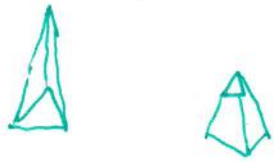
Chapter 4



Earthquake Response of Structures

One of the most interesting and important applications of the study of structural dynamics is analyzing the response of structures subjected to ground motion caused by earthquakes.

Analytical solution of the equation of motion is not always possible when the applied force (or in the case of earthquake excitation, the ground motion) varies arbitrarily with time. Therefore, we must introduce numerical methods of evaluating the dynamic response.

4.1 Numerical Integration - can't represent w/ sine function

$m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g$

We have seen in the previous sections that the problem of analyzing MDOF systems with proportional damping becomes a problem of analyzing N SDOF systems, where N is the number of degrees of freedom and the size of the mass, stiffness, and damping matrix.

When the forcing function is described by a simple analytical function, then the solution for the problem can be derived first by using modal transformation to uncouple the equations of motion into the modal domain. The response for each mode is then calculated using the Duhamel's Integral. Finally, the response in the physical coordinates is found by transforming the modal response using $\boldsymbol{x} = \boldsymbol{\Phi y}$.

When the forcing function is no longer a simple analytical function, the terms involving Duhamel's integral may be quite cumbersome to calculate analytically. In order to solve this problem, we employ numerical integration to calculate Duhamel's integral.

The problem involves computing Duhamel's integral (Equation 2.224), which for a damped SDOF system subjected to an arbitrary force, $P(t)$, can be expressed as

$$x(t) = \frac{1}{m\omega_D} \int_0^t P(\tau) e^{-\zeta\omega_n(t-\tau)} \sin \omega_D (t-\tau)\, d\tau \tag{4.1}$$

Note: $\sin(u-v) = \sin u \cos v - \cos u \sin v$

Also $e^{-\zeta\omega_n(t-\tau)} = e^{-\zeta\omega_n t} e^{\zeta\omega_n \tau}$

The above equation can be rewritten as

$$x(t) = \frac{1}{m\omega_D} \int_0^t P(\tau) e^{-\zeta\omega_n(t-\tau)} (\sin \omega_D t \cos \omega_D \tau - \cos \omega_D t \sin \omega_D \tau) d\tau \tag{4.2}$$

which can be rearranged as

$$x(t) = Ae^{-\zeta\omega_n t}\sin\omega_D t - Be^{-\zeta\omega_n t}\cos\omega_D t \tag{4.3}$$

where

Potential mistake

corrects mistake

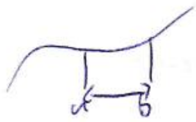
$$A = \frac{1}{m\omega_D}\int_0^t P(\tau)e^{+\zeta\omega_n\tau}\cos\omega_D\tau\, d\tau \tag{4.4a}$$

$$B = \frac{1}{m\omega_D}\int_0^t P(\tau)e^{+\zeta\omega_n\tau}\sin\omega_D\tau\, d\tau \tag{4.4b}$$

There are three simple ways to calculate the integrals described in Equations 4.4 numerically.

Rectangular rule The simplest method for numerical integration is by choosing an interpolation function that is a constant function over the time step. This is known as the *rectangular rule* or midpoint rule. For a function that passes directly through the point

$$(x, y) = \left[\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right]$$



The definite integral can be approximated as

$$\int_a^b f(x)dx \approx (b-a)\, f\left(\frac{a+b}{2}\right) \tag{4.5}$$

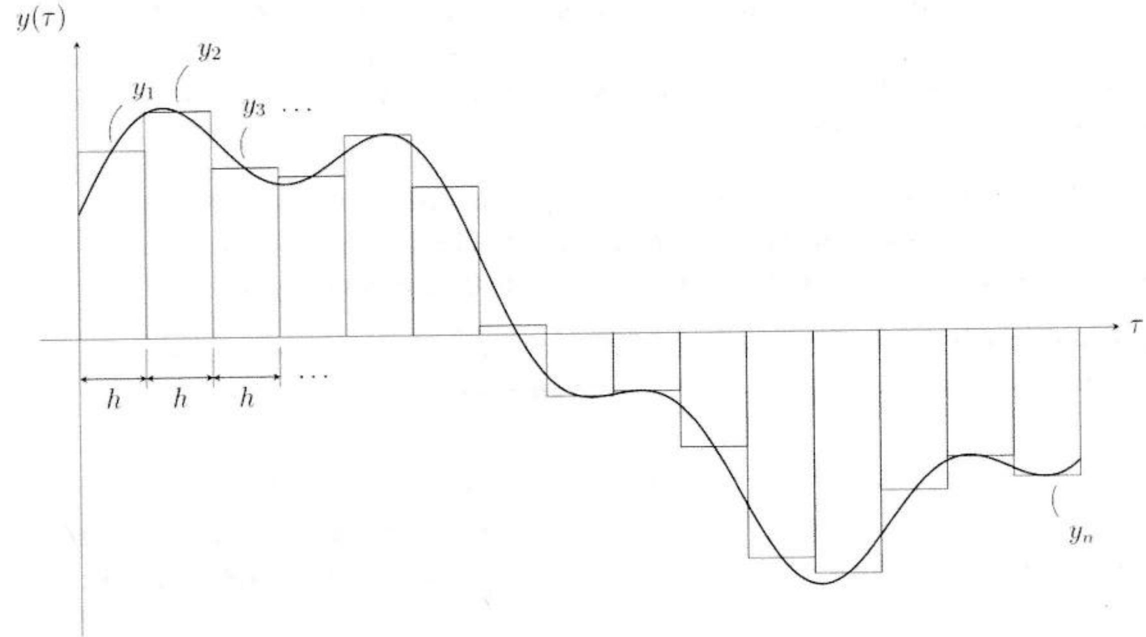


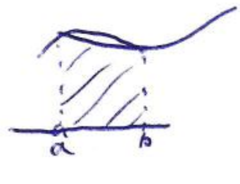
Figure 4.1 Numerical integration by rectangular rule

The integral of the function shown in Figure 4.1 from 0 to $\tau = t$ would be

$$\int_0^t y(\tau)d\tau \approx h\left(y_1 + y_2 + \dots + y_n\right) \tag{4.6}$$

Trapezoidal rule An improvement on the rectangular rule is using an interpolation function which passes through the points

$$(x_1, y_1) = [a, f(a)]$$
$$(x_2, y_2) = [b, f(b)]$$



The definite integral can be approximated as

$$\int_a^b f(x)dx \approx (b-a)\left(\frac{f(a)+f(b)}{2}\right) \tag{4.7}$$

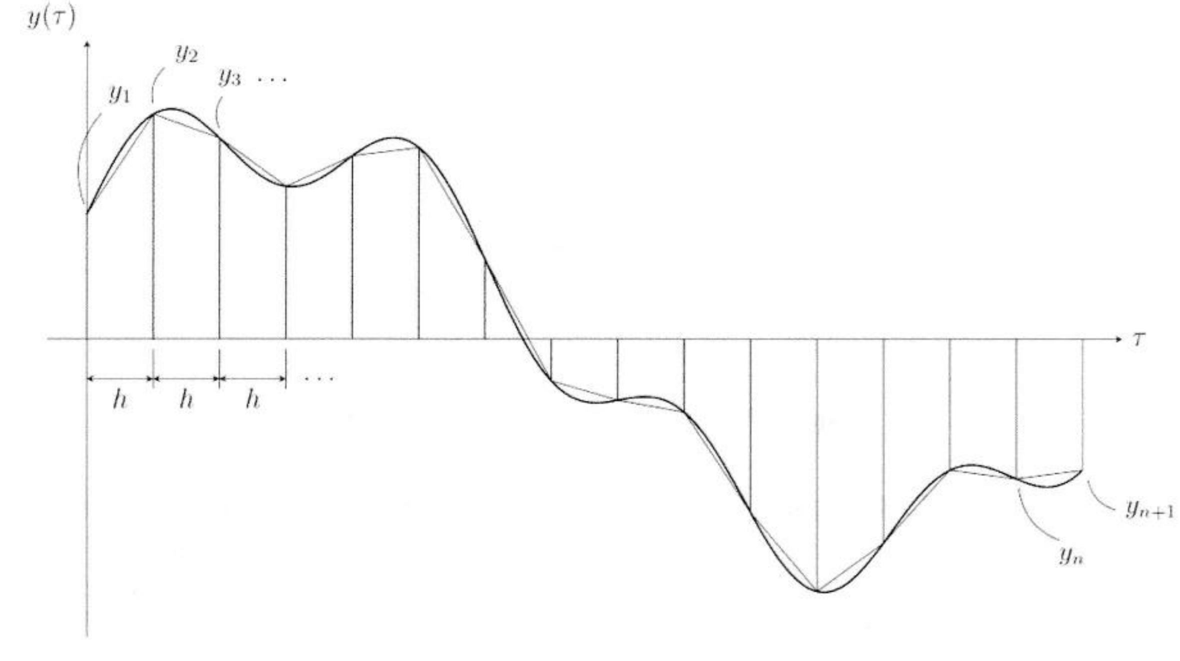


Figure 4.2 Numerical integration by trapezoidal rule

The integral of the function shown in Figure 4.2 from 0 to $\tau = t$ would be

$$\int_0^t y(\tau)d\tau \approx \frac{h}{2}\left(y_1 + 2y_2 + 2y_3 + \dots + 2y_n + y_{n+1}\right) \tag{4.8}$$

Simpson's rule The final method considered is Simpson's rule, which seeks to fit a quadratic polynomial function that takes the same values of $f(x)$ at the endpoints a and b, as well as the midpoint $a + b$. The integral can be approximated as follows:

3 points

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] \tag{4.9}$$

The integral of the function shown in Figure 4.2 from 0 to $\tau = t$ using Simpson's rule would be

$$\int_0^t y(\tau)\,d\tau = \frac{h}{3}\left(y_1+4y_2+2y_3+4y_4+\dots+4y_{n-1}+2y_n+y_{n+1}\right) \tag{4.10}$$

Example 4.1 Numerically evaluate the response to the half sine pulse shown in Figure 4.3 using the following system properties: $m = 2.53$ kip·s²/in, $\omega_n = 6.283$ rad/s, and $\zeta = 0$.

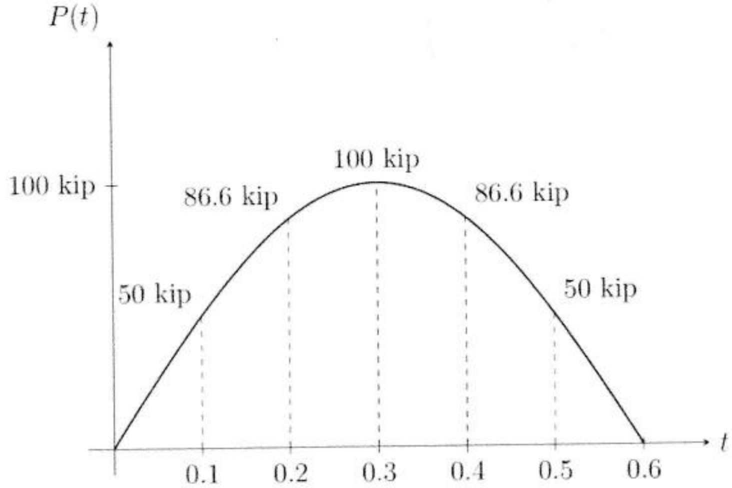


Figure 4.3 Half sine pulse forcing function

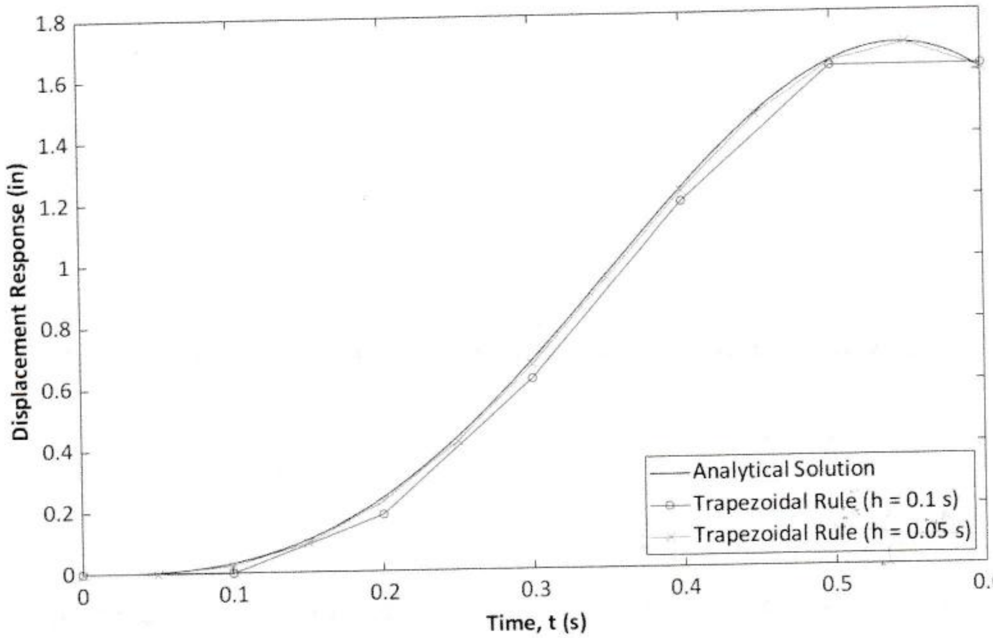
Solution: The MATLAB™ script used to perform numerical integration using the trapezoidal method is as follows:

```
dt=0.1; %s
d_tau=0.1;
m=2.53; % kip s^2/in
omega_n=6.283; % rad/s
zeta=0;
omega_D=omega_n*sqrt(1-zeta^2); % rad/s
A(1)=0;
B(1)=0;
t=0:dt:0.6;
tau=0:d_tau:0.6;
f_t=100*sin((2*pi/1.2)*tau); % kips
f_a=f_t.*exp(zeta*omega_n.*tau).*cos(omega_D.*tau);
```

$x(t) = A e^{-\zeta \omega_n t} \sin \omega_D t - B e^{-\zeta \omega_n t} \cos \omega_D t$

```
f_b=f_t.*exp(zeta*omega_n.*tau).*sin(omega_D.*tau);
for i=2:length(f_t)
    A(i)=A(i-1)+1/2*dt*(f_a(i-1)+f_a(i));
    B(i)=B(i-1)+1/2*dt*(f_b(i-1)+f_b(i));
    y(i)=1/(m*omega_D)*(A(i)*exp(-zeta*omega_n*t(i))*sin(omega_D*t(i))-...
        B(i)*exp(-zeta*omega_n*t(i))*cos(omega_D*t(i)));
end
```

The numerical solution is compared to the analytical solution below.



4.1.1 Newmark-β Method - looking at incremental Δs in equilibrium!! :)

We will now investigate an alternative, more powerful method of solving the system of equations for arbitrary force excitations that are difficult to solve analytically. The method is known as the *Newmark-β method* and is used in some analysis software packages.

We will first consider linear system. We begin by deriving the equations of motion in an incremental form. At the i^{th} time instant, t_i

$$m\ddot{x}_i + c\dot{x}_i + kx_i = P_i \tag{4.11}$$

For the $(i+1)^{\text{th}}$ time instant, t_{i+1},

$$m\ddot{x}_{i+1} + c\dot{x}_{i+1} + kx_{i+1} = P_{i+1} \tag{4.12}$$

Subtracting Equation 4.11 from Equation 4.12 gives,

$$m\Delta\ddot{x}_i + c\Delta\dot{x}_i + k\Delta x_i = \Delta P_i \tag{4.13}$$

where $\Delta\ddot{x}_i = \ddot{x}_{i+1} - \ddot{x}_i$, $\Delta\dot{x}_i = \dot{x}_{i+1} - \dot{x}_i$, $\Delta x_i = x_{i+1} - x_i$, and $\Delta P_i = P_{i+1} - P_i$.

The key to the Newmark-β method is how the change in acceleration is captured.

Average Acceleration The average acceleration model assumes the acceleration between the t_i and t_{i+1} is the average of the acceleration of the two instances.

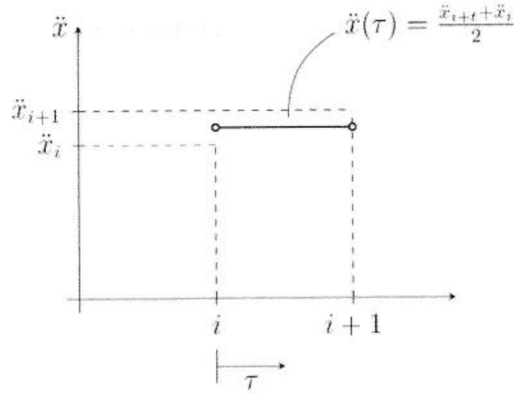


Figure 4.4 Average acceleration assumption in Newmark-β method

The acceleration is given by

$$\ddot{x}(\tau) = \frac{\ddot{x}_{i+1} + \ddot{x}_i}{2} \tag{4.14}$$

Integrating Equation 4.14, and solving for the constant using the boundary condition $\dot{x}(0) = \dot{x}_i$, we get

$$\dot{x}(\tau) = \dot{x}_i + \left(\frac{\ddot{x}_{i+1} + \ddot{x}_i}{2}\right)\tau \tag{4.15}$$

Integrating Equation 4.15, and applying the boundary condition $x(0) = x_i$, we get,

$$x(\tau) = x_i + \dot{x}_i\tau + \left(\frac{\ddot{x}_{i+1} + \ddot{x}_i}{2}\right)\frac{\tau^2}{2} \tag{4.16}$$

Hence, the velocity and displacement at t_{i+1} are (when $\tau = \Delta t$ in Equations 4.16 and 4.15)

$$\dot{x}_{i+1} = \dot{x}_i + \frac{\Delta t}{2}(\ddot{x}_{i+1} + \ddot{x}_i) \tag{4.17a}$$

$$x_{i+1} = x_i + \dot{x}_i\Delta t + \frac{(\Delta t)^2}{4}(\ddot{x}_{i+1} + \ddot{x}_i) \tag{4.17b}$$

Linear Acceleration The linear acceleration model assumes that the acceleration varies linearly between the i^{th} and the $(i+1)^{\text{th}}$ time step.

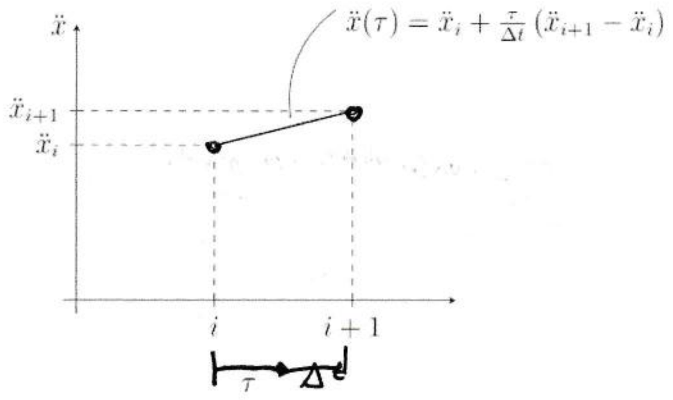


Figure 4.5 Linear acceleration assumption in Newmark-β method

The acceleration is given by

$$\ddot{x}(\tau) = \ddot{x}_i + \frac{\tau}{\Delta t}(\ddot{x}_{i+1} - \ddot{x}_i) \tag{4.18}$$

Integrating Equation 4.18, and solving for the constant using the boundary condition $\dot{x}(0) = \dot{x}_i$, we get

$$\dot{x}(\tau) = \dot{x}_i + \ddot{x}_i\tau + \left(\frac{\ddot{x}_{i+1} - \ddot{x}_i}{\Delta t}\right)\frac{\tau^2}{2} \tag{4.19}$$

Integrating Equation 4.19, and applying the boundary condition $x(0) = x_i$, we get,

$$x(\tau) = x_i + \dot{x}_i\tau + \ddot{x}_i\frac{\tau^2}{2} + \left(\frac{\ddot{x}_{i+1} - \ddot{x}_i}{\Delta t}\right)\frac{\tau^3}{6} \tag{4.20}$$

Hence, the velocity and displacement at t_{i+1} for the linear acceleration assumption are

Velocity
$$\dot{x}_{i+1} = \dot{x}_i + \ddot{x}_i\Delta t + \frac{\Delta t}{2}(\ddot{x}_{i+1} - \ddot{x}_i) \tag{4.21a}$$

Displacement
$$x_{i+1} = x_i + \dot{x}_i\Delta t + \ddot{x}_i\frac{(\Delta t)^2}{2} + \frac{(\Delta t)^2}{6}(\ddot{x}_{i+1} - \ddot{x}_i) \tag{4.21b}$$

The expressions for velocity and displacement can be generalized as

Similar to 4.17, except linear assumption

$$\dot{x}_{i+1} = \dot{x}_i + [(1-\gamma)\Delta t]\ddot{x}_i + (\gamma\Delta t)\ddot{x}_{i+1} \tag{4.22a}$$

$$x_{i+1} = x_i + \dot{x}_i\Delta t + \left[\left(\tfrac{1}{2}-\beta\right)(\Delta t)^2\right]\ddot{x}_i + \beta(\Delta t)^2\ddot{x}_{i+1} \tag{4.22b}$$

Setting $\gamma = 1/2$ and $\beta = 1/4$ yields Equations 4.17, the constant acceleration assumption. $\gamma = 1/2$ and $\beta = 1/6$ yields Equations 4.21, the linear acceleration assumption. The factors γ and β determine the stability and accuracy of the solution. In general,

$$\gamma = 1/2 \qquad (4.23a) \qquad\qquad \frac{1}{6} \leq \beta \leq \frac{1}{4} \qquad (4.23b)$$

uncontrollable error growth

In any numerical method, stability and accuracy are important considerations. For the purposes of this course, we simply note that regardless of which approximation is used (constant average or linear), using small time steps, Δt, is necessary to ensure the response can be captured accurately. As a general rule, $\Delta t \leq \min(T, T_n)/10$, where T and T_n are the excitation and natural periods respectively.

We can now use the approximations derived above to develop a solution to the incremental equation of motion given by Equation 4.13. Consider the general forms of velocity and displacement approximations. $m\Delta\ddot{x}_i + c\Delta\dot{x}_i + k\Delta x_i = \Delta P_i$

$$\dot{x}_{i+1} = \dot{x}_i + [(1-\gamma)\Delta t]\ddot{x}_i + (\gamma\Delta t)\ddot{x}_{i+1} \qquad (4.24a)$$

$$x_{i+1} = x_i + \dot{x}_i\Delta t + \left[\left(\frac{1}{2} - \beta\right)(\Delta t)^2\right]\ddot{x}_i + \beta(\Delta t)^2\ddot{x}_{i+1} \qquad (4.24b)$$

First, rearranging Equation 4.24a

$$\dot{x}_{i+1} - \dot{x}_i = [(1-\gamma)\Delta t]\ddot{x}_i + (\gamma\Delta t)\ddot{x}_{i+1}$$

$$\Delta\dot{x}_i = \Delta t\,\ddot{x}_i + (\gamma\Delta t)\Delta\ddot{x}_i \qquad (4.25)$$

Next, from Equation 4.24b,

$$x_{i+1} - x_i = \dot{x}_i\Delta t + \frac{(\Delta t)^2}{2}\ddot{x}_i - \beta(\Delta t)^2\ddot{u}_i + \beta(\Delta t)^2\ddot{x}_{i+1}$$

$$\Delta x_i = \dot{x}_i\Delta t + \frac{(\Delta t)^2}{2}\ddot{x}_i + \beta(\Delta t)^2(\ddot{x}_{i+1} - \ddot{x}_i) \qquad (4.27)$$

$$\Delta x_i = \dot{x}_i\Delta t + \frac{(\Delta t)^2}{2}\ddot{x}_i + \beta(\Delta t)^2\Delta\ddot{x}_i$$

Solving for $\Delta\ddot{x}_i$ in Equation 4.27 gives,

$$\Delta\ddot{x}_i = \frac{1}{\beta(\Delta t)^2}\Delta x_i - \frac{1}{\beta\Delta t}\dot{x}_i - \frac{1}{2\beta}\ddot{x}_i \qquad (4.28)$$

Substituting Equation 4.28 into Equation 4.25,

$$\Delta\dot{x}_i = \left(1-\frac{\gamma}{2\beta}\right)\Delta t\,\ddot{x}_i - \frac{\gamma}{\beta}\dot{x}_i + \frac{\gamma}{\beta\Delta t}\Delta x_i \tag{4.29}$$

Note that the change in acceleration and velocity from t_i to t_{i+1} given by Equation 4.28 and Equation 4.29 respectively, are dependent on the known acceleration and velocity at time i (i.e. $\ddot{x}_i$ and $\dot{x}_i$) and the change in displacement Δx_i. $m\Delta\ddot{x}_i + c\Delta\dot{x}_i + k\Delta x_i = \Delta P_i$

We can now substitute Equations 4.28 and 4.29 into the incremental equation of motion, Equation 4.13, and solve for the incremental displacement Δx_i.

Big Bad Equation

$$\left[k+\frac{1}{\beta(\Delta t)^2}m+\frac{\gamma}{\beta\Delta t}c\right]\Delta x_i = \Delta P_i + \left[\frac{1}{2\beta}m + \Delta t\left(\frac{\gamma}{2\beta}-1\right)c\right]\ddot{x}_i + \left(\frac{1}{\beta\Delta t}m+\frac{\gamma}{\beta}c\right)\dot{x}_i$$
$$\hat{k}\Delta x_i = \Delta\hat{P}_i \tag{4.30}$$

only unknown

$\hat{k}$ and $\Delta\hat{P}_i$ are only dependent on the system properties m, c, and k, the algorithm parameters γ, β, and the known state of the system at time i defined by x_i, $\dot{x}_i$, and $\ddot{x}_i$. Once the incremental displacement Δx_i is computed from Equation 4.30, the displacement at time $i+1$ is simply,

$$x_{i+1} = x_i + \Delta x_i \tag{4.31}$$

$\Delta\dot{x}_i$ and $\Delta\ddot{x}_i$ can be obtained from Equation 4.29 and 4.28, respectively, and the velocity and acceleration are updated in similar fashion

$$\dot{x}_{i+1} = \dot{x}_i + \Delta\dot{x}_i \tag{4.32}$$

$$\ddot{x}_{i+1} = \ddot{x}_i + \Delta\ddot{x}_i \tag{4.33}$$

After the response quantities have been computed at t_{i+1}, t_{i+1} becomes $t_i$as we move on to the next time step and start the process again. To initiate Newmark's method at $t=0$, we can compute the initial acceleration, $\ddot{x}_0$, from Equation 4.11 as

Specified

$$\ddot{x}_0 = \frac{P_0 - c\dot{x}_0 - kx_0}{m} \tag{4.34}$$

where x_0, $\dot{x}_0$, and P_0 are the displacement, velocity, and force initial conditions at $t=0$. Steps for analyzing linear SDOF systems using Newmark-β method is summarized below.

Newmark-β Method

Special cases:

- Constant average acceleration method ($\gamma = 1/2$, $\beta = 1/4$)
- Linear acceleration method ($\gamma = 1/2$, $\beta = 1/6$)

1.0 *Initial calculations*

1.1 $\ddot{x}_0 = \dfrac{P_0 - c\dot{x}_0 - kx_0}{m}$

1.2 Select Δt

1.3 $\hat{k} = k + \dfrac{1}{\beta(\Delta t)^2}m + \dfrac{\gamma}{\beta \Delta t}c$

2.0 *Calculations for each time step, $i = 0, 1, 2, \ldots$*

2.1 $\Delta\hat{P}_i = \Delta P_i + \left[\dfrac{1}{2\beta}m + \Delta t\left(\dfrac{\gamma}{2\beta} - 1\right)c\right]\ddot{x}_i + \left(\dfrac{1}{\beta\Delta t}m + \dfrac{\gamma}{\beta}c\right)\dot{x}_i$

2.2 $\Delta x_i = \dfrac{\Delta\hat{P}_i}{\hat{k}} \quad \Rightarrow \quad x_{i+1} = x_i + \Delta x_i$

2.3 $\Delta\dot{x}_i = \left(1 - \dfrac{\gamma}{2\beta}\right)\Delta t\ddot{x}_i - \dfrac{\gamma}{\beta}\dot{x}_i + \dfrac{\gamma}{\beta\Delta t}\Delta x_i \quad \Rightarrow \quad \dot{x}_{i+1} = \dot{x}_i + \Delta\dot{x}_i$

2.4 $\Delta\ddot{x}_i = \dfrac{1}{\beta(\Delta t)^2}\Delta x_i - \dfrac{1}{\beta\Delta t}\dot{x}_i - \dfrac{1}{2\beta}\ddot{x}_i \quad \Rightarrow \quad \ddot{x}_{i+1} = \ddot{x}_i + \Delta\ddot{x}_i$

3.0 *Move to the next time step. Set $i + 1$ to i and repeat from Step 2.1.*

4.1.2 Nonlinear Material Properties

For linear systems, implementing Newmark-β method is relatively straightforward. Newmark-β method can be applied to nonlinear systems but additional computations are needed. For nonlinear material behaviour, the equation of motion would have to be modified to

$$m\ddot{x} + c\dot{x} + f_s(x) = P(t) \tag{4.35}$$

where the spring force f_s is now nonlinearly related to the displacement x. Consider the nonlinear spring behaviour shown in Figure 4.6.

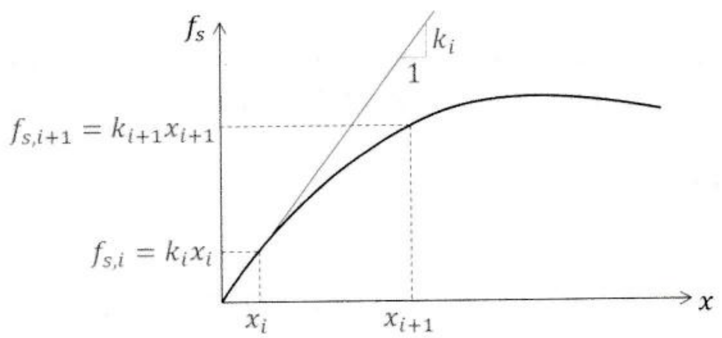


Figure 4.6 Nonlinear spring force

As can be seen, due to the varying stiffness, determining x_{i+1} becomes much more challenging. One approach to solving this problem is to use the *tangent stiffness* k_i at x_i and iteratively solving the dynamic equilibrium problem until the displacement approaches x_{i+1}, thereby, satisfying equilibrium at t_{i+1}. To do this, we need to add an additional calculation to calculate k_i at each time step, and use k_i in Equation 4.30. Nonlinear damping can be handled in a similar manner. Nonlinear systems will not be covered in this course.

4.1.3 First Order Methods

Rather than directly integrate second order equations, it is often easier to convert second order equations into a set of first order ordinary differential equations then integrate. Consider the equation of motion for forced vibration of a linear SDOF system.

$$m\ddot{x} + c\dot{x} + kx = P(t) \tag{4.36}$$

Solving for $\ddot{x}$, we get

$$\ddot{x} = \frac{P(t)}{m} - \frac{c\dot{x}}{m} - \frac{kx}{m} \tag{4.37}$$

Let

$$x_1 = x \tag{4.38a}$$

$$\dot{x}_1 = x_2 \tag{4.38b}$$

We can rewrite Equation 4.37 in terms of x_1 and x_2.

$$\dot{x}_2 = -\frac{c}{m}x_2 - \frac{k}{m}x_1 + \frac{1}{m}P(t) \tag{4.39}$$

Equation 4.38b and 4.39 can be written in matrix form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} P(t) \tag{4.40}$$

Equation 4.40 is the first order form of Equation 4.36, which can be readily solved in MATLAB™.

$\dot{z} = Az + Bu \leftarrow$ Input

Dynamics Matrix

State

Input Matrix

(State Space)

4.2 Earthquake Excitation

The time variation of the ground acceleration is the most useful way to define the ground motion due to an earthquake. Ground motion records are freely available electronically from many different databases such as the Pacific Earthquake Engineering Research Center (PEER) Ground Motion Database. There are many intricacies to analyzing and processing ground motion records but these are outside the scope of this course.

- Recall from our study of SDOF systems subjected to ground motion (Section 2.3.2), the equation of motion was

$$m\ddot{x} + c\dot{x} + kx = -m\ddot{x}_g \tag{4.41}$$

  where $\ddot{x}(t)$, $\dot{x}(t)$, and $x(t)$ are the acceleration, velocity, and displacement responses relative to the ground, respectively.
- The ground acceleration relative to the inertial reference frame, $\ddot{x}_g(t)$, appears on the right side of the equation. For a given ground acceleration, the problem is defined completely when the system's mass, stiffness, and damping are known properties.
- Figure 4.7 shows what is commonly referred to as the *El Centro* ground motion, so named because it was recorded at a site in El Centro, California during the Imperial Valley, California earthquake of May 18, 1940.

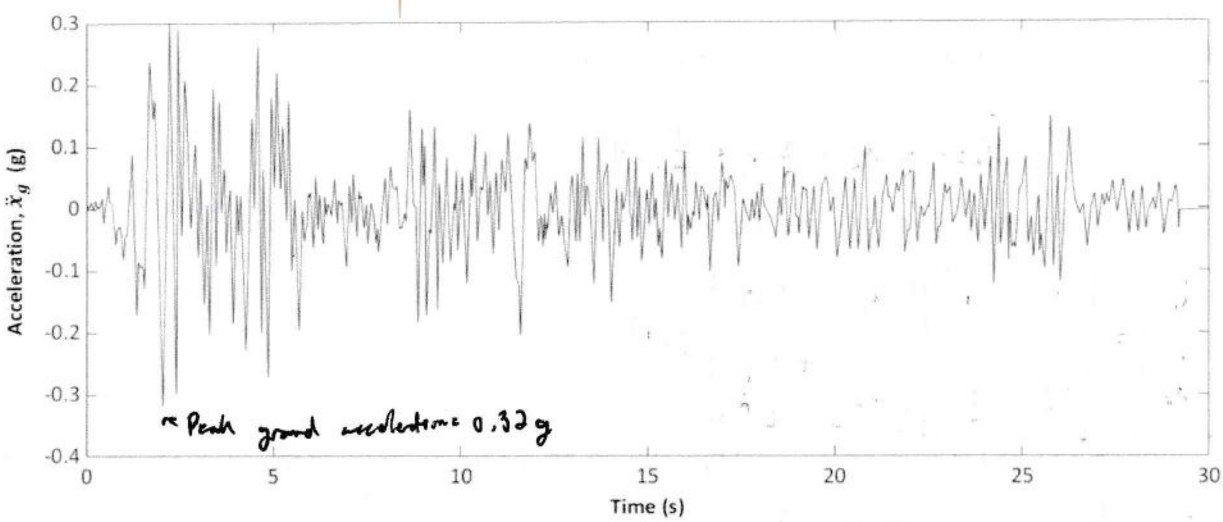


Figure 4.7 North-south component of horizontal ground acceleration recorded in El Centro, California during the Imperial Valley California earthquake on May 18, 1940

- The ground acceleration is defined by numerical values at discrete time instants. These time instants should be closely spaced to accurately capture the highly irregular variation acceleration with time. Typically, the time interval is 1/100 to 1/50 of a second.

- Acceleration is typically given in units of g, gravitational acceleration. The *peak ground acceleration* (PGA) in the El Centro ground motion is 0.319g.
- The equation of motion for earthquake ground motion in Equation 4.42 can be rearranged and expressed as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = -\ddot{x}_g \tag{4.42}$$

- The response of a SDOF system to a given earthquake ground motion is dependant only on the natural frequency (period) and the damping of the system.
- Newmark-β method can be easily implemented to simulate the earthquake response of structures. The MATLAB™ script below can be used to analyze SDOF systems under earthquake excitation.

Newmark Method

```
clear all
close all
clc

% SDOF system definition
m=1; % Mass
Tn=1; % Natural period
wn=(2*pi)/Tn; % Circular natural frequency
z=0.02; % Damping ratio

elcentro=load('elcentro.dat'); % Ground motion
F=-m*elcentro*9.8; % Force vector

% Newmark-beta method parameters
g=0.5; % gamma
b=0.25; % beta
dt=0.02; % Time step size

% Initialization
c=2*wn*z*m; % Damping coefficient
k=wn^2*m; % Stiffness
d(1)=0; % Initial displacement
v(1)=0; % Initial velocity
a(1)=(F(1)-c*v(1)-k*d(1))/m; % Initial acceleration
khat=k+(1/(b*dt^2))*m+(g/(b*dt))*c;
A=(1/(2*b))*m+dt*(g/(2*b)-1)*c;
B=(1/(b*dt))*m+(g/b)*c;

for i=1:(length(F)-1)
  dFhat=(F(i+1)-F(i))+A*a(i)+B*v(i);
  dd=dFhat/khat;
  dv=dt*(1-g/(2*b))*a(i)-(g/b)*v(i)+(g/(b*dt))*dd;
```

```
  da=(1/(b*dt^2))*dd-(1/(b*dt))*v(i)-(1/(2*b))*a(i);
  d(i+1)=d(i)+dd;
  v(i+1)=v(i)+dv;
  a(i+1)=a(i)+da;
end
```

The response for the SDOF to the El Centro ground motion is plotted in Figure 4.8 for several natural frequencies and damping levels.

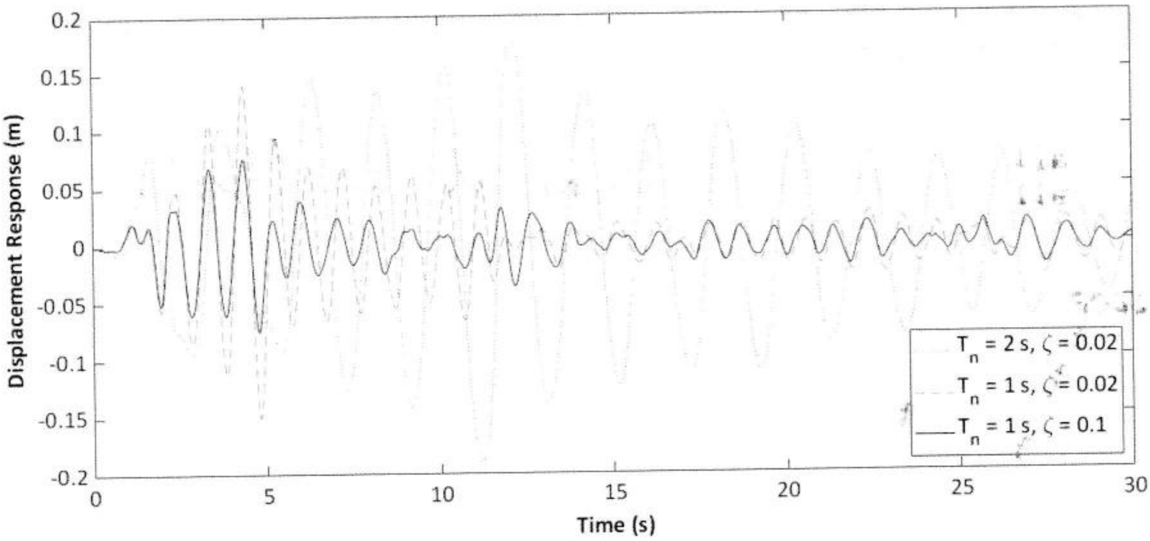


Figure 4.8 Response of a SDOF system to El Centro ground motion for various natural frequencies and damping ratios

4.2.1 Equivalent Static Force

For structural design, deformations, or the displacement of the structure relative to the ground, $x(t)$, are of primary interest.

- Deformations are linearly related to the internal forces in a structure.
- Consider a typical shear beam representation of a one-storey frame structure.

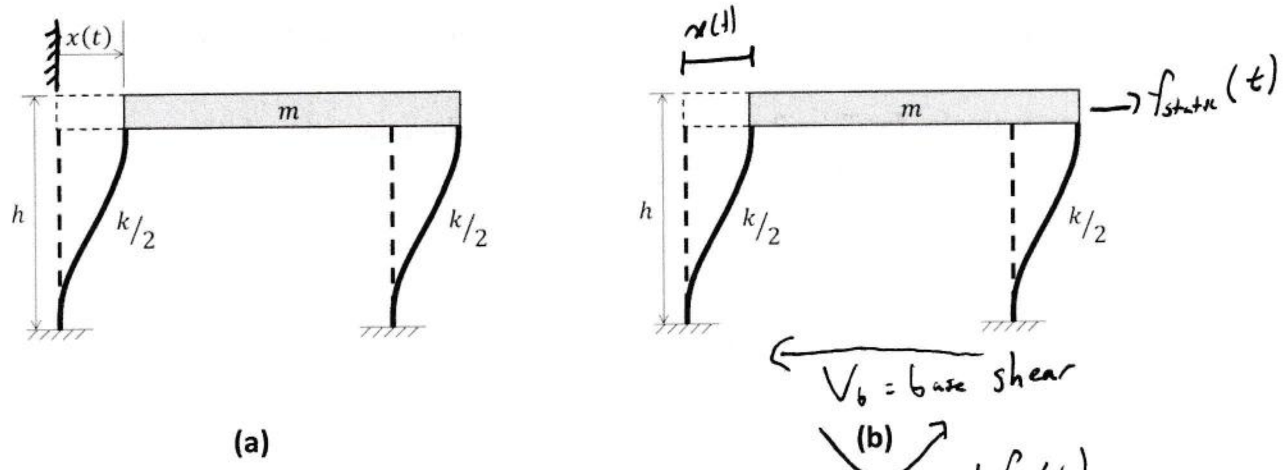


Figure 4.9 Equivalent static force

- The *equivalent static force* is the static force that must be applied to produce the same deformation $x(t)$ at time t.

$$f_{s_{static}}(t) = kx(t) \tag{4.43}$$

where k is the lateral stiffness of the frame.

- Once the static force $f_s(t)$ is determined, the base shear and base overturning moment can be found by a static analysis at each time instant. For the one-storey lateral frame, the base shear and overturning moment are

$$V_b(t) = f_{static}(t) \tag{4.44a}$$

$$M_b(t) = hf_{static}(t) \tag{4.44b}$$

where h is the height of the mass above the base.

- The maximum value of the equivalent static force can be used to determine the maximum base shear and base overturning moment during the event.

4.3 Elastic Response Spectrum

The response spectrum is perhaps the most important concept in the design of structures to withstand earthquake loading.

RS = plot of peak response as a function of natural frequency parameter (typ natural period)

We will investigate how to construct the deformation response spectrum for El Centro ground motion seen earlier.

- For example, for $T_n = 2$ s and $\zeta = 2\%$, the displacement time history computed using Nemark's method is plotted in Figure 4.10.

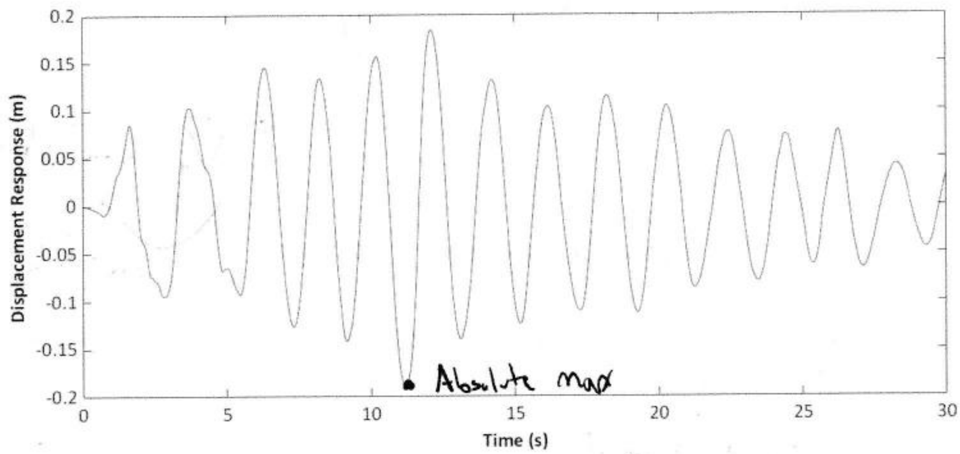


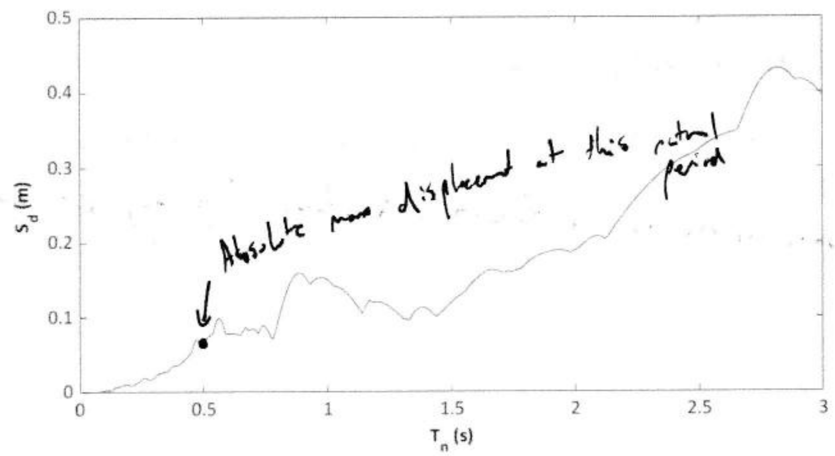
Figure 4.10 Displacement response to El Centro ground motion for SDOF system with $T_n = 2$ s and $\zeta = 2\%$

- The peak value, also called the *spectral displacement*, is

$$S_d = \max\left(\text{abs}\left(x(t \mid T_n = 2\text{ s}, \zeta = 0.02)\right)\right) = 0.1895\ m$$

and occurs at $t = 11.2$ s.

- This procedure is repeated for a range of natural periods, T_n, and the peak response for is plotted against T_n. The resulting plot is known as the *deformation response spectrum*. The deformation spectrum for the El Centro ground motion is shown in Figure 4.11.



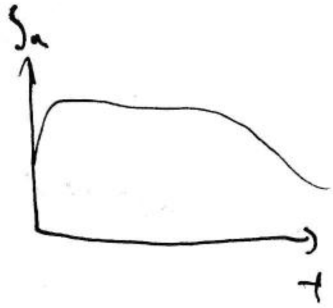


Figure 4.11 Elastic deformation response spectrum for El Centro ground motion for $\zeta = 2\%$

4.3.1 Pseudo- Velocity and Pseudo-Acceleration Spectra – Great as long as damping less than 10%.

As the prefix suggests, *pseudo-velocity* and *pseudo-acceleration* spectra do not represent the true response of a structure. Instead, they are related to the spectral displacement S_d through

Pseudo - Velocity

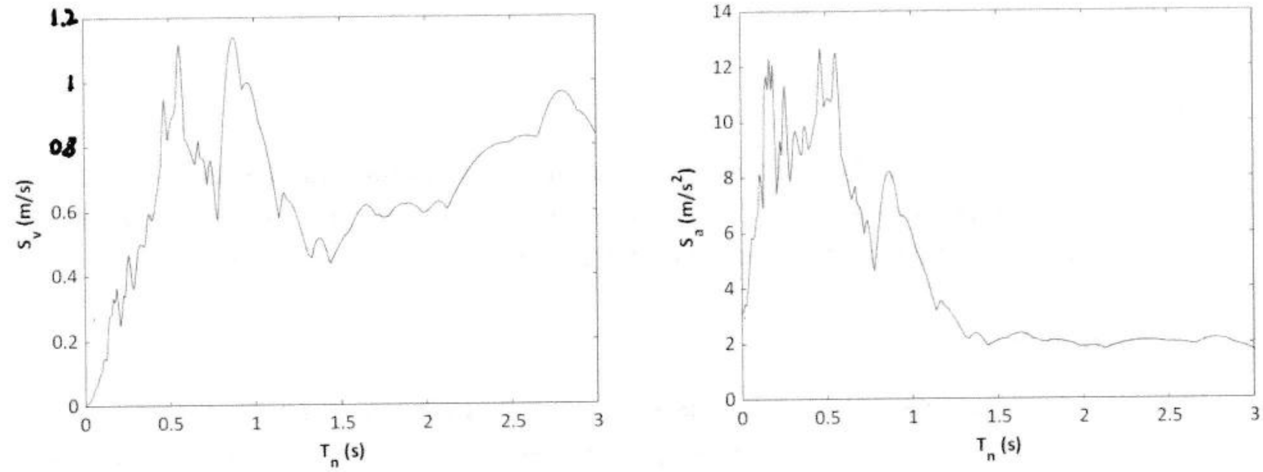
Pseudo - Acceleration

ω_n = rads/sec
S_d = m
since rads not too concrete, $S_v \sim$ m/s (velocity)

$$S_v = \omega_n S_d = \frac{2\pi}{T_n} S_d \quad (4.45a)$$

$$S_a = \omega_n^2 S_d = \left(\frac{2\pi}{T_n}\right)^2 S_d \quad (4.45b)$$

- The pseudo quantities have the correct units of velocity and acceleration, and for most structures with reasonable damping, they provide close approximations of the true response.
- The pseudo-velocity and pseudo-acceleration spectra for the El Centro ground motion are shown in Figure 4.12 for $\zeta = 2\%$.



(a) **(b)**

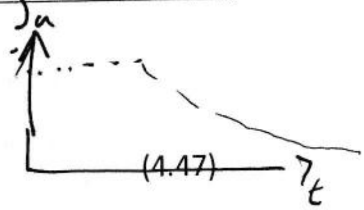
Figure 4.12 Pseudo spectra for El Centro ground motion for $\zeta = 2\%$
(a) Pseudo-velocity **(b)** Pseudo-acceleration

- Consider the expression for the equivalent static force discussed earlier. We can determine the maximum equivalent lateral force in an earthquake as

$$f_{s,max} = k|x(t)|_{max} = k S_d = k S_a/\omega_n^2 = m S_a \quad (4.46)$$

MAX EQUIVALENT LATERAL FORCE

- For structural design purposes, we can also compute the maximum base shear as

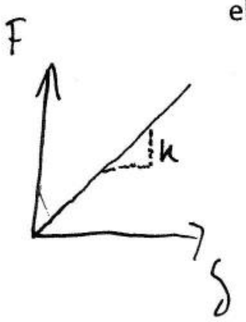


$$V_{b,max} = f_{s,max} = m S_a = \frac{W}{g} S_a = \frac{S_a}{g} W \quad (4.47)$$

Base shear coefficient

where W is the weight of the structure and g is the acceleration due to gravity. The non-dimensional ratio S_a/g is called the *base shear coefficient* or *lateral force coefficient*. It is used in building codes to represent the coefficient by which the structural weight is multiplied to obtain the peak base shear.

- Physically, the pseudo-velocity is related to the strain energy stored in the elastic elements of the structure. The maximum strain energy stored in the system is given by



$$V = \frac{1}{2} k S_d^2 = \frac{1}{2} m S_v^2 \quad (4.48)$$

- It must be emphasized that we only need to generate the deformation response spectrum, and the pseudo-velocity and the pseudo-acceleration spectra can be obtained by multiplying the values of the spectral displacement, S_d, by ω_n and ω_n^2, respectively.

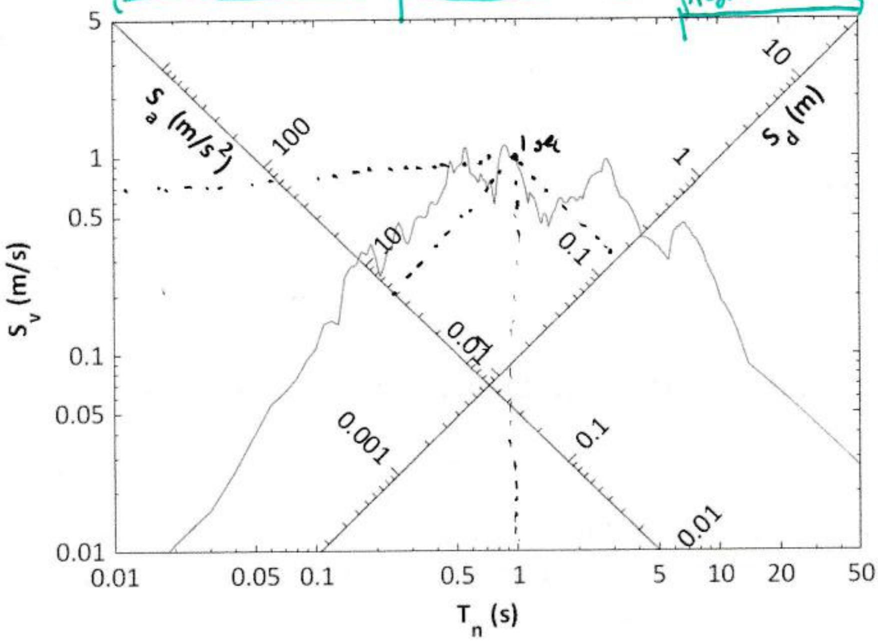
4.3.2 Tripartite Plot

The tripartite plot, also referred to as the combined D-V-A spectrum, is a compact, convenient way to plot the deformation, pseudo-velocity, and pseudo-acceleration spectra on the same graph. This is enabled by the fact that these three quantities are scalar multiples of each other. In tripartite plots, S_d, S_v, and S_a presented on a four-way logarithmic graph. The combined spectrum for the El Centro ground motion is shown in Figure 4.13.

- S_v is read off the vertical axis while S_d and S_a are on the diagonal axes sloping at +45° and -45° from the T_n-axis, respectively.
- In general, earthquake response spectra are very irregular. However, they are characterized by a general trapezoidal (tent) shape
- It is important to note that response spectra are earthquake-specific. Ground motion characteristics are affected by many different factors including earthquake magnitude, fault-to-site-distance, source-to-site geology, and the soil conditions at the site.
- As a result, different earthquakes recorded at the same site can yield distinctly different response spectra.

Very stiff   Typical stiffness   Not too stiff

Acceleration Sensitive | Velocity Sensitive | Displacement Sensitive Region



General Shape

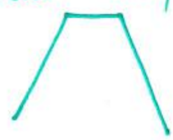


Figure 4.13 Combined deformation, pseudo-velocity, and pseudo-acceleration response spectrum for El Centro ground motion for $\zeta = 2\%$

Example 4.2 Determine the maximum equivalent static force for the frame show in Figure 4.14 subjected to El Centro ground motion. Assume 2% critical damping.

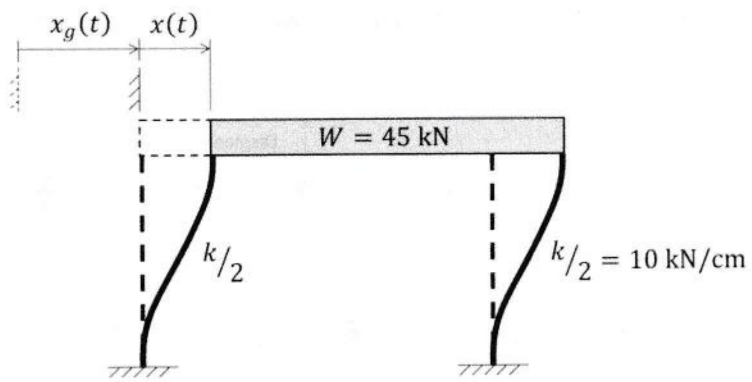


Figure 4.14 Lateral frame subjected to El Centro ground motion

<u>Solution</u>: The natural frequency of the frame for the lateral mode is

The peak displacement can be obtained from the tripartite plot in Figure 4.13 or the deformation spectrum in Figure 4.11

The maximum equivalent static force is

4.4 Earthquake Response of Linear MDOF Systems

Analysis of the earthquake response of linear MDOF systems can take two approaches.

- *Response history analysis* (RHA) is an accurate method which gives the response of a structure over time to the application of ground motion. It is able to accommodate general cases of structures and excitations.
- *Response spectrum analysis* seeks to compute the peak response of a structure directly from the earthquake response spectrum without the need for a response history analysis. The approach is approximate and generally used for design purposes.

4.4.1 Response History Analysis

Consider the general MDOF structure shown below in which the mass is lumped at the joints or nodes and each node has three degrees-of-freedom, two translational and one rotational.

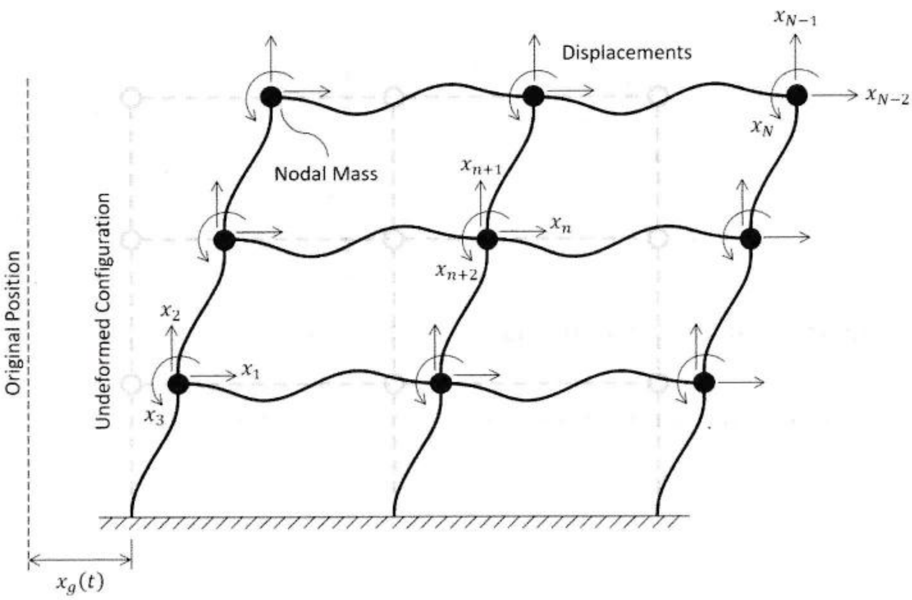


Figure 4.15 Plane frame modelled as a MDOF system subjected to earthquake ground motion