Marginal cost derivations for a Cobb-Douglas production function

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Imagine a producer wants to minimise costs from labour L_t and capital K_t, which come at factor prices w_t and r_t, respectively, subject to a production technology of the Cobb-Douglas function $Y_t = A_t L_t^{1-\alpha} K_t^{\alpha}$, where A_t is total factor productivity and α the capital share of production. The minimal cost is given as

$$C(w_t, r_t, Y_t, A_t) = \min_{L_t, K_t} w_t L_t + r_t K_t \text{ s.t. } Y_t = A_t L_t^{1-\alpha} K_t^{\alpha}.$$

Solving the constraint for capital, we obtain

$$K_t = \left(\frac{Y_t}{A_t L_t^{1-\alpha}} \right)^{\frac{1}{\alpha}},$$

so that

$$C(w_t, r_t, Y_t, A_t) = \min_{L_t} w_t L_t + r_{k,t} \left(\frac{Y_t}{A_t L_t^{1-\alpha}} \right)^{\frac{1}{\alpha}}.$$

The first-order condition of that problem is

$$w_t = \frac{1-\alpha}{\alpha} r_{k,t} \left(\frac{Y_t}{A_t L_t} \right)^{\frac{1}{\alpha}},$$

so the optimal use of labour in production, L_t^*, is given by

$$L_t^*(w_t, r_t, Y_t, A_t) = \left(\frac{1-\alpha}{\alpha} \frac{r_t}{w_t} \right)^{\alpha} \frac{Y_t}{A_t}.$$

Putting this back into the constraint, we obtain the optimal use of capital in production, K_t^*, as

$$K_t^*(w_t, r_t, Y_t, A_t) = \left(\frac{\alpha}{1-\alpha} \frac{w_t}{r_t} \right)^{1-\alpha} \frac{Y_t}{A_t}.$$

Now plugging $L_t^*$and K_t^* into the initial minimisation problem, we obtain

$$\begin{aligned}
C(w_t, r_t, Y_t, A_t) &= \left[\left(\frac{1-\alpha}{\alpha} \frac{r_t}{w_t} \right)^{\alpha} w_t + \left(\frac{\alpha}{1-\alpha} \frac{w_t}{r_t} \right)^{1-\alpha} r_t \right] \frac{Y_t}{A_t} \\
&= \left[\left(\frac{1-\alpha}{\alpha} \right)^{\alpha} + \left(\frac{\alpha}{1-\alpha} \right)^{1-\alpha} \right] w_t^{1-\alpha} r_t^{\alpha} \frac{Y_t}{A_t} \\
&= \left[\frac{1-\alpha+\alpha}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} \right] w_t^{1-\alpha} r_t^{\alpha} \frac{Y_t}{A_t} \\
&= \left(\frac{r_t}{\alpha} \right)^{\alpha} \left(\frac{w_t}{1-\alpha} \right)^{1-\alpha} \frac{Y_t}{A_t}
\end{aligned}$$