Dimensional analysis

Units and dimensions

Units

- Base units are defined as standards for the purpose of comparisons and measurement. The SI base units are:

Quantity	Name	Symbol
Mass	Kilogram	kg
Length	metre	m
Time	second	s
Electric current	Ampere	A
Absolute temperature	Kelvin	K
Luminous intensity	candela	cd
Amount of substance	Kilomole	kmol

- Derived units are formed from the base units, either by defn (ms^{-1}) or a physical law ($kgms^{-2}$).

Scales of units

- Within a system of measuring units, different scales are often used.
  - ↳ sometimes the relationship btwn the scales are a simple shift ($T = \theta + 273.15$)
  - ↳ In most cases a scale factor relates the scales (1ft = 12in, 1 mile = 1760yd = 5280ft)

Conversion of units

- Different systems of units have different defns. By comparing the defns in different systems of units, we can convert between the systems.
- eg: How fast is 14mph in ms^{-1}? (1 mile = 5280 ft, 1ft = 0.3048m)

$$14 \text{ mph} = 14 \frac{\text{mile}}{\text{hr}} \times \frac{1\text{hr}}{3600\text{s}} \times \frac{5280\text{ft}}{1\text{mile}} \times \frac{0.3048\text{m}}{1\text{ft}} = 6.26\,ms^{-1}.$$

Dimensions

- The dimensions of a quantity refers to the nature of the quantity, not the units it is measured in nor its value.
- Dimensions are conveniently chosen to correspond to the base units being used.

Quantity	Typical unit (SI)	Dimensions
Mass	kg	M
Length	m	L
Time	s	T
Electric current	A	I
Temperature	T	Θ

upper case

- For dimensional analysis, each dimension must be independent and we must have sufficient dimensions to define the problems we expect to encounter.
- Similar to derived units, we use defn or physical laws to find the dimensions of a quantity
  - ↳ By def'n : velocity , LT^{-1}
  - ↳ Physical law : Pressure , $ML^{-1}T^{-2}$ $\left(\frac{MLT^{-2}}{L^2}\right)$
- The choice of base units (and therefore dimensions) is arbitrary → sometimes it is more convenient to use to F-L-T system
  - ↳ By defn : Acceleration . LT^{-2}
  - ↳ Physical law : Mass . $FL^{-1}T^{2}$

Specification of dimensional quantities

- To completely specify any quantity that has a dimension, it is necessary to give 2 pieces of information — the numerical value and the units in which it is measured. We also leave a space between the value and the unit (e.g. $t = 2.5$ s)
- Often we refer to a dimensional quantity as an algebraic symbol, e.g. q, but occasionally we will have to formally remember that $q = q_{value}\, q_{unit}$ and has dimension $q_{dimension}$

The principle of dimensional consistency (homogeneity)

★ A complete statement of a physical law is independent of the system of measurement

★ Dimensional consistency is much stronger than just the dimensions matching. Dimensional consistency implies that the eqn will "look the same" even if we use a different system of units.

- e.g: Consider the periodic time of a simple pendulum.

Quantity	Symbol (SI)	Units (SI)	Symbol	Units	Dimensions
Periodic time	t	s	t'	hb	T
Length	l	m	l'	al	L
Gravitational accel'n	g	ms^{-2}	g'	$al\,hb^{-2}$	LT^{-2}

Conversion factors : $60s = 90hb \rightarrow \frac{60s}{90hb} = \frac{2s}{3hb}$

not true constants as they depend on units

$0.8m = 1\,al \rightarrow \frac{0.8m}{1al} = \frac{4m}{5al}$

also, we check the dimensions, LHS : T

RHS: $\left(\frac{L}{LT^{-2}}\right)^{1/2} = T$

Using SI units, periodic time is given by $t_{value} = 2\pi\sqrt{\frac{l_{value}}{g_{value}}}$ [1]

where $t = t_{value}$ s , $l = l_{value}$ m , $g = g_{value}$ ms^{-2}.

$\frac{9}{5} = \frac{4}{5}\cdot\left(\frac{3}{2}\right)^2$

Using the conversion factors, $t_{value} = \left[\frac{2}{3}\right] t'_{value}$; $l_{value} = \left[\frac{4}{5}\right] l'_{value}$; $g_{value} = \left[\frac{9}{5}\right] g'_{value}$

From eqn [1], $\frac{2}{3} t'_{value} = 2\pi\sqrt{\frac{4/5\, l'_{value}}{9/5\, g'_{value}}} \rightarrow t'_{value} = 2\pi\sqrt{\frac{l'_{value}}{g'_{value}}}$ [2]

∴ We can just say the periodic time for a simple pendulum is $t = 2\pi\sqrt{\frac{l}{g}}$.

true constant

eqn [1] and eqn [2] "look the same"

Non-dimensional relationships

Non-dimensional quantities / groups

- The following have no dimensions. They are non-dimensional quantities.
  - ↳ Pure numbers (e.g. 13.5)
  - ↳ True constants (e.g π, e)
  - ↳ Mathematical functions (e.g. sine, cosine, logarithms, exponentials).

* Note, since $\sin\theta = \theta - \frac{\theta^3}{3!} + \ldots$, both its argument θ and its value $\sin\theta$ must be non-dimensional.

- Ratio of various physical quantities can also be non-dimensional. They are non-dimensional groups
  - ↳ e.g: $\frac{l}{w}$ where l = length, w = width
  - $\frac{vt}{l}$ where v = speed, t = time, l = length.

* From dimensional analysis, the value of a non-dimensional quantity or group is independent of the system of units

Non-dimensional equations

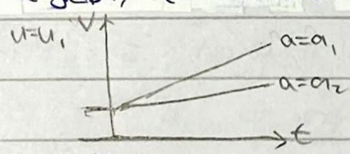
* From the principle of Dimensional Consistency, in a complete statement of a physical law, it is possible to rearrange the terms so that all groups/ quantities are non-dimensional.

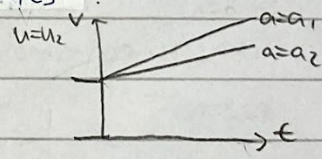
- e.g: $s = ut + \frac{1}{2}at^2 \rightarrow 1 = \frac{ut}{s} + \frac{1}{2}\frac{at^2}{s}$ — the non-dimensional eqn. is equivalent to the original eqn but just contains 2 non-dim. grps.

Non-dimensional graphs

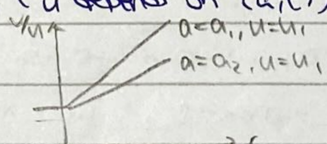
- Often results cannot be presented as algebraic expressions, instead they must be shown in graphical form.
- There are usually many parameters in a relationship → we need many graphs.
- We can reduce the no. of parameters by expressing the relationship in non-dimensional form.
- e.g: we know final velocity v depends on initial velocity u, acceleration a and time t.

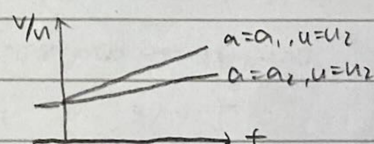
plotting this on a graph, (v depends on $\{u, a, t\}$).



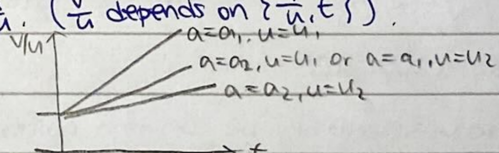


If we plot using $\frac{v}{u}$, ($\frac{v}{u}$ depends on $\{a, t\}$)

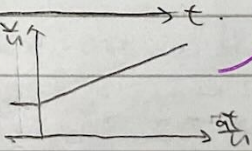




We can combine the plots by using $\frac{a}{u}$. ($\frac{v}{u}$ depends on $\{\frac{a}{u}, t\}$).



Now plot using $\frac{at}{u}$ ($\frac{v}{u}$ depends on $\{\frac{at}{u}\}$)



this universal graph contains all driving conditions.

Forming non-dimensional groups.

- We can use Dimensional Analysis to convert complex engineering problems into more manageable non-dimensional relationships using non-dimensional groups
- There are 3 methods in forming non-dimensional groups.

① Elimination method

- To produce non-dimensional groups, we rearrange the variables and discard those that are no longer needed so the physical dimensions are eliminated from the relationship.
- e.g: Assume we only know that v (dependent variable) is dependent on u, a, t (independent variables)

we can write this as	v depends on	$\{u, a, t\}$
Considering dimensions	LT^{-1}	$LT^{-1}\ LT^{-2}\ T$
Rearrange v & u into $\frac{v}{u}$ & u:	$\frac{v}{u}$ depends on	$\{u, a, t\}$
Considering dimensions	–	$LT^{-1}\ LT^{-2}\ T$
Rearrange u & a into u & $\frac{a}{u}$:	$\frac{v}{u}$ depends on	$\{\cancel{u}, \frac{a}{u}, t\}$
Considering dimensions	–	$\cancel{LT^{-1}}\ T^{-1}\ T$
Rearranging $\frac{a}{u}$ & t into $\frac{at}{u}$ & t:	$\frac{v}{u}$ depends on	$\{\frac{at}{u}, \cancel{t}\}$
Considering dimensions	–	$- \ \cancel{T}$

* we can only discard single quantities (X groups)

only one term u explicitly involves the dim. of length. → u can be discarded.

only one term t explicitly involves the dim. of time → t can be discarded.

→ we find that the non-dimensional relationship is between the 2 non-dimensional groups $\frac{v}{u}$ and $\frac{at}{u}$.

* What we have NOT obtained is the exact form of the non-dimensional relationship between $\frac{v}{u}$ and $\frac{at}{u}$. However since $\frac{v}{u}$ only depends on $\frac{at}{u}$, it can be plotted on a single curve on a single graph. [We can find this curve from experiments] (we only have 2 non-dim. grps.)

② Indicial method.

- This method is best suited to problems involving 1 non-dimensional group (X complex problems) (no. of indices (variables) greater than the no. of dim.)
- e.g: Assume we only know that t is dependent on h, g.

we assume the non-dimensional relationship is of the form

Relation	t	prop. to	$h^\alpha g^\beta$
Dimension	T	prop. to	$L^\alpha (LT^{-2})^\beta$

Compare the dimensions on each side of the eqn:

$T: \ 1 = -2\beta \quad ; \quad L: \ 0 = \alpha + \beta \quad \rightarrow \quad \alpha = 1/2, \ \beta = -1/2$

$\therefore t \propto \sqrt{\frac{h}{g}} \quad \rightarrow \quad t\sqrt{\frac{g}{h}}$ is constant.

③ By inspection.

- From experience, we can find non-dimensional groups by inspection.
- e.g: Assume we only know that t is dependent on h, g.

To eliminate dimension L, we use ratio g/h (dimensions T^{-2})

So it is clear that gt^2/h is a non-dimensional group.

Equivalence of apparently different results

- Provided there is at least one independent nondimensional group, there are an infinite no. of possible combinations on independent/dependent nondimensional groups. (i.e. non-dimensional groups are not unique.)
- eg: Assume we only know d is dependent on l, w, F
  Using elimination, we can find that $\frac{d}{l}$ depends on $\{\frac{F}{wl}\}$.
  If we are interested in l instead, we say l is dependent on d, w, F.
  Using elimination, we can find that $\frac{l}{d}$ depends on $\{\frac{F}{wd}\}$.
  This appears to be a different relationship but it is equivalent to the original one:
  Quantity $\left(\frac{l}{d}\right)_{new} = 1/(d/l)_{old}$
  Quantity $\left(\frac{F}{wd}\right)_{new} = \left(\frac{F}{wl}\right)_{old} \times \left(\frac{l}{d}\right)_{old}$.
  → The new independent group is a combination of old independent/dependent groups.

Buckingham's Pi theorem

- Buckingham's Pi theorem states that if we have N variables and M dimensions, we should expect $(N-M)$ or more non-dimensional groups.

* We count all the variables (independent + dependent).

- Let us assume we have N variables, where the statement

Quantity p_1 depends on $\{p_2, p_3 \dots p_N\}$

expresses a complete relationship between the N variables $p_1, p_2 \dots p_N$ which req. M dimensions.
The principle of Dimensional Consistency says any complete statement of a physical law can be rearranged s.t. all terms are non-dimensional.

Quantity Π_1 depends on $\{\Pi_2, \Pi_3, \dots \Pi_K\}$.

where there are K non-dimensional groups $\Pi_1, \Pi_2 \dots \Pi_K$

- In the production of producing non-dimensional groups, if a variable is discarded each time a dimension is eliminated, $K = N - M$.
- If discarding a variable "costs" more than eliminating a dimension, $K > N-M$.
- In general, we have Buckingham's Pi theorem.

$$\boxed{K \geq N-M}$$

Redundant or missing parameters

- sometimes, we have only 1 variable that includes a particular dimension → no combination of variables that include this variable can form a non-dimensional group.
- For simpler problems, the variable is redundant and can simply be discarded.
- For more complex problems, another variable (w/ appropriate dimensions) may be needed to obtain a complete statement of the physical law. → rethink approach

Geometric similarity and dynamic similarity

cannot tell them apart from a photograph

- 2 objects are geometrically similar when one object is an exact scale model of each other (i.e. ratios of lengths are fixed).
- This implies we only need 1 length scale among the independent variables.
- In cases where geometric similarity is true, only 1 non-dimensional group is req. to characterize the ratio of the geometric size of "full size" to "test model". (other non-dimensional groups may be req. to characterize other physical aspects)
- 2 situations are dynamically similar when all the independent non-dimensional groups are unchanged between the 2 situations.
- As a result, the numerical values of the dependent non-dimensional groups will also be unchanged between the 2 situations.

*The conditions for dynamic similarity must be satisfied when we build and test models.

Applications of dimensional analysis

Non-dimensional presentation of experimental data

- Experimental data is often expensive and time consuming to conform → we can use Dimensional Analysis to present experimental data in non-dimensional form so it is applicable to a wider range of experiments.

if we only have 2 non-dim. grps, we can plot them on a single curve on a single graph.

- Plotting a non-dimensional graph using non-dimensional groups, we have a universal curve that applies to another experiment w/ the parameters tweaked.
- we can find the non-dimentional relationships from experimental data (a plot of the non-dimensional groups)

Model testing and finding nondimensional relationships.

- eg: A manufacturer of a ship's propellers has been commissioned to produce a propeller for a sea ferry. to cruise at a speed of $12ms^{-1}$, the ferry will need a thrust of 25kN. The manufacturer produced a new but untested shape for the propeller. the propeller for the ferry should have a diameter of 0.6m and rotate at 800 rpm. In order to check the design, a 1/10th scale model is to be built and tested. The model propeller (0.06m diameter) will rotate at 8000 rpm in fresh water. We need to know (a) the speed of the model ferry

  (b) the thrust the model ferry should produce.

From the Q, we know the model and fullsize ferry/propeller are geometrically similar. The variables we shall use are.

Quantity	Symbol	Units	Dimension
Thrust	f	$kgms^{-2}$	MLT^{-2}
Diameter	d	m	L
Angular velocity	ω	$rads^{-1}$	T^{-1}
Ferry speed	v	ms^{-1}	LT^{-1}
Fluid density	ρ	kgm^{-3}	ML^{-3}

if the propellers were not geometrically similar, we would have more variables (now we only have diameter as lengthscale)

Say we are first interested in the thrust f. BPT: >5-3 non-dim. grps.

Quantity	f	depends on	$\{d, \omega, v, \rho\}$.
Dimensions	MLT^{-2}		$L\ T^{-1}\ LT^{-1}\ ML^{-3}$
Quantity	f/ρ	depends on	$\{d, \omega, v, \cancel{\rho}\}$
Dimensions	L^4T^{-2}		$L\ T^{-1}\ LT^{-1}\ \cancel{ML^{-3}}$
Quantity	$\frac{f}{\rho d^4}$	depends on	$\{\cancel{d}, \omega, \frac{v}{d}\}$
Dimensions	T^{-2}		$\cancel{L}\ T^{-1}\ T^{-1}$
Quantity	$\frac{f}{\rho d^4\omega^2}$	depends on	$\{\cancel{\omega}, \frac{v}{d\omega}\}$
Dimensions	-		$\cancel{T^{-1}}$ -

→ we find that $\frac{f}{\rho d^4\omega^2}$ depends on $\{\frac{v}{d\omega}\}$.

As we only have 2 nondimensional groups, we can plot them as a single curve on a single graph

For a test model, we require dynamic similarity → all independent model and full size nondimensional groups values match. → consequently the dependent model and full size nondimensional groups values match → same pt. on non-dim. curve.

Dynamic similarity → all independent non-dim. grp. values match.

$$\left(\frac{V}{\omega d}\right)_{model} = \left(\frac{V}{\omega d}\right)_{f-s}$$

$$\therefore V_{model} = V_{f-s} \cdot \left(\frac{\omega_{model}}{\omega_{f-s}}\right) \cdot \left(\frac{d_{model}}{d_{f-s}}\right) = 12\left(\frac{0.06}{0.6}\right)\left(\frac{8000}{800}\right) = 12\,ms^{-1}.$$

All independent non-dim. grp. values match → dependent non-dim grp. values match.

$$\left(\frac{f}{\rho d^4 \omega^2}\right)_{model} = \left(\frac{f}{\rho d^4 \omega^2}\right)_{f-s}.$$

$$\therefore f_{model} = f_{f-s} \cdot \left(\frac{\rho_{model}}{\rho_{f-s}}\right) \cdot \left(\frac{d_{model}}{d_{f-s}}\right)^4 \cdot \left(\frac{\omega_{model}}{\omega_{f-s}}\right)^2 = 25000\left(\frac{1000}{1050}\right)\left(\frac{0.06}{0.6}\right)^4 \cdot \left(\frac{8000}{800}\right)^2 = 243\,N.$$