UNIVERSITY OF WATERLOO

DEPARTMENT OF CIVIL AND ENVIRONMENTAL ENGINEERING

CivE 222: Differential Equations — S2020

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Lecture Hours:	**11:30 am – 12:50 pm EDT**	**Mondays and Wednesdays**
Wei-Chau's Office Hours:	**1:00 pm – 1:50 pm EDT**	**Mondays and Wednesdays**
Tutorial Hour:	**11:30 am – 12:20 pm EDT**	**Thursdays**

Special Notes

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Course Outline

Textbook: **Wei-Chau Xie,** 2010, **Differential Equations for Engineers,** Cambridge University Press, ISBN 978-0-521-19424-2, TA347.D45X54 2010.

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Marking Schemes

	Scheme 1	Scheme 2
Assignments	**30%**	**30%**
Midterm Exam	30%	20%
Final Exam	40%	50%
	100%	100%

Assignments

Assignment No. 1 Due: 6:00 p.m., Wednesday, May 27, 2020

Separation of variables; homogeneous equations, transformation of variables

2.2, 2.3, 2.7; 2.10, 2.12, 2.16

Assignment No. 2 Due: 6:00 p.m., Wednesday, June 3, 2020

Exact D.E.'s; integrating factors; linear first-order equations

2.26; 2.32, 2.34, 2.37; 2.63, 2.68;

Assignment No. 3 Due: 6:00 p.m., Wednesday, June 10, 2020

Bernoulli's D.E.'s; simple higher-order D.E.'s

2.73, 2.79; 2.99, 2.100, 2.102, 2.107

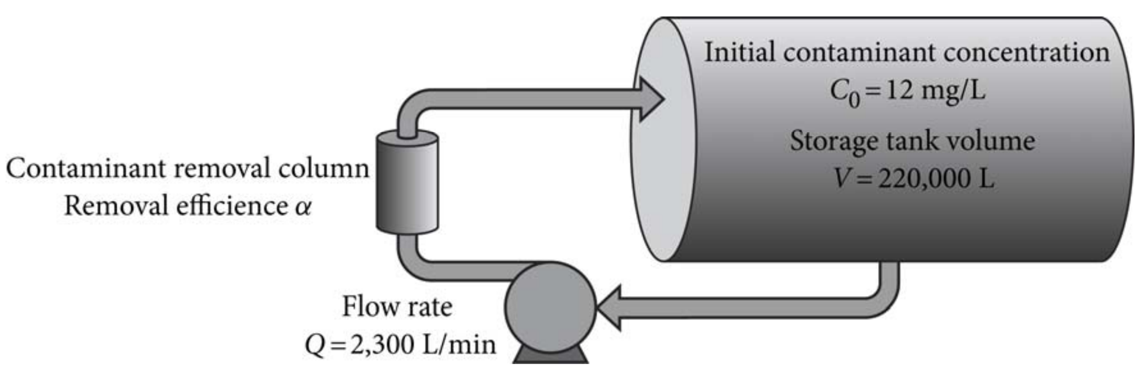
Assignment No. 4 Due: 6:00 p.m., Wednesday, June 17, 2020

Applications of first-order and simple higher-order D.E.'s

3.2, 3.5, 3.8, 3.13, 3.25

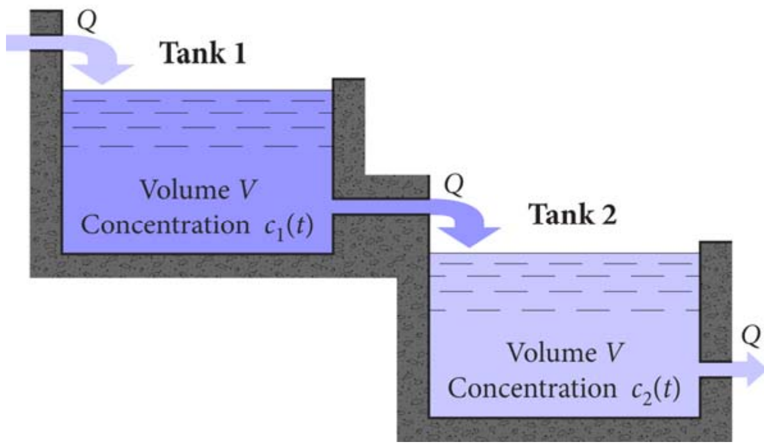
6. The storage tank of a water treatment facility is contaminated with a certain contaminant. Referring to the schematic diagram, the flow rate through the contaminant removal column is $Q = 2{,}300$ L/min, and the removal efficiency is $\alpha = 95\%$. Assume that the outflow from the contaminant removal column is well mixed with the water in the tank instantaneously. It is given that the volume of water in the tank is $V = 220{,}000$ L, with the initial concentration of contaminant being $C_0 = 12$ mg/L. Determine how long it will take to get the contaminant concentration below 0.01 mg/L.

ANS $t = 11$ hr 54 min



7. Tank 1 contains salty water of volume V; the total mass of salt is S at time $t = 0$. Pure water is poured in the tank at a rate of Q (volume per unit time) as shown. The water is well mixed instantly and discharged to Tank 2 at the same rate of Q. Tank 2 contains pure water of volume V at time $t = 0$. The salty water from Tank 1 is well mixed instantly in Tank 2 and discharged at the same rate of Q. Determine $c_2(t)$.

ANS $c_2(t) = \dfrac{QS}{V^2}\, t\, e^{-Qt/V}$

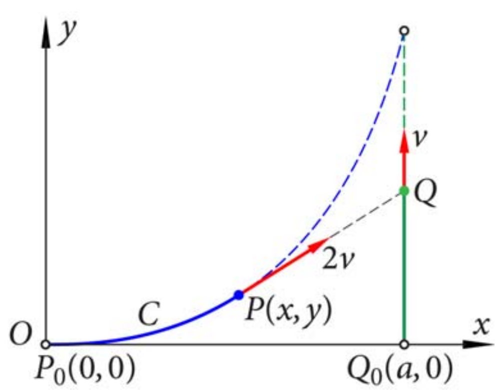


8. A mouse Q runs in the positive y-direction at the constant speed v starting from point $Q_0(a, 0)$ at time $t = 0$ as shown in Figure 1. A cat P chases the mouse at the constant speed $2v$ starting from the origin $P_0(0, 0)$. At any time instance, the cat is always aiming at the mouse, i.e., the velocity of the cat is in the direction PQ and tangent to the curve C along which the cat moves. Show that the differential equation governing the motion of cat $P(x, y)$ is

$$(a-x)\,\frac{d^2y}{dx^2} = \frac{1}{2}\sqrt{1+y'^2}$$

Determine the time T that it takes for the cat to catch the mouse.

ANS Time $T = \dfrac{2a}{3v}$



Assignment No. 5   Due: 6:00 p.m., Wednesday, July 8, 2020

Complementary solutions; method of operators

4.7, 4.8, 4.10; 4.27, 4.30, 4.35

Assignment No. 6   Due: 6:00 p.m., Wednesday, July 15, 2020

Method of operators; method of variation of parameters

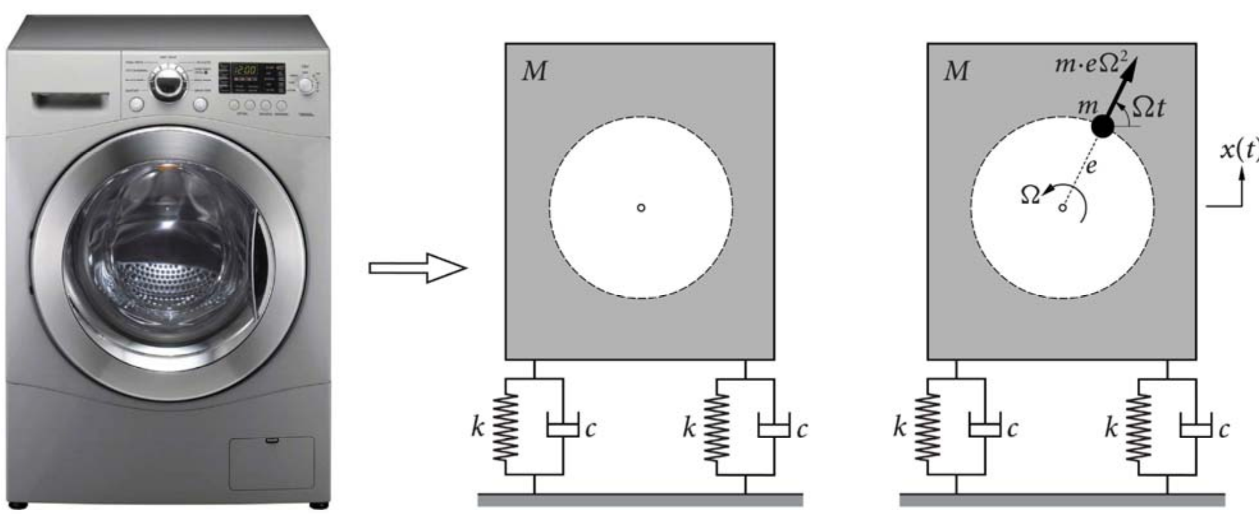
4.39, 4.44, 4.51; 4.57, 4.59, 4.62

Assignment No. 7   Due: 6:00 p.m., Wednesday, July 22, 2020

Applications of linear D.E.'s

5.3, 5.7, 5.10, 5.15, 5.16, 5.18, 5.20

8. Consider a dryer mounted on rubber supports; the drum rotates in the vertical plane with constant angular velocity Ω relative to the body of the machine, as shown in Figure (a). The combined mass of the body of the machine, the drum, and the clothes is M. The rubber supports can be modelled by two spring-dashpot systems, one for the left supports and the other for the right supports. When the clothes are spread uniformly around the drum, the mass of the drum and clothes is symmetric with respect to the axis of rotation. The rotation of the drum does not induce unbalanced loading on the system, as shown in Figure (b).



(a) Dryer   (b) **Uniform** spread of clothes   (c) **Nonuniform** spread of clothes

However, when the clothes are spread nonuniformly around the drum, the loading on the dryer due to the nonuniformity can be modelled by an excess mass m with eccentricity e that is rotating with angular velocity Ω, resulting in a centrifugal force $m\,e\,\Omega^2$ as shown in Figure (c). Note that the value of m depends on the degree of nonuniformity of the spread of the clothes (varying from 0 for uniform spread to m_{clothes} if all clothes are tangled into a tight block). The motion of the system is defined by the vertical displacement $x(t)$,measured from the position of static equilibrium. Suppose that the values of constants M, c, k, $m \neq 0$, e, and Ω are known. Determine the amplitude of the *steady-state* vibration of $x(t)$.

ANS Amplitude $= \dfrac{me\Omega^2}{\sqrt{(2k-M\Omega^2)^2+(2c\Omega)^2}}$

Assignment No. 8 Due: 6:00 p.m., Wednesday, July 29, 2020

Systems of linear D.E.'s; applications of systems of linear D.E.'s

7.7, 7.13, 7.15; 8.2, 8.4, 8.6

Assignment No. 9 Due: 6:00 p.m., Wednesday, August 5, 2020

Partial differential equations and applications

11.1, 11.5, 11.7

Table of Trigonometric Identities

Trigonometric Functions	Hyperbolic Functions
1. $\sin^2\theta + \cos^2\theta = 1$	$\cosh^2 x - \sinh^2 x = 1$
$\implies \quad 1 + \tan^2\theta = \sec^2\theta$	$\implies \quad 1 - \tanh^2 x = \operatorname{sech}^2 x$
$\implies \quad 1 + \cot^2\theta = \csc^2\theta$	$\implies \quad \coth^2 x - 1 = \operatorname{csch}^2 x$
2. $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$	$\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$
3. $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$	$\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$
4. $\sin 2\theta = 2\sin\theta\cos\theta$	$\sinh 2x = 2\sinh x\cosh x$
5. $\cos 2\theta = \cos^2\theta - \sin^2\theta$	$\cosh 2x = \cosh^2 x + \sinh^2 x$
$= 2\cos^2\theta - 1$	$= 2\cosh^2 x - 1$
$= 1 - 2\sin^2\theta$	$= 1 + 2\sinh^2 x$

6. $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$
7. $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$
8. $\sin A\sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)]$
9. $\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$
10. $\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$
11. $\cos A\sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$
12. $\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$
13. $\sin A - \sin B = 2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$
14. $\cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$
15. $\cos A - \cos B = -2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$
16. Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$
17. $z^n = a(\cos\theta + i\sin\theta)$

$$\implies z = \sqrt[n]{a}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right), \quad k = 0, 1, \ldots, n-1$$

Table of Derivatives

1. $\dfrac{d}{dx}x^n = nx^{n-1}$

2. $\dfrac{d}{dx}e^x = e^x \implies \dfrac{d}{dx}a^x = a^x \ln a, \qquad \because\ a^x = e^{x \ln a}$

3. $\dfrac{d}{dx}\ln x = \dfrac{1}{x} \implies \dfrac{d}{dx}\log_a x = \dfrac{1}{x \ln a}, \quad \because\ \log_a x = \dfrac{\ln x}{\ln a}$

Trigonometric Functions

4. $\dfrac{d}{dx}\sin x = \cos x$

5. $\dfrac{d}{dx}\cos x = -\sin x$

6. $\dfrac{d}{dx}\tan x = \dfrac{1}{\cos^2 x} = \sec^2 x$

7. $\dfrac{d}{dx}\cot x = -\dfrac{1}{\sin^2 x} = -\csc^2 x$

8. $\dfrac{d}{dx}\sec x = \dfrac{\sin x}{\cos^2 x} = \tan x \sec x$

9. $\dfrac{d}{dx}\csc x = -\dfrac{\cos x}{\sin^2 x} = -\cot x \csc x$

Hyperbolic Functions

$\dfrac{d}{dx}\sinh x = \cosh x$

$\dfrac{d}{dx}\cosh x = \sinh x$

$\dfrac{d}{dx}\tanh x = \dfrac{1}{\cosh^2 x} = \operatorname{sech}^2 x$

$\dfrac{d}{dx}\coth x = -\dfrac{1}{\sinh^2 x} = -\operatorname{csch}^2 x$

$\dfrac{d}{dx}\operatorname{sech} x = -\dfrac{\sinh x}{\cosh^2 x} = -\tanh x \operatorname{sech} x$

$\dfrac{d}{dx}\operatorname{csch} x = -\dfrac{\cosh x}{\sinh^2 x} = -\coth x \operatorname{csch} x$

Inverse Trigonometric Functions

10. $\dfrac{d}{dx}\sin^{-1} x = \dfrac{1}{\sqrt{1-x^2}}$ $\qquad \dfrac{d}{dx}\cos^{-1} x = -\dfrac{1}{\sqrt{1-x^2}}$

11. $\dfrac{d}{dx}\tan^{-1} x = \dfrac{1}{1+x^2}$ $\qquad \dfrac{d}{dx}\cot^{-1} x = -\dfrac{1}{1+x^2}$

12. $\dfrac{d}{dx}\sec^{-1} x = \dfrac{1}{x\sqrt{x^2-1}}$ $\qquad \dfrac{d}{dx}\csc^{-1} x = -\dfrac{1}{x\sqrt{x^2-1}}$

Table of Integrals

1. $\displaystyle\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1}, \quad n \neq -1$

2. $\displaystyle\int \frac{1}{x}\,\mathrm{d}x = \ln|x|, \quad x \neq 0$ $\qquad$ $\displaystyle\int \ln x\,\mathrm{d}x = x\left(\ln x - 1\right), \quad x > 0$

3. $\displaystyle\int \mathrm{e}^{ax}\,\mathrm{d}x = \frac{1}{a}\mathrm{e}^{ax}$ $\qquad$ $\displaystyle\int b^{ax}\,\mathrm{d}x = \frac{b^{ax}}{a\ln b}, \quad b > 0$

	Trigonometric Functions	**Hyperbolic Functions**
4.	$\displaystyle\int \sin x\,\mathrm{d}x = -\cos x$	$\displaystyle\int \sinh x\,\mathrm{d}x = \cosh x$
5.	$\displaystyle\int \cos x\,\mathrm{d}x = \sin x$	$\displaystyle\int \cosh x\,\mathrm{d}x = \sinh x$
6.	$\displaystyle\int \tan x\,\mathrm{d}x = -\ln\left|\cos x\right|$	$\displaystyle\int \tanh x\,\mathrm{d}x = \ln\cosh x$
7.	$\displaystyle\int \cot x\,\mathrm{d}x = \ln\left|\sin x\right|$	$\displaystyle\int \coth x\,\mathrm{d}x = \ln\left|\sinh x\right|$
8.	$\displaystyle\int \sec x\,\mathrm{d}x = \ln\left|\sec x + \tan x\right| = \ln\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| = \ln\left|\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)\right|$	$\displaystyle\int \operatorname{sech} x\,\mathrm{d}x = \tan^{-1}\left(\sinh x\right)$
9.	$\displaystyle\int \csc x\,\mathrm{d}x = \ln\left|\csc x - \cot x\right| = \ln\left|\tan\frac{x}{2}\right|$	$\displaystyle\int \operatorname{csch} x\,\mathrm{d}x = \ln\left|\tanh\frac{x}{2}\right|$
10.	$\displaystyle\int \sin^2 x\,\mathrm{d}x = \frac{x}{2} - \frac{1}{4}\sin 2x$	$\displaystyle\int \sinh^2 x\,\mathrm{d}x = \frac{1}{4}\sinh 2x - \frac{x}{2}$
11.	$\displaystyle\int \cos^2 x\,\mathrm{d}x = \frac{x}{2} + \frac{1}{4}\sin 2x$	$\displaystyle\int \cosh^2 x\,\mathrm{d}x = \frac{1}{4}\sinh 2x + \frac{x}{2}$
12.	$\displaystyle\int \tan^2 x\,\mathrm{d}x = \tan x - x$	$\displaystyle\int \tanh^2 x\,\mathrm{d}x = -\tanh x + \frac{1}{2}\ln\left|\frac{\tanh x + 1}{\tanh x - 1}\right|$
13.	$\displaystyle\int \cot^2 x\,\mathrm{d}x = -\cot x - x$	$\displaystyle\int \coth^2 x\,\mathrm{d}x = -\coth x + \frac{1}{2}\ln\left|\frac{\coth x + 1}{\coth x - 1}\right|$
14.	$\displaystyle\int \sec^2 x\,\mathrm{d}x = \tan x$	$\displaystyle\int \operatorname{sech}^2 x\,\mathrm{d}x = \tanh x$
15.	$\displaystyle\int \csc^2 x\,\mathrm{d}x = -\cot x$	$\displaystyle\int \operatorname{csch}^2 x\,\mathrm{d}x = -\coth x$

16. $$\int x \sin ax \, \mathrm{d}x = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax$$

17. $$\int x^2 \sin ax \, \mathrm{d}x = \frac{2x}{a^2} \sin ax - \frac{a^2x^2-2}{a^3} \cos ax$$

18. $$\int x^3 \sin ax \, \mathrm{d}x = \frac{3a^2x^2-6}{a^4} \sin ax - \frac{a^2x^3-6x}{a^3} \cos ax$$

19. $$\int x^4 \sin ax \, \mathrm{d}x = \frac{4a^2x^3-24x}{a^4} \sin ax - \frac{a^4x^4-12a^2x^2+24}{a^5} \cos ax$$

20. $$\int x \cos ax \, \mathrm{d}x = \frac{x}{a} \sin ax + \frac{1}{a^2} \cos ax$$

21. $$\int x^2 \cos ax \, \mathrm{d}x = \frac{a^2x^2-2}{a^3} \sin ax + \frac{2x}{a^2} \cos ax$$

22. $$\int x^3 \cos ax \, \mathrm{d}x = \frac{a^2x^3-6x}{a^3} \sin ax + \frac{3a^2x^2-6}{a^4} \cos ax$$

23. $$\int x^4 \cos ax \, \mathrm{d}x = \frac{a^4x^4-12a^2x^2+24}{a^5} \sin ax + \frac{4a^2x^3-24x}{a^4} \cos ax$$

24. $$\int \mathrm{e}^{bx} \sin ax \, \mathrm{d}x = \mathrm{e}^{bx} \, \frac{b \sin ax - a \cos ax}{a^2 + b^2}$$

25. $$\int \mathrm{e}^{bx} \cos ax \, \mathrm{d}x = \mathrm{e}^{bx} \, \frac{a \sin ax + b \cos ax}{a^2 + b^2}$$

26. $$\int x\mathrm{e}^{ax} \, \mathrm{d}x = \frac{ax-1}{a^2} \, \mathrm{e}^{ax}$$

27. $$\int x^2 \mathrm{e}^{ax} \, \mathrm{d}x = \frac{a^2x^2-2ax+2}{a^3} \, \mathrm{e}^{ax}$$

28. $$\int x^3 \mathrm{e}^{ax} \, \mathrm{d}x = \frac{a^3x^3-3a^2x^2+6ax-6}{a^4} \, \mathrm{e}^{ax}$$

29. $$\int x^4 \mathrm{e}^{ax} \, \mathrm{d}x = \frac{a^4x^4-4a^3x^3+12a^2x^2-24ax+24}{a^5} \, \mathrm{e}^{ax}$$

30. $$\int x^5 \mathrm{e}^{ax} \, \mathrm{d}x = \frac{a^5x^5-5a^4x^4+20a^3x^3-60a^2x^2+120ax-120}{a^6} \, \mathrm{e}^{ax}$$

31. $$\int \frac{1}{a^2 + x^2} \, \mathrm{d}x = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a > 0$$

32. $$\int \frac{1}{a^2 - x^2} \, \mathrm{d}x = \frac{1}{2a} \ln \frac{a+x}{a-x} \qquad \text{or} \qquad \frac{1}{a} \tanh^{-1} \frac{x}{a}, \quad |x| < |a|$$

33. $$\int \frac{1}{x^2-a^2}\,\mathrm{d}x = \frac{1}{2a}\ln\frac{x-a}{x+a} \qquad \text{or} \qquad -\frac{1}{a}\coth^{-1}\frac{x}{a}, \quad |x|>|a|$$

34. $$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \sin^{-1}\frac{x}{a} \qquad \text{or} \qquad -\cos^{-1}\frac{x}{a}$$

35. $$\int \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x = \cosh^{-1}\frac{x}{a} \qquad \text{or} \qquad \ln\left|x+\sqrt{x^2-a^2}\right|$$

36. $$\int \frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x = \sinh^{-1}\frac{x}{a} \qquad \text{or} \qquad \ln\left(x+\sqrt{x^2+a^2}\right)$$

37. $$\int \sqrt{a^2-x^2}\,\mathrm{d}x = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

38. $$\int \sqrt{x^2+a^2}\,\mathrm{d}x = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a}$$

$$\text{or} \qquad \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right)$$

39. $$\int \sqrt{x^2-a^2}\,\mathrm{d}x = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$$

$$\text{or} \qquad \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|$$

40. $$\int \frac{1}{x\sqrt{a^2-x^2}}\,\mathrm{d}x = -\frac{1}{a}\cosh^{-1}\frac{a}{x} \qquad \text{or} \qquad -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$$

41. $$\int \frac{1}{x\sqrt{x^2+a^2}}\,\mathrm{d}x = -\frac{1}{a}\sinh^{-1}\frac{a}{x} \qquad \text{or} \qquad -\frac{1}{a}\ln\left|\frac{a+\sqrt{x^2+a^2}}{x}\right|$$

42. $$\int \frac{1}{x\sqrt{x^2-a^2}}\,\mathrm{d}x = \frac{1}{a}\cos^{-1}\frac{a}{x} \qquad \text{or} \qquad \frac{1}{a}\sec^{-1}\frac{x}{a}$$

43. $$\int \frac{1}{\sqrt{2ax-x^2}}\,\mathrm{d}x = \cos^{-1}\left(1-\frac{x}{a}\right) \qquad \text{or} \qquad \sin^{-1}\left(\frac{x}{a}-1\right)$$

44. $$\int \sqrt{2ax-x^2}\,\mathrm{d}x = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}-1\right)$$

45. $$\int_0^{\frac{\pi}{2}} \begin{Bmatrix} \sin^n\theta \\ \cos^n\theta \end{Bmatrix} \mathrm{d}\theta = \frac{(n-1)!!}{n!!} \times \begin{cases} \frac{1}{2}\pi, & \text{if } n \text{ is an even integer} \\ 1, & \text{if } n \text{ is an odd integer} \end{cases}$$

46. $$\int_0^{\frac{\pi}{2}} \sin^n\theta\cos^m\theta\,\mathrm{d}\theta = \frac{(n-1)!!\,(m-1)!!}{(n+m)!!} \times \begin{cases} \frac{1}{2}\pi, & n, m \text{ even integers} \\ 1, & \text{otherwise} \end{cases}$$

✍ $$n!! = \begin{cases} n\cdot(n-2)\cdots 5\cdot 3\cdot 1, & n>0 \text{ odd integer} \\ n\cdot(n-2)\cdots 6\cdot 4\cdot 2, & n>0 \text{ even integer} \\ 1, & n=0 \end{cases}$$