 3x$

$y_p = \frac{1}{D^2-1}\left[12x^2e^x\right] + \frac{1}{D^2-1}\left[3e^{2x}\right] + \frac{1}{D^2-1}10\cos 3x$

Theorum 2 Theorum 1 Theorum 3

$= 12e^x \frac{1}{(D+1)^2-1}x^2 + 3\cdot\frac{1}{2^2-1}e^{2x} + 10\cdot\frac{1}{(-3^2)-1}\cos 3x$

$= 12e^x\cdot\frac{1}{D}\cdot\frac{1}{2}\cdot\frac{1}{1+1/2D}x^2 + e^{2x} - \cos 3x$

$= 6e^x\frac{1}{D}\cdot\left(1-\frac{1}{2}D+\frac{1}{4}D^2\right)x^2 + e^{2x} - \cos 3x$

$= 6e^x\cdot\frac{1}{D}\left[x^2 - x + \frac{1}{2}\right] + e^{2x} - \cos 3x$

$= 6e^x\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{2}x\right] + e^{2x} - \cos 3x$

$y_p = 2x^3e^x - 3x^2e^x + 3xe^x + e^{2x} - \cos 3x$

$y_c,\ D^2-1=0$

$D = \pm 1$

$\therefore y_c = C_1e^{-x} + C_2e^x$

$\therefore y = C_1e^{-x} + C_2e^x + e^x\left[2x^3 - 3x^2 + 3x\right] + e^{2x} - \cos 3x$

2) $(D^2-2D+2)y = 4x - 2 + 2e^x \sin x$

D^2-2D+2

$\frac{2 \pm \sqrt{4-4(2)}}{2}$

$= 1 \pm i1$

$y_p = 4 \cdot \frac{1}{D^2-2D+2} x - \frac{1}{D^2-2D+2} 2 + 2 \frac{1}{D^2-2D+2} e^x \sin x$

$= 2 \cdot \left(\frac{1}{1+(\frac{1}{2}D^2-D)} x - \frac{1}{1+(\frac{1}{2}D^2-D)} 1 + 2e^x \cdot \frac{1}{(D+1)^2-2(D+1)+2} \sin x \right)$ ← Theorum 2

$y_p = 2\left(\left(1-\frac{1}{2}D^2+D \dots 0\right)x - \left(1-\frac{1}{2}D^2+D\right)1 + 2e^x \cdot \frac{1}{D^2+2D+1-2D-2+2} \sin x\right)$ ← Theorum 3

$= 2[x+1] - 1 + 2e^x \cdot \frac{1}{D^2+1} \sin x$

$= 2x+2-1+2e^x \cdot \frac{1}{(-1^2)+1} \sin x$ ← Theorum 3

← 0, ∴ Theorum 4

$= 2x+2-1+2e^x \cdot \mathrm{Im}\left\{\frac{1}{D^2+1} e^{ix}\right\}$ $(\phi') \neq 0$

$= 2x+1+2e^x \cdot \mathrm{Im}\left\{\frac{1}{2i} x e^{ix}\right\}$

$= 2x+1+2e^x \mathrm{Im}\left(\frac{-i}{2} x (\cos x + i \sin x)\right)$

$= 2x+1+2e^x \mathrm{Im}\left\{\frac{-ix}{2} \cos x + \frac{\sin x}{2}\right\}$

$y_p = 2x+1 - xe^x \cos x$

, y_c F $1 \pm i1$ solns

∴ $y_c = e^x[A\cos x + B\sin x]$

∴ $y = e^x[A\cos x + B\sin x] + 2x + 1 - xe^x \cos x$

③ $y''' - 3y'' + 4y = 12e^{2x} + 4e^{3x}$

$(D^3 - 3D^2 + 4)y = 12e^{2x} + 4e^{3x}$

y_c, $D^3 - 3D^2 + 4 = 0$, -1 soln

$$\begin{array}{r|rrrr} -1 & 1 & -3 & 0 & 4 \\ & & -1 & 4 & -4 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

$(x+1)(x^2 - 4x + 4)$

$(x+1)(x-2)(x-2)$

$\therefore y_c = [C_0 + C_1 x]e^{2x} + C_2 e^{-x}$

$y_{p_1} = 12 \cdot \frac{1}{D^3 - 3D^2 + 4} e^{2x} + 4 \cdot \frac{1}{D^3 - 3D^2 + 4} e^{3x}$ ← Theorum 1

← Theorum 1

$= 12 \cdot \cancel{\frac{1}{0} e^{2x}}$ $\therefore$ Theorum 4 $+ e^{3x}$

$\phi'(D) = 0$

$\phi^2(D) \neq 0$

$= 12 \cdot \frac{1}{6D - 6} x^2 e^{2x} + e^{3x}$

$= 2x^2 e^{2x} + e^{3x}$

$\therefore y = [C_0 + C_1 x]e^{2x} + C_2 e^{-x} + 2x^2 e^{2x} + e^{3x}$

④ $y'' + 3y' + 2y = \sin(e^x)$

$(D^2 + 3D + 2)y = \sin(e^x)$

y_c, $D^2 + 3D + 2 = (D+2)(D+1)$,

② $\therefore y_c = C_1 e^{-2x} + C_2 e^{-x}$

③ y_p $y'_c = -2C_1 e^{-2x} - C_2 e^{-x}$

④ $y_p = C_1(x) e^{-2x} + C_2(x) e^{-x}$

⑤ $y_{p'} = -2C_1(x) e^{-2x} - C_2(x) e^{-x}$

⑥ $y_{p_2} = C_1'(x) e^{-2x} - 2C_1(x) e^{-2x} + C_2'(x) e^{-x} - C_2(x) e^{-x}$

⑦ Setting ⑤ = ⑥, $0 = C_1'(x) e^{-2x} + C_2'(x) e^{-x}$

$\frac{d⑤}{dx} = 4C_1(x) e^{-2x} - 2C_1'(x) e^{-2x} + C_2(x) e^{-x} - C_2'(x) e^{-x}$

$$\begin{array}{c|cccc} y'' & 4C_1(x)e^{-2x} & -2C_1'(x)e^{-2x} & +C_2(x)e^{-x} & -C_2'(x)e^{-x} \\ 3y' & -6C_1(x)e^{-2x} & 0 & -3C_2(x)e^{-x} & 0 \\ +\ 2y & 2C_1(x)e^{-2x} & 0 & 2C_2(x)e^{-x} & 0 \\ \hline \end{array}$$

⑧ $-2C_1'(x) e^{-2x} - C_2'(x) e^{-x} = \sin(e^x)$

$\underline{2 \cdot ⑦ + ⑧}$

$C_2'(x) e^{-x} = \sin(e^x)$

$\int C_2'(x) = \int e^x \sin(e^x)\,dx$, $d(e^x) = e^x\,dx$

$= \int \sin(e^x)\,d(e^x)$

$C_2(x) = -\cos(e^x)$, Sub into ⑦

$\to 0 = \frac{C_1'(x)}{e^x e^x} + \sin(e^x)$

$\int C_1'(x) = \int -\sin(e^x) e^x e^x\,dx$, $d(e^x) = e^x\,dx$

$C_1(x) = \int \underbrace{-\sin(e^x)}_{dv} \overbrace{e^x}^{u}\,d(e^x)$

$uv - \int v\,du$

$= e^x \cos(e^x) - \int \cos(e^x)\,d(e^x)$

$= e^x \cos(e^x) - \sin(e^x)$

$\therefore y_p = [e^x \cos(e^x) - \sin(e^x)] e^{-2x} - \cos(e^x) e^{-x}$

$= -\sin(e^x) e^{-2x}$

$y = C_1 e^{-2x} + C_2 e^{-x} - \sin(e^x) e^{-2x}$

5) $(D^2+1)y = \sec^3 x$

y_c: $D^2+1=0$

$\frac{-0 \pm \sqrt{-4}}{2}$

$= \frac{\pm 2i}{2}$

$= \pm i 1$

(2) $\therefore y_c = A\cos x + B\sin x$

(3) $y'_c = -A\sin x + B\cos x$

(4) $y_p = A(x)\cos x + B(x)\sin x$

(5) $y'_p = -A(x)\sin x + B(x)\cos x = 0$ $-A(x)\sin x + B'(x)\sin x + B(x)\cos x = 0$

(6) $y'_p = A'(x)\cos x - A(x)\sin x + B'(x)\sin x + B(x)\cos x$

$\therefore -A(x)\sin x + B(x)\cos x = A'(x)\cos x - A(x)\sin x + B'(x)\sin x + B(x)\cos x$

(7) $0 = A'(x)\cos x + B'(x)\sin x$

Now, $\frac{d(5)}{dx} = -A'(x)\sin x - A(x)\cos x + B'(x)\cos x - B(x)\sin x$

Plugging into (1),

$\therefore -A'(x)\sin x - A(x)\cos x + B'(x)\cos x - B(x)\sin x + A(x)\cos x + B(x)\sin x = \sec^3 x$

(8) $-A'(x)\sin x + B'(x)\cos x = \sec^3 x$

(7)* $\sin x$: $0 = A'(x)\sin x\cos x + B'(x)\sin^2 x$

(8)* $\cos x$: $\sec^2 x = -A'(x)\sin x\cos x + B'(x)\cos^2 x$

$+$

$\sec^2 x = B'(x)$

$\therefore B(x) = \tan x$

$0 = A'(x)\sin x\cos x + \frac{\sin^2 x}{\cos^2 x}$

$A'(x) = -\frac{\sin x}{\cos^3 x}$, $\frac{d(\cos x)}{dx} = -\sin x$

$A(x) = \int \frac{1}{\cos^3 x}\, d(\cos x)$

$A(x) = \frac{-1/2}{\cos^2 x}$ $\quad \frac{-u^{-2}}{2}$

$\therefore y_p = \frac{-1/2}{\cos x} + \frac{\sin^2 x}{\cos x} = \frac{-\frac{1}{2}}{\cos x} + \frac{1}{\cos x} - \frac{\cos^2 x}{\cos x}$, $y_p = \frac{1}{2}\sec x - \cos x$

$\therefore y = y_c + y_p$

$= A\cos x + B\sin x + \frac{1}{2}\sec x - \cos x,$ since A is arbitrary constant

$y = A\cos x + B\sin x + \frac{1}{2}\sec x$

⑥ $y'' + 2y' + y = 15e^{-x}\sqrt{x+1}$

$(D^2+2D+1)y = 15e^{-x}\sqrt{x+1}$

$(D+1)^2 y$

$\therefore y_c = [C_0 + C_1 x]e^{-x}$

$y'_c = -C_0 e^{-x} + C_1 e^{-x} - C_1 x e^{-x}$

$y_p = [C_0(x) + C_1(x)x]e^{-x}$

⑥ $y'_p = C_0'(x)e^{-x} - C_0(x)e^{-x} + C_1'(x)xe^{-x} + C_1(x)e^{-x} - C_1(x)xe^{-x}$

⑤ , $y'_p = -C_0(x)e^{-x} + C_1(x)e^{-x} - C_1(x)xe^{-x}$

⑦ ⑥-⑤ , $0 = C_0'(x)e^{-x} + C_1'(x)xe^{-x}$, $\therefore C_0'(x) + C_1'(x)x = 0$

$\frac{d⑤}{dx} = y''_p = -C_0'(x)e^{-x} + C_0(x)e^{-x} - C_1(x)e^{-x} + C_1'(x)e^{-x} - C_1'(x)xe^{-x} - C_1(x)e^{-x} + C_1(x)xe^{-x}$

y''_p	$C_0(x)e^{-x} - C_0'(x)e^{-x} - 2C_1(x)e^{-x} + C_1(x)xe^{-x} + C_1'(x)e^{-x} - C_1'(x)xe^{-x}$
$2y'$	$-2C_0(x)e^{-x} + 2C_1(x)e^{-x} - 2C_1(x)xe^{-x}$
y	$C_0(x)e^{-x} + C_1(x)xe^{-x}$

$-C_0'(x)e^{-x} + C_1'(x)e^{-x}(1-x) = 15e^{-x}\sqrt{x+1}$ $\qquad \frac{2}{5}u^{5/2}$

$u^{1/2}$ $\therefore -C_0'(x) + C_1'(x)(1-x) = 15\sqrt{x+1}$, subbing in ⑦ $\qquad u^{3/2}$

$C_1'(x)x + C_1'(x) - C_1'(x)x = 15\sqrt{x+1}$

$C_1'(x) = 15\sqrt{x+1}$ $\quad \therefore C_0'(x) = -15x\sqrt{x+1}$, $x+1=u$

$\int C_0'(x) = \int -15u\sqrt{u} + 15\sqrt{u}\,du$

$\int C_1'(x) = \int 15\sqrt{x+1}\,d(x+1)$ $\qquad C_0(x) = -6(x+1)^{5/2} + 10(x+1)^{3/2}$

$C_1(x) = \frac{10}{2}(x+1)^{3/2}$

$y_p = e^{-x}(x+1)^{3/2}[-6(x+1) + 10 + 10x]$ $\qquad y = e^{-x}[C_0 + C_1x + 4(x+1)^{5/2}]$

$= 4e^{-x}(x+1)^{3/2}[x+1] = 4e^{-x}(x+1)^{5/2}$