CHAPTER 6

EXERCISES 6.1

In trying to find the pattern by which terms are formed it is often beneficial to write the value of the index of summation above each term. We follow this suggestion when appropriate.

1. $\overset{k=1}{2 \cdot 3} + \overset{k=2}{3 \cdot 4} + \overset{k=3}{4 \cdot 5} + \overset{k=4}{5 \cdot 6} + \cdots + \overset{k=?}{99 \cdot 100} = \displaystyle\sum_{k=1}^{98} (k+1)(k+2)$

2. $\overset{k=1}{\dfrac{1}{2}} + \overset{k=2}{\dfrac{2}{4}} + \overset{k=3}{\dfrac{3}{8}} + \overset{k=4}{\dfrac{4}{16}} + \overset{k=5}{\dfrac{5}{32}} + \cdots + \overset{k=?}{\dfrac{10}{1024}} = \displaystyle\sum_{k=1}^{10} \frac{k}{2^k}$

3. $\overset{k=1}{\dfrac{16}{14+15}} + \overset{k=2}{\dfrac{17}{15+16}} + \overset{k=3}{\dfrac{18}{16+17}} + \cdots + \overset{k=?}{\dfrac{199}{197+198}} = \displaystyle\sum_{k=1}^{184} \frac{k+15}{(k+13)+(k+14)} = \sum_{k=1}^{184} \frac{k+15}{2k+27}$

4. Since each term is the square root of a positive integer, we write

$$1 + \sqrt{2} + \sqrt{3} + 2 + \sqrt{5} + \sqrt{6} + \sqrt{7} + \sqrt{8} + 3 + \cdots + 121 = \sum_{k=1}^{14641} \sqrt{k}.$$

5. $\overset{k=1}{1} + \overset{k=2}{\dfrac{1}{2}} + \overset{k=3}{\dfrac{1}{2 \cdot 3}} + \overset{k=4}{\dfrac{1}{2 \cdot 3 \cdot 4}} + \overset{k=5}{\dfrac{1}{2 \cdot 3 \cdot 4 \cdot 5}} + \cdots + \overset{k=?}{\dfrac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot \cdots \cdot 16}} = \displaystyle\sum_{k=1}^{16} \frac{1}{k!}$

6. Disregarding the sign changes, each term is an integer. To have even terms positive and odd terms negative, we use the fact that $(-1)^k$ is 1 when k is an even integer and -1 when k is an odd integer. In other words, $\quad -2 + 3 - 4 + 5 - 6 + 7 - 8 + \cdots - 1020 = \displaystyle\sum_{k=1}^{1019} (-1)^k (k+1).$

7. $\overset{k=1}{\dfrac{2 \cdot 3}{1 \cdot 4}} + \overset{k=2}{\dfrac{6 \cdot 7}{5 \cdot 8}} + \overset{k=3}{\dfrac{10 \cdot 11}{9 \cdot 12}} + \overset{k=4}{\dfrac{14 \cdot 15}{13 \cdot 16}} ++ \overset{k=?}{\dfrac{414 \cdot 415}{413 \cdot 416}} = \displaystyle\sum_{k=1}^{104} \frac{(4k-2)(4k-1)}{(4k-3)(4k)}$

8. $\overset{k=1}{\dfrac{\tan 1}{2}} + \overset{k=2}{\dfrac{\tan 2}{1+2^2}} + \overset{k=3}{\dfrac{\tan 3}{1+3^2}} + \overset{k=4}{\dfrac{\tan 4}{1+4^2}} + \cdots + \overset{k=?}{\dfrac{\tan 225}{1+225^2}} = \displaystyle\sum_{k=1}^{225} \frac{\tan k}{1+k^2}$

9. $\overset{k=1}{4^3} + \overset{k=2}{5^2} + \overset{k=3}{6} + \overset{k=4}{1} + \overset{k=5}{\dfrac{1}{8}} + \cdots + \overset{k=?}{\dfrac{1}{25^{18}}} = \displaystyle\sum_{k=1}^{22} (k+3)^{4-k}$

10. $0.9 + 0.99 + 0.999 + \cdots + 0.999\,999\,999 = \dfrac{9}{10} + \dfrac{99}{100} + \dfrac{999}{1000} + \cdots + \dfrac{999\,999\,999}{1\,000\,000\,000} = \displaystyle\sum_{k=1}^{9} \frac{10^k - 1}{10^k}$

11. If we set $i = n + 3$, then values of i corresponding to $n = 1$ and $n = 24$ are $i = 4$ and $i = 27$. Thus,

$$\sum_{n=1}^{24} \frac{n^2}{2n+1} = \sum_{i=4}^{27} \frac{(i-3)^2}{2(i-3)+1} = \sum_{i=4}^{27} \frac{i^2 - 6i + 9}{2i - 5}.$$

12. If we set $m = k - 2$, then values of m corresponding to $k = 2$ and $k = 101$ are $m = 0$ and $m = 99$. Thus,

$$\sum_{k=2}^{101} \frac{3k - k^2}{\sqrt{k+5}} = \sum_{m=0}^{99} \frac{3(m+2) - (m+2)^2}{\sqrt{(m+2)+5}} = \sum_{m=0}^{99} \frac{2 - m - m^2}{\sqrt{m+7}}.$$

13. If we set $j = n - 4$, then values of j corresponding to $n = 5$ and $n = 20$ are $j = 1$ and $j = 16$. Thus,

$$\sum_{n=5}^{20} (-1)^n \frac{2^n}{n^2+1} = \sum_{j=1}^{16} (-1)^{j+4} \frac{2^{j+4}}{(j+4)^2+1} = \sum_{j=1}^{16} 16(-1)^j \frac{2^j}{j^2 + 8j + 17}.$$

14. If we set $m = i + 2$, then values of m corresponding to $i = 0$ and $i = 37$ are $m = 2$ and $m = 39$. Thus,

$$\sum_{i=0}^{37} \frac{3^{3i}}{i!} = \sum_{m=2}^{39} \frac{3^{3(m-2)}}{(m-2)!} = \sum_{m=2}^{39} \frac{3^{3m}}{729(m-2)!}.$$

15. If we set $n = r - 5$, then values of n corresponding to $r = 15$ and $r = 225$ are $n = 10$ and $n = 220$. Thus,

$$\sum_{r=15}^{225} \frac{1}{r^2 - 10r} = \sum_{n=10}^{220} \frac{1}{(n+5)^2 - 10(n+5)} = \sum_{n=10}^{220} \frac{1}{n^2 - 25}.$$

16. $\displaystyle\sum_{n=1}^{12} (3n+2) = 3\sum_{n=1}^{12} n + 2\sum_{n=1}^{12} 1 = 3\left[\frac{(12)(13)}{2}\right] + 2(12) = 258$

17. $\displaystyle\sum_{j=1}^{21} (2j^2 + 3j) = 2\sum_{j=1}^{21} j^2 + 3\sum_{j=1}^{21} j = 2\left[\frac{21(22)(43)}{6}\right] + 3\left[\frac{21(22)}{2}\right] = 7315$

18.
$$\begin{aligned}\sum_{m=1}^{n} (4m-2)^2 &= \sum_{m=1}^{n} (16m^2 - 16m + 4) = 16\sum_{m=1}^{n} m^2 - 16\sum_{m=1}^{n} m + 4\sum_{m=1}^{n} 1 \\ &= 16\left[\frac{n(n+1)(2n+1)}{6}\right] - 16\left[\frac{n(n+1)}{2}\right] + 4n = \frac{4n(4n^2-1)}{3}\end{aligned}$$

19.
$$\begin{aligned}\sum_{k=2}^{29} (k^3 - 3k^2) &= \sum_{k=1}^{29} (k^3 - 3k^2) + 2 \\ &= \sum_{k=1}^{29} k^3 - 3\sum_{k=1}^{29} k^2 + 2 = \left[\frac{(29)^2(30)^2}{4}\right] - 3\left[\frac{29(30)(59)}{6}\right] + 2 = 163\,562\end{aligned}$$

20.
$$\begin{aligned}\sum_{n=1}^{25} (n+5)(n-4) &= \sum_{n=1}^{25} (n^2 + n - 20) = \sum_{n=1}^{25} n^2 + \sum_{n=1}^{25} n - 20\sum_{n=1}^{25} 1 \\ &= \frac{(25)(26)(51)}{6} + \frac{25(26)}{2} - 20(25) = 5350\end{aligned}$$

21.
$$\begin{aligned}\sum_{i=1}^{n} i(i-3)^2 &= \sum_{i=1}^{n} (i^3 - 6i^2 + 9i) = \sum_{i=1}^{n} i^3 - 6\sum_{i=1}^{n} i^2 + 9\sum_{i=1}^{n} i \\ &= \frac{n^2(n+1)^2}{4} - 6\left[\frac{n(n+1)(2n+1)}{6}\right] + 9\left[\frac{n(n+1)}{2}\right] = \frac{n(n+1)}{4}[n(n+1) - 4(2n+1) + 18] \\ &= \frac{n(n+1)(n^2 - 7n + 14)}{4}\end{aligned}$$

22. $\displaystyle\sum_{n=10}^{24} (n^2 - 5) = \sum_{n=1}^{24} n^2 - \sum_{n=1}^{9} n^2 - 5(15) = \frac{24(25)(49)}{6} - \frac{9(10)(19)}{6} - 75 = 4540$

23.
$$\begin{aligned}\sum_{i=7}^{17} (i^3 - 3i^2) &= \sum_{i=1}^{17} i^3 - \sum_{i=1}^{6} i^3 - 3\sum_{i=1}^{17} i^2 + 3\sum_{i=1}^{6} i^2 \\ &= \frac{17^2 \cdot 18^2}{4} - \frac{6^2 \cdot 7^2}{4} - 3\left[\frac{17(18)(35)}{6}\right] + 3\left[\frac{6(7)(13)}{6}\right] = 17\,886\end{aligned}$$

24.
$$\begin{aligned}\sum_{k=5}^{n} (k+3)(k+4) &= \sum_{k=1}^{n} (k^2 + 7k + 12) - \sum_{k=1}^{4} (k^2 + 7k + 12) \\ &= \sum_{k=1}^{n} k^2 + 7\sum_{k=1}^{n} k + 12n - \sum_{k=1}^{4} k^2 - 7\sum_{k=1}^{4} k - 12(4) \\ &= \frac{n(n+1)(2n+1)}{6} + 7\left[\frac{n(n+1)}{2}\right] + 12n - 30 - 70 - 48 = \frac{n^3 + 12n^2 + 47n - 444}{3}\end{aligned}$$

25. $$\sum_{i=n}^{2n}(i^2+2i-3)=\sum_{1=1}^{2n}(i^2+2i-3)-\sum_{i=1}^{n-1}(i^2+2i-3)$$
$$=\frac{2n(2n+1)(4n+1)}{6}+2\left[\frac{2n(2n+1)}{2}\right]-3(2n)$$
$$-\frac{(n-1)(n)(2n-1)}{6}-2\left[\frac{(n-1)n}{2}\right]+3(n-1)$$
$$=\frac{14n^3+33n^2+n-18}{6}$$

26. $$\sum_{i=m}^{n}[f(i)+g(i)]=[f(m)+g(m)]+[f(m+1)+g(m+1)]+\cdots+[f(n)+g(n)]$$
$$=[f(m)+f(m+1)+\cdots+f(n)]+[g(m)+g(m+1)+\cdots+g(n)]=\sum_{i=m}^{n}f(i)+\sum_{i=m}^{n}g(i)$$

$$\sum_{i=m}^{n}cf(i)=[c(f(m)]+[cf(m+1)]+\cdots+[cf(n)]=c[f(m)+f(m+1)+\cdots+f(n)]=c\sum_{i=m}^{n}f(i)$$

27. $$\sum_{k=1}^{n}\frac{1}{k(k+1)}=\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$
$$=1-\frac{1}{n+1}=\frac{n}{n+1}$$

28. $$\sum_{i=1}^{n}[f(i)-f(i-1)]=[f(1)-f(0)]+[f(2)-f(1)]+[f(3)-f(2)]+\cdots$$
$$+[f(n-1)-f(n-2)]+[f(n)-f(n-1)]$$
$$=f(n)-f(0)$$

29. Since $i^4-(i-1)^4=4i^3-6i^2+4i-1$ is valid for any integer i whatsoever,we may add the identity from $i=1$ to $i=n$,

$$\sum_{i=1}^{n}[i^4-(i-1)^4]=\sum_{i=1}^{n}(4i^3-6i^2+4i-1).$$

When we write the left-hand side in full, we find

$$\sum_{i=1}^{n}[i^4-(i-1)^4]=[1^4-0^4]+[2^4-1^4]+[3^4-2^4]+\cdots[n^4-(n-1)^4].$$

Most of these terms cancel one another, leaving only n^4; that is, $\sum_{i=1}^{n}[i^4-(i-1)^4]=n^4$. Thus,

$$n^4=\sum_{i=1}^{n}(4i^3-6i^2+4i-1)=4\sum_{i=1}^{n}i^3-6\sum_{i=1}^{n}i^2+4\sum_{i=1}^{n}i-\sum_{i=1}^{n}1$$
$$=4\sum_{i=1}^{n}i^3-6\left[\frac{n(n+1)(2n+1)}{6}\right]+4\left[\frac{n(n+1)}{2}\right]-n.$$

We can solve this equation for $\sum_{i=1}^{n}i^3$,

$$\sum_{i=1}^{n}i^3=\frac{1}{4}\left[n^4+n(n+1)(2n+1)-2n(n+1)+n\right]=\frac{n^2(n+1)^2}{4}.$$

30. No For example if $n = 2$, then the left side is $\sum_{i=1}^{2}[f(i)g(i)] = f(1)g(1) + f(2)g(2)$, whereas the right side is $\left[\sum_{i=1}^{2} f(i)\right]\left[\sum_{i=1}^{2} g(i)\right] = [f(1) + f(2)][g(1) + g(2)]$.

31. (a) $S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \sum_{i=1}^{n} ar^{i-1}$

(b) If we multiply S_n by r and subtract it from S_n, we obtain

$$S_n - rS_n = [a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}] - [ar + ar^2 + ar^3 + \cdots + ar^n].$$

Since all but two terms cancel on the right, this simplifies to

$$(1 - r)S_n = a - ar^n \quad \Longrightarrow \quad S_n = \frac{a(1 - r^n)}{1 - r}, \quad \text{provided } r \neq 1.$$

32. $\frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{1\,048\,576} = \frac{(1/8)[1 - (1/2)^{18}]}{1 - 1/2} = \frac{1 - 2^{-18}}{4}$

33. $1 - \frac{1}{3} + \frac{1}{9} + \cdots - \frac{1}{19\,683} = \frac{1 - (-1/3)^{10}]}{1 + 1/3} = \frac{3(1 - 3^{-10})}{4}$

34. $40(0.99) + 40(0.99)^2 + \cdots + 40(0.99)^{15} = \frac{40(0.99)[1 - (0.99)^{15}]}{1 - 0.99} = 3960[1 - (0.99)^{15}] = 554.2$

35. $\sqrt{0.99} + 0.99 + (0.99)^{3/2} + \cdots + (0.99)^{10} = \frac{\sqrt{0.99}[1 - (\sqrt{0.99})^{20}]}{1 - \sqrt{0.99}} = 18.98$

36. $\left|\sum_{i=1}^{n} f(i)\right| = |f(1) + f(2) + \cdots + f(n)| \leq |f(1)| + |f(2)| + \cdots + |f(n)| = \sum_{i=1}^{n} |f(i)|$

37. Without the signs, the terms can be represented by $1/2^k$ from $k = 0$ to $k = 12$. To obtain the correct signs, we use $\sqrt{2}\sin[(2k + 1)\pi/4]$. In sigma notation then, the summation is represented by $\sum_{k=0}^{12} \frac{\sqrt{2}\sin[(2k + 1)\pi/4]}{2^k}$.

EXERCISES 6.3

1. Since $f(x) = x$ is continuous for $0 \leq x \leq 1$, the definite integral exists, and we may choose any partition and star-points in its evaluation. For n equal subdivisions of length $1/n$, we use the points $x_i = i/n$, $i = 0, \ldots, n$. If we choose the right end of each subinterval as star-point, then $x_i^* = x_i = i/n$. Equation 6.10 now gives

$$\int_0^1 x\,dx = \lim_{\|\Delta x_i\| \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \lim_{\|\Delta x_i\| \to 0} \sum_{i=1}^{n} x_i^* \Delta x_i.$$

Since all subintervals have equal length $\Delta x_i = 1/n$, the norm of the partition is $\|\Delta x_i\| = 1/n$, and taking the limit as $\|\Delta x_i\| \to 0$ is tantamount to letting $n \to \infty$. Thus,

$$\int_0^1 x\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \lim_{n \to \infty} \frac{1}{n^2} \sum_{i=1}^{n} i = \lim_{n \to \infty} \frac{1}{n^2}\left[\frac{n(n + 1)}{2}\right] = \frac{1}{2}.$$

2. Since $f(x) = 3x$ is continuous for $0 \leq x \leq 2$, the definite integral exists, and we may choose any partition and star-points in its evaluation. For n equal subdivisions of length $2/n$, we use $x_i = 2i/n$, $i = 0, \ldots, n$. If we choose the right end of each subinterval as star-point, then $x_i^* = x_i = 2i/n$. Equation 6.10 now gives

$$\int_0^2 3x\,dx = \lim_{\|\Delta x_i\|\to 0}\sum_{i=1}^{n} f(x_i^*)\Delta x_i = \lim_{\|\Delta x_i\|\to 0}\sum_{i=1}^{n} 3x_i^*\Delta x_i.$$

Since all subintervals have equal length $\Delta x_i = 2/n$, the norm of the partition is $\|\Delta x_i\| = 2/n$, and taking the limit as $\|\Delta x_i\| \to 0$ is tantamount to letting $n \to \infty$. Thus,

$$\int_0^2 3x\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} 3\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{12}{n^2}\sum_{i=1}^{n} i = \lim_{n\to\infty}\frac{12}{n^2}\left[\frac{n(n+1)}{2}\right] = 6.$$

3. Since $f(x) = 3x + 2$ is continuous for $0 \le x \le 1$, the definite integral exists, and we may choose any partition and star-points in its evaluation. For n equal subdivisions of length $1/n$, we use the points $x_i = i/n$, $i = 0, \ldots, n$. If we choose the right end of each subinterval as star-point, then $x_i^* = x_i = i/n$. Equation 6.10 now gives

$$\int_0^1 (3x+2)\,dx = \lim_{\|\Delta x_i\|\to 0}\sum_{i=1}^{n} f(x_i^*)\Delta x_i = \lim_{\|\Delta x_i\|\to 0}\sum_{i=1}^{n} (3x_i^*+2)\Delta x_i.$$

Since all subintervals have equal length $\Delta x_i = 1/n$, the norm of the partition is $\|\Delta x_i\| = 1/n$, and taking the limit as $\|\Delta x_i\| \to 0$ is tantamount to letting $n \to \infty$. Thus,

$$\begin{aligned}\int_0^1 (3x+1)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n}\left[3\left(\frac{i}{n}\right)+2\right]\left(\frac{1}{n}\right) = \lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n}(3i+2n)\\ &= \lim_{n\to\infty}\frac{1}{n^2}\left(3\sum_{i=1}^{n} i + 2n^2\right) = \lim_{n\to\infty}\frac{1}{n^2}\left[\frac{3n(n+1)}{2}+2n^2\right] = \frac{7}{2}.\end{aligned}$$

4. Since $f(x) = x^3$ is continuous for $0 \le x \le 2$, the definite integral exists, and we may choose any partition and star-points in its evaluation. For n equal subdivisions of length $2/n$, we use $x_i = 2i/n$, $i = 0, \ldots, n$. If we choose the right end of each subinterval as star-point, then $x_i^* = x_i = 2i/n$. Equation 6.10 gives

$$\int_0^2 x^3\,dx = \lim_{\|\Delta x_i\|\to 0}\sum_{i=1}^{n} f(x_i^*)\Delta x_i = \lim_{\|\Delta x_i\|\to 0}\sum_{i=1}^{n} (x_i^*)^3\Delta x_i.$$

Since all subintervals have equal length $\Delta x_i = 2/n$, the norm of the partition is $\|\Delta x_i\| = 2/n$, and taking the limit as $\|\Delta x_i\| \to 0$ is tantamount to letting $n \to \infty$. Thus,

$$\int_0^2 x^3\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)^3\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{16}{n^4}\sum_{i=1}^{n} i^3 = \lim_{n\to\infty}\frac{16}{n^4}\left[\frac{n^2(n+1)^2}{4}\right] = 4.$$

5. Since $f(x) = x^2 + 2x$ is continuous for $1 \le x \le 2$, the definite integral exists, and we may choose any partition and star-points in its evaluation. For n equal subdivisions of length $1/n$, we use the points $x_i = 1 + i/n$, $i = 0, \ldots, n$. If we choose the right end of each subinterval as star-point, then $x_i^* = x_i = 1 + i/n$. Equation 6.10 now gives

$$\int_1^2 (x^2+2x)\,dx = \lim_{\|\Delta x_i\|\to 0}\sum_{i=1}^{n} f(x_i^*)\Delta x_i = \lim_{\|\Delta x_i\|\to 0}\sum_{i=1}^{n} [(x_i^*)^2 + 2x_i^*]\Delta x_i.$$

Since all subintervals have equal length $\Delta x_i = 1/n$, the norm of the partition is $\|\Delta x_i\| = 1/n$, and taking the limit as $\|\Delta x_i\| \to 0$ is tantamount to letting $n \to \infty$. Thus,

$$\begin{aligned}\int_1^2 (x^2+2x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(1+\frac{i}{n}\right)^2 + 2\left(1+\frac{i}{n}\right)\right]\left(\frac{1}{n}\right) = \lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n}(3n^2+4ni+i^2)\\ &= \lim_{n\to\infty}\frac{1}{n^3}\left(3n^3 + 4n\sum_{i=1}^{n} i + \sum_{i=1}^{n} i^2\right) = \lim_{n\to\infty}\frac{1}{n^3}\left[3n^3 + \frac{4n^2(n+1)}{2} + \frac{n(n+1)(2n+1)}{6}\right]\\ &= 3 + 2 + \frac{1}{3} = \frac{16}{3}.\end{aligned}$$

6. Since $f(x) = 1 - x$ is continuous for $-1 \le x \le 0$, the definite integral exists, and we may choose any partition and star-points in its evaluation. For n equal subdivisions of length $1/n$, we use the points $x_i = -1 + i/n$, $i = 0, \ldots, n$. If we choose the right end of each subinterval as star-point, then $x_i^* = x_i = -1 + i/n$. Equation 6.10 now gives

$$\int_{-1}^{0} (-x+1)\,dx = \lim_{\|\Delta x_i\|\to 0} \sum_{i=1}^{n} (-x_i^* + 1)\Delta x_i = \lim_{\|\Delta x_i\|\to 0} \sum_{i=1}^{n} \left(1 - \frac{i}{n} + 1\right)\left(\frac{1}{n}\right).$$

Since all subintervals have equal length $\Delta x_i = 1/n$, the norm of the partition is $\|\Delta x_i\| = 1/n$, and taking the limit as $\|\Delta x_i\| \to 0$ is tantamount to letting $n \to \infty$. Thus,

$$\int_{-1}^{0} (-x+1)\,dx = \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} (2n - i) = \lim_{n\to\infty} \frac{1}{n^2}\left[2n^2 - \frac{n(n+1)}{2}\right] = \lim_{n\to\infty} \frac{3n-1}{2n} = \frac{3}{2}.$$

7. Since $f(x) = x^2$ is continuous for $-1 \le x \le 1$, the definite integral exists, and we may choose any partition and star-points in its evaluation. For n equal subdivisions of length $2/n$, we use the points $x_i = -1 + 2i/n$, $i = 0, \ldots, n$. If we choose the right end of each subinterval as star-point, then $x_i^* = x_i = -1 + 2i/n$. Equation 6.10 now gives

$$\int_{-1}^{1} x^2\,dx = \lim_{\|\Delta x_i\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \lim_{\|\Delta x_i\|\to 0} \sum_{i=1}^{n} (x_i^*)^2 \Delta x_i.$$

Since all subintervals have equal length $\Delta x_i = 2/n$, the norm of the partition is $\|\Delta x_i\| = 2/n$, and taking the limit as $\|\Delta x_i\| \to 0$ is tantamount to letting $n \to \infty$. Thus,

$$\begin{aligned}\int_{-1}^{1} x^2\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(-1 + \frac{2i}{n}\right)^2 \left(\frac{2}{n}\right) = \lim_{n\to\infty} \frac{2}{n^3} \sum_{i=1}^{n} (n^2 - 4ni + 4i^2) \\ &= \lim_{n\to\infty} \frac{2}{n^3}\left[n^3 - \frac{4n^2(n+1)}{2} + \frac{4n(n+1)(2n+1)}{6}\right] = 2\left(1 - 2 + \frac{4}{3}\right) = \frac{2}{3}.\end{aligned}$$

8. Since $f(x) = x^3$ is continuous for $-1 \le x \le 1$, the definite integral exists, and we may choose any partition and star-points in its evaluation. For n equal subdivisions of length $2/n$, we use the points $x_i = -1 + 2i/n$, $i = 0, \ldots, n$. If we choose the right end of each subinterval as star-point, then $x_i^* = x_i = -1 + 2i/n$. Equation 6.10 now gives

$$\int_{-1}^{1} x^3\,dx = \lim_{\|\Delta x_i\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \lim_{\|\Delta x_i\|\to 0} \sum_{i=1}^{n} (x_i^*)^3 \Delta x_i.$$

Since all subintervals have equal length $\Delta x_i = 2/n$, the norm of the partition is $\|\Delta x_i\| = 2/n$, and taking the limit as $\|\Delta x_i\| \to 0$ is tantamount to letting $n \to \infty$. Thus,

$$\begin{aligned}\int_{-1}^{1} x^3\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(-1 + \frac{2i}{n}\right)^3 \left(\frac{2}{n}\right) = \lim_{n\to\infty} \frac{2}{n^4} \sum_{i=1}^{n} (-n^3 + 6n^2 i - 12ni^2 + 8i^3) \\ &= \lim_{n\to\infty} \frac{2}{n^4}\left[-n^4 + \frac{6n^3(n+1)}{2} - \frac{12n^2(n+1)(2n+1)}{6} + \frac{8n^2(n+1)^2}{4}\right] \\ &= 2(-1 + 3 - 4 + 2) = 0.\end{aligned}$$

9. The integrand x^{15} is an odd function. If we set up a partition with points symmetrically placed about $x = 0$, then terms in summation 6.10 cancel, and the value of the integral is zero.

10. (a) For n equal subdivisions of the interval $0 \le x \le 1$, we use the points $x_i = i/n$ where $i = 0, \cdots, n$. With star points chosen as $x_i^* = x_i = i/n$, equation 6.10 gives

$$\int_0^1 2^x dx = \lim_{\|\Delta x_i\| \to 0} \sum_{i=1}^n 2^{x_i^*} \Delta x_i = \lim_{n\to\infty} \sum_{i=1}^n 2^{i/n} \left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n 2^{i/n}.$$

(b) The formula in question 31(b) of Exercises 6.1 allows us to evaluate the sum in closed form,

$$\int_0^1 2^x dx = \lim_{n\to\infty} \left\{ \frac{1}{n} \frac{2^{1/n}\left[1 - (2^{1/n})^n\right]}{1 - 2^{1/n}} \right\} = \lim_{n\to\infty} \frac{2^{1/n}}{n(2^{1/n} - 1)}.$$

(c) The limit of the numerator is 1. The denominator is of the indeterminate form $0 \cdot \infty$, and we therefore use L'Hôpital's rule to evaluate

$$\lim_{n\to\infty} [n(2^{1/n} - 1)] = \lim_{n\to\infty} \frac{2^{1/n} - 1}{1/n} = \lim_{n\to\infty} \frac{2^{1/n}(-1/n^2)\ln 2}{-1/n^2} = \lim_{n\to\infty} (2^{1/n} \ln 2) = \ln 2.$$

Hence, $\displaystyle\int_0^1 2^x dx = \frac{1}{\ln 2} = \log_2 e.$

11. For n equal subdivisions of the interval $1 \le x \le 3$, we use the points $x_i = 1 + 2i/n$ where $i = 0, \cdots, n$. With star points chosen as $x_i^* = x_i = 1 + 2i/n$, equation 6.10 gives

$$\int_1^3 e^x\, dx = \lim_{\|\Delta x_i\| \to 0} \sum_{i=1}^n e^{x_i^*} \Delta x_i = \lim_{n\to\infty} \sum_{i=1}^n e^{1+2i/n} \left(\frac{2}{n}\right) = \lim_{n\to\infty} \frac{2e}{n} \sum_{i=1}^n e^{2i/n}.$$

The formula in Exercise 31(b) of Exercises 6.1 allows us to evaluate the sum in closed form,

$$\int_1^3 e^x\, dx = \lim_{n\to\infty} \left\{ \frac{2e}{n} \frac{e^{2/n}\left[1 - (e^{2/n})^n\right]}{1 - e^{2/n}} \right\} = 2(e - e^3) \lim_{n\to\infty} \frac{e^{2/n}}{n(1 - e^{2/n})}.$$

The limit of the numerator is 1. The denominator is of the indeterminate form $0 \cdot \infty$, and we therefore use L'Hôpital's rule to evaluate

$$\lim_{n\to\infty} [n(1 - e^{2/n})] = \lim_{n\to\infty} \frac{1 - e^{2/n}}{1/n} = \lim_{n\to\infty} \frac{-e^{2/n}(-2/n^2)}{-1/n^2} = \lim_{n\to\infty} (-2\, e^{2/n}) = -2.$$

Hence, $\displaystyle\int_1^3 e^x\, dx = \frac{2(e - e^3)}{-2} = e^3 - e.$

12. For n equal subdivisions of the interval $0 \le x \le \pi$, we use the points $x_i = i\pi/n$ where $i = 0, \cdots, n$. With star points chosen as $x_i^* = x_i$, equation 6.10 gives

$$\begin{aligned}
\int_0^\pi \sin x\, dx &= \lim_{\|\Delta x_i\| \to 0} \sum_{i=1}^n \sin x_i^* \Delta x_i = \lim_{n\to\infty} \sum_{i=1}^n \sin\left(\frac{i\pi}{n}\right)\left(\frac{\pi}{n}\right) = \lim_{n\to\infty} \frac{\pi}{n} \sum_{i=1}^n \sin\left(\frac{i\pi}{n}\right) \\
&= \lim_{n\to\infty} \frac{\pi}{n} \left[\frac{\sin\dfrac{(n+1)\pi}{2n} \sin\dfrac{n}{2}\left(\dfrac{\pi}{n}\right)}{\sin\dfrac{\pi}{2n}} \right] \qquad \text{(by given formula)} \\
&= \left[\lim_{n\to\infty} \frac{\dfrac{\pi}{n}}{\sin\dfrac{\pi}{2n}} \right] \left[\lim_{n\to\infty} \sin\frac{(n+1)\pi}{2n} \right] \qquad \text{(provided both limits exist)} \\
&= \left[\lim_{n\to\infty} \frac{-\pi/n^2}{\dfrac{-\pi}{2n^2}\cos\dfrac{\pi}{2n}} \right] \left[\sin\frac{\pi}{2} \right] \qquad \text{(using L'Hôpital's rule)} \\
&= 2.
\end{aligned}$$

13. For n equal subdivisions of the interval $0 \le x \le \pi/2$, we use the points $x_i = i\pi/(2n)$ where $i = 0, \cdots, n$. With star-points chosen as $x_i^* = x_i$, equation 6.10 gives

$$\int_0^{\pi/2} \cos x\,dx = \lim_{\|\Delta x_i\|\to 0} \sum_{i=1}^n \cos x_i^* \Delta x_i = \lim_{n\to\infty} \sum_{i=1}^n \cos\left(\frac{i\pi}{2n}\right)\left(\frac{\pi}{2n}\right) = \lim_{n\to\infty} \frac{\pi}{2n} \sum_{i=1}^n \cos\left(\frac{i\pi}{2n}\right)$$

$$= \lim_{n\to\infty} \frac{\pi}{2n}\left[\frac{\cos\dfrac{(n+1)\pi}{4n}\,\sin\dfrac{\pi}{4}}{\sin\dfrac{\pi}{4n}}\right] \qquad \text{(by given formula)}$$

$$= \frac{1}{\sqrt{2}}\left[\lim_{n\to\infty} \frac{\dfrac{\pi}{2n}}{\sin\dfrac{\pi}{4n}}\right]\left[\lim_{n\to\infty} \cos\frac{(n+1)\pi}{4n}\right] \qquad \text{(provided both limits exist)}$$

$$= \frac{1}{\sqrt{2}}\left[\lim_{n\to\infty} \frac{-\pi/(2n^2)}{\dfrac{-\pi}{4n^2}\cos\dfrac{\pi}{4n}}\right]\left[\cos\frac{\pi}{4}\right] \qquad \text{(using L'Hôpital's rule)}$$

$$= \frac{1}{\sqrt{2}}(2)\left(\frac{1}{\sqrt{2}}\right) = 1.$$

14. (a) With the suggested choice for x_i and x_i^*,

$$\int_a^b x^k\,dx = \lim_{\|\Delta x_i\|\to 0} \sum_{i=1}^n (x_i^*)^k \Delta x_i = \lim_{\|\Delta x_i\|\to 0} \sum_{i=1}^n (ah^i)^k (ah^i - ah^{i-1})$$

$$= \lim_{\|\Delta x_i\|\to 0} a^{k+1} \sum_{i=1}^n h^{ik}(h^i - h^{i-1}) = a^{k+1} \lim_{\|\Delta x_i\|\to 0} \sum_{i=1}^n h^{ik} h^{i-1}(h-1)$$

$$= a^{k+1} \lim_{\|\Delta x_i\|\to 0} \left(\frac{h-1}{h}\right) \sum_{i=1}^n (h^{k+1})^i.$$

Now, $\Delta x_i = ah^i - ah^{i-1} = ah^{i-1}(h-1)$, where $h = (b/a)^{1/n}$. Since $h > 1$, this is largest when i is largest; that is, $\|\Delta x_i\| = \Delta x_n = ah^{n-1}(h-1)$. This can be made to approach 0 by letting $n \to \infty$. Thus,

$$\int_a^b x^k\,dx = a^{k+1} \lim_{n\to\infty} \left(\frac{h-1}{h}\right) \sum_{i=1}^n (h^{k+1})^i.$$

(b) Since the sum on the right is a finite geometric series with $a = r = h^{k+1}$,

$$\int_a^b x^k\,dx = a^{k+1} \lim_{n\to\infty} \left(\frac{h-1}{h}\right)\left\{\frac{h^{k+1}[1-(h^{k+1})^n]}{1-h^{k+1}}\right\} = a^{k+1} \lim_{n\to\infty} (h-1)\left\{\frac{h^k[1-h^{n(k+1)}]}{1-h^{k+1}}\right\}.$$

We now set $h = (b/a)^{1/n}$,

$$\int_a^b x^k\,dx = a^{k+1} \lim_{n\to\infty} \left[\left(\frac{b}{a}\right)^{1/n} - 1\right]\left\{\frac{\left(\frac{b}{a}\right)^{k/n}\left[1-\left(\frac{b}{a}\right)^{k+1}\right]}{1-\left(\frac{b}{a}\right)^{(k+1)/n}}\right\}$$

$$= a^{k+1}\left(\frac{a^{k+1}-b^{k+1}}{a^{k+1}}\right) \lim_{n\to\infty} \frac{\left(\frac{b}{a}\right)^{k/n}\left[\left(\frac{b}{a}\right)^{1/n}-1\right]}{1-\left(\frac{b}{a}\right)^{(k+1)/n}}$$

$$= (b^{k+1}-a^{k+1}) \lim_{n\to\infty} \frac{\left(\frac{b}{a}\right)^{k/n}\left[\left(\frac{b}{a}\right)^{1/n}-1\right]}{\left(\frac{b}{a}\right)^{(k+1)/n}-1}.$$

(c) As $n \to \infty$, the factor $(b/a)^{k/n} \to 1$. The remainder of the limit is of the indeterminate form 0/0 and we therefore use L'Hôpital's rule to calculate that

$$\lim_{n\to\infty} \frac{\left(\frac{b}{a}\right)^{1/n} - 1}{\left(\frac{b}{a}\right)^{(k+1)/n} - 1} = \lim_{n\to\infty} \frac{\left(\frac{b}{a}\right)^{1/n}(-1/n^2)\ln(b/a)}{\left(\frac{b}{a}\right)^{(k+1)/n}[-(k+1)/n^2]\ln(b/a)}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{b}{a}\right)^{1/n}}{\left(\frac{b}{a}\right)^{(k+1)/n}(k+1)} = \frac{1}{k+1}.$$

Finally then, $\displaystyle\int_a^b x^k\,dx = (b^{k+1} - a^{k+1})\left(\frac{1}{k+1}\right)$.

15. Let the interval $a \le x \le b$ be subdivided into n subintervals in any manner whatsoever. If we choose star-points in each subinterval as rational numbers, then the limit of the summation in equation 6.10 will be $b - a$. On the other hand, if star-points are chosen as irrational numbers, the limit of the summation will be zero. Since the limit depends on the choice of star-points, the definite integral does not exist.

EXERCISES 6.4

1. $\displaystyle\int_3^4 (x^3+3)\,dx = \left\{\frac{x^4}{4} + 3x\right\}_3^4 = (64+12) - \left(\frac{81}{4} + 9\right) = \frac{187}{4}$

2. $\displaystyle\int_1^3 (x^2 - 2x + 3)\,dx = \left\{\frac{x^3}{3} - x^2 + 3x\right\}_1^3 = (9 - 9 + 9) - \left(\frac{1}{3} - 1 + 3\right) = \frac{20}{3}$

3. $\displaystyle\int_{-1}^1 (4x^3 + 2x)\,dx = \left\{x^4 + x^2\right\}_{-1}^1 = (1+1) - (1+1) = 0$

4. $\displaystyle\int_{-3}^{-1} \frac{1}{x^2}dx = \left\{-\frac{1}{x}\right\}_{-3}^{-1} = (1) - \left(\frac{1}{3}\right) = \frac{2}{3}$

5. $\displaystyle\int_4^2 \left(x^2 + \frac{3}{x^3}\right)dx = \left\{\frac{x^3}{3} - \frac{3}{2x^2}\right\}_4^2 = \left(\frac{8}{3} - \frac{3}{8}\right) - \left(\frac{64}{3} - \frac{3}{32}\right) = -\frac{1819}{96}$

6. $\displaystyle\int_0^{\pi/2} \sin x\,dx = \{-\cos x\}_0^{\pi/2} = (0) - (-1) = 1$

7. $\displaystyle\int_{-1}^1 (x^2 - 1 - x^4)\,dx = \left\{\frac{x^3}{3} - x - \frac{x^5}{5}\right\}_{-1}^1 = \left(\frac{1}{3} - 1 - \frac{1}{5}\right) - \left(-\frac{1}{3} + 1 + \frac{1}{5}\right) = -\frac{26}{15}$

8. $\displaystyle\int_{-1}^{-2} \left(\frac{1}{x^2} - 2x\right)dx = \left\{-\frac{1}{x} - x^2\right\}_{-1}^{-2} = \left(\frac{1}{2} - 4\right) - (1-1) = -\frac{7}{2}$

9. $\displaystyle\int_1^2 (x^4 + 3x^2 + 2)\,dx = \left\{\frac{x^5}{5} + x^3 + 2x\right\}_1^2 = \left(\frac{32}{5} + 8 + 4\right) - \left(\frac{1}{5} + 1 + 2\right) = \frac{76}{5}$

10. $\displaystyle\int_0^1 x(x^2+1)\,dx = \int_0^1 (x^3 + x)\,dx = \left\{\frac{x^4}{4} + \frac{x^2}{2}\right\}_0^1 = \left(\frac{1}{4} + \frac{1}{2}\right) - 0 = \frac{3}{4}$

11. $\displaystyle\int_0^1 x^2(x^2+1)^2\,dx = \int_0^1 (x^6 + 2x^4 + x^2)\,dx = \left\{\frac{x^7}{7} + \frac{2x^5}{5} + \frac{x^3}{3}\right\}_0^1 = \left(\frac{1}{7} + \frac{2}{5} + \frac{1}{3}\right) - (0) = \frac{92}{105}$

12. $\displaystyle\int_0^{2\pi} \cos 2x\,dx = \left\{\frac{1}{2}\sin 2x\right\}_0^{2\pi} = (0) - (0) = 0$

13. $\displaystyle\int_1^3 \frac{x^2+3}{x^2}dx = \int_1^3 \left(1 + \frac{3}{x^2}\right)dx = \left\{x - \frac{3}{x}\right\}_1^3 = (3-1) - (1-3) = 4$

14. $\displaystyle\int_0^1 (x^{2.2} - x^\pi)\,dx = \left\{\frac{x^{3.2}}{3.2} - \frac{x^{\pi+1}}{\pi+1}\right\}_0^1 = \left(\frac{1}{3.2} - \frac{1}{\pi+1}\right) - 0 = \frac{5}{16} - \frac{1}{\pi+1}$

15. $\int_{-1}^{1} x^2(x^3-x)\,dx = \int_{-1}^{1} (x^5-x^3)\,dx = \left\{\frac{x^6}{6}-\frac{x^4}{4}\right\}_{-1}^{1} = \left(\frac{1}{6}-\frac{1}{4}\right)-\left(\frac{1}{6}-\frac{1}{4}\right)=0$

16. $\int_{3}^{4} \frac{(x^2-1)^2}{x^2}dx = \int_{3}^{4}\left(x^2-2+\frac{1}{x^2}\right)dx = \left\{\frac{x^3}{3}-2x-\frac{1}{x}\right\}_{3}^{4} = \left(\frac{64}{3}-8-\frac{1}{4}\right)-\left(9-6-\frac{1}{3}\right)=\frac{125}{12}$

17. $\int_{1}^{2}\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)dx = \left\{\frac{2x^{3/2}}{3}-2\sqrt{x}\right\}_{1}^{2} = \left(\frac{4\sqrt{2}}{3}-2\sqrt{2}\right)-\left(\frac{2}{3}-2\right)=\frac{2(2-\sqrt{2})}{3}$

18. $\int_{-2}^{3}(x-1)^3\,dx = \left\{\frac{1}{4}(x-1)^4\right\}_{-2}^{3} = \frac{1}{4}(16)-\frac{1}{4}(81)=-\frac{65}{4}$

19. $\int_{2}^{4}\frac{(x^2-1)(x^2+1)}{x^2}dx = \int_{2}^{4}\left(x^2-\frac{1}{x^2}\right)dx = \left\{\frac{x^3}{3}+\frac{1}{x}\right\}_{2}^{4} = \left(\frac{64}{3}+\frac{1}{4}\right)-\left(\frac{8}{3}+\frac{1}{2}\right)=\frac{221}{12}$

20. $\int_{0}^{\pi/4} 3\cos x\,dx = \{3\sin x\}_{0}^{\pi/4} = \frac{3}{\sqrt{2}}-0=\frac{3}{\sqrt{2}}$ **21.** $\int_{0}^{\pi/4}\sec^2 x\,dx = \left\{\tan x\right\}_{0}^{\pi/4}=1$

22. $\int_{\pi/2}^{\pi}\sin x\,\cos x\,dx = \left\{\frac{1}{2}\sin^2 x\right\}_{\pi/2}^{\pi} = \frac{1}{2}(0)-\frac{1}{2}(1)=-\frac{1}{2}$

23. This integral does not exist since the integrand is undefined at $x=\pi/3$.

24. $\int_{-\pi/4}^{\pi/4}\sec x\,\tan x\,dx = \left\{\sec x\right\}_{-\pi/4}^{\pi/4} = (\sqrt{2})-(\sqrt{2})=0$

25. $\int_{0}^{2} 2^x\,dx = \left\{\frac{2^x}{\ln 2}\right\}_{0}^{2} = \frac{4}{\ln 2}-\frac{1}{\ln 2}=\frac{3}{\ln 2}$ **26.** $\int_{-1}^{2} e^x\,dx = \left\{e^x\right\}_{-1}^{2} = e^2-e^{-1}$

27. $\int_{0}^{1} e^{3x}\,dx = \left\{\frac{e^{3x}}{3}\right\}_{0}^{1} = \frac{e^3}{3}-\frac{1}{3}=\frac{e^3-1}{3}$ **28.** $\int_{-3}^{-2}\frac{1}{x}dx = \{\ln|x|\}_{-3}^{-2} = (\ln 2)-(\ln 3)=\ln(2/3)$

29. $\int_{1}^{3}\frac{(x+1)^2}{x}dx = \int_{1}^{3}\left(x+2+\frac{1}{x}\right)dx = \left\{\frac{x^2}{2}+2x+\ln|x|\right\}_{1}^{3} = \left(\frac{9}{2}+6+\ln 3\right)-\left(\frac{1}{2}+2\right)=8+\ln 3$

30. $\int_{0}^{1} 3^{4x}\,dx = \left\{\frac{1}{4}3^{4x}\log_3 e\right\}_{0}^{1} = \frac{1}{4}\log_3 e(3^4-3^0)=20\log_3 e$

31. $\int_{0}^{5}|x|\,dx = \int_{0}^{5}x\,dx = \left\{\frac{x^2}{2}\right\}_{0}^{5}=\frac{25}{2}$

32. Since $|x+1|$ is positive between $x=0$ and $x=4$,

$$\int_{0}^{4}x|x+1|\,dx = \int_{0}^{4}x(x+1)\,dx = \int_{0}^{4}(x^2+x)\,dx = \left\{\frac{x^3}{3}+\frac{x^2}{2}\right\}_{0}^{4} = \left(\frac{64}{3}+8\right)-0=\frac{88}{3}.$$

33. $\int_{-5}^{5}|x|\,dx = \int_{-5}^{0}-x\,dx+\int_{0}^{5}x\,dx = \left\{-\frac{x^2}{2}\right\}_{-5}^{0}+\left\{\frac{x^2}{2}\right\}_{0}^{5} = \frac{25}{2}+\frac{25}{2}=25$

34. Because $x+1$ changes sign at $x=-1$, we divide the integral into two parts,

$$\begin{aligned}\int_{-2}^{1}x|x+1|\,dx &= \int_{-2}^{-1}x(-x-1)\,dx+\int_{-1}^{1}x(x+1)\,dx = \left\{-\frac{x^3}{3}-\frac{x^2}{2}\right\}_{-2}^{-1}+\left\{\frac{x^3}{3}+\frac{x^2}{2}\right\}_{-1}^{1}\\ &= \left(\frac{1}{3}-\frac{1}{2}\right)-\left(\frac{8}{3}-2\right)+\left(\frac{1}{3}+\frac{1}{2}\right)-\left(-\frac{1}{3}+\frac{1}{2}\right)=-\frac{1}{6}.\end{aligned}$$

35. $\int_{-1/2}^{1/2}\frac{1}{\sqrt{1-x^2}}dx = \{\mathrm{Sin}^{-1}x\}_{-1/2}^{1/2} = \mathrm{Sin}^{-1}(1/2)-\mathrm{Sin}^{-1}(-1/2)=\frac{\pi}{3}$

36. $\int_{-}^{1}\frac{1}{1+x^2}dx = \{\mathrm{Tan}^{-1}x\}_{-1}^{1} = \mathrm{Tan}^{-1}1-\mathrm{Tan}^{-1}(-1)=\frac{\pi}{2}$

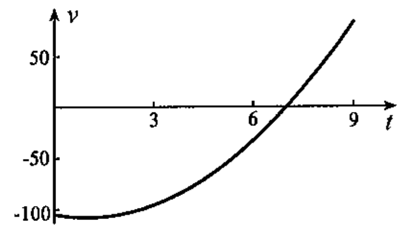
37. $\displaystyle\int_2^3 \frac{1}{x\sqrt{x^2-1}}dx = \left\{\mathrm{Sec}^{-1}x\right\}_2^3 = \mathrm{Sec}^{-1}3 - \mathrm{Sec}^{-1}(2) = \mathrm{Sec}^{-1}3 - \frac{\pi}{3}$

38. $\displaystyle\int_0^1 \cosh 2x\,dx = \left\{\frac{1}{2}\sinh 2x\right\}_0^1 = \frac{1}{2}\sinh 2$

39. This integral does not exist since $\mathrm{csch}x$ is not defined at $x = 0$.

40. $\displaystyle\int_0^{1/2} \frac{1}{1+4x^2}dx = \left\{\frac{1}{2}\mathrm{Tan}^{-1}2x\right\}_0^{1/2} = \frac{1}{2}\mathrm{Tan}^{-1}(1) = \frac{\pi}{8}$

41.
$$\begin{aligned}\int_0^9 v(t)\,dt &= \int_0^9 (3t^2-6t-105)\,dt\\ &= \left\{t^3-3t^2-105t\right\}_0^9\\ &= 729-243-945 = -459\end{aligned}$$

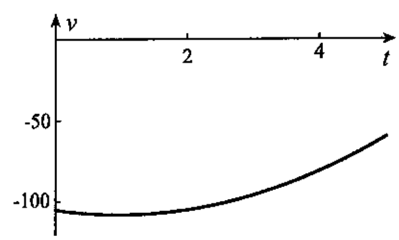


This is the displacement of the particle at $t = 9$ relative to its position at $t = 0$. Since $v(t) = 3(t-7)(t+5)$ changes sign at $t = 7$,

$$\begin{aligned}\int_0^9 |v(t)|\,dt &= \int_0^7 (-3t^2+6t+105)\,dt + \int_7^9 (3t^2-6t-105)\,dt\\ &= \left\{-t^3+3t^2+105t\right\}_0^7 + \left\{t^3-3t^2-105t\right\}_7^9\\ &= (-343+147+735)+(729-243-945)-(343-147-735) = 619.\end{aligned}$$

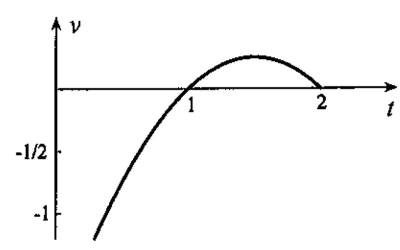
This is the distance travelled by the particle between $t = 0$ and $t = 9$.

42.
$$\begin{aligned}\int_0^5 v(t)\,dt &= \int_0^5 (3t^2-6t-105)\,dt = \left\{t^3-3t^2-105t\right\}_0^5\\ &= 125-75-525 = -475\end{aligned}$$



This is the displacement of the particle at $t = 5$ relative to its position at $t = 0$. Since $v(t) = 3(t-7)(t+5)$ is negative for $0 \le t \le 5$, the integral of $|v(t)|$ is 475. This is the distance travelled by the particle between $t = 0$ and $t = 5$.

43.
$$\begin{aligned}\int_0^2 v(t)\,dt &= \int_0^2 (-t^2+3t-2)\,dt\\ &= \left\{-\frac{t^3}{3}+\frac{3t^2}{2}-2t\right\}_0^2\\ &= -\frac{8}{3}+6-4 = -\frac{2}{3}\end{aligned}$$



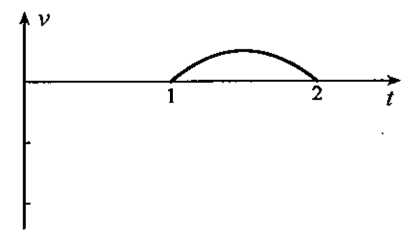
This is the displacement of the particle at $t = 2$ relative to its position at $t = 0$. Since $v(t) = -(t-1)(t-2)$ changes sign at $t = 1$,

$$\begin{aligned}\int_0^2 |v(t)|\,dt &= \int_0^1 (t^2-3t+2)\,dt + \int_1^2 (-t^2+3t-2)\,dt = \left\{\frac{t^3}{3}-\frac{3t^2}{2}+2t\right\}_0^1 + \left\{-\frac{t^3}{3}+\frac{3t^2}{2}-2t\right\}_1^2\\ &= \left(\frac{1}{3}-\frac{3}{2}+2\right)+\left(-\frac{8}{3}+6-4\right)-\left(-\frac{1}{3}+\frac{3}{2}-2\right) = 1.\end{aligned}$$

This is the distance travelled by the particle between $t = 0$ and $t = 2$.

44. $\displaystyle\int_1^2 v(t)\,dt = \int_1^2 (-t^2+3t-2)\,dt$

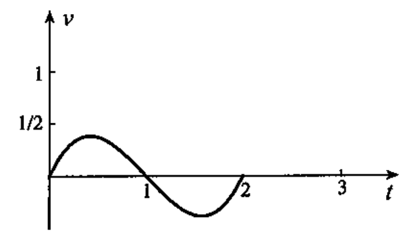
$$= \left\{-\frac{t^3}{3}+\frac{3t^2}{2}-2t\right\}_1^2$$
$$= \left(-\frac{8}{3}+6-4\right)-\left(-\frac{1}{3}+\frac{3}{2}-2\right)=\frac{1}{6}$$



This is the displacement of the particle at $t=2$ relative to its position at $t=1$. Since $v(t)=-(t-1)(t-2)$ is positive between $t=1$ and $t=2$, the integral of $|v(t)|$ is also $1/6$. It is the distance travelled by the particle between $t=1$ and $t=2$.

45. $\displaystyle\int_0^2 v(t)\,dt = \int_0^2 (t^3-3t^2+2t)\,dt$

$$= \left\{\frac{t^4}{4}-t^3+t^2\right\}_0^2$$
$$= 4-8+4=0$$



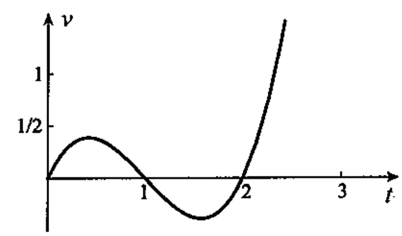
This means that the position of the particle is the same at $t=0$ and $t=2$. Since $v(t)=t(t-1)(t-2)$ changes sign at $t=1$,

$$\int_0^2 |v(t)|\,dt = \int_0^1 (t^3-3t^2+2t)\,dt + \int_1^2 (-t^3+3t^2-2t)\,dt = \left\{\frac{t^4}{4}-t^3+t^2\right\}_0^1 + \left\{-\frac{t^4}{4}+t^3-t^2\right\}_1^2$$
$$= \left(\frac{1}{4}-1+1\right)+(-4+8-4)-\left(-\frac{1}{4}+1-1\right)=\frac{1}{2}.$$

This the distance travelled by the particle between $t=0$ and $t=2$.

46. $\displaystyle\int_0^3 v(t)\,dt = \int_0^3 (t^3-3t^2+2t)\,dt$

$$= \left\{\frac{t^4}{4}-t^3+t^2\right\}_0^3$$
$$= \frac{81}{4}-27+9=\frac{9}{4}$$



This is the displacement of the particle at $t=3$ relative to its position at $t=0$. Since $v(t)=t(t-1)(t-2)$ changes sign at $t=1$ and $t=2$,

$$\int_0^3 |v(t)|\,dt = \int_0^1 (t^3-3t^2+2t)\,dt + \int_1^2 (-t^3+3t^2-2t)\,dt + \int_2^3 (t^3-3t^2+2t)\,dt$$
$$= \left\{\frac{t^4}{4}-t^3+t^2\right\}_0^1 + \left\{-\frac{t^4}{4}+t^3-t^2\right\}_1^2 + \left\{\frac{t^4}{4}-t^3+t^2\right\}_2^3$$
$$= \left(\frac{1}{4}-1+1\right)+(-4+8-4)-\left(-\frac{1}{4}+1-1\right)+\left(\frac{81}{4}-27+9\right)-(4-8+4)=\frac{11}{4}.$$

This is the distance travelled by the particle between $t=0$ and $t=3$.

47. The integrand is clearly nonnegative for $0\le x\le \pi/4$. For these values of x, the largest value of $\sin x$ is $1/\sqrt{2}$ and the smallest value of $1+x^2$ is 1. It follows that $\sin x/(1+x^2)$ cannot be larger than $1/\sqrt{2}$. By inequality 6.15 then

$$0\left(\frac{\pi}{4}\right)\le \int_0^{\pi/4}\frac{\sin x}{1+x^2}\,dx \le \frac{1}{\sqrt{2}}\left(\frac{\pi}{4}\right) \quad\Longrightarrow\quad 0\le \int_0^{\pi/4}\frac{\sin x}{1+x^2}dx\le \frac{\sqrt{2}\pi}{8}.$$

48. The integrand is clearly nonnegative for $0 \le x \le \pi/2$. For these values of x, the largest value of $\sin x$ is 1 and the smallest value of $1+x$ is 1. It follows that $\sin x/(1+x)$ cannot be larger than 1. By inequality 6.15 then

$$0\left(\frac{\pi}{2}\right) \le \int_0^{\pi/2} \frac{\sin x}{1+x}\, dx \le 1\left(\frac{\pi}{2}\right) \quad \Longrightarrow \quad 0 \le \int_0^{\pi/2} \frac{\sin x}{1+x} dx \le \frac{\pi}{2}.$$

49. The integrand cannot be negative for $0 \le x \le \pi$, and for these values of x, the largest value of $\sin x$ is 1 and the smallest value of $2+x^2$ is 2. It follows that the integrand cannot be larger than 1/2, and

$$0(\pi) \le \int_0^{\pi} \frac{\sin x}{2+x^2} dx \le \frac{1}{2}\pi \quad \Longrightarrow \quad 0 \le \int_0^{\pi} \frac{\sin x}{2+x^2} dx \le \frac{\pi}{2}.$$

50. The integrand is nonnegative for $\pi/4 \le x \le \pi/2$. For these values of x, the largest value of $\sin 2x$ is 1 and the smallest value of $10+x^2$ is $10+\pi^2/16$. It follows that $\sin 2x/(10+x^2)$ cannot be larger than $1/(10+\pi^2/16)$. By inequality 6.15 then

$$0\left(\frac{\pi}{4}\right) \le \int_{\pi/4}^{\pi/2} \frac{\sin 2x}{10+x^2}\, dx \le \frac{1}{10+\pi^2/16}\left(\frac{\pi}{4}\right) \quad \Longrightarrow \quad 0 \le \int_{\pi/4}^{\pi/2} \frac{\sin 2x}{10+x^2} dx \le \frac{4\pi}{160+\pi^2}.$$

51. Because $1 \le 1+4x^4 \le 5$ and $\cos 1 \le \cos x^2 \le 1$ for $0 \le x \le 1$, it follows that $\cos 1 \le (1+4x^4)\cos x^2 \le 5$, and

$$(\cos 1)1 \le \int_0^1 (1+4x^4)\cos x^2\, dx \le 5(1) \quad \Longrightarrow \quad \cos 1 \le \int_0^1 (1+4x^4)\cos x^2\, dx \le 5.$$

52. Because $\sqrt{5} \le \sqrt{4+x^3} \le \sqrt{31}$ for $1 \le x \le 3$, it follows that $\quad 2\sqrt{5} \le \displaystyle\int_1^3 \sqrt{4+x^3}\, dx \le 2\sqrt{31}.$

EXERCISES 6.5

1. By equation 6.19, $\displaystyle\frac{d}{dx}\int_0^x (3t^2+t)\, dt = 3x^2 + x.$

2. By equation 6.19, $\displaystyle\frac{d}{dx}\int_1^x \frac{1}{\sqrt{t^2+1}} dt = \frac{1}{\sqrt{x^2+1}}.$

3. By reversing limits, $\displaystyle\frac{d}{dx}\int_x^2 \sin(t^2)\, dt = -\frac{d}{dx}\int_2^x \sin(t^2)\, dt = -\sin(x^2).$

4. By reversing limits, $\displaystyle\frac{d}{dx}\int_x^{-1} t^3\cos t\, dt = -\frac{d}{dx}\int_{-1}^x t^3\cos t\, dt = -x^3\cos x.$

5. We set $u = 3x$ and use the chain rule,

$$\frac{d}{dx}\int_0^{3x} (2t-t^4)^2\, dt = \left[\frac{d}{du}\int_0^u (2t-t^4)^2\, dt\right]\frac{du}{dx} = (2u-u^4)^2(3) = 3(6x-81x^4)^2 = 27x^2(2-27x^3)^2.$$

6. We set $u = 2x$ and use the chain rule,

$$\frac{d}{dx}\int_1^{2x} \sqrt{t+1}\, dt = \left[\frac{d}{du}\int_1^u \sqrt{t+1}\, dt\right]\frac{du}{dx} = \sqrt{u+1}(2) = 2\sqrt{2x+1}.$$

7. We set $u = 3x^2$ and use the chain rule,

$$\frac{d}{dx}\int_4^{3x^2} \sin(3t+4)\, dt = \left[\frac{d}{du}\int_4^u \sin(3t+4)\, dt\right]\frac{du}{dx} = \sin(3u+4)(6x) = 6x\sin(9x^2+4).$$

8. We set $u = 5x+4$ and use the chain rule,

$$\frac{d}{dx}\int_{-2}^{5x+4} \sqrt{t^3+1}\, dt = \left[\frac{d}{du}\int_{-2}^u \sqrt{t^3+1}\, dt\right]\frac{du}{dx} = \sqrt{u^3+1}\,(5) = 5\sqrt{(5x+4)^3+1}.$$

9. When a is any number between x and $2x$, we may write

$$\begin{aligned}\int_x^{2x} (3\sqrt{t} - 2t)\,dt &= \int_x^a (3\sqrt{t} - 2t)\,dt + \int_a^{2x} (3\sqrt{t} - 2t)\,dt \\ &= -\int_a^x (3\sqrt{t} - 2t)\,dt + \int_a^{2x} (3\sqrt{t} - 2t)\,dt.\end{aligned}$$

In the second integral we set $u = 2x$ and use the chain rule,

$$\begin{aligned}\frac{d}{dx}\int_x^{2x} (3\sqrt{t} - 2t)\,dt &= -(3\sqrt{x} - 2x) + \left[\frac{d}{du}\int_a^u (3\sqrt{t} - 2t)\,dt\right]\frac{du}{dx} \\ &= -3\sqrt{x} + 2x + (3\sqrt{u} - 2u)(2) \\ &= -3\sqrt{x} + 2x + 6\sqrt{2x} - 8x = 3(2\sqrt{2} - 1)\sqrt{x} - 6x.\end{aligned}$$

10. When a is any number between $4x$ and $4x + 4$, we may write

$$\begin{aligned}\int_{4x}^{4x+4} \left(t^3 - \frac{1}{\sqrt{t}}\right) dt &= \int_{4x}^a \left(t^3 - \frac{1}{\sqrt{t}}\right) dt + \int_a^{4x+4} \left(t^3 - \frac{1}{\sqrt{t}}\right) dt \\ &= -\int_a^{4x} \left(t^3 - \frac{1}{\sqrt{t}}\right) dt + \int_a^{4x+4} \left(t^3 - \frac{1}{\sqrt{t}}\right) dt.\end{aligned}$$

In these integrals we set $u = 4x$ and $v = 4x + 4$ respectively, and use chain rules,

$$\begin{aligned}\frac{d}{dx}\int_{4x}^{4x+4} \left(t^3 - \frac{1}{\sqrt{t}}\right) dt &= -\left[\frac{d}{du}\int_a^u \left(t^3 - \frac{1}{\sqrt{t}}\right) dt\right]\frac{du}{dx} + \left[\frac{d}{dv}\int_a^v \left(t^3 - \frac{1}{\sqrt{t}}\right) dt\right]\frac{dv}{dx} \\ &= -\left[u^3 - \frac{1}{\sqrt{u}}\right](4) + \left[v^3 - \frac{1}{\sqrt{v}}\right](4) \\ &= -4\left[(4x)^3 - \frac{1}{\sqrt{4x}}\right] + 4\left[(4x+4)^3 - \frac{1}{\sqrt{4x+4}}\right] \\ &= -256x^3 + \frac{2}{\sqrt{x}} + 256(x+1)^3 - \frac{2}{\sqrt{x+1}} \\ &= 256(3x^2 + 3x + 1) + \frac{2}{\sqrt{x}} - \frac{2}{\sqrt{x+1}}.\end{aligned}$$

11. When a is any number between $-2x$ and x, we may write

$$\int_{-2x}^x \tan(3t+1)\,dt = \int_{-2x}^a \tan(3t+1)\,dt + \int_a^x \tan(3t+1)\,dt = -\int_a^{-2x} \tan(3t+1)\,dt + \int_a^x \tan(3t+1)\,dt.$$

In the first integral we set $u = -2x$ and use the chain rule,

$$\begin{aligned}\frac{d}{dx}\int_{-2x}^x \tan(3t+1)\,dt &= -\left[\frac{d}{du}\int_a^u \tan(3t+1)\,dt\right]\frac{du}{dx} + \tan(3x+1) \\ &= -\tan(3u+1)(-2) + \tan(3x+1) \\ &= 2\tan(1-6x) + \tan(3x+1).\end{aligned}$$

12. When a is a number between $-x^2$ and $-2x^2$, we may write

$$\begin{aligned}\int_{-x^2}^{-2x^2} \sec(1-t)\,dt &= \int_{-x^2}^a \sec(1-t)\,dt + \int_a^{-2x^2} \sec(1-t)\,dt \\ &= -\int_a^{-x^2} \sec(1-t)\,dt + \int_a^{-2x^2} \sec(1-t)\,dt.\end{aligned}$$

In these integrals we set $u = -x^2$ and $v = -2x^2$ respectively, and use chain rules,

$$\begin{aligned}\frac{d}{dx}\int_{-x^2}^{-2x^2} \sec(1-t)\,dt &= -\left[\frac{d}{du}\int_a^u \sec(1-t)\,dt\right]\frac{du}{dx} + \left[\frac{d}{dv}\int_a^v \sec(1-t)\,dt\right]\frac{dv}{dx}\\ &= -\sec(1-u)(-2x) + \sec(1-v)(-4x)\\ &= 2x\sec(1+x^2) - 4x\sec(1+2x^2).\end{aligned}$$

13. We set $u = \sin x$ and use the chain rule,

$$\frac{d}{dx}\int_0^{\sin x} \cos(t^2)\,dt = \left[\frac{d}{du}\int_0^u \cos(t^2)\,dt\right]\frac{du}{dx} = \cos(u^2)(\cos x) = \cos x\,\cos(\sin^2 x).$$

14. When a is a number between $\cos x$ and $\sin x$, we may write

$$\int_{\cos x}^{\sin x} \frac{1}{\sqrt{t+1}}dt = \int_{\cos x}^{a} \frac{1}{\sqrt{t+1}}dt + \int_a^{\sin x} \frac{1}{\sqrt{t+1}}dt = -\int_a^{\cos x} \frac{1}{\sqrt{t+1}}dt + \int_a^{\sin x} \frac{1}{\sqrt{t+1}}dt.$$

In these integrals we set $u = \cos x$ and $v = \sin x$ respectively, and use chain rules,

$$\begin{aligned}\frac{d}{dx}\int_{\cos x}^{\sin x} \frac{1}{\sqrt{t+1}}dt &= -\left[\frac{d}{du}\int_a^u \frac{1}{\sqrt{t+1}}dt\right]\frac{du}{dx} + \left[\frac{d}{dv}\int_a^v \frac{1}{\sqrt{t+1}}dt\right]\frac{dv}{dx}\\ &= -\frac{1}{\sqrt{u+1}}(-\sin x) + \frac{1}{\sqrt{v+1}}(\cos x) = \frac{\sin x}{\sqrt{\cos x+1}} + \frac{\cos x}{\sqrt{\sin x+1}}.\end{aligned}$$

15. We set $u = 2\sqrt{x}$ and use the chain rule,

$$\frac{d}{dx}\int_0^{2\sqrt{x}} \sqrt{t}\,dt = \left[\frac{d}{du}\int_0^u \sqrt{t}\,dt\right]\frac{du}{dx} = \sqrt{u}\left(\frac{1}{\sqrt{x}}\right) = \frac{\sqrt{2}}{x^{1/4}}.$$

16. If a is a number between $\sqrt{x}$ and $2\sqrt{x}$, we may write

$$\int_{\sqrt{x}}^{2\sqrt{x}} \sqrt{t}\,dt = \int_{\sqrt{x}}^{a} \sqrt{t}\,dt + \int_a^{2\sqrt{x}} \sqrt{t}\,dt = -\int_a^{\sqrt{x}} \sqrt{t}\,dt + \int_a^{2\sqrt{x}} \sqrt{t}\,dt.$$

In these integrals we set $u = \sqrt{x}$ and $v = 2\sqrt{x}$ respectively, and use chain rules,

$$\begin{aligned}\frac{d}{dx}\int_{\sqrt{x}}^{2\sqrt{x}} \sqrt{t}\,dt &= -\left[\frac{d}{du}\int_a^u \sqrt{t}\,dt\right]\frac{du}{dx} + \left[\frac{d}{dv}\int_a^v \sqrt{t}\,dt\right]\frac{dv}{dx}\\ &= -\sqrt{u}\left(\frac{1}{2\sqrt{x}}\right) + \sqrt{v}\left(\frac{1}{\sqrt{x}}\right) = -\frac{x^{1/4}}{2\sqrt{x}} + \frac{\sqrt{2}x^{1/4}}{\sqrt{x}} = \frac{2\sqrt{2}-1}{2x^{1/4}}.\end{aligned}$$

17. We set $u = x^2$ and use the chain rule,

$$\frac{d}{dx}\int_1^{x^2} t^2 e^{4t}\,dt = \left[\frac{d}{du}\int_1^u t^2 e^{4t}\,dt\right]\frac{du}{dx} = u^2 e^{4u}(2x) = 2x^5 e^{4x^2}.$$

18. $\displaystyle \frac{d}{dx}\int_x^2 \ln(t^2+1)\,dt = -\frac{d}{dx}\int_2^x \ln(t^2+1)\,dt = -\ln(x^2+1)$

19. If a is a number between x and $2x$, we may write

$$\int_x^{2x} t\,\ln t\,dt = \int_x^a t\,\ln t\,dt + \int_a^{2x} t\,\ln t\,dt = -\int_a^x t\,\ln t\,dt + \int_a^{2x} t\,\ln t\,dt.$$

In the second integral we set $u = 2x$ and use the chain rule,

$$\begin{aligned}\frac{d}{dx}\int_x^{2x} t\,\ln t\,dt &= -x\,\ln x + \left[\frac{d}{du}\int_a^u t\,\ln t\,dt\right]\frac{du}{dx}\\ &= -x\,\ln x + u\,\ln u\,(2) = -x\,\ln x + 4x\,\ln(2x) = (4\,\ln 2)x + 3x\,\ln x.\end{aligned}$$

20. If a is a number between $-2x$ and $3x$, then

$$\int_{-2x}^{3x} e^{-4t^2}\,dt = \int_{-2x}^{a} e^{-4t^2}\,dt + \int_{a}^{3x} e^{-4t^2}\,dt = -\int_{a}^{-2x} e^{-4t^2}\,dt + \int_{a}^{3x} e^{-4t^2}\,dt.$$

In these integrals we set $u = -2x$ and $v = 3x$ respectively, and use chain rules,

$$\begin{aligned}\frac{d}{dx}\int_{-2x}^{3x} e^{-4t^2}\,dt &= -\left[\frac{d}{du}\int_{a}^{u} e^{-4t^2}\,dt\right]\frac{du}{dx} + \left[\frac{d}{dv}\int_{a}^{v} e^{-4t^2}\,dt\right]\frac{dv}{dx}\\ &= -e^{-4u^2}(-2) + e^{-4v^2}(3) = 2\,e^{-16x^2} + 3\,e^{-36x^2}.\end{aligned}$$

21. If c is a number between $a(x)$ and $b(x)$, then

$$\int_{a(x)}^{b(x)} f(t)\,dt = \int_{a(x)}^{c} f(t)\,dt + \int_{c}^{b(x)} f(t)\,dt = -\int_{c}^{a(x)} f(t)\,dt + \int_{c}^{b(x)} f(t)\,dt.$$

In these integrals we set $u = a(x)$ and $v = b(x)$ respectively, and use chain rules,

$$\begin{aligned}\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt &= -\left[\frac{d}{du}\int_{c}^{u} f(t)\,dt\right]\frac{du}{dx} + \left[\frac{d}{dv}\int_{c}^{v} f(t)\,dt\right]\frac{dv}{dx}\\ &= -f(u)\,a'(x) + f(v)\,b'(x) = f[b(x)]\frac{db}{dx} - f[a(x)]\frac{da}{dx}.\end{aligned}$$

Equation 6.19 is the special case when $a(x)$ is a constant and $b(x) = x$.

22. $\displaystyle \frac{d}{dx}\int_{-2}^{5x+4} \sqrt{t^3+1}\,dt = \sqrt{(5x+4)^3+1}(5)$

23. $$\begin{aligned}\frac{d}{dx}\int_{4x}^{4x+4}\left(t^3 - \frac{1}{\sqrt{t}}\right)dt &= \left[(4x+4)^3 - \frac{1}{\sqrt{4x+4}}\right](4) - \left[(4x)^3 - \frac{1}{\sqrt{4x}}\right](4)\\ &= 256(x+1)^3 - \frac{2}{\sqrt{x+1}} - 256x^3 + \frac{2}{\sqrt{x}} = 256(3x^2+3x+1) + \frac{2}{\sqrt{x}} - \frac{2}{\sqrt{x+1}}\end{aligned}$$

24. $$\begin{aligned}\frac{d}{dx}\int_{-x^2}^{-2x^2} \sec(1-t)\,dt &= [\sec(1+2x^2)](-4x) - [\sec(1+x^2)](-2x)\\ &= 2x\sec(1+x^2) - 4x\sec(1+2x^2)\end{aligned}$$

25. $\displaystyle \frac{d}{dx}\int_{\cos x}^{\sin x} \frac{1}{\sqrt{t+1}}dt = \frac{\cos x}{\sqrt{\sin x+1}} - \frac{-\sin x}{\sqrt{\cos x+1}}$

26. $\displaystyle \frac{d}{dx}\int_{\sqrt{x}}^{2\sqrt{x}} \sqrt{t}\,dt = \sqrt{2\sqrt{x}}\left(\frac{1}{\sqrt{x}}\right) - \sqrt{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right) = \frac{2\sqrt{2}-1}{2x^{1/4}}$

27. $\displaystyle \frac{d}{dx}\int_{x}^{2} \ln(t^2+1)\,dt = -\ln(x^2+1)(1) = -\ln(x^2+1)$

28. $\displaystyle \frac{d}{dx}\int_{-2x}^{3x} e^{-4t^2}\,dt = e^{-4(3x)^2}(3) - e^{-4(-2x)^2}(-2) = 3\,e^{-36x^2} + 2\,e^{-16x^2}$

EXERCISES 6.6

1. The average value is $\displaystyle \frac{1}{2}\int_0^2 (x^2-2x)\,dx = \frac{1}{2}\left\{\frac{x^3}{3} - x^2\right\}_0^2 = \frac{1}{2}\left(\frac{8}{3}-4\right) = -\frac{2}{3}.$
2. Since the integrand is an odd function, its average value over the interval $-1 \le x \le 1$ is 0.
3. The average value is $\displaystyle \int_0^1 (x^3-x)\,dx = \left\{\frac{x^4}{4} - \frac{x^2}{2}\right\}_0^1 = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}.$
4. The average value is $\displaystyle \frac{1}{2-1}\int_1^2 x^4\,dx = \left\{\frac{x^5}{5}\right\}_1^2 = \frac{1}{5}(32-1) = \frac{31}{5}.$

5. The average value is $\int_0^1 \sqrt{x+1}\,dx = \left\{\frac{2}{3}(x+1)^{3/2}\right\}_0^1 = \frac{2}{3}(2\sqrt{2}-1)$.

6. The average value is $\frac{1}{2}\int_{-1}^1 \sqrt{x+1}\,dx = \frac{1}{2}\left\{\frac{2}{3}(x+1)^{3/2}\right\}_{-1}^1 = \frac{2\sqrt{2}}{3}$.

7. The average value is $\int_0^1 (x^4-1)\,dx = \left\{\frac{x^5}{5}-x\right\}_0^1 = \frac{1}{5}-1 = -\frac{4}{5}$.

8. The average value is $\frac{1}{2}\int_0^2 (x^4-1)\,dx = \frac{1}{2}\left\{\frac{x^5}{5}-x\right\}_0^2 = \frac{1}{2}\left(\frac{32}{5}-2\right) = \frac{11}{5}$.

9. The average value is $\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos x\,dx = \frac{1}{\pi}\left\{\sin x\right\}_{-\pi/2}^{\pi/2} = \frac{1}{\pi}(1+1) = \frac{2}{\pi}$.

10. The average value is $\frac{1}{\pi/2}\int_0^{\pi/2} \cos x\,dx = \frac{2}{\pi}\left\{\sin x\right\}_0^{\pi/2} = \frac{2}{\pi}$.

11. The average value is

$$\frac{1}{4}\int_{-2}^2 |x|\,dx = \frac{1}{4}\int_{-2}^0 (-x)\,dx + \frac{1}{4}\int_0^2 x\,dx = \frac{1}{4}\left\{-\frac{x^2}{2}\right\}_{-2}^0 + \frac{1}{4}\left\{\frac{x^2}{2}\right\}_0^2 = \frac{1}{4}(2)+\frac{1}{4}(2) = 1.$$

12. The average value is $\frac{1}{2}\int_0^2 |x|\,dx = \frac{1}{2}\int_0^2 x\,dx = \frac{1}{2}\left\{\frac{x^2}{2}\right\}_0^2 = 1$.

13. The average value is

$$\begin{aligned}\frac{1}{3}\int_0^3 |x^2-4|\,dx &= \frac{1}{3}\int_0^2 (4-x^2)\,dx + \frac{1}{3}\int_2^3 (x^2-4)\,dx \\ &= \frac{1}{3}\left\{4x-\frac{x^3}{3}\right\}_0^2 + \frac{1}{3}\left\{\frac{x^3}{3}-4x\right\}_2^3 = \frac{1}{3}\left(8-\frac{8}{3}\right) + \frac{1}{3}\left(9-12-\frac{8}{3}+8\right) = \frac{23}{9}.\end{aligned}$$

14. The average value is

$$\begin{aligned}\frac{1}{6}\int_{-3}^3 |x^2-4|\,dx &= \frac{1}{6}\left[\int_{-3}^{-2} (x^2-4)\,dx + \int_{-2}^2 (4-x^2)\,dx + \int_2^3 (x^2-4)\,dx\right] \\ &= \frac{1}{6}\left[\left\{\frac{x^3}{3}-4x\right\}_{-3}^{-2} + \left\{4x-\frac{x^3}{3}\right\}_{-2}^2 + \left\{\frac{x^3}{3}-4x\right\}_2^3\right] \\ &= \frac{1}{6}\left[\left(-\frac{8}{3}+8\right) - (-9+12) + \left(8-\frac{8}{3}\right) - \left(-8+\frac{8}{3}\right) + (9-12) - \left(\frac{8}{3}-8\right)\right] \\ &= \frac{23}{9}.\end{aligned}$$

15. Since the function is odd, its average value on $-1 \le x \le 1$ is zero.

16. The average value is $\frac{1}{4}\int_{-1}^3 \operatorname{sgn} x\,dx = \frac{1}{4}\left[\int_{-1}^0 (-1)\,dx + \int_0^3 (1)\,dx\right] = \frac{1}{4}\left[\{-x\}_{-1}^0 + \{x\}_0^3\right] = \frac{1}{2}$.

17. The average value is $\frac{1}{2}\int_0^2 h(x-1)\,dx = \frac{1}{2}\int_1^2 dx = \frac{1}{2}\left\{x\right\}_1^2 = \frac{1}{2}$.

18. The average value is $\frac{1}{2}\int_0^2 h(x-4)\,dx = \frac{1}{2}\int_0^2 (0)\,dx = 0$.

19. The average value is $\frac{1}{3}\int_0^3 \lfloor x \rfloor\,dx = \frac{1}{3}\left[\int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx\right] = \frac{1}{3}(0+1+2) = 1$.

20. The average value is

$$\frac{1}{3.5}\int_0^{3.5} \lfloor x \rfloor\,dx = \frac{1}{3.5}\left[\int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx + \int_3^{3.5} 3\,dx\right] = \frac{1}{3.5}\left[0+1+2+\frac{3}{2}\right] = \frac{9}{7}.$$

21. If $v(t)$ is the velocity function, its average value for $t_1 \le t \le t_2$ is

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} v(t)\,dt = \frac{1}{t_2 - t_1}\Big\{x(t)\Big\}_{t_1}^{t_2} = \frac{x(t_2) - x(t_1)}{t_2 - t_1} = \frac{x_2 - x_1}{t_2 - t_1}.$$

22. The average value is $\dfrac{1}{R}\displaystyle\int_0^R c(R^2 - r^2)\,dr = \frac{c}{R}\left\{R^2 r - \frac{r^3}{3}\right\}_0^R = \frac{c}{R}\left(R^3 - \frac{R^3}{3}\right) = \frac{2cR^2}{3}$.

23. According to 6.31, $2(2c - c^2) = \displaystyle\int_0^2 (2x - x^2)\,dx = \left\{x^2 - \frac{x^3}{3}\right\}_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$.
Value of c satisfying $6(2c - c^2) = 4 \implies 3c^2 - 6c + 2 = 0$ are $c = (6 \pm \sqrt{36 - 24})/6 = (3 \pm \sqrt{3})/3$.

24. According to 6.31, $4(c^3 - 8c) = \displaystyle\int_{-2}^2 (x^3 - 8x)\,dx = \left\{\frac{x^4}{4} - 4x^2\right\}_{-2}^2 = (4 - 16) - (4 - 16) = 0$.
The only value of c between -2 and 2 that satisfies this equation is $c = 0$.

25. According to 6.31, $\dfrac{\pi}{2}\cos c = \displaystyle\int_0^{\pi/2} \cos x\,dx = \Big\{\sin x\Big\}_0^{\pi/2} = 1$.
The only value of c between 0 and $\pi/2$ that satisfies this equation is $c = 0.881$ radians.

26. According to 6.31, $\pi \cos c = \displaystyle\int_0^{\pi} \cos x\,dx = \big\{\sin x\big\}_0^{\pi} = 0$. The only value of c between 0 and π which satisfies this equation is $c = \pi/2$.

27. According to 6.31, $2\sqrt{c+1} = \displaystyle\int_1^3 \sqrt{x+1}\,dx = \left\{\frac{2}{3}(x+1)^{3/2}\right\}_1^3 = \frac{4(4 - \sqrt{2})}{3}$.
The solution of this equation is $c = -1 + \dfrac{4}{9}(4 - \sqrt{2})^2 = \dfrac{63 - 32\sqrt{2}}{9}$.

28. From 6.31, $1(c^2)(c+1) = \displaystyle\int_0^1 x^2(x+1)\,dx = \left\{\frac{x^4}{4} + \frac{x^3}{3}\right\}_0^1 = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$.
Thus, c must satisfy $g(c) = 12c^3 + 12c^2 - 7 = 0$. There is only one solution of this equation between 0 and 1, which we can find by Newton's iterative procedure,

$$c_1 = 0.5, \qquad c_{n+1} = c_n - \frac{12c_n^3 + 12c_n^2 - 7}{36c_n^2 + 24c_n}.$$

Iteration gives $c_2 = 0.619$, $c_3 = 0.603\,50$, $c_4 = 0.603\,20$, $c_5 = 0.603\,20$. Since $g(0.603\,15) = -0.0015$ and $g(0.603\,25) = 0.0013$, we can say that to 4 decimals, $c = 0.6032$.

29. From 6.31, $2c\sqrt{c^2+1} = \displaystyle\int_0^2 x\sqrt{x^2+1}\,dx = \left\{\frac{1}{3}(x^2+1)^{3/2}\right\}_0^2 = \frac{1}{3}(5\sqrt{5} - 1)$.
Thus, c must satisfy $6c\sqrt{c^2+1} = 5\sqrt{5} - 1$ or $g(c) = 18c^4 + 18c^2 + 5\sqrt{5} - 63 = 0$. There is only one solution of this equation between 0 and 2, which we can find by Newton's iterative procedure,

$$c_1 = 1, \qquad c_{n+1} = c_n - \frac{18c_n^4 + 18c_n^2 + 5\sqrt{5} - 63}{72c_n^3 + 36c_n}.$$

Iteration gives $c_2 = 1.146$, $c_3 = 1.1268$, $c_4 = 1.1264$, $c_5 = 1.1264$. Since $g(1.1255) = -0.13$ and $g(1.1265) = 0.0090$, we can say that to 3 decimals, $c = 1.126$.

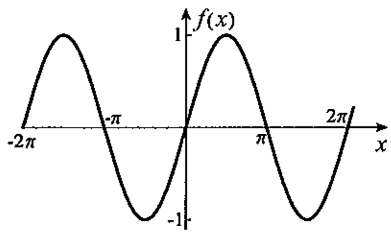
30. From 6.31, $1\left(\dfrac{1}{c^2} + \dfrac{1}{c^3}\right) = \displaystyle\int_1^2 \left(\frac{1}{x^2} + \frac{1}{x^3}\right)dx = \left\{-\frac{1}{x} - \frac{1}{2x^2}\right\}_1^2 = \left(-\frac{1}{2} - \frac{1}{8}\right) - \left(-1 - \frac{1}{2}\right) = \frac{7}{8}$.
Thus, c must satisfy $g(c) = 7c^3 - 8c - 8 = 0$. There is only one solution of this equation between 1 and 2, which we can find by Newton's iterative procedure,

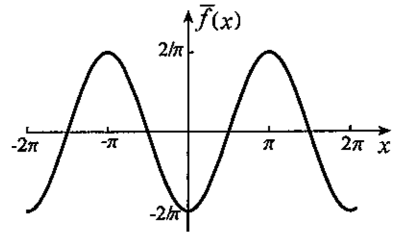
$$c_1 = 1.5, \qquad c_{n+1} = c_n - \frac{7c_n^3 - 8c_n - 8}{21c_n^2 - 8}.$$

Iteration gives $c_2 = 1.408$, $c_3 = 1.399\,8$, $c_4 = 1.399\,8$. Since $g(1.3995) = -0.0086$ and $g(1.4005) = 0.025$, we can say that to 3 decimals, $c = 1.400$.

31. The moving average is

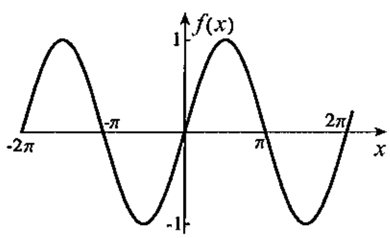
$$\overline{f}(x) = \frac{1}{\pi}\int_{x-\pi}^{x} \sin t\,dt = \frac{1}{\pi}\{-\cos t\}_{x-\pi}^{x} = \frac{1}{\pi}[\cos(x-\pi) - \cos x] = \frac{1}{\pi}(-\cos x - \cos x) = -\frac{2}{\pi}\cos x.$$

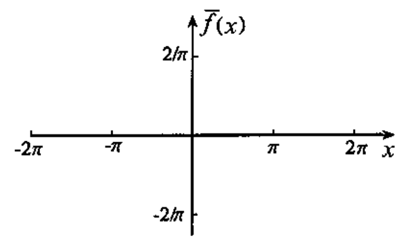




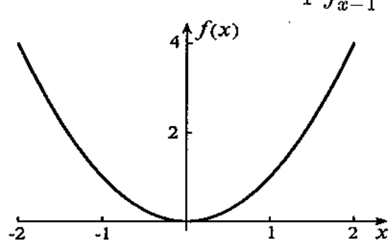
32. The moving average is

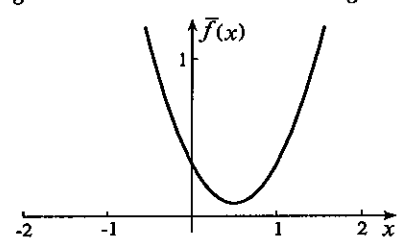
$$\overline{f}(x) = \frac{1}{2\pi}\int_{x-2\pi}^{x} \sin t\,dt = \frac{1}{2\pi}\{-\cos t\}_{x-2\pi}^{x} = \frac{1}{2\pi}[\cos(x-2\pi) - \cos x] = \frac{1}{2\pi}(\cos x - \cos x) = 0.$$



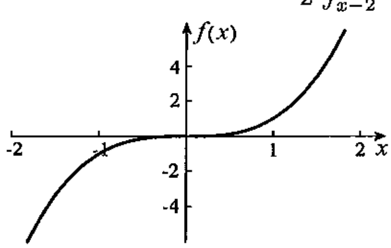


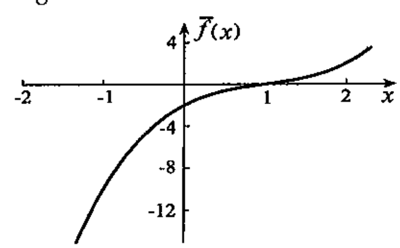
33. The moving average is $\overline{f}(x) = \dfrac{1}{1}\displaystyle\int_{x-1}^{x} t^2\,dt = \left\{\frac{t^3}{3}\right\}_{x-1}^{x} = \frac{1}{3}[x^3 - (x-1)^3] = x^2 - x + \frac{1}{3}.$





34. The moving average is $\overline{f}(x) = \dfrac{1}{2}\displaystyle\int_{x-2}^{x} t^3\,dt = \frac{1}{2}\left\{\frac{t^4}{4}\right\}_{x-2}^{x} = \frac{1}{8}[x^4 - (x-2)^4] = x^3 - 3x^2 + 4x - 2.$

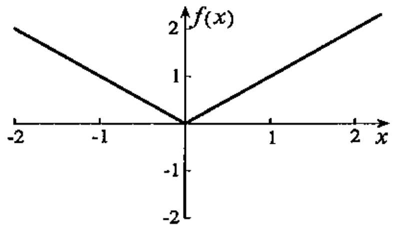


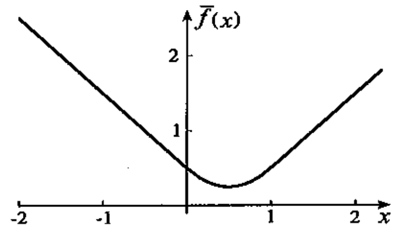


35. The moving average is $\overline{f}(x) = \dfrac{1}{1}\displaystyle\int_{x-1}^{x} |t|\,dt$. When $x \le 0$, $\overline{f}(x) = \displaystyle\int_{x-1}^{x} -t\,dt = \left\{-\frac{t^2}{2}\right\}_{x-1}^{x} = \frac{1}{2} - x.$

When $0 < x < 1$, $\overline{f}(x) = \displaystyle\int_{x-1}^{0} -t\,dt + \int_{0}^{x} t\,dt = \left\{-\frac{t^2}{2}\right\}_{x-1}^{0} + \left\{\frac{t^2}{2}\right\}_{0}^{x} = x^2 - x + \frac{1}{2}.$

When $x \ge 1$, $\overline{f}(x) = \displaystyle\int_{x-1}^{x} t\,dt = \left\{\frac{t^2}{2}\right\}_{x-1}^{x} = x - \frac{1}{2}.$





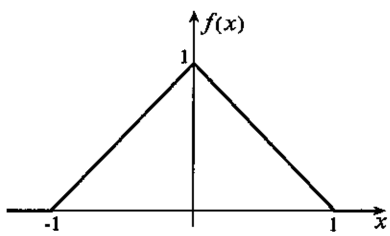
36. The moving average is $\dfrac{1}{1}\displaystyle\int_{x-1}^{x} f(t)\,dt.$ When $x \le -1$, $\overline{f}(x) = 0$.

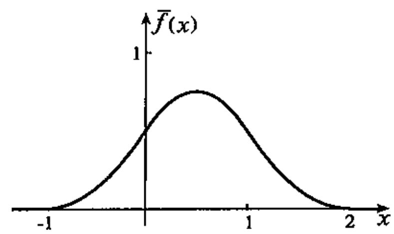
When $-1 < x < 0$, $\quad \overline{f}(x) = \displaystyle\int_{-1}^{x} (1+t)\,dt = \left\{t + \frac{t^2}{2}\right\}_{-1}^{x} = \frac{x^2}{2} + x + \frac{1}{2}.$

When $0 \le x < 1$, $\quad \overline{f}(x) = \displaystyle\int_{x-1}^{0} (1+t)\,dt + \int_{0}^{x} (1-t)\,dt = \left\{t + \frac{t^2}{2}\right\}_{x-1}^{0} + \left\{t - \frac{t^2}{2}\right\}_{0}^{x} = -x^2 + x + \frac{1}{2}.$

When $1 \le x < 2$, $\quad \overline{f}(x) = \displaystyle\int_{x-1}^{1} (1-t)\,dt = \left\{t - \frac{t^2}{2}\right\}_{x-1}^{1} = \frac{x^2}{2} - 2x + 2.$

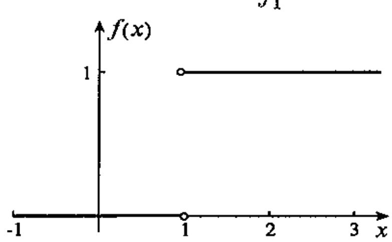
When $x \ge 2$, $\quad \overline{f}(x) = 0$.

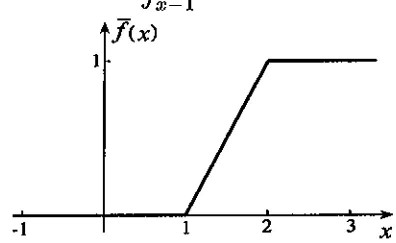




37. The moving average is $\dfrac{1}{1}\displaystyle\int_{x-1}^{x} h(t-1)\,dt.$ When $x < 1$, $\quad \overline{f}(x) = 0$.

When $1 \le x < 2$, $\quad \overline{f}(x) = \displaystyle\int_{1}^{x} 1\,dt = x - 1;$ $\quad$ when $x \ge 2$, $\quad \overline{f}(x) = \displaystyle\int_{x-1}^{x} 1\,dt = 1.$

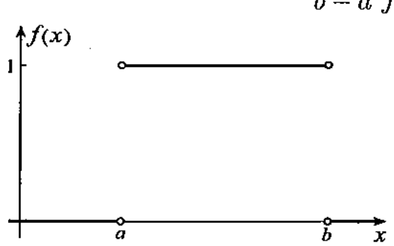


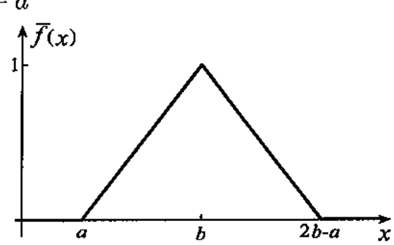


38. The moving average is $\quad \overline{f}(x) = \dfrac{1}{b-a}\displaystyle\int_{x-b+a}^{x} [h(x-a) - h(x-b)]\,dx.$ When $x < a$, $\overline{f}(x) = 0$.

When $a \le x < b$, $\quad \overline{f}(x) = \dfrac{1}{b-a}\displaystyle\int_{a}^{x} 1\,dt = \frac{x-a}{b-a}.$

When $b \le x < 2b - a$, $\quad \overline{f}(x) = \dfrac{1}{b-a}\displaystyle\int_{x-b+a}^{b} 1\,dt = \frac{2b - x - a}{b-a}.$ When $x \ge 2b - a$, $\quad \overline{f}(x) = 0$.





EXERCISES 6.7

1. $\displaystyle\int_1^2 x(3x^2-2)^4\,dx = \left\{\frac{1}{30}(3x^2-2)^5\right\}_1^2 = \frac{33\,333}{10}$

2. If we set $u = 1-z$, then $du = -dz$, and
$$\int_0^1 z\sqrt{1-z}\,dz = \int_1^0 (1-u)\sqrt{u}(-du) = \int_1^0 (u^{3/2}-\sqrt{u})\,du = \left\{\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right\}_1^0 = -\left(\frac{2}{5}-\frac{2}{3}\right) = \frac{4}{15}.$$

3. If we set $u = x+3$, then $du = dx$, and
$$\begin{aligned}\int_{-1}^0 \frac{x}{\sqrt{x+3}}dx &= \int_2^3 \frac{u-3}{\sqrt{u}}du = \int_2^3\left(\sqrt{u}-\frac{3}{\sqrt{u}}\right)du\\ &= \left\{\frac{2u^{3/2}}{3}-6\sqrt{u}\right\}_2^3 = (2\sqrt{3}-6\sqrt{3})-\left(\frac{4\sqrt{2}}{3}-6\sqrt{2}\right) = \frac{14\sqrt{2}}{3}-4\sqrt{3}.\end{aligned}$$

4. $\displaystyle\int_{\pi/4}^{\pi/3} \cos^5 x\,\sin x\,dx = \left\{-\frac{1}{6}\cos^6 x\right\}_{\pi/4}^{\pi/3} = -\frac{1}{6}\left(\frac{1}{2}\right)^6+\frac{1}{6}\left(\frac{1}{\sqrt{2}}\right)^6 = \frac{7}{384}$

5. If we set $u = 9-x^2$, then $du = -2x\,dx$, and
$$\begin{aligned}\int_1^3 x^3\sqrt{9-x^2}\,dx &= \int_1^3 x^2\sqrt{9-x^2}\,x\,dx = \int_8^0 (9-u)\sqrt{u}\left(\frac{du}{-2}\right) = \frac{1}{2}\int_0^8 (9\sqrt{u}-u^{3/2})\,du\\ &= \frac{1}{2}\left\{6u^{3/2}-\frac{2u^{5/2}}{5}\right\}_0^8 = \frac{1}{2}\left(96\sqrt{2}-\frac{256\sqrt{2}}{5}\right) = \frac{112\sqrt{2}}{5}.\end{aligned}$$

6. This definite integral does not exist because the integrand is not defined for $-2\sqrt{3}\le x\le 2\sqrt{3}$.

7. If we set $u = y-4$, then $du = dy$, and
$$\begin{aligned}\int_4^5 y^2\sqrt{y-4}\,dy &= \int_0^1 (u+4)^2\sqrt{u}\,du = \int_0^1 (u^{5/2}+8u^{3/2}+16\sqrt{u})\,du\\ &= \left\{\frac{2u^{7/2}}{7}+\frac{16u^{5/2}}{5}+\frac{32u^{3/2}}{3}\right\}_0^1 = \frac{2}{7}+\frac{16}{5}+\frac{32}{3} = \frac{1486}{105}.\end{aligned}$$

8. If we set $u = 1+x$, then $du = dx$, and
$$\begin{aligned}\int_{1/2}^1 \sqrt{\frac{x^2}{1+x}}dx &= \int_{1/2}^1 \frac{x}{\sqrt{1+x}}dx = \int_{3/2}^2 \frac{u-1}{\sqrt{u}}du = \int_{3/2}^2\left(\sqrt{u}-\frac{1}{\sqrt{u}}\right)du\\ &= \left\{\frac{2}{3}u^{3/2}-2\sqrt{u}\right\}_{3/2}^2 = \left(\frac{4\sqrt{2}}{3}-2\sqrt{2}\right)-\left(\sqrt{\frac{3}{2}}-2\sqrt{\frac{3}{2}}\right) = \sqrt{\frac{3}{2}}-\frac{2\sqrt{2}}{3}.\end{aligned}$$

9. If we set $v = 1+\sqrt{u}$, then $dv = \dfrac{du}{2\sqrt{u}}$, and
$$\int_1^4 \frac{\sqrt{1+\sqrt{u}}}{\sqrt{u}}du = \int_2^3 \sqrt{v}\,(2\,dv) = \left\{\frac{4v^{3/2}}{3}\right\}_2^3 = 4\sqrt{3}-\frac{8\sqrt{2}}{3} = \frac{4(3\sqrt{3}-2\sqrt{2})}{3}.$$

10. If we set $u = x^2+2x+2$, then $du = (2x+2)\,dx$, and
$$\int_{-2}^1 \frac{x+1}{(x^2+2x+2)^{1/3}}dx = \int_2^5 \frac{1}{u^{1/3}}\left(\frac{du}{2}\right) = \frac{1}{2}\left\{\frac{3}{2}u^{2/3}\right\}_2^5 = \frac{3}{4}\left(5^{2/3}-2^{2/3}\right).$$

11. If we set $u = x - 2$, then $du = dx$, and

$$\int_3^4 \frac{x^2}{(x-2)^4}\,dx = \int_1^2 \frac{(u+2)^2}{u^4}\,du = \int_1^2 \left(\frac{1}{u^2} + \frac{4}{u^3} + \frac{4}{u^4}\right) du$$
$$= \left\{-\frac{1}{u} - \frac{2}{u^2} - \frac{4}{3u^3}\right\}_1^2 = \left(-\frac{1}{2} - \frac{1}{2} - \frac{1}{6}\right) - \left(-1 - 2 - \frac{4}{3}\right) = \frac{19}{6}.$$

12. If we set $u = 2 + 3\sin x$, then $du = 3\cos x\,dx$, and

$$\int_0^{\pi/6} \sqrt{2 + 3\sin x}\cos x\,dx = \int_2^{7/2} \sqrt{u}\left(\frac{du}{3}\right) = \frac{1}{3}\left\{\frac{2}{3}u^{3/2}\right\}_2^{7/2} = \frac{2}{9}\left(\frac{7\sqrt{7}}{2\sqrt{2}} - 2\sqrt{2}\right) = \frac{\sqrt{2}}{18}(7\sqrt{7} - 8).$$

13. If we set $u = 1 + \cos x$, then $du = -\sin x\,dx$, and

$$\int_{\pi/4}^{\pi/2} \frac{\sin^3 x}{(1+\cos x)^4}\,dx = \int_{\pi/4}^{\pi/2} \frac{1 - \cos^2 x}{(1+\cos x)^4}\sin x\,dx = \int_{1+1/\sqrt{2}}^{1} \frac{1-(u-1)^2}{u^4}(-du)$$
$$= \int_{1+1/\sqrt{2}}^{1} \left(\frac{1}{u^2} - \frac{2}{u^3}\right) du = \left\{-\frac{1}{u} + \frac{1}{u^2}\right\}_{1+1/\sqrt{2}}^{1}$$
$$= (-1+1) - \left[\frac{-1}{1+1/\sqrt{2}} + \frac{1}{(1+1/\sqrt{2})^2}\right] = \frac{\sqrt{2}}{\sqrt{2}+1} - \frac{2}{3+2\sqrt{2}}$$
$$= \sqrt{2}(\sqrt{2}-1) - 2(3 - 2\sqrt{2}) = 3\sqrt{2} - 4.$$

14. $\displaystyle\int_1^4 \frac{(x+1)(x-1)}{\sqrt{x}}\,dx = \int_1^4 \left(x^{3/2} - \frac{1}{\sqrt{x}}\right) dx = \left\{\frac{2}{5}x^{5/2} - 2\sqrt{x}\right\}_1^4 = \left(\frac{64}{5} - 4\right) - \left(\frac{2}{5} - 2\right) = \frac{52}{5}$

15. If we set $u = 1 + \sqrt{x}$, then $du = \dfrac{dx}{2\sqrt{x}}$, and

$$\int_4^9 \sqrt{1+\sqrt{x}}\,dx = \int_3^4 \sqrt{u}\,2(u-1)\,du = 2\int_3^4 (u^{3/2} - \sqrt{u})\,du = 2\left\{\frac{2u^{5/2}}{5} - \frac{2u^{3/2}}{3}\right\}_3^4$$
$$= 2\left(\frac{64}{5} - \frac{16}{3} - \frac{18\sqrt{3}}{5} + 2\sqrt{3}\right) = \frac{16(14 - 3\sqrt{3})}{15}.$$

16. If we set $u = 1 + x$, then $du = dx$, and

$$\int_{-1/2}^{1} \sqrt{\frac{x^2}{1+x}}\,dx = \int_{-1/2}^{0} \frac{-x}{\sqrt{1+x}}\,dx + \int_0^1 \frac{x}{\sqrt{1+x}}\,dx = \int_{1/2}^{1} \frac{-(u-1)}{\sqrt{u}}\,du + \int_1^2 \frac{u-1}{\sqrt{u}}\,du$$
$$= \int_{1/2}^{1}\left(\frac{1}{\sqrt{u}} - \sqrt{u}\right) du + \int_1^2 \left(\sqrt{u} - \frac{1}{\sqrt{u}}\right) du = \left\{2\sqrt{u} - \frac{2}{3}u^{3/2}\right\}_{1/2}^{1} + \left\{\frac{2}{3}u^{3/2} - 2\sqrt{u}\right\}_1^2$$
$$= \left(2 - \frac{2}{3}\right) - \left(\sqrt{2} - \frac{1}{3\sqrt{2}}\right) + \left(\frac{4\sqrt{2}}{3} - 2\sqrt{2}\right) - \left(\frac{2}{3} - 2\right) = \frac{1}{6}(16 - 9\sqrt{2}).$$

17. If we set $u = x + 2$, then $du = dx$, and

$$\int_{-1}^{1} \frac{|x|}{(x+2)^3}\,dx = \int_{-1}^{0} \frac{-x}{(x+2)^3}\,dx + \int_0^1 \frac{x}{(x+2)^3}\,dx = \int_1^2 \frac{-(u-2)}{u^3}\,du + \int_2^3 \frac{u-2}{u^3}\,du$$
$$= \int_1^2 \left(\frac{2}{u^3} - \frac{1}{u^2}\right) du + \int_2^3 \left(\frac{1}{u^2} - \frac{2}{u^3}\right) du = \left\{-\frac{1}{u^2} + \frac{1}{u}\right\}_1^2 + \left\{-\frac{1}{u} + \frac{1}{u^2}\right\}_2^3$$
$$= \left(-\frac{1}{4} + \frac{1}{2} + 1 - 1\right) + \left(-\frac{1}{3} + \frac{1}{9} + \frac{1}{2} - \frac{1}{4}\right) = \frac{5}{18}.$$

18. If we set $u = x + 2$, then $du = dx$, and

$$\int_{-1}^{1}\left|\frac{x}{(x+2)^3}\right|dx = \int_{-1}^{0}\frac{-x}{(x+2)^3}dx + \int_{0}^{1}\frac{x}{(x+2)^3}dx = \int_{1}^{2}\left(\frac{2-u}{u^3}\right)du + \int_{2}^{3}\left(\frac{u-2}{u^3}\right)du$$
$$= \int_{1}^{2}\left(\frac{2}{u^3}-\frac{1}{u^2}\right)du + \int_{2}^{3}\left(\frac{1}{u^2}-\frac{2}{u^3}\right)du = \left\{-\frac{1}{u^2}+\frac{1}{u}\right\}_1^2 + \left\{-\frac{1}{u}+\frac{1}{u^2}\right\}_2^3$$
$$= \left(-\frac{1}{4}+\frac{1}{2}\right) - (-1+1) + \left(-\frac{1}{3}+\frac{1}{9}\right) - \left(-\frac{1}{2}+\frac{1}{4}\right) = \frac{5}{18}.$$

19. $\displaystyle\int_0^1 x^2 e^{x^3}\,dx = \left\{\frac{e^{x^3}}{3}\right\}_0^1 = \frac{e-1}{3}$

20. If we set $u = \ln x$, then $du = \dfrac{1}{x}dx$, and $\displaystyle\int_1^2 \frac{(\ln x)^2}{x}dx = \int_0^{\ln 2} u^2\,du = \left\{\frac{u^3}{3}\right\}_0^{\ln 2} = \frac{1}{3}(\ln 2)^3.$

21. $\displaystyle\int_2^4 \frac{1}{x\ln x}dx = \Big\{\ln|\ln x|\Big\}_2^4 = \ln(\ln 4) - \ln(\ln 2)$

22. If we set $u = 4 + 3\tan x$, then $du = 3\sec^2 x\,dx$, and

$$\int_{-\pi/4}^{\pi/4}\frac{\sec^2 x}{\sqrt{4+3\tan x}}dx = \int_1^7 \frac{1}{\sqrt{u}}\left(\frac{du}{3}\right) = \frac{1}{3}\Big\{2\sqrt{u}\Big\}_1^7 = \frac{2(\sqrt{7}-1)}{3}.$$

23. If $f(x)$ is odd, and we set $u = -x$ in the first of the integrals on the right in the following equation,

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = \int_a^0 f(-u)(-du) + \int_0^a f(x)\,dx = -\int_0^a f(u)\,du + \int_0^a f(x)\,dx = 0.$$

The proof is similar in the even case.

24. We can rewrite the integral as the sum of three integrals:

$$\int_a^{a+p} f(x)\,dx = \int_a^0 f(x)\,dx + \int_0^p f(x)\,dx + \int_p^{a+p} f(x)\,dx.$$

In the last integral we change variables according to $u = x - p$. Then $du = dx$, and

$$\int_a^{a+p} f(x)\,dx = -\int_0^a f(x)\,dx + \int_0^p f(x)\,dx + \int_0^a f(u+p)\,du$$
$$= -\int_0^a f(x)\,dx + \int_0^p f(x)\,dx + \int_0^a f(u)\,du = \int_0^p f(x)\,dx.$$

25. If we set $u = \dfrac{1}{\sqrt{x}}$, then $du = -\dfrac{1}{2x^{3/2}}dx$, and

$$\int_1^3 \frac{1}{x^{3/2}\sqrt{4-x}}dx = \int_1^{1/\sqrt{3}} \frac{1}{\sqrt{4-1/u^2}}(-2du) = -2\int_1^{1/\sqrt{3}}\frac{u}{\sqrt{4u^2-1}}du$$
$$= -2\left\{\frac{1}{4}\sqrt{4u^2-1}\right\}_1^{1/\sqrt{3}} = -\frac{1}{2}\sqrt{4/3-1} + \frac{1}{2}\sqrt{3} = \frac{1}{\sqrt{3}}.$$

26. If we set $u = 1/x$, then $du = -dx/x^2$, and

$$\int_{-6}^{-1}\frac{\sqrt{x^2-6x}}{x^4}dx = \int_{-1/6}^{-1}\frac{\sqrt{\dfrac{1}{u^2}-\dfrac{6}{u}}}{1/u^4}\left(-\frac{du}{u^2}\right) = -\int_{-1/6}^{-1}u^2\sqrt{\frac{1-6u}{u^2}}du = \int_{-1/6}^{-1}u\sqrt{1-6u}\,du.$$

We now set $v = 1 - 6u$, in which case $dv = -6\,du$, and

$$\int_{-1/6}^{-1} \frac{\sqrt{x^2-6x}}{x^4}dx = \int_2^7 \left(\frac{1-v}{6}\right)\sqrt{v}\left(\frac{dv}{-6}\right) = \frac{1}{36}\int_2^7 (v^{3/2}-\sqrt{v})\,dv$$
$$= \frac{1}{36}\left\{\frac{2v^{5/2}}{5} - \frac{2v^{3/2}}{3}\right\}_2^7 = \frac{1}{36}\left(\frac{98\sqrt{7}}{5} - \frac{14\sqrt{7}}{3} - \frac{8\sqrt{2}}{5} + \frac{4\sqrt{2}}{3}\right) = \frac{56\sqrt{7}-\sqrt{2}}{135}.$$

27. If we set $u^2 = \dfrac{1-x}{5+x}$, then $(5+x)u^2 = 1-x$, and $x = \dfrac{1-5u^2}{1+u^2}$.

Thus, $dx = \dfrac{(1+u^2)(-10u)-(1-5u^2)(2u)}{(1+u^2)^2}du = \dfrac{-12u}{(1+u^2)^2}du$, and

$$\int_{-4}^{0} \frac{x}{(5-4x-x^2)^{3/2}}dx = \int_{-4}^{0} \frac{x}{[(5+x)(1-x)]^{3/2}}dx$$
$$= \int_{\sqrt{5}}^{1/\sqrt{5}} \frac{(1-5u^2)/(1+u^2)}{\left[\left(5+\dfrac{1-5u^2}{1+u^2}\right)\left(1-\dfrac{1-5u^2}{1+u^2}\right)\right]^{3/2}}\frac{-12u}{(1+u^2)^2}du$$
$$= \int_{\sqrt{5}}^{1/\sqrt{5}} \frac{-12u(1-5u^2)}{\left[\dfrac{36u^2}{(u^2+1)^2}\right]^{3/2}}\frac{1}{(1+u^2)^3}du = \int_{\sqrt{5}}^{1/\sqrt{5}} -\frac{1-5u^2}{18u^2}du$$
$$= \frac{1}{18}\int_{1/\sqrt{5}}^{\sqrt{5}}\left(\frac{1}{u^2}-5\right)du = \frac{1}{18}\left\{-\frac{1}{u}-5u\right\}_{1/\sqrt{5}}^{\sqrt{5}}$$
$$= \frac{1}{18}\left(-\frac{1}{\sqrt{5}}-5\sqrt{5}\right) - \frac{1}{18}\left(-\sqrt{5}-\sqrt{5}\right) = -\frac{8\sqrt{5}}{45}.$$

28. If we set $\psi = \pi\cos\theta$, then $d\psi = -\pi\sin\theta\,d\theta$, and

$$\int_0^{\pi} \frac{\cos^2[(\pi/2)\cos\theta]}{\sin\theta}d\theta = \int_{\pi}^{-\pi}\frac{\cos^2(\psi/2)}{\sin\theta}\left(\frac{-d\psi}{\pi\sin\theta}\right) = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos^2(\psi/2)}{1-\psi^2/\pi^2}d\psi.$$

Since $\cos^2(\psi/2) = (1+\cos\psi)/2$, it follows that

$$\int_0^{\pi} \frac{\cos^2[(\pi/2)\cos\theta]}{\sin\theta}d\theta = \pi\int_{-\pi}^{\pi}\frac{(1+\cos\psi)/2}{\pi^2-\psi^2}d\psi = \frac{\pi}{2}\int_{-\pi}^{\pi}\frac{1+\cos\psi}{(\pi+\psi)(\pi-\psi)}d\psi.$$

We now set $\phi = \psi+\pi$, and $d\phi = d\psi$,

$$\int_0^{\pi} \frac{\cos^2[(\pi/2)\cos\theta]}{\sin\theta}d\theta = \frac{\pi}{2}\int_0^{2\pi}\frac{1+\cos(\phi-\pi)}{\phi(2\pi-\phi)}d\phi = \frac{\pi}{2}\int_0^{2\pi}\frac{1-\cos\phi}{\phi(2\pi-\phi)}d\phi.$$

REVIEW EXERCISES

1. $\displaystyle\int_0^3 (x^2+3x-2)\,dx = \left\{\frac{x^3}{3}+\frac{3x^2}{2}-2x\right\}_0^3 = 9+\frac{27}{2}-6 = \frac{33}{2}$

2. $\displaystyle\int_{-1}^1 (x^2-x^4)\,dx = \left\{\frac{x^3}{3}-\frac{x^5}{5}\right\}_{-1}^1 = \left(\frac{1}{3}-\frac{1}{5}\right)-\left(-\frac{1}{3}+\frac{1}{5}\right) = \frac{4}{15}$

3. $\displaystyle\int_{-1}^1 (x^3-3x)\,dx = \left\{\frac{x^4}{4}-\frac{3x^2}{2}\right\}_{-1}^1 = \left(\frac{1}{4}-\frac{3}{2}\right)-\left(\frac{1}{4}-\frac{3}{2}\right) = 0$

4. $\displaystyle\int_0^2 (x^2-2x)\,dx = \left\{\frac{x^3}{3}-x^2\right\}_0^2 = \frac{8}{3}-4 = -\frac{4}{3}$

5. $\displaystyle\int_1^2 (x+1)^2\,dx = \left\{\frac{(x+1)^3}{3}\right\}_1^2 = \frac{1}{3}(27-8) = \frac{19}{3}$

6. $\displaystyle\int_{-3}^{-2} \frac{1}{x^2}dx = \left\{-\frac{1}{x}\right\}_{-3}^{-2} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

7. $\displaystyle\int_4^9 \left(\frac{1}{\sqrt{x}} - \sqrt{x}\right) dx = \left\{2\sqrt{x} - \frac{2x^{3/2}}{3}\right\}_4^9 = (6-18) - \left(4 - \frac{16}{3}\right) = -\frac{32}{3}$

8. $\displaystyle\int_0^{\pi} \cos x\,dx = \left\{\sin x\right\}_0^{\pi} = 0$

9. $\displaystyle\int_{-1}^1 x(x+1)^2\,dx = \left\{\frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2}\right\}_{-1}^1 = \left(\frac{1}{4} + \frac{2}{3} + \frac{1}{2}\right) - \left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right) = \frac{4}{3}$

10. $\displaystyle\int_1^2 x^2(x^2+3)\,dx = \left\{\frac{x^5}{5} + x^3\right\}_1^2 = \left(\frac{32}{5} + 8\right) - \left(\frac{1}{5} + 1\right) = \frac{66}{5}$

11. $\displaystyle\int_0^3 \sqrt{x+1}\,dx = \left\{\frac{2(x+1)^{3/2}}{3}\right\}_0^3 = \frac{2}{3}(8-1) = \frac{14}{3}$

12. $\displaystyle\int_1^5 x\sqrt{x^2-1}\,dx = \left\{\frac{1}{3}(x^2-1)^{3/2}\right\}_1^5 = \frac{1}{3}(24)^{3/2} = 16\sqrt{6}$

13. $\displaystyle\int_1^4 \left(\frac{\sqrt{x}+1}{\sqrt{x}}\right) dx = \left\{x + 2\sqrt{x}\right\}_1^4 = (4+4) - (1+2) = 5$

14. If we set $u = x+1$, then $du = dx$, and

$$\int_{-1}^0 x\sqrt{x+1}\,dx = \int_0^1 (u-1)\sqrt{u}\,du = \int_0^1 (u^{3/2} - \sqrt{u})\,du = \left\{\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right\}_0^1 = \frac{2}{5} - \frac{2}{3} = -\frac{4}{15}.$$

15. If we set $u = x+1$, then $du = dx$, and

$$\begin{aligned}\int_1^2 \frac{x^2+1}{(x+1)^4}dx &= \int_2^3 \frac{(u-1)^2+1}{u^4}du = \int_2^3 \left(\frac{1}{u^2} - \frac{2}{u^3} + \frac{2}{u^4}\right) du \\ &= \left\{-\frac{1}{u} + \frac{1}{u^2} - \frac{2}{3u^3}\right\}_2^3 = \left(-\frac{1}{3} + \frac{1}{9} - \frac{2}{81}\right) - \left(-\frac{1}{2} + \frac{1}{4} - \frac{1}{12}\right) = \frac{7}{81}.\end{aligned}$$

16. If we set $u = 2-x$, then $du = -dx$, and

$$\begin{aligned}\int_{-4}^{-2} x^2\sqrt{2-x}\,dx &= \int_6^4 (2-u)^2\sqrt{u}(-du) = \int_4^6 (4\sqrt{u} - 4u^{3/2} + u^{5/2})\,du = \left\{\frac{8}{3}u^{3/2} - \frac{8}{5}u^{5/2} + \frac{2}{7}u^{7/2}\right\}_4^6 \\ &= \left(16\sqrt{6} - \frac{8(36)\sqrt{6}}{5} + \frac{12(36)\sqrt{6}}{7}\right) - \left(\frac{64}{3} - \frac{256}{5} + \frac{256}{7}\right) = \frac{2112\sqrt{6} - 704}{105}.\end{aligned}$$

17. If we set $u = 1 + \sin x$, then $du = \cos x\,dx$, and

$$\int_0^{\pi/4} \frac{\cos x}{(1+\sin x)^2}dx = \int_1^{1+1/\sqrt{2}} \frac{1}{u^2}du = \left\{-\frac{1}{u}\right\}_1^{1+1/\sqrt{2}} = \frac{-1}{1+1/\sqrt{2}} + 1 = \sqrt{2} - 1.$$

18. $\displaystyle\int_2^3 x(1+2x^2)^4\,dx = \left\{\frac{1}{20}(1+2x^2)^5\right\}_2^3 = \frac{1}{20}(19^5 - 9^5) = 120\,852.5$

19. If we set $u = x^{1/3}$, then $du = \dfrac{dx}{3x^{2/3}}$, and

$$\int_1^8 \frac{(1+x^{1/3})^2}{x^{2/3}}dx = \int_1^2 (1+u)^2(3\,du) = \left\{(1+u)^3\right\}_1^2 = 27 - 8 = 19.$$

20. $\displaystyle\int_{-4}^{4}|x+2|\,dx=\int_{-4}^{-2}-(x+2)\,dx+\int_{-2}^{4}(x+2)\,dx=-\left\{\frac{x^2}{2}+2x\right\}_{-4}^{-2}+\left\{\frac{x^2}{2}+2x\right\}_{-2}^{4}$
$=-(2-4)+(8-8)+(8+8)-(2-4)=20$

21. (a) Since $f(x)=x-5$ is continuous for $0\le x\le 2$, the definite integral exists, and we may choose any partition and star-points in its evaluation. For n equal subdivisions of length $2/n$, we use the points $x_i=2i/n$, $i=0,\ldots,n$. If we choose the right end of each subinterval as star-point, then $x_i^*=x_i=2i/n$. Equation 6.10 now gives

$$\begin{aligned}\int_0^2(x-5)\,dx&=\lim_{\|\Delta x_i\|\to 0}\sum_{i=1}^{n}(x_i^*-5)\Delta x_i=\lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left(\frac{2i}{n}-5\right)\\&=\lim_{n\to\infty}\frac{2}{n^2}\sum_{i=1}^{n}(2i-5n)=\lim_{n\to\infty}\frac{2}{n^2}\left[\frac{2n(n+1)}{2}-5n^2\right]=2-10=-8.\end{aligned}$$

(b) Since $f(x)=x^2+3$ is continuous for $0\le x\le 3$, the definite integral exists, and we may choose any partition and star-points in its evaluation. For n equal subdivisions of length $3/n$, we use the points $x_i=3i/n$, $i=0,\ldots,n$. If we choose the right end of each subinterval as star-point, then $x_i^*=x_i=3i/n$. Equation 6.10 now gives

$$\begin{aligned}\int_0^3(x^2+3)\,dx&=\lim_{\|\Delta x_i\|\to 0}\sum_{i=1}^{n}[(x_i^*)^2+3]\Delta x_i=\lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^2+3\right]\\&=\lim_{n\to\infty}\frac{9}{n^3}\sum_{i=1}^{n}(3i^2+n^2)=\lim_{n\to\infty}\frac{9}{n^3}\left[\frac{3n(n+1)(2n+1)}{6}+n^3\right]=9+9=18.\end{aligned}$$

22. (a) Since x^2+3x is nonnegative for $0\le x\le 4$, it is impossible for the limit summation of equation 6.10 to give a negative number.
(b) Since $1/x$ is negative for $-3\le x\le -2$, it is impossible for the limit summation of 6.10 to give a positive number.

23. The average value is $\displaystyle\frac{1}{1}\int_0^1\sqrt{x+4}\,dx=\left\{\frac{2(x+4)^{3/2}}{3}\right\}_0^1=\frac{2}{3}(5\sqrt{5}-8).$

24. The averge value is $\displaystyle\frac{1}{1}\int_{-2}^{-1}\left(\frac{1}{x^2}-x\right)dx=\left\{-\frac{1}{x}-\frac{x^2}{2}\right\}_{-2}^{-1}=\left(1-\frac{1}{2}\right)-\left(\frac{1}{2}-2\right)=2.$

25. If we set $u=x+1$, and $du=dx$, the average value is

$$\int_0^1 x\sqrt{x+1}\,dx=\int_1^2(u-1)\sqrt{u}\,du=\left\{\frac{2u^{5/2}}{5}-\frac{2u^{3/2}}{3}\right\}_1^2=\left(\frac{8\sqrt{2}}{5}-\frac{4\sqrt{2}}{3}\right)-\left(\frac{2}{5}-\frac{2}{3}\right)=\frac{4(\sqrt{2}+1)}{15}.$$

26. If we set $u=\sin x$ and $du=\cos x\,dx$, the average value is

$$\begin{aligned}\frac{1}{\pi/2}\int_0^{\pi/2}\cos^3x\,\sin^2x\,dx&=\frac{2}{\pi}\int_0^{\pi/2}(1-\sin^2x)\sin^2x\,\cos x\,dx=\frac{2}{\pi}\int_0^1(1-u^2)u^2\,du\\&=\frac{2}{\pi}\left\{\frac{u^3}{3}-\frac{u^5}{5}\right\}_0^1=\frac{2}{\pi}\left(\frac{1}{3}-\frac{1}{5}\right)=\frac{4}{15\pi}.\end{aligned}$$

27. By equation 6.19, $\displaystyle\frac{d}{dx}\int_1^x t\sqrt{t^3+1}\,dt=x\sqrt{x^3+1}$

28. When we reverse the limits, $\displaystyle\frac{d}{dx}\int_x^{-3}t^2(t+1)^3\,dt=-\frac{d}{dx}\int_{-3}^{x}t^2(t+1)^3\,dt=-x^2(x+1)^3.$

29. We set $u=x^2$ and use the chain rule,

$$\frac{d}{dx}\int_1^{x^2}\sqrt{t^2+1}\,dt=\left[\frac{d}{du}\int_1^u\sqrt{t^2+1}\,dt\right]\frac{du}{dx}=\sqrt{u^2+1}(2x)=2x\sqrt{x^4+1}.$$

30. When we reverse the limits, and set $u = 2x$,

$$\frac{d}{dx}\int_{2x}^{4} t\cos t\,dt = -\frac{d}{dx}\int_{4}^{2x} t\cos t\,dt = -\left[\frac{d}{du}\int_{4}^{u} t\cos t\,dt\right]\frac{du}{dx} = -u\cos u(2) = -4x\cos 2x.$$

31. When a is any number between $2x+3$ and $1-x$, we may write

$$\int_{2x+3}^{1-x}\frac{1}{t^2+1}dt = \int_{2x+3}^{a}\frac{1}{t^2+1}dt + \int_{a}^{1-x}\frac{1}{t^2+1}dt = -\int_{a}^{2x+3}\frac{1}{t^2+1}dt + \int_{a}^{1-x}\frac{1}{t^2+1}dt.$$

In these integrals we set $u = 2x+3$ and $v = 1-x$ respectively, and use chain rules,

$$\begin{aligned}\frac{d}{dx}\int_{2x+3}^{1-x}\frac{1}{t^2+1}dt &= -\left[\frac{d}{du}\int_{a}^{u}\frac{1}{t^2+1}dt\right]\frac{du}{dx} + \left[\frac{d}{dv}\int_{a}^{v}\frac{1}{t^2+1}dt\right]\frac{dv}{dx}\\ &= \frac{-1}{u^2+1}(2) + \frac{1}{v^2+1}(-1) = -\frac{2}{(2x+3)^2+1} - \frac{1}{(1-x)^2+1}.\end{aligned}$$

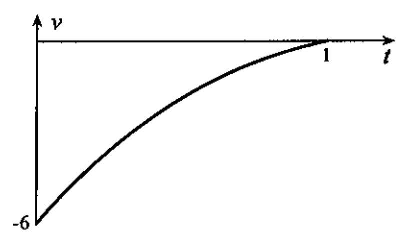
32. When a is a number between $-x^2$ and x^2,

$$\int_{-x^2}^{x^2}\sin^2 t\,dt = \int_{-x^2}^{a}\sin^2 t\,dt + \int_{a}^{x^2}\sin^2 t\,dt = -\int_{a}^{-x^2}\sin^2 t\,dt + \int_{a}^{x^2}\sin^2 t\,dt.$$

In these integrals we set $u = -x^2$ and $v = x^2$ respectively, and use chain rules,

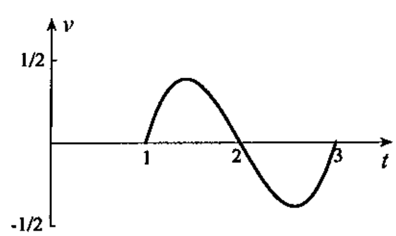
$$\begin{aligned}\frac{d}{dx}\int_{-x^2}^{x^2}\sin^2 t\,dt &= -\left[\frac{d}{du}\int_{a}^{u}\sin^2 t\,dt\right]\frac{du}{dx} + \left[\frac{d}{dv}\int_{a}^{v}\sin^2 t\,dt\right]\frac{dv}{dx}\\ &= -\sin^2 u(-2x) + \sin^2 v(2x) = 2x[\sin^2(-x^2) + \sin^2(x^2)] = 4x\sin^2 x^2.\end{aligned}$$

33. $$\begin{aligned}\int_0^1 v(t)\,dt &= \int_0^1 (t^3 - 6t^2 + 11t - 6)\,dt\\ &= \left\{\frac{t^4}{4} - 2t^3 + \frac{11t^2}{2} - 6t\right\}_0^1\\ &= \frac{1}{4} - 2 + \frac{11}{2} - 6 = -\frac{9}{4}\end{aligned}$$



This is the displacement of the particle at $t = 1$ relative to its position at $t = 0$. Since $v(t) = (t-1)(t-2)(t-3)$ is always negative between $t = 0$ and $t = 1$, it follows that the integral of $|v(t)|$ is equal to $9/4$. This is the distance travelled by the particle between $t = 0$ and $t = 1$.

34. $$\begin{aligned}\int_1^3 v(t)\,dt &= \int_1^3 (t^3 - 6t^2 + 11t - 6)\,dt\\ &= \left\{\frac{t^4}{4} - 2t^3 + \frac{11t^2}{2} - 6t\right\}_1^3\\ &= \left(\frac{81}{4} - 54 + \frac{99}{2} - 18\right)\\ &\quad - \left(\frac{1}{4} - 2 + \frac{11}{2} - 6\right) = 0\end{aligned}$$



This means that the particle is at the same position at times $t = 1$ and $t = 3$. Since $v(t) = (t-1)(t-2)(t-3)$ changes sign at $t = 2$,

$$\begin{aligned}\int_1^3 |v(t)|\,dt &= \int_1^2 (t^3 - 6t^2 + 11t - 6)\,dt + \int_2^3 (-t^3 + 6t^2 - 11t + 6)\,dt\\ &= \left\{\frac{t^4}{4} - 2t^3 + \frac{11t^2}{2} - 6t\right\}_1^2 + \left\{-\frac{t^4}{4} + 2t^3 - \frac{11t^2}{2} + 6t\right\}_2^3\end{aligned}$$

$$= (4 - 16 + 22 - 12) - \left(\frac{1}{4} - 2 + \frac{11}{2} - 6\right)$$
$$+ \left(-\frac{81}{4} + 54 - \frac{99}{2} + 18\right) - (-4 + 16 - 22 + 12)$$
$$= \frac{1}{2}.$$

This is the distance travelled by the particle between $t = 1$ and $t = 3$.

35. If we set $u = x^2 + 1$, then $du = 2x\,dx$, and

$$\int_0^1 \frac{x^3}{(x^2+1)^{3/2}}dx = \int_0^1 \frac{x^2}{(x^2+1)^{3/2}}x\,dx = \int_1^2 \frac{u-1}{u^{3/2}}\left(\frac{du}{2}\right) = \frac{1}{2}\int_1^2 \left(\frac{1}{\sqrt{u}} - \frac{1}{u^{3/2}}\right)du$$
$$= \frac{1}{2}\left\{2\sqrt{u} + \frac{2}{\sqrt{u}}\right\}_1^2 = \frac{1}{2}\left(2\sqrt{2} + \frac{2}{\sqrt{2}} - 2 - 2\right) = \frac{3\sqrt{2}-4}{2}.$$

36. If we set $u = 1 + \sin x$, then $du = \cos x\,dx$, and

$$\int_0^{\pi/6} \frac{\cos^3 x}{\sqrt{1+\sin x}}dx = \int_0^{\pi/6} \frac{1-\sin^2 x}{\sqrt{1+\sin x}}\cos x\,dx = \int_1^{3/2} \frac{1-(u-1)^2}{\sqrt{u}}du = \int_1^{3/2} (2\sqrt{u} - u^{3/2})\,du$$
$$= \left\{\frac{4}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right\}_1^{3/2} = \left(2\sqrt{\frac{3}{2}} - \frac{9}{10}\sqrt{\frac{3}{2}}\right) - \left(\frac{4}{3} - \frac{2}{5}\right) = \frac{33\sqrt{6}-56}{60}.$$

37. Because the integrand is an odd function, its integral over the interval $-1 \le x \le 1$ must be equal to 0.

38. If we set $u = x + 3$, then $du = dx$, and

$$\int_{-1}^2 \left|\frac{x}{\sqrt{3+x}}\right| du = \int_2^5 \left|\frac{u-3}{\sqrt{u}}\right| du = \int_2^3 \frac{3-u}{\sqrt{u}}du + \int_3^5 \frac{u-3}{\sqrt{u}}du$$
$$= \left\{6\sqrt{u} - \frac{2}{3}u^{3/2}\right\}_2^3 + \left\{\frac{2}{3}u^{3/2} - 6\sqrt{u}\right\}_3^5$$
$$= (6\sqrt{3} - 2\sqrt{3}) - \left(6\sqrt{2} - \frac{4\sqrt{2}}{3}\right) + \left(\frac{10\sqrt{5}}{3} - 6\sqrt{5}\right) - (2\sqrt{3} - 6\sqrt{3})$$
$$= \frac{1}{3}(24\sqrt{3} - 14\sqrt{2} - 8\sqrt{5}).$$

39. $\displaystyle\int_{-1}^2 x^2(4-x^3)^5\,dx = \left\{\frac{(4-x^3)^6}{-18}\right\}_{-1}^2 = \frac{4^6-5^6}{-18} = \frac{1281}{2}$

40. If we set $u = x^3 + 2x^2 + x$, then $du = (3x^2 + 4x + 1)\,dx$, and

$$\int_1^5 \frac{6x^2+8x+2}{\sqrt{x^3+2x^2+x}}dx = \int_4^{180} \frac{1}{\sqrt{u}}2du = \left\{4\sqrt{u}\right\}_4^{180} = 4(\sqrt{180} - 2) = 24\sqrt{5} - 8.$$

41. $\displaystyle\int_1^2 \frac{x-25}{\sqrt{x}-5}dx = \int_1^2 (\sqrt{x}+5)\,dx = \left\{\frac{2x^{3/2}}{3} + 5x\right\}_1^2 = \left(\frac{4\sqrt{2}}{3} + 10\right) - \left(\frac{2}{3} + 5\right) = \frac{4\sqrt{2}+13}{3}$

42. $\displaystyle\int_0^1 \frac{1}{\sqrt{2+x}+\sqrt{x}}dx = \int_0^1 \frac{1}{\sqrt{2+x}+\sqrt{x}}\frac{\sqrt{2+x}-\sqrt{x}}{\sqrt{2+x}-\sqrt{x}}dx = \int_0^1 \frac{\sqrt{2+x}-\sqrt{x}}{2+x-x}dx$

$$= \frac{1}{2}\left\{\frac{2}{3}(2+x)^{3/2} - \frac{2}{3}x^{3/2}\right\}_0^1 = \frac{1}{2}\left(2\sqrt{3} - \frac{2}{3}\right) - \frac{1}{2}\left(\frac{4\sqrt{2}}{3}\right) = \frac{1}{3}(3\sqrt{3} - 1 - 2\sqrt{2})$$