

## Fluid statics

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## Definition of a fluid

- A fluid is defined as a substance that cannot sustain shear stress when at rest.
  - shear stresses are sustained only when relative motion between fluid particles takes place.
- (On the other hand, solids are always capable of sustaining shear stresses.)
- The term fluid covers both liquids and gases.

## Pascal's law

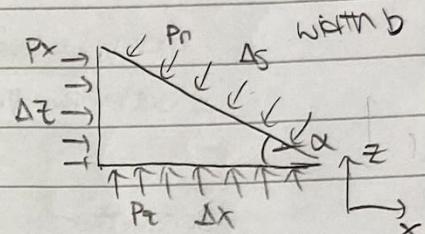
- Pascal's law states that the pressure of a fluid acts equally in all directions when it is at rest (as there are no shear stresses).
- The pressure force  $F$  is given by  $F = pA$  and always acts normal to the surface.

- Consider a small element of fluid at rest:

$$\text{N2L} \Leftrightarrow \sum F_x = p_x b \Delta z - p_n b \Delta S \sin \alpha = 0$$

$$\therefore \sum F_z = p_z b \Delta x - p_n b \Delta S \cos \alpha - \rho g \frac{b \Delta x \Delta z}{\Delta z} = 0.$$

Geometrically,  $\Delta z = \Delta S \sin \alpha$ ,  $\Delta z = \Delta S \cos \alpha$ .



$\therefore$

$$p_x = p_n$$

$$p_z = p_n + \frac{1}{2} \rho g \Delta z.$$

As  $\Delta z \rightarrow 0$ ,  $p_z \rightarrow p_n$ .

$\rightarrow p_x = p_n = p_z$ , i.e. pressure on all surfaces is the same.

## Variation of pressure with depth

- Consider a column of liquid at the free surface, of height  $z'$ .

The pressure at the free surface is atmospheric,  $p_a$ .

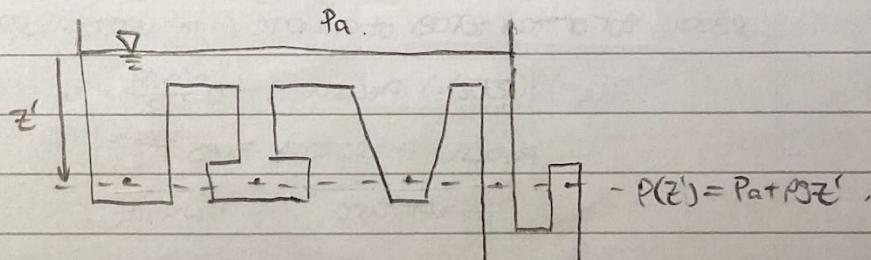
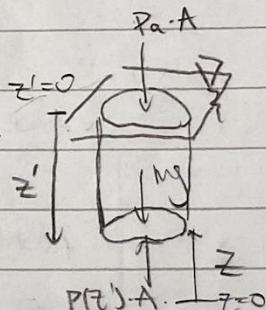
$$\text{N2L} \Leftrightarrow p_a A + \rho A z' g = p(z') A$$

$$p(z') = p_a + \rho g z'$$

$$\therefore \frac{dp}{dz'} = \rho g.$$

or in terms of the vertical coord  $z$ ,  $\frac{dp}{dz} = -\rho g$

- For constant density fluids, there is a linear variation of pressure with depth.



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## Manometer

- We can use a U-tube manometer to measure an unknown pressure  $P_m$ .
- The manometer is filled w/ a relatively dense fluid (e.g. coloured water / mercury).

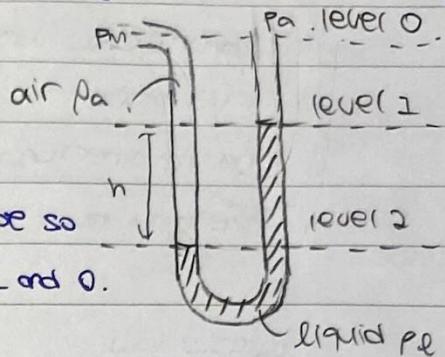
$$\text{Left : } P_2 = P_{i,L} + \rho g h$$

$$\text{Right : } P_2 = P_{i,R} + \rho g h$$

$$\therefore P_{i,L} - P_{i,R} = \rho g h - \rho g h$$

Above level 1, there is air both sides of the tube so the pressure varies by the same amount between levels 1 and 0.

$$\text{so } P_m - P_a = (\rho_e - \rho_a) gh$$



- Often, the manometer liquid is much denser than air so the effect of the air column,  $\rho a h$  can be ignored, so  $P_m = P_a + \rho_e g h$ .

- To increase the accuracy of height measurement  $h$ , the manometer tube can be inclined. (Same height difference  $\rightarrow$  longer length change).

\* Note the pressure rel. to atmospheric  $P_m - P_a$  is the gauge pressure  $P_g$ .

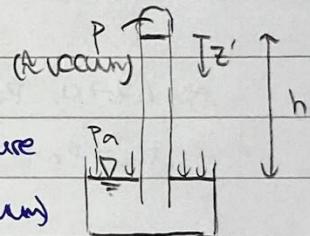
## Barometer

- A mercury barometer can be used to measure atmospheric pressure.  $P_a$ .

$$P(z') = P_a + \rho g z'$$

$$P_a = P_{v,Hg} + \rho_{Hg} g h$$

The top of the column would be at the vapour pressure of mercury  $P_{v,Hg}$ , which can be approximated as 0 (vacuum)



$$\therefore P_a = \rho_{Hg} g h.$$

\* We cannot achieve a pressure lower than the liquid's vapour pressure

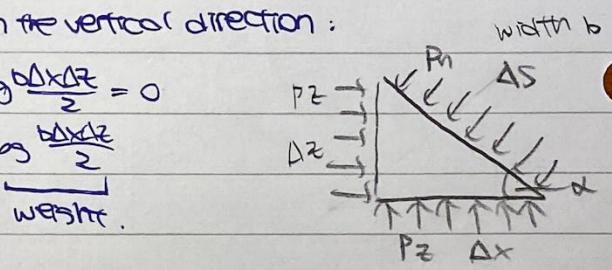
## Archimedes' principle.

- Archimedes' principle states that that a body wholly / partially in fluid receives an upthrust equal to the weight of fluid it displaces.  
(Consequence of integrating all pressure forces around the complete surface of the body)
- There can't be a residual moment so the upthrust must act through the C.O.I.
- Recall the forces of a fluid element in the vertical direction:

$$(\Sigma F_z) = P_2 b \Delta x - P_1 b \Delta x \cos \alpha - \rho g \frac{b \Delta x \Delta z}{2} = 0$$

$$P_2 b \Delta x - P_1 b \Delta x \cos \alpha = \rho g \frac{b \Delta x \Delta z}{2}$$

upthrust                      weight.



## Forces on submerged bodies.

- We can integrate all the pressure forces over the elemental areas of the surface to find the net force.

$$F = \int_S p dA$$

we integrate along the surface.

$$= \int_S \rho g z'(s) b(s) ds.$$

where  $z'(s)$  is the depth and  $b(s)$  is the width.

- Equivalently, we find that the

↳ vertical force  $F_z$  is weight of fluid above the surface.

↳ horizontal force  $F_x$  is the average force  $\times$  projected area.

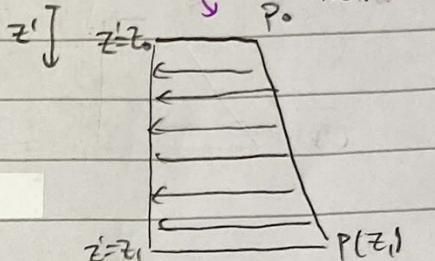
- Consider a trapezoidal pressure distribution.

$$F = \int_{z_0}^{z_1} p dA$$

$$= \int_{z_0}^{z_1} \rho g z' \cdot b dz'$$

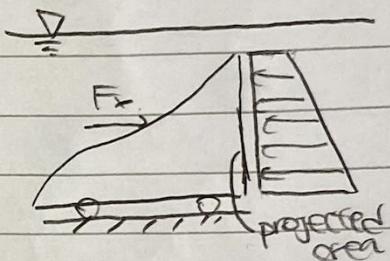
$$= \rho g b \left[ \frac{z_1^2}{2} - \frac{z_0^2}{2} \right]$$

Average  $= \rho g \cdot \frac{(z_1 + z_0)}{2} \cdot b(z_1 - z_0)$



- Consider a trolley submerged in a fluid:

Obviously, the trolley does not move, so the force  $F_x$  must equal the trapezoidal pressure force distribution  $\rightarrow F_x = \text{average force} \times \text{projected area.}$



## Point of action of the force on a submerged plane

- The distributed pressure force can be replaced by an equivalent force acting at the centre of pressure COP.

- We can find the COP by taking moments about  $s=0$ .

$$FS_{COP} = \int_S s dF = \int_S s \cdot p dA = \int_S \rho g s z'(s) b(s) ds.$$

$$S_{COP} \int_S p z'(s) b(s) ds = \int_S \rho g s z'(s) b(s) ds.$$

$$S_{COP} = \frac{\int_S \rho g s z'(s) b(s) ds}{\int_S \rho g z'(s) b(s) ds}.$$

- Equivalently, the COP is at the "COM" of the pressure force distribution (**\*NOT general!**, only valid for hydrostatics).

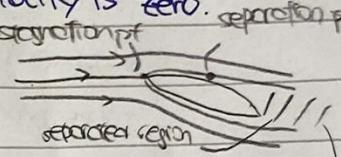
- Remember, the actual nature of the force is distributed, so this equivalent force at the COP is only meaningful in the context of the overall shape of the plane.

### Terms and definitions

#### Streamlines

- Streamlines are curves which are in line w/ the velocity vectors at each pt. in a flow. (so shows the trajectory of a fluid particle)
- The distance between streamlines is inversely proportional to velocity
- Streamlines cannot cross each other (the flow cannot have 2 different velocities)
- Streamlines cannot end in a flow (due to mass conservation) other than on a solid surface (stagnation pt.)
- Streamlines only make sense in the context of steady flow. ↴  
use streaklines for unsteady flow

#### Stagnation points.

- An object immersed in a flow has one or more stagnation pts.
  - A stagnation pt. is a pt. on the surface where the flow velocity is zero.
- 
- Flow separation
- Streamlines usually follow the surface of an object. However, flows will separate from the surface when they cannot follow its curvature or if the pressure gradient is too large
  - The velocity in the separated region is close to zero
  - The pressure in the separated region is close to that of adjacent flow ( $\nabla \neq 0 \rightarrow$  no momentum to support pressure gradient  $\rightarrow$  constant pressure).
  - The flow in the separated region is unsteady.

#### Steady and unsteady flow

- In steady flow, the trajectory of particles passing through each pt. in space do not change w/ time; In unsteady flow, the trajectory of particle varies w/ time.
- Sometimes, an unsteady flow can be treated as steady flow when a short timescale is considered  $\rightarrow$  quasi-steady flow.
- Note that steady  $\neq$  laminar; unsteady  $\neq$  turbulent.
  - $\hookrightarrow$  turbulent flow is always unsteady
  - $\hookrightarrow$  laminar flow may be steady or unsteady.

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## Compressible and incompressible flow

- Many flow problems are considered to be incompressible as it simplifies the eqns of motion (as we assume constant density)
- A rule of thumb is to assume that flow is incompressible when the flow velocity is below  $1/3$  the speed of sound (Mach no.  $\approx 0.33$ ), and the change in pressure is less than  $5\%$ .  $\rightarrow$  "limits of incompressible flow".  
(If we assume a constant density, a  $5\%$  density variation corresponds to an error in the pressure change of about  $17\%$ .)

## Viscous and inviscid flow

- Any object immersed in a flow will experience viscous friction as the fluid particles close to the surface interact w/ the surface and don't move w/ the bulk flow.
- At some distance away from the surface, the flow speed is similar to that far away from the body (freestream velocity) no-slip condition
- Therefore, there must be a velocity gradient near the surface (boundary layer), which causes shear forces in the fluid and friction forces of the surface.
- The effects of the boundary layer can often be ignored if it is small in extent rel. to the rest of the flow  $\rightarrow$  the flow is effectively inviscid. (viscous forces  $\ll$  pressure forces)
- Viscosity of a fluid affects flow separation. There is no separated region for inviscid flow. ( $\uparrow$  viscosity, separation pt. more upstream)
- The Reynolds number  $Re$  is a measure of the importance of inertial (pressure) forces rel. to viscous shear forces, given by

$$Re = \frac{\rho VL}{\mu}$$

where  $L$  is the characteristic length,  $V$  is the freestream velocity and  $\mu$  is the viscosity.

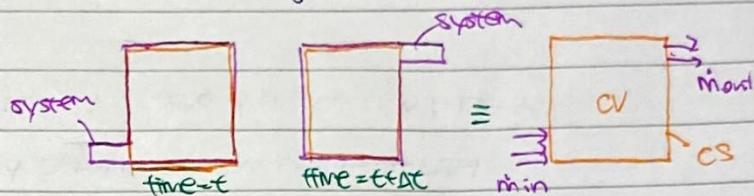
- A larger Reynolds number mean the viscous effects of the flow and hence the size of the boundary layer get less important.

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## Systems and control volumes

### SYSTEMS

- A system is defined as a device w/ fixed mass.  $\rightarrow$  no mass can cross a system boundary.
- only energy (work, heat) is permitted to cross the system boundary.
- The volume of a system may change (e.g. change of state) so the system boundaries can move.
- By defn, there is consu. of mass within a system.
- An external force  $F$  can be applied to the system boundaries or distributed through the system (body forces).
- The sum of these forces is equal to the rate of change of momentum of the system, (NCL for a system).



### Control volumes.

- Applying the principle of mass conservation to a moving system, we can conclude that the mass flow rate into and out of a reservoir volume  $V_0$  are equal.
- For control volumes (CV), we consider incoming and outgoing mass flow rates, rather than tracking a system w/ moving boundaries.  $\rightarrow$  steady CV has fixed boundaries. (in the selected frame of reference)
- For a CV, mass is allowed to cross the control surface (CS).

## Global technique: Conservation of mass

### Conservation of mass for a control volume.

- We can state the principle of mass conservation for a CV in multiple ways:
  - $\hookrightarrow 1) \sum_i m_i = 0$  where  $m_{in} < 0$  for flow into a CV  
 $m_{out} > 0$  for flow out of a CV
  - $\hookrightarrow 2) \sum m_{in} = \sum m_{out}$  where the mass flow rates are the
  - $\hookrightarrow 3) \oint_C \rho V \cdot dA = 0$  where all points out of the CV w/ dir  $\perp$  to the CS.
- This is also known as the continuity eqn. or steady flow mass eqn.
- For a single fluid w/ constant density (i.e. incompressible flow), we get conservation of volumetric flow rate. It is most commonly written as.

$$Q_{in} = Q_{out}$$

$$A_{in}V_{in} = A_{out}V_{out}$$

\* Note  $m = \rho Q$

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Mass flow rate - general inflows and outflows.

- In general, the mass flow rate on an area A is given by

$$\dot{m} = \int_A \rho v_n dA$$

or equivalently, using the vector form,

$$\dot{m} = \int_A \rho \mathbf{v} \cdot d\mathbf{A}$$

the A points outwards

- The quantity  $\rho v$  is the mass flux (units:  $\text{kg s}^{-1} \text{m}^{-2}$ ).

- The expression for mass flow rate can be simplified in special cases:

↳ Uniform + perpendicular flow :  $\dot{m} = \rho A V$

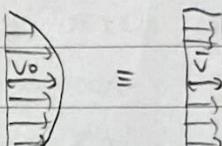
↳ Non-uniform + perpendicular flow :  $\dot{m} = \int_A \rho v dA$

↳ Uniform + collinear flow :  $\dot{m} = \rho A V_n$ .

- e.g.: The velocity in a pipe has a parabolic profile given by  $v(r) = V_0(1 - (\frac{r}{R})^2)$

Find the mean velocity in the pipe.

The mean velocity  $\bar{V}$  is the equivalent uniform velocity that carries the same amount of mass.

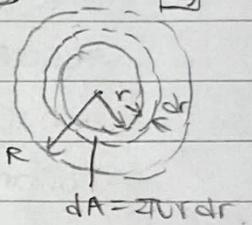


$$\dot{m} = \int_A \rho v dA$$

$$\rho \pi R^2 \bar{V} = \int_0^R \rho V_0(1 - (\frac{r}{R})^2) \cdot 2\pi r dr$$

$$\rho \pi R^2 \bar{V} = 2\pi V_0 R^2 \left[ \frac{R^2}{2} - \frac{R^4}{4R} \right]$$

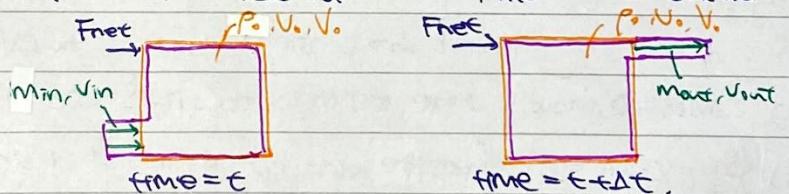
$$\bar{V} = \frac{V_0}{2}. \quad \leftarrow \text{standard result}$$



Global technique : N2L.

N2L - rate of change of momentum.

- Consider the system of a reservoir w/ an inflow and outflow



$\rightarrow +x$ .

Under steady conditions (so  $F_{net}$  is constant), we can apply N2L to the system.

$$F_{net}t = \dot{m}o = \frac{d(mV)}{dt} = \frac{\Delta mV}{\Delta t} = \frac{(mV)_{t+\Delta t} - (mV)_t}{\Delta t}$$

- Consider the momentum of the system at time  $t$  and time  $t + \Delta t$ .

$$\text{time } t : \quad (MV)_t = M_{in}V_{in} + P_0 V_o V_0 \quad \leftarrow \begin{matrix} \text{momentum} \\ \text{of inlet} \end{matrix} \quad \begin{matrix} \text{some as we have} \\ \text{steady flow} \end{matrix}$$

$$\text{time } t + \Delta t : \quad (MV)_{t+\Delta t} = M_{out}V_{out} + P_0 V_o V_0 \quad \leftarrow \begin{matrix} \text{momentum} \\ \text{of outlet} \end{matrix}$$

$$\therefore F_{net} = \frac{M_{out}V_{out} + P_0 V_o V_0 - M_{in}V_{in} - P_0 V_o V_0}{\Delta t} = \frac{M_{out}V_{out} - M_{in}V_{in}}{\Delta t}$$

(here we are only interested in  $\Delta P_x$ , but we can apply this to other directions as well!).

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- Using the defn of mass flow rate, we can move from a cyclic system (fixed mass, moving boundaries) to a control volume (fixed volume, flow through boundaries).

$$\frac{\dot{m}_{in}}{\Delta t} = \dot{m}_{in}$$

$$\frac{\dot{m}_{out}}{\Delta t} = \dot{m}_{out}$$

- We can rewrite NZL applied to control volumes.

$$F_{ext} = (\dot{m}v)_{out} - (\dot{m}v)_{in}$$

$\dot{m}v$  represents momentum flow rate

Momentum flow rate - general inflows and outflows,

- In general, the momentum flow rate of an area A is given by

$$\dot{m}v = \int_A (\rho v_n) v dA$$

or equivalently, using the vector form,

$$\dot{m}v = \int_A \rho v \cdot v dA$$

- The expression for momentum flow rate can be simplified in special cases.

↳ Uniform + perpendicular flow:  $\dot{m}v = \rho A v^2$

↳ Non-uniform + perpendicular flow:  $\dot{m}v = \int_A \rho v^2 dA$

↳ Uniform + oblique flow:  $\dot{m}v = \rho v_n v$

## NZL - forces

- There are a no. of ways a force can act on the fluid inside a CV.

↳ Internal / body forces (e.g.: gravity, electromagnetic),

↳ Forces transmitted via the boundaries (sum of mechanical, pressure and viscous forces along the complete boundary).

- For now, we neglect all internal body forces (inc. weight) and neglect viscous effects on the boundaries. → we can split the forces into 2 parts:

↳ Resultant of non-pressure forces  $\Sigma F$

↳ Pressure forces  $\Sigma P A$ . (we usually consider gauge pressures  $P_g$ ).

## Pressure forces - general

- In general, the pressure forces on an area A is given by

$$\text{pressure force} = \int_A P dA$$

equivalently, using the vector form,

$$\text{pressure force} = - \int_A P d\mathbf{A}$$

↑ -ve sign because we point outwards

- The expression for pressure force can be simplified in special cases.

↳ Uniform flow: Pressure force =  $PA$ .

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## Steady Flow Momentum Equation (SFME).

- Combining the expressions for net force and the rate of change of momentum, we get the steady flow momentum equation (SFME). simplified version:  
 $\sum F + \sum P A = (\dot{m}v)_{out} - (\dot{m}v)_{in}$

$$F_{net} = \sum F - \oint_{C_S} P dA = \oint_{C_S} (\rho v_n) V dA$$

equivalently, separating into scalar components

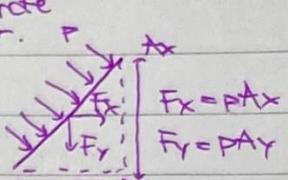
$$F_{net,x} = \sum F_x - \left( \oint_{C_S} P dA \right)_x = \oint_{C_S} (\rho v_n) V_x dA$$

$$F_{net,y} = \sum F_y - \left( \oint_{C_S} P dA \right)_y = \oint_{C_S} (\rho v_n) V_y dA$$

or using the dot product,

$$F_{net} = \sum F - \oint_{C_S} P dA = \oint_{C_S} \rho V \cdot (V \cdot dA)$$

\* Remember  $\sum F$  is the force exerted ON the fluid.



-eg: The velocity in a pipe has a parabolic profile given by  $v(r) = V_0 \left(1 - \left(\frac{r}{R}\right)^2\right)$

Calculate the momentum flow rate in the pipe.

$$\begin{aligned} \dot{m}V &= \int_A (\rho v_n) V dA \\ &= \int_0^R \rho V_0^2 \left(1 - \left(\frac{r}{R}\right)^2\right)^2 \cdot 2\pi r dr \\ &= 2\pi \rho V_0^3 \left[ \frac{R^2}{2} - \frac{R^4}{4R} + \frac{R^6}{6R^4} \right] \\ &= \frac{1}{3} \pi R^2 \rho V_0^3. \end{aligned}$$

\* note  $\dot{m} \bar{V}_{eq} = \frac{1}{2} \pi R^2 \rho V_0 \left(\frac{V_0}{2}\right) = \frac{1}{4} \pi R^2 \rho V_0^2 \neq (\dot{m}V)$ , i.e. equivalent uniform velocity inc carries the same mass as diff. to equivalent uniform velocity (not carries the same momentum).  $(\bar{V}_{mass}) = \frac{V_0}{2}; \bar{V}_{momentum} = \frac{2V_0}{3}.$

## Strategy for solving control volume problems

- ① Choose a CV and coord system. Label the force on the fluid in the CV.  
(sometimes, we choose a moving CV when considering a moving object to make the problem a steady flow problem) remember to add appropriate relative velocities.
- ② Evaluate the pressure forces on all of the boundaries use straight + parallel streamlines  
→ no pressure gradient.
- ③ Solve for only unknowns using the continuity eqn. (or eqn. of motion)
- ④ Calculate the momentum fluxes in the x-/y-directions for all inflows/outflows.  
(consider all sides of the surface - fluid entity (leaving at the top/bottom of a rectangular CV can have a contribution to the x-momentum flux)).
- ⑤ Assemble the SFME in the component directions, being careful to keep the correct sign (direction) for all forces.

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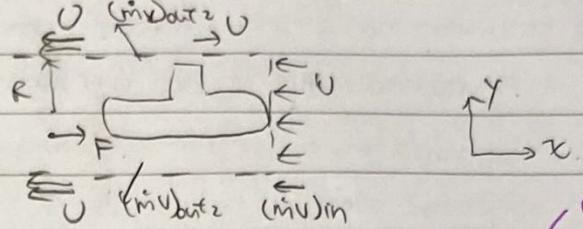
-e.g.: A submarine is towed horizontally at a steady speed  $U$  in deep, still water. Far behind at a fixed distance behind the submarine is an axially-symmetrical wake w/ velocity profile  $v(r) = U(1 - (\frac{r}{R})^2)$ .

The variation of water pressure w/ depth may be assumed to be unaffected by the presence of the submarine. What is the drag force of the submarine?

Method I (cylinder or CU)

①

$$v(r) = U(\frac{r}{R})^2$$



/ no need Bernoulli

②  $P_g$  = hydrostatic pressure, which is the same on both sides  $\rightarrow$  neglected.

$$\dot{m}_{in} = \rho A U = \rho \pi R^2 U$$

$$\begin{aligned} \dot{m}_{out,1} &= \int_A \rho v(r) dA \\ &= \frac{\rho U}{R^2} \int_0^R r^2 (2\pi r) dr \\ &= \frac{1}{3} \rho \pi R^3 U \end{aligned}$$

$\dot{m}_{in} > \dot{m}_{out,1} \rightarrow$  there must be outflow through the top/bottom,  $\dot{m}_{out,2}$ .  
By mass conservation,  $\dot{m}_{in} = \dot{m}_{out,1} + \dot{m}_{out,2}$   
 $\therefore \dot{m}_{out,2} = \rho \pi R^2 U - \frac{1}{3} \rho \pi R^3 U = \frac{2}{3} \rho \pi R^2 U$ .

$$\textcircled{4} \quad (\dot{m}v)_{in} = \dot{m}_{in}(-U) = -\rho \pi R^2 U^2$$

$$\begin{aligned} (\dot{m}v)_{out,1} &= - \int_A \rho (v(r))^2 dA \\ &= -\rho \frac{U^2}{R^4} \int_0^R r^4 (2\pi r) dr \\ &= -\frac{1}{3} \rho \pi R^2 U^2 \end{aligned}$$

$$\begin{aligned} (\dot{m}v)_{out,2} &= \dot{m}_{out,2}(-U) \\ &= -\frac{2}{3} \rho \pi R^2 U^2, \end{aligned}$$

$$\textcircled{5} \quad F = (\dot{m}v)_{out,1} + (\dot{m}v)_{out,2} - (\dot{m}v)_{in}$$

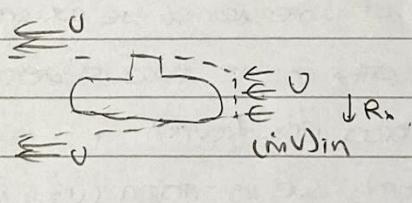
$$= -\frac{1}{3} \rho \pi R^2 U^2 - \frac{2}{3} \rho \pi R^2 U^2 - (-\rho \pi R^2 U^2)$$

$$= \frac{1}{6} \rho \pi R^2 U^2. \quad \therefore \text{Drag on submarine} = -\frac{1}{6} \rho \pi R^2 U^2$$

Method II (truncated cone CU)

②

$$v(r) = U(\frac{r}{R})^2$$



③  $P_g$  = hydrostatic pressure, which is the same on both sides  $\rightarrow$  neglected.

$$\textcircled{3} \quad \dot{m}_{in} = \rho A U = \rho \pi R_0^2 U$$

$$\dot{m}_{out} = \int_A \rho v(r) dA = \frac{1}{3} \rho \pi R^2 U$$

we set  $R_0 = \infty$ ,  $\dot{m}_{in} = \dot{m}_{out}$

$$\therefore \rho \pi R^2 U = \frac{1}{3} \rho \pi R^2 U \rightarrow R_0 = \frac{R}{\sqrt{2}}$$

$$\textcircled{4} \quad (\dot{m}v)_{in} = \dot{m}_{in}(-U) = -\rho \pi (\frac{R}{\sqrt{2}})^2 U^2$$

$$(\dot{m}v)_{out} = - \int_A \rho (v(r))^2 dA = -\frac{1}{3} \rho \pi R^2 U^2$$

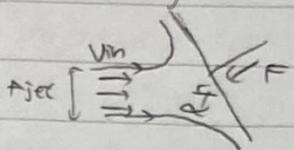
$$\textcircled{5} \quad F = (\dot{m}v)_{out} - (\dot{m}v)_{in}$$

$$= -\frac{1}{3} \rho \pi R^2 U^2 - (-\frac{1}{3} \rho \pi R^2 U^2)$$

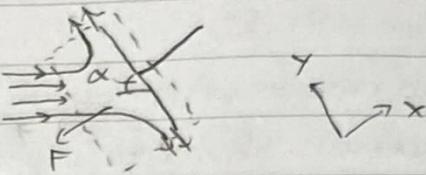
$$= \frac{1}{6} \rho \pi R^2 U^2 \quad \therefore \text{Drag on submarine} = -\frac{1}{6} \rho \pi R^2 U^2.$$

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- e.g.: A jet impinges on a plate at an angle  $\alpha$ . What is the force  $F_{\text{req.}}$  to hold the plate steady?



①



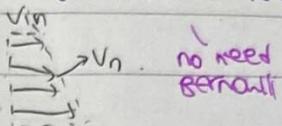
② Streamlines are straight + parallel  $\rightarrow$  no pressure gradient across the jet

$\therefore$  Atmospheric pressure acts everywhere  $\rightarrow$  neglect pressure forces.

$$\dot{m}_{\text{in}} = \dot{m}_{\text{out}}$$

$$\rho V_{\text{in}} A_{\text{jet}} = \rho V_{\text{in}} A_{\text{in}}$$

$$V_{\text{in}} A_{\text{in}} = V_{\text{in}} A_{\text{jet}}.$$



$$\text{④ } (\dot{m}V)_{x,m} = \dot{m}_{\text{in}} V_{x,m} = \dot{m} V_{\text{jet}} / \sin \alpha$$

$$(\dot{m}V)_{x,out1} = \dot{m}_{\text{out}} V_{x,out1}^0 = 0$$

$$(\dot{m}V)_{x,out2} = \dot{m}_{\text{out}} V_{x,out2}^0 = 0.$$

$$\text{⑤ } \Sigma F_x = (\dot{m}V)_{\text{out}}^0 - (\dot{m}V)_{\text{in}}$$

$$-F = -\rho V_{\text{in}} A_{\text{in}} |V_{\text{jet}}| / \sin \alpha$$

$$F = \rho A_{\text{jet}} |V_{\text{jet}}|^2 / \sin \alpha.$$

## Local technique: Forces on a fluid element

Fluid element:

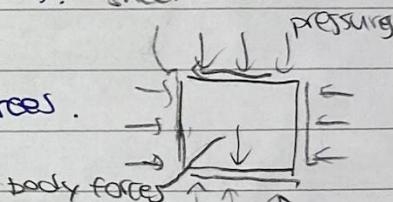
- Consider the local volume of fluid as a particle s.t. changes in velocity and pressure are small compared to global values (but big enough to contain a large no. of molecules so we can treat it as continuum). shear stress

- The forces acting on the fluid element include:

↳ Body forces (inc. gravity)

↳ Pressure forces.

↳ Shear forces due to friction (viscous force).



\* For now, we ignore fluid friction (valid for free stream; invalid for separation).

- Applying NCL on a fluid particle, we get

$$\mathbf{F}_{\text{net}} = m \mathbf{a}$$

- In the special case where the fluid is stationary,  $a = 0$  so pressure forces balance gravitational forces  $\rightarrow \Delta p = \rho g \Delta h$ .

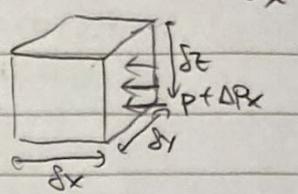
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## Pressure forces

- Consider a fluid element experiencing a pressure gradient  $\frac{\partial p}{\partial x}$ .

- The net force acting on the element in the x-direction is

$$F_{x\text{net}} = p \delta y \delta z - (p + \Delta p_x) \delta y \delta z \\ = -\Delta p_x \delta y \delta z$$



where the change in pressure across the element is given by

$$\Delta p_x = \frac{\partial p}{\partial x} \delta x.$$

Combining the eqns,

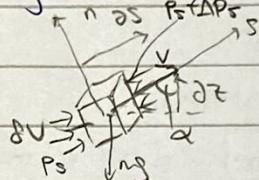
$$F_{x\text{net}} = -\frac{\partial p}{\partial x} \delta x \delta y \delta z = -\frac{\partial p}{\partial x} \delta V$$

- Similarly,  $F_{y\text{net}} = -\frac{\partial p}{\partial y} \delta V$ ;  $F_{z\text{net}} = -\frac{\partial p}{\partial z} \delta V$ . //  $F_{\text{net}} = -\frac{\partial p}{\partial s} \delta V$ , or  $\boxed{F = -\nabla p \delta V}$

**Local technique:** Tangential acceleration of a fluid element.

## Bernoulli's equation

- consider a fluid element travelling along a streamline.



NCL  $\rightarrow$

$$F_{\text{net}} = m \ddot{s}$$

$$-\frac{\partial p}{\partial s} \delta V - mg \cos \alpha = \rho \delta V \cdot \frac{d^2 v}{dt^2}$$

$$-\frac{\partial p}{\partial s} \delta V - \rho \delta V g \frac{\partial p}{\partial z} = \rho \delta V \cdot v \frac{dv}{ds}$$

$$\frac{\partial p}{\partial s} + \rho v \frac{dv}{ds} + \rho g \frac{\partial z}{\partial s} = 0$$

$$\frac{\partial p}{\partial s} + \rho \cdot \frac{1}{2} \frac{dv^2}{ds} + \rho g \frac{\partial z}{\partial s} = 0$$

For incompressible flow, we have constant density, so  $\frac{\partial p}{\partial s}(v) = \frac{1}{2} (\frac{\partial p v^2}{\partial s})$

$$\frac{\partial}{\partial s} (\rho + \frac{1}{2} \rho v^2 + \rho g z) = 0$$

$$\boxed{\rho + \frac{1}{2} \rho v^2 + \rho g z = \text{constant}} \quad [\text{Bernoulli's eqn}]$$

- Bernoulli's eqn. links the local flow pressure to the local flow velocity. Wherever the fluid velocity is high, the pressure is low and vice versa.

- Bernoulli eqn. can only be applied given the following conditions: not valid when there is considerable mixing, the flow is decelerating or near walls

↳ friction is negligible  $\rightarrow$  we can ignore the viscous forces

↳ steady flow  $\rightarrow$  we can ignore any temporal components,

↳ constant density (incompressible flow  $\rightarrow$  we req.  $\frac{\partial p}{\partial s}(v^2) = \frac{1}{2} (\frac{\partial p v^2}{\partial s})$ ) valid for low speeds

↳ 2 pts along a streamline  $\rightarrow$  we integrate over s (i.e. the streamline)

actually, we can use Bernoulli's eqn as long as the 2 pts have the same Bernoulli constant.

(Note: using stream pressure (velocity  $p_{\text{stream}}, v_{\text{stream}})$ ).

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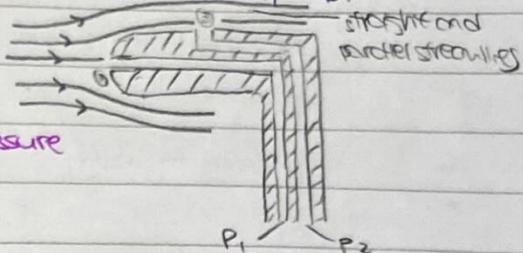
## Pitot-static tube

- The pitot-static tube measures the difference between the pressure and the pitot pressure. This can be used to measure the velocity of the flow.
- At infinity  $\infty$ , the pressure/velocity of the fluid is undisturbed by the presence of the tube  $\rightarrow$  we have freestream pressure  $p_{\infty}$  and freestream velocity  $v_{\infty}$ .
- Consider the streamline  $\infty \rightarrow \odot$ :

$$p_{\infty} + \frac{1}{2} \rho v_{\infty}^2 = p_{\odot} + \frac{1}{2} \rho v_{\odot}^2$$

Consider the streamline  $\infty \rightarrow \odot$ ,

$$p_{\infty} + \frac{1}{2} \rho v_{\infty}^2 = p_{\odot} + \frac{1}{2} \rho v_{\odot}^2$$



Consider the streamlines above  $\odot$ . They are straight and parallel  $\rightarrow$  no pressure gradient.

so we can assume  $p_2 \approx p_{\infty}$   $\rightarrow v_2 \approx v_{\infty}$ .

stagnation state dynamic

$$\therefore p_1 = p_2 + \frac{1}{2} \rho v_{\odot}^2$$

$\cancel{\text{KE of fluid per unit volume}}$   
(dynamic pressure / dynamic head)

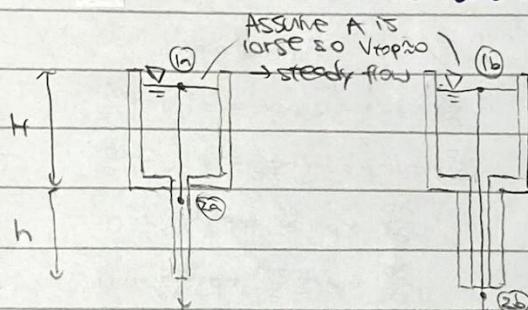
$$p_1 = p_2 + \frac{1}{2} \rho v_{\odot}^2$$

$$v_{\infty} = \sqrt{\frac{2}{\rho} (p_1 - p_2)}$$

Rearranging,

Discharge w/ and w/o a pipe.

- consider 2 tanks, both w/ a hole at the base but one of the holes is connected to a tube:



Consider the streamline  $\odot \rightarrow \odot$  (No pipe)

$$p_{\odot} + \frac{1}{2} \rho v_{\odot}^2 + \rho g H = p_{\odot} + \frac{1}{2} \rho v_{\odot}^2$$

stagnation + parallel streamlines  
at  $\odot \rightarrow p_{\odot} = p_{\infty}$

$$v_{\odot} = \sqrt{2gH}$$

Consider the streamline  $\odot \rightarrow \odot$  (W/ pipe)

$$p_{\odot} + \frac{1}{2} \rho v_{\odot}^2 + \rho g H = p_{\odot} + \frac{1}{2} \rho v_{\odot}^2 + \rho g (-h)$$

stagnation + parallel streamlines  
at  $\odot \rightarrow p_{\odot} = p_{\infty}$

$$v_{\odot} = \sqrt{2g(H-h)}$$

$\rightarrow v_{\odot} > v_{\odot}$  so tank w/ pipe empties out faster.

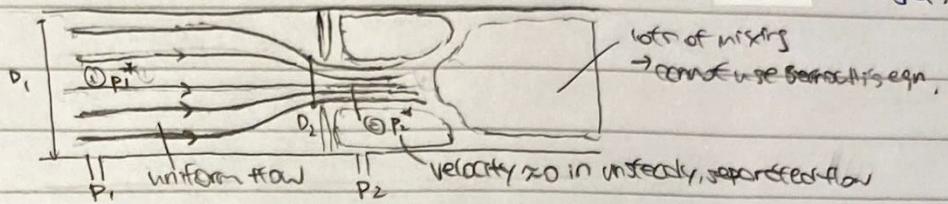
- The streamlines "meet" atmospheric pressure at different heights  $\rightarrow$  different flow rates.

- Moody diagram

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## Orifice plate flow meter

- An orifice plate flow meter measures the mass flow rate.
- A plate w/ a sharp edged hole is installed inside a pipe. Pressure ports are situated upstream and downstream of the plate.
- The pressure  $P_1$  is measured sufficiently far upstream from the orifice so that we can assume the flow velocity across the pipe  $V_1$  is uniform  $\rightarrow P_1$  is the pressure across the entire pipe.
- On the downstream side of the orifice, a cylindrical jet forms which is surrounded by fluid which is almost stationary  $\rightarrow$  no pressure gradient  $\rightarrow P_2$  is the same as the fluid in the jet.



Consider the control streamline ① → ②:

$$\begin{aligned} \text{straight + parallel streamlines} \quad p_1^* + \frac{1}{2}\rho V_1^2 &= p_2^* + \frac{1}{2}\rho V_2^2 \quad \text{Unsteady + Separated flow, Vang20} \\ \rightarrow \frac{dp}{dx} = 0 \quad \therefore p_1 = p_1^* & \quad \rightarrow \text{no momentum to system} \quad \frac{dp}{dx} \rightarrow 0 \\ p_1 + \frac{1}{2}\rho V_1^2 &= p_2 + \frac{1}{2}\rho V_2^2 \quad \therefore p_2 = p_2^* \end{aligned}$$

Applying the continuity eqn,

$$\dot{m}_1 = \dot{m}_2$$

$$\cancel{p_1 A_1^2 V_1} = \cancel{p_2 A_2^2 V_2}$$

$$\text{Define } \beta = \frac{D_1}{D_2}$$

$$V_2 = \beta^2 V_1$$

$$\therefore p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho \beta^4 V_1^2$$

$$V_1 = \sqrt{\frac{2(p_1 - p_2)}{\rho(\beta^4 - 1)}}$$

Therefore the mass flow rate is given by

$$\dot{m} = \rho A_2 V_2$$

$$= \rho \cdot A_2 \cdot \beta^2 \sqrt{\frac{2(p_1 - p_2)}{\rho(\beta^4 - 1)}} \quad \text{where } A_2 = \frac{\pi D_2^2}{4}$$

- Since the streamlines passing through the orifice can't change direction instantaneously, the jet area  $A_2$  is smaller than the orifice area.  $\rightarrow$  We set

$$A_2 = C_d A_{\text{orifice}}$$

*this would only apply on infinite  $\beta$ .*

where  $C_d$  is the discharge coefficient.

Therefore, the mass flow rate is given by

$$\boxed{\dot{m} = \rho C_d A_{\text{orifice}} \beta^2 \sqrt{\frac{2(p_1 - p_2)}{\rho(\beta^4 - 1)}}}$$

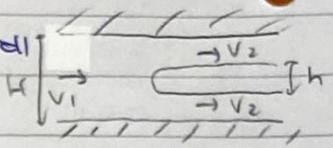
- The value of  $C_d$  is approximately 0.6

- Using a sharp edged orifice fixes the location at which the flow separates across a range of flow rates  $\rightarrow$  effects of friction are small  $\rightarrow C_d$  only has weak dependence on the Reynolds no. Re and contraction ratio  $\beta$ .

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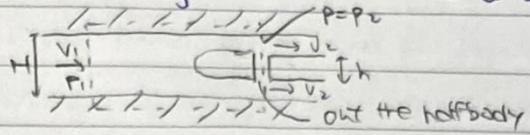
e.g.: Combined use of Bernoulli's eqn.

What is the drag on a half body of width  $h$  in a channel of overall width  $H$  and a depth (into the page) of  $w$ .



① Any CS would cut through the half body and the pressure acting ~~perp~~ is not obvious.

→ cut the half body of the CS, setting the exit pressure to be uniform at  $p_2$ .



$$\textcircled{2} \quad \sum(pA)_x = (p_1 - p_2)wh$$

③ continuity

$$m_{in} = m_{out}$$

$$\rho V_1 H w = \rho V_2 (H-h) w$$

$$V_2 = \frac{H}{H-h} V_1$$

Bernoulli

$$p_1 + \frac{1}{2} \rho V_1^2 = p_2 + \frac{1}{2} \rho V_2^2$$

$$p_1 - p_2 = \frac{1}{2} \rho (V_2^2 - V_1^2)$$

$$= \frac{1}{2} \rho V_1^2 \left( \left( \frac{H}{H-h} \right)^2 - 1 \right)$$

$$= \frac{1}{2} \rho V_1^2 \frac{h(2H-h)}{(H-h)^2}$$

$$\textcircled{4} \quad (\dot{m}V)_{in} = \rho V_1^2 H w \quad (\dot{m}V)_{out} = \rho V_2^2 (H-h) w = \rho V_1^2 \frac{H^2}{H-h} w$$

$$\textcircled{5} \quad F + \sum(pA)_x = (\dot{m}V)_{out} - (\dot{m}V)_{in}$$

$$F + (p_1 - p_2)wh = \rho V_1^2 \frac{H^2}{H-h} w - \rho V_1^2 H w$$

$$F = \rho V_1^2 H w \left( \frac{H}{H-h} - 1 \right) - \frac{1}{2} \rho V_1^2 \frac{w H (2H-h)}{(H-h)^2}$$

$$= \frac{1}{2} \rho V_1^2 w H \left[ \frac{2H(H-h) - 2(H-h)^2 - h(2H-h)}{(H-h)^2} \right]$$

$$= -\frac{1}{2} \rho V_1^2 w \frac{hH}{(H-h)^2}$$

$$\therefore \text{Force on half-body} = \frac{1}{2} \rho V_1^2 w \frac{hH}{(H-h)^2}.$$

\* When  $H \gg h$ ,  $F \rightarrow 0$

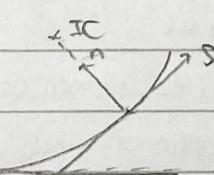
**Local technique: Normal acceleration of a fluid element**

Intrinsic coordinate convention,

- A different convention is used in fluids for the normal direction of intrinsic coords:
- ↳ In mechanics, the normal direction points towards the instantaneous centre.
- ↳ In fluids, the normal direction points away from the instantaneous centre.

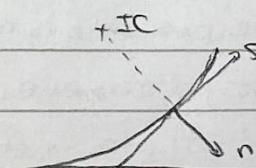
Mechanics

IC



Fluids

+ IC

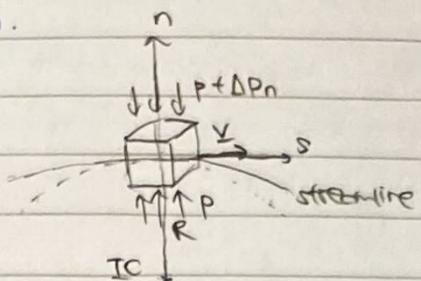


→ the  $\Delta n$  for an increase in radius  $\Delta R$ .

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streamline-curvature equation.

- consider a fluid element



NCL J

$$F_{\text{net}} = m \ddot{n}$$

$$-\frac{\partial p}{\partial n} \hat{s} \hat{v} = -\rho \ddot{s} V \frac{V^2}{R}$$

$$\boxed{\frac{\partial p}{\partial n} = \rho \frac{V^2}{R}}$$

[streamline-curvature eqn.]

- The streamline-curvature eqn./radial eqn. eqn. relates streamwise normal pressure gradient to streamline curvature.

- The eqn. shows that whenever there are curved streamlines, there must be a pressure gradient across them. There will always be lower pressure towards the centre of curvature.

parallel flow has no pressure gradient

alternatively, no change in direction  
→ no force, i.e. no pressure difference

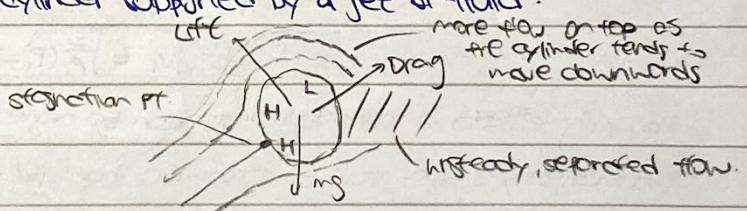
- For a straight line, the instantaneous radius of curvature  $R \rightarrow \infty$ , so  $\frac{\partial p}{\partial n} \rightarrow 0$ . This means whenever streamlines are straight and parallel, there cannot be a pressure gradient across them. (except for fluid static variations which occur even if the fluid is stationary).

- As such, the pressure in a free jet is always the same as ambient pressure.

## Coanda effect

- The Coanda effect is the tendency of a fluid jet to stay attached to a convex surface.

- Consider a "floating" cylinder supported by a jet of fluid.



- The self weight causes an asymmetry in the streamlines, resulting in a pressure difference and hence lift and drag.

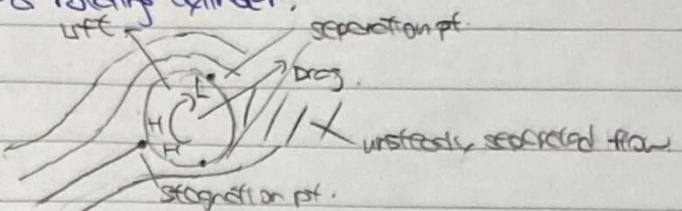
- Due to the Coanda effect, there is no separation pt. for inviscid flow.

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## Magnus effect

- The Magnus effect is associated w/ a spinning object moving through a fluid.

- Consider the streamlines near a rotating cylinder,

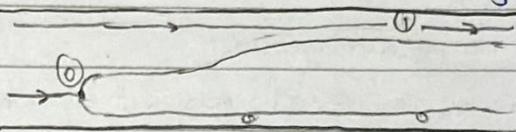


- The fluid at the top has a separation pt. further downstream than the fluid at the bottom due to the rotation. This asymmetry in streamlines result in a pressure difference and thus lift and drag.

- e.g.: Wind tunnel testing of a train in a tunnel.

the train is stationary, we blow 80ms⁻¹ winds at the train

A train model is tested in a wind tunnel at a free-stream velocity  $V_\infty$  of  $80\text{ms}^{-1}$ . At ①, the measured pressure  $P_1$  is  $98785\text{Pa}$ . The pressure at the stagnation pt. ⑥  $P_6$  is  $100000\text{Pa}$ .



What pressure would be expected at ① on a geometrically similar real train travelling in a similar tunnel at  $256\text{km/h}$ ?  $\rho_{air} = 1.2\text{kgm}^{-3}$  in both cases.

Consider the streamline  $\infty \rightarrow ⑥$  [model train]

$$P_6 + \frac{1}{2} \rho V_6^2 = P_\infty + \frac{1}{2} \rho V_\infty^2$$

this implies  $P_6$  is below atmospheric pressure in the wind tunnel!

Consider the streamline  $\infty \rightarrow ①$  [model train]

$$P_1 + \frac{1}{2} \rho V_1^2 = P_\infty + \frac{1}{2} \rho V_\infty^2$$

$$\therefore P_1 = P_\infty + \frac{1}{2} \rho V_1^2$$

$$\rightarrow V_1 = \sqrt{\frac{2}{\rho} (P_0 - P_1)} = \sqrt{\frac{2}{1.2} (100000 - 98785)} = 45\text{ms}^{-1}$$

Assuming effects of compressibility/friction is neglected, — decrease in pressure difference  $(P_0 - P_1)$  is  $\sim 1.2\%$ , < 5%

Geometrically similar train  $\rightarrow$  streamlines have the same shape  $\rightarrow$  same ratio of velocities.

$$\left(\frac{V_1}{V_\infty}\right)_{\text{model}} = \left(\frac{V_1}{V_\infty}\right)_{\text{train}}$$

$$\left(\frac{V_1}{V_\infty}\right)_{\text{train}} = \frac{45}{80} = \frac{3}{8}$$

The freestream pressure in the tunnel  $P_\infty$  would be  $100000\text{Pa}$  (atmospheric)

Consider the streamline  $\infty \rightarrow ①$  [real train]

$$P_1 + \frac{1}{2} \rho V_1^2 = P_\infty + \frac{1}{2} \rho V_\infty^2$$

$$P_1 = P_\infty + \frac{1}{2} \rho V_\infty^2 \left( 1 - \left(\frac{V_1}{V_\infty}\right)^2 \right)$$

$$= 100000 + \frac{1}{2} (1.2) \left(\frac{256}{80}\right)^2 \left( 1 - \left(\frac{3}{8}\right)^2 \right)$$

$$= 97300\text{Pa}$$

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## Pressure coefficient

- We can use Bernoulli's eqn to derive the pressure coefficient  $C_p$ .

$$P + \frac{1}{2} \rho V^2 = P_\infty + \frac{1}{2} \rho V_\infty^2$$

$$P - P_\infty = \frac{1}{2} \rho V_\infty^2 \left( 1 - \left( \frac{V}{V_\infty} \right)^2 \right)$$

$$C_p = 1 - \left( \frac{V}{V_\infty} \right)^2 = \frac{P - P_\infty}{\frac{1}{2} \rho V_\infty^2}$$

$$\frac{1}{2} \rho V_\infty^2$$

- The measured pressures are normalised by the freestream dynamic pressure/dynamic head.
- The freestream dynamic pressure can be measured in the undisturbed freestream conditions, well upstream of the flow using a Pitot static tube.
- The pitot (stagnation) pressure  $P_0$  is measured at the forward facing hole;  $P_\infty$  is measured at the sideways facing holes.

$$P_0 + \frac{1}{2} \rho V_0^2 = P_\infty + \frac{1}{2} \rho V_\infty^2$$

$$P_0 - P_\infty = \frac{1}{2} \rho V_\infty^2$$

- Therefore, we can express the pressure coefficient as:

$$\begin{aligned} C_p &= 1 \rightarrow V = 0, P = P_0 \\ C_p &= 0 \rightarrow V = V_\infty, P = P_\infty \\ C_p &< 0 \rightarrow V > V_\infty, P < P_\infty \end{aligned}$$

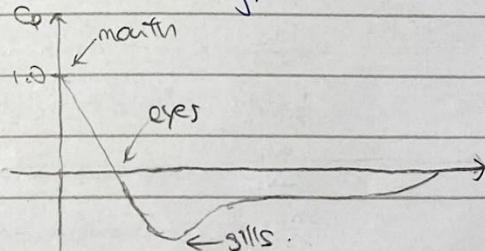
$$C_p = \frac{P - P_\infty}{P_0 - P_\infty}$$

all quantities are  
directly measured

- If 2 flows are geometrically similar, the effects of friction are small and the pressure difference is small compared to the average pressure, then regardless of absolute dimensions or absolute velocities, the pressure coefficients are the same.

## Pressure coefficient on a fish.

- Fish eyes are often located at a pt. where  $C_p \approx 0$ . This means the pressure on their eyes is always close to the freestream pressure  $\rightarrow$  no pressure change when they accelerate  $\rightarrow$  vision does not distort when swimming at different speeds.
- Their mouth is a stagnation pt., w/  $C_p = 1$ . This sets the max. pressure difference to the exit of their gills, which are located in the region of the min. pressure coefficient.  $\rightarrow$  help them breathe w/ minimal effort when swimming.

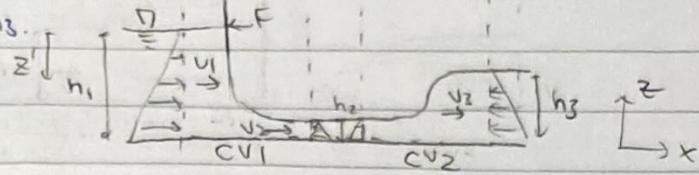


\* the profile of pressure coefficient is nondimensional  $\rightarrow$  independent of absolute size of fish

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- ej: Force on sluice gate and the hydraulic jump.

What is the force on the fluid per unit width  $F$  and the height of the free surface after the hydraulic jump  $h_3$ .



In order to apply the SPME, we need to assume

$\rightarrow V_1$  and

$\rightarrow$  the flow is steady

$\rightarrow$  the effects of friction can be ignored on the bottom of the channel. (CV boundary).

① Consider control volume 1 (has unit width)

$$\textcircled{2} \quad \sum(pA)_x = \int_0^{h_1} \rho g z' dz' - \int_0^{h_2} \rho g z' dz'$$

$$= \rho g \frac{h_1^2}{2} - \rho g \frac{h_2^2}{2}$$

$$\textcircled{3} \quad h_1 v_1 = h_2 v_2$$

define  $h'$  
$$h' = \frac{v_2}{v_1} = \frac{h_2}{h_1}$$

$$\textcircled{4} \quad (\text{inv})_{\text{in}} = \rho v_1^2 h_1 \quad (\text{inv})_{\text{out}} = \rho v_2^2 h_2$$

$$\textcircled{5} \quad \sum F_x + \sum(pA)_x = (\text{inv})_{\text{out}} - (\text{inv})_{\text{in}}$$

$$-F + \rho g \frac{h_1^2}{2} - \rho g \frac{h_2^2}{2} = \rho v_2^2 h_2 - \rho v_1^2 h_1$$

$$F = \frac{1}{2} \rho g h_1^2 \left( 1 - \left( \frac{h_2}{h_1} \right)^2 \right) + \rho v_1^2 h_1 \left( 1 - \frac{h_2}{h_1} \right)$$

force coefficient  
(force / dynamic pressure  $\rho v_1^2$ ) 
$$\frac{F}{\rho v_1^2 h_1} = \frac{gh_1}{v_1^2} \left( 1 - \frac{1}{h_1^2} \right) + 2 \left( 1 - \frac{h_2}{h_1} \right)$$

② Consider control volume 2 (has unit width)

$$\textcircled{2} \quad \sum(pA)_x = \int_0^{h_2} \rho g z' dz' - \int_0^{h_3} \rho g z' dz'$$

$$= \rho g \frac{h_2^2}{2} - \rho g \frac{h_3^2}{2}$$

$$\textcircled{3} \quad h_2 v_2 = h_3 v_3$$

define  $\alpha$  
$$\alpha = \frac{v_2}{v_3} = \frac{h_3}{h_2}$$

cannot use Bernoulli as there is considerable mixing in the hydraulic jump.

$$\textcircled{4} \quad (\text{inv})_{\text{in}} = \rho v_2^2 h_2 \quad (\text{inv})_{\text{out}} = \rho v_3^2 h_3$$

$$\textcircled{5} \quad \sum F_x + \sum(pA)_x = (\text{inv})_{\text{out}} - (\text{inv})_{\text{in}}$$

$$\rho g \frac{h_2^2}{2} - \rho g \frac{h_3^2}{2} = \rho v_3^2 h_3 - \rho v_2^2 h_2$$

$$\frac{1}{2} \rho g \frac{h_2^2}{2} \left( 1 - \left( \frac{h_3}{h_2} \right)^2 \right) = \rho v_3^2 h_3 \left( \left( \frac{v_3}{v_2} \right)^2 \cdot \left( \frac{h_3}{h_2} \right) - 1 \right)$$

$$\frac{g h_2}{v_2^2} \left( 1 - \alpha^2 \right) = 2 \left( \frac{1}{\alpha^2} - \alpha - 1 \right)$$

$$\frac{g h_2}{v_2^2} \left( 1 + \alpha \right) \left( 1 - \alpha \right) = \frac{2}{\alpha} \left( 1 - \alpha \right)$$

$$(1 - \alpha) \left[ \frac{g h_2}{v_2^2} \left( 1 + \alpha \right) - \frac{2}{\alpha} \right] = 0$$

$$\therefore \alpha = 1 \quad (\text{no jump})$$

or

$$\frac{g h_2}{v_2^2} \left( 1 + \alpha \right) - \frac{2}{\alpha} = 0$$

$$\alpha^2 + \alpha - 2 \frac{v_2^2}{g h_2} = 0$$

$$\alpha = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{8 v_2^2}{g h_2}} \quad (\text{only take the root})$$

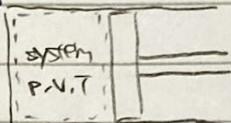
### Fundamental concepts and definitions

#### Thermodynamic systems.

- A thermodynamic system is defined as an arbitrary portion of the universe w/ fixed or moveable boundaries which may contain matter or energy, or both.
- A closed system is a fixed quantity of matter, around which a boundary can be drawn.
- Everything inside the boundary is the system, everything outside the boundary is the surroundings.
- Since no matter can cross the boundary, mass consv. is satisfied for a closed system.
- Energy can cross the boundary in the form of heat and work.
- If there is no heat or work exchange between the closed system and its surroundings, the system is an isolated system.

#### Thermodynamic properties and thermodynamic state

- Consider a system comprising a fixed mass of gas enclosed in a cylinder by a piston:



- The system possesses a no. of thermodynamic properties such as pressure p, volume V, and temp. T, which together define its thermodynamic state.
- Thermodynamic properties are quantifiable characteristics of a system that depend only on the state of the system and not on how it arrived at that state. (work and heat are not thermodynamic properties but process quantities)
- Thermodynamic properties are either extensive or intensive:
  - ↳ Extensive (ext/m) properties depend on the size/extent of the system (e.g.: V).
  - ↳ Intensive (int/int) properties do not depend on the size/extent of the system (e.g.: p, T).
- Specific properties are properties per unit mass – subset of intensive properties. By convention we use small letters for specific properties and capital letters for extensive properties.

#### The two property rule

- For simple compressible systems, (i) its intensive properties are uniform throughout, and (ii) the effects of electricity, capillarity, gravity and magnetism can be ignored.
- simple compressible systems are typically comprised of pure substances or non-reactive mixtures of gases
- For simple compressible systems at rest, 2 independent intensive properties and the mass are sufficient to define the state, provided the system is in eqm.  $\downarrow p \text{ and } V \text{ are dependent}$
- The two property rule is a consequence of there being only 2 ways to change the state of a simple compressible system – by heat and work interactions w/ the surroundings.

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## Thermodynamic equilibrium and quasi-equilibrium processes

- A thermodynamic system is in eqm when none of its thermodynamic properties are changing in time at a measurable rate / for non-quasi-eqm processes, only the end states can be related.
- The laws of thermodynamics can be applied to relate eqm states of a system but cannot in general tell us much about the intervening process.
- Therefore, it is not generally possible to plot the process on the p-v diagram (as the pressures and densities may vary from st. to st. in the system during the process)
- For a quasi-eqm./quasi-static process, where departures from eqm. are kept very small and the process effectively passes through a series of eqm. states, we can plot on the p-v diagram.  
↳ not in eqm  $\rightarrow$  not quasi-eqm.

**Zeroth law of thermodynamics**

Thermal equilibrium, and the zeroth law

- The zeroth law states if systems B and C are each separately in thermal eqm. w/ system A, then they would be in thermal eqm. w/ each other if brought into thermal contact.
- Bodies that are in thermal eqm. must have some common property, which we recognize as temperature. (i.e. bodies in thermal eqm. have a common temp.).

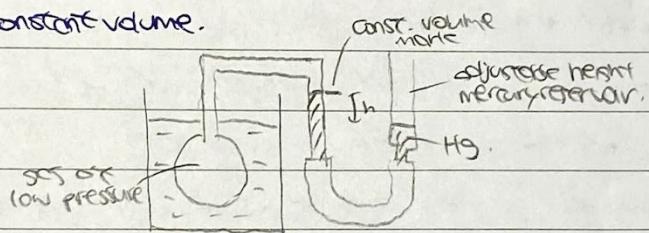
**Temperature scales****Empirical temperature scale**

each value of the thermometric property must only correspond to 1 temperature

- An empirical temperature scale is established by:
  - (i) selecting a thermometric substance for which some property (the thermometric property) varies w/ temp. in a well-behaved fashion.
  - (ii) specifying the values of temperature at 2 fixed, reproducible pts (e.g. ice/steam pt. temp.)
- Intermediate readings will depend on the selected thermometric property as it may not necessarily vary linearly w/ temp. hence empirical

**The ideal gas temperature scale**

- The ideal gas temperature scale is based on the (absolute) pressure of a sample of gas maintained at constant volume.



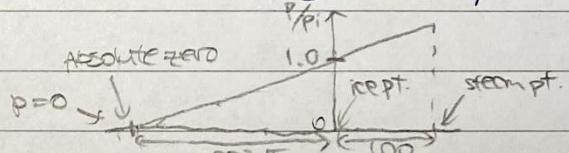
- This scale was considered superior to other empirical scales because many different gases would give the same readings for temperature, provided the density of gas was sufficiently low.
- Also temperature defined in this manner has a physical interpretation based on the kinetic theory of gases — temp. is prop. to the mean translational KE of molecules.
- The ideal gas scale permits the definition of a nonarbitrary absolute zero of temperature.

From the ideal gas eqn.,  $pV = nRT \rightarrow T = \frac{PV}{nR}$ , i.e.  $T \propto P$ .

$$\text{so } \frac{T_s}{T_i} = \frac{P_s}{P_i}$$

where "s" and "i" denote the steam and ice pt. respectively.

- By choosing the size of the unit of temp using  $T_s - T_i = 100$ , we can extrapolate to  $P=0$ .



- Absolute zero is found to be 273.15 degrees below the ice pt.

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## Thermodynamic temperature

- Lord Kelvin used the second law to define a temperature scale that is independent of any thermometric substance.

- consider a scale based on  $\frac{Q_1}{Q_2}$  for a reversible heat engine ( $\theta$  as new temp. scales)

Thm. 2 for cyclic heat engine states that all reversible heat engines operating between the same thermal reservoirs are equally efficient.  $\rightarrow \eta = f_n(\theta_1, \theta_2)$

$$\text{As } \eta = \frac{W}{Q_1} = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1}, \quad \frac{Q_1}{Q_2} = \frac{1 - \eta}{\eta} = f_n(\theta_1, \theta_2),$$

considering the dimensions,  $\frac{Q_1}{Q_2} = f_n(\frac{\theta_1}{\theta_2})$ .

$$\text{If we choose the simplest form of function, } \frac{Q_1}{Q_2} = \frac{\theta_1}{\theta_2}$$

- As we found  $\frac{Q_1}{Q_2} = \frac{T_1}{T_2}$  using the Carnot cycle and ideal gas temp. scale, we find that the thermodynamic and ideal gas temp. scales are identical, i.e.

$$\boxed{\frac{Q_1}{Q_2} = \frac{\theta_1}{\theta_2} = \frac{T_1}{T_2}}$$

- therefore the ideal gas temp. can be adopted as a thermodynamic temp. scale w/o dependency on ideal gas properties.  $\rightarrow$  we continue to use  $T$  to denote thermodynamic temp.

- A single fixed, reproducible pt. is req. to fix the scale — we chose the triple pt. of pure water.

$$\boxed{T_{TP} = 273.16 \text{ K}} \quad (= 0.01^\circ\text{C})$$

## Practical temperature measurement

- Measurement of thermodynamic temp. is hampered by the fact that no heat engine is perfectly reversible. The constant volume gas thermometer is bulky and cannot be applied in cases where the gas liquefies or dissociates.

- The following are examples of practical means of temp. measurement:

↳ Mercury-in-glass thermometers

↳ Thermocouples (Thermoelectric effect).

↳ platinum resistance thermometers (resistance increases w/ temp.)

↳ Thermistors

↳ Optical pyrometry (wavelength of emitted radiation depends on temp.)

- To calibrate these devices, an International Practical Scale of Temperature has been established based on 11 fixed pts.

## FIRST LAW OF THERMODYNAMICS

First law for closed systems.

- A closed system does not exchange matter w/ its surroundings, but it may exchange energy in the form of either heat or work.
- The FIRST LAW is a general statement of ENERGY CONSERVATION. For a closed system,

$$Q - W = \Delta E$$

where  $Q$  is the quantity of heat transferred to the system from the surroundings.

$W$  is the work transfer from the system to the surroundings.

$\Delta E$  is the change in energy of the system.

take care  
write  
sign  
convention

- Heat and Work are modes of energy transfer only – they are not possessed by the system, therefore they are not properties of a system.

Work transfer, / not a system property

- Work transfer between a system and its surroundings may take a variety of forms, inc. mechanical, electrical and magnetic work.
- Formally, work is done by a system on its surroundings if the sole effect of everything external to the system could have been the raising of a weight

Mechanical work

- Mechanical work transfer occurs when a system boundary moves by virtue of forces acting at the boundary.

$$W = \int F \cdot dS$$

where  $F$  is the force exerted by the system on an element of the boundary

$dS$  is the infinitesimal displacement of that element

Displacement work (pV work)

quasi-static process

- Displacement work is a type of mechanical work due to fully resisted expansion of a gas.
- The force exerted by the pressure of gas acting over the surface of a piston is  $F = pA$ . If the piston is displaced to the right a small distance  $dx$  whilst resisted by an opposing force, the work done by the gas is

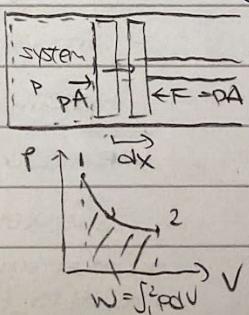
$$\delta W = pA dx = p dV$$

- The total work done between 2 states is

$$W = \int_{V_1}^{V_2} p dV$$

valid for  
only geometry

i.e. the area under the curve in the p-V diagram

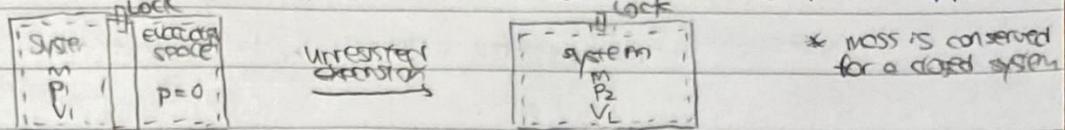


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Unrestricted expansion (free expansion)

quasi-static process

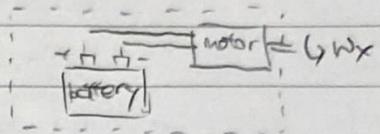
- The "pdV" expression is only valid if the expansion of gas is fully resisted. This is necessary in order that the pressure remains uniform throughout the system during the process.
- In the case of unrestricted expansion, no useful work would be extracted from the expansion;



simply draw the system boundary to enclose the final volume occupied by the gas and piston.

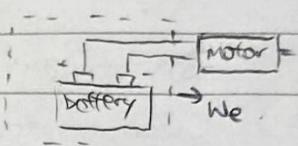
Shaft work

- Shaft work can be done by means of a rotating shaft



- If the torque exerted by the shaft is T, then for angular displacement dθ,

$$\delta W_x = T d\theta$$



Electrical work

- When electrical energy crosses a system boundary, it is always considered as a work transfer.
- If the end of the electrical supply is E, then the work done in passing an infinitesimal charge dQ around the circuit is

$$\delta We = EdQ$$

Other types of work.

- Other types of work inc. elastic work, magnetic work, capillary work.
- In most cases, we can write the infinitesimal work transfer in the form

$$\delta W = \phi dX$$

where  $\phi$  is some intensive property and X is the conjugate extensive property.

Heat transfer

not a system property

note that rate of heat transfer between 2 bodies increases w/ the temp. diff between them

- Heat transfer is energy transfer that is not work. It occurs by virtue of a temp. diff, from hotter bodies to cooler bodies (but may not necessarily result in a change in temp.)
- Heat Q can cross a system boundary by 2 main mechanisms:

↳ Conduction - occurs in solids, liquids, gases, the energy transfer from more energetic to less energetic particles due to interactions between them.

↳ Radiation - emitted by matter due to changes in electronic configurations of the atoms or molecules. The energy is radiated as EM waves and can pass through a vacuum.

\* Convection is strictly just a redistribution of energy within a system or its surroundings due to fluid motions.

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## Energy

- Energy  $E$  is a system property. Traditionally, changes in  $E$  are divided into 3 components:

$$\Delta E = \Delta KE + \Delta PE + \Delta U$$

$\Delta KE = \frac{1}{2}m(V_2^2 - V_1^2)$  is the change in kinetic energy of the system as a whole

$\Delta PE = mg(z_2 - z_1)$  is the change in gravitational potential energy of the system as a whole

$\Delta U = U_2 - U_1$ , i.e. the change in internal energy of the system

- In general,  $U$  may include contributions from magnetic, electrical and chemical energy

- For a simple compressible system,  $U$  is composed of random thermal energy (KE) of the atoms / molecules, plus only PE due to intermolecular forces.

- $E$ ,  $KE$ ,  $PE$  and  $U$  are all extensive properties. Corresponding specific properties are obtained by dividing by the system mass  $m$

## Other forms of the First Law

- The first law is generally more useful when expressed in terms of internal energy  $U$ .

For no change in GPE or KE of the system,

$$\Delta E = \Delta U$$

so the first law is reduced to

$$Q - W = \Delta U$$

- Dividing through by the system mass  $m$  gives

$$q - w = \Delta u$$

useful for finding relationship b/w properties during a quasistatic process.

- For an infinitesimally small change, we can use the differential form

$$dq - dw = du$$

L' imperfect differentials are used because heat and work depend on the process path, i.e. not system properties.

## Adiabatic processes

- An adiabatic process is one that involves no heat exchange between the system and the surroundings i.e.  $Q=0$ .

- It is an idealised concept, req. perfectly insulating walls or an infinitely short duration (so no time for heat transfer). However, many processes are approximately adiabatic.

- For an adiabatic process from state 1 to state 2, the first law becomes

$$W = -\Delta U = U_1 - U_2$$

- Since  $U$  is a property of the system, the net work done on/by the system during an adiabatic process is independent of the path taken

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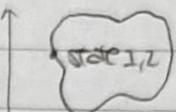
## Cyclic processes

- A cyclic process is one for which the system is returned to its original state
- Since U is a property of the system (function of state), the change in U for a cycle is zero:

$$\oint \delta Q = \oint \delta W$$

where the  $\oint$  sign denotes the summation of all the infinitesimal energy transfers around the cycle.

- The net work output thus equals the net heat input for a cyclic process



## Property relations and ideal gases

### Ideal gas

- For ideal gases, the average separation of molecules is such that the intermolecular forces are insignificant, except during molecular collisions. They obey the relation:

$$pV = n\bar{R}T \quad [\text{Eqn. of state}]$$

where  $n$  is the no. of moles of the gas.  $n = N/N_A$

and  $\bar{R}$  is the universal gas constant.  $\bar{R} = 8.3145 \text{ kJ mol}^{-1} \text{ K}^{-1}$ .

- As we usually work in terms of mass rather than moles, for a particular gas of molar mass  $M$  (measured in  $\text{kg/mol}$ ),

$$pV = mRT$$

$$pV = RT$$

$$P = pRT$$

where  $m$  is the mass of gas.  $m = NM$

$R$  is the specific gas constant.  $R = \bar{R}/M$ .

$$\begin{aligned} &\text{for air, 21% O}_2 \text{ (M=32), 79% N}_2 \text{ (M=28.15),} \\ &M_{air} = 0.21 \times 32 + 0.79 \times 28.15 = 28.76 \\ &\text{so } R = \bar{R}/M = 8.31/28.76 = 0.287 \text{ kJ kg}^{-1} \text{ K}^{-1} \end{aligned}$$

- For ideal gases, the specific internal energy  $U$  is a function of only  $T$ ,

$$U = f_n(T)$$

for an ideal gas, int. one negligible  
→ no PE from molecular interactions  
 $\therefore U$  doesn't depend on molecular separation

- An ideal gas can either be a semi-perfect gas or a perfect gas.

## Enthalpy

- In a constant pressure process, for which the only work is "pdV" work, so

$$W = \int pdV = p \int dV = p \Delta V$$

the FIRST law can thus be written in the form

$$\begin{aligned} Q &= \Delta U + p \Delta V \\ &= \Delta(U + pV) \end{aligned}$$

$Q = \Delta H$  only for isobaric processes  
but  $\Delta H = M \int pdT$  for all processes

- since  $U$ ,  $p$  and  $V$  are all properties of the system, the combination  $U + pV$  is also a property of the system. This is called the enthalpy  $H$ . Thus for constant pressure heat addition,

$$Q = \Delta H = m \Delta h$$

where

$$H = U + pV$$

$$h = U + pV$$

for an ideal gas,  $pV = RT$   
so  $h = U + RT$ , i.e.  $h = f_n(T)$

\* Enthalpy and energy have the same units but enthalpy is not energy

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## Specific heat capacities

- Heat transfer to a system usually increases its temp. (not necessarily true - change of state)
- The specific heat capacity is defined as the heat req. to raise the temp. of 1 kg of a substance by 1K.
- For a constant volume process,  $Q = \Delta U$  ( $W = 0$ )
- For a constant pressure process,  $Q = \Delta H$  ( $W = p\Delta V$ )
- The corresponding specific heat capacities are:

Isothermal specific heat capacity

$$C_V = \left( \frac{\partial U}{\partial T} \right)_V$$

Isobaric specific heat capacity

$$C_P = \left( \frac{\partial H}{\partial T} \right)_P$$

const. volume

const. pressure

These defns are valid for all substances.

- Although these defns are derived from the consideration of const. volume/pressure processes, given an ideal gas, for all processes  $C_V$  relates to  $\Delta U$ ,  $C_P$  relates to  $\Delta H$ :

For an ideal gas,  $U = f_n(T)$  and  $H = f_n(T)$ .

$$\therefore C_V = \frac{dU}{dT} \rightarrow dU = C_V dT \quad \text{and} \quad C_P = \frac{dH}{dT} \rightarrow dH = C_P dT.$$

Relationship between  $C_P$  and  $C_V$ .

- For an ideal gas,

$$PV = RT \quad \text{and} \quad U = f_n(T)$$

$$\text{From this, we find that} \quad C_V = \frac{dU}{dT} \quad \text{and} \quad C_P = \frac{dH}{dT}.$$

$$\text{By defin., } H = U + PV, \text{ so}$$

$$H = U + RT$$

Differentiate w.r.t. T,

$$\frac{dH}{dT} = \frac{dU}{dT} + \frac{d(RT)}{dT}$$

$$C_P = C_V + R$$

$$\therefore C_P - C_V = R$$

for perfect gases  
 $\gamma = 1.67$  monoatomic gas  
 $\gamma = 1.40$  diatomic gas.

- The ratio  $C_P/C_V$  crops up frequently and is given the symbol  $\gamma$ , so

$$\gamma = \frac{C_P}{C_V}$$

- Notice that  $\gamma - 1 = \frac{C_P}{C_V} - \frac{C_V}{C_V} = \frac{R}{C_V}$ , we can say

$$C_V = \frac{R}{\gamma - 1}$$

and

$$C_P = \frac{\gamma R}{\gamma - 1}$$

for perfect gases

$$U = U_0 + C_V T = U_0 + \frac{C_V}{R} PV = U_0 + \frac{1}{\gamma - 1} PV$$

$$H = U_0 + C_P T = U_0 + \frac{C_P}{R} PV = U_0 + \frac{\gamma}{\gamma - 1} PV$$

semi-perfect gases, and perfect gases.

- For an ideal gas, we find that

$$U = f_n(T)$$

;

$$H = f_n(T)$$

Using the defn of  $C_V$  and  $C_P$ ,

$$\Delta U = U_2 - U_1 = \int_{T_1}^{T_2} C_V dT$$

$$\Delta H = H_2 - H_1 = \int_{T_1}^{T_2} C_P dT$$

$\rightarrow C_V$  and  $C_P$  are functions of T only, i.e.

$$C_V = f_n(T)$$

$$C_P = f_n(T)$$

- Ideal gases for which specific heat capacities vary w/ temp are semi-perfect gases;

Ideal gases for which specific heat capacities are constant are perfect gases.

- Over large temp. diff, air/many gases should be treated as semi-perfect gases.

- Since the specific heat capacity for a perfect gas is constant,  $\Delta U = C_V \Delta T$  ;  $\Delta H = C_P \Delta T$

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## Non-ideal gases (Real gases)

- At intermediate pressures and low temp, attractive intermolecular forces are no longer negligible.
- Non-ideal gases can be modelled using the Van der Waals' eqn:

$$(P + \frac{n^2 a}{V^2})(V-nb) = nRT \quad [\text{Eqn. of state}]$$

- For simple compressible systems composed of pure substances, the two property rule applies  $\rightarrow$  relationships between properties of non-ideal gases are shown in tabular form.

e.g.  $u = f_n(p, T)$ ;  $v = f_n(p, T)$

## Summary of gas classifications

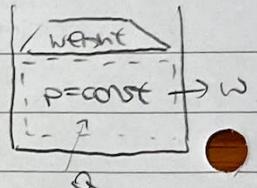
All gases	
Non-ideal gases	Ideal gases
$P = f_n(V, T)$	$PV = RT$
$u = f_n(V, T)$	$u = f_n(T)$
use property tables e.g.: steam, refrigerants	
	Semi-perfect gases
	$C_V, C_P = f_{ns}(T)$
	e.g.: air at high temp. products of combustion
	perfect gases
	$C_P, C_V = \text{constants}$
	e.g.: air at normal temp. monatomic and some diatomic

## Application of First law to perfect gases

### Isobaric process

- Isobaric means constant pressure. If the only work is "pdV" work, then

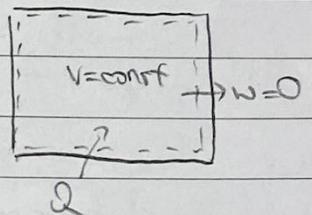
$$\begin{aligned} W &= \int_1^2 pdV = P \int_1^2 dV = P(V_2 - V_1) = mR(T_2 - T_1) \\ \text{Assume quasi-eqm.} \quad \Delta U &= MC_V(T_2 - T_1) \\ \therefore Q &= W + \Delta U = m(R + C_V)(T_2 - T_1) = mC_P(T_2 - T_1) \end{aligned}$$



### Isochoric process

- Isochoric means constant volume. In the absence of stirring, then

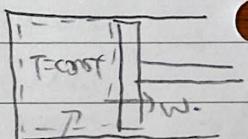
$$\begin{aligned} W &= \int_1^2 pdV = 0 \\ \text{Assume quasi-eqm.} \quad \Delta U &= MC_V(T_2 - T_1) \\ \therefore Q &= \Delta U = mC_V(T_2 - T_1) \end{aligned}$$



### Isothermal process

- Isothermal means constant temperature. If the only work is "pdV" work, then,

$$\begin{aligned} \text{Assume quasi-eqm.} \quad W &= \int_1^2 pdV = \int_1^2 \frac{MRT}{V} dV = MRT \ln\left(\frac{V_2}{V_1}\right) \\ pV &= MRT \quad \Delta U = 0 \\ i.e. PV &= \text{const.} \quad \therefore Q = \Delta U = 0 \\ W &= 0 \quad Q = 0 \quad \therefore Q = N = MRT \ln\left(\frac{V_2}{V_1}\right) \end{aligned}$$



\* Unrestricted adiabatic expansion of an ideal gas is also isothermal, but there is no work or heat transfer.  $\Delta U = Q = W = 0$  [ $u = f_n(T)$  so  $\Delta U = 0 \rightarrow \Delta T = 0$ ]

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## Adiabatic compression and expansion

- Consider a system comprising a perfect gas undergoing an infinitesimal adiabatic expansion or compression. The first law gives

$$\delta Q = dU + \delta W = 0$$

- If the process is quasi- $\text{eqm}$ , then the temp. and pressure are uniform throughout the system  $\rightarrow$  we can say  $dU = mc_v dT$  and  $\delta W = pdV$ , so

$$mc_v dT + pdV = 0$$

using  $p = \frac{mRT}{V}$  and dividing through by  $mT$  yields:

$$c_v \frac{dT}{T} + R \frac{dV}{V} = 0$$

$$\int_{T_1}^{T_2} \frac{dT}{T} = -\frac{R}{c_v} \int_{V_1}^{V_2} \frac{dV}{V}$$

$$\ln\left(\frac{T_2}{T_1}\right) = -(\gamma-1) \ln\left(\frac{V_2}{V_1}\right)$$

$$\frac{T_2}{T_1} = \left(\frac{V_1}{V_2}\right)^{\gamma-1}$$

$$\therefore \frac{T_2}{T_1} = \left(\frac{V_1}{V_2}\right)^{\gamma-1} = \left(\frac{P_1}{P_2}\right)^{\gamma-1} \quad \text{or} \quad T V^{\gamma-1} = \text{const.}$$

Using the substitution  $PV = RT$ ,

$$\frac{T_2}{T_1} = \frac{P_2 V_2 / R}{P_1 V_1 / R} = \left(\frac{V_1}{V_2}\right)^{\gamma-1} \rightarrow \frac{P_2}{P_1} = \left(\frac{V_1}{V_2}\right)^{\gamma}$$

$$\therefore \frac{P_2}{P_1} = \left(\frac{V_1}{V_2}\right)^{\gamma} = \left(\frac{V_1}{V_2}\right)^{\gamma} \quad \text{or} \quad P V^{\gamma} = \text{const.}$$

From these 2 relationships, we can see

$$\frac{P_2}{P_1} = \left(\frac{T_2}{T_1}\right)^{\gamma-1} \quad \text{or} \quad P/T^{\gamma-1} = \text{const.}$$

- A quasi- $\text{eqm}$ , adiabatic process has constant entropy  $\rightarrow$  isentropic process.

## Polytropic processes.

- For a polytropic process, the pressure and volume during compression/expansion are related by

$$P V^n = \text{const.}$$

where  $n$  is the polytropic index, assumed to remain constant during the process.

- Usually,  $n$  will be in the range 1.0 to  $\gamma$ . For perfect gases,

$\hookrightarrow n=1$  represents an isothermal process

note  $n=0$  : isobaric  
 $n=\infty$  : isochoric

$\hookrightarrow n=\gamma$  represents an isentropic process (adiabatic + quasi- $\text{eqm}$ .)

$\hookrightarrow 1 < n < \gamma$  usually (but not always) represents a process w/ heat transfer. In this case,

the relationship is purely empirical and cannot be derived from first principles.

- The work done on/by a system during a polytropic process is given by

$$W = \int_{V_1}^{V_2} P dV = \int_{V_1}^{V_2} \frac{K}{V^n} dV = K \left[ \frac{V^{1-n}}{1-n} \right]_{V_1}^{V_2}, \quad \text{valid for } n \neq 1$$

Assume quasi- $\text{eqm}$ :

$$\therefore W = \frac{P_1 V_1 - P_2 V_2}{n-1}$$

where  $K = P_1 V_1^n = P_2 V_2^n$

## Second law of thermodynamics

## Second law of thermodynamics

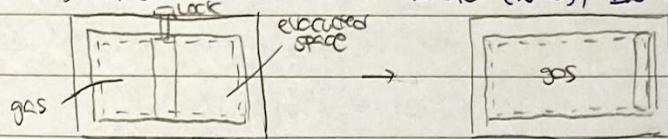
## Irreversible processes

- Irreversible processes are associated w/ departures from eqm. They can be viewed as thermodynamically irresponsible since they are lost opportunity for doing work.
- In general, after a system undergoes an irreversible process, it can be restored to its initial state through suitable flows of heat and work from the surroundings. However, in doing so, the surroundings are changed permanently

## Unrestricted expansion (free expansion)

- Consider the unrestricted expansion of a gas. Assume the container walls are perfectly insulating.

→ adiabatic ( $Q=0$ ). Since there is no work transfer ( $W=0$ ),  $\Delta U = Q - W = 0$ .



- To return the system back to its initial state, we might do the following:

- (i) Adiabatic compression back to the original volume. This involves doing work on the system → increase internal energy → gas becomes hotter than before
- (ii) Cooling at constant volume. The piston is now locked into place and the insulation is removed to allow the gas to cool down to its original temp.

- Although the system has returned to its original state, the surrounding has not - work has been extracted and an equal amt of received back in return.
- As there is no way of restoring the system to its initial state w/o affecting the surroundings, the unrestricted expansion is an irreversible process.

## Heat transfer across a finite temperature difference.

- When 2 blocks w/ a finite temp. diff. is brought into contact, heat transfer occurs irreversibly. Once eqm. is reached and the blocks are at the same temp, we wouldn't expect a spontaneous flow of heat back to the initially hotter block.
- If the 2 blocks were brought to eqm. while driving a reversible heat-engine, it would produce the work → irreversibility is equivalent to a lost opportunity for doing useful work.
- It is possible to restore both blocks to their original temp., but this req. a work input and heat rejection from/to the surroundings.

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## Reversible processes

- A process is reversible if the system and its surroundings can be returned to their initial state.
- A common feature of reversible processes is that they must not involve any substantial departures from eqm. → reversible processes are quasistatic, processes and must take place slowly (Quasistatic process also implies we have uniform conditions)
- Heat transfer over an infinitesimally small temp. diff. is reversible. Since the temp. diff. is infinitesimally small, the process is very slow.

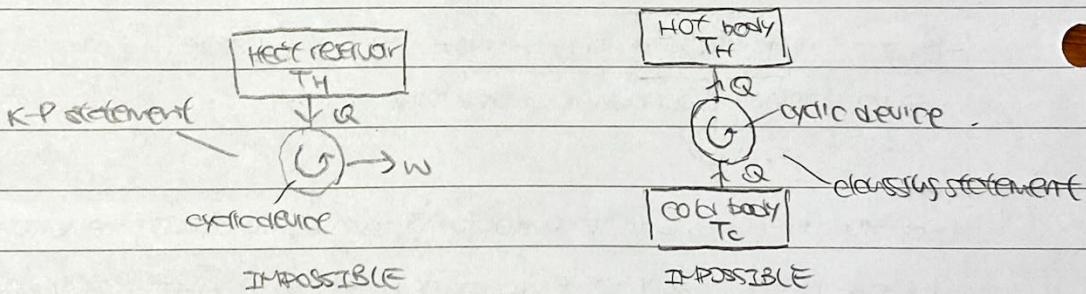
statements of the Second Law.

### ① The Kelvin-Planck statement

- The K-P statement states it is impossible to construct a cyclic device whose sole purpose is to produce the work whilst receiving heat from a single thermal reservoir.
- In other words, the complete conversion of heat into useful work is impossible. there is a limit of efficiency of the process
- The reverse process - the complete conversion of work into heat is perfectly acceptable.

### ② The Clausius statement

- The Clausius statement states it is impossible to construct a cyclic device whose sole purpose is the transfer of heat from a colder body to a hotter body.
- In other words, heat cannot spontaneously pass from a cold body to a hot body.
- If there is a work input to the cyclic process, it is perfectly possible to transfer heat from the cold reservoir to the hot reservoir.



- A cyclic device is a device that uses a system undergoing cyclic processes.
- A heat reservoir is a body which acts as a source or sink of heat. They are often assumed to have infinite thermal capacity, i.e. temp. remains const. as it gives out/receives heat.
- Neither of these impossible processes would violate the First Law

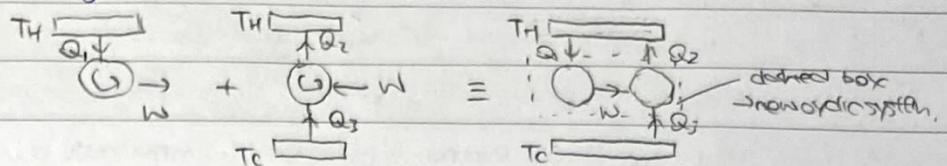
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Equivalence of the KEP and Clausius statements.  $\rightarrow$  Violation of one implies violation of the other.

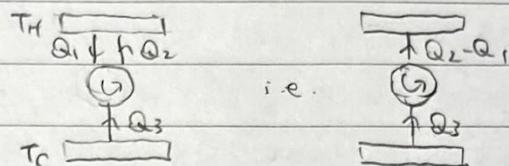
① KEP  $\rightarrow$  Clausius:

- We violate the KEP statement, i.e. we assume.

- Consider adding a heat pump, we get



If we consider the dashed box as a cyclic system, we get

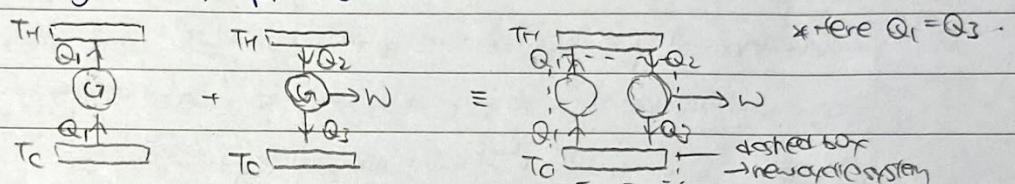


i.e. violation of the Clausius statement.

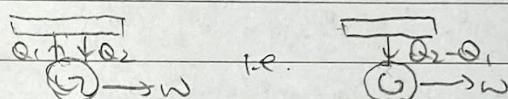
② Clausius  $\rightarrow$  KEP

- We violate the Clausius statement, i.e. we assume

- Consider adding a heat pump, we get



If we consider the dashed box as a cyclic system, we get



i.e. validation of the KEP statement

## Heat engines, refrigerators and heat pumps

### Heat engines

- A heat engine is a cyclic device for converting heat into work.

- According to the KEP statement, complete conversion is not possible and some heat input is rejected.

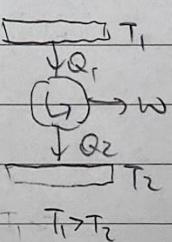
- Consider the simplest configuration of the heat engine

Since the device is cyclic,  $\Delta U=0$ , so the First Law gives.

$$W = Q_1 - Q_2.$$

- The thermal/cyclic efficiency  $\eta$  is defined as the fraction of the heat input that is converted to useful work.

$$\eta = \frac{W}{Q_1} = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1}$$



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## The Carnot cycle

- For the Carnot cycle, all processes within the cycle must be reversible. All heat transfer to the cycle is at the temp. of the hot reservoir and all heat transfer from the cycle should be at the temp. of the cold reservoir.
- The cycle consists of the following processes:
  - A → B : The gas is brought into contact w/ a heat source at temp.  $T_1$  and expands isothermally ( $Q_1$  from heat source)
  - B → C : The heat source is removed and the expansion continues adiabatically until the temperature is  $T_2$ .
  - C → D : The gas is brought into contact w/o heat sink at temp.  $T_2$  and is compressed isothermally ( $Q_2$  to heat sink)
  - D → A : the heat sink is removed and the gas is returned to its initial state by adiabatic compression.

## Carnot cycle efficiency.

- Note that all 4 of the processes in the cycle involve work transfer, whereas only A → B and C → D involve heat transfer → easiest to compute efficiency using  $\eta = 1 - \frac{Q_2}{Q_1}$ ,

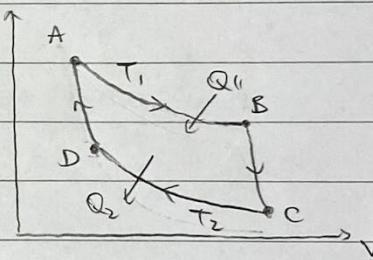
$$A \rightarrow B : W_{AB} = \int_A^B pdV = \int_A^B \frac{MRT}{V} dV = MRT_1 \ln\left(\frac{V_B}{V_A}\right)$$

$$Q_1 = Q_{AB} = W_{AB} = MRT_1 \ln\left(\frac{V_B}{V_A}\right)$$

$$C \rightarrow D : W_{CD} = \int_C^D pdV = \int_C^D \frac{MRT_2}{V} dV = MRT_2 \ln\left(\frac{V_D}{V_C}\right)$$

$$Q_2 = -Q_{CD} = -W_{CD} = MRT_2 \ln\left(\frac{V_D}{V_C}\right)$$

$$\therefore \eta = 1 - \frac{Q_2}{Q_1} = 1 - \frac{MRT_2 \ln\left(\frac{V_D}{V_C}\right)}{MRT_1 \ln\left(\frac{V_B}{V_A}\right)}$$



since processes B → C and D → A are adiabatic and reversible (which implies they are also quasi-erm), we can use  $TV^\gamma = \text{constant}$ .

$$\left(\frac{V_C}{V_B}\right) = \left(\frac{T_1}{T_2}\right)^{\frac{1}{\gamma-1}} = \left(\frac{V_B}{V_A}\right) \quad \rightarrow \quad \left(\frac{V_C}{V_B}\right) = \left(\frac{V_B}{V_A}\right)$$

$$\therefore \boxed{\eta = 1 - \frac{T_2}{T_1}}$$

- The expression for the Carnot efficiency depends only upon the reservoir temp. and does not involve gas properties such as  $C_p$  and  $\gamma$ .

- The Carnot efficiency is the max. thermal efficiency of any cyclic heat engine operating between temperatures  $T_1$  and  $T_2$ . (Proved in the next 2 pages)

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Two theorems concerning cyclic heat engines.

- Carnot deduced the following statements concerning the efficiency of cyclic heat engines.

(i) The max. efficiency of a cyclic heat engine operating between 2 thermal reservoirs is attained when the cycle is reversible.

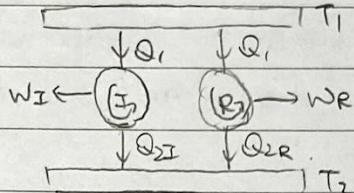
(ii) All reversible heat engines operating between the same thermal reservoirs are equally efficient.

- Violation of any of these theorems is equivalent to violating the Second Law (K-P statement).

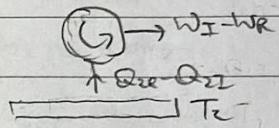
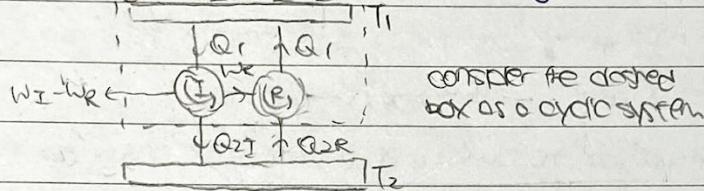
① (i)  $\rightarrow$  K-P.

- We violate thm. (i), i.e.  $\eta_I > \eta_R$ .

- Consider a reversible and irreversible heat engine operating between  $T_1$  and  $T_2$ .



since  $\eta_I > \eta_R$ ,  $W_I > W_R$ . Now reversing the reversible heat engine,

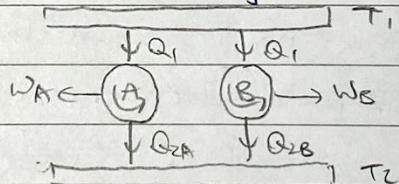


i.e. violation of the K-P statement.

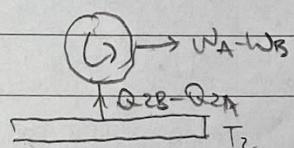
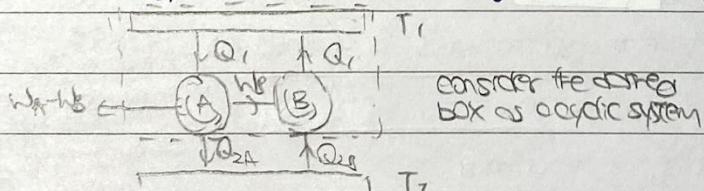
② (ii)  $\rightarrow$  K-P.

- We violate thm. (ii), i.e.  $\eta_A \neq \eta_B$ , so  $\eta_A > \eta_B$ .

- Consider 2 reversible heat engines A, B operating between  $T_1$  and  $T_2$ .



since  $\eta_A > \eta_B$ ,  $WA > WB$ . Now reversing heat engine B,



i.e. violation of the K-P statement.

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## reversible cyclic heat engines

- Since all reversible cyclic heat engines operating between the same 2 fixed temp. have the same efficiency, and since a Carnot engine is such a reversible engine,
- $\therefore$  The max. thermal efficiency of a cyclic heat engine operating between  $T_1$  and  $T_2$  is :

$$\eta_{\max} = 1 - \frac{T_2}{T_1}$$

and the efficiency is attained when the cycle is reversible.

- AS efficiency is also given by  $\eta = 1 - \frac{Q_2}{Q_1}$ , for all reversible cyclic heat engines operating between  $T_1$  and  $T_2$ ,

$$\frac{Q_1}{T_1} = \frac{Q_2}{T_2}$$

\* Every cyclic engine must reject a large portion of the input energy to a low temp. sink. This energy, originally at a high temp.  $T_1$ , is degraded to a less useful form at a lower temp.  $T_2$

## Refrigerators and heat pumps.

- Refrigerators and heat pumps are devices for extracting heat from a cold thermal reservoir and delivering heat to a hotter thermal reservoir.
- Such devices do not violate the classical statement of the second law because they have a work input.
- The only distinction between the 2 types of device is that the purpose of a refrigerator is to remove as much heat as possible from a cold space, whereas the purpose of a heat pump is to deliver as much heat as possible to a hot space.
- For a refrigerator, we pay for the work done and we want to maximise  $Q_2$ .

Accordingly the coefficient of performance is defined as .

[Refrigerator]

$$COP_R = \frac{Q_2}{W} = \frac{Q_2}{Q_1 - Q_2} *$$

\* cyclic device so  $\Delta U = 0$   
 $\therefore W = Q_1 - Q_2$

(usually greater than 1)

- For a heat pump, we pay for the work done and we want to maximise  $Q_1$

Accordingly the coefficient of performance is defined as .

[Heat pump]

$$COP_P = \frac{Q_1}{W} = \frac{Q_1}{Q_1 - Q_2} *$$

(always greater than 1)

- Comparing both terms, we find that

$$COP_P = 1 + COP_R$$

- As w/ heat engines, the best performance of a refrigerator or heat pump is achieved when the cycle is reversible. For reversible cycles,

$$\frac{Q_1}{T_1} = \frac{Q_2}{T_2}$$

which can be applied to find the max. COP of a refrigerator or heat pump .

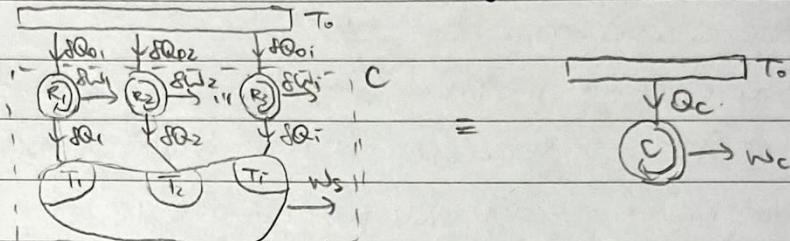
$$COP_{R,\max} = \frac{T_2}{T_1 - T_2}$$

$$COP_{P,\max} = \frac{T_1}{T_1 - T_2}$$

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The Clausius inequality.

- The Clausius inequality is the most concise mathematical statement of the second law and can be applied to any system undergoing a cyclic process w/ any no. of heat sources/sinks.
- consider a system  $S$  undergoing an irreversible cycle whilst receiving heat at a range of temps ( $\delta Q_i$  at  $T_i$ ) and producing net work  $w_c$ .
- We imagine each of the  $\delta Q_i$  being supplied via reversible cyclic devices  $R_i$  which in turn extract heat from a single thermal reservoir at  $T_0$ .
- We also define a composite system  $C$ , comprising of  $S$  and all the  $R_i$ .



- Since  $R_i$  is reversible, the heat input to  $R_i$  is  $-\frac{\delta Q_i}{\delta Q_{0i}} = 1 - \frac{T_i}{T_0} \rightarrow \delta Q_{0i} = T_0 \frac{\delta Q_i}{T_i}$
- The net heat supplied to  $C$  is therefore  $Q_C = \sum \delta Q_{0i} = T_0 \sum \frac{\delta Q_i}{T_i}$
- From the 1st law, the total net work produced by  $C$  is  $W_C = Q_C = T_0 \sum \frac{\delta Q_i}{T_i}$
- $C$  would violate the K-P statement unless heat and work flows are -ve.

$$W_C = T_0 \sum \frac{\delta Q_i}{T_i} \leq 0$$

$$\sum \frac{\delta Q_i}{T_i} \leq 0 \quad (\text{since } T > 0)$$

AS  $\delta Q_i \rightarrow 0$   $\oint \frac{dQ}{T} \leq 0$  [The Clausius inequality]

where the equality applies to a reversible cycle.

$$\text{so } \oint \frac{dQ_{\text{rev}}}{T} = \oint \frac{dQ_{\text{irr}}}{T}$$

- Using our convention,  $Q$  is +ve when the heat transfer to the system.  $T$  is the temp. of the boundary of the system at which heat is transferred (usually taken as the temp. of the corresponding thermal reservoir as any heat transfer outside is reversible)
- The  $\oint$  symbol indicates that the integral is performed over the entire cycle and over all parts of the system boundary
- The integral can be evaluated if the heat transfers occur at a constant temp. or if we can express  $dQ$  as a function of  $T$ .

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## Entropy

Definition of entropy  $S$

specific entropy  $s$  is defined as  $s = S/m$

- The entropy of a system  $S$  is an extensive thermodynamic property, defined s.t. the change in entropy in taking the system from state 1 to state 2 is given by

$$\Delta S = S_2 - S_1 = \int_1^2 \left( \frac{dQ}{T} \right)_{\text{rev}}$$

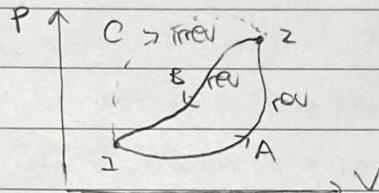
where integration must be carried out over a reversible path

- \* only changes of entropy  $\Delta S$  are defined.

## Entropy as a thermodynamic property

- To qualify as a thermodynamic property, the value of entropy must depend only on the state of the system, and not how it arrived at that state.

- Consider the reversible cyclic path 1-A-2-B-1



Clausius inequality:

$$\oint dS = \oint \left( \frac{dQ}{T} \right)_{\text{rev}} = 0$$

$$\therefore \int_A^2 dS + \int_B^1 dS = 0 \rightarrow \int_A^2 dS = - \int_B^1 dS = \int_B^2 dS$$

$$\rightarrow (S_2 - S_1)_A = (S_2 - S_1)_B = S_2 - S_1 = \Delta S$$

- This shows that the change in entropy between state 1 and state 2 is independent of whether we take the arbitrary path A or B  $\rightarrow$  entropy is a property.
- Note the irreversible process between states 1 and 2 (1-C-2) do not pass through a series of equilibrium states  $\rightarrow$  cannot strictly be plotted on a p-v diagram.
- However since the process starts and ends in eqm, the change in entropy is still  $\Delta S$  as entropy  $S$  is a function of state. However,  $\Delta S$  cannot be computed by integrating along path C.  $\hookrightarrow$  to find  $\Delta S$  of an irreversible process, find  $\Delta S$  of the reversible process w/ the same end states. (best if adiabatic).

## Entropy vs energy

- A comparison of entropy and energy is shown below:

	The first law	The second law
For a cyclic process.	$\oint dE = \oint (dQ - dW) = 0$	$\oint dS = \oint \left( \frac{dQ}{T} \right)_{\text{rev}} = 0$
For a process from state 2 to state 2	$\int^2_1 (dQ - dW) \text{ indep. of path taken}$	$\int^2_1 \left( \frac{dQ}{T} \right)_{\text{rev}} \text{ indep. of reversible path taken}$
A property can therefore be defined	$\Delta E = E_2 - E_1 = \int_1^2 (dQ - dW)$	$\Delta S = S_2 - S_1 = \int_1^2 \left( \frac{dQ}{T} \right)_{\text{rev}}$

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## Entropy changes for reversible processes

- The basic defn of entropy  $\Delta S = S_2 - S_1 = \int_1^2 \left(\frac{dQ}{T}\right)_{rev}$  can't be applied to an infinitesimal process:

$$dS = \left(\frac{dQ}{T}\right)_{rev} \quad \text{or} \quad dQ_{rev} = TdS$$

- These expressions show that

↳ If a process is both reversible and adiabatic ( $dQ=0$ ), it must also be isentropic ( $dS=0$ )

↳ Addition of heat to a system will increase its entropy and vice versa.

↳ For a reversible process, the heat added is equal to the area under the curve on a T-S diagram.  $Q_{rev} = \int_1^2 TdS$ .

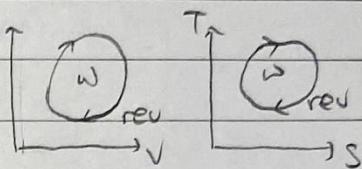
- For a reversible, cyclic process, applying the first law,

[Cyclic]

$$\oint dQ = \oint dW$$

[Reversible]

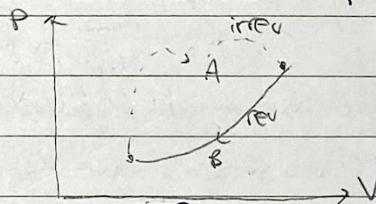
$$\oint TdS = \oint pdV$$



- This means the areas enclosed on the T-S and p-V diagrams are the same and both equal to the net work done / heat gain for a reversible cyclic process.

## Entropy changes for irreversible processes

- Consider the cycle below - it comprises an irreversible process between states 1 and 2, followed by a return to state 1 via a reversible path



Clausius inequality:

$$\oint \frac{dQ}{T} = \int_1^2 \left(\frac{dQ}{T}\right)_A + \int_2^1 \left(\frac{dQ}{T}\right)_B \leq 0$$

$$\int_1^2 \left(\frac{dQ}{T}\right)_A + (S_1 - S_2) \leq 0$$

$$\Delta S \geq \int_1^2 \left(\frac{dQ}{T}\right)_A$$

- Applying this to an infinitesimal process,

$$dS \geq \frac{dQ}{T}$$

or alternatively, we can write

$$dS = \frac{dQ}{T} + dS_{irrev} \quad \text{where } dS_{irrev} \geq 0$$

note  $dS_{irrev}$  is prop. to the amt. of lost potential work.  
i.e.  $dS_{irrev} \propto W_{ref-W_{max}}$

- The term  $dS_{irrev}$  is the infinitesimal entropy production due to irreversibility within the system. It is this term and this term only which cannot be -ve.

- The only way to reduce the entropy of a closed system is to extract heat to the surroundings ( $dQ$  can be -ve).

\* Note that  $T$  is the temperature of the boundary of the system of which heat is transferred.

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- e.g.: Irreversible entropy change of free expansion.

Find the irreversible entropy change of free expansion of gas from volume  $V_i$  to volume  $V_f$ . The initial temp./pressure is  $p_i, T_i$  and the system is insulated ( $Q=0$ ).

- Consider a reversible process w/ the same end states - fully resisted expansion.

\* Note for a free expansion,  $W=0$ . Since the system is insulated,  $Q=0$ , by 1st law,  $\Delta U=0$ .

This means  $\Delta U=0$ , i.e.,  $\Delta T=0 \rightarrow T_f=T_i \rightarrow$  consider a fully resisted isothermal expansion.

- Fully resisted isothermal expansion:  $\Delta U=0 \rightarrow Q=W = \int_1^2 p dV = \int_1^2 \frac{RT}{V} dV = p_i V_i \ln\left(\frac{V_f}{V_i}\right)$

$$\Delta S = \int_1^2 \left(\frac{\partial Q}{\partial T}\right)_{rev} = \frac{Q}{T_i} = \frac{p_i V_i}{T_i} \ln\left(\frac{V_f}{V_i}\right)$$

- Free expansion (adiabatic):  $\Delta S = \int \frac{dQ}{T} + \Delta S_{irrev}$

$$\Delta S_{irrev} = \frac{p_i V_i}{T_i} \ln\left(\frac{V_f}{V_i}\right)$$

- e.g.: Heat transfer across a finite temperature difference.

Find the irreversible entropy change when 2 identical blocks of mass  $m$  of diff. temps  $T_1, T_2$  are brought into contact and reach thermal eqm. Assume they have a specific heat capacity  $C_V$  and are insulated from the surroundings ( $Q=0$ )

- Consider a reversible process w/ the same end states - heat transfer via infinitesimal temp. diff. (between  $T_1, T_2$ )

- Reversible heat transfer:  $dQ_{rev} - \frac{dQ}{T} = dU = mC_V dT$

$$\Delta S = \int_A^B \left(\frac{\partial Q}{\partial T}\right)_{rev} = \int_A^B m C_V dT = mC_V \ln\left(\frac{T_B}{T_A}\right)$$

\* Note the expression for  $\Delta S$  is in terms of properties only  $\rightarrow$  valid for irreversible heat transfer.

- consider the 2 blocks as a system,  $dQ=dU=0 \rightarrow dU=0$ .

$$mC_V(T_{eq}-T_2) + mC_V(T_{eq}-T_1) = 0 \rightarrow T_{eq} = \frac{T_1+T_2}{2}$$

$$\therefore \Delta S = \Delta S_1 + \Delta S_2 = mC_V \ln\left(\frac{T_{eq}}{T_1}\right) + mC_V \ln\left(\frac{T_{eq}}{T_2}\right) = mC_V \ln\left(\frac{T_{eq}^2}{T_1 T_2}\right) = mC_V \ln\left(\frac{(T_1+T_2)^2}{4T_1 T_2}\right).$$

$$\Delta S = \int \frac{dQ}{T} + \Delta S_{irrev} \rightarrow \Delta S_{irrev} = mC_V \ln\left(\frac{(T_1+T_2)^2}{4T_1 T_2}\right)$$

$$\star (T_1+T_2)^2 = 4T_1 T_2 + (T_1-T_2)^2, \quad (T_1-T_2)^2 > 0, \quad \text{so } \frac{(T_1+T_2)^2}{4T_1 T_2} > 1, \quad \text{i.e. } \Delta S_{irrev} > 0.$$

## Applications of entropy

### The TdS equations

- The entropy of a system cannot be measured directly and so it must be inferred from values of other properties like  $p, V, T \rightarrow$  TdS equations.

- Consider a simple compressible system undergoing an infinitesimal change of state by heat and work interactions w/ the surroundings. The first law gives

$$dU = \delta Q - \delta W$$

- If the process is reversible, then  $\delta Q = TdS$  and  $\delta W = pdV$ , so

$$\boxed{TdS = dU + pdV} \quad [\text{TdS eqn. 1}]$$

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- Recall that entropy  $H$  is defined as

$$H = U + PV$$

so for an infinitesimal change in entropy,

$$dH = dU + PDV + VdP$$

$$dH = TdS + VdP \quad (\text{used } TdS \text{ eqn. 1})$$

$$TdS = dH - VdP \quad [TdS \text{ eqn. 2}]$$

- \* Although both eqns are derived for a reversible process, TdS only involves thermodynamic properties. (2nd property rule  $\rightarrow dS$  is fixed)  $\rightarrow$  eqn. valid for any process, reversible or irreversible, provided the end states are in eqm.

- Dividing by the system mass gives the TdS eqns in terms of specific properties.

$$TdS = du + pdv$$

$$TdS = dh - vdP$$

- These 2 relations embody the first and second laws together and are valid for simple compressible systems in the absence of capillary and EM fields.

Entropy changes for a perfect gas.

- For a perfect gas,  $du = cvdT$  and  $PV = RT$ , from TdS eqn. 1,

$$ds = \frac{1}{T}du + \frac{P}{T}dv = \frac{cv}{T}dT + \frac{R}{V}dv$$

As  $c_v$  and  $R$  are constants, integrating directly,

$$S_2 - S_1 = c_v \ln\left(\frac{T_2}{T_1}\right) + R \ln\left(\frac{V_2}{V_1}\right)$$

- Using  $\frac{P_1V_1}{P_2V_2} = \frac{RT_1}{RT_2}$ , the entropy change can be expressed in the alternative forms.

$$S_2 - S_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{P_1}{P_2}\right)$$

$$S_2 - S_1 = c_p \ln\left(\frac{V_2}{V_1}\right) + c_v \ln\left(\frac{P_2}{P_1}\right)$$

- The isentropic relations for a perfect gas can be obtained by setting  $S_2 - S_1 = 0$ .

constant pressure and constant volume lines in the T-s diagram. (perfect gas)

- Starting w/ TdS eqn. 2, and setting  $dP = 0$ ,  $dh = C_p dT$ ,

$$TdS = dh - vdP \rightarrow TdS = C_p dT$$

$$\therefore \left(\frac{\partial T}{\partial S}\right)_P = \frac{C_p}{I} \quad (\text{slope increases w/ increasing } T)$$

- Alternatively, we can integrate to find an expression for  $T$  in terms of  $S$ .

$$\int_{T_0}^T \frac{1}{T} dT = \int_{S_0}^S \frac{1}{C_p} dS$$

$$T = T_0 \exp\left(\frac{S - S_0}{C_p}\right)$$

$\therefore$  Exponential relationship between temp. and entropy for perfect gases, at constant  $P$

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- starting w/  $Tds = dQ - pdV$ , and setting  $dv=0$ ,  $du=CvdT$ ,

$$Tds = du + pdv \rightarrow Tds = CvdT$$

$$\therefore \left(\frac{\partial T}{\partial S}\right)_V = \frac{1}{C_V} \quad (\text{slope increases w/ increasing } T)$$

- Alternatively, we can integrate to find an expression for  $T$  in terms of  $s$ .

$$\int_{T_0}^T \frac{1}{T} dT = \int_{S_0}^S \frac{1}{C_V} ds$$

$$T = T_0 \exp\left(\frac{S-S_0}{C_V}\right)$$

. Exponential relationship between temp. and entropy for perfect gases at constant  $V$ .

\* since  $C_p = C_v + R$ , i.e.  $C_p > C_v$ , so the constant volume lines are steeper than the isobars.

## The principle of maximum entropy

- From the fundamental differential expression for entropy

$$dS = \frac{dQ}{T} + dS_{\text{irrev}} \quad \text{where } dS_{\text{irrev}} > 0,$$

. we can see that any 2 of the words "adiabatic" ( $dQ=0$ ), "reversible" ( $dS_{\text{irrev}}=0$ ), "isentropic" ( $ds=0$ ) imply the third.

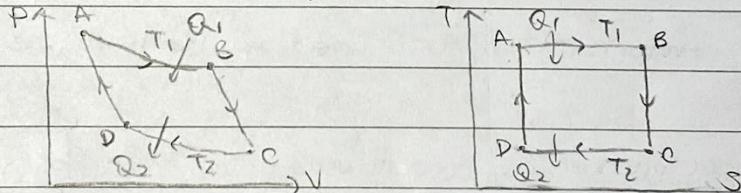
- If a system is isolated ( $dQ=0$ ), then  $dS = dS_{\text{irrev}} > 0$ , so only processes that increase the entropy can take place  $\rightarrow$  principle of maximum entropy.

- This tells us that as an isolated system approaches eqm, its entropy tends towards a max value (for fluid systems in the absence of EM effects, there are 4 main types of eqm that can be disturbed - mechanical, thermal, compositional and chemical eqm.)

## Reciprocating engines

Practical deficiencies of the Carnot engine.

- Below is the Carnot cycle plotted on  $p-v$  and  $T-s$  diagrams.



- The Carnot engine has the highest possible efficiency for any cyclic device operating between fixed temp. limits  $T_1$  and  $T_2$ , but they are never built because of several practical difficulties:  
 ↳ Isothermal heat transfer to and from external reservoirs req. a very large contact area or a very long time  $\rightarrow$  power can only be extracted slowly

↳ The cycle has a poor work ratio (skinny on the  $p-v$  diagram)

The work ratio is defined as  $WR = \frac{\text{Net work output}}{\text{Total positive work}} = \frac{\text{Enclosed area}}{\text{Area under ABC}}$

The poor work ratio makes the cycle susceptible to irreversibilities

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The Stirling engine / closely related to the Ericsson engine.

- A Stirling engine is an example of a reciprocating external combustion engine - heat is transferred from an external source other than by burning fuel internally.

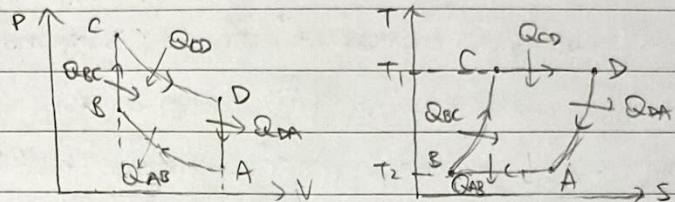
- The ideal Stirling cycle consists of the following 4 reversible processes in series:

A → B : Isothermal compression at temperature  $T_2$  (same as Carnot)

B → C : Constant volume heat addition

C → D : Isothermal expansion at temperature  $T_1$  (same as Carnot)

D → A : constant volume heat rejection



- Theoretically, all the heat rejected in the process D → A could be recycled as heat input in process B → C using a heat exchanger (regenerator).
- In this case, all the heat addition would take place in process C → D at temp.  $T_1$  and all the heat rejection would take place in process A → B at temp.  $T_2$ .
- This recycling of heat would req. a regenerator that is 100% effective, so in practice some additional heat input is req. and the cycle efficiency is much lower than the equiv. Carnot cycle.
- The efficiency of the ideal Stirling cycle can be calculated as follows:

$$A \rightarrow B : W_{AB} = \int_A^B p dV = \int_A^B \frac{mRT_2}{V} dV = mRT_2 \ln\left(\frac{V_B}{V_A}\right)$$

$$Q_{AB} = W_{AB} = mRT_2 \ln\left(\frac{V_B}{V_A}\right) = -mRT_2 \ln\left(\frac{V_A}{V_B}\right) \quad [-ve]$$

$$B \rightarrow C : \Delta U_{BC} = mc_v(T_1 - T_2)$$

$$Q_{BC} = \Delta U_{BC} = mc_v(T_1 - T_2) \quad [+ve]$$

$$C \rightarrow D : W_{CD} = \int_C^D p dV = \int_C^D \frac{mRT_1}{V} dV = mRT_1 \ln\left(\frac{V_D}{V_C}\right)$$

$$Q_{CD} = W_{CD} = mRT_1 \ln\left(\frac{V_D}{V_C}\right) \quad [+ve] \quad Q_{BC} = -Q_{CD}$$

$$D \rightarrow A : \Delta U_{DA} = mc_v(T_2 - T_1)$$

$$Q_{DA} = \Delta U_{DA} = mc_v(T_2 - T_1) = -mc_v(T_1 - T_2) \quad [-ve]$$

$$\therefore \eta_{ideal} = \frac{W_{net}}{Q_{in}} = \frac{mR(T_1 - T_2) \ln\left(\frac{V_A}{V_B}\right)}{mRT_1 \ln\left(\frac{V_A}{V_B}\right) + mc_v(T_1 - T_2)} \quad [\text{ideal cycle, no regenerator}]$$

$$\eta_{ideal} = \frac{W_{net}}{Q_{in}} = \frac{mR(T_1 - T_2) \ln\left(\frac{V_A}{V_B}\right)}{mRT_1 \ln\left(\frac{V_A}{V_B}\right)} = 1 - \frac{T_2}{T_1} = \eta_{Carnot} \quad [\text{ideal cycle, w/ regenerator}]$$

- Stirling engines are used in some low-power applications, particularly where the combustion of fuel does not provide the heat source.

- Their performance is limited as they req. a regenerator and the real cycle is irreversible. Similar to the Carnot engine, they tend to have a poor work ratio.

(In practice, the work output is just sufficient to overcome friction in the piston/cylinder parts)

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spark ignition and compression ignition engines.

- The practical issues of the Carnot/stirling engines can be overcome by burning fuel within the cylinders of the engine, rather than transferring heat from an external source.

(Burning fuel is not heat transfer - it is a change in the form of internal energy)

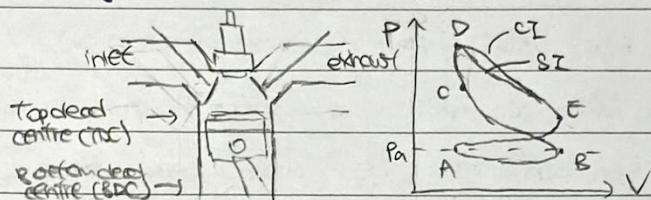
- For reciprocating engines, this can be done in 2 ways:

↳ spark ignition (SI) in which a mixture of air + vaporized fuel is introduced into the cylinder and ignited by a spark (petrol engines)

↳ compression ignition (CI) in which air is compressed and reaches a high enough temp. that fuel ignites spontaneously. The fuel is injected directly into the cylinder in the form of a liquid droplet spray (diesel engines)

- Both types can operate on either a 4-stroke or 2-stroke cycle.

- A 4-stroke engine along w/ its indicator diagram is shown below:



① Induction stroke (A→B): The inlet valve is opened and the piston moves downward, drawing the air/fuel mixture/air into the cylinder. Pressure drops below  $P_0$  due to frictional losses.

② Compression stroke (B→C): Both valves are closed and the piston moves upward, compressing the mixture/air. Spark occurs / Fuel is injected which initiates a combustion at almost constant volume, causing a rapid increase in temperature and pressure.

③ Power stroke (C→D): The valves remain closed as the piston returns to BDC. The pressure initially continues to rise as combustion is completed (D), and then falls as the hot gas expands. Work is done on the piston and transferred via the connecting rod to the crankshaft.

④ Exhaust stroke (D→A): The exhaust valve opens and the products of combustion are expelled as the piston returns to TDC.

\* There is only 1 power stroke for 2 revolutions of the crank in a 4-stroke engine.

\* For both SI and CI engines, piston speeds are unlikely to exceed  $20 \text{ m/s}$  → much less than the speed of sound → pressure is uniform within the cylinder →  $W_D = \text{area under pV diagram}$ .

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## The air-standard cycles.

- Air-standard cycles provide a very simplified model of the processes above. The main approximations involved are
  - ↳ The working fluid is assumed to be air throughout, w/ constant  $C_V, C_p$ .
  - ↳ The air is treated as a closed system and the combustion of fuel is modelled as heat addition from an external source.
  - ↳ The compression and expansion processes are assumed adiabatic and reversible (isentropic).
  - ↳ Heat losses may be significant in reality, esp. during the power stroke.

## The air-standard Otto cycle.

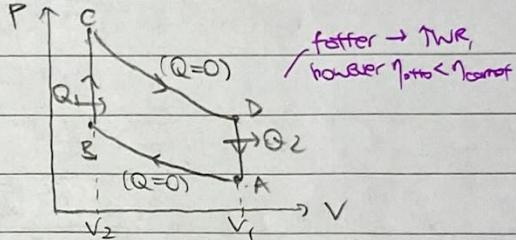
- The air-standard Otto cycle provides an approx model for SI engines and consists:

A → B : Isentropic compression

B → C : Heat addition at constant volume

C → D : Isentropic expansion

D → A : Heat rejection at constant volume



- The final process (D→A) replaces "expulsion of gas and renewal w/ fresh charge" in real engine.

- The compression ratio  $r_v$  is defined as  $r_v = \frac{V_1}{V_2}$

- The efficiency of the Otto cycle can be calculated as follows:

$$A \rightarrow B : \frac{T_B}{T_A} = \left( \frac{V_A}{V_B} \right)^{\gamma-1} = r_v^{\gamma-1} \rightarrow T_B = r_v^{\gamma-1} T_A$$

$$B \rightarrow C : q_1 = \Delta U = C_V \Delta T \rightarrow \Delta T = \frac{q_1}{C_V} \rightarrow T_C = T_B + \Delta T = r_v^{\gamma-1} T_A + \frac{q_1}{C_V}$$

$$C \rightarrow D : \frac{T_D}{T_C} = \left( \frac{V_C}{V_D} \right)^{\gamma-1} = \left( \frac{1}{r_v} \right)^{\gamma-1} \rightarrow T_D = r_v^{1-\gamma} T_C = T_A + \frac{1}{C_V} r_v^{1-\gamma}$$

$$D \rightarrow A : q_2 = \Delta U = C_V \Delta T = C_V (T_D - T_A)$$

$$\therefore \eta = \frac{w}{q_{in}} = \frac{q_1 - q_2}{q_1} = 1 - \frac{C_V (T_D - T_A)}{q_1} = 1 - \frac{C_V \cdot \frac{1}{C_V} r_v^{1-\gamma}}{q_1} = 1 - \frac{1}{r_v^{\gamma-1}}$$

- According to this expression, the thermal efficiency  $\eta$  increases w/ compression ratio  $r_v$ .

This trend applies to real engines, but the efficiencies are much lower than predicted by the expression due to irreversibilities and heat losses.

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## The air-standard diesel cycle

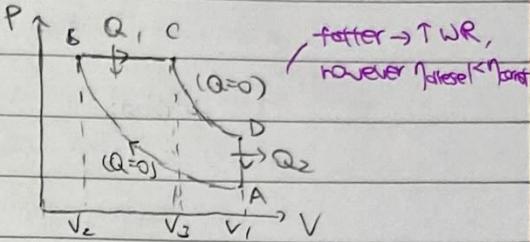
- The air-standard diesel cycle provides an approx model for CI engines and consists:

A → B: Isentropic compression

B → C: Heat addition at constant pressure

C → D: Isentropic expansion

D → A: Heat rejection at constant volume



- The constant pressure heat addition is intended to model the slower combustion process

- The final process (D → A) replaces "expulsion of gas and renewal w/ fresh charge" in real engine.

- The compression ratio  $r_v$  is defined as  $r_v = \frac{V_1}{V_2}$ ; The cut-off ratio  $\alpha$  is defined as  $\alpha = \frac{V_3}{V_2}$

- The efficiency of the Diesel cycle can be calculated as follows:

$$A \rightarrow B: \frac{T_B}{T_A} = \left( \frac{V_A}{V_B} \right)^{\gamma-1} = r_v^{\gamma-1} \rightarrow T_B = r_v^{\gamma-1} T_A$$

$$B \rightarrow C: \frac{T_C}{T_B} = \frac{V_C}{V_B} = \alpha \rightarrow T_C = \alpha T_B = \alpha r_v^{\gamma-1} T_A$$

$$q_1 = C_p \Delta T = C_p (T_C - T_A) = C_p T_A r_v^{\gamma-1} (\alpha - 1)$$

$$C \rightarrow D: \frac{T_D}{T_C} = \left( \frac{V_C}{V_D} \right)^{\gamma-1} = \left( \frac{\alpha}{r_v} \right)^{\gamma-1} \rightarrow T_D = \left( \frac{\alpha}{r_v} \right)^{\gamma-1} T_C = \alpha^{\gamma} T_A$$

$$D \rightarrow A: q_2 = \Delta U = C_V \Delta T = C_V (T_D - T_A)$$

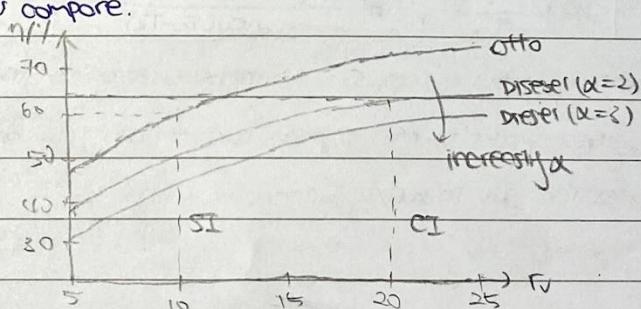
$$\therefore \eta = \frac{W_{out}}{q_{in}} = \frac{q_1 - q_2}{q_1} = 1 - \frac{C_V (T_D - T_A)}{C_p T_A r_v^{\gamma-1} (\alpha - 1)} = 1 - \frac{T_A (\alpha^{\gamma} - 1)}{\gamma (r_v^{\gamma-1}) (\alpha - 1) T_A} = 1 - \frac{1}{r_v^{\gamma-1}} \frac{(\alpha^{\gamma} - 1)}{\gamma (\alpha - 1)}$$

- Note that  $\frac{(\alpha^{\gamma} - 1)}{\gamma (\alpha - 1)} > 1$  so  $1 - \frac{1}{r_v^{\gamma-1}} > 1 - \frac{1}{r_v^{\gamma-1}} \frac{(\alpha^{\gamma} - 1)}{\gamma (\alpha - 1)}$ , i.e.  $\eta_{Otto} > \eta_{diesel}$ , for a given  $r_v$ .

- In real engines, the efficiencies are much lower than predicted by the expression due to irreversibilities and heat losses.

## Practical aspects of the SI and CI engines

- Despite being crude approximations, the Otto and Diesel cycle give some insight into how SI and CI engines compare.



- The Otto cycle has a higher thermal efficiency than the Diesel cycle for a given compression ratio  $r_v$ , and the Diesel efficiency falls w/ increasing cut-off ratio  $\alpha$  (as we increase  $Q_{in}$ )

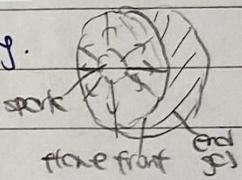
- In practice, SI engines are limited to  $r_v < 12$ , o/w preignition/knocking may occur.

CI engines operate at higher compression ratios, typically  $r_v \sim 20 \rightarrow$  tend to be more efficient

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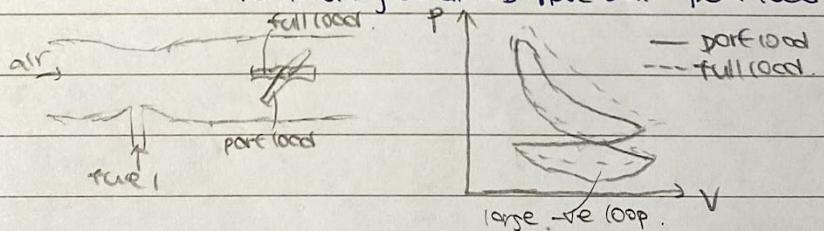
## Power output and cylinder size

- Aside from compression ratio  $r_v$ , the main parameters affecting power output are (i) cylinder volume, (ii) rotational speed and (iii) air-fuel ratio
- For conventional SI engines, the air-fuel ratio is effectively fixed s.t. all  $O_2$ /fuel are consumed. (too much air  $\rightarrow$  mixture won't ignite; too much fuel  $\rightarrow$  unburnt fuel)
- For CI engines, fuel droplets must first evaporate to produce a locally combustible mixture which will then ignite. In practice only about ~80% of  $O_2$  can be consumed.
- This means less fuel is burnt per unit cylinder volume in a diesel engine  $\rightarrow$  power output is less than that for an SI engine of the same size.
- Also, the rotational speed of diesels is limited to allow enough time for droplet evaporation and combustion  $\rightarrow$  power output is less than that for an SI engine.
- However, the max. cylinder volume of SI engines are limited by "end gas" pre ignition.
  - $\hookrightarrow$  Flame front spreads outward from the spark  $\rightarrow$  pressure + temp. of combustible end gas increases  $\rightarrow$  may preignite if too far from the spark  $\rightarrow$  knocking.



## Part-load efficiency.

- Since the air-fuel ratio is fixed for conventional SI engines, power is controlled by regulating the amount of mixture entering the cylinder (throttling).
- The pressure losses through the throttle increases the  $-ve$  area loop on the pV diagram so SI engines have very poor part-load efficiency.
- For diesel engines, the power is regulated by controlling the amount of fuel sprayed into the cylinder  $\rightarrow$  no need for throttling so diesels have better part-load efficiency.



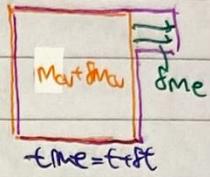
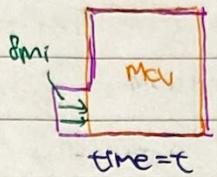
## Summary of pros:

- SI engines :  $N_{max} > N_{diesel}$  for a given  $r_v$ .
  - more fuel burnt per unit cylinder volume
  - higher rotational speeds
- CI engines : higher  $r_v/cylinder\ volume$  (fuel does not preignite/no knocking)
  - higher part load efficiency.

## Conservation of mass for control volumes

conservation of mass for control volumes (inc. non-steady flow).

- when we have accumulation/depletion of mass (non-steady flow), or a compressible gas, the mass within the CV changes w/ time.
- consider the system of a reservoir w/ an inflow and outflow



- consider the mass of the system at time = t and time = t+Δt

$$\text{time } t : \quad M_{cv} + \Delta m_i$$

$$\text{time } t+\Delta t : \quad M_{cv} + \Delta M_{cv} + \Delta m_o$$

- since no mass crosses the system boundary,

$$\Delta m_i + \Delta m_o = \Delta M_{cv} + f M_{cv} \rightarrow \Delta m_i - f M_{cv} = \Delta M_{cv}$$

- dividing through by Δt + let Δt → 0 and generalizing for multiple inlet/exit streams,

$$\sum_{in} \dot{m}_i - \sum_{out} \dot{m}_o = \frac{dM_{cv}}{dt}$$

$$\dot{m} = \rho AV, \quad P = \frac{P}{RT} \text{ for an ideal gas.}$$

conservation of mass for control volumes (steady flow).

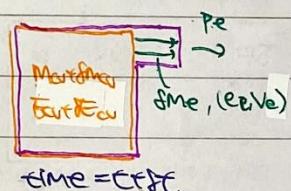
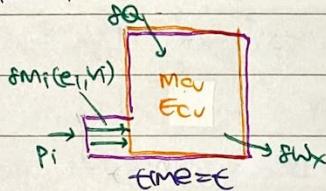
- If steady state conditions prevail, i.e. no accumulation/depletion of mass(energy/entropy), or if the flow is incompressible, the  $\frac{dM_{cv}}{dt}$  term is zero.

$$\sum_{in} \dot{m}_i = \sum_{out} \dot{m}_o$$

## The FIRST law for control volumes

The first law for control volumes (inc. non-steady flow)

- consider the system of a reservoir w/ an inflow and outflow



- consider the energy of the system at time = t and time = t+Δt

$$\text{time } t : \quad E_{cv} + \Delta m_i \cdot e_i$$

$$\text{time } t+\Delta t : \quad E_{cv} + \Delta E_{cv} + \Delta m_o \cdot e_o$$

- In considering the work done by the system, we must account for both displacement (ΔW<sub>x</sub>) work and shaft work.

flow work - work req.  
to push fluid  
across the CS.

$$\text{during time } \Delta t : \quad \Delta W = \Delta W_x + p_e(V_e - \Delta m_o) - p_i(V_i - \Delta m_i)$$

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- Applying the first law to the system over the interval  $\Delta t$  gives

$$\dot{Q} - (\dot{m}_x \cdot h_e + p_e V_e \cdot \dot{m}_e) = (\dot{E}_{cv} + \dot{m}_e \cdot e_e) - (\dot{E}_{cv} + \dot{m}_x \cdot e_i)$$

$$\dot{Q} - \dot{m}_x h_e = \dot{E}_{cv} + \dot{m}_e (e_e + p_e V_e) - \dot{m}_x (e_i + p_e V_i)$$

- Noting that the specific energy  $e$  is the sum of internal, kinetic and potential energy terms,

$$e + pV = u + PV + \frac{1}{2}V^2 + gz = h + \frac{1}{2}V^2 + gz$$

therefore, substituting back into the eqn. above and generalising to multiple inlet/exit streams,

$$\dot{Q} - \dot{m}_x h_e = \dot{E}_{cv} + \sum_{out} \dot{m}_e (h_e + \frac{1}{2}V_e^2 + gz_e) - \sum_{in} \dot{m}_i (h_i + \frac{1}{2}V_i^2 + gz_i)$$

- Dividing through by  $\dot{E}_{cv}$  let  $\dot{E}_{cv} \rightarrow 0$ , the system and CV become closer together and eventually occupy the same space.

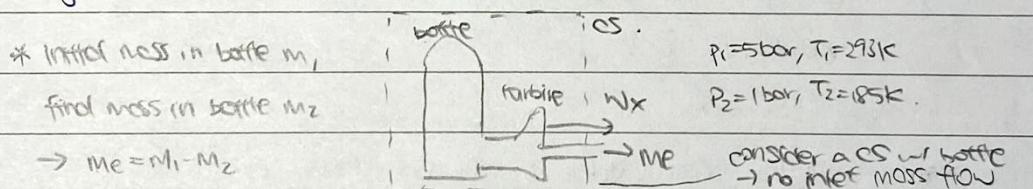
$$\dot{Q} - \dot{W}_x = \frac{d\dot{E}_{cv}}{dt} + \sum_{out} \dot{m}_e (h_e + \frac{1}{2}V_e^2 + gz_e) - \sum_{in} \dot{m}_i (h_i + \frac{1}{2}V_i^2 + gz_i) \quad [\text{N-SSEE}]$$

where  $\dot{Q}$  is the rate of heat transfer to the CV;

$\dot{W}_x$  is the shaft power from the CV.

Non-steady flow processes.

- e.g.: A fully insulated bottle is initially filled w/ dry air at a pressure of 5 bar and a temp. of 293K. The bottle is connected to a turbine which exhausts to the atmosphere at a pressure of 1 bar. The flow through the turbine is adiabatic and the temp. of turbine exit is 185K. At the end of the process, the temp./pressure within the bottle is also 185K/1 bar. Determine the shaft work extracted by the turbine per kg of air initially in the bottle. Changes in KE/PE can be neglected and the volume of the pipework is small.



Note that  $U = U_0 + C_v T$ ;  $h = h_0 + C_p T = h_0 + p_0 V_0 + C_p T = h_0 + p_1 T_0 + C_p T$   $\rightarrow T_0 = 0 \text{ K}$ .

N-SSEE :  $\dot{Q} - \dot{W}_x = \frac{d\dot{E}_{cv}}{dt} + m_e (h_e + \frac{1}{2}V_e^2 + gz_e) - \dot{W}_x (h_i + \frac{1}{2}V_i^2 + gz_i)$

$$- \dot{W}_x = \frac{d\dot{E}_{cv}}{dt} + m_e h_e$$

Integrating over time,

$$- \dot{W}_x = \Delta E_{cv} + M_e h_e$$

\* KE/PE negligible  $\rightarrow \Delta E_{cv} \approx \Delta U_{cv}$

$$- \dot{W}_x = (m_1 h_2 - m_1 h_1) + (M_e - M_1) h_e$$

\* using  $U = U_0 + C_v T$  ;  $h = h_0 + C_p T$

$$- \dot{W}_x = M_2 (h_0 + C_p T_2) - M_1 (h_0 + C_v T_1) + (M_1 - M_2) (U_0 + C_p T_0)$$

$$\therefore \frac{\dot{W}_x}{M_1} = U_0 + C_v T_1 - \frac{M_2}{M_1} (h_0 + C_p T_2) - \left(1 - \frac{M_2}{M_1}\right) (h_0 + C_p T_0)$$

$$= C_v T_1 - \frac{M_2}{M_1} C_p T_2 - \left(1 - \frac{M_2}{M_1}\right) C_p T_0 + U_0 \left(1 - \frac{M_2}{M_1} \right) \left(1 - \frac{M_2}{M_1}\right)$$

Note that  $\frac{M_2}{M_1} = \frac{P_2 V}{P_1 V} \times \frac{T_1}{T_2} = \frac{P_2 T_1}{P_1 T_2} = \frac{1}{5} \left(\frac{293}{185}\right) = 0.317$

$$\therefore \frac{\dot{W}_x}{M_1} = 7.8(293) - 0.317(7.8)(185) - (1 - 0.317)(1005)(185) = 41.3 \text{ kJ kg}^{-1}$$

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The first law for control volume (steady flow)

- If steady state conditions prevail (i.e. no accumulation/depletion of mass/energy/entropy), the  $\frac{\partial}{\partial t}$  term (in the N-SSEE) is zero. For 1 inflow + 1 outflow,

$$\dot{Q} - \dot{W}_X = \dot{m} [(h_e + \frac{1}{2} V_e^2 + g z_e) - (h_i + \frac{1}{2} V_i^2 + g z_i)]$$

- Dividing by the mass flow rate  $\dot{m}$ ,

$$q - w_X = (h_e + \frac{1}{2} V_e^2 + g z_e) - (h_i + \frac{1}{2} V_i^2 + g z_i) \quad [\text{SSEE}]$$

where  $q$  is the net heat input per unit mass of fluid passing through the CV

$w_X$  is the net shaft work output per unit mass of fluid passing through the CV.

Neglect of kinetic and potential energy terms.

- KE and PE terms can often be neglected and it is worth considering when this is realistic.

- Consider the speed of the flow / height change for the KE/PE terms to be equivalent to a temp. change of  $\pm K$ .

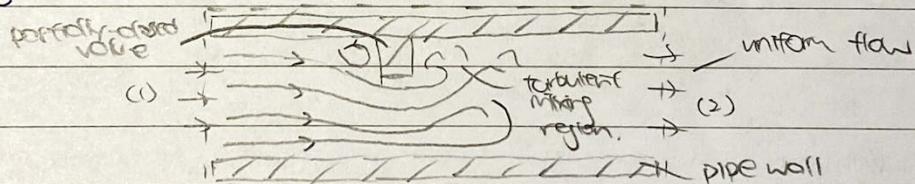
$$\text{KE} : \frac{1}{2} V^2 \sim C_p \Delta T = 100 \rightarrow V \sim 45 \text{ m s}^{-1}$$

$$\text{PE} : g \Delta z \sim C_p \Delta T = 100 \rightarrow \Delta z \sim 100 \text{ m.}$$

- As a general rule of thumb, we can neglect KE/PE terms when we have large temp changes, b/w, there is a small but non-negligible error.

The throttling process

- Throttling is flow through a partially closed valve or a constriction in a pipe.



- Immediately downstream of the valve, the flow will be strongly turbulent and contain rapidly recirculating eddies.

- Although the flow is locally unsteady, when averaged over a period of time, there will be no accumulation of mass/energy within the CV  $\rightarrow$  globally steady.

- Far enough downstream of (2), the eddies will have died out  $\rightarrow$  flow uniform and // to pipe axis.

- The flow can usually be considered adiabatic as there is not enough time for heat transfer.

$$\text{SSEE} \quad \dot{Q} - \dot{W}_X = (h_2 + \frac{1}{2} V_2^2 + g z_2) - (h_1 + \frac{1}{2} V_1^2 + g z_1)$$

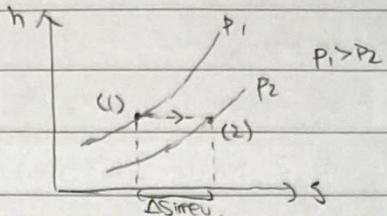
In almost all cases, PE term can be neglected; In many cases, KE term can be neglected.

$$h_1 + \frac{1}{2} V_1^2 + g z_1 = h_2 + \frac{1}{2} V_2^2 + g z_2 \rightarrow h_1 = h_2.$$

- Under the above assumptions (adiabatic, KE/PE terms negligible), the throttling process is isenthalpic.

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- Consider the throttling process on an h-s diagram.

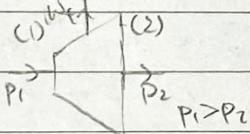
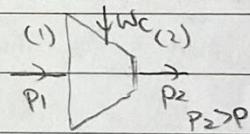


The throttling process reduces the pressure at constant enthalpy, resulting in an increase in entropy.

- Since the flow was assumed adiabatic, the entropy increase must have come from irreversibility, in this case is from viscous dissipation in the turbulent region downstream of the valve.
- The irreversibility above constitutes a lost work opportunity — the pressure drop could have been achieved through a turbine, producing shaft work.
- For ideal gases, enthalpy is a function of temp. only (i.e.,  $h = f(T)$ ), so the isenthalpic throttling process is also isothermal.
- \* For perfect gases,  $h \propto T$  so the T-s and h-s diagrams have the same shape.

turbines and compressors.

- Turbines and compressors are globally steady devices that exchange shaft work w/ the fluid passing through them. (L21)

Turbines	Compressors
<ul style="list-style-type: none"> <li>- Produce shaft work.</li> <li>- Have a pressure drop through them</li> </ul> 	<ul style="list-style-type: none"> <li>- Produce a pressure rise in the flow.</li> <li>- Absorb shaft work</li> </ul> 

- e.g.: Combustion gases enter the turbine of a jet engine at 15 bar and 1250K w/o velocity of  $50 \text{ m/s}^2$ . At exit, the pressure, temp. and velocity are 3 bar, 850K and  $100 \text{ m/s}^2$ , respectively.

If the exit area is  $0.6 \text{ m}^2$ , calculate the mass flow rate and the power output of the turbine.

Assume perfect gas relations w/  $R = 287 \text{ J kg}^{-1} \text{ K}^{-1}$  and  $C_p = 1100 \text{ J kg}^{-1} \text{ K}^{-1}$ .

$$\dot{m} = P_2 A_2 V_2 = \frac{P_2}{R T_2} A_2 V_2 = \frac{3 \times 10^5}{287 (850)} (0.6)(100) = 73.5 \text{ kg s}^{-1}. \quad * P_2 = \frac{P_2}{R T_2}$$

SFEE:  $\dot{W}_x = (h_2 + \frac{1}{2} V_2^2 + g Z_2) - (h_1 + \frac{1}{2} V_1^2 + g Z_1)$

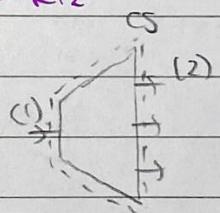
$$W_x = C_p(T_1 - T_2) + \frac{1}{2}(V_1^2 - V_2^2)$$

$$= 1200(1250 - 850) + \frac{1}{2}(50^2 - 100^2)$$

REL. SNO!!

$$= 476.5 \text{ kJ kg}^{-1}$$

$$\therefore \dot{W}_x = \dot{m} W_x = 73.5(476.5) = 35 \text{ MW}.$$

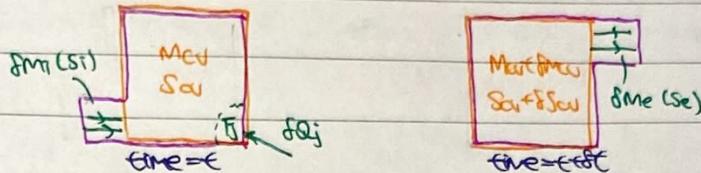


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## The Second Law for control volumes

The Second law for control volumes (inc. nonsteady, irreversible flow)

- consider the system of a reservoir w/ an inflow and outflow



- consider the entropy of the system at time = t and time = t + Δt.

$$\text{time } t : \quad S_{\text{CV}} + \delta m_i \cdot s_i$$

$$\text{time } t + \Delta t : \quad S_{\text{CV}} + \delta S_{\text{CV}} + \delta m_e \cdot s_e$$

- Applying the second law to the system over the interval Δt gives:

$$\delta S_{\text{sys}} = \sum \frac{\delta Q_j}{T_j} + \delta S_{\text{irrev}} \quad \delta S_{\text{irrev}} \geq 0$$

$$(S_{\text{CV}} + \delta S_{\text{CV}} + \delta m_e \cdot s_e) - (S_{\text{CV}} + \delta m_i \cdot s_i) = \sum \frac{\delta Q_j}{T_j} + \delta S_{\text{irrev}}$$

$$\delta S_{\text{CV}} + \delta m_e \cdot s_e + \delta m_i \cdot s_i = \sum \frac{\delta Q_j}{T_j} + \delta S_{\text{irrev}}$$

- Dividing through by Δt + let Δt → 0, and accounting for multiple inlet/exit streams,

$$\frac{\delta S_{\text{CV}}}{\Delta t} + \sum_{\text{out}} m_e s_e + \sum_{\text{in}} m_i s_i = \sum \frac{\dot{Q}_j}{T_j} + \dot{S}_{\text{irrev}}$$

## The second law for control volumes (steady, irreversible flow)

- If steady state conditions prevail, i.e. no accumulation/depletion of mass/energy/entropy),

the  $\frac{\delta S_{\text{CV}}}{\Delta t}$  term is zero. For 1 inflow + 1 outflow,

$$m(s_e - s_i) = \sum \frac{\dot{Q}_j}{T_j} + \dot{S}_{\text{irrev}}.$$

- Dividing by the mass flow rate  $m$ ,

$$s_e - s_i = \sum \frac{\dot{Q}_j}{T_j} + \dot{S}_{\text{irrev}}$$

- Note the summation term is often difficult to evaluate as we don't know  $T_j$ , except in very simplified cases (e.g. isothermal processes)

can also be solved by considering air at friction forces using SSEE

- e.g.: Isothermal flow in a pipe - Air flows in a pipe w/ a high velocity of which friction is significant. Heat transfer occurs through the pipe wall. At the temp. inside remains at 50°C. consv. of mass:  $P_A f_A V_A = P_B f_B V_B$

$$\frac{P_A}{RT_A} V_A = \frac{P_B}{RT_B} V_B \rightarrow V_B = \frac{P_A}{P_B} V_A = \frac{31}{3} \cdot 300 = 310 \text{ m/s}.$$

$$\text{SSEE: } q - \dot{w}_{\text{fr}}^0 = C_p(T_B - T_A) + \frac{1}{2}(V_B^2 - V_A^2) \rightarrow q = \frac{1}{2}(310^2 - 300^2) = 3050 \text{ J/kg}.$$

$$\text{2nd law for CV: } s_B - s_A = \frac{q}{T} + \dot{S}_{\text{irrev}}$$

$$\dot{S}_{\text{irrev}} = C_p \ln \left( \frac{T_B}{T_A} \right) - R \ln \left( \frac{P_B}{P_A} \right) \frac{1}{T} \rightarrow \dot{S}_{\text{irrev}} = -287 \ln \left( \frac{31}{3} \right) - \frac{8050}{305} = -0.51 \text{ J/kgK}.$$

As we assumed flow A → B and  $\dot{S}_{\text{irrev}} < 0$ , the flow must be B → A.

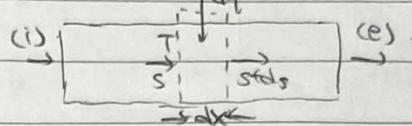
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The Second law for control volumes (steady, reversible flow)

- If in addition to being steady, the flow is also reversible, then the  $\Delta S_{\text{inver}}$  term is zero.

$$S_e - S_i = \sum \frac{q_i}{T_i}$$

- This can be applied to an infinitesimal section of the control volume



which gives

$$dS = \frac{dq}{T}$$

\* This looks similar to the definition for entropy change for a system, but it is different.  $dS$  here is the change in specific entropy between the inlet and exit of the infinitesimal CV, not a change over time.

Integration of the TdS equation.

- For the flow of a pure substance, the eqn. above can be combined w/ TdS eqn. 2 and integrated to give the total heat addition per unit mass of flow.

$$dh = TdS = dh - vdp$$

$$q = \int_i^e dq = \int_i^e (dh - vdp) = (h_e - h_i) - \int_i^e vdp$$

- This expression for  $q$  can be substituted into the SPEC.

$$q - w_x = (h_e + \frac{1}{2}V_e^2 + gz_e) - (h_i + \frac{1}{2}V_i^2 + gz_i)$$

$$(h_e - h_i) - \int_i^e vdp - w_x = (h_e + \frac{1}{2}V_e^2 + gz_e) - (h_i + \frac{1}{2}V_i^2 + gz_i)$$

$$\therefore -w_x = \int_i^e vdp + (\frac{1}{2}V_e^2 + gz_e) - (\frac{1}{2}V_i^2 + gz_i)$$

where  $-w_x$  is the shaft work input per unit mass of flow.

Incompressible reversible flow.

- The above expression can be evaluated if the flow is incompressible, as the density  $\rho$ , hence the specific volume  $v$  will be constant.

note  $v = \frac{1}{\rho}$

$$-w_x = \frac{(P_e - P_i)}{\rho} + (\frac{1}{2}V_e^2 + gz_e) - (\frac{1}{2}V_i^2 + gz_i)$$

- In the case where  $w_x$  is zero, we have Bernoulli's equation

$$\frac{P_e}{\rho} + \frac{1}{2}V_e^2 + gz_e = \frac{P_i}{\rho} + \frac{1}{2}V_i^2 + gz_i$$

applicable if flow is steady, incompressible (reversible) and incompressible!

$$P_e + \frac{1}{2}\rho V_e^2 + \rho g z_e = P_i + \frac{1}{2}\rho V_i^2 + \rho g z_i$$

- The flow may be considered compressible when (i) the working fluid is a liquid or (ii) the working fluid is a gas and the velocity is less than 30% the speed of sound.  $\rightarrow \frac{\Delta P}{P} \approx 5\%$ .

(For a perfect gas,  $V_{\text{sound}} = \sqrt{\gamma RT}$ )

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- e.g.: A small low-speed electric fan blows air through a horizontal duct w/ constant area at a volume flow rate of 3000 L per min. The pressure diff. b/wn inlet/exit of the fan is 290 Pa.

(a) Calculate the min. shaft power req. to drive the fan

(b) Show that the diff. between the actual power and the min. power is prop. to the increase in specific entropy of the air passing through the fan.

(c). Low speed  $\rightarrow$  incompressible flow (const.  $p$ ).

Duct has constant area + horizontal  $\rightarrow$  no change in KE/PE.

Minimum power  $\rightarrow$  reversible.

$$\therefore -W_x = \int_i^e v dp = \frac{P_e - P_i}{\rho} \quad Q \text{ is volumetric flow rate}$$

$$\text{so } \dot{W}_{\min} = \dot{v}(-W_x) = \rho \dot{Q} \cdot \frac{P_e - P_i}{\rho} = \dot{Q} \Delta P$$

$$= \frac{3000 \times 10^{-3}}{60} \times 290 = 14.5 \text{ W.}$$

(b). The actual power is greater than the min. shaft power as the real process is irreversible.

QFEE  $Q - W_x = m \left[ (h_e + \frac{1}{2} \dot{v}^2 + g z_e) - (h_i + \frac{1}{2} \dot{v}^2 + g z_i) \right]$

$$W_{act} = \dot{m}(h_e - h_i) = \dot{m}\Delta h$$

Integrating Tds eqn 2, assuming  $T$  and  $\gamma$  are roughly constant,

$$Tds = h - vdp \quad \rightarrow \quad T\Delta s = \Delta h - \frac{\Delta P}{\rho}$$

Therefore, the diff. b/wn the actual and min. power input is.

$$W_{act} - W_{\min} = \dot{m}\Delta h - \dot{Q}\Delta P = \dot{m}(\Delta h - \frac{\Delta P}{\rho}) = \dot{m}\Delta S$$

since the process is adiabatic,  $\Delta S > \Delta S_{irrev} \rightarrow W_{act} - W_{\min} = \dot{m}\Delta S_{irrev} = S_{irrev}$ .

The second law for control volumes (steady, reversible, adiabatic flow)

- If in addition to being steady and reversible, the flow is also adiabatic, then the  $\frac{\sum \dot{Q}_i}{\dot{V}}$  term is zero.

$$S_e = S_i$$

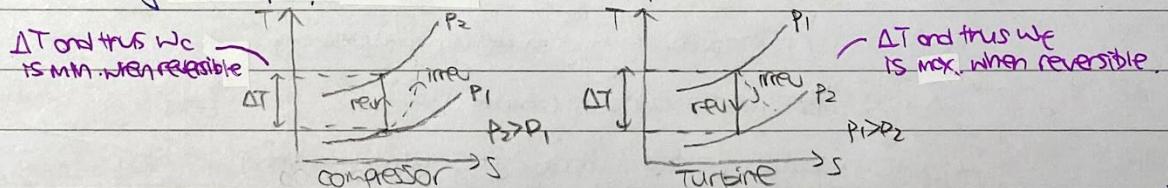
i.e. For CV, steady + adiabatic + reversible  $\rightarrow$  isentropic.

Turbines and compressors.

- Flow through compressor and turbine is sufficiently fast for heat transfer  $\rightarrow$  reversible. [adiabatic]

- As an approximation (for now), the flow is reversible.

- When operating in steady state under these conditions, the flow is isentropic.



- To calculate the shaft work, (i) apply the isentropic relations to relate the temp. at inlet/exit  $\frac{T_2}{T_1} = \left( \frac{P_2}{P_1} \right)^{\frac{\gamma}{\gamma-1}}$

(ii) Apply the SFEE, (usually neglecting KE/PE terms):  $(W_x)_{is} = \Delta h = C_p \Delta T$ ,  $(W_x) \propto \Delta T$

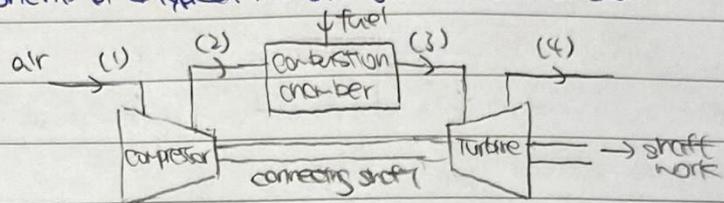
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## Gas turbines and jet engines

### Gas turbines

- The function of a gas turbine (GT) is to produce shaft work to drive a generator, or a pump, or some other piece of rotating machinery.

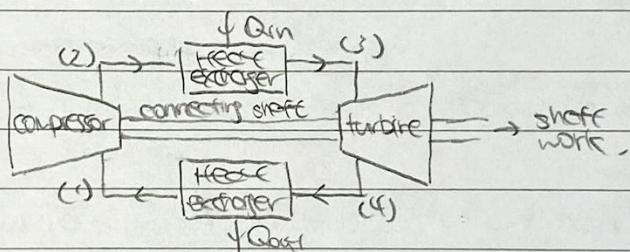
- The main components of a typical industrial gas turbine is as shown:



- Air is taken in at (1) and compressed to state (2), of which pressure is 12-40x that at (1)
- Fuel is then added and combustion occurs at roughly constant pressure. Hot, high pressure combustion gases enter the turbine at (3) and expand down to exhaust pressure at (4),  $\approx P_1$ .
- The shaft work produced by the turbine goes partly to driving the compressor, but there is some leftover, which constitutes the useful shaft work output.

### The air-standard Joule cycle

- The air-standard Joule cycle (Brayton cycle) is an approximate model of the gas turbine that replaces the open-circuit "cycle" w/ a closed-circuit one as shown:



↳ Each device operates as a steady flow device  $\rightarrow$  use SFEE (1st law for cv).

- Here, we make the following assumptions:

$\hookrightarrow$  The working fluid is air throughout, w/ constant  $\gamma$  and  $C_p$ .

$\hookrightarrow$  The fuel mass flow rate is neglected and energy addition is modelled as heat input.

$\hookrightarrow$  KE / PE changes are neglected throughout

$\hookrightarrow$  The turbine and compressor are assumed adiabatic + reversible  $\rightarrow$  isentropic.

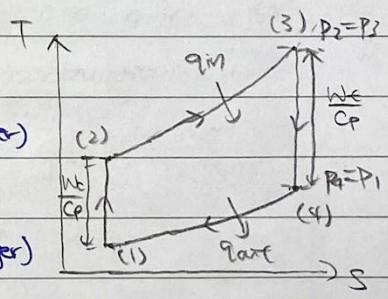
- The air-standard Joule cycle consists:

(1) $\rightarrow$ (2): Isentropic compression (compressor)

(2) $\rightarrow$ (3): Constant pressure heat addition (heat exchanger)

(3) $\rightarrow$ (4): Isentropic expansion (turbine)

(4) $\rightarrow$ (1): Constant pressure heat rejection (heat exchanger)



- The final process (4) $\rightarrow$ (1) has no parallel in the real GT and merely closes the cycle.

- The pressure ratio  $r_p$  is defined as  $r_p = \frac{P_2}{P_1} = \frac{T_2}{T_1}$

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- The efficiency of the Joule cycle can be calculated as follows:

$$(1) \rightarrow (2): \quad \frac{T_2}{T_1} = \left( \frac{P_2}{P_1} \right)^{\frac{\gamma-1}{\gamma}} = r_p^{\frac{\gamma-1}{\gamma}}$$

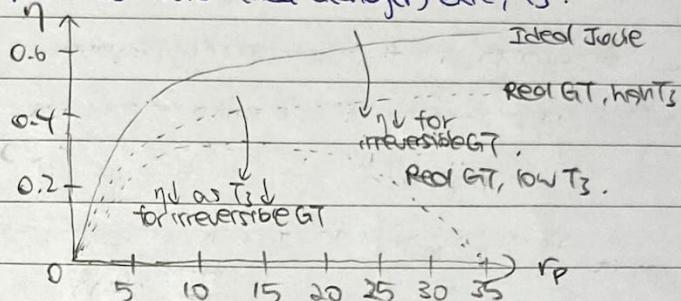
$$(2) \rightarrow (3): \quad q_{in} = C_p(T_2 - T_3)$$

$$(3) \rightarrow (4): \quad \frac{T_4}{T_3} = \left( \frac{P_4}{P_3} \right)^{\frac{\gamma-1}{\gamma}} = r_p^{\frac{1}{\gamma}}$$

$$(4) \rightarrow (1): \quad q_{out} = C_p(T_4 - T_1)$$

$$\therefore \eta = 1 - \frac{q_{out}}{q_{in}} = 1 - \frac{C_p(T_4 - T_1)}{C_p(T_2 - T_3)} = 1 - \frac{T_1 \left( \frac{T_2}{T_1}^{\frac{\gamma-1}{\gamma}} - 1 \right)}{T_2 \left( \frac{T_4}{T_3}^{\frac{\gamma-1}{\gamma}} - 1 \right)} = 1 - \frac{1}{r_p^{\frac{\gamma-1}{\gamma}}}$$

- The efficiency  $\eta$  of the (ideal) Joule cycle depends only on the pressure ratio  $r_p$  and does not depend on the temp. at "combustor" (heat exchanger) exit,  $T_3$ .



- However, for a "real" GT, taking account of irreversibility in the compressor and turbine, the efficiency of the turbine  $\eta$  decreases w/ decreasing  $T_3$  (i.e.  $\eta$  also depends on  $T_3$ ).

- The net work output per unit mass of flow depends on  $T_3$ . When  $= W_T - W_C$

↪ Assuming the turbines are adiabatic and neglecting KE/PE terms, SFT gives

$$|W_C| = C_p \Delta T_C$$

$$|W_T| = C_p \Delta T_T$$

where  $\Delta T_C$  and  $\Delta T_T$  are the compressor temp. rise and turbine temp. drop respectively.

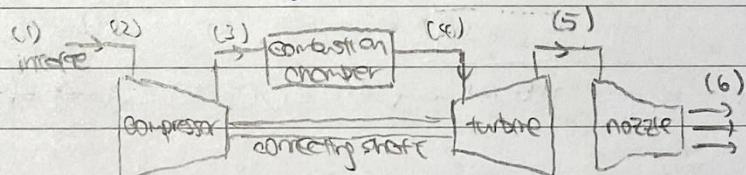
↪ The constant pressure lines on the T-s diagram diverge  $\rightarrow \Delta T_T > \Delta T_C \rightarrow W_T > W_C$

- Since the cycle is effectively reversible, the area enclosed on the T-s diagram is equal to the net heat input ( $q_{in} - q_{out}$ ), which from the 1st law must equal the net shaft work output.

## The jet engine

- The jet engine produces no shaft work. Its function is to generate thrust by means of a high velocity exhaust jet.

- The main components of a turbojet are shown below.



- Rather than producing net shaft work, the turbine work output for a jet engine just balances the compressor work input.

- The hot exhaust gas is then expanded further through a nozzle, producing a high velocity jet and hence thrust.

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- Applying the SFEE to the exhaust nozzle, assuming the flow is adiabatic,

$$q^0 - w_x^0 = (h_6 + \frac{1}{2}V_6^2) - (h_5 + \frac{1}{2}V_5^2)$$

We can usually assume that the KE term is small at inlet (5) to the nozzle, so

$$\frac{1}{2}V_6^2 = C_p(T_5 - T_6).$$

- If we assume the flow in the nozzle is reversible (+ adiabatic)  $\rightarrow$  isentropic relations.

$$\frac{T_5}{T_6} = \left(\frac{P_5}{P_6}\right)^{\frac{k}{k-1}} \quad P_6 = P_a$$

- Applying the SFEE to the intake, assuming the flow is adiabatic,

$$q - w_x = (h_2 + \frac{1}{2}V_2^2) - (h_1 + \frac{1}{2}V_1^2)$$

When the aircraft is in flight, the incoming airstream at the engine intake has significant KE rel.

to the engine. Assume the KE is small at the compressor inlet (2).

$V_1$  is the flight velocity  $\rightarrow \frac{1}{2}V_1^2 = C_p(T_2 - T_1)$

- If we assume the flow at the intake is reversible (adiabatic)  $\rightarrow$  isentropic relations

$$\frac{P_2}{P_1} = \left(\frac{T_2}{T_1}\right)^{\frac{k}{k-1}} \quad T_1 = \text{ambient temp.}$$

There is a pressure rise at the intake, which is achieved by stagnating the flow rel. to

the engine (ram effect)

(1) is same  $T$  as (2)  
when flight velocity  $V_1 = 0$

- The processes involved in a jet engine are:

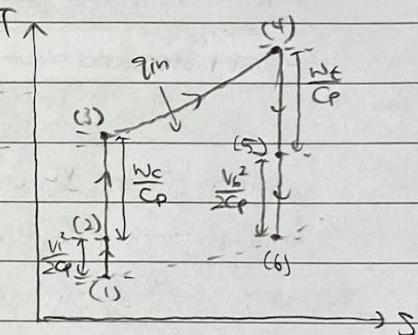
(1)  $\rightarrow$  (2) : isentropic stagnation (intake)

(2)  $\rightarrow$  (3) : isentropic compression (compressor)

(3)  $\rightarrow$  (4) : constant pressure heat input (combustor)

(4)  $\rightarrow$  (5) : isentropic expansion (turbine)

(5)  $\rightarrow$  (6) : isentropic acceleration (nozzle)



\* Note that  $w_e = w_c$  so  $(T_4 - T_5) = (T_3 - T_2)$  — key for jet engines

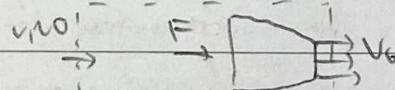
- It is not usually necessary to close the "cycle" via a frictional process (6)  $\rightarrow$  (1) as we are normally interested in calculating the engine thrust, rather than efficiency.

- To calculate the engine thrust, apply the SFME for the whole engine.

$\Rightarrow$  Draw the CV s.t.  $V_1 \neq 0$

SFME:  $F + \sum \delta A = m(V_6 - V_1)$

$$F = m V_6$$



$$P = P_a \rightarrow \sum \delta A \text{ term cancels out}$$

(where  $V_6$  is calculated using  $\frac{1}{2}V_6^2 = C_p(T_5 - T_6)$  and  $\frac{T_5}{T_6} = \left(\frac{P_5}{P_6}\right)^{\frac{k}{k-1}}$ )