CHAPTER 15

EXERCISES 15.1

1. For the given function, $\dfrac{dy}{dx} + 2xy = -2Cxe^{-x^2} + 2x(2 + Ce^{-x^2}) = 4x.$

2. For the given function, $\dfrac{dy}{dx} - \dfrac{y^2}{x^2} = \dfrac{(1+Cx)(1) - x(C)}{(1+Cx)^2} - \dfrac{x^2}{x^2(1+Cx)^2} = 0.$

3. For the given function,

$$x^3\frac{dy}{dx} + (2-3x^2)y = x^3\left[\frac{3x^2}{2} + 3Cx^2e^{1/x^2} + Cx^3e^{1/x^2}\left(-\frac{2}{x^3}\right)\right] + (2-3x^2)\left(\frac{x^3}{2} + Cx^3e^{1/x^2}\right) = x^3.$$

4. For the given function, $\dfrac{d^2y}{dx^2} + 9y = (-9C_1\sin 3x - 9C_2\cos 3x) + 9(C_1\sin 3x + C_2\cos 3x) = 0.$

5. If we write the function in the form $y = \dfrac{1}{2C_1}(C_1^2e^x + e^{-x})$, then

$$\begin{aligned}\left(\frac{d^2y}{dx^2}\right)^2 - 1 - \left(\frac{dy}{dx}\right)^2 &= \left[\frac{1}{2C_1}(C_1^2e^x + e^{-x})\right]^2 - 1 - \left[\frac{1}{2C_1}(C_1^2e^x - e^{-x})\right]^2\\ &= \frac{1}{4C_1^2}(C_1^4e^{2x} + 2C_1^2 + e^{-2x}) - 1 - \frac{1}{4C_1^2}(C_1^4e^{2x} - 2C_1^2 + e^{-2x}) = 0.\end{aligned}$$

6. For the given function,

$$\begin{aligned}2\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 9y = {}& 2[4C_1e^{2x}\cos(x/\sqrt{2}) - (4/\sqrt{2})C_1e^{2x}\sin(x/\sqrt{2}) - (C_1/2)e^{2x}\cos(x/\sqrt{2})\\ &+ 4C_2e^{2x}\sin(x/\sqrt{2}) + (4/\sqrt{2})C_2e^{2x}\cos(x/\sqrt{2}) - (C_2/2)e^{2x}\sin(x/\sqrt{2})]\\ &- 8[2C_1e^{2x}\cos(x/\sqrt{2}) - (C_1/\sqrt{2})e^{2x}\sin(x/\sqrt{2}) + 2C_2e^{2x}\sin(x/\sqrt{2})\\ &+ (C_2/\sqrt{2})e^{2x}\cos(x/\sqrt{2})] + 9[C_1e^{2x}\cos(x/\sqrt{2}) + C_2e^{2x}\sin(x/\sqrt{2})] = 0.\end{aligned}$$

7. For the given function,

$$\begin{aligned}\frac{d^4y}{dx^4} + 5\frac{d^2y}{dx^2} + 4y = {}& (16C_1\cos 2x + 16C_2\sin 2x + C_3\cos x + C_4\sin x)\\ &+ 5(-4C_1\cos 2x - 4C_2\sin 2x - C_3\cos x - C_4\sin x)\\ &+ 4(C_1\cos 2x + C_2\sin 2x + C_3\cos x + C_4\sin x) = 0.\end{aligned}$$

8. For the given function,

$$\begin{aligned}2\frac{d^2y}{dx^2} - 16\frac{dy}{dx} + 32y = {}& 2[-(1/2)e^{4x} + 2(C_2 - x/2)4e^{4x} + 16(C_1 + C_2x - x^2/4)e^{4x}]\\ &- 16[(C_2 - x/2)e^{4x} + 4(C_1 + C_2x - x^2/4)e^{4x}] + 32(C_1 + C_2x - x^2/4)e^{4x} = -e^{4x}.\end{aligned}$$

9. For the given function,

$$\begin{aligned}x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 4y = {}& x^2\left[\frac{2C_1}{x^2}\sin(2\ln x) - \frac{4C_1}{x^2}\cos(2\ln x) - \frac{2C_2}{x^2}\cos(2\ln x) - \frac{4C_2}{x^2}\sin(2\ln x)\right]\\ &+ x\left[-\frac{2C_1}{x}\sin(2\ln x) + \frac{2C_2}{x}\cos(2\ln x)\right] + 4[C_1\cos(2\ln x) + C_2\sin(2\ln x) + 1/4]\\ ={}& 1.\end{aligned}$$

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10. For the given function,

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-1/4)y=x^2\left(-\frac{C_1\sin x}{\sqrt{x}}-\frac{C_1\cos x}{x^{3/2}}+\frac{3C_1\sin x}{4x^{5/2}}-\frac{C_2\cos x}{\sqrt{x}}\right.$$
$$\left.+\frac{C_2\sin x}{x^{3/2}}+\frac{3C_2\cos x}{4x^{5/2}}\right)+x\left(\frac{C_1\cos x}{\sqrt{x}}-\frac{C_1\sin x}{2x^{3/2}}-\frac{C_2\sin x}{\sqrt{x}}-\frac{C_2\cos x}{2x^{3/2}}\right)$$
$$+x^2\left(\frac{C_1\sin x}{\sqrt{x}}+\frac{C_2\cos x}{\sqrt{x}}\right)-\frac{1}{4}\left(\frac{C_1\sin x}{\sqrt{x}}+\frac{C_2\cos x}{\sqrt{x}}\right)=0.$$

11. For $y(0)=1$ and $y'(0)=6$, constants C_1 and C_2 must satisfy $1=C_2$ and $6=3C_1$. Thus, $y(x)=2\sin 3x+\cos 3x$.

12. For $y(0)=2$ and $y(\pi/2)=3$, constants C_1 and C_2 must satisfy $2=C_2$ and $3=-C_1$. Thus, $y(x)=-3\sin 3x+2\cos 3x$.

13. For $y(\pi/12)=0$ and $y'(\pi/12)=1$, constants C_1 and C_2 must satisfy $0=C_1/\sqrt{2}+C_2/\sqrt{2}$ and $1=3C_1/\sqrt{2}-3C_2/\sqrt{2}$. These give $C_1=-C_2=\sqrt{2}/6$, and $y(x)=(\sqrt{2}/6)(\sin 3x-\cos 3x)$.

14. For $y(1)=1$ and $y(2)=2$, constants C_1 and C_2 must satisfy

$$1=C_1\sin 3+C_2\cos 3 \qquad \text{and} \qquad 2=C_1\sin 6+C_2\cos 6.$$

The solution of these equations is $C_1=(2\cos 3-\cos 6)/\sin 3$ and $C_2=(\sin 6-2\sin 3)/\sin 3$, and therefore $y(x)=[(2\cos 3-\cos 6)\sin 3x+(\sin 6-2\sin 3)\cos 3x]/\sin 3$.

15. Integration with respect to x gives a general solution $y(x)=2x^3+x^2+C$.

16. Integration with respect to x gives $y(x)=\displaystyle\int\frac{1}{9+x^2}dx$, and if we set $x=3\tan\theta$ and $dx=3\sec^2\theta\,d\theta$, then, $y(x)=\displaystyle\int\frac{1}{9\sec^2\theta}3\sec^2\theta\,d\theta=\frac{1}{3}\theta+C=\frac{1}{3}\mathrm{Tan}^{-1}(x/3)+C$.

17. Integration with respect to x gives $\dfrac{dy}{dx}=x^2+e^x+C$. A second integration gives a general solution $y(x)=\dfrac{x^3}{3}+e^x+Cx+D$.

18. Integration with respect to x gives $\dfrac{dy}{dx}=\displaystyle\int x\ln x\,dx=\frac{x^2}{2}\ln x-\frac{x^2}{4}+C$. A further integration now yields $y(x)=\displaystyle\int\left(\frac{x^2}{2}\ln x-\frac{x^2}{4}+C\right)dx=\frac{x^3}{6}\ln x-\frac{5x^3}{36}+Cx+D$.

19. Integration with respect to x gives $\dfrac{d^2y}{dx^2}=-\dfrac{1}{12x^4}+C$. From a second integration, $\dfrac{dy}{dx}=\dfrac{1}{36x^3}+Cx+D$. Finally, one more integration gives a general solution $y(x)=-\dfrac{1}{72x^2}+\dfrac{Cx^2}{2}+Dx+E$.

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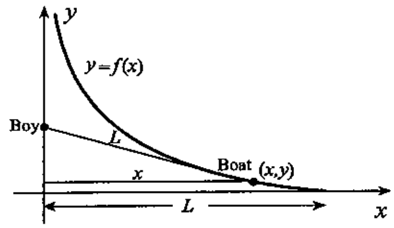
20. (a) Since the slope of the curve is also the slope of the string,

$$\frac{dy}{dx} = -\frac{\sqrt{L^2 - x^2}}{x}.$$



(b) If we integrate with respect to x,

$$y = -\int \frac{\sqrt{L^2 - x^2}}{x} dx.$$

We now set $x = L\sin\theta$ and $dx = L\cos\theta\, d\theta$,

$$y = -\int \frac{L\cos\theta}{L\sin\theta} L\cos\theta\, d\theta = -L\int \frac{1-\sin^2\theta}{\sin\theta} d\theta = L\int (\sin\theta - \csc\theta)\, d\theta$$

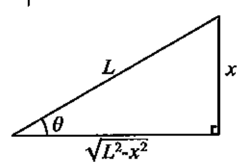
$$= L[-\cos\theta - \ln|\csc\theta - \cot\theta|] + C = -L\left(\frac{\sqrt{L^2 - x^2}}{L} + \ln\left|\frac{L}{x} - \frac{\sqrt{L^2 - x^2}}{x}\right|\right) + C$$

$$= -\sqrt{L^2 - x^2} - L\ln\left|\frac{L - \sqrt{L^2 - x^2}}{x}\right| + C = -\sqrt{L^2 - x^2} - L\ln\left|\frac{L - \sqrt{L^2 - x^2}}{x}\frac{L + \sqrt{L^2 - x^2}}{L + \sqrt{L^2 - x^2}}\right| + C$$

$$= -\sqrt{L^2 - x^2} - L\ln\left|\frac{x}{L + \sqrt{L^2 - x^2}}\right| + C = L\ln\left|\frac{L + \sqrt{L^2 - x^2}}{x}\right| - \sqrt{L^2 - x^2} + C.$$

Since $y = 0$ when $x = L$, it follows that $C = 0$, and

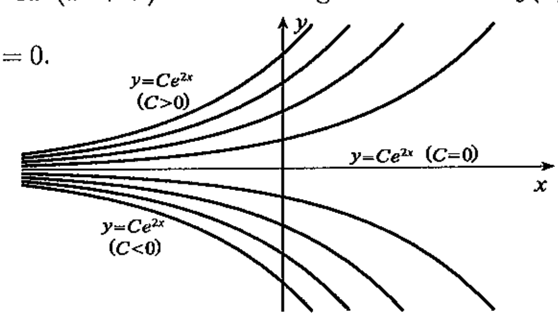
$$y = L\ln\left(\frac{L + \sqrt{L^2 - x^2}}{x}\right) - \sqrt{L^2 - x^2}.$$



21. For this function, $\dfrac{dy}{dx} - 2xy^2 = \dfrac{2x}{(x^2+C)^2} - 2x\left(\dfrac{-1}{x^2+C}\right)^2 = 0$. A singular solution is $y(x) \equiv 0$.

22. For this function, $\dfrac{dy}{dx} - 3x^2(y-1)^2 = \dfrac{3x^2}{(x^3+C)^2} - 3x^2(x^3+C)^{-2} = 0$. A singular solution is $y(x) \equiv 1$.

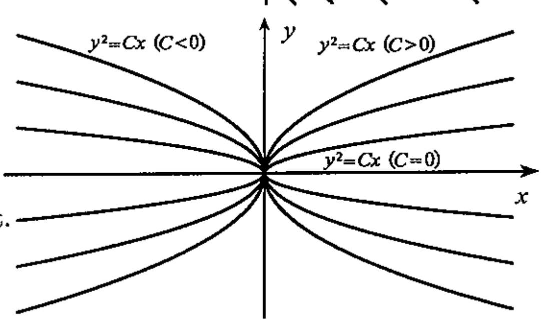
23. (a) For this function, $\dfrac{dy}{dx} - 2y = 2Ce^{2x} - 2(Ce^{2x}) = 0$.
(b) Graphs of curves in the family are shown to the right.
(c) For that solution passing through (x_0, y_0), C must satisfy $y_0 = Ce^{2x_0}$. Thus, $C = y_0e^{-2x_0}$, and the required solution is $y(x) = y_0e^{2(x-x_0)}$.



24. (a) Differentiation of $y^2 = Cx$ (implicitly) with respect to x gives $2y\dfrac{dy}{dx} = C$. If we substitute $C = y^2/x$, we obtain

$$2y\frac{dy}{dx} = \frac{y^2}{x} \implies 2x\frac{dy}{dx} = y.$$

(b) The curves are the parabolas shown to the right.
(c) For that solution passing through (x_0, y_0), C must satisfy $y_0^2 = Cx_0$. Thus, $C = y_0^2/x_0$, except when $x_0 = 0$.



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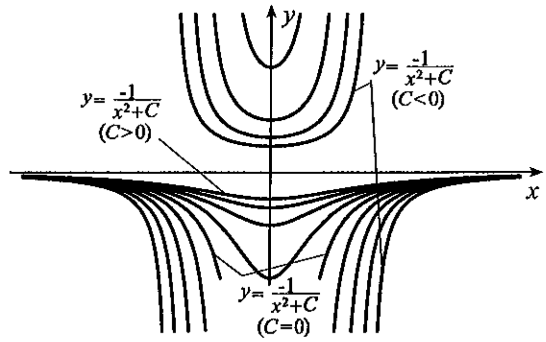
25. (a) Graphs are shown to the right.

(b) For the solution to pass through a point (x_0, y_0), C must satisfy $y_0 = -1/(x_0^2 + C)$. When we solve for C, we obtain $C = -1/y_0 - x_0^2$, provided, of course, that $y_0 \neq 0$.



26. If we integrate both sides of the differential equation with respect to x, we obtain the general solution

$$y(x) = \begin{cases} -1/x + C, & x < 0 \\ -1/x + D, & x > 0 \end{cases}.$$

(a) For the solution to satisfy $y(1) = 1$, it is necessary for $D = 2$. Constant C is undetermined.
(b) For the solution to satisfy $y(-1) = 2$, it is necessary for $C = 1$. Constant D is undetermined.
(c) For the solution to satisfy $y(1) = 1$ and $y(-1) = 2$, we choose $C = 1$ and $D = 2$.

27. (a) Since arc length along a circle is given by $s = r\theta$, it follows that $ds/dt = r d\theta/dt + \theta dr/dt$. Now dr/dt is very small compared to $d\theta/dt$ so that we set $ds/dt = r d\theta/dt$. Since $dn/dt = (2\pi)^{-1} d\theta/dt$, and $v = ds/dt$, it follows that $dn/dt = v/(2\pi r)$.
(b) If we set $\pi(r^2 - r_0^2) = wvt$, then $r = \sqrt{wvt/\pi + r_0^2}$. Hence,

$$\frac{dn}{dt} = \frac{v}{2\pi\sqrt{wvt/\pi + r_0^2}} = \frac{v/(2\pi r_0)}{\sqrt{wvt/(\pi r_0^2) + 1}}.$$

(c) Integration gives

$$n(t) = \frac{v}{2\pi r_0}\left(\frac{2\pi r_0^2}{wv}\right)\sqrt{\frac{wvt}{\pi r_0^2} + 1} + C = \frac{r_0}{w}\sqrt{\frac{wvt}{\pi r_0^2} + 1} + C.$$

Since $n(0) = 0$, we obtain $0 = r_0/w + C$, and therefore

$$n(t) = \frac{r_0}{w}\sqrt{\frac{wvt}{\pi r_0^2} + 1} - \frac{r_0}{w} = \frac{r_0}{w}\left(\sqrt{\frac{wvt}{\pi r_0^2} + 1} - 1\right).$$

EXERCISES 15.2

1. This equation can be separated, $\dfrac{dy}{y^2} = \dfrac{dx}{x^2}$, (provided $y \neq 0$), and therefore a one-parameter family of solutions is defined implicitly by $-1/y = -1/x + C$. This equation can be solved for $y(x) = x/(1 - Cx)$. The function $y = 0$ is a solution of the differential equation, and because it is not contained in the one-parameter family of solutions, it is a singular solution.

2. This equation can be separated, $\dfrac{1}{2 - y}\,dy = 2x\,dx$, (provided $y \neq 2$), and therefore a one-parameter family of solutions is defined implicitly by $-\ln|2 - y| = x^2 + C$. This equation can be solved for $y(x) = 2 + De^{-x^2}$, where $D = \pm e^{-C}$. The function $y = 2$ is a solution of the differential equation. If we allow D to be equal to zero, this solution is contained in the family, and it is not, therefore, a singular solution.

3. When the equation is separated, $\dfrac{dy}{y} = \dfrac{-2x\,dx}{x^2 + 1}$, (provided $y \neq 0$). A one-parameter family of solutions is defined implicitly by $\ln|y| = -\ln|x^2 + 1| + C$. When we solve for x, the result is $y = D/(x^2 + 1)$, where $D = \pm e^C$. The function $y = 0$ is a solution of the differential equation. If we allow D to be equal to zero, this solution is contained in the family, and it is not, therefore, a singular solution.

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4. When this equation is separated, $\dfrac{1}{3y+2}\,dy = dx$, (provided $y \neq -2/3$). A one-parameter family of solutions is defined implicitly by $(1/3)\ln|3y+2| = x + C$. This equation can be solved for $y(x) = De^{3x} - 2/3$, where $D = \pm(1/3)e^{3C}$. The function $y = -2/3$ is a solution of the differential equation. If we allow D to be equal to zero, this solution is contained in the family, and it is not, therefore, a singular solution.

5. Separation gives $\dfrac{4y\,dy}{y^2+2} = \dfrac{3\,dx}{x-1}$, and therefore a one-parameter family of solutions is defined implicitly by $2\ln|y^2+2| = 3\ln|x-1| + C$. When we solve for the explicit solution, we obtain $y(x) = \pm\sqrt{D|x-1|^{3/2} - 2}$ where $D = \pm e^{C/2}$.

6. This equation can be separated, $\dfrac{x^2\,dx}{1-x} = \dfrac{y^2\,dy}{1+y}$, (provided $y \neq -1$), and a one-parameter family of solutions is therefore defined implicitly by $\displaystyle\int \frac{x^2}{1-x}\,dx = \int \frac{y^2}{1+y}\,dy$. To integrate, we write

$$\int\left(-x - 1 + \frac{1}{1-x}\right)dx = \int\left(y - 1 + \frac{1}{y+1}\right)dy,$$

and hence, $-\dfrac{x^2}{2} - x - \ln|x-1| + C = \dfrac{y^2}{2} - y + \ln|y+1|$. The function $y = -1$ is a solution of the differential equation, and because it is not contained in the one-parameter family, it is a singular solution.

7. Separation gives $\sec y\,dy = -\csc x\,dx$, (provided $y \neq (2n+1)\pi/2$, where n is an integer). A one-parameter family of solutions is defined implicitly by $\ln|\sec y + \tan y| = -\ln|\csc x - \cot x| + C$. When we exponentiate, $(\csc x - \cot x)(\sec y + \tan y) = D$, where $D = \pm e^C$. The functions $y = (2n+1)\pi/2$ are solutions of the differential equation, and because they are not contained in the one-parameter family, they are singular solutions.

8. When we separate this equation, $y^2\,dy = \left(\dfrac{1 - x^2e^x}{x}\right)dx$, and a one-parameter family of solutions is defined implicitly by $\dfrac{y^3}{3} = \ln|x| - \displaystyle\int xe^x\,dx = \ln|x| - xe^x + e^x + C$. Explicitly we obtain $y(x) = [D + 3\ln|x| - 3xe^x + 3e^x]^{1/3}$, where $D = 3C$.

9. When we separate, $y\,dy = -(x\sec x\tan x + \sec x)\,dx$, and a one-parameter family of solutions is therefore defined implicitly by $y^2/2 = -x\sec x + C$. When we solve, $y(x) = \pm\sqrt{D - 2x\sec x}$, where $D = 2C$.

10. When we separate this equation, $\dfrac{dy}{1+y^2} = \dfrac{dx}{1+x^2}$, and a one-parameter family of solutions is defined implicitly by $\mathrm{Tan}^{-1}y = \mathrm{Tan}^{-1}x + C$. If we apply the tangent function to both sides of $C = \mathrm{Tan}^{-1}y - \mathrm{Tan}^{-1}x$, we find

$$\tan C = D = \tan(\mathrm{Tan}^{-1}y - \mathrm{Tan}^{-1}x) = \frac{\tan(\mathrm{Tan}^{-1}y) - \tan(\mathrm{Tan}^{-1}x)}{1 + \tan(\mathrm{Tan}^{-1}y)\tan(\mathrm{Tan}^{-1}x)} = \frac{y-x}{1+xy}.$$

When this equation is solved for y the result is $y(x) = \dfrac{x+D}{1-Dx}$.

11. When we separate the equation, $\dfrac{dy}{y} = \dfrac{-2\,dx}{x+1}$, (provided $y \neq 0$). A one-parameter family of solutions is defined implicitly by $\ln|y| = -2\ln|x+1| + C$. When we solve for y, the result is $y(x) = D/(x+1)^2$, where $D = \pm e^C$. For the solution to satisfy $y(1) = 2$, we must have $2 = D/4$. Thus, $D = 8$, and $y(x) = 8/(x+1)^2$.

12. We separate this equation, $\left(\dfrac{y-1}{y}\right)dy = \left(\dfrac{x+1}{x}\right)dx$, (provided $y \neq 0$). A one-parameter family of solutions is defined implicitly by $y - \ln|y| = x + \ln|x| + C$. When we take exponentials, $xy = De^{y-x}$, where $D = \pm e^{-C}$. To satisfy $y(1) = 2$, we must have $(1)(2) = De^{2-1}$. Thus, $D = 2/e$, and $xy = 2e^{y-x-1}$.

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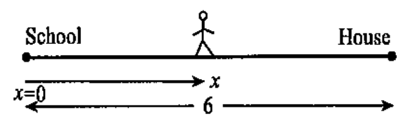
13. When we separate the equation, $e^{-y}dy = e^x dx$. A one-parameter family of solutions is defined implicitly by $-e^{-y} = e^x + C$. For the solution to satisfy $y(0) = 0$, we must have $-1 = 1 + C$. Thus, $C = -2$, and $-e^{-y} = e^x - 2 \Longrightarrow y = -\ln(2 - e^x)$.

14. When we separate variables, $\dfrac{1}{1+y^2}\,dy = 2x\,dx$. A one-parameter family of solutions is defined implicitly by $\mathrm{Tan}^{-1}y = x^2 + C$. Thus, $y(x) = \tan(x^2 + C)$. To satisfy $y(2) = 4$, we must have $4 = \tan(4 + C)$. This implies that $C = \mathrm{Tan}^{-1}4 - 4 + n\pi$, where n is an integer, and hence

$$y(x) = \tan(x^2 + \mathrm{Tan}^{-1}4 - 4 + n\pi) = \frac{\tan(x^2-4) + \tan(\mathrm{Tan}^{-1}4 + n\pi)}{1 - \tan(x^2-4)\,\tan(\mathrm{Tan}^{-1}4 + n\pi)} = \frac{4 + \tan(x^2-4)}{1 - 4\tan(x^2-4)}.$$

15. When we separate the equation, $\csc^2 y\,dy = \sec^2 x\,dx$. A one-parameter family of solutions is defined implicitly by $-\cot y = \tan x + C$. For the solution to satisfy $y(0) = \pi/2$, we must have $C = 0$. Thus, $\cot y = -\tan x$. This can be simplified by writing $\cot y = -\cot(\pi/2 - x) = \cot(x - \pi/2)$. This implies that $y = x - \pi/2 + n\pi$ for some integer n. The condition $y(0) = \pi/2$ requires $n = 1$, and the solution of the differential equation is $y = x + \pi/2$.

16. (a) Since the girl's speed is proportional to x^2, $dx/dt = kx^2$. Separation of this equation gives



$$\frac{1}{x^2}\,dx = k\,dt,$$

a one-parameter family of solutions of which is defined implicitly by $-1/x = kt + C$. Thus, $x = -1/(kt + C)$. If we choose time $t = 0$ when $x = 6$, then $6 = -1/C$ and $x = 6/(1 - 6kt)$ km.

(b) The girl reaches school when $x = 0$, but this only happens after an infinitely long time.

17. When we separate the equation, $\dfrac{y^2\,dy}{y^3+1} = \dfrac{-dx}{x(x^2+1)} = \left(\dfrac{x}{x^2+1} - \dfrac{1}{x}\right)dx$. A one-parameter family of solutions is defined implicitly by $(1/3)\ln|y^3+1| = (1/2)\ln|x^2+1| - \ln|x| + C$. When we solve this equation for y, we obtain an explicit definition of the solution $y(x) = \left[\dfrac{D(x^2+1)^{3/2}}{|x|^3} - 1\right]^{1/3}$, where $D = \pm e^{3C}$.

18. If $T(t)$ represents the temperature of the water as a function of time t, then according to Newton's law of cooling, $\dfrac{dT}{dt} = k(T - 20)$, where $k < 0$ is a constant. Separation of variables leads to $\dfrac{1}{T-20}\,dT = k\,dt$, and therefore $\ln|T - 20| = kt + C$. When we solve for T, the result is $T(t) = 20 + De^{kt}$. If we choose time $t = 0$ when $T = 80$, then $80 = 20 + D$. Thus, $D = 60$, and $T(t) = 20 + 60e^{kt}$. Because $T(2) = 60$, it follows that $60 = 20 + 60e^{2k}$, from which $k = (1/2)\ln(2/3) = -0.203$. Finally then, $T(t) = 20 + 60e^{-0.203t}$, or, $T(t) = 20 + 60e^{(t/2)\ln(2/3)} = 20 + 60(2/3)^{t/2}$.

19. If $T(t)$ represents the temperature of the mercury in the thermometer as a function of time t, then according to Newton's law of cooling, $\dfrac{dT}{dt} = k(T + 20)$, where $k < 0$ is a constant. Separation of variables leads to $\dfrac{1}{T+20}\,dT = k\,dt$, and therefore $\ln|T + 20| = kt + C$. When we solve for T, the result is $T(t) = -20 + De^{kt}$. If we choose time $t = 0$ when $T = 23$, then $23 = -20 + D$. Thus, $D = 43$, and $T(t) = -20 + 43e^{kt}$. Because $T(4) = 0$, it follows that $0 = -20 + 43e^{4k}$, from which $k = (1/4)\ln(20/43)$. The temperature is $-19°$ C when $-19 = -20 + 43e^{kt} \Longrightarrow t = (1/k)\ln(1/43) = 19.7$ minutes.

20. If $A(t)$ represents the amount of drug in the body at time t (in hours), then $\dfrac{dA}{dt} = kA$, where $k < 0$ is a constant. Separation of variables gives $\dfrac{1}{A}\,dA = k\,dt$, and therefore $\ln|A| = kt + C$, or, $A = De^{kt}$. If A_0 is the size of the original dose injected at time $t = 0$, then $A_0 = D$, and $A = A_0e^{kt}$. Since $A(1) = 0.95A_0$, it follows that $0.95A_0 = A_0e^k$. Thus, $k = \ln(0.95)$, and $A = A_0e^{t\ln(0.95)}$. The dose decreases to $A_0/2$ when $A_0/2 = A_0e^{t\ln(0.95)}$, the solution of which is $t = -\ln 2/\ln(0.95) = 13.51$ h.

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21. Since the rate of change of the amount of nitrogen is proportional to $\overline{N} - N$, we can write that $\dfrac{dN}{dt} = k(\overline{N} - N)$, where $k > 0$ is a constant. If we separate this equation $\dfrac{1}{\overline{N} - N}dN = k\,dt$, a one-parameter family of solutions is defined implicitly by

$$-\ln(\overline{N} - N) = kt + C \implies \overline{N} - N = e^{-kt-C} = De^{-kt} \implies N = \overline{N} - De^{-kt},$$

where $D = e^{-C}$ is a constant. The initial value $N(0) = N_0$ requires $N_0 = \overline{N} - D$, and therefore $N = \overline{N} - (\overline{N} - N_0)e^{-kt} = N_0e^{-kt} + \overline{N}(1 - e^{-kt})$.

22. Because the total rate of change dA/dt of the amount of glucose in the bloodstream is the rate at which it is added less the rate at which it is used up, $\dfrac{dA}{dt} = R - kA$, where $k > 0$ is a constant. This equation can be separated, $\dfrac{1}{R - kA}dA = dt$, and therefore a one-parameter family of solutions is defined implicitly by $-(1/k)\ln|R - kA| = t + C$. When this equation is solved for A, the result is $A(t) = R/k + De^{-kt}$. If we choose time $t = 0$ when $A = A_0$, then $A_0 = R/k + D$, and

$$A(t) = \frac{R}{k} + \left(A_0 - \frac{R}{k}\right)e^{-kt} = \frac{R}{k}(1 - e^{-kt}) + A_0e^{-kt}.$$

23. The equation of the normal line to the curve $y = f(x)$ at any point $P(x_0, y_0)$ is

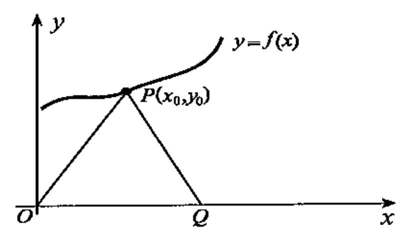
$$y - y_0 = -\frac{1}{f'(x_0)}(x - x_0).$$

The x-intercept of this line is

$$x = x_0 + f'(x_0)y_0.$$

Triangle OPQ is isosceles if $\|OP\|^2 = \|PQ\|^2$, and this can be expressed in the form

$$x_0{}^2 + y_0{}^2 = [f'(x_0)y_0]^2 + y_0{}^2.$$



Since this must be true at each point on the curve, we drop subscripts and replace $f'(x)$ with dy/dx, the result being $x^2 = (dy/dx)^2y^2$. This equation can be separated, $y\,dy = \pm x\,dx$, and a one-parameter family of solutions is defined implicitly by $y^2/2 = \pm x^2/2 + C$. In the case of the circles $x^2 + y^2 = 2C$, the normal always passes through the origin, and the triangle OPQ degenerates to a straight line. Hence the only nondegenerate case is the hyperbolas $y^2 - x^2 = 2C$.

24. If x is the number of grams of C at time t, then $x/2$ came from each of A and B. This means that there is $10 - x/2$ grams of A and $15 - x/2$ grams of B remaining. Hence, the rate dx/dt at which C is formed is related to x by

$$\frac{dx}{dt} = K\left(10 - \frac{x}{2}\right)\left(15 - \frac{x}{2}\right) = \frac{K}{4}(20 - x)(30 - x) = k(20 - x)(30 - x),$$

where we have set $k = K/4$. If we choose time $t = 0$ when A and B are brought together, then $x(t)$ must also satisfy $x(0) = 0$. We separate the equation and use partial fractions to write it in the form

$$k\,dt = \frac{1}{(20 - x)(30 - x)}dx = \left(\frac{1/10}{20 - x} - \frac{1/10}{30 - x}\right)dx.$$

A one-parameter family of solutions is defined implicitly by

$$kt + C = \frac{1}{10}\left[-\ln(20 - x) + \ln(30 - x)\right].$$

Absolute values are unnecessary because x cannot exceed 20. We now solve this equation for x by writing

$$10(kt + C) = \ln\left(\frac{30 - x}{20 - x}\right), \quad \text{and exponentiating,} \quad \frac{30 - x}{20 - x} = De^{10kt},$$

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where $D = e^{10C}$. Cross multiplying gives $(20 - x)De^{10kt} = 30 - x$, and therefore $x = \dfrac{30 - 20De^{10kt}}{1 - De^{10kt}}$. The initial condition $x(0) = 0$ requires $D = 3/2$, in which case

$$x = \frac{30 - 30e^{10kt}}{1 - (3/2)e^{10kt}} = \frac{60\left(1 - e^{10kt}\right)}{2 - 3e^{10kt}} \text{ g}.$$

25. If $x(t)$ represents the amount of C in the mixture at time t, then

$$\frac{dx}{dt} = k\left(10 - \frac{x}{2}\right)\left(10 - \frac{x}{2}\right) = \frac{k}{4}(20 - x)^2,$$

where factors $10 - x/2$ represent the amounts of A and B in the mixture at time t, and k is a constant. We can separate this equation, $\dfrac{dx}{(20 - x)^2} = \dfrac{k\,dt}{4}$, and a one-parameter family of solutions is defined implicitly by $1/(20 - x) = kt/4 + D$. When this equation is solved for x, the result is $x(t) = 20 - 4/(kt + 4D)$. If we choose time $t = 0$ when the reaction begins, then $0 = 20 - 1/D$. Thus, $D = 1/20$, and

$$x(t) = 20 - \frac{4}{kt + 1/5} = \frac{100kt}{5kt + 1} \text{ g}.$$

26. In Section 5.5 we showed that if V is the volume of water in the left sphere, then

$$\frac{dV}{dt} = A(y)\frac{dy}{dt}$$

where $A(y)$ is the surface area of the water. Since water leaves this container at rate

$$\frac{a}{3}\sqrt{2gh} = \frac{a}{3}\sqrt{2g(2y)} = \frac{2a}{3}\sqrt{gy},$$

it follows that

$$A(y)\frac{dy}{dt} = -\frac{2a}{3}\sqrt{gy}.$$

Because $A(y) = \pi x^2 = \pi(R^2 - y^2)$,

$$\pi(R^2 - y^2)\frac{dy}{dt} = -\frac{2a}{3}\sqrt{gy} \quad \Longrightarrow \quad \frac{R^2 - y^2}{\sqrt{y}}\,dy = -\frac{2a\sqrt{g}}{3\pi}\,dt,$$

a separated equation. A one-parameter family of solutions is defined implicitly by

$$2R^2\sqrt{y} - \frac{2}{5}y^{5/2} = -\frac{2a\sqrt{g}t}{3\pi} + C.$$

If we choose $t = 0$ when $y = R$, then $2R^2\sqrt{R} - \dfrac{2}{5}R^{5/2} = C$, and

$$2R^2\sqrt{y} - \frac{2}{5}y^{5/2} = -\frac{2a\sqrt{g}t}{3\pi} + \frac{8}{5}R^{5/2}.$$

The water levels are the same in both spheres when $y = 0$, in which case

$$0 = -\frac{2a\sqrt{g}t}{3\pi} + \frac{8}{5}R^{5/2} \qquad \Longrightarrow \qquad t = \frac{12\pi R^{5/2}}{5a\sqrt{g}}.$$

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27. The differential equation describing the height y of the surface is

$$A(y)\frac{dy}{dt} = -0.6a\sqrt{2g(R+y)},$$

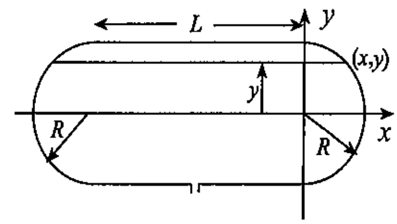
where $A(y)$, the surface area of the water is

$A(y) = 2xL + \pi x^2 = 2L\sqrt{R^2-y^2} + \pi(R^2-y^2)$.

Thus,

$$[2L\sqrt{R^2-y^2} + \pi(R^2-y^2)]\frac{dy}{dt} = -0.6a\sqrt{2g(R+y)}.$$

This is separable,



$$\left[\frac{2L\sqrt{(R+y)(R-y)}}{\sqrt{R+y}} + \pi\frac{(R+y)(R-y)}{\sqrt{R+y}}\right] dy = -0.6a\sqrt{2g}\,dt,$$

or,

$$\left[2L\sqrt{R-y} + \pi(R-y)\sqrt{R+y}\right] dy = -0.6a\sqrt{2g}\,dt.$$

Integration leads to the following equation that implicitly defines a one-parameter family of solutions

$$-\frac{4L}{3}(R-y)^{3/2} + \pi\int (R-y)\sqrt{R+y}\,dy = -0.6a\sqrt{2g}t + C.$$

If we set $u = R+y$ and $du = dy$ in the integral, then

$$\int (R-y)\sqrt{R+y}\,dy = \int (2R-u)\sqrt{u}\,du = \frac{4R}{3}u^{3/2} - \frac{2}{5}u^{5/2} = \frac{4R}{3}(R+y)^{3/2} - \frac{2}{5}(R+y)^{5/2}.$$

Hence, solutions of the differential equation are defined implicitly by

$$-\frac{4L}{3}(R-y)^{3/2} + \frac{4\pi R}{3}(R+y)^{3/2} - \frac{2\pi}{5}(R+y)^{5/2} = -0.6a\sqrt{2g}t + C.$$

If we choose $t=0$ when $y=R$, then

$$C = \frac{4\pi R}{3}(2R)^{3/2} - \frac{2\pi}{5}(2R)^{5/2} = \frac{16\sqrt{2}\pi R^{5/2}}{15}.$$

The tank empties when $y=-R$ in which case $-0.6a\sqrt{2g}t + C = -\frac{4L}{3}(2R)^{3/2}$. The emptying time is therefore

$$t = \frac{1}{0.6a\sqrt{2g}}\left(C + \frac{8\sqrt{2}L}{3}R^{3/2}\right) = \frac{1}{0.6a\sqrt{2g}}\left(\frac{16\sqrt{2}\pi R^{5/2}}{15} + \frac{8\sqrt{2}LR^{3/2}}{3}\right) = \frac{8R^{3/2}}{9a\sqrt{g}}(2\pi R + 5L).$$

28. Volume of water in the lock above the downstream level is $V = (8)(16)(2-h)$. Consequently,

$$\frac{dV}{dt} = -128\frac{dh}{dt}.$$

Since water enters the lock at $0.04\sqrt{2gh}$ m^3/s, it follows that

$$-128\frac{dh}{dt} = 0.04\sqrt{2gh} \quad\Longrightarrow\quad \frac{1}{\sqrt{h}}\,dh = -\frac{\sqrt{2g}}{3200}\,dt,$$

a separated equation. A one-parameter family of solutions is defined implicitly by

$$2\sqrt{h} = -\frac{\sqrt{2g}t}{3200} + C.$$

If valve A is opened at $t=0$ (when $h=2$), then $2\sqrt{2} = C$, and

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$$2\sqrt{h} = -\frac{\sqrt{2g}t}{3200} + 2\sqrt{2}.$$

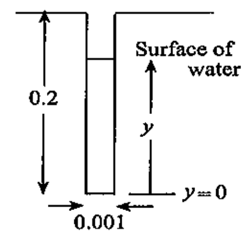
The upstream gate is opened when $h = 0.02$ in which case

$$2\sqrt{0.02} = -\frac{\sqrt{2g}t}{3200} + 2\sqrt{2} \quad \Longrightarrow \quad t = -\frac{3200}{\sqrt{2g}}(2\sqrt{0.02} - 2\sqrt{2}) = 1839.$$

Operation of the lock takes a little over 30 minutes.

29. When the height of the water above the bottom of the slit is y, Exercise 30 in Section 7.9 with $H = 0$ indicates that the volume of water per unit time through the slit is

$$\frac{2\sqrt{2g}c(10^{-3})}{3}y^{3/2} = \frac{\sqrt{2g}cy^{3/2}}{1500}.$$



But the rate of change of the volume of the water in the container is $A\,dy/dt$ where $A = 1$ is the cross-sectional area of the container. Hence

$$-\frac{dy}{dt} = \frac{\sqrt{2g}cy^{3/2}}{1500} \qquad \Longrightarrow \qquad \frac{1}{y^{3/2}}\,dy = -\frac{\sqrt{2g}c}{1500}\,dt,$$

a separate differential equation. A one-parameter family of solutions is defined implicitly by

$$-\frac{2}{\sqrt{y}} = -\frac{\sqrt{2g}ct}{1500} + C.$$

The initial condition $y(0) = 0.2$ requires $C = -2/\sqrt{0.2}$, and therefore

$$-\frac{2}{\sqrt{y}} = -\frac{\sqrt{2g}ct}{1500} - \frac{2}{\sqrt{0.2}}.$$

The water level has dropped 10 cm when

$$-\frac{2}{\sqrt{0.1}} = -\frac{\sqrt{2g}ct}{1500} - \frac{2}{\sqrt{0.2}} \quad \Longrightarrow \quad t = \frac{1500}{\sqrt{2g}(0.6)}\left(\frac{2}{\sqrt{0.1}} - \frac{2}{\sqrt{0.2}}\right) = 1046 \text{ s}.$$

30. The differential equation is separable, $\frac{1}{2gh - v^2}dv = \frac{1}{2L}dt$, so that a one-parameter of solutions is defined implicitly by

$$\frac{t}{2L} + C = \int \frac{1}{2gh - v^2}dv = \int \left(\frac{\frac{1}{2\sqrt{2gh}}}{\sqrt{2gh} + v} + \frac{\frac{1}{2\sqrt{2gh}}}{\sqrt{2gh} - v} \right) dv$$

$$= \frac{1}{2\sqrt{2gh}}\left(\ln|\sqrt{2gh} + v| - \ln|\sqrt{2gh} - v|\right) = \frac{1}{2\sqrt{2gh}}\ln\left|\frac{\sqrt{2gh} + v}{\sqrt{2gh} - v}\right|.$$

Exponentiation gives

$$\frac{\sqrt{2gh} + v}{\sqrt{2gh} - v} = De^{\sqrt{2gh}t/L} \quad \Longrightarrow \quad \sqrt{2gh} + v = De^{\sqrt{2gh}t/L}(\sqrt{2gh} - v) \quad \Longrightarrow \quad v = \frac{D\sqrt{2gh}e^{\sqrt{2gh}t/L} - \sqrt{2gh}}{1 + De^{\sqrt{2gh}t/L}}.$$

Since $v(0) = 0$,

$$0 = \frac{D\sqrt{2gh} - \sqrt{2gh}}{1 + D} \qquad \Longrightarrow \qquad D = 1.$$

Thus, $v(t) = \dfrac{\sqrt{2gh}(e^{\sqrt{2gh}t/L} - 1)}{e^{\sqrt{2gh}t/L} + 1}$.

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31. (a) When $a = b = c$,

$$\frac{dx}{dt} = k(a-x)^3 \qquad \Longrightarrow \qquad \frac{1}{(a-x)^3}\,dx = k\,dt,$$

a separated equation. A one-parameter family of solutions is defined implicitly by

$$\frac{1}{2(a-x)^2} = kt + C \quad \Longrightarrow \quad (a-x)^2 = \frac{1}{2(kt+C)}.$$

Square roots give $\quad x = a \pm \dfrac{1}{\sqrt{2(kt+C)}}$. If, as is normal, the initial condition is $x(0) = 0$, we would choose the negative sign, in which case $x(t) = a - 1/\sqrt{2(kt+C)}$.

(b) When $a = b$,

$$\frac{dx}{dt} = k(a-x)^2(c-x) \quad \Longrightarrow \quad \frac{1}{(a-x)^2(c-x)}\,dx = k\,dt.$$

A one-parameter family of solutions is defined implicitly by

$$\begin{aligned} kt + C &= \int \left[\frac{-1/(a-c)^2}{a-x} + \frac{1/(c-a)}{(a-x)^2} + \frac{1/(a-c)^2}{c-x}\right] dx \\ &= \frac{1}{(a-c)^2}\ln|a-x| + \frac{1}{(c-a)(a-x)} - \frac{1}{(a-c)^2}\ln|c-x|. \end{aligned}$$

(c) When $a \neq b \neq c$,

$$\frac{1}{(a-x)(b-x)(c-x)}\,dx = k\,dt.$$

A one-parameter family of solutions is defined implicitly by

$$\begin{aligned} kt + C &= \int \left[\frac{\dfrac{1}{(b-a)(c-a)}}{a-x} + \frac{\dfrac{1}{(a-b)(c-b)}}{b-x} + \frac{\dfrac{1}{(a-c)(b-c)}}{c-x}\right] dx \\ &= \frac{-1}{(b-a)(c-a)}\ln|a-x| - \frac{1}{(a-b)(c-b)}\ln|b-x| - \frac{1}{(a-c)(b-c)}\ln|c-x|. \end{aligned}$$

32. The differential equation can be separated, and partial fractions leads to

$$\frac{1}{A}\,dA = \frac{M^2-1}{M\left[\left(\dfrac{k-1}{2}\right)M^2+1\right]}\,dM = \left[\frac{(k+1)M/2}{\left(\dfrac{k-1}{2}\right)M^2+1} - \frac{1}{M}\right] dM.$$

A one-parameter family of solutions is defined implicitly by

$$\ln A = \left(\frac{k+1}{2}\right)\left(\frac{1}{k-1}\right)\ln\left[\left(\frac{k-1}{2}\right)M^2+1\right] - \ln M + C,$$

from which

$$A = \frac{D\left[\left(\dfrac{k-1}{2}\right)M^2+1\right]^{(k+1)/(2k-2)}}{M}.$$

The condition $A(1) = A_0$ gives

$$A_0 = D\left[\left(\frac{k-1}{2}\right)+1\right]^{(k+1)/(2k-2)} = D\left(\frac{k+1}{2}\right)^{(k+1)/(2k-2)},$$

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from which

$$A = \frac{A_0}{M} \frac{\left[\left(\frac{k-1}{2}\right) M^2 + 1\right]^{(k+1)/(2k-2)}}{\left(\frac{k+1}{2}\right)^{(k+1)/(2k-2)}} = \frac{A_0}{M}\left[\frac{(k-1)M^2+2}{k+1}\right]^{(k+1)/(2k-2)}.$$

33. (a) When H is a constant, we can separate the differential equation

$$\frac{v}{v+H}dv = \frac{K}{n}dt.$$

A one-parameter family of solutions is defined implicitly by

$$\int\left(1+\frac{-H}{v+H}\right)dv = \frac{K}{n}t + C \qquad \Longrightarrow \qquad v - H\ln(v+H) = \frac{K}{n}t + C.$$

For $v(0) = 0$, we find $0 - H\ln H = C$, and therefore the solution is defined implicitly by

$$v - H\ln\left(\frac{v+H}{H}\right) = \frac{Kt}{n}.$$

(b) When $H(t) = qt - nv$, the differential equation becomes

$$\frac{dv}{dt} = \frac{K}{n}\left(\frac{v+qt-nv}{v}\right) \qquad \Longrightarrow \qquad \frac{dv}{dt} = \frac{K}{n}\left[(1-n)+\frac{qt}{v}\right].$$

If we try a solution of the form $v(t) = At$, and substitute into the differential equation,

$$A = \frac{K}{n}\left[(1-n)+\frac{qt}{At}\right] \qquad \Longrightarrow \qquad A^2 - \frac{K}{n}(1-n)A - \frac{qK}{n} = 0.$$

Solutions of this quadratic are

$$A = \frac{\frac{K}{n}(1-n) \pm \sqrt{\frac{K^2}{n^2}(1-n)^2 + \frac{4qK}{n}}}{2}.$$

Since the differential equation requires dv/dt to be positive, we much choose the positive radical, in which case

$$v(t) = \left[\frac{K}{n}(1-n) + \sqrt{\frac{K^2}{n^2}(1-n)^2 + \frac{4qK}{n}}\right]t.$$

34. If we set $v = y/x$ or $y = vx$, then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$. Substitution into the differential equation gives

$$v + x\frac{dv}{dx} = f(v) \qquad \Longrightarrow \qquad \frac{1}{f(v)-v}\,dv = \frac{1}{x}\,dx,$$

which is separated.

35. The functions should be positively homogeneous of the same degree (see Section 12.6 for a definition of homogeneous functions).

36. We write $\dfrac{dy}{dx} = \dfrac{x^2-y^2}{xy} = \dfrac{1-(y/x)^2}{y/x}$, a homogeneous differential equation. If we set $v = y/x$, or $y = vx$, then $dy/dx = v + xdv/dx$, and $v + x\dfrac{dv}{dx} = \dfrac{1-v^2}{v}$. This can be separated in the form

$$\frac{1}{x}\,dx = \frac{1}{\frac{1-v^2}{v} - v}\,dv = \frac{v}{1-2v^2}\,dv.$$

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A one-parameter family of solutions is defined implicitly by $-(1/4)\ln|1-2v^2| = \ln|x| + C$. Exponentiation of both sides leads to $2v^2 = (x^4 - D)/x^4$, and when we substitute $v = y/x$, the solution reduces to $x^2(x^2 - 2y^2) = D$.

37. We write $\dfrac{dy}{dx} = \dfrac{2y + \sqrt{x^2+4y^2}}{2x} = \dfrac{2(y/x) + \sqrt{1+4(y/x)^2}}{2}$, a homogeneous differential equation. If we set $v = y/x$, or $y = vx$, then $dy/dx = v + x dv/dx$, and

$$v + x\frac{dv}{dx} = \frac{2v + \sqrt{1+4v^2}}{2} \quad \Longrightarrow \quad \frac{dv}{\sqrt{1+4v^2}} = \frac{dx}{2x},$$

a separated differential equation, with one-parameter family of solutions defined implicitly by

$$\int \frac{1}{\sqrt{1+4v^2}} dv = \int \frac{1}{2x} dx = \frac{1}{2}\ln|x| + C.$$

If we set $v = (1/2)\tan\theta$ and $dv = (1/2)\sec^2\theta\, d\theta$,

$$\frac{1}{2}\ln|x| + C = \int \frac{1}{\sec\theta}\frac{1}{2}\sec^2\theta\, d\theta = \frac{1}{2}\ln|\sec\theta + \tan\theta| = \frac{1}{2}\ln|\sqrt{1+4v^2} + 2v|.$$

When this equation is solved for v in terms of x, the result is

$$v = \frac{D^2x^2 - 1}{4Dx} \qquad \Longrightarrow \qquad y(x) = \frac{D^2x^2-1}{4D}.$$

38. When we write $\dfrac{dy}{dx} = \dfrac{y/x + 1}{y/x - 1}$, the differential equation is clearly homogeneous. We therefore set $v = y/x$, or, $y = vx$, in which case $dy/dx = v + x dv/dx$, and $v + x\dfrac{dv}{dx} = \dfrac{v+1}{v-1}$. This can be separated in the form $\dfrac{v-1}{-v^2+2v+1}\, dv = \dfrac{1}{x}\, dx$. A one-parameter family of solutions of this equation is defined implicitly by $-(1/2)\ln|-v^2+2v+1| = \ln|x| + C$. When this equation is exponentiated, $-v^2 + 2v + 1 = D/x^2$, and substitution of $v = y/x$ gives $x^2 + 2xy - y^2 = D$.

39. We write $\dfrac{dy}{dx} = \dfrac{y + x\cos(y/x)}{x} = \dfrac{y}{x} + \cos(y/x)$, a homogeneous differential equation. If we set $v = y/x$, or $y = vx$, then $dy/dx = v + x dv/dx$, and $v + x\dfrac{dv}{dx} = v + \cos v \implies \sec v\, dv = \dfrac{dx}{x}$, a separated differential equation, with one-parameter family of solutions defined implicitly by $\ln|\sec v + \tan v| = \ln|x| + C$. Exponentiation gives $\sec v + \tan v = Dx$, and therefore $\sec(y/x) + \tan(y/x) = Dx$.

40. We write $\dfrac{dy}{dx} = \dfrac{e^{-y/x} + (y/x)^2}{y/x}$, a homogeneous differential equation. When we set $v = y/x$, or, $y = vx$, then $dy/dx = v + x dv/dx$, and $v + x\dfrac{dv}{dx} = \dfrac{e^{-v}+v^2}{v} \implies ve^v\, dv = \dfrac{1}{x}\, dx$, a separated differential equation with one-parameter family of solutions defined implicitly by $ve^v - e^v = \ln|x| + C$. Substitution of $v = y/x$ leads to $e^{y/x}(y - x) = x\ln|x| + Cx$.

41. We write $\dfrac{dy}{dx} = -\dfrac{x^2y + y^3}{x^3} = -\dfrac{y}{x} - \left(\dfrac{y}{x}\right)^3$, a homogeneous differential equation. If we set $v = y/x$, or $y = vx$, then $dy/dx = v + x dv/dx$, and

$$v + x\frac{dv}{dx} = -v - v^3 \quad \Longrightarrow \quad -\frac{dx}{x} = \frac{dv}{v(2+v^2)} = \frac{1}{2}\left(\frac{1}{v} - \frac{v}{2+v^2}\right) dv.$$

A one-parameter family of solutions of this separated equation is defined implicitly by $-\ln|x| + C = (1/2)\ln|v| - (1/4)\ln|2+v^2|$. When we exponentiate and solve for v^2, the result is $v^2 = 2D/(x^4 - D)$. Setting $v = y/x$ leads to $x^4y^2 = D(2x^2 + y^2)$.

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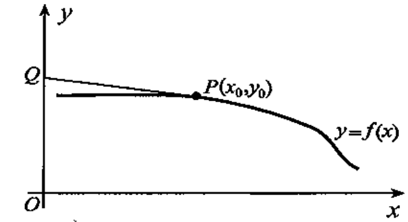
42. Let $P(x_0, y_0)$ be any point on the required curve $y = f(x)$. The equation of the tangent line at (x_0, y_0) is



$$y - y_0 = f'(x_0)(x - x_0).$$

The y-intercept of this line is $y_0 - f'(x_0)x_0$.
Since $\|OQ\|^2 = \|PQ\|^2$,

$$[y_0 - f'(x_0)x_0]^2 = x_0^2 + [f'(x_0)x_0]^2, \text{ or,}$$

$$y_0^2 - 2f'(x_0)x_0y_0 + [f'(x_0)]^2x_0^2 = x_0^2 + [f'(x_0)x_0]^2.$$

Thus, $y_0^2 - x_0^2 = 2x_0y_0f'(x_0)$. Since this must be valid at every point on the curve, we drop the subscripts and set $f'(x) = dy/dx$,

$$y^2 - x^2 = 2xy\frac{dy}{dx} \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}.$$

In this homogeneous differential equation, we set $y = vx$ and $dy/dx = v + xdv/dx$,

$$v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2vx^2} = \frac{v^2 - 1}{2v}.$$

Thus, $x\dfrac{dv}{dx} = \dfrac{v^2 - 1}{2v} - v = -\dfrac{v^2 + 1}{2v} \implies \dfrac{2v}{v^2 + 1}\,dv = -\dfrac{1}{x}\,dx$. A one-parameter family of solutions is defined implicitly by $\ln(v^2 + 1) = -\ln|x| + C$, or, $v^2 + 1 = D/x$. Substitution of $v = y/x$ now gives $x^2 + y^2 = Dx$. Since $(1, 2)$ is on the curve, it follows that $1 + 4 = D$, and the required curve is $y = \sqrt{5x - x^2}$.

43. The differential equation is separable $\sin y\,dy = dx$ with a one-parameter family of solutions defined implicitly by $-\cos y = x + C$.
(a) For $y(0) = \pi/4$, we must set $C = -1/\sqrt{2}$. The solution is then defined implicitly by $\cos y = 1/\sqrt{2} - x$, or explicitly by $y = \mathrm{Cos}^{-1}(1/\sqrt{2} - x)$.
(b) For $y(0) = 7\pi/4$, we again find $C = -1/\sqrt{2}$, and $\cos y = 1/\sqrt{2} - x$. Because $y = 7\pi/4$ is not in the principal value range for the inverse cosine function, we write that $\cos(2\pi - y) = 1/\sqrt{2} - x$, and now take inverse cosines, $2\pi - y = \mathrm{Cos}^{-1}(1/\sqrt{2} - x)$, from which $y = 2\pi - \mathrm{Cos}^{-1}(1/\sqrt{2} - x)$.

44. If $y(t)$ represents the amount of trypsin at any given time, then

$$\frac{dy}{dt} = ky[A - (y - y_0)] = ky(A + y_0 - y),$$

where k is a constant. This equation can be separated,

$$k\,dt = \frac{1}{y(A + y_0 - y)}\,dy = \left(\frac{\dfrac{1}{A + y_0}}{y} + \frac{\dfrac{1}{A + y_0}}{A + y_0 - y}\right)dy.$$

Thus, $\left(\dfrac{1}{y} + \dfrac{1}{A + y_0 - y}\right)dy = k(A + y_0)\,dt$, a one-parameter family of solutions of which is defined implicitly by $\ln y - \ln(A + y_0 - y) = k(A + y_0)t + C$. When this equation is exponentiated and solved for y, the result is

$$y(t) = \frac{D(A + y_0)e^{k(A + y_0)t}}{1 + De^{k(A + y_0)t}}.$$

Since $y(0) = y_0$, it follows that $y_0 = \dfrac{D(A + y_0)}{1 + D}$, from which $D = y_0/A$. Hence,

$$y(t) = \frac{\dfrac{y_0}{A}(A + y_0)e^{k(A + y_0)t}}{1 + \dfrac{y_0}{A}e^{k(A + y_0)t}} = \frac{y_0(A + y_0)}{y_0 + Ae^{-k(A + y_0)t}}.$$

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45. If we set $v = ax + by$, then $\dfrac{dv}{dx} = a + b\dfrac{dy}{dx}$. Substitution of these into the differential equation gives

$$\frac{1}{b}\frac{dv}{dx} - \frac{a}{b} = f(v) \qquad \Longrightarrow \qquad \frac{dv}{bf(v) + a} = dx,$$

a separated differential equation.

46. If we set $v = x + y$, then $dv/dx = 1 + dy/dx$, and $\dfrac{dv}{dx} - 1 = v$. This equation can be separated, $\dfrac{1}{v+1}\,dv = dx$, a one-parameter family of solutions of which is defined implicitly by $\ln|v+1| = x + C$. Exponentiation and substitution of $v = x + y$ leads to $y = De^x - x - 1$.

47. If we set $v = x + y$, then $dv/dx = 1 + dy/dx$, and $\dfrac{dv}{dx} - 1 = v^2$. This equation can be separated, $\dfrac{1}{1+v^2}\,dv = dx$, a one-parameter family of solutions of which is defined implicitly by $\mathrm{Tan}^{-1}v = x + C$. If we take tangents and substitute $v = x + y$, we obtain $x + y = \tan(x + C)$. Thus, $y = -x + \tan(x + C)$.

48. If we set $v = 2x + 3y$, then $dv/dx = 2 + 3dy/dx$, and $\dfrac{1}{3}\dfrac{dv}{dx} - \dfrac{2}{3} = \dfrac{1}{v}$. This equation can be separated, $\dfrac{v}{3+2v}\,dv = dx$, a one-parameter family of solutions of which is defined implicitly by

$$x + C = \int\left(\frac{1}{2} - \frac{3/2}{2v+3}\right)dv = \frac{v}{2} - \frac{3}{4}\ln|2v+3|.$$

Substitution of $v = 2x + 3y$ gives $6y - 3\ln|4x + 6y + 3| = D$.

49. If we set $v = x - y$, then $dv/dx = 1 - dy/dx$, and $1 - \dfrac{dv}{dx} = \sin^2 v$. This equation can be separated, $\sec^2 v\,dv = dx$, a one-parameter family of solutions of which is defined implicitly by $\tan v = x + C$. When we take inverse tangents and set $v = x - y$, the result is $x - y = \mathrm{Tan}^{-1}(x + C) + n\pi$, where n is an integer. Thus, $y = x - \mathrm{Tan}^{-1}(x + C) - n\pi$.

50. When we substitute for k and C, and separate variables,

$$\frac{dN}{N(1 - N/10^6)} = dt \quad \Longrightarrow \quad \frac{10^6 dN}{N(10^6 - N)} = dt \quad \Longrightarrow \quad \left(\frac{1}{N} + \frac{1}{10^6 - N}\right)dN = dt.$$

A one-parameter family of solutions is defined implicitly by

$$\ln|N| - \ln|10^6 - N| = t + D \quad \Longrightarrow \quad \ln\left|\frac{N}{10^6 - N}\right| = t + D \quad \Longrightarrow \quad \frac{N}{10^6 - N} = \pm e^{t+D} = Ee^t,$$

where $E = \pm e^D$. Consequently, $N = (10^6 - N)Ee^t \Longrightarrow N = \dfrac{10^6 Ee^t}{1 + Ee^t} = \dfrac{10^6}{1 + Fe^{-t}}$, where $F = 1/E$. From $N(0) = 100$, we obtain $100 = 10^6/(1 + F) \Longrightarrow F = 9999$. The number of bacteria is therefore $N(t) = 10^6/(1 + 9999e^{-t})$.

51. When we separate variables,

$$\frac{dN}{N(1 - N/C)} = k\,dt \quad \Longrightarrow \quad \frac{C\,dN}{N(C - N)} = k\,dt \quad \Longrightarrow \quad \left(\frac{1}{N} + \frac{1}{C - N}\right)dN = k\,dt.$$

A one-parameter family of solutions is defined implicitly by

$$\ln|N| - \ln|C - N| = kt + D \quad \Longrightarrow \quad \ln\left|\frac{N}{C - N}\right| = kt + D \quad \Longrightarrow \quad \frac{N}{C - N} = \pm e^{kt+D} = Ee^{kt},$$

where $E = \pm e^D$. Consequently, $N = (C - N)Ee^{kt} \Longrightarrow N = \dfrac{CEe^{kt}}{1 + Ee^{kt}} = \dfrac{C}{1 + Fe^{-kt}}$, where $F = 1/E$. From $N(0) = N_0$, we obtain $N_0 = C/(1 + F) \Longrightarrow F = (C - N_0)/N_0$. The population is therefore $N(t) = C/\{1 + [(C - N_0)/N_0]e^{-kt}\}$.

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52. The differential equation is separable, $\dfrac{dw}{aw^{2/3} - bw} = dt$, in which case a one-parameter family of solutions is defined implicitly by

$$\int \frac{1}{aw^{2/3} - bw} dw = t + C.$$

If we set $u = w^{1/3} \Longrightarrow w = u^3$, and $dw = 3u^2\, du$, then

$$t + C = \int \frac{3u^2}{au^2 - bu^3} du = 3\int \frac{1}{a - bu} du = -\frac{3}{b}\ln|a - bu|.$$

Consequently,

$$\ln|a - bu| = -\frac{b}{3}(t + C) \quad \Longrightarrow \quad |a - bu| = e^{-b(t+C)/3} \quad \Longrightarrow \quad u = \frac{a}{b} + De^{-bt/3},$$

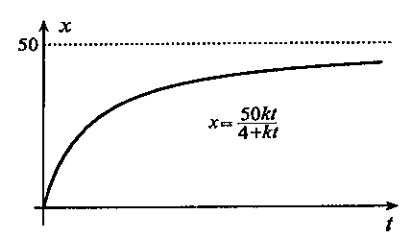
where $D = \pm(1/b)e^{-bC/3}$. Since $u = w^{1/3}$, we find $w = (a/b + De^{-bt/3})^3$. From $w(0) = w_0$, we obtain $w_0 = (a/b + D)^3 \Longrightarrow D = w_0^{1/3} - a/b$. Finally, then

$$w(t) = \left[\frac{a}{b} + \left(w_0^{1/3} - \frac{a}{b}\right)e^{-bt/3}\right]^3 = \left[\frac{a}{b}\left(1 - e^{-bt/3}\right) + w_0^{1/3}e^{-bt/3}\right]^3.$$

53. If $x(t)$ represents the number of grams of dissolved chemical at time t, then

$$\frac{dx}{dt} = k(50 - x)\left(\frac{25}{100} - \frac{x}{200}\right) = \frac{k}{200}(50 - x)^2,$$

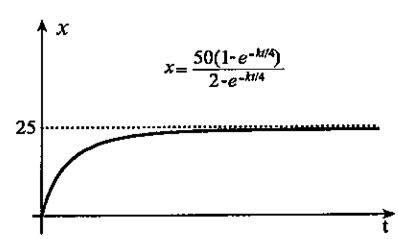
where k is a constant. This equation can be separated, $\dfrac{1}{(50 - x)^2}\, dx = \dfrac{k}{200}\, dt$, and a one-parameter family of solutions is defined implicitly by $\dfrac{1}{50 - x} = \dfrac{kt}{200} + C$. Since $x(0) = 0$, it follows that $1/50 = C$, and $\dfrac{1}{50 - x} = \dfrac{kt}{200} + \dfrac{1}{50}$. When we solve this equation for x, we obtain $x(t) = \dfrac{50kt}{4 + kt}$ g.



(b) In this case

$$\frac{dx}{dt} = k(50 - x)\left(\frac{25}{100} - \frac{x}{100}\right) = \frac{k}{100}(50 - x)(25 - x).$$

Once again we separate variables,
$\dfrac{k}{100}\, dt = \dfrac{1}{(50 - x)(25 - x)}\, dx = \left(\dfrac{-1/25}{50 - x} + \dfrac{1/25}{25 - x}\right) dx.$
Integration gives $\dfrac{kt}{4} + C = \ln(50 - x) - \ln(25 - x)$.
Since $x(0) = 0$, it follows that $C = \ln 50 - \ln 25 = \ln 2$, and $\dfrac{kt}{4} + \ln 2 = \ln\left(\dfrac{50 - x}{25 - x}\right)$. When we solve this equation for x, the result is $x(t) = \dfrac{50(1 - e^{-kt/4})}{2 - e^{-kt/4}}$ g.



(c) In this case

$$\frac{dx}{dt} = k(10 - x)\left(\frac{25}{100} - \frac{x}{100}\right) = \frac{k}{100}(10 - x)(25 - x).$$

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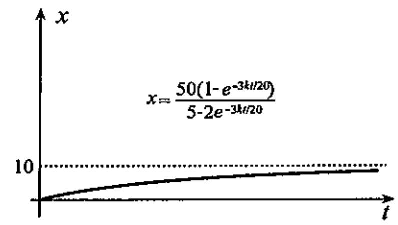
We separate variables again,

$$\frac{k}{100}\,dt = \frac{1}{(10-x)(25-x)}\,dx = \left(\frac{1/15}{10-x} + \frac{-1/15}{25-x}\right)dx,$$

and integrate for $\frac{3kt}{20} + C = -\ln(10-x) + \ln(25-x)$.
Since $x(0) = 0$, it follows that $C = -\ln 10 + \ln 25$
$= \ln(5/2)$, and $\frac{3kt}{20} + \ln\left(\frac{5}{2}\right) = \ln\left(\frac{25-x}{10-x}\right)$.
When this equation is solved for x, the result is
$x(t) = \dfrac{50(1-e^{-3kt/20})}{5-2e^{-3kt/20}}$ g.



54. When H is a constant, the differential becomes

$$\frac{dv}{dt} = \frac{K}{n}\left[\frac{v^2-(D-\gamma-H)v-DH}{v(v-D)}\right] \qquad \Longrightarrow \qquad \frac{v(v-D)}{v^2-(D-\gamma-H)v-DH}\,dv = \frac{K}{n}dt.$$

a separated differential equation. A one-parameter family of solutions is defined implicitly by

$$\int\left[1+\frac{-(H+\gamma)v+DH}{v^2-(D-\gamma-H)v-DH}\right]dv = \frac{K}{n}t + C.$$

The roots of the quadratic $v^2-(D-\gamma-H)v-DH = 0$ are

$$v = \frac{(D-\gamma-H) \pm \sqrt{(D-\gamma-H)^2+4DH}}{2}.$$

Let us denote them by r_1 and r_2 where r_1 uses the positive radical and r_2 the negative one. Then

$$\frac{Kt}{n} + C = \int\left[1+\frac{DH-(H+\gamma)v}{(v-r_1)(v-r_2)}\right]dv = v + \int\left[\frac{\frac{DH-(H+\gamma)r_1}{r_1-r_2}}{v-r_1} + \frac{\frac{DH-(H+\gamma)r_2}{r_2-r_1}}{v-r_2}\right]dv$$

$$= v + \frac{1}{r_1-r_2}\{[DH-(H+\gamma)r_1]\ln|v-r_1| - [DH-(H+\gamma)r_2]\ln|v-r_2|\}.$$

For $v(0) = 0$,

$$\frac{1}{r_1-r_2}\{[DH-(H+\gamma)r_1]\ln r_1 - [DH-(H+\gamma)r_2]\ln r_2] = C.$$

Thus, $v(t)$ is defined implicitly by

$$\begin{aligned}\frac{Kt}{n} &= v + \frac{1}{r_1-r_2}\{[DH-(H+\gamma)r_1]\ln|v-r_1| - [DH-(H+\gamma)r_2]\ln|v-r_2|\} \\ &\quad - \frac{1}{r_1-r_2}\{[DH-(H+\gamma)r_1]\ln r_1 - [DH-(H+\gamma)r_2]\ln r_2] \\ &= v + \frac{1}{r_1-r_2}\left\{[DH-(H+\gamma)r_1]\ln\left|\frac{v-r_1}{r_1}\right| - [DH-(H+\gamma)r_2]\ln\left|\frac{v-r_2}{r_2}\right|\right\}.\end{aligned}$$

55. Suppose that the snow started falling T hours before 12:00. We assume first that snow falls at a constant rate of r metres per hour. The depth of snow on the ground at time t (if we choose $t = 0$ at 12:00) is $d = r(t+T)$. Next we assume that the speed v of the plow (in kilometres per hour) is inversely proportional to the depth of snow (in metres),

$$v = \frac{k}{d} = \frac{k}{r(t+T)} = \frac{R}{t+T}, \quad R = \frac{k}{r}.$$

Since $v = dx/dt$ (where x measures distance travelled), integration of this equation gives $x(t) = R\ln(t+T) + C$. We now use the conditions $x(0) = 0$, $x(1) = 2$, and $x(2) = 3$:

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$$0 = R\ln T + C, \quad 2 = R\ln(1+T) + C, \quad 3 = R\ln(2+T) + C.$$

When each of these is solved for C and results equated,

$$-R\ln T = 2 - R\ln(1+T), \quad -R\ln T = 3 - R\ln(2+T).$$

If these are solved for R and results equated, the equation $T^2 + T - 1 = 0$ is obtained. Solutions are $T = (-1 \pm \sqrt{5})/2$. Since T must be positive, $T = (\sqrt{5} - 1)/2$ hours, or 37 minutes. Snow therefore started falling at 11:23.

56. If $S(t)$ represents the amount of drug in the dog as a function of time t, then $\dfrac{dS}{dt} = kS$, where $k < 0$ is a constant. Separation of this equation gives $\dfrac{1}{S}\,dS = k\,dt$, a one-parameter family of solutions of which is defined by $\ln S = kt + C$. Exponentiation gives $S = De^{kt}$. If S_0 represents the amount injected at time $t = 0$, then $S_0 = D$, and $S = S_0e^{kt}$. Since $S = S_0/2$ when $t = 5$, it follows that $S_0/2 = S_0e^{5k}$, and this implies that $k = -(1/5)\ln 2$. At the end of the one hour operation, the amount of drug in the body must be 400 mg, and therefore $400 = S_0e^{k}$. Thus, $S_0 = 400e^{-k} = 459.5$ mg.

57. It is a change of variable of integration $y = y(x)$.

58. We can separate this equation, $\left(\dfrac{y^6 - 1}{y^4}\right) dy = (x^2 + 2)\,dx$, and therefore a one-parameter family of solutions is defined implicitly by $\dfrac{y^3}{3} + \dfrac{1}{3y^3} = \dfrac{x^3}{3} + 2x + C$. For $y(1) = 1$, we must have $1/3 + 1/3 = 1/3 + 2 + C$. Thus, $C = -5/3$, and $\dfrac{y^3}{3} + \dfrac{1}{3y^3} = \dfrac{x^3}{3} + 2x - \dfrac{5}{3}$. Multiplication by $3y^3$ gives $y^3(x^3 + 6x - 5) = 1 + y^6$.

59. (a) If $x(t)$ represents the amount of C in the mixture at time t, then

$$\frac{dx}{dt} = k\left(20 - \frac{2x}{3}\right)\left(10 - \frac{x}{3}\right) = \frac{2k}{9}(30 - x)^2.$$

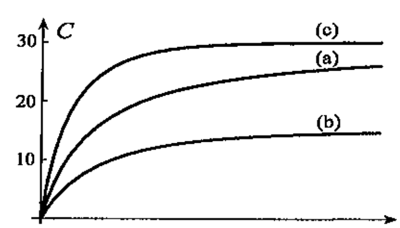
We can separate this equation, $\dfrac{dx}{(30-x)^2} = \dfrac{2k\,dt}{9} \implies \dfrac{1}{30 - x} = \dfrac{2kt}{9} + C$. For $x(0) = 0$, we must have $1/30 = C$, and therefore $\dfrac{1}{30 - x} = \dfrac{2kt}{9} + \dfrac{1}{30} \implies x(t) = \dfrac{600kt}{20kt + 3}$ g.

(b) In this case the differential equation is $\dfrac{dx}{dt} = k\left(20 - \dfrac{2x}{3}\right)\left(5 - \dfrac{x}{3}\right) = \dfrac{2k}{9}(30 - x)(15 - x)$. It is separable, $\dfrac{2k\,dt}{9} = \dfrac{dx}{(30-x)(15-x)} = \dfrac{1}{15}\left(\dfrac{1}{15 - x} - \dfrac{1}{30 - x}\right) dx$, and a one-parameter family of solutions is defined implicitly by $10kt/3 + C = \ln|30 - x| - \ln|15 - x|$. Since x cannot exceed 15, we may drop absolute values. The condition $x(0) = 0$ implies that $C = \ln 30 - \ln 15 = \ln 2$. Hence, $10kt/3 + \ln 2 = \ln(30 - x) - \ln(15 - x)$. When we exponentiate and solve for x, the result is $x(t) = \dfrac{30(1 - e^{-10kt/3})}{2 - e^{-10kt/3}}$ g.

(c) In this case the differential equation is

$$\frac{dx}{dt} = k\left(20 - \frac{2x}{3}\right)\left(20 - \frac{x}{3}\right) = \frac{2k}{9}(30 - x)(60 - x).$$

The equation is separable and the solution is similar to that in part (b). The result is $x(t) = \dfrac{60(1 - e^{-20kt/3})}{2 - e^{-20kt/3}}$ g. Graphs of all three functions are shown to the right.



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60. If $x(t)$ and $y(t)$ are x and y coordinates of the bird as functions of time t, then

$$\frac{dx}{dt} = -V\cos\theta, \quad \frac{dy}{dt} = v - V\sin\theta.$$

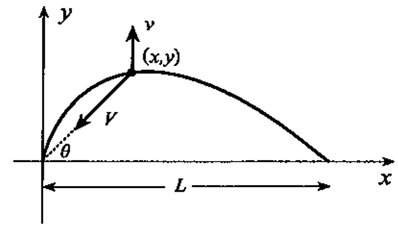
Division of these gives

$$\frac{dy}{dx} = \frac{v - V\sin\theta}{-V\cos\theta}.$$

Since $\tan\theta = y/x$ it follows that

$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}}, \qquad \cos\theta = \frac{x}{\sqrt{x^2+y^2}},$$

and therefore



$$\frac{dy}{dx} = \frac{v - \dfrac{Vy}{\sqrt{x^2+y^2}}}{\dfrac{-Vx}{\sqrt{x^2+y^2}}} = \frac{Vy - v\sqrt{x^2+y^2}}{Vx}.$$

This is a homogeneous differential equation so that we set $y = px$ and $dy/dx = p + xdp/dx$,

$$p + x\frac{dp}{dx} = \frac{Vpx - v\sqrt{x^2+p^2x^2}}{Vx} = p - \frac{v\sqrt{1+p^2}}{V}.$$

We can now separate, $\dfrac{1}{\sqrt{1+p^2}}\,dp = -\dfrac{v}{Vx}\,dx$, and a one-parameter family of solutions is defined implicitly by $\displaystyle\int \frac{1}{\sqrt{1+p^2}}\,dp = -\frac{v}{V}\ln x + C$. If we set $p = \tan\phi$ and $dp = \sec^2\phi\,d\phi$, then

$$C - \frac{v}{V}\ln x = \int \frac{\sec^2\phi}{\sec\phi}d\phi = \ln|\sec\phi + \tan\phi| = \ln|\sqrt{1+p^2} + p|.$$

When we exponentiate, $p + \sqrt{1+p^2} = Dx^{-v/V} \implies \sqrt{1+p^2} = Dx^{-v/V} - p$. Squaring gives $1 + p^2 = D^2x^{-2v/V} - 2pDx^{-v/V} + p^2$. This equation can be solved for p, and therefore

$$y = px = \frac{x}{2}\left(Dx^{-v/V} - \frac{1}{D}x^{v/V}\right).$$

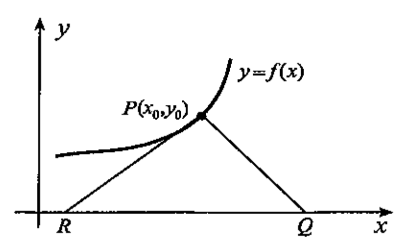
Since $y(L) = 0$, it follows that $0 = \dfrac{L}{2}\left(DL^{-v/V} - \dfrac{1}{D}L^{v/V}\right)$, and therefore $D = L^{v/V}$. The curve followed by the bird is

$$y = \frac{x}{2}\left(L^{v/V}x^{-v/V} - L^{-v/V}x^{v/V}\right) = \frac{x}{2}\left[\left(\frac{L}{x}\right)^{v/V} - \left(\frac{x}{L}\right)^{v/V}\right] = \frac{L}{2}\left[\left(\frac{x}{L}\right)^{1-v/V} - \left(\frac{x}{L}\right)^{1+v/V}\right].$$

61. Equations of the tangent and normal lines at any point $P(x_0, y_0)$ on the required curve $y = f(x)$ are $y - y_0 = f'(x_0)(x - x_0)$ and $y - y_0 = -\dfrac{1}{f'(x_0)}(x - x_0)$. The x-coordinates of Q and R are

$$x_Q = x_0 + f'(x_0)y_0, \quad x_R = x_0 - \frac{y_0}{f'(x_0)}.$$

Since the area of ΔPQR is equal to the slope of the curve,



$$f'(x_0) = \frac{1}{2}\|PQ\|\|PR\| = \frac{1}{2}\sqrt{[f'(x_0)y_0]^2 + y_0{}^2}\sqrt{\left[\frac{y_0}{f'(x_0)}\right]^2 + y_0{}^2}.$$

Since $P(x_0, y_0)$ is any point on the curve, we may drop subscripts,

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$$2f'(x) = y^2\sqrt{[f'(x)]^2+1}\sqrt{\frac{1}{[f'(x)]^2}+1} = \frac{y^2\{[f'(x)]^2+1\}}{|f'(x)|}.$$

Since $f'(x)$ must be nonnegative, we may write that

$$2[f'(x)]^2 = y^2\{[f'(x)]^2+1\} \quad \Longrightarrow \quad \frac{dy}{dx} = \sqrt{\frac{y^2}{2-y^2}} = \frac{\pm y}{\sqrt{2-y^2}}.$$

If we choose the positive sign so that dy/dx will be positive at $(1,1)$, then

$$\frac{\sqrt{2-y^2}}{y}dy = dx \quad \Longrightarrow \quad x + C = \int \frac{\sqrt{2-y^2}}{y}dy.$$

We set $y = \sqrt{2}\sin\theta$ and $dy = \sqrt{2}\cos\theta\,d\theta$,

$$\begin{aligned} x + C &= \int \frac{\sqrt{2}\cos\theta}{\sqrt{2}\sin\theta}\sqrt{2}\cos\theta\,d\theta = \sqrt{2}\int(\csc\theta - \sin\theta)\,d\theta \\ &= \sqrt{2}[\ln|\csc\theta - \cot\theta| + \cos\theta] = \sqrt{2}\ln\left|\frac{\sqrt{2}}{y} - \frac{\sqrt{2-y^2}}{y}\right| + \sqrt{2-y^2}. \end{aligned}$$

For $y(1) = 1$, we must have $1 + C = \sqrt{2}\ln(\sqrt{2}-1) + 1$, and therefore the curve is defined implicitly by

$$x = \sqrt{2}\ln\left|\frac{\sqrt{2}-\sqrt{2-y^2}}{y}\right| + \sqrt{2-y^2} - \sqrt{2}\ln(\sqrt{2}-1).$$

EXERCISES 15.3

1. An integrating factor is $e^{\int 2x\,dx} = e^{x^2}$. When the differential equation is multiplied by e^{x^2},

$$e^{x^2}\frac{dy}{dx} + 2xye^{x^2} = 4xe^{x^2} \qquad \Longrightarrow \qquad \frac{d}{dx}(ye^{x^2}) = 4xe^{x^2}.$$

Integration now gives $ye^{x^2} = 2e^{x^2} + C \implies y = 2 + Ce^{-x^2}$.

2. An integrating factor is $e^{\int 2/x\,dx} = e^{2\ln|x|} = x^2$. When the differential equation is multiplied by x^2,

$$x^2\frac{dy}{dx} + 2xy = 6x^5 \qquad \Longrightarrow \qquad \frac{d}{dx}(yx^2) = 6x^5.$$

Integration now gives $yx^2 = x^6 + C \implies y = x^4 + C/x^2$.

3. If we write $\dfrac{dy}{dx} + 2y = x$, an integrating factor is $e^{\int 2\,dx} = e^{2x}$. When we multiply the differential equation by e^{2x},

$$e^{2x}\frac{dy}{dx} + 2ye^{2x} = xe^{2x} \qquad \Longrightarrow \qquad \frac{d}{dx}(ye^{2x}) = xe^{2x}.$$

Integration now gives $ye^{2x} = \dfrac{x}{2}e^{2x} - \dfrac{1}{4}e^{2x} + C \implies y = x/2 - 1/4 + Ce^{-2x}$.

4. An integrating factor is $e^{\int \cot x\,dx} = e^{\ln|\sin x|} = |\sin x|$. For either $\sin x < 0$ or $\sin x > 0$, multiplication of the differential equation by $|\sin x|$ gives

$$\sin x\frac{dy}{dx} + y\cos x = 5\sin x e^{\cos x} \qquad \Longrightarrow \qquad \frac{d}{dx}(y\sin x) = 5\sin x\,e^{\cos x}.$$

Integration gives $y\sin x = -5e^{\cos x} + C \implies y = \csc x(C - 5e^{\cos x})$.

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5. If we write $\dfrac{dy}{dx}+\dfrac{2xy}{x^2+1}=\dfrac{-x^2}{x^2+1}$, an integrating factor is $e^{\int 2x/(x^2+1)\,dx}=e^{\ln(x^2+1)}=x^2+1$. When we multiply the differential equation by x^2+1,

$$(x^2+1)\frac{dy}{dx}+2xy=-x^2 \quad\Longrightarrow\quad \frac{d}{dx}[y(x^2+1)]=-x^2.$$

Integration now gives $y(x^2+1)=-x^3/3+C \implies y=(3C-x^3)/(3x^2+3)$.

6. If we write $\dfrac{dy}{dx}-\dfrac{2}{x+1}y=2$, an integrating factor is $e^{\int -2/(x+1)\,dx}=e^{-2\ln|x+1|}=1/(x+1)^2$. When we multiply the differential equation by $1/(x+1)^2$,

$$\frac{1}{(x+1)^2}\frac{dy}{dx}-\frac{2}{(x+1)^3}y=\frac{2}{(x+1)^2} \quad\Longrightarrow\quad \frac{d}{dx}\left[\frac{y}{(x+1)^2}\right]=\frac{2}{(x+1)^2}.$$

Integration now gives $\dfrac{y}{(x+1)^2}=\dfrac{-2}{x+1}+C \implies y=-2(x+1)+C(x+1)^2$.

7. The differential equation can be expressed in the form $\dfrac{d}{dx}\left(\dfrac{y}{x}\right)=\dfrac{1}{x^3}$, so that $\dfrac{y}{x}=-\dfrac{1}{2x^2}+C$, and $y=\dfrac{-1}{2x}+Cx$.

8. Since $dy/dx-y=e^{2x}$, an integrating factor is $e^{\int -dx}=e^{-x}$. When we multiply the differential equation by this integrating factor,

$$e^{-x}\frac{dy}{dx}-ye^{-x}=e^{x} \quad\Longrightarrow\quad \frac{d}{dx}(ye^{-x})=e^{x}.$$

Integration gives $ye^{-x}=e^x+C \implies y=e^{2x}+Ce^x$.

9. An integrating factor is $e^{\int dx}=e^x$. When we multiply the differential equation be e^x,

$$e^x\frac{dy}{dx}+ye^x=2e^x\cos x \quad\Longrightarrow\quad \frac{d}{dx}(ye^x)=2e^x\cos x.$$

Integration now gives

$$ye^x=2\int e^x\cos x\,dx=e^x(\cos x+\sin x)+C \quad\Longrightarrow\quad y=\cos x+\sin x+Ce^{-x}.$$

10. Since $\dfrac{dy}{dx}+\left(\dfrac{2-3x^2}{x^3}\right)y=1$, an integrating factor is $e^{\int \frac{2-3x^2}{x^3}dx}=e^{-1/x^2-3\ln|x|}=\dfrac{e^{-1/x^2}}{|x|^3}$. For either $x<0$ or $x>0$, multiplication of the differential equation by this factor gives

$$\frac{1}{x^3}e^{-1/x^2}\frac{dy}{dx}+\left(\frac{2-3x^2}{x^6}\right)e^{-1/x^2}y=\frac{1}{x^3}e^{-1/x^2} \quad\Longrightarrow\quad \frac{d}{dx}\left(\frac{y}{x^3}e^{-1/x^2}\right)=\frac{1}{x^3}e^{-1/x^2}.$$

Integration gives $\dfrac{y}{x^3}e^{-1/x^2}=\dfrac{1}{2}e^{-1/x^2}+C \implies y=x^3/2+Cx^3e^{1/x^2}$.

11. An integrating factor is $e^{\int 1/(x\ln x)dx}=e^{\ln(\ln x)}=\ln x$. When we multiply the differential equation by $\ln x$,

$$\ln x\frac{dy}{dx}+\frac{1}{x}y=x^2\ln x \quad\Longrightarrow\quad \frac{d}{dx}(y\ln x)=x^2\ln x.$$

Integration now gives

$$y\ln x=\int x^2\ln x\,dx=\frac{x^3}{3}\ln x-\frac{x^3}{9}+C \quad\Longrightarrow\quad y=\frac{x^3}{3}+\frac{9C-x^3}{9\ln x}.$$

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12. If we write $\dfrac{dy}{dx} - (2\cot 2x)y = 1 - 2x\cot 2x - 2\csc 2x$, an integrating factor is $e^{\int -2\cot 2x\,dx} = e^{-\ln|\sin 2x|} = |\csc 2x|$. For either $\csc 2x < 0$ or $\csc 2x > 0$, multiplication of the differential equation by $|\csc 2x|$ leads to

$$\csc 2x\frac{dy}{dx} - 2y\cot 2x\csc 2x = \csc 2x - 2x\cot 2x\ \csc 2x - 2\csc^2 2x,$$

or,

$$\frac{d}{dx}(y\csc 2x) = \csc 2x - 2x\cot 2x\ \csc 2x - 2\csc^2 2x.$$

Integration gives $y\csc 2x = x\csc 2x + \cot 2x + C \implies y = x + \cos 2x + C\sin 2x$.

13. An integrating factor is $e^{\int 3x^2\,dx} = e^{x^3}$. When we multiply the differential equation by e^{x^3},

$$e^{x^3}\frac{dy}{dx} + 3x^2e^{x^3}y = x^2e^{x^3} \qquad \Longrightarrow \qquad \frac{d}{dx}(ye^{x^3}) = x^2e^{x^3}.$$

Integration now gives $ye^{x^3} = \frac{1}{3}e^{x^3} + C \implies y = \frac{1}{3} + Ce^{-x^3}$. For $y(1) = 2$, we must have $2 = 1/3 + Ce^{-1}$. Hence, $C = 5e/3$, and $y = (1 + 5e^{1-x^3})/3$.

14. Since $dy/dx + y = e^x \sin x$, an integrating factor is $e^{\int dx} = e^x$. Multiplication of the differential equation by this factor gives

$$e^x\frac{dy}{dx} + ye^x = e^{2x}\sin x \qquad \Longrightarrow \qquad \frac{d}{dx}(ye^x) = e^{2x}\sin x.$$

Integration now yields $ye^x = \frac{1}{5}(2e^{2x}\sin x - e^{2x}\cos x) + C \implies y = e^x(2\sin x - \cos x)/5 + Ce^{-x}$. For $y(0) = -1$, we must have $-1 = -1/5 + C$. Thus, $y = e^x(2\sin x - \cos x)/5 - (4/5)e^{-x}$.

15. An integrating factor is $e^{\int x^3/(x^4+1)\,dx} = e^{(1/4)\ln(x^4+1)} = (x^4+1)^{1/4}$. When we multiply the differential equation by $(x^4+1)^{1/4}$,

$$(x^4+1)^{1/4}\frac{dy}{dx} + \frac{x^3y}{(x^4+1)^{3/4}} = x^7(x^4+1)^{1/4} \qquad \Longrightarrow \qquad \frac{d}{dx}[y(x^4+1)^{1/4}] = x^7(x^4+1)^{1/4}.$$

Integration now gives $y(x^4+1)^{1/4} = \int x^7(x^4+1)^{1/4}\,dx$. If we set $u = x^4 + 1$ and $du = 4x^3\,dx$, then

$$\begin{aligned} y(x^4+1)^{1/4} &= \int (u-1)u^{1/4}\left(\frac{du}{4}\right) = \frac{1}{4}\left(\frac{4}{9}u^{9/4} - \frac{4}{5}u^{5/4}\right) + C \\ &= \frac{1}{9}(x^4+1)^{9/4} - \frac{1}{5}(x^4+1)^{5/4} + C. \end{aligned}$$

For $y(0) = 1$, we must have $1 = 1/9 - 1/5 + C \implies C = 49/45$. Hence, $y = (x^4+1)^2/9 - (x^4+1)/5 + (49/45)(x^4+1)^{-1/4}$.

16. If we write $dx/dy + (1/y)x = y^2$, the differential equation is linear in $x = x(y)$. An integrating factor is $e^{\int (1/y)\,dy} = e^{\ln|y|} = |y|$. For either $y < 0$ or $y > 0$, multiplication by $|y|$ yields

$$y\frac{dx}{dy} + x = y^3 \qquad \Longrightarrow \qquad \frac{d}{dy}(yx) = y^3.$$

Integration gives $yx = \dfrac{y^4}{4} + C$, an implicit definition for solutions.

17. If we set $z = y^{1-n}$, then $dz/dx = (1-n)y^{-n}dy/dx$. If we divide all terms in the differential equation by y^n, and substitute

$$\frac{1}{y^n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x) \qquad \Longrightarrow \qquad \frac{1}{1-n}\frac{dz}{dx} + P(x)z = Q(x).$$

Multiplication by $1 - n$ gives a linear first order equation.

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18. If we set $z = 1/y$, then $dz/dx = (-1/y^2)dy/dx$, and $-y^2\dfrac{dz}{dx} + y = y^2e^x$. Division by $-y^2$ gives $\dfrac{dz}{dx} - \dfrac{1}{y} = -e^x \Longrightarrow \dfrac{dz}{dx} - z = -e^x$. An integrating factor for this equation is $e^{\int -dx} = e^{-x}$. When we multiply the differential equation by this factor,

$$e^{-x}\frac{dz}{dx} - ze^{-x} = -1 \qquad \Longrightarrow \qquad \frac{d}{dx}(ze^{-x}) = -1.$$

Integration now yields

$$ze^{-x} = -x + C \quad \Longrightarrow \quad z = (C-x)e^x \quad \Longrightarrow \quad \frac{1}{y} = (C-x)e^x \quad \Longrightarrow \quad y = \frac{e^{-x}}{C-x}.$$

19. If we set $z = 1/y$, then $dz/dx = (-1/y^2)dy/dx$, and $-y^2\dfrac{dz}{dx} + \dfrac{y}{x} = \dfrac{y^2}{x^2}$. Division by $-y^2$ gives $\dfrac{dz}{dx} - \dfrac{1}{xy} = -\dfrac{1}{x^2} \Longrightarrow \dfrac{dz}{dx} - \dfrac{z}{x} = -\dfrac{1}{x^2}$. An integrating factor for this equation is $e^{\int -(1/x)\,dx} = 1/|x|$. For $x > 0$ or $x < 0$, multiplication of the differential equation by this factor gives

$$\frac{1}{x}\frac{dz}{dx} - \frac{z}{x^2} = -\frac{1}{x^3} \qquad \Longrightarrow \qquad \frac{d}{dx}\left(\frac{z}{x}\right) = -\frac{1}{x^3}.$$

Integration now yields

$$\frac{z}{x} = \frac{1}{2x^2} + C \Longrightarrow z = \frac{1}{2x} + Cx \quad \Longrightarrow \quad \frac{1}{y} = \frac{1}{2x} + Cx \Longrightarrow y = \frac{2x}{2Cx^2+1}.$$

20. If we set $z = 1/y$, then $dz/dx = -(1/y^2)dy/dx$, and $-y^2\dfrac{dz}{dx} - y = -(x^2+2x)y^2$. Division by $-y^2$ gives $\dfrac{dz}{dx} + \dfrac{1}{y} = x^2 + 2x \Longrightarrow \dfrac{dz}{dx} + z = x^2 + 2x$. An integrating factor for this equation is $e^{\int dx} = e^x$. Multiplication by this factor now yields

$$e^x\frac{dz}{dx} + ze^x = (x^2+2x)e^x \qquad \Longrightarrow \qquad \frac{d}{dx}(ze^x) = (x^2+2x)e^x.$$

This can be integrated to give

$$ze^x = \int (x^2+2x)e^x\,dx = x^2e^x + C \quad \Longrightarrow \quad z = \frac{1}{y} = x^2 + Ce^{-x} \quad \Longrightarrow \quad y = \frac{1}{x^2+Ce^{-x}}.$$

21. In the form $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2y^5$, we see that the differential equation is Bernoulli. We set $z = y^{-4}$ and $dz/dx = -4y^{-5}dy/dx$. Substitution gives $-\dfrac{1}{4}y^5\dfrac{dz}{dx} + \dfrac{y}{x} = x^2y^5$. Multiplication by $-4y^{-5}$ leads to

$$\frac{dz}{dx} - \frac{4}{xy^4} = -4x^2 \quad \Longrightarrow \quad \frac{dz}{dx} - \frac{4}{x}z = -4x^2.$$

An integrating factor for this linear equation is $e^{\int -4/x\,dx} = e^{-4\ln|x|} = 1/x^4$. Multiplication by this factor now yields

$$\frac{1}{x^4}\frac{dz}{dx} - \frac{4}{x^5}z = -\frac{4}{x^2} \qquad \Longrightarrow \qquad \frac{d}{dx}\left(\frac{z}{x^4}\right) = -\frac{4}{x^2}.$$

Integration gives

$$\frac{z}{x^4} = \frac{4}{x} + C \quad \Longrightarrow \quad z = 4x^3 + Cx^4.$$

When we set $z = 1/y^4$, we obtain $\dfrac{1}{y^4} = 4x^3 + Cx^4 \Longrightarrow y = \dfrac{\pm 1}{(4x^3+Cx^4)^{1/4}}$.

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22. If we set $z = y^{-3}$, then $dz/dx = -(3/y^4)dy/dx$, and $-\dfrac{y^4}{3}\dfrac{dz}{dx} + y\tan x = y^4 \sin x$. Multiplication by $-3y^{-4}$ gives

$$\frac{dz}{dx} - \frac{3\tan x}{y^3} = -3\sin x \qquad \Longrightarrow \qquad \frac{dz}{dx} - (3\tan x)z = -3\sin x.$$

An integrating factor for this equation is $e^{\int -3\tan x\,dx} = e^{3\ln|\cos x|} = |\cos x|^3$. For either $\cos x < 0$ or $\cos x > 0$, multiplication by $|\cos x|^3$ gives

$$\cos^3 x\frac{dz}{dx} - 3z\cos^2 x\sin x = -3\cos^3 x\sin x \qquad \Longrightarrow \qquad \frac{d}{dx}(z\cos^3 x) = -3\sin x\,\cos^3 x.$$

Integration now yields

$$z\cos^3 x = \frac{3}{4}\cos^4 x + C \quad \Longrightarrow \quad z = \frac{3}{4}\cos x + C\sec^3 x.$$

When we replace z with $1/y^3$,

$$\frac{1}{y^3} = \frac{3}{4}\cos x + C\sec^3 x \quad \Longrightarrow \quad y = \left(\frac{3}{4}\cos x + C\sec^3 x\right)^{-1/3}.$$

23. If glucose is added at rate $R(t)$, the differential equation becomes

$$\frac{dA}{dt} = R(t) - kA \qquad \Longrightarrow \qquad \frac{dA}{dt} + kA = R(t).$$

An integrating factor is $e^{\int k\,dt} = e^{kt}$. When we multiply the differential equation by e^{kt},

$$e^{kt}\frac{dA}{dt} + ke^{kt}A = R(t)e^{kt} \qquad \Longrightarrow \qquad \frac{d}{dt}(Ae^{kt}) = R(t)e^{kt}.$$

Integration now gives

$$Ae^{kt} = \int R(t)e^{kt}\,dt + C \quad \Longrightarrow \quad A(t) = e^{-kt}\int R(t)e^{kt}\,dt + Ce^{-kt}.$$

In order to incorporate the initial condition $A(0) = A_0$, we rewrite the antiderivative as a definite integral with a variable upper limit, $A(t) = e^{-kt}\displaystyle\int_0^t R(u)e^{ku}\,du + Ce^{-kt}$. The initial condition now implies that $A_0 = C$, and therefore $A(t) = \displaystyle\int_0^t R(u)e^{k(u-t)}\,du + A_0e^{-kt}$.

24. If $S(t)$ represents the number of grams of sugar in the tank as a function of time t, then dS/dt, the rate of change of S with respect to t, must be the rate at which sugar is added less the rate at which it is removed. The rate at which sugar is added is 2 g/min. Since the concentration of sugar at time t is $S/(10^5 + 100t)$ g/mL, the rate at which sugar is removed at time t is $100S/(10^5 + 100t)$ g/min. Thus,

$$\frac{dS}{dt} = 2 - \frac{100S}{10^5 + 100t}, \qquad \Longrightarrow \qquad \frac{dS}{dt} + \frac{S}{1000 + t} = 2.$$

An integrating factor for this equation is $e^{\int 1/(1000+t)\,dt} = e^{\ln|1000+t|} = 1000 + t$. Multiplication of the differential equation by $1000 + t$ gives

$$(1000+t)\frac{dS}{dt} + S = 2(1000+t) \quad \Longrightarrow \quad \frac{d}{dt}[(1000+t)S] = 2(1000+t) \quad \Longrightarrow \quad (1000+t)S = (1000+t)^2 + C.$$

Since $S(0) = 4000$, it follows that $(1000)(4000) = (1000)^2 + C$. Thus, $C = 3\times 10^6$, and

$$S = 1000 + t + \frac{3\times 10^6}{1000+t}\ \text{g}.$$

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25. If $S(t)$ represents the number of grams of sugar in the tank as a function of time t, then dS/dt, the rate of change of S with respect to t, must be the rate at which sugar is added less the rate at which it is removed. The rate at which sugar is added is 2 g/min. Since the concentration of sugar at time t is $S/(10^5 - 100t)$ g/mL, the rate at which sugar is removed at time t is $300S/(10^5 - 100t)$ g/min. Thus,

$$\frac{dS}{dt} = 2 - \frac{300S}{10^5 - 100t}, \qquad \Longrightarrow \qquad \frac{dS}{dt} + \frac{3S}{1000 - t} = 2.$$

An integrating factor for this equation is $e^{\int 3/(1000-t)\,dt} = e^{-3\ln|1000-t|} = 1/(1000-t)^3$. Multiplication by $1/(1000-t)^3$ gives

$$\frac{1}{(1000-t)^3}\frac{dS}{dt} + \frac{3S}{(1000-t)^4} = \frac{2}{(1000-t)^3} \quad \Longrightarrow \quad \frac{d}{dt}\left[\frac{S}{(1000-t)^3}\right] = \frac{2}{(1000-t)^3}.$$

Integration now yields

$$\frac{S}{(1000-t)^3} = \frac{1}{(1000-t)^2} + C \quad \Longrightarrow \quad S(t) = 1000 - t + C(1000-t)^3.$$

Since $S(0) = 4000$, it follows that $4000 = 1000 + C(1000)^3$. Thus, $C = 3 \times 10^{-6}$, and $S = 1000 - t + 3 \times 10^{-6}(1000-t)^3$ g. This solution is valid only until the tank empties $0 \le t \le 1000$.

26. If $S(t)$ represents the number of grams of salt in the tank as a function of time t, then dS/dt, the rate of change of S with respect to t, must be the rate at which salt is added less the rate at which it is removed. The rate at which salt is added is 0.2 g/s. Since the concentration of salt at time t is $S/(10^6 + 5t)$ g/mL, the rate at which salt is removed at time t is $5S/(10^6 + 5t)$ g/s. Thus,

$$\frac{dS}{dt} = \frac{1}{5} - \frac{5S}{10^6 + 5t} \quad \Longrightarrow \quad \frac{dS}{dt} + \frac{5S}{10^6 + 5t} = \frac{1}{5}.$$

An integrating factor for this equation is $e^{\int 5/(10^6+5t)\,dt} = e^{\ln|10^6+5t|} = 10^6 + 5t$. Multiplication by $10^6 + 5t$ gives

$$(10^6 + 5t)\frac{dS}{dt} + 5S = \frac{1}{5}(10^6 + 5t) \quad \Longrightarrow \quad \frac{d}{dt}\left[(10^6 + 5t)S\right] = \frac{1}{5}(10^6 + 5t).$$

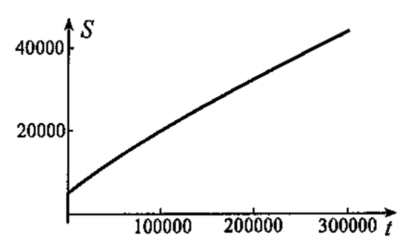
Integration now yields

$$(10^6 + 5t)S = \frac{1}{50}(10^6 + 5t)^2 + C \quad \Longrightarrow \quad S(t) = \frac{1}{50}(10^6 + 5t) + \frac{C}{10^6 + 5t}.$$

Since $S(0) = 5000$, it follows that
$5000 = \dfrac{10^6}{50} + \dfrac{C}{10^6}$. This gives
$C = -15 \times 10^9$, and therefore

$$S(t) = \frac{1}{50}(10^6 + 5t) - \frac{15 \times 10^9}{10^6 + 5t} \text{ g}.$$

A graph is shown to the right; it is asymptotic to the line $S = (10^6 + 5t)/50$.



27. If $S(t)$ represents the number of grams of salt in the tank as a function of time t, then dS/dt, the rate of change of S with respect to t, must be the rate at which salt is added less the rate at which it is removed. The rate at which salt is added is 0.2 g/s. Since the concentration of salt at time t is $S/10^6$ g/mL, the rate at which salt is removed at time t is $10S/10^6$ g/s. Thus,

$$\frac{dS}{dt} = \frac{1}{5} - \frac{10S}{10^6} \quad \Longrightarrow \quad \frac{dS}{dt} + \frac{S}{10^5} = \frac{1}{5}.$$

An integrating factor for this equation is $e^{\int 1/10^5\,dt} = e^{t/10^5}$. Multiplication by $e^{t/10^5}$ gives

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$$e^{t/10^5}\frac{dS}{dt}+\frac{Se^{t/10^5}}{10^5}=\frac{1}{5}e^{t/10^5} \implies \frac{d}{dt}(Se^{t/10^5})=\frac{1}{5}e^{t/10^5}.$$

Integration now yields

$$Se^{t/10^5}=20\,000e^{t/10^5}+C \implies S(t)=20\,000+Ce^{-t/10^5}.$$

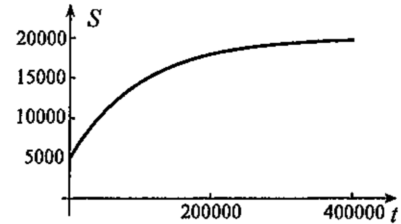
Since $S(0)=5000$, it follows that

$$5000=20\,000+C.$$

Thus, $C=-15\,000$, and $S(t)$ is

$$S(t)=20\,000-15\,000e^{-t/10^5}\text{ g.}$$

A graph of this function is shown to the right. The limit as $t\to\infty$ is 20 000.



28. If $S(t)$ represents the number of grams of salt in the tank as a function of time t, then dS/dt, the rate of change of S with respect to t, must be the rate at which salt is added less the rate at which it is removed. The rate at which salt is added is 0.2 g/s. Since the concentration of salt at time t is $S/(10^6-10t)$ g/mL, the rate at which salt is removed at time t is $20S/(10^6-10t)$ g/s. Thus,

$$\frac{dS}{dt}=\frac{1}{5}-\frac{20S}{10^6-10t} \implies \frac{dS}{dt}+\frac{2S}{10^5-t}=\frac{1}{5},$$

valid for $0<t<10^5$. An integrating factor is $e^{\int 2/(10^5-t)\,dt}=e^{-2\ln(10^5-t)}=\dfrac{1}{(10^5-t)^2}$. When we multiply by this factor, the differential equation becomes

$$\frac{1}{(10^5-t)^2}\frac{dS}{dt}+\frac{2S}{(10^5-t)^3}=\frac{1}{5(10^5-t)^2} \implies \frac{d}{dt}\left[\frac{S}{(10^5-t)^2}\right]=\frac{1}{5(10^5-t)^2}.$$

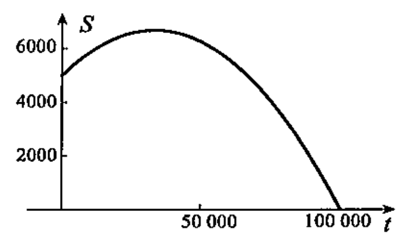
Integration now gives

$$\frac{S}{(10^5-t)^2}=\frac{1}{5(10^5-t)}+C \implies S(t)=\frac{10^5-t}{5}+C(10^5-t)^2.$$

Since $S(0)=5000$, it follows that

$5000=\dfrac{10^5}{5}+C(10^{10})$, and therefore $C=-15/10^7$. Thus,

$$S(t)=20\,000-\frac{t}{5}-\frac{15}{10^7}(10^5-t)^2.$$



29. If $C(t)$ represents the number of cubic metres of carbon dioxide in the room as a function of time t, then dC/dt, the rate of change of C with respect to t, must be the rate at which carbon dioxide is added less the rate at which it is removed. The rate at which carbon dioxide is added is $1/400$ m^3/min. Since the concentration of carbon dioxide at time t is $C/100$, the rate at which carbon dioxide is removed is $C/20$ m^3/min. Thus,

$$\frac{dC}{dt}=\frac{1}{400}-\frac{C}{20}, \implies \frac{dC}{dt}+\frac{C}{20}=\frac{1}{400}.$$

An integrating factor for this equation is $e^{\int 1/20\,dt}=e^{t/20}$. Multiplication by $e^{t/20}$ gives

$$e^{t/20}\frac{dC}{dt}+\frac{Ce^{t/20}}{20}=\frac{e^{t/20}}{400} \implies \frac{d}{dt}(Ce^{t/20})=\frac{1}{400}e^{t/20}.$$

Integration now yields

$$Ce^{t/20}=\frac{1}{20}e^{t/20}+D \implies C=\frac{1}{20}+De^{-t/20}.$$

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Since $C(0) = 1/10$, it follows that $1/10 = 1/20 + D \implies D = 1/20$, and $C(t) = 1/20 + (1/20)e^{-t/20}$. The limit of this function as $t \to \infty$ is $1/20$.

30. If $t = 0$ when the oven is turned on, its temperature is $T_o(t) = 20 + 36t$ for $0 \le t \le 5$, and 200 for $t > 5$. If $T(t)$ is the temperature of the potato, then

$$\frac{dT}{dt} = k(T_o - T), \quad T(0) = 20.$$

For $0 \le t \le 5$, this becomes $dT/dt = -kT + k(20 + 36t)$. An integrating factor for this linear first-order differential equation is e^{kt} so that

$$e^{kt}\frac{dT}{dt} + ke^{kt}T = k(20+36t)e^{kt} \quad \implies \quad \frac{d}{dt}(Te^{kt}) = k(20+36t)e^{kt} \quad \implies \quad Te^{kt} = k\int(20+36t)e^{kt}dt + C.$$

Integration by parts leads to

$$T = ke^{-kt}\left(\frac{20e^{kt}}{k} - \frac{36e^{kt}}{k^2} + \frac{36te^{kt}}{k}\right) + Ce^{-kt} = 20 - \frac{36}{k} + 36t + Ce^{-kt}.$$

For $T(0) = 20$, we have $20 = 20 - 36/k + C \implies C = 36/k$, and

$$T(t) = 20 - \frac{36}{k} + 36t + \frac{36}{k}e^{-kt} = 20 + 36t + \frac{36}{k}(e^{-kt} - 1).$$

For $t > 5$,

$$\frac{dT}{dt} = k(200 - T), \quad T(5) = 200 + \frac{36}{k}(e^{-5k} - 1).$$

This equation is separable,

$$\frac{dT}{200 - T} = k\,dt \quad \implies \quad -\ln|200 - T| = kt + C \quad \implies \quad T = 200 + De^{-kt},$$

where $D = \pm e^{-C}$. The temperature at $t = 5$ requires

$$200 + De^{-5k} = 200 + \frac{36}{k}(e^{-5k} - 1) \quad \implies \quad D = \frac{36}{k}(1 - e^{5k}).$$

Hence, for $t > 5$, temperature is $T(t) = 200 + (36/k)(1 - e^{5k})e^{-kt}$.

31. The energy balance equation is

$$(0.03)(4190)(10) + 2000 = (0.03)(4190)T + 100(4190)\frac{dT}{dt} \quad \implies \quad \frac{dT}{dt} + \frac{3T}{10\,000} = \frac{3257}{419\,000}.$$

An integrating factor is $e^{\int(3/10\,000)\,dt} = e^{3t/10\,000}$, so that the differential equation can be expressed in the form

$$e^{3t/10\,000}\frac{dT}{dt} + \frac{3Te^{3t/10\,000}}{10\,000} = \frac{3257}{419\,000}e^{3t/10\,000} \quad \implies \quad \frac{d}{dt}(Te^{3t/10\,000}) = \frac{3257}{419\,000}e^{3t/10\,000}.$$

Integration now gives

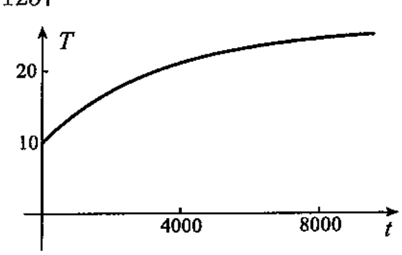
$$Te^{3t/10\,000} = \frac{32\,570}{1257}e^{3t/10\,000} + C \implies T = \frac{32\,570}{1257} + Ce^{-3t/10\,000}.$$

Since $T(0) = 10$, we find

$$10 = \frac{32\,570}{1257} + C \implies C = -\frac{20\,000}{1257}.$$

Temperature of the water is therefore

$$T(t) = \frac{32\,570}{1257} - \frac{20\,000}{1257}e^{-3t/10\,000}.$$



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32. The energy balance equation is

$$\left(\frac{100}{t+1}\right)(4190)(10)+2000=\left(\frac{100}{t+1}\right)(4190)T+100(4190)\frac{dT}{dt} \quad\Longrightarrow\quad \frac{dT}{dt}+\frac{T}{t+1}=\frac{2}{419}+\frac{10}{t+1}.$$

An integrating factor is $e^{\int[1/(t+1)]\,dt}=e^{\ln(t+1)}=t+1$, so that the differential equation can be expressed in the form

$$(t+1)\frac{dT}{dt}+T=\frac{2}{419}(t+1)+10 \quad\Longrightarrow\quad \frac{d}{dt}[(t+1)T]=\frac{2}{419}(t+1)+10.$$

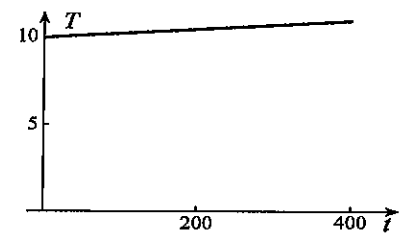
Integration now gives

$$(t+1)T=10t+\frac{1}{419}(t+1)^2+C \quad\Longrightarrow\quad T(t)=\frac{10t}{t+1}+\frac{1}{419}(t+1)+\frac{C}{t+1}.$$

Since $T(0)=10$, we find $10=\dfrac{1}{419}+C\Longrightarrow C=\dfrac{4189}{419}$.

Temperature of the water is therefore

$$T(t)=\frac{10t}{t+1}+\frac{t+1}{419}+\frac{4189}{419(t+1)}=\frac{4190t+4189}{419(t+1)}+\frac{t+1}{419}.$$



33. The energy balance equation is

$$(0.03)(4190)(10)+20t=(0.03)(4190)T+100(4190)\frac{dT}{dt} \quad\Longrightarrow\quad \frac{dT}{dt}+\frac{3T}{10\,000}=\frac{t}{20\,950}+\frac{3}{1000}.$$

An integrating factor is $e^{\int(3/10\,000)\,dt}=e^{3t/10\,000}$, so that the differential equation can be expressed in the form

$$e^{3t/10\,000}\frac{dT}{dt}+\frac{3e^{3t/10\,000}T}{10\,000}=\left(\frac{t}{20\,950}+\frac{3}{1000}\right)e^{3t/10\,000} \quad\Longrightarrow\quad \frac{d}{dt}(Te^{3t/10\,000})=\left(\frac{t}{20\,950}+\frac{3}{1000}\right)e^{3t/10\,000}.$$

Integration by parts on the right leads to

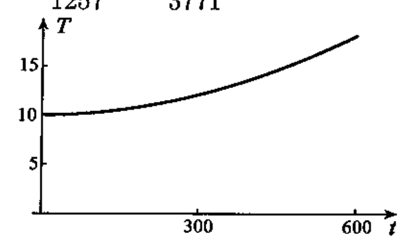
$$Te^{3t/10\,000}=\left(\frac{200t}{1257}-\frac{1\,962\,290}{3771}\right)e^{3t/10\,000}+C \quad\Longrightarrow\quad T(t)=\frac{200t}{1257}-\frac{1\,962\,290}{3771}+Ce^{-3t/10\,000}.$$

Since $T(0)=10$, we find

$$10=-\frac{1\,962\,290}{3771}+C\Longrightarrow C=\frac{2\times10^6}{3771}.$$

Temperature of the water is therefore

$$T(t)=\frac{200t}{1257}-\frac{1\,962\,290}{3771}+\frac{2\times10^6}{3771}e^{-3t/10\,000}.$$



34. The energy balance equation is

$$(0.03)(4190)(10e^{-t})+2000=(0.03)(4190)T+100(4190)\frac{dT}{dt} \quad\Longrightarrow\quad \frac{dT}{dt}+\frac{3T}{10\,000}=\frac{2}{419}+\frac{3e^{-t}}{1000}.$$

An integrating factor is $e^{\int(3/10\,000)\,dt}=e^{3t/10\,000}$, so that the differential equation can be expressed in the form

$$e^{3t/10\,000}\frac{dT}{dt}+\frac{3Te^{3t/10\,000}}{10\,000}=\left(\frac{2}{419}+\frac{3e^{-t}}{1000}\right)e^{3t/10\,000} \quad\Longrightarrow\quad \frac{d}{dt}(Te^{3t/10\,000})=\left(\frac{2}{419}+\frac{3e^{-t}}{1000}\right)e^{3t/10\,000}.$$

Integration gives

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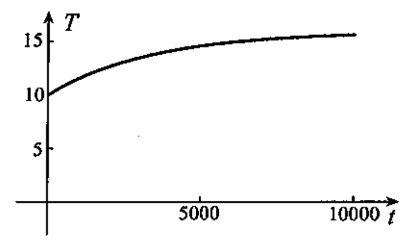
$$Te^{3t/10\,000} = \frac{20\,000}{1257}e^{3t/10\,000} - \frac{30}{9997}e^{-9997t/10\,000} + C \quad \Longrightarrow \quad T(t) = \frac{20\,000}{1257} - \frac{30}{9997}e^{-t} + Ce^{-3t/10\,000}.$$

Since $T(0) = 10$, we find $10 = \dfrac{20\,000}{1257} - \dfrac{30}{9997} + C$,
and this implies that $C = -\dfrac{74\,240\,000}{12\,566\,229}$.
Temperature of the water is therefore
$T(t) = \dfrac{20\,000}{1257} - \dfrac{30}{9997}e^{-t} - \dfrac{74\,240\,000}{12\,566\,229}e^{-3t/10\,000}$.



35. The differential equation is linear first-order,

$$\frac{dV}{dt} + \frac{V}{RC} = \frac{1}{RC}h(t),$$

with integrating factor $e^{\int [1/(RC)]dt} = e^{t/(RC)}$. Multiplying the differential equation by this factor leads to

$$\frac{d}{dt}\left[Ve^{t/(RC)}\right] = \frac{1}{RC}h(t)e^{t/(RC)}.$$

Integration for $t > 0$ yields

$$Ve^{t/(RC)} = \frac{1}{RC}\int h(t)e^{t/(RC)}dt = \frac{1}{RC}\int e^{t/(RC)}dt = e^{t/(RC)} + D.$$

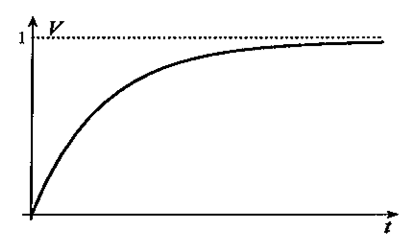
Division by $e^{t/(RC)}$ gives

$$V(t) = 1 + De^{-t/(RC)}.$$

The initial condition $V(0) = 0$ requires $0 = 1 + D$, so that the step response is

$$V(t) = 1 - e^{-t/(RC)}.$$

Its graph is shown to the right.



36. The differential equation is linear first-order,

$$\frac{di}{dt} + \frac{Ri}{L} = \frac{R}{L}[h(t) - h(t-1)],$$

with integrating factor $e^{\int (R/L)dt} = e^{Rt/L}$. Multiplying the differential equation by this factor leads to

$$\frac{d}{dt}\left[ie^{Rt/L}\right] = \frac{R}{L}[h(t) - h(t-1)]e^{Rt/L}.$$

Integration for $0 < t < 1$ yields

$$ie^{Rt/L} = \frac{R}{L}\int e^{Rt/L}dt = e^{Rt/L} + D.$$

Division by $e^{Rt/L}$ gives $i(t) = 1 + De^{-Rt/L}$. The initial condition $i(0) = 0$ gives $0 = 1 + D$, so that for $0 < t < 1$,

$$i(t) = 1 - e^{-Rt/L}.$$

When $t > 1$, integration gives

$$ie^{Rt/L} = \frac{R}{L}\int (1-1)e^{Rt/L}dt = E \qquad \Longrightarrow \qquad i = Ee^{-Rt/L}.$$

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To evaluate E we demaind that the current be continuous at $t = 1$. This requires

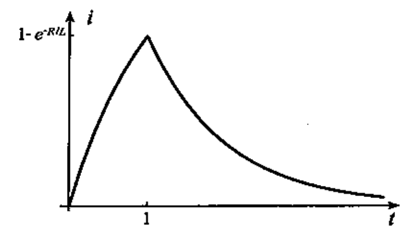
$\lim_{t\to 1^-} i(t) = \lim_{t\to 1^+} i(t)$, or

$1 - e^{-R/L} = Ee^{-R/L} \implies E = e^{R/L} - 1.$

The pulse response function is

$$i(t) = \begin{cases} 1 - e^{-Rt/L}, & 0 \le t \le 1 \\ (e^{R/L} - 1)e^{-Rt/L}, & t > 1 \end{cases}.$$

Its graph is shown to the right.



37. (a) Since $\dfrac{dI}{dt} + \dfrac{R}{L}I = \dfrac{E}{L}$ an integrating factor is $e^{\int (R/L)\,dt} = e^{Rt/L}$. Multiplication of the diferential equation by this factor leads to

$$\frac{d}{dt}\left[Ie^{Rt/L}\right] = \frac{E_0}{L}e^{Rt/L}\sin(\omega t).$$

Integration now yields

$$Ie^{Rt/L} = \frac{E_0 e^{Rt/L}}{L(R^2/L^2 + \omega^2)}\left[\frac{R}{L}\sin(\omega t) - \omega\cos(\omega t)\right] + A$$

or,

$$I(t) = Ae^{-Rt/L} + \frac{E_0}{R^2 + \omega^2 L^2}[R\sin(\omega t) - \omega L\cos(\omega t)].$$

If we set

$$R\sin(\omega t) - \omega L\cos(\omega t) = B\sin(\omega t - \phi) = B\sin(\omega t)\cos\phi - B\cos(\omega t)\sin\phi,$$

then $B\cos\phi = R$ and $B\sin\phi = \omega L$. These equations imply that

$$B = \sqrt{R^2 + \omega^2 L^2} \qquad \text{and} \qquad \tan\phi = \frac{\omega L}{R} \implies \phi = \mathrm{Tan}^{-1}\left(\frac{\omega L}{R}\right).$$

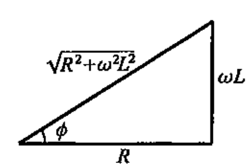
Hence, $I(t) = Ae^{-Rt/L} + \dfrac{E_0}{R^2 + \omega^2 L^2}\sqrt{R^2 + \omega^2 L^2}\sin(\omega t - \phi) = Ae^{-Rt/L} + \dfrac{E_0}{Z}\sin(\omega t - \phi)$,

where $Z = \sqrt{R^2 + \omega^2 L^2}$.

(b) If $I(0) = I_0$, then

$$I_0 = A + \frac{E_0}{Z}\sin(-\phi) = A - \frac{E_0}{Z}\frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}} = A - \frac{E_0\omega L}{Z^2},$$

and therefore $A = I_0 + E_0\omega L/Z^2$.



38. (a) Since $\dfrac{dI}{dt} + \dfrac{1}{RC}I = \dfrac{1}{R}\dfrac{dE}{dt} = \dfrac{\omega}{R}E_0\cos(\omega t)$, an integrating factor is $e^{\int [1/(RC)]\,dt} = e^{t/(RC)}$. Multiplication by this factor gives

$$e^{t/(RC)}\frac{dI}{dt} + \frac{1}{RC}Ie^{t/(RC)} = \frac{\omega}{R}E_0 e^{t/(RC)}\cos(\omega t) \implies \frac{d}{dt}\left[Ie^{t/(RC)}\right] = \frac{\omega}{R}E_0 e^{t/(RC)}\cos(\omega t).$$

Integration now yields

$$Ie^{t/(RC)} = \frac{\omega E_0}{R}\left\{\frac{e^{t/(RC)}}{\omega^2 + 1/(RC)^2}[(1/(RC))\cos(\omega t) + \omega\sin(\omega t)]\right\} + A.$$

Thus, $I(t) = Ae^{-t/(RC)} + \dfrac{\omega E_0}{R[\omega^2 + 1/(RC)^2]}\{[1/(RC)]\cos(\omega t) + \omega\sin(\omega t)\}$. If we set

$$\frac{1}{RC}\cos(\omega t) + \omega\sin(\omega t) = B\sin(\omega t - \phi) = B\sin(\omega t)\cos\phi - B\cos(\omega t)\sin\phi,$$

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then $B\cos\phi = \omega$ and $-B\sin\phi = 1/(RC)$. These equations imply that

$$B = \sqrt{\omega^2 + \frac{1}{(RC)^2}} \qquad \text{and} \qquad \tan\phi = \frac{-1}{\omega CR} \quad \Longrightarrow \quad \phi = \mathrm{Tan}^{-1}\left(\frac{-1}{\omega CR}\right).$$

Hence, $I(t) = Ae^{-t/(RC)} + \dfrac{\omega E_0}{R[\omega^2 + 1/(RC)^2]}\sqrt{\omega^2 + 1/(RC)^2}\sin(\omega t - \phi)$

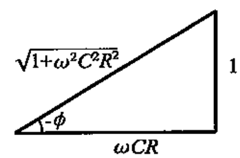
$$= Ae^{-t/(RC)} + \frac{E_0}{Z}\sin(\omega t - \phi),$$

where $Z = \dfrac{R\sqrt{\omega^2 + 1/(RC)^2}}{\omega} = \sqrt{R^2 + \dfrac{1}{\omega^2 C^2}}$.

(b) If $I(0) = I_0$, then

$$I_0 = A + \frac{E_0}{Z}\sin(-\phi) = A + \frac{E_0}{Z}\frac{1}{\sqrt{1 + \omega^2 C^2 R^2}}$$

$$= A + \frac{E_0}{\omega CZ\sqrt{R^2 + 1/(\omega C)^2}} = A + \frac{E_0}{\omega CZ^2},$$



and therefore $A = I_0 - E_0/(\omega CZ^2)$.

39. For $0 < t < 5$, the solution is as before, $T(t) = 20 + 36t + (36/k)(e^{-kt} - 1)$. For $t > 5$, the temperature of the oven is $T_o(t) = 200 + 10\sin[\pi(t-5)/5] = 200 - 10\sin(\pi t/5)$, in which case $T(t)$ must satisfy

$$\frac{dT}{dt} = k[200 - 10\sin(\pi t/5) - T], \quad T(5) = 200 + \frac{36}{k}(e^{-5k} - 1).$$

An integrating factor is e^{kt} so that

$$e^{kt}\frac{dT}{dt} + ke^{kt}T = k[200 - 10\sin(\pi t/5)]e^{kt} \quad \Longrightarrow \quad \frac{d}{dt}(Te^{kt}) = k[200 - 10\sin(\pi t/5)]e^{kt}.$$

Integration by parts on the right leads to

$$T(t) = 200 - \frac{10k}{k^2 + \pi^2/25}[k\sin(\pi t/5) - (\pi/5)\cos(\pi t/5)] + Ce^{-kt}.$$

For $T(5) = 200 + (36/k)(e^{-5k} - 1)$,

$$200 + \frac{36}{k}(e^{-5k} - 1) = 200 - \frac{10k}{k^2 + \pi^2/25}(\pi/5) + Ce^{-5k} \quad \Longrightarrow \quad C = \frac{36}{k}(1 - e^{5k}) + \frac{2k\pi e^{5k}}{k^2 + \pi^2/25}.$$

40. If we substitute $Q = \dfrac{S}{K(1-x)} - \dfrac{xI}{1-x}$ into the differential equation,

$$\frac{dS}{dt} = I - \frac{S}{K(1-x)} + \frac{xI}{1-x} \quad \Longrightarrow \quad \frac{dS}{dt} + \frac{S}{K(1-x)} = \frac{I}{1-x} = \frac{I_0 e^{-\lambda t}}{1-x}.$$

An integrating factor is $e^{t/b}$, where $b = K(1-x)$. Multiplication of the differential equation by $e^{t/b}$ gives

$$e^{t/b}\frac{dS}{dt} + \frac{Se^{t/b}}{b} = \frac{I_0 e^{t/b}e^{-\lambda t}}{1-x} \quad \Longrightarrow \quad \frac{d}{dt}[Se^{t/b}] = \frac{I_0}{1-x}e^{(-\lambda + 1/b)t}.$$

Integration gives

$$Se^{t/b} = \frac{I_0}{(1-x)(-\lambda + 1/b)}e^{(-\lambda+1/b)t} + C \quad \Longrightarrow \quad S(t) = Ce^{-t/b} + \frac{I_0}{(1-x)(-\lambda + 1/b)}e^{-\lambda t}.$$

Thus,

$$Q(t) = \frac{C}{b}e^{-t/b} + \frac{I_0}{b(1-x)(-\lambda + 1/b)}e^{-\lambda t} - \frac{xI_0 e^{-\lambda t}}{1-x}.$$

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Since $Q(0) = 0$,

$$0 = \frac{C}{b} + \frac{I_0}{b(1-x)(-\lambda + 1/b)} - \frac{xI_0}{1-x}.$$

This gives C, and

$$\begin{aligned} Q(t) &= \left[\frac{xI_0}{1-x} - \frac{I_0}{b(1-x)(-\lambda+1/b)}\right]e^{-t/b} + \left[\frac{-xI_0}{1-x} + \frac{I_0}{b(1-x)(-\lambda+1/b)}\right]e^{-\lambda t} \\ &= I_0\left[\frac{1}{(1-x)(1-\lambda b)} - \frac{x}{1-x}\right](e^{-\lambda t} - e^{-t/b}). \end{aligned}$$

41. Since $\overline{N}$ is proportional to depth y below the surface, $\overline{N} = \overline{N}(y) = ky$, $0 \le y \le Y$, where $k > 0$ is a constant and Y is the depth of the water. Since dN/dt is proportional to $\overline{N} - N$, $dN/dt = l(\overline{N} - N) = l(ky - N)$, where $l > 0$ is a constant. Because the diver drops to the bottom in time T, $y = Yt/T$, where $t = 0$ on entry, and therefore $dN/dt = l(kYt/T - N)$, $0 \le t \le T$. When we write the differential equation in the form $\dfrac{dN}{dt} + lN = \dfrac{klYt}{T}$, an integrating factor is e^{lt}. Multiplication by e^{lt} yields

$$e^{lt}\frac{dN}{dt} + le^{lt}N = \frac{klYt}{T}e^{lt} \quad \Longrightarrow \quad \frac{d}{dt}(Ne^{lt}) = \frac{klY}{T}te^{lt} \quad \Longrightarrow \quad Ne^{lt} = \frac{klY}{T}\left(\frac{t}{l}e^{lt} - \frac{1}{l^2}e^{lt}\right) + C.$$

Thus, $N(t) = \dfrac{klY}{T}\left(\dfrac{t}{l} - \dfrac{1}{l^2}\right) + Ce^{-lt} = \dfrac{kY}{lT}(lt-1) + Ce^{-lt}$. Since $N(0) = 0$, we obtain $0 = -\dfrac{kY}{lT} + C$, and therefore

$$N(t) = \frac{kY}{lT}(lt-1) + \frac{kY}{lT}e^{-lt}, \quad 0 \le t \le T.$$

When the diver reaches the bottom, the nitrogen pressure in his body is $N(T) = \dfrac{kY}{lT}(lT-1) + \dfrac{kY}{lT}e^{-lT}$. For time $t > T$, nitrogen pressure must satisfy the differential equation

$$\frac{dN}{dt} = l(\tilde{N} - N), \quad \text{where } \tilde{N} = \overline{N}(Y),$$

a differential equation that can be separated, $\dfrac{dN}{\tilde{N} - N} = l\,dt$. A one-parameter family of solutions is defined implicitly by $-\ln|\tilde{N} - N| = lt + C \implies N = \tilde{N} + De^{-lt}$. Since $N(T) = \dfrac{kY}{lT}(lT-1) + \dfrac{kY}{lT}e^{-lT}$,

$$\frac{kY}{lT}(lT-1) + \frac{kY}{lT}e^{-lT} = \tilde{N} + De^{-lT} \quad \Longrightarrow \quad D = \left[\frac{kY}{lT}(lT-1) - \tilde{N}\right]e^{lT} + \frac{kY}{lT}.$$

Thus, $N(t) = \tilde{N} + \left[\dfrac{kY}{lT}(lT-1) - \tilde{N}\right]e^{l(T-t)} + \dfrac{kY}{lT}e^{-lt}$, $\quad t > T$.

EXERCISES 15.4

1. Since y is explicitly missing, we set $\dfrac{dy}{dx} = v$ and $\dfrac{d^2y}{dx^2} = \dfrac{dv}{dx}$,

$$x\frac{dv}{dx} + v = 4x \quad \Longrightarrow \quad \frac{d}{dx}(vx) = 4x.$$

Integration gives $vx = 2x^2 + C$. When we solve for v and set $v = dy/dx$,

$$v = \frac{dy}{dx} = 2x + \frac{C}{x} \quad \Longrightarrow \quad y = x^2 + C\ln|x| + D.$$

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2. Since x is explicitly missing, we set $\dfrac{dy}{dx} = v$ and $\dfrac{d^2y}{dx^2} = \dfrac{dv}{dx} = \dfrac{dv}{dy}\dfrac{dy}{dx} = v\dfrac{dv}{dy}$,

$$2yv\,\frac{dv}{dy} = 1 + v^2.$$

This equation can be separated, $\dfrac{2v}{1+v^2}\,dv = \dfrac{1}{y}\,dy$, and a one-parameter family of solutions is defined implicitly by $\ln(1+v^2) = \ln|y| + C$. When this equation is solved for v, $v = \dfrac{dy}{dx} = \pm\sqrt{Dy-1}$. This equation can also be separated, $\dfrac{1}{\sqrt{Dy-1}}\,dy = \pm dx$, and a two-parameter family of solutions is defined implicitly by $2\sqrt{Dy-1} = \pm Dx + F$.

3. Since y is explicitly missing, we set $\dfrac{dy}{dx} = v$ and $\dfrac{d^2y}{dx^2} = \dfrac{dv}{dx}$,

$$\frac{dv}{dx} = v + 2x \qquad \Longrightarrow \qquad \frac{dv}{dx} - v = 2x.$$

An integrating factor for this linear first-order equation is e^{-x}, so that the differential equation can be expressed in the form

$$e^{-x}\frac{dv}{dx} - ve^{-x} = 2xe^{-x} \quad \Longrightarrow \quad \frac{d}{dx}(ve^{-x}) = 2xe^{-x} \quad \Longrightarrow \quad ve^{-x} = \int 2xe^{-x}\,dx = -2(x+1)e^{-x} + C.$$

When we solve for v and set it equal to dy/dx,

$$v = \frac{dy}{dx} = -2(x+1) + Ce^{x} \quad \Longrightarrow \quad y = -x^2 - 2x + Ce^{x} + D.$$

4. Since y is explicitly missing, we set $\dfrac{dy}{dx} = v$ and $\dfrac{d^2y}{dx^2} = \dfrac{dv}{dx}$, giving $x^2\dfrac{dv}{dx} = v^2$. This equation can be separated, $\dfrac{1}{v^2}\,dv = \dfrac{1}{x^2}\,dx$, and a one-parameter family of solutions is defined implicitly by $-1/v = -1/x - C$. We now solve for v, $v = \dfrac{dy}{dx} = \dfrac{x}{Cx+1}$, and integrate to obtain

$$\begin{aligned} y &= \int \frac{x}{Cx+1}\,dx = \int\left(\frac{1}{C} + \frac{-1/C}{Cx+1}\right)dx = \frac{x}{C} - \frac{1}{C^2}\ln|Cx+1| + D \\ &= Ex - E^2\ln\left|\frac{x}{E} + 1\right| + D = Ex - E^2\ln|x+E| + F. \end{aligned}$$

5. Since y is explicitly missing, we set $\dfrac{dy}{dx} = v$ and $\dfrac{d^2y}{dx^2} = \dfrac{dv}{dx}$,

$$\sin x\frac{dv}{dx} + v\cos x = \sin x \quad \Longrightarrow \quad \frac{d}{dx}(v\sin x) = \sin x.$$

Integration gives $v\sin x = -\cos x + C$. When we solve for v and set $v = dy/dx$,

$$v = \frac{dy}{dx} = -\cot x + C\csc x \quad \Longrightarrow \quad y = \ln|\csc x| + C\ln|\csc x - \cot x| + D.$$

6. When we substitute $\dfrac{dy}{dx} = v$ and $\dfrac{d^2y}{dx^2} = \dfrac{dv}{dx} = \dfrac{dv}{dy}\dfrac{dy}{dx} = v\dfrac{dv}{dy}$, we obtain $v\dfrac{dv}{dy} = (1+v^2)^{3/2}$. This equation can be separated, $\dfrac{v}{(1+v^2)^{3/2}}\,dv = dy$, a one-parameter family of solutions of which is defined implicitly by $-1/\sqrt{1+v^2} = y + C$. When this is solved for v,

$$v = \frac{dy}{dx} = \pm\frac{\sqrt{1-(y+C)^2}}{y+C} \quad \Longrightarrow \quad \frac{y+C}{\sqrt{1-(y+C)^2}}\,dy = \pm dx.$$

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Integration gives $-\sqrt{1-(y+C)^2} = \pm x + D$. By squaring this equation, the solution can be rewritten in the form $(x+E)^2 + (y+C)^2 = 1$.

7. Since x is explicitly missing, we set $\dfrac{dy}{dx} = v$ and $\dfrac{d^2y}{dx^2} = \dfrac{dv}{dx} = \dfrac{dv}{dy}\dfrac{dy}{dx} = v\,\dfrac{dv}{dy}$, giving $v\dfrac{dv}{dy} + 4y = 0$. When we separate, $v\,dv = -4y\,dy$, a one-parameter family of solutions then being defined by $v^2/2 = -2y^2 + C$. We now solve for v and set it equal to dy/dx,

$$v = \frac{dy}{dx} = \pm\sqrt{2C - 4y^2} \qquad \Longrightarrow \qquad \frac{dy}{\sqrt{D-4y^2}} = \pm dx,$$

where we have replaced $2C$ with D. When we substitute $y = (\sqrt{D}/2)\sin\theta$ and $dy = (\sqrt{D}/2)\cos\theta\,d\theta$,

$$\pm x + E = \int \frac{1}{\sqrt{D-4y^2}}dy = \int \frac{1}{\sqrt{D}\cos\theta}\frac{\sqrt{D}}{2}\cos\theta\,d\theta = \frac{\theta}{2} = \frac{1}{2}\mathrm{Sin}^{-1}\left(\frac{2y}{\sqrt{D}}\right).$$

Thus, $\mathrm{Sin}^{-1}\left(\dfrac{2y}{\sqrt{D}}\right) = \pm 2x + 2E \Longrightarrow \dfrac{2y}{\sqrt{D}} = \sin(2E \pm 2x)$ and this implies that

$$y = \frac{\sqrt{D}}{2}(\sin 2E\cos 2x \pm \cos 2E \sin 2x) = F\cos 2x + G\sin 2x,$$

where $F = (1/2)\sqrt{D}\sin 2E$ and $G = \pm(1/2)\sqrt{D}\cos 2E$.

8. Since x is explicitly missing, we set $\dfrac{dy}{dx} = v$ and $\dfrac{d^2y}{dx^2} = \dfrac{dv}{dx} = \dfrac{dv}{dy}\dfrac{dy}{dx} = v\,\dfrac{dv}{dy}$, giving $v\,\dfrac{dv}{dy} = vy$. Division by v and integration gives

$$v = \frac{y^2}{2} + C \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{y^2}{2} + \frac{D}{2} \qquad (D = 2C).$$

Separation now yields $\dfrac{1}{D+y^2}\,dy = \dfrac{1}{2}\,dx \Longrightarrow \dfrac{x}{2} + E = \displaystyle\int \frac{1}{D+y^2}\,dy$. If $D = 0$, then $x/2 + E = -1/y$. If $D > 0$, we set $y = \sqrt{D}\tan\theta$ and $dy = \sqrt{D}\sec^2\theta\,d\theta$,

$$\frac{x}{2} + E = \int \frac{\sqrt{D}\sec^2\theta}{D\sec^2\theta}\,d\theta = \frac{1}{\sqrt{D}}\theta = \frac{1}{\sqrt{D}}\mathrm{Tan}^{-1}\left(\frac{y}{\sqrt{D}}\right) = F\,\mathrm{Tan}^{-1}(Fy).$$

If $D < 0$, we set $D = -F^2$ and use partial fractions to write

$$\frac{x}{2} + E = \int \frac{1}{y^2 - F^2}\,dy = \int \left(\frac{\frac{-1}{2F}}{y+F} + \frac{\frac{1}{2F}}{y-F}\right)dy$$
$$= -\frac{1}{2F}\ln|y+F| + \frac{1}{2F}\ln|y-F| = \frac{1}{2F}\ln\left|\frac{y-F}{y+F}\right|.$$

9. Since y is explicitly missing, we set $\dfrac{dy}{dx} = v$ and $\dfrac{d^2y}{dx^2} = \dfrac{dv}{dx}$,

$$\frac{dv}{dx} + v^2 = 1 \quad \Longrightarrow \quad \frac{dv}{1-v^2} = dx \quad \Longrightarrow \quad \frac{1}{2}\left(\frac{1}{1-v} + \frac{1}{1+v}\right)dv = dx.$$

Integration gives $-\ln|1-v| + \ln|1+v| = 2x + C$. When we solve for v we obtain

$$v = \frac{dy}{dx} = \frac{De^{2x}-1}{De^{2x}+1} \quad \Longrightarrow \quad y = \int \frac{De^{2x}-1}{De^{2x}+1}dx = \frac{1}{2}\ln(De^{2x}+1) - \int \frac{1}{De^{2x}+1}dx.$$

If we set $e^x = (1/\sqrt{D})\tan\theta$ and $e^x\,dx = (1/\sqrt{D})\sec^2\theta\,d\theta$, then

$$y = \frac{1}{2}\ln(De^{2x}+1) - \int \frac{1}{\sec^2\theta}\frac{\sec^2\theta}{\tan\theta}d\theta = \frac{1}{2}\ln(De^{2x}+1) + \ln|\csc\theta| + E$$

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$$= \frac{1}{2}\ln(De^{2x}+1) + \ln\left|\frac{\sqrt{De^{2x}+1}}{\sqrt{D}e^{x}}\right| + E$$
$$= \ln(De^{2x}+1) - x + F, \quad F = E - \frac{1}{2}\ln D.$$

10. If we set $\frac{dy}{dx} = v$ and $\frac{d^2y}{dx^2} = \frac{dv}{dx}$, then $\left(\frac{dv}{dx}\right)^2 = 1 + v^2$. This equation can be separated, $\frac{1}{\sqrt{1+v^2}}\,dv = \pm dx$, and a one-parameter family of solutions is defined implicitly by

$$\pm x + C = \int \frac{1}{\sqrt{1+v^2}}\,dv \quad \text{(and if we set } v = \tan\theta,\ dv = \sec^2\theta\,d\theta),$$
$$= \int \frac{\sec^2\theta}{\sec\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| = \ln|\sqrt{1+v^2} + v|.$$

We now exponentiate, $\sqrt{1+v^2} + v = De^{\pm x}$, transpose the v and square, $1 + v^2 = D^2e^{\pm 2x} - 2Dve^{\pm x} + v^2$. We can now solve for v,

$$v = \frac{dy}{dx} = \frac{D^2e^{\pm 2x} - 1}{2De^{\pm x}} = \frac{1}{2}\left(De^{\pm x} - \frac{1}{D}e^{\mp x}\right).$$

In either case dy/dx is of the form $\frac{dy}{dx} = \frac{1}{2}\left(De^{x} - \frac{1}{D}e^{-x}\right)$, and hence, $y = \frac{1}{2}\left(De^{x} + \frac{1}{D}e^{-x}\right) + E$.

11. We could use substitution 15.28 but it simpler to write

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0 \quad \Longrightarrow \quad r\frac{dT}{dr} = C \quad \Longrightarrow \quad \frac{dT}{dr} = \frac{C}{r} \quad \Longrightarrow \quad T = C\ln r + D.$$

For $T(a) = T_a$ and $T(b) = T_b$, C and D must satisfy $T_a = C\ln a + D$ and $T_b = C\ln b + D$. These can be solved for $C = (T_b - T_a)/\ln(b/a)$ and $D = (T_a\ln b - T_b\ln a)/\ln(b/a)$.

12. We could use substitution 15.28 but it simpler to write

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = k \quad \Longrightarrow \quad r\frac{dT}{dr} = kr + C \quad \Longrightarrow \quad \frac{dT}{dr} = k + \frac{C}{r} \quad \Longrightarrow \quad T = kr + C\ln r + D.$$

For $T(a) = T_a$ and $T(b) = T_b$, C and D must satisfy $T_a = ka + C\ln a + D$ and $T_b = kb + C\ln b + D$. These can be solved for $C = [T_b - T_a - k(b-a)]/\ln(b/a)$ and $D = [T_a\ln b - T_b\ln a + k(b\ln a - a\ln b)]/\ln(b/a)$.

13. We could use substitution 15.28 but it simpler to write

$$\frac{d}{dr}\left(r\frac{dh}{dr}\right) = 0 \quad \Longrightarrow \quad r\frac{dh}{dr} = C \quad \Longrightarrow \quad \frac{dh}{dr} = \frac{C}{r} \quad \Longrightarrow \quad h = C\ln r + D.$$

The boundary conditions require

$$C\ln r_w + D = h_w, \quad C\ln r_i + D = h_i.$$

These imply that $C = \frac{h_i - h_w}{\ln(r_i/r_w)}$ and $D = h_w - \left[\frac{h_i - h_w}{\ln(r_i/r_w)}\right]\ln r_w$, and therefore

$$h = \frac{h_i - h_w}{\ln(r_i/r_w)}\ln r + h_w - \left[\frac{h_i - h_w}{\ln(r_i/r_w)}\right]\ln r_w = h_w + \left[\frac{h_i - h_w}{\ln(r_i/r_w)}\right]\ln\left(\frac{r}{r_w}\right).$$

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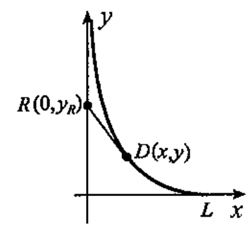
14. We let $y = y(x)$ be the equation of the required curve, and use two principles. First, if $D(x, y)$ is any position of the dog, then the line DR joining the positions of the rabbit and dog is always tangent to $y = y(x)$. This can be expressed in the form

$$(y - y_R)/(x - 0) = y'(x) \implies y_R = y - xy'(x).$$

Secondly, distance run by the dog and rabbit are always the same. Distance run by the rabbit is y_R. Distance



run by the dog in the same time interval is represented by the integral

$$\int_x^L \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

When we equate these distances, we obtain

$$y - xy'(x) = \int_x^L \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

We differentiate this equation to eliminate the integral,

$$\frac{dy}{dx} - x\frac{d^2y}{dx^2} - \frac{dy}{dx} = -\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \implies x\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Since y is explicitly missing, we set $v = dy/dx$ and $dv/dx = d^2y/dx^2$,

$$x\frac{dv}{dx} = \sqrt{1 + v^2} \implies \int \frac{dv}{\sqrt{1 + v^2}} = \int \frac{dx}{x} = \ln x + C.$$

We set $v = \tan\theta$ and $dv = \sec^2\theta\,d\theta$,

$$\ln x + C = \int \frac{\sec^2\theta}{\sec\theta}d\theta = \ln|\sec\theta + \tan\theta| = \ln|\sqrt{1 + v^2} + v|.$$

Exponentiation gives

$$Dx = \sqrt{1 + v^2} + v \implies Dx - v = \sqrt{1 + v^2} \implies 1 + v^2 = D^2x^2 - 2Dxv + v^2,$$

where $D = e^C$. When we solve for v,

$$v = \frac{dy}{dx} = \frac{D^2x^2 - 1}{2Dx} = \frac{Dx}{2} - \frac{1}{2Dx}.$$

Since $y'(L) = 0$, we obtain $D = 1/L$, and therefore

$$\frac{dy}{dx} = \frac{x}{2L} - \frac{L}{2x} \implies y = \frac{x^2}{4L} - \frac{L}{2}\ln x + C.$$

The condition $y(L) = 0$ implies that $0 = L/4 - (L/2)\ln L + C$. Consequently,

$$y = \frac{x^2}{4L} - \frac{L}{2}\ln x + \frac{L}{2}\ln L - \frac{L}{4} = \frac{1}{4L}(x^2 - L^2) + \frac{L}{2}\ln\left(\frac{L}{x}\right).$$

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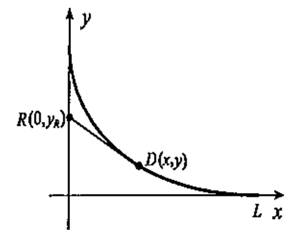
15. We let $y = y(x)$ be the equation of the required curve, and use two principles. First, if $D(x,y)$ is any position of the dog, then the line DR joining the positions of the rabbit and dog is always tangent to $y = y(x)$. This can be expressed in the form

$$(y - y_R)/(x - 0) = y'(x) \implies y_R = y - xy'(x).$$



Secondly, distance run by the dog is twice that run by the rabbit. Distance run by the rabbit is y_R. Distance run by the dog in the same time is represented by the integral

$$\int_x^L \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

When we equate this to $2y_R$, we obtain

$$2y - 2xy'(x) = \int_x^L \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

We differentiate this equation to eliminate the integral,

$$2\frac{dy}{dx} - 2x\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = -\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \quad \implies \quad 2x\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Since y is explicitly missing, we set $v = dy/dx$ and $dv/dx = d^2y/dx^2$,

$$2x\frac{dv}{dx} = \sqrt{1 + v^2} \quad \implies \quad \int \frac{dv}{\sqrt{1 + v^2}} = \int \frac{dx}{2x} = \frac{1}{2}\ln x + C.$$

We set $v = \tan\theta$ and $dv = \sec^2\theta\, d\theta$,

$$\frac{1}{2}\ln x + C = \int \frac{\sec^2\theta}{\sec\theta} d\theta = \ln|\sec\theta + \tan\theta| = \ln|\sqrt{1 + v^2} + v|.$$

Exponentiation gives

$$D\sqrt{x} = \sqrt{1 + v^2} + v \quad \implies \quad D\sqrt{x} - v = \sqrt{1 + v^2} \quad \implies \quad 1 + v^2 = D^2x - 2Dv\sqrt{x} + v^2,$$

where $D = e^C$. When we solve for v,

$$v = \frac{dy}{dx} = \frac{D^2x - 1}{2D\sqrt{x}} = \frac{D\sqrt{x}}{2} - \frac{1}{2D\sqrt{x}}.$$

Since $y'(L) = 0$, we obtain $D = 1/\sqrt{L}$, and therefore

$$\frac{dy}{dx} = \frac{\sqrt{x}}{2\sqrt{L}} - \frac{\sqrt{L}}{2\sqrt{x}} \quad \implies \quad y = \frac{x^{3/2}}{3\sqrt{L}} - \sqrt{Lx} + C.$$

The condition $y(L) = 0$ implies that $0 = L/3 - L + C \implies C = 2L/3$. Consequently,

$$y = \frac{x^{3/2}}{3\sqrt{L}} - \sqrt{Lx} + \frac{2L}{3}.$$

The dog catches the rabbit if, and when, $x = 0$, and this occurs when $y = 2L/3$.

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16. (a) We let $y = y(x)$ be the equation of the required curve, and use two principles. First, if $H(x, y)$ is any position of the hawk, then the line PH joining the positions of the pigeon and hawk is always tangent to $y = y(x)$. If $P(0, y_p)$ is the position of the pigeon, then this requirement can be expressed in the form

$$\frac{y - y_p}{x - 0} = y'(x) \quad \Longrightarrow \quad y_p = y - xy'(x).$$

Secondly, distance flown by the hawk is V/v times that by the pigeon. Distance flown by the pigeon is y_p. Distance flown by the hawk in the same time is represented by the integral

$$\int_x^L \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

When we equate this to $(V/v)y_p$, we obtain

$$\frac{V}{v}[y - xy'(x)] = \int_x^L \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \Longrightarrow \quad x\frac{dy}{dx} - y = \frac{v}{V}\int_L^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

(b) Differentiation of this equation gives

$$\frac{dy}{dx} + x\frac{d^2y}{dx^2} - \frac{dy}{dx} = \frac{v}{V}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \quad \Longrightarrow \quad x\frac{d^2y}{dx^2} = \frac{v}{V}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

(c) If we set $p = dy/dx$ and $dp/dx = d^2y/dx^2$, then $x\dfrac{dp}{dx} = \dfrac{v}{V}\sqrt{1 + p^2}$, a separable equation, $\dfrac{1}{\sqrt{1 + p^2}}\, dp = \dfrac{v}{Vx}\, dx$. A one-parameter family of solutions is defined implicitly by

$$\begin{aligned}\frac{v}{V}\ln x + C &= \int \frac{1}{\sqrt{1 + p^2}} dp \qquad \text{(and if we set } p = \tan\theta,\ dp = \sec^2\theta\, d\theta\text{)}\\ &= \int \frac{\sec^2\theta}{\sec\theta} d\theta = \ln|\sec\theta + \tan\theta| = \ln|\sqrt{1 + p^2} + p|.\end{aligned}$$

Since $p(L) = y'(L) = 0$, it follows that $\dfrac{v}{V}\ln L + C = \ln 1 = 0$. Thus,

$$\frac{v}{V}\ln x - \frac{v}{V}\ln L = \ln|\sqrt{1 + p^2} + p|.$$

Exponentiation gives $\left(\dfrac{x}{L}\right)^{v/V} = \sqrt{1 + p^2} + p$. When this equation is solved for p, the result is

$$p = \frac{dy}{dx} = \frac{1}{2}\left[\left(\frac{x}{L}\right)^{v/V} - \left(\frac{x}{L}\right)^{-v/V}\right].$$

Integration yields $y = \dfrac{1}{2}\left[\dfrac{L}{v/V + 1}\left(\dfrac{x}{L}\right)^{v/V+1} - \dfrac{L}{-v/V + 1}\left(\dfrac{x}{L}\right)^{-v/V+1}\right] + D$. Because $y(L) = 0$ we obtain $0 = \dfrac{1}{2}\left[\dfrac{L}{v/V + 1} - \dfrac{L}{-v/V + 1}\right] + D$. This equation can be solved for $D = LVv/(V^2 - v^2)$, and therefore

$$y(x) = \frac{LV}{2}\left[\frac{1}{V + v}\left(\frac{x}{L}\right)^{v/V+1} - \frac{1}{V - v}\left(\frac{x}{L}\right)^{-v/V+1}\right] + \frac{LVv}{V^2 - v^2}.$$

(d) The hawk catches the pigeon when $x = 0$, in which case $y(0) = \dfrac{LVv}{V^2 - v^2}$.

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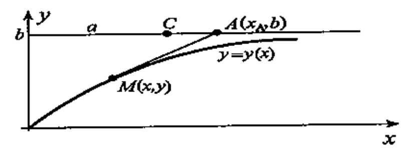
17. (a) We let $y = y(x)$ be the equation of the required curve, and use two principles. First, if $M(x, y)$ is any position of the missile, then the line MA joining the positions of the missile and aircraft is always tangent to $y = y(x)$. If $A(x_A, b)$ is the position of the aircraft, then this requirement can be expressed in the form



$$\frac{y-b}{x-x_A} = y'(x) \implies x_A = x - \frac{y-b}{y'(x)}.$$

Secondly, distance flown by the missile is V/v times that by the aircraft. Distance flown by the aircraft after the missile takes off is $x_A - a = x - a - (y-b)/y'(x)$. Distance flown by the missile in the same time is represented by the integral

$$\int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

When we equate this to $(V/v)[x - a - (y-b)/y'(x)]$, we obtain

$$\frac{V}{v}\left[x - a - \frac{y-b}{y'(x)}\right] = \int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \implies a + \frac{v}{V}\int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \frac{b-y}{y'(x)} + x.$$

If we differentiate with respect to x to eliminate the integral,

$$\frac{v}{V}\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{y'(-y') - (b-y)y''}{(y')^2} + 1 = \frac{y''(y-b)}{(y')^2} \implies \frac{v}{V}(y')^2\sqrt{1+(y')^2} = y''(y-b).$$

If we set $u = y'$ and $y'' = u\,du/dy$, then

$$\frac{v}{V}u^2\sqrt{1+u^2} = u\frac{du}{dy}(y-b) \implies \frac{v/V}{y-b}dy = \frac{1}{u\sqrt{1+u^2}}du,$$

a separated differential equation. A one-parameter family of solutions is defined implicitly by

$$\frac{v}{V}\ln|y-b| = \int \frac{1}{u\sqrt{1+u^2}}du.$$

The trigonometric substitution $u = \tan\theta$ leads to

$$\frac{v}{V}\ln|y-b| = \ln\left(\frac{\sqrt{1+u^2}-1}{u}\right) + C.$$

Exponentiation gives $D(b-y)^{v/V} = \dfrac{\sqrt{1+u^2}-1}{u}$. Since $u(0) = b/a$, we obtain

$$Db^{v/V} = \frac{\sqrt{1+b^2/a^2}-1}{b/a} \implies D = \frac{\sqrt{a^2+b^2}-a}{b^{1+v/V}}.$$

If we square $1 + Du(b-y)^{v/V} = \sqrt{1+u^2}$, we obtain

$$1 + 2Du(b-y)^{v/V} + D^2u^2(b-y)^{2v/V} = 1 + u^2 \implies \frac{dy}{dx} = u = \frac{2D(b-y)^{v/V}}{1-D^2(b-y)^{2v/V}}.$$

This equation can be separated,

$$dx = \frac{1-D^2(b-y)^{2v/V}}{2D(b-y)^{v/V}}dy = \left[\frac{1}{2D}(b-y)^{-v/V} - \frac{D}{2}(b-y)^{v/V}\right]dy,$$

a one-parameter family of solutions being defined by

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$$-\frac{(b-y)^{1-v/V}}{2D(1-v/V)}+\frac{D(b-y)^{1+v/V}}{2(1+v/V)}=x+C.$$

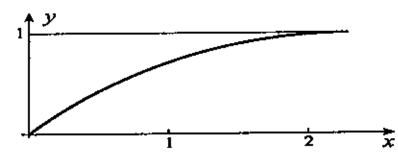
Since $y(0)=0$, we obtain $C=-\dfrac{b^{1-v/V}}{2D(1-v/V)}+\dfrac{Db^{1+v/V}}{2(1+v/V)}$. When these values of C and D are substituted into the implicit definition of the curve, it simplifies to

$$x=\frac{V}{2}\left\{\frac{\sqrt{a^2+b^2}+a}{V-v}\left[1-\left(\frac{b-y}{b}\right)^{1-v/V}\right]-\frac{\sqrt{a^2+b^2}-a}{V+v}\left[1-\left(\frac{b-y}{b}\right)^{1+v/V}\right]\right\}.$$

(b) When $a=b=1$ and $V=2v$, the curve simplifies to

$$x=\frac{2}{3}(2+\sqrt{2})-(\sqrt{2}+1)\sqrt{1-y}+\frac{\sqrt{2}-1}{3}(1-y)^{3/2}.$$

A plot is shown to the right. The misile catches the aircraft when
$y=1\Longrightarrow x=-C=2(2+\sqrt{2})/3.$



18. (a) If we set $v=dy/dT$ and $d^2y/dT^2=v\,dv/dy$,

$$v\frac{dv}{dy}=\frac{5v^2}{4y}-ay+\frac{b}{y}-\frac{c}{y^3}\quad\Longrightarrow\quad\frac{d}{dy}(v^2)-\frac{5v^2}{2y}=-2ay+\frac{2b}{y}-\frac{2c}{y^3}.$$

This is linear in v^2 with integrating factor $e^{\int -5/(2y)dy}=e^{-(5/2)\ln|y|}=1/y^{5/2}$. Hence,

$$\frac{d}{dy}\left(\frac{v^2}{y^{5/2}}\right)=-\frac{2a}{y^{3/2}}+\frac{2b}{y^{7/2}}-\frac{2c}{y^{11/2}}\quad\Longrightarrow\quad\frac{v^2}{y^{5/2}}=\frac{4a}{\sqrt{y}}-\frac{4b}{5y^{5/2}}+\frac{4c}{9y^{9/2}}+D.$$

Solving for $v=dy/dT$ gives

$$\frac{dy}{dT}=\pm\sqrt{4ay^2-\frac{4b}{5}+\frac{4c}{9y^2}+Dy^{5/2}}.$$

Since dy/dT must be negative, and when we set $D=0$, $\dfrac{dy}{dT}=-2\sqrt{ay^2-\dfrac{b}{5}+\dfrac{c}{9y^2}}$.

(b) The above differential equation is separable,

$$\frac{1}{\sqrt{ay^2-\frac{b}{5}+\frac{c}{9y^2}}}=-2\,dT\quad\Longrightarrow\quad\frac{y}{\sqrt{\left(y^2-\frac{b}{10a}\right)^2+\left(\frac{c}{9a}-\frac{b^2}{100a^2}\right)}}dy=-2\sqrt{a}\,dT.$$

If we let $y^2-\dfrac{b}{10a}=\sqrt{\dfrac{c}{9a}-\dfrac{b^2}{100a^2}}\tan\theta$, then $2y\,dy=\sqrt{\dfrac{c}{9a}-\dfrac{b^2}{100a^2}}\sec^2\theta\,d\theta$, and

$$\begin{aligned}-2\sqrt{a}T+E&=\int\frac{\frac{1}{2}\sqrt{\frac{c}{9a}-\frac{b^2}{100a^2}}\sec^2\theta}{\sqrt{\frac{c}{9a}-\frac{b^2}{100a^2}}\sec\theta}d\theta=\frac{1}{2}\ln|\sec\theta+\tan\theta|\\&=\frac{1}{2}\ln\left|\frac{\sqrt{y^4-\frac{by^2}{5a}+\frac{c}{9a}}}{\sqrt{\frac{c}{9a}-\frac{b^2}{100a^2}}}+\frac{y^2-\frac{b}{10a}}{\sqrt{\frac{c}{9a}-\frac{b^2}{100a^2}}}\right|.\end{aligned}$$

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This can be expressed in the form $\ln\left|\sqrt{y^4 - \dfrac{by^2}{5a} + \dfrac{c}{9a}} + y^2 - \dfrac{b}{10a}\right| = -4\sqrt{a}T + F$. Since $y(0) = 1$, we obtain $F = \ln\left|\sqrt{1 - \dfrac{b}{5a} + \dfrac{c}{9a}} + 1 - \dfrac{b}{10a}\right|$, so that $y(T)$ is defined implicitly by

$$\ln\left|\sqrt{y^4 - \frac{by^2}{5a} + \frac{c}{9a}} + y^2 - \frac{b}{10a}\right| = -4\sqrt{a}T + \ln\left|\sqrt{1 - \frac{b}{5a} + \frac{c}{9a}} + 1 - \frac{b}{10a}\right|.$$

EXERCISES 15.5

1. The resistive force is of the form $F_R = -\beta\sqrt{v}$, where β is a constant and v is speed. Since $-F^* = -\beta\sqrt{v^*}$, it follows that $\beta = F^*/\sqrt{v^*}$. The differential equation describing motion is

$$m\frac{dv}{dt} = F + F_R = F - \beta\sqrt{v}.$$

Terminal velocity is attained when acceleration is zero, and this occurs when $F - \beta\sqrt{v} = 0$. This gives $v = F^2/\beta^2 = F^2 v^{*2}/F^{*2}$.

2. The condition $v(0) = v_0$ implies that $C = (1/k)\ln|kv_0 - mg|$. Hence

$$\frac{1}{k}\ln|kv - mg| = -\frac{t}{m} + \frac{1}{k}\ln|kv_0 - mg| \quad\Longrightarrow\quad \ln\left|\frac{kv - mg}{kv_0 - mg}\right| = -\frac{kt}{m} \quad\Longrightarrow\quad \left|\frac{kv - mg}{kv_0 - mg}\right| = e^{-kt/m}.$$

Since terminal velocity occurs when $dv/dt = 0$, it follows from the differential equation that $mg - kv = 0$. Terminal velocity is therefore $v = mg/k$. If the initial velocity v_0 is less than terminal velocity, then v will always be less than terminal velocity. In this case, both $kv - mg < 0$ and $kv_0 - mg < 0$, the quotient being positive. If v_0 is greater than terminal velocity, v will always be greater than terminal velocity. In this case, both $kv - mg > 0$ and $kv_0 - mg > 0$, as is their quotient. In both cases then, we may write

$$\frac{kv - mg}{kv_0 - mg} = e^{-kt/m} \quad\Longrightarrow\quad kv - mg = (kv_0 - mg)e^{-kt/m} \quad\Longrightarrow\quad v = \frac{mg}{k} + \left(v_0 - \frac{mg}{k}\right)e^{-kt/m}.$$

3. (a) Let us take x as positive in the direction of motion of the boat with $x = 0$ and $t = 0$ when motion commences. Since the resistive force is proportional to velocity and is 200 N when the speed is 30 km/hr or 25/2 m/s, its magnitude is 16 times the velocity. Newton's second law for the acceleration of the boat is

$$250\frac{dv}{dt} = 250 - 16v \quad\Longrightarrow\quad \frac{dv}{125 - 8v} = \frac{dt}{125}.$$

A one-parameter family of solutions of this separated equation is defined implicitly by $-(1/8)\ln|125 - 8v| = t/125 + C$. When we solve for v, we get $v = 125/8 - De^{-8t/125}$. Since $v(0) = 0$, we find $0 = 125/8 - D$, and therefore $v(t) = (125/8)(1 - e^{-8t/125})$ m/s.

(b) $\lim_{t\to\infty} v(t) = 125/8$ m/s

4. Let us measure x as positive in the direction of motion taking $x = 0$ and $t = 0$ at the instant the brakes are applied. Because the coefficient of kinetic friction is less than one, we can say that the x-component of the force of friction has magnitude less than $9.81m$, where m is the mass of the car. If we use this as the magnitude of the frictional force, then because this is the maximum possible, we will be finding the maximum possible speed before the brakes were applied. In other words, we are testifying for the defence. Newton's second law for the x-component of the acceleration dv/dt of the car gives

$$m\frac{dv}{dt} = -9.81m,$$

and this can be integrated for $v(t) = -9.81t + C$. If we set $v = v_0$ at time $t = 0$, then $v_0 = C$, and

$$v = \frac{dx}{dt} = -9.81t + v_0.$$

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Integration gives $x(t) = -4.905t^2 + v_0 t + D$. Because $x(0) = 0$, it follows that $D = 0$, and

$$x(t) = -4.905t^2 + v_0 t.$$

The car comes to a stop when $0 = v = -9.81t + v_0$, and this implies that $t = v_0/9.81$. Since $x = 9$ at this instant,

$$9 = -4.905\left(\frac{v_0}{9.81}\right)^2 + v_0\left(\frac{v_0}{9.81}\right).$$

The solution of this equation is $v_0 = 13.29$ m/s or $v_0 = 47.8$ km/hr. Thus, the maximum possible speed of the car was 47.8 km/hr.

5. Let us take x as positive in the direction of motion of the car with $x = 0$ and $t = 0$ when motion commences. Newton's second law for the acceleration of the car is

$$1500\frac{dv}{dt} = 2500 - v^2 \quad \Longrightarrow \quad \frac{dv}{v^2 - 2500} = -\frac{dt}{1500}.$$

A one-parameter family of solutions of this separated differential equation is defined implicitly by

$$-\frac{t}{1500} + C = \int \frac{1}{v^2 - 2500}dv = \frac{1}{100}\int\left(\frac{-1}{v+50} + \frac{1}{v-50}\right)dv$$
$$= \frac{1}{100}(\ln|v-50| - \ln|v+50|) = \frac{1}{100}\ln\left|\frac{v-50}{v+50}\right|.$$

Exponentiation gives

$$\frac{v-50}{v+50} = De^{-t/15} \quad \Longrightarrow \quad v - 50 = D(v+50)e^{-t/15} \quad \Longrightarrow \quad v(t) = \frac{50(1 + De^{-t/15})}{1 - De^{-t/15}}.$$

The initial condition $v(0) = 0$ gives $0 = 50(1+D)/(1-D) \Longrightarrow D = -1$. Hence,

$$v(t) = \frac{50(1 - e^{-t/15})}{1 + e^{-t/15}} \quad \text{and} \quad v(10) = \frac{50(1 - e^{-10/15})}{1 + e^{-10/15}} = 16.1 \text{ m/s}.$$

Integration of the velocity gives

$$x(t) = 50\int\frac{1 - e^{-t/15}}{1 + e^{-t/15}}dt = 50\int\left(1 - \frac{2e^{-t/15}}{1 + e^{-t/15}}\right)dt = 50[t + 30\ln(1 + e^{-t/15})] + C.$$

Since $x(0) = 0$, we find $0 = 50[30\ln 2] + C \Longrightarrow C = -1500\ln 2$. Thus,

$$x(t) = 50t + 1500[\ln(1 + e^{-t/15}) - \ln 2] = 50t + 1500\ln\left[\frac{1}{2}(1 + e^{-t/15})\right].$$

For $t = 10$, we find $x(10) = 500 + 1500\ln\left[\frac{1}{2}(1 + e^{-10/15})\right] = 81.8$ m.

6. (a) Let us take y as positive downward with $y = 0$ and time $t = 0$ when motion begins. Newton's second gives

$$m\frac{dv}{dt} = mg - kv \quad \Longrightarrow \quad \frac{1}{mg - kv}dv = \frac{1}{m}dt,$$

a separated differential equation. A one-parameter family of solutions is defined implicitly by

$$-\frac{1}{k}\ln|mg - kv| = \frac{t}{m} + C \quad \Longrightarrow \quad \ln|mg - kv| = -\frac{kt}{m} - kC \quad \Longrightarrow \quad mg - kv = De^{-kt/m}.$$

For $v(0) = 0$, we find $mg = D$, and therefore

$$mg - kv = mge^{-kt/m} \quad \Longrightarrow \quad v(t) = \frac{mg}{k}(1 - e^{-kt/m}).$$

Terminal velocity is mg/k. Velocity is 95% of this when

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$$0.95\frac{mg}{k} = \frac{mg}{k}(1 - e^{-kt/m}) \quad \Longrightarrow \quad e^{-kt/m} = 1 - 0.95 \quad \Longrightarrow \quad t = \frac{m}{k}\ln 20.$$

(b) Integration of velocity gives

$$y(t) = \frac{mg}{k}\left(t + \frac{m}{k}e^{-kt/m}\right) + D.$$

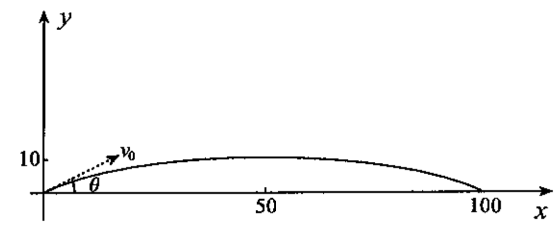
Since $y(0) = 0$, it follows that $0 = (mg/k)(m/k) + D$. Hence, $D = -m^2g/k^2$ and

$$y(t) = \frac{mg}{k}\left(t + \frac{m}{k}e^{-kt/m}\right) - \frac{m^2g}{k^2} = \frac{mgt}{k} - \frac{m^2g}{k^2}(1 - e^{-kt/m}).$$

If we substitute $t = (m/k)\ln 20$, we obtain distance fallen as

$$y = \frac{m^2g}{k^2}\ln 20 - \frac{m^2g}{k^2}(1 - e^{-\ln 20}) = \frac{m^2g}{k^2}\left(\ln 20 - \frac{19}{20}\right).$$

7. We can work with separate equations for x- and y-components of motion or in vectors. We choose the latter. The acceleration of the arrow is $\mathbf{a} = -g\hat{\mathbf{j}}$, so that $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{C}$. If we take $t = 0$ to be the time when the arrow leaves the bow, then when the bow is held at angle θ, and the initial speed of the arrow is v_0,



$\mathbf{v}(0) = v_0(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}})$. This gives $v_0(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) = \mathbf{C}$. Integration of $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{C}$ gives $\mathbf{r} = -gt^2\hat{\mathbf{j}}/2 + \mathbf{C}t + \mathbf{D}$. If the arrow starts from the origin, then $\mathbf{r}(0) = \mathbf{0}$, from which $\mathbf{D} = \mathbf{0}$, and therefore

$$\mathbf{r} = -\frac{1}{2}gt^2\hat{\mathbf{j}} + v_0t(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) = (v_0t\cos\theta)\hat{\mathbf{i}} + \left(-\frac{1}{2}gt^2 + v_0t\sin\theta\right)\hat{\mathbf{j}}.$$

If T is the time for the arrow to reach maximum height, we can say that $\mathbf{R}(T) = 50\hat{\mathbf{i}} + 10\hat{\mathbf{j}}$ and $\mathbf{r}(2T) = 100\hat{\mathbf{i}}$. These imply that

$$50\hat{\mathbf{i}} + 10\hat{\mathbf{j}} = (v_0T\cos\theta)\hat{\mathbf{i}} + \left(-\frac{1}{2}gT^2 + v_0T\sin\theta\right)\hat{\mathbf{j}}, \quad 100\hat{\mathbf{i}} = [v_0(2T)\cos\theta]\hat{\mathbf{i}} + \left[-\frac{1}{2}g(2T)^2 + v_0(2T)\sin\theta\right]\hat{\mathbf{j}}.$$

When we equate components, we obtain four equations, three of which are independent,

$$50 = v_0T\cos\theta, \quad 10 = -\frac{1}{2}gT^2 + v_0T\sin\theta, \quad 0 = -2gT^2 + 2v_0T\sin\theta.$$

We eliminate T and solve for v_0 and θ. The result is $v_0 = 37.7$ m/s and $\theta = 0.381$ radians.

8. (a) When M is at position x, and is moving to the left, the spring force is $-kx$ and the frictional force is μMg. Newton's second law therefore gives

$$M\frac{d^2x}{dt^2} = -kx + \mu Mg, \quad x(0) = x_0, \quad v(0) = -v_0,$$

and this equation is valid until M comes to an instantaneous stop for the first time.

(b) If we set $\dfrac{dx}{dt} = v$ and $\dfrac{d^2x}{dt^2} = \dfrac{dv}{dt} = \dfrac{dv}{dx}\dfrac{dx}{dt} = v\dfrac{dv}{dx}$, then $Mv\dfrac{dv}{dx} = -kx + \mu Mg$, subject to $v(x_0) = -v_0$. This equation can be separated, $Mv\,dv = (-kx + \mu Mg)\,dx$, and a one-parameter family of solutions is defined implicitly by $\dfrac{Mv^2}{2} = -\dfrac{kx^2}{2} + \mu Mgx + C$. The initial condition requires $\dfrac{Mv_0^2}{2} = -\dfrac{kx_0^2}{2} + \mu Mgx_0 + C$. Thus,

$$\frac{Mv^2}{2} = -\frac{kx^2}{2} + \mu Mgx + \frac{Mv_0^2}{2} + \frac{kx_0^2}{2} - \mu Mgx_0 \implies \frac{k}{2}(x_0^2 - x^2) = \frac{M}{2}(v^2 - v_0^2) + \mu Mg(x_0 - x).$$

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The left side is the loss of stored energy in the spring at x relative to that initially; $M(v^2 - v_0^2)/2$ is the gain in kinetic energy at x; and $\mu Mg(x_0 - x)$ is the work done against friction when m moves from x_0 to x.

(c) If we set $x = x^*$ when $v = 0$, then x^* is defined implicitly by

$$\frac{k}{2}(x_0^2 - x^{*2}) = -\frac{Mv_0^2}{2} + \mu Mg(x_0 - x^*).$$

This is a quadratic equation in x^*, $kx^{*2} - 2\mu Mgx^* + (2\mu Mgx_0 - kx_0^2 - Mv_0^2) = 0$, with solutions

$$\begin{aligned} x^* &= \frac{2\mu Mg \pm \sqrt{4\mu^2M^2g^2 - 4k(2\mu Mgx_0 - kx_0^2 - Mv_0^2)}}{2k} \\ &= \frac{\mu Mg \pm \sqrt{\mu^2M^2g^2 - k(2\mu Mgx_0 - kx_0^2 - Mv_0^2)}}{k}. \end{aligned}$$

Whether the mass stops to the left of, to the right of, or on the origin depends on the quantity $2\mu Mgx_0 - kx_0^2 - Mv_0^2$. When it is negative, the sum $(1/2)kx_0^2 + (1/2)Mv_0^2$ of the initial energy stored in the spring $(1/2)kx_0^2$ and the initial kinetic energy of the mass $(1/2)Mv_0^2$ is greater than the work done against friction μMgx_0 as the mass travels from $x = x_0$ to $x = 0$. The mass therefore stops at position

$$x^* = \frac{\mu Mg - \sqrt{\mu^2M^2g^2 + k(kx_0^2 + Mv_0^2 - 2\mu Mgx_0)}}{k}$$

to the left of the origin. When $2\mu Mgx_0 - kx_0^2 - Mv_0^2 = 0$, initial spring energy and kinetic energy are just sufficient to bring the mass back to the origin ($x^* = 0$). Finally, when $2\mu Mgx_0 - kx_0^2 - Mv_0^2 > 0$, there is not sufficient initial energy to return the mass to the origin; it stops at position

$$x^* = \frac{\mu Mg - \sqrt{\mu^2M^2g^2 - k(2\mu Mgx_0 - kx_0^2 - Mv_0^2)}}{k} > 0.$$

9. Let us choose y as positive downward taking $y = 0$ and $t = 0$ when the mass is released. Newton's second law for the acceleration of the mass is

$$(1)\frac{dv}{dt} = g - \frac{v^2}{500},$$

where $g = 9.81$. This equation is separable, $\dfrac{1}{v^2 - 500g}\,dv = -\dfrac{1}{500}\,dt$, and a one-parameter family of solutions is defined implicitly by

$$\begin{aligned} -\frac{t}{500} + C &= \int \frac{1}{v^2 - 500g}\,dv = \int \left(\frac{\frac{1}{2\sqrt{500g}}}{v - \sqrt{500g}} + \frac{\frac{-1}{2\sqrt{500g}}}{v + \sqrt{500g}} \right) dv \\ &= \frac{1}{2\sqrt{500g}}\left(\ln|v - \sqrt{500g}| - \ln|v + \sqrt{500g}|\right). \end{aligned}$$

When this equation is solved for v, the result is $v = \dfrac{\sqrt{500g}[1 + De^{-2\sqrt{g/500t}}]}{1 - De^{-2\sqrt{g/500t}}}$. Since $v(0) = 20$, it follows that $20 = \dfrac{\sqrt{500g}[1 + D]}{1 - D}$, and therefore $D = (20 - \sqrt{500g})/(20 + \sqrt{500g}) = -0.556$. Thus, $v(t) = 70.0\left(\dfrac{1 - 0.556e^{-0.280t}}{1 + 0.556e^{-0.280t}}\right)$ m/s.

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10. Let us choose y as positive downward taking $y = 0$ and $t = 0$ when the mass is released. Newton's second law for the acceleration of the mass is

$$(1)\frac{dv}{dt} = g - \frac{v^2}{500},$$

where $g = 9.81$. This equation is separable, $\dfrac{1}{v^2 - 500g}\,dv = -\dfrac{1}{500}\,dt$, and a one-parameter family of solutions is defined implicitly by

$$-\frac{t}{500} + C = \int \frac{1}{v^2 - 500g}\,dv = \int \left(\frac{\frac{1}{2\sqrt{500g}}}{v - \sqrt{500g}} + \frac{\frac{-1}{2\sqrt{500g}}}{v + \sqrt{500g}} \right) dv$$
$$= \frac{1}{2\sqrt{500g}} \left(\ln|v - \sqrt{500g}| - \ln|v + \sqrt{500g}| \right).$$

When this equation is solved for v, the result is $v = \dfrac{\sqrt{500g}[1 + De^{-2\sqrt{g/500}t}]}{1 - De^{-2\sqrt{g/500}t}}$. Since $v(0) = 100$, it follows that $100 = \dfrac{\sqrt{500g}[1 + D]}{1 - D}$, and therefore $D = (100 - \sqrt{500g})/(100 + \sqrt{500g}) = 0.176$. Thus, $v(t) = 70.0 \left(\dfrac{1 + 0.176e^{-0.280t}}{1 - 0.176e^{-0.280t}} \right)$.

11. Let us choose y as positive upward taking $y = 0$ and $t = 0$ when the mass is released. Newton's second law for the acceleration is

$$(1)\frac{dv}{dt} = -g - \frac{v^2}{500},$$

where $g = 9.81$. We can separate this equation, $\dfrac{1}{v^2 + 500g}\,dv = -\dfrac{1}{500}\,dt$, and a one-parameter family of solutions is defined implicitly by

$$\frac{1}{\sqrt{500g}} \operatorname{Tan}^{-1} \left(\frac{v}{\sqrt{500g}} \right) = -\frac{t}{500} + C.$$

When we solve this for v, $v(t) = \sqrt{500g} \tan \left[\sqrt{500g} \left(C - \dfrac{t}{500} \right) \right]$. Since $v(0) = 20$, it follows that $20 = \sqrt{500g} \tan [C\sqrt{500g}]$, and this equation can be solved for $C = (1/\sqrt{500g})\operatorname{Tan}^{-1}(20/\sqrt{500g}) = 0.00397$. The velocity is

$$v(t) = 70.0 \tan \left[70.0 \left(0.00397 - \frac{t}{500} \right) \right].$$

Maximum height is atained when $v = 0$ and this occurs when $t = 1.99$ s.

12. If we choose y as positive downward, then integration of the differential equation $m dv/dt = mg - kv^2$ as in Example 15.9 leads to

$$v(t) = \frac{V(1 + De^{-2kVt/m})}{1 - De^{-2kVt/m}},$$

where $V = \sqrt{mg/k}$ is the terminal velocity of the body. The initial velocity $v(0) = v_0$ requires

$$v_0 = \frac{V(1 + D)}{1 - D} \quad \Longrightarrow \quad v_0(1 - D) = V(1 + D) \quad \Longrightarrow \quad D = \frac{v_0 - V}{v_0 + V}.$$

Hence, $$v(t) = \frac{V\left[1 + \left(\dfrac{v_0 - V}{v_0 + V}\right) e^{-2kVt/m}\right]}{1 - \left(\dfrac{v_0 - V}{v_0 + V}\right) e^{-2kVt/m}} = \frac{V\left[1 - \left(\dfrac{V - v_0}{V + v_0}\right) e^{-2kVt/m}\right]}{1 + \left(\dfrac{V - v_0}{V + v_0}\right) e^{-2kVt/m}}.$$

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13. If we choose y as positive downward, then integration of the differential equation $m dv/dt = mg - kv^2$ as in Example 15.9 leads to

$$v(t) = \frac{V(1 + De^{-2kVt/m})}{1 - De^{-2kVt/m}},$$

where $V = \sqrt{mg/k}$ is the terminal velocity of the body. The initial velocity $v(0) = v_0$ requires

$$v_0 = \frac{V(1+D)}{1-D} \implies v_0(1-D) = V(1+D) \implies D = \frac{v_0 - V}{v_0 + V}.$$

Hence, $$v(t) = \frac{V\left[1 + \left(\dfrac{v_0 - V}{v_0 + V}\right)e^{-2kVt/m}\right]}{1 - \left(\dfrac{v_0 - V}{v_0 + V}\right)e^{-2kVt/m}}.$$

14. If we choose y as positive upward, the differential equation describing the velocity of the body is

$$m\frac{dv}{dt} = -mg - kv^2 \implies \frac{dv}{v^2 + mg/k} = -\frac{k}{m}dt \implies \sqrt{\frac{k}{mg}}\,\mathrm{Tan}^{-1}\left(\frac{v}{\sqrt{mg/k}}\right) = -\frac{kt}{m} + C.$$

Hence, $v(t) = \sqrt{\dfrac{mg}{k}}\tan\left(D - \sqrt{\dfrac{kg}{m}}t\right)$, where $D = \sqrt{mg/k}C$. The initial condition $v(0) = v_0$ requires $v_0 = \sqrt{mg/k}\tan D \implies D = \mathrm{Tan}^{-1}[\sqrt{k/(mg)}v_0]$. Thus,

$$v(t) = \sqrt{\frac{mg}{k}}\tan\left[\mathrm{Tan}^{-1}\left(\sqrt{\frac{k}{mg}}v_0\right) - \sqrt{\frac{kg}{m}}t\right].$$

Maximum height is attained when

$$0 = v(t) = \sqrt{\frac{mg}{k}}\tan\left[\mathrm{Tan}^{-1}\left(\sqrt{\frac{k}{mg}}v_0\right) - \sqrt{\frac{kg}{m}}t\right] \implies \mathrm{Tan}^{-1}\left(\sqrt{\frac{k}{mg}}v_0\right) - \sqrt{\frac{kg}{m}}t = n\pi,$$

where n is an integer. Solving for t gives $t = \sqrt{\dfrac{m}{kg}}\left[\mathrm{Tan}^{-1}\left(\sqrt{\dfrac{k}{mg}}v_0\right) - n\pi\right]$. When we choose $n = 0$ to obtain the smallest positive solution, $t = \sqrt{\dfrac{m}{kg}}\,\mathrm{Tan}^{-1}\left(\sqrt{\dfrac{k}{mg}}v_0\right)$.

15. If we choose y as positive upward, the differential equation describing the velocity of the rock is

$$\frac{dv}{dt} = -9.81 - \frac{v^2}{10} \implies \frac{dv}{v^2 + 98.1} = -\frac{dt}{10} \implies \frac{1}{\sqrt{98.1}}\,\mathrm{Tan}^{-1}\left(\frac{v}{\sqrt{98.1}}\right) = -\frac{t}{10} + C.$$

Hence, $v(t) = \sqrt{98.1}\tan\left(\dfrac{-\sqrt{98.1}t}{10} + D\right)$. The initial velocity $v(0) = 20$ requires $20 = \sqrt{98.1}\tan D \implies D = \mathrm{Tan}^{-1}(20/\sqrt{98.1})$. Thus, $v(t) = \sqrt{98.1}\tan\left[\mathrm{Tan}^{-1}\left(\dfrac{20}{\sqrt{98.1}}\right) - \dfrac{\sqrt{98.1}t}{10}\right]$. Integration gives $y(t) = 10\ln\left\{\cos\left[\mathrm{Tan}^{-1}\left(\dfrac{20}{\sqrt{98.1}}\right) - \dfrac{\sqrt{98.1}t}{10}\right]\right\} + C$. If we take $y(0) = 0$, then $0 = 10\ln\left\{\cos\left[\mathrm{Tan}^{-1}\left(\dfrac{20}{\sqrt{98.1}}\right)\right]\right\} + C$, which defines C. Hence,

$$y(t) = 10\ln\left\{\cos\left[\mathrm{Tan}^{-1}\left(\frac{20}{\sqrt{98.1}}\right) - \frac{\sqrt{98.1}t}{10}\right]\right\} - 10\ln\left\{\cos\left[\mathrm{Tan}^{-1}\left(\frac{20}{\sqrt{98.1}}\right)\right]\right\}.$$

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For maximum height we set $v = 0$, and this implies that $\operatorname{Tan}^{-1}\left(\dfrac{20}{\sqrt{98.1}}\right) - \dfrac{\sqrt{98.1}t}{10} = n\pi$, where n is an integer. When we choose $n = 0$ for the smallest positive solution, $t = (10/\sqrt{98.1})\operatorname{Tan}^{-1}(20/\sqrt{98.1})$. For this t, the height of the rock is $-10\ln\{\cos[\operatorname{Tan}^{-1}(20/\sqrt{98.1})]\} = 8.1$ m.

16. (a) Let us choose y as positive upward taking $y = 0$ and $t = 0$ when the mass is released. Newton's second law for acceleration during ascent is $m\dfrac{dv}{dt} = -mg - kv^2$. This is valid only during ascent since air resistance during descent is kv^2. If we set $V = \sqrt{mg/k}$, then

$$-\frac{m}{k}\frac{dv}{dt} = V^2 + v^2 \quad \Longrightarrow \quad \frac{1}{v^2 + V^2}\,dv = -\frac{k}{m}\,dt \quad \Longrightarrow \quad \frac{1}{V}\operatorname{Tan}^{-1}\left(\frac{v}{V}\right) = -\frac{kt}{m} + C.$$

Since $v(0) = v_0$, we obtain $\dfrac{1}{V}\operatorname{Tan}^{-1}\left(\dfrac{v_0}{V}\right) = C$. Thus,

$$\frac{1}{V}\operatorname{Tan}^{-1}\left(\frac{v}{V}\right) = -\frac{kt}{m} + \frac{1}{V}\operatorname{Tan}^{-1}\left(\frac{v_0}{V}\right).$$

When we solve this equation for v, the result is $v(t) = V\tan\left[\operatorname{Tan}^{-1}\left(\dfrac{v_0}{V}\right) - \dfrac{kVt}{m}\right]$. Once again this is only valid during ascent since air resistance is kv^2 (rather than $-kv^2$) on descent.

(b) Integration of the velocity gives $y = \dfrac{m}{k}\ln\left|\cos\left[\operatorname{Tan}^{-1}\left(\dfrac{v_0}{V}\right) - \dfrac{kVt}{m}\right]\right| + D$. Since $y(0) = 0$, we find that $0 = \dfrac{m}{k}\ln\left|\cos\left[\operatorname{Tan}^{-1}\left(\dfrac{v_0}{V}\right)\right]\right| + D$, and therefore $D = (m/k)\ln(\sqrt{v_0^2 + V^2}/V)$. The height of the mass is

$$y = \frac{m}{k}\ln\left|\cos\left[\operatorname{Tan}^{-1}\left(\frac{v_0}{V}\right) - \frac{kVt}{m}\right]\right| + \frac{m}{k}\ln\left(\frac{\sqrt{v_0^2 + V^2}}{V}\right).$$

Maximum height occurs when $0 = v = V\tan\left[\operatorname{Tan}^{-1}\left(\dfrac{v_0}{V}\right) - \dfrac{kVt}{m}\right]$, and the solution of this equation is $t = \dfrac{m}{kV}\operatorname{Tan}^{-1}\left(\dfrac{v_0}{V}\right)$. Maximum height is therefore

$$\frac{m}{k}\ln\left|\cos\left[\operatorname{Tan}^{-1}\left(\frac{v_0}{V}\right) - \operatorname{Tan}^{-1}\left(\frac{v_0}{V}\right)\right]\right| + \frac{m}{k}\ln\left(\frac{\sqrt{v_0^2 + V^2}}{V}\right) = \frac{m}{k}\ln\left(\frac{\sqrt{v_0^2 + V^2}}{V}\right).$$

17. If we take x as positive in the direction of motion, the differential equation describing motion is

$$m\frac{dv}{dt} = -F - \beta v \quad \Longrightarrow \quad \frac{dv}{F + \beta v} = -\frac{dt}{m} \quad \Longrightarrow \quad \frac{1}{\beta}\ln|F + \beta v| = -\frac{t}{m} + C.$$

Exponentiation gives $F + \beta v = De^{-\beta t/m} \Longrightarrow v = (1/\beta)(De^{-\beta t/m} - F)$. The initial velocity $v(0) = v_0$ requires $v_0 = (1/\beta)(D - F) \Longrightarrow D = F + \beta v_0$, and $v(t) = (1/\beta)(F + \beta v_0)e^{-\beta t/m} - F/\beta$. Integration of this gives

$$x(t) = -\frac{m}{\beta^2}(F + \beta v_0)e^{-\beta t/m} - \frac{Ft}{\beta} + C.$$

If we take $x(0) = 0$, then $0 = -(m/\beta^2)(F + \beta v_0) + C \Longrightarrow C = (m/\beta^2)(F + \beta v_0)$, and

$$x(t) = -\frac{m}{\beta^2}(F + \beta v_0)e^{-\beta t/m} - \frac{Ft}{\beta} + \frac{m}{\beta^2}(F + \beta v_0) = -\frac{Ft}{\beta} + \frac{m}{\beta^2}(F + \beta v_0)(1 - e^{-\beta t/m}).$$

The object comes to rest when $0 = v(t) = \dfrac{1}{\beta}(F + \beta v_0)e^{-\beta t/m} - \dfrac{F}{\beta}$. This can be solved for t,

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$$e^{-\beta t/m} = \frac{F}{F+\beta v_0} \quad \Longrightarrow \quad t = -\frac{m}{\beta}\ln\left(\frac{F}{F+\beta v_0}\right) = \frac{m}{\beta}\ln\left(1+\frac{\beta v_0}{F}\right).$$

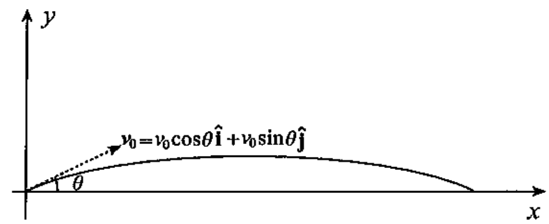
The position of the object at this time is

$$x = -\frac{F}{\beta}\left(\frac{m}{\beta}\right)\ln\left(1+\frac{\beta v_0}{F}\right) + \frac{m}{\beta^2}(F+\beta v_0)\left(1-\frac{F}{F+\beta v_0}\right) = -\frac{Fm}{\beta^2}\ln\left(1+\frac{\beta v_0}{F}\right) + \frac{mv_0}{\beta}.$$

18. The initial-value problem for the motion of the projectile (figure to the right) is

$$m\frac{d^2\mathbf{r}}{dt^2} = -mg\hat{\mathbf{j}} - \beta\mathbf{v},$$

subject to initial displacement $\mathbf{r}(0) = \mathbf{0}$ and initial velocity $\mathbf{r}'(0) = \mathbf{v}_0 = v_0\cos\theta\,\hat{\mathbf{i}} + v_0\sin\theta\,\hat{\mathbf{j}}$. We can solve this differential equation in vector form, or in component form. The component scalar differential equations are identical, so let us save space by integrating vectorially. When we write



$$\frac{d\mathbf{v}}{dt} + \frac{\beta}{m}\mathbf{v} = -g\hat{\mathbf{j}},$$

we have a linear first-order differential equation with integrating factor $e^{\int(\beta/m)dt} = e^{\beta t/m}$. When we multiply the differential equation by $e^{\beta t/m}$, it can be expressed in the form

$$\frac{d}{dt}\left(\mathbf{v}e^{\beta t/m}\right) = -ge^{\beta t/m}\hat{\mathbf{j}} \quad \Longrightarrow \quad \mathbf{v}e^{\beta t/m} = -\frac{mg}{\beta}e^{\beta t/m}\hat{\mathbf{j}} + \mathbf{C} \quad \Longrightarrow \quad \mathbf{v} = -\frac{mg}{\beta}\hat{\mathbf{j}} + \mathbf{C}e^{-\beta t/m}.$$

The initial velocity condition requires $\mathbf{v}_0 = -\frac{mg}{\beta}\hat{\mathbf{j}} + \mathbf{C}$, and therefore

$$\mathbf{v}(t) = -\frac{mg}{\beta}\hat{\mathbf{j}} + \left(\mathbf{v}_0 + \frac{mg}{\beta}\hat{\mathbf{j}}\right)e^{-\beta t/m}.$$

Integrating once again gives

$$\mathbf{r}(t) = -\frac{mgt}{\beta}\hat{\mathbf{j}} - \frac{m}{\beta}\left(\mathbf{v}_0 + \frac{mg}{\beta}\hat{\mathbf{j}}\right)e^{-\beta t/m} + \mathbf{D}.$$

For $\mathbf{r}(0) = \mathbf{0}$, we must have $\mathbf{0} = -\frac{m}{\beta}\left(\mathbf{v}_0 + \frac{mg}{\beta}\hat{\mathbf{j}}\right) + \mathbf{D}$. Thus,

$$\mathbf{r}(t) = -\frac{mgt}{\beta}\hat{\mathbf{j}} - \frac{m}{\beta}\left(\mathbf{v}_0 + \frac{mg}{\beta}\hat{\mathbf{j}}\right)e^{-\beta t/m} + \frac{m}{\beta}\left(\mathbf{v}_0 + \frac{mg}{\beta}\right)\hat{\mathbf{j}} = -\frac{mgt}{\beta}\hat{\mathbf{j}} + \frac{m}{\beta}\left(\mathbf{v}_0 + \frac{mg}{\beta}\hat{\mathbf{j}}\right)(1 - e^{-\beta t/m}).$$

19. When resistance is proportional to the square of velocity, Newton's second law can be expressed in the form

$$m\frac{d\mathbf{v}}{dt} = -mg\hat{\mathbf{j}} - \beta|\mathbf{v}|\mathbf{v}.$$

In vector form it is perhaps not clear that we cannot integrate this equation. If we separate it into x- and y-components, however, we obtain

$$m\frac{dv_x}{dt} = -\beta v_x\sqrt{v_x^2 + y_y^2}, \qquad m\frac{dv_y}{dt} = -mg - \beta v_y\sqrt{v_x^2 + v_y^2}.$$

Both equations contain both unknowns v_x and v_y, unlike the previous exercise where component equations would be uncoupled.

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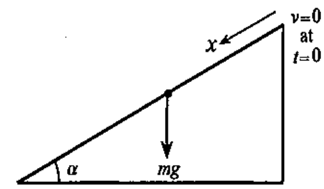
20. Newton's second law for acceleration gives

$$m\frac{dv}{dt} = mg\sin\alpha.$$



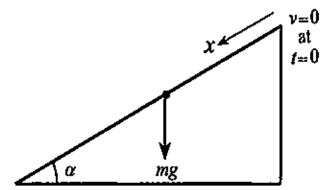
Integration gives $v = (g\sin\alpha)t + C$. Since $v(0) = 0$, it follows that $C = 0$, and

$$v = \frac{dx}{dt} = (g\sin\alpha)t.$$

Integration now gives $x = (g\sin\alpha)t^2/2 + E$. If we choose $x = 0$ when motion begins, then $E = 0$, and $x = (g\sin\alpha)t^2/2$. The time for the mass to travel distance D is given by $D = (g\sin\alpha)t^2/2 \implies t = \sqrt{2D/(g\sin\alpha)}$. The speed of the mass at this time is $g\sin\alpha\sqrt{\dfrac{2D}{g\sin\alpha}} = \sqrt{2gD\sin\alpha}$.

21. When air resistance proportional to velocity acts on the mass, Newton's second law gives

$$m\,\frac{dv}{dt} = mg\sin\alpha - kv.$$



This equation can be separated,

$$\frac{1}{v - \dfrac{mg}{k}\sin\alpha}\,dv = -\frac{k}{m}\,dt,$$

and a one-parameter family of solutions is defined implicitly by $\ln\left|v - \dfrac{mg}{k}\sin\alpha\right| = -\dfrac{kt}{m} + C$. When we solve for v, the result is $v = (mg/k)\sin\alpha + De^{-kt/m}$. Since $v(0) = 0$, it follows that $0 = (mg/k)\sin\alpha + D$. Hence, $v = (mg/k)\sin\alpha[1 - e^{-kt/m}]$. Integration now gives $x = \dfrac{mg}{k}\sin\alpha\left[t + \dfrac{m}{k}e^{-kt/m}\right] + E$. Since $x(0) = 0$, we obtain $0 = \dfrac{m^2g}{k^2}\sin\alpha + E$. Thus,

$$x = \frac{m^2g}{k^2}\sin\alpha\left[\frac{kt}{m} + e^{-kt/m}\right] - \frac{m^2g}{k^2}\sin\alpha = \frac{m^2g}{k^2}\sin\alpha\left[\frac{kt}{m} + e^{-kt/m} - 1\right].$$

22. Since $dy/dt = 2x + 4$ and $dx/dt = (3 - y)/2$, it follows that

$$\frac{dy}{dx} = \frac{2x+4}{(3-y)/2} \quad\Longrightarrow\quad (3-y)\,dy = 4(x+2)\,dx,$$

a separated equation. A one-parameter family of solutions is defined implicitly by

$$3y - \frac{y^2}{2} = 4\left(\frac{x^2}{2} + 2x\right) + C.$$

Since the electron passes through $(0, 3)$, we must have $9 - 9/2 = C$, and therefore

$$3y - \frac{y^2}{2} = 2x^2 + 8x + \frac{9}{2} \quad\Longrightarrow\quad 4x^2 + y^2 + 16x - 6y + 9 = 0.$$

This is an ellipse.

23. If we take the positive direction away from the earth's surface, then Newton's second law for the acceleration of a projectile gives $m\dfrac{dv}{dt} = -\dfrac{GMm}{r^2}$. If we set $\dfrac{dv}{dt} = \dfrac{dv}{dr}\dfrac{dr}{dt} = v\dfrac{dv}{dr}$, then

$$v\frac{dv}{dr} = -\frac{GM}{r^2} \quad\Longrightarrow\quad v\,dv = -\frac{GM\,dr}{r^2}.$$

A one-parameter family of solutions of this separated differential equation is defined implicitly by $v^2/2 = GM/r + C$. If R is the radius of the earth, and v_0 is the initial velocity of the projectile, then $v(R) = v_0$, and this implies that ${v_0}^2/2 = GM/R + C$. Thus, $v^2/2 = GM/r + {v_0}^2/2 - GM/R$. The projectile escapes the gravitational pull of the earth if its velocity approaches 0 as r becomes infinite. This requires ${v_0}^2/2 - GM/R = 0$, and therefore the initial velocity of the projectile must be $v_0 = \sqrt{2GM/R}$. For an

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alternative expression, we note that on the earth's surface where $r = R$, the force of gravity on the mass is $-mg = -GMm/R^2 \Longrightarrow GM/R = gR$. Thus, $v_0 = \sqrt{2gR}$.

24. (a) At the surface of the earth, the magnitude of the force of gravity on m is

$$9.81m = \frac{GmM}{(6.37\times 10^6)^2} \quad\Longrightarrow\quad GM = 9.81(6.37\times 10^6)^2 = 3.98\times 10^{14}.$$

(b) If we set $\dfrac{dv}{dt} = \dfrac{dv}{dr}\dfrac{dr}{dt} = v\dfrac{dv}{dr}$, then $v\dfrac{dv}{dr} = -\dfrac{GM}{r^2}$. Variables are separable, $v\,dv = -(GM/r^2)dr$, with solutions defined implicitly by $v^2/2 = GM/r + C$. Since the object is dropped from height $r = 11\,370$, $0 = GM/(11.37\times 10^6) + C \Longrightarrow C = -GM/(11.37\times 10^6)$. Thus,

$$\frac{v^2}{2} = \frac{GM}{r} - \frac{GM}{11.37\times 10^6} \Longrightarrow v(r) = -\sqrt{\frac{2GM}{11.37\times 10^6}}\sqrt{\frac{11.37\times 10^6 - r}{r}} = -8.37\times 10^3\sqrt{\frac{11.37\times 10^6 - r}{r}}.$$

(c)If we substitute $v = dr/dt$,

$$\frac{dr}{dt} = -8.37\times 10^3\sqrt{\frac{11.37\times 10^6 - r}{r}} \quad\Longrightarrow\quad \sqrt{\frac{r}{11.37\times 10^6 - r}}dr = -8.37\times 10^3 dt,$$

from which

$$-8.37\times 10^3 t + C = \int \sqrt{\frac{r}{11.37\times 10^6 - r}}dr.$$

If we set $r = 11.37\times 10^6 \sin^2\theta$ and $dr = 22.74\times 10^6 \sin\theta\cos\theta\,d\theta$, then

$$\begin{aligned}
-8.37\times 10^3 t + C &= \int \frac{\sqrt{11.37\times 10^6}\sin\theta}{\sqrt{11.37\times 10^6}\cos\theta} 22.74\times 10^6 \sin\theta\cos\theta\,d\theta = 22.74\times 10^6 \int \sin^2\theta\,d\theta \\
&= 22.74\times 10^6 \int \left(\frac{1-\cos 2\theta}{2}\right)d\theta = 11.37\times 10^6\left(\theta - \frac{1}{2}\sin 2\theta\right) \\
&= 11.37\times 10^6(\theta - \sin\theta\cos\theta) \\
&= 11.37\times 10^6\left(\mathrm{Sin}^{-1}\sqrt{\frac{r}{11.37\times 10^6}} - \sqrt{\frac{r}{11.37\times 10^6}}\sqrt{1 - \frac{r}{11.37\times 10^6}}\right) \\
&= 11.37\times 10^6\ \mathrm{Sin}^{-1}\sqrt{\frac{r}{11.37\times 10^6}} - \sqrt{11.37\times 10^6 r - r^2}.
\end{aligned}$$

If the object is dropped at time $t = 0$, then $C = 11.37\times 10^6\ \mathrm{Sin}^{-1}\sqrt{\dfrac{11.37\times 10^6}{11.37\times 10^6}} = 5.685\times 10^6\pi$, and the following equation defines r as a function of t,

$$-8.37\times 10^3 t + 5.685\times 10^6\pi = 11.37\times 10^6\ \mathrm{Sin}^{-1}\sqrt{\frac{r}{11.37\times 10^6}} - \sqrt{11.37\times 10^6 r - r^2}.$$

(d) The object hits the earth when

$$-8.37\times 10^3 t + 5.685\times 10^6\pi = 11.37\times 10^6\ \mathrm{Sin}^{-1}\sqrt{\frac{6.37\times 10^6}{11.37\times 10^6}} - \sqrt{11.37\times 10^6(6.37\times 10^6) - (6.37\times 10^6)^2}.$$

When this is solved for t, the result is approximately 2000 s.

25. We take y positive upward with $y = 0$ and $t = 0$ at the point of release. The differential equation describing the motion of the stone during ascent is

$$\frac{1}{10}\frac{dv}{dt} = -\frac{9.81}{10} - \frac{v^2}{1000} \quad\Longrightarrow\quad \frac{dv}{v^2 + 981} = -\frac{dt}{100} \quad\Longrightarrow\quad \frac{1}{\sqrt{981}}\mathrm{Tan}^{-1}\left(\frac{v}{\sqrt{981}}\right) = -\frac{t}{100} + C.$$

The initial condition $v(0) = 20$ implies that $C = (1/\sqrt{981})\mathrm{Tan}^{-1}(20/\sqrt{981})$, and therefore

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$$\frac{1}{\sqrt{981}}\text{Tan}^{-1}\left(\frac{v}{\sqrt{981}}\right) = \frac{-t}{100} + \frac{1}{\sqrt{981}}\text{Tan}^{-1}\left(\frac{20}{\sqrt{981}}\right) \quad \Longrightarrow \quad v = \sqrt{981}\tan\left[\text{Tan}^{-1}\left(\frac{20}{\sqrt{981}}\right) - \frac{\sqrt{981}t}{100}\right].$$

Integration of this gives $y(t) = 100\ln\left|\cos\left[\text{Tan}^{-1}\left(\frac{20}{\sqrt{981}}\right) - \frac{\sqrt{981}t}{100}\right]\right| + D.$

Since $y(0) = 0$, we find $0 = 100\ln|\cos[\text{Tan}^{-1}(20/\sqrt{981})]| + D$, from which $D = -100\ln(\sqrt{981}/\sqrt{1381}) = 50\ln(1381/981)$. The height of the stone during ascent is therefore

$$y(t) = 100\ln\left|\cos\left[\text{Tan}^{-1}\left(\frac{20}{\sqrt{981}}\right) - \frac{\sqrt{981}t}{100}\right]\right| + 50\ln\left(\frac{1381}{981}\right).$$

Maximum height occurs when $0 = v = \sqrt{981}\tan\left[\text{Tan}^{-1}\left(\frac{20}{\sqrt{981}}\right) - \frac{\sqrt{981}t}{100}\right]$, and the solution of this equation is $t = (100/\sqrt{981})\text{Tan}^{-1}(20/\sqrt{981}) = 1.81$ s. The height of the stone at this time is

$$100\ln\left|\cos\left[\text{Tan}^{-1}\left(\frac{20}{\sqrt{981}}\right) - \text{Tan}^{-1}\left(\frac{20}{\sqrt{981}}\right)\right]\right| + 50\ln\left(\frac{1381}{981}\right) = 50\ln\left(\frac{1381}{981}\right).$$

Maintaining the same coordinate system, the differential equation describing the descent of the stone is

$$\frac{1}{10}\frac{dv}{dt} = -\frac{9.81}{10} + \frac{v^2}{1000} \quad \Longrightarrow \quad \frac{dv}{v^2 - 981} = \frac{dt}{100} \quad \Longrightarrow \quad \left(\frac{1}{v - \sqrt{981}} - \frac{1}{v + \sqrt{981}}\right)dv = \frac{\sqrt{981}}{50}dt.$$

Integration gives

$$\ln|v - \sqrt{981}| - \ln|v + \sqrt{981}| = \frac{\sqrt{981}t}{50} + E \quad \Longrightarrow \quad \ln\left|\frac{v - \sqrt{981}}{v + \sqrt{981}}\right| = \frac{\sqrt{981}t}{50} + E.$$

Exponentiation gives

$$\frac{v - \sqrt{981}}{v + \sqrt{981}} = Fe^{\sqrt{981}t/50} \Longrightarrow v - \sqrt{981} = (v + \sqrt{981})Fe^{\sqrt{981}t/50} \Longrightarrow v = \frac{\sqrt{981}(1 + Fe^{\sqrt{981}t/50})}{1 - Fe^{\sqrt{981}t/50}}.$$

Let us reset the time to $t = 0$ when the stone begins its downward motion. Then $v(0) = 0$, and this implies that $0 = \sqrt{981}(1 + F)/(1 - F) \Longrightarrow F = -1$. Thus $v(t) = \frac{\sqrt{981}(1 - e^{\sqrt{981}t/50})}{1 + e^{\sqrt{981}t/50}}$. Integration gives the position of the stone on its descent,

$$y(t) = \sqrt{981}\int\left(1 - \frac{2e^{\sqrt{981}t/50}}{1 + e^{\sqrt{981}t/50}}\right)dt = \sqrt{981}\left[t - \frac{100}{\sqrt{981}}\ln|1 + e^{\sqrt{981}t/50}|\right] + G.$$

Since $y(0) = 50\ln(1381/981)$,

$$50\ln(1381/981) = \sqrt{981}\left(-\frac{100}{\sqrt{981}}\ln 2\right) + G \quad \Longrightarrow \quad G = 50\ln\left(\frac{1381}{981}\right) + 100\ln 2 = 50\ln\left(\frac{5524}{981}\right).$$

The stone hits the ground when $0 = y(t) = \sqrt{981}\left[t - \frac{100}{\sqrt{981}}\ln(1 + e^{\sqrt{981}t/50})\right] + 50\ln\left(\frac{5524}{981}\right)$. This equation can be solved numerically for $t = 1.92$ s. When we add this to the ascent time, total time in the air is $1.81 + 1.92 = 3.73$ s.

If air resistance is ignored, acceleration of the stone is given by $dv/dt = -9.81 \Longrightarrow v = -9.81t + C$. With $v(0) = 20$, we find $C = 20$. A second integration gives $y = -4.905t^2 + 20t + D$. Since $y(0) = 0$, it follows that $D = 0$, and $y(t) = -4.905t^2 + 20t$. The stone returns to the ground when

$$0 = y = -4.905t^2 + 20t = t(-4.905t + 20) \quad \Longrightarrow \quad t = 20/4.905 = 4.08 \text{ s}.$$

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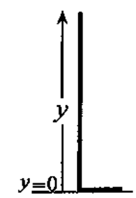
26. (a) According to Newton's second law $m\dfrac{d^2r}{dt^2} = -\dfrac{GMm}{(r+R)^2} + \dfrac{GM^*m}{(a+R^*-r)^2}$. On the earth's surface we know that the magnitude of the force of gravity of the earth on m is mg, and therefore $mg = \dfrac{GMm}{R^2} \Longrightarrow GM = gR^2$. Similarly, on the moon's surface, we obtain $mg^* = \dfrac{GM^*m}{R^{*2}} \Longrightarrow GM^* = g^*R^{*2}$. Thus,

$$\frac{d^2r}{dt^2} = \frac{-gR^2}{(r+R)^2} + \frac{g^*R^{*2}}{(a+R^*-r)^2}.$$

(b) If we set $\dfrac{dr}{dt} = v$ and $\dfrac{d^2r}{dt^2} = \dfrac{dv}{dt} = \dfrac{dv}{dr}\dfrac{dr}{dt} = v\dfrac{dv}{dr}$, then $v\dfrac{dv}{dr} = \dfrac{-gR^2}{(r+R)^2} + \dfrac{g^*R^{*2}}{(a+R^*-r)^2}$. This equation can be separated, $v\,dv = \left[\dfrac{-gR^2}{(r+R)^2} + \dfrac{g^*R^{*2}}{(a+R^*-r)^2}\right]dr$, and a one-parameter family of solutions is defined implicitly by $\dfrac{v^2}{2} = \dfrac{gR^2}{r+R} + \dfrac{g^*R^{*2}}{a+R^*-r} + C$. Since $v(0) = v_0$, it follows that $\dfrac{v_0^2}{2} = \dfrac{gR^2}{R} + \dfrac{g^*R^{*2}}{a+R^*} + C$, and therefore $v^2 = \dfrac{2gR^2}{r+R} + \dfrac{2g^*R^{*2}}{a+R^*-r} + v_0^2 - 2gR - \dfrac{2g^*R^{*2}}{a+R^*}$.

27. If y is the length of chain still falling at any given time t, then the mass of falling chain at this time is $m = 2y$, and the force of gravity on this chain has y-component $-2gy$. Newton's second law requires



$$\frac{d}{dt}\left(2y\frac{dy}{dt}\right) = -2gy \quad\Longrightarrow\quad y\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^2 = -gy.$$

Because t is explicitly missing, we set $\dfrac{dy}{dt} = v$ and $\dfrac{d^2y}{dt^2} = \dfrac{dv}{dt} = \dfrac{dv}{dy}\dfrac{dy}{dt} = v\dfrac{dv}{dy}$,

$$yv\frac{dv}{dy} + v^2 = -gy \quad\Longrightarrow\quad \frac{dv}{dy} + \frac{v}{y} = -\frac{g}{v}.$$

This is a Bernoulli equation (see Exercise 15.3-17) so that we set $w = v^2$ and $dw/dy = 2v\,dv/dy$,

$$\frac{1}{2v}\frac{dw}{dy} + \frac{v}{y} = -\frac{g}{v} \quad\Longrightarrow\quad \frac{dw}{dy} + \frac{2}{y}w = -2g.$$

An integrating factor is $e^{\int (2/y)\,dy} = y^2$, and therefore

$$\frac{d}{dy}(wy^2) = -2gy^2 \quad\Longrightarrow\quad wy^2 = -\frac{2gy^3}{3} + C \quad\Longrightarrow\quad v^2 = -\frac{2gy}{3} + \frac{C}{y^2}.$$

Since $v(3) = 0$, we find $0 = -2g + C/9$, and therefore $v^2 = -\dfrac{2gy}{3} + \dfrac{18g}{y^2}$. Since v must be negative,

$$v = -\sqrt{\left(-\frac{2y}{3} + \frac{18}{y^2}\right)g} = -\frac{1.81}{y}\sqrt{54 - 2y^3} \quad \text{m/s}.$$

The velocity of the end when it hits the floor is $\lim_{y\to 0^+} v(y) = -\infty$.

EXERCISES 15.6

1. Since $L(y_1 + y_2) = 5(y_1 + y_2) = 5y_1 + 5y_2 = L(y_1) + L(y_2)$, and
$L(cy_1) = 5(cy_1) = c(5y_1) = cL(y_1)$,
the operator L is linear.

2. Since $L(y_1 + y_2) = 15x(y_1 + y_2) = 15xy_1 + 15xy_2 = L(y_1) + L(y_2)$, and
$L(cy_1) = 15x(cy_1) = c(15xy_1) = cL(y_1)$,

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the operator L is linear.

3. Since $\quad L(y_1+y_2)=y_1+y_2+z(x)$, but
$$L(y_1)+L(y_2)=[y_1+z(x)]+[y_2+z(x)],$$
the operator L is not linear.

4. Since $L(y_1+y_2)=\lim_{x\to 3}(y_1+y_2)=\lim_{x\to 3}y_1+\lim_{x\to 3}y_2=L(y_1)+L(y_2)$, and
$$L(cy_1)=\lim_{x\to 3}cy_1=c\lim_{x\to 3}y_1=cL(y_1),$$
the operator L is linear.

5. Since $L(y_1+y_2)=\lim_{x\to\infty}(y_1+y_2)=\lim_{x\to\infty}y_1+\lim_{x\to\infty}y_2=L(y_1)+L(y_2)$, and
$$L(cy_1)=\lim_{x\to\infty}cy_1=c\lim_{x\to\infty}y_1=cL(y_1),$$
the operator L is linear.

6. Since $L(y_1+y_2)=\dfrac{d}{dx}(y_1+y_2)=\dfrac{dy_1}{dx}+\dfrac{dy_2}{dx}=L(y_1)+L(y_2)$, and
$$L(cy_1)=\frac{d}{dx}(cy_1)=c\frac{dy_1}{dx}=cL(y_1),$$
the operator L is linear.

7. Since $L(y_1+y_2)=\dfrac{d^3}{dx^3}(y_1+y_2)=\dfrac{d^3y_1}{dx^3}+\dfrac{d^3y_2}{dx^3}=L(y_1)+L(y_2)$, and
$$L(cy_1)=\frac{d^3}{dx^3}(cy_1)=c\frac{d^3y_1}{dx^3}=cL(y_1),$$
the operator L is linear.

8. Since $L(y_1+y_2)=\displaystyle\int(y_1+y_2)\,dx=\int y_1\,dx+\int y_2\,dx=L(y_1)+L(y_2)$, and
$$L(cy_1)=\int cy_1\,dx=c\int y_1\,dx=cL(y_1),$$
the operator L is linear.

9. Since $L(y_1+y_2)=\displaystyle\int_{-1}^{4}(y_1+y_2)\,dx=\int_{-1}^{4}y_1\,dx+\int_{-1}^{4}y_2\,dx=L(y_1)+L(y_2)$, and
$$L(cy_1)=\int_{-1}^{4}cy_1\,dx=c\int_{-1}^{4}y_1\,dx=cL(y_1),$$
the operator L is linear.

10. Since $L(y_1+y_2)=(y_1+y_2)^{1/3}$, but $L(y_1)+L(y_2)=y_1^{1/3}+y_2^{1/3}$, the operator L is not linear.

11. This equation is linear, and in operator notation
$$\phi(x,D)y=x^2+5 \qquad \text{where } \phi(x,D)=2xD^2+x^3.$$

12. This equation is linear, and in operator notation
$$\phi(x,D)y=x^2 \qquad \text{where } \phi(x,D)=2xD^2+(x^3-5).$$

13. Because of the term $5y^2$, the equation is not linear.

14. This equation is linear, and in operator notation
$$\phi(x,D)y=10\sin x \qquad \text{where } \phi(x,D)=xD^3+3xD^2-2D+1.$$

15. Because of the term y^2, the equation is not linear.

16. Because of the term $y\,d^3y/dx^3$, the equation is not linear.

17. This equation is linear, and in operator notation
$$\phi(D)y=9\sec^2 x \qquad \text{where } \phi(D)=D^2-3D-2.$$

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18. Because of the yy'' term, this equation is not linear.

19. Since the equation can be expressed in the form $\dfrac{dy}{dx} = (4-x^2)^2 - 1$, the equation is linear. In operator notation, $Dy = (4-x^2)^2 - 1$.

20. This equation is linear, and in operator notation

$$\phi(D)y = \ln x \qquad \text{where } \phi(D) = D^4 + D^2 - 1.$$

21. If $y_1(t)$ and $y_2(t)$ are any two functions in S, and c is a constant,

$$\begin{aligned} L(y_1+y_2) &= \int_0^\infty [y_1(t)+y_2(t)]e^{-st}\,dt = \int_0^\infty y_1(t)e^{-st}\,dt + \int_0^\infty y_2(t)e^{-st}\,dt \\ &= L(y_1) + L(y_2), \\ L(cy_1) &= \int_0^\infty cy_1(t)e^{-st}\,dt = c\int_0^\infty y_1(t)e^{-st}\,dt = cL(y_1). \end{aligned}$$

Thus, L is a linear operator.

22. If $y_1(x)$ and $y_2(x)$ are any two functions in S, and c is a constant,

$$\begin{aligned} L(y_1+y_2) &= \int_0^{2\pi} [y_1(x)+y_2(x)]\cos nx\,dx = \int_0^{2\pi} y_1(x)\,\cos nx\,dx + \int_0^{2\pi} y_2(x)\,\cos nx\,dx \\ &= L(y_1) + L(y_2), \\ L(cy_1) &= \int_0^{2\pi} cy_1(x)\,\cos nx\,dx = c\int_0^{2\pi} y_1(x)\,\cos nx\,dx = cL(y_1). \end{aligned}$$

Thus, L is a linear operator.

EXERCISES 15.7

1. Since $y_1'' + y_1' - 6y_1 = 4e^{2x} + 2e^{2x} - 6e^{2x} = 0$, and similarly, $y_2'' + y_2' - 6y_2 = 0$, $y_1(x)$ and $y_2(x)$ are solutions of the equation. Because the equation is linear and homogeneous, a general solution is $y(x) = C_1e^{2x} + C_2e^{-3x}$.

2. Since $y_1' + y_1\,\tan x = -\sin x + \cos x\,\tan x = 0$, $y_1(x)$ is a solution. Because the equation is linear and homogeneous, a general solution is $y(x) = C\cos x$.

3. Since $y_1'''' + 5y_1'' + 4y_1 = 16\cos 2x - 20\cos 2x + 4\cos 2x = 0$, $y_1(x)$ is a solution of the differential equation. Similarly, $y_2(x)$, $y_3(x)$, and $y_4(x)$ are solutions. Because the equation is linear and homogeneous, a general solution is $y(x) = C_1\cos 2x + C_2\sin 2x + C_3\cos x + C_4\sin x$.

4. Since $2y_1'' - 16y_1' + 32y_1 = 96e^{4x} - 192e^{4x} + 96e^{4x} = 0$, and

$$2y_2'' - 16y_2' + 32y_2 = -4(16xe^{4x} + 8e^{4x}) + 32(4xe^{4x} + e^{4x}) - 64xe^{4x} = 0,$$

$y_1(x)$ and $y_2(x)$ are solutions of the equation. Because the equation is linear and homogeneous, a general solution is $y(x) = D_1(3e^{4x}) + D_2(-2xe^{4x}) = (C_1 + C_2x)e^{4x}$.

5. The equation is clearly satisfied by $y_1(x) = 10$. Since $y_2''' - 3y_2'' + 2y_2' = 3e^x - 9e^x + 6e^x = 0$, $y_2(x)$ is also a solution. Similarly, $y_3(x)$ is a solution. Because the equation is linear and homogeneous, a general solution is $y(x) = D_1(1) + D_2(3e^x) + D_3(4e^{2x}) = C_1 + C_2e^x + C_3e^{2x}$.

6. Since
$$\begin{aligned} 2y_1'' - 8y_1' + 9y_1 &= 2[4e^{2x}\cos(x/\sqrt{2}) - (2\sqrt{2})e^{2x}\sin(x/\sqrt{2}) - (1/2)e^{2x}\cos(x/\sqrt{2})] \\ &\quad -8[2e^{2x}\cos(x/\sqrt{2}) - (1/\sqrt{2})e^{2x}\sin(x/\sqrt{2})] + 9e^{2x}\cos(x/\sqrt{2}) \\ &= 0, \end{aligned}$$

and similarly, $2y_2'' - 8y_2' + 9y_2 = 0$, $y_1(x)$ and $y_2(x)$ are solutions of the equation. Because the equation is linear and homogeneous, a general solution is $y(x) = C_1e^{2x}\cos(x/\sqrt{2}) + C_2e^{2x}\sin(x/\sqrt{2})$.

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7. Since $x^2y_1'' + xy_1' + (x^2 - 1/4)y_1 = x^2\left(-\frac{\sin x}{\sqrt{x}} - \frac{\cos x}{x^{3/2}} + \frac{3\sin x}{4x^{5/2}}\right)$

$$+x\left(\frac{\cos x}{\sqrt{x}} - \frac{\sin x}{2x^{3/2}}\right) + \left(x^2 - \frac{1}{4}\right)\left(\frac{\sin x}{\sqrt{x}}\right)$$
$$= 0,$$

$y_1(x)$ is a solution of the differential equation. Similarly, $y_2(x)$ is a solution. Because the equation is linear and homogeneous, a general solution is $y(x) = (C_1/\sqrt{x})\sin x + (C_2/\sqrt{x})\cos x$.

8. Since $4y_1 + xy_1' + x^2y_1'' = 4\cos(2\ln x) + x[(-2/x)\sin(2\ln x)]$

$$+x^2[(2/x^2)\sin(2\ln x) - (4/x^2)\cos(2\ln x)]$$
$$= 0,$$

and similarly, $4y_2 + xy_2' + x^2y_2'' = 0$, $y_1(x)$ and $y_2(x)$ are solution of the equation. Because the equation is linear and homogeneous, a general solution is $y(x) = C_1\cos(2\ln x) + C_2\sin(2\ln x)$.

9. Since $y_1'' - y_1y_1' = \frac{-4}{(x+1)^3} - \left(\frac{-2}{x+1}\right)\left[\frac{2}{(x+1)^2}\right] = 0$, $y_1(x)$ is a solution of the differential equation. Similarly, $y_2(x)$ is a solution. Because

$$\begin{aligned}(y_1+y_2)'' - (y_1+y_2)(y_1'+y_2') &= (y_1'' - y_1y_1') + (y_2'' - y_2y_2') - (y_1y_2' + y_1'y_2)\\ &= -(y_1y_2' + y_1'y_2) = -\left(\frac{-2}{x+1}\right)\left[\frac{2}{(x+2)^2}\right] - \left[\frac{2}{(x+1)^2}\right]\left(\frac{-2}{x+2}\right) \neq 0,\end{aligned}$$

$y_1 + y_2$ is not a solution. We would not expect $y_1 + y_2$ to be a solution because the equation is not linear.

10. If the $y_i(x)$ are linearly dependent, there exists constants C_i, not all zero, such that

$$C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x) = 0.$$

When we differentiate this equation $n-1$ times, we obtain $n-1$ more equations:

$$\begin{array}{ccccccccc} C_1y_1'(x) & + & C_2y_2'(x) & + & \cdots & + & C_ny_n'(x) & = & 0,\\ C_1y_1''(x) & + & C_2y_2''(x) & + & \cdots & + & C_ny_n''(x) & = & 0,\\ \vdots & & \vdots & & & & \vdots & & \\ C_1y_1^{(n-1)}(x) & + & C_2y_2^{(n-1)}(x) & + & \cdots & + & C_ny_n^{(n-1)}(x) & = & 0.\end{array}$$

Because this system of n linear equations in $C_1, \ldots, C_n$ has a nontrivial solution, it follows that the determinant of its coefficients must vanish; that is,

$$W(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n\\ y_1' & y_2' & \cdots & y_n'\\ \vdots & \vdots & & \vdots\\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)}\end{vmatrix} = 0 \quad \text{on } I.$$

11. Since $W(1, x, x^2) = \begin{vmatrix} 1 & x & x^2\\ 0 & 1 & 2x\\ 0 & 0 & 2\end{vmatrix} = 2$, the functions are linearly independent.

12. Since $W(x, 2x - 3x^2, x^2) = \begin{vmatrix} x & 2x - 3x^2 & x^2\\ 1 & 2 - 6x & 2x\\ 0 & -6 & 2\end{vmatrix} = x[2(2 - 6x) + 12x] - [2(2x - 3x^2) + 6x^2] = 0$, the functions are linearly dependent.

13. Since $W(\sin x, \cos x) = \begin{vmatrix} \sin x & \cos x\\ \cos x & -\sin x\end{vmatrix} = 1$, the functions are linearly independent.

14. Since $W(x, xe^x, x^2e^x) = \begin{vmatrix} x & xe^x & x^2e^x\\ 1 & (x+1)e^x & (x^2+2x)e^x\\ 0 & (x+2)e^x & (x^2+4x+2)e^x\end{vmatrix}$, which at $x = 1$ reduces to $\begin{vmatrix} 1 & e & e\\ 1 & 2e & 3e\\ 0 & 3e & 7e\end{vmatrix} = e^2$, the functions are therefore linearly independent.

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15. Since $W(x\sin x, e^{2x}) = \begin{vmatrix} x\sin x & e^{2x} \\ \sin x + x\cos x & 2e^{2x} \end{vmatrix} = e^{2x}[(2x-1)\sin x - x\cos x] \not\equiv 0$, the functions are linearly independent.

EXERCISES 15.8

1. The auxiliary equation is $0 = m^2 + m - 6 = (m+3)(m-2)$ with solutions $m = -3, 2$. A general solution of the differential equation is therefore $y(x) = C_1e^{-3x} + C_2e^{2x}$.

2. The auxiliary equation is $0 = 2m^2 - 16m + 32 = 2(m-4)^2$ with solutions $m = 4, 4$. A general solution of the differential equation is therefore $y(x) = (C_1 + C_2x)e^{4x}$.

3. The auxiliary equation is $0 = 2m^2 + 16m + 82$ with solutions $m = -4 \pm 5i$. A general solution of the differential equation is therefore $y(x) = e^{-4x}(C_1\cos 5x + C_2\sin 5x)$.

4. The auxiliary equation is $0 = m^2 + 2m - 2$ with solutions $m = -1 \pm \sqrt{3}$. A general solution of the differential equation is therefore $y(x) = C_1e^{-(1+\sqrt{3})x} + C_2e^{(-1+\sqrt{3})x}$.

5. The auxiliary equation is $0 = m^2 - 4m + 5$ with solutions $m = 2 \pm i$. A general solution of the differential equation is therefore $y(x) = e^{2x}(C_1\cos x + C_2\sin x)$.

6. The auxiliary equation is $0 = m^3 - 3m^2 + m - 3 = (m-3)(m^2+1)$ with solutions $m = 3, \pm i$. A general solution of the differential equation is therefore $y(x) = C_1e^{3x} + C_2\cos x + C_3\sin x$.

7. The auxiliary equation is $0 = m^4 + 2m^2 + 1 = (m^2+1)^2$ with solutions $m = \pm i, \pm i$. A general solution of the differential equation is therefore $y(x) = (C_1 + C_2x)\cos x + (C_3 + C_4x)\sin x$.

8. The auxiliary equation is $0 = m^3 - 6m^2 + 12m - 8 = (m-2)^3$ with solutions $m = 2, 2, 2$. A general solution of the differential equation is therefore $y(x) = (C_1 + C_2x + C_3x^2)e^{2x}$.

9. The auxiliary equation is $0 = 3m^3 - 12m^2 + 18m - 12 = 3(m-2)(m^2-2m+2)$ with solutions $m = 2, 1\pm i$. A general solution of the differential equation is therefore $y(x) = C_1e^{2x} + e^x(C_2\cos x + C_3\sin x)$.

10. The auxiliary equation is $0 = m^4 + 5m^2 + 4 = (m^2+1)(m^2+4)$ with solutions $m = \pm i, \pm 2i$. A general solution of the differential equation is therefore $y(x) = C_1\cos x + C_2\sin x + C_3\cos 2x + C_4\sin 2x$.

11. The auxiliary equation is $0 = m^3 - 3m^2 + 2m = m(m-1)(m-2)$ with solutions $m = 0, 1, 2$. A general solution of the differential equation is therefore $y(x) = C_1 + C_2e^x + C_3e^{2x}$.

12. The auxiliary equation is $0 = m^4 + 16 = (m^2+4i)(m^2-4i)$. To solve $m^2 = 4i$, we set $m = a + bi$, so that $4i = (a+bi)^2 = (a^2 - b^2) + 2abi$. When we equate real and imaginary parts, $a^2 - b^2 = 0$ and $2ab = 4$, These imply that $a = b = \pm\sqrt{2}$. Thus, $m = \pm\sqrt{2}(1+i)$. From $m^2 = -4i$, we obtain $m = \pm\sqrt{2}(1-i)$. A general solution of the differential equation is therefore

$$y(x) = e^{\sqrt{2}x}[C_1\cos(\sqrt{2}x) + C_2\sin(\sqrt{2}x)] + e^{-\sqrt{2}x}[C_3\cos(\sqrt{2}x) + C_4\sin(\sqrt{2}x)].$$

13. For this general solution, roots of the auxiliary equation had to be $m = 1, -4, -4$, and therefore

$$\phi(m) = (m-1)(m+4)^2 = m^3 + 7m^2 + 8m - 16.$$

A possible differential equation is therefore $y''' + 7y'' + 8y' - 16y = 0$.

14. For this general solution, roots of the auxiliary equation had to be $m = -2 \pm 4i$, and therefore

$$\phi(m) = (m + 2 + 4i)(m + 2 - 4i) = m^2 + 4m + 20.$$

A possible differential equation is therefore $y'' + 4y' + 20y = 0$.

15. For this general solution, roots of the auxiliary equation had to be $m = 0, \pm\sqrt{3}$, and therefore

$$\phi(m) = m(m - \sqrt{3})(m + \sqrt{3}) = m^3 - 3m.$$

A possible differential equation is therefore $y''' - 3y' = 0$.

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16. For this general solution, the roots of the auxiliary equation had to be $m = 1 \pm \sqrt{2}i$, $1 \pm \sqrt{2}i$, and therefore

$$\phi(m) = (m - 1 + \sqrt{2}i)^2(m - 1 - \sqrt{2}i)^2 = (m^2 - 2m + 3)^2 = m^4 - 4m^3 + 10m^2 - 12m + 9.$$

A possible differential equation is therefore $y'''' - 4y''' + 10y'' - 12y' + 9y = 0$.

17. When we substitute $e^{ax}\sin bx$ into the differential equation,

$$\begin{aligned} 0 &= 10e^{ax}\sin bx + 2(ae^{ax}\sin bx + be^{ax}\cos bx) + (a^2e^{ax}\sin bx + 2abe^{ax}\cos bx - b^2e^{ax}\sin bx) \\ &= e^{ax}[(10 + 2a + a^2 - b^2)\sin bx + (2b + 2ab)\cos bx]. \end{aligned}$$

Since $\cos bx$ and $\sin bx$ are linearly independent functions, it follows that $10 + 2a + a^2 - b^2 = 0$ and $2b + 2ab = 0$. These equations imply that $a = -1$ and $b = \pm 3$. When we substitute $e^{-x}\cos 3x$ into the differential equation,

$$\begin{aligned} y'' + 2y' + 10y &= (e^{-x}\cos 3x + 6e^{-x}\sin 3x - 9e^{-x}\cos 3x) \\ &\qquad + 2(-e^{-x}\cos 3x - 3e^{-x}\sin 3x) + 10e^{-x}\cos 3x \\ &= 0. \end{aligned}$$

A general solution of the differential equation is therefore $e^{-x}(C_1\cos 3x + C_2\sin 3x)$.

18. For this $y(x)$ to be a solution, the roots of the auxiliary equation must be $m = -1$, $-2\pm 4i$, and therefore $\phi(m) = (m + 1)(m + 2 + 4i)(m + 2 - 4i) = m^3 + 5m^2 + 24m + 20$. It follows that

$$m^3 + 5m^2 + 24m + 20 = m^3 + am^2 + bm + c,$$

and we conclude that $a = 5$, $b = 24$, and $c = 20$.

19. The auxiliary equation is $m^2 + \lambda = 0$. Solutions depend on whether λ is positive, negative, or zero. We consider all three cases. When $\lambda < 0$, solutions are $m = \pm\sqrt{-\lambda}$, in which case $y(x) = C_1e^{\sqrt{-\lambda}x} + C_2e^{-\sqrt{-\lambda}x}$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y(0) = C_1 + C_2, \qquad 0 = y(3) = C_1e^{3\sqrt{-\lambda}} + C_2e^{-3\sqrt{-\lambda}}.$$

The only solution of these is $C_1 = C_2 = 0$, and therefore $y(x) = 0$.

When $\lambda = 0$, the auxiliary equation has a double root $m = 0$, in which case $y(x) = C_1 + C_2x$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y(0) = C_1, \qquad 0 = y(3) = C_1 + 3C_2.$$

Once again, the only solution of these is $C_1 = C_2 = 0$, from which $y(x) = 0$.

When $\lambda > 0$, roots of the auxiliary equation are $m = \pm\sqrt{\lambda}\,i$, in which case $y(x) = C_1\cos\sqrt{\lambda}x + C_2\sin\sqrt{\lambda}x$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y(0) = C_1, \qquad 0 = y(3) = C_1\cos 3\sqrt{\lambda} + C_2\sin 3\sqrt{\lambda}.$$

With $C_1 = 0$, the second of these implies that $C_2\sin 3\sqrt{\lambda} = 0$. Since we cannot set $C_2 = 0$, else $y(x)$ would again be zero, we must set $\sin 3\sqrt{\lambda} = 0 \Longrightarrow 3\sqrt{\lambda} = n\pi$, where $n \neq 0$ is an integer. Thus, eigenvalues of the Sturm-Liouville system are $\lambda_n = n^2\pi^2/9$, with corresponding eigenfunctions $y_n(x) = C_2\sin(n\pi x/3)$.

20. The auxiliary equation is $m^2 + \lambda = 0$. Solutions depend on whether λ is positive, negative, or zero. We consider all three cases. When $\lambda < 0$, solutions are $m = \pm\sqrt{-\lambda}$, in which case $y(x) = C_1e^{\sqrt{-\lambda}x} + C_2e^{-\sqrt{-\lambda}x}$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y'(0) = \sqrt{-\lambda}\,C_1 - \sqrt{-\lambda}\,C_2, \qquad 0 = y'(4) = \sqrt{-\lambda}\,C_1e^{4\sqrt{-\lambda}} - \sqrt{-\lambda}\,C_2e^{-4\sqrt{-\lambda}}.$$

The only solution of these is $C_1 = C_2 = 0$, and therefore $y(x) = 0$.

When $\lambda = 0$, the auxiliary equation has a double root $m = 0$, in which case $y(x) = C_1 + C_2x$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y'(0) = C_2, \qquad 0 = y'(4) = C_2.$$

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Thus, $\lambda_0 = 0$ is an eigenvalue with eigenfunction $y_0(x) = C_1$.

When $\lambda > 0$, roots of the auxiliary equation are $m = \pm\sqrt{\lambda}\,i$, in which case $y(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y'(0) = \sqrt{\lambda}\,C_2, \qquad 0 = y'(4) = -\sqrt{\lambda}\,C_1 \sin 4\sqrt{\lambda} + \sqrt{\lambda}\,C_2 \cos 4\sqrt{\lambda}.$$

With $C_2 = 0$, the second of these implies that $C_1 \sin 4\sqrt{\lambda} = 0$. Since we cannot set $C_1 = 0$, else $y(x)$ would again be zero, we must set $\sin 4\sqrt{\lambda} = 0 \Longrightarrow 4\sqrt{\lambda} = n\pi$, where $n \neq 0$ is an integer. Thus, additional eigenvalues of the Sturm-Liouville system are $\lambda_n = n^2\pi^2/16$, with corresponding eigenfunctions $y_n(x) = C_1 \cos(n\pi x/4)$.

21. The auxiliary equation is $m^2 + \lambda = 0$. Solutions depend on whether λ is positive, negative, or zero. We consider all three cases. When $\lambda < 0$, solutions are $m = \pm\sqrt{-\lambda}$, in which case $y(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y(0) = C_1 + C_2, \qquad 0 = y'(2) = \sqrt{-\lambda}\,C_1 e^{2\sqrt{-\lambda}} - \sqrt{-\lambda}\,C_2 e^{-2\sqrt{-\lambda}}.$$

The only solution of these is $C_1 = C_2 = 0$, and therefore $y(x) = 0$.

When $\lambda = 0$, the auxiliary equation has a double root $m = 0$, in which case $y(x) = C_1 + C_2 x$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y(0) = C_1, \qquad 0 = y'(2) = C_2.$$

Once again, the only solution is $y(x) = 0$.

When $\lambda > 0$, roots of the auxiliary equation are $m = \pm\sqrt{\lambda}\,i$, in which case $y(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y(0) = C_1, \qquad 0 = y'(2) = -\sqrt{\lambda}\,C_1 \sin 2\sqrt{\lambda} + \sqrt{\lambda}\,C_2 \cos 2\sqrt{\lambda}.$$

With $C_1 = 0$, the second of these implies that $C_2 \cos 2\sqrt{\lambda} = 0$. Since we cannot set $C_2 = 0$, else $y(x)$ would again be zero, we must set $\cos 2\sqrt{\lambda} = 0 \Longrightarrow 2\sqrt{\lambda} = (2n-1)\pi/2$, where n is an integer. Thus, eigenvalues of the Sturm-Liouville system are $\lambda_n = (2n-1)^2\pi^2/16$, with corresponding eigenfunctions $y_n(x) = C_2 \sin[(2n-1)\pi x/4)]$.

22. The auxiliary equation is $m^2 + \lambda = 0$. Solutions depend on whether λ is positive, negative, or zero. We consider all three cases. When $\lambda < 0$, solutions are $m = \pm\sqrt{-\lambda}$, in which case $y(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y'(0) = \sqrt{-\lambda}\,C_1 - \sqrt{-\lambda}\,C_2, \qquad 0 = y(5) = C_1 e^{5\sqrt{-\lambda}} + C_2 e^{-5\sqrt{-\lambda}}.$$

The only solution of these is $C_1 = C_2 = 0$, and therefore $y(x) = 0$.

When $\lambda = 0$, the auxiliary equation has a double root $m = 0$, in which case $y(x) = C_1 + C_2 x$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y'(0) = C_2, \qquad 0 = y(5) = C_1 + 5C_2.$$

Once again, the only solution is $C_1 = C_2 = 0$, from which $y(x) = 0$.

When $\lambda > 0$, roots of the auxiliary equation are $m = \pm\sqrt{\lambda}\,i$, in which case $y(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y'(0) = \sqrt{\lambda}\,C_2, \qquad 0 = y(5) = C_1 \cos 5\sqrt{\lambda} + C_2 \sin 5\sqrt{\lambda}.$$

With $C_2 = 0$, the second of these implies that $C_1 \cos 5\sqrt{\lambda} = 0$. Since we cannot set $C_1 = 0$, else $y(x)$ would again be zero, we must set $\cos 5\sqrt{\lambda} = 0 \Longrightarrow 5\sqrt{\lambda} = (2n-1)\pi/2$, where n is an integer. Thus, eigenvalues of the Sturm-Liouville system are $\lambda_n = (2n-1)^2\pi^2/100$, with corresponding eigenfunctions $y_n(x) = C_1 \cos[(2n-1)\pi x/10)]$.

23. The auxiliary equation is $m^2 - m + \lambda = 0$ with solutions $m = (1 \pm \sqrt{1-4\lambda})/2$. The form of the solutions depends on whether $\lambda < 1/4$, $\lambda = 1/4$, or $\lambda > 1/4$. We consider all three cases. When $\lambda < 1/4$, we set $\omega = \sqrt{1-4\lambda}$, in which case roots of the auxiliary equation are $m = (1 \pm \omega)/2$, and $y(x) = C_1 e^{(1+\omega)x/2} + C_2 e^{(1-\omega)x/2}$. The boundary conditions require C_1 and C_2 to satisfy

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$$0 = y(0) = C_1 + C_2, \qquad 0 = y(1) = C_1 e^{(1+\omega)/2} + C_2 e^{(1-\omega)/2}.$$

The only solution of these is $C_1 = C_2 = 0$, and therefore $y(x) = 0$.

When $\lambda = 1/4$, the auxiliary equation has a double root $m = 1/2$, in which case $y(x) = (C_1 + C_2 x)e^{x/2}$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y(0) = C_1, \qquad 0 = y(1) = (C_1 + C_2)e^{1/2}.$$

Once again, the only solution of these is $C_1 = C_2 = 0$, from which $y(x) = 0$.

When $\lambda > 1/4$, we set $\omega = \sqrt{4\lambda - 1}$, in which case roots of the auxiliary equation are $m = (1 \pm \omega i)/2$, and $y(x) = e^{x/2}[C_1 \cos(\omega x/2) + C_2 \sin(\omega x/2)$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y(0) = C_1, \qquad 0 = y(1) = e^{1/2}[C_1 \cos(\omega/2) + C_2 \sin(\omega/2)].$$

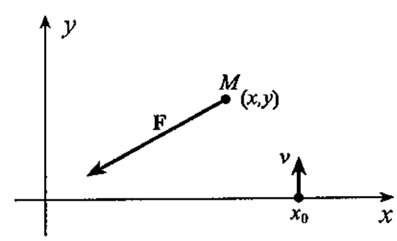
With $C_1 = 0$, the second of these implies that $C_2 \sin(\omega/2) = 0$. Since we cannot set $C_2 = 0$, else $y(x)$ would again be zero, we must set $\sin(\omega/2) = 0 \Longrightarrow \omega/2 = n\pi$, where $n \neq 0$ is an integer. Thus,

$$\omega = 2n\pi \quad \Longrightarrow \quad \sqrt{4\lambda - 1} = 2n\pi \quad \Longrightarrow \quad \lambda = n^2\pi^2 + \frac{1}{4}.$$

Eigenvalues of the Sturm-Liouville system are $\lambda_n = n^2\pi^2 + 1/4$ with corresponding eigenfunctions $y_n(x) = C_2 e^{x/2} \sin(n\pi x)$.

24. (a) When the mass is at position (x, y), the force acting on it is

$$\mathbf{F} = k\sqrt{x^2 + y^2}\left(\frac{-x\hat{\mathbf{i}} - y\hat{\mathbf{j}}}{\sqrt{x^2 + y^2}}\right) = -k(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}).$$



According to Newton's second law, the acceleration is given by

$$M\frac{d^2\mathbf{r}}{dt^2} = -k(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}).$$

When we equate components,

$$M\frac{d^2x}{dt^2} = -kx, \quad M\frac{d^2y}{dt^2} = -ky.$$

The auxiliary equation for each of these differential equations is $Mm^2 + k = 0$ with solutions $m = \pm\sqrt{k/M}\, i$. If we set $\omega = \sqrt{k/M}$, then

$$x(t) = A\cos\omega t + B\sin\omega t, \qquad y(t) = C\cos\omega t + D\sin\omega t.$$

With the initial conditions $x(0) = x_0$, $x'(0) = 0$, $y(0) = 0$, and $y'(0) = v$,

$$x_0 = A, \quad 0 = \omega B, \quad 0 = C, \quad v = \omega D.$$

Thus, $x = x_0 \cos\omega t$ and $y = (v/\omega)\sin\omega t$ define the path of the mass parametrically. Eliminating t gives

$$\left(\frac{x}{x_0}\right)^2 + \left(\frac{\omega y}{v}\right)^2 = 1 \quad \Longrightarrow \quad \frac{x^2}{x_0^2} + \frac{ky^2}{Mv^2} = 1,$$

an ellipse.

(b) In this case, the differential equations are

$$M\frac{d^2x}{dt^2} = kx, \quad M\frac{d^2y}{dt^2} = ky.$$

The auxiliary equation for each of these differential equations is $Mm^2 - k = 0$ with solutions $m = \pm\sqrt{k/M}$. If we set $\omega = \sqrt{k/M}$, then

$$x(t) = Ae^{\omega t} + Be^{-\omega t}, \qquad y(t) = Ce^{\omega t} + De^{-\omega t}.$$

The initial conditions require

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$$x_0 = A + B, \quad 0 = \omega A - \omega B, \quad 0 = C + D, \quad v = \omega C - \omega D.$$

These give $A = B = x_0/2$ and $C = -D = v/(2\omega)$. Thus, parametric equations for the path of the mass are

$$x = \frac{x_0}{2}e^{\omega t} + \frac{x_0}{2}e^{-\omega t} = \frac{x_0}{2}(e^{\omega t} + e^{-\omega t}), \quad y = \frac{v}{2\omega}e^{\omega t} - \frac{v}{2\omega}e^{-\omega t} = \frac{v}{2\omega}(e^{\omega t} - e^{-\omega t}).$$

When t is eliminated, we obtain

$$\left(\frac{2x}{x_0}\right)^2 - \left(-\frac{2\omega y}{v}\right)^2 = (e^{\omega t} + e^{-\omega t})^2 - (e^{\omega t} - e^{-\omega t})^2 = 4 \quad \Longrightarrow \quad \frac{x^2}{x_0^2} - \frac{ky^2}{Mv^2} = 1,$$

a hyperbola. The mass moves along the right half of this hyperbola.

25. $D\{e^{px}f(x)\} = e^{px}f'(x) + pe^{px}f(x) = e^{px}\{f'(x) + pf(x)\} = e^{px}\{(D+p)f(x)\}$
The result $D^k\{e^{px}f(x)\} = e^{px}\{(D+p)^k f(x)\}$ has just been proven for $k = 1$. Suppose it is valid for some integer r; that is, suppose that $D^r\{e^{px}f(x)\} = e^{px}\{(D+p)^r f(x)\}$. Then,

$$\begin{aligned}
D^{r+1}\{e^{px}f(x)\} &= D[D^r\{e^{px}f(x)\}] = D[e^{px}\{(D+p)^r f(x)\}] \\
&= e^{px}\{(D+p)^r f'(x)\} + pe^{px}\{(D+p)^r f(x)\} \\
&= e^{px}(D+p)^r[f'(x) + pf(x)] = e^{px}(D+p)^r(D+p)f(x) \\
&= e^{px}\{(D+p)^{r+1}f(x)\}.
\end{aligned}$$

Consequently, the result is valid for $r+1$, and by mathematical induction it is valid for all $k \geq 1$.
If $\phi(D) = \sum_{i=0}^{n} a_{n-i}D^i$, (see 15.45), then

$$\begin{aligned}
\phi(D)\{e^{px}f(x)\} &= \left(\sum_{i=0}^{n} a_{n-i}D^i\right)\{e^{px}f(x)\} = \sum_{i=0}^{n} a_{n-i}D^i\{e^{px}f(x)\} \\
&= \sum_{i=0}^{n} a_{n-i}e^{px}\{(D+p)^i f(x)\} = e^{px}\left[\sum_{i=0}^{n} a_{n-i}(D+p)^i\right] f(x) \\
&= e^{px}\{\phi(D+p)f(x)\}.
\end{aligned}$$

26. (a) If $m = m_0$ is a root of multiplicity k of $\phi(m) = 0$, then $\phi(m) = (m - m_0)^k\psi(m)$, and therefore $\phi(D) = (D - m_0)^k\psi(D)$. We now calculate that

$$\begin{aligned}
&\phi(D)[(C_1 + C_2x + \cdots + C_kx^{k-1})e^{m_0x}] \\
&\quad = e^{m_0x}[\phi(D + m_0)(C_1 + C_2x + \cdots + C_kx^{k-1})] \\
&\quad = e^{m_0x}[(D + m_0 - m_0)^k\psi(D + m_0)(C_1 + C_2x + \cdots + C_kx^{k-1})] \\
&\quad = e^{m_0x}[\psi(D + m_0)D^k(C_1 + C_2x + \cdots + C_kx^{k-1})] \\
&\quad = 0.
\end{aligned}$$

Thus, $(C_1 + C_2x + \cdots + C_kx^{k-1})e^{m_0x}$ is a solution of $\phi(D)y = 0$.
(b) If $a \pm bi$ are complex conjugate roots of multiplicity k of $\phi(m) = 0$, then

$$\phi(m) = (m - a - bi)^k(m - a + bi)^k\psi(m) \quad \Longrightarrow \quad \phi(D) = (D - a - bi)^k(D - a + bi)^k\psi(D).$$

We now calculate that

$$\begin{aligned}
&\phi(D)\{e^{ax}[(C_1 + C_2x + \cdots + C_kx^{k-1})\cos bx + (D_1 + D_2x + \cdots + D_kx^{k-1})\sin bx]\} \\
&\quad = \phi(D)\{\mathrm{Re}[e^{(a+bi)x}](C_1 + C_2x + \cdots + C_kx^{k-1}) + \mathrm{Im}[e^{(a+bi)x}](D_1 + D_2x + \cdots + D_kx^{k-1})\} \\
&\quad = \mathrm{Re}\{\phi(D)[e^{(a+bi)x}(C_1 + C_2x + \cdots + C_kx^{k-1})]\} \\
&\qquad + \mathrm{Im}\{\phi(D)[e^{(a+bi)x}(D_1 + D_2x + \cdots + D_kx^{k-1})]\} \\
&\quad = \mathrm{Re}\{e^{(a+bi)x}\phi(D + a + bi)(C_1 + C_2x + \cdots + C_kx^{k-1})\} \\
&\qquad + \mathrm{Im}\{e^{(a+bi)x}\phi(D + a + bi)(D_1 + D_2x + \cdots + D_kx^{k-1})\}
\end{aligned}$$

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$$= \text{Re}\{e^{(a+bi)x}D^k(D+2bi)^k\psi(D+a+bi)(C_1+C_2x+\cdots+C_kx^{k-1})\}$$
$$+\,\text{Im}\{e^{(a+bi)x}D^k(D+2bi)^k\psi(D+a+bi)(D_1+D_2x+\cdots+D_kx^{k-1})\}$$
$$= \text{Re}\{e^{(a+bi)x}\psi(D+a+bi)(D+2bi)^k[D^k(C_1+C_2x+\cdots+C_kx^{k-1})]\}$$
$$+\,\text{Im}\{e^{(a+bi)x}\psi(D+a+bi)(D+2bi)^k[D^k(D_1+D_2x+\cdots+D_kx^{k-1})]\}$$
$$= 0.$$

It follows that $e^{ax}[(C_1+C_2x+\cdots+C_kx^{k-1})\cos bx+(D_1+D_2x+\cdots+D_kx^{k-1})\sin bx]$ is a solution of $\phi(D)y=0$.

27. The auxiliary equation $Mm^2+\beta m+k=0$ has solutions $m=\dfrac{-\beta\pm\sqrt{\beta^2-4kM}}{2M}$. We consider three cases:
Case 1: $\beta^2-4kM=0$
In this case, the auxiliary equation has equal roots $m=-\beta/(2M)$, and a general solution of the differential equation is $x(t)=(C_1+C_2t)e^{-\beta t/(2M)}$.
Case 2: $\beta^2-4kM>0$
In this case, the auxiliary equation has real and distinct roots, and a general solution of the differential equation is

$$x(t)=C_1e^{(-\beta+\sqrt{\beta^2-4kM})t/(2M)}+C_2e^{(-\beta-\sqrt{\beta^2-4kM})t/(2M)}.$$

Case 3: $\beta^2-4kM<0$

In this case, the auxiliary equation has complex conjugate roots $-\dfrac{\beta}{2M}\pm\dfrac{\sqrt{4kM-\beta^2}}{2M}i$. A general solution of the differential equation is

$$x(t)=e^{-\beta t/(2M)}\left[C_1\cos\frac{\sqrt{4kM-\beta^2}}{2M}t+C_2\sin\frac{\sqrt{4kM-\beta^2}}{2M}t\right].$$

EXERCISES 15.9

1. The auxiliary equation is $0=2m^2-16m+32=2(m-4)^2$ with solutions $m=4,\,4$. A general solution of the associated homogeneous equation is $y_h(x)=(C_1+C_2x)e^{4x}$. By operators,

$$y_p=\frac{1}{2D^2-16D+32}(-e^{4x})=-e^{4x}\frac{1}{2(D+4)^2-16(D+4)+32}(1)=-e^{4x}\frac{1}{2D^2}(1)=-\frac{x^2}{4}e^{4x}.$$

By undetermined coefficients, $y_p=Ax^2e^{4x}$. Subsititution into the differential equation gives

$$2(16Ax^2e^{4x}+16Axe^{4x}+2Ae^{4x})-16(4Ax^2e^{4x}+2Axe^{4x})+32Ax^2e^{4x}=-e^{4x},$$

and this simplifies to $4Ae^{4x}=-e^{4x}$. Thus, $A=-1/4$, and $y_p=-(x^2/4)e^{4x}$. A general solution of the differential equation is therefore $y(x)=(C_1+C_2x)e^{4x}-(x^2/4)e^{4x}$.

2. The auxiliary equation is $0=m^2+2m-2$ with solutions $m=-1\pm\sqrt{3}$. A general solution of the associated homogeneous equation is $y_h(x)=C_1e^{-(1+\sqrt{3})x}+C_2e^{(-1+\sqrt{3})x}$. By operators,

$$y_p=\frac{1}{D^2+2D-2}(x^2e^{-x})=e^{-x}\frac{1}{(D-1)^2+2(D-1)-2}(x^2)=e^{-x}\frac{1}{D^2-3}(x^2)$$
$$=\frac{e^{-x}}{-3}\frac{1}{1-D^2/3}(x^2)=\frac{e^{-x}}{-3}\left(1+\frac{D^2}{3}+\cdots\right)x^2=\frac{e^{-x}}{-3}\left(x^2+\frac{2}{3}\right).$$

By undetermined coefficients, $y_p=Ax^2e^{-x}+Bxe^{-x}+Ce^{-x}$. Subsititution into the differential equation gives

$$(2Ae^{-x}-4Axe^{-x}+Ax^2e^{-x}-2Be^{-x}+Bxe^{-x}+Ce^{-x})+2(2Axe^{-x}-Ax^2e^{-x}+Be^{-x}$$
$$-Bxe^{-x}-Ce^{-x})-2(Ax^2e^{-x}+Bxe^{-x}+Ce^{-x})=x^2e^{-x}.$$

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When we equate coefficients of x^2e^{-x}, xe^{-x}, and e^{-x}:

$$-3A = 1, \qquad -3B = 0, \qquad 2A - 3C = 0.$$

Thus, $A = -1/3$, $B = 0$, and $C = -2/9$ and once again $y_p = (-1/3)x^2e^{-x} - (2/9)e^{-x}$. A general solution of the differential equation is therefore $y(x) = C_1e^{-(1+\sqrt{3})x} + C_2e^{(-1+\sqrt{3})x} - e^{-x}(3x^2+2)/9$.

3. The auxiliary equation is $0 = m^3 - 3m^2 + m - 3 = (m-3)(m^2+1)$ with solutions $m = 3, \pm i$. A general solution of the associated homogeneous equation is $y_h(x) = C_1e^{3x} + C_2\cos x + C_3\sin x$. By operators,

$$\begin{aligned}
y_p &= \frac{1}{D^3-3D^2+D-3}(3xe^x+2) = 3e^x\frac{1}{(D+1)^3-3(D+1)^2+(D+1)-3}(x) + \frac{1}{D^3-3D^2+D-3}(2)\\
&= 3e^x\frac{1}{D^3-2D-4}(x) + \frac{1}{-3[1-(D^3-3D^2+D)/3]}(2)\\
&= -\frac{3e^x}{4}\frac{1}{1-(D^3-2D)/4}(x) - \frac{1}{3}[1+\cdots](2)\\
&= -\frac{3e^x}{4}\left[1+\left(\frac{D^3-2D}{4}\right)+\cdots\right]x - \frac{2}{3} = -\frac{3e^x}{4}\left(x-\frac{1}{2}\right) - \frac{2}{3}.
\end{aligned}$$

By undetermined coefficients, $y_p = (Ax+B)e^x + C$. Subsititution into the differential equation gives

$$\begin{aligned}
(3Ae^x + Axe^x + Be^x) - 3(2Ae^x + Axe^x + Be^x) + (Ae^x + Axe^x + Be^x)&\\
- 3(Axe^x + Be^x + C) = 3xe^x + 2.&
\end{aligned}$$

When we equate coefficients of xe^x, e^x, and 1:

$$-4A = 3, \qquad -2A - 4B = 0, \qquad -3C = 2.$$

Thus, $A = -3/4$, $B = 3/8$, and $C = -2/3$, and once again $y_p = 3e^x(1-2x)/8 - 2/3$. A general solution of the differential equation is therefore $y(x) = C_1e^{3x} + C_2\cos x + C_3\sin x + 3e^x(1-2x)/8 - 2/3$.

4. The auxiliary equation is $0 = m^4 + 2m^2 + 1 = (m^2+1)^2$ with solutions $m = \pm i, \pm i$. A general solution of the associated homogeneous equation is $y_h(x) = (C_1 + C_2x)\cos x + (C_3 + C_4x)\sin x$. By operators,

$$\begin{aligned}
y_p &= \frac{1}{D^4+2D^2+1}\cos 2x = \mathrm{Re}\left(\frac{1}{D^4+2D^2+1}e^{2ix}\right)\\
&= \mathrm{Re}\left[e^{2ix}\frac{1}{(D+2i)^4+2(D+2i)^2+1}(1)\right] = \mathrm{Re}\left[e^{2ix}\frac{1}{9+\cdots}(1)\right] = \mathrm{Re}\left[\frac{e^{2ix}}{9}\right] = \frac{1}{9}\cos 2x.
\end{aligned}$$

By undetermined coefficients, $y_p = A\cos 2x + B\sin 2x$. Substitution into the differential equation gives

$$(16A\cos 2x + 16B\sin 2x) + 2(-4A\cos 2x - 4B\sin 2x) + (A\cos 2x + B\sin 2x) = \cos 2x.$$

When we equate coefficients of $\cos 2x$ and $\sin 2x$, we obtain $9A = 1$, and $9B = 0$. Thus, $A = 1/9$ and $B = 0$, and $y_p = (1/9)\cos 2x$. A general solution of the differential equation is

$$y(x) = (C_1 + C_2x)\cos x + (C_3 + C_4x)\sin x + (1/9)\cos 2x.$$

5. The auxiliary equation is $0 = m^3 - 6m^2 + 12m - 8 = (m-2)^3$ with solutions $m = 2, 2, 2$. A general solution of the associated homogeneous equation is $y_h(x) = (C_1 + C_2x + C_3x^2)e^{2x}$. By operators,

$$y_p = \frac{1}{D^3-6D^2+12D-8}2e^{2x} = 2e^{2x}\frac{1}{(D+2)^3-6(D+2)^2+12(D+2)-8}(1) = 2e^{2x}\frac{1}{D^3}(1) = \frac{x^3}{3}e^{2x}.$$

By undetermined coefficients, $y_p = Ax^3e^{2x}$. Substitution into the differential equation gives

$$\begin{aligned}
(6Ae^{2x} + 36Axe^{2x} + 36Ax^2e^{2x} + 8Ax^3e^{2x}) - 6(6Axe^{2x} + 12Ax^2e^{2x} + 4Ax^3e^{2x})&\\
+ 12(3Ax^2e^{2x} + 2Ax^3e^{2x}) - 8Ax^3e^{2x} = 2e^{2x}.&
\end{aligned}$$

This simplifies to $6Ae^{2x} = 2e^{2x}$, so that $A = 1/3$. Once again $y_p = (x^3/3)e^{2x}$. A general solution of the differential equation is $y(x) = (C_1 + C_2x + C_3x^2)e^{2x} + (x^3/3)e^{2x}$.

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6. The auxiliary equation is $0 = m^4 + 5m^2 + 4 = (m^2+4)(m^2+1)$ with solutions $m = \pm i$, $\pm 2i$. A general solution of the associated homogeneous equation is $y_h(x) = C_1\cos x + C_2\sin x + C_3\cos 2x + C_4\sin 2x$. By operators,

$$y_p = \frac{1}{D^4+5D^2+4}e^{-2x} = e^{-2x}\frac{1}{(D-2)^4+5(D-2)^2+4}(1) = e^{-2x}\frac{1}{40+\cdots}(1) = \frac{1}{40}e^{-2x}.$$

By undetermined coefficients, $y_p = Ae^{-2x}$. Substitution into the differential equation gives

$$(16Ae^{-2x}) + 5(4Ae^{-2x}) + 4(Ae^{-2x}) = e^{-2x}.$$

This equation implies that $A = 1/40$, and a general solution of the differential equation is

$$y(x) = C_1\cos x + C_2\sin x + C_3\cos 2x + C_4\sin 2x + (1/40)e^{-2x}.$$

7. The auxiliary equation is $0 = m^3 - 3m^2 + 2m = m(m-1)(m-2)$ with solutions $m = 0, 1, 2$. A general solution of the associated homogeneous equation is $y_h(x) = C_1 + C_2e^x + C_3e^{2x}$. By operators,

$$\begin{aligned}
y_p &= \frac{1}{D^3-3D^2+2D}(x^2+e^{-x}) = \frac{1}{D^3-3D^2+2D}x^2 + e^{-x}\frac{1}{(D-1)^3-3(D-1)^2+2(D-1)}(1)\\
&= \frac{1}{D(D^2-3D+2)}(x^2) + e^{-x}\frac{1}{D^3-6D^2+11D-6}(1)\\
&= \frac{1}{2D[1+(D^2-3D)/2]}(x^2) + e^{-x}\frac{1}{-6[1-(D^3-6D^2+11D)/6]}(1)\\
&= \frac{1}{2D}\left[1-\left(\frac{D^2-3D}{2}\right)+\left(\frac{D^2-3D}{2}\right)^2+\cdots\right]x^2 - \frac{e^{-x}}{6}[1+\cdots](1)\\
&= \frac{1}{2D}\left(x^2+3x+\frac{7}{2}\right) - \frac{e^{-x}}{6} = \frac{1}{2}\left(\frac{x^3}{3}+\frac{3x^2}{2}+\frac{7x}{2}\right) - \frac{e^{-x}}{6}.
\end{aligned}$$

By undetermined coefficients, $y_p = Ae^{-x} + Bx^3 + Cx^2 + Dx$. Substitution into the differential equation gives

$$(-Ae^{-x}+6B) - 3(Ae^{-x}+6Bx+2C) + 2(-Ae^{-x}+3Bx^2+2Cx+D) = x^2 + e^{-x}.$$

When we equate coefficients of e^{-x}, x^2, x, and 1:

$$-6A = 1, \quad 6B = 1, \quad -18B + 4C = 0, \quad 6B - 6C + 2D = 0.$$

Thus, $A = -1/6$, $B = 1/6$, $C = 3/4$, and $D = 7/4$, and once again $y_p = (2x^3+9x^2+21x)/12 - (1/6)e^{-x}$. A general solution of the differential equation is

$$y(x) = C_1 + C_2e^x + C_3e^{2x} + (2x^3+9x^2+21x)/12 - (1/6)e^{-x}.$$

8. The auxiliary equation is $0 = 2m^2 + 16m + 82$ with solutions $m = -4 \pm 5i$. A general solution of the associated homogeneous equation is $y_h(x) = e^{-4x}(C_1\cos 5x + C_2\sin 5x)$. By operators,

$$\begin{aligned}
y_p &= \frac{1}{2D^2+16D+82}(-2e^{2x}\sin x) = -\frac{1}{D^2+8D+41}\mathrm{Im}[e^{(2+i)x}] = -\mathrm{Im}\left[\frac{1}{D^2+8D+41}e^{(2+i)x}\right]\\
&= -\mathrm{Im}\left[e^{(2+i)x}\frac{1}{(D+2+i)^2+8(D+2+i)+41}(1)\right] = -\mathrm{Im}\left[e^{(2+i)x}\frac{1}{60+12i+\cdots}(1)\right]\\
&= -\frac{1}{12}\mathrm{Im}\left[e^{(2+i)x}\frac{1}{5+i}\frac{5-i}{5-i}\right] = \frac{-1}{12}\mathrm{Im}\left[e^{(2+i)x}\frac{5-i}{26}\right]\\
&= \frac{-1}{312}\mathrm{Im}\left[e^{2x}(\cos x + i\sin x)(5-i)\right] = -\frac{e^{2x}}{312}(-\cos x + 5\sin x).
\end{aligned}$$

By undetermined coefficients, $y_p = Ae^{2x}\sin x + Be^{2x}\cos x$. Substitution into the differential equation gives

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$$\begin{aligned}2(4Ae^{2x}\sin x + 4Ae^{2x}\cos x - Ae^{2x}\sin x + 4Be^{2x}\cos x - 4Be^{2x}\sin x - Be^{2x}\cos x)\\ + 16(2Ae^{2x}\sin x + Ae^{2x}\cos x + 2Be^{2x}\cos x - Be^{2x}\sin x)\\ + 82(Ae^{2x}\sin x + Be^{2x}\cos x) = -2e^{2x}\sin x.\end{aligned}$$

When we equate coefficients of $e^{2x}\sin x$ and $e^{2x}\cos x$:

$$120A - 24B = -2, \qquad 120B + 24A = 0.$$

These imply that $A = -5/312$ and $B = 1/312$, and once again $y_p = e^{2x}(\cos x - 5\sin x)/312$. A general solution of the differential equation is therefore

$$y(x) = e^{-4x}(C_1\cos 5x + C_2\sin 5x) + e^{2x}(\cos x - 5\sin x)/312.$$

9. The auxiliary equation is $0 = m^2 + m - 6 = (m-2)(m+3)$ with solutions $m = -3, 2$. A general solution of the associated homogeneous equation is $y_h(x) = C_1e^{-3x} + C_2e^{2x}$. By operators,

$$\begin{aligned}y_p &= \frac{1}{D^2+D-6}(x+\cos x) = \frac{1}{D^2+D-6}(x) + \mathrm{Re}\left(\frac{1}{D^2+D-6}e^{ix}\right)\\ &= \frac{1}{-6[1-(D^2+D)/6]}(x) + \mathrm{Re}\left[e^{ix}\frac{1}{(D+i)^2+(D+i)-6}(1)\right]\\ &= -\frac{1}{6}\left[1+\left(\frac{D^2+D}{6}\right)+\cdots\right]x + \mathrm{Re}\left[e^{ix}\frac{1}{D^2+(1+2i)D-7+i}(1)\right]\\ &= -\frac{1}{6}\left(x+\frac{1}{6}\right) + \mathrm{Re}\left[e^{ix}\left(\frac{1}{-7+i}\right)\right]\\ &= -\frac{1}{36}(6x+1) + \mathrm{Re}\left[e^{ix}\left(\frac{-7-i}{50}\right)\right] = -\frac{1}{36}(6x+1) - \frac{7}{50}\cos x + \frac{1}{50}\sin x.\end{aligned}$$

By undetermined coefficients, $y_p = Ax + B + C\cos x + D\sin x$. Substitution into the differential equation gives

$$(-C\cos x - D\sin x) + (A - C\sin x + D\cos x) - 6(Ax + B + C\cos x + D\sin x) = x + \cos x.$$

When we equate coefficients of x, 1, $\cos x$, and $\sin x$, we obtain

$$-6A = 1, \quad A - 6B = 0, \quad -7C + D = 1, \quad -C - 7D = 0.$$

Thus, $A = -1/6$, $B = -1/36$, $C = -7/50$, and $D = 1/50$. Once again $y_p = -(6x+1)/36 + (\sin x - 7\cos x)/50$. A general solution of the differential equation is

$$y(x) = C_1e^{-3x} + C_2e^{2x} - (6x+1)/36 + (\sin x - 7\cos x)/50.$$

10. The auxiliary equation is $0 = m^2 - 4m + 5$ with solutions $m = 2 \pm i$. A general solution of the associated homogeneous equation is $y_h(x) = e^{2x}(C_1\cos x + C_2\sin x)$. By operators,

$$\begin{aligned}y_p &= \frac{1}{D^2-4D+5}(x\cos x) = \mathrm{Re}\left[\frac{1}{D^2-4D+5}xe^{ix}\right]\\ &= \mathrm{Re}\left[e^{ix}\frac{1}{(D+i)^2-4(D+i)+5}(x)\right] = \mathrm{Re}\left[e^{ix}\frac{1}{D^2+(-4+2i)D+(4-4i)}(x)\right]\\ &= \mathrm{Re}\left[\frac{e^{ix}}{4-4i}\frac{1}{1+\dfrac{(-4+2i)D+D^2}{4-4i}}(x)\right] = \mathrm{Re}\left\{\frac{e^{ix}}{4-4i}\left[1-\left(\frac{(-4+2i)D+D^2}{4-4i}\right)+\cdots\right]x\right\}\\ &= \mathrm{Re}\left[\frac{e^{ix}}{4(1-i)}\left(x+\frac{4-2i}{4-4i}\right)\right] = \mathrm{Re}\left[\frac{e^{ix}}{4}\left(\frac{x}{1-i}\frac{1+i}{1+i}+\frac{4-2i}{-8i}\right)\right]\\ &= \mathrm{Re}\left[\frac{\cos x + i\sin x}{4}\left(\frac{x(1+i)}{2}+\frac{1+2i}{4}\right)\right] = \frac{x}{8}(\cos x - \sin x) + \frac{1}{16}(\cos x - 2\sin x).\end{aligned}$$

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By undetermined coefficients, $y_p = Ax\cos x + Bx\sin x + C\cos x + D\sin x$. Substitution into the differential equation gives

$$\begin{aligned}&(-2A\sin x - Ax\cos x + 2B\cos x - Bx\sin x - C\cos x - D\sin x)\\ &\quad - 4(A\cos x - Ax\sin x + B\sin x + Bx\cos x - C\sin x + D\cos x)\\ &\quad + 5(Ax\cos x + Bx\sin x + C\cos x + D\sin x) = x\cos x.\end{aligned}$$

When we equate coefficients of $x\cos x$, $x\sin x$, $\cos x$, and $\sin x$:

$$4A - 4B = 1, \quad 4A + 4B = 0, \quad -4A + 2B + 4C - 4D = 0, \quad -2A - 4B + 4C + 4D = 0.$$

These imply that $A = 1/8$, $B = -1/8$, $C = 1/16$, and $D = -1/8$, giving the same y_p as above. A general solution of the differential equation is therefore

$$y(x) = e^{2x}(C_1\cos x + C_2\sin x) + x(\cos x - \sin x)/8 + (\cos x - 2\sin x)/16.$$

11. The auxiliary equation is $0 = 3m^3 - 12m^2 + 18m - 12 = 3(m-2)(m^2 - 2m + 2)$ with solutions $m = 2,\ 1 \pm i$. A general solution of the associated homogeneous equation is $y_h(x) = C_1e^{2x} + e^x(C_2\cos x + C_3\sin x)$. By operators,

$$\begin{aligned}y_p &= \frac{1}{3D^3 - 12D^2 + 18D - 12}(x^2 + 3x - 4) = \frac{1}{-12[1 - (D^3 - 4D^2 + 6D)/4]}(x^2 + 3x - 4)\\ &= -\frac{1}{12}\left[1 + \left(\frac{D^3 - 4D^2 + 6D}{4}\right) + \left(\frac{D^3 - 4D^2 + 6D}{4}\right)^2 + \cdots\right](x^2 + 3x - 4)\\ &= -\frac{1}{12}\left[(x^2 + 3x - 4) + \frac{3}{2}(2x + 3) + \frac{5}{4}(2)\right] = -\frac{1}{12}(x^2 + 6x + 3).\end{aligned}$$

By undetermined coefficients, $y_p = Ax^2 + Bx + C$. Subsititution into the differential equation gives

$$3(0) - 12(2A) + 18(2Ax + B) - 12(Ax^2 + Bx + C) = x^2 + 3x - 4.$$

When we equate coefficients of x^2, x, and 1:

$$-12A = 1, \qquad 36A - 12B = 3, \qquad -24A + 18B - 12C = -4.$$

Thus, $A = -1/12$, $B = -1/2$, and $C = -1/4$. Once again $y_p = -(x^2 + 6x + 3)/12$. A general solution of the differential equation is therefore $y(x) = C_1e^{2x} + e^x(C_2\cos x + C_3\sin x) - (x^2 + 6x + 3)/12$.

12. The auxiliary equation is $0 = m^3 + 9m^2 + 27m + 27 = (m+3)^2$ with solutions $m = -3, -3, -3$. A general solution of the associated homogeneous equation is $y_h(x) = (C_1 + C_2x + C_3x^2)e^{-3x}$. Undetermined coefficients suggests $y_p(x) = Axe^{3x} + Be^{3x} + Cx\cos x + Dx\sin x + E\cos x + F\sin x$.

13. The auxiliary equation is $0 = m^3 + 4m^2 + m + 4 = (m+4)(m^2+1)$ with solutions $m = -4, \pm i$. A general solution of the associated homogeneous equation is $y_h(x) = C_1e^{-4x} + C_2\cos x + C_3\sin x$. Undetermined coefficients suggests $y_p(x) = Axe^x\sin x + Bxe^x\cos x + Ce^x\sin x + De^x\cos x$.

14. The auxiliary equation is $0 = 2m^3 - 6m^2 - 12m + 16 = 2(m-1)(m-4)(m+2)$ with solutions $m = 1, -2, 4$. A general solution of the associated homogeneous equation is $y_h(x) = C_1e^x + C_2e^{-2x} + C_3e^{4x}$. Undetermined coefficients suggests $y_p(x) = Ax^2e^x + Bxe^x + Cx^3 + Dx^2 + Ex + F + G\cos x + H\sin x$.

15. The auxiliary equation is $0 = 2m^2 - 4m + 10$ with solutions $m = 1 \pm 2i$. A general solution of the associated homogeneous equation is $y_h(x) = e^x(C_1\cos 2x + C_2\sin 2x)$. Undetermined coefficients suggests

$$y_p(x) = Axe^x\sin 2x + Bxe^x\cos 2x.$$

16. According to the operator shift theorem, $\phi(D)\{e^{px}g(x)\} = e^{px}[\phi(D+p)g(x)]$, and therefore $e^{px}g(x) = \frac{1}{\phi(D)}\{e^{px}[\phi(D+p)g(x)]\}$. If we set $f(x) = \phi(D+p)g(x)$, in which case $g(x) = \dfrac{1}{\phi(D+p)}f(x)$, then

$$e^{px}\frac{1}{\phi(D+p)}f(x) = \frac{1}{\phi(D)}\{e^{px}f(x)\}.$$

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17. The auxiliary equation is $0 = m^2 + 2m - 4$ with solutions $m = -1 \pm \sqrt{5}$. A general solution of the associated homogeneous equation is $y_h(x) = C_1 e^{-(1+\sqrt{5})x} + C_2 e^{(-1+\sqrt{5})x}$. Since $\cos^2 x = (1 + \cos 2x)/2$, undetermined coefficients suggests $y_p = A + B\cos 2x + C\sin 2x$. Substitution into the differential equation gives

$$(-4B\cos 2x - 4C\sin 2x) + 2(-2B\sin 2x + 2C\cos 2x) - 4(A + B\cos 2x + C\sin 2x) = \frac{1}{2} + \frac{1}{2}\cos 2x.$$

When we equate coefficients of $\cos 2x$, $\sin 2x$, and 1:

$$-8B + 4C = 1/2, \qquad -8C - 4B = 0, \qquad -4A = 1/2.$$

Thus, $A = -1/8$, $B = -1/20$, and $C = 1/40$. A particular solution is $y_p = -1/8 + (\sin 2x - 2\cos 2x)/40$, and a general solution of the differential equation is

$$y(x) = C_1 e^{-(1+\sqrt{5})x} + C_2 e^{(-1+\sqrt{5})x} - 1/8 + (\sin 2x - 2\cos 2x)/40.$$

18. The auxiliary equation is $0 = 2m^2 - 4m + 3$ with solutions $m = 1 \pm (1/\sqrt{2})i$. A general solution of the associated homogeneous equation is $y_h(x) = e^x[C_1\cos(x/\sqrt{2}) + C_2\sin(x/\sqrt{2})]$. By operators,

$$\begin{aligned}
y_p &= \frac{1}{2D^2 - 4D + 3}(\cos x\,\sin 2x) = \frac{1}{2D^2 - 4D + 3}\left[\frac{1}{2}(\sin 3x + \sin x)\right] \\
&= \frac{1}{2}\mathrm{Im}\left[\frac{1}{2D^2 - 4D + 3}e^{3ix}\right] + \frac{1}{2}\mathrm{Im}\left[\frac{1}{2D^2 - 4D + 3}e^{ix}\right] \\
&= \frac{1}{2}\mathrm{Im}\left[e^{3ix}\frac{1}{2(D+3i)^2 - 4(D+3i) + 3}(1)\right] + \frac{1}{2}\mathrm{Im}\left[e^{ix}\frac{1}{2(D+i)^2 - 4(D+i) + 3}(1)\right] \\
&= \frac{1}{2}\mathrm{Im}\left[\frac{e^{3ix}}{-15 - 12i}\right] + \frac{1}{2}\mathrm{Im}\left[\frac{e^{ix}}{1 - 4i}\right] = -\frac{1}{6}\mathrm{Im}\left[\frac{e^{3ix}}{5 + 4i}\frac{5 - 4i}{5 - 4i}\right] + \frac{1}{2}\mathrm{Im}\left[\frac{e^{ix}}{1 - 4i}\frac{1 + 4i}{1 + 4i}\right] \\
&= \frac{-1}{6}\mathrm{Im}\left[\frac{(\cos 3x + i\sin 3x)(5 - 4i)}{41}\right] + \frac{1}{2}\mathrm{Im}\left[\frac{(\cos x + i\sin x)(1 + 4i)}{17}\right] \\
&= -\frac{1}{246}(-4\cos 3x + 5\sin 3x) + \frac{1}{34}(4\cos x + \sin x).
\end{aligned}$$

A general solution of the differential equation is therefore

$$y(x) = e^x[C_1\cos(x/\sqrt{2}) + C_2\sin(x/\sqrt{2})] + (4\cos 3x - 5\sin 3x)/246 + (4\cos x + \sin x)/34.$$

19. The auxiliary equation $0 = m^2 - 3m + 2 = (m-1)(m-2)$ has solutions $m = 1, 2$. A general solution of the associated homogeneous equation is $y_h(x) = C_1 e^x + C_2 e^{2x}$. By operators,

$$\begin{aligned}
y_p &= \frac{1}{D^2 - 3D + 2}(8x^2 + 12e^{-x}) = \frac{8}{2[1 + (D^2 - 3D)/2]}(x^2) + 12e^{-x}\frac{1}{(D-1)^2 - 3(D-1) + 2}(1) \\
&= 4\left[1 - \left(\frac{D^2 - 3D}{2}\right) + \left(\frac{D^2 - 3D}{2}\right)^2 + \cdots\right]x^2 + 12e^{-x}\frac{1}{D^2 - 5D + 6}(1) \\
&= 4\left[x^2 + \frac{3}{2}(2x) + \frac{7}{4}(2)\right] + \frac{12e^{-x}}{6} = 4x^2 + 12x + 14 + 2e^{-x}.
\end{aligned}$$

A general solution of the differential equation is $y(x) = C_1 e^x + C_2 e^{2x} + 4x^2 + 12x + 14 + 2e^{-x}$. To satisfy the conditions $y(0) = 0$ and $y'(0) = 2$, we must have $0 = C_1 + C_2 + 14 + 2$ and $2 = C_1 + 2C_2 + 12 - 2$. These imply that $C_1 = -24$ and $C_2 = 8$.

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20. The auxiliary equation $m^2 + 9 = 0$ has solutions $m = \pm 3i$. A general solution of the associated homogeneous equation is $y_h(x) = C_1 \cos 3x + C_2 \sin 3x$. By operators,

$$y_p = \frac{1}{D^2+9}\{x[\mathrm{Im}(e^{3ix}) + \mathrm{Re}(e^{3ix})]\} = \mathrm{Im}\left[\frac{1}{D^2+9}(xe^{3ix})\right] + \mathrm{Re}\left[\frac{1}{D^2+9}(xe^{3ix})\right].$$

Consider then
$$\frac{1}{D^2+9}(xe^{3ix}) = e^{3ix}\frac{1}{(D+3i)^2+9}(x) = e^{3ix}\frac{1}{D^2+6iD}(x)$$
$$= e^{3ix}\frac{1}{6iD[1+D/(6i)]}(x) = e^{3ix}\frac{1}{6iD}\left(1 - \frac{D}{6i} + \cdots\right)x$$
$$= e^{3ix}\frac{1}{6iD}\left(x - \frac{1}{6i}\right) = -\frac{i}{6}e^{3ix}\left(\frac{x^2}{2} + \frac{ix}{6}\right) = \frac{1}{36}e^{3ix}(x - 3ix^2).$$

Thus, $y_p(x) = \dfrac{1}{36}(x \sin 3x - 3x^2 \cos 3x) + \dfrac{1}{36}(x \cos 3x + 3x^2 \sin 3x)$, and

$$y(x) = C_1 \cos 3x + C_2 \sin 3x + \frac{x}{36}(\cos 3x + \sin 3x) + \frac{x^2}{12}(\sin 3x - \cos 3x).$$

For $y(0) = 0$ and $y'(0) = 0$, we must have $0 = C_1$ and $0 = 3C_2 + 1/36$. Hence,
$y(x) = -\dfrac{1}{108}\sin 3x + \dfrac{x}{36}(\cos 3x + \sin 3x) + \dfrac{x^2}{12}(\sin 3x - \cos 3x)$.

21. The auxiliary equation is $Jm^4 + k = 0 \implies m^2 = \pm\sqrt{k/J}i$. If we set $\lambda = (1/\sqrt{2})(k/J)^{1/4}$, then $m^2 = \pm 2\lambda^2 i$. If we now set $m = a + bi$ in $m^2 = 2\lambda^2 i$, then $a^2 - b^2 + 2abi = 2\lambda^2 i$. When we equate real and imaginary parts, $a^2 - b^2 = 0$ and $2ab = 2\lambda^2$. These give $a = b = \pm\lambda$; that is, $m = \pm\lambda(1 + i)$. From $m^2 = -2\lambda^2 i$, we obtain $m = \pm\lambda(1 - i)$. A general solution of the associated homogeneous equation is $y_h(x) = e^{\lambda x}(C_1 \cos \lambda x + C_2 \sin \lambda x) + e^{-\lambda x}(C_3 \cos \lambda x + C_4 \sin \lambda x)$. Since a particular solution is $y_p(x) = w/k$, a general solution of the differential equation is $y(x) = e^{\lambda x}(C_1 \cos \lambda x + C_2 \sin \lambda x) + e^{-\lambda x}(C_3 \cos \lambda x + C_4 \sin \lambda x) + w/k$.

22. Since $\dfrac{dy}{dx} = \dfrac{dy}{dz}\dfrac{dz}{dx} = \dfrac{dy}{dz}\Big/\dfrac{dx}{dz} = \dfrac{1}{x}\dfrac{dy}{dz}$, we obtain

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}.$$

Thus, $x\dfrac{dy}{dx} = \dfrac{dy}{dz}$ and $x^2\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dz^2} - \dfrac{dy}{dz}$. When we substitute these into the differential equation,

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} + a\frac{dy}{dz} + by = F(e^z) \quad \Longrightarrow \quad \frac{d^2y}{dz^2} + (a-1)\frac{dy}{dz} + by = F(e^z),$$

a linear differential equation with constant coefficients.

23. If we set $r = e^z$ and use the results of Exercise 22 on $r^2\dfrac{d^2u}{dr^2} + r\dfrac{du}{dr} - u = 0$,

$$0 = \frac{d^2u}{dz^2} - \frac{du}{dz} + \frac{du}{dz} - u = \frac{d^2u}{dz^2} - u.$$

The auxiliary equation is $m^2 - 1 = 0$ with solutions $m = \pm 1$. A general solution of the differential equation is therefore $u(z) = C_1e^z + C_2e^{-z}$, and therefore $u(r) = C_1r + C_2/r$.

24. If we set $x = e^z$ and use the results of Exercise 22, $1 = \dfrac{d^2y}{dz^2} - \dfrac{dy}{dz} + \dfrac{dy}{dz} + 4y = \dfrac{d^2y}{dz^2} + 4y$. The auxiliary equation is $m^2 + 4 = 0$ with solutions $m = \pm 2i$. A general solution of the associated homogeneous equation is $y_h(z) = C_1 \cos 2z + C_2 \sin 2z$. Since $y_p(z) = 1/4$,

$$y(z) = C_1 \cos 2z + C_2 \sin 2z + 1/4 \quad \Longrightarrow \quad y(x) = C_1 \cos(2\ln x) + C_2 \sin(2\ln x) + 1/4.$$

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25. The solution of the associated homogeneous equation was discussed in Exercise 15.8-27. Substituting a particular solution of the form $x_p(t) = B\sin\omega t + C\cos\omega t$ into the differential equation gives

$$M(-\omega^2 B\sin\omega t - \omega^2 C\cos\omega t) + \beta(\omega B\cos\omega t - \omega C\sin\omega t) + k(B\sin\omega t + C\cos\omega t) = A\sin\omega t.$$

When we equate coefficients of $\sin\omega t$ and $\cos\omega t$,

$$(k - M\omega^2)B - \beta\omega C = A, \qquad \beta\omega B + (k - M\omega^2)C = 0.$$

Solutions of these are $B = \dfrac{A(k - M\omega^2)}{(k - M\omega^2)^2 + \beta^2\omega^2}$, $C = \dfrac{-A\beta\omega}{(k - M\omega^2)^2 + \beta^2\omega^2}$. The particular solution is therefore $x_p(t) = \dfrac{A}{(k - M\omega^2)^2 + \beta^2\omega^2}[(k - M\omega^2)\sin\omega t - \beta\omega\cos\omega t]$.

This solution is unacceptable when $\beta = 0$ and $\omega^2 = k/M$. In this case, the general solution of the associated homogeneous equation is $C_1\cos\omega t + C_2\sin\omega t$. A particular solution is of the form $x_p(t) = Bt\sin\omega t + Ct\cos\omega t$. Substitution into the differential equation gives

$$\begin{aligned} A\sin\omega t &= M(2\omega B\cos\omega t - \omega^2 Bt\sin\omega t - 2\omega C\sin\omega t - \omega^2 Ct\cos\omega t) + k(Bt\sin\omega t + Ct\cos\omega t) \\ &= B(k - M\omega^2)t\sin\omega t + C(k - M\omega^2)t\cos\omega t + 2M\omega B\cos\omega t - 2M\omega C\sin\omega t \\ &= 2M\omega B\cos\omega t - 2M\omega C\sin\omega t. \end{aligned}$$

Thus, $B = 0$ and $C = -A/(2M\omega)$. The particular solution is $x_p(t) = -At/(2M\omega)\cos\omega t$.

26. (a) Since the auxiliary equation $0 = m^2 - 3m + 2 = (m-1)(m-2)$ has solutions $m = 1, 2$, a general solution of the associated homogeneous equation is $y_h(x) = C_1e^x + C_2e^{2x}$. When we assume a particular solution of the form $y_p = Ax + B$ and substitute into the differential equation, $-3A + 2(Ax + B) = x \implies A = 1/2$, $B = 3/4$. Thus, $y(x) = C_1e^x + C_2e^{2x} + x/2 + 3/4$. To satisfy the initial conditions, we must have $2 = C_1 + C_2 + 3/4$ and $-1/2 = C_1 + 2C_2 + 1/2$. These imply that $C_1 = 7/2$ and $C_2 = -9/4$. The solution for $0 \le x \le 1$ is $y(x) = (7/2)e^x - (9/4)e^{2x} + x/2 + 3/4$.
(b) Since the differential equation is homogeneous for $x > 1$, a general solution on this interval is $y(x) = D_1e^x + D_2e^{2x}$.
(c) For $y(x)$ and $y'(x)$ to be continuous at $x = 1$, we require

$$\frac{7e}{2} - \frac{9e^2}{4} + \frac{5}{4} = D_1e + D_2e^2, \quad \frac{7e}{2} - \frac{9e^2}{2} + \frac{1}{2} = D_1e + 2D_2e^2.$$

These can be solved for $D_1 = \dfrac{7e+4}{2e}$ and $D_2 = -\dfrac{9e^2+3}{4e^2}$.
(d) The function will not satisfy the differential equation at $x = 1$ since its second derivative does not exist there.

27. Since roots of the auxiliary equation $m^2 + 1 = 0$ are $m = \pm i$, a general solution of the associated homogeneous equation is $y_h(x) = C_1\cos x + C_2\sin x$. A particular solution on the interval $0 \le x \le \pi$ is $x - 1$, so that on this interval $y(x) = C_1\cos x + C_2\sin x + x - 1$. To satisfy the initial conditions, we must have $0 = C_1 - 1$ and $0 = C_2 + 1$. Thus, $y(x) = \cos x - \sin x + x - 1$. On the interval $x > \pi$, we substitute a particular solution of the form $y_p = Ae^{-x}$ into the differential equation, obtaining $e^{-x} = Ae^{-x} + Ae^{-x} \implies A = 1/2$. Thus, on $x > \pi$, a general solution is $y(x) = D_1\cos x + D_2\sin x + (1/2)e^{-x}$. For the solution to be continuous and have a continuous first derivative at $x = \pi$, we require

$$-1 + \pi - 1 = -D_1 + \frac{e^{-\pi}}{2}, \qquad 1 + 1 = -D_2 - \frac{e^{-\pi}}{2}.$$

Solutions of these are $D_1 = (1/2)e^{-\pi} + 2 - \pi$ and $D_2 = -(1/2)e^{-\pi} - 2$.

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28. If we change dependent variables according to $y = d\Phi/dr$, then $r^3y''' + 2r^2y'' - ry' + y = 0$. If we now change independent variables with $r = e^z$, then as in Exercise 22,

$$r\frac{dy}{dr} = \frac{dy}{dz} \qquad \text{and} \qquad r^2\frac{d^2y}{dr^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}.$$

Furthermore,

$$\frac{d^3y}{dr^3} = \frac{d}{dr}\left(\frac{d^2y}{dr^2}\right) = \frac{d}{dr}\left[\frac{1}{r^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)\right] = -\frac{2}{r^3}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) + \frac{1}{r^2}\frac{d}{dz}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)\frac{dz}{dr}$$
$$= -\frac{2}{r^3}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) + \frac{1}{r^2}\left(\frac{d^3y}{dz^3} - \frac{d^2y}{dz^2}\right)\frac{1}{r}.$$

Hence, $r^3\dfrac{d^3y}{dr^3} = \dfrac{d^3y}{dz^3} - 3\dfrac{d^2y}{dz^2} + 2\dfrac{dy}{dz}$. Substitution of these into the differential equation for $y(r)$ gives

$$0 = \left(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right) + 2\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) - \frac{dy}{dz} + y = \frac{d^3y}{dz^3} - \frac{d^2y}{dz^2} - \frac{dy}{dz} + y.$$

The auxiliary equation is $0 = m^3 - m^2 - m + 1 = (m-1)^2(m+1)$ with solutions $m = -1, 1, 1$. Thus, $y(z) = (C_1 + C_2z)e^z + C_3e^{-z}$, from which $y(r) = (C_1 + C_2\ln r)r + C_3/r$.
Integration now gives

$$\Phi(r) = \int\left[r(C_1 + C_2\ln r) + \frac{C_3}{r}\right]dr + C_4 = \frac{C_1r^2}{2} + C_2\int r\ln r\,dr + C_3\ln r + C_4$$
$$= \frac{C_1r^2}{2} + C_2\left(\frac{r^2}{2}\ln r - \frac{r^2}{4}\right) + C_3\ln r + C_4 = C_5r^2 + C_6r^2\ln r + C_3\ln r + C_4.$$

29. (a) If we substitute $\Phi(r,\theta) = f(r)\cos n\theta$ into the biharmonic partial differential equation,

$$0 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left[f''(r)\cos n\theta + \frac{f'(r)}{r}\cos n\theta - \frac{n^2f(r)}{r^2}\cos n\theta\right]$$
$$= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left\{\left[f''(r) + \frac{f'(r)}{r} - \frac{n^2f(r)}{r^2}\right]\cos n\theta\right\}$$
$$= \left[f''''(r) + \frac{f'''(r)}{r} - \frac{2f''(r)}{r^2} + \frac{2f'(r)}{r^3} - \frac{n^2f''(r)}{r^2} + \frac{4n^2f'(r)}{r^3} - \frac{6n^2f(r)}{r^4}\right]\cos n\theta$$
$$+ \frac{1}{r}\left[f'''(r) + \frac{f''(r)}{r} - \frac{f'(r)}{r^2} - \frac{n^2f'(r)}{r^2} + \frac{2n^2f(r)}{r^3}\right]\cos n\theta$$
$$- \frac{n^2}{r^2}\left[f''(r) + \frac{f'(r)}{r} - \frac{n^2f(r)}{r^2}\right]\cos n\theta.$$

Thus, $f(r)$ must satisfy

$$0 = f''''(r) + \frac{2f'''(r)}{r} + \frac{f''(r)}{r^2}(-2 - n^2 + 1 - n^2) + \frac{f'(r)}{r^3}(2 + 4n^2 - 1 - n^2 - n^2)$$
$$+ \frac{f(r)}{r^4}(-6n^2 + 2n^2 + n^4)$$
$$= f''''(r) + \frac{2f'''(r)}{r} - \frac{(1+2n^2)f''(r)}{r^2} + \frac{(1+2n^2)f'(r)}{r^3} + \frac{(n^4 - 4n^2)f(r)}{r^4}.$$

If we set $r = e^z$ and $y = f(r)$, then as in Exercise 22 and 28,

$$r\frac{dy}{dr} = \frac{dy}{dz}, \qquad r^2\frac{d^2y}{dr^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}, \qquad r^3\frac{d^3y}{dr^3} = \frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}.$$

Furthermore,

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$$\frac{d^4y}{dr^4} = \frac{d}{dr}\left(\frac{d^3y}{dr^3}\right) = \frac{d}{dr}\left[\frac{1}{r^3}\left(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right)\right]$$
$$= -\frac{3}{r^4}\left(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right) + \frac{1}{r^3}\frac{d}{dz}\left(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right)\frac{dz}{dr}$$
$$= -\frac{3}{r^4}\left(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right) + \frac{1}{r^3}\left(\frac{d^4y}{dz^4} - 3\frac{d^3y}{dz^3} + 2\frac{d^2y}{dz^2}\right)\frac{1}{r}.$$

Hence, $r^4\dfrac{d^4y}{dr^4} = \dfrac{d^4y}{dz^4} - 6\dfrac{d^3y}{dz^3} + 11\dfrac{d^2y}{dz^2} - 6\dfrac{dy}{dz}$. Substitution of these into the differential equation for $y = f(r)$ gives

$$0 = \left(\frac{d^4y}{dz^4} - 6\frac{d^3y}{dz^3} + 11\frac{d^2y}{dz^2} - 6\frac{dy}{dz}\right) + 2\left(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right) - (1+2n^2)\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$
$$+ (1+2n^2)\frac{dy}{dz} + (n^4 - 4n^2)y$$
$$= \frac{d^4y}{dz^4} - 4\frac{d^3y}{dz^3} + (4-2n^2)\frac{d^2y}{dz^2} + 4n^2\frac{dy}{dz} + (n^4 - 4n^2)y.$$

The auxiliary equation is $0 = m^4 - 4m^3 + (4-2n^2)m^2 + 4n^2m + (n^4 - 4n^2)$. When $n = 1$, this becomes $0 = m^4 - 4m^3 + 2m^2 + 4m - 3 = (m-1)^2(m+1)(m-3)$, in which case $f(z) = (C_1 + C_2z)e^z + C_3e^{-z} + C_4e^{3z}$, and $f(r) = C_1r + C_2r\ln r + C_3/r + C_4r^3$. When $n > 1$, $0 = (m-n)(m+n)(m-n-2)(m+n-2)$, in which case

$$f(z) = C_1e^{nz} + C_2e^{-nz} + C_3e^{(n+2)z} + C_4e^{(2-n)z} \Longrightarrow f(r) = C_1r^n + C_2r^{-n} + C_3r^{n+2} + C_4r^{2-n}.$$

EXERCISES 15.10

1. (a) With the coordinate system of Figure 15.11, the differential equation describing the position $x(t)$ of the mass is

$$(1)\frac{d^2x}{dt^2} + 16x = 0, \qquad x(0) = -1/10, \quad x'(0) = 0.$$

The auxiliary equation is $m^2 + 16 = 0$ with solutions $m = \pm 4i$. A general solution of the differential equation is $x(t) = C_1\cos 4t + C_2\sin 4t$. To satisfy the initial conditions, we must have $-1/10 = C_1$ and $0 = 4C_2$. Thus, $x(t) = -(1/10)\cos 4t$ m.

(b) In this case the differential equation describing the position $x(t)$ of the mass is

$$(1)\frac{d^2x}{dt^2} + \frac{1}{10}\frac{dx}{dt} + 16x = 0 \quad\Longrightarrow\quad 10x'' + x' + 160x = 0.$$

The auxiliary equation is $10m^2 + m + 160 = 0$ with solutions $m = (-1 \pm 9\sqrt{79}i)/20$. A general solution of the differential equation is $x(t) = e^{-t/20}[C_1\cos(9\sqrt{79}t/20) + C_2\sin(9\sqrt{79}t/20)$. To satisfy the initial conditions, we must have $-1/10 = C_1$ and $0 = -C_1/20 + 9\sqrt{79}C_2/20$. These give

$$x(t) = e^{-t/20}\left(-\frac{1}{10}\cos\frac{9\sqrt{79}t}{20} - \frac{\sqrt{79}}{7110}\sin\frac{9\sqrt{79}t}{20}\right) \text{ m.}$$

(c) In this case the differential equation describing the position $x(t)$ of the mass is

$$(1)\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 16x = 0.$$

The auxiliary equation is $m^2 + 10m + 16 = 0$ with solutions $m = -2, -8$. A general solution of the differential equation is $x(t) = C_1e^{-2t} + C_2e^{-8t}$. The initial conditions require $-1/10 = C_1 + C_2$ and $0 = -2C_1 - 8C_2$. These give $C_1 = -2/15$ and $C_2 = 1/30$. Thus, $x(t) = (e^{-8t} - 4e^{-2t})/30$ m.

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2. With the coordinate system of Figure 15.11, the differential equation describing the position $x(t)$ of the mass is

$$\frac{1}{5}\frac{d^2x}{dt^2}+\frac{3}{2}\frac{dx}{dt}+10x=4\sin 10t \quad\Longrightarrow\quad 2x''+15x'+100x=40\sin 10t,$$

subject to $x(0)=0$, $x'(0)=0$. The auxiliary equation is $0=2m^2+15m+100$ with solutions $m=(-15\pm 5\sqrt{23}i)/4$. A general solution of the associated homogeneous equation is

$$x_h(t)=e^{-15t/4}[C_1\cos(5\sqrt{23}t/4)+C_2\sin(5\sqrt{23}t/4)].$$

A particular solution of the differential equation is

$$\begin{aligned}x_p(t)&=\frac{1}{2D^2+15D+100}[40\,\mathrm{Im}(e^{10it})]=40\,\mathrm{Im}\left[\frac{1}{2D^2+15D+100}(e^{10it})\right]\\&=40\,\mathrm{Im}\left[e^{10it}\frac{1}{2(D+10i)^2+15(D+10i)+100}(1)\right]=40\,\mathrm{Im}\left(\frac{e^{10it}}{-100+150i}\right)\\&=\frac{4}{5}\mathrm{Im}\left(\frac{e^{10it}}{-2+3i}\frac{-2-3i}{-2-3i}\right)=-\frac{4}{5}\mathrm{Im}\left[\frac{(2+3i)(\cos 10t+i\sin 10t)}{13}\right]\\&=-\frac{4}{65}(3\cos 10t+2\sin 10t).\end{aligned}$$

Thus, $x(t)=e^{-15t/4}[C_1\cos(5\sqrt{23}t/4)+C_2\sin(5\sqrt{23}t/4)]-(4/65)(3\cos 10t+2\sin 10t)$. To satisfy the initial conditions, we must have $0=C_1-12/65$ and $0=-15C_1/4+5\sqrt{23}C_2/4-16/13$. These imply that $C_1=12/65$ and $C_2=20/(13\sqrt{23})$, and therefore

$$\begin{aligned}x(t)=e^{-15t/4}\{&(12/65)\cos(5\sqrt{23}t/4)+[20/(13\sqrt{23})]\sin(5\sqrt{23}t/4)\}\\&-(4/65)(3\cos 10t+2\sin 10t)\text{ m}.\end{aligned}$$

3. The differential equation describing charge $Q(t)$ on the capacitor is

$$2\frac{d^2Q}{dt^2}+\frac{1}{0.001}Q=20 \quad\Longrightarrow\quad Q''+500Q=10,$$

subject to $Q(0)=0$ and $Q'(0)=0$. The auxiliary equation is $0=m^2+500$ with solutions $m=\pm 10\sqrt{5}i$. A general solution of the differential equation is therefore $Q(t)=C_1\cos 10\sqrt{5}t+C_2\sin 10\sqrt{5}t+1/50$. To satisfy the initial conditions, we must have $0=C_1+1/50$ and $0=10\sqrt{5}C_2$. Thus, $Q(t)=-(1/50)\cos 10\sqrt{5}t+1/50$, and the current in the circuit is $I(t)=(1/\sqrt{5})\sin 10\sqrt{5}t$ A.

4. The differential equation describing charge $Q(t)$ on the capacitor is

$$(1)\frac{d^2Q}{dt^2}+100\frac{dQ}{dt}+\frac{1}{0.02}Q=0 \quad\Longrightarrow\quad Q''+100Q'+50Q=0,$$

subject to $Q(0)=5$ and $Q'(0)=0$. The auxiliary equation is $m^2+100m+50=0$ with solutions $m=-0.50,\,-99.50$. A general solution of the differential equation is therefore $Q(t)=C_1e^{-0.50t}+C_2e^{-99.50t}$. To satisfy the initial conditions, we must have $5=C_1+C_2$ and $0=-0.50C_1-99.50C_2$. These imply that $C_1=5.03$ and $C_2=-0.0253$, and therefore $Q(t)=5.03e^{-0.50t}-0.0253e^{-99.50t}$ C.

5. The differential equation describing the current $I(t)$ in the circuit is

$$5\frac{d^2I}{dt^2}+20\frac{dI}{dt}=20\cos 2t,\qquad I(0)=0,\quad I'(0)=0.$$

The auxiliary equation is $5m^2+20m=0$ with solutions $m=0,\,-4$. A general solution of the associated homogeneous differential equation is therefore $I(t)=C_1+C_2e^{-4t}$. Substituting a particular solution of the form $I_p=A\cos 2t+B\sin 2t$,

$$5(-4A\cos 2t-4B\sin 2t)+20(-2A\sin 2t+2B\cos 2t)=20\cos 2t.$$

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This implies that $-20A + 40B = 20$ and $-20B - 40A = 0$, from which $A = -1/5$ and $B = 2/5$. The current is therefore $I(t) = C_1 + C_2e^{-4t} + (2\sin 2t - \cos 2t)/5$. The initial conditions require $0 = C_1 + C_2 - 1/5$ and $0 = -4C_2 + 4/5$, from which $C_1 = 0$ and $C_2 = 1/5$. The transient part of the current is $(1/5)e^{-4t}$ A, and the steady-state part is $(2\sin 2t - \cos 2t)/5$ A.

6. With the coordinate system of Figure 15.11, the differential equation describing the position of M is $M\frac{d^2x}{dt^2} + kx = 0$. According to equation 15.63, the stretch in the spring at equilibrium is Mg/k, and therefore the initial conditions are $x(0) = Mg/k$ and $x'(0) = 0$. The auxiliary equation is $0 = Mm^2 + k$ with solutions $m = \pm\sqrt{k/M}i$. Thus, $x(t) = C_1\cos(\sqrt{k/M}t) + C_2\sin(\sqrt{k/M}t)$. To satisfy the initial conditions, $Mg/k = C_1$ and $0 = \sqrt{k/M}C_2$. Thus, $x(t) = (Mg/k)\cos(\sqrt{k/M}t)$.

7. (a) Since the x-component of the force of friction when the mass is moving to the left is 1/2 N, the differential equation describing the position $x(t)$ of the mass from the time it starts until it comes to a stop for the first time is

$$\frac{1}{2}\frac{d^2x}{dt^2} + 18x = \frac{1}{2} \quad \Longrightarrow \quad x'' + 36x = 1,$$

subject to $x(0) = 0.05$ and $x'(0) = 0$.
(b) The auxiliary equation is $m^2 + 36 = 0$ with solutions $m = \pm 6i$, and therefore $x(t) = C_1\cos 6t + C_2\sin 6t + 1/36$. To satisfy the initial conditions, we must have $1/20 = C_1 + 1/36$ and $0 = 6C_2$. Thus, $x(t) = (1/45)\cos 6t + 1/36$. Since $v(t) = (-2/15)\sin 6t$, the mass comes to rest for the first time when $6t = \pi$, and at this time, its position is $x = (1/45)\cos\pi + 1/36 = 1/180$ m. Since this is to the right of the equilibrium position, further motion will not occur.

8. (a) Since the x-component of the force of friction when the mass is moving to the left is 1/2 N, the differential equation describing the position $x(t)$ of the mass from the time it starts until it comes to a stop for the first time is

$$\frac{1}{2}\frac{d^2x}{dt^2} + 18x = \frac{1}{2} \quad \Longrightarrow \quad x'' + 36x = 1,$$

subject to $x(0) = 1/4$ and $x'(0) = 0$.
(b) The auxiliary equation is $m^2 + 36 = 0$ with solutions $m = \pm 6i$, and therefore $x(t) = C_1\cos 6t + C_2\sin 6t + 1/36$. To satisfy the initial conditions, we must have $1/4 = C_1 + 1/36$ and $0 = 6C_2$. Thus, $x(t) = (2/9)\cos 6t + 1/36$. Since $v(t) = (-4/3)\sin 6t$, the mass comes to rest for the first time when $6t = \pi$, and at this time, its position is $x = (2/9)\cos\pi + 1/36 = -7/36$ m. At this position, the spring force is $18(7/36) = 7/2$ N. Because this is greater than the 1/2 N friction force, further motion will occur.

9. The differential equation describing charge $Q(t)$ on the capacitor is

$$\frac{1}{2}\frac{d^2Q}{dt^2} + 3\frac{dQ}{dt} + \frac{1}{0.1}Q = 0 \quad \Longrightarrow \quad Q'' + 6Q + 20Q = 0,$$

subject to $Q(0) = 0$ and $Q'(0) = 1$. The auxiliary equation $m^2 + +6m + 20 = 0$ has solutions $m = -3 \pm \sqrt{11}i$. A general solution of the differential equation is therefore $Q(t) = e^{-3t}(C_1\cos\sqrt{11}t + C_2\sin\sqrt{11}t)$. To satisfy the initial conditions, we must have $0 = C_1$ and $1 = -3C_1 + \sqrt{11}C_2$. Thus, $Q(t) = (1/\sqrt{11})e^{-3t}\sin\sqrt{11}t$. To find the maximum charge on the capacitor, we find critical points for $Q(t)$,

$$0 = Q'(t) = \frac{1}{\sqrt{11}}(-3e^{-3t}\sin\sqrt{11}t + \sqrt{11}e^{-3t}\cos\sqrt{11}t).$$

The smallest positive solution of this equation is $t = (1/\sqrt{11})\mathrm{Tan}^{-1}(\sqrt{11}/3)$, and the charge on the capacitor at this time is 0.105 C.

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10. (a) If $\beta = 0$, then $M\dfrac{d^2x}{dt^2} + kx = 0$. The auxiliary equation is $Mm^2+k = 0$ with solutions $m = \pm\sqrt{k/M}i$. Thus, $x(t) = C_1\cos(\sqrt{k/M}t) + C_2\sin(\sqrt{k/M}t)$.
(b) If $\beta \neq 0$ and $\beta^2 - 4kM < 0$, the auxiliary equation $Mm^2 + \beta m + k = 0$ has solutions

$$m = \frac{-\beta \pm \sqrt{\beta^2 - 4kM}}{2M} = -\frac{\beta}{2M} \pm \frac{\sqrt{4kM - \beta^2}}{2M}i.$$

Thus, $x(t) = e^{-\beta t/(2M)}(C_1\cos\omega t + C_2\sin\omega t)$, where $\omega = \sqrt{4kM - \beta^2}/(2M)$.
(c) If $\beta \neq 0$ and $\beta^2 - 4kM > 0$, the auxiliary equation has solutions $m = \dfrac{-\beta \pm \sqrt{\beta^2 - 4kM}}{2M}$, and therefore

$$x(t) = C_1e^{(-\beta+\sqrt{\beta^2-4kM})t/(2M)} + C_2e^{(-\beta-\sqrt{\beta^2-4kM})t/(2M)} = e^{-\beta t/(2M)}(C_1e^{\omega t} + C_2e^{-\omega t}),$$

where $\omega = \sqrt{\beta^2 - 4kM}/(2M)$.
(d) If $\beta \neq 0$ and $\beta^2 - 4kM = 0$, the auxiliary equation has solutions $m = -\beta/(2M),\ -\beta/(2M)$, and therefore $x(t) = (C_1 + C_2t)e^{-\beta t/(2M)}$.

11. The steady-state solution is the particular solution of the form $x_p = B\sin\omega t + C\cos\omega t$. When we substitute into the differential equation,

$$m(-B\omega^2\sin\omega t - C\omega^2\cos\omega t) + \beta(B\omega\cos\omega t - C\omega\sin\omega t) + k(B\sin\omega t + C\cos\omega t) = AP_0\sin\omega t.$$

Equating coefficients of $\sin\omega t$ and $\cos\omega t$ gives

$$-m\omega^2B - \beta\omega C + kB = AP_0, \qquad -m\omega^2C + \beta B\omega + kC = 0,$$

the solution of which is $B = \dfrac{AP_0(k - m\omega^2)}{(k - m\omega^2)^2 + \beta^2\omega^2}$, $\quad C = \dfrac{-AP_0\beta\omega}{(k - m\omega^2)^2 + \beta^2\omega^2}$. The steady-state solution is therefore $x_p(t) = \dfrac{AP_0[(k - m\omega^2)\sin\omega t - \beta\omega\cos\omega t]}{(k - m\omega^2)^2 + \beta^2\omega^2}$.

12. With the coordinate system of Figure 15.11, the differential equation describing the position $x(t)$ of the mass is

$$\frac{1}{10}\frac{d^2x}{dt^2} + 4000x = 3\cos 200t \quad\Longrightarrow\quad x'' + 40\,000x = 30\cos 200t,$$

subject to $x(0) = 0,\ x'(0) = 10$. The auxiliary equation is $m^2 + 40\,000 = 0$ with solutions $m = \pm 200i$. A general solution of the associated homogeneous equation is $x_h(t) = C_1\cos 200t + C_2\sin 200t$. Substituting a particular solution of the form $x_p = At\cos 200t + Bt\sin 200t$ into the differential equation gives

$$\begin{aligned}(-400A\sin 200t - 40\,000At\cos 200t + 400B\cos 200t - 40\,000Bt\sin 200t)&\\ + 40\,000(At\cos 200t + Bt\sin 200t) &= 30\cos 200t.\end{aligned}$$

This implies that $A = 0$ and $B = 3/40$, so that $x(t) = C_1\cos 200t + C_2\sin 200t + (3t/40)\sin 200t$. The initial conditions require $0 = C_1$ and $10 = 200C_2$. Thus, $x(t) = (1/20 + 3t/40)\sin 200t$ m. Displacements become unbounded as t gets large.

13. With the coordinate system of Figure 15.11, the differential equation describing the position of M is $(1)\dfrac{d^2x}{dt^2} + 64x = 2\sin 8t$, subject to $x(0) = 0$ and $x'(0) = 0$. The auxiliary equation is $0 = m^2 + 64$ with solutions $m = \pm 8i$. The general solution of the associated homogeneous differential equation is $x(t) = C_1\cos 8t + C_2\sin 8t$. A particular solution is

$$\begin{aligned}x_p(t) &= \frac{1}{D^2 + 64}(2\sin 8t) = 2\frac{1}{D^2 + 64}\mathrm{Im}(e^{8it}) = 2\,\mathrm{Im}\left[\frac{1}{D^2 + 64}(e^{8it})\right]\\
&= 2\,\mathrm{Im}\left[e^{8it}\frac{1}{(D + 8i)^2 + 64}(1)\right] = 2\,\mathrm{Im}\left[e^{8it}\frac{1}{D^2 + 16iD}(1)\right] = 2\,\mathrm{Im}\left[e^{8it}\frac{1}{D}\frac{1}{D + 16i}(1)\right]\\
&= 2\,\mathrm{Im}\left[e^{8it}\frac{1}{D}\left(\frac{-i}{16}\right)\right] = -\frac{1}{8}\mathrm{Im}(ite^{8it}) = -\frac{t}{8}\cos 8t.\end{aligned}$$

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Thus, $x(t) = C_1 \cos 8t + C_2 \sin 8t - (t/8)\cos 8t$. To satisfy the initial conditions, we must have $0 = C_1$ and $0 = 8C_2 - 1/8$. Thus, $x(t) = (1/64)\sin 8t - (t/8)\cos 8t$ m. For large t, the oscillations become unbounded.

14. The differential equation describing the position of the mass is $M\dfrac{d^2x}{dt^2} + kx = A\cos\omega t$. Solutions of the auxiliary equation $Mm^2 + k = 0$ are $m = \pm\sqrt{k/M}\,i$. Hence the general solution of the associated homogeneous equation is $x(t) = C_1 \cos\sqrt{k/M}\,t + C_2 \sin\sqrt{k/M}\,t$. Resonance occurs when $\sqrt{k/M} = \omega$.

15. The differential equation describing the current in the circuit is

$$\frac{25}{9}\frac{d^2I}{dt^2} + \frac{1}{0.04}I = -45\sin 3t \quad\Longrightarrow\quad 5I'' + 45I = -81\sin 3t,$$

subject to $I(0) = I'(0) = 0$. The auxiliary equation $5m^2 + 45 = 0$ has solutions $m = \pm 3i$, and therefore $I_h(t) = C_1\cos 3t + C_2 \sin 3t$. A particular solution is

$$\begin{aligned} I_p(t) &= \frac{1}{5D^2+45}[-81\,\mathrm{Im}(e^{3it})] = -\frac{81}{5}\mathrm{Im}\left[\frac{1}{D^2+9}(e^{3it})\right] = -\frac{81}{5}\mathrm{Im}\left[e^{3it}\frac{1}{(D+3i)^2+9}(1)\right] \\ &= -\frac{81}{5}\mathrm{Im}\left[e^{3it}\frac{1}{D(D+6i)}(1)\right] = -\frac{81}{5}\mathrm{Im}\left(\frac{e^{3it}}{6i}t\right) = \frac{27t}{10}\mathrm{Im}(ie^{3it}) = \frac{27}{10}t\cos 3t. \end{aligned}$$

Thus, $I(t) = C_1 \cos 3t + C_2 \sin 3t + (27t/10)\cos 3t$. To satisfy the initial conditions, we must have $0 = C_1$ and $0 = 3C_2 + 27/10$, and the solution becomes $I(t) = -(9/10)\sin 3t + (27/10)t\cos 3t$ A. Resonance does indeed occur.

16. (a) Substituting a particular solution of the form $x_p(t) = B\cos\omega t + C\sin\omega t$ into the differential equation gives

$$M(-\omega^2 B\cos\omega t - \omega^2 C\sin\omega t) + \beta(-\omega B\sin\omega t + \omega C\cos\omega t) + k(B\cos\omega t + C\sin\omega t) = A\cos\omega t.$$

When we equate coefficients of $\cos\omega t$ and $\sin\omega t$, we obtain

$$(k - M\omega^2)B + \beta\omega C = A, \qquad -\beta\omega B + (k - M\omega^2)C = 0.$$

Solutions of these are $B = \dfrac{A(k-M\omega^2)}{(k-M\omega^2)^2+\beta^2\omega^2}$, $\quad C = \dfrac{A\beta\omega}{(k-M\omega^2)^2+\beta^2\omega^2}$. The particular solution is therefore

$$x_p(t) = \frac{A}{(k-M\omega^2)^2+\beta^2\omega^2}[(k-M\omega^2)\cos\omega t + \beta\omega\sin\omega t].$$

(b) If we set $(k - M\omega^2)\cos\omega t + \beta\omega\sin\omega t = R\sin(\omega t + \phi) = R(\sin\omega t\cos\phi + \cos\omega t\sin\phi)$, and equate coefficients of $\sin\omega t$ and $\cos\omega t$,

$$k - M\omega^2 = R\sin\phi, \quad \beta\omega = R\cos\phi.$$

These imply that $R^2 = (k - M\omega^2)^2 + \beta^2\omega^2$, and therefore

$$\sin\phi = \frac{k-M\omega^2}{\sqrt{(k-M\omega^2)^2+\beta^2\omega^2}}, \quad \cos\phi = \frac{\beta\omega}{\sqrt{(k-M\omega^2)^2+\beta^2\omega^2}}.$$

(c) The amplitude $x(t)$ is a maximum when $(k - M\omega^2)^2 + \beta^2\omega^2$ is smallest. To determine the value of ω that yields the minimum, we solve

$$0 = 2(k - M\omega^2)(-2M\omega) + 2\beta^2\omega = 2\omega[-2M(k - M\omega^2) + \beta^2].$$

The nonzero solution is $\omega = \sqrt{k/M - \beta^2/(2M^2)}$. The amplitude at this value of ω is

$$\frac{A}{\sqrt{\left[k - M\left(\dfrac{k}{M} - \dfrac{\beta^2}{2M^2}\right)\right]^2 + \beta^2\left(\dfrac{k}{M} - \dfrac{\beta^2}{2M^2}\right)}} = \frac{2AM}{\beta\sqrt{4kM - \beta^2}}.$$

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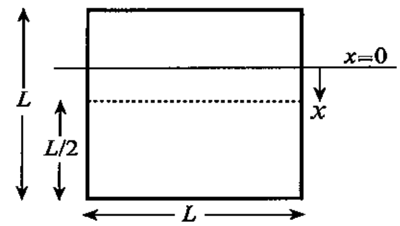
17. (a) Since the cube floats half submerged, its density is one-half that of water, namely 500 kg/m^3. Suppose we let x denote the distance of the midpoint of the cube below the surface of the water. When the midpoint is x m below the surface, the force on the cube is the buoyant force due to Archimedes' principle less the force of gravity,



$$-9810L^2\left(\frac{L}{2}+x\right)+4905L^3=-9810L^2x.$$

The differential equation describing oscillations of the cube is therefore

$$500L^3\frac{d^2x}{dt^2}=-9810L^2x \quad\Longrightarrow\quad x''+\frac{981}{50L}x=0.$$

(b) The auxiliary equation $m^2+981/(50L)=0$ has solutions $m=\pm\sqrt{981/(50L)}i$, and therefore

$$x(t)=C_1\cos\sqrt{\frac{981}{50L}}t+C_2\sin\sqrt{\frac{981}{50L}}t.$$

The frequency of the oscillations is $\dfrac{\sqrt{981/(50L)}}{2\pi}=\dfrac{0.705}{\sqrt{L}}$.

18. If x measures displacement of the platform from its equilibrium position, then the differential equation for the combined motion is

$$\left(\frac{W+w}{g}\right)\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}+kx=0.$$

The auxiliary equation is $\left(\dfrac{W+w}{g}\right)m^2+\beta m+k=0$ with solutions

$$m=\frac{-\beta\pm\sqrt{\beta^2-4k(W+w)/g}}{2(W+w)/g}.$$

Oscillations occur for large w, and for small values of w no oscillations occur. The largest value of w for no oscillations occurs when

$$\beta^2-\frac{4k(W+w)}{g}=0 \quad\Longrightarrow\quad w=\frac{\beta^2g}{4k}-W.$$

19. The differential equation for vertical motion of the trailer is

$$400\frac{d^2y}{dt^2}+40\,000y=40\,000A\cos\frac{\pi vt}{5}.$$

Resonance occurs when the forcing frequency is equal to the natural frequency of the trailer; that is, when

$$\frac{\pi v}{5}=\sqrt{\frac{40\,000}{400}} \quad\Longrightarrow\quad v=\frac{50}{\pi}\text{ m/s}.$$

This is equivalent to 57.3 km/hr.

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20. (a) The differential equation is

$$200\frac{d^2y}{dt^2} + 3000\frac{dy}{dt} + 50\,000y = 5000\sin\frac{\pi vt}{40} \quad\Longrightarrow\quad \frac{d^2y}{dt^2} + 15\frac{dy}{dt} + 250y = 25\sin\frac{\pi vt}{40}.$$

The auxiliary equation is $m^2 + 15m + 250 = 0$ with solutions

$$m = \frac{-15 \pm \sqrt{15^2 - 4(250)}}{2} = \frac{-15 \pm 5\sqrt{31}i}{2}.$$

Hence, $y_h(t) = e^{-15t/2}\left(C_1\cos\frac{5\sqrt{31}t}{2} + C_2\sin\frac{5\sqrt{31}t}{2}\right)$. A particular solution is of the form $y_p(t) = A\cos\frac{\pi vt}{40} + B\sin\frac{\pi vt}{40}$. Substituting into the differential equation,

$$\left(-\frac{\pi^2v^2A}{1600}\cos\frac{\pi vt}{40} - \frac{\pi^2v^2B}{1600}\sin\frac{\pi vt}{40}\right) + 15\left(\frac{-\pi vA}{40}\sin\frac{\pi vt}{40} + \frac{\pi vB}{40}\cos\frac{\pi vt}{40}\right)$$
$$+ 250\left(A\cos\frac{\pi vt}{40} + B\sin\frac{\pi vt}{40}\right) = 25\sin\frac{\pi vt}{40}.$$

Equating coefficients gives

$$-\frac{\pi^2v^2A}{1600} + \frac{3\pi vB}{8} + 250A = 0, \qquad -\frac{\pi^2v^2B}{1600} - \frac{3\pi vA}{8} + 250B = 25.$$

The solution is

$$A = \frac{-24\,000\,000\pi v}{(400\,000 - \pi^2v^2)^2 + 360\,000\pi^2v^2}, \quad B = \frac{40\,000(400\,000 - \pi^2v^2)}{(400\,000 - \pi^2v^2)^2 + 360\,000\pi^2v^2}.$$

Thus, $y(t) = e^{-15t/2}\left(C_1\cos\frac{5\sqrt{31}t}{2} + C_2\sin\frac{5\sqrt{31}t}{2}\right) + A\cos\frac{\pi vt}{40} + B\sin\frac{\pi vt}{40}$. The initial conditions $y(0) = 0$ and $y'(0) = 0$ require

$$0 = C_1 + A, \qquad 0 = -\frac{15}{2}C_1 + \frac{5\sqrt{31}}{2}C_2 + \frac{\pi vB}{40}.$$

These give $C_1 = -A$ and $C_2 = -(300A + \pi vB)/(100\sqrt{31})$. When $v = 10$, the solution is

$$y(t) = e^{-15t/2}\left(0.00473\cos\frac{5\sqrt{31}t}{2} - 0.00310\sin\frac{5\sqrt{31}t}{2}\right) - 0.00473\cos\frac{\pi t}{4} + 0.100\sin\frac{\pi t}{4}.$$

The amplitude of the steady-state part is

$$\sqrt{(0.00473)^2 + (0.100)^2} = 0.100.$$

(b) When $v = 20$, the solution is

$$y(t) = e^{-15t/2}\left(0.00953\cos\frac{5\sqrt{31}t}{2} - 0.00616\sin\frac{5\sqrt{31}t}{2}\right) - 0.00953\cos\frac{\pi t}{2} + 0.100\sin\frac{\pi t}{2}.$$

The amplitude of the steady-state part is

$$\sqrt{(0.00953)^2 + (0.100)^2} = 0.100.$$

21. The auxiliary equation is

$$0 = m^2 + 3m + 2 = (m+1)(m+2) \quad\Longrightarrow\quad m = -1,\ -2.$$

A general solution of the differential equation is $x(t) = C_1e^{-t} + C_2e^{-2t} + a/2$. The initial conditions require

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$$0 = C_1 + C_2 + \frac{a}{2}, \qquad \frac{2a}{3} = -C_1 - 2C_2.$$

The solution is $C_1 = -a/3$ and $C_2 = -a/6$, so that

$$x(t) = -\frac{a}{3}e^{-t} - \frac{a}{6}e^{-2t} + \frac{a}{2} = \frac{a}{6}(3 - 2e^{-t} - e^{-2t}).$$

22. (a) The force field associated with the given potential is $\mathbf{F} = -\nabla V = (-36y + 96x)\hat{\mathbf{i}} - 36x\hat{\mathbf{j}}$. By Newton's second law,

$$3\left(\frac{d^2x}{dt^2}\hat{\mathbf{i}} + \frac{d^2y}{dt^2}\hat{\mathbf{j}}\right) = (-36y + 96x)\hat{\mathbf{i}} - 36x\hat{\mathbf{j}}.$$

When we equate components,

$$\frac{d^2x}{dt^2} = -12y + 32x, \qquad \frac{d^2y}{dt^2} = -12x.$$

(b) If we substitute from the second of the differential equations in part (a) into the first, we obtain

$$-\frac{1}{12}\frac{d^4y}{dt^4} = -12y + 32\left(-\frac{1}{12}\frac{d^2y}{dt^2}\right) \qquad \Longrightarrow \qquad \frac{d^4y}{dt^4} - 32\frac{d^2y}{dt^2} - 144y = 0.$$

The auxiliary equation is $0 = m^4 - 32m^2 - 144 = (m^2 - 36)(m^2 + 4)$ with solutions $m = \pm 6, \pm 2i$. Hence,

$$y(t) = C_1e^{6t} + C_2e^{-6t} + C_3\cos 2t + C_4\sin 2t.$$

The second equation gives

$$x(t) = -\frac{1}{12}\frac{d^2y}{dt^2} = -3C_1e^{6t} - 3C_2e^{-6t} + \frac{1}{3}C_3\cos 2t + \frac{1}{3}C_4\sin 2t.$$

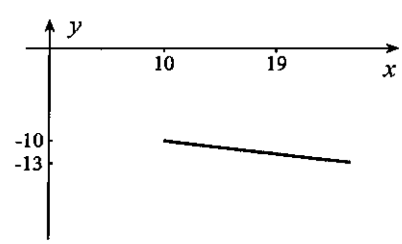
The initial conditions $x(0) = 10$, $x'(0) = 0$, $y(0) = -10$, and $y'(0) = 0$ require

$$10 = -3C_1 - 3C_2 + \frac{C_3}{3}, \qquad 0 = -18C_1 + 18C_2 + \frac{2C_4}{3},$$
$$-10 = C_1 + C_2 + C_3, \qquad 0 = 6C_1 - 6C_2 + 2C_4.$$

These give $C_1 = -2$, $C_2 = -2$, $C_3 = -6$, and $C_4 = 0$. Thus,

$$x(t) = 6e^{6t} + 6e^{-6t} - 2\cos 2t, \qquad y(t) = -2e^{6t} - 2e^{-6t} - 6\cos 2t.$$

(c) A plot of the curve is shown to the right. It appears to be a straight line with slope -3. This is not the case, however. It looks this way only because the exponential function e^{6t} is so dominant, the oscillations of the cosine terms are obliterated.



23. If we take $\mathbf{E} = E\hat{\mathbf{j}}$ and $\mathbf{B} = -B\hat{\mathbf{i}}$, then

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = E\hat{\mathbf{j}} + \mathbf{v} \times (-B\hat{\mathbf{i}}) = E\hat{\mathbf{j}} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ v_x & v_y & v_z \\ -B & 0 & 0 \end{vmatrix} = E\hat{\mathbf{j}} - Bv_z\hat{\mathbf{j}} + Bv_y\hat{\mathbf{k}}.$$

Newton's second law gives

$$m\frac{d\mathbf{v}}{dt} = q[(E - Bv_z)\hat{\mathbf{j}} + Bv_y\hat{\mathbf{k}}].$$

When we equate components, we obtain

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$$m\frac{dv_x}{dt} = 0, \qquad m\frac{dv_y}{dt} = q(E - Bv_z), \qquad m\frac{dv_z}{dt} = qBv_y.$$

With $x(0) = x'(0) = 0$, the first of these gives $x(t) = 0$ for all t. If we substitute from the third differential equation into the second, we find

$$m\left(\frac{m}{qB}\frac{d^2v_z}{dt^2}\right) = q(E - Bv_z) \qquad \Longrightarrow \qquad m^2\frac{d^2v_z}{dt^2} + q^2B^2v_z = q^2BE.$$

The auxiliary equation is $m^2 + q^2B^2 = 0$ with solutions $m = \pm qBi$. Since a particular solution is E/B, the general solution is

$$v_z(t) = C_1\cos\left(\frac{qBt}{m}\right) + C_2\sin\left(\frac{qBt}{m}\right) + \frac{E}{B}.$$

The initial condition $v_z(0) = 0$ requires $0 = C_1 + E/B$, and therefore

$$v_z(t) = \frac{E}{B}\left[1 - \cos\left(\frac{qBt}{m}\right)\right] + C_2\sin\left(\frac{qBt}{m}\right).$$

Since $v_y = [m/(qB)]dv_z/dt$, we find that

$$v_y = \frac{m}{qB}\left[\frac{Eq}{m}\sin\left(\frac{qBt}{m}\right) + \frac{qBC_2}{m}\cos\left(\frac{qBt}{m}\right)\right].$$

The initial condition $v_y(0) = 0$ requires $C_2 = 0$. Thus,

$$v_y(t) = \frac{E}{B}\sin\left(\frac{qBt}{m}\right), \qquad v_z = \frac{E}{B}\left[1 - \cos\left(\frac{qBt}{m}\right)\right].$$

Integration of these gives

$$y = -\frac{Em}{qB^2}\cos\left(\frac{qBt}{m}\right) + D_1, \qquad z = \frac{E}{B}\left[t - \frac{m}{qB}\sin\left(\frac{qBt}{m}\right)\right] + D_2.$$

The initial conditions $y(0) = 0$ and $z(0) = 0$, yield $0 = -Em/(qB^2) + D_1$ and $0 = D_2$. Thus,

$$y(t) = \frac{Em}{qB^2}\left[1 - \cos\left(\frac{qBt}{m}\right)\right], \qquad z(t) = \frac{Em}{qB^2}\left[\frac{qBt}{m} - \sin\left(\frac{qBt}{m}\right)\right].$$

24. (a) We write $\dfrac{d^2h}{dx^2} - \dfrac{K_v}{KbB}h = -\dfrac{K_vH_0}{KbB}$. The auxiliary equation $m^2 - K_v/(KbB) = 0$ has real roots $\pm\sqrt{K_v/(KbB)}$. If we let $m = \sqrt{K_v/(KbB)}$, then a general solution of the differential equation is

$$h(x) = C_1e^{mx} + C_2e^{-mx} + H_0.$$

The boundary conditions require

$$H_0 = C_1 + C_2 + H_0, \qquad b = C_1e^{mL} + C_2e^{-mL} + H_0,$$

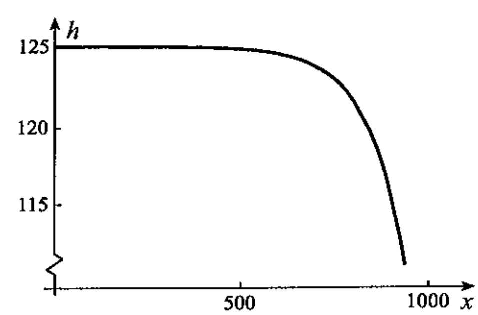
from which $C_1 = -C_2 = \dfrac{b - H_0}{e^{mL} - e^{-mL}}$. Thus,

$$h(x) = \frac{b - H_0}{e^{mL} - e^{-mL}}(e^{mx} - e^{-mx}) + H_0.$$

(b) With $K_v = 10^{-8}$, $K = 10^{-6}$, $b = 100$, $B = 1$, $H_0 = 125$, and $L = 1000$,

$$h(x) = 125 - \frac{25}{e^{10} - e^{-10}}(e^{x/100} - e^{-x/100}).$$

The plot is indeed relatively flat for $0 \le x \le 600$.



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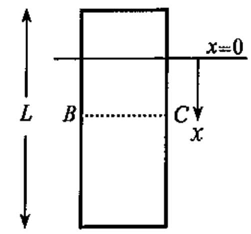
25. Let BC be the line on the cylinder that resides in the surface of the water when the cylinder is at equilibrium. If x represents the depth of BC below the surface when the cylinder is in motion, then Newton's second law for the acceleration of the cylinder is



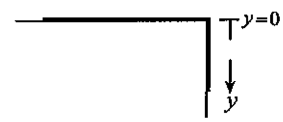
$$M\frac{d^2x}{dt^2} = -9.81(1000)\rho(Ax),$$

where M is the mass of the cylinder, A is its cross-sectional area, and ρ is its density. Since $M = \rho AL$, where L is the length of the cylinder,

$$\rho AL\frac{d^2x}{dt^2} = -9810\rho Ax \implies L\frac{d^2x}{dt^2} + 9810x = 0.$$

The auxiliary equation $Lm^2 + 9810 = 0$ has roots $m = \pm\sqrt{9810/L}i$, so that $x(t) = C_1 \cos\sqrt{9810/L}t + C_2 \sin\sqrt{9810/L}t$. Since the period of the oscillations is 4 s, it follows that $2\pi\sqrt{L/9810} = 4 \implies L = 39\,240/\pi^2$. The mass of the cylinder is therefore $\rho AL = \rho(\pi/100)(39\,240/\pi^2) = 124.9\rho$ kg.

26. Suppose the mass of the chain is M so that its mass per unit length is M/a. When the length of chain hanging from the edge of the table is y, then



$$M\frac{d^2y}{dt^2} = \frac{Mgy}{a}.$$

This differential equation is subject to the initial conditions $y(0) = b$ and $y'(0) = 0$, provided $t = 0$ is taken at the instant motion begins. The differential equation is linear with auxiliary equation $m^2 - g/a = 0 \implies m = \pm\sqrt{g/a}$. A general solution is therefore $y(t) = C_1 e^{\sqrt{g/a}t} + C_2 e^{-\sqrt{g/a}t}$. The initial conditions require

$$b = C_1 + C_2, \quad 0 = \sqrt{\frac{g}{a}}C_1 - \sqrt{\frac{g}{a}}C_2 \quad \implies \quad C_1 = C_2 = b/2.$$

Thus, $y(t) = \frac{b}{2}(e^{\sqrt{g/a}t} + e^{-\sqrt{g/a}t})$. The chain slides off the table when $y = a$ in which case

$$a = \frac{b}{2}(e^{\sqrt{g/a}t} + e^{-\sqrt{g/a}t}) \quad \implies \quad e^{2\sqrt{g/a}t} - \frac{2a}{b}e^{\sqrt{g/a}t} + 1 = 0.$$

This is a quadratic in $e^{\sqrt{g/a}t}$ with solutions

$$e^{\sqrt{g/a}t} = \frac{2a/b \pm \sqrt{4a^2/b^2 - 4}}{2} = \frac{1}{b}(a \pm \sqrt{a^2 - b^2}) \quad \implies t = \sqrt{\frac{a}{g}}\ln\left(\frac{a \pm \sqrt{a^2 - b^2}}{b}\right).$$

It is straightforward to verify that $(a - \sqrt{a^2 - b^2})/b < 1$ in which case t would be negative, an unacceptable value. Hence, $t = \sqrt{\frac{a}{g}}\ln\left(\frac{a + \sqrt{a^2 - b^2}}{b}\right)$.

27. (a) The auxiliary equation $m^4 + P/(EI)m^2 = 0$ has solutions $m = 0, 0, \pm\sqrt{P/(EI)}i$. A general solution of the differential equation is

$$y(x) = C_1 \cos\sqrt{\frac{P}{EI}}x + C_2 \sin\sqrt{\frac{P}{EI}}x + C_3 x + C_4.$$

(b) The boundary conditions require

$$0 = y(0) = C_1 + C_4, \quad 0 = y'(0) = \sqrt{\frac{P}{EI}}C_2 + C_3, \quad 0 = y''(L) = -\frac{C_1 P}{EI}\cos\sqrt{\frac{P}{EI}}L - \frac{C_2 P}{EI}\sin\sqrt{\frac{P}{EI}}L,$$

$$0 = y'''(L) + \frac{P}{EI}y'(L) = \left(\frac{P}{EI}\right)^{3/2} C_1 \sin\sqrt{\frac{P}{EI}}L - \left(\frac{P}{EI}\right)^{3/2} C_2 \cos\sqrt{\frac{P}{EI}}L$$

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$$+\frac{P}{EI}\left(-\sqrt{\frac{P}{EI}}C_1\sin\sqrt{\frac{P}{EI}}L+\sqrt{\frac{P}{EI}}C_2\cos\sqrt{\frac{P}{EI}}L+C_3\right).$$

These imply that $C_2 = C_3 = 0$, $0 = C_1 + C_4$, and $0 = C_1\cos\sqrt{\frac{P}{EI}}L$. Since $C_1 \neq 0$, else $y(x) \equiv 0$, we must set

$$0=\cos\sqrt{\frac{P}{EI}}L=0 \qquad \Longrightarrow \qquad \sqrt{\frac{P}{EI}}L=\frac{(2n+1)\pi}{2},$$

where n is an integer. Consequently, $P = (2n+1)^2\pi^2 EI/(4L^2)$, and

$$y(x)=-C_4\cos\frac{(2n+1)\pi x}{2L}+C_4=C_4\left[1-\cos\frac{(2n+1)\pi x}{2L}\right].$$

(c) Euler's buckling load is the smallest buckling load $\pi^2 EI/(4L^2)$ occurring when $n = 0$.

28. (a) The auxiliary equation $m^4 + P/(EI)m^2 = 0$ has solutions $m = 0, 0 \pm \sqrt{P/(EI)}i$. A general solution of the differential equation is

$$y(x)=C_1\cos\sqrt{\frac{P}{EI}}x+C_2\sin\sqrt{\frac{P}{EI}}x+C_3x+C_4.$$

(b) If we set $\mu = \sqrt{P/(EI)}$ for simplicity in notation, the boundary conditions require

$$0=y(0)=C_1+C_4,\quad 0=y'(0)=\mu C_2+C_3,\quad 0=y(L)=C_1\cos\mu L+C_2\sin\mu L+C_3L+C_4,$$

$$0=y'(L)=-\mu C_1\sin\mu L+\mu C_2\cos\mu L+C_3.$$

If we solve the first two for C_4 and C_3 and substitute into the last two, the result is

$$0=C_1\cos\mu L+C_2\sin\mu L-\mu LC_2-C_1,\quad 0=-\mu C_1\sin\mu L+\mu C_2\cos\mu L-\mu C_2,$$

or,

$$C_1(\cos\mu L-1)+C_2(\sin\mu L-\mu L)=0,\quad C_1(\sin\mu L)+C_2(1-\cos\mu L)=0.$$

If we solve each for C_1 and equate results,

$$-\frac{C_2(\sin\mu L-\mu L)}{\cos\mu L-1}=-\frac{C_2(1-\cos\mu L)}{\sin\mu L},$$

or,

$$\begin{aligned}0&=C_2(\sin^2\mu L-\mu L\sin\mu L+\cos^2\mu L-2\cos\mu L+1)=C_2(2-2\cos\mu L-\mu L\sin\mu L)\\&=C_2\left[2-2\left(1-2\sin^2\frac{\mu L}{2}\right)-2\mu L\sin\frac{\mu L}{2}\cos\frac{\mu L}{2}\right]=2C_2\sin\frac{\mu L}{2}\left(2\sin\frac{\mu L}{2}-\mu L\cos\frac{\mu L}{2}\right).\end{aligned}$$

There are three possibilities:
Case 1: $C_2 = 0$ — In this case, $C_3 = 0$, $C_4 = -C_1$, and $C_1(\cos\mu L - 1) = 0 = C_1\sin\mu L$. We cannot set $C_1 = 0$, else $C_4 = 0$ also, and $y(x) \equiv 0$. Hence we must set $\cos\mu L = 1$ and $\sin\mu L = 0$. These imply that $\mu L = 2n\pi$, where n is an integer, and therefore $\sqrt{P/(EI)}L = 2n\pi \Longrightarrow P = 4n^2\pi^2EI/L^2$. The smallest positive P is $P = 4\pi^2EI/L^2$.
Case 2: $\sin\frac{\mu L}{2} = 0$ — In this case, $\mu L/2 = n\pi$, where n is an integer, and $\sqrt{P/(EI)}L/2 = n\pi \Longrightarrow P = 4n^2\pi^2EI/L^2$. The smallest positive P is once again $P = 4\pi^2EI/L^2$.
Case 3: $2\sin\frac{\mu L}{2} - \mu L\cos\frac{\mu L}{2} = 0$ — In this case, $\mu L/2$ must satisfy the equation

$$\tan\frac{\mu L}{2}=\frac{\mu L}{2}.$$

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This equation must be solved numerically. The smallest positive solution is $\mu L/2 = 4.49$. Hence, $\sqrt{P/(EI)}L/2 = 4.49 \Longrightarrow P = 4(4.49)^2 EI/L^2 = 80.6EI/L^2$. This value is larger than that in Cases 1 and 2. Thus, the Euler buckling load is $P = 4\pi^2 EI/L^2$.

29. (a)

$$\begin{aligned}\frac{d^2N_A}{dt^2} &= -k_1\frac{dN_A}{dt} + k_2\frac{dN_B}{dt} = -k_1\frac{dN_A}{dt} + k_2[-(k_2+k_3)N_B + k_1N_A + k_4N_C]\\ &= -k_1\frac{dN_A}{dt} + k_1k_2N_A - k_2(k_2+k_3)N_B + k_2k_4(1-N_A-N_B)\\ &= -k_1\frac{dN_A}{dt} + (k_1k_2 - k_2k_4)N_A - k_2(k_2+k_3+k_4)N_B + k_2k_4\\ &= -k_1\frac{dN_A}{dt} + (k_1k_2 - k_2k_4)N_A - (k_2+k_3+k_4)\left(\frac{dN_A}{dt} + k_1N_A\right) + k_2k_4\\ &= -(k_1+k_2+k_3+k_4)\frac{dN_A}{dt} + (k_1k_2 - k_2k_4 - k_1k_2 - k_1k_3 - k_1k_4)N_A + k_2k_4.\end{aligned}$$

Thus,

$$\frac{d^2N_A}{dt^2} + (k_1+k_2+k_3+k_4)\frac{dN_A}{dt} + (k_1k_3 + k_2k_4 + k_1k_4)N_A = k_2k_4.$$

(b) When $k_1 = 2$, $k_2 = 1$, $k_3 = 4$, and $k_4 = 3$,

$$\frac{d^2N_A}{dt^2} + 10\frac{dN_A}{dt} + 17N_A = 3.$$

The auxiliary equation $m^2 + 10m + 17 = 0$ has roots $m = \dfrac{-10 \pm \sqrt{100-68}}{2} = -5 \pm 2\sqrt{2}$. The general solution for $N_A(t)$ is therefore

$$N_A(t) = C_1e^{(-5+2\sqrt{2})t} + C_2e^{-(5+2\sqrt{2})t} + \frac{3}{17}.$$

The initial conditions require

$$C_1 + C_2 + \frac{3}{17} = 1, \qquad (-5+2\sqrt{2})C_1 - (5+2\sqrt{2})C_2 = 0.$$

These give

$$N_A(t) = \frac{7}{68}(4+5\sqrt{2})e^{(-5+2\sqrt{2})t} + \frac{7}{68}(4-5\sqrt{2})e^{-(5+2\sqrt{2})t} + \frac{3}{17}.$$

Its limit is $\lim\limits_{t\to\infty} N_A(t) = \dfrac{3}{17}$.

(c) The differential equation for $N_B(t)$ is

$$\frac{dN_B}{dt} = -(k_2+k_3)N_B + k_1N_A + k_4(1-N_A-N_B)$$

or,

$$\frac{dN_B}{dt} + (k_2+k_3+k_4)N_B = (k_1-k_4)N_A + k_4.$$

With the given values for k_1, k_2, k_3, and k_4,

$$\frac{dN_B}{dt} + 8N_B = -N_A + 3.$$

This is a first-order linear differential equation with integating factor e^{8t}. We write therefore that

$$\frac{d}{dt}[e^{8t}N_B] = 3e^{8t} - \frac{7}{68}(4+5\sqrt{2})e^{(3+2\sqrt{2})t} + \frac{7}{68}(4-5\sqrt{2})e^{(3-2\sqrt{2})t} - \frac{3}{17}e^{8t}.$$

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Integration gives

$$e^{8t}N_B = \frac{6}{17}e^{8t} - \frac{7(4+5\sqrt{2})}{68(3+2\sqrt{2})}e^{(3+2\sqrt{2})t} + \frac{7(4-5\sqrt{2})}{68(3-2\sqrt{2})}e^{(3-2\sqrt{2})t} + D,$$

from which

$$N_B(t) = \frac{6}{17} - \frac{7(7\sqrt{2}-8)}{68}e^{(-5+2\sqrt{2})t} - \frac{7(7\sqrt{2}+8)}{68}e^{-(5+2\sqrt{2})t} + De^{-8t}.$$

The initial condition $N_B(0) = 0$ requires

$$0 = \frac{6}{17} - \frac{7(7\sqrt{2}-8)}{68} - \frac{7(7\sqrt{2}+8)}{68} + D \quad \Longrightarrow \quad D = \frac{49\sqrt{2}-12}{34}.$$

Thus,

$$N_B(t) = \frac{6}{17} - \frac{7(7\sqrt{2}-8)}{68}e^{(-5+2\sqrt{2})t} - \frac{7(7\sqrt{2}+8)}{68}e^{-(5+2\sqrt{2})t} + \frac{49\sqrt{2}-12}{34}e^{-8t}.$$

Its limit is $\lim_{t\to\infty} N_B(t) = \dfrac{6}{17}$.

(d) $N_C(t) = 1 - N_A - N_B = \dfrac{8}{17} - \dfrac{7(6-\sqrt{2})}{34}e^{(-5+2\sqrt{2})t} + \dfrac{7(6+\sqrt{2})}{34}e^{-(5+2\sqrt{2})t} - \dfrac{49\sqrt{2}-12}{34}e^{-8t}.$

Its limit is $\lim_{t\to\infty} N_C(t) = \dfrac{8}{17}$.

30. (b) The auxiliary equation $m^2 + (k_1+k_2+k_3+k_4)m + (k_1k_3+k_2k_4+k_1k_4) = 0$ has solutions

$$m = \frac{-(k_1+k_2+k_3+k_4) \pm \sqrt{(k_1+k_2+k_3+k_4)^2 - 4(k_1k_3+k_2k_4+k_1k_4)}}{2}.$$

We denote these roots by ω_1 and ω_2 ($\omega_1 > \omega_2$), where it is important to note that both ω_1 and ω_2 are negative. The general solution of the differential equation is

$$N_A(t) = C_1e^{\omega_1 t} + C_2e^{\omega_2 t} + K,$$

where $K = k_2k_4/(k_1k_3+k_2k_4+k_1k_4)$. The initial conditions require

$$1 = C_1 + C_2 + K, \qquad 0 = \omega_1C_1 + \omega_2C_2,$$

solutions of which are

$$C_1 = \frac{\omega_2(1-K)}{\omega_2-\omega_1}, \qquad C_2 = \frac{\omega_1(1-K)}{\omega_1-\omega_2}.$$

Thus,

$$N_A(t) = \frac{\omega_2(1-K)}{\omega_2-\omega_1}e^{\omega_1 t} + \frac{\omega_1(1-K)}{\omega_1-\omega_2}e^{\omega_2 t} + K.$$

Since ω_1 and ω_2 are negative, $\lim_{t\to\infty} N_A(t) = K$.

(c) From Exercise 29, the differential equation for $N_B(t)$ is

$$\frac{dN_B}{dt} + (k_2+k_3+k_4)N_B = (k_1-k_4)N_A + k_4.$$

This is a first-order linear equation with integrating factor $e^{(k_2+k_3+k_4)t}$. We write therefore that

$$\begin{aligned}\frac{d}{dt}[e^{(k_2+k_3+k_4)t}N_B] &= k_4e^{(k_2+k_3+k_4)t} + (k_1-k_4)e^{(k_2+k_3+k_4)t}N_A(t)\\ &= k_4e^{(k_2+k_3+k_4)t} + (k_1-k_4)Ke^{(k_2+k_3+k_4)t}\\ &\quad + \frac{(k_1-k_4)(1-K)}{\omega_2-\omega_1}\left[\omega_2e^{(\omega_1+k_2+k_3+k_4)t} - \omega_1e^{(\omega_2+k_2+k_3+k_4)t}\right].\end{aligned}$$

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Integration gives

$$e^{(k_2+k_3+k_4)t}N_B = \frac{k_4}{k_2+k_3+k_4}e^{(k_2+k_3+k_4)t} + \left[\frac{(k_1-k_4)K}{k_2+k_3+k_4}\right]e^{(k_2+k_3+k_4)t}$$
$$+\frac{(k_1-k_4)(1-K)}{\omega_2-\omega_1}\left[\frac{\omega_2 e^{(\omega_1+k_2+k_3+k_4)t}}{\omega_1+k_2+k_3+k_4} - \frac{\omega_1 e^{(\omega_2+k_2+k_3+k_4)t}}{\omega_2+k_2+k_3+k_4}\right] + D.$$

Thus,

$$N_B(t) = \frac{k_4}{k_2+k_3+k_4} + \frac{(k_1-k_4)K}{k_2+k_3+k_4} + \frac{(k_1-k_4)(1-K)}{\omega_2-\omega_1}\left[\frac{\omega_2 e^{\omega_1 t}}{\omega_1+k_2+k_3+k_4} - \frac{\omega_1 e^{\omega_2 t}}{\omega_2+k_2+k_3+k_4}\right]$$
$$+ De^{-(k_2+k_3+k_4)t}.$$

The initial condition $N_B(0) = 0$ requires

$$0 = \frac{k_4}{k_2+k_3+k_4} + \frac{(k_1-k_4)K}{k_2+k_3+k_4} + \frac{(k_1-k_4)(1-K)}{\omega_2-\omega_1}\left[\frac{\omega_2}{\omega_1+k_2+k_3+k_4} - \frac{\omega_1}{\omega_2+k_2+k_3+k_4}\right] + D,$$

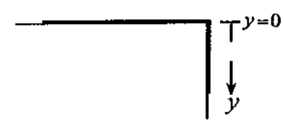
and this defines D. Since again ω_1 and ω_2 are negative,

$$\lim_{t\to\infty} N_B(t) = \frac{k_4}{k_2+k_3+k_4} + \frac{(k_1-k_4)K}{k_2+k_3+k_4}.$$

(d) $N_C(t) = 1 - N_A(t) - N_B(t)$ Its limit as $t \to \infty$ is

$$\lim_{t\to\infty} N_C(t) = 1 - \lim_{t\to\infty} N_A(t) - \lim_{t\to\infty} N_B(t) = 1 - K - \frac{k_4}{k_2+k_3+k_4} + \frac{(k_1-k_4)K}{k_2+k_3+k_4}.$$

31. Suppose the mass of the chain is M so that its mass per unit length is M/a. When the length of chain hanging from the edge of the table is y, then



$$M\frac{d^2y}{dt^2} = \frac{Mgy}{a} - \frac{\mu Mg}{a}(a-y) \implies \frac{d^2y}{dt^2} - \frac{g}{a}(1+\mu)y = -\mu g.$$

This differential equation is subject to the initial conditions $y(0) = b$ and $y'(0) = 0$, provided $t = 0$ is taken at the instant motion begins. The differential equation is linear with auxiliary equation $m^2 - (g/a)(1+\mu) = 0 \implies m = \pm\sqrt{g(1+\mu)/a}$. A general solution is therefore $y(t) = C_1 e^{\sqrt{g(1+\mu)/a}t} + C_2 e^{-\sqrt{g(1+\mu)/a}t} + a\mu/(1+\mu)$. The initial conditions require

$$b = C_1 + C_2 + \frac{a\mu}{1+\mu}, \quad 0 = \sqrt{\frac{g(1+\mu)}{a}}C_1 - \sqrt{\frac{g(1+\mu)}{a}}C_2 \implies C_1 = C_2 = \frac{1}{2}\left(b - \frac{a\mu}{1+\mu}\right).$$

Thus, $y(t) = \dfrac{1}{2}\left(b - \dfrac{a\mu}{1+\mu}\right)(e^{\sqrt{g(1+\mu)/a}t} + e^{-\sqrt{g(1+\mu)/a}t}) + \dfrac{a\mu}{1+\mu}$. The chain slides off the table when $y = a$,

$$a = \frac{1}{2}\left(b - \frac{a\mu}{1+\mu}\right)(e^{\sqrt{g(1+\mu)/a}t} + e^{-\sqrt{g(1+\mu)/a}t}) + \frac{a\mu}{1+\mu},$$

which can be expressed in the form

$$e^{2\sqrt{g(1+\mu)/a}t} - \frac{2a}{b(1+\mu)-a\mu}e^{\sqrt{g(1+\mu)/a}t} + 1 = 0.$$

This is a quadratic in $e^{\sqrt{g(1+\mu)/a}t}$ with solutions

$$e^{\sqrt{g(1+\mu)/a}t} = \frac{1}{2}\left[\frac{2a}{b(1+\mu)-a\mu} \pm \sqrt{\frac{4a^2}{[b(1+\mu)-a\mu]^2} - 4}\right] = \frac{a \pm \sqrt{a^2 - [b(1+\mu)-a\mu]^2}}{b(1+\mu)-a\mu},$$

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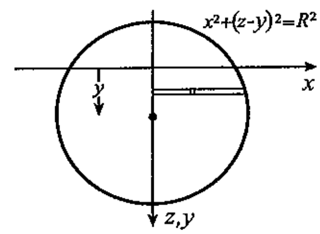
$$t = \sqrt{\frac{a}{g(1+\mu)}} \ln\left\{\frac{a \pm \sqrt{a^2 - [b(1+\mu) - a\mu]^2}}{b(1+\mu) - a\mu}\right\}.$$

It can be shown that the negative root is less than 1, in which case $t < 0$. Hence,

$$t = \sqrt{\frac{a}{g(1+\mu)}} \ln\left\{\frac{a + \sqrt{a^2 - [b(1+\mu) - a\mu]^2}}{b(1+\mu) - a\mu}\right\}.$$

32. Because the sphere floats half submerged, its density is one-half that of water, namely 500 kg/m^3. The resultant vertical force on the sphere when its centre is y m below the surface is the buoyant force due to the water displaced by the sphere less the force of gravity on the sphere,



$$-9810V + 4905\left(\frac{4}{3}\right)\pi R^3,$$

where V is the volume of water displaced by the sphere when its centre is y m below the surface. We can calculate V with the following double iterated integral,

$$\begin{aligned} V &= \int_0^{R+y}\int_0^{\sqrt{R^2-(z-y)^2}} 2\pi x\,dx\,dz = 2\pi\int_0^{R+y}\left\{\frac{x^2}{2}\right\}_0^{\sqrt{R^2-(z-y)^2}} dz \\ &= \pi\int_0^{R+y}[R^2 - (z-y)^2]\,dz = \pi\left\{R^2 z - \frac{(z-y)^3}{3}\right\}_0^{R+y} = \frac{\pi}{3}(2R^3 + 3R^2 y - y^3). \end{aligned}$$

The resultant force on the sphere when its centre is at depth y is therefore

$$\frac{-9810\pi}{3}(2R^3 + 3R^2 y - y^3) + \frac{19\,620}{3}\pi R^3 = \frac{9810\pi}{3}(y^3 - 3R^2 y).$$

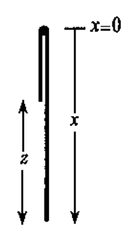
Newton's second law now gives

$$\frac{4}{3}\pi R^3(500)\frac{d^2y}{dt^2} = \frac{9810\pi}{3}(y^3 - 3R^2 y) \quad\Longrightarrow\quad \frac{d^2y}{dt^2} = -\frac{3(9.81)}{2R^3}\left(R^2 y - \frac{y^3}{3}\right).$$

33. (a) If x is the length of the longer piece of cable, then Newton's second law for acceleration of the cable is

$$25\rho\frac{d^2x}{dt^2} = 9.81\rho z,$$

where ρ is the mass per unit length of the cable. Since $x + (x - z) = 25$, it follows that $z = 2x - 25$ and



$$25\frac{d^2x}{dt^2} = 9.81(2x - 25),$$

or,

$$25\frac{d^2x}{dt^2} - 19.62x = -245.25.$$

The auxiliary equation $25m^2 - 19.62 = 0$ has roots $\pm\sqrt{19.62/25}$. If we denote the positive root by m, then $x(t) = C_1 e^{mt} + C_2 e^{-mt} + 245.25/19.62$. The initial conditions $x(0) = 15$ and $x'(0) = 0$ require $15 = C_1 + C_2 + 245.25/19.62$ and $0 = mC_1 - mC_2$. These imply that $C_1 = C_2 = 1.25$. The cable slides off the peg when $25 = 1.25(e^{mt} + e^{-mt}) + 245.25/19.62$ and the solution of this equation is 2.59 s.
(b) In this case Newton's second is

$$25\rho\frac{d^2x}{dt^2} = 9.81\rho z - 9.81\rho \quad\Longrightarrow\quad 25\frac{d^2x}{dt^2} - 19.62x = -255.06.$$

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The solution of this differential equation is $x(t) = C_1e^{mt} + C_2e^{-mt} + 255.06/19.62$, where m is as in part (a). The initial conditions require $15 = C_1 + C_2 + 255.06/19.62$ and $0 = mC_1 - mC_2$, and these gives $C_1 = C_2 = 1$. The cable slides off the peg when $25 = e^{mt} + e^{-mt} + 255.06/19.62$ and the solution of this equation is 2.80 s.

REVIEW EXERCISES

1. This equation can be separated $\dfrac{dy}{y} = \dfrac{dx}{x^2}$ (provided $y \neq 0$), and a one-parameter family of solutions is therefore defined implicitly by $\ln|y| = -\dfrac{1}{x} + C \implies y = De^{-1/x}$. By letting $D = 0$, we include the solution $y = 0$.

2. This equation can be separated $y\,dy = \left(\dfrac{x+1}{x}\right)dx$, and a one-parameter family of solutions is therefore defined implicitly by $\dfrac{y^2}{2} = x + \ln|x| + C \implies y = \pm\sqrt{2(x + \ln|x|) + D}$.

3. An integrating factor for this linear first-order equation is $e^{\int 3x\,dx} = e^{3x^2/2}$. When we multiply the differential equation by this factor,
$$e^{3x^2/2}\frac{dy}{dx} + 3xe^{3x^2/2}y = 2xe^{3x^2/2} \implies \frac{d}{dx}(ye^{3x^2/2}) = 2xe^{3x^2/2}.$$
Integration now gives
$$ye^{3x^2/2} = \int 2xe^{3x^2/2}\,dx = \frac{2}{3}e^{3x^2/2} + C \implies y(x) = Ce^{-3x^2/2} + 2/3.$$

4. An integrating factor for this linear first-order equation is $e^{\int 4\,dx} = e^{4x}$. When we multiply the differential equation by this factor,
$$e^{4x}\frac{dy}{dx} + 4ye^{4x} = x^2e^{4x} \implies \frac{d}{dx}(ye^{4x}) = x^2e^{4x}.$$
Integration now gives
$$ye^{4x} = \int x^2e^{4x}\,dx = \frac{1}{4}x^2e^{4x} - \frac{x}{8}e^{4x} + \frac{1}{32}e^{4x} + C \implies y(x) = Ce^{-4x} + x^2/4 - x/8 + 1/32.$$

5. The auxiliary equation is $0 = m^2 + 4m + 3$ with solutions $m = -1, -3$, and therefore $y(x) = C_1e^{-x} + C_2e^{-3x} + 2/3$.

6. The auxiliary equation is $0 = m^2 + 3m + 4$ with solutions $m = (-3 \pm \sqrt{7}i)/2$, and therefore $y(x) = e^{-3x/2}[C_1\cos(\sqrt{7}x/2) + C_2\sin(\sqrt{7}x/2)] + 1/2$.

7. This equation can be separated $\dfrac{y\,dy}{\sqrt{1+y^2}} = dx$, and a one-parameter family of solutions is therefore defined implicitly by $\sqrt{1+y^2} = x + C \implies y = \pm\sqrt{(x+C)^2 - 1}$.

8. If we set $dy/dx = v$ and $d^2y/dx^2 = dv/dx$, then $\dfrac{dv}{dx} + \dfrac{1}{x}v = x$, and multiplication by x results in $x^2 = x\dfrac{dv}{dx} + v = \dfrac{d}{dx}(xv)$. Integration now gives $xv = x^3/3 + C$, and hence $\dfrac{dy}{dx} = v = \dfrac{x^2}{3} + \dfrac{C}{x}$. Integration with respect to x now gives $y(x) = \dfrac{x^3}{9} + C\ln|x| + D$.

9. The auxiliary equation $m^2 + 6m + 3 = 0$ has solutions $m = -3 \pm \sqrt{6}$. A general solution of the associated homogeneous equation is $y_h(x) = C_1e^{(-3+\sqrt{6})x} + C_2e^{-(3+\sqrt{6})x}$. A particular solution is
$$y_p(x) = \frac{1}{D^2 + 6D + 3}xe^x = e^x\frac{1}{(D+1)^2 + 6(D+1) + 3}(x) = e^x\frac{1}{D^2 + 8D + 10}(x)$$

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$$= e^x \frac{1}{10[1 + (D^2 + 8D)/10]}(x) = \frac{1}{10}e^x \left[1 - \left(\frac{D^2 + 8D}{10}\right) + \cdots\right] x = \frac{1}{10}e^x(x - 4/5).$$

A general solution is therefore $y(x) = C_1 e^{(-3+\sqrt{6})x} + C_2 e^{-(3+\sqrt{6})x} + e^x(5x - 4)/50$.

10. An integrating factor for this linear first-order equation is $e^{\int 2x\,dx} = e^{x^2}$. When we multiply the differential equation by this factor,

$$e^{x^2/2}\frac{dy}{dx} + 2xe^{x^2/2}y = 2x^3 e^{x^2/2} \quad \Longrightarrow \quad \frac{d}{dx}(ye^{x^2}) = 2x^3 e^{x^2}.$$

Integration now gives

$$ye^{x^2} = \int 2x^3 e^{x^2}\,dx = x^2 e^{x^2} - e^{x^2} + C \quad \Longrightarrow \quad y(x) = x^2 - 1 + Ce^{-x^2}.$$

11. Since the dependent variable x is missing from the differential equation, we use substitutions 15.28 to write $y^2 v\dfrac{dv}{dy} = v \Longrightarrow dv = \dfrac{dy}{y^2}$. A one-parameter family of solutions of this separated differential equation is $v = -1/y + C$, from which $dy/dx = C - 1/y$. This equation can also be separated,

$$dx = \frac{y\,dy}{Cy - 1} = \frac{1}{C}\left(1 + \frac{1}{Cy - 1}\right) dy \quad \Longrightarrow \quad Cx + D = y + \frac{1}{C}\ln|Cy - 1|.$$

12. The auxiliary equation $0 = m^2 - 4m + 4 = (m - 2)^2$ has solutions $m = 2, 2$. A general solution of the associated homogeneous equation is $y_h(x) = (C_1 + C_2 x)e^{2x}$. A particular solution is

$$\begin{aligned} y_p(x) &= \frac{1}{(D-2)^2}\mathrm{Im}(e^{ix}) = \mathrm{Im}\left[\frac{1}{(D-2)^2}e^{ix}\right] = \mathrm{Im}\left[e^{ix}\frac{1}{(D - 2 + i)^2}(1)\right] \\ &= \mathrm{Im}\left(e^{ix}\frac{1}{3 - 4i}\right) = \mathrm{Im}\left(\frac{e^{ix}}{3 - 4i}\frac{3 + 4i}{3 + 4i}\right) = \frac{1}{25}(4\cos x + 3\sin x). \end{aligned}$$

Thus, $y(x) = (C_1 + C_2 x)e^{2x} + (4\cos x + 3\sin x)/25$.

13. The auxiliary equation $0 = m^2 - 4m + 4 = (m - 2)^2$ has solutions $m = 2, 2$. A general solution of the associated homogeneous equation is $y_h(x) = (C_1 + C_2 x)e^{2x}$. A particular solution is

$$y_p(x) = \frac{1}{(D-2)^2}x^2 e^{2x} = e^{2x}\frac{1}{D^2}(x^2) = \frac{x^4}{12}e^{2x}.$$

Thus, $y(x) = (C_1 + C_2 x + x^4/12)e^{2x}$.

14. The auxiliary equation $0 = m^2 + 4$ has solutions $m = \pm 2i$. A general solution of the associated homogeneous equation is $y_h(x) = C_1\cos 2x + C_2\sin 2x$. A particular solution is

$$\begin{aligned} y_p(x) &= \frac{1}{D^2 + 4}\mathrm{Im}(e^{2ix}) = \mathrm{Im}\left[\frac{1}{D^2 + 4}e^{2ix}\right] = \mathrm{Im}\left[e^{2ix}\frac{1}{(D + 2i)^2 + 4}(1)\right] \\ &= \mathrm{Im}\left[e^{2ix}\frac{1}{D(D + 4i)}(1)\right] = \mathrm{Im}\left(\frac{e^{2ix}}{4i}x\right) = -\frac{x}{4}\mathrm{Im}(ie^{2ix}) = -\frac{x}{4}\cos 2x. \end{aligned}$$

Thus, $y(x) = (C_1 - x/4)\cos 2x + C_2\sin 2x$.

15. The auxiliary equation $0 = m^2 + 4m = m(m + 4)$ has solutions $m = 0, -4$. A general solution of the associated homogeneous equation is $y_h(x) = C_1 + C_2 e^{-4x}$. A particular solution is

$$\begin{aligned} y_p(x) &= \frac{1}{D^2 + 4D}(x^2) = \frac{1}{4D(1 + D/4)}(x^2) = \frac{1}{4D}\left(1 - \frac{D}{4} + \frac{D^2}{16} + \cdots\right)x^2 \\ &= \frac{1}{4D}\left(x^2 - \frac{x}{2} + \frac{1}{8}\right) = \frac{1}{4}\left(\frac{x^3}{3} - \frac{x^2}{4} + \frac{x}{8}\right). \end{aligned}$$

Thus, $y(x) = C_1 + C_2 e^{-4x} + x^3/12 - x^2/16 + x/32$.

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16. This equation can be separated $\dfrac{2}{y}\,dy = -\dfrac{(x+1)^2}{x}\,dx$, (provided $y \neq 0$), and a one-parameter family of solutions is defined implicitly by

$$2\ln|y| = -\int\left(x+2+\frac{1}{x}\right)dx = -\left(\frac{x^2}{2}+2x+\ln|x|\right)+C.$$

Exponentiation leads to $xy^2 = De^{-2x-x^2/2}$. The function $y=0$ is a solution of the differential equation, but because it can be obtained if we permit $D=0$, it is not a singular solution.

17. The auxiliary equation $0 = m^3+3m^2+3m+1 = (m+1)^3$ has solutions $m=-1,-1,-1$. A general solution of the associated homogeneous equation is $y_h(x) = (C_1+C_2x+C_3x^2)e^{-x}$. A particular solution is

$$y_p(x) = \frac{1}{(D+1)^3}(2e^{-x}) = 2e^{-x}\frac{1}{D^3}(1) = \frac{x^3}{3}e^{-x}.$$

Thus, $y(x) = (C_1+C_2x+C_3x^2+x^3/3)e^{-x}$.

18. The auxiliary equation $0 = m^2+2m+4$ has solutions $m=-1\pm\sqrt{3}i$. A general solution of the associated homogeneous equation is $y_h(x) = e^{-x}[C_1\cos(\sqrt{3}x)+C_2\sin(\sqrt{3}x)]$. A particular solution is

$$\begin{aligned}
y_p(x) &= \frac{1}{D^2+2D+4}\mathrm{Re}(e^{-x}e^{\sqrt{3}ix}) = \mathrm{Re}\left[\frac{1}{D^2+2D+4}e^{(-1+\sqrt{3}i)x}\right]\\
&= \mathrm{Re}\left[e^{(-1+\sqrt{3}i)x}\frac{1}{(D-1+\sqrt{3}i)^2+2(D-1+\sqrt{3}i)+4}(1)\right]\\
&= \mathrm{Re}\left[e^{(-1+\sqrt{3}i)x}\frac{1}{D(D+2\sqrt{3}i)}(1)\right] = \mathrm{Re}\left[\frac{e^{(-1+\sqrt{3}i)x}}{2\sqrt{3}i}x\right]\\
&= -\frac{xe^{-x}}{2\sqrt{3}}\mathrm{Re}\big(ie^{\sqrt{3}ix}\big) = \frac{x}{2\sqrt{3}}e^{-x}\sin(\sqrt{3}x).
\end{aligned}$$

Thus, $y(x) = e^{-x}[C_1\cos(\sqrt{3}x)+C_2\sin(\sqrt{3}x)]+(\sqrt{3}/6)xe^{-x}\sin(\sqrt{3}x)$.

19. An integrating factor for this linear first-order equation is $e^{\int -\tan x\,dx} = e^{\ln|\cos x|} = |\cos x|$. For $\cos x>0$ or $\cos x<0$, multiplication by $\cos x$ leads to

$$\cos x\frac{dy}{dx} - y\sin x = \cos^2 x \quad\Longrightarrow\quad \frac{d}{dx}(y\cos x) = \cos^2 x.$$

Integration now gives

$$y\cos x = \int\left(\frac{1+\cos 2x}{2}\right)dx = \frac{x}{2}+\frac{1}{4}\sin 2x + C \quad\Longrightarrow\quad y(x) = (C+x/2)\sec x + (1/2)\sin x.$$

20. An integrating factor for this linear first-order equation in $x(y)$, $\dfrac{dx}{dy}+3x=-2y^2$, is $e^{\int 3\,dy} = e^{3y}$. Multiplication by e^{3y} gives

$$e^{3y}\frac{dx}{dy}+3xe^{3y} = -2y^2e^{3y} \quad\Longrightarrow\quad \frac{d}{dy}(xe^{3y}) = -2y^2e^{3y}.$$

Integrate with respect to y now yields

$$xe^{3y} = \int -2y^2e^{3y}\,dy = -\frac{2y^2}{3}e^{3y}+\frac{4y}{9}e^{3y}-\frac{4}{27}e^{3y}+C \quad\Longrightarrow\quad x = -2y^2/3+4y/9-4/27+Ce^{-3y}.$$

21. A one-parameter family of solutions of the separated equation $\dfrac{dy}{y^2} = -\dfrac{dx}{x+1}$ is given by $-1/y = -\ln|x+1|+C$. For $y(0)=3$, we must have $-1/3 = C$, and therefore the required solution is $y = 3/(3\ln|x+1|+1)$.

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22. The auxiliary equation $0 = m^2 - 8m - 9 = (m-9)(m+1)$ has solutions $m = -1, 9$. A general solution of the associated homogeneous equation is $y_h(x) = C_1e^{-x} + C_2e^{9x}$. A particular solution is

$$y_p(x) = \frac{1}{D^2 - 8D - 9}(2x+4) = \frac{1}{-9\left(1 + \dfrac{8D - D^2}{9}\right)}(2x+4)$$
$$= -\frac{1}{9}\left[1 - \left(\frac{8D - D^2}{9}\right) + \cdots\right](2x+4) = -\frac{1}{9}\left[(2x+4) - \frac{16}{9}\right].$$

Thus, $y(x) = C_1e^{-x} + C_2e^{9x} - 2x/9 - 20/81$. For $y(0) = 3$ and $y'(0) = 7$, we must have $3 = C_1 + C_2 - 20/81$ and $7 = -C_1 + 9C_2 - 2/9$. These imply that $C_1 = 11/5$ and $C_2 = 424/405$, and therefore $y(x) = \dfrac{11}{5}e^{-x} + \dfrac{424}{405}e^{9x} - \dfrac{2x}{9} - \dfrac{20}{81}$.

23. The auxiliary equation $m^2 + 9 = 0$ has solutions $m = \pm 3i$. A general solution of the associated homogeneous equation is $y_h(x) = C_1 \cos 3x + C_2 \sin 3x$. A particular solution is

$$y_p(x) = \frac{1}{D^2+9}e^x = e^x\frac{1}{(D+1)^2+9}(1) = e^x\frac{1}{D^2+2D+10}(1) = \frac{1}{10}e^x.$$

Thus, $y(x) = C_1 \cos 3x + C_2 \sin 3x + (1/10)e^x$. For $y(0) = 0$ and $y(\pi/2) = 4$, we must have $0 = C_1 + 1/10$ and $4 = -C_2 + (1/10)e^{\pi/2}$. These imply that $C_1 = -1/10$ and $C_2 = -4 + (1/10)e^{\pi/2}$, and therefore $y(x) = -(1/10)\cos 3x - (4 - e^{\pi/2}/10)\sin 3x + (1/10)e^x$.

24. An integrating factor for this linear first-order equation is $e^{\int 2/x\,dx} = e^{2\ln|x|} = x^2$. Multiplication of the differential equation by this factor gives

$$x^2\frac{dy}{dx} + 2xy = x^2 \sin x \quad \Longrightarrow \quad \frac{d}{dx}(x^2y) = x^2 \sin x.$$

We now integrate to obtain

$$x^2y = \int x^2 \sin x\,dx = -x^2\cos x + 2x \sin x + 2\cos x + C.$$

For $y(1) = 1$, we must have $1 = -\cos 1 + 2\sin 1 + 2\cos 1 + C$. Thus,

$$y(x) = -\cos x + \frac{2}{x}\sin x + \frac{2}{x^2}\cos x + \frac{1}{x^2}(1 - \cos 1 - 2\sin 1).$$

25. If $A(t)$ represents the amount of radioactive material in the sample, then $dA/dt = kA$, where k is a constant. This equation is separable, $dA/A = k\,dt$, and a one-parameter family of solutions is defined implicitly by $\ln A = kt + C \Longrightarrow A = De^{kt}$. If A_0 is the original size of the sample (at time $t = 0$), then $D = A_0$, and $A = A_0e^{kt}$. Since $A(5) = 3A_0/4$, it follows that $3A_0/4 = A_0e^{5k} \Longrightarrow k = (1/5)\ln(3/4)$. The sample is reduced to $A_0/10$ when $A_0/10 = A_0e^{kt}$. When we solve for t, the result is $t = -k^{-1}\ln 10 = 40$ years.

26. (a) According to Archimedes' principle, the buoyant force due to fluid pressure is the weight of fluid displaced by the wood, $(1.0 \times 10^{-6})(900)(9.81) = 8.829 \times 10^{-3}$ N. The total force due to fluid and gravity has magnitude $8.829 \times 10^{-3} - (1.0 \times 10^{-6})(500)(9.81) = 3.924 \times 10^{-3}$ N.

(b) If y measures distance from the bottom, then $\dfrac{1}{2000}\dfrac{d^2y}{dt^2} = 3.924 \times 10^{-3} - 2\dfrac{dy}{dt}$. We may integrate this equation, $\dfrac{1}{2000}\dfrac{dy}{dt} + 2y = 3.924 \times 10^{-3}t + C$. Since $y'(0) = 0 = y(0)$, it follows that C must also be zero, and $\dfrac{dy}{dt} + 4000y = 7.848t$. An integrating factor for this equation is $e^{\int 4000\,dt} = e^{4000t}$. Multiplication by this factor gives

$$e^{4000t}\frac{dy}{dt} + 4000e^{4000t}y = 7.848te^{4000t} \quad \Longrightarrow \quad \frac{d}{dt}(ye^{4000t}) = 7.848te^{4000t}.$$

Integration yields

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$$ye^{4000t} = 7.848\left(\frac{t}{4000}e^{4000t} - \frac{1}{16\times 10^6}e^{4000t}\right) + D.$$

Consequently, $y(t) = 1.962\times 10^{-3}t - 4.905\times 10^{-7} + De^{-4000t}$. Because $y(0) = 0$, it follows that $0 = -4.905\times 10^{-7} + D$, and therefore $y(t) = 1.962\times 10^{-3}t + 4.905\times 10^{-7}(e^{-4000t} - 1)$ m.

27. (a) With the coordinate system of Figure 15.11, the differential equation describing the position $x(t)$ of the mass is

$$\frac{1}{10}\frac{d^2x}{dt^2} + x = 0 \quad\Longrightarrow\quad \frac{d^2x}{dt^2} + 10x = 0,$$

subject to $x(0) = 1/25$ and $x'(0) = 0$. The auxiliary equation is $m^2 + 10 = 0$ with solutions $m = \pm\sqrt{10}i$. A general solution of the differential equation is $x(t) = C_1\cos\sqrt{10}t + C_2\sin\sqrt{10}t$. To satisfy the initial conditions, we must have $1/25 = C_1$ and $0 = \sqrt{10}C_2$. Thus, $x(t) = (1/25)\cos\sqrt{10}t$ m.
(b) In this case the differential equation describing the position $x(t)$ of the mass is

$$\frac{1}{10}\frac{d^2x}{dt^2} + \frac{1}{5}\frac{dx}{dt} + x = 0 \quad\Longrightarrow\quad \frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 0.$$

The auxiliary equation is $m^2 + 2m + 10 = 0$ with solutions $m = -1\pm 3i$. A general solution of the differential equation is $x(t) = e^{-t}(C_1\cos 3t + C_2\sin 3t)$. To satisfy the initial conditions, we must have $1/25 = C_1$ and $0 = -C_1 + 3C_2$. These give $x(t) = e^{-t}(3\cos 3t + \sin 3t)/75$ m.
(c) In this case the differential equation describing the position $x(t)$ of the mass is

$$\frac{1}{10}\frac{d^2x}{dt^2} + \sqrt{2/5}\frac{dx}{dt} + x = 0 \quad\Longrightarrow\quad \frac{d^2x}{dt^2} + 2\sqrt{10}\frac{dx}{dt} + 10x = 0.$$

The auxiliary equation is $0 = m^2 + 2\sqrt{10}m + 10 = (m + \sqrt{10})^2$ with solutions $m = -\sqrt{10}, -\sqrt{10}$. A general solution of the differential equation is $x(t) = (C_1 + C_2t)e^{-\sqrt{10}t}$. To satisfy the initial conditions, we must have $1/25 = C_1$ and $0 = -\sqrt{10}C_1 + C_2$. Thus, $x(t) = (1/25)(1 + \sqrt{10}t)e^{-\sqrt{10}t}$ m.

28. (a),(b) Let us take $y = 0$ on the bridge and y positive downward. During free fall, $\dfrac{1}{100}\dfrac{dv}{dt} = \dfrac{1}{100}(9.81)$. Thus, $v(t) = 9.81t + C$, and the condition $v(0) = 0$ requires $C = 0$. Integration now gives $y(t) = 4.905t^2 + D$. Because $y(0) = 0$, we obtain $D = 0$, and $y(t) = 4.905t^2$. The stone strikes the water when $50 = 4.905t^2$, and this equation implies that $t = \sqrt{50/4.905}$ s. At this instant, its velocity is $v = 9.81\sqrt{50/4.905} = \sqrt{50(19.62)}$. When the stone is in the water,

$$\frac{1}{100}\frac{dv}{dt} = \frac{1}{100}(9.81) - \frac{v}{5} \quad\Longrightarrow\quad \frac{dv}{dt} + 20v = 9.81.$$

An integrating factor for this equation is $e^{\int 20\,dt} = e^{20t}$, so that

$$e^{20t}\frac{dv}{dt} + 20ve^{20t} = 9.81e^{20t} \quad\Longrightarrow\quad \frac{d}{dt}(ve^{20t}) = 9.81e^{20t}.$$

Integration gives

$$ve^{20t} = 0.4905e^{20t} + C \quad\Longrightarrow\quad v(t) = 0.4905 + Ce^{-20t}.$$

Since $v(\sqrt{50/4.905}) = 0.9\sqrt{50(19.62)}$,

$$0.9\sqrt{50(19.62)} = 0.4905 + Ce^{-20\sqrt{50/4.905}},$$

and this equation implies that $C = 1.494\times 10^{29}$. Integration of $v(t)$ yields $y(t) = 0.4905t - \dfrac{C}{20}e^{-20t} + D$. Because $y(\sqrt{50/4.905}) = 50$, we have $50 = 0.4905\sqrt{50/4.905} - \dfrac{1.494\times 10^{29}}{20}e^{-20\sqrt{50/4.905}} + D$, and this requires $D = 49.82$. Thus, $y(t) = 0.4905t - 7.47\times 10^{27}e^{-20t} + 49.82$.
(c) The stone reaches the bottom when $60 = 0.4905t - 7.47\times 10^{27}e^{-20t} + 49.82$. The solution of this equation is 20.75 seconds.

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29. (a),(b) The velocity and position functions in Exercise 28 remain valid when the stone is falling in the air. When the stone is in the water,

$$\frac{1}{100}\frac{dv}{dt} = \frac{1}{100}(9.81) - \frac{v}{5} - 1000(9.81)\left(\frac{3}{10^6}\right) \quad \Longrightarrow \quad \frac{dv}{dt} + 20v = 6.867.$$

An integrating factor for this equation is $e^{\int 20\,dt} = e^{20t}$, so that

$$e^{20t}\frac{dv}{dt} + 20ve^{20t} = 6.867e^{20t} \quad \Longrightarrow \quad \frac{d}{dt}(ve^{20t}) = 6.867e^{20t}.$$

Integration gives

$$ve^{20t} = 0.343\,35e^{20t} + C \quad \Longrightarrow \quad v(t) = 0.343\,35 + Ce^{-20t}.$$

Since $v(\sqrt{50/4.905}) = 0.9\sqrt{50(19.62)}$,

$$0.9\sqrt{50(19.62)} = 0.343\,35 + Ce^{-20\sqrt{50/4.905}},$$

and this implies that $C = 1.502 \times 10^{29}$. Integration of $v(t)$ yields $y(t) = 0.343\,35t - \dfrac{C}{20}e^{-20t} + D$. Because $y(\sqrt{50/4.905}) = 50$, we have $50 = 0.343\,35\sqrt{50/4.905} - \dfrac{1.502 \times 10^{29}}{20}e^{-20\sqrt{50/4.905}} + D$, and this requires $D = 50.30$. Thus, $y(t) = 0.343\,35t - 7.51 \times 10^{27}e^{-20t} + 50.30$.
(c) The stone reaches the bottom when $60 = 0.343\,35t - 7.51 \times 10^{27}e^{-20t} + 50.30$. The solution of this equation is 28.25 seconds.

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