Transmission lines

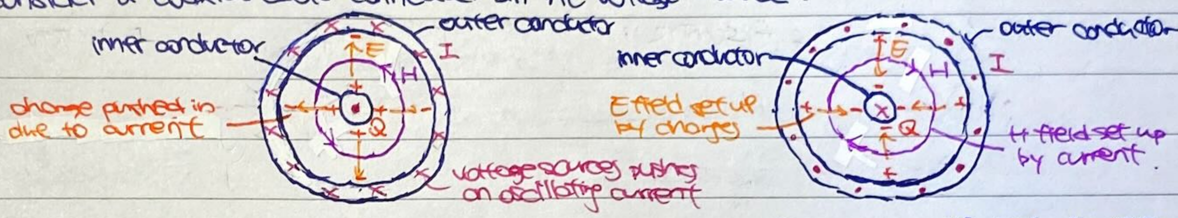
Transmission line equivalent circuit

Transmission lines

- A transmission line guides the flow of energy in the form of an EM wave from one place to another. (energy flowing → transmission of data is possible)
- Examples of transmission lines of diff. length scales:
  - ↳ Electricity power transmission lines
  - ↳ coaxial cables for low noise signals
  - ↳ Microstrip lines for data transfer on PCBs.

Qualitative view of transmission lines

- Consider a coaxial cable connected an AC voltage source.

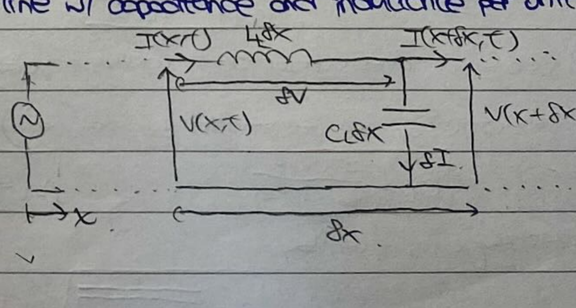


The above sketches show the cross-section of the coaxial cable at two diff. instances. The EM wave propagates out of the page (⊥ to both E-field and H-field).

- As the wave propagates along in the dielectric, it induces charges on the conductors, such charges sustain the E-field and H-field, guiding them along the transmission line.
- e^- in the conductors are only oscillating locally, moving at the drift velocity (resulting in the net space charge close to the conductor surface).
- However, the e^- in the conductors can guide the surrounding EM wave → they can travel at high speeds, determined by the dielectric.

Ideal transmission line equivalent circuit

- All transmission lines must consist of at least two conductors (out + return current path), separated by an insulator (dielectric)
- conductors separated by a gap has capacitance; loop of wire has inductance → we can model a transmission line w/ capacitance and inductance per unit length, C_L, L_L.



The Telegrapher's equations

- consider the voltage drop across the inductor δV,

$$\delta V = V(x+\delta x, t) - V(x,t) = -L_L \delta x \frac{\partial I}{\partial t}$$

$$\boxed{\frac{\partial V}{\partial x} = -L_L \frac{\partial I}{\partial t}}$$

consider the current flowing through the capacitor δI,

$$\delta I = I(x+\delta x, t) - I(x,t) = -C_L \delta x \frac{\partial V}{\partial t}$$

$$\boxed{\frac{\partial I}{\partial x} = -C_L \frac{\partial V}{\partial t}}$$

- The Telegrapher's eqns (TE) describe the relationship btwn the time and position dependent voltage and current of any pt. on a transmission line.

Wave equation solution to the Telegrapher's equations.

- Differentiating both sides of the TE wrt x,

$$\frac{\partial^2 V}{\partial x^2} = -L_L \frac{\partial}{\partial x}\frac{\partial I}{\partial t} = -L_L \frac{\partial}{\partial t}\frac{\partial I}{\partial x}$$

$$\boxed{\frac{\partial^2 V}{\partial x^2} = L_L C_L \frac{\partial^2 V}{\partial t^2}}$$

$$\frac{\partial^2 I}{\partial x^2} = -C_L \frac{\partial}{\partial x}\frac{\partial V}{\partial t} = -C_L \frac{\partial}{\partial t}\frac{\partial V}{\partial x}$$

$$\boxed{\frac{\partial^2 I}{\partial x^2} = L_L C_L \frac{\partial^2 I}{\partial t^2}}$$

we get the generic form of a wave eqn. in 1D.

- In general, the wave eqn. in 1D for some arbitrary function ψ is as follows.

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$$

one sol'n to the generic wave eqn in 1D is

$$\psi = \psi_0 \cos(\omega t - \beta x).$$

which represents a wave moving in the +ve x-direction w/ time,

and ω is the angular frequency, $\omega = \frac{2\pi}{T}$; β is the propagation const., $\beta = \frac{2\pi}{\lambda}$,

- substituting the sol'n above into the wave eqn, we get.

$$\beta^2 \psi = \frac{1}{c^2}\omega^2 \psi.$$

$$\therefore \boxed{c = \frac{\omega}{\beta} = \frac{2\pi f}{2\pi/\lambda} = f\lambda = \text{wave velocity}}.$$

- For a transmission line, the wave velocity c is given by.

$$\boxed{c = \frac{1}{\sqrt{L_L C_L}}}$$

- For a coaxial cable w/ inner conductor of radius a, outer conductor of radius b, and intermediate dielectric of rel permittivity ε_r, rel permeability μ_r,

$$\boxed{C_L = \frac{2\pi\varepsilon_0\varepsilon_r}{\ln(b/a)}} \qquad \boxed{L_L = \frac{\mu_0\mu_r \ln(b/a)}{2\pi}}$$

so the wave velocity c is given by

$$\boxed{c = \frac{1}{\sqrt{\varepsilon_0\varepsilon_r\mu_0\mu_r}}}$$

→ speed of wave along a transmission line is dependent only on the property of the dielectric medium btwn the conductors (ε_r, μ_r).

- If the dielectric is air, $\varepsilon_r = \mu_r = 1$, then $c = (\varepsilon_0\mu_0)^{-1/2} = 2.998 \times 10^8 \text{ ms}^{-1}$.

Expressions for current and voltage waves

- The general sol'n to the wave eqn. for voltage is,

$$V = \overset{\text{forward}}{V_F\cos(\omega t - \beta x + \phi_F)} + \overset{\text{backward}}{V_B\cos(\omega t + \beta x + \phi_B)}.$$

where V_F/ϕ_F and V_B/ϕ_B are the amplitude/phases of the forward and backward travelling voltage waves.

- Using complex notation, we have,

$$\boxed{V = \bar{V}_F e^{j(\omega t - \beta x)} + \bar{V}_B e^{j(\omega t + \beta x)}} \qquad \bar{V}_i = V_i e^{j\phi_i}$$

- The general sol'n to the wave eqn. for current is

$$I = \overset{\text{forward}}{I_F\cos(\omega t - \beta x + \phi_F)} + \overset{\text{backward}}{I_B\cos(\omega t + \beta x + \phi_B)}.$$

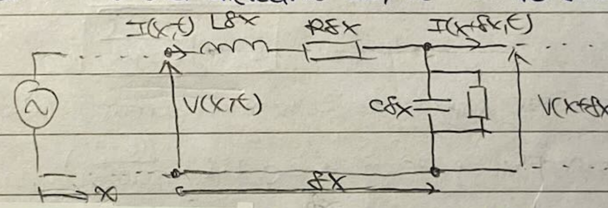
where I_F/ϕ_F and I_B/ϕ_B are the amplitude/phases of the forward and backward travelling current waves.

- Using complex notation, we have,

$$\boxed{I = \bar{I}_F e^{j(\omega t - \beta x)} + \bar{I}_B e^{j(\omega t + \beta x)}} \qquad \bar{I}_i = I_i e^{j\phi_i}$$

Lossy transmission lines

- In reality, in the transmission line, the conductors have a resistance per unit length R_L, and the dielectric is not a ~~perfect~~ insulator and has conductance per unit length G_L.
- Factoring in the conductor R_L and dielectric G_L, the new transmission line equivalent circuit is



- The losses result in the amplitude of voltage/current to decay exponentially w/ distance.

$$V = \bar{V}_F e^{j(\omega t - \beta x)} e^{-\alpha x} + \bar{V}_B e^{j(\omega t + \beta x)} e^{\alpha x} = \bar{V}_F e^{j\omega t - \gamma x} + \bar{V}_B e^{j\omega t + \gamma x}$$

$$I = \bar{I}_F e^{j(\omega t - \beta x)} e^{-\alpha x} + \bar{I}_B e^{j(\omega t + \beta x)} e^{\alpha x} = \bar{I}_F e^{j\omega t - \gamma x} + \bar{I}_B e^{j\omega t + \gamma x}$$

where α is the attenuation constant

and γ is the propagation constant, defined as

$$\boxed{\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}}$$

Wavelength.

- If the wavelength λ is long compared to the physical length of the system concerned, then we can consider the current to be behaving as an incompressible fluid (no spatial phase difference). (There is minimal phase diff. between pts → wave effects not significant).

Wave analysis of transmission lines

Characteristic impedance

- Assuming an ideal transmission line, and substituting the soln for V, I into the TE (w/ capacitance per unit length C_L), we have

$$\frac{\partial}{\partial x}\left(\bar{I}_F e^{j(\omega t-\beta x)} + \bar{I}_B e^{j(\omega t+\beta x)}\right) = -C\frac{\partial}{\partial t}\left(\bar{V}_F e^{j(\omega t-\beta x)} + \bar{V}_B e^{j(\omega t+\beta x)}\right)$$

$$\left(-\beta \bar{I}_F e^{-j\beta x} + \beta \bar{I}_B e^{j\beta x}\right) = -C\left(\omega \bar{V}_F e^{-j\beta x} + \omega \bar{V}_B e^{j\beta x}\right)$$ cancelled $je^{j\omega t}$

Equating the forward ($e^{-j\beta x}$) and backward ($e^{j\beta x}$) terms separately, we get

$$\frac{\bar{V}_F}{\bar{I}_F} = \frac{\beta}{\omega C} \qquad \frac{\bar{V}_B}{-\bar{I}_B} = \frac{\beta}{\omega C}.$$

* voltages defined as voltage of top conductor wrt bottom conductor; currents defined flowing in the +ve x-direction → wave travelling in -ve x direction, current will be -ve.

- We can defined the characteristic impedance Z_0 as the ratio btwn voltage and current of a unidirectional wave at any pt on the transmission line.

$$\boxed{Z_0 = \frac{\bar{V}_F}{\bar{I}_F} = \frac{\bar{V}_B}{-\bar{I}_B}} = \frac{\beta}{\omega C}$$

Recall that $\frac{\beta}{\omega} = \frac{1}{f\lambda} = \frac{1}{c} = \sqrt{LC}$, so

$$\boxed{Z_0 = \sqrt{\frac{L}{C}}}$$

Z_0 is real for an ideal, lossless transmission line.

- Substituting expressions for L and C for a transmission line gives a form

$$\boxed{Z_0 = \text{geometry factor} \times \sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}}}$$

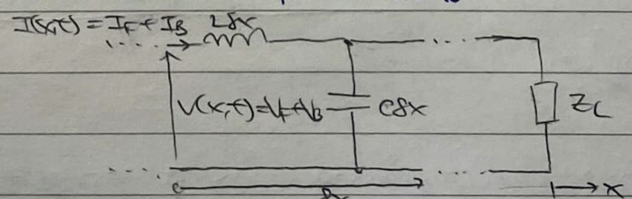
- For a lossy transmission line, then the characteristic impedance Z_0 gains frequency dependence

$$\boxed{Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}}}$$

- The characteristic impedance Z_0 is the apparent impedance that is "seen" if looking into an infinitely long line at $x=0$ (i.e. $\frac{V(0,t)}{I(0,t)}$)
- Although characteristic impedance Z_0 has units of Ω, it does <u>NOT</u> dissipate power as V and I are not btwn the same pts.

Reflections from a load impedance

- Consider a wave from some source travelling in the forward direction down a transmission line which is terminated in a load impedance at $x=0$



Other than the forward travelling wave, we should allow for the fact that some of the wave may be reflected by the load to give a backwards travelling wave, so

$$V(x,t) = V_F + V_B \qquad I(x,t) = I_F + I_B$$

- As we have defined the load as being at $x=0$,

$$V(0,t) = \bar{V}_F e^{j(\omega t - \beta \cdot 0)} + \bar{V}_B e^{j(\omega t + \beta \cdot 0)} = (\bar{V}_F + \bar{V}_B) e^{j\omega t}$$

$$I(0,t) = \bar{I}_F e^{j(\omega t - \beta \cdot 0)} + \bar{I}_B e^{j(\omega t + \beta \cdot 0)} = (\bar{I}_F + \bar{I}_B) e^{j\omega t}$$

Since these are the voltage and current across the load Z_L,

$$Z_L = \frac{V(0,t)}{I(0,t)} = \frac{\bar{V}_F + \bar{V}_B}{\bar{I}_F + \bar{I}_B}$$

Expressing $\bar{I}_F, \bar{I}_B$ in terms of $\bar{V}_F, \bar{V}_B$ using $Z_0 = \frac{\bar{V}_F}{\bar{I}_F} = -\frac{\bar{V}_B}{\bar{I}_B}$, we have

$$Z_L = \frac{\bar{V}_F + \bar{V}_B}{\bar{V}_F / Z_0 - \bar{V}_B / Z_0} = Z_0 \cdot \frac{\bar{V}_F + \bar{V}_B}{\bar{V}_F - \bar{V}_B}$$

Rearranging for $\bar{V}_B$,

$$\bar{V}_B = \bar{V}_F \frac{Z_L - Z_0}{Z_L + Z_0}$$

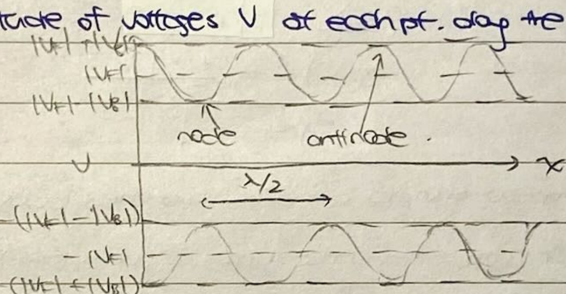
- We define the voltage reflection coefficient ρ_L as

$$\boxed{\rho_L = \frac{\bar{V}_B}{\bar{V}_F} = \frac{Z_L - Z_0}{Z_L + Z_0}}$$

- The forward wave V_F and the backward wave V_B sum together to produce a standing wave

$$\boxed{v(x,t) = \bar{V}_F (e^{-j\beta x} + \rho_L e^{j\beta x}) e^{j\omega t}}$$

Plotting the amplitude of voltages V at each pt. along the wave x, we get.



- We can define the voltage standing wave ratio (VSWR) as the ratio of the max. voltage amplitude to the min. voltage amplitude.

$$\boxed{VSWR = \frac{|\bar{V}_F| + |\bar{V}_B|}{|\bar{V}_F| - |\bar{V}_B|} = \frac{1 + |\rho_L|}{1 - |\rho_L|}}$$

- Since some of the forward travelling wave is reflected, some power is reflected too. As $P \propto V^2$, and $V_B \propto \rho_L$, the proportion of incident power that is reflected is $|\rho_L|^2$.

$$\boxed{\frac{P_{reflect}}{P_{incident}} = \left(\frac{V_B}{V_F}\right)^2 = |\rho_L|^2}$$

- There are three special case for Z_L:

① open circuit, $Z_L = \infty$, $\rho_L = 1$.

↳ incident wave reflected w/o change in phase for voltage at reflection pt.

↳ a perfect standing wave (VSWR $= \infty$) is formed w/ an antinode at the reflection pt.

② short circuit, $Z_L = 0$, $\rho_L = -1$.

↳ incident wave reflected w/ a π phase change for voltage at reflection pt.

↳ a perfect standing wave (VSWR $= \infty$) is formed w/ a node at the reflection pt.

③ matching impedance, $Z_L = Z_0$, $\rho_L = 0$

↳ no wave is reflected and all of the power is dissipated in the load

Input impedance of a terminated line.

- Consider a cable w/ characteristic impedance Z_0 that has length l and is terminated w/ an impedance of Z_L at $x=0$ (so input at $x=-l$)
- the apparent impedance at any pt x in the line is the ratio of voltage to current at that pt.

$$Z(x) = \frac{\bar{V}_F e^{j(\omega t-\beta x)} + \bar{V}_B e^{j(\omega t+\beta x)}}{\bar{I}_F e^{j(\omega t-\beta x)} + \bar{I}_B e^{j(\omega t+\beta x)}}$$

Losing a factor of $e^{j\omega t}$ and expressing $\bar{I}_F, \bar{I}_B$ in terms of $\bar{V}_F, \bar{V}_B$ using Z_0

$$Z(x) = \frac{\bar{V}_F e^{-j\beta x} + \bar{V}_B e^{j\beta x}}{\bar{V}_F/Z_0\, e^{-j\beta x} - \bar{V}_B/Z_0\, e^{j\beta x}}$$

$$= Z_0 \cdot \frac{e^{-j\beta x} + \bar{V}_B/\bar{V}_F\, e^{j\beta x}}{e^{-j\beta x} - \bar{V}_B/\bar{V}_F\, e^{j\beta x}}$$

$$= Z_0 \cdot \frac{e^{-j\beta x} + \rho_L e^{j\beta x}}{e^{-j\beta x} - \rho_L e^{j\beta x}}$$

$$\rho_L = \frac{\bar{V}_B}{\bar{V}_F} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$= Z_0 \frac{(Z_L+Z_0)e^{-j\beta x} + (Z_L - Z_0)e^{j\beta x}}{(Z_L+Z_0)e^{-j\beta x} + (Z_L-Z_0)e^{j\beta x}}$$

$$= Z_0 \cdot \frac{Z_L\cos\beta x - jZ_0 \sin\beta x}{-jZ_L \sin\beta x + Z_0\cos\beta x}$$

so at the input $x=-l$, the apparent impedance Z is given by

$$\boxed{Z(-l) = Z_0 \frac{Z_L + jZ_0\tan\beta l}{Z_0 + jZ_L\tan\beta l}}$$

- when the length of the line $l = \lambda/4$, $\beta l = \frac{2\pi}{\lambda}\cdot\frac{\lambda}{4} = \frac{\pi}{2} \rightarrow \tan(\beta l) = \infty$, so

$$Z(-\lambda/4) = Z_0 \frac{Z_L + jZ_0\infty}{Z_0 + jZ_L\infty}$$

$$\boxed{Z(-\lambda/4) = \frac{Z_0^2}{Z_L}}$$

quarter-wave matching.

- we can use this phenomenon to connect mismatched loads to a transmission line — say we have a source w/ output impedance (input of our system) Z_{in}, and load impedance Z_L. we connect them w/ a transmission line of length $\lambda/4$ and characteristic impedance Z_0 given by

$$\boxed{Z_0 = \sqrt{Z_L Z_{in}}}$$

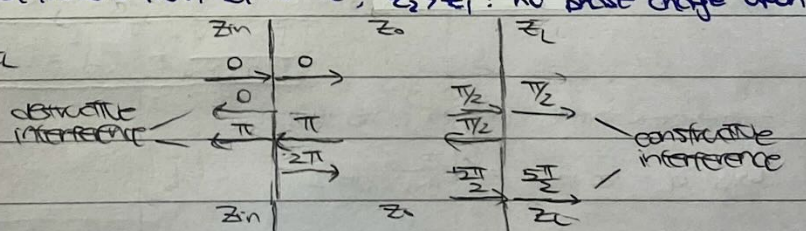
geometric mean

no reflections, all power transmitted

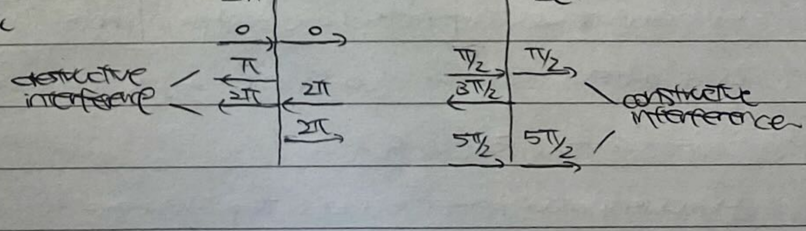
(apparent impedance at the input side of the wire would be $Z(-\lambda/4) = \frac{Z_0^2}{Z_L} = Z_{in}$)

- Quarter-wave matching can also be explained qualitatively as follows:
  - Note when a signal travels from Z_1 to Z_2, $Z_1 > Z_2$: π phase change upon reflection; $Z_2 > Z_1$: no phase change upon reflection

case I: $Z_{in} < Z_0 < Z_L$

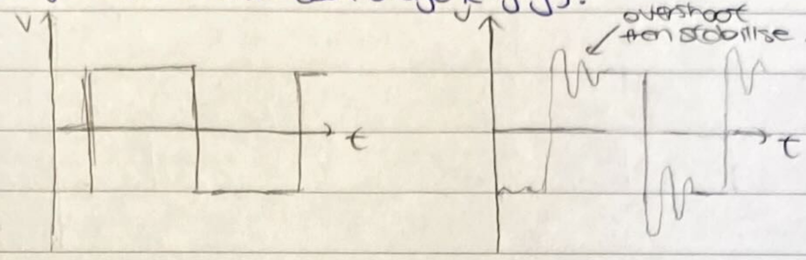


case II: $Z_{in} > Z_0 > Z_L$

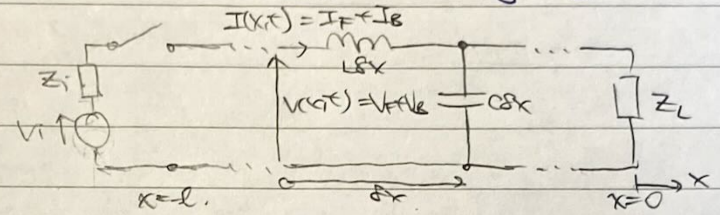


Ringing.

- When sending digital signals (square wave) down a transmission line, it takes some time for the voltage to settle after each change (ringing).



- Consider applying our input signal using a simple voltage source w/ impedance Z_i and a switch connected to the transmission line of length ℓ.



Say the switch closes at $t=0$,

- At this moment, the load impedance can have no effect on the input to the line, so the voltage transmitted into the line is given by a simple potential divider,

$$V(x=-\ell, t=0) = V_i \frac{Z_0}{Z_i + Z_0}$$

- This voltage is transmitted along the line at the wave velocity $c = \frac{1}{\sqrt{LC}}$, arriving at the load some time T later, where

$$T = \frac{\ell}{c} = \ell\sqrt{LC}.$$

- Upon reaching the load, a fraction of the wave is reflected back, so the total voltage at the load,

$$V(x=0, t=T) = V_i \frac{Z_0}{Z_i + Z_0}(1 + \rho_L)$$

where ρ_L is the reflection coefficient at the load $\left[\rho_L = \frac{Z_L - Z_0}{Z_L + Z_0}\right]$

- This will take a further time T to travel back to the input end of the transmission line, which will be reflected again, so the total voltage at the input,

$$V(x=-\ell, t=2T) = V_i \frac{Z_0}{Z_i + Z_0}(1 + \rho_L + \rho_L \rho_i)$$

where ρ_i is the reflection coefficient at the input $\left[\rho_i = \frac{Z_i - Z_0}{Z_i + Z_0}\right]$

- In general, after n round trips, the voltage at the input is given by

$$V(x=-\ell, t=2nT) = V_i \frac{Z_0}{Z_i + Z_0}\left[1 + \sum_{i=1}^{n}\left(\rho_L^n \rho_i^{n-1} + \rho_L^n \rho_i^n\right)\right]$$

as the geometric series converges w/ time.

Electromagnetic waves

The Maxwell equations of electromagnetism

The Maxwell equations of electromagnetism

- The four Maxwell eqns of electromagnetism each describe the origin + nature of electric/magnetic fields.

	Differential form	Integral form
Gauss Law (Electric fields)	$\nabla \cdot \underline{D} = \rho$	$\oiint_S \underline{D} \cdot d\underline{A} = Q$
Gauss Law (Magnetic fields)	$\nabla \cdot \underline{B} = 0$	$\oiint_S \underline{B} \cdot d\underline{A} = 0$
Faraday's Law	$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$	$\oint_C \underline{E} \cdot d\underline{r} = -\frac{d}{dt}\iint_S \underline{B} \cdot d\underline{A}$
Ampère-Maxwell Law	$\nabla \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t}$	$\oint_C \underline{H} \cdot d\underline{r} = \iint_S \underline{J} \cdot d\underline{A} + \frac{d}{dt}\iint_S \underline{D} \cdot d\underline{A}$

Gauss Law for electric fields

- Gauss Law for electric fields states that the total electric flux of $\underline{D}$ through a Gaussian surface S equals the charge enclosed by the surface.

$$\boxed{\oiint_S \underline{D} \cdot d\underline{A} = Q}$$

To convert into differential form, apply divergence thm. and use $Q = \iiint_V \rho \, dV$

$$\iiint_V (\nabla \cdot \underline{D}) \, dV = \iiint_V \rho \, dV$$

$$\boxed{\nabla \cdot \underline{D} = \rho}$$

- Gauss Law for electric fields simply states that charge Q produces electric fields $\underline{D}$.

Gauss Law for magnetic fields.

- Gauss Law for magnetic fields states that the total magnetic flux of $\underline{B}$ through a Gaussian surface S equals zero.

$$\boxed{\oiint_S \underline{B} \cdot d\underline{A} = 0}$$

To convert into differential form, apply divergence thm.

$$\iiint_V (\nabla \cdot \underline{B}) \, dV = 0$$

$$\boxed{\nabla \cdot \underline{B} = 0}$$

- Gauss Law for magnetic fields simply states that there are no magnetic monopoles — only dipoles, so magnetic flux density $\underline{B}$ form closed loops.

Faraday's Law

- Faraday's law states that a changing magnetic flux Φ_B through a coil of wire produces a emf V that acts to oppose the changing flux.

$$\boxed{V = -\frac{d\Phi_B}{dt}}$$

Integrating the E field around the loop of wire gives the p.d. around the closed loop V

$$V = \oint_C \underline{E} \cdot d\underline{r}$$

Using the definition of magnetic flux Φ_B,

$$\Phi_B = \iint_S \underline{B} \cdot d\underline{A}$$

We can state Faraday's law as

$$\boxed{\oint_C \underline{E} \cdot d\underline{r} = -\frac{d}{dt}\iint_S \underline{B} \cdot d\underline{A}}$$

To convert into differential form, apply Stokes thm,

$$\iint_S (\nabla \times \underline{E}) \cdot d\underline{A} = -\iint_S \frac{\partial \underline{B}}{\partial t} \cdot d\underline{A}$$

$$\boxed{\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}}$$

- Faraday's Law simply states a changing magnetic field $\underline{B}$ produces a circulating electric field $\underline{E}$

Ampère-Maxwell Law.

- Ampère's law states that circulation of $\underline{H}$ of an Amperian loop C equals the current enclosed in that loop.

$$\oint_C \underline{H} \cdot d\underline{r} = I$$

To convert into differential form, apply Stokes thm, and use $I = \iint_S \underline{J} \cdot d\underline{A}$

$$\iint_S (\nabla \times \underline{H}) \cdot d\underline{A} = \iint_S \underline{J} \cdot d\underline{A}$$

$$\nabla \times \underline{H} = \underline{J}$$

- Maxwell spotted a flaw — if we have a capacitor in a circuit, then although a current flows through the circuit, no current flows through the two plates of the capacitor
- Therefore, if we put an Amperian loop btwn the plates of the capacitor, we would have a magnetic field $\underline{H}$ (due to current in circuit) despite not having any enclosed current
- To fix this, Maxwell added a displacement current term $\underline{J}_{disp}$ to give Ampère-Maxwell Law

$$\boxed{\nabla \times \underline{H} = \underline{J} + \underline{J}_{disp}}$$

- Consider a small volume of space w/ charge density $\rho(\underline{r})$. The total charge Q is given by

$$Q = \iiint_V \rho(\underline{r})\, dV$$

If this charge is moving, we can express the rate of loss of total charge $-\frac{dQ}{dt}$ as

$$-\frac{dQ}{dt} = \iint_S \underline{J} \cdot d\underline{A}$$

$$\rightarrow \iint_S \underline{J} \cdot d\underline{A} = -\frac{d}{dt}\iiint_V \rho(\underline{r})\, dV.$$

- Applying divergence thm, we get

$$\iiint_V (\nabla \cdot \underline{J}) dV = -\frac{d}{dt}\iiint_V \rho(\underline{r}) dV$$

$$\boxed{\nabla \cdot \underline{J} = -\frac{\partial \rho(\underline{r})}{\partial t}} \quad \text{[continuity of charge]}$$

since Gauss' Law for electric field states that $\nabla \cdot \underline{D} = \rho$, we can say

$$\nabla \cdot \underline{J} = -\nabla \cdot \left(\frac{\partial \underline{D}}{\partial t}\right)$$

Taking the divergence of the curl of $\underline{H}$, $\nabla \times \underline{H} = \underline{J} + \underline{J}_{disp}$,

$$\nabla \cdot (\nabla \times \underline{H}) = \nabla \cdot (\underline{J} + \underline{J}_{disp})$$

$$0 = \nabla \cdot \underline{J} + \nabla \cdot \underline{J}_{disp}$$

$$\nabla \cdot \underline{J}_{disp} = \nabla \cdot \left(\frac{\partial \underline{D}}{\partial t}\right)$$

$$\boxed{\underline{J}_{disp} = \frac{\partial \underline{D}}{\partial t}}$$

so Ampère-Maxwell is

$$\boxed{\nabla \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t}}$$

To convert into integral form, integrate around a surface and apply Stokes' thm

$$\iint_S (\nabla \times \underline{H}) \cdot d\underline{A} = \iint_S \left(\underline{J} + \frac{\partial \underline{D}}{\partial t}\right) \cdot d\underline{A}$$

$$\boxed{\oint_C \underline{H} \cdot d\underline{r} = \iint_S \underline{J} \cdot d\underline{A} + \frac{d}{dt}\iint_S \underline{D} \cdot d\underline{A}}$$

Electromagnetic waves in dielectrics

Electromagnetic waves in dielectrics

- In a dielectric (insulating) medium such as air, glass, plastics, there is no free charge, so

$$\rho = 0 \qquad \underline{J} = 0$$

and Maxwell's eqns become

$$\nabla \cdot \underline{D} = 0 \ [1a] \qquad \nabla \cdot \underline{B} = 0 \ [1b]$$

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \ [1c] \qquad \nabla \times \underline{H} = \frac{\partial \underline{D}}{\partial t} \ [1d]$$

- Taking the curl of [1c] / [1d]

$$\nabla \times (\nabla \times \underline{E}) = -\nabla \times \left(\frac{\partial \underline{B}}{\partial t}\right) \qquad \nabla \times (\nabla \times \underline{H}) = \nabla \times \left(\frac{\partial \underline{D}}{\partial t}\right)$$

$$\nabla(\nabla \cdot \underline{E}) - \nabla^2 \underline{E} = -\frac{\partial}{\partial t}(\nabla \times \underline{B}) \qquad \nabla(\nabla \cdot \underline{H}) - \nabla^2 \underline{H} = \frac{\partial}{\partial t}(\nabla \times \underline{D})$$

$$-\nabla\left(\tfrac{1}{\varepsilon}\cancel{\nabla \cdot \underline{D}}^{0}\right) + \nabla^2 \underline{E} = \mu \frac{\partial}{\partial t}(\nabla \times \underline{H}) \qquad -\nabla\left(\tfrac{1}{\mu}\cancel{\nabla \cdot \underline{B}}^{0}\right) + \nabla^2 \underline{H} = -\varepsilon \frac{\partial}{\partial t}(\nabla \times \underline{E})$$

$$\nabla^2 \underline{E} = \mu \frac{\partial^2 \underline{D}}{\partial t^2} \qquad \nabla^2 \underline{H} = \varepsilon \frac{\partial^2 \underline{B}}{\partial t^2}$$

$$\boxed{\nabla^2 \underline{E} = \mu\varepsilon \frac{\partial^2 \underline{E}}{\partial t^2}} \qquad \boxed{\nabla^2 \underline{H} = \mu\varepsilon \frac{\partial^2 \underline{H}}{\partial t^2}}$$

- This is just the wave eqn. $\nabla^2 \psi = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$, where c is the velocity of the wave. Therefore, an electric/magnetic field can propagate as a wave through a dielectric w/ a velocity c.

$$\boxed{c = \frac{1}{\sqrt{\mu\varepsilon}}}$$

which is identical to the velocity of the wave c on a transmission line

- The expression for the velocity of the wave c also shows that the velocity should decrease in a dielectric. The refractive index n is defined st.

$$\boxed{c_{medium} = \frac{c_{air}}{n}}$$

For non-magnetic dielectrics (glass, plastic, water), $\mu_r = 1$, so

$$c_{medium} = \frac{1}{\sqrt{\mu_0 \varepsilon_0 \varepsilon_r}} = \frac{c_{air}}{\sqrt{\varepsilon_r}}$$

Therefore, the refractive index n is related to the rel. permittivity by

$$\boxed{n = \sqrt{\varepsilon_r}}$$

Plane waves in dielectrics

- consider a plane wave – a wave that propagates in a specific direction, which is uniform in the plane ⊥ to the propagation direction
- A valid plane wave soln to the wave eqn. for an $\underline{E}$ field / $\underline{H}$ field is

$$\underline{E} = (E_{0x}\underline{i} + E_{0y}\underline{j} + E_{0z}\underline{k})\exp(j(\omega t - \beta z)) \qquad \underline{H} = (H_{0x}\underline{i} + H_{0y}\underline{j} + H_{0z}\underline{k})\exp(j(\omega t - \beta z))$$

which is a wave travelling in the +ve z direction

- starting from [1a] / [1b],

$$\nabla \cdot \underline{D} = 0, \ \underline{D} = \varepsilon \underline{E} \rightarrow \nabla \cdot \underline{E} = 0 \qquad \nabla \cdot \underline{B} = 0, \ \underline{B} = \mu \underline{H} \rightarrow \nabla \cdot \underline{H} = 0$$

Expressing the divergence of the field in cartesian coords,

$$\nabla \cdot \underline{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \qquad \nabla \cdot \underline{H} = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0$$

For a plane wave which is uniform in the xy-plane, ⊥ to propagation,

$$\frac{\partial E_x}{\partial x} = \frac{\partial E_y}{\partial y} = 0 \qquad \frac{\partial H_x}{\partial x} = \frac{\partial H_y}{\partial y} = 0$$

$$\therefore \frac{\partial E_z}{\partial z} = 0 \qquad \therefore \frac{\partial H_z}{\partial z} = 0$$

Substituting the expression for the plane wave gives

$$-j\beta E_{0z}\exp(j(\omega t - \beta z)) = 0 \qquad -j\beta H_{0z}\exp(j(\omega t - \beta z)) = 0.$$

so either $\beta = 0 \rightarrow$ infinite λ, which is not a wave

or $E_{0z} = 0$ / $H_{0z} = 0$ → no component of field in the dir. of propagation of a plane EM wave.

- Now assume the plane EM wave is polarised s.t. $\underline{E}_0 = E_{0x}\underline{i}$, i.e.,

$$\underline{E} = E_{0x}\underline{i}\exp(j(\omega t - \beta z))$$

To find the magnetic field, we use [1c]

$$\nabla \times \underline{E} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_{0x}\exp(j(\omega t - \beta z)) & 0 & 0 \end{vmatrix} = -\frac{\partial \underline{B}}{\partial t}$$

$$\underline{j}\frac{\partial}{\partial z}[E_{0x}\exp(j(\omega t - \beta z))] - \underline{k}\frac{\partial}{\partial y}\left[\cancel{E_{0x}\exp(j(\omega t - \beta z))}\right]^{0} = -\frac{\partial \underline{B}}{\partial t}$$

$$\underline{j}(j\beta E_{0x}\exp(j(\omega t - \beta z))) = \frac{\partial \underline{B}}{\partial t}$$

$$\therefore \underline{B} = \underline{j}\frac{\beta}{\omega}E_{0x}\exp(j(\omega t - \beta z))$$

$$\underline{H} = \underline{j}H_{0y}\exp(j(\omega t - \beta z)) \quad \text{where } H_{0y} = \frac{\beta}{\omega}\frac{E_{0x}}{\mu}$$

→ $\underline{H}$ is ⊥ to both $\underline{E}$ and the dir. of propagation of the wave.

- The amplitude of the magnetic field H_{oy} can be rewritten in terms of the amplitude of the electric field E_{ox} as follows:

$$H_{oy} = \frac{\beta}{\omega}\frac{E_{ox}}{\mu} = \frac{1}{\sqrt{\varepsilon\mu}}\frac{E_{ox}}{\mu} = \frac{1}{c}\frac{E_{ox}}{\mu} = \sqrt{\frac{\varepsilon}{\mu}}E_{ox}$$

$$\therefore H_{oy} = \sqrt{\frac{\varepsilon_0\varepsilon_r}{\mu_0\mu_r}}E_{ox}$$

- The ratio of the two amplitudes is only dependent on the properties of the medium through which the wave is travelling → this is defined as the impedance η of the medium

$$\boxed{\eta = \frac{|E|}{|H|} = \sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}}}$$

For air, $\eta = 377\Omega$; For water, $\eta = 42\Omega$.

* Note that this is the characteristic impedance of an ideal transmission line w/o geometry factor.

The power in a wave and the Poynting vector

- The Poynting vector $\underline{N}$ is defined as

If a field occupies a volume of space, then there must be PE stored

$$\boxed{\underline{N} = \underline{E}\times\underline{H}}$$

The direction of $\underline{N}$ is the dir. of propagation of the wave

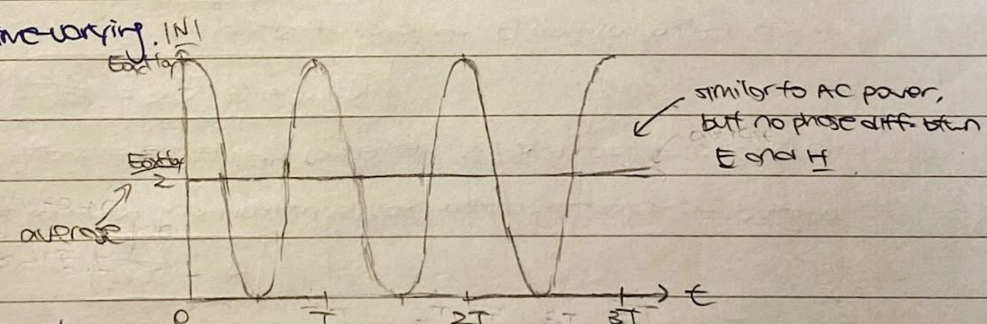
The magnitude of $\underline{N}$ is the instantaneous power density (power per unit area) in the wave, which will be time-varying.

- For a planar wave,

$$\underline{E} = E_{ox}\underline{i}\cos(\omega t-\beta z) \qquad \underline{H} = H_{oy}\underline{j}\cos(\omega t-\beta z)$$

$$\therefore \boxed{|\underline{N}| = |\underline{E}\times\underline{H}| = E_{ox}H_{oy}\cos^2(\omega t-\beta z)}$$

- Plotting $|\underline{N}|$ at a particular pt. in space (say $z=0$), then we find that the instantaneous power is time-varying.



i.e. intensity

The average power per unit area $\overline{|\underline{N}|}$ is only half the peak power, so

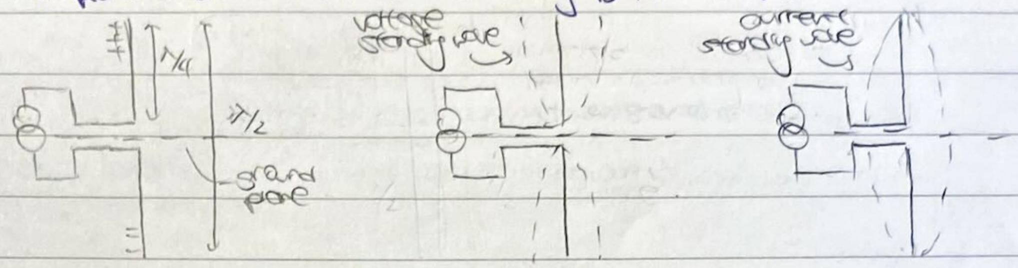
$$\boxed{\overline{|\underline{N}|} = \frac{|E||H|}{2} = \frac{E\times H^*}{2}}$$

- Alternatively, if we define rms values of E and H, then this becomes

$$\boxed{\overline{|\underline{N}|} = E_{rms}H_{rms}}$$

Antennas

- Antennas can both generate a free-space EM wave from an electrical signal and convert an EM wave back into an electrical signal again.
- The type of antenna selected to send/receive a particular signal depends on the wavelength, distance and directionality req.
- One of the simplest antenna designs for broadcasting is the half-dipole/ground plane antenna.



- An AC signal is applied to the antenna, resulting in an oscillating dipole and current. This produces an E-field in the dir. in the antenna; the current produces a H-field $\perp$ to the E-field → this sets up an EM wave that propagate in a dir. $\perp$ to both waves.
- The wave is preferentially transmitted in the ground plane for the half-dipole antenna. (which is ideal for a radio transmitter — we don't want to waste power transmitting to no one)
- A good engineering figure of merit is the gain of the antenna, G defined as

$$\boxed{G = \frac{\text{maximum power density}}{\text{isotropic power density}}}$$

i.e. say a transmitting antenna is outputting a total power P, if the power was radiated isotropically, then at some distance r from the antenna

$$\boxed{|\bar{N}|_{iso} = \frac{P}{4\pi r^2}}$$

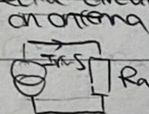
If the antenna has gain G, the power density in the max direction of propagation will be

$$\boxed{|\bar{N}|_{max} = G \cdot |\bar{N}|_{iso} = \frac{G \cdot P}{4\pi r^2}}$$

- The radiation resistance R_a is defined as the resistance that would have to be put in place of the transmitting antenna to dissipate as much power as the antenna radiates.

$$\boxed{R_a = \frac{P}{I_{rms}^2}}$$

effective circuit of an antenna



where I_{rms} is the rms current driving the antenna.

- Another good engineering figure of merit is the effective area A_{eff}, which is the area of the EM wave that would give the power P_{abs} absorbed by the antenna.

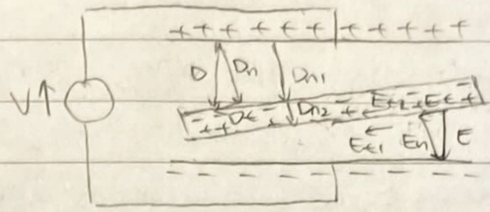
$$\boxed{A_{eff} = \frac{P_{abs}}{|\bar{N}|}}$$

(useful when doing calculations for receiving antennas, which harvest energy from an EM wave and convert this back into an electrical signal again)

Reflection and transmission of electromagnetic waves in dielectrics

Boundary conditions

- Consider a block of material "2" which is btwn the plates of a capacitor that is o/w filled w/ air (material "1")

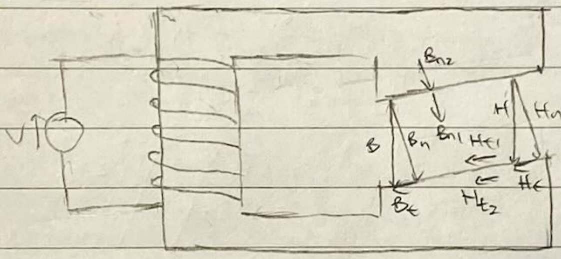


- After resolving both E and D fields into components tangential/normal to the interface, applying Maxwell's eqns, it can be shown that

$$\boxed{D_{n1} = D_{n2}} \qquad \boxed{E_{t1} = E_{t2}}$$

* makes sense since the charge on the surface of the dielectric acts like a line of charge, which has a purely radial electric field → tangential component conserved.

- Consider an iron (material "2") toroid w/ an air (material "1") gap.



- After resolving both H and B fields into components tangential/normal to the interface, applying Maxwell's eqns, it can be shown that

$$\boxed{B_{n1} = B_{n2}} \qquad \boxed{H_{t1} = H_{t2}}$$

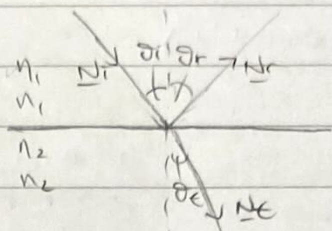
→ normal components of D and B ; tangential components of E and H conserved at interfaces

Polarised plane electromagnetic waves

- In an arbitrary plane EM wave, (e.g. sunlight), the electric field's dir. varies randomly within the plane ⊥ to the propagation direction
- However, the electric field of a linearly polarised wave (e.g. dipole antenna), always points in one direction (and its opposite direction).
- Circular polarisation can be achieved by superposing two linear polarisations at 90° to each other, and w/ a 90° phase shift.
- Any arbitrary EM wave can be resolved into two components w/ their electric fields ⊥ to each other.

Reflection and refraction of plane waves – direction

- consider the general case of a plane EM wave arriving at a flat interface btwn two materials w/ some angle of incidence θ_i



$$\eta = \frac{|E|}{|H|} = \sqrt{\frac{\mu_r\mu_0}{\varepsilon_0\varepsilon_r}} \quad (\eta \propto \tfrac{1}{n})$$

$$n = \sqrt{\varepsilon_r}$$

Assume both materials 1 and 2 to be infinite in thickness

into the page? thickness

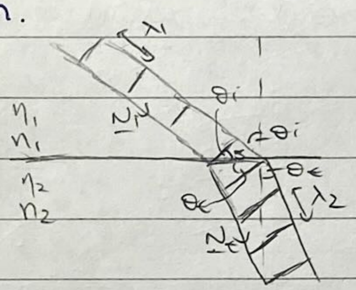
- The law of reflection states that

$$\boxed{\theta_i = \theta_r}$$

- Snell's law of reflection states that

$$\boxed{n_1 \sin\theta_i = n_2 \sin\theta_t}$$

which can be proved using geometry and the fact the speed of wave changes due to diff. refractive indices n.



To preserve the planes of peaks/troughs, it must take the same time for the wave to travel a distance λ_1 in medium 1 and a distance λ_2 in medium 2, so

$$s = \frac{\lambda_1}{\sin\theta_i} = \frac{\lambda_2}{\sin\theta_t}$$

Frequency is the same in both media

$$f\lambda_1 = \frac{c}{n_1} \qquad f\lambda_2 = \frac{c}{n_2}$$

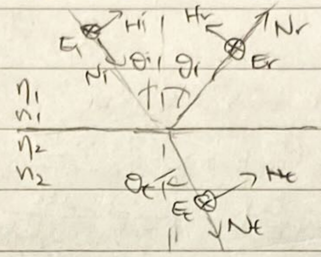
$$\therefore n_1 \sin\theta_i = n_2 \sin\theta_t$$

- Note that if a wave travels from one material into one w/ a lower refractive index n, (i.e. $n_2 < n_1$), then if the angle of incidence θ_i is greater than the critical angle θ_c, there can be no transmission, and the wave is totally reflected.

$$\boxed{\theta_c = \sin^{-1}(n_2/n_1)}$$

Reflection and refraction of plane waves – proportion

- Since we can resolve any arbitrary wave into two ⊥ components, we just have to consider two scenarios – E field ⊥ and // to the plane of incidence (they have diff. BC)
- Consider the case where the E field is ⊥ to the plane of incidence (s-polarisation)



- Conservation of the tangential components of $\underline{E}$ and $\underline{H}$ gives.

$$E_{t1} = E_{t2} \qquad H_{t1} = H_{t2}$$

Applying the geometry of the system,

$$E_i + E_r = E_t \qquad H_i\cos\theta_i - H_r\cos\theta_i = H_t\cos\theta_t$$

Converting expressions in terms of H into E using the characteristic impedance η,

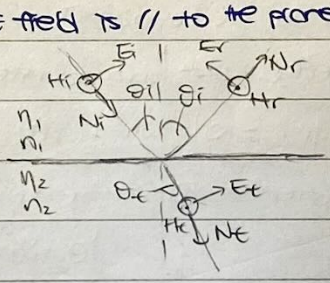
$$H_i = \frac{E_i}{\eta_1} \qquad H_r = \frac{E_r}{\eta_1} \qquad H_t = \frac{E_t}{\eta_2}$$

Substituting and rearranging, we get the Fresnel equations. (⊥)

$$\boxed{\left(\frac{E_t}{E_i}\right)_\perp = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}} \quad [\times E_r]$$

$$\boxed{\left(\frac{E_r}{E_i}\right)_\perp = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}} \quad [\times E_t]$$

- Consider the case where the E field is // to the plane of incidence. (p-polarisation)



- Conservation of the tangential components of $\underline{E}$ and $\underline{H}$ give

$$E_{t1} = E_{t2} \qquad H_{t1} = H_{t2}$$

Applying the geometry of the system,

$$E_i\cos\theta_i - E_r\cos\theta_i = E_t\cos\theta_t \qquad H_i + H_r = H_t$$

Converting expressions in terms of H into E using the characteristic impedance η,

$$H_i = \frac{E_i}{\eta_1} \qquad H_r = \frac{E_r}{\eta_1} \qquad H_t = \frac{E_t}{\eta_2}$$

Substituting and rearranging, we get the Fresnel equations (//)

$$\boxed{\left(\frac{E_t}{E_i}\right)_{//} = \frac{2\eta_2\cos\theta_i}{\eta_1\cos\theta_i + \eta_2\cos\theta_t}} \quad [\times E_r]$$

$$\boxed{\left(\frac{E_r}{E_i}\right)_{//} = \frac{\eta_1\cos\theta_i - \eta_2\cos\theta_t}{\eta_1\cos\theta_i + \eta_2\cos\theta_t}} \quad [\times E_t]$$

Alternate form of the Fresnel equations

- Instead of using the characteristic impedance η, we can use the refractive index n.

$$\eta \propto \frac{1}{\sqrt{\varepsilon_r}}, \quad n = \sqrt{\varepsilon_r} \quad \rightarrow \quad \eta \propto \frac{1}{n} \quad \rightarrow \quad \frac{\eta_1}{\eta_2} = \frac{n_2}{n_1}$$

- Applying Snell's law, and letting $\theta_i = \theta_1$; $\theta_t = \theta_2$, we have

$$\frac{n_2}{n_1} = \frac{\sin\theta_1}{\sin\theta_2}.$$

- Rearranging the Fresnel eqns, we have

$$t_\perp = \left(\frac{E_t}{E_i}\right)_\perp = \frac{2\eta_2\cos\theta_1}{\eta_2\cos\theta_1 + \eta_1\cos\theta_2} = \frac{2n_1\cos\theta_1}{n_1\cos\theta_1 + n_2\cos\theta_2} = \frac{2\sin\theta_2\cos\theta_1}{\sin(\theta_1+\theta_2)}$$

$$t_{/\!/} = \left(\frac{E_t}{E_i}\right)_{/\!/} = \frac{2\eta_2\cos\theta_1}{\eta_1\cos\theta_1 + \eta_2\cos\theta_2} = \frac{2n_1\cos\theta_1}{n_2\cos\theta_1 + n_1\cos\theta_2} = \frac{2\sin\theta_2\cos\theta_1}{\sin(\theta_1+\theta_2)\cos(\theta_1-\theta_2)}$$

$$r_\perp = \left(\frac{E_r}{E_i}\right)_\perp = \frac{\eta_2\cos\theta_1 - \eta_1\cos\theta_2}{\eta_2\cos\theta_1 + \eta_1\cos\theta_2} = \frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2} = -\frac{\sin(\theta_1-\theta_2)}{\sin(\theta_1+\theta_2)}$$

$$r_{/\!/} = \left(\frac{E_r}{E_i}\right)_{/\!/} = \underbrace{\frac{\eta_1\cos\theta_1 - \eta_2\cos\theta_2}{\eta_1\cos\theta_1 + \eta_2\cos\theta_2}}_{\text{using characteristic impedance } \eta} = \underbrace{\frac{n_2\cos\theta_1 - n_1\cos\theta_2}{n_2\cos\theta_1 + n_1\cos\theta_2}}_{\text{using refractive index } n} = \underbrace{\frac{\tan(\theta_1-\theta_2)}{\tan(\theta_1+\theta_2)}}_{\text{combined w/ Snell's law}}$$

Polarisation by reflection – the Brewster angle.

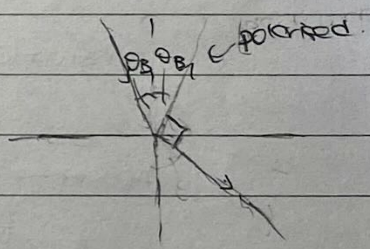
- When unpolarised light is incident at the Brewster angle θ_B, the light that is reflected from the surface is perfectly polarised.
- When light w/ a particular polarisation is incident at the Brewster angle θ_B, the light is transmitted perfectly through the surface, w/ no reflection.
- At the Brewster angle θ_B, the component of the E field // to the plane of incidence is not reflected → reflected wave plane polarised w/ E field ⊥ to the plane of incidence, i.e.

$$\boxed{r_{/\!/} = 0} \quad \text{when} \quad \theta_1 = \theta_B$$

By inspection of the alternate form of the Fresnel eqn, we can see that this occurs when

$$\boxed{\theta_1 + \theta_2 = 90^\circ}$$

Starting w/ Snell's law of refraction, we get.



$$n_1\sin\theta_1 = n_2\sin\theta_2$$

$$n_1\sin\theta_1 = n_2\sin(90^\circ - \theta_1)$$

$$\tan\theta_1 = \frac{n_1}{n_2}$$

$$\therefore \boxed{\theta_B = \arctan\left(\frac{n_1}{n_2}\right)}$$

Anti-reflection coatings

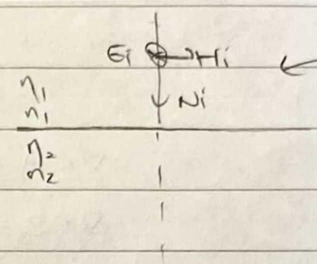
- consider a special case where the incident EM wave has $\theta_I = 0°$. The Fresnel eqns reduce to

$$t_\perp = \frac{2\eta_2}{\eta_1+\eta_2} \qquad r_\perp = \frac{\eta_2-\eta_1}{\eta_1+\eta_2}$$

$$t_{//} = \frac{2\eta_2}{\eta_1+\eta_2} \qquad r_{//} = \frac{\eta_1-\eta_2}{\eta_1+\eta_2}$$

As for the transmission line, an incident wave at 0° will be reflected when there is a change in characteristic impedance η (i.e. $\eta_1 \neq \eta_2$).

- consider the effective characteristic impedance some distance x from an interface $\eta(x)$.



in this derivation, we use the case the E field is ⊥ to the plane of incidence.

In the case above (E field ⊥ to the plane of incidence), $\eta(x)$ is given by

$$\eta(x) = \frac{\bar{E}_i e^{j(\omega t-\beta x)} + \bar{E}_r e^{j(\omega t+\beta x)}}{\bar{H}_i e^{j(\omega t-\beta x)} - \bar{H}_r e^{j(\omega t+\beta x)}}$$

* Note the change in sign for $\bar{H}_r$ in this case as the direction is changed on reflection. (In the other polarisation case where E field // to plane of incidence, $\bar{E}_r$ would change sign).

- Using the characteristic impedance to express H in terms of E. $\eta_1 = \frac{E_i}{H_i}$.

$$\eta(x) = \frac{\bar{E}_i e^{-j\beta x} + \bar{E}_r e^{j\beta x}}{\bar{E}_i/\eta_1\, e^{-j\beta x} - \bar{E}_r/\eta_1\, e^{j\beta x}}$$

$$= \eta_1 \frac{e^{-j\beta x} + \rho e^{j\beta x}}{e^{-j\beta x} - \rho e^{j\beta x}}$$

where $\rho = \frac{\bar{E}_r}{\bar{E}_i}$ is defined as the electric field reflection coefficient. $\rho = \frac{\bar{E}_r}{\bar{E}_i} = r_\perp = \frac{\eta_2-\eta_1}{\eta_2+\eta_1}$

so $\frac{1-\rho}{1+\rho} = \frac{\eta_1}{\eta_2}$

- Performing quarter wave matching by setting $x = -\lambda/4$ ($\beta x = -\pi/2$), we have

$$\eta(x=-\lambda/4) = \eta_1 \frac{1-\rho}{1+\rho} = \eta_1\left(\frac{\eta_1}{\eta_2}\right)$$

- we can therefore make an anti-reflection coating whose characteristic impedance is

$$\eta_0 = \sqrt{\eta_1\eta_2}$$

* A disadvantage of this approach is that it is tuned to one frequency (ok for lasers)

- An alternative approach is to put nanostructured material on the surface where the physical size of the nanostructures is much less than λ. → EM wave sees a spatially averaged structure
- The nanostructure then appears to modulate the characteristic impedance smoothly as a function of depth into the layer. → reflections minimised across a broad range of wavelengths.

Electromagnetic waves in conductors

Electromagnetic waves in conducting media.

- In a conducting media, we can no longer say $\underline{J}=0$ as in the dielectric case, as there can clearly be a current.
- However, we will assume we cannot get a net charge in any volume of space, so $\rho=0$.
- The Maxwell eqns in a conducting medium become

$$\nabla\cdot\underline{D}=0 \quad [2a] \qquad \nabla\cdot\underline{B}=0 \quad [2b]$$

$$\nabla\times\underline{E}=-\frac{\partial \underline{B}}{\partial t} \quad [2c] \qquad \nabla\times\underline{H}=\underline{J}+\frac{\partial \underline{D}}{\partial t} \quad [2d]$$

- Starting from [2d],

$$\nabla\times\underline{H}=\underline{J}+\frac{\partial \underline{D}}{\partial t}$$

$$\nabla\times\underline{B}=\mu\underline{J}+\mu\varepsilon\frac{\partial \underline{E}}{\partial t}.$$

Applying Ohm's law, $\boxed{\underline{J}=\sigma\underline{E}}$, we get

$$\nabla\times\underline{B}=\mu\sigma\underline{E}+\mu\varepsilon\frac{\partial \underline{E}}{\partial t}$$

Taking the curl of [2c],

$$\nabla\times(\nabla\times\underline{E})=\nabla\times\left(-\frac{\partial \underline{B}}{\partial t}\right)$$

$$\nabla(\nabla\cdot\underline{E})-\nabla^2\underline{E}=-\frac{\partial}{\partial t}(\nabla\times\underline{B})$$

$$-\nabla\big(\tfrac{1}{\varepsilon}\cancelto{0}{\nabla\cdot\underline{D}}\big)+\nabla^2\underline{E}=\frac{\partial}{\partial t}\left(\mu\sigma\underline{E}+\mu\varepsilon\frac{\partial \underline{E}}{\partial t}\right)$$

$$\boxed{\nabla^2\underline{E}=\mu\sigma\frac{\partial \underline{E}}{\partial t}+\mu\varepsilon\frac{\partial^2 \underline{E}}{\partial t^2}}$$

- Starting from [2c],

$$\nabla\times\underline{E}=-\frac{\partial \underline{B}}{\partial t}$$

$$\nabla\times\underline{D}=-\mu\varepsilon\frac{\partial \underline{H}}{\partial t}$$

Taking the curl of [2d],

$$\nabla\times(\nabla\times\underline{H})=\nabla\times\underline{J}+\nabla\times\left(\frac{\partial \underline{D}}{\partial t}\right)$$

$$\nabla(\nabla\cdot\underline{H})-\nabla^2\underline{H}=\nabla\times\underline{J}+\frac{\partial}{\partial t}(\nabla\times\underline{D})$$

$$-\nabla\big(\tfrac{1}{\mu}\cancelto{0}{\nabla\cdot\underline{B}}\big)+\nabla^2\underline{H}=-\nabla\times\underline{J}-\frac{\partial}{\partial t}(\nabla\times\underline{D})$$

Applying Ohm's law, $\boxed{\underline{J}=\sigma\underline{E}}$, we get

$$\nabla^2\underline{H}=-\sigma\nabla\times\underline{E}-\frac{\partial}{\partial t}(\nabla\times\underline{D})$$

$$=-\frac{1}{\varepsilon}\sigma\nabla\times\underline{D}-\frac{\partial}{\partial t}(\nabla\times\underline{D})$$

$$=\mu\sigma\frac{\partial \underline{H}}{\partial t}+\frac{\partial}{\partial t}\left(\mu\varepsilon\frac{\partial \underline{H}}{\partial t}\right)$$

$$\boxed{\nabla^2\underline{H}=\mu\sigma\frac{\partial \underline{H}}{\partial t}+\mu\varepsilon\frac{\partial^2 \underline{H}}{\partial t^2}}$$

- This is very similar to the wave eqn. for EM waves in dielectrics, but there is an extra term $\mu\sigma\frac{\partial \underline{E}}{\partial t}$ / $\mu\sigma\frac{\partial \underline{H}}{\partial t}$.

Plane waves in conducting media

- A valid plane wave sol'n to the modified wave eqn. for an $\underline{E}$ field / $\underline{H}$ field is

$$\underline{E} = E_{0x}\,\underline{i}\,\exp(j\omega t)\exp(-\gamma z) \qquad \underline{H} = H_{0x}\,\underline{i}\,\exp(j\omega t)\exp(-\gamma z)$$

which is a wave travelling in the +ve z direction

- γ is the propagation constant, and is a complex no. of the form

$$\boxed{\gamma = \alpha + j\beta}$$

- Expanding the propagation constant, we get

$$\underline{E} = \underbrace{E_{0x}\,\underline{i}\,\exp(j(\omega t - \beta z))}_{\text{forward wave}}\underbrace{\exp(-\alpha z)}_{\text{attenuation term}} \qquad \underline{H} = \underbrace{H_{0x}\,\underline{i}\,\exp(j(\omega t - \beta z))}_{\text{forward wave}}\underbrace{\exp(-\alpha z)}_{\text{attenuation term}}$$

- We should expect the wave to attenuate as it travels through the conductor as it must be inducing a current flow → dissipates energy due to finite resistivity → EM wave attenuated.

The skin effect

- Substituting the plane wave sol'n for an $\underline{E}$ field into the modified wave eqn, we get

$$(\alpha + j\beta)^2\,\underline{E} = j\omega\mu\sigma\,\underline{E} - \mu\varepsilon\omega^2\,\underline{E}$$

$$\therefore \boxed{\gamma = \alpha + j\beta = \sqrt{j\mu\omega(\sigma + j\omega\varepsilon)}}$$

This form shows that the magnitude of σ rel. to $\omega\varepsilon$ determines whether the attenuation effect can be neglected or not.

- If the material is highly resistive, $\sigma \ll \omega\varepsilon$,

$$\gamma = \sqrt{j\omega j\mu\varepsilon} = j\omega\sqrt{\mu\varepsilon} \quad \rightarrow \quad \alpha = 0,\ \beta = \omega\sqrt{\mu\varepsilon}.$$

i.e. the wave is not attenuated and travels at $c = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}$ as in the dielectric case.

- If the material is highly conductive, $\sigma \gg \omega\varepsilon$,

$$\gamma = \sqrt{j\mu\omega\sigma} = \sqrt{\frac{\mu\omega\sigma}{2}}(1+j) \quad \rightarrow \quad \alpha = \beta = \sqrt{\frac{\mu\omega\sigma}{2}}.$$

The attenuation term in the eqn. for the EM wave is $\exp(-\alpha z)$, so the wave is attenuated by a factor of e over a distance $\frac{1}{\alpha}$.

- We define this distance to be the skin depth δ,

$$\boxed{\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\mu\omega\sigma}}}$$

which is a measure of the distance that an EM wave can penetrate into a conductive medium.

- We would expect the skin depth δ to decrease w/ increasing frequency ω as a greater oscillation frequency change would result in greater energy loss → unable to penetrate in.

Skin effect on transmission lines

- When we drive an AC current down a wire of circular cross section in a coaxial cable, an EM wave exists around the wire. If δ is large compared to the radius of the wire, there is current through the entire cross section. → $\boxed{R = \rho \frac{l}{A}}$.
- However, if δ is less than the radius, then the EM wave only penetrates a small distance into the wire → conduction is only limited to the surface "skin" → $\boxed{R = \rho \frac{l}{\pi[R^2 - (R-\delta)^2]}}$
- The skin effect means that the effective resistance of the wire increases w/ frequency. This results in attenuation in the wire → signal amplitude decays over shorter distance at high frequencies (so we use optical fibre over copper wires for high speed data transmission over long distances)

Intrinsic impedance of a conductive medium

- Assume the plane EM wave is polarised s.t. $\underline{E}_0 = E_{0x}\underline{i}$, i.e.

$$\underline{E} = E_{0x}\underline{i}\exp(j(\omega t - \beta z))\exp(-\alpha z)$$

To find the magnetic field, we use [2c],

$$\nabla \times \underline{E} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_{0x}\exp(j(\omega t-\beta z))\exp(-\alpha z) & 0 & 0 \end{vmatrix} = -\frac{\partial \underline{B}}{\partial t}$$

$$\underline{j}\frac{\partial}{\partial z}[E_{0x}\exp(j(\omega t-\beta z))\exp(-\alpha z)] - \underline{k}\cancel{\frac{\partial}{\partial y}[E_{0x}\exp(j(\omega t-\beta z))\exp(-\alpha z)]} = -\frac{\partial \underline{B}}{\partial t}$$

$$\underline{j}\left[(\alpha + j\beta)E_{0x}\exp(j(\omega t-\beta z))\exp(-\alpha z)\right] = \frac{\partial \underline{B}}{\partial t}$$

$$\therefore \underline{B} = \underline{j}\frac{\alpha + j\beta}{j\omega}E_{0x}\exp(j(\omega t-\beta z))\exp(-\alpha z)$$

$$\underline{H} = \underline{j}H_{0y}\exp(j(\omega t-\beta z))\exp(-\alpha z), \text{ where } H_{0y} = \frac{\alpha + j\beta}{j\omega}\frac{E_{0x}}{\mu}$$

→ $\underline{H}$ is ⊥ to both $\underline{E}$ and the dir. of propagation of the wave.

- The amplitude of the magnetic field H_{0y} can be rewritten in terms of the amplitude of the electric field E_{0x} as follows:

$$H_{0y} = \frac{\alpha + j\beta}{j\omega}\frac{E_{0x}}{\mu} = \frac{\gamma}{j\omega}\frac{E_{0x}}{\mu} = \sqrt{\frac{\sigma + j\omega\varepsilon}{j\omega\mu}}E_{0x}$$

- The ratio of the two amplitudes is only dependent on the properties of the medium through which the wave is travelling → this is defined as the impedance of the medium

$$\boxed{\eta = \left|\frac{\underline{E}}{\underline{H}}\right| = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}}$$

- Note that this is real for dielectrics and complex for conductors.
- For conductors, there is a phase diff. between $\underline{E}$ and $\underline{H}$ fields. → peaks occur at diff. times.
- For a highly conductive medium, $\sigma \gg \omega\varepsilon$, we have

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{2\sigma}}(1+j)$$

so the electric field $\underline{E}$ leads the magnetic field $\underline{H}$ by $\pi/4$.

- The characteristic impedance η for a typical metal is much smaller than that for dielectrics → wave dominated by the magnetic field.

Waves at conducting interfaces.

based on conservation of D_n, B_n and E_t, H_t at interfaces.

- The Fresnel eqns derived for EM waves in dielectrics are still valid for conductors, but now the characteristic impedance η is complex (instead of real).
- In practice, the most common situation we are interested in is when a wave in a dielectric medium "1" is incident on a conducting medium "2". Consider the case $\theta_i = 0°$.

$$t_\perp = t_{\parallel} = \frac{2\eta_2}{\eta_1 + \eta_2}$$

- For a typical metal, $|\eta_2| \ll |\eta_1|$, so we can say that almost all of the wave is reflected. (act like a perfect reflector).
- Even less highly conducting media (eg water) will be highly reactive