Differential Equations Assignment 4

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①

$k = 1/hr$

$T_m = \Delta T = 35°$ (2pm), $15°$ (2am), $10°$ w/ base 25

- period 24 hours $= \frac{2\pi}{24} = \pi/12$
- starting point time zero = 8:00 am, gives no horizontal shift

$\therefore T_m = 10\sin\left[\pi/12 (t)\right] + 25$, since adding 8, −8 in equation

Since Newton's law of cooling states $\frac{dT}{dm} = -k(T - T_m)$

a) $\frac{dT}{dt} + \underbrace{kT}_{P(t)} = \underbrace{k\left(25 + 10\sin\left[\pi/12 (t)\right]\right)}_{Q(t)}$, 1st order linear

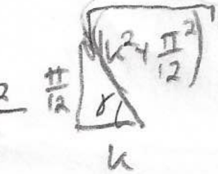
b) $T = e^{-\int k\,dt}\left[\int\left[25k + 10k\sin(\pi/12\, t)\right]e^{\int k\,dt}\,dt + C\right]$

$T = e^{-kt}\left[\int 25k e^{kt} + \int 10k\sin(\pi/12\, t)e^{kt} + C\right]$

$T = \frac{25ke^{kt}}{e^{kt}k} + e^{-kt}10k\int \sin(\pi/12\, t)e^{kt}\,dt$, using ㉔ from table $+ \frac{C}{e^{kt}}$

$= 25 + \frac{10ke^{kt}}{e^{kt}}\left[\frac{k\sin\pi/12\, t - \pi/12\cos\pi/12\, t}{k^2 + (\pi/12)^2}\right] + \frac{C}{e^{kt}}$ ← since steady state, this term goes to zero as $t \to \infty$

$= 25 + 10k\left[\frac{k\sin\pi/12\, t - \pi/12\cos\pi/12\, t}{k^2 + (\pi/12)^2}\right]$



, let $\tan\gamma = \frac{\pi/12}{k}$

$\cos\gamma = \frac{k}{\sqrt{k^2 + \frac{\pi}{12}^2}}$

$\sin\gamma = \frac{\left(\frac{\pi}{12}\right)}{\sqrt{k^2 + \left(\frac{\pi}{12}\right)^2}}$

$= 25 + \frac{10k}{k^2 + (\frac{\pi}{12})^2}\sqrt{k^2 + (\frac{-\pi}{12})^2}\left[\frac{k\sin\frac{\pi}{12}t}{\sqrt{k^2 + (\frac{-\pi}{12})^2}} - \frac{\frac{\pi}{12}\cos\frac{\pi}{12}t}{\sqrt{k^2 + (\frac{-\pi}{12})^2}}\right]$

$$T = 25 + \frac{10k}{k^2+(\frac{\pi}{12})^2}\sqrt{k^2+(\tfrac{\pi}{12})^2}\left(\sin\frac{\pi t}{12k}\cos\gamma - \cos\frac{\pi t}{12k}\sin\gamma\right)$$, since $\sin(A \pm B) = \sin A\cos B \pm \cos A \sin B$

$$T = 25 + \frac{10k}{k^2+(\frac{\pi}{12})^2}\sqrt{k^2+(\tfrac{\pi}{12})^2}\sin\left(\frac{\pi t}{12k} - \gamma\right)$$, since $\tan\gamma = \frac{\pi/12}{k}$

$\gamma = \tan^{-1}\left(\frac{\pi}{12k}\right)$

$$T = 25 + \frac{10\sqrt{k^2+(\frac{\pi}{12})^2}}{\left(\sqrt{k^2+(\frac{\pi}{12})^2}\right)^2}\sin\left(\frac{\pi t}{12k} - \tan^{-1}\left(\frac{\pi}{12k}\right)\right)$$

$$T = 25 + \frac{10k}{\sqrt{k^2+(\frac{\pi}{12})^2}}\sin\left(\frac{\pi t}{12k} - \tan^{-1}\left(\frac{\pi}{12k}\right)\right)$$, multiply top and bottom by 12

$$T = 25 + \frac{120k}{\sqrt{144k^2+\pi^2}}\sin\left(\frac{\pi t}{12k} - \tan^{-1}\left(\frac{\pi}{12k}\right)\right)$$

c) $$25 + \frac{120(0.2)}{\sqrt{144(0.2)^2+\pi^2}}\sin\left(\frac{\pi t}{12k} - \tan^{-1}\left(\frac{\pi}{12k}\right)\right)$$

$$25 + 6.07\,(\pm 1)$$ max and min

$\therefore$ Max = 31.1°C,

Min = 18.9°C

(2)

$\Sigma F = ma,$

$T - (a + bv^2) - ma = 0$

$T - a - bv^2 - \frac{m\,dv}{dt} = 0$, Case 1: $T - a - bv^2 = 0$

$V = \sqrt{\frac{T-a}{b}}$ (Terminal)

Case 2, $\int dt = m \int \frac{1}{T-a-bv^2}\,dv$, $T - a - bv^2$ $b\left(\frac{T}{b} - \frac{a}{b} - v^2\right)$, $= b\left(\sqrt{\frac{T-a}{b}}^{\,2} - v^2\right)$

$t = \frac{m}{b}\int \frac{1}{\left(\sqrt{\frac{T-a}{b}}\right)^2 - v^2}\,dv$, from equation (32)

$t = \frac{m}{b} \cdot \frac{1}{\sqrt{\frac{T-a}{b}}} \tanh^{-1}\left(\frac{V}{\sqrt{\frac{T-a}{b}}}\right) + C_1$, , since @ $v=0, t=0$, $C_1 = 0$

$\frac{tb}{m} \cdot \sqrt{\frac{T-a}{b}} = \tanh^{-1}\left(\frac{V}{\sqrt{\frac{T-a}{b}}}\right)$ $\sqrt{\frac{T-a}{b}} = \alpha$, $\beta = \frac{b\alpha}{m}$

$\beta t = \tanh^{-1}\left(\frac{V}{\alpha}\right)$

$\tanh(\beta t) = V/\alpha$

$V = \alpha \tanh(\beta t)$, includes Case 1

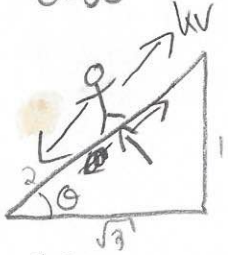
$\frac{dx}{dt} = \alpha \tanh(\beta t)$

$x = \alpha \int \tanh(\beta t)\,dt$, $\cosh(0) = \frac{1+e^0}{2e^0} = 1$, $\ln 1 = 0$

$x = \frac{\alpha}{\beta} \ln\cosh(\beta t) + C_2$ $\therefore$ since $t=0$, $x=0$, $C_2 = 0$

$x = \frac{\alpha}{\beta} \ln\cosh(\beta t)$

3.8 $\theta = 30$



$\Sigma F = ma = 0, \quad mg\sin\theta - \mu mg\cos\theta - kv - m\frac{dv}{dt} = 0$

$$\frac{1}{2}mg - \frac{\sqrt{3}}{2}\mu mg - kv - m\frac{dv}{dt} = 0$$

$$mg\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\mu\right) - kv - m\frac{dv}{dt} = 0$$

$$\frac{dv}{dt} + \underbrace{\frac{k}{m}}_{=B} v = \underbrace{g\left[\frac{1}{2} - \frac{\sqrt{3}}{2}\mu\right]}_{\alpha}$$

$$v = e^{-\int B dt}\left[\int \alpha e^{\int B dt} dt + C_1\right]$$

$$v = \frac{1}{e^{Bt}} \frac{\alpha e^{Bt}}{B} + \frac{C_1}{e^{Bt}}$$

$$v = \frac{\alpha}{B} + \frac{C_1}{e^{Bt}}, \text{ since from rest } C_1 = \frac{-\alpha}{B}$$

@ $t=0$

$$\frac{dx}{dt} = \frac{\alpha}{B} - \frac{\alpha}{Be^{Bt}}$$

$$x = \frac{\alpha}{B}\int 1 + e^{-Bt} dt$$

$$x = \frac{\alpha t}{B} + \frac{\alpha_1 e^{-Bt}}{B^2} + C_2, \text{ @ } x=0, t=0$$

$$0 = 0 + \frac{\alpha}{B^2} + C_2$$

$$C_2 = -\frac{\alpha}{B^2}$$

@ $t = 3.61\,s$, $x = 25\,m$

less effort, more reward

$$25 = \frac{\alpha}{B}(3.61) + \frac{\alpha e^{-B(3.61)}}{B^2} - \frac{\alpha}{B^2}$$

$$① \; 25 = \frac{\alpha}{B}\left[3.61 + \frac{e^{-3.61B}}{B} - \frac{1}{B}\right] = \alpha$$

Disregard

$$\alpha = \frac{1}{3.61}\left[25 + C_2\left(e^{-B\cdot 3.61} - 1\right)\right]$$

$$① \; \alpha = \frac{1}{3.61}\left[25 + \frac{\alpha}{B^2}\left(e^{-B\cdot 3.61} - 1\right)\right]$$

@ $t = 5.3$, $x = 50\,m$

$$50 = \frac{\alpha}{B}(5.3) + \frac{\alpha e^{-5.3B}}{B^2} + C_2$$

$$② \; 50 = \frac{\alpha}{B}\left[5.3 + \frac{e^{-5.3B}}{B} - \frac{1}{B}\right]$$

$$\frac{②}{①} = 2 = \frac{\left[5.3 + \frac{e^{-5.3B}}{B} - \frac{1}{B}\right]}{\left[3.61 + \frac{e^{-3.61B}}{B} - \frac{1}{B}\right]}$$

Disregard

$$② \; \alpha = \frac{50}{5.3} + \frac{\alpha}{5.3B^2}\left(\frac{1}{e^{5.3B}} - 1\right)$$

From Matlab, $B = 0.14847$

$$\alpha = \frac{50\,B}{\left[5.3 - \frac{e^{-5.3B}}{B} + 1/B\right]}, \quad \alpha = 4.5516$$

From definition,

$$g\left[\frac{1}{2} - \frac{\sqrt{3}}{2}\mu\right] = \alpha$$

$$\mu = 0.041105$$

$$k = mB = 70 \cdot 0.14847$$

$$k = 10.395 \; \frac{N\cdot sec}{m}$$

@ $x = 100$

$$100 = \frac{\alpha}{B}\left[t_3 + \frac{1}{B}\left(e^{-Bt} - 1\right)\right]$$

$$t_3 = 7.92\ sec$$

$$\therefore v_3 = \frac{\alpha}{B}\left(1 - e^{-Bt}\right)$$

$$= 21.2\ m/sec$$

```
B = linspace(0,10,1000);
y = @(B) (5.3 + exp(-5.3.*B)./B - 1./B)./(3.61 + exp(-3.61.*B)./B - 1./B) - 2;
Beta = fzero(y,0.1)
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    dequestion4 at line 3 column 6
Beta = -0.39212
>> dequestion4

Beta =  0.14847
>>
```
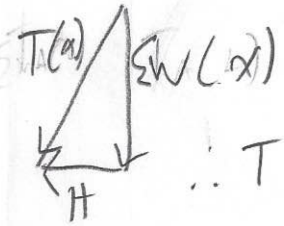
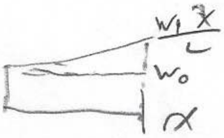

Command Window

④



$\therefore T^2 = H^2 + W(x)^2$

T_0?



$\therefore w(x) = w_0 + \frac{(w_1 - w_0)x}{L}$

From class, $\frac{dy}{dx} = \frac{1}{H}\int_0^x w(x)\,dx$

$$\frac{dy}{dx} = \frac{1}{H}\left[w_0 x + \frac{(w_1 - w_0)x^2}{2L}\right] + C_1$$

@ $x=0$, $\frac{dy}{dx} = 0$ (tan θ)

$\therefore C_1 = 0$

Immediately integrable

$$y = \frac{1}{H}\left[\frac{w_0 x^2}{2} + \frac{(w_1 - w_0)x^3}{6L}\right] + C_2$$

@ $x=0$, $y=0$

$\therefore C_2 = 0$

To find H

$y(L) = h$

$$h = \frac{1}{H}\left[\frac{w_0 L^2}{2} + \frac{(w_1 - w_0)L^3}{6L}\right]$$

$$H = \frac{1}{h}\left[\frac{w_0 L^2}{2} + \frac{w_1 L^2}{6} - \frac{w_0 L^2}{6}\right]$$

$$H = \frac{1}{h}\left[\frac{w_1 L^2}{3} - \frac{w_0 L^2}{6}\right]$$

To find $W(L)$

$$W(L) = \int_0^L w_0 + \frac{(w_1 - w_0)x}{L}\,dx = w_0 L + \frac{w_1 L^2}{2L} - \frac{w_0 L^2}{2L}$$

$\rightarrow w_0(L) + \frac{w_1 L}{2} - \frac{w_0 L}{2}$

$$W(L) = \frac{w_0 L}{2} + \frac{w_1 L}{2} = \frac{L}{2}(w_0 + w_1)$$

To find $F_T(A)$

$T^2 = H^2 + W(x)^2$

$$T^2 = \left(\frac{1}{h}\left[\frac{w_1 L^2}{3} - \frac{w_0 L^2}{6}\right]\right)^2 + \frac{L}{2}(w_0 + w_1)^2$$

$$\therefore T = \sqrt{\left[\frac{1}{h}\left(\frac{w_1 L^2}{3} - \frac{w_0 L^2}{6}\right)\right]^2 + \left[\frac{L}{2}(w_0 + w_1)\right]^2}$$

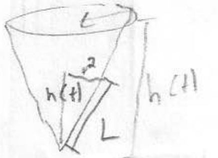
3.25

$$\frac{r}{h(t)} = \frac{R}{H},\ r = \frac{R\,h(t)}{H}$$

V = water in − water out

$$\frac{1}{3}\pi r^2 h(t) = q\,dt - \pi r^2 dt$$

$$\frac{1}{3}\pi\left(\frac{R\,h(t)}{H}\right)^2 h = q\,dt - \pi k\left(\frac{R^2 h(t)^2}{H^2}\right)dt$$



$$\frac{dV}{dh} = \frac{\pi R^2}{H^2} h^2$$

$$\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt}$$

$$\frac{dV}{dt} = q - \frac{\pi k R^2 h^2}{H^2}$$

$$(a^2 - b^2) = (a-b)(a+b)$$

$$\frac{\pi R^2 h^2}{H^2}\frac{dh}{dt} = q - \frac{\pi k R^2 h^2}{H^2}$$

$$\frac{dh}{dt}h^2 = k\mu^2 - kh^2$$, Variable separable

$$\int \frac{h^2}{(\mu^2 - h^2)}\,dh = \int k\,dt \quad ; \quad \frac{A}{(\mu-h)} + \frac{B}{(\mu+h)} = 0,\ A = B = 1$$

$$\int -1 + \frac{\mu^2}{\mu^2 - h^2}\,dh$$

$$\mu = 0,\ A + B = 0,\ A = -B$$

$$-h + \frac{\mu^2}{2\mu}\int \frac{1}{\mu+h} + \frac{1}{\mu-h}\,dh = kt + C_1$$

$$-h + \frac{\mu}{2}\ln\left|\frac{\mu+h}{\mu-h}\right| = kt + C_1$$

Since

$t = 0, h = 0,\ C_1 = 0$

$$-h + \frac{\mu}{2}\ln\left|\frac{\mu+h}{\mu-h}\right| = kt$$

⑥ From pg. 129 , $\left[V_0 + (Q_{in} - Q_{out})A\right]\frac{dc}{dt} + Q_{in}c = Q_{in}c_{in}$

$\Delta Q = 0$

$\therefore 220000\frac{dC}{dt} = -2185C$, Variable separable

$-\frac{44000}{C \cdot 437} dC = \int dt$

$\therefore \frac{-44000}{437} \ln|c| = t + C_1$, since $C(0) = 12$

$-100.6864989 \ln|12| = C_1$

$C_1 = -250.1966$

$\therefore t = -\frac{44000}{437} \ln(c) + 250.1966$, @ $C = 0.01$

$t = \frac{-44000}{437} \ln(0.01) + 250.1966$

$t = \frac{713.875 \text{ min}}{60}$

$= 11\text{hr } 54 \text{ min}$

$m = C \cdot V$ $\quad C = \frac{m}{V}$

Benjamin Klassen

7. Tank 1

$Q_{in} = Q_{out} = Q$

$$V \frac{dc(t)}{dt} = \left[C_{in} - C(t)\right] Q$$

$$\frac{dc(t)}{dt} + \frac{Q}{V} C(t) = C_{in}\frac{Q}{V} \quad \text{let } \frac{Q}{V} = \alpha,$$

$$C(t) = e^{-\int \alpha dt} \cdot \left[\int C_{in}\alpha e^{\int \alpha dt} dt + C\right]$$

$$C(t) = \frac{1}{e^{\alpha t}} \cdot \left[\int C_{in}\alpha e^{\alpha t} dt + C\right]$$

$$C(t) = C_{in} + \frac{C}{e^{\alpha t}}, \quad @ t=0, \ C(0) = \frac{S}{V}. \quad \text{Also, } C_{in} = 0$$

$$\frac{S}{V} = C$$

$$C(t) = \frac{S/V}{e^{\alpha t}}$$

Tank 2

$$V \frac{dc(t)}{dt} = \left[1 + \frac{S/V}{e^{\alpha t}} - C_2\right] Q, \quad \frac{Q}{V} = \alpha, \quad S/V = B$$

$$\frac{dc(t)}{dt} = \alpha\left[1 + \frac{B}{e^{\alpha t}} - C_2(t)\right]$$

$$\frac{dc(t)}{dt} + \alpha C_2(t) = \alpha\left(\frac{B}{e^{\alpha t}}\right)$$

$$C(t) = e^{-\int \alpha dt}\left[\int \frac{\alpha B}{e^{\alpha t}} e^{\int \alpha dt} + C\right]$$

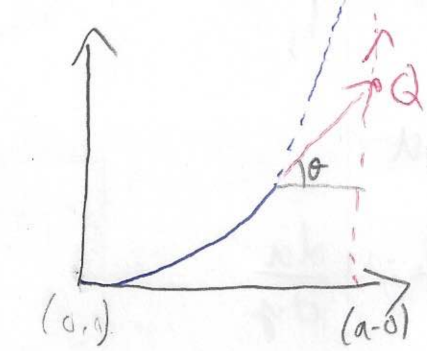
$$C(t) = \frac{1}{e^{\alpha t}}\left[\int \alpha B dt + C\right]$$

$$C(t) = \frac{\alpha B t}{e^{\alpha t}} + \frac{C}{e^{\alpha t}}, \ @ \ t=0, \ C(t)=0 \ \therefore C=0$$

$$C(t) = \frac{\alpha B t}{e^{\alpha t}}$$

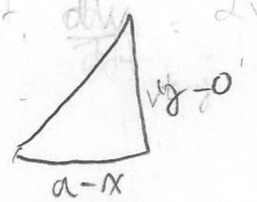
$$C_2 = \frac{QSt}{V^2} e^{-\frac{Q}{V}t}$$

(8)



$V_x = V\cos\theta = V\cdot\frac{(a-x)}{\sqrt{(a-x)^2+(vt-y)^2}}$, $V_y = V\cdot\frac{(vt-y)}{\sqrt{(a-x)^2+(vt-y)^2}}$

$V_y = V\sin\theta$



$\frac{vt-0}{a-x} = \frac{dy}{dx}$, $vt = -(a-x)\frac{dy}{dx}$

From arc length, total distance travelled by cat se below:

$\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ ∴ since v of cat × 2 v mouse

∴ $\frac{2v}{v}(a-x)\frac{dy}{dx} = \int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$, derivative of both sides

$2(a-x)\frac{d^2y}{dx^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$(a-x)\frac{d^2y}{dx^2} = \frac{1}{2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$, second order, y absent

b) $(a-x)\frac{dv}{dt} = \frac{1}{2}\sqrt{1+v^2}$

$$(a-x)\frac{d^2y}{dx^2} = \frac{1}{2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$\frac{dy}{dx} = u$

$\frac{dy^2}{dx^2} = u\frac{du}{dy}$

$$(a-x)\frac{du}{dx} = \frac{1}{2}\sqrt{1+u^2}$$

$$\frac{1}{2}\int (a-x)\,dx = \int \frac{1}{\sqrt{1^2+u^2}}\,du$$

$$\frac{1}{2}ax - \frac{x^2}{4} = \sin^{-1}\left(\frac{u}{1}\right)$$

$$\sin\left(\frac{1}{2}x\left(a-\frac{x}{2}\right)\right) = \frac{dy}{dx}$$

New approach

$$\frac{dx}{dt} = \frac{a}{t}$$

$$\int dx = \int \frac{a}{t}\,dt$$

$x = a\ln|t| + C$, when $x=a$

$1 = \ln(t) + C$

Unable to finish