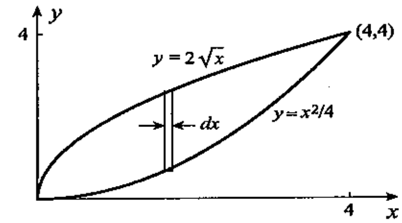
CHAPTER 7

EXERCISES 7.1

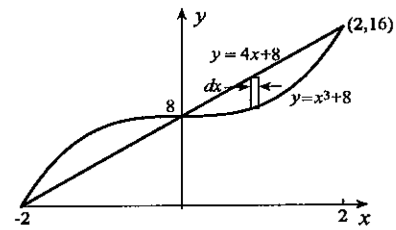
1. $A = \int_0^4 (2\sqrt{x} - x^2/4)\,dx$

$$= \left\{ \frac{4x^{3/2}}{3} - \frac{x^3}{12} \right\}_0^4 = \frac{16}{3}$$



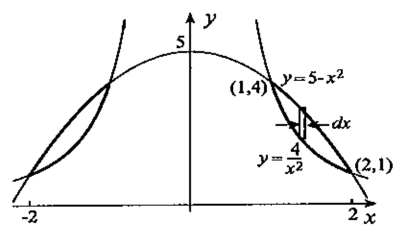
2. $A = 2\int_0^2 [(4x+8) - (x^3+8)]\,dx$

$$= 2\left\{ 2x^2 - \frac{x^4}{4} \right\}_0^2 = 8$$



3. $A = 2\int_1^2 \left[(5 - x^2) - \frac{4}{x^2} \right] dx$

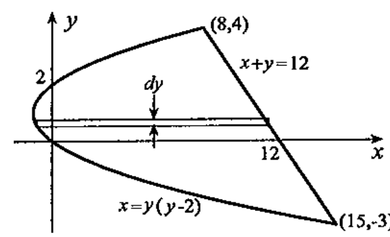
$$= 2\left\{ 5x - \frac{x^3}{3} + \frac{4}{x} \right\}_1^2 = \frac{4}{3}$$



4. $A = \int_{-3}^4 [(12 - y) - y(y-2)]\,dy$

$$= \int_{-3}^4 (12 + y - y^2)\,dy$$

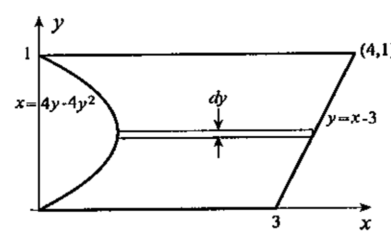
$$= \left\{ 12y + \frac{y^2}{2} - \frac{y^3}{3} \right\}_{-3}^4 = \frac{343}{6}$$



5. $A = \int_0^1 [(y+3) - (4y - 4y^2)]\,dy$

$$= \int_0^1 (3 - 3y + 4y^2)\,dy$$

$$= \left\{ 3y - \frac{3y^2}{2} + \frac{4y^3}{3} \right\}_0^1 = \frac{17}{6}$$

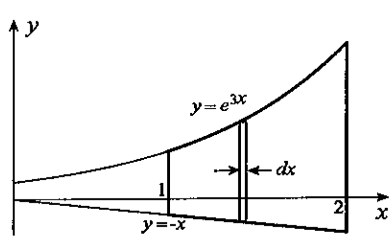


6. $A = \int_1^2 (e^{3x} + x)\,dx$

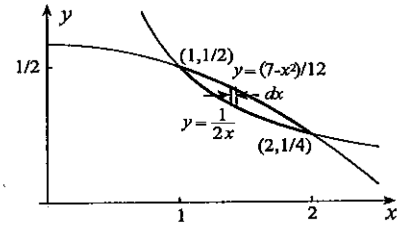
$$= \left\{ \frac{e^{3x}}{3} + \frac{x^2}{2} \right\}_1^2$$

$$= \left(\frac{e^6}{3} + 2 \right) - \left(\frac{e^3}{3} + \frac{1}{2} \right)$$

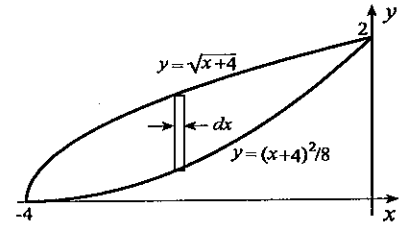
$$= \frac{1}{3}(e^6 - e^3) + \frac{3}{2}$$



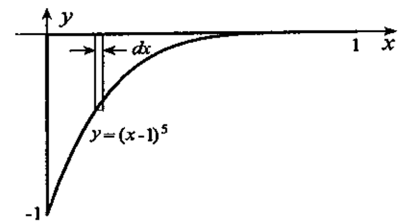
7. $$A = \int_1^2 \left(\frac{7-x^2}{12} - \frac{1}{2x}\right) dx$$
$$= \left\{\frac{7x}{12} - \frac{x^3}{36} - \frac{1}{2}\ln|x|\right\}_1^2 = \frac{7-9\ln 2}{18}$$



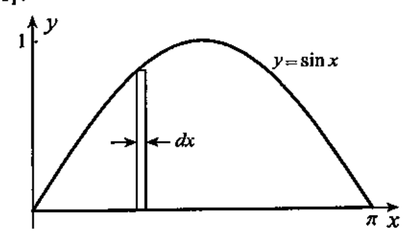
8. $$A = \int_{-4}^0 \left[\sqrt{x+4} - \frac{(x+4)^2}{8}\right] dx$$
$$= \left\{\frac{2}{3}(x+4)^{3/2} - \frac{1}{24}(x+4)^3\right\}_{-4}^0$$
$$= \frac{8}{3}$$



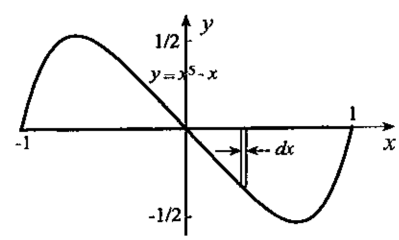
9. $$A = \int_0^1 -(x-1)^5\, dx$$
$$= \left\{-\frac{(x-1)^6}{6}\right\}_0^1 = \frac{1}{6}$$



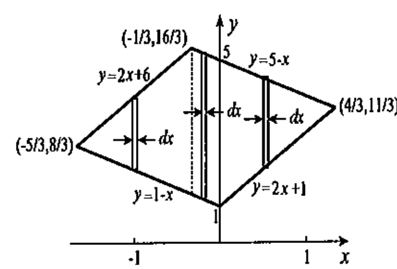
10. $$A = \int_0^\pi \sin x\, dx = \{-\cos x\}_0^\pi = 2$$



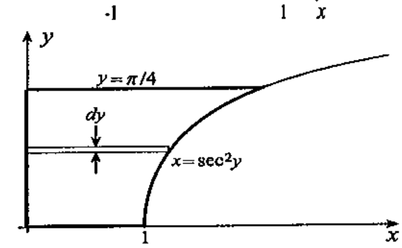
11. $$A = 2\int_0^1 (-x^5 + x)\, dx$$
$$= 2\left\{-\frac{x^6}{6} + \frac{x^2}{2}\right\}_0^1 = \frac{2}{3}$$



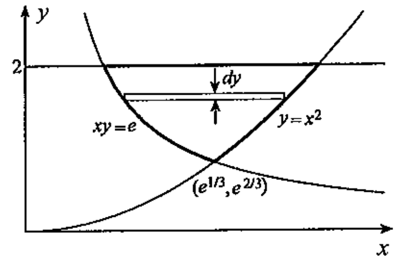
12. $$A = \int_{-5/3}^{-1/3} [(2x+6) - (1-x)]\, dx + \int_{-1/3}^{0} [(5-x) - (1-x)]\, dx$$
$$+ \int_0^{4/3} [(5-x) - (2x+1)]\, dx$$
$$= \int_{-5/3}^{-1/3} (3x+5)\, dx + \int_{-1/3}^{0} 4\, dx + \int_0^{4/3} (4-3x)\, dx$$
$$= \left\{\frac{3x^2}{2} + 5x\right\}_{-5/3}^{-1/3} + \{4x\}_{-1/3}^{0} + \left\{4x - \frac{3x^2}{2}\right\}_0^{4/3} = \frac{20}{3}$$



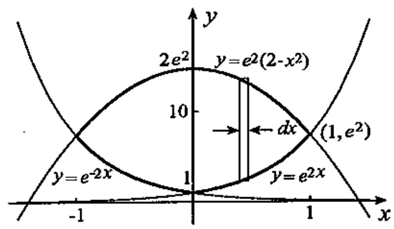
13. $$A = \int_0^{\pi/4} \sec^2 y\, dy$$
$$= \left\{\tan y\right\}_0^{\pi/4} = 1$$



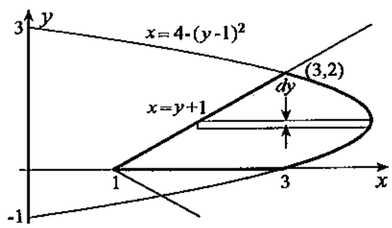
14. $$A = \int_{e^{2/3}}^{2}\left(\sqrt{y} - \frac{e}{y}\right)dy$$
$$= \left\{\frac{2}{3}y^{3/2} - e\ln|y|\right\}\Bigg|_{e^{2/3}}^{2}$$
$$= \left(\frac{4\sqrt{2}}{3} - e\ln 2\right) - \left(\frac{2e}{3} - \frac{2e}{3}\right)$$
$$= \frac{4\sqrt{2}}{3} - e\ln 2$$



15. $$A = 2\int_0^1 [e^2(2-x^2) - e^{2x}]\,dx$$
$$= 2\left\{2e^2x - \frac{e^2x^3}{3} - \frac{e^{2x}}{2}\right\}\Bigg|_0^1 = \frac{7e^2+3}{3}$$



16. $$A = \int_0^2 [4 - (y-1)^2 - (y+1)]\,dy$$
$$= \int_0^2 [3 - y - (y-1)^2]\,dy$$
$$= \left\{3y - \frac{y^2}{2} - \frac{(y-1)^3}{3}\right\}\Bigg|_0^2$$
$$= \left(6 - 2 - \frac{1}{3}\right) - \left(\frac{1}{3}\right) = \frac{10}{3}$$



17. (a) The area is $A = \displaystyle\int_1^3 (16 - x^2)\,dx = \left\{16x - \frac{x^3}{3}\right\}\Bigg|_1^3 = \frac{70}{3}$.

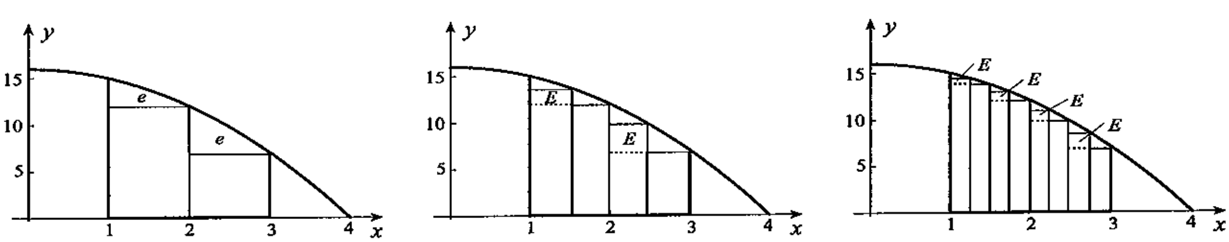
(b) $A_2 = f(2)(1) + f(3)(1) = 12 + 7 = 19$ The error is $70/3 - 19 = 13/3$. It is the areas marked with e in the left figure.

(c) $A_4 = f(3/2)(1/2) + f(2)(1/2) + f(5/2)(1/2) + f(3)(1/2) = \dfrac{1}{2}\left(\dfrac{55}{4} + 12 + \dfrac{39}{4} + 7\right) = \dfrac{85}{4}$

The error is $70/3 - 85/4 = 25/12$. The extra precision is the addition of the two rectangles marked with an E in the middle figure.

(d) $A_8 = \dfrac{1}{4}[f(5/4) + f(3/2) + f(7/4) + f(2) + f(9/4) + f(5/2) + f(11/4) + f(3)] = \dfrac{357}{16}$

The error is $70/3 - 357/16 = 49/48$. The extra precision is the addition of the four rectangles marked with an E in the right figure.



18. (a) The area is $A = \int_1^3 (x^3+1)\,dx = \left\{\frac{x^4}{4} + x\right\}_1^3 = 22.$

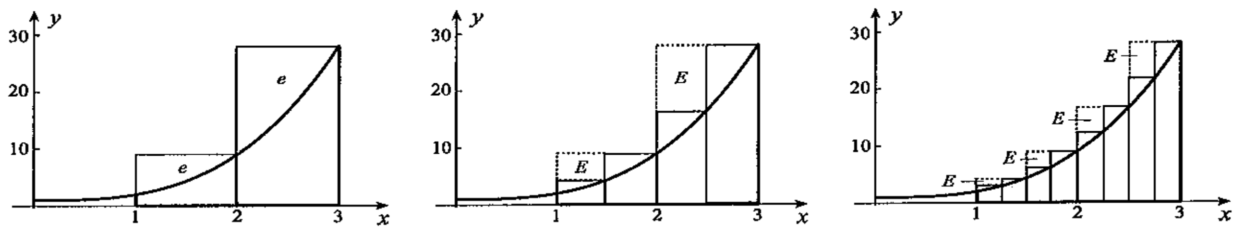
(b) $A_2 = f(2)(1) + f(3)(1) = 9 + 28 = 37$ The error is $37 - 22 = 15$. It is the areas marked with e in the left figure.

(c) $A_4 = f(3/2)(1/2) + f(2)(1/2) + f(5/2)(1/2) + f(3)(1/2) = \frac{1}{2}\left(\frac{35}{8} + 9 + \frac{133}{8} + 28\right) = 29$

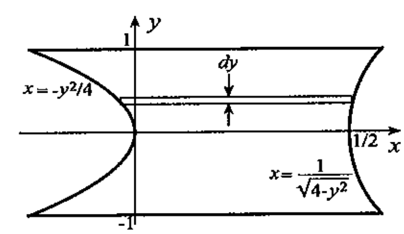
The error is $29 - 22 = 7$. The extra precision is the deletion of the two rectangles marked with an E in the middle figure.

(d) $A_8 = \frac{1}{4}[f(5/4) + f(3/2) + f(7/4) + f(2) + f(9/4) + f(5/2) + f(11/4) + f(3)] = \frac{203}{8}$

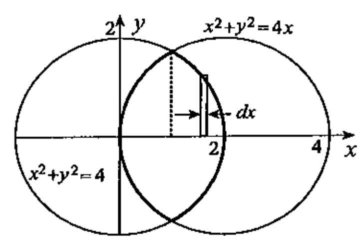
The error is $203/8 - 22 = 27/8$. The extra precision is the deletion of the four rectangles marked with an E in the right figure.



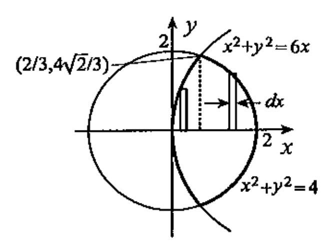
19. $A = 2\int_0^1 \left(\frac{1}{\sqrt{4-y^2}} + \frac{y^2}{4}\right) dy$



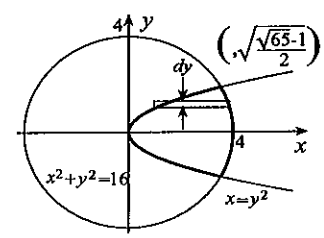
20. $A = 4\int_1^2 \sqrt{4-x^2}\,dx$



21. $A = 2\int_0^{2/3} \sqrt{6x - x^2}\,dx$
$+2\int_{2/3}^2 \sqrt{4-x^2}\,dx$

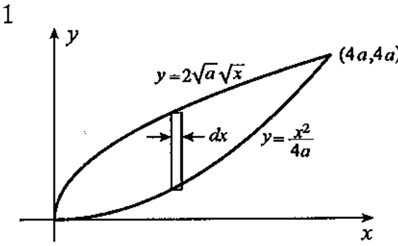


22. $A = 2\int_0^{\sqrt{(\sqrt{65}-1)/2}} (\sqrt{16-y^2} - y^2)\,dy$



23. $A = \int_0^{4a} \left(2\sqrt{a}\sqrt{x} - \frac{x^2}{4a}\right) dx$

$= \left\{\frac{4\sqrt{a}\,x^{3/2}}{3} - \frac{x^3}{12a}\right\}_0^{4a} = \frac{16a^2}{3}$

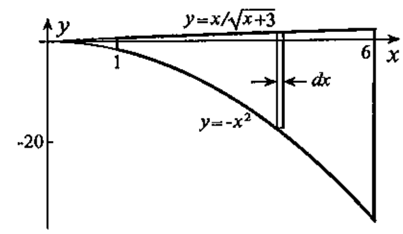


24. $A = \int_1^6 \left(\frac{x}{\sqrt{x+3}} + x^2\right) dx$

If we set $u = x + 3$ in the first term, then $du = dx$, and

$A = \int_4^9 \frac{u-3}{\sqrt{u}} du + \left\{\frac{x^3}{3}\right\}_1^6$

$= \left\{\frac{2}{3}u^{3/2} - 6\sqrt{u}\right\}_4^9 + \frac{215}{3} = \frac{235}{3}$

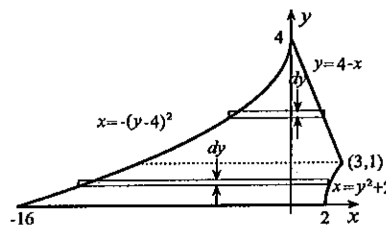


25. $A = \int_0^1 [(y^2+2) + (y-4)^2]\, dy$

$+ \int_1^4 [(4-y) + (y-4)^2]\, dy$

$= \left\{\frac{y^3}{3} + 2y + \frac{1}{3}(y-4)^3\right\}_0^1$

$+ \left\{4y - \frac{y^2}{2} + \frac{1}{3}(y-4)^3\right\}_1^4 = \frac{169}{6}$

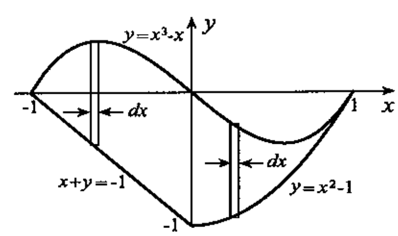


26. $A = \int_{-1}^0 [(x^3 - x) - (-x-1)]\, dx$

$+ \int_0^1 [(x^3 - x) - (x^2 - 1)]\, dx$

$= \left\{\frac{x^4}{4} + x\right\}_{-1}^0 + \left\{\frac{x^4}{4} - \frac{x^2}{2} - \frac{x^3}{3} + x\right\}_0^1$

$= \frac{7}{6}$



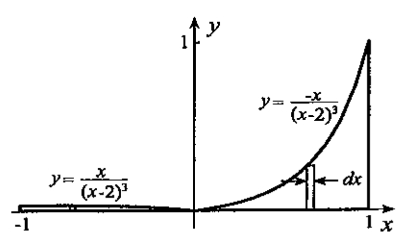
27. $A = \int_{-1}^0 \frac{x}{(x-2)^3} dx + \int_0^1 \frac{-x}{(x-2)^3} dx$

If we set $u = x - 2$ and $du = dx$ in these integrals,

$A = \int_{-3}^{-2} \frac{u+2}{u^3} du - \int_{-2}^{-1} \frac{u+2}{u^3} du$

$= \left\{-\frac{1}{u} - \frac{1}{u^2}\right\}_{-3}^{-2} - \left\{-\frac{1}{u} - \frac{1}{u^2}\right\}_{-2}^{-1}$

$= \frac{5}{18}.$

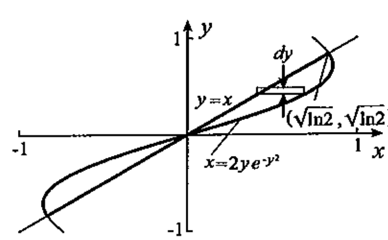


28. $A = 2\int_0^{\sqrt{\ln 2}} (2ye^{-y^2} - y)\, dy$

$= 2\left\{-e^{-y^2} - \frac{y^2}{2}\right\}_0^{\sqrt{\ln 2}}$

$= 2\left(-e^{-\ln 2} - \frac{\ln 2}{2}\right) - 2(-1)$

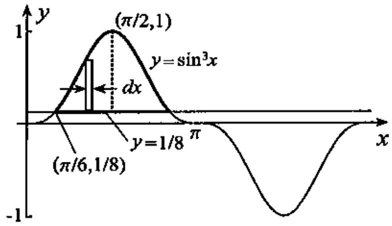
$= 1 - \ln 2$



29. $A = 2\int_{\pi/6}^{\pi/2} (\sin^3 x - 1/8)\, dx$

$= 2\int_{\pi/6}^{\pi/2} [\sin x(1-\cos^2 x) - 1/8]\, dx$

$= 2\left\{-\cos x + \frac{1}{3}\cos^3 x - \frac{x}{8}\right\}_{\pi/6}^{\pi/2} = \frac{9\sqrt{3}-\pi}{12}$

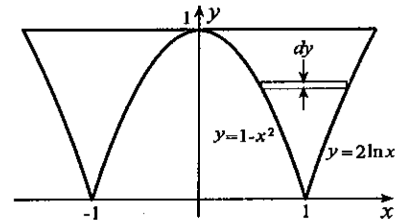


30. $A = 2\int_0^1 (e^{y/2} - \sqrt{1-y})\, dy$

$= 2\left\{2e^{y/2} + \frac{2}{3}(1-y)^{3/2}\right\}_0^1$

$= 2(2\sqrt{e}) - 2\left(2 + \frac{2}{3}\right)$

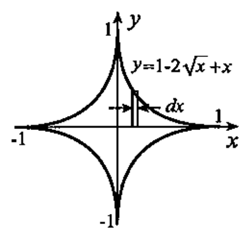
$= 4\sqrt{e} - \frac{16}{3}$



31. $A = 4\int_0^1 (1 - 2\sqrt{x} + x)\, dx$

$= 4\left\{x - \frac{4x^{3/2}}{3} + \frac{x^2}{2}\right\}_0^1$

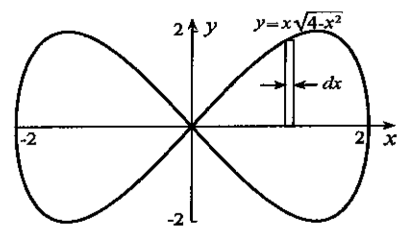
$= \frac{2}{3}$



32. $A = 4\int_0^2 x\sqrt{4-x^2}\, dx$

$= 4\left\{-\frac{1}{3}(4-x^2)^{3/2}\right\}_0^2$

$= \frac{4}{3}(4)^{3/2} = \frac{32}{3}$



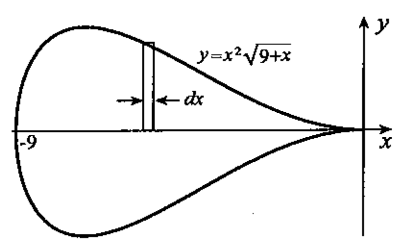
33. $A = 2\int_{-9}^{0} x^2\sqrt{9+x}\, dx$

If we set $u = 9 + x$ and $du = dx$, then

$A = 2\int_0^9 (u-9)^2\sqrt{u}\, du$

$= 2\int_0^9 (u^{5/2} - 18u^{3/2} + 81\sqrt{u})\, du$

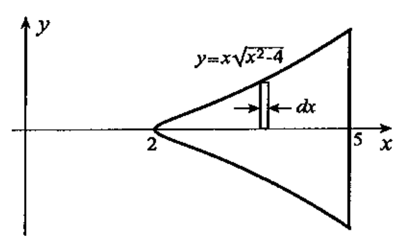
$= 2\left\{\frac{2u^{7/2}}{7} - \frac{36u^{5/2}}{5} + \frac{162u^{3/2}}{3}\right\}_0^9 = \frac{23\,328}{35}.$



34. $A = 2\int_2^5 x\sqrt{x^2-4}\, dx$

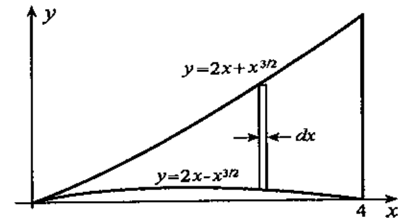
$= 2\left\{\frac{1}{3}(x^2-4)^{3/2}\right\}_2^5$

$= \frac{2}{3}(21^{3/2}) = 14\sqrt{21}$



35. $A = \int_0^4 [(2x + x^{3/2}) - (2x - x^{3/2})]\,dx$

$= 2\int_0^4 x^{3/2}\,dx = 2\left\{\frac{2x^{5/2}}{5}\right\}_0^4 = \frac{128}{5}$



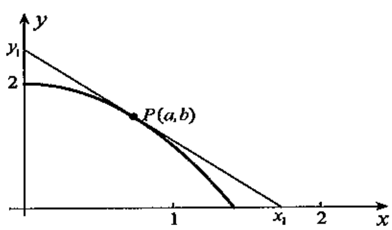
36. The equation of the tangent line at any point $P(a,b)$ on the first quadrant part of the parabola is $y - b = -2a(x-a)$ (left figure below). The x- and y-intercepts of this line are $x_1 = a + b/(2a)$, and $y_1 = b + 2a^2$. The area of the triangle is $A = \frac{1}{2}x_1y_1 = \frac{1}{2}\left(a + \frac{b}{2a}\right)(b + 2a^2)$. Since $b = 2 - a^2$, we can express A in the form

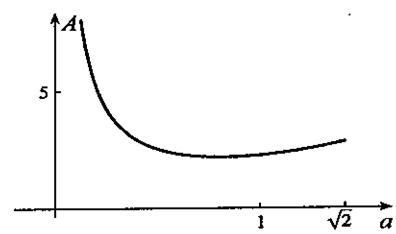
$$A = \frac{1}{2}\left(a + \frac{2-a^2}{2a}\right)(2 - a^2 + 2a^2) = \frac{1}{4a}(2+a^2)^2, \quad 0 < a \le \sqrt{2}.$$

The plot of this function in the right figure indicates that its minimum occurs at the critical point. To find it we solve

$$0 = \frac{dA}{da} = \frac{1}{4}\left[\frac{a(2)(2+a^2)(2a) - (2+a^2)^2}{a^2}\right] = \frac{(2+a^2)(3a^2-2)}{4a^2}.$$

The only positive solution is $a = \sqrt{2/3}$. Hence the required point is $(\sqrt{2/3}, 4/3)$.





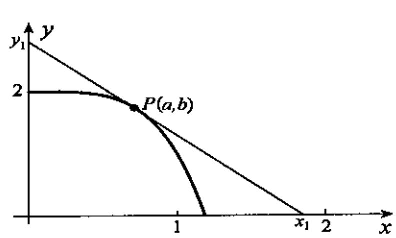
37. The equation of the tangent line at any point $P(a,b)$ on the first quadrant part of the curve is $y - b = -4a^3(x-a)$ (left figure below). The x- and y-intercepts of this line are $x_1 = a + b/(4a^3)$, and $y_1 = b + 4a^4$. The area of the triangle is $A = \frac{1}{2}x_1y_1 = \frac{1}{2}\left(a + \frac{b}{4a^3}\right)(b + 4a^4)$. Since $b = 2 - a^4$, we can express A in the form

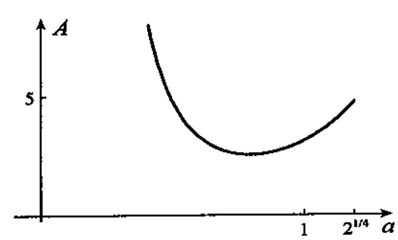
$$A = \frac{1}{2}\left(a + \frac{2-a^4}{4a^3}\right)(2 - a^4 + 4a^4) = \frac{1}{8a^3}(2+3a^4)^2, \quad 0 < a \le 2^{1/4}.$$

The plot of this function in the right figure indicates that its minimum occurs at the critical point. To find it we solve

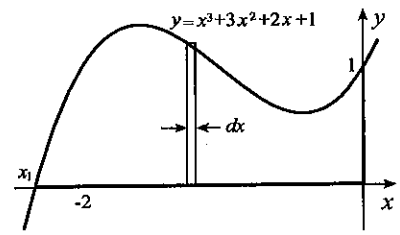
$$0 = \frac{dA}{da} = \frac{1}{8}\left[\frac{a^3(24a^3)(2+3a^4) - 3a^2(2+3a^4)^2}{a^6}\right] = \frac{3(2+3a^4)(5a^4-2)}{8a^4}.$$

The only positive solution is $a = (2/5)^{1/4}$. Hence the required point is $\left((2/5)^{1/4}, 8/5\right)$.





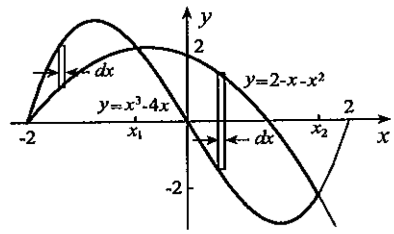
38. Newton's iterative procedure gives the x-intercept of the cubic as $x_1 = -2.324\,718$. Hence,



$$\begin{aligned} A &= \int_{x_1}^{0} (x^3 + 3x^2 + 2x + 1)\,dx \\ &= \left\{ \frac{x^4}{4} + x^3 + x^2 + x \right\}_{x_1}^{0} \\ &= 2.182. \end{aligned}$$

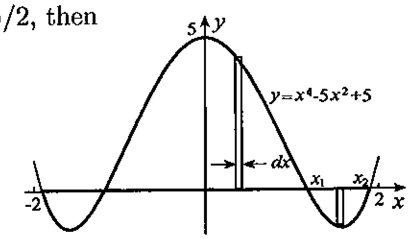
39. The x-coordinates of points of intersection of the curves are defined by the equation $x^3 + x^2 - 3x - 2 = 0$. Newton's iterative procedure gives $x_1 = -0.618\,034$ and $x_2 = 1.618\,034$ as solutions of this equation. Hence,

$$\begin{aligned} A &= \int_{-2}^{x_1} [(x^3 - 4x) - (2 - x - x^2)]\,dx \\ &\quad + \int_{x_1}^{x_2} [(2 - x - x^2) - (x^3 - 4x)]\,dx \\ &= \left\{ \frac{x^4}{4} + \frac{x^3}{3} - \frac{3x^2}{2} - 2x \right\}_{-2}^{x_1} \\ &\quad + \left\{ -\frac{x^4}{4} - \frac{x^3}{3} + \frac{3x^2}{2} + 2x \right\}_{x_1}^{x_2} = 5.946. \end{aligned}$$



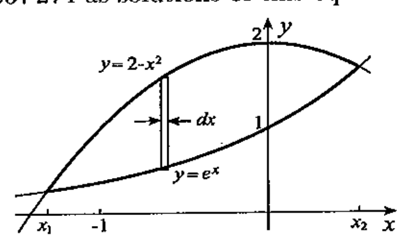
40. Since $x^4 - 5x^2 + 5$ is a quadratic in x^2, x-intercepts can be found exactly; they are $\pm\sqrt{(5+\sqrt{5})/2}$ and $\pm\sqrt{(5-\sqrt{5})/2}$. If we set $x_1 = \sqrt{(5-\sqrt{5})/2}$, and $x_2 = \sqrt{(5+\sqrt{5})/2}$, then

$$\begin{aligned} A &= 2\int_{0}^{x_1} (x^4 - 5x^2 + 5)\,dx \\ &\quad + 2\int_{x_1}^{x_2} (-x^4 + 5x^2 - 5)\,dx \\ &= 2\left\{ \frac{x^5}{5} - \frac{5x^3}{3} + 5x \right\}_{0}^{x_1} + 2\left\{ -\frac{x^5}{5} + \frac{5x^3}{3} - 5x \right\}_{x_1}^{x_2} = 8.436. \end{aligned}$$



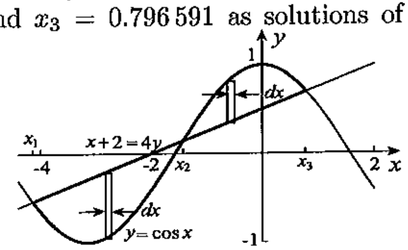
41. The x-coordinates of points of intersection of the curves are defined by the equation $e^x = 2 - x^2$. Newton's method gives $x_1 = -1.315\,974$, and $x_2 = 0.537\,274$ as solutions of this equation. Thus,

$$\begin{aligned} A &= \int_{x_1}^{x_2} (2 - x^2 - e^x)\,dx \\ &= \left\{ 2x - \frac{x^3}{3} - e^x \right\}_{x_1}^{x_2} \\ &= 1.452. \end{aligned}$$



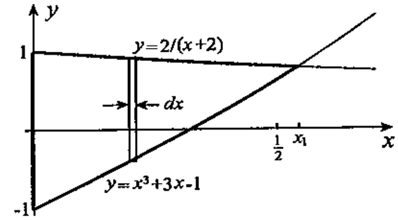
42. The x-coordinates of points of intersection of the curves are defined by the equation $\cos x = (x+2)/4$. Newton's method gives $x_1 = -4.146\,081$, $x_2 = -1.427\,069$, and $x_3 = 0.796\,591$ as solutions of this equation. Thus,

$$\begin{aligned} A &= \int_{x_1}^{x_2} \left[\frac{1}{4}(x+2) - \cos x \right] dx + \int_{x_2}^{x_3} \left[\cos x - \frac{1}{4}(x+2) \right] dx \\ &= \left\{ \frac{1}{8}(x+2)^2 - \sin x \right\}_{x_1}^{x_2} + \left\{ \sin x - \frac{1}{8}(x+2)^2 \right\}_{x_2}^{x_3} \\ &= 2.067. \end{aligned}$$



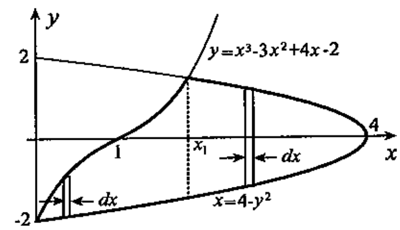
43. The x-coordinate of the point of intersection of the curves is defined by the equation $2/(x+2) = x^3 + 3x - 1 \Longrightarrow x^4 + 2x^3 + 3x^2 + 5x - 4 = 0$. Newton's method gives $x_1 = 0.542\,373$ as the solution of this equation. Thus,

$$A = \int_0^{x_1} \left(\frac{2}{x+2} - x^3 - 3x + 1 \right) dx$$
$$= \left\{ 2\ln|x+2| - \frac{x^4}{4} - \frac{3x^2}{2} + x \right\}_0^{x_1}$$
$$= 0.559.$$



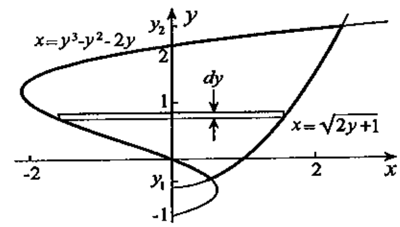
44. Points of intersection of the curves have y-coordinates satisfying $y = (4-y^2)^3 - 3(4-y^2)^2 + 4(4-y^2) - 2$, and this equation simplifies to $y^6 - 9y^4 + 28y^2 + y - 30 = 0$. The negative solution is $y = -2$ and the positive solution can be obtained by Newton's method as $y = 1.466\,078$. The equation of the parabola gives the corresponding x-coordinate as $x_1 = 1.850\,616$. The area is

$$A = \int_0^{x_1} (x^3 - 3x^2 + 4x - 2 + \sqrt{4-x})\, dx$$
$$+2\int_{x_1}^4 \sqrt{4-x}\, dx$$
$$= \left\{ \frac{x^4}{4} - x^3 + 2x^2 - 2x - \frac{2}{3}(4-x)^{3/2} \right\}_0^{x_1}$$
$$+2\left\{ -\frac{2}{3}(4-x)^{3/2} \right\}_{x_1}^4 = 7.177.$$



45. Points of intersection of the curves have y-coordinates satisfying $y^3 - y^2 - 2y = \sqrt{2y+1}$, or, $(y^3 - y^2 - 2y)^2 = 2y + 1$. The solutions can be obtained by Newton's method as $y_1 = -0.354\,740$ and $y_2 = 2.310\,040$. The area is

$$A = \int_{y_1}^{y_2} [\sqrt{2y+1} - (y^3 - y^2 - 2y)]\, dy$$
$$= \left\{ \frac{1}{3}(2y+1)^{3/2} - \frac{y^4}{4} + \frac{y^3}{3} + y^2 \right\}_{y_1}^{y_2} = 6.608.$$



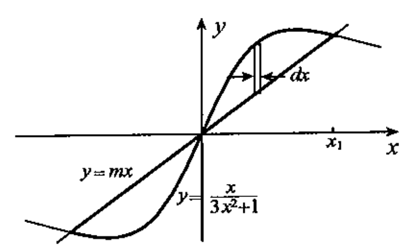
46. For points of intersection of the curves, we solve

$$mx = \frac{x}{3x^2+1}$$

This equation simplifies to $3mx^3 + mx = x$, one solution of which is $x = 0$. Other solutions must satisfy $3mx^2 + m - 1 = 0$, from which

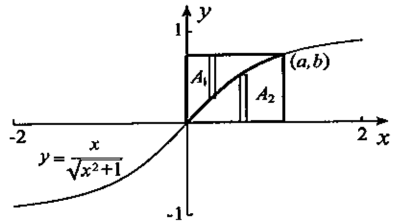
$$x = \pm\sqrt{\frac{1-m}{3m}}.$$

Consequently, an area is defined when $0 < m < 1$. If we set $x_1 = \sqrt{(1-m)/(3m)}$, the required area is



$$A = 2\int_0^{x_1} \left(\frac{x}{3x^2+1} - mx \right) dx = 2\left\{ \frac{1}{6}\ln(3x^2+1) - \frac{mx^2}{2} \right\}_0^{x_1} = \frac{1}{3}\ln(3x_1^2+1) - mx_1^2$$
$$= \frac{1}{3}\ln\left[3\left(\frac{1-m}{3m} \right) + 1 \right] - m\left(\frac{1-m}{3m} \right) = \frac{1}{3}\ln\left[\frac{1-m+m}{m} \right] + \frac{m-1}{3} = -\frac{1}{3}\ln m + \frac{m-1}{3}.$$

47. For $A_2 = 2A_1$,



$$\int_0^a \frac{x}{\sqrt{x^2+1}}dx = 2\int_0^a \left(b - \frac{x}{\sqrt{x^2+1}}\right)dx$$
$$\Longrightarrow$$
$$\left\{\sqrt{x^2+1}\right\}_0^a = 2\left\{bx - \sqrt{x^2+1}\right\}_0^a$$
$$\Longrightarrow$$
$$\sqrt{a^2+1} - 1 = 2(ab - \sqrt{a^2+1} + 1).$$

Since (a,b) is on the curve, $b = a/\sqrt{a^2+1}$, and the above equation can be expressed in the form

$$3\sqrt{a^2+1} = \frac{2a^2}{\sqrt{a^2+1}} + 3 \quad\Longrightarrow\quad 3(a^2+1) = 2a^2 + 3\sqrt{a^2+1} \quad\Longrightarrow\quad a^2 + 3 = 3\sqrt{a^2+1}.$$

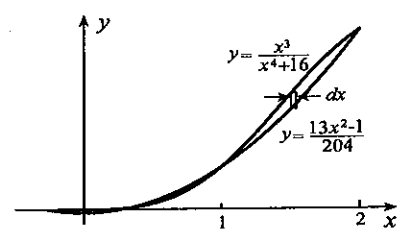
When this is squared, it simplifies to $a^4 - 3a^2 = 0$ with solutions $a = 0, \pm\sqrt{3}$. Thus, there are three points $(0,0)$ and $(\pm\sqrt{3}, \pm\sqrt{3}/2)$.

48. If x_{i-1} and x_i are the ends of the i^{th} rectangle, then $\Delta x_i = 2/2^n = 1/2^{n-1}$, and $x_i = 1 + i/2^{n-1}$. It follows that

$$A_{2^n} = \sum_{i=1}^{2^n} f(x_i)\Delta x_i = \sum_{i=1}^{2^n}\left[16 - \left(1 + \frac{i}{2^{n-1}}\right)^2\right]\left(\frac{1}{2^{n-1}}\right) = \frac{1}{2^{n-1}}\sum_{i=1}^{2^n}\left[15 - \frac{i}{2^{n-2}} - \frac{i^2}{2^{2n-2}}\right]$$
$$= \frac{1}{2^{n-1}}\left[15(2^n) - \frac{1}{2^{n-2}}\frac{2^n(2^n+1)}{2} - \frac{1}{2^{2n-2}}\frac{2^n(2^n+1)(2^{n+1}+1)}{6}\right] \quad \text{(see equations 6.3,6.4)}$$
$$= 30 - \left(4 + \frac{1}{2^{n-2}}\right) - \frac{1}{6}\left(1 + \frac{1}{2^n}\right)\left(16 + \frac{1}{2^{n-3}}\right) = \frac{70}{3} - \frac{1}{2^{n-2}} - \frac{1}{6}\left(\frac{3}{2^{n-3}} + \frac{1}{2^{2n-3}}\right).$$

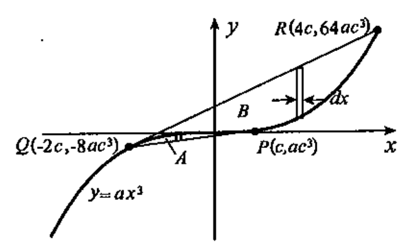
Since $A = 70/3$, we may write $A_{2^n} = A - \left[\frac{1}{2^{n-2}} + \frac{1}{6}\left(\frac{3}{2^{n-3}} + \frac{1}{2^{2n-3}}\right)\right]$, and clearly, $\lim_{n\to\infty} A_{2^n} = A$.

49. The figure indicates that there are indeed three areas bounded by the curves. The curves intersect at $x = -0.249, 0.340, 1, 2$. The largest of the three areas is



$$A = \int_1^2\left(\frac{x^3}{x^4+16} - \frac{13x^2-1}{204}\right)dx$$
$$= \left\{\frac{1}{4}\ln|x^4+16| - \frac{13x^3}{612} + \frac{x}{204}\right\}_1^2 = 0.014.$$

50. If the coordinates of P are (c, ac^3), then the tangent line at P has equation



$$y - ac^3 = 3ac^2(x - c),$$

and the x-coordinate of Q is defined by

$$ax^3 = ac^3 + 3ac^2(x - c).$$

The solution of this equation is $x = -2c$. The equation of the tangent line at Q is $y + 8ac^3 = 12ac^2(x + 2c)$, and the intersection of this line with $y = ax^3$ is the point $R(4c, 64ac^3)$. Now,

$$A = \int_{-2c}^{c}[ax^3 - ac^3 - 3ac^2(x - c)]\,dx = \left\{\frac{ax^4}{4} - \frac{3ac^2x^2}{2} + 2ac^3x\right\}_{-2c}^{c} = \frac{27}{4}ac^4,$$

and

$$B = \int_{-2c}^{4c}[-8ac^3 + 12ac^2(x + 2c) - ax^3]\,dx = \left\{16ac^3x + 6ac^2x^2 - \frac{ax^4}{4}\right\}_{-2c}^{4c} = 108ac^4.$$

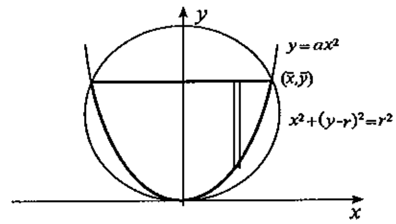
Thus, $B = 16A$.

51. To find the points of intersection of the curves, we solve

$$\frac{y}{a} + (y-r)^2 = r^2 \implies \frac{y}{a} + y^2 - 2ry + r^2 = r^2.$$

Solutions are $y = 0$ and $y = 2r - 1/a$. The x-coordinate for the second of these is $x = \sqrt{2r/a - 1/a^2}$. Let us denote the point of intersection by

$$(\overline{x}, \overline{y}) = \left(\sqrt{\frac{2r}{a} - \frac{1}{a^2}}, 2r - \frac{1}{a}\right).$$



The area inside the parabola and below the line $y = \overline{y}$ is

$$A = 2\int_0^{\overline{x}} (\overline{y} - ax^2)\,dx = 2\left\{\overline{y}x - \frac{ax^3}{3}\right\}_0^{\overline{x}} = 2\left(\overline{x}\overline{y} - \frac{a\overline{x}^3}{3}\right) = 2\overline{x}\left(a\overline{x}^2 - \frac{a\overline{x}^2}{3}\right)$$

$$= \frac{4a\overline{x}^3}{3} = \frac{4a}{3}\left(\frac{2r}{a} - \frac{1}{a^2}\right)^{3/2} = \frac{4}{3a^2}(2ar-1)^{3/2}.$$

The domain of this function $A(a)$ is $\frac{1}{2r} \le a < \infty$. For critical points we solve

$$0 = \frac{dA}{da} = \frac{4}{3}\left[-\frac{2}{a^3}(2ar-1)^{3/2} + \frac{1}{a^2}\left(\frac{3}{2}\right)(2ar-1)^{1/2}(2r)\right]$$

$$= \frac{4\sqrt{2ar-1}}{3a^3}[-2(2ar-1) + 3ar] = \frac{4\sqrt{2ar-1}(2-ar)}{3a^3}.$$

Solutions are $a = 2/r$ and $a = 1/(2r)$. Since

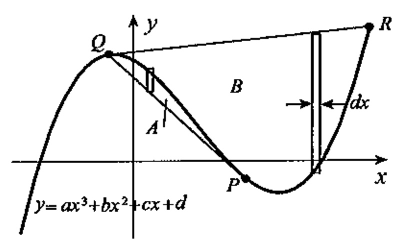
$$A\left(\frac{1}{2r}\right) = 0, \qquad A\left(\frac{2}{r}\right) > 0, \qquad \lim_{a\to\infty} A(a) = 0,$$

it follows that area is maximized when $a = 2/r$.

52. If the coordinates of P are $(e, ae^3 + be^2 + ce + d)$, the tangent line at P has equation

$$y - (ae^3 + be^2 + ce + d) = (3ae^2 + 2be + c)(x - e).$$

The intersection point Q of this line with the cubic has x-coordinate $x = -(b + 2ae)/a$. The equation of the tangent line at Q is



$$y - \left(-\frac{2b^2e}{a} - 8be^2 - 8ae^3 - \frac{bc}{a} - 2ce + d\right)$$
$$= \left(\frac{b^2}{a} + 8be + 12ae^2 + c\right)\left(x + \frac{b}{a} + 2e\right).$$

and the intersection of this line with the cubic has x-coordinate $x = (b + 4ae)/a$. Now,

$$A = \int_{-(b+2ae)/a}^{e} [(ax^3 + bx^2 + cx + d) - (ae^3 + be^2 + ce + d) - (3ae^2 + 2be + c)(x - e)]\,dx$$

$$= \left\{\frac{ax^4}{4} + \frac{bx^3}{3} - (ae^3 + be^2)x - \frac{(3ae^2 + 2be)(x - e)^2}{2}\right\}_{-(b+2ae)/2}^{e}$$

$$= \frac{27ae^4}{4} + 9be^3 + \frac{b^4}{12a^3} + \frac{b^3e}{a^2} + \frac{9b^2e^2}{2a}.$$

$$B = \int_{-(b+2ae)/a}^{(b+4ae)/a} \Big[-\frac{2b^2e}{a} - 8be^2 - 8ae^3 - \frac{bc}{a} - 2ce + d$$

$$+\left(\frac{b^2}{a}+8be+12ae^2+c\right)\left(x+\frac{b}{a}+2e\right)-(ax^3+bx^2+cx+d)\Big]\,dx$$

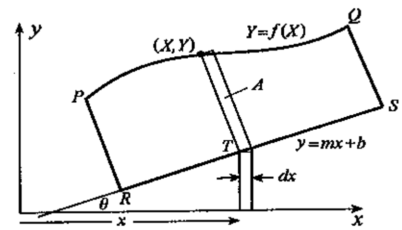
$$=\int_{-(b+2ae)/a}^{(b+4ae)/a}\left[20be^2+16ae^3+\frac{b^3}{a^2}+\frac{8b^2e}{a}+x\left(\frac{b^2}{a}+8be+12ae^2\right)-ax^3-bx^2\right]dx$$

$$=\left\{\left(20be^2+16ae^3+\frac{b^3}{a^2}+\frac{8b^2e}{a}\right)x+\left(\frac{b^2}{a}+8be+12ae^2\right)\frac{x^2}{2}-\frac{ax^4}{4}-\frac{bx^3}{3}\right\}_{-(b+2ae)/a}^{(b+4ae)/a}$$

$$=108ae^4+144be^3+\frac{4b^4}{3a^3}+\frac{16b^3e}{a^2}+\frac{72b^2e^2}{a}=16A.$$

53. The width of the rectangle A shown is $\sec\theta\,dx=\sqrt{1+\tan^2\theta}\,dx=\sqrt{1+m^2}\,dx$. Using formula 1.16, the length of the rectangle is the distance from (X,Y) to $y=mx+b$,

$$\frac{|Y-mX-b|}{\sqrt{1+m^2}}=\frac{|f(X)-mX-b|}{\sqrt{1+m^2}}.$$



Hence the required area is

$$\text{Area }=\int_{x_R}^{x_S}\frac{|f(X)-mX-b|}{\sqrt{1+m^2}}\sqrt{1+m^2}\,dx$$

$$=\int_{x_R}^{x_S}|f(X)-mX-b|\,dx.$$

We now need to relate x and X. Since the slope of the line joining $(X,Y)=(X,f(X))$ and T is $-1/m$,

$$\frac{Y-mx-b}{X-x}=-\frac{1}{m}\quad\Longrightarrow\quad mf(X)-m^2x-bm=x-X\quad\Longrightarrow\quad x=\frac{1}{1+m^2}[X+mf(X)-bm].$$

Instead of trying to solve this for X in terms of x, we treat it as a change of variables in the area integral. Then $dx=\dfrac{1}{1+m^2}[1+mf'(X)]\,dX$. Since $X=x_P$ when $x=x_R$ and $X=x_Q$ when $x=x_S$,

$$\text{Area}=\int_{x_P}^{x_Q}|f(X)-mX-b|\frac{1}{1+m^2}[1+mf'(X)]\,dX.$$

We now replace X with x, and note that $f(X)>mX+b$,

$$\text{Area}=\frac{1}{1+m^2}\int_{x_P}^{x_Q}[f(x)-mx-b][1+mf'(x)]\,dx.$$

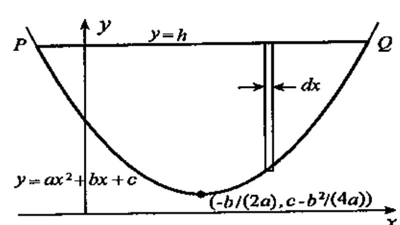
54. If x_P and x_Q are x-coordinates of P and Q, the area is

$$A=\int_{x_P}^{x_Q}(h-ax^2-bx-c)\,dx$$

$$=\left\{(h-c)x-\frac{ax^3}{3}-\frac{bx^2}{2}\right\}_{x_P}^{x_Q}$$

$$=(h-c)(x_Q-x_P)-\frac{a}{3}(x_Q^3-x_P^3)-\frac{b}{2}(x_Q^2-x_P^2)$$

$$=(x_Q-x_P)\left[(h-c)-\frac{a}{3}(x_Q^2+x_Qx_P+x_P^2)-\frac{b}{2}(x_Q+x_P)\right].$$



To find the x-coordinates of P and Q we solve

$$h=ax^2+bx+c\quad\Longrightarrow\quad ax^2+bx+(c-h)=0\quad\Longrightarrow\quad x=\frac{-b\pm\sqrt{b^2-4a(c-h)}}{2a}.$$

Thus, $x_P=\dfrac{-b-\sqrt{b^2-4a(c-h)}}{2a}$ and $x_Q=\dfrac{-b+\sqrt{b^2-4a(c-h)}}{2a}$. Hence,

$$\begin{aligned}x_P^2 + x_P x_Q + x_Q^2 &= \frac{1}{4a^2}[b^2 + 2b\sqrt{b^2 - 4a(c-h)} + b^2 - 4a(c-h) + b^2 - b^2 + 4a(c-h) + b^2 \\ &\qquad - 2b\sqrt{b^2 - 4a(c-h)} + b^2 - 4a(c-h)] \\ &= \frac{1}{a^2}[b^2 - a(c-h)],\end{aligned}$$

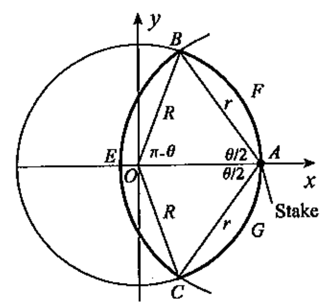
and

$$\begin{aligned}A &= (x_Q - x_P)\left\{(h-c) - \frac{1}{3a}[b^2 - a(c-h)] - \frac{b}{2}\left(-\frac{b}{a}\right)\right\} \\ &= (x_Q - x_P)\left[(h-c) - \frac{b^2}{3a} - \frac{1}{3}(h-c) + \frac{b^2}{2a}\right] \\ &= (x_Q - x_P)\left[\frac{2}{3}(h-c) + \frac{b^2}{6a}\right] = \frac{2}{3}(x_Q - x_P)\left(h - c + \frac{b^2}{4a}\right),\end{aligned}$$

where $h - c + b^2/(4a)$ is the distance from the vertex to the line $y = h$.

55. If the length of the rope is r, then the cow can graze on half the pasture if

$$\begin{aligned}\frac{1}{2}\pi R^2 &= \text{Area } ABF + \text{Area } ACG \\ &\quad + \text{Area } ABEC \\ &= 2(\text{Area } ABF) + \text{Area } ABEC.\end{aligned}$$



Using the formula $R^2(\theta - \sin\theta)/2$ for the area of a sector of a circle subtended by angle θ,

$$\frac{1}{2}\pi R^2 = 2(R^2/2)[\pi - \theta - \sin(\pi - \theta)] + \frac{1}{2}r^2\theta,$$

and this equation can be written in the form

$$r^2\theta = R^2(2\theta + 2\sin\theta - \pi).$$

But from the sine law applied to triangle ABO, $\dfrac{\sin(\pi - \theta)}{r} = \dfrac{\sin(\theta/2)}{R}$, from which

$$r = \frac{R\sin\theta}{\sin(\theta/2)} = \frac{2R\sin(\theta/2)\cos(\theta/2)}{\sin(\theta/2)} = 2R\cos(\theta/2).$$

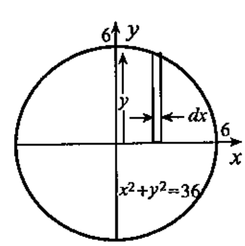
Thus, $4R^2\theta\cos^2(\theta/2) = R^2(2\theta + 2\sin\theta - \pi)$, or,

$$\pi = 2\theta + 2\sin\theta - 4\theta\left(\frac{1+\cos\theta}{2}\right) = 2(\sin\theta - \theta\cos\theta).$$

Newton's interative procedure can be used to solve this equation for $\theta = 1.906$ radians, and therefore $r = 2R\cos(1.906/2) = 1.158R$.

EXERCISES 7.2

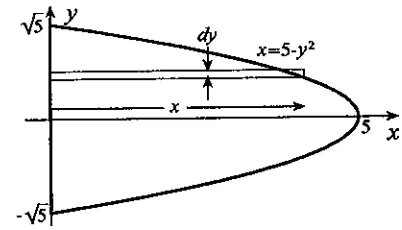
1. $$\begin{aligned}V &= 2\int_0^6 \pi y^2\,dx = 2\pi\int_0^6 (36 - x^2)\,dx \\ &= 2\pi\left\{36x - \frac{x^3}{3}\right\}_0^6 \\ &= 288\pi\end{aligned}$$



2. $V = 2\int_0^{\sqrt{5}} \pi x^2\, dy = 2\pi \int_0^{\sqrt{5}} (5-y^2)^2\, dy$

$$= 2\pi \int_0^{\sqrt{5}} (25 - 10y^2 + y^4)\, dy$$

$$= 2\pi \left\{ 25y - \frac{10y^3}{3} + \frac{y^5}{5} \right\}_0^{\sqrt{5}} = \frac{80\sqrt{5}\pi}{3}$$

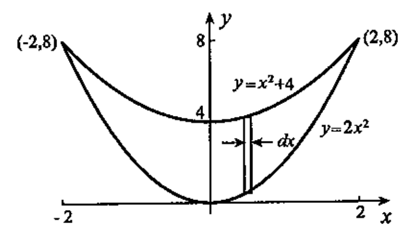


3. $V = 2\int_0^2 [\pi(x^2+4)^2 - \pi(2x^2)^2]\, dx$

$$= 2\pi \int_0^2 (16 + 8x^2 - 3x^4)\, dx$$

$$= 2\pi \left\{ 16x + \frac{8x^3}{3} - \frac{3x^5}{5} \right\}_0^2$$

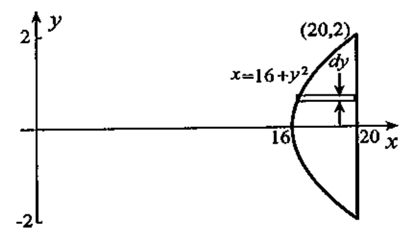
$$= \frac{1024\pi}{15}$$



4. $V = 2\int_0^2 [\pi(20)^2 - \pi(16+y^2)^2]\, dy$

$$= 2\pi \int_0^2 (144 - 32y^2 - y^4)\, dy$$

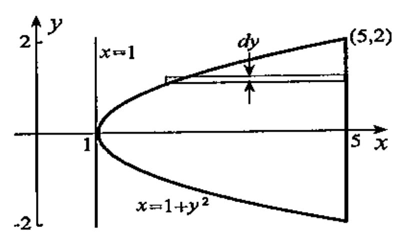
$$= 2\pi \left\{ 144y - \frac{32y^3}{3} - \frac{y^5}{5} \right\}_0^2 = \frac{5888\pi}{15}$$



5. $V = 2\int_0^2 [\pi(5-1)^2 - \pi(1+y^2-1)^2]\, dy$

$$= 2\pi \int_0^2 (16 - y^4)\, dy$$

$$= 2\pi \left\{ 16y - \frac{y^5}{5} \right\}_0^2 = \frac{256\pi}{5}$$

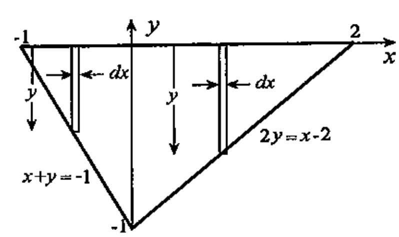


6. $V = \int_{-1}^0 \pi(-y)^2\, dx + \int_0^2 \pi(-y)^2\, dx$

$$= \pi \int_{-1}^0 (1+x)^2\, dx + \pi \int_0^2 \frac{1}{4}(2-x)^2\, dx$$

$$= \pi \left\{ \frac{1}{3}(1+x)^3 \right\}_{-1}^0 + \pi \left\{ -\frac{1}{12}(2-x)^3 \right\}_0^2$$

$$= \pi$$

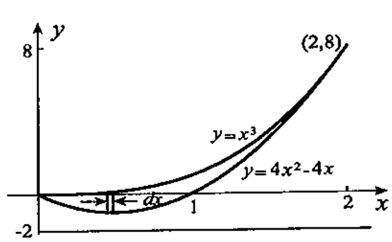


7. $V = \int_0^2 [\pi(x^3+2)^2 - \pi(4x^2-4x+2)^2]\, dx$

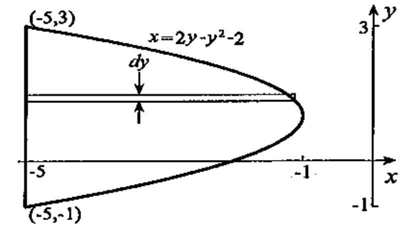
$$= \pi \int_0^2 (x^6 - 16x^4 + 36x^3 - 32x^2 + 16x)\, dx$$

$$= \pi \left\{ \frac{x^7}{7} - \frac{16x^5}{5} + 9x^4 - \frac{32x^3}{3} + 8x^2 \right\}_0^2$$

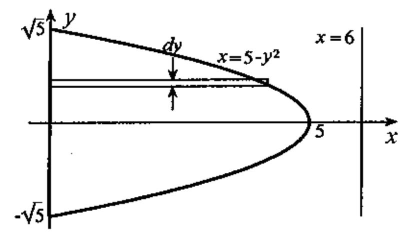
$$= \frac{688\pi}{105}$$



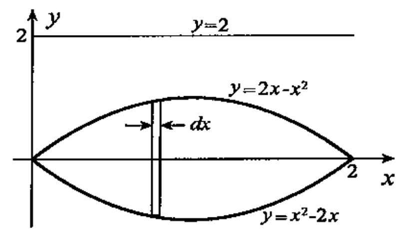
8. $$V = \int_{-1}^{3} [\pi(5)^2 - \pi(-2y + y^2 + 2)^2]\,dy$$
$$= \pi \int_{-1}^{3} (21 + 8y - 8y^2 + 4y^3 - y^4)\,dy$$
$$= \pi \left\{ 21y + 4y^2 - \frac{8y^3}{3} + y^4 - \frac{y^5}{5} \right\}_{-1}^{3}$$
$$= \frac{1088\pi}{15}$$



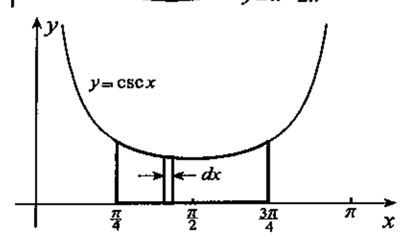
9. $$V = 2\int_{0}^{\sqrt{5}} [\pi(6)^2 - \pi(6 - 5 + y^2)^2]\,dy$$
$$= 2\pi \int_{0}^{\sqrt{5}} (35 - 2y^2 - y^4)\,dy$$
$$= 2\pi \left\{ 35y - \frac{2y^3}{3} - \frac{y^5}{5} \right\}_{0}^{\sqrt{5}}$$
$$= \frac{160\sqrt{5}\pi}{3}$$



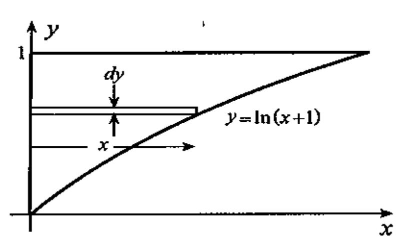
10. $$V = \int_{0}^{2} [\pi(2 - x^2 + 2x)^2 - \pi(2 - 2x + x^2)^2]\,dx$$
$$= 8\pi \int_{0}^{2} (2x - x^2)\,dx$$
$$= 8\pi \left\{ x^2 - \frac{x^3}{3} \right\}_{0}^{2} = \frac{32\pi}{3}$$



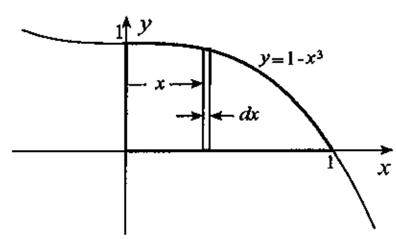
11. $$V = 2\int_{\pi/4}^{\pi/2} \pi \csc^2 x\,dx$$
$$= 2\pi \Big\{ -\cot x \Big\}_{\pi/4}^{\pi/2}$$
$$= 2\pi$$



12. $$V = \int_{0}^{1} \pi x^2\,dy = \pi \int_{0}^{1} (e^y - 1)^2\,dy$$
$$= \pi \int_{0}^{1} (e^{2y} - 2e^y + 1)\,dy$$
$$= \pi \left\{ \frac{e^{2y}}{2} - 2e^y + y \right\}_{0}^{1}$$
$$= \pi \left(\frac{e^2}{2} - 2e + 1 - \frac{1}{2} + 2 \right)$$
$$= \frac{\pi}{2}(e^2 - 4e + 5)$$



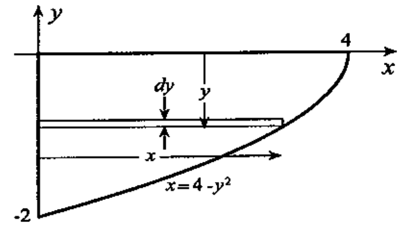
13. $$V = \int_{0}^{1} 2\pi x(1 - x^3)\,dx$$
$$= 2\pi \left\{ \frac{x^2}{2} - \frac{x^5}{5} \right\}_{0}^{1}$$
$$= \frac{3\pi}{5}$$



14. $V = \int_{-2}^{0} 2\pi(-y)x\,dy$

$= -2\pi \int_{-2}^{0} y(4-y^2)\,dy$

$= -2\pi \left\{ 2y^2 - \frac{y^4}{4} \right\}_{-2}^{0} = 8\pi$

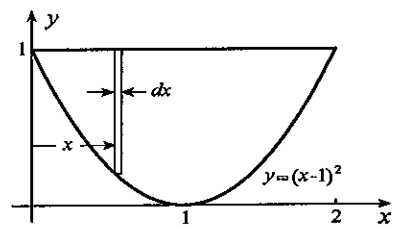


15. $V = \int_{0}^{2} 2\pi x[1-(x-1)^2]\,dx$

$= 2\pi \int_{0}^{2} (2x^2 - x^3)\,dx$

$= 2\pi \left\{ \frac{2x^3}{3} - \frac{x^4}{4} \right\}_{0}^{2}$

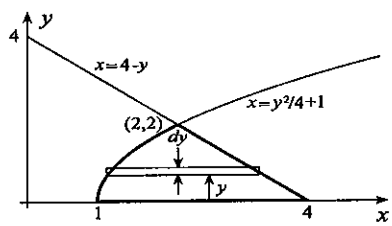
$= \frac{8\pi}{3}$



16. $V = \int_{0}^{2} 2\pi y \left(4 - y - \frac{y^2}{4} - 1 \right) dy$

$= \frac{\pi}{2} \int_{0}^{2} (-y^3 - 4y^2 + 12y)\,dy$

$= \frac{\pi}{2} \left\{ -\frac{y^4}{4} - \frac{4y^3}{3} + 6y^2 \right\}_{0}^{2} = \frac{14\pi}{3}$

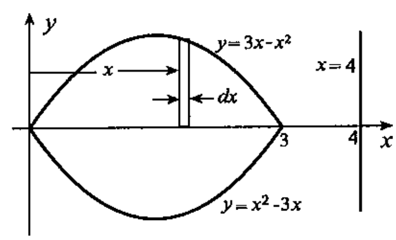


17. $V = 2\int_{0}^{3} 2\pi(4-x)(3x-x^2)\,dx$

$= 4\pi \int_{0}^{3} (12x - 7x^2 + x^3)\,dx$

$= 4\pi \left\{ 6x^2 - \frac{7x^3}{3} + \frac{x^4}{4} \right\}_{0}^{3}$

$= 45\pi$

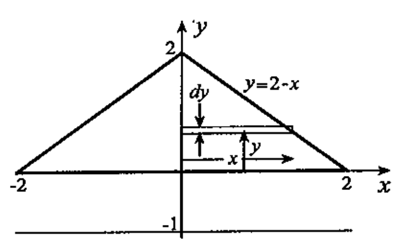


18. $V = 2\int_{0}^{2} 2\pi(y+1)x\,dy$

$= 4\pi \int_{0}^{2} (y+1)(2-y)\,dy$

$= 4\pi \int_{0}^{2} (2 + y - y^2)\,dy$

$= 4\pi \left\{ 2y + \frac{y^2}{2} - \frac{y^3}{3} \right\}_{0}^{2} = \frac{40\pi}{3}$

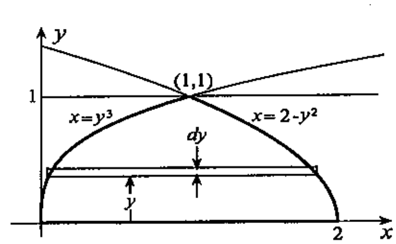


19. $V = \int_{0}^{1} 2\pi(1-y)(2-y^2-y^3)\,dy$

$= 2\pi \int_{0}^{1} (2 - 2y - y^2 + y^4)\,dy$

$= 2\pi \left\{ 2y - y^2 - \frac{y^3}{3} + \frac{y^5}{5} \right\}_{0}^{1}$

$= \frac{26\pi}{15}$

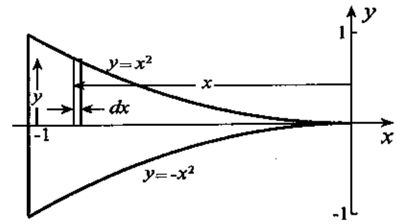


20. $V = 2\int_{-1}^{0} 2\pi(x+1)y\,dx$

$= 4\pi \int_{-1}^{0} (x+1)x^2\,dx$

$= 4\pi \int_{-1}^{0} (x^3 + x^2)\,dx$

$= 4\pi \left\{ \frac{x^4}{4} + \frac{x^3}{3} \right\}_{-1}^{0} = \frac{\pi}{3}$

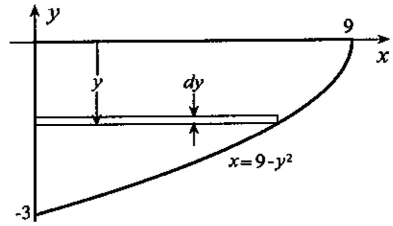


21. $V = \int_{-3}^{0} 2\pi(-y)(9-y^2)\,dy$

$= 2\pi \int_{-3}^{0} (y^3 - 9y)\,dy$

$= 2\pi \left\{ \frac{y^4}{4} - \frac{9y^2}{2} \right\}_{-3}^{0}$

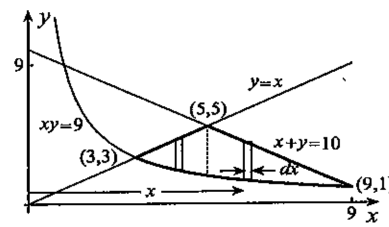
$= \frac{81\pi}{2}$



22. $V = \int_{3}^{5} 2\pi x \left(x - \frac{9}{x} \right) dx + \int_{5}^{9} 2\pi x \left(10 - x - \frac{9}{x} \right) dx$

$= 2\pi \left\{ \frac{x^3}{3} - 9x \right\}_{3}^{5} + 2\pi \left\{ 5x^2 - \frac{x^3}{3} - 9x \right\}_{5}^{9}$

$= \frac{344\pi}{3}$

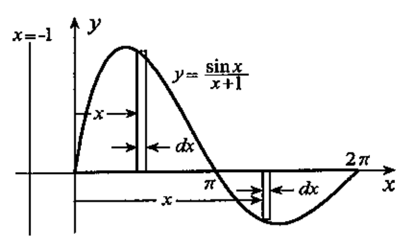


23. $V = \int_{0}^{\pi} 2\pi(x+1) \left(\frac{\sin x}{x+1} \right) dx$

$+ \int_{\pi}^{2\pi} 2\pi(x+1) \left(-\frac{\sin x}{x+1} \right) dx$

$= 2\pi \Big\{ -\cos x \Big\}_{0}^{\pi} + 2\pi \Big\{ \cos x \Big\}_{\pi}^{2\pi}$

$= 8\pi$



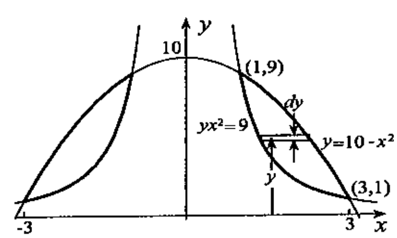
24. $V = 2\int_{1}^{9} 2\pi y \left(\sqrt{10-y} - \frac{3}{\sqrt{y}} \right) dy$

If we set $u = 10 - y$ and $du = -dy$ in the first term,

$V = 4\pi \int_{9}^{1} (10-u)\sqrt{u}(-du) + 4\pi\{-2y^{3/2}\}_{1}^{9}$

$= 4\pi \left\{ \frac{2}{5}u^{5/2} - \frac{20}{3}u^{3/2} \right\}_{9}^{1} - 8\pi(27-1)$

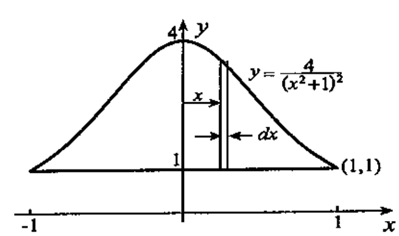
$= \frac{1472\pi}{15}.$



25. $V = \int_{0}^{1} 2\pi x \left[\frac{4}{(x^2+1)^2} - 1 \right] dx$

$= 2\pi \left\{ \frac{-2}{x^2+1} - \frac{x^2}{2} \right\}_{0}^{1}$

$= \pi$

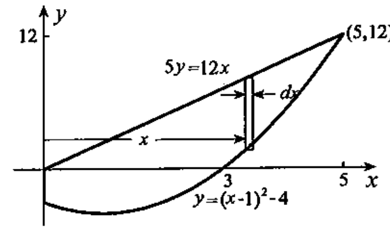


26. $V = \int_0^5 2\pi x \left[\frac{12x}{5} - (x-1)^2 + 4\right] dx$

$= \frac{2\pi}{5}\int_0^5 (-5x^3 + 22x^2 + 15x)\, dx$

$= \frac{2\pi}{5}\left\{-\frac{5x^4}{4} + \frac{22x^3}{3} + \frac{15x^2}{2}\right\}_0^5$

$= \frac{775\pi}{6}$



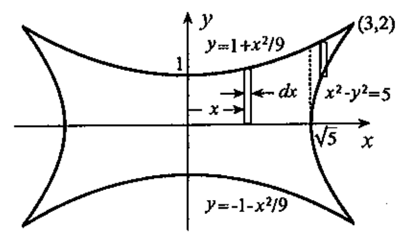
27. $V = 2\int_0^{\sqrt{5}} 2\pi x \left(1 + \frac{x^2}{9}\right) dx$

$+2\int_{\sqrt{5}}^3 2\pi x \left(1 + \frac{x^2}{9} - \sqrt{x^2-5}\right) dx$

$= 4\pi\left\{\frac{x^2}{2} + \frac{x^4}{36}\right\}_0^{\sqrt{5}}$

$+4\pi\left\{\frac{x^2}{2} + \frac{x^4}{36} - \frac{1}{3}(x^2-5)^{3/2}\right\}_{\sqrt{5}}^3$

$= \frac{49\pi}{3}$



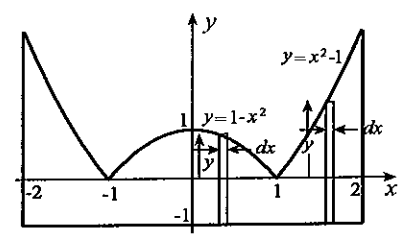
28. $V = 2\int_0^1 \pi(y+1)^2\, dx + 2\int_1^2 \pi(y+1)^2\, dx$

$= 2\pi\int_0^1 (1 - x^2 + 1)^2\, dx + 2\pi\int_1^2 (x^2 - 1 + 1)^2\, dx$

$= 2\pi\int_0^1 (4 - 4x^2 + x^4)\, dx + 2\pi\int_1^2 x^4\, dx$

$= 2\pi\left\{4x - \frac{4x^3}{3} + \frac{x^5}{5}\right\}_0^1 + 2\pi\left\{\frac{x^5}{5}\right\}_1^2$

$= \frac{272\pi}{15}$

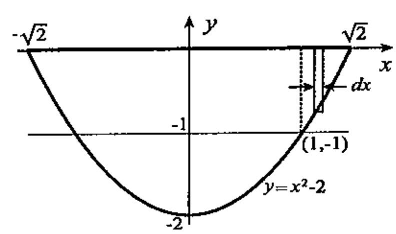


29. To eliminate duplications, we consider only the area above $y = -1$.

$V = 2\pi(1)^2(1) + 2\int_1^{\sqrt{2}} [\pi(1)^2 - \pi(x^2 - 2 + 1)^2]\, dx$

$= 2\pi + 2\pi\int_1^{\sqrt{2}} (2x^2 - x^4)\, dx$

$= 2\pi + 2\pi\left\{\frac{2x^3}{3} - \frac{x^5}{5}\right\}_1^{\sqrt{2}} = \frac{16\pi(\sqrt{2}+1)}{15}$

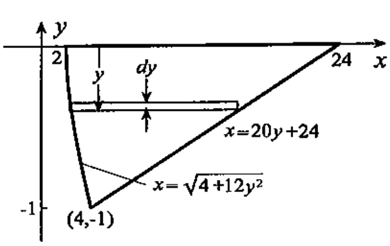


30. $V = \int_{-1}^0 2\pi(-y)(20y + 24 - \sqrt{4+12y^2})\, dy$

$= -2\pi\int_{-1}^0 \left(20y^2 + 24y - y\sqrt{4+12y^2}\right) dy$

$= -2\pi\left\{\frac{20y^3}{3} + 12y^2 - \frac{1}{36}(4+12y^2)^{3/2}\right\}_{-1}^0$

$= \frac{68\pi}{9}$



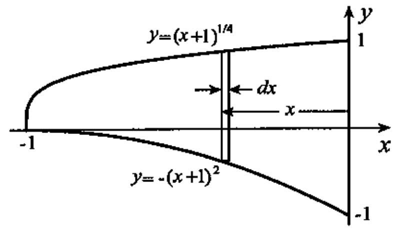
31. $V = \int_{-1}^{0} 2\pi(-x)[(x+1)^{1/4} + (x+1)^2]\,dx$

If we set $u = x+1$ and $du = dx$ in the term involving $(x+1)^{1/4}$,

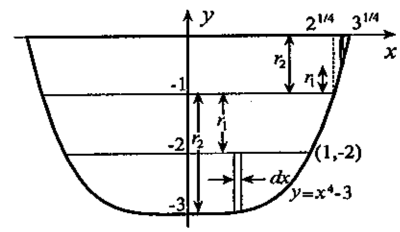
$$V = -2\pi \int_0^1 (u-1)u^{1/4}\,du - 2\pi \int_{-1}^{0} (x^3 + 2x^2 + x)\,dx$$

$$= -2\pi \left\{ \frac{4u^{9/4}}{9} - \frac{4u^{5/4}}{5} \right\}_0^1 - 2\pi \left\{ \frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} \right\}_{-1}^{0}$$

$$= \frac{79\pi}{90}$$



32. To eliminate duplications we reject that part of the area between $y = -2$ and $y = -1$. We rotate that part of the area to the right of the y-axis and double the result. The rectangle $0 \le x \le 2^{1/4}$, $-1 \le y \le 0$ gives a cylinder, and the other two parts of the area require integrations,



$$V = 2[\pi(1)^2(2^{1/4})] + 2\int_{2^{1/4}}^{3^{1/4}} (\pi r_2^2 - \pi r_1^2)\,dx + 2\int_0^1 (\pi r_2^2 - \pi r_1^2)\,dx$$

$$= 2^{5/4}\pi + 2\pi \int_{2^{1/4}}^{3^{1/4}} [(1)^2 - (x^4 - 3 + 1)^2]\,dx + 2\pi \int_0^1 [(-1 - x^4 + 3)^2 - 1^2]\,dx$$

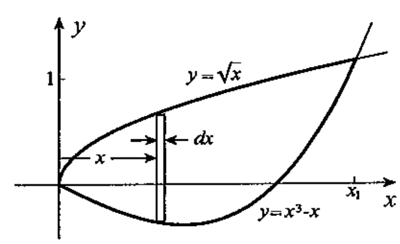
$$= 2^{5/4}\pi + 2\pi \int_{2^{1/4}}^{3^{1/4}} (-3 + 4x^4 - x^8)\,dx + 2\pi \int_0^1 (3 - 4x^4 + x^8)\,dx$$

$$= 2^{5/4}\pi + 2\pi \left\{ -3x + \frac{4x^5}{5} - \frac{x^9}{9} \right\}_{2^{1/4}}^{3^{1/4}} + 2\pi \left\{ 3x - \frac{4x^5}{5} + \frac{x^9}{9} \right\}_0^1 = \frac{16\pi}{45}(13 + 2^{17/4} - 3^{9/4}).$$

33. Newton's method can be used to solve the equation $x^3 - x = \sqrt{x}$ for the x-coordinate of the point of intersection of the curves. The result is $x_1 = 1.362\,599$. The volume of the solid of revolution is

$$V = \int_0^{x_1} 2\pi x(\sqrt{x} - x^3 + x)\,dx$$

$$= 2\pi \left\{ \frac{2x^{5/2}}{5} - \frac{x^5}{5} + \frac{x^3}{3} \right\}_0^{x_1} = 4.843.$$



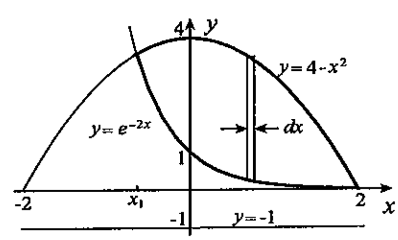
34. Newton's method can be used to solve the equation $e^{-2x} + x^2 - 4 = 0$ for x-coordinates of the points of intersection of the two curves. The results are $x_1 = -0.639\,263$ and $x_2 = 1.995\,373$. The volume of the solid of revolution is

$$V = \int_{x_1}^{x_2} [\pi(4 - x^2 + 1)^2 - \pi(e^{-2x} + 1)^2]\,dx$$

$$= \pi \int_{x_1}^{x_2} (24 - 10x^2 + x^4 - e^{-4x} - 2e^{-2x})\,dx$$

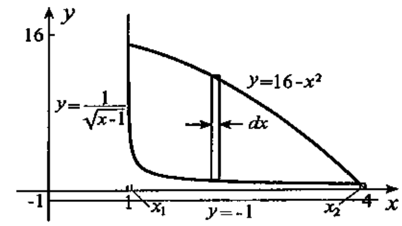
$$= \pi \left\{ 24x - \frac{10x^3}{3} + \frac{x^5}{5} + \frac{e^{-4x}}{4} + e^{-2x} \right\}_{x_1}^{x_2}$$

$$= 111.303.$$



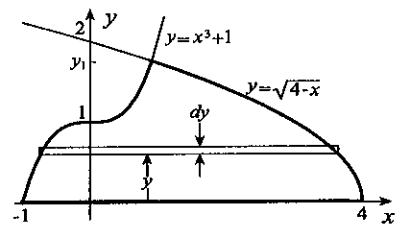
35. Newton's method can be used to solve the equation $16 - x^2 = 1/\sqrt{x-1}$ for x-coordinates of the points of intersection of the two curves. The results are $x_1 = 1.004\,450$ and $x_2 = 3.926\,248$. The volume of the solid of revolution is

$$\begin{aligned} V &\int_{x_1}^{x_2} \left[\pi(16 - x^2 + 1)^2 - \pi\left(\frac{1}{\sqrt{x-1}} + 1\right)^2\right] dx \\ &= \pi \int_{x_1}^{x_2} \left(288 - 34x^2 + x^4 - \frac{1}{x-1} - \frac{2}{\sqrt{x-1}}\right) dx \\ &= \pi \left\{288x - \frac{34x^3}{3} + \frac{x^5}{5} - \ln|x-1| - 4\sqrt{x-1}\right\}_{x_1}^{x_2} \\ &= 1069.241. \end{aligned}$$



36. The y-coordinate of the point of intersection of the curves can be obtained by solving the equation $y = (4 - y^2)^3 + 1$. Newton's iterative procedure leads to the solution $y_1 = 1.757\,401\,58$. The volume of the solid of revolution is

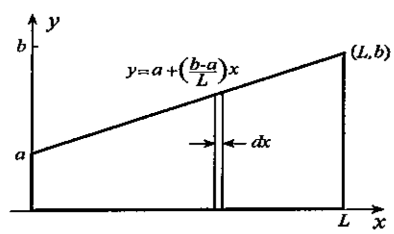
$$\begin{aligned} V &= \int_0^{y_1} 2\pi y[(4 - y^2) - (y-1)^{1/3}]\, dy \\ &= 2\pi \int_0^{y_1} [4y - y^3 - y(y-1)^{1/3}]\, dy. \end{aligned}$$



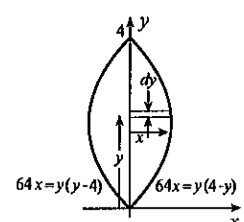
We set $u = y - 1$ and $du = dy$ in the last term,

$$\begin{aligned} V &= 2\pi\left\{2y^2 - \frac{y^4}{4}\right\}_0^{y_1} - 2\pi\int_{-1}^{y_1-1}(u+1)u^{1/3}\, du = 2\pi\left\{2y^2 - \frac{y^4}{4}\right\}_0^{y_1} - 2\pi\left\{\frac{3}{7}u^{7/3} + \frac{3}{4}u^{4/3}\right\}_{-1}^{y_1-1} \\ &= 2\pi\left\{2y^2 - \frac{y^4}{4} - \frac{3}{7}(y-1)^{7/3} - \frac{3}{4}(y-1)^{4/3}\right\}_0^{y_1} = 21.186. \end{aligned}$$

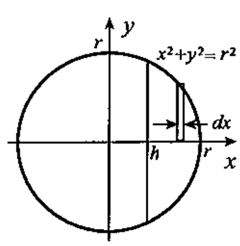
37.
$$\begin{aligned} V &= \int_0^L \pi\left[a + \left(\frac{b-a}{L}\right)x\right]^2 dx \\ &= \pi\left\{\frac{L}{3(b-a)}\left[a + \left(\frac{b-a}{L}\right)x\right]^3\right\}_0^L \\ &= \frac{\pi L(a^2 + ab + b^2)}{3} \end{aligned}$$



38.
$$\begin{aligned} V &= 2\int_0^4 2\pi yx\, dy = 4\pi\int_0^4 y\left[\frac{y(4-y)}{64}\right] dy \\ &= \frac{\pi}{16}\left\{\frac{4y^3}{3} - \frac{y^4}{4}\right\}_0^4 = \frac{4\pi}{3} \end{aligned}$$



39. (a)
$$\begin{aligned} V &= \int_h^r \pi(r^2 - x^2)\, dx \\ &= \pi\left\{r^2x - \frac{x^3}{3}\right\}_h^r \\ &= \pi\left(r^3 - \frac{r^3}{3} - r^2h + \frac{h^3}{3}\right) \\ &= \frac{\pi}{3}(r-h)^2(2r+h) \end{aligned}$$



(b) For the volume in part (a) to be one-third that of the sphere,

$$\frac{\pi}{3}(r-h)^2(2r+h)=\frac{1}{3}\left(\frac{4}{3}\pi r^3\right) \implies 3h^3-9r^2h+2r^3=0 \implies 3\left(\frac{h}{r}\right)^3-9\left(\frac{h}{r}\right)+2=0.$$

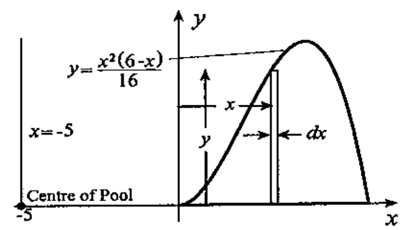
The ratio $z=h/r$ must satisfy the cubic $3z^3-9z+2=0$. Newton's iterative procedure yields the solution $z=0.2261$.

40. (a) For the points $(0,0)$, $(4,2)$, and $(6,0)$ to be on the cubic, a, b, c, and d must satisfy

$$0=d, \quad 2=64a+16b+4c+d, \quad 0=216a+36b+6c+d.$$

In addition, because $y'(4)=0$, we must have $0=48a+8b+c$. These four equations imply that $a=-1/16$, $b=3/8$, $c=d=0$, and therefore $y=-x^3/16+3x^2/8=x^2(6-x)/16$.
(b) The amount of fill required is

$$\begin{aligned}V&=\int_0^6 2\pi(x+5)y\,dx\\&=2\pi\int_0^6(x+5)\frac{x^2(6-x)}{16}dx\\&=\frac{\pi}{8}\int_0^6(30x^2+x^3-x^4)\,dx\\&=\frac{\pi}{8}\left\{10x^3+\frac{x^4}{4}-\frac{x^5}{5}\right\}_0^6=\frac{1161\pi}{10}\ \text{m}^3.\end{aligned}$$

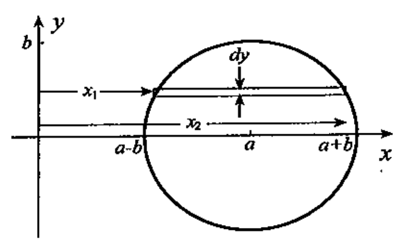


41. With horizontal rectangles, washers give

$$V=2\int_0^b(\pi x_2^2-\pi x_1^2)\,dy.$$

By solving the equation $(x-a)^2+y^2=b^2$ for x, we obtain $x_2=a+\sqrt{b^2-y^2}$ and $x_1=a-\sqrt{b^2-y^2}$. Thus,

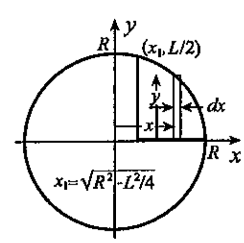
$$\begin{aligned}V&=2\pi\int_0^b[(a+\sqrt{b^2-y^2})^2-(a-\sqrt{b^2-y^2})^2]\,dy\\&=2\pi\int_0^b 4a\sqrt{b^2-y^2}\,dy=8\pi a\int_0^b\sqrt{b^2-y^2}\,dy\end{aligned}$$



This integral, less the constant out front, is the area of one-quarter of the circle. It follows then that $V=8\pi a(1/4)\pi(b^2)=2\pi^2ab^2$.

42. The volume remaining is

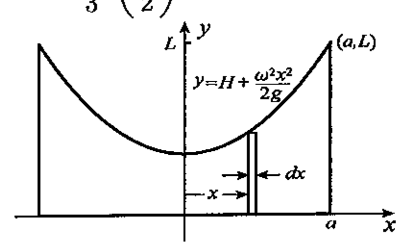
$$\begin{aligned}V&=2\int_{x_1}^R 2\pi xy\,dx=4\pi\int_{x_1}^R x\sqrt{R^2-x^2}\,dx\\&=4\pi\left\{-\frac{1}{3}(R^2-x^2)^{3/2}\right\}_{x_1}^R\\&=\frac{4\pi}{3}(R^2-x_1^2)^{3/2}.\end{aligned}$$



Since $x_1=\sqrt{R^2-L^2/4}$, the volume is $V=\dfrac{4\pi}{3}\left[R^2-R^2+\dfrac{L^2}{4}\right]^{3/2}=\dfrac{4\pi}{3}\left(\dfrac{L}{2}\right)^3$, and this is the volume of a sphere with radius $L/2$.

43. The volume occupied by the water as it reaches the top is

$$\begin{aligned}V&=\int_0^a 2\pi x\left(H+\frac{\omega^2x^2}{2g}\right)dx=2\pi\left\{\frac{Hx^2}{2}+\frac{\omega^2x^4}{8g}\right\}_0^a\\&=\pi a^2\left(H+\frac{\omega^2a^2}{4g}\right).\end{aligned}$$



Because the volume of water in the pail does not change, and it was half full originally,

$$\pi a^2\left(\frac{L}{2}\right) = \pi a^2\left(H + \frac{\omega^2 a^2}{4g}\right) \quad \Longrightarrow \quad L = 2H + \frac{\omega^2 a^2}{2g}.$$

When the water just reaches the top, the point (a, L) is on the parabola, and therefore

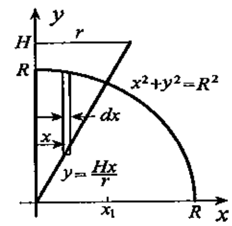
$$L = H + \frac{\omega^2 a^2}{2g} \quad \Longrightarrow \quad H = L - \frac{\omega^2 a^2}{2g}.$$

Thus, $L = 2\left(L - \dfrac{\omega^2 a^2}{2g}\right) + \dfrac{\omega^2 a^2}{2g}$, and when this equation is solved for ω, the result is $\omega = \sqrt{2gL}/a$.

44. The volume can be obtained by rotating the area bounded by the circle $x^2 + y^2 = R^2$, the line $y = Hx/r$, and the y-axis about the y-axis. The x-coordinate of the point of intersection of the line and circle, call it x_1, must satisfy

$$x^2 + \frac{H^2 x^2}{r^2} = R^2.$$

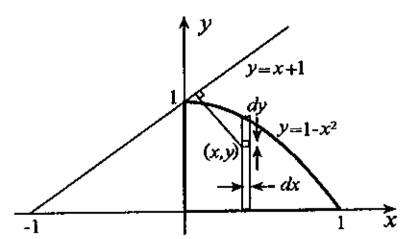
The solution of this equation is $x_1 = rR/\sqrt{r^2 + H^2}$.



The required volume is

$$\begin{aligned}
V &= \int_0^{x_1} 2\pi x\left(\sqrt{R^2 - x^2} - \frac{Hx}{r}\right)dx = 2\pi\left\{-\frac{1}{3}(R^2 - x^2)^{3/2} - \frac{H}{r}\frac{x^3}{3}\right\}_0^{x_1} \\
&= -\frac{2\pi}{3}\left[(R^2 - x_1^2)^{3/2} + \frac{Hx_1^3}{r} - R^3\right] = -\frac{2\pi}{3}\left[\left(R^2 - \frac{r^2R^2}{r^2 + H^2}\right)^{3/2} + \frac{H}{r}\frac{r^3R^3}{(r^2 + H^2)^{3/2}} - R^3\right] \\
&= -\frac{2\pi}{3}\left[\frac{R^3H^3}{(r^2 + H^2)^{3/2}} + \frac{Hr^2R^3}{(r^2 + H^2)^{3/2}} - R^3\right] = \frac{2\pi R^3}{3}\left[1 - \frac{H(r^2 + H^2)}{(r^2 + H^2)^{3/2}}\right] \\
&= \frac{2\pi R^3}{3}\left[1 - \frac{H}{\sqrt{r^2 + H^2}}\right].
\end{aligned}$$

45. If we divide the vertical rectangle of width dx into smaller rectangles of width dy, the perpendicular distance from this small rectangle to the line $y = x + 1$ is given by distance formula 1.16 as $|x - y + 1|/\sqrt{2} = (x - y + 1)/\sqrt{2}$. When the small rectangle is rotated around $y = x + 1$, it produces a ring with volume approximately equal to $2\pi[(x - y + 1)/\sqrt{2}]\,dx\,dy$. By adding over all the small rectangles in the vertical rectangle, we obtain the volume resulting from rotating the vertical rectangle around the line as

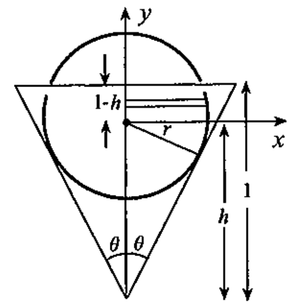


$$\int_0^{1-x^2} \frac{2\pi(x - y + 1)}{\sqrt{2}}dx\,dy = \sqrt{2}\,\pi\,dx\left\{-\frac{(x - y + 1)^2}{2}\right\}_0^{1-x^2} = \frac{\pi}{\sqrt{2}}(1 + 2x - 2x^3 - x^4)\,dx.$$

The volume of the solid of revolution is therefore

$$V = \int_0^1 \frac{\pi}{\sqrt{2}}(1 + 2x - 2x^3 - x^4)\,dx = \frac{\pi}{\sqrt{2}}\left\{x + x^2 - \frac{x^4}{2} - \frac{x^5}{5}\right\}_0^1 = \frac{13\sqrt{2}\pi}{20}.$$

46. Let h be the height of the centre of a sphere lying within the cone above the vertex of the cone. The resulting radius of the sphere is $r = h\sin\theta$. We now calculate the volume of the sphere inside the cone using volumes of solids of revolution,

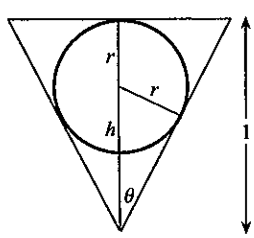


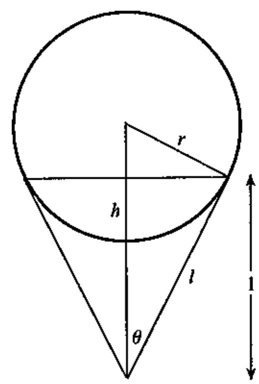
$$\begin{aligned} V &= \int_{-r}^{1-h} \pi(r^2 - y^2)\,dy = \pi\left\{r^2 y - \frac{y^3}{3}\right\}_{-r}^{1-h} \\ &= \pi\left[r^2(1-h) - \frac{1}{3}(1-h)^3 + r^3 - \frac{r^3}{3}\right]. \end{aligned}$$

When we substitute $r = h\sin\theta$, we obtain the function to be maximized

$$V(h) = \pi\left[h^2\sin^2\theta(1-h) - \frac{1}{3}(1-h)^3 + \frac{2h^3}{3}\sin^3\theta\right].$$

Certainly h must be nonnegative, but we can do better for a lower bound. Once the sphere is completely within the cone (left figure below), the volume decreases as h decreases. Consequently, the smallest value of h to use is that when the top of the sphere is level with the top of the cone. This occurs when $h + r = 1$ so that $h = 1 - r = 1 - h\sin\theta \Longrightarrow h = 1/(1+\sin\theta)$. The maximum value of h occurs when the sphere is tangent to the cone at its top edge (right figure below). For this situation $h = l\sec\theta = \sec^2\theta$. Thus, the domain of $V(h)$ is $1/(1+\sin\theta) \le h \le \sec^2\theta$.





For critical points of $V(h)$, we solve

$$0 = V'(h) = \pi[\sin^2\theta(2h - 3h^2) + (1-h)^2 + 2h^2\sin^3\theta].$$

This implies that

$$h^2(1 - 3\sin^2\theta + 2\sin^3\theta) + h(2\sin^2\theta - 2) + 1 = 0.$$

Solutions of this quadratic are

$$h = \frac{2 - 2\sin^2\theta \pm \sqrt{(2\sin^2\theta - 2)^2 - 4(1 - 3\sin^2\theta + 2\sin^3\theta)}}{2(1 - 3\sin^2\theta + 2\sin^3\theta)}.$$

Now,

$$\begin{aligned} (2\sin^2\theta - 2)^2 - 4(1 - 3\sin^2\theta + 2\sin^3\theta) &= 4\sin^4\theta - 8\sin^2\theta + 4 - 4 + 12\sin^2\theta - 8\sin^3\theta \\ &= 4\sin^2\theta(\sin^2\theta - 2\sin\theta + 1) \\ &= 4\sin^2\theta(1 - \sin\theta)^2, \end{aligned}$$

and

$$\begin{aligned}2\sin^3\theta - 3\sin^2\theta + 1 &= (\sin\theta - 1)(2\sin^2\theta - \sin\theta - 1)\\ &= (1-\sin\theta)(1+\sin\theta - 2\sin^2\theta)\\ &= (1-\sin\theta)(1-\sin\theta)(1+2\sin\theta).\end{aligned}$$

Thus, critical points are

$$\begin{aligned}h &= \frac{2(1-\sin\theta)(1+\sin\theta) \pm 2\sin\theta(1-\sin\theta)}{2(1-\sin\theta)^2(1+2\sin\theta)}\\ &= \frac{1}{1-\sin\theta}, \quad \frac{1}{(1-\sin\theta)(1+2\sin\theta)}.\end{aligned}$$

Since $\dfrac{1}{1-\sin\theta} > \dfrac{1}{1-\sin^2\theta} = \dfrac{1}{\cos^2\theta} = \sec^2\theta$, the first critical point is not in the domain of $V(h)$. The second critical point, call it $\tilde{h}$, is in the domain of $V(h)$ because

$$\frac{1}{1+\sin\theta} \le \frac{1}{1+\sin\theta - 2\sin^2\theta} = \frac{1}{(1-\sin\theta)(1+2\sin\theta)} \le \frac{1}{(1-\sin\theta)(1+\sin\theta)} = \frac{1}{\cos^2\theta} = \sec^2\theta.$$

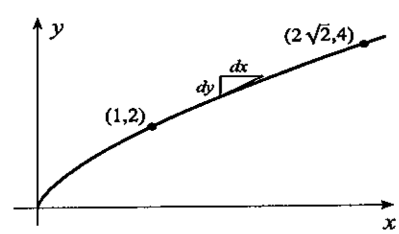
To verify that $\tilde{h}$ gives an absolute maximum, we could evaluate V at $\tilde{h}$ and at the end points of its domain of definition. It is simpler, however, to notice that $V(h)$ is a cubic polynomial in h, with two critical points, $\tilde{h}$ being the smaller one. If the coefficient of h^3 in $V(h)$ is positive, then $\tilde{h}$ must yield a relative maximum that is also the absolute maximum. The coefficient of h^3 is

$$\pi\left(-\sin^2\theta + \frac{1}{3} + \frac{2}{3}\sin^3\theta\right) = \frac{\pi}{3}(2\sin^3\theta - 3\sin^2\theta + 1) = \frac{\pi}{3}(1-\sin\theta)^2(1+2\sin\theta) > 0.$$

Thus, $\tilde{h}$ maximizes V, and the radius of the sphere is $r = \tilde{h}\sin\theta = \dfrac{\sin\theta}{(1-\sin\theta)(1+2\sin\theta)}$.

EXERCISES 7.3

1. Small lengths along the curve are approximated by

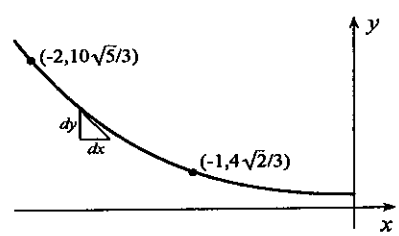


$$\begin{aligned}\sqrt{(dx)^2 + (dy)^2} &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx\\ &= \sqrt{1 + \left(\frac{4x^{-1/3}}{3}\right)^2}\,dx = \frac{\sqrt{9x^{2/3}+16}}{3x^{1/3}}\,dx.\end{aligned}$$

Total length of the curve is

$$L = \int_1^{2\sqrt{2}} \frac{\sqrt{9x^{2/3}+16}}{3x^{1/3}}\,dx = \frac{1}{3}\left\{\frac{(9x^{2/3}+16)^{3/2}}{9}\right\}_1^{2\sqrt{2}} = \frac{34\sqrt{34} - 125}{27}.$$

2. Small lengths along the curve are approximated by

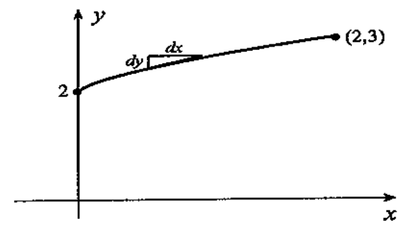


$$\begin{aligned}\sqrt{(dx)^2 + (dy)^2} &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx\\ &= \sqrt{1 + [2x(x^2+1)^{1/2}]^2}\,dx = \sqrt{1 + 4x^2(x^2+1)}\,dx\\ &= (2x^2+1)\,dx.\end{aligned}$$

Total length of the curve is

$$L = \int_{-2}^{-1} (2x^2+1)\,dx = \left\{\frac{2x^3}{3} + x\right\}_{-2}^{-1} = \frac{17}{3}.$$

3. Small lengths along the curve are approximated by

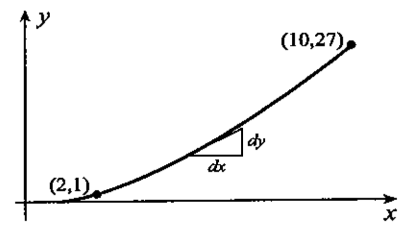


$$\sqrt{(dx)^2+(dy)^2}=\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$
$$=\sqrt{1+(3\sqrt{y-2})^2}dy=\sqrt{9y-17}\,dy.$$

Total length of the curve is

$$L=\int_2^3\sqrt{9y-17}\,dy=\left\{\frac{2}{27}(9y-17)^{3/2}\right\}_2^3=\frac{2(10\sqrt{10}-1)}{27}.$$

4. We approximate small lengths along the curve by



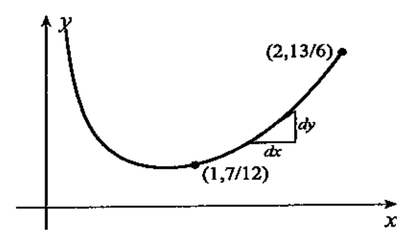
$$\sqrt{(dx)^2+(dy)^2}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$=\sqrt{1+\left(\frac{3}{2}\sqrt{x-1}\right)^2}\,dx=\frac{1}{2}\sqrt{9x-5}\,dx.$$

Total length of the curve is

$$L=\int_2^{10}\frac{1}{2}\sqrt{9x-5}\,dx=\frac{1}{2}\left\{\frac{2}{27}(9x-5)^{3/2}\right\}_2^{10}=\frac{85^{3/2}-13^{3/2}}{27}.$$

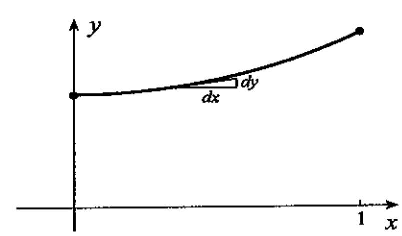
5. Small lengths along the curve are approximated by

$$\sqrt{(dx)^2+(dy)^2}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$=\sqrt{1+\left(\frac{3x^2}{4}-\frac{1}{3x^2}\right)^2}\,dx=\sqrt{1+\left(\frac{9x^4}{16}-\frac{1}{2}+\frac{1}{9x^4}\right)}dx$$
$$=\sqrt{\left(\frac{3x^2}{4}+\frac{1}{3x^2}\right)^2}\,dx=\left(\frac{3x^2}{4}+\frac{1}{3x^2}\right)dx.$$



Total length of the curve is $L=\displaystyle\int_1^2\left(\frac{3x^2}{4}+\frac{1}{3x^2}\right)dx=\left\{\frac{x^3}{4}-\frac{1}{3x}\right\}_1^2=\frac{23}{12}.$

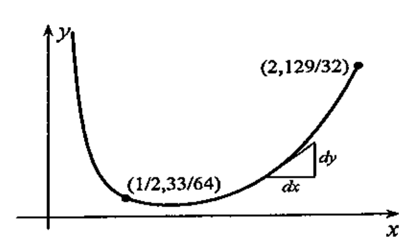
6. Small lengths along the curve are approximated by



$$\sqrt{(dx)^2+(dy)^2}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$=\sqrt{1+\left(\frac{e^x-e^{-x}}{2}\right)^2}\,dx=\sqrt{\frac{e^{2x}+2+e^{-2x}}{4}}dx$$
$$=\frac{1}{2}(e^x+e^{-x})\,dx.$$

Total length of the curve is $L=\displaystyle\int_0^1\frac{1}{2}(e^x+e^{-x})\,dx=\frac{1}{2}\left\{e^x-e^{-x}\right\}_0^1=\frac{1}{2}(e-e^{-1}).$

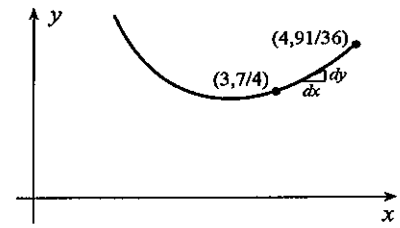
7. Small lengths along the curve are approximated by



$$\sqrt{(dx)^2+(dy)^2}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$=\sqrt{1+\left(x^3-\frac{1}{4x^3}\right)^2}\,dx=\sqrt{1+\left(x^6-\frac{1}{2}+\frac{1}{16x^6}\right)}dx$$
$$\sqrt{\left(x^3+\frac{1}{4x^3}\right)^2}\,dx=\left(x^3+\frac{1}{4x^3}\right)dx.$$

Total length of the curve is $L = \int_{1/2}^{2}\left(x^3 + \frac{1}{4x^3}\right)dx = \left\{\frac{x^4}{4} - \frac{1}{8x^2}\right\}_{1/2}^{2} = \frac{285}{64}$.

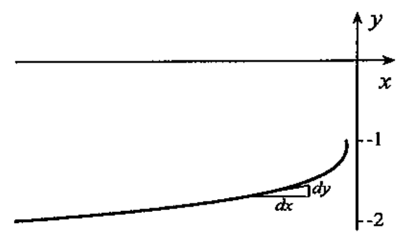
8. We approximate small lengths along the curve by



$$\sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$
$$= \sqrt{1 + \left(\frac{x^2}{12} - \frac{3}{x^2}\right)^2}\,dx = \sqrt{1 + \frac{x^4}{144} - \frac{1}{2} + \frac{9}{x^4}}\,dx$$
$$= \sqrt{\left(\frac{x^2}{12} + \frac{3}{x^2}\right)^2}\,dx = \left(\frac{x^2}{12} + \frac{3}{x^2}\right)dx.$$

Total length of the curve is $L = \int_3^4 \left(\frac{x^2}{12} + \frac{3}{x^2}\right)dx = \left\{\frac{x^3}{36} - \frac{3}{x}\right\}_3^4 = \frac{23}{18}$.

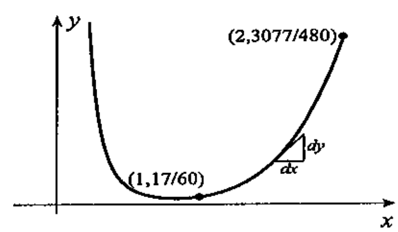
9. We approximate small lengths along the curve by



$$\sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$
$$= \sqrt{1 + \left(\frac{7y^6}{20} - \frac{5}{7y^6}\right)^2}\,dy = \sqrt{1 + \frac{49y^{12}}{400} - \frac{1}{2} + \frac{25}{49y^{12}}}\,dy$$
$$= \sqrt{\left(\frac{7y^6}{20} + \frac{5}{7y^6}\right)^2}\,dy = \left(\frac{7y^6}{20} + \frac{5}{7y^6}\right)dy.$$

Total length of the curve is $L = \int_{-2}^{-1}\left(\frac{7y^6}{20} + \frac{5}{7y^6}\right)dy = \left\{\frac{y^7}{20} - \frac{1}{7y^5}\right\}_{-2}^{-1} = \frac{7267}{1120}$.

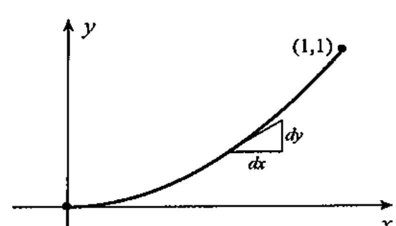
10. We approximate small lengths along the curve by



$$\sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$
$$= \sqrt{1 + \left(x^4 - \frac{1}{4x^4}\right)^2}\,dx = \sqrt{1 + x^8 - \frac{1}{2} + \frac{1}{16x^8}}\,dx$$
$$= \sqrt{\left(x^4 + \frac{1}{4x^4}\right)^2}\,dx = \left(x^4 + \frac{1}{4x^4}\right)dx.$$

Total length of the curve is $L = \int_1^2 \left(x^4 + \frac{1}{4x^4}\right)dx = \left\{\frac{x^5}{5} - \frac{1}{12x^3}\right\}_1^2 = \frac{3011}{480}$.

11. With small lengths along the curve approximated by

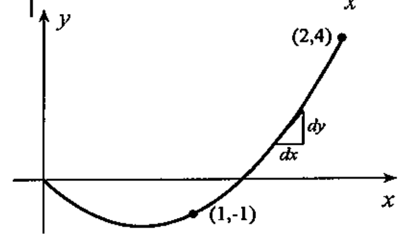


$$\sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + 4x^2}\,dx,$$

the length of the curve is given by

$$L = \int_0^1 \sqrt{1 + 4x^2}\,dx.$$

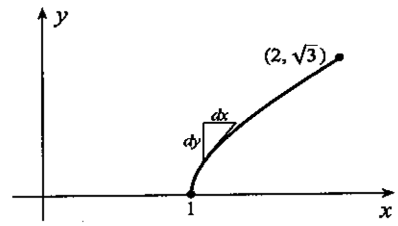
12. We approximate small lengths along the curve by



$$\sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$
$$= \sqrt{1 + (6x - 4)^2}\,dx = \sqrt{36x^2 - 48x + 17}\,dx.$$

Total length of the curve is given by $L = \int_1^2 \sqrt{36x^2 - 48x + 17}\,dx$.

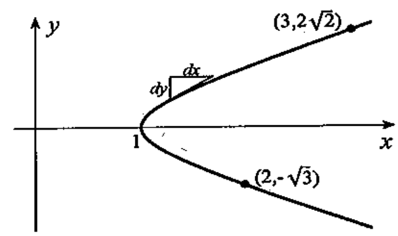
13. We approximate small lengths along the curve by



$$\sqrt{(dx)^2+(dy)^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$= \sqrt{1+\left(\frac{x}{\sqrt{x^2-1}}\right)^2}\,dx = \sqrt{\frac{2x^2-1}{x^2-1}}\,dx.$$

Total length of the curve is given by $L = \int_1^2 \sqrt{\frac{2x^2-1}{x^2-1}}dx.$

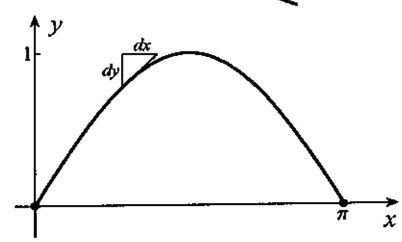
14. We approximate small lengths along the curve by



$$\sqrt{(dx)^2+(dy)^2} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$
$$= \sqrt{1+\left(\frac{y}{\sqrt{1+y^2}}\right)^2}\,dy = \sqrt{\frac{1+2y^2}{1+y^2}}\,dy.$$

Total length of the curve is given by $L = \int_{-\sqrt{3}}^{2\sqrt{2}} \sqrt{\frac{1+2y^2}{1+y^2}}dy.$

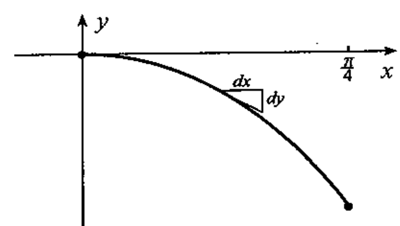
15. We approximate small lengths along the curve by



$$\sqrt{(dx)^2+(dy)^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+\cos^2 x}\,dx.$$

Total length of the curve is given by $L = \int_0^{\pi} \sqrt{1+\cos^2 x}\,dx.$

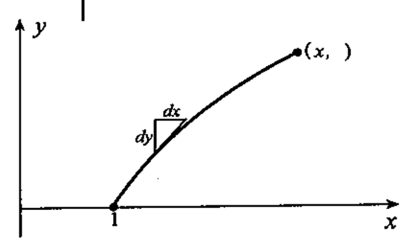
16. We approximate small lengths along the curve by



$$\sqrt{(dx)^2+(dy)^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$= \sqrt{1+\left(\frac{-\sin x}{\cos x}\right)^2}\,dx = \sec x\,dx.$$

Total length of the curve is given by $L = \int_0^{\pi/4} \sec x\,dx.$

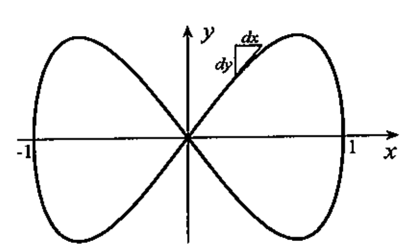
17. We approximate small lengths along the curve by



$$\sqrt{(dx)^2+(dy)^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$= \sqrt{1+(1/x)^2}\,dx = \frac{\sqrt{x^2+1}}{x}dx.$$

The length of the curve from $(1,0)$ to any point with x-coordinate equal to x is $L = \int_1^x \frac{\sqrt{t^2+1}}{t}dt.$

18. Differentiation of $8y^2 = x^2 - x^4$ with respect to x gives $16y\frac{dy}{dx} = 2x - 4x^3$, and from this equation $\frac{dy}{dx} = \frac{x-2x^3}{8y}$. Small lengths along that portion of the curve in the first quadrant are approximated by



$$\sqrt{(dx)^2+(dy)^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+\left(\frac{x-2x^3}{8y}\right)^2}\,dx = \frac{\sqrt{64y^2+x^2-4x^4+4x^6}}{8y}\,dx$$

$$= \frac{\sqrt{8x^2 - 8x^4 + x^2 - 4x^4 + 4x^6}}{8y}\,dx = \frac{3x - 2x^3}{2\sqrt{2}x\sqrt{1-x^2}}\,dx = \frac{3-2x^2}{2\sqrt{2}\sqrt{1-x^2}}\,dx.$$

Total length of the curve is four times that in the first quadrant,

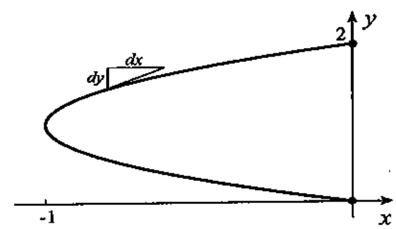
$$L = 4\int_0^1 \frac{3-2x^2}{2\sqrt{2}\sqrt{1-x^2}}dx = \sqrt{2}\int_0^1 \frac{3-2x^2}{\sqrt{1-x^2}}dx.$$

19. We approximate small lengths along the curve by

$$\sqrt{(dx)^2+(dy)^2} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$
$$= \sqrt{1+(2y-2)^2}dy = \sqrt{4y^2-8y+5}\,dy.$$

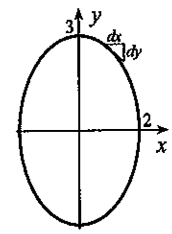
Total length of the curve is given by

$$L = \int_0^2 \sqrt{4y^2-8y+5}\,dy.$$



20. We approximate small lengths along that part of the ellipse in the first quadrant by

$$\sqrt{(dx)^2+(dy)^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$= \sqrt{1+\left(\frac{-3x}{2\sqrt{4-x^2}}\right)^2}dx = \sqrt{\frac{16+5x^2}{4(4-x^2)}}\,dx.$$

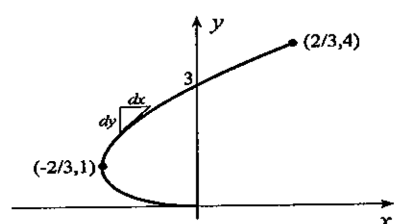


Total length of the curve is four times that in the first quadrant

$$L = 4\int_0^2 \sqrt{\frac{16+5x^2}{4(4-x^2)}}\,dx = 2\int_0^2 \sqrt{\frac{16+5x^2}{4-x^2}}\,dx.$$

21. We approximate small lengths along the curve by

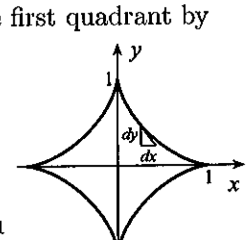
$$\sqrt{(dx)^2+(dy)^2} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$
$$= \sqrt{1+\left(\frac{\sqrt{y}}{2}-\frac{1}{2\sqrt{y}}\right)^2}dy = \sqrt{1+\left(\frac{y}{4}-\frac{1}{2}+\frac{1}{4y}\right)}dy$$
$$= \sqrt{\left(\frac{\sqrt{y}}{2}+\frac{1}{2\sqrt{y}}\right)^2}dy = \left(\frac{\sqrt{y}}{2}+\frac{1}{2\sqrt{y}}\right)dy$$



Total length of the curve is $L = \int_1^4 \left(\frac{\sqrt{y}}{2}+\frac{1}{2\sqrt{y}}\right)dy = \left\{\frac{y^{3/2}}{3}+\sqrt{y}\right\}_1^4 = \frac{10}{3}.$

22. Differentiation of $x^{2/3}+y^{2/3} = 1$ with respect to x gives $\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$, and therefore $\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$. We approximate small lengths along the curve in the first quadrant by

$$\sqrt{(dx)^2+(dy)^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$= \sqrt{1+\left(\frac{y}{x}\right)^{2/3}}dx = \frac{\sqrt{x^{2/3}+y^{2/3}}}{x^{1/3}}dx = x^{-1/3}\,dx.$$



Total length of the curve is therefore $L = 4\int_0^1 x^{-1/3}\,dx = 4\left\{\frac{3}{2}x^{2/3}\right\}_0^1 = 6.$

23. Small lengths along the curve are approximated by

$$\sqrt{(dx)^2+(dy)^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+\left(x^n-\frac{1}{4x^n}\right)^2}\,dx$$
$$= \sqrt{1+\left(x^{2n}-\frac{1}{2}+\frac{1}{16x^{2n}}\right)}dx = \sqrt{\left(x^n+\frac{1}{4x^n}\right)^2}\,dx = \left(x^n+\frac{1}{4x^n}\right)dx.$$

The length of the curve is therefore

$$L = \int_a^b \left(x^n+\frac{1}{4x^n}\right)dx = \left\{\frac{x^{n+1}}{n+1}-\frac{1}{4(n-1)x^{n-1}}\right\}_a^b = \frac{b^{n+1}-a^{n+1}}{n+1}+\frac{a^{1-n}-b^{1-n}}{4(n-1)}.$$

24. Small lengths along the curve are approximated by

$$\sqrt{(dx)^2+(dy)^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+\left[\frac{(2n+1)x^{2n}}{4(2n-1)}-\frac{(2n-1)}{(2n+1)x^{2n}}\right]^2}\,dx$$
$$= \sqrt{1+\frac{(2n+1)^2x^{4n}}{16(2n-1)^2}-\frac{1}{2}+\frac{(2n-1)^2}{(2n+1)^2x^{4n}}}\,dx$$
$$= \sqrt{\left[\frac{(2n+1)x^{2n}}{4(2n-1)}+\frac{2n-1}{(2n+1)x^{2n}}\right]^2}\,dx = \left[\frac{(2n+1)x^{2n}}{4(2n-1)}+\frac{2n-1}{(2n+1)x^{2n}}\right]dx.$$

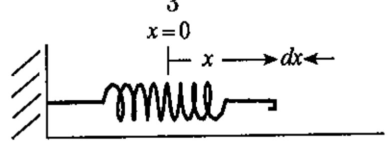
The length of the curve is therefore

$$L = \int_a^b \left[\frac{(2n+1)x^{2n}}{4(2n-1)}+\frac{2n-1}{(2n+1)x^{2n}}\right]dx = \left\{\frac{x^{2n+1}}{4(2n-1)}-\frac{1}{(2n+1)x^{2n-1}}\right\}_a^b$$
$$= \frac{b^{2n+1}-a^{2n+1}}{4(2n-1)}+\frac{1}{2n+1}\left(\frac{1}{a^{2n-1}}-\frac{1}{b^{2n-1}}\right).$$

EXERCISES 7.4

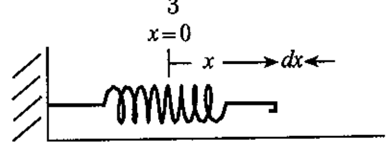
1. Let the spring be stretched in the positive x-direction and $x=0$ correspond to the free end of the spring in the unstretched position (figure below). The restoring force of the spring is $F_s=-kx$. Since $F_s=-10$ N when $x=0.03$ m, it follows that $-10=-0.03k$, and therefore $k=1000/3$ N/m. The force required to counteract the spring force when the spring is stretched an amount x is $F(x)=1000x/3$. The work done by this force in stretching the spring a further distance dx is $\frac{1000x}{3}dx$ J. The total work in stretching the spring from 5 cm to 7 cm is

$$W = \int_{0.05}^{0.07}\frac{1000}{3}x\,dx = \frac{1000}{3}\left\{\frac{x^2}{2}\right\}_{0.05}^{0.07} = \frac{2}{5}\text{ J}.$$

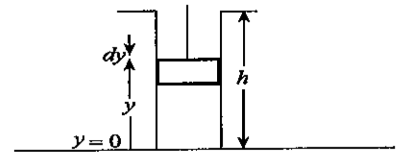


2. Let the spring be stretched in the positive x-direction and $x=0$ correspond to the free end of the spring in the unstretched position (figure below). The restoring force of the spring is $F_s=-kx$. Since $F_s=-10$ N when $x=0.03$ m, it follows that $-10=-0.03k$, and therefore $k=1000/3$ N/m. The force required to counteract the spring force when the spring is stretched an amount x is $F(x)=1000x/3$. The work done by this force in stretching the spring a further distance dx is $\frac{1000x}{3}dx$ J. The total work in stretching the spring from 7 cm to 9 cm is

$$W = \int_{0.07}^{0.09}\frac{1000}{3}x\,dx = \frac{1000}{3}\left\{\frac{x^2}{2}\right\}_{0.07}^{0.09} = \frac{8}{15}\text{ J}.$$

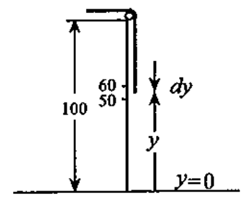


3. When the cage is y m from the bottom of the shaft, the force of gravity on the cage and that part of the cable lifting the cage is $-9.81[M + m(h - y)]$ N. The force required to counter gravity is therefore $9.81[M + m(h - y)]$ N. The work done in raising the cable a further distance dy is $9.81[M + m(h - y)]\,dy$ J. The total work to raise the cage from the bottom of the shaft is



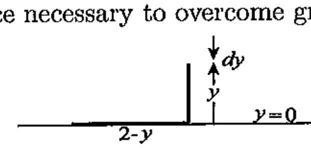
$$W = \int_0^h 9.81[M + m(h - y)]\,dy = 9.81\left\{My - \frac{m}{2}(h - y)^2\right\}_0^h = 9.81(Mh + mh^2/2) \text{ J.}$$

4. When the lower end of the cable is y m above the ground, the force of gravity on that part of the cable hanging from the building is $-9.81(2)(100 - y)$ N. The force required to counter gravity is therefore $19.62(100 - y)$ N. The work done by this force in raising the cable a further distance dy is $19.62(100 - y)\,dy$ J. The total work to raise the cable the 10 m is



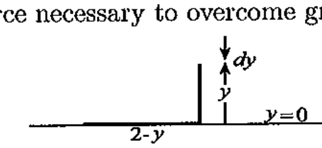
$$W = \int_{50}^{60} 19.62(100 - y)\,dy = 19.62\left\{100y - \frac{y^2}{2}\right\}_{50}^{60} = 8829 \text{ J.}$$

5. When the end of the chain has been lifted a distance y, the force necessary to overcome gravity and hold the chain in this position is $9.81(10)y$ N. The work done by this force in lifting the end of the chain an additional amount dy is $98.1y\,dy$ J. The total work to lift the end of the chain 2 m is



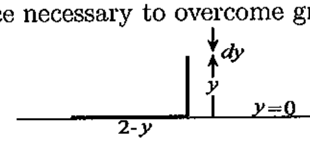
therefore $W = \int_0^2 98.1y\,dy = 98.1\left\{\frac{y^2}{2}\right\}_0^2 = 196.2$ J.

6. When the end of the chain has been lifted a distance y, the force necessary to overcome gravity and hold the chain in this position is $9.81(10)y$ N. The work done by this force in lifting the end of the chain an additional amount dy is $98.1y\,dy$ J. The total work to lift the end of the chain 1 m is



therefore $W = \int_0^1 98.1y\,dy = 98.1\left\{\frac{y^2}{2}\right\}_0^1 = 49.05$ J.

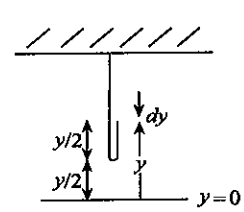
7. When the end of the chain has been lifted a distance y, the force necessary to overcome gravity and hold the chain in this position is $9.81(10)y$ N. The work done by this force in lifting the end of the chain an additional amount dy is $98.1y\,dy$ J. The total work to lift the end of the chain 4 m is



therefore $W = \int_0^2 98.1y\,dy + \int_2^4 (9.81)(20)\,dy = 98.1\left\{\frac{y^2}{2}\right\}_0^2 + 196.2(2) = 588.6$ J.

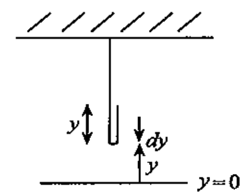
8. (a) When the lower end has been lifted a distance y, the force necessary to overcome gravity is $9.81(3)(y/2)$ N. The work done by this force in lifting the lower end of the chain an additional amount dy is $(29.43y/2)\,dy$ J. The total work done in lifting the end the 5 m is

$$W = \int_0^5 \frac{29.43y}{2}\,dy = \frac{29.43}{2}\left\{\frac{y^2}{2}\right\}_0^5 = 183.9 \text{ J.}$$



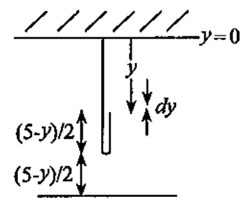
(b) In this case the force necessary to counter gravity at the position shown is $9.81(3)y$ N. To move the bend in the cable up a distance dy, the end of the cable must be lifted a distance $2dy$, requiring $29.43y(2dy)$ J of work. Hence, the total work done is

$$W = \int_0^{2.5} 58.86y\,dy = 58.86\left\{\frac{y^2}{2}\right\}_0^{2.5} = 183.9 \text{ J.}$$



(c) When the chain is at the position shown, the force to overcome gravity is $F(y) = -9.81(3)(5-y)/2$ N. The work to raise the chain through a displacement dy which is in the negative y-direction is $[-29.43(5-y)/2]dy$. The total work to raise the chain is therefore

$$W = \int_5^0 -\frac{29.43}{2}(5-y)\,dy = -\frac{29.43}{2}\left\{-\frac{1}{2}(5-y)^2\right\}_5^0 = 183.9 \text{ J.}$$

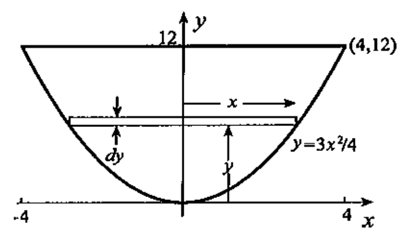


9. The force of gravity on the disc of water shown is

$$F_g = -9.81(1000)\pi x^2\,dy = -9810\pi\left(\frac{4y}{3}\right)dy \text{ N.}$$

The work that an equal and opposite force would do in raising this disc to the top of the tank is

$$(12-y)9810\pi\left(\frac{4y}{3}\right)dy \text{ J.}$$



The total work to empty the tank is therefore

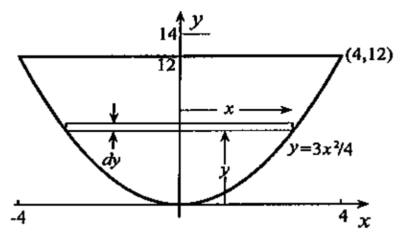
$$W = \int_0^{12}(12-y)9810\pi\left(\frac{4y}{3}\right)dy = \frac{4(9810)\pi}{3}\left\{6y^2 - \frac{y^3}{3}\right\}_0^{12} = 1.18\times 10^7 \text{ J.}$$

10. The force of gravity on the disc of water shown is

$$F_g = -9.81(1000)\pi x^2\,dy = -9810\pi\left(\frac{4y}{3}\right)dy \text{ N.}$$

The work that an equal and opposite force would do in raising this disc to a level 2 m above the top of the tank is

$$(14-y)9810\pi\left(\frac{4y}{3}\right)dy \text{ J.}$$



The total work to empty the tank is therefore

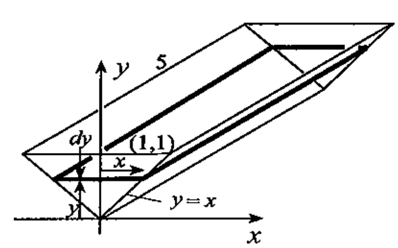
$$W = \int_0^{12}(14-y)9810\pi\left(\frac{4y}{3}\right)dy = \frac{4(9810)\pi}{3}\left\{7y^2 - \frac{y^3}{3}\right\}_0^{12} = 1.78\times 10^7 \text{ J.}$$

11. The force of gravity on a slab of water dy m thick is

$$-9.81(1000)2x(5)dy = -98\,100y\,dy \text{ N.}$$

To lift this slab to the top of the trough requires $(1-y)(98\,100)y\,dy$ J of work. Hence the work required to empty the trough is

$$W = \int_0^1 98\,100y(1-y)\,dy$$
$$= 98\,100\left\{\frac{y^2}{2} - \frac{y^3}{3}\right\}_0^1 = 16\,350 \text{ J.}$$



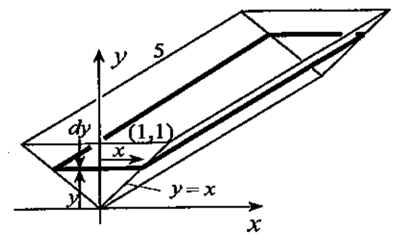
12. The force of gravity on a slab of water dy m thick is

$$-9.81(1000)2x(5)\,dy = -98\,100y\,dy \text{ N}.$$

To lift this slab to a height 2 m above the trough requires $(3-y)98\,100y\,dy$ J of work. Hence the work required to empty the trough is

$$W = \int_0^1 98\,100y(3-y)\,dy$$

$$= 98\,100\left\{\frac{3y^2}{2} - \frac{y^3}{3}\right\}\Big|_0^1 = 114\,450 \text{ J}.$$



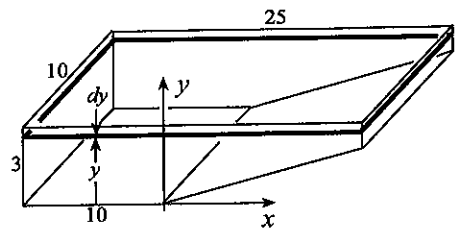
13. The force of gravity on slabs of water dy m thick for $2 \le y \le 3$ is

$$-9.81(1000)(10)(25)\,dy = -2\,452\,500\,dy \text{ N}.$$

The work to lower the level of the water by 1/2 m is

$$W = \int_{5/2}^3 (3-y)(2\,452\,500)\,dy$$

$$= 2\,452\,500\left\{-\frac{1}{2}(3-y)^2\right\}\Big|_{5/2}^3 = 3.07 \times 10^5 \text{ J}.$$



14. The force of gravity on slabs of water dy m thick for $0 \le y \le 2$ is

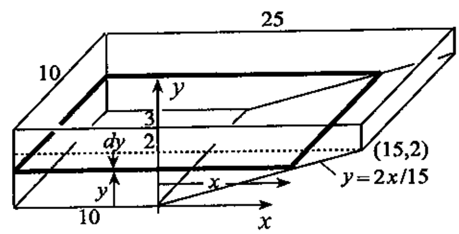
$$-9.81(1000)(10)(x+10)\,dy$$

$$= -98\,100\left(\frac{15y}{2} + 10\right)dy \text{ N}.$$

For slabs above $y = 2$, the force is

$$-9.81(1000)(10)(25)\,dy$$

$$= -2\,452\,500\,dy \text{ N}.$$



The work to empty the pool over its edge is therefore

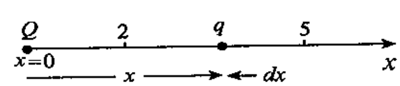
$$W = \int_0^2 98\,100\left(\frac{15y}{2} + 10\right)(3-y)\,dy + \int_2^3 2\,452\,500(3-y)\,dy$$

$$= 245\,250\int_0^2 (12 + 5y - 3y^2)\,dy + 2\,452\,500\int_2^3 (3-y)\,dy$$

$$= 245\,250\left\{12y + \frac{5y^2}{2} - y^3\right\}\Big|_0^2 + 2\,452\,500\left\{3y - \frac{y^2}{2}\right\}\Big|_2^3 = 7.60 \times 10^6 \text{ J}.$$

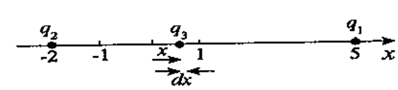
15. When q is at position x, the force on it is $qQ/(4\pi\epsilon_0 x^2)$. The work done by this force as q moves from $x = 2$ to $x = 5$ is

$$W = \int_2^5 \frac{qQ}{4\pi\epsilon_0 x^2}\,dx = \frac{qQ}{4\pi\epsilon_0}\left\{-\frac{1}{x}\right\}\Big|_2^5 = \frac{3qQ}{40\pi\epsilon_0}.$$



16. When q_3 is at position x, the total force on it due to q_1 and q_2 is

$$F(x) = \frac{-q_1q_3}{4\pi\epsilon_0(5-x)^2} + \frac{q_2q_3}{4\pi\epsilon_0(x+2)^2}.$$

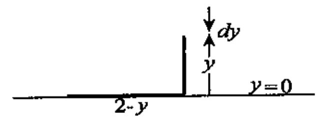


The work done by this force as q_3 moves from $x = 1$ to $x = -1$ is

$$W = \int_1^{-1}\left[\frac{-q_1q_3}{4\pi\epsilon_0(5-x)^2} + \frac{q_2q_3}{4\pi\epsilon_0(x+2)^2}\right]dx = \frac{-q_1q_3}{4\pi\epsilon_0}\left\{\frac{1}{5-x}\right\}\Big|_1^{-1} + \frac{q_2q_3}{4\pi\epsilon_0}\left\{\frac{-1}{x+2}\right\}\Big|_1^{-1}$$

$$= \frac{q_1q_3}{48\pi\epsilon_0} - \frac{q_2q_3}{6\pi\epsilon_0} = \frac{q_3}{48\pi\epsilon_0}(q_1 - 8q_2).$$

17. When the end of the chain has been lifted a distance y, the force necessary to overcome gravity on the hanging part and friction on that part on the floor is $9.81(10)y + 0.01(9.81)(10)(2-y) = 1.962 + 97.119y$ N.

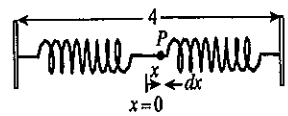


The work done by this force in lifting the end of chain 2 m is therefore

$$W = \int_0^2 (1.962 + 97.119y)\,dy = \left\{1.962y + \frac{97.119y^2}{2}\right\}_0^2 = 198.162 \text{ J.}$$

18. When P is moved x m to the right, the resultant force of the two springs on P is

$-k(1+x) + k(1-x) = -2kx$ N.



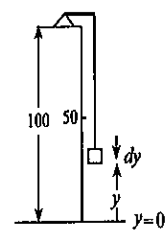
The work done by an equal and opposite force in moving P a distance b m to the right is

$$W = \int_0^b 2kx\,dx = \left\{kx^2\right\}_0^b = kb^2 \text{ J.}$$

19. The force necessary to maintain a draw of x m is $F = kx$. Since $F = 200$ when $x = 0.5$, it follows that $k = 400$ N/m. The work to fully draw the bow is $W = \int_0^{1/2} 400x\,dx = \left\{200x^2\right\}_0^{1/2} = 50$ J.

20. When the bucket has been raised y m ($y < 50$), the force of gravity on what remains in the bucket and the hanging cable is

$-9.81[(100 - y/5) + 5(100 - y)] = -9.81(600 - 26y/5)$ N.



When $y \geq 50$, the force of gravity on bucket, cable and pigeon is

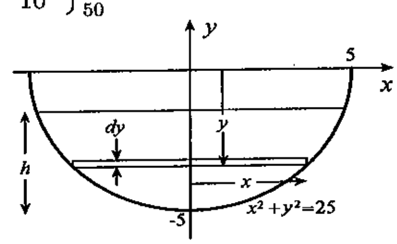
$$-9.81[(100 - y/5) + 5(100 - y) + 2 - (y - 50)]$$
$$= -9.81(652 - 31y/5) \text{ N.}$$

The work to overcome these forces is

$$W = \int_0^{50} 9.81\left(600 - \frac{26y}{5}\right)dy + \int_{50}^{100} 9.81\left(652 - \frac{31y}{5}\right)dy$$
$$= 9.81\left\{600y - \frac{13y^2}{5}\right\}_0^{50} + 9.81\left\{652y - \frac{31y^2}{10}\right\}_{50}^{100} = 3.22 \times 10^5 \text{ J.}$$

21. (a) When the depth of oil is h m, the volume of oil in the tank is

$$V = \int_{-5}^{h-5} \pi(25 - y^2)\,dy = \pi\left\{25y - \frac{y^3}{3}\right\}_{-5}^{h-5}$$
$$= \frac{\pi}{3}(15h^2 - h^3).$$



This volume will be $(1/3)\pi(125)$ when

$$\frac{125\pi}{3} = \frac{\pi}{3}(15h^2 - h^3),$$

and this equation reduces to $h^3 - 15h^2 + 125 = 0$. Newton's iterative procedure with $h_1 = 3$ and $h_{n+1} = h_n - \dfrac{h_n^3 - 15h_n^2 + 125}{3h_n^2 - 30h_n}$ leads to $h = 3.2635$ m.

(b) The work to empty the half-full tank is

$$W = \int_{-5}^{-1.7365} 9.81(750)(-y)(\pi x^2)\,dy = -9.81(750)\pi \int_{-5}^{-1.7365} y(25 - y^2)\,dy$$
$$= -9.81(750)\pi\left\{\frac{25y^2}{2} - \frac{y^4}{4}\right\}_{-5}^{-1.7365} = 2.79 \times 10^6 \text{ J.}$$

22. (a) Assuming that the earth is a sphere

$$F = \frac{6.67 \times 10^{-11}(m)(4/3)\pi(6.37\times 10^6)^3(5.52\times 10^3)}{(6.37\times 10^6)^2} = 9.82m \text{ N.}$$

(b) The work required is $W = \int_{6.37\times 10^6}^{6.38\times 10^6} \frac{G(10)M}{r^2}\,dr = 10GM\left\{-\frac{1}{r}\right\}_{6.37\times 10^6}^{6.38\times 10^6}$

$$= 10(6.67\times 10^{-11})\frac{4}{3}\pi(6.37\times 10^6)^3(5.52\times 10^3)\left(\frac{-1}{6.38\times 10^6}+\frac{1}{6.37\times 10^6}\right)$$

$$= 9.8087\times 10^5 \text{ J.}$$

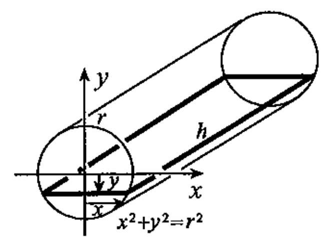
(c) With a constant $F = 9.82m$, $\quad W = \int_{6.37\times 10^6}^{6.38\times 10^6} 9.82(10)\,dy = 9.82\times 10^5$ J.

23. The force of gravity on a slab of oil dy m thick is $-9.81(\rho)(2x)(h)dy$ N. To lift this slab to the top of the tank requires $(r-y)(19.62\rho hx\,dy)$ J of work. Hence the work required to empty the tank is

$$W = \int_{-r}^{r}(r-y)(19.62\rho hx)\,dy = 19.62\rho h\int_{-r}^{r}(r-y)\sqrt{r^2-y^2}\,dy$$

$$= 19.62\rho h\left[\int_{-r}^{r} r\sqrt{r^2-y^2}\,dy - \int_{-r}^{r} y\sqrt{r^2-y^2}\,dy\right]$$

$$= 19.62\rho rh\int_{-r}^{r}\sqrt{r^2-y^2}\,dy - 19.62\rho h\left\{-\frac{1}{3}(r^2-y^2)^{3/2}\right\}_{-r}^{r}.$$



Since the remaing integral represents one-half the area of the end of the tank,

$$W = 19.62\rho hr\left(\frac{1}{2}\pi r^2\right) = 9.81\pi\rho r^3 h \text{ J.}$$

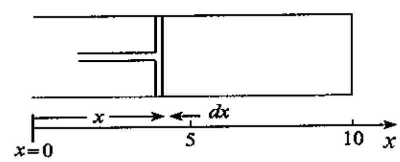
24. At position x, the force exerted by the piston is

$$F(x) = PA = \frac{C}{V}A = \frac{CA}{A(10-x)} = \frac{C}{10-x}.$$

The work done is therefore

$$W = \int_0^5 \frac{C}{10-x}\,dx = C\Big\{-\ln|10-x|\Big\}_0^5$$

$$= C(-\ln 5 + \ln 10) = C\ln 2.$$



25. We divide the work into four parts as shown,

$$W = \int_0^{70}\rho g(y)\pi\left(\frac{1}{20}\right)^2 dy + \int_{70}^{73}\rho g(y)\pi[9-(y-73)^2]\,dy$$

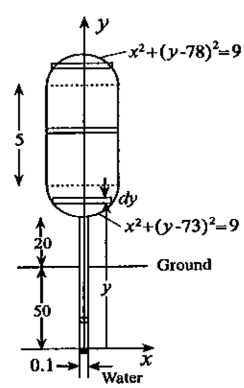
$$+ \int_{73}^{78}\rho g(y)\pi(3)^2\,dy + \int_{78}^{81}\rho g(y)\pi[9-(y-78)^2]\,dy$$

$$= \rho g\pi\left[\frac{1}{400}\int_0^{70} y\,dy + \int_{70}^{73}(-y^3+146y^2-5320y)\,dy\right.$$

$$\left.+9\int_{73}^{78} y\,dy + \int_{78}^{81}(-y^3+156y^2-6075y)\,dy\right]$$

$$= \rho g\pi\left[\frac{1}{400}\left\{\frac{y^2}{2}\right\}_0^{70} + \left\{-\frac{y^4}{4}+\frac{146y^3}{3}-2660y^2\right\}_{70}^{73}\right.$$

$$\left.+9\left\{\frac{y^2}{2}\right\}_{73}^{78} + \left\{-\frac{y^4}{4}+\frac{156y^3}{3}-\frac{6075y^2}{2}\right\}_{78}^{81}\right] = 1.89\times 10^8 \text{ J.}$$



26. Using points A and D we obtain $k_2 = 12\,000$ and $k_1 = 20\,000$. Since the work is the area bounded by the curves

$$W = \int_{1/5}^{3/5} \left(\frac{k_1}{V} - \frac{k_2}{V}\right) dV = (k_1 - k_2)\{\ln V\}_{1/5}^{3/5} = 8.8 \times 10^3 \text{ J}.$$

27. Using points A and D we obtain $k_2 = 1000$ and $k_1 = 20\,000$. Since the work is the area bounded by the curves

$$W = \int_{10\,000}^{100\,000} \left(\frac{k_1}{P} - \frac{k_2}{P}\right) dP = (k_1 - k_2)\{\ln P\}_{10\,000}^{100\,000} = 4.4 \times 10^4 \text{ J}.$$

28. Using points B and C we obtain $k_2 = 4.64$ and $k_1 = 6.89$. Since the work is the area bounded by the curves

$$W = \int_{2\times 10^{-4}}^{8\times 10^{-4}} \left(\frac{k_1}{V^{1/4}} - \frac{k_2}{V^{1.4}}\right) dV = (k_1 - k_2)\left\{\frac{-1}{0.4V^{0.4}}\right\}_{2\times 10^{-4}}^{8\times 10^{-4}} = 72 \text{ J}.$$

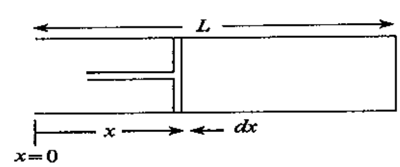
29. Using points A and D we obtain $k_2 = 15.0$ and $k_1 = 66.6$. Since the work is the area bounded by the curves

$$\begin{aligned} W &= \int_{2\times 10^{-4}}^{5.75\times 10^{-4}} \left(23 \times 10^5 - \frac{k_2}{V^{1.4}}\right) dV + \int_{5.75\times 10^{-4}}^{20\times 10^{-4}} \left(\frac{k_1}{V^{1.4}} - \frac{k_2}{V^{1.4}}\right) dV \\ &= \left\{23 \times 10^5 V + \frac{k_2}{0.4V^{0.4}}\right\}_{2\times 10^{-4}}^{5.75\times 10^{-4}} + (k_1 - k_2)\left\{\frac{-1}{0.4V^{0.4}}\right\}_{5.75\times 10^{-4}}^{20\times 10^{-4}} = 1.5 \times 10^3 \text{ J}. \end{aligned}$$

30. Using points A and D we obtain $k_2 = 2.62$ and $k_1 = 32.2$. Since the work is the area bounded by the curves

$$W = \int_{150\,000}^{1\,040\,000} \left[\left(\frac{k_1}{P}\right)^{5/7} - \left(\frac{k_2}{P}\right)^{5/7}\right] dP = (k_1^{5/7} - k_2^{5/7})\left\{\frac{7}{2}P^{2/7}\right\}_{150\,000}^{1\,040\,000} = 7.8 \times 10^2 \text{ J}.$$

31. Let $x = 0$ represent the position of the piston face when $V = V_0$. At position x, the force exerted by the piston is



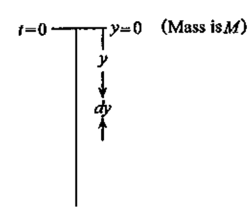
$$F(x) = PA = \frac{CA}{V^{7/5}} = \frac{CA}{[A(L-x)]^{7/5}} = \frac{C}{A^{2/5}(L-x)^{7/5}}.$$

The work done is therefore

$$W = \int_0^{L/2} \frac{C}{A^{2/5}(L-x)^{7/5}} dx = \frac{C}{A^{2/5}}\left\{\frac{5}{2(L-x)^{2/5}}\right\}_0^{L/2} = \frac{5C}{2A^{2/5}}\left[\frac{1}{(L/2)^{2/5}} - \frac{1}{L^{2/5}}\right] = \frac{5(2^{2/5}-1)C}{2(AL)^{2/5}}.$$

Since $V_0 = AL$, it follows that $W = \dfrac{5(2^{2/5}-1)C}{2V_0^{2/5}}$.

32. After time t, the mass of liquid remaining in the drop is $M - mt$, so that it completely disappears after time $t = M/m$. If we choose y as positive downward, then the work done by gravity in a small distance dy is $g(M - mt)\,dy$. Total work done is therefore



$$W = \int_0^{t=M/m} g(M - mt)\,dy = \int_0^{M/m} g(M - mt)\frac{dy}{dt}\,dt.$$

To find the velocity $v = dy/dt$ of the drop, we use Newton's second law in the form $F = d[(M - mt)v]/dt$. It requires

$$(M - mt)g = \frac{d}{dt}[(M - mt)v].$$

Integration gives

$$-\frac{g}{2m}(M-mt)^2 = (M-mt)v + C.$$

Since the initial velocity at time $t=0$ is $v=0$, we find that $C=-M^2g/(2m)$, and therefore

$$-\frac{g}{2m}(M-mt)^2 = (M-mt)v - \frac{M^2g}{2m} \quad\Longrightarrow\quad v=\frac{dy}{dt} = -\frac{g}{2m}(M-mt) + \frac{M^2g}{2m(M-mt)}.$$

The work done by gravity can now be calculated

$$\begin{aligned} W &= \int_0^{M/m} g(M-mt)\left[-\frac{g}{2m}(M-mt) + \frac{M^2g}{2m(M-mt)}\right]dt \\ &= \frac{g^2}{2m}\int_0^{M/m}[-(M-mt)^2+M^2]\,dt = \frac{g^2}{2m}\left\{\frac{(M-mt)^3}{3m}+M^2t\right\}_0^{M/m} = \frac{g^2M^3}{3m^2}\text{ J.}\end{aligned}$$

33. During the time that the bell is completely submerged ($0 \le y \le 98$), the force exerted by the winch is

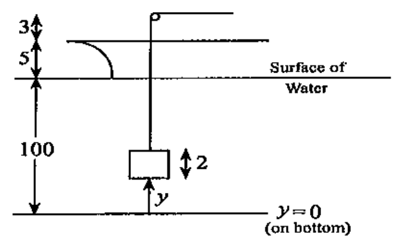
$$\begin{aligned} F(y) &= 9.81[(10\,000-8\,000)+5(106-y)-(98-y)] \\ &= 9.81(2432-4y)\text{ N.}\end{aligned}$$

When the bell is partially submerged ($98 \le y \le 100$),

$$\begin{aligned} F(y) &= 9.81[10\,000-1000(2)(2)(100-y)+5(106-y)] \\ &= 9.81(3995y-389\,470)\text{ N.}\end{aligned}$$

When the bell is out of the water ($100 \le y \le 105$),

$$F(y) = 9.81[10\,000+5(106-y)] = 9.81(10\,530-5y)\text{ N.}$$



The work to lift the bell is therefore

$$\begin{aligned} W &= \int_0^{98} 9.81(2432-4y)\,dy + \int_{98}^{100} 9.81(3995y-389\,470)\,dy + \int_{100}^{105} 9.81(10\,530-5y)\,dy \\ &= 9.81\left\{2432y-2y^2\right\}_0^{98} + 9.81\left\{\frac{3995y^2}{2}-389\,470y\right\}_{98}^{100} + 9.81\left\{10\,530y-\frac{5y^2}{2}\right\}_{100}^{105} \\ &= 2.76\times10^6\text{ J.}\end{aligned}$$

EXERCISES 7.5

1. According to Example 7.20, the energy stored in a spring when it is stretched an amount x is $kx^2/2$. If the stretch is doubled the energy is $k(2x)^2/2 = 2kx^2$. The energy has therefore been quadrupled.

2. (a) If x represents the stretch in the spring, then the sum of the potential energy in the spring and the kinetic energy of the mass must always be constant, $C = kx^2/2 + mv^2/2$. Initially, $C = kx_0^2/2 + mv_0^2/2$. Consequently, x and v are related thereafter by the equation $kx_0^2/2 + mv_0^2/2 = kx^2/2 + mv^2/2 \Longrightarrow kx^2+mv^2 = kx_0^2+mv_0^2$.

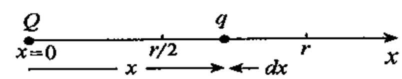
(b) Maximum stretch occurs when $v=0$ in which case $kx^2 = kx_0^2+mv_0^2$. This equation can be solved for $x=\sqrt{x_0^2+mv_0^2/k}$.

(c) Maximum speed occurs when $x=0$ in which case $mv^2 = kx_0^2+mv_0^2$. This equation can be solved for $v=\sqrt{v_0^2+kx_0^2/m}$.

3. From Exercise 19 in Section 7.4, the potential energy stored in the fully-drawn crossbow is 50 J. When we equate this to $mv^2/2$, and substitute for m, we obtain $50 = \frac{1}{2}\left(\frac{20}{1000}\right)v^2 \quad\Longrightarrow\quad v = 50\sqrt{2}$ m/s.

4. At position x, the force of repulsion on q is $qQ/(4\pi\epsilon_0 x^2)$. The gain in potential energy as q moves from $x = r$ to $x = r/2$ is the work done by an equal and opposite force in causing the motion,

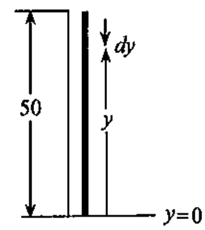
$$W = \int_r^{r/2} \frac{-qQ}{4\pi\epsilon_0 x^2}dx = \left\{\frac{qQ}{4\pi\epsilon_0 x}\right\}_r^{r/2} = \frac{qQ}{4\pi\epsilon_0 r}.$$



5. (a) A length dy of the chain at height y has gravitational potential energy $9.81(2dy)y = 19.62y\,dy$. The total gravitational potential energy of the chain is therefore

$$\int_0^{50} 19.62y\,dy = \left\{9.81y^2\right\}_0^{50} = 24\,525 \text{ J}.$$

(b) The work to lift the chain to the top of the building is the gravitational potential energy that it has on the top of the building less its present potential energy, $9.81(100)(50) - 24\,525 = 24\,525$ J.



6. (a) The ultimate compression x occurs when the energy stored in the spring is equal to the original gravitational potential energy of the mass relative to this position,

$$\frac{1}{2}kx^2 = 9.81mx \quad \Longrightarrow \quad x = \frac{19.62m}{k}.$$

(b) During the oscillations of the mass, the sum of spring potential energy, gravitational potential energy, and kinetic energy will be constant. If we equate initial values of these energies (taking $x = 0$ at the uncompressed position of the spring, and x positive upward) and values at maximum compression, denoted by x,

$$\frac{1}{2}mv_0^2 = \frac{1}{2}kx^2 + 9.81mx \quad \Longrightarrow \quad kx^2 + 19.62mx - mv_0^2 = 0.$$

Solutions of this quadratic equation are $x = \dfrac{-19.62m \pm \sqrt{(19.62m)^2 + 4kmv_0^2}}{2k}$. Maximum compression of the spring is therefore $\left(9.81m + \sqrt{(9.81m)^2 + kmv_0^2}\right)/k$.

7. The work done from $x = a$ to $x = b$ is $W = \displaystyle\int_a^b F(x)\,dx$. Newton's second law states that force F and acceleration a are related by $F = ma = m\dfrac{dv}{dt}$. Hence,

$$W = \int_a^b m\frac{dv}{dt}\,dx = \int_{x=a}^{x=b} m\frac{dv}{dt}\frac{dx}{dt}\,dt = \int_{x=a}^{x=b} mv\frac{dv}{dt}\,dt = \int_{x=a}^{x=b} \frac{d}{dt}\left(\frac{1}{2}mv^2\right)dt = \left\{\frac{1}{2}mv^2\right\}_{x=a}^{x=b}.$$

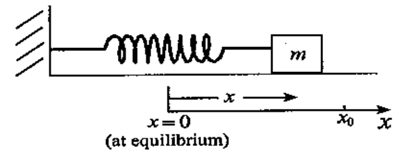
This is the difference in kinetic energies at $x = b$ and $x = a$.

8. (a) The magnitude of the force of attraction on the mass at distance x from the centre of the earth is GmM/x^2. The work done by a force equal and opposite to this in raising the mass from the earth's surface to height 10^5 m is $W = \displaystyle\int_{6.37\times10^6}^{6.47\times10^6} \frac{GmM}{x^2}dx = GmM\left\{-\frac{1}{x}\right\}_{6.37\times10^6}^{6.47\times10^6} = 9.67 \times 10^6$ J.

(b) If the mass is dropped from this height, this gravitational potential energy is converted into kinetic energy. If it strikes the earth with speed v, then

$$\frac{1}{2}mv^2 = 9.67 \times 10^6 \quad \Longrightarrow \quad v = \sqrt{\frac{2(9.67 \times 10^6)}{10}} = 1.4 \times 10^3 \text{ m/s}.$$

9. (a) When the mass is at position x, its kinetic energy plus spring potential energy plus the work done against friction is equal to its initial spring potential energy,



$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 + \mu mg(x_0 - x) = \frac{1}{2}kx_0^2$$
$$\Longrightarrow$$
$$kx_0^2 = mv^2 + kx^2 + 2\mu mg(x_0 - x).$$

(b) When the mass comes to a stop for the first time $v = 0$, in which case

$$kx_0^2 = kx^2 + 2\mu mg(x_0 - x) = 0 \quad \Longrightarrow \quad kx^2 - 2\mu mgx + (2\mu mgx_0 - kx_0^2) = 0.$$

Solutions of this quadratic equation are

$$x = \frac{2\mu mg \pm \sqrt{4\mu^2m^2g^2 - 4k(2\mu mgx_0 - kx_0^2)}}{2k} \quad \Longrightarrow \quad x = x_0,\ \frac{2\mu mg}{k} - x_0.$$

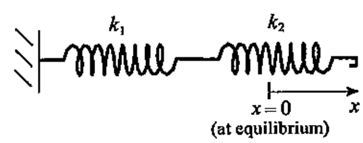
Thus, the mass stops at position $x = 2\mu mg/k - x_0$.

(c) As $\mu \to 0$, the stopping position approaches $-x_0$. This is to be expected, because without friction, the mass oscillates back and forth between $\pm x_0$.

(d) As $\mu \to 1$, the stopping position approaches $2mg/k - x_0$. This is to the left of its starting position.

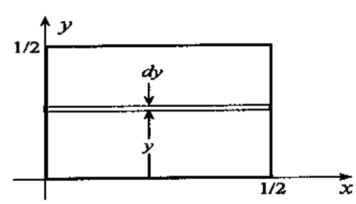
10. When the right end of the spring is moved a distance x, then $x_1 + x_2 = x$, where $x_2/x_1 = k_1/k_2$. These equations imply that $x_1 = k_2x/(k_1 + k_2)$, and $x_2 = k_1x/(k_1 + k_2)$. Since the force necessary to maintain a combined stretch x is $F(x) = k_1x_1 + k_2x_2 = \dfrac{2k_1k_2x}{k_1 + k_2}$, the work to produce total stretch L is

$$W = \int_0^L \frac{2k_1k_2x}{k_1 + k_2}dx = \frac{2k_1k_2}{k_1 + k_2}\left\{\frac{x^2}{2}\right\}_0^L = \frac{k_1k_2L^2}{k_1 + k_2}.$$



EXERCISES 7.6

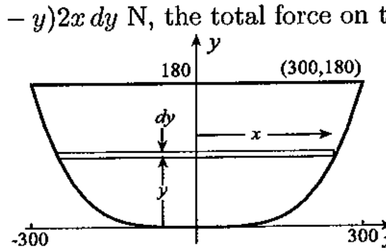
1. The force on the bottom is simply the weight of water in the tank, namely, $(1/4)(1000)(9.81) = 2452.5$ N. On each end, the force is



$$F = \int_0^{1/2} 9.81(1000)(1/2 - y)(1/2)\,dy$$
$$= \frac{4905}{2}\int_0^{1/2}(1 - 2y)\,dy = \frac{4905}{2}\left\{\frac{(1-2y)^2}{-4}\right\}_0^{1/2} = \frac{4905}{8}\text{ N}.$$

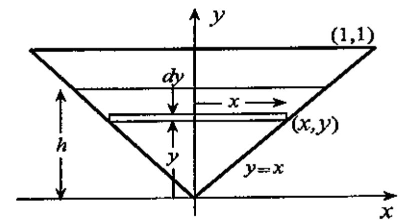
The force on the side is twice that on the ends, 4905/4 N.

2. Since the force on the representative rectangle is $9.81(1000)(180 - y)2x\,dy$ N, the total force on the dam is



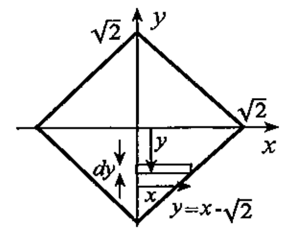
$$F = \int_0^{180} 19\,620(180 - y)(45 \times 10^6)^{1/4}y^{1/4}\,dy$$
$$= 196\,200(4500)^{1/4}\left\{144y^{5/4} - \frac{4}{9}y^{9/4}\right\}_0^{180}$$
$$= 6.78 \times 10^{10}\text{ N}.$$

3. If h is the depth of water in the trough when it is half-full, then $h^2 = 1/2 \implies h = 1/\sqrt{2}$. Since the force on the representative rectangle is $9.81(1000)(1/\sqrt{2} - y)(2x)\,dy$, the total force on the end of the trough is



$$F = \int_0^{1/\sqrt{2}} 19\,620 \left(\frac{1}{\sqrt{2}} - y\right) y\,dy$$
$$= 19\,620 \left\{\frac{y^2}{2\sqrt{2}} - \frac{y^3}{3}\right\}_0^{1/\sqrt{2}} = \frac{1635}{\sqrt{2}} \text{ N.}$$

4. The force on the rectangle is $9810(1000)(-y)x\,dy$ N. The total force on the plate is



$$F = 2\int_{-\sqrt{2}}^{0} 9810(-y)(y + \sqrt{2})\,dy$$
$$= -19\,620 \left\{\frac{y^3}{3} + \frac{\sqrt{2}y^2}{2}\right\}_{-\sqrt{2}}^{0} = 9.25 \times 10^3 \text{ N.}$$

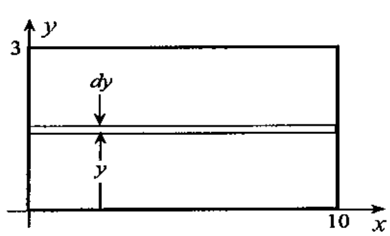
5. The force is the weight of oil in the tank, $\rho g(\pi r^2 h)$, where $g = 9.81$.

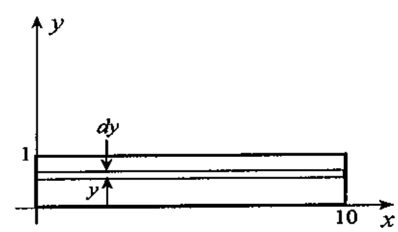
6. The force on the deep end of the pool (left figure below) is

$$F = \int_0^3 9810(3 - y)(10)\,dy = 98\,100 \left\{3y - \frac{y^2}{2}\right\}_0^3 = 4.41 \times 10^5 \text{ N.}$$

The force on the shallow end (right figure) is

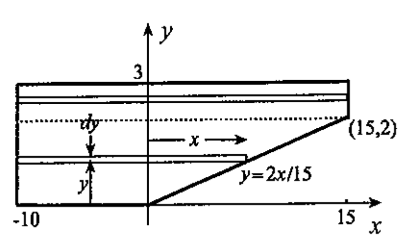
$$F = \int_0^1 9810(1 - y)(10)\,dy = 98\,100 \left\{y - \frac{y^2}{2}\right\}_0^1 = 4.91 \times 10^4 \text{ N.}$$





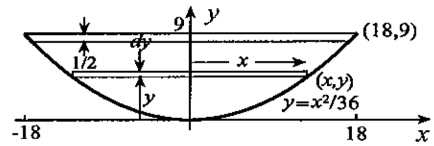
The force on each side of the pool is

$$F = \int_0^2 9810(3 - y)(x + 10)\,dy + \int_2^3 9810(3 - y)(25)\,dy$$
$$= 9810\int_0^2 (3 - y)\left(\frac{15y}{2} + 10\right) dy + 25(9810)\int_2^3 (3 - y)\,dy$$
$$= \frac{5(9810)}{2}\int_0^2 (12 + 5y - 3y^2)\,dy + 25(9810)\int_2^3 (3 - y)\,dy$$
$$= \frac{5(9810)}{2}\left\{12y + \frac{5y^2}{2} - y^3\right\}_0^2 + 25(9810)\left\{3y - \frac{y^2}{2}\right\}_2^3$$
$$= 7.60 \times 10^5 \text{ N.}$$



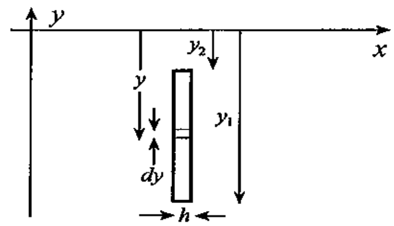
7. Since the force on the representative rectangle is $9.81(1000)(17/2 - y)(2x)\,dy$, the force on the dam is

$$\begin{aligned}F &= 19\,620\int_0^{17/2}\left(\frac{17}{2} - y\right)6\sqrt{y}\,dy\\ &= 58\,860\int_0^{17/2}(17\sqrt{y} - 2y^{3/2})\,dy\\ &= 58\,860\left\{\frac{34y^{3/2}}{3} - \frac{4y^{5/2}}{5}\right\}_0^{17/2} = 6.61 \times 10^6 \text{ N.}\end{aligned}$$



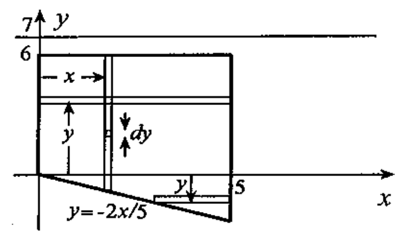
8. Since the force on the tiny rectangle shown is $9.81\rho(-y)(h\,dy)$ N, the force on the long rectangle is

$$\begin{aligned}F &= \int_{y_1}^{y_2} -9.81\rho h y\,dy = -9.81\rho h\left\{\frac{y^2}{2}\right\}_{y_1}^{y_2}\\ &= \frac{9.81\rho h}{2}(y_1^2 - y_2^2)\text{ N.}\end{aligned}$$



9. With horizontal rectangles,

$$\begin{aligned}F &= \int_{-2}^{0} 9.81\rho(7 - y)(5 - x)\,dy + \int_0^6 9.81\rho(7 - y)(5)\,dy\\ &= 9.81\rho\int_{-2}^{0}(7 - y)\left(5 + \frac{5y}{2}\right)dy + 49.05\rho\left\{-\frac{1}{2}(7 - y)^2\right\}_0^6\\ &= 24.525\rho\int_{-2}^{0}(14 + 5y - y^2)\,dy + 1177.2\rho\\ &= 24.525\rho\left\{14y + \frac{5y^2}{2} - \frac{y^3}{3}\right\}_{-2}^{0} + 1177.2\rho = 1553.25\rho\text{ N.}\end{aligned}$$



There is a temptation to use Exercise 8 as a formula for vertical rectangles. This would be a mistake since the formula was based on a coordinate system different from the one in this problem. Instead, we divide the long vertical rectangle in the figure into smaller rectangles of width dy. Since the force on this rectangle is $9.81\rho(7 - y)dx\,dy$, the force on the long vertical rectangle is

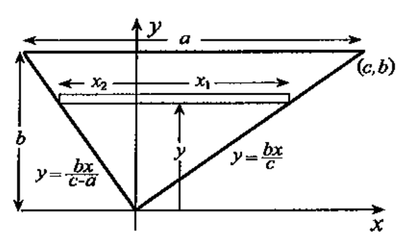
$$\int_{-2x/5}^{6} 9.81\rho(7 - y)dx\,dy = 9.81\rho\,dx\left\{-\frac{(7 - y)^2}{2}\right\}_{-2x/5}^{6} = 4.905\rho\,dx\left[\left(7 + \frac{2x}{5}\right)^2 - 1\right].$$

The force on the plate is therefore

$$F = \int_0^5 4.905\rho\left[\left(7 + \frac{2x}{5}\right)^2 - 1\right]dx = 4.905\rho\left\{\frac{5}{6}\left(7 + \frac{2x}{5}\right)^3 - x\right\}_0^5 = 1553.25\rho\text{ N.}$$

10. The force on the rectangle is

$$\begin{aligned}&9.81\rho(b - y)(x_1 - x_2)\,dy\\ &= 9.81\rho(b - y)\left[\frac{cy}{b} - \left(\frac{c - a}{b}\right)y\right]dy\\ &= \frac{9.81\rho a}{b}(b - y)y\,dy\text{ N.}\end{aligned}$$

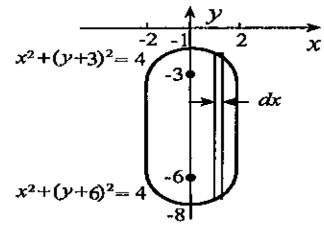


The total force on the plate is

$$F = \int_0^b \frac{9.81\rho a}{b}(by - y^2)\,dy = \frac{9.81\rho a}{b}\left\{\frac{by^2}{2} - \frac{y^3}{3}\right\}_0^b = \frac{9.81\rho ab^2}{6}\text{ N.}$$

11. Since the x-axis is in the surface of the water, we may use the formula of Exercise 8 with vertical rectangles,

$$F = 2\int_0^2 \frac{9.81(1000)}{2}\left[\left(-6-\sqrt{4-x^2}\right)^2 - \left(-3+\sqrt{4-x^2}\right)^2\right] dx$$
$$= 9810\int_0^2 \left(27+18\sqrt{4-x^2}\right) dx$$
$$= 88\,290\int_0^2 \left(3+2\sqrt{4-x^2}\right) dx \text{ N}.$$

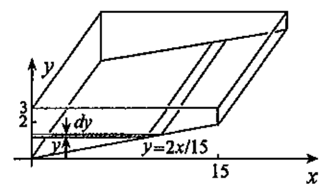


12. The force on the horizontal part of the bottom is the weight of the water directly above it:

$9810(10)(3)(10) = 2.943 \times 10^6$ N.

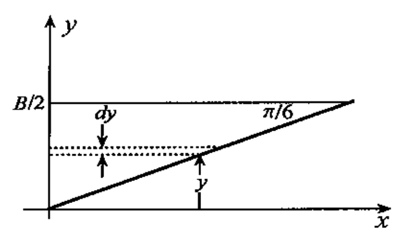
For the slanted part of the bottom we notice that differential dy gives rise to a rectangle of width $\sqrt{229}dy/2$ across the bottom. The force on this part is therefore

$$F = \int_0^2 9810(3-y)(10)\left(\frac{\sqrt{229}}{2}dy\right)$$
$$= 5\sqrt{229}(9810)\left\{3y - \frac{y^2}{2}\right\}_0^2 = 2.969 \times 10^6 \text{ N}.$$



13. Differential dy gives rise to a rectangle of width $2\,dy$. The force on the bow is

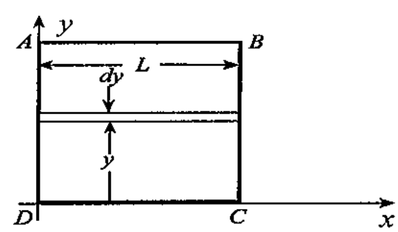
$$F = \int_0^{B/2} 9.81(1000)\left(\frac{B}{2}-y\right)A(2\,dy)$$
$$= 9810A\left\{-\frac{1}{4}(B-2y)^2\right\}_0^{B/2}$$
$$= \frac{4905AB^2}{2} \text{ N}.$$



14. If lengths of AB and BC are L m, the force on face $ABCD$ is

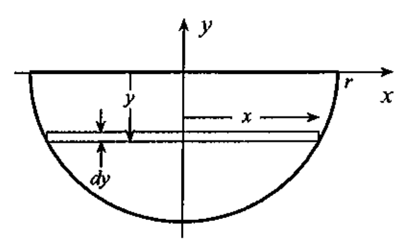
$$F = \int_0^L 9810(L-y)L\,dy$$
$$= 9810L\left\{Ly - \frac{y^2}{2}\right\}_0^L = 4905L^3 \text{ N}.$$

When we set this equal to 20 000 and solve, the result is $L = (20\,000/4905)^{1/3} = 1.60$ m.



15. The force is

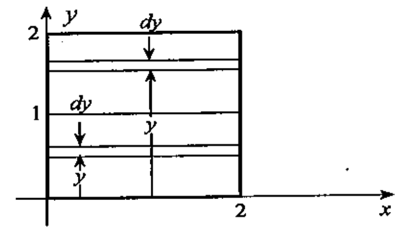
$$F = \int_{-r}^0 9.81\rho(-y)(2x)\,dy$$
$$= -19.62\rho\int_{-r}^0 y\sqrt{r^2-y^2}\,dy$$
$$= -19.62\rho\left\{-\frac{1}{3}(r^2-y^2)^{3/2}\right\}_{-r}^0 = 6.54\rho r^3.$$



16. If the top of the cylinder is at depth d, then the magnitude of the force on this face is $9.81\rho d(\pi r^2)$. Since the force on the bottom is $9.81\rho(d+h)\pi r^2$, the resultant vertical force on the cylinder is $9.81\rho(d+h)\pi r^2 - 9.82\rho d(\pi r^2) = 9.81\rho(\pi r^2 h)$. This is the weight of the fluid displaced by the cylinder.

17. (a) The force F_1 on the upper half is straightforward,

$$F_1 = \int_1^2 9.81(900)(2-y)(2)\,dy$$
$$= 17\,658\left\{-\frac{1}{2}(2-y)^2\right\}_1^2 = 8829 \text{ N}.$$



Pressure at a point y on the lower half of the surface is that due to the weight of fluid of unit cross-sectional area above the point,

$$9.81[900 + 1000(1-y)] = 981(19-10y).$$

The force on the lower half is therefore

$$F_2 = \int_0^1 981(19-10y)(2)\,dy = 1962\left\{-\frac{1}{20}(19-10y)^2\right\}_0^1 = 27\,468 \text{ N}.$$

The total force is $F_1 + F_2 = 36\,297$ N.

(b) If the water and oil create a mixture with density 0.95 gm/cm^3, the force on each side is

$$F = \int_0^2 9.81(950)(2-y)(2)\,dy = 18\,639\left\{-\frac{1}{2}(2-y)^2\right\}_0^2 = 37\,278 \text{ N}.$$

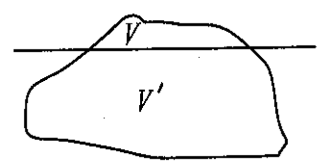
Thus, the force increases by 9810 N.

18. (a) Suppose V and V' are the volumes of the object above and below the surface respectively. The buoyant force on the object is therefore $9.81\rho_w V'$. Because the object is floating at this position, this force is equal to that of gravity on the object;

$$9.81\rho_w V' = 9.81\rho_o(V+V').$$

Thus, $\dfrac{V'}{V+V'} = \dfrac{\rho_o}{\rho_w}$. The percentage of the volume of the object above water is

$$100\frac{V}{V+V'} = 100\left(1 - \frac{V'}{V+V'}\right) = 100\left(1 - \frac{\rho_o}{\rho_w}\right) = 100\frac{\rho_w - \rho_o}{\rho_w}.$$

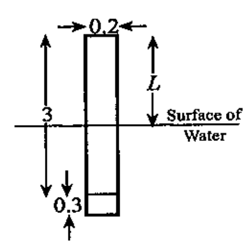


(b) For an iceberg, this percentage is $100\dfrac{1000-915}{1000} = 8.5$.

19. Suppose L represents the length of buoy above water. Archimedes' principle states that the weight of the buoy is equal to the weight of water displaced by the buoy,

$$\pi(0.1)^2(3)(500)g + \pi(0.1)^2(0.3)(3000)g = \pi(0.1)^2(3.3-L)(1000)g.$$

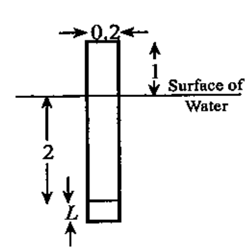
The solution of this equation is $L = 0.9$ m.



20. Suppose L is the length of the concrete attachment. Archimedes' principle states that the weight of the buoy must be equal to the weight of water displaced by the buoy,

$$\pi(0.1)^2(3)(500)g + \pi(0.1)^2(L)(3000)g = \pi(0.1)^2(2+L)(1000)g.$$

The solution of this equation is $L = 1/4$ m.

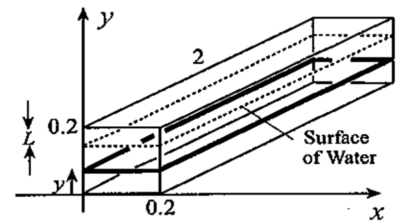


21. Suppose L denotes the height of log above water.
The density of the log is $\rho(y) = 1000 - 2500y$ kg/m^3.
The weight of the log is

$$W_{\text{log}} = \int_0^{1/5} (1000 - 2500y)g\left(\frac{1}{5}\right)(2)\,dy$$

$$= 200g\int_0^{1/5}(2-5y)\,dy$$

$$= 200g\left\{2y - \frac{5y^2}{2}\right\}_0^{1/5} = 60g \text{ N}.$$



The weight of the water displaced by the log is

$$W_{\text{water}} = 1000g\left(\frac{1}{5} - L\right)\left(\frac{1}{5}\right)(2) = 80g(1-5L) \text{ N}.$$

Archimedes' principle requires $W_{\text{log}} = W_{\text{water}}$ so that $60g = 80g(1-5L) \Longrightarrow L = 1/20$ m.

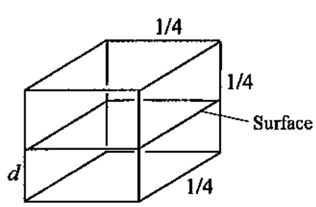
22. (a) The pressure at A due to the mercury in the tube is equal to the weight of a column of mercury of unit cross-sectional area and height h. With density of mercury equal to 13.6×10^3 kg/m^3, this weight is $(9.81)(13.6 \times 10^3)h = 1.33 \times 10^5 h$ N/m^2.
(b) When $h = .761$, atmospheric pressure is $(1.33 \times 10^5)(.761) = 1.01 \times 10^5$ N/m^2.

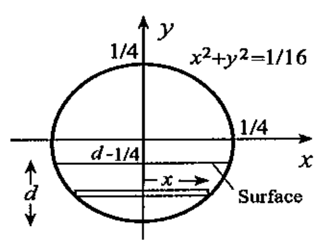
23. (a) If d is the depth of the lower face below the surface (left figure below), then the buoyant force due to water pressure is the weight of water displaced:

$$F = 9810\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)d = \frac{9810d}{16}.$$

Because the block is floating, this force is equal to the weight of the block; that is,

$$\frac{9810d}{16} = \left(\frac{1}{4}\right)^3(400)(9.81) \quad\Longrightarrow\quad d = 10 \text{ cm}.$$





(b) When the lowest point of the sphere is at depth d (right figure above), the volume of water displaced is

$$V = \int_{-1/4}^{d-1/4} \pi x^2\,dy = \pi\int_{-1/4}^{d-1/4}\left(\frac{1}{16} - y^2\right)dy = \pi\left\{\frac{y}{16} - \frac{y^3}{3}\right\}_{-1/4}^{d-1/4} = \frac{\pi}{12}(3d^2 - 4d^3).$$

At this position, the weight of water displaced is equal to the weight of the object:

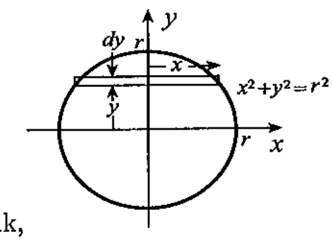
$$\frac{\pi}{12}(3d^2 - 4d^3)(9810) = \frac{4}{3}\pi\left(\frac{1}{4}\right)^3(9.81)(400).$$

This equation simplifies to $40d^3 - 30d^2 + 1 = 0$. Newton's iterative procedure

$$d_{n+1} = d_n - \frac{40d_n^3 - 30d_n^2 + 1}{120d_n^2 - 60d_n}$$

with an initial approximation $d_1 = 1/8$ gives $d = 0.2165$. The lowest point is therefore 21.65 cm below the surface.

24. The force on the end of the tank when the radius is r is



$$F = \int_{-r}^{r} 9.81\rho(r-y)2\sqrt{r^2-y^2}\,dy$$
$$= 19.62\rho r\int_{-r}^{r}\sqrt{r^2-y^2}\,dy + 19.62\rho\left\{\frac{1}{3}(r^2-y^2)^{3/2}\right\}_{-r}^{r}.$$

Since the integral represents half the area of the end of the tank,

$$F = 19.62\rho r\left(\frac{1}{2}\pi r^2\right) = 9.81\pi\rho r^3.$$

The radius of the end of the tank must satisfy $40\,000 - 2\pi r(1000) = 9.81\pi\rho r^3$. We use Newton's iterative procedure with initial approximation $r = 1$ to solve this equation,

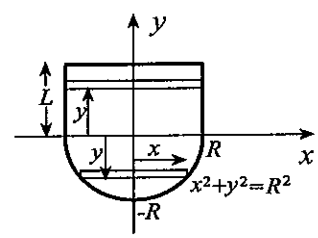
$$r_1 = 1, \qquad r_{n+1} = r_n - \frac{9.81(1019)\pi r_n^3 - 2000\pi r_n - 40\,000}{29.43(1019)\pi r_n^2 - 2000\pi}.$$

The result is $r = 1.02$ m.

25. The force on the rectangular part is

$$F_R = \int_0^L 9.81\rho(L-y)2R\,dy = 19.62\rho R\left\{Ly - \frac{y^2}{2}\right\}_0^L = 9.81\rho RL^2.$$

The force on the semicircular part is



$$F_S = \int_{-R}^{0} 9.81\rho(L-y)2x\,dy$$
$$= 19.62\rho\int_{-R}^{0}(L-y)\sqrt{R^2-y^2}\,dy$$
$$= 19.62\rho L\int_{-R}^{0}\sqrt{R^2-y^2}\,dy - 19.62\rho\int_{-R}^{0} y\sqrt{R^2-y^2}\,dy.$$

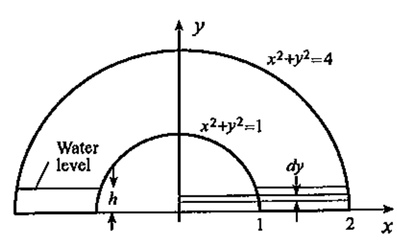
The first integral is the area of half the semicircular part, and therefore

$$F_S = 19.62\rho L\left(\frac{1}{4}\pi R^2\right) - 19.62\rho\left\{-\frac{1}{3}(R^2-y^2)^{3/2}\right\}_{-R}^{0} = 4.905\pi\rho LR^2 + 6.54\rho R^3.$$

These forces are equal when $9.81\rho RL^2 = 4.905\pi\rho LR^2 + 6.54\rho R^3$, and this equation simplifies to $6\left(\frac{L}{R}\right)^2 - 3\pi\left(\frac{L}{R}\right) - 4 = 0$. The positive solution of this quadratic equation in L/R is $L/R = 1.92$.

26. (a) If the shell is placed carefully on the water, no water will penetrate the hemispherical cavity. If the depth of the flat edge is h (figure to the right), then the volume of water displaced by the shell is

$$V = \int_0^h \pi x^2\,dy = \pi\int_0^h (4-y^2)\,dy$$
$$= \pi\left\{4y - \frac{y^3}{3}\right\}_0^h = \pi\left(4h - \frac{h^3}{3}\right)$$
$$= \frac{\pi h(12-h^2)}{3}.$$



According to Archimedes' principle, the weight of water displaced must be equal to the weight of the shell; that is,

$$9.81\left(\frac{2}{3}\pi\right)(2^3-1^3)(2) = 9.81(1000)\frac{\pi h(12-h^2)}{3}.$$

This equation reduces to $250h^3 - 3000h + 7 = 0$. With Newton's iterative procedure

$$h_1 = 0.0025, \qquad h_{n+1} = h_n - \frac{250h_n^3 - 3000h_n + 7}{750h_n^2 - 3000},$$

we obtain $h = 0.00233$. Hence, the shell sinks 2.33 mm.

(b) With a hole in the top of the shell, air will escape from the cavity and the water level inside the cavity will be the same as outside; only the water displaced by the shell itself is taken into account. The volume of water displaced is

$$V = \int_0^h \pi[(4 - y^2 - (1 - y^2)]\,dy = \pi \int_0^h 3\,dy = 3\pi h.$$

In this case, Archimedes' principle requires

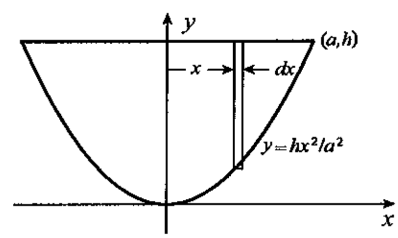
$$9.81\left(\frac{2}{3}\pi\right)(2^3 - 1^3)(2) = 9.81(1000)3\pi h,$$

and therefore $h = 0.00311$. The shell now sinks 3.11 mm.

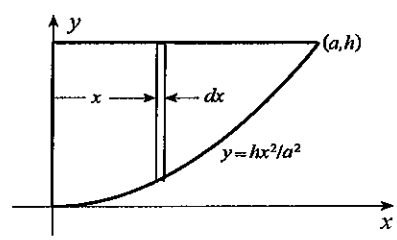
EXERCISES 7.7

1. By symmetry, $\overline{x} = 0$. $\quad M = 2\int_0^a \rho\left(h - \frac{hx^2}{a^2}\right)dx = 2\rho h\left\{x - \frac{x^3}{3a^2}\right\}_0^a = \frac{4\rho a h}{3}$

Since
$$M\overline{y} = 2\int_0^a \rho\left(\frac{h + hx^2/a^2}{2}\right)\left(h - \frac{hx^2}{a^2}\right)dx$$
$$= \rho h^2 \int_0^a \left(1 - \frac{x^4}{a^4}\right)dx$$
$$= \rho h^2\left\{x - \frac{x^5}{5a^4}\right\}_0^a = \frac{4\rho a h^2}{5},$$
we find $\overline{y} = \frac{4\rho a h^2}{5}\frac{3}{4\rho a h} = \frac{3h}{5}$.



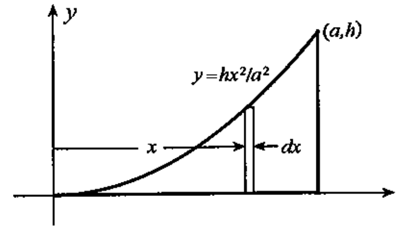
2. $$M = \int_0^a \rho\left(h - \frac{hx^2}{a^2}\right)dx$$
$$= \rho h\left\{x - \frac{x^3}{3a^2}\right\}_0^a = \frac{2\rho a h}{3}$$
Since
$$M\overline{x} = \int_0^a x\rho\left(h - \frac{hx^2}{a^2}\right)dx$$
$$= \rho h\left\{\frac{x^2}{2} - \frac{x^4}{4a^2}\right\}_0^a = \frac{\rho h a^2}{4},$$



it follows that $\overline{x} = \frac{\rho h a^2}{4}\frac{3}{2\rho a h} = \frac{3a}{8}$. Since

$$M\overline{y} = \int_0^a \rho\left(\frac{h + hx^2/a^2}{2}\right)\left(h - \frac{hx^2}{a^2}\right)dx = \frac{\rho h^2}{2}\int_0^a\left(1 - \frac{x^4}{a^4}\right)dx = \frac{\rho h^2}{2}\left\{x - \frac{x^5}{5a^4}\right\}_0^a = \frac{2\rho a h^2}{5},$$

we find $\overline{y} = \frac{2\rho a h^2}{5}\frac{3}{2\rho a h} = \frac{3h}{5}$.

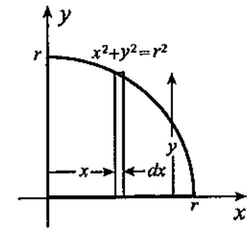


3. $M = \int_0^a \rho\left(\frac{hx^2}{a^2}\right)dx = \frac{\rho h}{a^2}\left\{\frac{x^3}{3}\right\}_0^a = \frac{\rho ha}{3}$

Since $M\overline{x} = \int_0^a \rho\left(\frac{hx^2}{a^2}\right)x\,dx = \frac{\rho h}{a^2}\left\{\frac{x^4}{4}\right\}_0^a = \frac{\rho ha^2}{4}$,

it follows that $\overline{x} = \frac{\rho ha^2}{4}\frac{3}{\rho ha} = \frac{3a}{4}$.

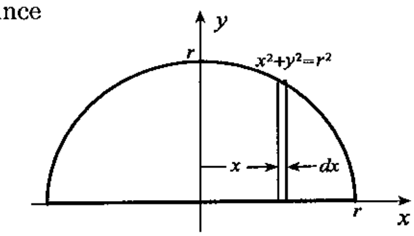
Since $M\overline{y} = \int_0^a \rho\left(\frac{hx^2}{a^2}\right)\left(\frac{hx^2}{2a^2}\right)dx = \frac{\rho h^2}{2a^4}\left\{\frac{x^5}{5}\right\}_0^a = \frac{\rho h^2 a}{10}$, we obtain $\overline{y} = \frac{\rho h^2 a}{10}\frac{3}{\rho ha} = \frac{3h}{10}$.



4. The mass of the plate is $M = \rho\pi r^2/4$. Since

$$M\overline{x} = \int_0^r x\rho y\,dx = \rho\int_0^r x\sqrt{r^2-x^2}\,dx$$
$$= \rho\left\{-\frac{1}{3}(r^2-x^2)^{3/2}\right\}_0^r = \frac{\rho r^3}{3},$$

it follows that $\overline{x} = \frac{\rho r^3}{3}\frac{4}{\pi\rho r^2} = \frac{4r}{3\pi}$. By symmetry, $\overline{y} = \overline{x} = 4r/(3\pi)$.

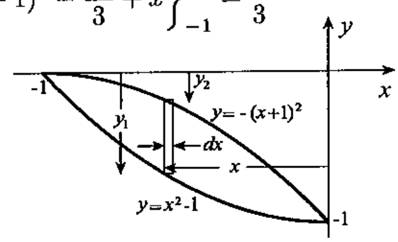


5. By symmetry, $\overline{x} = 0$. The mass of the plate is $M = \rho\pi r^2/2$. Since

$$M\overline{y} = 2\int_0^r \rho\sqrt{r^2-x^2}\left(\frac{\sqrt{r^2-x^2}}{2}\right)dx$$
$$= \rho\left\{r^2x - \frac{x^3}{3}\right\}_0^r = \frac{2\rho r^3}{3},$$

it follows that $\overline{y} = \frac{2\rho r^3}{3}\frac{2}{\rho\pi r^2} = \frac{4r}{3\pi}$.

6. $A = \int_{-1}^0 (y_2-y_1)\,dx = \int_{-1}^0 [-(x+1)^2-(x^2-1)]\,dx = \left\{-\frac{1}{3}(x+1)^3 - \frac{x^3}{3} + x\right\}_{-1}^0 = \frac{1}{3}$



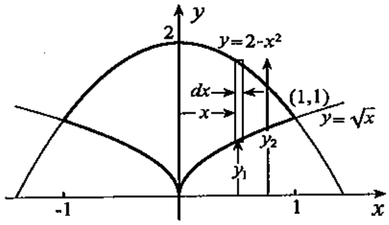
Since $A\overline{x} = \int_{-1}^0 x(y_2-y_1)\,dx = \int_{-1}^0 x[-(x+1)^2-(x^2-1)]\,dx$

$$= \int_{-1}^0 (-2x^3-2x^2)\,dx = \left\{-\frac{x^4}{2} - \frac{2x^3}{3}\right\}_{-1}^0 = -\frac{1}{6},$$

it follows that $\overline{x} = -\frac{1}{6}(3) = -\frac{1}{2}$. Since

$$A\overline{y} = \int_{-1}^0 \left(\frac{y_1+y_2}{2}\right)(y_2-y_1)\,dx = \frac{1}{2}\int_{-1}^0 (y_2^2-y_1^2)\,dx$$
$$= \frac{1}{2}\int_{-1}^0 [(x+1)^4-(x^4-2x^2+1)]\,dx = \frac{1}{2}\left\{\frac{1}{5}(x+1)^5 - \frac{x^5}{5} + \frac{2x^3}{3} - x\right\}_{-1}^0 = -\frac{1}{6},$$

we find $\overline{y} = -3/6 = -1/2$.



7. By symmetry, $\overline{x} = 0$. The area is

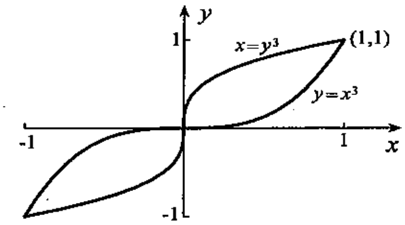
$$A = 2\int_0^1 (y_2-y_1)\,dx = 2\int_0^1 (2-x^2-\sqrt{x})\,dx$$
$$= 2\left\{2x - \frac{x^3}{3} - \frac{2x^{3/2}}{3}\right\}_0^1 = 2.$$

Since $A\overline{y} = 2\int_0^1 \left(\frac{y_1+y_2}{2}\right)(y_2-y_1)\,dx = \int_0^1 [(2-x^2)^2-x]\,dx$

$$= \int_0^1 (4-x-4x^2+x^4)\,dx = \left\{4x - \frac{x^2}{2} - \frac{4x^3}{3} + \frac{x^5}{5}\right\}_0^1 = \frac{71}{30},$$

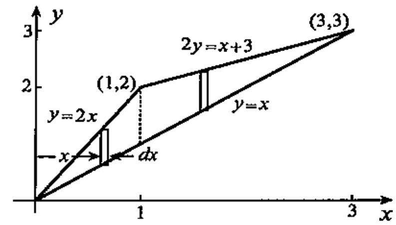
it follows that $\overline{y} = (71/30)(1/2) = 71/60$.

8. By symmetry, $\overline{x} = \overline{y} = 0$.



9. $A = \displaystyle\int_0^1 (2x - x)\,dx + \int_1^3 \left(\frac{x+3}{2} - x\right) dx$

$$= \left\{\frac{x^2}{2}\right\}_0^1 + \left\{\frac{3x}{2} - \frac{x^2}{4}\right\}_1^3 = \frac{3}{2}$$



Since $A\overline{x} = \displaystyle\int_0^1 x(2x - x)\,dx + \int_1^3 x\left(\frac{x+3}{2} - x\right) dx$

$$= \left\{\frac{x^3}{3}\right\}_0^1 + \left\{\frac{3x^2}{4} - \frac{x^3}{6}\right\}_1^3 = 2,$$

it follows that $\overline{x} = 2(2/3) = 4/3$. Since

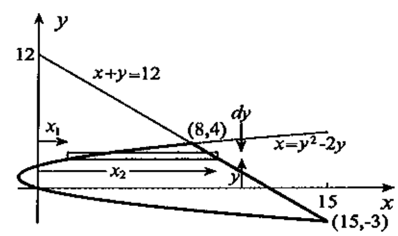
$$A\overline{y} = \int_0^1 \left(\frac{2x+x}{2}\right)(2x - x)\,dx + \int_1^3 \left[\frac{(x+3)/2 + x}{2}\right)\left(\frac{x+3}{2} - x\right) dx$$

$$= \frac{1}{2}\int_0^1 3x^2\,dx + \frac{1}{2}\int_1^3 \left[\left(\frac{x+3}{2}\right)^2 - x^2\right] dx = \frac{1}{2}\left\{x^3\right\}_0^1 + \frac{1}{2}\left\{\frac{2}{3}\left(\frac{x+3}{2}\right)^3 - \frac{x^3}{3}\right\}_1^3 = \frac{5}{2},$$

we find $\overline{y} = (5/2)(2/3) = 5/3$.

10. $A = \displaystyle\int_{-3}^4 (x_2 - x_1)\,dy = \int_{-3}^4 [(12 - y) - (y^2 - 2y)]\,dy$

$$= \left\{12y + \frac{y^2}{2} - \frac{y^3}{3}\right\}_{-3}^4 = \frac{343}{6}$$



Since $A\overline{x} = \displaystyle\int_{-3}^4 \left(\frac{x_2 + x_1}{2}\right)(x_2 - x_1)\,dy$

$$= \frac{1}{2}\int_{-3}^4 (x_2^2 - x_1^2)\,dy$$

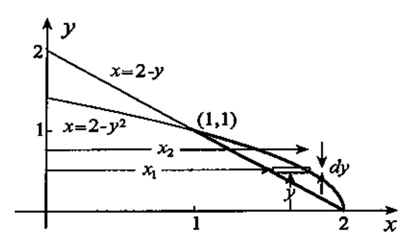
$$= \frac{1}{2}\int_{-3}^4 [(12 - y)^2 - (y^4 - 4y^3 + 4y^2)]\,dy = \frac{1}{2}\left\{-\frac{1}{3}(12 - y)^3 - \frac{y^5}{5} + y^4 - \frac{4y^3}{3}\right\}_{-3}^4 = \frac{3773}{10},$$

it follows that $\overline{x} = (3773/10)(6/343) = 33/5$. Since

$$A\overline{y} = \int_{-3}^4 y(x_2 - x_1)\,dy = \int_{-3}^4 y(12 + y - y^2)\,dy = \left\{6y^2 + \frac{y^3}{3} - \frac{y^4}{4}\right\}_{-3}^4 = \frac{343}{12},$$

we find $\overline{y} = (343/12)(6/343) = 1/2$.

11. $A = \displaystyle\int_0^1 (x_2 - x_1)\,dy = \int_0^1 [(2 - y^2) - (2 - y)]\,dy = \left\{\frac{y^2}{2} - \frac{y^3}{3}\right\}_0^1 = \frac{1}{6}$



Since $A\overline{x} = \displaystyle\int_0^1 \left(\frac{x_1 + x_2}{2}\right)(x_2 - x_1)\,dy = \frac{1}{2}\int_0^1 (x_2^2 - x_1^2)\,dy$

$$= \frac{1}{2}\int_0^1 [(2 - y^2)^2 - (2 - y)^2]\,dy$$

$$= \frac{1}{2}\int_0^1 (y^4 - 5y^2 + 4y)\,dy = \frac{1}{2}\left\{\frac{y^5}{5} - \frac{5y^3}{3} + 2y^2\right\}_0^1 = \frac{4}{15},$$

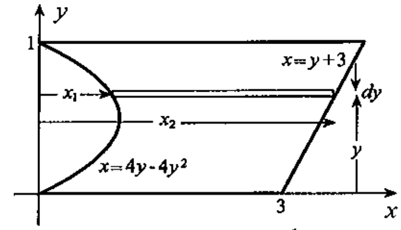
it follows that $\overline{x} = (4/15)(6) = 8/5$. Since

$$A\bar{y} = \int_0^1 y[(2-y^2)-(2-y)]\,dy = \int_0^1 (y^2-y^3)\,dy = \left\{\frac{y^3}{3}-\frac{y^4}{4}\right\}_0^1 = \frac{1}{12},$$

we find that $\bar{y} = (1/12)(6) = 1/2$.

12. $A = \int_0^1 (x_2 - x_1)\,dy = \int_0^1 [(y+3)-(4y-4y^2)]\,dy$

$$= \left\{3y - \frac{3y^2}{2} + \frac{4y^3}{3}\right\}_0^1 = \frac{17}{6}$$



Since $A\bar{x} = \int_0^1 \left(\frac{x_2+x_1}{2}\right)(x_2-x_1)\,dy = \frac{1}{2}\int_0^1 (x_2^2 - x_1^2)\,dy$

$$= \frac{1}{2}\int_0^1 [(y+3)^2 - (16y^2 - 32y^3 + 16y^4)]\,dy = \frac{1}{2}\left\{\frac{1}{3}(y+3)^3 - \frac{16y^3}{3} + 8y^4 - \frac{16y^5}{5}\right\}_0^1 = \frac{59}{10},$$

we find that $\bar{x} = (59/10)(6/17) = (177/85)$. Because

$$A\bar{y} = \int_0^1 y(x_2-x_1)\,dy = \int_0^1 y(3-3y+4y^2)\,dy = \left\{\frac{3y^2}{2} - y^3 + y^4\right\}_0^1 = \frac{3}{2},$$

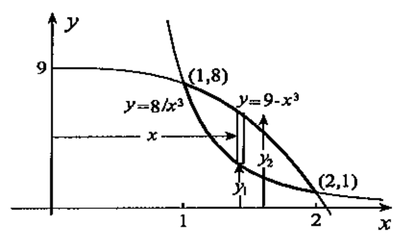
it follows that $\bar{y} = (3/2)(6/17) = (9/17)$.

13. $A = \int_1^2 (y_2 - y_1)\,dx = \int_1^2 \left(9 - x^3 - \frac{8}{x^3}\right)dx$

$$= \left\{9x - \frac{x^4}{4} + \frac{4}{x^2}\right\}_1^2 = \frac{9}{4}$$

Since $A\bar{x} = \int_1^2 x\left(9 - x^3 - \frac{8}{x^3}\right)dx$

$$= \left\{\frac{9x^2}{2} - \frac{x^5}{5} + \frac{8}{x}\right\}_1^2 = \frac{33}{10},$$



it follows that $\bar{x} = (33/10)(4/9) = 22/15$. Since

$$A\bar{y} = \int_1^2 \left(\frac{y_1+y_2}{2}\right)(y_2-y_1)\,dx = \frac{1}{2}\int_1^2 (y_2^2 - y_1^2)\,dx = \frac{1}{2}\int_1^2 \left[(9-x^3)^2 - \frac{64}{x^6}\right]dx$$

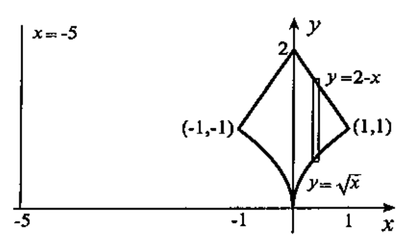
$$= \frac{1}{2}\int_1^2 \left(81 - 18x^3 + x^6 - \frac{64}{x^6}\right)dx = \frac{1}{2}\left\{81x - \frac{9x^4}{2} + \frac{x^7}{7} + \frac{64}{5x^5}\right\}_1^2 = \frac{4041}{420},$$

we find that $\bar{y} = (4041/420)(4/9) = 449/105$.

14. Since the centre of mass is on the y-axis, the first moment is $5M$ where M is the mass of the plate. Because the plate is symmetric about the y-axis, we find the mass of the right half and double the result:

$$\text{Moment} = 5(2)\int_0^1 \rho[(2-x)-\sqrt{x}]\,dx$$

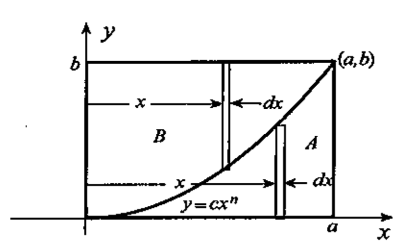
$$= 10\rho\left\{2x - \frac{x^2}{2} - \frac{2}{3}x^{3/2}\right\}_0^1 = \frac{25\rho}{3}.$$



15. First we notice that $b = ca^n \implies c = b/a^n$.
Areas of the regions are

$$A = \int_0^a cx^n\,dx = c\left\{\frac{x^{n+1}}{n+1}\right\}_0^a = \frac{ca^{n+1}}{n+1}, \quad B = ab - \frac{ca^{n+1}}{n+1}.$$

We now calculate moments of A and B about the axes,



$$A\overline{x}_A = \int_0^a x(cx^n)\,dx = c\left\{\frac{x^{n+2}}{n+2}\right\}_0^a = \frac{ca^{n+2}}{n+2},$$
$$B\overline{x}_B = \int_0^a x(b-cx^n)\,dx = \left\{\frac{bx^2}{2} - \frac{cx^{n+2}}{n+2}\right\}_0^a = \frac{ba^2}{2} - \frac{ca^{n+2}}{n+2},$$
$$A\overline{y}_A = \int_0^a \frac{1}{2}(cx^n)(cx^n)\,dx = \frac{c^2}{2}\left\{\frac{x^{2n+1}}{2n+1}\right\}_0^a = \frac{c^2a^{2n+1}}{2(2n+1)},$$
$$B\overline{y}_B = \int_0^a \frac{1}{2}(b+cx^n)(b-cx^n)\,dx = \frac{1}{2}\int_0^a (b^2 - c^2x^{2n})\,dx = \frac{1}{2}\left\{b^2x - \frac{c^2x^{2n+1}}{2n+1}\right\}_0^a = \frac{1}{2}\left(b^2a - \frac{c^2a^{2n+1}}{2n+1}\right).$$

We can now use $c = b/a^n$ to find

$$\overline{x}_A = \frac{ca^{n+2}}{n+2}\frac{n+1}{ca^{n+1}} = \left(\frac{n+1}{n+2}\right)a, \quad \overline{x}_B = \left(\frac{ba^2}{2} - \frac{ca^{n+2}}{n+2}\right)\left[\frac{n+1}{ab(n+1) - ca^{n+1}}\right] = \left(\frac{n+1}{2n+4}\right)a,$$

$$\overline{y}_A = \frac{c^2a^{2n+1}}{2(2n+1)}\frac{n+1}{ca^{n+1}} = \left(\frac{n+1}{4n+2}\right)b, \quad \overline{y}_B = \frac{1}{2}\left(ab^2 - \frac{c^2a^{2n+1}}{2n+1}\right)\left[\frac{n+1}{ab(n+1) - ca^{n+1}}\right] = \left(\frac{n+1}{2n+1}\right)b.$$

16. The mass of the seesaw itself is $2\rho L$, and this mass may be considered to act at its centre of mass, the midpoint of the seesaw. For balance to occur, the total first moment about the fulcrum must vanish:

$$0 = \sum_{i=1}^{6} m_i(x_i - \overline{x}) + 2\rho L(L - \overline{x}).$$

If we set $M = \sum_{i=1}^{6} m_i$, then $\overline{x}(M + 2\rho L) = \sum_{i=1}^{6} m_i x_i + 2\rho L^2$, and therefore

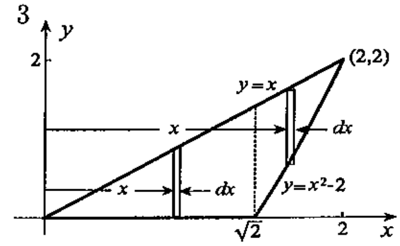
$$\overline{x} = \frac{1}{M + 2\rho L}\left(\sum_{i=1}^{6} m_i x_i + 2\rho L^2\right).$$

17. $A = \frac{1}{2}(2) + \int_{\sqrt{2}}^2 (x - x^2 + 2)\,dx = 1 + \left\{\frac{x^2}{2} - \frac{x^3}{3} + 2x\right\}_{\sqrt{2}}^2 = \frac{10 - 4\sqrt{2}}{3}$

Since $A\overline{x} = \int_0^{\sqrt{2}} x(x)\,dx + \int_{\sqrt{2}}^2 x(x - x^2 + 2)\,dx$

$$= \left\{\frac{x^3}{3}\right\}_0^{\sqrt{2}} + \left\{\frac{x^3}{3} - \frac{x^4}{4} + x^2\right\}_{\sqrt{2}}^2 = \frac{5}{3},$$

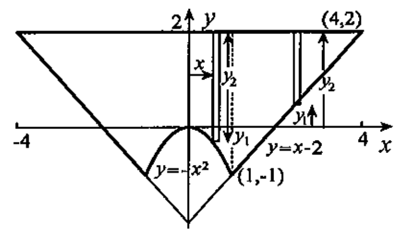
we obtain $\overline{x} = \frac{5}{3}\frac{3}{10 - 4\sqrt{2}} = \frac{5}{10 - 4\sqrt{2}}$. Since



$$A\overline{y} = \int_0^{\sqrt{2}} \left(\frac{x}{2}\right)(x)\,dx + \int_{\sqrt{2}}^2 \frac{1}{2}(x + x^2 - 2)(x - x^2 + 2)\,dx = \left\{\frac{x^3}{6}\right\}_0^{\sqrt{2}} + \frac{1}{2}\int_{\sqrt{2}}^2 (-4 + 5x^2 - x^4)\,dx$$
$$= \frac{\sqrt{2}}{3} + \frac{1}{2}\left\{-4x + \frac{5x^3}{3} - \frac{x^5}{5}\right\}_{\sqrt{2}}^2 = \frac{16\sqrt{2} - 8}{15},$$

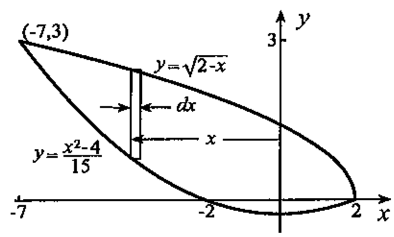
we find $\overline{y} = \frac{16\sqrt{2} - 8}{15}\frac{3}{10 - 4\sqrt{2}} = \frac{8\sqrt{2} - 4}{25 - 10\sqrt{2}}$.

18. $$A = 2\int_0^1 (y_2 - y_1)\,dx + 2\int_1^4 (y_2 - y_1)\,dx$$
$$= 2\int_0^1 (2 + x^2)\,dx + 2\int_1^4 [2 - (x-2)]\,dx$$
$$= 2\left\{2x + \frac{x^3}{3}\right\}_0^1 + 2\left\{4x - \frac{x^2}{2}\right\}_1^4 = \frac{41}{3}$$



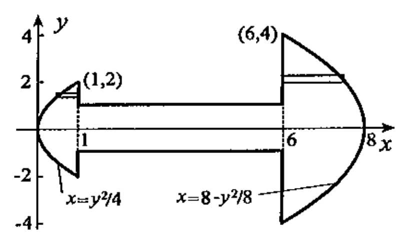
By symmetry, $\overline{x} = 0$, and because
$$A\overline{y} = 2\int_0^1 \left(\frac{y_2 + y_1}{2}\right)(y_2 - y_1)\,dx + 2\int_1^4 \left(\frac{y_2 + y_1}{2}\right)(y_2 - y_1)\,dx$$
$$= \int_0^1 (y_2^2 - y_1^2)\,dx + \int_1^4 (y_2^2 - y_1^2)\,dx = \int_0^1 (4 - x^4)\,dx + \int_1^4 [4 - (x-2)^2]\,dx$$
$$= \left\{4x - \frac{x^5}{5}\right\}_0^1 + \left\{4x - \frac{1}{3}(x-2)^3\right\}_1^4 = \frac{64}{5},$$
it follows that $\overline{y} = \dfrac{64}{5}\dfrac{3}{41} = \dfrac{192}{205}$.

19. $$A = \int_{-7}^2 \left(\sqrt{2-x} - \frac{x^2}{15} + \frac{4}{15}\right) dx$$
$$= \left\{-\frac{2}{3}(2-x)^{3/2} - \frac{x^3}{45} + \frac{4x}{15}\right\}_{-7}^2 = \frac{63}{5}$$



If we set $u = 2 - x$ and $du = -dx$ in the first term of
$$A\overline{x} = \int_{-7}^2 \left(x\sqrt{2-x} - \frac{x^3}{15} + \frac{4x}{15}\right) dx, \text{ then}$$
$$A\overline{x} = \int_9^0 (2-u)\sqrt{u}\,(-du) + \left\{\frac{-x^4}{60} + \frac{2x^2}{15}\right\}_{-7}^2 = \left\{-\frac{4u^{3/2}}{3} + \frac{2u^{5/2}}{5}\right\}_9^0 + \left\{\frac{-x^4}{60} + \frac{2x^2}{15}\right\}_{-7}^2 = -\frac{549}{20}.$$
Thus, $\overline{x} = -(549/20)(5/63) = -61/28$. Since
$$A\overline{y} = \int_{-7}^2 \left(\frac{\sqrt{2-x} + x^2/15 - 4/15}{2}\right)\left(\sqrt{2-x} - \frac{x^2}{15} + \frac{4}{15}\right) dx = \frac{1}{2}\int_{-7}^2 \left(2 - x - \frac{x^4}{225} + \frac{8x^2}{225} - \frac{16}{225}\right) dx$$
$$= \frac{1}{2}\left\{2x - \frac{x^2}{2} - \frac{x^5}{1125} + \frac{8x^3}{675} - \frac{16x}{225}\right\}_{-7}^2 = \frac{7263}{500},$$
it follows that $\overline{y} = (7263/500)(5/63) = 807/700$.

20. If ρ is the mass per unit area,
$$M = 2\int_0^2 \rho\left(1 - \frac{y^2}{4}\right) dy + 10\rho + 2\int_0^4 \rho\left(8 - \frac{y^2}{8} - 6\right) dy$$
$$= 2\rho\left\{y - \frac{y^3}{12}\right\}_0^2 + 10\rho + 2\rho\left\{2y - \frac{y^3}{24}\right\}_0^4 = \frac{70\rho}{3}.$$

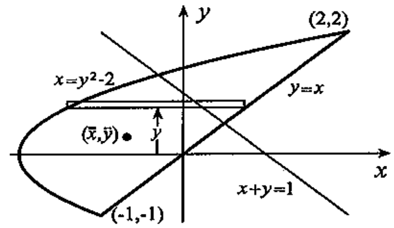


Clearly $\overline{y} = 0$, and because
$$M\overline{x} = 2\int_0^2 \frac{1}{2}\left(1 + \frac{y^2}{4}\right)\rho\left(1 - \frac{y^2}{4}\right) dy + 10\rho\left(\frac{7}{2}\right) + 2\int_0^4 \frac{1}{2}\left(8 - \frac{y^2}{8} + 6\right)\rho\left(8 - \frac{y^2}{8} - 6\right) dy$$
$$= \rho\int_0^2 \left(1 - \frac{y^4}{16}\right) dy + 35\rho + \rho\int_0^4 \left(28 - 2y^2 + \frac{y^4}{64}\right) dy = \rho\left\{y - \frac{y^5}{80}\right\}_0^2 + 35\rho + \rho\left\{28y - \frac{2y^3}{3} + \frac{y^5}{320}\right\}_0^4$$
$$= \frac{1637\rho}{15},$$
it follows that $\overline{x} = (1637\rho/15)[3/(70\rho)] = 1637/350$.

21. We find the centre of mass of the plate. Since $M = \int_{-1}^{2} \rho(y - y^2 + 2)\,dy = \rho\left\{\frac{y^2}{2} - \frac{y^3}{3} + 2y\right\}_{-1}^{2} = \frac{9\rho}{2}$, and

$$M\overline{x} = \int_{-1}^{2} \frac{\rho}{2}(y + y^2 - 2)(y - y^2 + 2)\,dy = \frac{\rho}{2}\int_{-1}^{2}(-y^4 + 5y^2 - 4)\,dy$$

$$= \frac{\rho}{2}\left\{-\frac{y^5}{5} + \frac{5y^3}{3} - 4y\right\}_{-1}^{2} = -\frac{9\rho}{5},$$



it follows that $\overline{x} = -\frac{9\rho}{5}\cdot\frac{2}{9\rho} = -\frac{2}{5}$. With

$$M\overline{y} = \int_{-1}^{2} \rho y(y - y^2 + 2)\,dy = \rho\left\{\frac{y^3}{3} - \frac{y^4}{4} + y^2\right\}_{-1}^{2} = \frac{9\rho}{4},$$

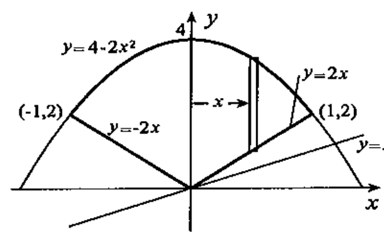
we find $\overline{y} = \frac{9\rho}{4}\cdot\frac{2}{9\rho} = \frac{1}{2}$. If we concentrate the mass at its centre of mass, and use formula 1.16 for the distance from the centre of mass to the line $x + y = 1$, we obtain for the required moment

$$\left(\frac{9\rho}{2}\right)\frac{|-2/5 + 1/2 - 1|}{\sqrt{2}} = \frac{81\sqrt{2}\rho}{40}.$$

22. We find the centre of mass of the plate. Since $M = 2\int_{0}^{1} \rho(4 - 2x^2 - 2x)\,dx = 2\rho\left\{4x - \frac{2x^3}{3} - x^2\right\}_{0}^{1} = \frac{14\rho}{3}$, and

$$M\overline{y} = 2\int_{0}^{1} \frac{\rho}{2}(4 - 2x^2 + 2x)(4 - 2x^2 - 2x)\,dx = 4\rho\int_{0}^{1}(4 - 5x^2 + x^4)\,dx$$

$$= 4\rho\left\{4x - \frac{5x^3}{3} + \frac{x^5}{5}\right\}_{0}^{1} = \frac{152\rho}{15},$$



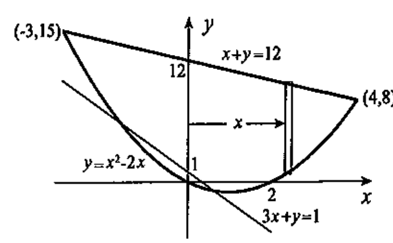
it follows that $\overline{y} = \frac{152\rho}{15}\cdot\frac{3}{14\rho} = \frac{76}{35}$. Symmetry of the plate indicates that $\overline{x} = 0$. If we concentrate the mass at its centre of mass, and use formula 1.16 for the distance from the centre of mass to the line $y = x$, we obtain for the required moment

$$\left(\frac{14\rho}{3}\right)\frac{|76/35 - 0|}{\sqrt{2}} = \frac{76\sqrt{2}\rho}{15}.$$

23. We find the centre of mass of the plate. Since $M = \int_{-3}^{4} \rho(12 - x - x^2 + 2x)\,dx = \rho\left\{12x + \frac{x^2}{2} - \frac{x^3}{3}\right\}_{-3}^{4} = \frac{343\rho}{6}$, and

$$M\overline{x} = \int_{-3}^{4} \rho x(12 - x^2 + x)\,dx = \rho\left\{6x^2 - \frac{x^4}{4} + \frac{x^3}{3}\right\}_{-3}^{4} = \frac{343\rho}{12},$$

it follows that $\overline{x} = \frac{343\rho}{12}\cdot\frac{6}{343\rho} = \frac{1}{2}$. With



$$M\overline{y} = \int_{-3}^{4} \frac{\rho}{2}(12 - x + x^2 - 2x)(12 - x - x^2 + 2x)\,dx$$

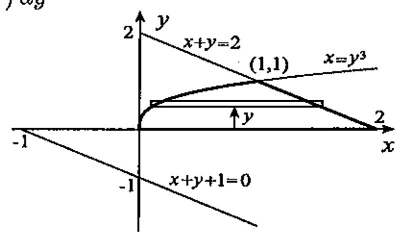
$$= \frac{\rho}{2}\int_{-3}^{4}(144 - 24x - 3x^2 + 4x^3 - x^4)\,dx$$

$$= \frac{\rho}{2}\left\{144x - 12x^2 - x^3 + x^4 - \frac{x^5}{5}\right\}_{-3}^{4} = \frac{3773\rho}{10},$$

we find $\overline{y} = \dfrac{3773\rho}{10} \cdot \dfrac{6}{343\rho} = \dfrac{33}{5}$. If we concentrate the mass at its centre of mass, and use formula 1.16 for the distance from the centre of mass to the line $3x + y = 1$, we obtain for the required moment

$$\left(\frac{343\rho}{6}\right) \frac{|3(1/2) + 33/5 - 1|}{\sqrt{10}} = \frac{24353\sqrt{10}\rho}{600}.$$

24. We find the centre of mass of the plate. Since $M = \displaystyle\int_0^1 \rho(2 - y - y^3)\,dy = \rho\left\{2y - \frac{y^2}{2} - \frac{y^4}{4}\right\}_0^1 = \frac{5\rho}{4}$, and



$$M\overline{x} = \int_0^1 \frac{\rho}{2}(2 - y + y^3)(2 - y - y^3)\,dy = \frac{\rho}{2}\int_0^1 (4 - 4y + y^2 - y^6)\,dy$$
$$= \frac{\rho}{2}\left\{4y - 2y^2 + \frac{y^3}{3} - \frac{y^7}{7}\right\}_0^1 = \frac{23\rho}{21},$$

it follows that $\overline{x} = \dfrac{23\rho}{21} \cdot \dfrac{4}{5\rho} = \dfrac{92}{105}$. With

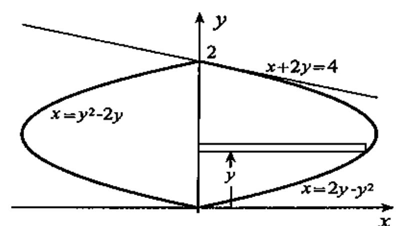
$$M\overline{y} = \int_0^1 \rho y(2 - y - y^3)\,dy = \rho\left\{y^2 - \frac{y^3}{3} - \frac{y^5}{5}\right\}_0^1 = \frac{7\rho}{15},$$

we find $\overline{y} = \dfrac{7\rho}{15} \cdot \dfrac{4}{5\rho} = \dfrac{28}{75}$. If we concentrate the mass at its centre of mass, and use formula 1.16 for the distance from the centre of mass to the line $x + y + 1 = 0$, we obtain for the required moment

$$\left(\frac{5\rho}{4}\right) \frac{|92/105 + 28/75 + 1|}{\sqrt{2}} = \frac{1181\sqrt{2}\rho}{840}.$$

25. Symmetry of the plate indicates that its centre of mass is $(0, 1)$. It mass is

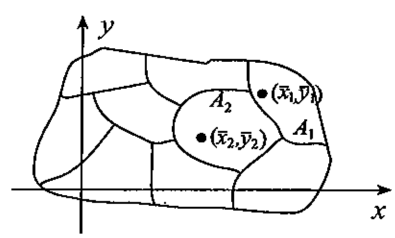
$$M = 2\int_0^2 \rho(2y - y^2)\,dy = 2\rho\left\{y^2 - \frac{y^3}{3}\right\}_0^2 = \frac{8\rho}{3}.$$



If we concentrate the mass at its centre of mass, and use formula 1.16 for the distance from the centre of mass to the line $x + 2y - 4 = 0$, we obtain for the required moment

$$\left(\frac{8\rho}{3}\right) \frac{|0 + 2(1) - 4|}{\sqrt{5}} = \frac{16\sqrt{5}\rho}{15}.$$

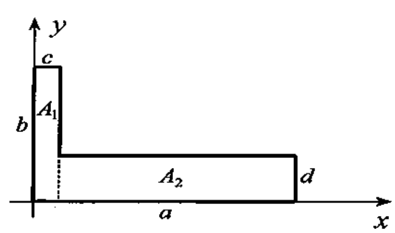
26. The first moment of A about the y-axis is $\quad A\overline{x} = \displaystyle\sum_{i=1}^{n} A_i\overline{x}_i,$



and therefore $\quad \overline{x} = \dfrac{1}{A}\displaystyle\sum_{i=1}^{n} A_i\overline{x}_i.$

Similarly, $\overline{y} = \dfrac{1}{A}\displaystyle\sum_{i=1}^{n} A_i\overline{y}_i.$

27. If we divide A into two parts as shown, then

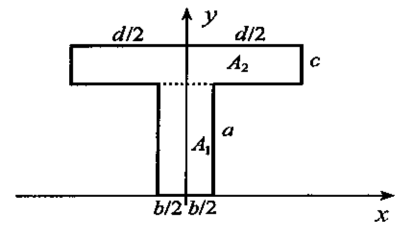


$$\overline{x} = \frac{1}{A}(A_1\overline{x}_1 + A_2\overline{x}_2)$$
$$= \frac{1}{bc + d(a - c)}\left[\frac{c}{2}(bc) + \frac{a + c}{2}(ad - cd)\right]$$
$$= \frac{c^2(b - d) + a^2 d}{2(ad + bc - cd)};$$

and

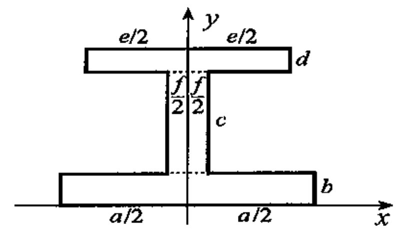
$$\overline{y} = \frac{1}{A}(A_1\overline{y}_1 + A_2\overline{y}_2) = \frac{1}{bc + d(a - c)}\left[\frac{b}{2}(bc) + \frac{d}{2}(ad - cd)\right] = \frac{c(b^2 - d^2) + ad^2}{2(ad + bc - cd)}.$$

28. Symmetry gives $\overline{x}=0$. If we divide the plate into two parts as shown, then



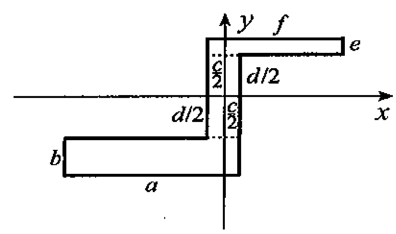
$$\overline{y}=\frac{1}{A}(A_1\overline{y}_1+A_2\overline{y}_2)=\frac{1}{ab+cd}\left[\frac{a}{2}(ab)+\left(a+\frac{c}{2}\right)(cd)\right]$$
$$=\frac{a^2b+cd(2a+c)}{2(ab+cd)}.$$

29. Symmetry give $\overline{x}=0$. If we divide the plate into three parts as shown, then



$$\overline{y}=\frac{1}{ab+cf+de}\left[\frac{b}{2}(ab)+\left(b+\frac{c}{2}\right)(cf)+\left(b+c+\frac{d}{2}\right)(de)\right]$$
$$=\frac{ab^2+cf(2b+c)+de(2b+2c+d)}{2(ab+cf+de)}.$$

30. If we divide the plate into three parts as shown, then

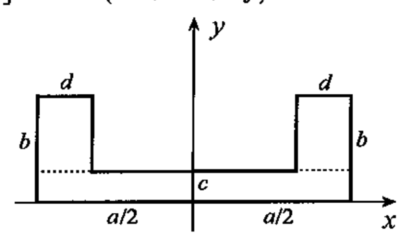


$$\overline{x}=\frac{1}{ab+cd+ef}\left[\left(\frac{c/2-a+c/2}{2}\right)(ab)+\left(\frac{f-c/2-c/2}{2}\right)(ef)\right]$$
$$=\frac{ab(c-a)+ef(f-c)}{2(ab+cd+ef)};$$

and

$$\overline{y}=\frac{1}{ab+cd+ef}\left[\left(-\frac{d}{2}-\frac{b}{2}\right)(ab)+\left(\frac{d}{2}+\frac{e}{2}\right)(ef)\right]=\frac{ef(d+e)-ab(b+d)}{2(ab+cd+ef)}.$$

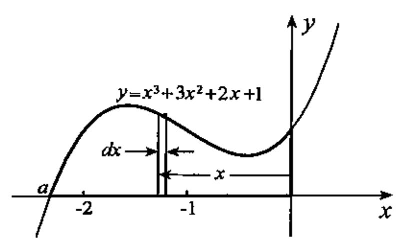
31. Symmetry gives $\overline{x}=0$. If we divide the plate into three parts as shown, then



$$\overline{y}=\frac{1}{ac+2d(b-c)}\left[\frac{c}{2}(ac)+\left(\frac{b+c}{2}\right)(2d)(b-c)\right]$$
$$=\frac{ac^2+2d(b^2-c^2)}{2ac+4d(b-c)}.$$

32. To six decimal places, the x-intercept of the curve is $a=-2.324\,718$. The area of the region is

$$A=\int_a^0(x^3+3x^2+2x+1)\,dx$$
$$=\left\{\frac{x^4}{4}+x^3+x^2+x\right\}\Bigg|_a^0=2.182\,258.$$



Since $A\overline{x}=\displaystyle\int_a^0 x(x^3+3x^2+2x+1)\,dx$

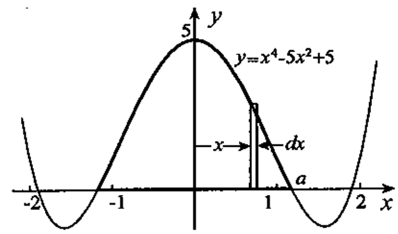
$$=\left\{\frac{x^5}{5}+\frac{3x^4}{4}+\frac{2x^3}{3}+\frac{x^2}{2}\right\}\Bigg|_a^0=-2.652\,017,$$

we get $\overline{x}=-2.652\,017/2.182\,258=-1.215$. Since

$$A\overline{y}=\int_a^0\frac{1}{2}(x^3+3x^2+2x+1)^2\,dx=\frac{1}{2}\int_a^0(x^6+6x^5+13x^4+14x^3+10x^2+4x+1)\,dx$$
$$=\frac{1}{2}\left\{\frac{x^7}{7}+x^6+\frac{13x^5}{5}+\frac{7x^4}{2}+\frac{10x^3}{3}+2x^2+x\right\}\Bigg|_a^0=1.140\,899,$$

it follows that $\overline{y}=1.140\,899/2.182\,258=0.523$.

33. The positive x-intercept closest to the origin is $a = \sqrt{(5-\sqrt{5})/2}$. Symmetry makes it clear that $\overline{x} = 0$. The area of the plate is



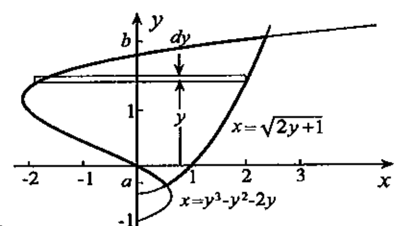
$$A = 2\int_0^a (x^4 - 5x^2 + 5)\,dx = 2\left\{\frac{x^5}{5} - \frac{5x^3}{3} + 5x\right\}_0^a = 7.238\,433.$$

Since

$$A\overline{y} = 2\int_0^a \frac{1}{2}(x^4 - 5x^2 + 5)^2\,dx = \int_0^a (x^8 - 10x^6 + 35x^4 - 50x^2 + 25)\,dx$$
$$= \left\{\frac{x^9}{9} - \frac{10x^7}{7} + 7x^5 - \frac{50x^3}{3} + 25x\right\}_0^a = 14.072\,588,$$

it follows that $\overline{y} = \dfrac{14.072\,588}{7.238\,433} = 1.944$.

34. The y-coordinates of the points of intersection are $a = -0.354740$ and $b = 2.310040$. The area of the region is



$$A = \int_a^b (\sqrt{2y+1} - y^3 + y^2 + 2y)\,dy = \left\{\frac{1}{3}(2y+1)^{3/2} - \frac{y^4}{4} + \frac{y^3}{3} + y^2\right\}_a^b = 6.608\,233.$$

Since

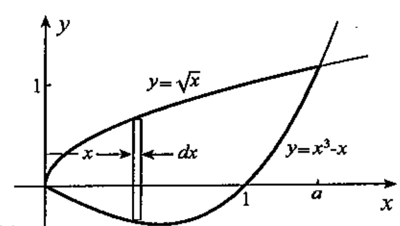
$$A\overline{x} = \int_a^b \left(\frac{\sqrt{2y+1} + y^3 - y^2 - 2y}{2}\right)(\sqrt{2y+1} - y^3 + y^2 + 2y)\,dy$$
$$= \frac{1}{2}\int_a^b (1 + 2y - y^6 + 2y^5 + 3y^4 - 4y^3 - 4y^2)\,dy = \frac{1}{2}\left\{y + y^2 - \frac{y^7}{7} + \frac{y^6}{3} + \frac{3y^5}{5} - y^4 - \frac{4y^3}{3}\right\}_a^b$$
$$= 1.448\,074,$$

it follows that $\overline{x} = 1.448\,074/6.608\,233 = 0.219$. If we set $u = 2y+1$ and $du = 2\,dy$ in the first term of the following integral, then

$$A\overline{y} = \int_a^b y(\sqrt{2y+1} - y^3 + y^2 + 2y)\,dy = \int_{2a+1}^{2b+1} \left(\frac{u-1}{2}\right)\sqrt{u}\left(\frac{du}{2}\right) + \left\{-\frac{y^5}{5} + \frac{y^4}{4} + \frac{2y^3}{3}\right\}_a^b$$
$$= \frac{1}{4}\left\{\frac{2u^{5/2}}{5} - \frac{2u^{3/2}}{3}\right\}_{2a+1}^{2b+1} + \left\{-\frac{y^5}{5} + \frac{y^4}{4} + \frac{2y^3}{3}\right\}_a^b = 7.494\,397.$$

Hence, $\overline{y} = 7.494\,397/6.608\,233 = 1.134$.

35. The x-coordinate of the point of intersection of the curves is $a = 1.362\,599$. The area of the region is



$$A = \int_0^a (\sqrt{x} - x^3 + x)\,dx = \left\{\frac{2}{3}x^{3/2} - \frac{x^4}{4} + \frac{x^2}{2}\right\}_0^a = 1.126\,905.$$

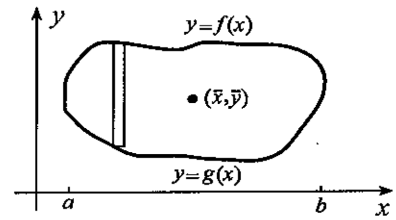
Since $A\overline{x} = \displaystyle\int_0^a x(\sqrt{x} - x^3 + x)\,dx = \left\{\frac{2}{5}x^{5/2} - \frac{x^5}{5} + \frac{x^3}{3}\right\}_0^a = 0.770\,781$, it follows that $\overline{x} = 0.770\,781/1.126\,905 = 0.684$. Since

$$A\overline{y} = \int_0^a \frac{1}{2}(\sqrt{x} + x^3 - x)(\sqrt{x} - x^3 + x)\,dx = \frac{1}{2}\int_0^a (x - x^2 + 2x^4 - x^6)\,dx$$
$$= \frac{1}{2}\left\{\frac{x^2}{2} - \frac{x^3}{3} + \frac{2x^5}{5} - \frac{x^7}{7}\right\}_0^a = 0.359\,018,$$

we obtain $\overline{y} = 0.359\,018/1.126\,905 = 0.319$.

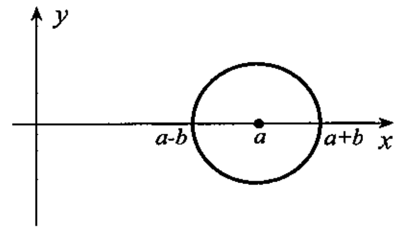
36. If we choose the y-axis along the axis of rotation, the volume generated is

$$V = \int_a^b 2\pi x[f(x) - g(x)]\,dx$$
$$= 2\pi \int_a^b x[f(x) - g(x)]\,dx$$
$$= 2\pi(A\overline{x}) = (2\pi\overline{x})A.$$



37. Since the centroid of the circle is at $(a, 0)$, the volume of the donut is

$$V = (\pi b^2)(2\pi a) = 2\pi^2 ab^2.$$



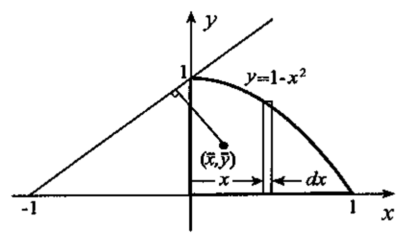
38. We find the centroid of the area. Since

$$A = \int_0^1 (1 - x^2)\,dx = \left\{x - \frac{x^3}{3}\right\}_0^1 = \frac{2}{3},$$

and

$$A\overline{x} = \int_0^1 x(1 - x^2)\,dx = \left\{\frac{x^2}{2} - \frac{x^4}{4}\right\}_0^1 = \frac{1}{4},$$

we find $\overline{x} = (1/4)(3/2) = 3/8$. Since



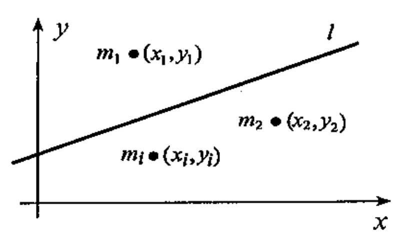
$$A\overline{y} = \int_0^1 \frac{1}{2}(1 - x^2)^2\,dx = \frac{1}{2}\int_0^1 (1 - 2x^2 + x^4)\,dx = \frac{1}{2}\left\{x - \frac{2x^3}{3} + \frac{x^5}{5}\right\}_0^1 = \frac{4}{15},$$

we get $\overline{y} = (4/15)(3/2) = 2/5$. Using the result of Exercise 36 and distance formula 1.16, the volume is

$$V = 2\pi\left(\frac{|3/8 - 2/5 + 1|}{\sqrt{2}}\right)\left(\frac{2}{3}\right) = \frac{13\sqrt{2}\pi}{20}.$$

39. (a) If $Ax + By + C = 0$ is the equation of the line l, then the first moment of the system about l is $\sum_{i=1}^{n} m_i d_i$ where d_i is the distance from the m_i to l.

Using formula 1.16, $d_i = \dfrac{Ax_i + By_i + C}{\sqrt{A^2 + B^2}}$, and therefore



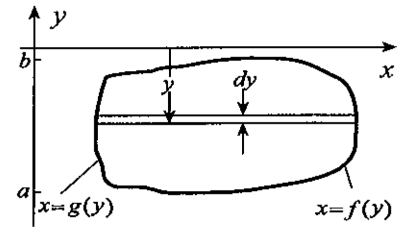
$$\sum_{i=1}^{n} m_i d_i = \sum_{i=1}^{n} \frac{m_i(Ax_i + By_i + C)}{\sqrt{A^2 + B^2}}$$
$$= \frac{1}{\sqrt{A^2 + B^2}}\left[A\sum_{i=1}^{n} m_i x_i + B\sum_{i=1}^{n} m_i y_i + C\sum_{i=1}^{n} m_i\right]$$
$$= \frac{1}{\sqrt{A^2 + B^2}}[A(M\overline{x}) + B(M\overline{y}) + CM] = M\left[\frac{A\overline{x} + B\overline{y} + C}{\sqrt{A^2 + B^2}}\right] = Md$$

where d is the distance from $(\overline{x}, \overline{y})$ to l.

(b) Yes, since for such a line $A\overline{x} + B\overline{y} + C = 0$.

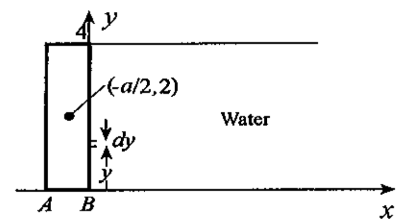
40. For the plate shown,

$$F = \int_a^b 9.81\rho(-y)[f(y) - g(y)]\,dy$$
$$= -9.81\rho \int_a^b y[f(y) - g(y)]\,dy$$
$$= -9.81\rho(A\overline{y}) = 9.81\rho A(-\overline{y}).$$



41. (a) For equilibrium, moments of concrete dam and water on the face of the dam about the line through A (perpendicular to the page) must sum to zero,

$$0 = -4(10)a(2400g)\left(\frac{a}{2}\right) + \int_0^4 (y)\rho g(4 - y)(10)\,dy$$
$$= -48\,000ga^2 + 10\rho g\left\{2y^2 - \frac{y^3}{3}\right\}_0^4$$
$$= -48\,000ga^2 + \frac{320\,000g}{3}.$$



This implies that $a = 2\sqrt{5}/3$ m.

(b) In this case we include the moment due to water pressure on AB,

$$0 = -4(10)a(2400g)\left(\frac{a}{2}\right) + \int_0^4 (y)\rho g(4 - y)(10)\,dy + \int_0^a (x)\rho g(4)(10)\,dx$$
$$= -48\,000ga^2 + \frac{320\,000g}{3} + 40\,000g\left\{\frac{x^2}{2}\right\}_0^a = -28\,000ga^2 + \frac{320\,000g}{3}.$$

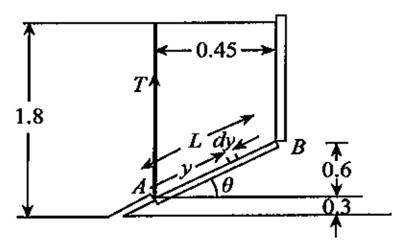
This implies that $a = 4\sqrt{105}/21$ m.

42. The sum of the moments of the tension T in the cable and the force of the water on the gate about B must be zero,

$$0 = -T(0.45) + \int_0^L (L - y)(1.5 - y\sin\theta)\rho g(0.525)\,dy.$$

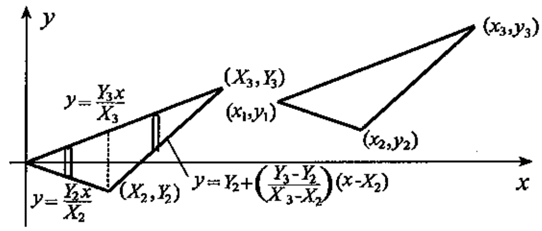
Since $\sin\theta = 600/750 = 4/5$, and $L = \sqrt{(0.6)^2 + (0.45)^2} = 3/4$,

$$T = \frac{0.525\rho g}{0.45}\int_0^{3/4}\left(\frac{3}{4} - y\right)\left(\frac{3}{2} - \frac{4y}{5}\right)dy$$
$$= \frac{0.525\rho g}{0.45}\left\{\frac{9y}{8} - \frac{21y^2}{20} + \frac{4y^3}{15}\right\}_0^{3/4}$$
$$= 4185 \text{ N}.$$



43. If the triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) in the figure is translated so that the vertex (x_1, y_1) coincides with the origin, then vertices of the new triangle corresponding to (x_2, y_2) and (x_3, y_3) are $(X_2, Y_2) = (x_2 - x_1, y_2 - y_1)$ and $(X_3, Y_3) = (x_3 - x_1, y_3 - y_1)$ respectively. We first find the centroid of the translated triangle. Its area is

$$A = \int_0^{X_2}\left(\frac{Y_3x}{X_3} - \frac{Y_2x}{X_2}\right)dx + \int_{X_2}^{X_3}\left[\frac{Y_3x}{X_3} - Y_2 - \left(\frac{Y_3 - Y_2}{X_3 - X_2}\right)(x - X_2)\right]dx$$
$$= \left\{\left(\frac{Y_3}{X_3} - \frac{Y_2}{X_2}\right)\frac{x^2}{2}\right\}_0^{X_2} + \left\{\frac{Y_3}{X_3}\frac{x^2}{2} - Y_2x - \frac{1}{2}\left(\frac{Y_3 - Y_2}{X_3 - X_2}\right)(x - X_2)^2\right\}_{X_2}^{X_3}$$
$$= \frac{1}{2}(X_2Y_3 - X_3Y_2).$$



The first moment of the triangle about the y-axis is

$$\begin{aligned}A\overline{X} &= \int_0^{X_2}\left(\frac{Y_3}{X_3}-\frac{Y_2}{X_2}\right)x^2\,dx+\int_{X_2}^{X_3}\left\{\left(\frac{Y_3}{X_3}-\frac{Y_3-Y_2}{X_3-X_2}\right)x^2+\left[-Y_2+X_2\left(\frac{Y_3-Y_2}{X_3-X_2}\right)\right]x\right\}dx\\ &= \left(\frac{Y_3}{X_3}-\frac{Y_2}{X_2}\right)\left\{\frac{x^3}{3}\right\}_0^{X_2}+\left(\frac{Y_3}{X_3}-\frac{Y_3-Y_2}{X_3-X_2}\right)\left\{\frac{x^3}{3}\right\}_{X_2}^{X_3}+\left[-Y_2+X_2\left(\frac{Y_3-Y_2}{X_3-X_2}\right)\right]\left\{\frac{x^2}{2}\right\}_{X_2}^{X_3}\\ &= \frac{1}{6}(X_2Y_3-X_3Y_2)(X_2+X_3).\end{aligned}$$

Thus, $\overline{X}=\dfrac{(X_2Y_3-X_3Y_2)(X_2+X_3)}{6}\dfrac{2}{X_2Y_3-X_3Y_2}=\dfrac{X_2+X_3}{3}$. The first moment of the triangle about the y-axis is

$$\begin{aligned}A\overline{Y} &= \int_0^{X_2}\frac{1}{2}\left[\left(\frac{Y_3}{X_3}\right)^2-\left(\frac{Y_2}{X_2}\right)^2\right]x^2\,dx+\frac{1}{2}\int_{X_2}^{X_3}\left[\left(\frac{Y_3}{X_3}\right)^2x^2-Y_2^2-\left(\frac{Y_3-Y_2}{X_3-X_2}\right)^2(x-X_2)^2\right.\\ &\qquad \left.-2Y_2\left(\frac{Y_3-Y_2}{X_3-X_2}\right)(x-X_2)\right]dx\\ &= \frac{1}{2}\left[\left(\frac{Y_3}{X_3}\right)^2-\left(\frac{Y_2}{X_2}\right)^2\right]\left\{\frac{x^3}{3}\right\}_0^{X_2}+\frac{1}{2}\left(\frac{Y_3}{X_3}\right)^2\left\{\frac{x^3}{3}\right\}_{X_2}^{X_3}-\frac{1}{2}Y_2^2\Big\{x\Big\}_{X_2}^{X_3}\\ &\qquad -\frac{1}{6}\left(\frac{Y_3-Y_2}{X_3-X_2}\right)^2\left\{(x-X_2)^3\right\}_{X_2}^{X_3}-Y_2\left(\frac{Y_3-Y_2}{X_3-X_2}\right)\left\{(x-X_2)^2\right\}_{X_2}^{X_3}\\ &= \frac{1}{6}(X_2Y_3-X_3Y_2)(Y_2+Y_3).\end{aligned}$$

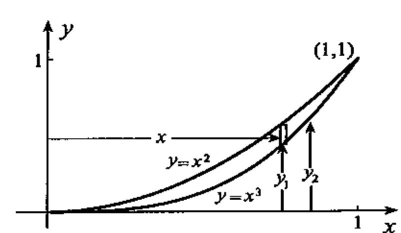
Thus, $\overline{Y}=\dfrac{(X_2Y_3-X_3Y_2)(Y_2+Y_3)}{6}\dfrac{2}{X_2Y_3-X_3Y_2}=\dfrac{Y_2+Y_3}{3}$. The centroid $(\overline{x},\overline{y})$ of the original triangle is therefore

$$\begin{aligned}(\overline{x},\overline{y}) &= (\overline{X}+x_1,\overline{Y}+y_1)=\left(\frac{X_2+X_3}{3}+x_1,\frac{Y_2+Y_3}{3}+y_1\right)\\ &= \left(\frac{x_2-x_1+x_3-x_1+3x_1}{3},\frac{y_2-y_1+y_3-y_1+3y_1}{3}\right)=\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right).\end{aligned}$$

EXERCISES 7.8

1. $I=\displaystyle\int_0^1 x^2\rho(y_2-y_1)\,dx=\rho\int_0^1 x^2(x^2-x^3)\,dx$

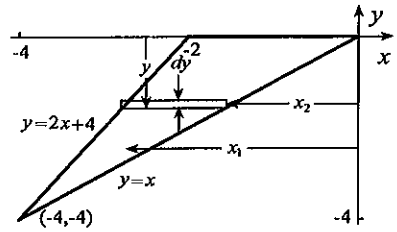
$\displaystyle=\rho\int_0^1(x^4-x^5)\,dx=\rho\left\{\frac{x^5}{5}-\frac{x^6}{6}\right\}_0^1=\frac{\rho}{30}$



2. $I = \int_{-4}^{0} (-y)^2 \rho (x_2 - x_1)\, dy$

$$= \rho \int_{-4}^{0} y^2 \left[y - \left(\frac{y-4}{2} \right) \right] dy = \frac{\rho}{2} \int_{-4}^{0} (y^3 + 4y^2)\, dy$$

$$= \frac{\rho}{2} \left\{ \frac{y^4}{4} + \frac{4y^3}{3} \right\}_{-4}^{0} = \frac{32\rho}{3}$$

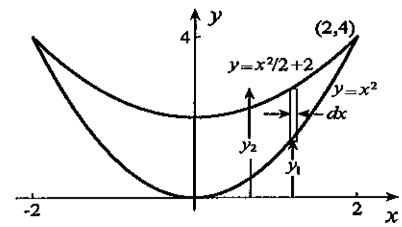


3. Using formula 7.42,

$$I = 2 \int_0^2 \frac{\rho}{3} (y_2^3 - y_1^3)\, dx = \frac{2\rho}{3} \int_0^2 [(2 + x^2/2)^3 - (x^2)^3]\, dx$$

$$= \frac{2\rho}{3} \int_0^2 \left(8 + 6x^2 + \frac{3x^4}{2} - \frac{7x^6}{8} \right) dx$$

$$= \frac{2\rho}{3} \left\{ 8x + 2x^3 + \frac{3x^5}{10} - \frac{x^7}{8} \right\}_0^2 = \frac{256\rho}{15}$$

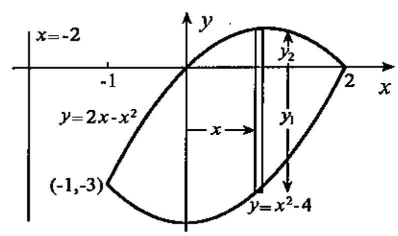


4. $I = \int_{-1}^{2} (x+2)^2 \rho (y_2 - y_1)\, dx$

$$= \rho \int_{-1}^{2} (x+2)^2 (2x - x^2 - x^2 + 4)\, dx$$

$$= \rho \int_{-1}^{2} (16 + 24x + 4x^2 - 6x^3 - 2x^4)\, dx$$

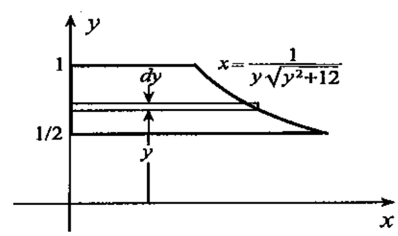
$$= \rho \left\{ 16x + 12x^2 + \frac{4x^3}{3} - \frac{3x^4}{2} - \frac{2x^5}{5} \right\}_{-1}^{2} = \frac{603\rho}{10}$$



5. $I = \int_{1/2}^{1} y^2 \rho \left(\frac{1}{y\sqrt{y^2+12}} \right) dy$

$$= \rho \int_{1/2}^{1} \frac{y}{\sqrt{y^2+12}} dy$$

$$= \rho \left\{ \sqrt{y^2+12} \right\}_{1/2}^{1} = \rho(\sqrt{13} - 7/2)$$

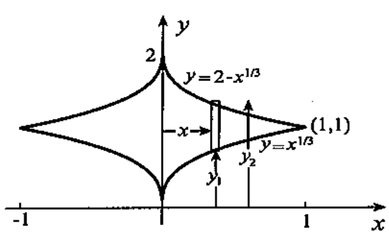


6. $I = 2 \int_0^1 x^2 \rho (y_2 - y_1)\, dx$

$$= 2\rho \int_0^1 x^2 (2 - x^{1/3} - x^{1/3})\, dx$$

$$= 4\rho \int_0^1 (x^2 - x^{7/3})\, dx$$

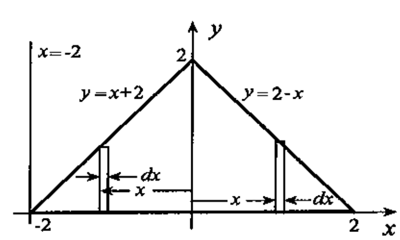
$$= 4\rho \left\{ \frac{x^3}{3} - \frac{3}{10} x^{10/3} \right\}_0^1 = \frac{2\rho}{15}$$



7. $I = \int_{-2}^{0} (x+2)^2 \rho (x+2)\, dx + \int_0^2 (x+2)^2 \rho (2-x)\, dx$

$$= \rho \int_{-2}^{0} (x+2)^3\, dx + \rho \int_0^2 (8 + 4x - 2x^2 - x^3)\, dx$$

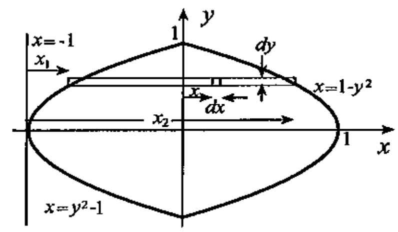
$$= \rho \left\{ \frac{(x+2)^4}{4} \right\}_{-2}^{0} + \rho \left\{ 8x + 2x^2 - \frac{2x^3}{3} - \frac{x^4}{4} \right\}_0^2 = \frac{56\rho}{3}$$



8. If we divide the long horizontal rectangle with width dy into tiny rectangles with length dx, then the moment of inertia of the long rectangle about the line $x = -1$ is

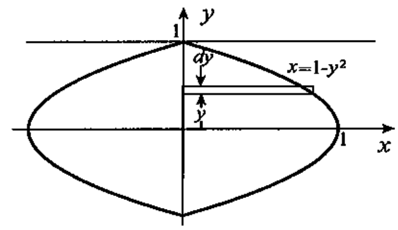
$$\int_{x_1}^{x_2} (x+1)^2 \rho dy\, dx = \rho\, dy \left\{ \frac{1}{3}(x+1)^3 \right\}_{x_1}^{x_2} = \frac{\rho}{3}[(x_2+1)^3 - (x_1+1)^3]\, dy.$$

The moment of inertia of the plate is



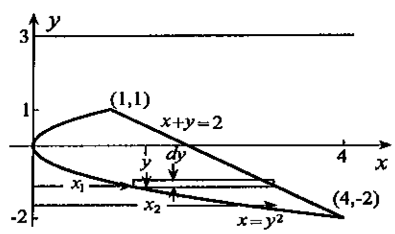
$$\begin{aligned} I &= \frac{2\rho}{3}\int_0^1 [(1-y^2+1)^3 - (y^2-1+1)^3]\, dy \\ &= \frac{2\rho}{3}\int_0^1 (8 - 12y^2 + 6y^4 - 2y^6)\, dy \\ &= \frac{2\rho}{3}\left\{ 8y - 4y^3 + \frac{6y^5}{5} - \frac{2y^7}{7} \right\}_0^1 = \frac{344\rho}{105} \end{aligned}$$

9.



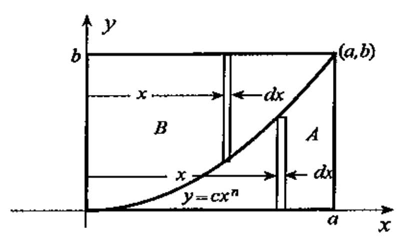
$$\begin{aligned} I &= 2\int_{-1}^1 (y-1)^2 \rho (1-y^2)\, dy \\ &= 2\rho \int_{-1}^1 (-y^4 + 2y^3 - 2y + 1)\, dy \\ &= 2\rho \left\{ -\frac{y^5}{5} + \frac{y^4}{2} - y^2 + y \right\}_{-1}^1 = \frac{16\rho}{5} \end{aligned}$$

10.



$$\begin{aligned} I &= \int_{-2}^1 (y-3)^2 \rho (x_2 - x_1)\, dy = \rho \int_{-2}^1 (y-3)^2[(2-y) - y^2]\, dy \\ &= \rho \int_{-2}^1 (18 - 21y - y^2 + 5y^3 - y^4)\, dy \\ &= \rho \left\{ 18y - \frac{21y^2}{2} - \frac{y^3}{3} + \frac{5y^4}{4} - \frac{y^5}{5} \right\}_{-2}^1 = \frac{1143\rho}{20} \end{aligned}$$

11. (a) Using formula 7.42,



$$\begin{aligned} I_{Bx} &= \int_0^a \frac{1}{3}[b^3 - (cx^n)^3]\, dx = \frac{1}{3}\int_0^a (b^3 - c^3x^{3n})\, dx \\ &= \frac{1}{3}\left\{ b^3x - \frac{c^3x^{3n+1}}{3n+1} \right\}_0^a = \frac{1}{3}\left(b^3a - \frac{c^3a^{3n+1}}{3n+1} \right) \\ &= \frac{nb^3a}{3n+1}, \text{ (using } b = ca^n) \end{aligned}$$

$$I_{Ax} = \int_0^a \frac{1}{3}(cx^n)^3\, dx = \frac{c^3}{3}\int_0^a x^{3n}\, dx = \frac{c^3}{3}\left\{ \frac{x^{3n+1}}{3n+1} \right\}_0^a = \frac{c^3a^{3n+1}}{3(3n+1)} = \frac{b^3a}{3(3n+1)}$$

This shows that $I_{Bx} = 3nI_{Ax}$.

(b) The following calculations show that $nI_{Ay} = 3I_{By}$.

$$I_{Ay} = \int_0^a x^2(cx^n)\, dx = c\int_0^a x^{n+2}\, dx = c\left\{ \frac{x^{n+3}}{n+3} \right\}_0^a = \frac{ca^{n+3}}{n+3} = \frac{ba^3}{n+3}$$

$$I_{By} = \int_0^a x^2(b - cx^n)\, dx = \int_0^a (bx^2 - cx^{n+2})\, dx = \left\{ \frac{bx^3}{3} - \frac{cx^{n+3}}{n+3} \right\}_0^a = \frac{ba^3}{3} - \frac{ca^{n+3}}{n+3} = \frac{ba^3n}{3(n+3)}$$

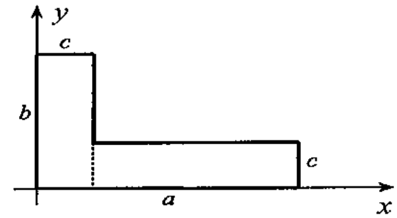
12. The product of the mass and the square of the distance from the x-axis to the centre of mass of the rectangle is $\rho(y_2 - y_1)h\left(\frac{y_1+y_2}{2}\right)^2$. This is not the same as 7.42. For example, if $y_1 = 0$, so that the rectangle has its base on the x-axis, then 7.42 gives $\rho h y_2^3/3$ whereas the above expression gives $\rho h y_2^3/4$.

13. If we divide the section into two parts as shown, and use formula 7.42 with $\rho = 1$, we obtain

$$I_x = \frac{c}{3}(b^3) + \frac{a-c}{3}(c^3) = \frac{c}{3}(b^3 + ac^2 - c^3),$$

and

$$I_y = \frac{b}{3}(c^3) + \frac{c}{3}(a^3 - c^3) = \frac{c}{3}(a^3 + bc^2 - c^3).$$

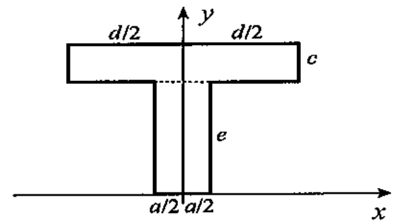


14. If we divide the right half of the section into two parts as shown, and use formula 7.42 with $\rho = 1$, we obtain

$$I_x = \frac{2}{3}\left(\frac{d}{2}\right)[(c+e)^3 - e^3] + \frac{2}{3}\left(\frac{a}{2}\right)e^3$$
$$= \frac{1}{3}[ae^3 + cd(c^2 + 3ce + 3e^2),$$

and

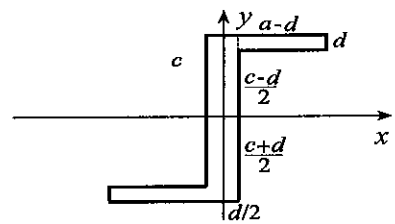
$$I_y = 2\left(\frac{c}{3}\right)\left(\frac{d}{2}\right)^3 + 2\left(\frac{e}{3}\right)\left(\frac{a}{2}\right)^3 = \frac{1}{12}(cd^3 + ea^3),$$



15. If we divide the first quadrant part of the section into subareas as shown, then formula 7.42 with $\rho = 1$ gives

$$I_x = \frac{2}{3}(a-d)\left[\left(\frac{c+d}{2}\right)^3 - \left(\frac{c-d}{2}\right)^3\right] + \frac{4}{3}\left(\frac{d}{2}\right)\left(\frac{c+d}{2}\right)^3,$$

and this simplifies to $I_x = \dfrac{ad}{6}(3c^2 + d^2) + \dfrac{d}{12}(c-d)^3$.



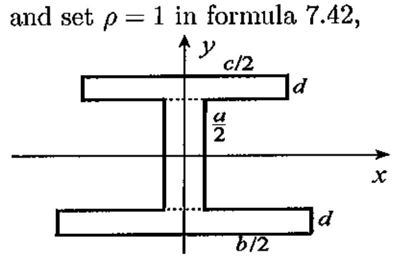
The second moment about the y-axis is $I_y = \dfrac{2}{3}(d)\left[\left(a - \dfrac{d}{2}\right)^3 - \left(\dfrac{d}{2}\right)^3\right] + \dfrac{4}{3}\left(\dfrac{c+d}{2}\right)\left(\dfrac{d}{2}\right)^3$, and this simplifies to $I_y = \dfrac{d}{12}(2a-d)^3 + \dfrac{cd^3}{12}$.

16. If we divide the right half of the section into subareas as shown, and set $\rho = 1$ in formula 7.42,

$$I_x = \frac{2}{3}\left(\frac{c}{2}\right)\left[\left(d + \frac{a}{2}\right)^3 - \left(\frac{a}{2}\right)^3\right] + \frac{4}{3}\left(\frac{d}{2}\right)\left(\frac{a}{2}\right)^3$$
$$+\frac{2}{3}\left(\frac{b}{2}\right)\left[\left(d + \frac{a}{2}\right)^3 - \left(\frac{a}{2}\right)^3\right]$$
$$= \frac{1}{24}[2a^3d + (b+c)(6a^2d + 12ad^2 + 8d^3)],$$

and

$$I_y = \frac{2}{3}(d)\left(\frac{c}{2}\right)^3 + \frac{4}{3}\left(\frac{a}{2}\right)\left(\frac{d}{2}\right)^3 + \frac{2}{3}(d)\left(\frac{b}{2}\right)^3 = \frac{d}{12}(ad^2 + b^3 + c^3).$$

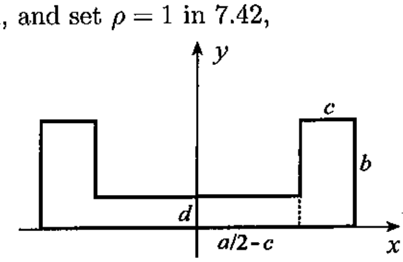


17. If we divide the right half of the section into subareas as shown, and set $\rho = 1$ in 7.42,

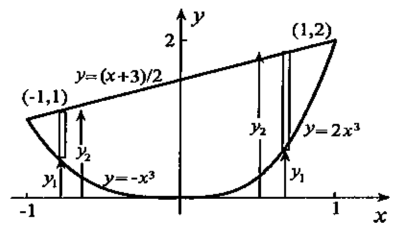
$$I_x = \frac{2}{3}(c)(b)^3 + \frac{2}{3}\left(\frac{a}{2} - c\right)(d)^3 = \frac{1}{3}(ad^3 + 2b^3c - 2cd^3),$$

and

$$I_y = \frac{2}{3}(b)\left[\left(\frac{a}{2}\right)^3 - \left(\frac{a}{2} - c\right)^3\right] + \frac{2}{3}(d)\left(\frac{a}{2} - c\right)^3$$
$$= \frac{1}{12}[a^3d + (b-d)(6a^2c - 12ac^2 + 8c^3)].$$



18. The mass of the plate is



$$\begin{aligned}M &= \int_{-1}^{0} \rho(y_2 - y_1)\,dx + \int_0^1 \rho(y_2 - y_1)\,dx \\ &= \rho\int_{-1}^{0}\left(\frac{x+3}{2} + x^3\right)dx + \rho\int_0^1\left(\frac{x+3}{2} - 2x^3\right)dx \\ &= \rho\left\{\frac{(x+3)^2}{4} + \frac{x^4}{4}\right\}_{-1}^{0} + \rho\left\{\frac{(x+3)^2}{4} - \frac{x^4}{2}\right\}_0^1 = \frac{9\rho}{4}.\end{aligned}$$

The moment of inertia about the x-axis is

$$\begin{aligned}I_x &= \int_{-1}^{0}\frac{\rho}{3}(y_2^3 - y_1^3)\,dx + \int_0^1 \frac{\rho}{3}(y_2^3 - y_1^3)\,dx \\ &= \frac{\rho}{3}\int_{-1}^{0}\left[\left(\frac{x+3}{2}\right)^3 - (-x^3)^3\right]dx + \frac{\rho}{3}\int_0^1\left[\left(\frac{x+3}{2}\right)^3 - (2x^3)^3\right]dx \\ &= \frac{\rho}{3}\left\{\frac{(x+3)^4}{32} + \frac{x^{10}}{10}\right\}_{-1}^{0} + \frac{\rho}{3}\left\{\frac{(x+3)^4}{32} - \frac{4x^{10}}{5}\right\}_0^1 = \frac{11\rho}{5}.\end{aligned}$$

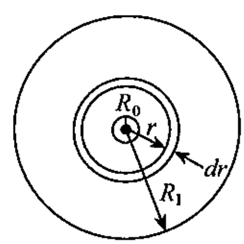
If r_x is the moment of gyration about the x-axis, then $\dfrac{11\rho}{5} = \dfrac{9\rho}{4}r_x^2 \implies r_x = \sqrt{44/45}$. The moment of inertia about the y-axis is

$$\begin{aligned}I_y &= \int_{-1}^{0} x^2\rho\left(\frac{x+3}{2} + x^3\right)dx + \int_0^1 x^2\rho\left(\frac{x+3}{2} - 2x^3\right)dx \\ &= \frac{\rho}{2}\int_{-1}^{0}(x^3 + 3x^2 + 2x^5)\,dx + \frac{\rho}{2}\int_0^1(x^3 + 3x^2 - 4x^5)\,dx \\ &= \frac{\rho}{2}\left\{\frac{x^4}{4} + x^3 + \frac{x^6}{3}\right\}_{-1}^{0} + \frac{\rho}{2}\left\{\frac{x^4}{4} + x^3 - \frac{2x^6}{3}\right\}_0^1 = \frac{\rho}{2}.\end{aligned}$$

If r_y is the radius of gyration about the y-axis, then $\dfrac{\rho}{2} = \dfrac{9\rho}{4}r_y^2 \implies r_y = \sqrt{2}/3$. The radius of gyration is the distance from a line at which a single particle of mass equal to that of the plate has the same moment of inertia as the plate itself.

19. Let inner and outer radii of the record be denoted by R_0 and R_1 respectively. Suppose the mass per unit area of the material in the record is a constant ρ, and the record rotates with angular speed ω (radians per unit time). We divide the record into thin rings of width dr. The ring with inner radius r has area approximately equal to $2\pi r\,dr$. Each point in this ring moves with speed $v = \omega r$, and therefore the kinetic energy of the ring is approximately

$$\frac{1}{2}(2\pi r\,dr)\rho(\omega r)^2 = \pi\rho\omega^2 r^3\,dr.$$



The kinetic energy of the record is therefore

$$K = \int_{R_0}^{R_1}\pi\rho\omega^2 r^3\,dr = \pi\rho\omega^2\left\{\frac{r^4}{4}\right\}_{R_0}^{R_1} = \frac{1}{4}\pi\rho\omega^2(R_1^4 - R_0^4).$$

The moment of inertia of the ring at radius r about a line through the centre of the record and perpendicular to its face is $(2\pi r\,dr)\rho(r^2) = 2\pi\rho r^3\,dr$. Consequently, the moment of inertia of the record about this line is

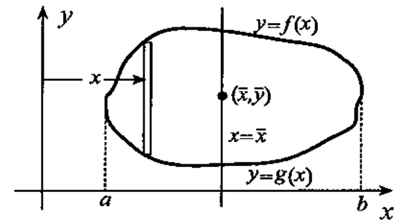
$$I = \int_{R_0}^{R_1} 2\pi\rho r^3\,dr = 2\pi\rho\left\{\frac{r^4}{4}\right\}_{R_0}^{R_1} = \frac{\pi\rho}{2}(R_1^4 - R_0^4).$$

Thus, $K = \dfrac{1}{4}\pi\rho\omega^2(R_1^4 - R_0^4) = \dfrac{1}{2}\left[\dfrac{\pi\rho}{2}(R_1^4 - R_0^4)\right]\omega^2 = \dfrac{1}{2}I\omega^2.$

20. Let us take the coplanar line to be the y-axis. The moment of inertia about the y-axis is

$$I_y = \int_a^b x^2\rho[f(x) - g(x)]\,dx.$$

The moment of inertia about the line $x = \overline{x}$ is



$$\begin{aligned}I &= \int_a^b (x - \overline{x})^2\rho[f(x) - g(x)]\,dx\\ &= \int_a^b x^2\rho[f(x) - g(x)]\,dx - 2\overline{x}\int_a^b x\rho[f(x) - g(x)]\,dx + \overline{x}^2\int_a^b \rho[f(x) - g(x)]\,dx\\ &= I_y - 2\overline{x}(M\overline{x}) + \overline{x}^2 M.\end{aligned}$$

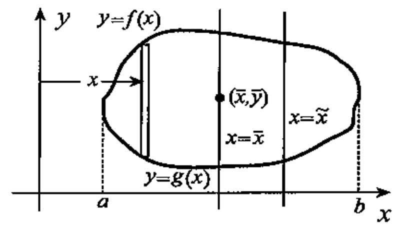
Thus, $I_y = I + M\overline{x}^2$.

21. $I_{\tilde{x}} = \displaystyle\int_a^b (x - \tilde{x})^2[f(x) - g(x)]\,dx$

According to the parallel axis theorem in Exercise 20, we may write

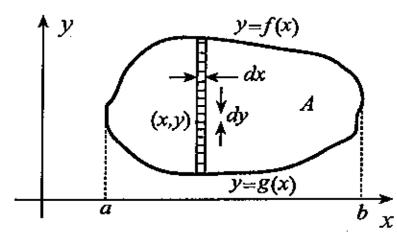
$$I_{\tilde{x}} = I_{\overline{x}} + (\tilde{x} - \overline{x})^2 A,$$

where A is the area of the plate. Since $I_{\overline{x}}$ is a fixed quantity, it follows that $I_{\tilde{x}}$ is a minimum when $\tilde{x} = \overline{x}$.



22. We divide the area A into vertical rectangles of width dx, and then divide this rectangle further into horizontal rectangles of width dy. Because the polar moment of inertia of the tiny rectangle is $(x^2 + y^2)\rho\,dy\,dx$, the polar moment of inertia of the vertical rectangle is

$$\begin{aligned}\int_{g(x)}^{f(x)} [(x^2 + y^2)\rho\,dx]\,dy &= \left\{\rho\,dx\left(x^2 y + \frac{y^3}{3}\right)\right\}_{g(x)}^{f(x)}\\ &= \left\{x^2[f(x) - g(x)] + \frac{1}{3}\{[f(x)]^3 - [g(x)]^3\}\right\}\rho\,dx.\end{aligned}$$



The polar moment of inertia of the plate is therefore

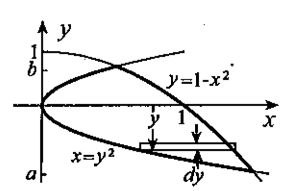
$$\begin{aligned}J_0 &= \int_a^b \left\{x^2[f(x) - g(x)] + \frac{1}{3}\{[f(x)]^3 - [g(x)]^3\}\right\}\rho\,dx\\ &= \int_a^b x^2\rho[f(x) - g(x)]\,dx + \int_a^b \frac{1}{3}\rho\{[f(x)]^3 - [g(x)]^3\}\,dx = I_x + I_y.\end{aligned}$$

23. The y-coordinates of the points of intersection of the curves are $a = -1.220\,744$ and $b = 0.724\,492$. The moment of inertia is

$$I = \int_a^b 2y^2(\sqrt{1-y} - y^2)\,dy.$$

If we set $u = 1 - y$ and $du = -dy$, then

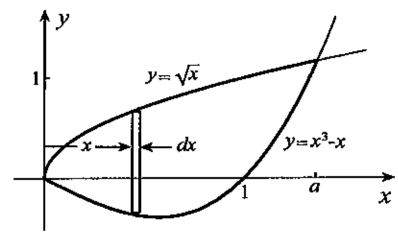
$$\begin{aligned}\int y^2\sqrt{1-y}\,dy &= \int (1-u)^2\sqrt{u}(-du)\\ &= -\int(\sqrt{u} - 2u^{3/2} + u^{5/2})\,du\\ &= -\frac{2}{3}u^{3/2} + \frac{4}{5}u^{5/2} - \frac{2}{7}u^{7/2} + C\\ &= -\frac{2}{3}(1-y)^{3/2} + \frac{4}{5}(1-y)^{5/2} - \frac{2}{7}(1-y)^{7/2} + C.\end{aligned}$$



Thus, $I = 2\left\{-\dfrac{2}{3}(1-y)^{3/2} + \dfrac{4}{5}(1-y)^{5/2} - \dfrac{2}{7}(1-y)^{7/2} - \dfrac{y^5}{5}\right\}_a^b = 0.680.$

24. The x-coordinate of the point of intersection of the curves is $a = 1.362\,599$. The moment of inertia is

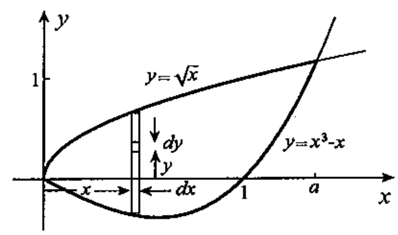
$$I = \int_0^a x^2(2)(\sqrt{x} - x^3 + x)\,dx$$
$$= 2\int_0^a (x^{5/2} - x^5 + x^3)\,dx$$
$$= 2\left\{\frac{2x^{7/2}}{7} - \frac{x^6}{6} + \frac{x^4}{4}\right\}_0^a = 1.278.$$



25. Since the moment of inertia of the tiny rectangle about the x-axis is $2y^2\,dx\,dy$, the moment of inertia of the long vertical rectangle is

$$\int_{x^3-x}^{\sqrt{x}} 2y^2\,dx\,dy = 2\left\{\frac{y^3}{3}\right\}_{x^3-x}^{\sqrt{x}} dx$$
$$= \frac{2}{3}[x^{3/2} - (x^3-x)^3]\,dx.$$

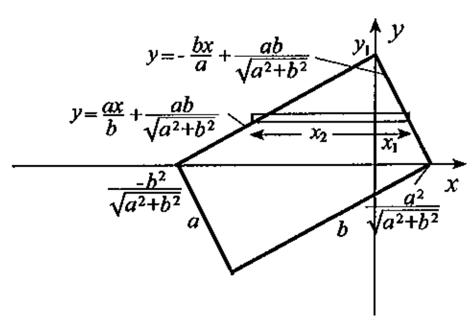
The x-coordinate of the point of intersection of the two curves is $a = 1.362\,599$. The moment of inertia of the plate about the x-axis is



$$I = \int_0^a \frac{2}{3}[x^{3/2} - (x^3-x)^3]\,dx = \frac{2}{3}\int_0^a (x^{3/2} - x^9 + 3x^7 - 3x^5 + x^3)\,dx$$
$$= \frac{2}{3}\left\{\frac{2}{5}x^{5/2} - \frac{x^{10}}{10} + \frac{3x^8}{8} - \frac{x^6}{2} + \frac{x^4}{4}\right\}_0^a = 0.519.$$

26. If we choose the coordinate system shown, the second moment of area about the x-axis is

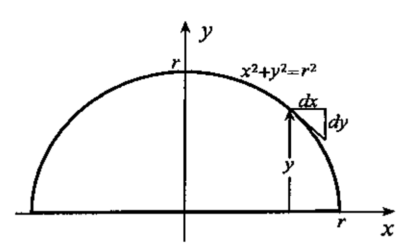
$$I = 2\int_0^{y_1} y^2(x_1 - x_2)\,dy \qquad (\text{where } y_1 = ab/\sqrt{a^2+b^2})$$
$$= 2\int_0^{y_1} y^2\left[\frac{a}{b}\left(\frac{ab}{\sqrt{a^2+b^2}} - y\right) - \frac{b}{a}\left(y - \frac{ab}{\sqrt{a^2+b^2}}\right)\right]dy$$
$$= 2\int_0^{y_1} y^2\left[\frac{a^2}{\sqrt{a^2+b^2}} + \frac{b^2}{\sqrt{a^2+b^2}} - y\left(\frac{a}{b} + \frac{b}{a}\right)\right]dy$$
$$= 2\left\{\sqrt{a^2+b^2}\,\frac{y^3}{3} - \frac{a^2+b^2}{ab}\frac{y^4}{4}\right\}_0^{y_1}$$
$$= 2\left[\frac{1}{3}\sqrt{a^2+b^2}\,\frac{a^3b^3}{(a^2+b^2)^{3/2}} - \frac{a^2+b^2}{ab}\,\frac{a^4b^4}{4(a^2+b^2)^2}\right]$$
$$= \frac{a^3b^3}{6(a^2+b^2)}.$$



EXERCISES 7.9

1. The sphere can be formed by rotating the semicircle $y = \sqrt{r^2 - x^2}$ around the x-axis. Small lengths along the curve corresponding to lengths dx along the x-axis are given by

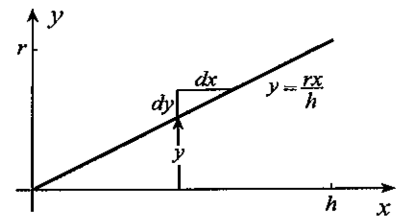
$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}\,dx$$
$$= \frac{r}{\sqrt{r^2-x^2}}dx.$$



The area of the sphere is therefore $A = 2\int_0^r 2\pi y\left(\frac{r}{\sqrt{r^2-x^2}}\right)dx = 4\pi r\int_0^r dx = 4\pi r\left\{x\right\}_0^r = 4\pi r^2.$

2. The cone can be formed by rotating the straight line segment shown about the x-axis. Small lengths along the curve corresponding to lengths dx along the x-axis are given by

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+(r/h)^2}\,dx.$$

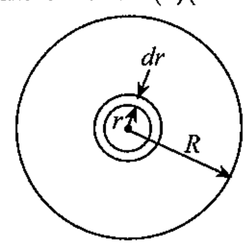


The area of the curved portion of the cone is therefore

$$A = \int_0^h 2\pi y\sqrt{1+(r/h)^2}\,dx = 2\pi\sqrt{1+(r/h)^2}\int_0^h \frac{rx}{h}dx = \frac{2\pi r}{h^2}\sqrt{r^2+h^2}\left\{\frac{x^2}{2}\right\}_0^h = \pi r\sqrt{r^2+h^2}.$$

3. (a) Since the amount of blood flowing through the ring per unit time is $v(r)(2\pi r\,dr)$, the total flow through the vessel is

$$\begin{aligned} F &= \int_0^R cR\sqrt{R^2-r^2}(2\pi r)\,dr \\ &= 2\pi cR\left\{-\frac{1}{3}(R^2-r^2)^{3/2}\right\}_0^R = \frac{2\pi cR^4}{3}. \end{aligned}$$



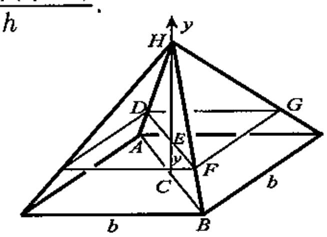
(b) With $v(r) = (c/R^2)(R^2-r^2)^2$,

$$F = \int_0^R \frac{c}{R^2}(R^2-r^2)^2(2\pi r)\,dr = \frac{2\pi c}{R^2}\left\{-\frac{1}{6}(R^2-r^2)^3\right\}_0^R = \frac{\pi cR^4}{3}.$$

4. Cross sections of the pyramid parallel to the base are always square. At height y above the base, similar triangles give $\|DE\|/\|EH\| = \|AC\|/\|HC\|$, or $\|DE\| = \dfrac{(h-y)(b/\sqrt{2})}{h}$.

Consequently,

$$\begin{aligned} \|FG\| &= \frac{\|DF\|}{\sqrt{2}} = \frac{2\|DE\|}{\sqrt{2}} \\ &= \sqrt{2}\left[\frac{b}{\sqrt{2}h}(h-y)\right] = \frac{b}{h}(h-y). \end{aligned}$$



The area of the square at height y is therefore $b^2(h-y)^2/h^2$, and the volume of the pyramid is

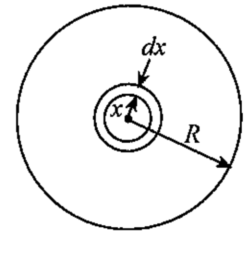
$$V = \int_0^h \frac{b^2}{h^2}(h-y)^2\,dy = \frac{b^2}{h^2}\left\{-\frac{1}{3}(h-y)^3\right\}_0^h = \frac{b^2h}{3}.$$

5. The total amount of water consumed is

$$\begin{aligned} \int_0^6 f(t)\,dt &= \int_0^6 (5000 + 21.65t^2 - 249.7t^3 + 97.52t^4 - 9.680t^5)\,dt \\ &= \left\{5000t + \frac{21.65t^3}{3} - \frac{249.7t^4}{4} + \frac{97.52t^5}{5} - \frac{9.680t^6}{6}\right\}_0^6 = 2.70\times 10^4\ \text{m}^3. \end{aligned}$$

6. (a) The number of bees in a ring of width dx at distance x from the hive is $\rho(2\pi x\,dx)$. The total number of bees in the colony is therefore

$$\begin{aligned} N &= \int_0^R \rho(2\pi x\,dx) \\ &= 2\pi\int_0^R x\left[\frac{600\,000}{31\pi R^5}(R^3+2R^2x-Rx^2-2x^3)\right]dx \\ &= \frac{1\,200\,000}{31R^5}\left\{\frac{R^3x^2}{2}+\frac{2R^2x^3}{3}-\frac{Rx^4}{4}-\frac{2x^5}{5}\right\}_0^R = 20\,000. \end{aligned}$$



(b) The number of bees within $R/2$ of the hive is

$$\overline{N} = \int_0^{R/2} \rho(2\pi x\, dx) = \frac{1\,200\,000}{31R^5}\left\{\frac{R^3x^2}{2} + \frac{2R^2x^3}{3} - \frac{Rx^4}{4} - \frac{2x^5}{5}\right\}_0^{R/2} = 6976,$$

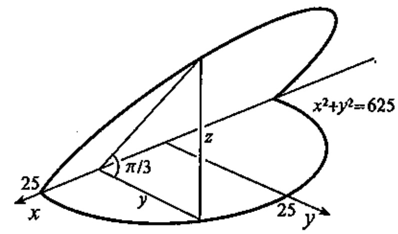
or approximately 7000.

7. The amount of blood flowing through the hardened vessel is

$$\int_0^{R/2} c(R^2 - 4r^2)(2\pi r)\, dr = 2\pi c\left\{\frac{R^2r^2}{2} - r^4\right\}_0^{R/2} = \frac{\pi c R^4}{8}.$$

Thus, only 25% of the flow gets through the hardened vessel.

8. Cross sections of the wedge parallel to the axis of the trunk and perpendicular to the diameter are triangles. At distance x from the centre of the wedge, $y = \sqrt{625 - x^2}$, and $z = y\tan(\pi/3) = \sqrt{3}\sqrt{625 - x^2}$. The area of the triangle at position x is therefore

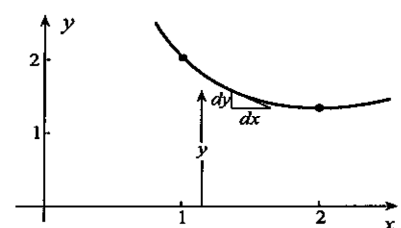


$$\frac{1}{2}yz = \frac{\sqrt{3}}{2}(625 - x^2),$$

and the volume of the wedge is

$$V = 2\int_0^{25} \frac{\sqrt{3}}{2}(625 - x^2)\, dx = \sqrt{3}\left\{625x - \frac{x^3}{3}\right\}_0^{25} = \frac{31\,250}{\sqrt{3}}\ \text{cm}^3.$$

9. Small lengths along the curve corresponding to lengths dx along the x-axis are given by

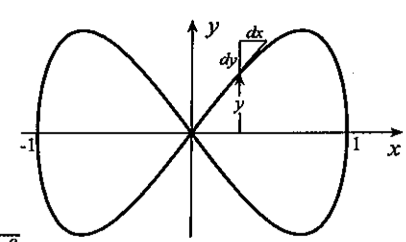


$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \left(\frac{x^2}{8} - \frac{2}{x^2}\right)^2}\, dx$$
$$= \sqrt{1 + \frac{x^4}{64} - \frac{1}{2} + \frac{4}{x^4}}\, dx$$
$$= \sqrt{\left(\frac{x^2}{8} + \frac{2}{x^2}\right)^2}\, dx = \left(\frac{x^2}{8} + \frac{2}{x^2}\right) dx.$$

The area of the surface is

$$A = \int_1^2 2\pi y\left(\frac{x^2}{8} + \frac{2}{x^2}\right) dx = 2\pi\int_1^2 \left(\frac{x^3}{24} + \frac{2}{x}\right)\left(\frac{x^2}{8} + \frac{2}{x^2}\right) dx = 2\pi\int_1^2 \left(\frac{x^5}{192} + \frac{x}{3} + \frac{4}{x^3}\right) dx$$
$$= 2\pi\left\{\frac{x^6}{1152} + \frac{x^2}{6} - \frac{2}{x^2}\right\}_1^2 = \frac{263\pi}{64}.$$

10. If we differentiate $8y^2 = x^2(1 - x^2)$ with respect to x,



$$16y\frac{dy}{dx} = 2x - 4x^3 \implies \frac{dy}{dx} = \frac{x - 2x^3}{8y}.$$

Small lengths along the curve in the first quadrant corresponding to lengths dx along the x-axis are given by

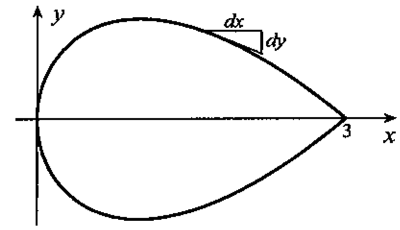
$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \left(\frac{x - 2x^3}{8y}\right)^2}\, dx = \frac{\sqrt{64y^2 + x^2 - 4x^4 + 4x^6}}{8y}\, dx$$
$$= \frac{\sqrt{8x^2 - 8x^4 + x^2 - 4x^4 + 4x^6}}{8y}\, dx = \frac{3x - 2x^3}{2\sqrt{2}x\sqrt{1 - x^2}}\, dx = \frac{3 - 2x^2}{2\sqrt{2}\sqrt{1 - x^2}}\, dx.$$

The area of the surface of revolution is therefore

$$A = 2\int_0^1 2\pi y\left(\frac{3-2x^2}{2\sqrt{2}\sqrt{1-x^2}}\right)dx = \sqrt{2}\pi\int_0^1 \frac{x\sqrt{1-x^2}}{2\sqrt{2}}\left(\frac{3-2x^2}{\sqrt{1-x^2}}\right)dx$$
$$= \frac{\pi}{2}\int_0^1 (3x-2x^3)\,dx = \frac{\pi}{2}\left\{\frac{3x^2}{2} - \frac{x^4}{2}\right\}_0^1 = \frac{\pi}{2}.$$

11. If we differentiate $9y^2 = 9x - 6x^2 + x^3$ with respect to x,

$$18y\frac{dy}{dx} = 9 - 12x + 3x^2 \implies \frac{dy}{dx} = \frac{3-4x+x^2}{6y}.$$



Small lengths along the curve in the first quadrant corresponding to lengths dx along the x-axis are given by

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = \sqrt{1+\left(\frac{3-4x+x^2}{6y}\right)^2}dx$$
$$= \sqrt{1+\frac{(3-x)^2(1-x)^2}{4x(3-x)^2}}dx = \sqrt{\frac{4x+(1-x)^2}{4x}}dx = \sqrt{\frac{(1+x)^2}{4x}}dx = \frac{1+x}{2\sqrt{x}}dx.$$

(a) The area of the surface of revolution for rotation around the y-axis is

$$2\int_0^3 2\pi x\left(\frac{1+x}{2\sqrt{x}}\right)dx = 2\pi\int_0^3(\sqrt{x}+x^{3/2})\,dx = 2\pi\left\{\frac{2x^{3/2}}{3}+\frac{2x^{5/2}}{5}\right\}_0^3 = \frac{56\sqrt{3}\pi}{5}.$$

(b) For rotation around the x-axis, the area is

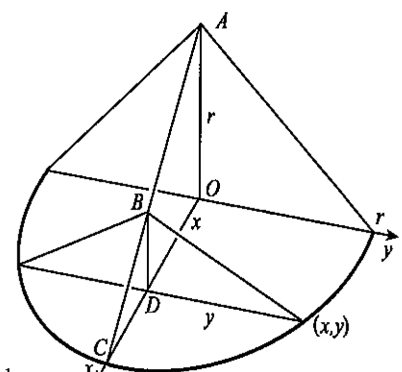
$$\int_0^3 2\pi y\left(\frac{1+x}{2\sqrt{x}}\right)dx = \pi\int_0^3\frac{1}{3}(3-x)\sqrt{x}\left(\frac{1+x}{\sqrt{x}}\right)dx = \frac{\pi}{3}\int_0^3(3+2x-x^2)\,dx = \frac{\pi}{3}\left\{3x+x^2-\frac{x^3}{3}\right\}_0^3 = 3\pi.$$

12. For the triangle at position x, we note that by similar triangles $\|BD\|/\|CD\| = \|AO\|/\|CO\|$, or,

$$\|BD\| = (r-x)\frac{r}{r} = r - x.$$

Because the base of the triangle has length $2y = 2\sqrt{r^2-x^2}$, the area of the triangle at x is

$$\frac{1}{2}(r-x)2\sqrt{r^2-x^2} = (r-x)\sqrt{r^2-x^2}.$$



The volume of the solid is therefore

$$V = 2\int_0^r (r-x)\sqrt{r^2-x^2}\,dx$$
$$= 2r\int_0^r\sqrt{r^2-x^2}\,dx - 2\int_0^r x\sqrt{r^2-x^2}\,dx.$$

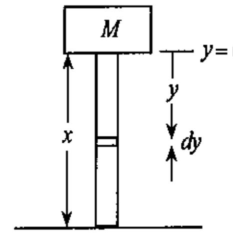
Since the first integral is one-quarter of the area of the base circle,

$$V = 2r\left(\frac{1}{4}\pi r^2\right) - 2\left\{-\frac{1}{3}(r^2-x^2)^{3/2}\right\}_0^r = \frac{\pi r^3}{2} - \frac{2r^3}{3} = \frac{(3\pi-4)r^3}{6}.$$

13. According to formula 7.43, the amount of stretch in the rod is $FL/(AE)$. Hence, the length of the rod is $L + FL/(AE)$.

14. If x is the original length of the rod, then according to equation 7.43, the compression when M is placed on top is $Mgx/(AE)$, where $g = 9.81$. It follows that $x - \dfrac{Mgx}{AE} = L \implies x = \dfrac{AEL}{AE - Mg}$.

15. Let x and ρ be the unconstrained length and density of the rod. The force on each cross section in an element dy is $Mg + \rho g A y$, and therefore its compression is $\dfrac{(M + \rho A y)g}{AE}\,dy$.



Total compression of the rod is

$$\int_0^x \frac{(M+\rho Ay)g}{AE}\,dy = \frac{g}{AE}\left\{\frac{(M+\rho Ay)^2}{2\rho A}\right\}_0^x = \frac{g[(M+\rho Ax)^2 - M^2]}{2\rho EA^2}.$$

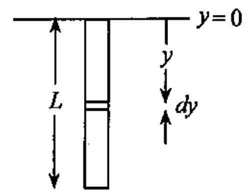
The length of the compressed rod is x minus this value and therefore

$$x - \frac{g[(M+\rho Ax)^2 - M^2]}{2\rho EA^2} = L \quad\Longrightarrow\quad \frac{\rho g}{E}x^2 + 2\left(\frac{Mg}{EA} - 1\right)x + 2L = 0.$$

Of the two solutions to this quadratic equation, we choose

$$x = \frac{-2\left(\dfrac{Mg}{EA} - 1\right) + \sqrt{4\left(\dfrac{Mg}{EA} - 1\right)^2 - \dfrac{8\rho gL}{E}}}{2\rho g/E} = \frac{E}{\rho g}\left[1 - \frac{Mg}{EA} + \sqrt{\left(1 - \frac{Mg}{EA}\right)^2 - \frac{2\rho gL}{E}}\right].$$

16. If we consider a small length dy at position y, the force on each cross section in this element is approximately the same, and equal to the weight of that part of the rod below it plus F, $\rho g(L-y)A + F$, where ρ is the density of the material in the rod. According to equation 7.43, the element dy stretches by



$$\frac{[\rho g(L-y)A + F]\,dy}{AE}.$$

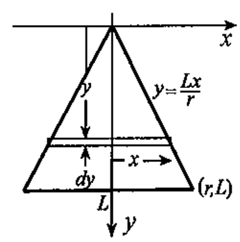
Total stretch in the rod is therefore

$$\int_0^L \frac{\rho g(L-y)A + F}{AE}dy = \frac{1}{AE}\left\{-\frac{\rho gA}{2}(L-y)^2 + Fy\right\}_0^L = \frac{1}{AE}\left(FL + \frac{\rho gAL^2}{2}\right).$$

The length of the rod is therefore $L + \dfrac{FL}{AE} + \dfrac{\rho gL^2}{2E}$.

17. The force on the cross section of width dy at position y is the weight of the rod below it,

$$\begin{aligned} W &= \rho g\left(\frac{1}{3}\pi Lr^2 - \frac{1}{3}\pi x^2 y\right) \\ &= \frac{\rho g\pi}{3}\left[Lr^2 - y\left(\frac{ry}{L}\right)^2\right] \\ &= \frac{\rho g\pi r^2}{3L^2}(L^3 - y^3). \end{aligned}$$



The element dy therefore stretches by

$$\frac{\rho g\pi r^2}{3L^2AE}(L^3 - y^3)\,dy = \frac{\rho g\pi r^2}{3L^2E\pi(ry/L)^2}(L^3 - y^3)\,dy.$$

To find total stretch we should integrate this function from 0 to L. But this cannot be done because the function has an infinite discontinuity at $y = 0$.

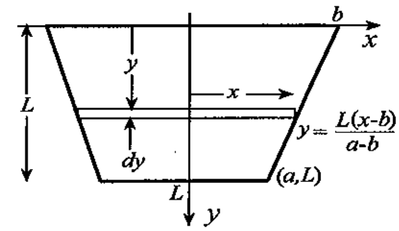
18. Since the force on each cross section is Mg, the stretch of the element of width dy at position y is $[Mg/(AE)]\,dy$. Since the cross-sectional area at y is $A = \pi x^2 = \pi r^2(1 - y/L)^2$, the stretch of element dy is

$$\frac{Mg}{E\pi r^2(1 - y/L)^2}\,dy.$$

To find total stretch we should integrate this function from 0 to L. But this cannot be done because the function has an infinite discontinuity at $y = L$.

19. The answer to this problem is the same as that in Exercise 20 with a and b interchanged.

20. The force on the cross section of width dy at position y is the weight of the rod below it. To find this weight we calculate the volume by slicing,



$$\int_y^L \left[b + \frac{(a-b)}{L}y\right]^2 dy = \left\{\frac{L}{3(a-b)}\left[b + \frac{(a-b)}{L}y\right]^3\right\}_y^L$$
$$= \frac{1}{3(a-b)L^2}\left\{a^3L^3 - [bL + (a-b)y]^3\right\}.$$

The element dy therefore stretches by

$$\frac{\rho g\{a^3L^3 - [bL + (a-b)y]^3\}}{3(a-b)L^2[b + (a-b)y/L]^2E}\,dy = \frac{\rho g\{a^3L^3 - [bL + (a-b)y]^3\}}{3E(a-b)[bL + (a-b)y]^2}\,dy.$$

Total stretch of the rod is

$$\int_0^L \frac{\rho g\{a^3L^3 - [bL + (a-b)y]^3\}}{3E(a-b)[bL + (a-b)y]^2}\,dy = \frac{\rho g}{3E(a-b)}\int_0^L \left\{\frac{a^3L^3}{[bL + (a-b)y]^2} - [bL + (a-b)y]\right\}dy$$
$$= \frac{\rho g}{3E(a-b)}\left\{\frac{-a^3L^3}{(a-b)[bL + (a-b)y]} - \frac{[bL + (a-b)y]^2}{2(a-b)}\right\}_0^L$$
$$= \frac{\rho g L^2(2a^3 - 3a^2b + b^3)}{6bE(a-b)^2}.$$

The total length of the rod is this quantity plus L.

21. The answer to this problem is the same as that in Exercise 22 with a and b interchanged.

22. Without the extra weight, the answer to the stretch of the rod in Exercise 19 is the same as that in Exercise 20 with a and b interchanged. With M added, we must add Mg to the weight of each cross section. Stretch is therefore given by the answer in Exercise 20 with a and b interchanged plus the following integral

$$\int_0^L \frac{Mg}{[a + (b-a)y/L]^2E}\,dy = \frac{MgL^2}{E}\int_0^L \frac{1}{[aL + (b-a)y]^2}\,dy$$
$$= \frac{MgL^2}{E}\left\{\frac{-1}{(b-a)[aL + (b-a)y]}\right\}_0^L = \frac{MgL}{abE}.$$

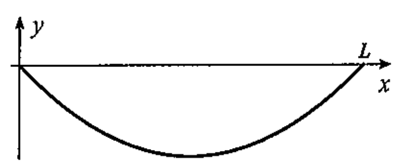
Thus, the length of the rod is $L + \dfrac{MgL}{abE} + \dfrac{\rho g L^2(2b^3 - 3ab^2 + a^3)}{6bE(a-b)^2}$.

23. The probability of an electron striking a ring of width dx at position x is $f(x)(2\pi x)\,dx$. The percentage of electrons striking within distance r from the centre of the target is

$$100\int_0^r 2\pi x\,f(x)\,dx = 200\pi\int_0^r \frac{5x}{3\pi R^5}(R^3 - x^3)\,dx = \frac{1000}{3R^5}\left\{\frac{R^3x^2}{2} - \frac{x^5}{5}\right\}_0^r = \frac{100r^2(5R^3 - 2r^3)}{3R^5}.$$

24. (a) According to equation 7.46, deflections are given by

$$y(x) = \int_0^L \frac{1}{L\tau}[x(L-X)h(X-x) + X(L-x)h(x-X)]k\,dX$$
$$= \frac{k}{L\tau}\int_0^x X(L-x)\,dX + \frac{k}{L\tau}\int_x^L x(L-X)\,dX$$
$$= \frac{k(L-x)}{L\tau}\left\{\frac{X^2}{2}\right\}_0^x + \frac{kx}{L\tau}\left\{\frac{-(L-X)^2}{2}\right\}_x^L$$
$$= \frac{kx(L-x)}{2\tau}.$$



(b) The graph of deflections is the parabola shown above. It is symmetric about $x = L/2$ with minimum at $x = L/2$. We would expect this for constant loading.

25. With $F(x) = k - F\delta(x - L/2)$, formla 7.46 gives

$$\begin{aligned}
y(x) &= \int_0^L \frac{1}{L\tau}[x(L-X)h(X-x) + X(L-x)h(x-X)][k - F\delta(X - L/2)]\,dX \\
&= \frac{k}{L\tau}\int_0^L [x(L-X)h(X-x) + X(L-x)h(x-X)]\,dX \\
&\qquad - \frac{F}{L\tau}\int_0^L [x(L-X)h(X-x) + X(L-x)h(x-X)]\delta(X - L/2)\,dX \\
&= \frac{k}{L\tau}\int_0^x X(L-x)\,dX + \frac{k}{L\tau}\int_x^L x(L-X)\,dX \\
&\qquad - \frac{F}{L\tau}\left[x\left(L - \frac{L}{2}\right)h\left(\frac{L}{2} - x\right) + \frac{L}{2}(L-x)\,h\left(x - \frac{L}{2}\right)\right] \\
&= \frac{k(L-x)}{L\tau}\left\{\frac{X^2}{2}\right\}_0^x + \frac{kx}{L\tau}\left\{\frac{-(L-X)^2}{2}\right\}_x^L - \frac{F}{2\tau}\left[x\,h\left(\frac{L}{2} - x\right) + (L-x)\,h\left(x - \frac{L}{2}\right)\right] \\
&= \frac{kx(L-x)}{2\tau} - \frac{F}{2\tau}\begin{cases} x & 0 < x < L/2 \\ L - x, & L/2 < x < L \end{cases} \\
&= \frac{1}{2\tau}\begin{cases} -kx^2 + (kL - F)x, & 0 < x \le L/2 \\ (L-x)(kx - F), & L/2 < x < L \end{cases},
\end{aligned}$$

where we have removed the discontinuity at $x = L/2$.

26. (a) According to equation 7.46,

$$\begin{aligned}
y(x) &= \int_0^L \frac{1}{6EI}[(x-X)^3h(x-X) - x^3 + 3Xx^2]k\,dX \\
&= \frac{k}{6EI}\int_0^x [(x-X)^3 - x^3 + 3Xx^2]\,dX + \frac{k}{6EI}\int_x^L (3Xx^2 - x^3)\,dX \\
&= \frac{k}{6EI}\left\{\frac{-(x-X)^4}{4} - x^3X + \frac{3X^2x^2}{2}\right\}_0^x + \frac{k}{6EI}\left\{\frac{3X^2x^2}{2} - x^3X\right\}_x^L \\
&= \frac{kx^2(x^2 - 4Lx + 6L^2)}{24EI}.
\end{aligned}$$

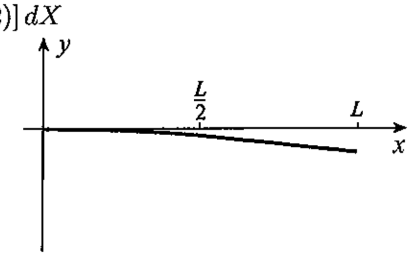
(b) With uniform loading and a free end at $x = L$, maximum deflection, meaning minimum y, should occur at $x = L$. We can verify this by finding critical points of $y(x)$,

$$0 = y'(x) = \frac{k}{24EI}(4x^3 - 12Lx^2 + 12L^2x) = \frac{4kx}{24EI}(x^2 - 3Lx + 3L^2).$$

Since $x = 0$ is the only solution (as expected), minimum $y(x)$ must occur at either $x = 0$ or $x = L$. Since $y(0) = 0$ and $y(L) < 0$, maximum deflection occurs at $x = L$.

27. According to equation 7.46,

$$\begin{aligned}
y(x) &= \int_0^L \left[\frac{1}{6EI}(x-X)^3h(x-X) - \frac{x^3}{6EI} + \frac{Xx^2}{2EI}\right][-F\delta(X - L/2)]\,dX \\
&= -F\left[\frac{1}{6EI}(x - L/2)^3h(x - L/2) - \frac{x^3}{6EI} + \frac{Lx^2}{4EI}\right] \\
&= \frac{F}{48EI}\begin{cases} 8x^3 - 12Lx^2, & 0 < x \le L/2 \\ L^3 - 6L^2x, & L/2 < x \le L \end{cases}.
\end{aligned}$$



We have removed the discontinuity at $x = L/2$.

(b) A graph is shown to the right. It is straight for $L/2 < x \le L$ since there is no loading on this part of the board.

28. (a) According to equation 7.46,

$$\begin{aligned}
y(x) &= \int_0^L \left[\frac{1}{6EI}(x-X)^3h(x-X) + \frac{x^3}{6EIL^3}(-L^3+3LX^2-2X^3) + \frac{x^2}{2EIL^2}(X^3-2LX^2+L^2X)\right] k\,dX \\
&= \frac{k}{6EIL^3}\int_0^x \left[L^3(x-X)^3 + x^3(-L^3+3LX^2-2X^3) + 3x^2L(X^3-2LX^2+L^2X)\right] dX \\
&\quad + \frac{k}{6EIL^3}\int_x^L \left[x^3(-L^3+3LX^2-2X^3) + 3x^2L(X^3-2LX^2+L^2X)\right] dX \\
&= \frac{k}{6EIL^3}\left\{\frac{-L^3(x-X)^4}{4} + x^3\left(-L^3X+LX^3-\frac{X^4}{2}\right) + 3x^2L\left(\frac{X^4}{4}-\frac{2LX^3}{3}+\frac{L^2X^2}{2}\right)\right\}_0^x \\
&\quad + \frac{k}{6EIL^3}\left\{x^3\left(-L^3X+LX^3-\frac{X^4}{2}\right) + 3x^2L\left(\frac{X^4}{4}-\frac{2LX^3}{3}+\frac{L^2X^2}{2}\right)\right\}_x^L \\
&= \frac{x^2(L-x)^2}{24EI}.
\end{aligned}$$

(b) Phsically we expect maximum deflection, meaning minimum y, at $x = L/2$. To confirm this, we find critical points of $y(x)$,

$$0 = y'(x) = \frac{1}{24EI}[2x(L-x)^2 - 2x^2(L-x)] = \frac{1}{12EI}x(L-x)(L-2x).$$

This gives the expected $x = 0$ and $x = L$ (the beam is horizontal at its ends), and the hoped for $x = L/2$.

29. (a) According to equation 7.46,

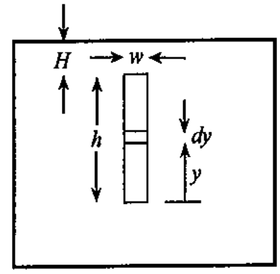
$$\begin{aligned}
y(x) &= \int_0^L \left[\frac{1}{6EI}(x-X)^3h(x-X) + \frac{x^3}{6EIL^3}(-L^3+3LX^2-2X^3)\right. \\
&\qquad \left. + \frac{x^2}{2EIL^2}(X^3-2LX^2+L^2X)\right](-F)\delta(x-L/2)\,dX \\
&= -F\left[\frac{1}{6EI}(x-L/2)^3h(x-L/2) + \frac{x^3}{6EIL^3}\left(-L^3+\frac{3L^3}{4}-\frac{L^3}{4}\right)\right. \\
&\qquad \left. + \frac{x^2}{2EIL^2}\left(\frac{L^3}{8}-\frac{L^3}{2}+\frac{L^3}{2}\right)\right] \\
&= \frac{F}{48EI}\left[-8(x-L/2)^3h(x-L/2) - 4x^3 + 3Lx^2\right].
\end{aligned}$$

(b) Physically, we expect maximum deflection, meaning minimum y, at $x = L/2$. To confirm this we find critical points of $y(x)$ on the intervals $0 \le x < L/2$ and $L/2 < x \le L$. For $0 \le x < L/2$, the shape is $y(x) = F(3Lx^2 - 4x^3)/(48EI)$. For critical points, we solve $0 = y'(x) = F(6Lx - 12x^2)/(48EI) = Fx(L-2x)/(8EI)$, giving $x = 0$, as expected. In addition, the derivative approaches zero as $x \to L/2^-$. Similarly, for $L/2 < x \le L$, the critical point is $x = L$, and $y'(x) \to 0$ as $x \to L/2^+$.

30. According to the modified Torricelli law, the velocity of water through a vertical rectangle of height dy at a distance y above the bottom of the slit (see figure) is

$$v = c\sqrt{2g(H+h-y)}.$$

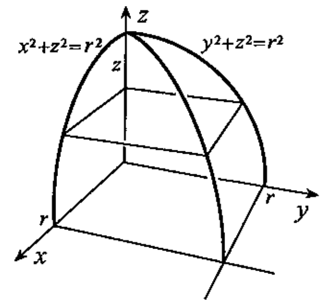
The rate at which water passes through this tiny rectangle is $v(w\,dy)$, and therefore the volume of water passing through the slit per unit time is



$$Q = \int_0^h c\sqrt{2g(H+h-y)}\,w\,dy = \sqrt{2g}cw\left\{-\frac{2}{3}(H+h-y)^{3/2}\right\}_0^h = \frac{2\sqrt{2g}cw}{3}[(H+h)^{3/2} - H^{3/2}].$$

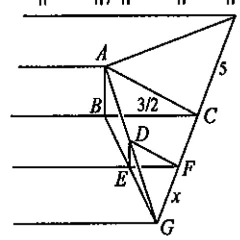
31. The volume is eight times that shown in the figure to the right. Horizontal cross sections of this volume are squares with length and width $\sqrt{r^2 - z^2}$. Consequently, the required volume is

$$V = 8\int_0^r (r^2 - z^2)\,dz = 8\left\{r^2 z - \frac{z^3}{3}\right\}_0^r = \frac{16r^3}{3}.$$



32. Vertical cross sections of the attic parallel to the length of the roof are trapezoids. Because BGC and EGF are similar triangles, ratios of corresponding sides are equal: $\|EF\|/\|FG\| = \|BC\|/\|GC\|$, or

$$\|EF\| = \frac{x(3/2)}{5} = \frac{3x}{10}.$$



Since ABC and DEF are similar triangles, we can say that $\|DE\|/\|EF\| = \|AB\|/\|BC\|$, or

$$\|DE\| = \frac{3x}{10}\left(\frac{2}{3/2}\right) = \frac{2x}{5}.$$

The area of the trapezoid at position x is therefore

$$\|DE\|\left(\frac{1}{2}\right)[15 + (15 - 2\|EF\|)] = \frac{2x}{5}\left(\frac{1}{2}\right)\left[30 - 2\left(\frac{3x}{10}\right)\right] = \frac{3x}{25}(50 - x).$$

The volume in the attic can now be calculated as

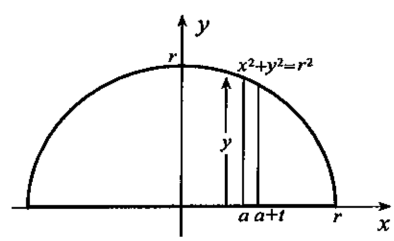
$$V = 2\int_0^5 \frac{3x}{25}(50 - x)\,dx = \frac{6}{25}\left\{25x^2 - \frac{x^3}{3}\right\}_0^5 = 140 \text{ m}^3.$$

33. In a small interval dt of time at time t before T, the number of births is $r(t)\,dt$. The number of these that survive until time T is $p(T-t)r(t)\,dt$. Integration of this gives the number of individuals born after $t = 0$ that survive to time T,

$$N(T) = \int_0^T p(T - t)r(t)\,dt.$$

34. (a) The area of the peel for a slice of thickness t at any value $x = a$ is

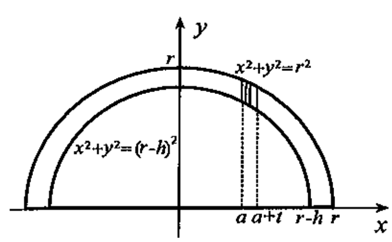
$$\begin{aligned}\text{Area} &= \int_a^{a+t} 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \\ &= 2\pi\int_a^{a+t} \sqrt{r^2 - x^2}\sqrt{1 + \frac{x^2}{r^2 - x^2}}\,dx \\ &= 2\pi\int_a^{a+t} r\,dx = 2\pi rt.\end{aligned}$$



Since this is independent of a, any slice of width t has the same peel area. If there are n slices, each of width $2r/n$, the area of peel is $2\pi r(2r/n) = 4\pi r^2/n$. This is reasonable in that this is the area of the sphere divided by the number of slices.

(b) The volume in the peel for a slice of thickness t at $x = a$ is

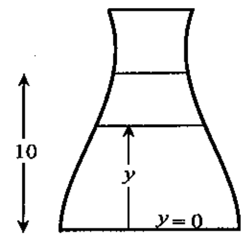
$$\begin{aligned}\text{Volume} &= \int_a^{a+t} \{\pi(r^2 - x^2) - \pi[(r - h)^2 - x^2]\}\,dx \\ &= \pi\int_a^{a+t} (2rh - h^2)\,dx = \pi(2rh - h^2)t.\end{aligned}$$



Since this is independent of a, the volume of peel is the same for each slice. This is true, however, only for slices which have holes. Clearly, slices between $r - h$ and r do not all have the same volume.

35. Suppose that $A(y)$ is the cross-sectional area of the container when the depth is y. Then, $dV/dt = k\,A(y)$. But the volume of water in the container when the depth is y is

$$V = \int_0^y A(y)\,dy.$$



The derivative of this equation with respect to t (using 6.19) is

$$\frac{dV}{dt} = \frac{dV}{dy}\frac{dy}{dt} = A(y)\frac{dy}{dt}.$$

We now equate the two expressions for dV/dt,

$$A(y)\frac{dy}{dt} = kA(y).$$

Hence $\dfrac{dy}{dt} = k$, and the solution of this equation is $y = kt + C$. The conditions $y(0) = 10$ and $y(5) = 9$ require $k = -1/5$ and $C = 10$. Consequently $y(t) = 10 - t/5$. The container empties when $y = 0$, and this occurs when $t = 50$ days.

36. (a) Yearly ordering costs are $F(N/x) + Nf$.

(b) The stocking cost at time t (in years) from the beginning of one stock period for a period dt is $p(x - Nt)\,dt$, where $x - Nt$ represents the number of refrigerators at time t. Total yearly stocking charges are

$$\frac{N}{x}\int_0^{x/N} p(x - Nt)\,dt = \frac{Np}{x}\left\{xt - \frac{Nt^2}{2}\right\}_0^{x/N} = \frac{px}{2}.$$

(c) Total yearly inventory costs are $I(x) = \dfrac{FN}{x} + Nf + \dfrac{px}{2}$. For critical points of this function, we solve

$$0 = I'(x) = -\frac{FN}{x^2} + \frac{p}{2} \quad\Longrightarrow\quad x = \sqrt{\frac{2FN}{p}}.$$

Since $\displaystyle\lim_{x\to 0^+} I(x) = \infty = \lim_{x\to\infty} I(x)$, it follows that $I(x)$ is minimized by $x = \sqrt{2FN/p}$.

EXERCISES 7.10

1. $$\int_3^\infty \frac{1}{(x+4)^2}dx = \lim_{b\to\infty}\int_3^b \frac{1}{(x+4)^2}\,dx = \lim_{b\to\infty}\left\{\frac{-1}{x+4}\right\}_3^b = \lim_{b\to\infty}\left(\frac{-1}{b+4} + \frac{1}{7}\right) = \frac{1}{7}$$

2. $$\int_3^\infty \frac{1}{(x+4)^{1/3}}\,dx = \lim_{b\to\infty}\int_3^b \frac{1}{(x+4)^{1/3}}\,dx = \lim_{b\to\infty}\left\{\frac{3}{2}(x+4)^{2/3}\right\}_3^b = \lim_{b\to\infty}\left[\frac{3}{2}(b+4)^{2/3} - \frac{3}{2}(7)^{2/3}\right] = \infty$$

3. $$\int_{-\infty}^{-4} \frac{x}{\sqrt{x^2-2}}\,dx = \lim_{a\to-\infty}\int_a^{-4} \frac{x}{\sqrt{x^2-2}}\,dx = \lim_{a\to-\infty}\left\{\sqrt{x^2-2}\right\}_a^{-4} = \lim_{a\to-\infty}(\sqrt{14} - \sqrt{a^2-2}) = -\infty$$

4. $$\int_{-\infty}^{-4} \frac{x}{(x^2-2)^4}\,dx = \lim_{a\to-\infty}\int_a^{-4} \frac{x}{(x^2-2)^4}\,dx = \lim_{a\to-\infty}\left\{\frac{-1}{6(x^2-2)^3}\right\}_a^{-4}$$
$$= \lim_{a\to-\infty}\left[-\frac{1}{6(14)^3} + \frac{1}{6(a^2-2)^3}\right] = \frac{-1}{16\,464}$$

5. $$\int_{-\infty}^\infty \frac{10^{10}x^3}{(x^4+5)^2}\,dx = 10^{10}\lim_{a\to-\infty}\int_a^0 \frac{x^3}{(x^4+5)^2}\,dx + 10^{10}\lim_{b\to\infty}\int_0^b \frac{x^3}{(x^4+5)^2}\,dx$$
$$= 10^{10}\lim_{a\to-\infty}\left\{\frac{-1}{4(x^4+5)}\right\}_a^0 + 10^{10}\lim_{b\to\infty}\left\{\frac{-1}{4(x^4+5)}\right\}_0^b$$
$$= 10^{10}\lim_{a\to-\infty}\left[-\frac{1}{20} + \frac{1}{4(a^4+5)}\right] + 10^{10}\lim_{b\to\infty}\left[\frac{-1}{4(b^4+5)} + \frac{1}{20}\right] = -\frac{10^{10}}{20} + \frac{10^{10}}{20} = 0$$

6. $$\int_{-\infty}^{\infty} \frac{x^3}{(x^4+5)^{1/4}}\,dx = \lim_{a\to-\infty}\int_a^0 \frac{x^3}{(x^4+5)^{1/4}}\,dx + \lim_{b\to\infty}\int_0^b \frac{x^3}{(x^4+5)^{1/4}}\,dx$$
$$= \lim_{a\to-\infty}\left\{\frac{1}{3}(x^4+5)^{3/4}\right\}_a^0 + \lim_{b\to\infty}\left\{\frac{1}{3}(x^4+5)^{3/4}\right\}_0^b$$
$$= \lim_{a\to-\infty}\left[\frac{5^{3/4}-(a^4+5)^{3/4}}{3}\right] + \lim_{b\to\infty}\left[\frac{(b^4+5)^{3/4}-5^{3/4}}{3}\right]$$
Since neither of these limits exists, the integral diverges.

7. $$\int_0^1 \frac{1}{(1-x)^{5/3}}\,dx = \lim_{c\to1^-}\int_0^c \frac{1}{(1-x)^{5/3}}\,dx = \lim_{c\to1^-}\left\{\frac{3}{2(1-x)^{2/3}}\right\}_0^c = \lim_{c\to1^-}\left[\frac{3}{2(1-c)^{2/3}}-\frac{3}{2}\right] = \infty$$

8. $$\int_0^1 \frac{1}{\sqrt{1-x}}\,dx = \lim_{c\to1^-}\int_0^c \frac{1}{\sqrt{1-x}}\,dx = \lim_{c\to1^-}\left\{-2\sqrt{1-x}\right\}_0^c = \lim_{c\to1^-}(-2\sqrt{1-c}+2) = 2$$

9. $$\int_1^{\infty} x\sqrt{x^2-1}\,dx = \lim_{b\to\infty}\int_1^b x\sqrt{x^2-1}\,dx = \lim_{b\to\infty}\left\{\frac{1}{3}(x^2-1)^{3/2}\right\}_1^b = \frac{1}{3}\lim_{b\to\infty}(b^2-1)^{3/2} = \infty$$

10. $$\int_2^5 \frac{x}{\sqrt{x^2-4}}\,dx = \lim_{c\to2^+}\int_c^5 \frac{x}{\sqrt{x^2-4}}\,dx = \lim_{c\to2^+}\left\{\sqrt{x^2-4}\right\}_c^5 = \lim_{c\to2^+}(\sqrt{21}-\sqrt{c^2-4}) = \sqrt{21}$$

11. $$\int_{-1}^1 \frac{x}{(1-x^2)^2}\,dx = \lim_{c\to-1^+}\int_c^0 \frac{x}{(1-x^2)^2}\,dx + \lim_{d\to1^-}\int_0^d \frac{x}{(1-x^2)^2}\,dx$$
$$= \lim_{c\to-1^+}\left\{\frac{1}{2(1-x^2)}\right\}_c^0 + \lim_{d\to1^-}\left\{\frac{1}{2(1-x^2)}\right\}_0^d$$
$$= \lim_{c\to-1^+}\left[\frac{1}{2}-\frac{1}{2(1-c^2)}\right] + \lim_{d\to1^-}\left[\frac{1}{2(1-d^2)}-\frac{1}{2}\right]$$
Since neither of these limits exists, the integral diverges.

12. $$\int_{-\infty}^{\infty} \frac{1}{x^2}\,dx = \lim_{a\to-\infty}\int_a^{-1} \frac{1}{x^2}\,dx + \lim_{b\to0^-}\int_{-1}^b \frac{1}{x^2}\,dx + \lim_{c\to0^+}\int_c^1 \frac{1}{x^2}\,dx + \lim_{d\to\infty}\int_1^d \frac{1}{x^2}\,dx$$
Since $\lim_{c\to0^+}\int_c^1 \frac{1}{x^2}\,dx = \lim_{c\to0^+}\left\{-\frac{1}{x}\right\}_c^1 = \lim_{c\to0^+}\left(-1+\frac{1}{c}\right) = \infty$, the integral diverges.

13. $$\int_0^{\infty} \frac{1}{\sqrt{x}}\,dx = \lim_{c\to0^+}\int_c^1 \frac{1}{\sqrt{x}}\,dx + \lim_{b\to\infty}\int_1^b \frac{1}{\sqrt{x}}\,dx = \lim_{c\to0^+}\left\{2\sqrt{x}\right\}_c^1 + \lim_{b\to\infty}\left\{2\sqrt{x}\right\}_1^b$$
$$= 2\lim_{c\to0^+}(1-\sqrt{c}) + 2\lim_{b\to\infty}(\sqrt{b}-1)$$
Since the second limit does not exist, the integral diverges.

14. $$\int_{-\infty}^{\pi/2} \frac{x}{(x^2-4)^2}\,dx = \lim_{a\to-\infty}\int_a^{-3} \frac{x}{(x^2-4)^2}\,dx + \lim_{b\to-2^-}\int_{-3}^b \frac{x}{(x^2-4)^2}\,dx + \lim_{c\to-2^+}\int_c^{\pi/2} \frac{x}{(x^2-4)^2}\,dx$$
Since $\lim_{b\to-2^-}\int_{-3}^b \frac{x}{(x^2-4)^2}\,dx = \lim_{b\to-2^-}\left\{\frac{-1}{2(x^2-4)}\right\}_{-3}^b = \lim_{b\to-2^-}\left[\frac{-1}{2(b^2-4)}+\frac{1}{10}\right] = -\infty$,
the integral diverges.

15. $\int_4^{\infty} \cos x\,dx = \lim_{b\to\infty}\int_4^b \cos x\,dx = \lim_{b\to\infty}\left\{\sin x\right\}_4^b = \lim_{b\to\infty}(\sin b - \sin 4)$, and this limit does not exist.

16. $$\int_{-\infty}^{\infty} \sin x\,dx = \lim_{a\to-\infty}\int_a^0 \sin x\,dx + \lim_{b\to\infty}\int_0^b \sin x\,dx$$
Since $\lim_{b\to\infty}\int_0^b \sin x\,dx = \lim_{b\to\infty}\left\{-\cos x\right\}_0^b = \lim_{b\to\infty}(-\cos b + 1)$, and this limit does not exist, neither does the integral.

17. If we set $u = x + 3$ and $du = dx$, then

$$\int \frac{x}{\sqrt{x+3}}\,dx = \int \frac{u-3}{\sqrt{u}}\,du = \frac{2u^{3/2}}{3} - 6\sqrt{u} + C = \frac{2}{3}(x+3)^{3/2} - 6\sqrt{x+3} + C.$$

Thus $\displaystyle\int_0^\infty \frac{x}{\sqrt{x+3}}\,dx = \lim_{b\to\infty}\int_0^b \frac{x}{\sqrt{x+3}}\,dx = \lim_{b\to\infty}\left\{\frac{2}{3}(x+3)^{3/2} - 6\sqrt{x+3}\right\}_0^b$

$$= \lim_{b\to\infty}\left[\frac{2}{3}(b+3)^{3/2} - 6\sqrt{b+3} - 2\sqrt{3} + 6\sqrt{3}\right] = \infty.$$

18. If we set $u = x^2 - 4$, then $du = 2x\,dx$, and

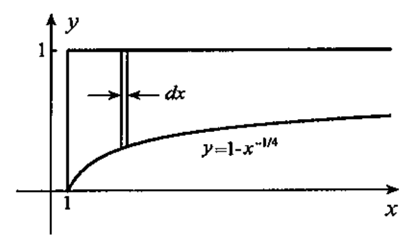
$$\int \frac{x^3}{\sqrt{x^2-4}}\,dx = \int \frac{(u+4)}{\sqrt{u}}\frac{du}{2} = \frac{1}{2}\left\{\frac{2}{3}u^{3/2} + 8\sqrt{u}\right\} + C = \frac{1}{3}(x^2-4)^{3/2} + 4\sqrt{x^2-4} + C.$$

Thus $\displaystyle\int_2^3 \frac{x^3}{\sqrt{x^2-4}}\,dx = \lim_{c\to 2+}\int_c^3 \frac{x^3}{\sqrt{x^2-4}}\,dx = \lim_{c\to 2+}\left\{\frac{1}{3}(x^2-4)^{3/2} + 4\sqrt{x^2-4}\right\}_c^3$

$$= \lim_{c\to 2+}\left[\frac{5\sqrt{5}}{3} + 4\sqrt{5} - \frac{1}{3}(c^2-4)^{3/2} - 4\sqrt{c^2-4}\right] = \frac{17\sqrt{5}}{3}.$$

19. No. The sine function is odd in Exercise 16 but the integral diverges.

20. (a) If it is possible, the area must be defined by



$$\int_1^\infty \left[1 - (1 - x^{-1/4})\right]dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^{1/4}}\,dx$$

$$= \lim_{b\to\infty}\left\{\frac{4}{3}x^{3/4}\right\}_1^b = \lim_{b\to\infty}\left(\frac{4}{3}b^{3/4} - \frac{4}{3}\right) = \infty.$$

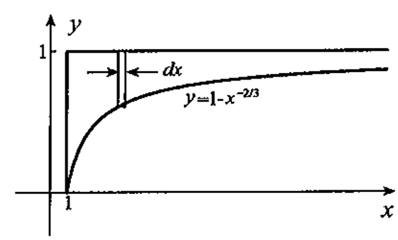
Consequently, we cannot assign an area to the region.

(b) If it is possible, the volume must be defined by

$$\int_1^\infty \pi\left[1 - (1 - x^{-1/4})\right]^2 dx = \lim_{b\to\infty}\pi\int_1^b \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty}\pi\{2\sqrt{x}\}_1^b = \lim_{b\to\infty}\pi(2\sqrt{b} - 2) = \infty.$$

Thus, no volume can be assigned to the solid of revolution.

21. (a) If it is possible, the area must be defined by



$$\int_1^\infty \left[1 - (1 - x^{-2/3})\right]dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^{2/3}}\,dx$$

$$= \lim_{b\to\infty}\left\{3x^{1/3}\right\}_1^b = 3\lim_{b\to\infty}(b^{1/3} - 1) = \infty.$$

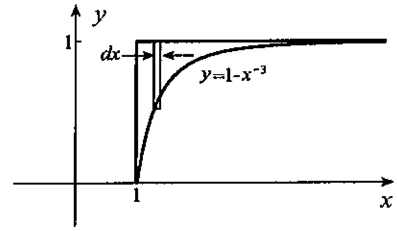
Consequently, we cannot assign an area to the region.

(b) If it is possible, the volume must be defined by

$$\int_1^\infty \pi\left[1 - (1 - x^{-2/3})\right]^2 dx = \pi\lim_{b\to\infty}\int_1^b \frac{1}{x^{4/3}}\,dx = \pi\lim_{b\to\infty}\left\{\frac{-3}{x^{1/3}}\right\}_1^b = 3\pi\lim_{b\to\infty}\left(1 - \frac{1}{b^{1/3}}\right) = 3\pi.$$

Thus, the volume is 3π.

22. (a) If it is possible, the area must be defined by



$$\int_1^\infty [1 - (1 - x^{-3})]\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^3}\,dx$$

$$= \lim_{b\to\infty}\left\{\frac{-1}{2x^2}\right\}_1^b = \lim_{b\to\infty}\left(\frac{-1}{2b^2} + \frac{1}{2}\right) = \frac{1}{2}.$$

(b) If it is possible, the volume must be defined by

$$\int_1^\infty \pi\left[1 - (1 - x^{-3})\right]^2 dx = \lim_{b\to\infty}\pi\int_1^b \frac{1}{x^6}\,dx = \lim_{b\to\infty}\pi\left\{\frac{-1}{5x^5}\right\}_1^b = \lim_{b\to\infty}\pi\left(\frac{-1}{5b^5} + \frac{1}{5}\right) = \frac{\pi}{5}.$$

23. (a) Both functions are nonnegative for $x \ge 0$. They are pdf's since

$$\int_0^\infty \frac{6x}{(1+3x^2)^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{6x}{(1+3x^2)^2}\,dx = \lim_{b\to\infty}\left\{\frac{-1}{1+3x^2}\right\}_0^b = \lim_{b\to\infty}\left(1-\frac{1}{1+3b^2}\right) = 1,$$

$$\begin{aligned}\int_0^\infty \frac{2x}{(1+x)^3}\,dx &= \lim_{b\to\infty}\int_0^b \frac{2(x+1)-2}{(1+x)^3}\,dx = \lim_{b\to\infty}\int_0^b\left[\frac{2}{(1+x)^2}-\frac{2}{(1+x)^3}\right]dx\\ &= \lim_{b\to\infty}\left\{\frac{-2}{1+x}+\frac{1}{(1+x)^2}\right\}_0^b = \lim_{b\to\infty}\left[\frac{-2}{1+b}+\frac{1}{(1+b)^2}+2-1\right] = 1.\end{aligned}$$

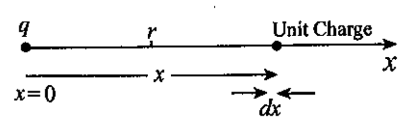
(b) $P(x \ge 3)$ for these pdf's are:

$$\begin{aligned}P(x\ge 3) &= \int_3^\infty \frac{6x}{(1+3x^2)^2}dx = \lim_{b\to\infty}\int_3^b \frac{6x}{(1+3x^2)^2}dx = \lim_{b\to\infty}\left\{\frac{-1}{1+3x^2}\right\}_3^b\\ &= \lim_{b\to\infty}\left(\frac{1}{28}-\frac{1}{1+3b^2}\right) = \frac{1}{28};\\ P(x\ge 3) &= \int_3^\infty \frac{2x}{(1+x)^3}dx = \lim_{b\to\infty}\int_3^b \frac{2x}{(1+x)^3}dx = \lim_{b\to\infty}\int_3^b\left[\frac{2}{(1+x)^2}-\frac{2}{(1+x)^3}\right]dx\\ &= \lim_{b\to\infty}\left\{\frac{-2}{1+x}+\frac{1}{(1+x)^2}\right\}_3^b = \lim_{b\to\infty}\left[\frac{-2}{1+b}+\frac{1}{(1+b)^2}+\frac{2}{4}-\frac{1}{16}\right] = \frac{7}{16}.\end{aligned}$$

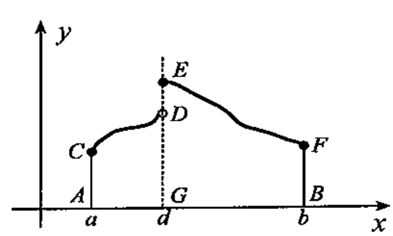
24. $$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx = \lim_{b\to\infty}\left\{\frac{1}{(1-p)x^{p-1}}\right\}_1^b = \lim_{b\to\infty}\left[\frac{1}{(1-p)b^{p-1}}+\frac{1}{p-1}\right] = \begin{cases}\infty, & p<1\\ \dfrac{1}{p-1}, & p>1\end{cases}$$

25. The potential is

$$\begin{aligned}V &= \int_\infty^r \frac{-q}{4\pi\epsilon_0 x^2}\,dx = \frac{q}{4\pi\epsilon_0}\lim_{b\to\infty}\int_r^b \frac{1}{x^2}\,dx\\ &= \frac{q}{4\pi\epsilon_0}\lim_{b\to\infty}\left\{-\frac{1}{x}\right\}_r^b = \frac{q}{4\pi\epsilon_0}\lim_{b\to\infty}\left(\frac{1}{r}-\frac{1}{b}\right)\\ &= \frac{q}{4\pi\epsilon_0 r}.\end{aligned}$$



26. $$\int_a^b f(x)\,dx = \lim_{c\to d^-}\int_a^c f(x)\,dx + \lim_{c\to d^+}\int_c^b f(x)\,dx$$
The first limit can be interpreted as area $ACDG$ were the hole to be filled in at D. The second limit is area $GEFB$. Since both areas are clearly defined, the improper integral does indeed exist.

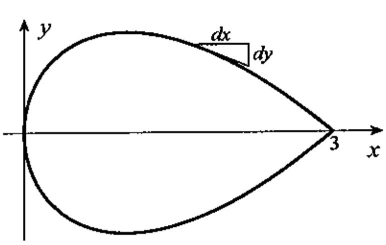


27. If we differentiate $9y^2 = 9x - 6x^2 + x^3$ with respect to x,

$$18y\frac{dy}{dx} = 9 - 12x + 3x^2 \Longrightarrow \frac{dy}{dx} = \frac{3-4x+x^2}{6y}.$$

Small lengths along the curve in the first quadrant corresponding to lengths dx along the x-axis are given by

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+\left(\frac{3-4x+x^2}{6y}\right)^2}\,dx$$



$$= \sqrt{1 + \frac{(3-x)^2(1-x)^2}{4x(3-x)^2}}\,dx = \sqrt{\frac{4x+(1-x)^2}{4x}}\,dx = \sqrt{\frac{(1+x)^2}{4x}}\,dx = \frac{1+x}{2\sqrt{x}}\,dx.$$

The length of the loop is

$$2\int_0^3 \frac{1+x}{2\sqrt{x}}\,dx = \lim_{c\to 0^+}\int_c^3 \left(\frac{1}{\sqrt{x}} + \sqrt{x}\right)dx = \lim_{c\to 0^+}\left\{2\sqrt{x} + \frac{2x^{3/2}}{3}\right\}_c^3$$
$$= \lim_{c\to 0^+}\left(2\sqrt{3} + 2\sqrt{3} - 2\sqrt{c} - \frac{2c^{3/2}}{3}\right) = 4\sqrt{3}.$$

28. No. For example, according to definition 7.49 the integral in Exercise 16 does not exist. But were we to use this definition:

$$\int_{-\infty}^{\infty} \sin x\,dx = \lim_{a\to\infty}\int_{-a}^{a} \sin x\,dx = \lim_{a\to\infty}\{-\cos x\}_{-a}^{a} = \lim_{a\to\infty}[-\cos a + \cos(-a)] = \lim_{a\to\infty}(0) = 0,$$

and the improper integral would exist.

29. Since $\dfrac{x^2}{\sqrt{x^2-1}} \ge \dfrac{x}{\sqrt{x^2-1}}$ for $x \ge 2$, we can say that

$$\int_2^{\infty} \frac{x^2}{\sqrt{x^2-1}}\,dx \ge \int_2^{\infty} \frac{x}{\sqrt{x^2-1}}\,dx = \lim_{b\to\infty}\int_2^b \frac{x}{\sqrt{x^2-1}}\,dx$$
$$= \lim_{b\to\infty}\left\{\sqrt{x^2-1}\right\}_2^b = \lim_{b\to\infty}(\sqrt{b^2-1} - \sqrt{3}) = \infty.$$

30. Since $\dfrac{x^3}{(27-x^3)^2} \ge \dfrac{x^2}{(27-x^3)^2} > 0$ for $1 \le x \le 3$, we can say that

$$\int_1^3 \frac{x^3}{(27-x^3)^2}\,dx \ge \int_1^3 \frac{x^2}{(27-x^3)^2}\,dx = \lim_{c\to 3^-}\int_1^c \frac{x^2}{(27-x^3)^2}\,dx$$
$$= \lim_{c\to 3^-}\left\{\frac{1}{3(27-x^3)}\right\}_1^c = \lim_{c\to 3^-}\left[\frac{1}{3(27-c^3)} - \frac{1}{78}\right] = \infty.$$

31. Since $\dfrac{x^2}{\sqrt{1-x^2}} \le \dfrac{x}{\sqrt{1-x^2}}$ for $0 \le x \le 1$, we can say that

$$\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx \le \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx = \lim_{c\to 1^-}\int_0^c \frac{x}{\sqrt{1-x^2}}\,dx$$
$$= \lim_{c\to 1^-}\left\{-\sqrt{1-x^2}\right\}_0^c = \lim_{c\to 1^-}\left(1 - \sqrt{1-c^2}\right) = 1.$$

The given intergal therefore converges.

32. Since $\dfrac{\sqrt{-x}}{(x^2+5)^2} < \dfrac{-x}{(x^2+5)^2}$ for $x \le -2$, we can say that

$$\int_{-\infty}^{-2} \frac{\sqrt{-x}}{(x^2+5)^2}\,dx < \int_{-\infty}^{-2} \frac{-x}{(x^2+5)^2}\,dx = \lim_{a\to-\infty}\int_a^{-2} \frac{-x}{(x^2+5)^2}\,dx$$
$$= \lim_{a\to-\infty}\left\{\frac{1}{2(x^2+5)}\right\}_a^{-2} = \lim_{a\to-\infty}\left[\frac{1}{18} - \frac{1}{2(a^2+5)}\right] = \frac{1}{18}.$$

The given integral therefore converges.

33. By changing the value of c, to d say, where $d > c$,we are simply moving the integral of a continuous function $f(x)$ from the second integral on the right of 7.49 to the first integral on the right. This does not affect whether either integral converges or diverges.

34. $$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = \int_{-\infty}^{\infty} f(x)\left[\lim_{\epsilon\to 0} P_\epsilon(x-a)\right]dx = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} f(x)\,P_\epsilon(x-a)\,dx$$

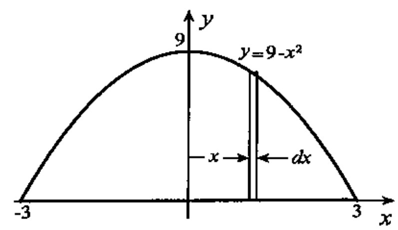
$$= \lim_{\epsilon\to 0}\int_a^{a+\epsilon} f(x)\frac{1}{\epsilon}dx = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_a^{a+\epsilon} f(x)\,dx.$$

If we let $F(x)$ be an antiderivative of $f(x)$, then

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = \lim_{\epsilon\to 0}\frac{1}{\epsilon}[F(a+\epsilon)-F(a)] = F'(a) = f(a).$$

REVIEW EXERICSES

1. (a) $A = 2\int_0^3 (9-x^2)\,dx = 2\left\{9x - \frac{x^3}{3}\right\}_0^3 = 36$



(b) $V_x = 2\int_0^3 \pi(9-x^2)^2\,dx = 2\pi\int_0^3 (81-18x^2+x^4)\,dx$

$$= 2\pi\left\{81x - 6x^3 + \frac{x^5}{5}\right\}_0^3 = \frac{1296\pi}{5}$$

$$V_y = \int_0^3 2\pi x(9-x^2)\,dx = 2\pi\left\{\frac{9x^2}{2} - \frac{x^4}{4}\right\}_0^3 = \frac{81\pi}{2}$$

(c) By symmetry, $\overline{x} = 0$. Since

$$A\overline{y} = 2\int_0^3 \frac{1}{2}(9-x^2)^2\,dx = \int_0^3 (81-18x^2+x^4)\,dx = \left\{81x - 6x^3 + \frac{x^5}{5}\right\}_0^3 = \frac{648}{5},$$

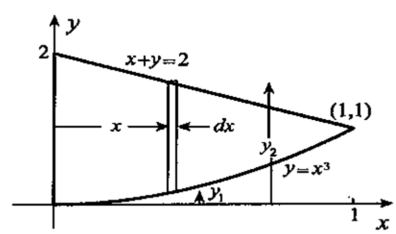
it follows that $\overline{y} = \frac{648}{5}\frac{1}{36} = \frac{18}{5}$.

(d) $I_x = 2\int_0^3 \frac{1}{3}(9-x^2)^3\,dx = \frac{2}{3}\int_0^3 (729-243x^2+27x^4-x^6)\,dx$

$$= \frac{2}{3}\left\{729x - 81x^3 + \frac{27x^5}{5} - \frac{x^7}{7}\right\}_0^3 = \frac{23\,328}{35}$$

$$I_y = 2\int_0^3 x^2(9-x^2)\,dx = 2\left\{3x^3 - \frac{x^5}{5}\right\}_0^3 = \frac{324}{5}$$

2. (a) $A = \int_0^1 (2-x-x^3)\,dx = \left\{2x - \frac{x^2}{2} - \frac{x^4}{4}\right\}_0^1 = \frac{5}{4}$



(b) $V_x = \int_0^1 \pi(y_2^2 - y_1^2)\,dx = \pi\int_0^1 [(2-x)^2 - (x^3)^2]\,dx$

$$= \pi\left\{-\frac{1}{3}(2-x)^3 - \frac{x^7}{7}\right\}_0^1 = \frac{46\pi}{21}$$

$$V_y = \int_0^1 2\pi x(y_2-y_1)\,dx = 2\pi\int_0^1 x(2-x-x^3)\,dx = 2\pi\left\{x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right\}_0^1 = \frac{14\pi}{15}$$

(c) Since $A\overline{x} = \int_0^1 x(y_2-y_1)\,dx = \frac{1}{2\pi}V_y = \frac{7}{15}$, we find $\overline{x} = \frac{7}{15}\frac{4}{5} = \frac{28}{75}$. Since

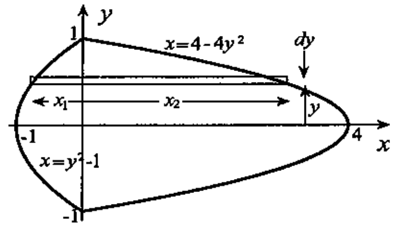
$A\overline{y} = \int_0^1 \frac{1}{2}(y_2+y_1)(y_2-y_1)\,dx = \frac{1}{2}\int_0^1 (y_2^2-y_1^2)\,dx = \frac{1}{2\pi}V_x = \frac{23}{21}$, it follows that $\overline{y} = \frac{23}{21}\frac{4}{5} = \frac{92}{105}$.

(d) $I_x = \int_0^1 \frac{1}{3}(y_2^3 - y_1^3)\,dx = \frac{1}{3}\int_0^1 [(2-x)^3 - (x^3)^3]\,dx = \frac{1}{3}\left\{-\frac{1}{4}(2-x)^4 - \frac{x^{10}}{10}\right\}_0^1 = \frac{73}{60}$

$$I_y = \int_0^1 x^2(2-x-x^3)\,dx = \left\{\frac{2x^3}{3} - \frac{x^4}{4} - \frac{x^6}{6}\right\}_0^1 = \frac{1}{4}$$

3. (a) $A = 2\int_0^1 [(4-4y^2)-(y^2-1)]\,dy$

$$= 2\left\{5y - \frac{5y^3}{3}\right\}_0^1 = \frac{20}{3}$$



(b) $V_x = \int_0^1 2\pi y(x_2-x_1)\,dy = 2\pi\int_0^1 y(5-5y^2)\,dy$

$$= 2\pi\left\{\frac{5y^2}{2} - \frac{5y^4}{4}\right\}_0^1 = \frac{5\pi}{2}$$

$$V_y = 2\int_0^1 \pi(x_2)^2\,dy = 2\pi\int_0^1 (4-4y^2)^2\,dy = 32\pi\int_0^1 (1-2y^2+y^4)\,dy = 32\pi\left\{y - \frac{2y^3}{3} + \frac{y^5}{5}\right\}_0^1 = \frac{256\pi}{15}$$

(c) By symmetry, $\overline{y} = 0$. Since

$$A\overline{x} = 2\int_0^1 \frac{1}{2}(x_1+x_2)(x_2-x_1)\,dy = \int_0^1 [(4-4y^2)^2-(y^2-1)^2]\,dy = 15\int_0^1 (1-2y^2+y^4)\,dy$$

$$= 15\left\{y - \frac{2y^3}{3} + \frac{y^5}{5}\right\}_0^1 = 8,$$

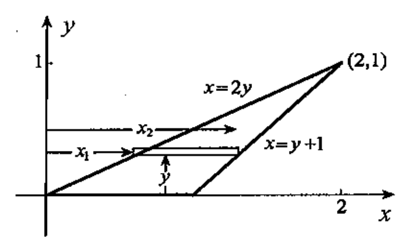
it follows that $\overline{x} = 8(3/20) = 6/5$.

(d) $I_x = 2\int_0^1 y^2(x_2-x_1)\,dy = 2\int_0^1 y^2(5-5y^2)\,dy = 2\left\{\frac{5y^3}{3} - y^5\right\}_0^1 = \frac{4}{3}$

$$I_y = 2\int_0^1 \frac{1}{3}(x_2^3 - x_1^3)\,dy = \frac{2}{3}\int_0^1 [(4-4y^2)^3-(y^2-1)^3]\,dy = \frac{130}{3}\int_0^1 (1-3y^2+3y^4-y^6)\,dy$$

$$= \frac{130}{3}\left\{y - y^3 + \frac{3y^5}{5} - \frac{y^7}{7}\right\}_0^1 = \frac{416}{21}$$

4. (a) $A = \int_0^1 (y+1-2y)\,dy = \left\{y - \frac{y^2}{2}\right\}_0^1 = \frac{1}{2}$



(b) $V_x = \int_0^1 2\pi y(x_2-x_1)\,dy = 2\pi\int_0^1 y(1-y)\,dy$

$$= 2\pi\left\{\frac{y^2}{2} - \frac{y^3}{3}\right\}_0^1 = \frac{\pi}{3}$$

$$V_y = \int_0^1 \pi(x_2^2 - x_1^2)\,dy$$

$$= \pi\int_0^1 [(y+1)^2-(2y)^2]\,dy = \pi\left\{\frac{1}{3}(y+1)^3 - \frac{4y^3}{3}\right\}_0^1 = \pi$$

(c) Since $A\overline{x} = \int_0^1 \frac{1}{2}(x_2+x_1)(x_2-x_1)\,dy = \frac{1}{2}\int_0^1 (x_2^2-x_1^2)\,dy = \frac{1}{2\pi}V_y = \frac{1}{2}$, we find $\overline{x} = \frac{1}{2}(2) = 1$.

Since $A\overline{y} = \int_0^1 y(x_2-x_1)\,dy = \frac{1}{2\pi}V_x = \frac{1}{6}$, it follows that $\overline{y} = \frac{1}{6}(2) = \frac{1}{3}$.

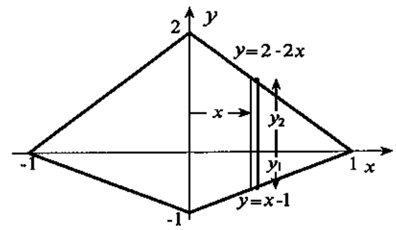
(d) $I_x = \int_0^1 y^2(1-y)\,dy = \left\{\frac{y^3}{3} - \frac{y^4}{4}\right\}_0^1 = \frac{1}{12}$

$$I_y = \int_0^1 \frac{1}{3}(x_2^3 - x_1^3)\,dy = \frac{1}{3}\int_0^1 [(y+1)^3-(2y)^3]\,dy = \frac{1}{3}\left\{\frac{1}{4}(y+1)^4 - 2y^4\right\}_0^1 = \frac{7}{12}$$

5. (a) Area $= 2 + 1 = 3$

(b) $V_x = (2/3)\pi(2)^2(1) = 8\pi/3, \quad V_y = (1/3)\pi(1)^2(2) + (1/3)\pi(1)^2(1) = \pi$

(c) By symmetry, $\overline{x} = 0$. Since



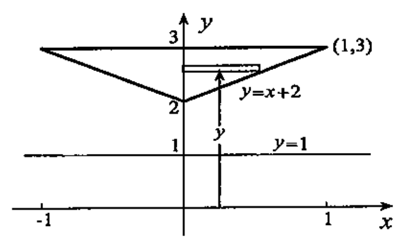
$$A\overline{y} = 2\int_0^1 \frac{1}{2}(y_2 + y_1)(y_2 - y_1)\,dx$$
$$= \int_0^1 [(2-2x)^2 - (x-1)^2]\,dx$$
$$= \left\{\frac{(2-2x)^3}{-6} - \frac{(x-1)^3}{3}\right\}_0^1 = 1,$$

it follows that $\overline{y} = 1/3$.

(d) $I_x = 2\displaystyle\int_0^1 \frac{1}{3}(y_2^3 - y_1^3)\,dx = \frac{2}{3}\int_0^1 [(2-2x)^3 - (x-1)^3]\,dx = \frac{2}{3}\left\{\frac{(2-2x)^4}{-8} - \frac{(x-1)^4}{4}\right\}_0^1 = \frac{3}{2}$

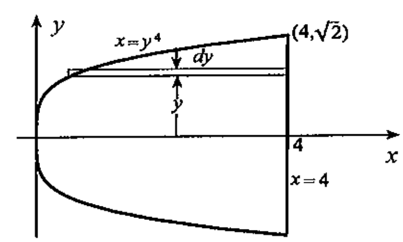
$$I_y = 2\int_0^1 x^2(y_2 - y_1)\,dx = 2\int_0^1 x^2(3-3x)\,dx = 2\left\{x^3 - \frac{3x^4}{4}\right\}_0^1 = \frac{1}{2}$$

6. $V = 2\displaystyle\int_2^3 2\pi(y-1)(y-2)\,dy$



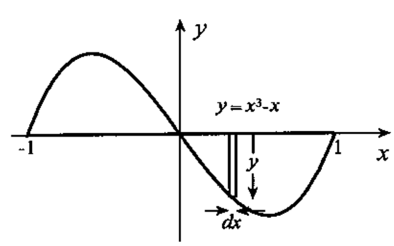
$$= 4\pi\int_2^3 (y^2 - 3y + 2)\,dy$$
$$= 4\pi\left\{\frac{y^3}{3} - \frac{3y^2}{2} + 2y\right\}_2^3 = \frac{10\pi}{3}$$

7. $V = 2\displaystyle\int_0^{\sqrt{2}} \pi(4 - y^4)^2\,dy$



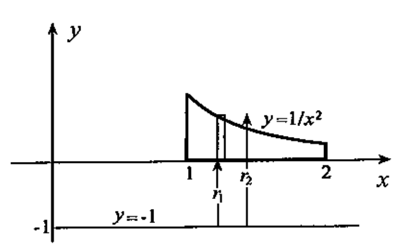
$$= 2\pi\int_0^{\sqrt{2}} (16 - 8y^4 + y^8)\,dy$$
$$= 2\pi\left\{16y - \frac{8y^5}{5} + \frac{y^9}{9}\right\}_0^{\sqrt{2}} = \frac{1024\sqrt{2}\pi}{45}$$

8. $V = 2\displaystyle\int_0^1 \pi(-y)^2\,dx = 2\pi\int_0^1 (x^3 - x)^2\,dx$



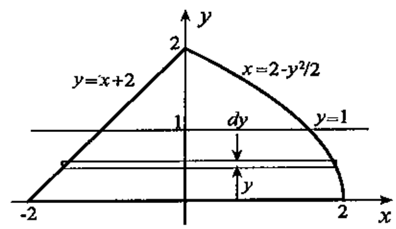
$$= 2\pi\int_0^1 (x^6 - 2x^4 + x^2)\,dx$$
$$= 2\pi\left\{\frac{x^7}{7} - \frac{2x^5}{5} + \frac{x^3}{3}\right\}_0^1 = \frac{16\pi}{105}$$

9. $V = \displaystyle\int_1^2 (\pi r_2^2 - \pi r_1^2)\,dx = \pi\int_1^2 [(1/x^2 + 1)^2 - (1)^2]\,dx$



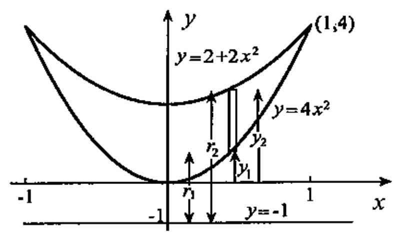
$$= \pi\int_1^2 \left(\frac{1}{x^4} + \frac{2}{x^2}\right)dx$$
$$= \pi\left\{-\frac{1}{3x^3} - \frac{2}{x}\right\}_1^2 = \frac{31\pi}{24}$$

10. We reject the area above $y = 1$.



$$V = \int_0^1 2\pi(1-y)\left(2 - \frac{y^2}{2} - y + 2\right) dy$$
$$= \pi \int_0^1 (y^3 + y^2 - 10y + 8)\, dy$$
$$= \pi \left\{ \frac{y^4}{4} + \frac{y^3}{3} - 5y^2 + 8y \right\}_0^1 = \frac{43\pi}{12}$$

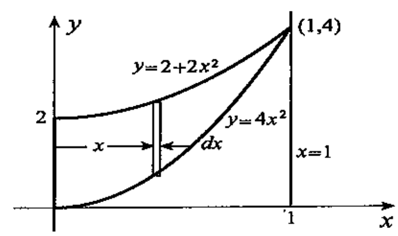
11. The first moment is



$$2\int_0^1 \left[\frac{y_1 + y_2}{2} + 1\right](y_2 - y_1)\, dx$$
$$= \int_0^1 [(y_2^2 - y_1^2 + 2(y_2 - y_1)]\, dx$$
$$= \int_0^1 [(2+2x^2)^2 - (4x^2)^2 + 2(2 - 2x^2)]\, dx$$
$$= 4\int_0^1 (2 + x^2 - 3x^4)\, dx = 4\left\{2x + \frac{x^3}{3} - \frac{3x^5}{5}\right\}_0^1 = \frac{104}{15}$$

$$I = 2\int_0^1 \frac{1}{3}(r_2^3 - r_1^3)\, dx = \frac{2}{3}\int_0^1 [(2+2x^2+1)^3 - (4x^2+1)^3]\, dx = \frac{4}{3}\int_0^1 (13 + 21x^2 - 6x^4 - 28x^6)\, dx$$
$$= \frac{4}{3}\left\{13x + 7x^3 - \frac{6x^5}{5} - 4x^7\right\}_0^1 = \frac{296}{15}$$

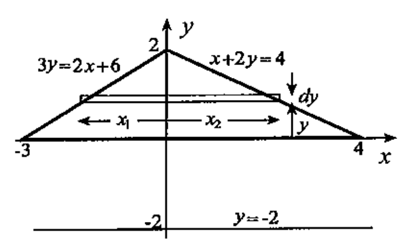
12. The first moment is



$$\int_0^1 (x-1)(2 + 2x^2 - 4x^2)\, dx$$
$$= 2\int_0^1 (-1 + x + x^2 - x^3)\, dx$$
$$= 2\left\{-x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\right\}_0^1 = -\frac{5}{6}$$

$$I = \int_0^1 (x-1)^2(2 + 2x^2 - 4x^2)\, dx = 2\int_0^1 (1 - 2x + 2x^3 - x^4)\, dx = 2\left\{x - x^2 + \frac{x^4}{2} - \frac{x^5}{5}\right\}_0^1 = \frac{3}{5}$$

13. The first moment is



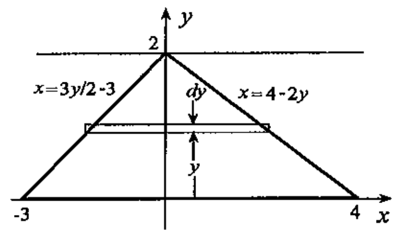
$$\int_0^2 (y+2)(x_2 - x_1)\, dy$$
$$= \int_0^2 (y+2)[(4-2y) - (3y/2 - 3)]\, dy$$
$$= \frac{7}{2}\int_0^2 (4 - y^2)\, dy = \frac{7}{2}\left\{4y - \frac{y^3}{3}\right\}_0^2 = \frac{56}{3}$$

$$I = \int_0^2 (y+2)^2(x_2 - x_1)\, dy = \int_0^2 (y+2)^2[(4-2y) - (3y/2 - 3)]\, dy$$
$$= \frac{7}{2}\int_0^2 (8 + 4y - 2y^2 - y^3)\, dy = \frac{7}{2}\left\{8y + 2y^2 - \frac{2y^3}{3} - \frac{y^4}{4}\right\}_0^2 = \frac{154}{3}$$

14. The first moment is

$$\int_0^2 (y-2)\left(4-2y-\frac{3y}{2}+3\right)dy = -\frac{7}{2}\int_0^2 (y-2)^2\,dy$$

$$= -\frac{7}{2}\left\{\frac{1}{3}(y-2)^3\right\}_0^2 = -\frac{28}{3}$$



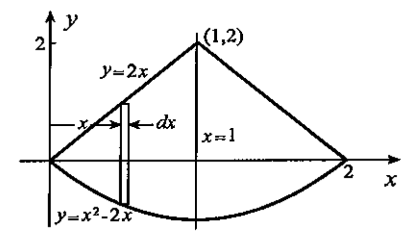
$$I = \int_0^2 (y-2)^2\left(4-2y-\frac{3y}{2}+3\right)dy = -\frac{7}{2}\int_0^2 (y-2)^3\,dy = -\frac{7}{2}\left\{\frac{1}{4}(y-2)^4\right\}_0^2 = 14$$

15. Because the area is symmetric about $x=1$, its first moment about this line is zero. Its moment of inertia is

$$I = 2\int_0^1 (x-1)^2(2x-x^2+2x)\,dx$$

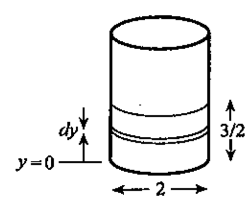
$$= 2\int_0^1 (4x-9x^2+6x^3-x^4)\,dx$$

$$= 2\left\{2x^2-3x^3+\frac{3x^4}{2}-\frac{x^5}{5}\right\}_0^1 = \frac{3}{5}$$



16. $W = \int_0^{3/2} (3-y)1000(9.81)\pi(1)^2\,dy$

$$= 9810\pi\left\{3y-\frac{y^2}{2}\right\}_0^{3/2} = 1.04\times 10^5 \text{ J.}$$



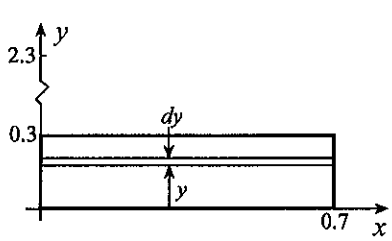
17. The force on the rear window (left figure below) is

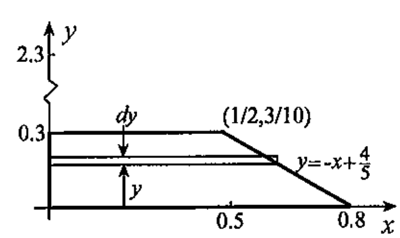
$$F = \int_0^{3/10} 9.81(1000)(23/10-y)(7/10)\,dy = 6867\left\{-\frac{1}{2}\left(\frac{23}{10}-y\right)^2\right\}_0^{3/10} = 4.43\times 10^3 \text{ N.}$$

The force on the front window (right figure below) is

$$F = \int_0^{3/10} 9.81(1000)(23/10-y)(4/5-y)\,dy = \frac{981}{5}\int_0^{3/10}(92-155y+50y^2)\,dy$$

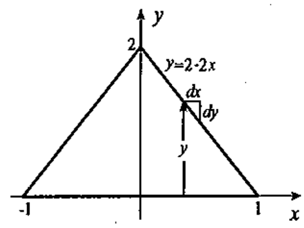
$$= \frac{981}{5}\left\{92y-\frac{155y^2}{2}+\frac{50y^3}{3}\right\}_0^{3/10} = 4.13\times 10^3 \text{ N.}$$





18. The surface area is twice the area of the surface of revolution obtained by rotating that part of $y = 2 - 2x$ in the first quadrant about the x-axis. Small lengths along the line are given by

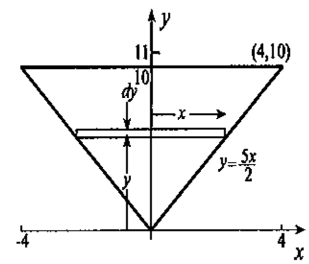
$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+(-2)^2}\,dx = \sqrt{5}\,dx.$$



The required area is therefore

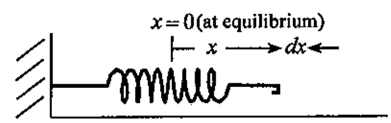
$$A = 2\int_0^1 2\pi y\sqrt{5}\,dx = 4\sqrt{5}\pi\int_0^1 (2-2x)\,dx = 8\sqrt{5}\pi\left\{x - \frac{x^2}{2}\right\}_0^1 = 4\sqrt{5}\pi.$$

19. $$\begin{aligned} W &= \int_8^{10} (11-y)(9.81)(1000)(\pi x^2)\,dy \\ &= 9810\pi\int_8^{10}(11-y)\left(\frac{4y^2}{25}\right)dy \\ &= 1569.6\pi\left\{\frac{11y^3}{3} - \frac{y^4}{4}\right\}_8^{10} = 1.55\times 10^6 \text{ J} \end{aligned}$$



20. Suppose x_0 is the stretch that requires W units of work. When the spring is stretched x, the force required to maintain this stretch is kx. The work to stretch the spring x_0 is

$$W = \int_0^{x_0} kx\,dx = k\left\{\frac{x^2}{2}\right\}_0^{x_0} = \frac{1}{2}kx_0^2.$$



The work required to obtain a stretch of $2x_0$ is $\displaystyle\int_0^{2x_0} kx\,dx = k\left\{\frac{x^2}{2}\right\}_0^{2x_0} = 2kx_0^2 = 4W$. Thus, it requires $3W$ more unit of work to increase the stretch from x_0 to $2x_0$.

21. $$\int_1^\infty \frac{1}{x^{3/2}}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^{3/2}}dx = \lim_{b\to\infty}\left\{\frac{-2}{x^{1/2}}\right\}_1^b = \lim_{b\to\infty}\left(2 - \frac{2}{\sqrt{b}}\right) = 2$$

22. $$\int_0^3 \frac{1}{\sqrt{3-x}}\,dx = \lim_{c\to 3^-}\int_0^c \frac{1}{\sqrt{3-x}}\,dx = \lim_{c\to 3^-}\left\{-2\sqrt{3-x}\right\}_0^c = \lim_{c\to 3^-}(-2\sqrt{3-c} + 2\sqrt{3}) = 2\sqrt{3}$$

23. $$\int_{-1}^0 \frac{1}{(x+1)^2}dx = \lim_{c\to -1^+}\int_c^0 \frac{1}{(x+1)^2}dx = \lim_{c\to -1^+}\left\{\frac{-1}{x+1}\right\}_c^0 = \lim_{c\to -1^+}\left(-1 + \frac{1}{c+1}\right) = \infty$$

24. $$\begin{aligned} \int_{-2}^2 \frac{x}{\sqrt{4-x^2}}\,dx &= \lim_{c\to -2^+}\int_c^0 \frac{x}{\sqrt{4-x^2}}\,dx + \lim_{d\to 2^-}\int_0^d \frac{x}{\sqrt{4-x^2}}\,dx \\ &= \lim_{c\to -2^+}\left\{-\sqrt{4-x^2}\right\}_c^0 + \lim_{d\to 2^-}\left\{-\sqrt{4-x^2}\right\}_0^d \\ &= \lim_{c\to -2^+}(\sqrt{4-c^2} - 2) + \lim_{d\to 2^-}(2 - \sqrt{4-d^2}) = -2 + 2 = 0 \end{aligned}$$

25. $$\begin{aligned} \int_{-\infty}^\infty \frac{1}{(x+3)^3}dx &= \lim_{a\to -\infty}\int_a^{-4} \frac{1}{(x+3)^3}dx + \lim_{c\to -3^-}\int_{-4}^c \frac{1}{(x+3)^3}dx + \lim_{d\to -3^+}\int_d^0 \frac{1}{(x+3)^3}dx \\ &\quad + \lim_{b\to\infty}\int_0^b \frac{1}{(x+3)^3}dx \end{aligned}$$

Since $\lim_{d\to -3^+}\int_d^0 \frac{1}{(x+3)^3}dx = \lim_{d\to -3^+}\left\{\frac{-1}{2(x+3)^2}\right\}_d^0 = \lim_{d\to -3^+}\left[\frac{1}{2(d+3)^2}-\frac{1}{18}\right] = \infty$, the original improper integral diverges.

26. $\int_{-\infty}^{-3}\frac{1}{\sqrt{-x}}\,dx = \lim_{a\to -\infty}\int_a^{-3}\frac{1}{\sqrt{-x}}\,dx = \lim_{a\to -\infty}\left\{-2\sqrt{-x}\right\}_a^{-3} = \lim_{a\to -\infty}(2\sqrt{-a}-2\sqrt{3}) = \infty$

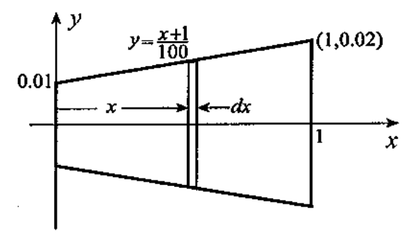
27. $\int_{-6}^{\infty}x\sqrt{x^2+4}\,dx = \lim_{b\to\infty}\int_{-6}^{b}x\sqrt{x^2+4}\,dx = \lim_{b\to\infty}\left\{\frac{1}{3}(x^2+4)^{3/2}\right\}_{-6}^{b} = \frac{1}{3}\lim_{b\to\infty}[(b^2+4)^{3/2}-40^{3/2}] = \infty$

28. $$\int_{-\infty}^{\infty}\frac{x}{(x^2-1)^2}\,dx = \lim_{a\to -\infty}\int_a^{-2}\frac{x}{(x^2-1)^2}\,dx + \lim_{b\to -1^-}\int_{-2}^{b}\frac{x}{(x^2-1)^2}\,dx + \lim_{c\to -1^+}\int_c^0\frac{x}{(x^2-1)^2}\,dx$$
$$+\lim_{d\to 1^-}\int_0^d\frac{x}{(x^2-1)^2}\,dx + \lim_{e\to 1^+}\int_e^2\frac{x}{(x^2-1)^2}\,dx + \lim_{f\to\infty}\int_2^f\frac{x}{(x^2-1)^2}\,dx.$$

Since $\lim_{e\to 1^+}\int_e^2\frac{x}{(x^2-1)^2}\,dx = \lim_{e\to 1^+}\left\{\frac{-1}{2(x^2-1)}\right\}_e^2 = \lim_{e\to 1^+}\left[\frac{1}{2(e^2-1)}-\frac{1}{6}\right] = \infty$, we conclude that the given improper integral diverges.

29. (a) Cross sections are circles with radius y where $y=(x+1)/100$. With the volume of a slab of width dx as $\pi y^2\,dx$, we obtain

$$V = \int_0^1 \pi\left[\frac{1}{100}(x+1)\right]^2 dx = \frac{\pi}{10\,000}\left\{\frac{(x+1)^3}{3}\right\}_0^1 = \frac{7\pi}{30\,000}\ \text{m}^3.$$



(b) Using the disc method, the volume integral is identical.

30. The force of gravity on the slab shown is

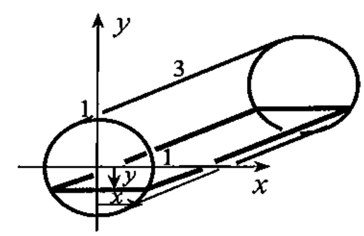
$$-9.81(1000)(2x)(3)\,dy\ \text{N}.$$

Since the work to move this slab from its present position to the top of the tank is

$$(1-y)(9810)(6x)\,dy\ \text{J},$$

the total work to empty the tank is

$$W = \int_{-1}^{0}6(9810)(1-y)\sqrt{1-y^2}\,dy$$
$$= 58\,860\int_{-1}^{0}\sqrt{1-y^2}\,dy - 58\,860\int_{-1}^{0}y\sqrt{1-y^2}\,dy.$$



Since the first integral represents one-quarter of the area of the circle on the end of the tank,

$$W = 58\,860\left(\frac{1}{4}\right)\pi(1)^2 - 58\,860\left\{-\frac{1}{3}(1-y^2)^{3/2}\right\}_{-1}^{0} = 14\,715\pi + 19\,620 = 6.58\times 10^4\ \text{J}.$$