Fundamentals of fluid mechanics

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The discrete and continuum descriptions of fluids.

- Fluids are made up of a large no. of molecules, where their motion is governed by Newton's laws. [Discrete] ← use ordinary derivatives
- In any real life situation, it would be impossible to follow every molecule → we zoom out and look at the average properties of the fluid, at a pt. in space. [Continuum] ← use partial derivatives

* we think the fluid as a continuous lump of stuff.

particle $\underline{v}_i$ — averaging → $\underline{v}(x,y,z) = \frac{1}{N}\sum_{i=1}^{N} \underline{v}_i$ — field $\underline{v} = \underline{v}(x,y,z)$.

Macroscopic properties of a fluid.

- The properties of a fluid all arise from its molecular nature – by considering the microscopic molecular motion, we can work out macroscopic properties (μ, λ, c etc.)
- The Knudsen no. Kn, is defined as the ratio of the mean free path λ to the size of the region L we are considering.

$$\boxed{Kn = \frac{\lambda}{L}}$$

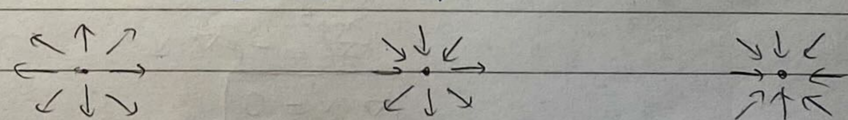
- The continuum model only works when averaging over a large no. of molecules and collisions, i.e. it req. $Kn \ll 1$.

Gradient of a scalar field.

- The gradient of a scalar field p, ∇p, points in the direction in which the scalar field increases the most, and is orthogonal to the contour lines (lines of const. p).
- $|\nabla p|$ gives the magnitude of the slope along the ∇p direction.

Divergence of a vector field.

- The divergence of a vector field $\underline{v}$, $\nabla \cdot \underline{v}$, produces a scalar field equal to the net flux of $\underline{v}$ out of each pt. in space.



$\nabla\cdot\underline{v} > 0$ $\qquad$ $\nabla\cdot\underline{v} = 0$ $\qquad$ $\nabla\cdot\underline{v} < 0$

Curl of a vector field (vorticity)

- The curl of a vector field $\underline{v}$, $\nabla \times \underline{v}$, is the vorticity ω, which is twice the angular velocity of the fluid. (oriented along the local axis of rotation of the flow)

$$\boxed{\underline{\omega} = \nabla \times \underline{v}}.$$

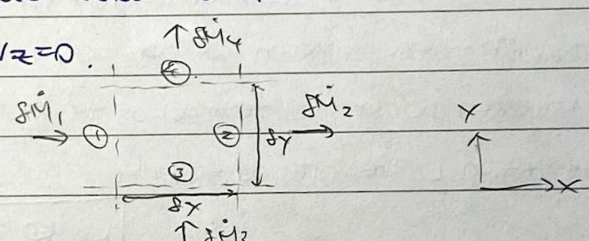
- Fluids can move along curved paths w/o rotating ($\underline{\omega} = 0$), e.g. plughole vortex.

Advection of a scalar/vector field

- The advection of a scalar/vector field, $\rho/\underline{v}$, $\underline{v}\cdot\nabla\rho/(\underline{v}\cdot\nabla)\underline{v}$ represents the changes in ~~the field~~

The law of conservation of mass (differential form).

- Consider a small rectangular volume of space w/ depth δz into the page. For simplicity, we assume $V_z = 0$.



$$\delta\dot{M}_1 = \rho V_x \,\delta y\,\delta z \qquad \delta\dot{M}_2 = \left(\rho + \frac{\partial\rho}{\partial x}\delta x\right)\left(V_x + \frac{\partial V_x}{\partial x}\right)\delta y\,\delta z$$

$$\delta\dot{M}_3 = \rho V_y \,\delta x\,\delta z \qquad \delta\dot{M}_4 = \left(\rho + \frac{\partial\rho}{\partial y}\delta y\right)\left(V_y + \frac{\partial V_y}{\partial y}\right)\delta x\,\delta z$$

- In a time δt, the change in mass δM is given by

$$\delta M = (\delta\dot{M}_1 - \delta\dot{M}_2 + \delta\dot{M}_3 - \delta\dot{M}_4)\delta t$$

(ignoring second order terms)
$$= -\left(\frac{\partial(\rho V_x)}{\partial x} + \frac{\partial(\rho V_y)}{\partial y}\right)\delta x\,\delta y\,\delta z.$$

Since $\delta M = \rho\,\delta x\,\delta y\,\delta z$, and taking the limit $\delta t \to 0$, we have

$$\frac{\partial}{\partial t}(\rho\,\delta x\,\cancel{\delta y\,\delta z}) = -\left(\frac{\partial(\rho V_x)}{\partial x} + \frac{\partial(\rho V_y)}{\partial y}\right)\cancel{\delta x\,\delta y\,\delta z}$$

$$\boxed{\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\underline{v})}$$

true in general.

where $\rho\underline{v}$ is the mass flux.

- In this course, we will assume $\rho = \text{constant}$, i.e. incompressible flow. In this case the continuity eqn. simplifies to

$$\cancel{\frac{\partial\rho}{\partial t}}^{\,0} = -\rho\nabla\cdot\underline{v}$$

$$\boxed{\nabla\cdot\underline{v} = 0}$$

* We would not be able to apply the ideal gas eqn, $\rho = \frac{p}{RT}$, since here we assumed $\rho = \text{constant}$, instead of $\rho = fn(p, T)$.

Inviscid flow.

The material derivative

- As a fluid particle moves w/ the flow, the change in field it experiences is given by the advection

$$\overset{\text{particle}}{\frac{dp}{dt}} = (\underline{v}\cdot\nabla)\overset{\text{field}}{p}$$

- For unsteady flow, the field changes w/ time so we add an unsteady component.

$$\overset{\text{particle}}{\frac{dp}{dt}} = \overset{\text{field}}{\left.\frac{\partial p}{\partial t}\right|_{(x,y,z)=\text{const}}} + (\underline{v}\cdot\nabla)\overset{\text{field}}{p}$$

- This derivative is the material derivative, and is denoted by

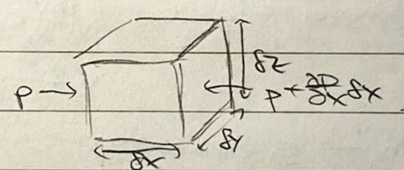
$$\boxed{\frac{D}{Dt} = \Big(\overset{\text{partial wrt } t}{\frac{\partial}{\partial t}} + \overset{\text{advection}}{\underline{v}\cdot\nabla}\Big)}$$

(this definition is coordinate free)

- The description of fluids
  - ↳ i) following fluid particles → Lagrangian description $\frac{d}{dt}$
  - ↳ ii) looking at fixed pts in space → Eulerian description $\frac{D}{Dt}$.

The Euler equation.

- Consider a neutrally buoyant solid cube in a fluid flow and ignore all the viscous forces.



In the x-direction,
$$F_x = p\,\delta y\,\delta z - \Big(p + \frac{\partial p}{\partial x}\delta x\Big)\delta y\,\delta z$$
$$= -\frac{\partial p}{\partial x}\delta x\,\delta y\,\delta z$$

We can obtain similar expressions for the y, z directions, so we write

$$\underline{F} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = -\begin{bmatrix} \partial p/\partial x \\ \partial p/\partial y \\ \partial p/\partial z \end{bmatrix}\delta x\,\delta y\,\delta z = -\nabla p\,\delta x\,\delta y\,\delta z$$

- Applying N2L to the cube, $\underline{F} = m\underline{a}$,

$$-\nabla p\,\delta x\,\delta y\,\delta z = \rho\,\delta x\,\delta y\,\delta z\,\frac{d\underline{v}_{\text{particle}}}{dt}$$
$$-\nabla p = \rho\frac{d\underline{v}_{\text{particle}}}{dt} \leftarrow \text{held by particle.}$$

To express the above expression in terms of the fluid's velocity field $\underline{v}$,

$$-\nabla p = \rho\frac{D\underline{v}}{Dt} \leftarrow \text{held by field}$$

$$\boxed{\rho\Big(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\Big) = -\nabla p}$$

- When gravity acts on a fluid, $-\rho g\hat{\underline{e}}_z$, its expression as a potential $-\nabla\rho g z$ allows us to embed it in the pressure gradient term, i.e. $-\nabla p \to -\nabla(p+\rho g z)$.

$$-\nabla\rho g z = -\frac{\partial}{\partial z}(\rho g z)\cdot\hat{\underline{e}}_z = -\rho g\hat{\underline{e}}_z$$

$$\rho\Big(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\Big) = -\rho g\hat{\underline{e}}_z - \nabla p$$
$$= -\nabla\rho g z - \nabla p$$
$$= -\nabla(p + \rho g z)$$

Link between Euler equations and SFME

- While the SFME gives info. overall in the CV, the Euler eqns give info. at each pt. in space.

$$\text{SFME: } \frac{\partial}{\partial t}\int_{vol} \rho \underline{v}\, dV + \int_{surf} \rho \underline{v}(\underline{v}\cdot d\underline{A}) = \underline{F} - \int_{surf} p\, d\underline{A} \qquad [1]$$

$$\text{Euler: } \frac{\partial \rho \underline{v}}{\partial t} + \rho(\underline{v}\cdot\nabla)\underline{v} = \text{body forces} - \nabla p \qquad [2]$$

- We can derive one from the other via the Divergence thm.

$$[1] = \int_{vol} [2]\, dV$$

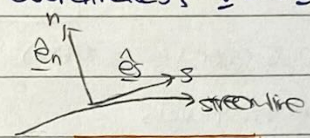
Euler's equations applied along a straight streamline.

- Consider the velocity in intrinsic coordinates, $\underline{v} = V \underline{e_s}$

$$\frac{\partial e_s}{\partial t} = \frac{\partial e_s}{\partial s}\cdot\frac{\partial s}{\partial t}$$

$$-\omega\, \underline{e_n} = \frac{\partial e_s}{\partial s}\cdot V$$

$$\frac{\partial e_s}{\partial s} = -\frac{\omega}{V}\underline{e_n} = -\frac{1}{R}\underline{e_n}$$



Note that $\nabla \equiv e_s \frac{\partial}{\partial s} + e_n \frac{\partial}{\partial n}$, $\boxed{\frac{\partial e_s}{\partial s} = -\frac{1}{R} e_n}$

- Applying Euler's eqn. for steady flow,

$$\rho\left(\cancel{\frac{\partial \underline{v}}{\partial t}}^{0} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla p$$

$$\left[V e_s \cdot \left(e_s \frac{\partial}{\partial s} + e_n \frac{\partial}{\partial n}\right)\right] V e_s = -\frac{1}{\rho}\nabla p$$

$\underline{e_s}\cdot\underline{e_s} = 1 \,;\; \underline{e_s}\cdot\underline{e_n} = 0$

$$V\frac{\partial(V e_s)}{\partial s} = -\frac{1}{\rho}\nabla p$$

$$V\left(\frac{\partial V}{\partial s} e_s + V\frac{\partial e_s}{\partial t}\right) = -\frac{1}{\rho}\nabla p$$

$$V\frac{\partial V}{\partial s} e_s - V^2\frac{1}{R} e_n = -\frac{1}{\rho}\left(\frac{\partial p}{\partial s} e_s + \frac{\partial p}{\partial n} e_n\right)$$

Resolving in the $\underline{e_s}, \underline{e_n}$ directions,

↳ $\underline{e_s}$ direction (along a streamline):

$$V\frac{\partial V}{\partial s} = -\frac{1}{\rho}\frac{\partial p}{\partial s}$$

$$\frac{\partial}{\partial s}\left(\tfrac{1}{2}V^2\right) = -\frac{\partial}{\partial s}\left(\frac{p}{\rho}\right)$$

$$\frac{\partial}{\partial s}\left(\frac{p}{\rho} + \tfrac{1}{2}V^2\right) = 0$$

use density of streamlines to find velocity, then find p. field → $\boxed{p + \tfrac{1}{2}\rho V^2 = \text{constant}}$ [Bernoulli eqn]

↳ $\underline{e_n}$ direction (across a streamline):

$$-\frac{V^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial n}$$

✓ can be used to find p. field directly → $\boxed{\frac{\partial p}{\partial n} = \rho\frac{V^2}{R}}$ [streamline curvature eqn.]

- If there is no vorticity (i.e. $\underline{\omega} = \nabla\times\underline{v} = 0$), then the total pressure is uniform and we can apply Bernoulli's eqn. across streamlines.

$$\text{Euler: } \cancel{\frac{\partial \underline{v}}{\partial t}}^{0} + \underline{v}\cdot\nabla\underline{v} = -\nabla\left(\frac{p}{\rho} + gz\right)$$

$$\cancel{(\nabla\times\underline{v})}^{0}\times\underline{v} + \tfrac{1}{2}\nabla(\underline{v}\cdot\underline{v}) = -\nabla\left(\frac{p}{\rho} + gz\right)$$

$\underline{v}\cdot\nabla\underline{v} = (\nabla\times\underline{v})\times\underline{v} + \tfrac{1}{2}\nabla(\underline{v}\cdot\underline{v})$

$$\nabla\left(\frac{p}{\rho} + \tfrac{1}{2}\underline{v}\cdot\underline{v} + gz\right) = 0.$$

$$\int_A^B \nabla\left(\frac{p}{\rho} + \tfrac{1}{2}\underline{v}\cdot\underline{v} + gz\right) ds = 0.$$

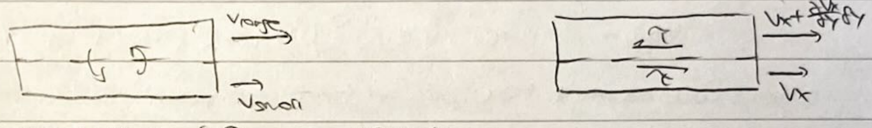
$$\left.\frac{p}{\rho} + \tfrac{1}{2}\underline{v}\cdot\underline{v} + gz\right|_A - \left.\frac{p}{\rho} + \tfrac{1}{2}\underline{v}\cdot\underline{v} + gz\right|_B = 0.$$

A, B can be any pt in space → $\frac{p}{\rho} + \tfrac{1}{2}\underline{v}\cdot\underline{v} + gz$ const. everywhere.

Viscous flow

Viscosity and shear.

- In viscous flow, the molecules in the fluid do not have defined positions. When one layer is displaced, the molecules flow over each other to accomodate the displacement.
- A shear stress τ is sustained when there is rel. motion between fluid particles. (i.e. no shear stresses when the fluid is in static eqm.).
- Experimentally, it is observed that adj. layers of conventional fluids exchange momentum at a rate prop. to the velocity gradient, $\frac{dV_x}{dy}$.



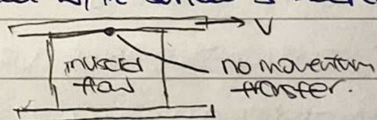
molecules in the two layers mix → momentum exchange → shear force/stress.

- The rate of momentum exchange per unit area is the shear stress τ, given by

$$\boxed{\tau = \mu \frac{dV_x}{dy}}$$

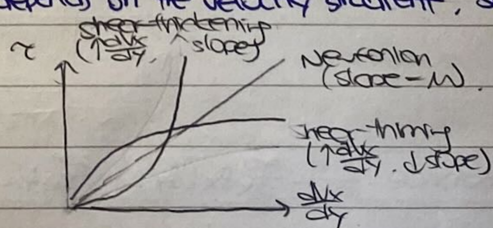
where μ is the viscosity.

- If the flow is inviscid (i.e. $\mu = 0$), the flow cannot support any shear stress → free slip, i.e. the velocity of the fluid in contact w/ the surface is independent from that of the surface



Newtonian and non-Newtonian fluids.

- For most fluids, the rate of momentum exchange is prop. to the velocity gradient. Such fluids are Newtonian fluids.
- Long chain molecules or fluids containing suspended solids may align/distort w/ the flow direction → viscosity depends on the velocity gradient, such fluids are non-Newtonian fluids.

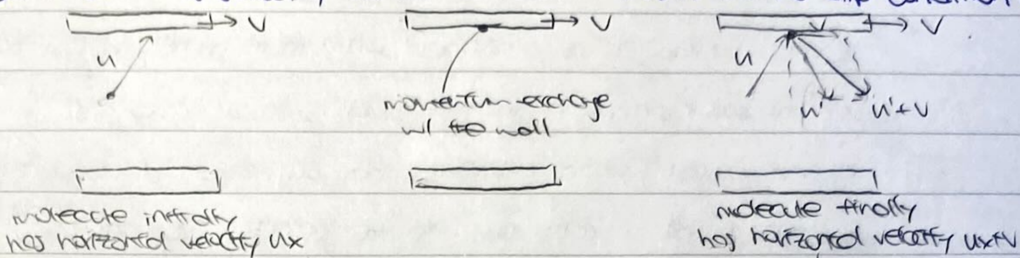


Viscosity and temperature.

- Viscosity varies strongly w/ temp. since it is closely linked to molecular motion:
  - ↳ Gases : $\uparrow T$ → ↑ avg molecular speeds → ↑ rate of momentum transfer → $\uparrow \mu$.
  - ↳ Liquids : $\uparrow T$ → ↓ temporary bonds. → ↓ rate of momentum transfer → $\downarrow \mu$.

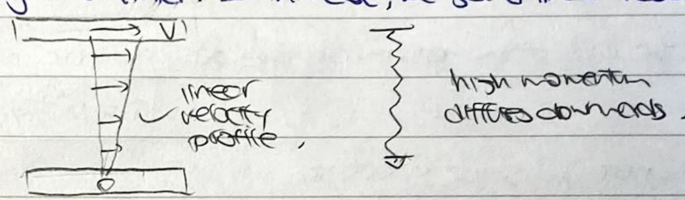
The no-slip condition and momentum transfer.

- Consider gas molecules btwn two plates, where the top plate moves at a velocity $V \rightarrow$. Experimentally, we find that after the gas molecules collide w/ the top plate, they have, on average, the same x-velocity as the surface — this is the no-slip condition.



∴ stationary top plate: $\bar{v} = \frac{1}{N}\sum_{i=1}^{N}(u_x)_i = 0$; Moving top plate: $\bar{v} = \frac{1}{N}\sum_{i=1}^{N}(u_x)_i + V = V$.

- The molecules that have just left the top plate collide into molecules nearby → momentum diffuses downwards through the fluid. In this case, we get a linear velocity profile.



Viscosity between parallel plates

- Consider steady, incompressible, viscous flows btwn two // plates.

streamlines straight and aligned w/ plates → V_x const. along x, $\boxed{V_x = V_x(y)}$.

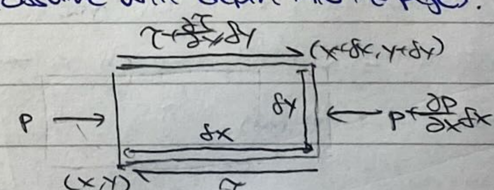
→ $\boxed{V_y = 0}$. ← satisfies continuity. $\nabla \cdot \underline{V} = 0$

$$\therefore \frac{d\underline{v}}{dt} = \frac{D\underline{v}}{Dt} = \frac{\partial \underline{v}}{\partial t} + \underline{V}\cdot\nabla\underline{V} = \cancelto{0}{\frac{\partial \underline{v}}{\partial t}} + V_x \cancelto{0}{\frac{\partial \underline{v}}{\partial x}} + \cancelto{0}{V_y}\frac{\partial \underline{v}}{\partial y} = 0.$$

i.e. the flow acceleration is zero everywhere.

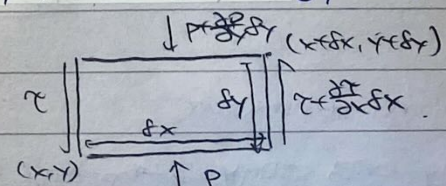
- As there is no acceleration, the forces on a CV must sum to zero. Consider an elemental fluid as follows (assume unit depth into the page).

x-direction



N2L ↔ : $-\frac{\partial p}{\partial x}\delta x(\delta y) + \frac{\partial \tau}{\partial y}\delta y(\delta x) = 0 \rightarrow \frac{\partial \tau}{\partial y} = \frac{\partial p}{\partial x}$.

y-direction



N2L ↕: $-\frac{\partial p}{\partial y}\delta y(\delta x) + \frac{\partial \tau}{\partial x}\delta x(\delta y) = 0 \rightarrow \frac{\partial \tau}{\partial x} = \frac{\partial p}{\partial y}$

- The shear stress τ, is a function of the velocity gradient, $\frac{dV_x}{dy}$. Since $V_x = V_x(y)$,

$$\tau = \tau(y) \rightarrow \frac{\partial \tau}{\partial x} = \frac{\partial p}{\partial y} = 0 \rightarrow p = p(x).$$

- Now consider the force balance in the x-direction, we have.

$$\frac{d\tau}{dy} = \frac{dp}{dx}$$ — total derivatives since we know $\tau = \tau(y)$ and $p = p(x)$.

Substituting the expression for the shear stress τ, $\tau = \mu \frac{dv_x}{dy}$, we get

$$\mu \frac{d^2 v_x}{dy^2} = \frac{dp}{dx}.$$

The eqn. above can only be possible if both sides are equal to the same constant, ie.

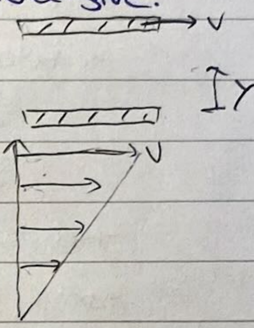
$$\boxed{\mu \frac{d^2 v_x}{dy^2} = \frac{dp}{dx} = \text{const}}.$$

Couette flow.

- when one of the two // plates moves // to the other w/ velocity v, the flow is Couette flow.

In this case, the pressure gradient is zero so the balance of forces give.

$$\mu \frac{d^2 v_x}{dy^2} = \frac{dp}{dx} = 0.$$



Integrating, we get a sol'n of the form

$$v_x = By + C.$$

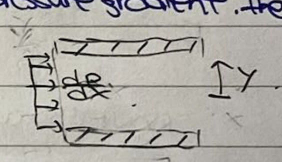
Applying the BCs, $v_x(0) = 0$, $v_x(H) = V$, we get.

$$\boxed{v_x = \frac{V}{H} y}$$

Poiseuille flow.

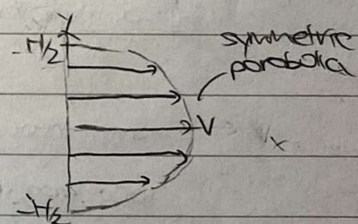
- when the two // plates are stationary but there is a constant pressure gradient, the flow is Poiseuille flow. The balance of forces give

$$\mu \frac{d^2 v_x}{dy^2} = \frac{dp}{dx}$$ ← const.



Integrating, we get a sol'n of the form.

$$v_x = \left(\frac{1}{2\mu}\frac{dp}{dx}\right) y^2 + By + C.$$



Applying the BCs, $v_x(-\frac{H}{2}) = 0$, $v_x(\frac{H}{2}) = 0$, we get.

$$\boxed{v_x = -\left(\frac{1}{2\mu}\frac{dp}{dx}\right)\left(\frac{H^2}{4} - y^2\right)}$$

- The driving pressure gradient must be -ve (i.e. $P_{upstream} > P_{downstream}$)

Combined Couette and Poiseuille flow.

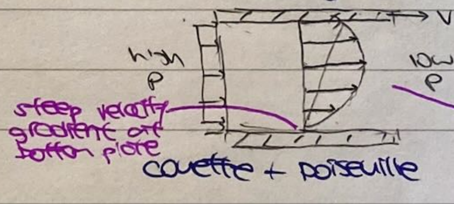
- when we combine Couette and Poiseuille flow, we get the same eqn. of motion but diff. BC.

$$\mu \frac{d^2 v_x}{dy^2} = \frac{dp}{dx} \quad \rightarrow \quad v_x = \left(\frac{1}{2\mu}\frac{dp}{dx}\right) y^2 + By + C.$$

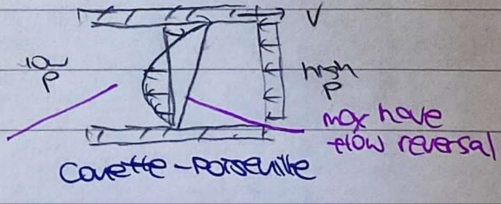
- depending on the sign of $\frac{dp}{dx}$, there are two types of sol'ns;

① Favourable pressure gradient ($\frac{dp}{dx} < 0$)

② Adverse pressure gradient ($\frac{dp}{dx} > 0$)

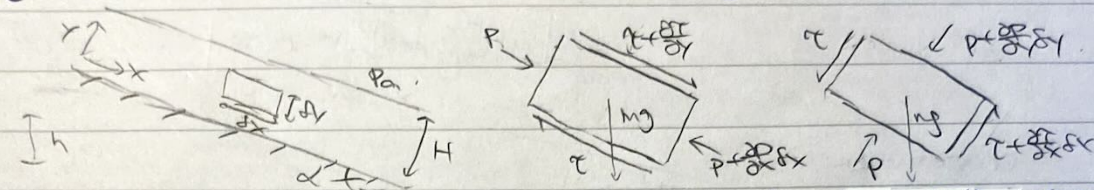


both parabolic profiles.



Viscous flow down a slope.

- Consider a steady, viscous flow down a slope. Assume we have a constant thickness, the streamlines must be straight and // to the slope.
- As w/ the analysis for two // plates, we can conclude that the flow acceleration is zero everywhere and apply a force balance on an elemental fluid.



N2L $\searrow$: $-\frac{\partial p}{\partial x}\delta x\delta y + \frac{\partial \tau}{\partial y}\delta x\delta y + \rho\delta x\delta y g\sin\alpha = 0 \rightarrow \frac{\partial \tau}{\partial y} = \frac{\partial(p-\rho g x\sin\alpha)}{\partial x} = \frac{\partial(p+\rho g h)}{\partial x}$.

N2L $\nearrow$: $-\frac{\partial p}{\partial y}\delta x\delta y + \frac{\partial \tau}{\partial x}\delta x\delta y - \rho\delta x\delta y g\cos\alpha = 0 \rightarrow \frac{\partial \tau}{\partial x} = \frac{\partial(p+\rho g y\cos\alpha)}{\partial y} = \frac{\partial(p+\rho g h)}{\partial y}$.

As before, $\tau = \tau(y)$, $p = p(x)$, so

at $y=H$, $p = p_a$

$$\frac{d\tau}{dx} = \frac{d(p+\rho g h)}{dy} = 0 \quad \rightarrow \quad p+\rho g h = \text{const} = p_a + \rho g h(x)$$

$$\frac{d\tau}{dy} = \frac{d(p+\rho g h)}{dx} \quad \rightarrow \quad \mu\frac{d^2 v_x}{dy^2} = \frac{d(p_a+\rho g h(x))}{dx} = \rho g\frac{dh}{dx} = -\rho g\sin\alpha$$

Integrating, we get $v_x = -\frac{\rho g\sin\alpha}{2\mu}y^2 + By + C$

free surface

Applying BCs, $v_x(0) = 0$, $\left.\frac{dv_x}{dy}\right|_{y=H} = 0$ $\quad v_x = \frac{\rho g\sin\alpha}{2\mu}(2yh - y^2)$

* We could include gravity by replacing p w/ $p+\rho g h$ in $\frac{dp}{dx}$ or $\frac{dp}{dy}$ directly.

- Consider a CV enclosing $y=0$ to $y=H$. Applying a force balance, we find that the shear force by the wall τ_w equals the x-component of the weight (indep. of μ).

this can be found via $\tau_w = \mu\left.\frac{dv_x}{dy}\right|_{wall}$

Navier-Stokes equations.

- For inviscid fluids, we have the Euler eqns,

$$-\nabla p = \rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v} \qquad \text{[Euler eqn]}$$

- When shear stresses are included, there are extra terms on the LHS. For an incompressible Newtonian fluid, we have

$$\mu\frac{\partial^2 \underline{v}}{\partial x^2} + \mu\frac{\partial^2 \underline{v}}{\partial y^2} + \mu\frac{\partial^2 \underline{v}}{\partial z^2} - \nabla p = \rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}$$

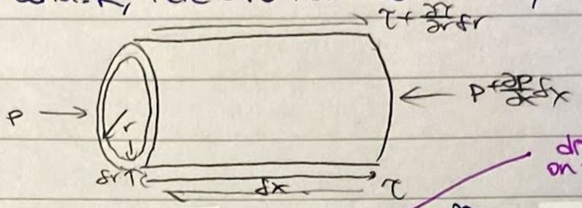
$$\mu\left(\frac{\partial^2 \underline{v}}{\partial x^2} + \frac{\partial^2 \underline{v}}{\partial y^2} + \frac{\partial^2 \underline{v}}{\partial z^2}\right) - \nabla p = \rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}$$

$$\boxed{\mu\nabla^2\underline{v} - \nabla p = \rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}} \qquad \text{[Navier-Stokes eqn]}$$

Example — viscous pipe flow

- Consider a pipe of radius R w/ pressure gradient $\frac{dp}{dx}$. Use a cylindrical shell of radius r, thickness δr and length δx as our fluid element.

For steady flow and const. R, there is no flow acceleration anywhere (as before).



different surface are on the inside / outside.

N2L (→) : $(2\pi r\delta r)p - (2\pi r\delta r)(p+\frac{\partial p}{\partial x}\delta x) + 2\pi(r+\delta r)(\tau+\frac{\partial \tau}{\partial r}\delta r)\delta x - 2\pi r\cdot\tau\cdot\delta x$.

Ignoring HOT, $$-\frac{\partial p}{\partial x}+\frac{\partial \tau}{\partial r}+\frac{\tau}{r}=0.$$

As before, we can say $V_x = V_x(r)$, $\tau=\tau(r)$ and $p=p(x)$.

$$\therefore -\frac{dp}{dx}+\frac{d\tau}{dr}+\frac{\tau}{r}=0$$

$$-\frac{dp}{dx}+\mu\frac{d^2V_x}{dr^2}+\frac{\mu}{r}\frac{dV_x}{dr}=0.$$

key step: $\frac{d}{dr}\left(r\frac{dV_x}{dr}\right) = r\frac{d^2V_x}{dr^2}+\frac{dV_x}{dr}$

$$-\frac{dp}{dx}+\frac{\mu}{r}\frac{d}{dr}\left(r\frac{dV_x}{dr}\right)=0.$$

$$\rightarrow \quad r\frac{dV_x}{dr} = \int\frac{dp}{dx}\frac{r}{\mu}dr = \frac{dp}{dx}\frac{r^2}{2\mu}+B.$$

$$\therefore V_x = \int \frac{dp}{dx}\frac{r}{2\mu}+\frac{B}{r}\,dr$$

$$= \frac{dp}{dx}\frac{r^2}{4\mu}+B\ln r + C.$$

Applying BCs, $v(R)=0$, $v(0)$ is finite,

$$\boxed{V_x = -\frac{dp}{dx}\left(\frac{R^2-r^2}{4\mu}\right)}.$$

Internal flows

Pipe flow

Reynold's number.

- starting from the Navier-Stokes eqn, and assume that the flow is steady,

$$-\nabla p + \mu \underbrace{\nabla^2 \underline{v}}_{\text{viscous forces}} = \rho \cancelto{0}{\frac{\partial \underline{v}}{\partial t}} + \underbrace{\rho \underline{v} \cdot \nabla \underline{v}}_{\text{advective forces}}$$

Applying the order-of-magnitude analysis,

$$\mu \nabla^2 \underline{v} \sim \mu \frac{V}{D^2} \quad ; \quad \underline{v} \cdot \nabla \underline{v} \sim \rho \frac{V^2}{D}.$$

- The Reynold's no., Re, is defined as the ratio of advective forces to viscous forces.

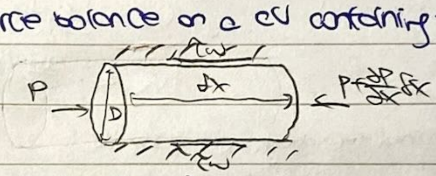
$$\boxed{Re_D = \frac{\rho V^2 / D}{\mu V / D^2} = \frac{\rho V D}{\mu}}$$

where D is the characteristic length.

* The Reynold's no. can be defined based on diff. characteristic lengths.

Friction and pressure loss.

- Consider a flow inside a pipe w/ diameter D. As before, there is no flow acceleration everywhere, so applying a force balance on a CV containing the whole pipe,



N2L (→) :

$$p(\pi R^2) - \left(p + \frac{dp}{dx} dx\right)(\pi R^2) - \tau_w (2\pi R dx) = 0.$$

$$\tau_w = -\frac{R}{2}\frac{dp}{dx}.$$

As before, we know that $v_x = u(y)$, $\tau = \tau(y)$ and $p = p(x)$, so the wall shear stress τ_w is

$$\boxed{\tau_w = -\frac{R}{2}\frac{dp}{dx}}$$

- we can use the bulk velocity V as our characteristic velocity, defined as,

$$\boxed{V = \frac{\text{volumetric flow rate}}{\text{CSA}} = \frac{Q}{\pi R^2} = \frac{\int_0^R v \cdot 2\pi r dr}{\pi R^2}}$$

* Note that V is const. for incompressible flow in a const. CSA pipe.

- Applying dimensional analysis on the wall shear stress τ_w, we can construct a dimensionless friction, the friction coefficient C_f.

$$\tau_w = fn(\rho, V, D, \mu)$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho V^2} = fn\left(\frac{\rho V D}{\mu}\right)$$

$$\rightarrow \quad \boxed{C_f = \frac{\tau_w}{\frac{1}{2}\rho V^2}} \quad \text{and} \quad C_f = fn(Re)$$

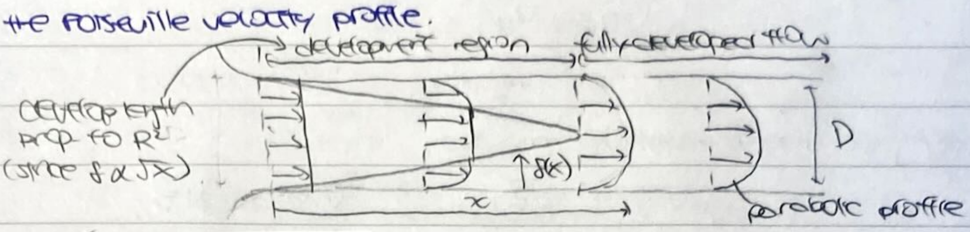
- For pipe flow, the friction coefficient C_f is given by,

$$C_f = \frac{-\frac{R}{2}\frac{dp}{dx}}{\frac{1}{2}\rho V^2}$$

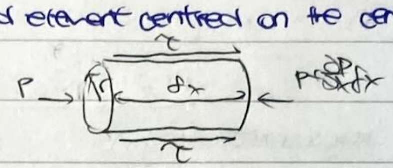
$$\boxed{C_f = -\frac{R}{\rho V^2}\frac{dp}{dx}}$$

Laminar flow in a circular pipe.

- When a fluid flows down a pipe, it takes some length from the entrance to develop fully, then the velocity profile becomes independent of x, and adopts the Poiseuille velocity profile.



Consider a cylindrical fluid element centred on the centreline. Applying a force balance,



N2L →: $$p(\pi r^2) - \left(p+\frac{\partial p}{\partial x}\delta x\right)(\pi r^2) + 2\tau\cdot(2\pi r\delta x) = 0.$$

$$\tau = \frac{r}{2}\frac{\partial p}{\partial x}.$$

$$\mu\frac{dv_x}{dr} = \frac{r}{2}\frac{dp}{dx}$$

$p=p(x)$ so $\frac{\partial p}{\partial x} = \frac{dp}{dx}$.

Applying the BC $v_x(R)=0$ and integrating,

$$\int_0^{v_x} dv_x = \frac{1}{2\mu}\frac{dp}{dx}\int_R^r r\,dr$$

$$\boxed{v_x = -\frac{dp}{dx}\left(\frac{R^2-r^2}{4\mu}\right)}.$$

- We can then find the bulk velocity V.

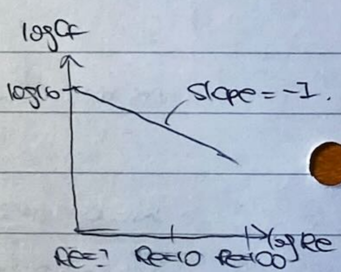
$$V = \frac{\int_0^R v_x \cdot 2\pi r\,dr}{\pi R^2} = \frac{1}{\pi R^2}\int_0^R -\frac{dp}{dx}\left(\frac{R^2-r^2}{4\mu}\right)2\pi r\,dr$$

$$\therefore \quad V = -\frac{R^2}{8\mu}\frac{dp}{dx} \quad \rightarrow \quad \boxed{\frac{dp}{dx} = -\frac{8\mu}{R^2}V}$$

- We can then find the friction coefficient c_f.

$$c_f = -\frac{R}{\rho V^2}\frac{dp}{dx} = +\frac{R}{\rho V^2}\left(\frac{8\mu}{R^2}V\right) = \frac{16\mu}{\rho V D}$$

$$\therefore \boxed{c_f = \frac{16}{Re_D}}$$



* Experimental results match well w/ theory for laminar flow.

- Previously, we formed the friction coefficient c_f by dividing the wall shear stress τ_w by the dynamic pressure $\frac{1}{2}\rho V^2$. If we divided by the order-of-magnitude value of shear stress $\frac{\mu V}{D}$ instead, we would get $c_f^* = \frac{\tau_w}{\mu V/D}$. For laminar pipe flow,

$$c_f^* = c_f \cdot \frac{\frac{1}{2}\rho V^2}{\mu V/D} = c_f \cdot \frac{\rho V D}{2\mu} = \frac{16}{Re_D}\cdot\frac{Re_D}{2} = 8.$$

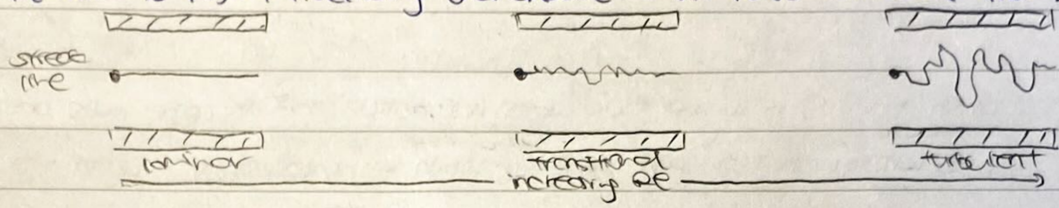
We can see this choice of friction coefficient is more convenient for low Re, where viscous forces dominate.

Turbulence, mixing and friction

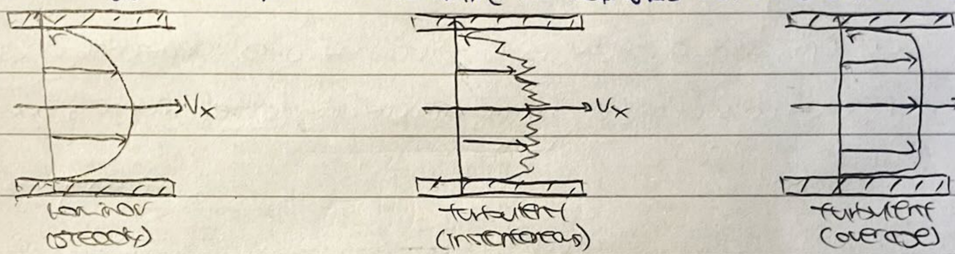
- If the flow were to remain laminar, the friction coefficient C_f would always decrease w/ increasing Reynold's no. Re.
- However, as the Reynold's no. Re increases, the flow is not viscous enough to dissipate all the energy put into it merely by viscous friction → breaks down into smaller and smaller eddies, where viscosity can act.

  for a given pipe, increasing the flow velocity

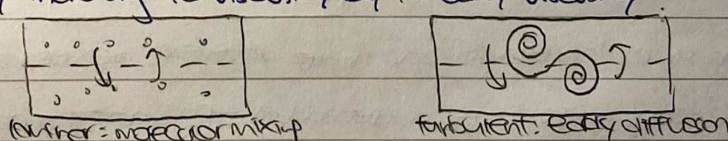
- The flow is highly fluctuating and chaotic — we have turbulent flow.



- When turbulent, the flow is highly unsteady — the pressure and velocity at a pt. in space vary in time. However, the time-averaged quantities do have steady values in pipe flow → use these quantities to define the Reynold's no. Re and friction coefficient C_f.



- In laminar flow, molecular diffusion is the only transport process between layers of fluid → hard to mix — they need to be folded rather than stirred.
- In turbulent flow, pockets of fluid move between layers of fluid in turbulent eddies. This occurs on a much larger scale than molecular diffusion → increased mixing rate.
- Therefore, the rate of momentum transport in a turbulent flow is much greater than that in a laminar flow. → higher shear stress and a greater pressure drop.
- Just as in the laminar case where we modelled the transport of momentum due to molecular diffusion by viscosity, we can model the increased transport of momentum due to eddies by increasing the viscosity by the eddy viscosity.



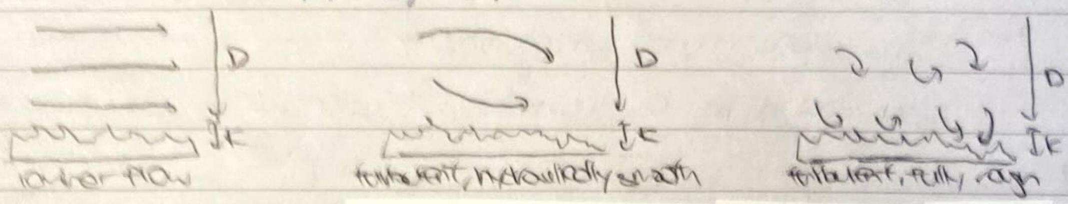
$$\tau = \mu \frac{dV_x}{dy} \qquad \tau = \mu_T \frac{dV_x}{dy}$$

where $\mu_T = \mu +$ eddy viscosity, and eddy viscosity $\gg \mu$.

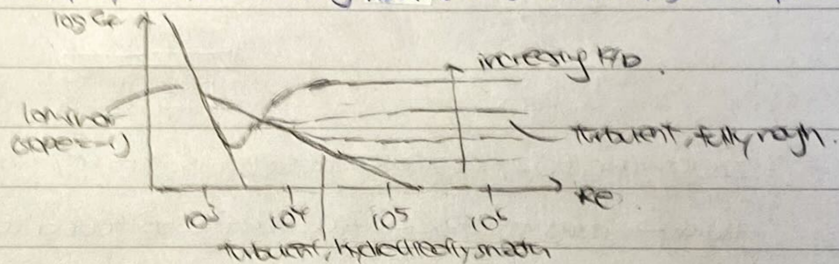
- The value of the eddy viscosity depends on the eddy size and intensity (which may vary throughout the flow), and it is difficult to find a universal model.

Roughness

- Roughness increases friction, so we typically try to eliminate it from pipes. However, no matter how smooth a pipe is, it will always have a certain roughness.
- The roughness of a pipe can be characterised by the ratio of the average bump size k, to the diameter of the pipe D, k/D.



- When $k/D \ll 1$, a laminar flow / turbulent flow at a moderately low Reynold's no. Re do not notice that the pipe walls are rough. → hydraulically smooth flow. (eddies too large to notice surface roughness)
- As the Reynold's no. Re increases, the flow breaks down into eddies small enough to interact w/ the rough bumps. → friction increases compared to hydraulically smooth flow.
- For v. large Reynold's no. Re, the eddies are so small that viscosity becomes irrelevant at the length scale of the bumps k. → friction becomes independent of viscosity μ and Reynold's no Re (friction caused entirely by the pressure drag from the roughness bumps) → fully rough flow.



Network analysis

Static, dynamic, stagnation and total pressures.

- The static pressure p is the actual pressure at any given pt in the flow. It is the pressure that we measure w/ a sensor that did not alter the flow velocity.
- The stagnation pressure p_0 is defined as the quantity that would be conserved for an inviscid flow in the absence of gravitational effects.

$$\boxed{p_0 = p + \tfrac{1}{2}\rho V^2}$$

- The dynamic pressure $\frac{1}{2}\rho V^2$ is the difference between stagnation and static pressures.
- The total pressure p_T is defined as the quantity that would be conserved for an inviscid flow when the work done by gravity is relevant.

$$\boxed{p_T = p + \tfrac{1}{2}\rho V^2 + \rho g h}.$$

- Hydraulics engineers often talk about heads instead of pressures, as any pressure diff. can be transformed into a height diff. in a static column of liquid.

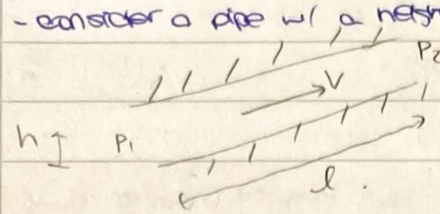
→ Pressure head $= \frac{p - p_a}{\rho_L g}$ ↳ Net head $= \frac{p_0 - p_a}{\rho_L g}$

→ Velocity head $= \frac{\frac{1}{2}\rho V^2}{\rho_L g}$ ↳ Total head $= \frac{p_T}{\rho_L g}$.

where ρ_L is the density of the liquid in the column.

Loss of total pressure along a pipe.

- consider a pipe w/ a height diff. across the ends.



$$P_{T1} = P_1 + \rho g h_1 + \tfrac{1}{2}\rho V_1^2 = P_1' + \tfrac{1}{2}\rho V_1^2$$

$$P_{T2} = P_2 + \rho g h_2 + \tfrac{1}{2}\rho V_2^2 = P_2' + \tfrac{1}{2}\rho V_2^2$$

→ we can embed $\rho g h$ term into P'. ($P_i' = P_i + \rho g h_i$)

- Recall that the friction coefficient C_f can be expressed in terms of the pressure gradient $\frac{dp}{dx}$,

$$C_f = -\frac{R}{\rho V^2}\frac{dp}{dx}.$$

Making the pressure gradient $\frac{dp}{dx}$ the subject, and expressing in terms of the diameter D,

$$\frac{dp}{dx} = -\frac{4}{D}C_f\left(\tfrac{1}{2}\rho V^2\right) = -\frac{f}{D}\left(\tfrac{1}{2}\rho V^2\right)$$

where f is the friction factor, defined as $f = 4C_f$.

- If the CSA is uniform along the pipe, D, V and Re are const, so we have.

$$\frac{dp}{dx} = \frac{P_2' - P_1'}{L} = \frac{(P_2 + \rho g h_2) - (P_1 + \rho g h_1)}{L} = -\frac{4}{D}C_f\left(\tfrac{1}{2}\rho V^2\right)$$

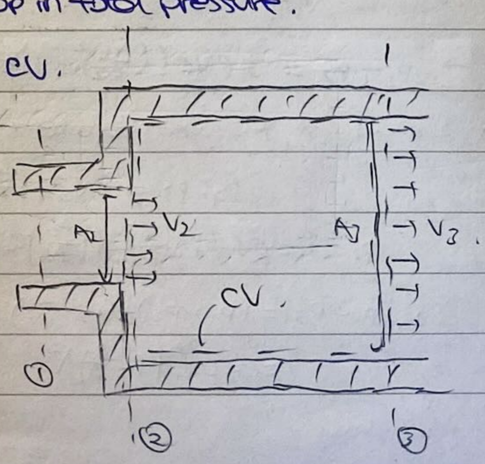
$$\therefore \quad P_{T2} - P_{T1} = \left(P_2 + \rho g h_2 + \tfrac{1}{2}\rho V^2\right) - \left(P_1 + \rho g h_1 + \tfrac{1}{2}\rho V^2\right) \qquad [\text{Note that } V_1 = V_2 = V]$$

$$= P_2 - P_1 + \rho g (h_2 - h_1)$$

$$\boxed{P_{T2} - P_{T1} = -\frac{4L}{D}C_f\left(\tfrac{1}{2}\rho V^2\right)}$$

Loss of total pressure at a pipe discharge.

- when a pipe of const. section ends in a sudden expansion (e.g. directly into open air), the out-coming flow forms a jet.
- It is generally safe to assume that the jet streamlines are straight → no normal pressure gradients → static pressure of the jet is the same as the quiescent fluid surrounding it.
- If the surrounding fluid is air and the discharging one is much denser (e.g. water), the friction between the two, and the associated energy loss can typically be neglected.
  (they do not mix turbulently → only dissipate energy via viscous dissipation → small at these high Reynolds no. Re → little loss of mechanical energy at the nozzle).
- When the discharge is into the same fluid, viscous friction is not negligible + turbulent mixing results in a drop in total pressure.
- Consider the following CV.



continuity: $V_2A_2 = V_3A_3 \rightarrow \frac{V_3}{V_2} = \frac{A_2}{A_3}$.

SFME: P_2 acts on A_3, not A_2

$P_2 A_3 + \dot{m}V_2 = P_3A_3 + \dot{m}V_3$.

can't use Bernoulli due to turbulent mixing

$$\rightarrow P_2 - P_3 = \frac{\dot{m}}{A_3}(V_3 - V_2) = \frac{\rho A_2 V_2}{A_3}(V_3 - V_2) = \rho V_2^2\left(\frac{A_2 V_3}{A_3 V_2} - \frac{A_2}{A_3}\right)$$

$$= \rho V_2^2\left(\left(\frac{A_2}{A_3}\right)^2 - \frac{A_2}{A_3}\right)$$

Assuming changes in $\rho g h$ are negligible, the drop in total pressure is given by

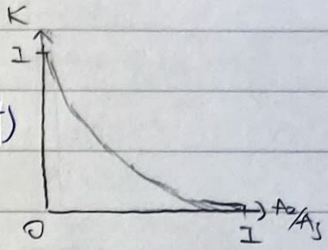
if changes in $\rho g h$ are not negligible, we simply replace P_i w/ $P_i + \rho g h_i$

$$P_{T2} - P_{T3} = \left(P_2 + \tfrac{1}{2}\rho V_2^2\right) - \left(P_3 + \tfrac{1}{2}\rho V_3^2\right)$$

$$= P_2 - P_3 + \tfrac{1}{2}\rho V_2^2\left(1 - \left(\frac{V_3}{V_2}\right)^2\right)$$

$$= \rho V_2^2\left(\left(\frac{A_2}{A_3}\right)^2 - \frac{A_2}{A_3}\right) + \tfrac{1}{2}\rho V_2^2\left(1 - \left(\frac{A_2}{A_3}\right)^2\right)$$

$$= \tfrac{1}{2}\rho V_2^2\left(1 - \frac{A_2}{A_3}\right)^2$$



$$\boxed{P_{T2} - P_{T3} = \tfrac{1}{2}\rho V_2^2 K}$$

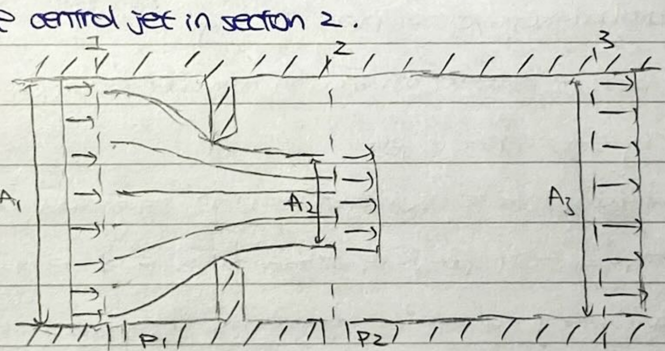
where K is the loss coefficient for the sudden expansion.

- The model is still valid when a pipe discharges into a reservoir. In this case, $A_3 \rightarrow \infty$, so $A_2/A_3 \rightarrow 0$ and $K = 1$.

Loss of total pressure across an orifice plate

used to measure flow velocity at the cost of introducing a pressure loss.

- Consider an orifice plate. Assume the velocity is uniform and steady across section 1, section 3 and the central jet in section 2.



section 2 is chosen s.t. the jet streamlines are locally straight → no normal pressure gradient. → static pressure is the same across all of section 2

- Assume there are no viscous losses between section 1 and 2,

Bernoulli: $P_1 + \frac{1}{2}\rho V_1^2 = P_2 + \frac{1}{2}\rho V_2^2$

continuity. $V_1A_1 = V_2A_2 \rightarrow \frac{V_2}{V_1} = \frac{A_1}{A_2}$

$$\rightarrow P_1 - P_2 = \tfrac{1}{2}\rho V_1^2\left(\left(\frac{A_1}{A_2}\right)^2 - 1\right)$$

- Between section 2 and 3, the jet mixes turbulently as in the pipe discharge case.

$$P_2 - P_3 = \rho V_2^2\left(\left(\frac{A_2}{A_3}\right)^2 - \frac{A_2}{A_3}\right)$$

since $A_3 = A_1$,

$$P_2 - P_3 = \tfrac{1}{2}\rho V_1^2\left(2\left(\frac{A_1}{A_2}\right)^2\right)\left(\left(\frac{A_2}{A_1}\right)^2 - \frac{A_2}{A_1}\right) = \tfrac{1}{2}\rho V_1^2\left(2 - 2\frac{A_1}{A_2}\right)$$

$$\therefore P_1 - P_3 = (P_1 - P_2) + (P_2 - P_3) = \tfrac{1}{2}\rho V_1^2\left[\left(\frac{A_1}{A_2}\right)^2 - 1 + 2 - 2\frac{A_1}{A_2}\right]$$

$$= \tfrac{1}{2}\rho V_1^2\left(1 - \frac{A_1}{A_2}\right)^2$$

Assuming changes in $\rho g h$ are negligible, the drop in total pressure is given by

$$P_{T1} - P_{T3} = (P_1 + \tfrac{1}{2}\rho V_1^2) - (P_3 + \tfrac{1}{2}\rho V_3^2)$$

If changes in $\rho g h$ are not negligible, we simply replace P_i w/ $P_i + \rho g h_i$

since $V_3 = V_1$,

$$= P_1 - P_3$$

$$= \tfrac{1}{2}\rho V_1^2 \left(1 - \tfrac{A_1}{A_2}\right)^2$$

$$\boxed{P_{T1} - P_{T3} = \tfrac{1}{2}\rho V_1^2 K}$$

where K is the loss coefficient for the orifice plate.

- The difficulty to determine the loss coefficient K lies in determining the position and value of A_2. Applying dimensional analysis on the pressure difference ΔP,

$$\Delta p = fn(\rho, V, D, \mu, d)$$

$$K = \frac{\Delta P}{\tfrac{1}{2}\rho V^2} = fn\left(\frac{\rho V D}{\mu}, d/D\right).$$

At v. large Reynold's no. of pipes ($Re \sim 10^6$), we have highly turbulent flow, and the problem becomes independent of the actual value of $Re \rightarrow K = fn(d/D)$ [K found from tables/plots]

Changes in total pressure across other network components.

- Other network elements like entrance to a pipe, valves, junctions and bends cause a similar pressure loss. Dimensional analysis shows that the loss is in the form.

$$\boxed{\Delta P_T = \tfrac{1}{2}\rho V_1^2 K}$$

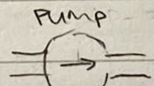
where the loss coefficient K is a function of the Reynold's no. Re and the geometry.

- At v. high Reynold's no. ($Re \sim 10^6$), the flow is highly turbulent and the loss coefficient K becomes Reynold's no. Re independent → only geometry dependent [K found from tables].

Mechanical work, pumps and turbines.

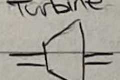
- Pumps do mechanical work on a fluid to produce a rise in total pressure. Turbines extract mechanical work from a fluid and cause a total pressure loss

Pump



$\Delta P_T > 0,\ \dot{W}_x < 0$

Turbine



$\Delta P_T < 0,\ \dot{W}_x > 0$

- For a process w/ irreversibilities, we have

$$\boxed{\frac{\dot{m}}{\rho}(P_{T,out} - P_{T,in}) + \text{irreversibilities} = -\dot{W}_x}$$

volumetric flow rate — change in mechanical energy per unit volume — work done on the component.

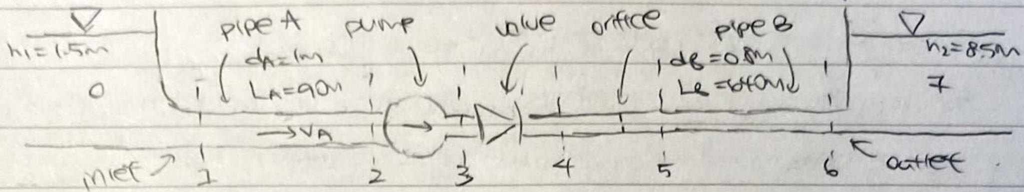
* Note that $\dot{W}_x$ is the mechanical work done by/extracted from the system

Network analysis – worked example

– The network below has the following friction and loss coefficients (for large Re)

$C_{f,pipe} = 3.75 \times 10^{-3}$, $K_{inlet} = 0.1$, $K_{valve} = 0.5$, $K_{orifice} = 0.5$, $K_{outlet} = 1.0$.

Find the power req. for the pump to drive $Q = 2.75\,m^3s^{-1}$ of water from the left tank to the right tank. Assume that both tanks are at atmospheric pressure, P_a.



Consider the changes in total pressure across diff. network components.

Left tank: $\quad P_{T0} = P_a + \rho g h_1$

Inlet: $\quad \Delta P_{T0\to1} = -\frac{1}{2}\rho V_A^2 K_{inlet}$

Pipe A: $\quad \Delta P_{T1\to2} = -\frac{1}{2}\rho V_A^2 \left(\frac{4L_A}{d_A} C_{f,pipe}\right)$

Pump: $\quad \Delta P_{T2\to3} = -\frac{\dot{W}_x}{Q}$

Valve: $\quad \Delta P_{T3\to4} = -\frac{1}{2}\rho V_B^2 K_{valve}$

Orifice plate: $\quad \Delta P_{T4\to5} = -\frac{1}{2}\rho V_B^2 K_{orifice}$

Pipe B: $\quad \Delta P_{T5\to6} = -\frac{1}{2}\rho V_B^2 \left(\frac{4L_B}{d_B} C_{f,pipe}\right)$

Outlet: $\quad \Delta P_{T6\to7} = -\frac{1}{2}\rho V_B^2 K_{outlet}$

Right tank: $\quad P_{T7} = P_a + \rho g h_2$

The volumetric flow rate gives us the values of V_A and V_B via continuity.

$$Q = \int_{section} V_x\,dA = \frac{\pi d_A^2}{4} V_A = \frac{\pi d_B^2}{4} V_B.$$

Consider the drop in total pressure across the network,

$$P_7 - P_0 = \sum_{i=0}^{6} \Delta P_{Ti\to i+1} = \Delta P_{T0\to1} + \Delta P_{T1\to2} + \ldots + \Delta P_{T6\to7}$$

$$\therefore \quad \rho g (h_2 - h_1) = -\frac{1}{2}\rho V_A^2 \left(K_{inlet} + \frac{4L_A}{d_A} C_{f,pipe}\right) - \frac{\dot{W}_x}{Q} - \frac{1}{2}\rho V_B^2 \left(K_{valve} + K_{orifice} + \frac{4L_B}{d_B} C_{f,pipe} + K_{outlet}\right)$$

$$\rightarrow -\dot{W}_x = 789\,kW.$$

External flows

Boundary layers

Boundary layers — inviscid vs high Re flows.

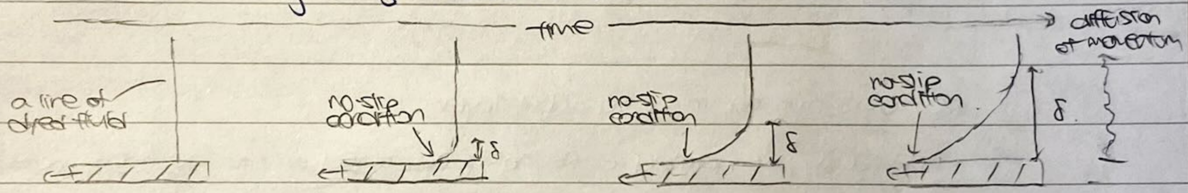
- The Reynold's no. Re is a measure of the rel. importance of advective vs viscous effects. As the Reynold's no. Re tends to infinity, we would expect the flow to become inviscid.
- However, this isn't entirely true — an inviscid flow would slip over a solid surface, whereas a flow w/ non-zero viscosity will satisfy the no-slip condition (since viscous effects can never be negligible near a wall).
- Previously, we assumed that variations in u for the viscous term occured over distances D, but in high-Re flows, they occur in v. thin layers near the walls — boundary layer, of thickness δ. We can define a Reynold's no. based on the boundary layer thickness, δ, Re_δ.

$$\boxed{Re_\delta = \frac{\rho V \delta}{\mu}}$$

- In laminar flows, Re_δ turns out to be small enough that viscous effects are important — but only up to a distance $\sim \delta$ from the wall.

Boundary layer growth.

- Consider a plate moving through a viscous liquid.



Due to the no-slip condition, the flow in immediate contact w/ the plate must have the same velocity as the plate. Through shear, the fluid near the plate will exchange momentum w/ the layer above, which will begin to move.

- In the frame of reference of the moving plate, for a fixed x, we can observe a y-diffusion of momentum away from the plate over time.
- If we think of the plate as stationary w/ the flow moving w/ respect to x, we can observe the effect of the no-slip condition propagate further in y as we move downstream along x.

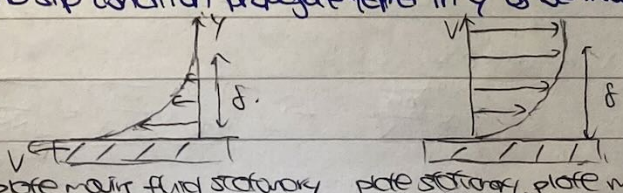


plate moving, fluid stationary   plate stationary, plate moving

- The velocity is not parabolic or linear (not Couette or Poiseuille flow). There is no top BC → no physical bound limiting the diffusion of momentum from the plate.
- In a 1D diffusion problem, the diffusion distance increases w/ $\sqrt{t}$. In this case, w/ a uniform convective velocity, this translates to $\sqrt{x}$ dependence for the thickness δ.

- consider the flow as a free stream over a fixed plate, w/ a steady, 2D boundary layer, and no pressure gradients.

① Continuity, $\nabla \cdot \underline{V} = 0$.

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0.$$

Applying scaling analysis, $\frac{\partial V_x}{\partial x} \sim \frac{V}{x}$, $\frac{\partial V_y}{\partial y} \sim \frac{V_y}{\delta}$

$$\therefore \boxed{V_y \sim \frac{\delta}{x} V} \ll V \quad (\text{since } \delta \ll x)$$

② Navier-Stokes (x):

$$\left[\rho \cancelto{0}{\frac{\partial \underline{V}}{\partial t}} + \rho(\underline{V} \cdot \nabla \underline{V}) = -\nabla p + \mu \nabla^2 \underline{V}\right]_x$$

$$\rho\left(V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y}\right) = \cancelto{0}{\frac{\partial p}{\partial x}} + \mu\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2}\right)$$

Applying scaling analysis, $\rho\left(V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y}\right) \sim \rho\left(\frac{V^2}{x} + \frac{V_y V}{\delta}\right) \sim \rho \frac{V^2}{x}$.

$$\mu\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2}\right) \sim \mu\left(\frac{V}{x^2} + \frac{V}{\delta^2}\right) \sim \mu \frac{V}{\delta^2} \quad \left(\text{since } \frac{1}{x^2} \ll \frac{1}{\delta^2}\right)$$

$$\therefore \boxed{\delta^2 \sim \frac{\mu x}{\rho V}} \longleftarrow \delta \sim \sqrt{x}.$$

③ Navier-Stokes (y):

$$\left[\rho \cancelto{0}{\frac{\partial \underline{V}}{\partial t}} + \rho(\underline{V} \cdot \nabla \underline{V}) = -\nabla p + \mu \nabla^2 \underline{V}\right]_y$$

$$\rho\left(V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y}\right) = \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2}\right)$$

Applying scaling analysis, $\rho\left(V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y}\right) \sim \rho\left(\frac{V V_y}{x} + \frac{V_y^2}{\delta}\right) \sim \rho \frac{V^2 \delta}{x^2}$, $\frac{\partial p}{\partial y} \sim \frac{\Delta p_y}{\delta}$.

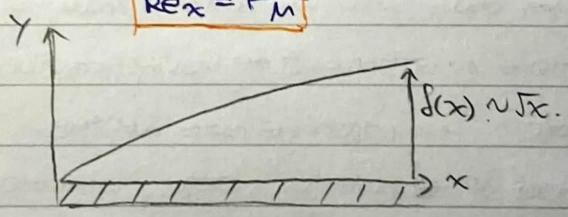
$$\mu\left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2}\right) \sim \mu\left(\frac{V_y}{x^2} + \frac{V_y}{\delta^2}\right) \sim \mu \frac{V}{\delta x}$$

Note that $\rho \frac{V^2 \delta}{x^2} = \frac{1}{\delta} \rho \frac{V^2}{x^2} \delta^2 = \frac{1}{\delta} \rho \frac{V^2}{x^2} \frac{\mu x}{\rho V} = \mu \frac{V}{\delta x} \rightarrow \therefore \boxed{\frac{\partial p}{\partial y} \sim 0}$

Reynold's number in a boundary layer.

- Previously, we defined a Reynold's no. based on the boundary layer thickness, Re_δ. However, it is often difficult to determine δ → use a Reynold's no. based on the stream distance from the beginning of the boundary layer x, Re_x

$$\boxed{Re_x = \frac{\rho V x}{\mu}}$$

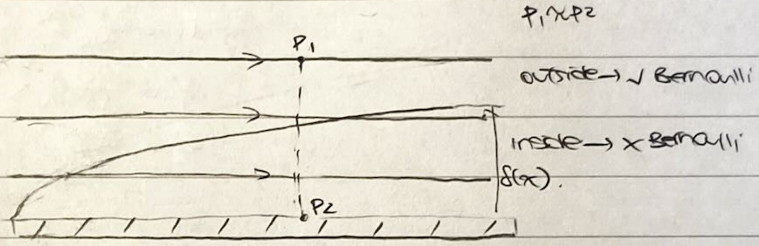


- There is a one-to-one correspondence between Re_δ and Re_x, as the relationship between δ and x is univocal.

$$\delta \sim \sqrt{\frac{\mu x}{\rho V}} \rightarrow \boxed{\frac{\delta}{x} \sim \sqrt{\frac{\mu}{\rho V x}} = \sqrt{\frac{1}{Re_x}}}$$

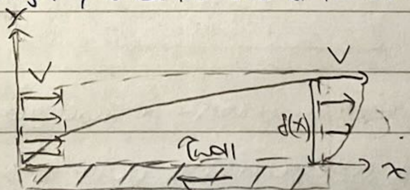
Bernoulli and streamline-curvature.

- within a boundary layer, viscous forces are important → Bernoulli doesn't hold. However by definition, outside the boundary layer, viscous effects are negligible → Bernoulli holds.
- For flows w/ relatively low viscosity, we can solve the flow field assuming that it is inviscid (using Euler), neglecting the thin boundary layers. Then, we use this external solution as the BC to solve the boundary layer region.
- As seen previously, the wall-normal velocity in the boundary layer is very small ($V_y \sim \frac{\delta}{x} V \ll V$). so the streamlines are almost wall-parallel and have negligible curvature.
- Moreover, we have seen that the wall-normal pressure-gradient is also negligible, so the pressure at a given pt. in the boundary layer is essentially the same as the free stream just above it.



Momentum loss.

- As the boundary layer grows, the flow loses momentum because of the friction shear stress exerted by the wall.
- The momentum loss can be related to the shear stress by considering a CV delimited by the streamline that passes through $y=\delta$ at the end of the CV.



per unit length into the page

SFME (x): $$-\int_0^x \tau_w \, dx = \int_0^\delta \rho V_x^2 \, dy - \rho V^2 h$$

continuity: $\tau_w = \mu \frac{dV_x}{dy}\Big|_{y=0}$ $$\dot m = \rho V h = \int_0^\delta \rho V_x \, dy$$

$$\therefore \underbrace{\int_0^x \tau_w \, dx}_{\text{net force on plate}} = \underbrace{V\int_0^\delta \rho V_x \, dy}_{\text{momentum flux in}} - \underbrace{\int_0^\delta \rho V_x^2 \, dy}_{\text{momentum flux out}}$$

→ we can work out the friction at the wall if we know the velocity profile $V_x(y)$.

* Note that we have made no assumptions on the type of flow → works for both the laminar and turbulent cases, can be easily extended to the case w/ a pressure gradient.

Example — Laminar boundary layer.

- An approximation for the laminar boundary layer is as follows.

$$\boxed{v_x(y) = V\left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]}$$

wall shear stress: $\tau_w = \mu \frac{dv_x}{dy}\Big|_{y=0} = \frac{3}{2}\frac{\mu V}{\delta}$.

momentum flux in: $V\int_0^\delta \rho v_x \, dy = \frac{5}{8}\rho V^2 \delta$.

momentum flux out: $\int_0^\delta \rho v_x^2 \, dy = \frac{17}{35}\rho V^2 \delta$.

considering the CV above

SFME:

$$\int_0^x \tau_w \, dx = V\int_0^\delta \rho v_x \, dy - \int_0^\delta \rho v_x^2 \, dy$$

$$\frac{3\mu V}{2}\int_0^x \frac{1}{\delta} dx = \frac{5}{8}\rho V^2\delta - \frac{17}{35}\rho V^2 \delta.$$

$$\int_0^x \frac{1}{\delta} dx = \frac{13}{140}\frac{\rho V}{\mu}\delta.$$

Differentiating wrt x,

$$\frac{1}{\delta} = \frac{13}{140}\frac{\rho V}{\mu}\frac{d\delta}{dx}.$$

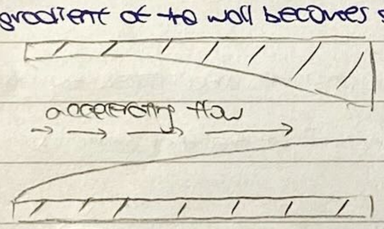
$$\int_0^x dx = \frac{13}{140}\frac{\rho V}{\mu}\int_0^\delta \delta \, d\delta$$

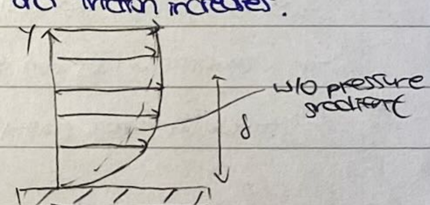
$$\delta^2 = \frac{280}{13}\frac{\mu}{\rho V}x. \quad \leftarrow \delta \sim \sqrt{x}.$$

$$\therefore \frac{\delta}{x} = \frac{4.64}{\sqrt{Re_x}}.$$

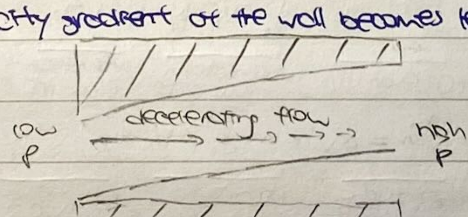
Boundary layers in flows w/ a pressure gradient

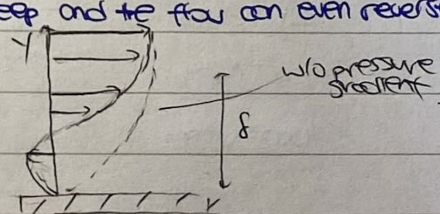
- In the presence of a favourable pressure gradient, as the flow moves towards low pressure it accelerates, and the velocity profile becomes fuller
- The velocity gradient at the wall becomes steeper and friction increases.



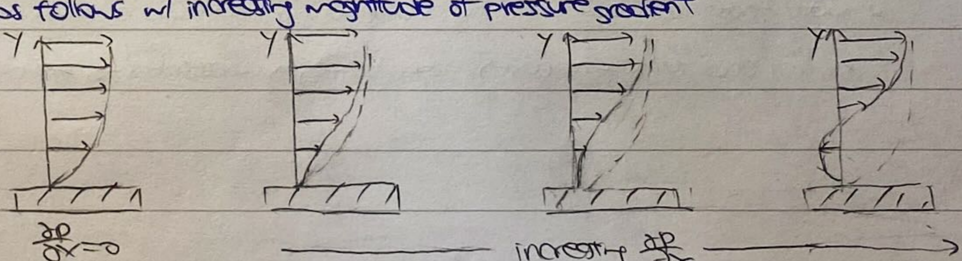


- In the presence of an adverse pressure gradient, as the flow moves towards high pressure it decelerates, and the velocity profile is depleted.
- The velocity gradient at the wall becomes less steep and the flow can even reverse.



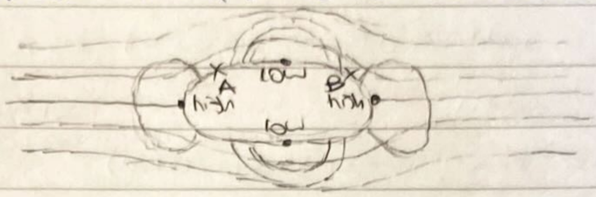


- Note in the adverse pressure gradient case, the velocity profile in the boundary layer varies as follows w/ increasing magnitude of pressure gradient



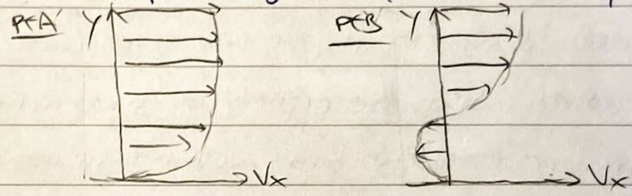
Boundary layer separation

- The streamline and pressure field of inviscid flow around an ellipse are shown below.

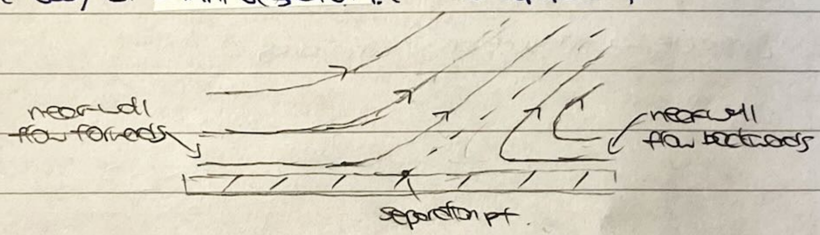


All real fluids are viscous and obey the no-slip condition → a thin boundary layer forms around the surface of the ellipse, growing as the fluid moves downstream

- Around the front of the ellipse there is a favourable pressure gradient (pt A); Around the back of the ellipse, there is an adverse pressure gradient, and will eventually cause flow reversal (pt B).

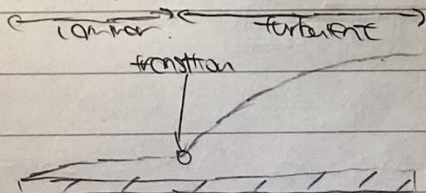


- The flow reversal completely changes the flow — the reversing fluid has to go somewhere but cannot reverse all the way to the front as the fluid is moving forward there → it separates from the body at a mini stagnation pt — the separation pt.



Transition to turbulence in boundary layers.

- In a boundary layer, the Reynold's no. Re_x increases as the flow moves downstream (the viscous terms become comparatively less important).
- In the early stages of a boundary layer, the Reynold's no. Re_x is rel. low, the viscous forces are strong enough to dissipate all the energy fed into the flow (remains laminar)
- As the Reynold's no. Re_x increases, the viscosity is unable to support the flow shear, and the flow breaks down into smaller, disorganised eddies — the boundary layer becomes turbulent.



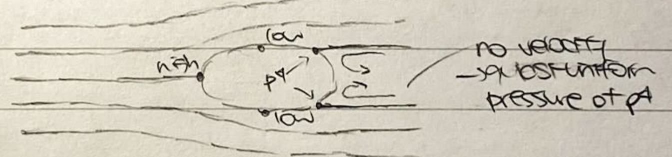
- In a v. controlled environment (wind tunnel w/ low noise), the transition to turbulence occurs suddenly for a given value of Re_x. In real life applications, the transition is less organised, and turbulent spots form irregularly.

Effect of turbulence on the boundary layer.

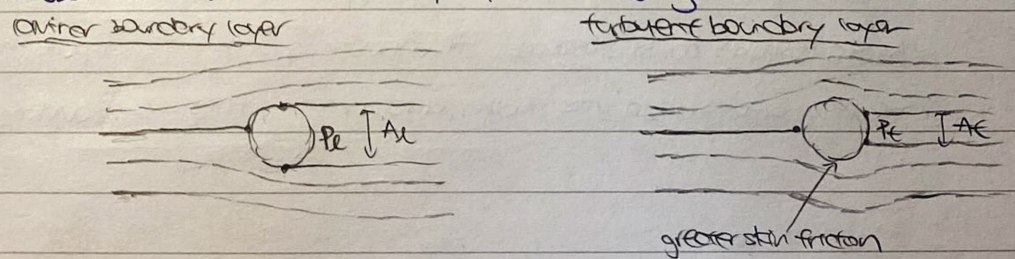
- Turbulence increases the rate of momentum transfer between the surface and the free stream (eddy viscosity >> viscosity). This has the following direct effects:
  ↳ Faster growth ↳ profile fuller than laminar.
  ↳ Gradient at wall steeper (higher friction).
- The fact that the velocity profile of the turbulent boundary layer is fuller makes it more robust against flow reversal → separation is delayed.

Effect of separation on pressure distribution.

- Downstream of separation pts, the flow in recirculation regions is usually much quieter than upstream → can comparatively be considered stationary.
- The pressure distribution on the wall in these regions is therefore roughly const, and its value is approx. that of the separation pt.
- This results in the pressure at the back of obstacles not recovering fully to its corresponding high value for the idealised inviscid sol'n → lower pressure than at the front of the obstacle → pressure drag / form drag.



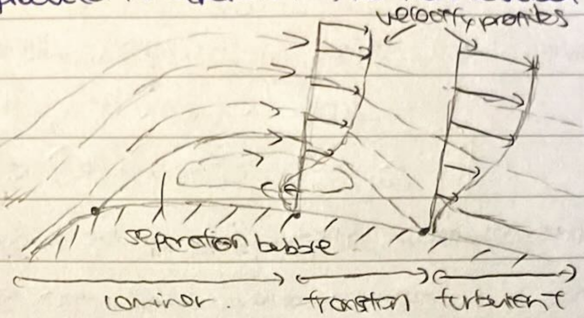
- Given that turbulent boundary layers can delay separation, they can produce smaller separated regions / wakes behind obstacles, so they can mitigate the above lack of pressure recovery.
- Therefore, an obstacle w/ turbulent boundary layers will have less form drag than the same obstacle w/ laminar boundary layers → triggering turbulence can reduce form drag.



∗ $p_l < p_t \rightarrow \Delta p_l > \Delta p_t$, $A_l > A_t \rightarrow F_{D,l} > F_{D,t}$.

Boundary layer reattachment

- when a boundary layer separates and subsequently becomes turbulent, the velocity profile becomes fuller, due to greater rate of diffusion of momentum.
- The new velocity profile may be robust enough to overcome the adverse pressure gradient that had originally produced the separation → no flow reversal → flow reattaches.



- separation bubbles can often be observed at the leading edge of wings at high angles of attack.
- Aerofoils are design to ensure that, once separated, the flow transitions to turbulence quickly, so that the separation bubble is short.
- If the transition occurs too far downstream, the bubble will cover a large area of the wing, or the flow may not be able to reattach → dangerous in planes as it would cause stalling.

Comparison of separation and transition to turbulence.

- Boundary layer separation and boundary layer transition to turbulence are entirely diff. phenomena, but transition to turbulence often occurs in boundary layers that have just separated.
- The reason that transition to turbulence occurs rapidly in laminar boundary layers that have just separated is that their velocity profiles have an inflection pt, which makes them unstable.
- However, in v. viscous separated flow, the viscosity is high enough to damp down any perturbations, and the flow remains laminar.
- If there is little or no adverse pressure gradient, the boundary layer will become turbulent w/o undergoing separation.

separation increases drag

Delaying boundary layer separation.

- Under adverse pressure gradients, whether the flow near the wall reverses is determined by the competing effects of the pressure gradient and the rate of diffusion of momentum.
- A more viscous flow has higher rate of diffusion of momentum → fuller velocity profile → more robust against separation.
- However, increasing viscosity would increase drag by increasing the wall shear stress. We can delay separation using other ways (Refer to the next section)

External flows and drag

Lift and drag.

- When a fluid flows around an object, the flow exerts a force on the object. We are usually interested in the components of the force // to (drag) and ⊥ to (lift) the freestream velocity V_∞.
- Applying dimensional analysis on lift/drag, we find that

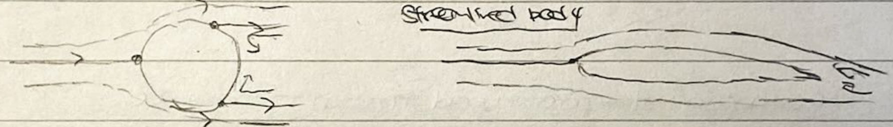
$$L \text{ or } D = fn(\rho, V, D, \mu, \alpha, V, a, \text{geometry})$$

$$\boxed{C_L = \frac{L}{\frac{1}{2}\rho V^2 A} \text{ or } C_D = \frac{D}{\frac{1}{2}\rho V^2 A} = fn\left(\frac{\rho V D}{\mu}, \alpha, \frac{V}{a}, \text{geometry}\right)}$$

* For incompressible flows, $M = \frac{V}{a} << 1$ → can be ignored).

- As seen previously, there are two phenomena that give rise to contributions to drag:
  - ↳ skin friction: real fluids w/ non-zero viscosity produce boundary layers, which generate friction and shear stress
  - ↳ form drag: when a flow separates, a low pressure region forms behind the stagnation pt. → pressure diff. compared to the front.
- The comparison of drag in bluff and streamlined bodies are as follows:

bluff body   streamlined body

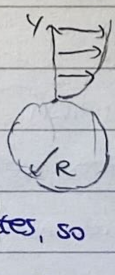


lower skin friction, higher form drag   higher skin friction, lower form drag.

- For sufficiently large Reynolds no. Re, form drag dominates in bluff bodies; skin friction dominates in streamlined bodies. (An aerofoil w/ a large α acts like a bluff body).

Flows at very low Reynold's number (Re << 1)

- When the Reynold's no. Re is vanishingly small, the flow is perfectly attached and does not separate → no form drag. (Purely viscous flow/ Stoke's flow/ Creeping flow).
- The velocity increases w/ distance from the surface, and the whole flow resembles a v. thick boundary layer.



- For stoke's flow, skin friction dominates, so $D \sim \tau_w A \sim \mu \frac{V}{D} A$.

since the drag coefficient C_D is defined as $C_D = \frac{D}{\frac{1}{2}\rho V^2}$, we have

$$C_D = \frac{D}{\frac{1}{2}\rho V^2 A} \sim \frac{\mu V/D\, A}{\rho V^2 A} \sim \frac{\mu}{\rho V D}$$

$$\therefore \boxed{C_D \sim \frac{1}{Re}}.$$

→ Scaling analysis shows that $C_D \sim \frac{1}{Re}$ for stoke's flow.

- To find the exact sol'n for the drag coefficient, C_D, we start w/ the Navier-Stokes eqn.

$$\rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v} = -\nabla p + \mu\nabla^2\underline{v}$$

Since viscous forces dominate over inertial forces, $\rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}$ is negligible, so

$$\nabla p = \mu\nabla^2\underline{v}.$$

For a sphere, the analytical sol'n is:

radial: $\frac{v_r}{V} = \left(1 - \frac{3}{2}\left[\frac{r}{R}\right]^{-1} + \frac{1}{2}\left[\frac{r}{R}\right]^{-3}\right)\cos\theta$

azimuthal: $\frac{v_\theta}{V} = -\left(1 - \frac{3}{4}\left[\frac{r}{R}\right]^{-1} - \frac{1}{4}\left[\frac{r}{R}\right]^{-3}\right)\sin\theta$.

$$\frac{p - p_\infty}{\mu V/R} = -\left(\frac{3}{2}\left[\frac{r}{R}\right]^{-2}\right)\cos\theta.$$

- The differential contributions of the drag can be integrated along the sphere's surface, and the resulting drag is given by

$$\boxed{D = 6\pi\mu RV}$$

The drag coefficient is therefore

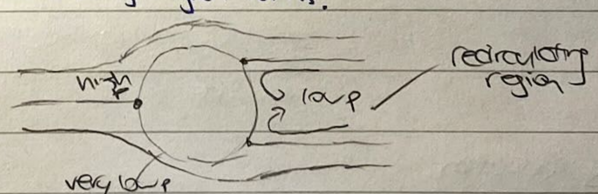
$$C_D = \frac{D}{\frac{1}{2}\rho V^2\pi R^2} = \frac{6\pi\mu RV}{\frac{1}{2}\rho V^2\pi R^2} = \frac{12\mu}{\rho VR} = \frac{24\mu}{\rho VD}$$

$$\therefore \boxed{C_D = \frac{24}{Re}}.$$

- Stokes flows are quantitatively v. diff from inviscid flows — even though their streamlines appear to be similar, Stokes flows satisfy no slip at solid surfaces.
- Stokes flows only exist at v. small length scales / v. viscous flows → rarely seen in real life.

Flows at moderately low Reynold's number ($Re \sim 10-1000$)

- In the flow around a sphere at $Re \approx 100$, the boundary layer separates just behind the shoulder and a toroidal recirculating region forms.



- The pressure in the recirculating region is approx the same as that of the separation pt, which is lower than the corresponding region at the front of the sphere → form drag.
- As the Reynold's no. Re increases, the rel. importance of viscous effects decreases → there is a lower rate of diffusion of momentum from the freestream → less robust against adverse pressure gradient → the separation pt. moves upstream towards the shoulder.
- As the separation pt. moves upstream towards the shoulder, the pressure diff. btwn the front and back and the wake area increase → increased form drag.

Flows of moderately high Reynold's number ($Re \sim 10^3 - 10^5$)

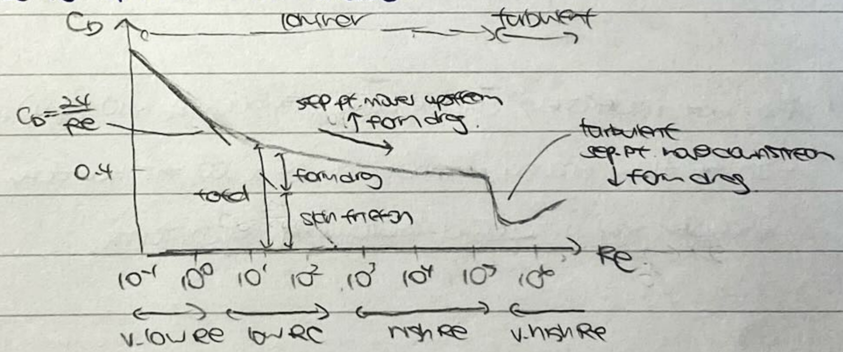
- As the Reynold's no. Re increases, the form drag becomes more dominant; beyond $Re \sim 10^3$, the skin friction is negligible.
- Between $Re \sim 10^3 - 2\times10^5$, the separation pt. remains v. near the shoulder (doesn't move past the shoulder as there would no longer be an adverse pressure gradient), so the drag coefficient remains approx. const. at $C_D = 0.4$.

Flows of very high Reynold's number ($Re \gtrsim 10^6$).

- For Reynold's no. Re above 2×10^5, the boundary layer becomes turbulent before it reaches the shoulder.
- Turbulence in the boundary layer increases the transport of momentum from the free stream and thus delays separation
- As the separation pt. moves downstream away from the shoulder, the pressure diff. b/tw the front and back and the wake area decrease → decreased form drag.

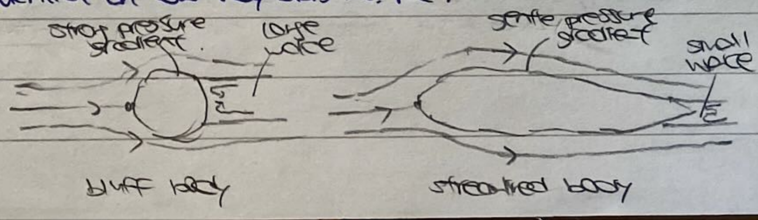
The C_D-Re plot for spheres.

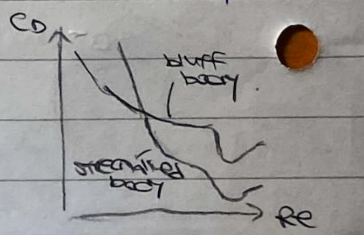
- The C_D-Re plot for spheres is as follows:



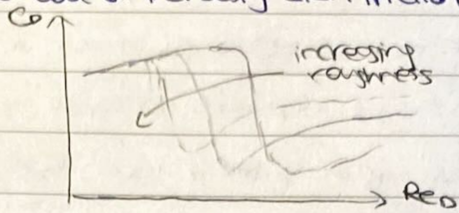
Drag reduction

- Form drag increases in proportion to the cross-sectional area of the separated region behind a body, and ρV^2; skin friction increases in proportion to the surface area of the body and $\mu \frac{V}{\delta}$.
  ↳ Form drag ~ wake area × $\frac{1}{2}\rho V^2$ ↳ skin friction ~ surface area × $\mu \frac{V}{\delta}$.
- For applications like planes, the Reynold's no. Re is high, so form drag > skin friction. → reduce form drag by having gentle adverse pressure gradients to delay separation (streamlining)
- Streamlining often has the side effect of increasing the skin friction, but this is only influential at low Reynold's no. Re.

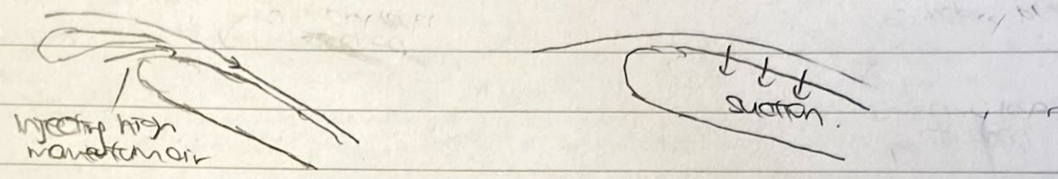




- Another way to delay boundary layer separation is to trigger turbulence by roughening the surface of the body (at the cost of increasing skin friction)

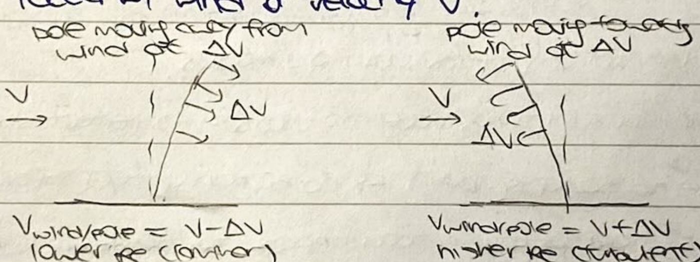


- There are two active methods (req. power) to delay boundary layer separation.
  ↳ Inject high momentum air into the boundary layer to overcome flow reversal.
  ↳ Drain away the layer of slow-moving air at the bottom of the boundary layer.



Resonance due to laminar/turbulent transition

- The sudden reduction of drag for slightly larger velocities due to laminar/turbulent transition in the boundary layer, and the corresponding separation delay, can result in resonance.
- This phenomena is usually observed in slender objects like poles/chimneys.
- Consider a pole faced w/ wind of velocity V.



If the pole is allowed to rock back and forth, its velocity rel. to the flow may oscillate enough that the boundary layers are laminar or turbulent at diff. times of the oscillating cycle.

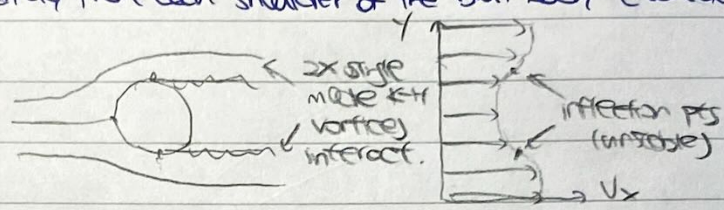
- Pole moves away from wind → velocity rel. to flow decreases → laminar boundary layers → increased separation region → larger form drag.

Pole moves towards wind → velocity rel. to flow increases → turbulent boundary layers → reduced separation region → smaller form drag.

→ Rocking motion enhanced, and resonance occurs.

Flow instability and vortex shedding.

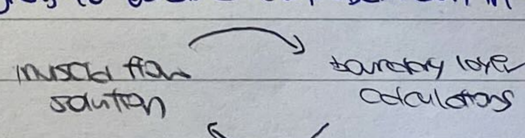
- When boundary layers separate, they create a shear layer, which is inherently unstable due to the inflection pt in their velocity profile.
- The shear layers develop waves that roll up into Kelvin-Helmholtz vortices.
- There are two approx. // shear layers behind a bluff body. They feed on each other and resonate, and the resulting flow is more unstable than a single shear layer.
- The shear layers start by undulating up and down together, then roll up into vortices that are shed alternately from each shoulder of the bluff body (vortex shedding).



- The vortex shedding frequency f is a function of the velocity of the flow, V, and the distance btwn the shear layers D.
- Experimentally, the Strouhal no, $\boxed{\frac{fD}{V} \text{ is approx } 0.2}$ for moderate and high Reynold's no. Re.
- For slender objects, there may be problems if the vortex shedding frequency is close to the resonant frequency of the structure.

Use and limitations of inviscid flow models.

- By defn, inviscid flows have no viscosity → perfect slip w/ solid boundaries, so there can be no boundary layers → no boundary layer separation (even w/ v. large adverse $\frac{dp}{dx}$).
- Inviscid flow is only a good approx. to flows w/ v. large Reynold's no. Re if boundary layers are thin and closely attached to the solid surfaces.
- As soon as the boundary layer separates, the morphology of external, inviscid-like flow changes completely
- Therefore inviscid models only work in regions w/ zero/favourable pressure gradients.
- However, since the inviscid-flow sol'n is usually much easier to obtain than the N-S sol'n, it can serve as a first approach to determining the real sol'n.
- We can use the resulting pressure gradients to estimate where the flow will separate, and include the separated regions to obtain a new inviscid sol'n in the modified domain.



Dimensional analysis.

Dimensional analysis in thermofluids.

Buckingham's Pi theorem

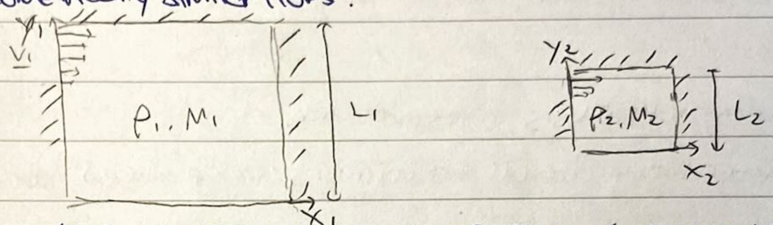
- Buckingham's Pi theorem states that if we have N variables (indep.+dep.) and M dimensions, then we should expect (N−M) or more NDGs.

$$\boxed{K \geq N-M}$$

where K is the no. of NDGs.

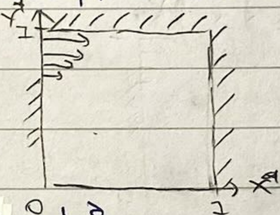
The dimensionless form of the Navier-Stokes equation.

- consider two geometrically similar flows.



We can define equivalent reference lengths and reference velocities in both situations, and measure all distances in these units. In these units, the BC for both problems are identical, but the fluid properties ρ, μ are not.

$x_1 = x^* L_1$
$y_1 = y^* L_1$
$\underline{V}_1 = \underline{v}^* V_1$
$t_1 = t^* L_1/V_1$
$p_1 = p^* \rho_1 V_1^2$
physical prop.: ρ_1, μ_1



$x_2 = x^* L_2$
$y_2 = y^* L_2$
$\underline{V}_2 = \underline{v}^* V_2$
$t_2 = t^* L_2/V_2$
$p_2 = p^* \rho_2 V_2^2$
physical prop. : ρ_2, μ_2

Noting that $\frac{\partial x_i}{\partial x^*} = L_i \rightarrow \frac{\partial}{\partial x_i} = \frac{1}{L_i}\frac{\partial}{\partial x^*}$,

$$\nabla_i = \begin{pmatrix} \partial/\partial x_i \\ \partial/\partial y_i \end{pmatrix} = \frac{1}{L_i}\begin{pmatrix} \partial/\partial x^* \\ \partial/\partial y^* \end{pmatrix} = \frac{1}{L_i}\nabla^*$$

Starting from the Navier-Stokes eqn. from one of the cases,

$$\rho_i \frac{D\underline{V}_i}{Dt} = -\nabla_i p_i + \mu_i \nabla_i^2 \underline{V}_i$$

$$\rho_i \frac{D(\underline{v}^* V_i)}{D(t^* L_i/V_i)} = -\frac{1}{L_i}\nabla^*(p^* \rho_i V_i^2) + \mu_i \frac{1}{L_i^2}\nabla^{*2}(\underline{v}^* V_i)$$

$$\rho_i \frac{V_i^2}{L_i}\frac{D\underline{v}^*}{Dt^*} = -\frac{\rho_i V_i^2}{L_i}\nabla^* p^* + \mu_i \frac{V_i}{L_i^2}\nabla^{*2}\underline{v}^*$$

$$\frac{D\underline{v}^*}{Dt^*} = -\nabla^* p^* - \frac{\mu_i}{\rho_i V_i L_i}\nabla^{*2}\underline{v}^*$$

$$\boxed{\frac{D\underline{v}^*}{Dt^*} = -\nabla^* p^* - \frac{1}{Re_i}\nabla^{*2}\underline{v}^*}$$

Given identical BC in * variables, the two eqns must have the same Reynold's no. Re_i for them to have the same sol'n → then we would have dynamical similarity.

- In this scenario, the Reynold's no. Re_i is the only control parameter for geometrically similar objects. For compressible flows, the Mach no. M_i would be an additional control parameter.

Order-of-magnitude analysis.

- We can perform order-of-magnitude analysis (scaling analysis on an eqn. to estimate the rel. importance of each term.
- We can perform the analysis by choosing some characteristic parameters.
- For example, consider the Navier-Stokes eqn.

$$\rho \frac{\partial \underline{v}}{\partial t} + \rho \underline{v} \cdot \nabla \underline{v} = -\nabla P + \mu \nabla^2 \underline{v}$$

scaling analysis, $\rho \frac{\partial \underline{v}}{\partial t} \sim \rho \cdot \frac{V}{t}$, $\rho \underline{v} \cdot \nabla \underline{v} \sim \rho V \cdot \frac{V}{L} = \rho \frac{V^2}{L}$

$-\nabla P \sim \frac{\Delta P}{L} = \frac{\rho V^2}{L}$, $\mu \nabla^2 \underline{v} \sim \mu \frac{V}{L^2}$

For steady flows, the characteristic time t is infinite $\rightarrow$ ignore the $\rho \frac{\partial \underline{v}}{\partial t}$ term.

Dimensional analysis in thermofluids.

- After listing out all the variables (indep. + dep.), form indep. NDGs directly by looking at the NDGs in the databook.

↳ $Re = \frac{\rho V D}{\mu} = \frac{VD}{\nu}$

↳ $C_D = \frac{D}{\frac{1}{2}\rho V^2 A}$

↳ $C_f = \frac{\tau_w}{\frac{1}{2}\rho V^2}$

↳ $Pr = \frac{\mu c_p}{\lambda} = \frac{\nu}{\alpha}$

↳ $St = \frac{h}{\rho V c_p} = \frac{Nu}{RePr}$

↳ $Bi = \frac{hs}{\lambda}$

↳ $Fr = \frac{V}{\sqrt{gz}}$

↳ $C_L = \frac{L}{\frac{1}{2}\rho V^2 A}$

↳ $M = \frac{V}{a}$.

↳ $Nu = \frac{hd}{\lambda}$

↳ $Gr = \frac{gd^3 \beta \Delta T}{\nu^2}$

↳ $Fo = \frac{\alpha t}{s^2}$