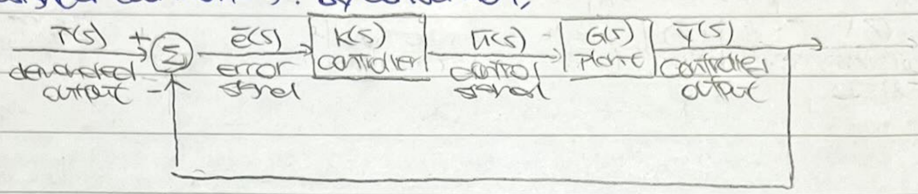
signals, system and feedback

Signal, systems and feedback

Systems and block diagrams

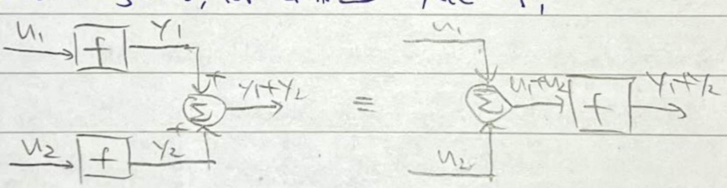
- Signals are functions of time and are represented by the connections (they must be quantities and thus have units, eg ✓ steam pressure ; ✗ steam)
- Systems have eqns and are represented by blocks, which have signals as their inputs and outputs.
- For the control engineer, some blocks are fixed (plant) while other blocks are to be designed (controller). By convention,



* Systems : i/p signals → o/p signals ; functions : i/p value → o/p value (signals are functions of time).

Linear systems.

- Consider a system f mapping dynamic inputs u to outputs y, i.e. $y_i = f(u_i)$. The system f is linear if superposition holds, i.e. $y_1 + y_2 = f(u_1) + f(u_2) = f(u_1 + u_2)$.
- In terms of block diagrams, for a linear system f,



* From the above, we can conclude for a linear system f, $\alpha f(u_1) + \beta f(u_2) = f(\alpha u_1 + \beta u_2)$

- In addition, we shall assume that all systems are
  - ↳ Causal: the o/p at time T, $y(T)$, only depends on the i/p up to time T (i.e. $y(T)$ is independent of $u(t)$, $t \geq T$)
  - ↳ Time invariant: the response of the system to a particular i/p doesn't depend when that i/p is applied (i.e. if $u(t) \to y(t)$, then $u(t-\tau) \to y(t-\tau)$).

* Causality is satisfied by all physical systems but time invariance can be violated in some advanced models

- If a system can be described by linear DEs w/ const. coefficients and possibly delays, then it is necessarily linear

Linearisation

- All real systems are actually non-linear, but many of these behave approx. linearly for small perturbations from eqm.
- consider a system that is described by an ODE of the form

$$\dot{x} = f(x, u)$$

where f is some smooth function.

- Assume that this system has an eqm. at (x_0, u_0), i.e.

$$\dot{x}_0 = f(x_0, u_0) = 0$$

If we consider a small change from the eqm, i.e. $x = x_0 + \delta x$ and $u = u_0 + \delta u$.

$$\frac{d}{dt}(x_0 + \delta x) = f(x_0 + \delta x, u_0 + \delta u)$$

$$\cancel{\dot{x}_0}^{\,0} + \delta\dot{x} = \cancel{f(x_0, u_0)}^{\,0} + \underbrace{\frac{\partial f}{\partial x}\Big|_{x=x_0}}_{A} \delta x + \underbrace{\frac{\partial f}{\partial u}\Big|_{u=u_0}}_{B} \delta u + \overset{\text{neglect}}{\cancel{H.O.T.}}$$

which results in the linear ODE

$$\boxed{\delta\dot{x} = A\delta x + B\delta u}$$

(this is a simple example of the state-space model which can be generalised for higher order systems w/ many i/ps and o/ps).

* If we want to design a controller to keep a system near eqm, then we can ensure that perturbations are small → behaviour is approx. linear.

Laplace transforms (Refer to y1 notes for more details)

- The Laplace transform is defined as

$$\boxed{\bar{y}(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\,dt}$$

provided the integral converges for sufficiently large and +ve values of s.

- Linearity: $L[\alpha f(t) + \beta g(t)] = \alpha \bar{f}(s) + \beta \bar{g}(s)$
- Inverse: $g(t) = L^{-1}[\bar{g}(s)]$
- Common Laplace transforms: $L[1] = \frac{1}{s}$; $L[e^{\alpha t}] = \frac{1}{s-\alpha}$; $L[\delta(t)] = 1$

$L[\sin \alpha t] = \frac{\alpha}{s^2+\alpha^2}$; $L[\cos \alpha t] = \frac{s}{s^2+\alpha^2}$

- Derivatives: $L\left[\frac{d^n g}{dt^n}\right] = s^n \bar{g}(s) - s^{n-1} g(0) - s^{n-2} \dot{g}(0) - \ldots - \frac{d^{n-1} g}{dt^{n-1}}\Big|_{t=0}$
- Multiplication by t: $L[t g(t)] = -\frac{d}{ds}\bar{g}(s)$
- Translation property (shift in s): $L[e^{\alpha t} g(t)] = \bar{g}(s-\alpha)$
- Shift in t: $L[g(t-\tau)H(t-\tau)] = e^{-s\tau}\bar{g}(s)$
- Convolution: $L[f(t) * g(t)] = \bar{f}(s)\bar{g}(s)$

Poles and zeros

- Suppose $G(s)$ is a rational function of s, i.e.

$$G(s) = \frac{n(s)}{d(s)}$$

where $n(s)$ and $d(s)$ are polynomials in s.

- Then the roots of $n(s)$ are the zeros of $G(s)$ and the roots of $d(s)$ are the poles of $G(s)$.

Initial and final value theorems

- If $\bar{y}(s) = \mathcal{L}[y(t)]$, then whenever the indicated limits exist, we have
  - ↳ Final Value Theorem: $\boxed{\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\bar{y}(s)}$
  - ↳ Initial value theorem: $\boxed{\lim_{t\to 0^+} y(t) = \lim_{s\to\infty} s\bar{y}(s)}$

* These theorems are not valid for signals that do not tend to a const. value as $t\to\infty$.

- For rational functions of s, it is easy to demonstrate these relationships hold:
  ↳ transfer functions of systems w/ ODEs are rational functions of s.

Let a partial fraction of $\bar{y}(s)$ be given as

$$\bar{y}(s) = \frac{b_0}{s} + \sum_{i=1}^{n} \frac{b_i}{s+a_i} \quad \text{so} \quad y(t) = b_0 + \sum_{i=1}^{n} b_i e^{-a_i t}$$

Consider $y(t) = b_0 + \sum_{i=1}^{n} b_i e^{-a_i t}$, $y(0) = b_0 + \sum_{i=1}^{n} b_i$ and $y(\infty) = b_0$ (provided $a_i > 0$)

Consider $s\bar{y}(s) = b_0 + \sum_{i=1}^{n} \frac{s b_i}{s+a_i}$, $s\bar{y}(s)\big|_{s=0} = b_0$ and $s\bar{y}(s)\big|_{s=\infty} = b_0 + \sum_{i=1}^{n} b_i$

step/impulse responses and transfer functions

step and impulse responses.

- The step function $H(t)$ is defined as $H(t) = 1$ for $t>0$, 0 for $t<0$.
- The delta function $\delta(t)$ is defined as $\delta(t) = 0$ except at $t=0$, $\int_a^b \delta(t)\,dt = 1$, $0 \in [a,b]$.

* The delta function could be defined in multiple ways but it doesn't really matter

① ②

as $\Delta \to 0$
(both areas are 1)

- Sifting theorem: $\boxed{\int_a^c f(t)\,\delta(t-b)\,dt = f(b)}$, $b\in[a,c]$. If $b \notin [a,c]$, the integral evaluates to 0.
- The step response $r(t)$ is the o/p of the system when the i/p is a step $H(t)$, and all initial conditions are zero
- The impulse response $g(t)$ is the o/p of the system, when the i/p is an impulse $\delta(t)$, and all initial conditions are zero.

The convolution integral

- Consider a system w/ i/p $u(t)$, o/p $y(t)$ and impulse response $g(t)$.

input	output
$\delta(t-\tau)$	$g(t-\tau)$
$\int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)d\tau$	$\int_{-\infty}^{\infty} u(\tau)g(t-\tau)d\tau$

Applying the sifting thm, $u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)d\tau$, therefore $y(t) = \int_{-\infty}^{\infty} u(\tau)g(t-\tau)d\tau$

- The convolution integral is therefore

$$\boxed{y(t) = \int_{-\infty}^{\infty} u(\tau)g(t-\tau)d\tau}$$

which can be abbreviated as $y(t) = u(t) * g(t)$

- Letting $T = t-\tau$, it follows that

$$\boxed{y(t) = \int_{-\infty}^{\infty} u(t-T)g(T)dT}$$

which can be abbreviated as $y(t) = g(t) * u(t)$.

commutativity, ie. $\boxed{u(t) * g(t) = g(t) * u(t)}$

- For functions f, g supported on only $[0,\infty)$, (i.e. zero for negative arguments), the integral can be truncated as

$$\boxed{y(t) = \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t g(\tau)f(t-\tau)d\tau}$$

refer to y1 notes for more details.

The transfer function

we usually assume initial conditions are zero if not specified.

- As an alternative to convolution, the response of a linear system to arbitrary inputs can be determined using Laplace transforms.
- Applying the Laplace transform on the convolution integral,

$$L[y(t)] = L[g(t) * u(t)]$$

$$L[y(t)] = L[g(t)] \times L[u(t)]$$

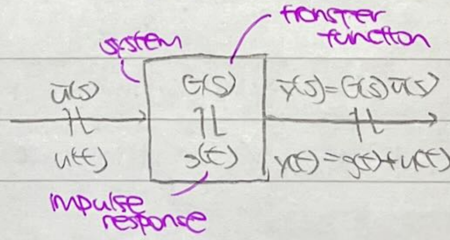
$$\boxed{\bar{y}(s) = G(s)\,\bar{u}(s)}$$

Laplace transform turns convolution (hard) into multiplication (easy)

lowercase + bar: signals
uppercase: transfer func.

where $G(s)$ is the transfer of the system $[\boxed{G(s) = L[g(t)]}]$

- All LTI systems have transfer functions (given by the Laplace transform of the impulse response)



- If a system has multiple i/ps, the transfer function from a particular i/p to a particular o/p is defined as the Laplace transform of that o/p when an impulse is applied to the given i/p, all other i/ps are zero and all initial conditions are zero.
- For a LTI system w/ i/p u and o/p y, we can write $\bar{y}(s) = G(s)\bar{u}(s)$ + other terms indep. of u

Here, $G(s)$ is the transfer function from $\bar{u}(s)$ to $\bar{y}(s)$

from non-zero initial conditions or other non-zero inputs.

Transforms of signals vs transfer functions of systems

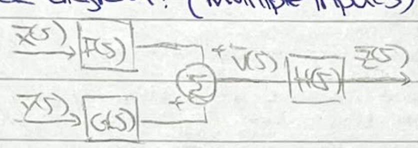
- Mathematically, there is no distinction, but in practice they have diff. interpretations

$G(s)$	Signals $\mathcal{L}^{-1}[G(s)]$	Systems $Y(s) = G(s)U(s)$
1	$\delta(t)$	$y(t) = u(t)$
$\frac{1}{s}$	$H(t)$	$y(t) = \int_0^t u(\tau)d\tau$ [integrators]
$\frac{1}{s+a}$	e^{-at}	$\dot{y}(t) + ay(t) = u(t)$ [1st order lag]
$\frac{\omega}{s^2+\omega^2}$	$\sin(\omega t)$	$\ddot{y}(t) + \omega^2 y(t) = \omega u(t)$
e^{-sT}	$\delta(t-T)$	$y(t) = u(t-T)$ [delay]

- In particular, a transfer function corresponds to a system, and in the time domain it is represented by a DE relating the i/p and the o/p.
- On the other hand, a signal in the time domain is a function of time.

Interconnections of LTI systems.

- Consider the following block diagram: (multiple inputs)

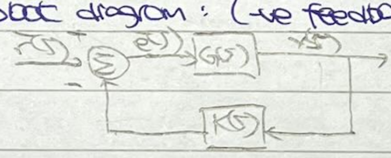


$V(s) = F(s)X(s) + G(s)Y(s)$ and $Z(s) = H(s)V(s)$

so $Z(s) = H(s)\left(F(s)X(s) + G(s)Y(s)\right)$

$= \underbrace{H(s)F(s)}_{\text{TF from } X(s) \text{ to } Z(s)} X(s) + \underbrace{H(s)G(s)}_{\text{TF from } Y(s) \text{ to } Z(s)} Y(s)$

- Consider the following block diagram: (-ve feedback loop)



$E(s) = R(s) - K(s)Y(s)$ and $Y(s) = G(s)E(s)$

so $Y(s) = G(s)\left(R(s) - K(s)Y(s)\right)$

$\left(1 + G(s)K(s)\right)Y(s) = G(s)R(s)$

$$Y(s) = \frac{G(s)}{1 + G(s)K(s)} R(s)$$

results in closed-loop systems.

Open-loop systems

Stability and pole-locations

Asymptotically stable, marginally stable and unstable systems.

- An LTI system is asymptotically stable if its impulse response $g(t)$ satisfies

$$\boxed{\int_0^\infty |g(t)|\,dt < \infty}$$

(this defn of asymptotic stability guarantees that $\lim_{t\to\infty} g(t) = 0$ as o/w the integral above $\to\infty$)

- An LTI system is marginally stable if it is not asymptotically stable, but there exist numbers $A, B < \infty$ s.t. its impulse response $g(t)$ satisfies

$$\boxed{\int_0^T |g(t)|\,dt < A + BT \quad \text{for all } T}$$

- An LTI system is unstable if it is neither asymptotically stable nor marginally stable.
(The impulse response $g(t)$ of an unstable system will tend to ∞ as $t\to\infty$).

Poles and the impulse response.

- Although stability is most easily defined in terms of the impulse response, it is most easily determined in terms of pole locations
(at least for systems w/ rational transfer functions, the ones that come from ODEs).
- Consider a general LTI system described by an ODE, and consequently having a rational transfer function $G(s)$, i.e. it can be written as the ratio of two polynomials.

$$G(s) = \frac{n(s)}{d(s)}$$

where the coefficients of $d(s)$ come from the LHS of the underlying ODE; the coefficients of $n(s)$ come from the RHS of the underlying ODE.
underlying ODE in the form $\alpha\frac{d^k y}{dt^k} + \beta\frac{d^{k-1} y}{dt^{k-1}} + \dots + \gamma\frac{dy}{dt} + \varepsilon y = a\frac{d^{k-1} u}{dt^{k-1}} + b\frac{d^{k-2} u}{dt^{k-2}} + \dots + c\frac{du}{dt} + e u$,
and the transfer function would be $G(s) = \frac{a s^{k-1} + b s^{k-2} + \dots + c s + e}{\alpha s^k + \beta s^{k-1} + \dots + \gamma s + \varepsilon}$)

- We can factorise the denominator to give

p_i are the poles $\quad G(s) = \frac{n(s)}{(s-p_1)(s-p_2)\dots(s-p_n)}$ $\quad$ always satisfied by physically realisable systems.

Assuming that $G(s)$ is proper, i.e. $\deg[n(s)] \le \deg[d(s)]$, we can use partial fractions,

$$G(s) = \frac{\alpha_1}{s-p_1} + \frac{\alpha_2}{s-p_2} + \dots + \frac{\alpha_n}{s-p_n} + C$$

$\alpha_i = \lim_{s\to p_i} (s-p_i)G(s)$ is the residue at $s = p_i$

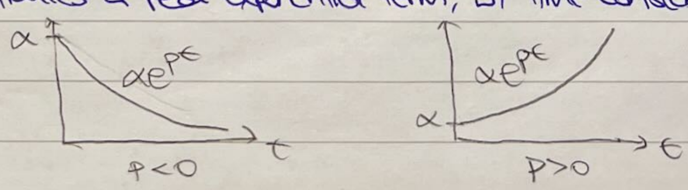
By taking the inverse Laplace transform, we can write the impulse response in the form

$$g(t) = \alpha_1 e^{p_1 t} + \alpha_2 e^{p_2 t} + \dots + \alpha_n e^{p_n t} + C\delta(t).$$

- The contribution of each term $\alpha_i e^{p_i t}$ to $g(t)$ depends on whether p is real or complex:

① p is real

- Each pole contributes a real exponential term, w/ time constant $\left|\frac{1}{p}\right|$

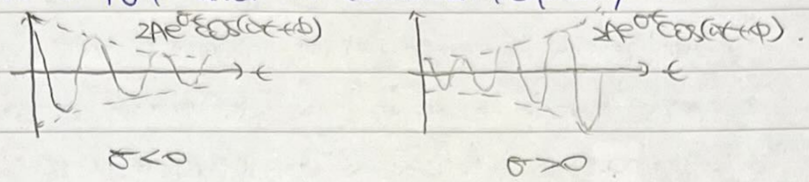


② p is complex

- Each pair of complex poles contributes a term of the form

$$2Ae^{\sigma t}\cos(\omega t+\phi) \quad \text{where } \sigma=\mathrm{Re}[p],\ \omega=\mathrm{Im}[p].$$

the time constant is $\left|\frac{1}{\sigma}\right|$ and the oscillation frequency is ω.



↳ complex poles always appear in conjugate pairs since they are roots of a real polynomial.
so for a pair of complex poles, the contribution is

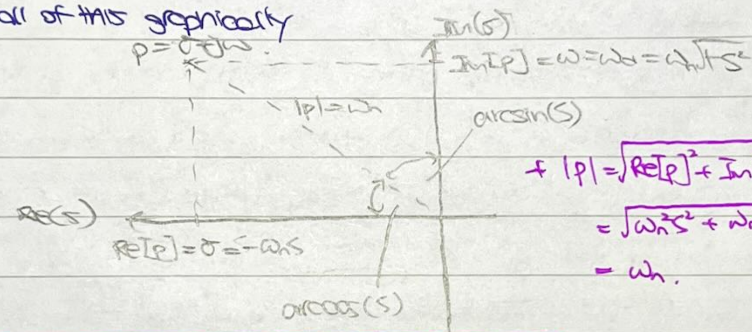
$$\alpha e^{pt}+\alpha^* e^{p^* t} = Ae^{j\phi}e^{(\sigma+j\omega)t}+Ae^{-j\phi}e^{(\sigma-j\omega)t} = Ae^{\sigma t}\left[e^{j(\omega t+\phi)}+e^{-j(\omega t+\phi)}\right] = Ae^{\sigma t}\cdot 2\cos(\omega t+\phi)$$

- comparing this w/ the impulse response of a second order system (in Mechanics 1B)

$$2Ae^{\sigma t}\cos(\omega t+\phi) \Leftrightarrow Ce^{-\omega_n \zeta t}\sin(\omega_d t)$$

$$\rightarrow \quad \boxed{\sigma=-\omega_n\zeta} \quad \text{and} \quad \boxed{\omega=\omega_d=\omega_n\sqrt{1-\zeta^2}}$$

- we can represent all of this graphically



$$+\ |p|=\sqrt{\mathrm{Re}[p]^2+\mathrm{Im}[p]^2} = \sqrt{\omega_n^2\zeta^2+\omega_n^2(1-\zeta^2)} = \omega_n.$$

* locus of $\boxed{\begin{array}{l}\text{const. } \omega_n \rightarrow \text{circle of radius } \omega_n \\ \text{const. } \zeta \rightarrow \text{half-lines w/ argument } \pi \pm \arccos(\zeta) \\ \text{const. } \omega_n\zeta \rightarrow \text{vertical line w/ x-coord} = -\omega_n\zeta.\end{array}}$ (inverse of the time constant)

so given the pole locations, in the complex plane, of a second order system, we can read off the natural frequency ω_n, the damping ratio ζ and the reciprocal of the time const $\frac{1}{\omega_n\zeta}$

→ Any polynomial can be broken down to 1st/2nd order terms in complex space → the impulse response of any rational system can be regarded as a combination of 1st/2nd order terms.

→ For high order systems, the poles closest to the imaginary axis are the dominant poles as their contributions die away most slowly, and we can still read off the time constant of decay (and natural frequency, damping ratio) for each "mode" of the system.

* Above, we have assumed there are no repeated poles. Repeated poles give rise to terms of the form $t^m e^{pt}$ ↳ same general characteristics as the exponential dominates.

Poles and the transient response

- The transient response, to a large extent, is a characteristic of the system itself rather than the input.
- Consider a system w/ transfer function

$$G(s) = \frac{n(s)}{(s-p_1)(s-p_2)\dots(s-p_n)}$$

If the system is given an input $u(t)$, then the response is given by

$$\bar{y}(s) = G(s)\bar{u}(s) = \frac{n(s)}{(s-p_1)(s-p_2)\cdots(s-p_n)}\bar{u}(s).$$

$$= \frac{\gamma_1}{s-p_1} + \frac{\gamma_2}{s-p_2} + \dots + \frac{\gamma_n}{s-p_n} + \text{other terms}.$$

and therefore $\quad y(t) = \gamma_1 e^{p_1 t} + \gamma_2 e^{p_2 t} + \dots + \gamma_n e^{p_n t} + \text{other terms}.$

→ the response $y(t)$ contains the same terms as the impulse response (w/ diff. amplitudes) plus some extra terms due to particular characteristics of the input.

Stability theorem.

- For systems w/ proper rational transfer functions, it can be shown that.
  - ↳ 1) A system is asymptotically stable if ALL its poles have -ve real parts.
  - ↳ 2) A system is marginally stable if it has one or more distinct poles on the imaginary axis, and any remaining poles have -ve real parts
  - ↳ 3) A system is unstable if ANY poles has a +ve real parts, or if there are any repeated poles on the imaginary axis.

Asymptotic stability and pole locations (Proof).

- An LTI system w/ rational transfer function $G(s)$ is asymptotically stable iff all poles of $G(s)$ lie in the LHP.

① All poles have a -ve real part → the system is asymptotically stable.

Initially, assume the poles of $G(s)$ are distinct, then we can write.

$$G(s) = \frac{n(s)}{(s-p_1)(s-p_2)\dots(s-p_n)} = \alpha_0 + \frac{\alpha_1}{s-p_1} + \frac{\alpha_2}{s-p_2} + \dots + \frac{\alpha_n}{s-p_n}$$

$$\therefore \; g(t) = \alpha_0 \delta(t) + \alpha_1 e^{p_1 t} + \alpha_2 e^{p_2 t} + \dots + \alpha_n e^{p_n t}$$

For each p_i, we can write $p_i = \sigma_i + j\omega_i$, and we know that

$$|e^{p_i t}| = |e^{(\sigma_i + j\omega_i)t}| = |e^{\sigma_i t} e^{j\omega_i t}| = |e^{\sigma_i t}|\cancel{|e^{j\omega_i t}|}^{1} = e^{\sigma_i t}.$$

Using the triangle inequality, $|a+b| \le |a| + |b|$

$$|g(t)| \le |\alpha_0|\delta(t) + |\alpha_1|e^{\sigma_1 t} + |\alpha_2|e^{\sigma_2 t} + \dots + |\alpha_n|e^{\sigma_n t}$$

As we know $\int_0^\infty e^{\sigma t}\,dt = -\frac{1}{\sigma}$ for $\sigma < 0$, ∞ o/w. As we are given all $\sigma_i < 0$,

$$\int_0^\infty |g(t)|\,dt \le |\alpha_0| + \left|\frac{\alpha_1}{\sigma_1}\right| + \left|\frac{\alpha_2}{\sigma_2}\right| + \dots + \left|\frac{\alpha_n}{\sigma_n}\right| < \infty. \rightarrow \text{asymptotically stable.}$$

* For repeated poles, we use $\int_0^\infty |t^{k-1} e^{pt}|\,dt = \int_0^\infty t^{k-1} e^{\sigma t}\,dt < \infty$ instead.

② A system is asymptotically stable → all poles have a -ve part.

- For all values of s for which $Re\{s\} \geq 0$, (s in RHP), we have

By defn of Laplace transform: $|G(s)| = |\int_0^\infty e^{-st} g(t) dt|$

Using $|\int A \cdot B dx| \leq \int |A||B| dx$: $|\int_0^\infty e^{-st} g(t) dt| \leq \int_0^\infty |e^{-st}||g(t)| dt$.

since $|e^{-st}| \leq$ for $Re\{s\} \geq 0$: $\int_0^\infty |e^{-st}||g(t)| dt \leq \int_0^\infty |g(t)| dt$.

For asymptotically stable systems: $\int_0^\infty |g(t)| dt = A < \infty$.

→ $|G(s)|$ must be finite in RHP, i.e. no poles in RHP

note $s = \sigma + j\omega$ so $|e^{-st}| = |e^{-\sigma t - j\omega t}| = |e^{-\sigma t}||e^{-j\omega t}| = e^{-\sigma t} \leq 1$ for $Re\{s\} \geq 0$.

the frequency response $G(j\omega)$

for a linear system, if the input is a sinusoid, the output is a sinusoid of the same freq. but diff. amplitude/phase.

Derivation of the gain and phase shift (frequency response)

- consider an asymptotically stable system w/ input $\bar{u}(s)$, output $\bar{y}(s)$, at rest until $t=0$, and a rational transfer function $G(s) = \frac{n(s)}{d(s)}$, so.

$$\bar{y}(s) = G(s)\bar{u}(s).$$

For a sinusoidal input, let $u(t) = e^{j\omega t}$, and take the real part at the end.

$\bar{u}(s) = \frac{1}{s - j\omega}$ and since $G(s)$ cannot have a pole at $s = j\omega$,

$$\bar{y}(s) = \frac{n(s)}{(s-p_1)(s-p_2)\dots(s-p_n)} \cdot \frac{1}{s-j\omega} = \frac{\lambda_1}{s-p_1} + \frac{\lambda_2}{s-p_2} + \dots + \frac{\lambda_n}{s-p_n} + \frac{\lambda_0}{s-j\omega}.$$

To find λ_0, we multiply both sides by $s - j\omega$

$$\bar{y}(s)(s-j\omega) = G(s) = \frac{\lambda_1(s-j\omega)}{s-p_1} + \frac{\lambda_2(s-j\omega)}{s-p_2} + \dots + \frac{\lambda_n(s-j\omega)}{s-p_n} + \lambda_0$$

setting $s = j\omega$, we have $G(j\omega) = \lambda_0$.

Therefore, $$y(t) = \underbrace{\lambda_1 e^{p_1 t} + \lambda_2 e^{p_2 t} + \dots + \lambda_n e^{p_n t}}_{y_{transient}} + \underbrace{G(j\omega) e^{j\omega t}}_{y_{steady\text{-}state}}$$

- since all the poles of $G(s)$, p_i have a -ve real part (asymptotically stable system), then $y_{transient} \to 0$ as $t \to \infty$, leaving us w/ the steady state response $y_{steady\text{-}state}$.

Taking the real part of the steady-state response,

$$Re\{y_{ss}(t)\} = Re\{G(j\omega) e^{j\omega t}\}$$
$$= Re\{|G(j\omega)| e^{j(\omega t + \angle G(j\omega))}\}$$
$$= \underbrace{|G(j\omega)|}_{\text{gain } A} \cos(\omega t + \underbrace{\angle G(j\omega)}_{\text{phase shift } \phi}).$$

here, $G(j\omega) = |G(j\omega)| e^{j\angle G(j\omega)}$

- We can conclude that for a system w/ a rational transfer function $G(s)$, then.

$$\boxed{\begin{aligned} \text{gain } A &= |G(j\omega)| \\ \text{phase shift } \phi &= \angle G(j\omega) \end{aligned}}$$

Frequency response of an arbitrary linear system.

- The above result is not just true for systems described by ODEs (rational G(s)), but also any arbitrary linear system that is asymptotically stable.
- Consider a general, asymptotically stable system w/ a transfer function $G(s)$, subject to a sinusoidal input that "turns on" suddenly at $t=0$.

Let $u(t) = \cos\omega t = \mathrm{Re}\{e^{j\omega t}\}$, so.

$$y(t) = \int_0^t u(t-\tau)g(\tau)\,d\tau = \int_0^t \mathrm{Re}\{e^{j\omega(t-\tau)}\}g(\tau)\,d\tau = \mathrm{Re}\{\int_0^t e^{j\omega(t-\tau)}g(\tau)\,d\tau\}.$$

$$= \underbrace{\mathrm{Re}\{\int_0^\infty e^{j\omega(t-\tau)}g(\tau)\,d\tau\}}_{y_{steady\text{-}state}} - \underbrace{\mathrm{Re}\{\int_t^\infty e^{j\omega(t-\tau)}g(\tau)\,d\tau\}}_{y_{transient}}.$$

- Consider the second term,

$$|y_{tr}(t)| \leq \left|\int_t^\infty e^{j\omega(t-\tau)}g(\tau)\,d\tau\right| \leq \int_t^\infty \cancelto{1}{|e^{j\omega(t-\tau)}|}\,|g(\tau)|\,d\tau = \int_t^\infty |g(\tau)|\,d\tau.$$

but for asymptotically stable systems, $\int_0^\infty |g(\tau)|d\tau$ is finite, so $\lim_{t\to\infty}|y_{tr}(t)| = 0$.

- Consider the first term,

$$y_{ss}(t) = \mathrm{Re}\{e^{j\omega t}\int_0^\infty e^{-j\omega\tau}g(\tau)\,d\tau\} = \mathrm{Re}\{e^{j\omega t}G(j\omega)\} = \mathrm{Re}\{|G(j\omega)|e^{j(\omega t+\angle G(j\omega))}\}$$

$$= \underbrace{|G(j\omega)|}_{\text{gain } A}\cos(\omega t + \underbrace{\angle G(j\omega)}_{\text{phase shift } \phi}).$$

- We can conclude that for any asymptotically stable linear system w/ transfer function $G(s)$,

$$\boxed{\begin{aligned}\text{gain } A &= |G(j\omega)| \\ \text{phase shift } \phi &= \angle G(j\omega)\end{aligned}}$$

Plotting the frequency response

- The frequency response $G(j\omega)$ is a complex-valued function of frequency ω. At each frequency ω, the complex no. $G(j\omega)$ can be expressed in terms of real/imag parts or gain/phase.
- The main ways of representing this information are as follows.
  - ↳ The Bode diagram: Two separate graphs, $|G(j\omega)|$ vs ω (log-log) and $\angle G(j\omega)$ vs ω (lin-log)
  - ↳ The Nyquist diagram: One single parametric plot, $\mathrm{Im}\{G(j\omega)\}$ vs $\mathrm{Re}\{G(j\omega)\}$ (lin-lin) as ω varies.
  - ↳ The Nichols diagram: One single parametric plot, $|G(j\omega)|$ vs $\angle G(j\omega)$ (log-lin) as ω varies.
- The Bode diagram is easy to sketch to a high degree of accuracy and gives an indication of the frequency ranges in which diff. levels of performance are achieved.
- The Nyquist diagram provides a rigorous way of determining the stability of a feedback system
- The Nichols diagram combines some of the advantages of both these (beyond the scope of this course).

Sketching Bode diagrams

- consider a transfer function written as a ratio of factorised polynomials,

$$\text{eg.} \quad G(s) = \frac{a_1(s)a_2(s)}{b_1(s)b_2(s)}$$

we can see the gain is given by

$$\log_{10}|G(j\omega)| = \log_{10}|a_1(j\omega)| + \log_{10}|a_2(j\omega)| - \log_{10}|b_1(j\omega)| - \log_{10}|b_2(j\omega)|$$

and the phase is given by

$$\angle G(j\omega) = \angle a_1(j\omega) + \angle a_2(j\omega) - \angle b_1(j\omega) - \angle b_2(j\omega)$$

- since a polynomial can always be written as a product of terms of the type

$$K,\ sT,\ 1+sT,\ 1+2\zeta Ts+s^2T^2 \quad (\text{complex } s).$$

we just need to know how to sketch Bode diagrams for these terms.

(For the Bode plot of the complex system, we add the gains/phases of each individual term)

① constant K.

- consider a transfer function of the form

$$G(s) = K.$$

- Gain: $\log_{10}|G(j\omega)| = \log_{10}|K| = K'$ [In dB, $20K'$] — shift in y-axis in the gain plot.

Phase: $\angle G(j\omega) = \angle K = 0.$

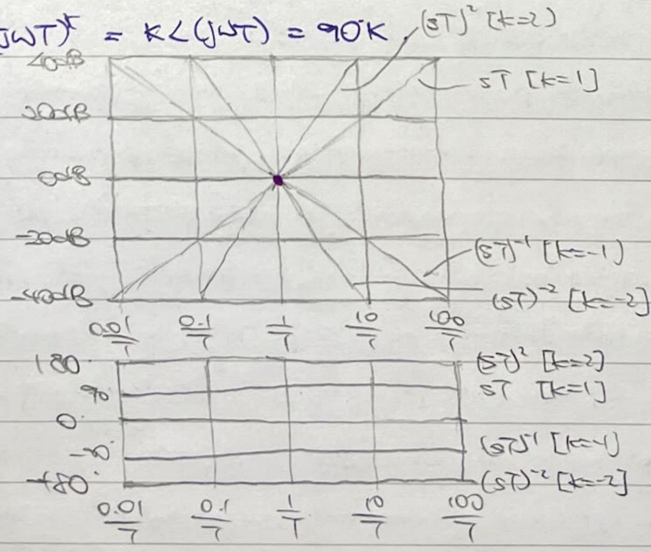
② Powers of s $(sT)^k$.

- consider a transfer function of the form

$$G(s) = (sT)^k.$$

- Gain: $\log_{10}|G(j\omega)| = \log_{10}|(j\omega T)^k| = \log_{10}(\omega T)^k = k\log_{10}(\omega T)$ [In dB, $20k\log_{10}(\omega T)$].

Phase: $\angle G(j\omega) = \angle(j\omega T)^k = k\angle(j\omega T) = 90°k$



- The gain curve is a straight line w/ slope k decades/decade / 20k dB/decade, intersecting the 0dB line at $\omega T = 1$; the phase curve is a constant at $90k°$.

③ First order terms $(1+sT)$

- Consider a transfer function of the form

$$G(s) = 1+sT \quad (T>0)$$

- Gain: $\log_{10}|G(j\omega)| = \log_{10}|1+j\omega T| = \log_{10}(1+\omega^2T^2)^{1/2}$ [In dB, $20\log_{10}(1+\omega^2T^2)^{1/2}$].

Phase: $\angle G(j\omega) = \angle 1+j\omega T = \arctan(\omega T)$

- Asymptotes [Gain in dB, i.e. Gain $= 20\log_{10}|G(j\omega)|$].

$\omega \to 0$ (i.e. $\omega \ll 1/T$): $20\log_{10}|G(j\omega)| \to 20\log_{10}1 = 0\,dB$

low freq asymptote

$\angle G(j\omega) \to \angle 1 = 0°$

straight line w/ slope 20dB/decade and x-axis intercept at $\omega = 1/T$.

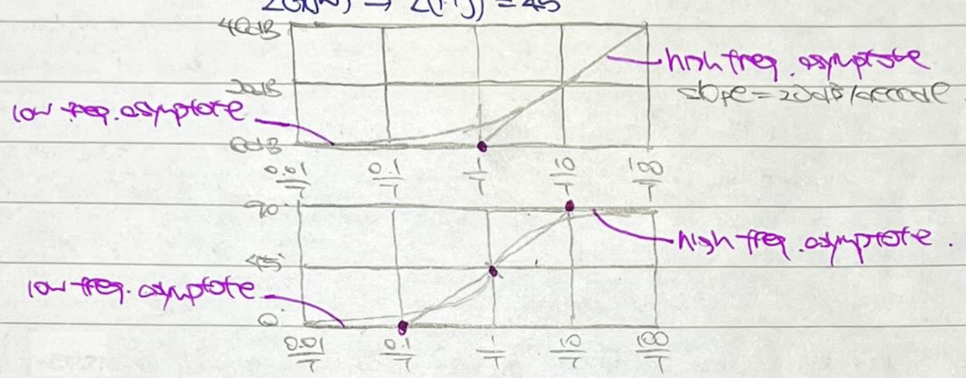
$\omega \to \infty$ (i.e. $\omega \gg 1/T$): $20\log_{10}|G(j\omega)| \to 20\log_{10}(\omega T) = (20\log_{10}\omega - 20\log_{10}1/T)\,dB$

high freq. asymptote

$\angle G(j\omega) \to \angle(j\omega T) = 90°$

$\omega = 1/T$: $20\log_{10}|G(j\omega)| \to 20\log_{10}\sqrt{2} = 3\,dB$

$\angle G(j\omega) \to \angle(1+j) = 45°$



- In the gain plot, the asymptotes intersect at $\omega = 1/T$. ; In the phase plot, the straight line passes through $(\omega = 1/T, \angle G(j\omega) = 45°)$ and intersects the high-freq. asymptote at $\omega = 10/T$, the low freq. asymptote at $\omega = 0.1/T$.

④ second order terms $(1+2\zeta sT + s^2T^2)$.

- Consider a transfer function of the form

$$G(s) = \frac{1}{1+2\zeta sT+s^2T^2} \quad (T>0,\ 0\leq\zeta\leq 1).$$

- Gain: $\log_{10}|G(j\omega)| = -\log_{10}|1-\omega^2T^2+2\zeta j\omega T|$ [In dB, $-20\log_{10}|1-\omega^2T^2+2\zeta j\omega T|$]

Phase: $\angle G(j\omega) = -\angle 1-\omega^2T^2+2\zeta j\omega T$

- Asymptotes: [Gain in dB, i.e. Gain $= 20\log_{10}|G(j\omega)|$].

$\omega \to 0$ (i.e. $\omega \ll 1/T$): $20\log_{10}|G(j\omega)| \to -20\log_{10}1 = 0\,dB$

low freq. asymptote

$\angle G(j\omega) \to -\angle 1 = 0°$

straight line w/ slope −40dB/decade, and x-axis intercept at $\omega = 1/T$.

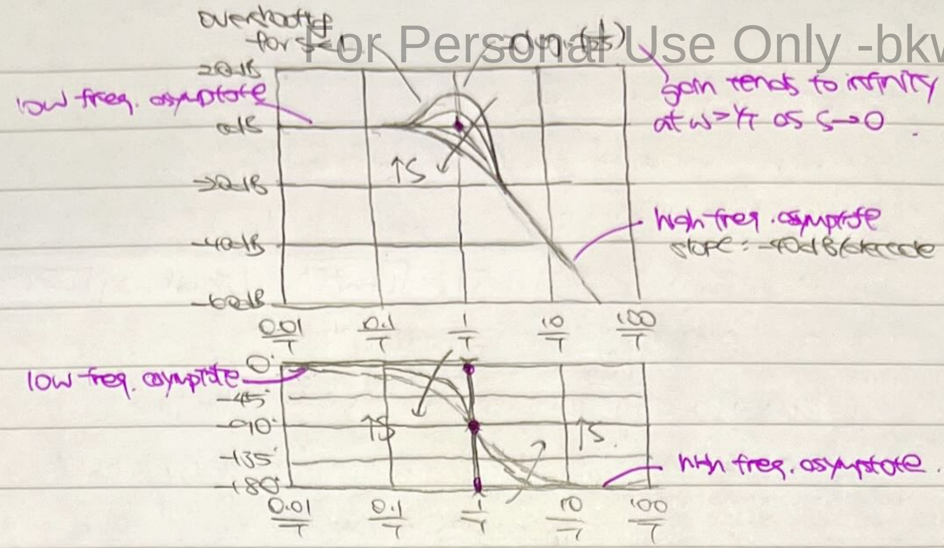
$\omega \to \infty$ (i.e. $\omega \gg 1/T$): $20\log_{10}|G(j\omega)| \to -20\log_{10}|-\omega^2T^2| = -40\log_{10}(\omega T) = (40\log_{10}1/T - 40\log_{10}\omega)\,dB$.

high freq. asymptote

$\angle G(j\omega) \to -\angle(-\omega^2T^2) = -180°$

$\omega = 1/T$: $20\log_{10}|G(j\omega)| = -20\log_{10}|2\zeta j| = 20\log_{10}\frac{1}{2\zeta}$

$\angle G(j\omega) = -\angle(2\zeta j) = -90°$



- In the gain plot, the asymptotes intersect at $\omega = 1/T$; In the phase plot, the asymptotes intersect via a vertical line at $\omega = 1/T$. For all ζ, the line passes through $(\omega = 1/T, \angle G(j\omega) = -90°)$

* These plots (but lin-lin scale) are in the mechanics DB p9 (case (a) of harmonic response)

↳ Here, the governing ODE is $\frac{\ddot{y}}{\omega_n^2} + 2\zeta\frac{\dot{y}}{\omega_n} + y = x$.

Taking $\omega_n = 1/T$, we have $T^2\ddot{y} + 2\zeta T\dot{y} + y = x$.

Taking the Laplace transform, $T^2 s^2 \bar{y} + 2\zeta T s\bar{y} + \bar{y} = \bar{x}$

Rearranging, $G(s) = \frac{\bar{y}}{\bar{x}} = \frac{1}{1 + 2\zeta Ts + s^2T^2}$

Final notes on Bode plot sketching.

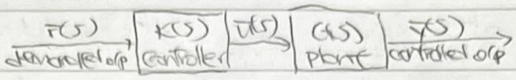
It is possible to have a divergent impulse response but convergent freq. response

- For RHP poles and zeros, i.e. if we have a term $(1 - sT)$ instead of $(1 + sT)$, then the gain plot is unchanged, but the term's contribution to the phase plot is reversed in sign.
- To sketch a Bode plot, we draw the straight line asymptotes first, then give a rough approximation to the true gain/phase by rounding the corners appropriately.

closed-loop systems

Feedback control systems

Open-loop control



- In principle, we could choose a "desired" transfer function F(s) and use $K(s)=\frac{F(s)}{G(s)}$

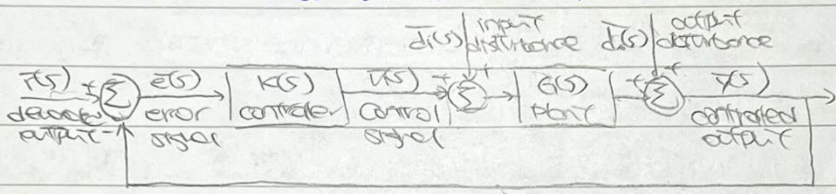
  so we would get $Y(s)=K(s)G(s)r(s)=\frac{F(s)}{G(s)}G(s)r(s)=F(s)r(s)$

  but in practice, this will NOT work.

- this is because it req. an exact model of the plant and that there be no disturbances.

Closed-loop control (Feedback control)

- Feedback is used to combat the effects of uncertainty.
- A typical feedback control configuration is as follows.



$$Y(s)=d_o(s)+G(s)[d_i(s)+U(s)]=d_o(s)+G(s)[d_i(s)+K(s)E(s)] \quad [1]$$

$$E(s)=F(s)-Y(s) \quad [2]$$

Sub [2] into [1]: $$Y(s)=d_o(s)+G(s)[d_i(s)+K(s)(F(s)-Y(s))]$$

$$[1+G(s)K(s)]Y(s)=d_o(s)+G(s)d_i(s)+G(s)K(s)F(s)$$

$$\boxed{Y(s)=\frac{G(s)K(s)}{1+G(s)K(s)}F(s)+\frac{G(s)}{1+G(s)K(s)}d_i(s)+\frac{1}{1+G(s)K(s)}d_o(s)} \quad [3]$$

Sub [3] into [2]: $$E(s)=F(s)-\frac{G(s)K(s)}{1+G(s)K(s)}F(s)-\frac{G(s)}{1+G(s)K(s)}d_i(s)-\frac{1}{1+G(s)K(s)}d_o(s)$$

$$\boxed{E(s)=\left(1-\frac{G(s)K(s)}{1+G(s)K(s)}\right)F(s)-\frac{G(s)}{1+G(s)K(s)}d_i(s)-\frac{1}{1+G(s)K(s)}d_o(s)}$$

* A shortcut for determining the transfer function b/wn two pts in a closed loop.

$$\boxed{\text{transfer function}=\frac{\text{product of all terms b/wn two pts}}{1+\text{product of all the terms around loop}}}$$

The closed-loop characteristic equation.

- The return ratio of a loop (loop transfer function) L(s) is defined as −1 times the product of all the terms around the loop.
- The closed-loop characteristic eqn. (CLCE) is then

$$\boxed{1+L(s)=0}$$

L(s) is not always equal to K(s)G(s).

where the roots of the eqn. are the closed-loop poles.

* All the closed-loop transfer functions (CLTF) have the same denominator of 1+L(s)

- The closed-loop poles determine the stability of the closed-loop system and the characteristics of the closed-loop system's transient response (speed of response, resonance)

Sensitivity and complementary sensitivity.

- The sensitivity function $S(s)$ is defined as

$$\boxed{S(s) = \frac{1}{1+L(s)}}$$

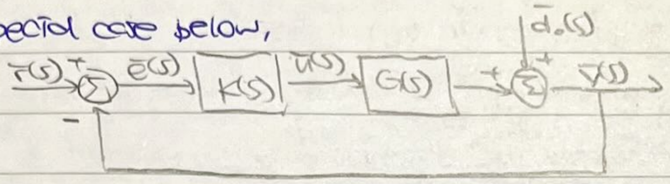
which characterises the sensitivity of a control system to disturbances appearing at the output of the plant.

- The complementary sensitivity function $T(s)$ is defined as

$$\boxed{T(s) = \frac{L(s)}{1+L(s)}}$$

It is "complementary" because $S(s)+T(s) = 1$.

* In the special case below,



the complementary sensitivity appears as the transfer function from the demanded output $\bar{r}(s)$ to the controlled output $\bar{y}(s)$.

Here, $\bar{y}(s) = \frac{G(s)K(s)}{1+G(s)K(s)}\bar{r}(s) + \frac{1}{1+G(s)K(s)}\bar{d}_o(s) = T(s)\bar{r}(s) + S(s)\bar{d}_o(s)$.

in this example $L(s) = G(s)K(s)$

Steady-state response

- Consider an asymptotically stable system w/ impulse response $g(t)$ and transfer function $G(s)$.

Let $y(t) = \int_0^t \overbrace{H(t-\tau)}^{1 \text{ for } \tau \le t} g(\tau)\,d\tau = \int_0^t g(\tau)\,d\tau$ denote the step response of this system

and note that $\bar{y}(s) = G(s)\, \overbrace{L[H(s)]}^{1/s} = \frac{G(s)}{s}$.

- By taking limits as $t \to \infty$ on the expression for $y(t)$, the final value is given by

$$\lim_{t\to\infty} y(t) = \int_0^\infty g(\tau)\,d\tau = \int_0^\infty e^{-0\tau} g(\tau)\,d\tau = L[g(t)]\Big|_{s=0} = G(0)$$

Alternatively, using the final value theorem,

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\bar{y}(s) = \lim_{s\to 0} s\cdot\frac{G(s)}{s} = G(0)$$

→ $\boxed{\text{Final-value of step response, i.e. steady-state gain / DC-gain, is given by } G(0)}$

* Note that the term "steady-state response" means two diff. things depending on the input.

For an asymptotically stable system w/ transfer function $G(s)$:

↳ The steady-state response of the system to a step input is equal to U for $t \geq 0$, is a constant $U \cdot G(0)$.

↳ The steady-state response of the system to a sinusoidal input $\cos(\omega t)$ is the sinusoid $|G(j\omega)| \cos(\omega t + \angle G(j\omega))$

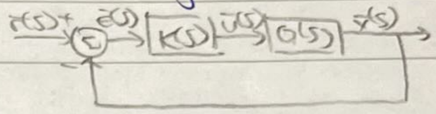
The steady-state gain of a system to a a step input, $G(0)$, is the same as the frequency response evaluated at $\omega = 0$ (so it is also known as DC gain)

Proportional (P) control

- For P control, the controller block is given by

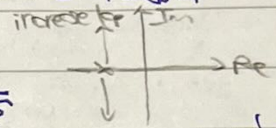
$$\boxed{K(s) = K_p}$$

- Typical result of increasing the gain K_p (when $G(s)$ is itself stable):
  - ↳ increased accuracy of control
  - ↳ increased control action.
  - ↳ possible loss of closed loop stability for large K_p
  - ↳ reduced damping
- consider the example $K(s)=K_p$, $G(s)=\frac{1}{(s+1)^2}$



$$\frac{Y(s)}{r(s)} = \frac{K_pG(s)}{1+K_pG(s)} = \frac{K_p\frac{1}{(s+1)^2}}{1+K_p\frac{1}{(s+1)^2}} = \frac{K_p}{s^2+2s+1+K_p}$$

CLCE: $s^2+2s+1+K_p=0$ → CLP are $s=\frac{-2\pm\sqrt{2^2-4(1)(1+K_p)}}{2(1)} = -1\pm j\sqrt{K_p}$.



⇒ we can then determine ω_n and ζ using two methods

i) using CLP, $\omega_n = |-1+j\sqrt{K_p}| = \sqrt{1+K_p}$, $\zeta = \frac{-Re(-1+j\sqrt{K_p})}{|-1+j\sqrt{K_p}|} = \frac{1}{\sqrt{1+K_p}}$

ii) comparing coeff w/ DB, after setting coeff of s^2 to 1, [coeff of s : $2\zeta\omega_n$ → $\zeta = \frac{1}{\sqrt{1+K_p}}$; const term : ω_n^2 → $\omega_n = \sqrt{1+K_p}$

⇒ we can determine the steady state values.

$$Y(s) = \frac{K_pG(s)}{1+K_pG(s)}r(s) = \frac{K_p}{s^2+2s+1+K_p}r(s) \; ; \; e(s) = \frac{1}{1+K_pG(s)}r(s) = \frac{(s+1)^2}{s^2+2s+1+K_p}r(s).$$

If $r(t)=H(t)$, then $\lim_{t\to\infty} y(t) = G(0)$. so

$$\lim_{t\to\infty} y(t) = \frac{K_p}{s^2+2s+1+K_p}\Big|_{s=0} = \frac{K_p}{1+K_p} \quad \text{[final value of step response]}.$$

$$\lim_{t\to\infty} e(t) = \frac{(s+1)^2}{s^2+2s+1+K_p}\Big|_{s=0} = \frac{1}{1+K_p} \quad \text{[steady state error]}$$

∴ Here, we can see $\uparrow K_p$ gives $\downarrow$ steady-state errors, but $\downarrow\zeta$ ($\uparrow$ oscillatory transient response)

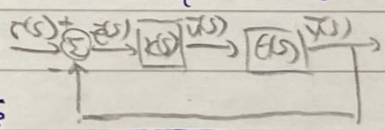
(In more complex controllers, we can use derivative action to increase damping, and integral action to remove steady-state errors).

Proportional + Derivative (PD) control

- For PD control, the controller block is given by

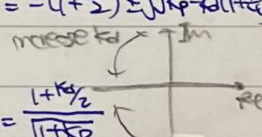
$$\boxed{K(s) = K_p + K_ds}$$

- Typical result of increasing the gain K_d (when $G(s)$ is itself stable):
  - ↳ increased damping
  - ↳ greater sensitivity to noise.
- consider the example $K(s)=K_p+K_ds$, $G(s)=\frac{1}{(s+1)^2}$



$$\frac{Y(s)}{r(s)} = \frac{(K_p+K_ds)G(s)}{1+(K_p+K_ds)G(s)} = \frac{(K_p+K_ds)\frac{1}{(s+1)^2}}{1+(K_p+K_ds)\frac{1}{(s+1)^2}} = \frac{K_p+K_ds}{s^2+(2+K_d)s+1+K_p}$$

CLCE: $s^2+(2+K_d)s+1+K_p=0$ → CLP are $s=\frac{-(2+K_d)\pm\sqrt{(2+K_d)^2-4(1)(1+K_p)}}{2(1)} = -\left(1+\frac{K_d}{2}\right)\pm j\sqrt{K_p-K_d\left(1+\frac{K_d}{4}\right)}$



⇒ we can then determine ω_n and ζ using two methods

i) using CLP, $\omega_n = \left|-\left(1+\frac{K_d}{2}\right)+j\sqrt{K_p-K_d\left(1+\frac{K_d}{4}\right)}\right| = \sqrt{1+K_p}$, $\zeta = \frac{-Re\left(-\left(1+\frac{K_d}{2}\right)+j\sqrt{K_p-K_d\left(1+\frac{K_d}{4}\right)}\right)}{\left|-\left(1+\frac{K_d}{2}\right)+j\sqrt{K_p-K_d\left(1+\frac{K_d}{4}\right)}\right|} = \frac{1+K_d/2}{\sqrt{1+K_p}}$

ii) Comparing coeff w/ DB, after setting coeff of s^2 to 1, [coeff of s : $2\zeta\omega_n$ → $\zeta = \frac{2+K_d}{2\sqrt{1+K_p}}$; const term: ω_n^2 → $\omega_n = \sqrt{1+K_p}$.

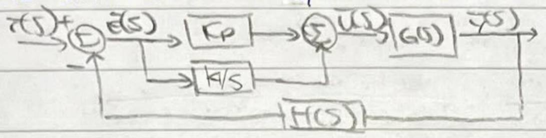
* The steady-state values are the same as P control.

Proportional + Integral (PI) control

- For PI control, the controller block is given by

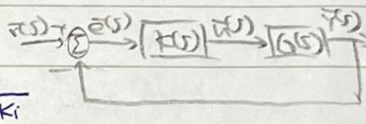
$$\boxed{K(s) = K_p + \frac{K_i}{s}}$$

- The presence of an integrator in the controller ensures that there is no steady-state error if the closed-loop system is asymptotically stable. (i.e. $\lim_{t\to\infty} e(t) = 0$)
  ↳ Proof by contradiction. Assume the system settles down to an eqm w/ $\lim_{t\to\infty} e(t) = A \neq 0$.



If the error $e(t)$ tends to a const. value that is non-zero, the output of the integrator would tend to infinity → contradict asymptotic stability of the system.

- Consider the example $K(s) = K_p + \frac{K_i}{s}$, $G(s) = \frac{1}{(s+1)^2}$



$$\frac{y(s)}{r(s)} = \frac{(K_p + K_i/s)G(s)}{1+(K_p+K_i/s)G(s)} = \frac{(K_p+K_i/s)\cdot\frac{1}{(s+1)^2}}{1+(K_p+K_i/s)\cdot\frac{1}{(s+1)^2}} = \frac{K_p s + K_i}{s(s^2+2s+1+K_p)+K_i}$$

⇒ we can determine the steady state values.

$$y(s) = \frac{(K_p+K_i/s)G(s)}{1+(K_p+K_i/s)G(s)}r(s) = \frac{K_p s + K_i}{s(s^2+2s+1+K_p)+K_i}r(s) \; ; \; e(s) = \frac{1}{1+(K_p+K_i/s)G(s)}r(s) = \frac{s(s+1)^2}{s(s^2+2s+1+K_p)+K_i}r(s)$$

If $r(t) = H(t)$, then $\lim_{t\to\infty} y(t) = G(0)$, so

$$\lim_{t\to\infty} y(t) = \left.\frac{K_p s + K_i}{s(s^2+2s+1+K_p)+K_i}\right|_{s=0} = 1 \qquad \text{[final value of step response]}$$

$$\lim_{t\to\infty} e(t) = \left.\frac{s(s+1)^2}{s(s^2+2s+1+K_p)+K_i}\right|_{s=0} = 0 \qquad \text{[steady state error]}$$

∴ Here, we can see there is no steady-state error.

* We cannot find ω_n and ζ as the denominator is now a cubic instead of a quadratic.

Proportional + Integral + Derivative (PID) control

- For PID control, the controller block is given by

$$\boxed{K(s) = K_p + \frac{K_i}{s} + K_d s}$$

- PID control can potentially combine the advantages of both derivative and integral action, but can be difficult to tune (many empirical rules such as Ziegler-Nichols).

Feedback stability and the Nyquist diagram

Sketching Nyquist diagrams

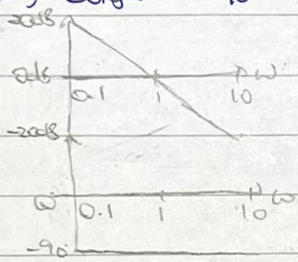
- Unlike the Bode diagram, there are no detailed rules for sketching Nyquist diagrams.
- It suffices to determine the asymptotic behaviour as $\omega \to 0$ and $\omega \to \infty$ and then calculate a few pts in between.
- If $G(0)$ is finite and non-zero, then the Nyquist locus will always start off by leaving the real axis at right angles to it.
- If $G(0)$ is infinite (pole at $s=0$), then we must find the 1st order term of the Taylor series expansion of $G(j\omega)$ about $\omega=0$.

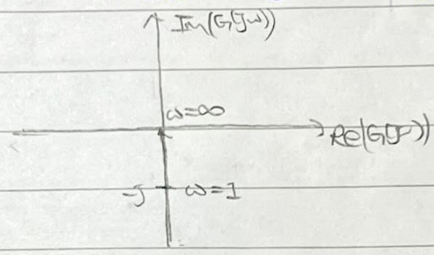
- e.g: Integrator

$G(s) = 1/s$, $G(j\omega) = 1/j\omega$.

$|G(j\omega)| = 1/\omega$, $\angle G(j\omega) = -90°$

	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
$\omega=0$	∞	$-90°$
$\omega=1$	1	$-90°$
$\omega\to\infty$	0	$-90°$



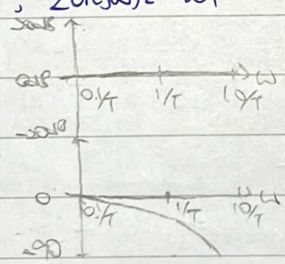


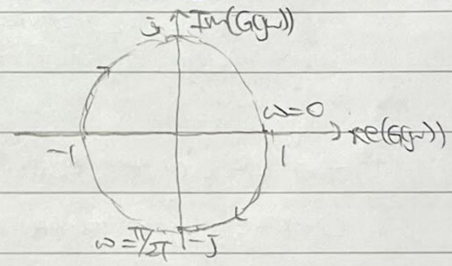
- e.g: Time delay

$G(s) = e^{-sT}$. $G(j\omega) = e^{-j\omega T}$

$|G(j\omega)| = 1$, $\angle G(j\omega) = -\omega T$

	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
$\omega=0$	1	0
$\omega=\frac{\pi}{2T}$	1	$-\pi/2$
$\omega\to\infty$	1	undef.



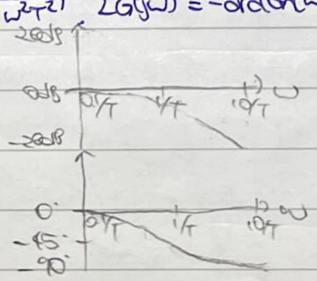


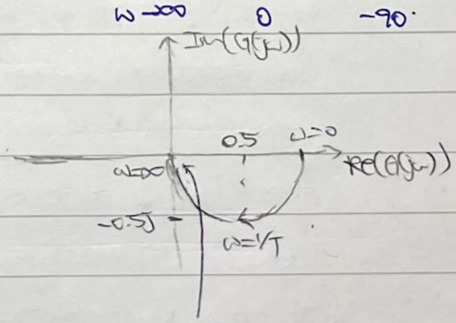
- e.g. First order lag

$G(s) = \frac{1}{1+sT}$, $G(j\omega) = \frac{1}{1+j\omega T}$

$|G(j\omega)| = \frac{1}{\sqrt{1+\omega^2T^2}}$ $\angle G(j\omega) = -\arctan(\omega T)$

	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
$\omega=0$	1	$0°$
$\omega=1/T$	$1/\sqrt{2}$	$-45°$
$\omega\to\infty$	0	$-90°$





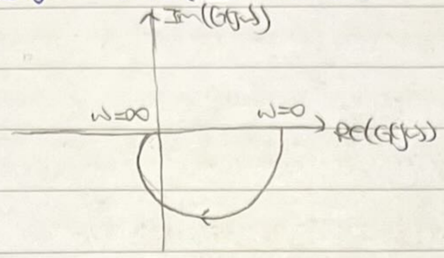
phase tends to $-90°$ as ω tends to ∞.

- e.g.: second order lag

$G(s) = \frac{1}{(1+sT_1)(1+sT_2)}$, $G(j\omega) = \frac{1}{(1+j\omega T_1)(1+j\omega T_2)}$

$|G(j\omega)| = \frac{1}{\sqrt{1+\omega^2 T_1^2}\sqrt{1+\omega^2 T_2^2}}$, $\angle G(j\omega) = -\arctan(\omega T_1) - \arctan(\omega T_2)$

	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
$\omega = 0$	1	0
$\omega \to \infty$	0	$-180°$



- e.g.: Time delay w/ lag and integrator

$G(s) = \frac{e^{-sT_1}}{s(1+sT_2)}$, $G(j\omega) = \frac{e^{-j\omega T_1}}{j\omega(1+j\omega T_2)}$

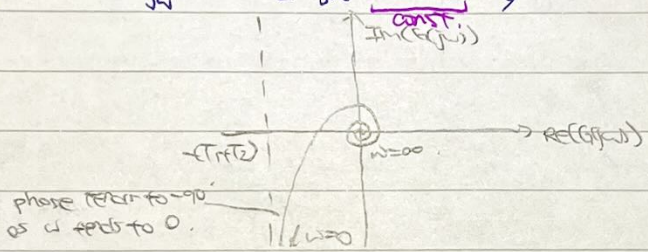
$|G(j\omega)| = \cancel{|e^{-j\omega T_1}|}^{1} \times \frac{1}{|j\omega|} \times \left|\frac{1}{1+j\omega T_2}\right|$, $\angle G(j\omega) = \underbrace{\cancel{\angle e^{-j\omega T_1}}}_{-\omega T_1} - \underbrace{\cancel{\angle j\omega}}_{90°} - \angle(1+j\omega T_2)$

	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
$\omega = 0$	∞	$-90°$
$\omega \to \infty$	0	undef

$G(0)$ is infinite → Taylor expansion around $\omega = 0$.

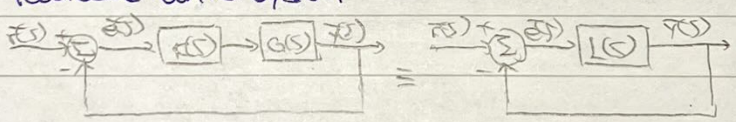
$e^{-j\omega T_1} \to 1 - j\omega T_1$; $\frac{1}{1+j\omega T_2} \to 1 - j\omega T_2$.

so $G(j\omega) \to \frac{(1-j\omega T_1)(1-j\omega T_2)}{j\omega} = \frac{1}{j\omega} \underbrace{-(T_1+T_2)}_{\text{const.}} + \cancel{j\omega T_1 T_2}^{0}$



Feedback stability

- consider a feedback control system



The CLP are the poles of $\frac{G(s)K(s)}{1+G(s)K(s)}$ / roots of $1+G(s)K(s) = 0$.

→ it is difficult to see how $K(s)$ should be chosen to ensure that all the CLP are in LHP.

- Nyquist's stability thm allows us to deduce closed loop properties (location of CLP) from open-loop properties (freq. response of the return ratio, $L(j\omega) = G(j\omega)K(j\omega)$).
- An informal statement of Nyquist's stability thm: If a feedback system has an asymptotically stable return ratio $L(s)$, then the feedback system is also asymptotically stable if the Nyquist diagram of $L(j\omega)$ leaves the pt. $-1+0j$ on its left.
- The informal statement above is unambiguous in most cases, and usually still works if $L(s)$ has poles at the origin / is unstable.

* To apply Nyquist's stability thm, we usually try to find where the locus of $L(j\omega)$ crosses the real axis (or what ω will $L(j\omega)$ be purely real).

Intuition for Nyquist's stability theorem

- Negative feedback is used to reduce the size of the error $e(t)$. If $y(t)$ is too large (i.e. $y(t) > r(t)$), then $e(t)$ is -ve, which will tend to reduce $y(t)$.
- However, for any real system, the phase lag from the input to the output, $-\angle L(j\omega)$, will tend to increase w/ frequency, eventually reaching $180°$. (-ve feedback → +ve feedback).
- If the gain $|L(j\omega)|$ has not decreased to less than 1 by this frequency, then the feedback loop would be unstable.
- If the Nyquist locus passes through the pt. $-1+0j$, i.e. $L(j\omega_1) = -1$ for some ω_1, then the closed-loop frequency response $\frac{L(j\omega)}{1+L(j\omega)}$ becomes infinite when $\omega = \omega_1$.
- Also, when $\omega = \omega_1$ s.t. $L(j\omega_1) = -1$, say $e(t) = \cos(\omega_1 t)$.

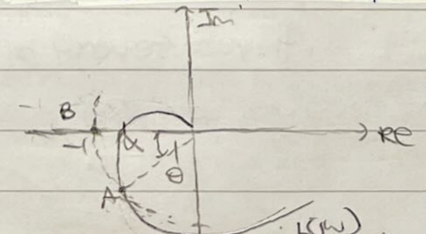
  Then in steady-state, $y(t) = |L(j\omega_1)| \cos(\omega_1 t + \angle L(j\omega_1)) = \cos(\omega_1 t + \pi) = -\cos(\omega_1 t)$.

  and $r(t) = y(t) + e(t) = -\cos(\omega_1 t) + \cos(\omega_1 t) = 0$.

  i.e. there is a sustained oscillation of the feedback system even when there is no input.

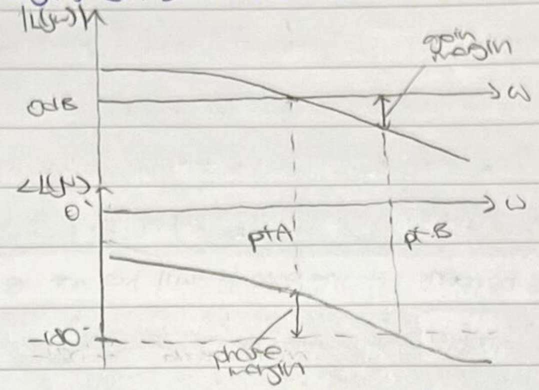
Gain and phase margins.

- $L(j\omega)$ encircling or going through the pt. $-1+0j$ is clearly bad, leading to the feedback system not being asymptotically stable. However, $L(j\omega)$ coming close to the pt $-1+0j$ w/o encircling it is also undesirable:
  - ↳ it implies that a CLP will be close to the imaginary axis and the closed-loop system will be oscillatory (close to imag. axis → large argument → $\downarrow \zeta$).
  - ↳ if $G(s)$ is the transfer function of an inaccurate model, then the "true" Nyquist diagram might actually encircle the pt. $-1+0j$.
- Gain and phase margins are measures of how close the return ratio $L(j\omega)$ gets to $-1+0j$.
  - ↳ The gain margin measures how much the gain of the return ratio can be increased before the closed-loop system becomes unstable.
  - ↳ The phase margin measures how much phase lag can be added to the return ratio before the closed-loop system becomes unstable.
- The gain and phase margins can be deduced from the Nyquist plot of $L(j\omega)$.



Gain margin $= 1/\alpha$ Phase margin $= \theta$

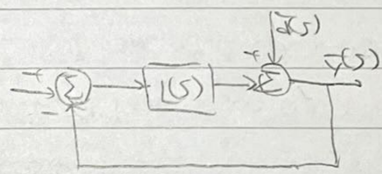
- The gain and phase margins can be deduced from the Bode plot of $L(j\omega)$.



- Given a Nyquist diagram of $L'(s) = kL(s)$ for $k=1$, it is easy to find the gain and phase margins for $k \neq 1$ by considering the pt $-1/k + 0j$ instead of $-1+0j$. (and by extension, the circle centred at the origin w/ radius $1/k$).

Performance of feedback systems

- A small magnitude in the frequency response of the sensitivity function, $\left|\frac{1}{1+L(j\omega)}\right|$, leads to good closed-loop properties.
  ↳ 1) Rejection of disturbances.



$$y(s) = \frac{1}{1+L(s)} d(s)$$
$$= S(s)\, d(s)$$

  ↳ 2) Reducing the effects of uncertainty. (Say $L(s)$ depends on λ).

$$\frac{d/d\lambda \; 1/(1+L)}{1/(1+L)} = \frac{(1+L)\frac{dL}{d\lambda} - L\frac{dL}{d\lambda}}{(1+L)^2} \Big/ \frac{L}{1+L} = \frac{1}{1+L} \frac{d/d\lambda\, L}{L}$$

relative change in closed-loop.

$$= S \frac{d/d\lambda\, L}{L}$$

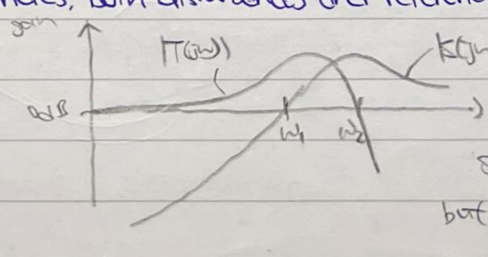
relative change in open-loop

due to the fundamental limit of performance.

- Good designs aim for sensitivity reduction over an appropriate range of frequencies (as we cannot make S arbitrarily small for all frequencies)
- Typically, we req. that $\left|\frac{1}{1+L(s)}\right| \ll 1$ for $\omega < \omega_1$, where ω_1 is the desired control BW.

The relationship between open and closed loop frequency responses.

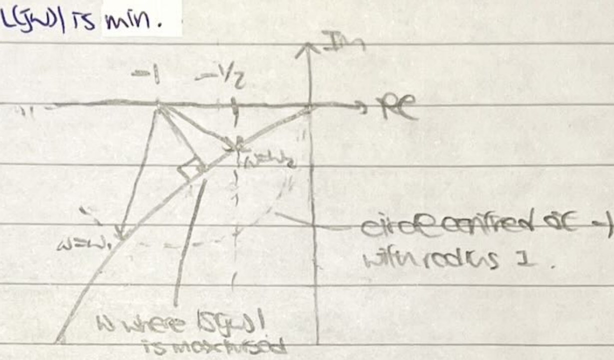
- Feedback can reduce the effect of disturbances at low frequencies, up to ω_1. ω_1 is defined as the lowest frequency at which $|S(j\omega)| = 1$.
- The closed-loop system will respond to reference inputs at frequencies up to ω_2. ω_2 is defined as the highest frequency at which $|T(j\omega)| = 1$.
- Between these frequencies, both disturbances and reference signals are amplified.



$S(j\omega) + T(j\omega) = 1$

but $|S(j\omega)| + |T(j\omega)| \neq 1$

- The actual value of the frequencies ω_1 and ω_2, and the size of the peaks of $|S(j\omega)|$, and $|T(j\omega)|$ can be determined directly from the open-loop frequency response:
  ↳ 1) $|S(j\omega)| = \left|\frac{1}{1+L(j\omega)}\right| = 1$ when $|1+L(j\omega)| = 1$, which is when the distance from the pt. -1 to the Nyquist locus equals 1 ($\omega = \omega_1$)
  ↳ 2) $|T(j\omega)| = \left|\frac{L(j\omega)}{1+L(j\omega)}\right| = 1$ when $|L(j\omega)| = |1+L(j\omega)|$, which is when the distances from the pt. -1 and the origin to the Nyquist locus are equal ($\omega = \omega_2$). i.e. $L(j\omega) = -1/2$.
  ↳ 3) $|S(j\omega)| = \left|\frac{1}{1+L(j\omega)}\right|$ is max. when $|1+L(j\omega)|$ is min., which is when the distance from the pt. -1 to the Nyquist locus is min.
  ↳ 4) There is no geometric representation of the max value of $|T(j\omega)| = \left|\frac{L(j\omega)}{1+L(j\omega)}\right|$ → try pts around where $|1+L(j\omega)|$ is min.



- If $L(j\omega)$ has frequencies where it comes close to -1 (when gain/phase margins are small), then $|1+L(j\omega)|$ is small → $|S(j\omega)|$ is large → poor performance.

The Nyquist stability theorem (formal)

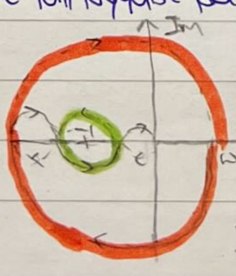
since $(-j\omega) = (j\omega)^*$, it follows that $L(-j\omega) = L(j\omega)^*$ → reflect about real axis for $-$ve ω.

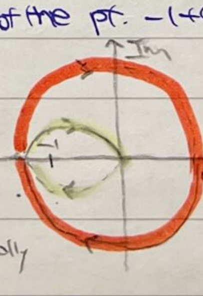
- The Nyquist stability thm. states for a feedback system w/ asymptotically stable return ratio $L(s)$, the feedback system is itself asymptotically stable iff the pt. $-1+0j$ is not encircled by the "full" Nyquist diagram of $L(j\omega)$, $\omega \in (-\infty, \infty)$.

* The simplifying assumption that $L(s)$ is asymptotically stable guarantees that $L(j\omega)$ is finite for all ω (no $j\omega$-axis poles), and $L(\infty)$ is finite ($L(s)$ must be proper)
  → Full Nyquist plot is a closed curve (since $L(j\infty) = L(-j\infty) = L(\infty)$).
* The no. of encirclements = no. of CW encirclements − no. of CCW encirclements.

- If $L(s)$ has n poles at 0 (marginally stable/unstable), the thm. still works after closing the Nyquist plot by adding a large $n \times 180°$ arc in the CW direction.
- If $L(s)$ is unstable, and has n unstable poles, then the feedback system is asymptotically stable iff the full Nyquist plot makes $-n$ encirclements of the pt. $-1+0j$.

e.g.

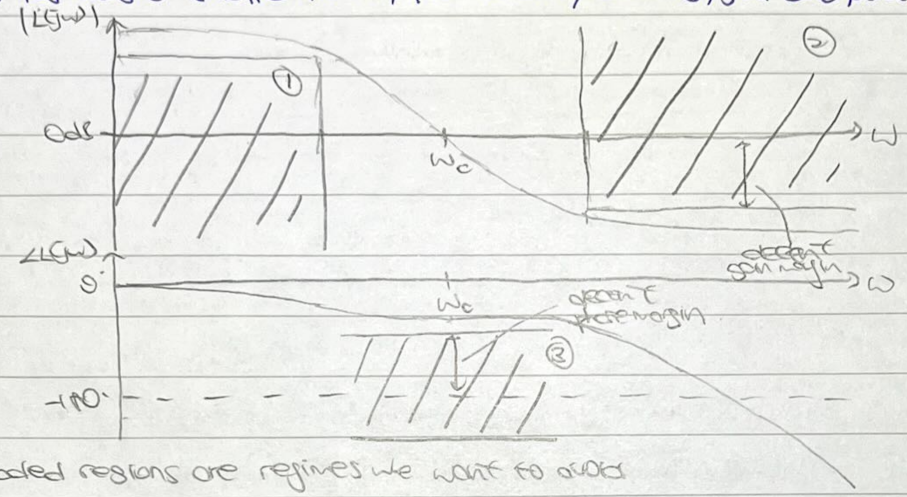




The design of feedback systems

Feedback system design, a loop-shaping approach.

- In the loop-shaping approach for control design, the controller $K(s)$ is chosen s.t. the frequency response of the return ratio $L(j\omega)$ has an appropriate shape for good properties.
  - ↳ 1) $|K(j\omega)G(j\omega)| \gg 1$ for frequency ranges where the benefits of feedback are sought, typically $\omega < \omega_c$. [small steady-state error, $|S(j\omega)| \ll 1$]
  - ↳ 2) $|K(j\omega)G(j\omega)| \ll 1$ at other frequencies, typically $\omega > \omega_c$ [$|T(j\omega)| \ll 1$] (high freq → large phase lags → small gain to ensure we do not encircle -1).
  - ↳ 3) $K(j\omega)G(j\omega)$ satisfies the Nyquist stability criterion, w/ adequate gain/phase margins.



* shaded regions are regimes we want to avoid

- To achieve a shape to the above, we can use a combination of phase lag and phase lead compensators (phase lag – generalised PI ; phase lead – generalised PD).

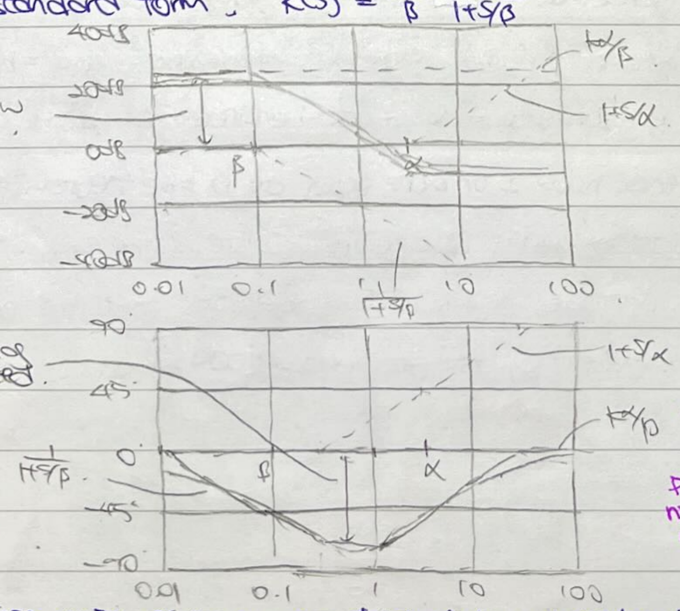
Phase lag compensators. (generalised PI)

- Phase lag compensators have a transfer function

PI controller when $\beta = 0$.

$$K(s) = K\frac{s+\alpha}{s+\beta} \quad \text{for } \beta < \alpha \; (\ll \omega_c \text{ typically})$$

and in the standard form, $K(s) = \dfrac{K\alpha}{\beta}\dfrac{1+s/\alpha}{1+s/\beta}$



✓ improve low freq. gain.

✗ phase lag introduced.

reduce steady state errors and sensitivity

not an issue if $\beta < \alpha \ll \omega_c$. phase lags are bad near ω_c as we would reduce the phase margin.

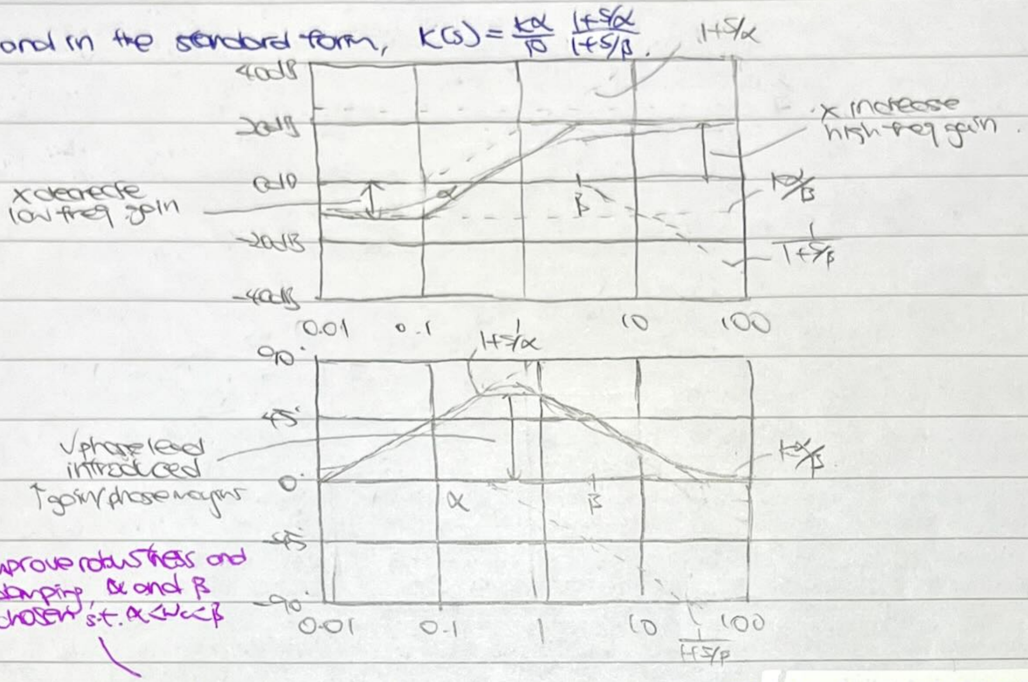
- Improves low freq. gain at the expense of introducing a phase lag between $\omega = \beta$ and $\omega = \alpha$.

phase lead compensators (generalised PD)

- phase lead compensators have a transfer function

$$K(s) = K\frac{s+\alpha}{s+\beta} \quad \text{for } \alpha<\beta \;\; (\alpha<\omega_c<\beta \text{ typically})$$

and in the standard form, $K(s) = \frac{K\alpha}{\beta}\frac{1+s/\alpha}{1+s/\beta}$.



improve robustness and damping, α and β chosen s.t. $\alpha<\omega_c<\beta$

- Improves gain/phase margins by introducing a phase lead between $\omega=\alpha$ and $\omega=\beta$, at the expensive of decreasing the low freq. gain and increasing the high freq. gain

increasing steady-state errors and sensitivity. higher risk of encircling -1.

* Typically, we use a phase lead compensator to increase the gain/phase margin near ω_c. so $\alpha_{PL} < \omega_c < \beta_{PL}$. Then, we use a phase lag compensator to compensate for the decrease in low freq./increase in high freq. (due to phase lead compensator), w/ $\beta<\alpha<<\omega_c$. This would introduce a phase lag where it does not matter.