Chapter 3

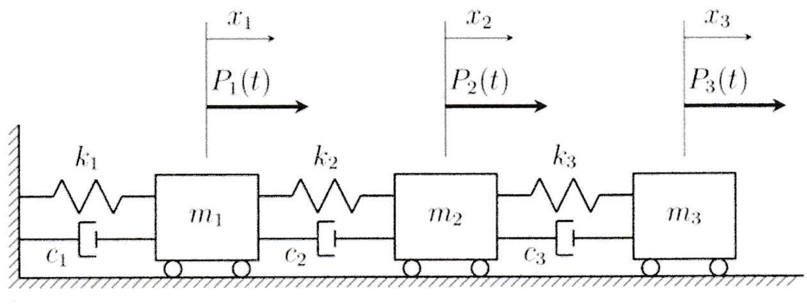
Multi-Degree-of-Freedom Systems

For an accurate description of the displacement configuration of a structure subjected to a dynamic loading, often displacements along more than one coordinate are necessary. Such a system is known as a *multi-degree-of-freedom system*.

We begin by discussing the formulation of the equations of motion for multi-degree-of-freedom (MDOF) systems.

3.1 Formulation of the Equation of Motion for an MDOF System

We begin with an example of a MDOF system with three degrees-of-freedom, as shown in Figure 3.1. Note that the displacements are time-varying, but are simply designated as x rather than $x(t)$ for succinctness.



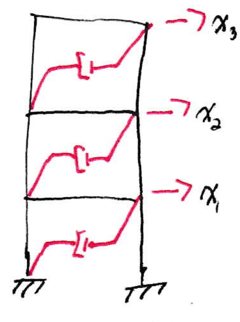
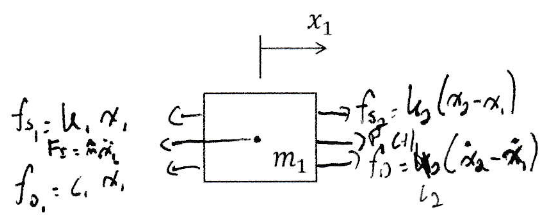


Figure 3.1 Three degree-of-freedom spring-mass-damper model

Example 3.1 Derive the equations of motion for the 3DOF system shown in Figure 3.1.

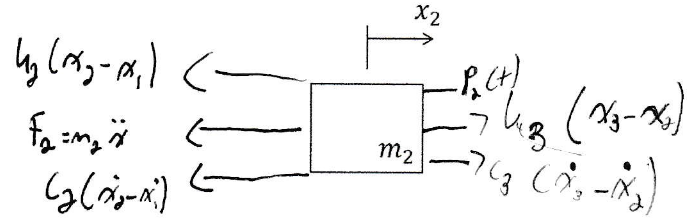
Solution: First, consider the free body diagram of m_1.



The equation of motion is

$$m_1\ddot{x}_1 + k_1 x_1 + c_1\dot{x}_1 + k_2(x_2-x_1) - c_2(\dot{x}_2-\dot{x}_1) = P_1(t) \tag{3.1}$$

Next, consider the free body diagram of m_2.

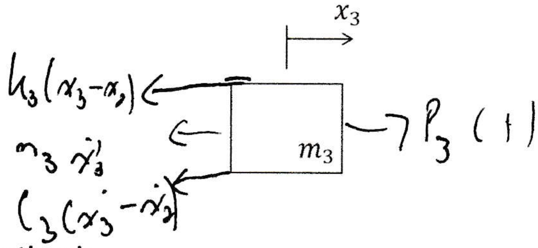


The equation of motion is

$$x_2\ddot{x}_2 + k_2(x_2-x_1) + c_2(\dot{x}_2-\dot{x}_1) - k_3(x_3-x_2) - c_3(\dot{x}_3-\dot{x}_2) = P_2 \tag{3.2}$$

$$m_2\ddot{x}_2 + [c_2+c_3]\dot{x}_2 + [k_2+k_3]x_2 - c_2\dot{x}_1 - k_2x_1 - k_3x_3 - c_3\dot{x}_3 = P_2$$

Finally, consider the free body diagram of m_3.



The equation of motion is

$$m_3\ddot{x}_3 + c_3\dot{x}_3 - c_3\dot{x}_2 + k_3x_3 - k_3x_2 = P_3 \tag{3.3}$$

Equations 3.1 through 3.3 can be written in matrix form as

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{3.4}$$

$$= \begin{bmatrix} P_1(t) \\ P_2(t) \\ P_3(t) \end{bmatrix} \tag{3.5}$$

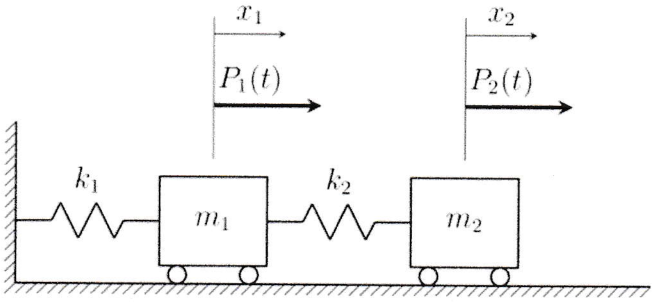
where $\boldsymbol{M}$ is the *mass matrix*, $\boldsymbol{C}$ is the *damping matrix*, and $\boldsymbol{K}$ is the *stiffness matrix*.

$$M\ddot{x} + C\dot{x} + kx = P(t)$$

We can also formulate MDOF equations of motion using an analytical mechanics approach. *Lagrange's Equation* is a powerful method but will not be covered in this course.

3.2 Solution for a 2-DOF System

We will now consider the solution for a 2-DOF system. We begin by finding the equation of motion for a 2-DOF system with no damping.



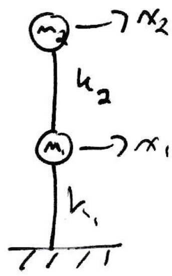
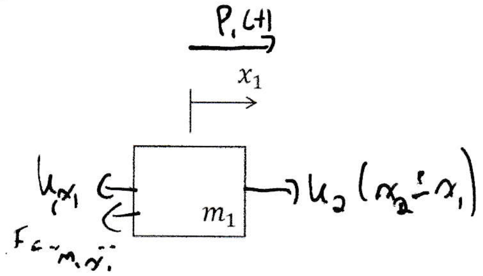


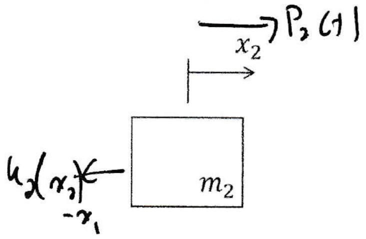
Figure 3.2 Two degree-of-freedom spring-mass model

Using Newtonian Mechanics, we can derive the equation of motion. For mass m_1, the free body diagram is:

using Blueplan



Similarly, for the mass m_2, the free body diagram is



The equations of motion are

$$P_1(t) = k_1x_1 - k_2(x_2 - x_1) + m_1\ddot{x}_1 \quad , \; m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = P_1 \tag{3.6a}$$

$$P_2(t) = k_2x_2 + m_2\ddot{x}_2 \tag{3.6b}$$

which can be written in matrix form as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix} \tag{3.7}$$

$$M\ddot{x} + Kx = P$$

Several observations can be made from this result.

- Equation 3.7 is uncoupled in the mass terms, but coupled in the stiffness terms.
- Coupling is not an inherent property of the system; rather, it is a consequence of the coordinate system used.
- Both the mass and stiffness matrices are symmetric.

We will now turn to solving this system. Assuming the solution is of the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sin(\omega_n t + \alpha) \quad , \tag{3.8}$$

where X_1 and X_2 are the amplitudes of x_1 and x_2, respectively. As before, ω_n is the circular natural frequency, and α is the phase angle. Substituting Equation 3.8 into Equations 3.6 gives,

$$-m_1\omega_n^2 X_1 \sin(\omega_n t + \alpha) + (k_1 + k_2)X_1 \sin(\omega_n t + \alpha) - k_2 X_2 \sin(\omega_n t + \alpha) = P_1(t) \tag{3.9a}$$

$$-m_2\omega_n^2 X_2 \sin(\omega_n t + \alpha) - k_2 X_1 \sin(\omega_n t + \alpha) + k_2 X_2 \sin(\omega_n t + \alpha) = P_2(t) \tag{3.9b}$$

3.2.1 Free Vibration Response

For the free vibration response ($P_1(t) = P_2(t) = 0$), the solution becomes

$$[-m_1\omega_n^2 X_1 + (k_1 + k_2)X_1 - k_2 X_2] \sin(\omega_n t + \alpha) = 0 \tag{3.10a}$$

Homogeneous system

$$[-m_2\omega_n^2 X_2 - k_2 X_1 + k_2 X_2] \sin(\omega_n t + \alpha) = 0 \tag{3.10b}$$

For a nontrivial solution, X_1 and $X_2 \neq 0$ @ same time

∴ Non invertible, determinate = 0 . $\begin{bmatrix} (k_1+k_2)-m_1\omega_n^2 & -k_2 \\ -k_2 & k_2-m_2\omega_n^2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (3.11)

Therefore,

AD - BC = 0

$\begin{vmatrix} A & B \\ C & D \end{vmatrix}$ (3.12)

Expanding the determinant gives

$$\omega_n^4 - \left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right)\omega_n^2 + \frac{k_1 k_2}{m_1 m_2} = 0 \tag{3.13}$$

Equation 3.13 is known as the characteristic equation of the system. The roots are

$$\omega_n^2 = \frac{1}{2}\left[\left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right) \pm \sqrt{\left(\left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right)^2 - 4\frac{k_1 k_2}{m_1 m_2}\right)}\right] \tag{3.14}$$

The roots $\omega_{n,1}$ and $\omega_{n,2}$ are real and positive. The lower of $\omega_{n,1}$ and $\omega_{n,2}$ is known as the first or ***fundamental*** *circular natural frequency* of the system.

∴ has 2 undamped natural frequencies; can never exactly determine amplitudes