Electricity

Vectors and coordinate systems

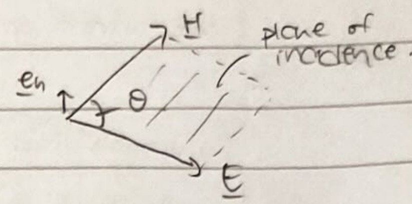
Vectors

- A vector has both magnitude and direction. The magnitude of $\underline{E}$ is a scalar quantity written as E / $|\underline{E}|$. A unit vector $\underline{e}_E$ has the same direction as $\underline{E}$.

$$\underline{E} = E\underline{e}_E .$$

- A vector can be represented in any suitable coord system as a series of component values. eg in Cartesian coords, $\underline{E} = E_x \underline{e}_x + E_y \underline{e}_y + E_z \underline{e}_z$.
- The dot product of $\underline{E}$ and $\underline{H}$ is given by

$$\underline{E} \cdot \underline{H} = |\underline{E}||\underline{H}|\cos\theta .$$



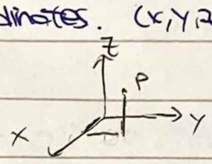
- The cross product of $\underline{E}$ and $\underline{H}$ is given by

$$\underline{E} \times \underline{H} = |\underline{E}||\underline{H}|\sin\theta \, \underline{e}_n$$

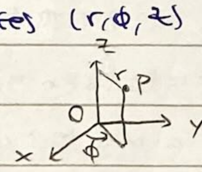
Coordinate systems.

- The key to solving any electromagnetic problem is to use the correct symmetry to simplify the maths. → use the correct coordinate system.

① Cartesian / rectangular coordinates. (x, y, z)



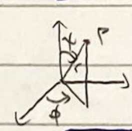
② circular cylindrical coordinates (r, ϕ, z) [wires]



ϕ : azimuthal

$r = \sqrt{x^2+y^2}$, $\phi = \arctan(\frac{y}{x})$. $z = z$. // $x = r\cos\phi$, $y = r\sin\phi$, $z = z$.

③ Spherical coordinates (r, ϕ, ψ) [balls, pts, spheres].



ϕ : azimuthal

ψ : zenithal

$r = \sqrt{x^2+y^2+z^2}$, $\phi = \arctan(\frac{y}{x})$, $\psi = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right)$ // $x = r\sin\psi\cos\phi$, $y = r\sin\psi\sin\phi$, $z = r\cos\psi$

Electric force and electric field.

Electric charge.

- The charge of an e^- is -1.6×10^{-19}C. → a unit charge $e = 1.6\times10^{-19}$C.
- The interaction of charges generates force and force fields.
  - ↳ charges of the same sign — repel
  - ↳ charges of the opposite sign — attract.

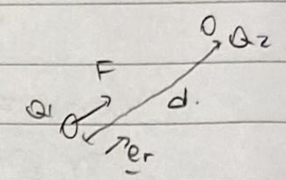
Electrical conductors and insulators.

- In insulators (i.e. plastics, ceramics, dielectrics), charge motion is restricted within the material → charge is assumed to be distributed throughout the material (unless told o/w)
- However, in some cases, there may still be an imbalance of e^- throughout the molecules in the material → net overall charge. (dielectrics).
- In conductors, (i.e. metals), charges are free to move throughout the material → charge is assumed to be uniformly distributed on the surface of the material (since charges are furthest apart in this configuration).

Coulomb's law

- Coulomb's law states that the electrostatic force F is given by.

$$\boxed{\underline{F} = k_e \frac{Q_1 Q_2}{r^2} \underline{e_r} = \frac{1}{4\pi\varepsilon_0} \frac{Q_1 Q_2}{r^2} \underline{e_r}}$$



where r is the distance between the 2 charges Q_1, Q_2.

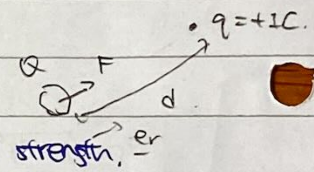
& k_e is Coulomb's constant. $k_e = 8.988 \times 10^9 \ Nm^2C^{-2}$.

ε_0 is the dielectric permittivity of free space, $\varepsilon_0 = 8.854 \times 10^{-12} Fm^{-1}$.

Electric field.

- The electric field $\underline{E}$ represents the lines of force surrounding charges in an electrostatic system. It is the electrostatic force experienced by a unit charge.

$$\boxed{\underline{E} = k_e \frac{Q}{r^2} \underline{e_r} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \underline{e_r}}$$



and $\boxed{\underline{F} = q\underline{E}}$, where q is a unit test charge.

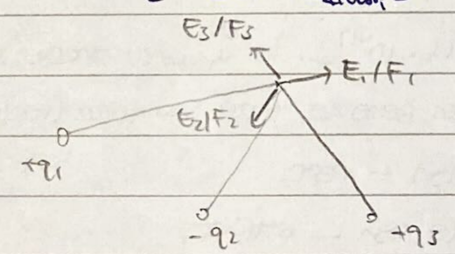
- The density of electric field lines represents the electric field strength.
- Electric field lines always begin and end at a charged particle / pt. on a surface (infinity).

Superposition

- The electrostatic force F and electric field E are vector quantities → the net force and electric field can be found by the vector sum of the contributions of each charge.

$$\underline{F} = \Sigma \underline{F_i} = \Sigma \frac{Q_i Q_i}{4\pi\varepsilon_0 r_i^2} \underline{e_{ri}}$$

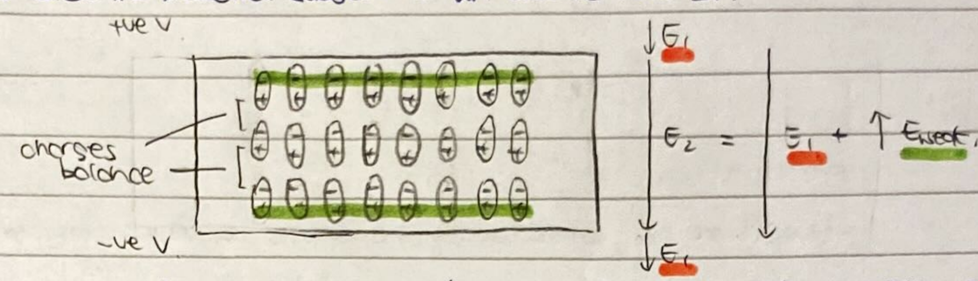
$$\underline{E} = \Sigma \underline{E_i} = \Sigma \frac{Q_i}{4\pi\varepsilon_0 r_i^2} \underline{e_{ri}}$$



Dielectrics and electric flux density

Dielectrics

- When charges within a material are subjected to an electric field,
  - ↳ metal : charges are free to move → current flow
  - ↳ dielectric : charges bound inside the molecules but can move within each molecule.
    - → charge imbalance (polarisation).
- Consider a dielectric material subject to an electric field $\underline{E}_1$.



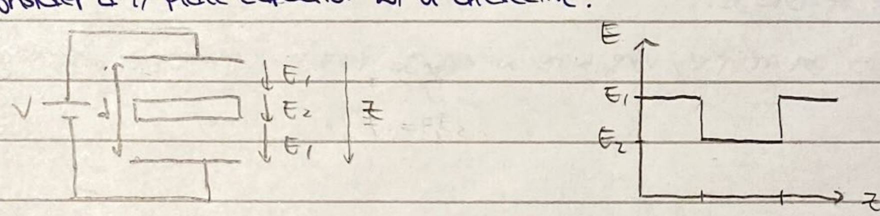
- The electric field polarises the molecules within the dielectric, slightly separating the +ve and -ve charges.
- In the bulk of the dielectric, the +ve/-ve charges cancel out, but on the surface, a net +ve and -ve charge appears.
- The net charge at the edge sets up weak internal E field E_{weak}, which is the opposite dir. of the ext. E field E_1. $E_{weak} = -\chi_e \underline{E}_2$, χ_e is the electrical susceptibility.
- The net electric field E_2 in the dielectric material is therefore given by
  $(\underline{E}_2 = \underline{E}_1 + \underline{E}_{weak} = \underline{E}_1 - \chi_e \underline{E}_2)$ $\quad E_2 = E_1 - E_{weak}$. $\quad$ so $|E_2| < |E_1|$
- For a linear material, the E field inside the dielectric material is proportional to the ext. E field E_1, so (across air: E_1; across dielectric: E_2)

$$E_1 = \varepsilon_r E_2$$

where ε_r is the rel. permittivity / dielectric constant of the material.

$*$ $\varepsilon_r \geq 1$. It is assumed that $\varepsilon_r = 1$ for vacuum/air. [permittivity $\varepsilon = \varepsilon_0 \varepsilon_r$].

- This assumes that materials are perfect dielectrics. However in reality, when the E field is large, the e^- will be pulled out of the molecules → material starts conducting (Dielectric breakdown). The dielectric strength E_{max} of a material is defined as the max. E field it can tolerate before breakdown occurs. (E_{max} air $= 3 \times 10^6 \, Vm^{-1}$).
- Consider a // plate capacitor w/ a dielectric.



→ E-z graph is discontinuous → define a new quantity : Electric flux density D.

Electric flux density

- It is more convenient to use electric flux density $\underline{D}$ rather than E field $\underline{E}$ to analyse dielectrics.
- The electric flux density $\underline{D}$ is defined as.

D is continuous
E is directly related to voltage.

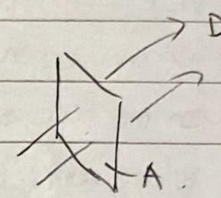
$$\boxed{\underline{D} = \varepsilon \underline{E} = \varepsilon_0 \varepsilon_r \underline{E}}$$

- The normal component of electric flux density $\underline{D}$ takes the same value on passing from one dielectric to another → flux conservation. (always continuous!)

Gauss's law and electric potential.

Electric flux.

- Flux is the idea of evaluating the amt. of a vector field that passes through a surface of area A.
- Electric flux Φ_D is given by.



$$\Phi_D = \iint_S \underline{D} \cdot d\underline{A}.$$

- If the field is always ⊥ to the surface, $\Phi_D = DA$.

Gauss's law.

- Gauss's law states that the total electric flux of D through any closed surface S equals the charge enclosed by the surface.

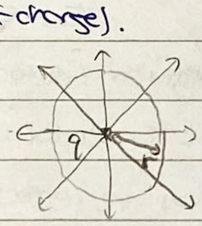
Gaussian surface.

$$\boxed{Q_{enc} = \oint\!\!\oint_S \underline{D} \cdot d\underline{S} = \Phi_D}$$

- Gauss's law is very useful because if we know the charge Q, we can calculate D.

① A point charge (or sphere of charge).

consider a pt. charge. +q



G/L

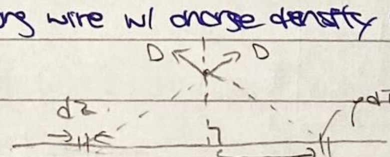
$$Q_{enc} = \oint\!\!\oint_S \underline{D} \cdot d\underline{S}$$

$$q = D \cdot 4\pi r^2$$ $\underline{D}$ always ⊥ $\underline{S}$.

$$\boxed{D = \frac{q}{4\pi r^2}}$$ consistent w/ Coulomb's law.

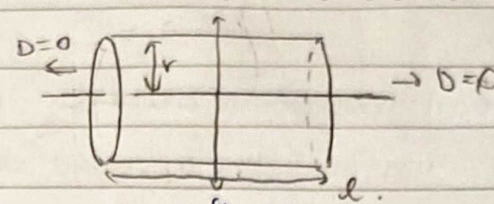
② A line of charge.

Consider an infinitely long wire w/ charge density (charge per unit length) ρ.



Horizontal component of D cancels out.

By symmetry, the resultant $\underline{D}$ must act radially outwards at every pt. along the line.



G/L $$Q_{enc} = \oiint_S \underline{D} \cdot d\underline{S}$$

$$\rho \cancel{\ell} = D \cdot (2\pi r \cancel{\ell})$$

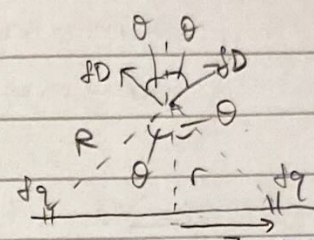
$$\boxed{D = \frac{\rho}{2\pi r}}$$

the ends of the cylinder are ignored as flux through them is 0.

Alternatively, this could be solved using Coulomb's law.

$R = \sqrt{r^2+z^2}$. $\cos\theta = \frac{r}{R} = \frac{r}{\sqrt{r^2+z^2}}$

Consider 2 elements either side a distance z away.



$$\delta q = \rho \delta z.$$

Coulomb's law.
$$\frac{\delta D_v}{\cancel{\varepsilon}} = \frac{1}{4\pi \cancel{\varepsilon}} \frac{\delta q}{R^2} \cos\theta \times 2$$

$$\delta D_v = \frac{1}{2\pi} \frac{\rho r}{(r^2+z^2)^{3/2}} \delta z$$

$$D = \frac{\rho r}{2\pi} \int_0^\infty \frac{1}{(r^2+z^2)^{3/2}} dz$$

$$= \frac{\rho r}{2\pi} \int_0^{\pi/2} \frac{1}{(r^2 \sec^2\theta)^{3/2}} r\sec^2\theta \, d\theta$$

$$= \frac{\rho r}{2\pi} \int_0^{\pi/2} \frac{\cos\theta}{r^2} d\theta$$

$$= \frac{\rho}{2\pi r}$$

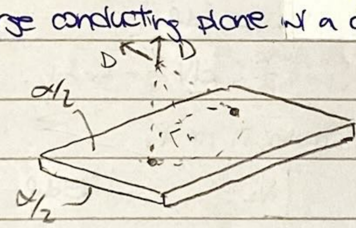
$z = r\tan\theta$

$dz = r\sec^2\theta \, d\theta$

z	θ
∞	$\pi/2$
0	0

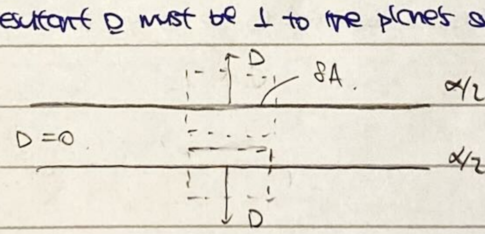
(3) Thin uniform metal plane.

Consider an infinitely large conducting plane w/ a charge density (charge per unit area) α.



components of D // to plane cancels out.

By symmetry, the resultant $\underline{D}$ must be $\perp$ to the plane's surface



By symmetry, $D_{top} = D_{bottom}$. Consider the top surface.

G/L $$Q_{enc} = \oiint_S \underline{D} \cdot d\underline{S}$$

$$\frac{\alpha}{2} \cancel{\delta A} = D \cancel{\delta A}$$

$$\boxed{D = \frac{\alpha}{2}}$$

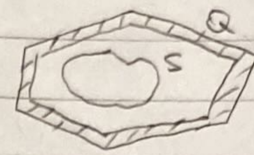
Key results: Point/Sphere : $D = \frac{q}{4\pi r^2}$ $E = \frac{q}{4\pi \varepsilon r^2}$

Line : $D = \frac{\rho}{2\pi r}$ $E = \frac{\rho}{2\pi \varepsilon r}$

Plane : $D = \frac{\alpha}{2}$ $E = \frac{\alpha}{2\varepsilon}$

The Faraday cage.

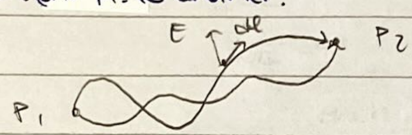
- A simple but important consequence of Gauss's law is the absence of an electric field inside a hollow, charged conducting thinned wall shell.
- When considering a conducting object where charges are free to move, we always assume the charge resides on the surface.
- A Gaussian surface S inside must have 0 flux through it, since it encloses 0 charge.



- Given 0 flux for a finite surface area → E-field is 0 everywhere inside the hollow conducting shell.

Potential difference.

- Consider the work req. to move a unit charge from pt. P_1 to P_2 in the presence of an E-field E. At any pt. along the path, we must exert a force $-E$ to hold the charge in eqm.
- The electrostatic potential difference btwn 2 pts. is the work done when a unit +ve charge is moved from one pt. to another.



same work done for both paths as the force field is conservative.

- Work done in moving distance δl is:

work done on the system: $\delta V = -\overbrace{E}^{\text{ext. force}} \cdot \delta \underline{l}$

So total work to move from P_1 to P_2:

$$\boxed{V_2 - V_1 = -\int_1^2 \underline{E} \cdot d\underline{l}}$$ [Eqn. w/ no name].

- Electrostatic potential is defined to be 0 at infinity.

Capacitance

Capacitance.

- Capacitance exists between any pair of conductors which are electrically insulated from each other (Actually everything has a capacitance but it may be extremely small).
- Capacitance C is the ratio of charge to potential difference applied (units F (m^{-1}/m^{-2})).

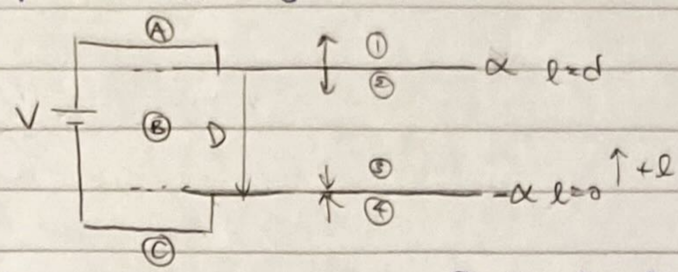
$\boxed{Q = CV}$	$\boxed{\rho = C_l V}$	$\boxed{\alpha = C_A V}$
point	line	plane.

- Strategy to find capacitance:

use superposition for combinations of pts, lines, planes.

↳ ① Specify voltages / charges
↳ ② Calculate D (using Gauss' law)
↳ ③ Calculate E (using $D = \varepsilon_0 \varepsilon_r E$).
↳ ④ Calculate V (using EWN)
↳ ⑤ Calculate C (using $C = \frac{Q}{V}$).

Capacitance of a pair of // conducting plates.



At Ⓐ $\Sigma D = D_1 + D_3$

$= \frac{\alpha}{2}\uparrow + \frac{\alpha}{2}\downarrow = 0$

At Ⓒ $\Sigma D = D_2 + D_4$

$= \frac{\alpha}{2}\downarrow + \frac{\alpha}{2}\uparrow = 0.$

At Ⓑ $\Sigma D = D_2 + D_3$

$= \frac{\alpha}{2}\downarrow + \frac{\alpha}{2}\downarrow = \alpha\downarrow.$

$\therefore$ $D=0$ above top plate / below bot. plate ; $D = \alpha\downarrow$ between the plates.

Find E $\quad \varepsilon_0 E = D$

$E = \alpha/\varepsilon_0$

EVN. $\quad V_2 - V_1 = -\int_1^2 \underline{E}\cdot d\underline{l}$

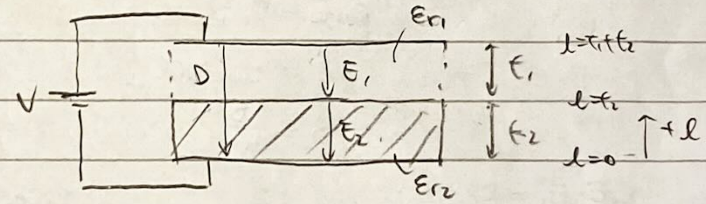
$V - 0 = -\int_0^d (-E)\, dl$

$V = Ed = \frac{\alpha}{\varepsilon_0} d. = \frac{Qd}{A\varepsilon_0}.$

Defn of C $\quad C = \frac{Q}{V}$

$\boxed{C = \frac{\varepsilon_0 A}{d}.}$

Capacitance of a pair of // conducting plates w/ dielectric.



$D = \alpha$

Find E $\quad \varepsilon_0 \cancel{\varepsilon_{r1}}^{1} E_1 = D \qquad \varepsilon_0 \varepsilon_{r2} E_2 = D$

$E_1 = \frac{D}{\varepsilon_0} \qquad E_2 = \frac{D}{\varepsilon_0 \varepsilon_{r2}} = \frac{E_1}{\varepsilon_{r2}}$

EVN $\quad V_2 - V_1 = -\int_1^2 \underline{E}\cdot d\underline{l}$

$V - 0 = -\int_0^{t_2} (-E_2)\cdot dl - \int_{t_2}^{t_1+t_2} (-E_1)\cdot dl$

$V = E_2 t_2 + E_1 t_1 + \cancel{E_1 t_2} - \cancel{E_1 t_2}$

$= \frac{E_1}{\varepsilon_{r2}} t_2 + E_1 t_1$

$= \frac{D}{\varepsilon_0}\left(\frac{t_2}{\varepsilon_{r2}} + t_1\right)$

$= \frac{\alpha}{\varepsilon_0}\left(\frac{t_2}{\varepsilon_{r2}} + t_1\right) = \frac{Q}{A\varepsilon_0}\left(\frac{t_2}{\varepsilon_{r2}} + t_1\right).$

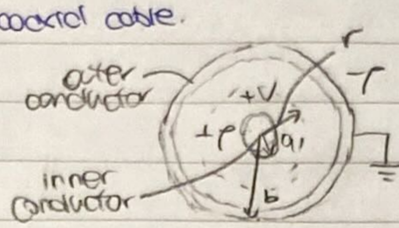
Defn of C. $\quad C = \frac{Q}{V}$

$\boxed{C = \frac{\varepsilon_0 A}{t_1 + \frac{t_2}{\varepsilon_{r2}}} = \frac{\varepsilon_0 \varepsilon_{r2} A}{\varepsilon_{r2} t_1 + t_2}}$

check $t_1 = 0$, $C = \frac{\varepsilon_0 \varepsilon_{r2} A}{t_2}$

$t_2 = 0$, $C = \frac{\varepsilon_0 A}{t_1}$

Capacitance of air-filled coaxial cable.



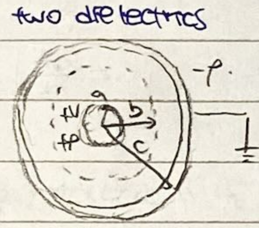
G/L
$$Q_{enc} = \oiint_S \underline{D} \cdot d\underline{S}.$$
$$\rho \cancel{l} = D \cdot (2\pi r \cancel{l})$$
$$D = \frac{\rho}{2\pi r}.$$

Find E
$$\varepsilon_0 E = D$$
$$E = \frac{\rho}{2\pi\varepsilon_0 r}$$

ENN
$$V_2 - V_1 = -\int_1^2 E\,dl$$
$$0 - V = -\int_a^b \frac{\rho}{2\pi\varepsilon_0 r} dr.$$
$$V = \frac{\rho}{2\pi\varepsilon_0} \ln\left(\frac{b}{a}\right).$$

Defn of C.
$$C_l = \frac{Q}{Vl} = \frac{\rho l}{Vl}$$
$$\boxed{C_l = \frac{2\pi\varepsilon_0}{\ln\left(\frac{b}{a}\right)}}.$$

Capacitance of coaxial cable w/ two dielectrics



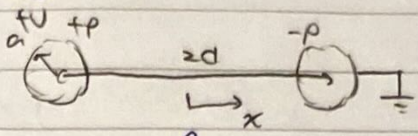
G/L
$$Q_{enc} = \oiint_S \underline{D} \cdot d\underline{S}.$$
$$\rho l = D(2\pi r \cancel{l})$$
$$D = \frac{\rho}{2\pi r}.$$

Find E $\quad \varepsilon_0 \varepsilon_{r1} E_1 = D \qquad \qquad \varepsilon_0 \varepsilon_{r2} E_2 = D$
$$E_1 = \frac{D}{2\pi\varepsilon_0\varepsilon_{r1} r} \qquad \qquad E_2 = \frac{D}{2\pi\varepsilon_0\varepsilon_{r2} r}$$

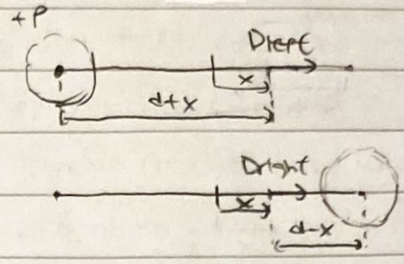
ENN
$$V_2 - V_1 = -\int_1^2 E\,dl$$
$$0 - V = -\left[\int_a^b \frac{D}{2\pi\varepsilon_0\varepsilon_{r1} r} dr + \int_b^c \frac{D}{2\pi\varepsilon_0\varepsilon_{r2} r} dr\right]$$
$$V = \frac{\rho}{2\pi\varepsilon_0}\left[\frac{1}{\varepsilon_{r1}} \ln\left(\frac{b}{a}\right) + \frac{1}{\varepsilon_{r2}} \ln\left(\frac{c}{b}\right)\right]$$

Defn of C
$$C_l = \frac{Q}{Vl} = \frac{\rho \cancel{l}}{V \cancel{l}}$$
$$\boxed{C_l = \frac{2\pi\varepsilon_0}{1/\varepsilon_{r1} \ln(b/a) + 1/\varepsilon_{r2} \ln(c/b)}}$$

capacitence of two parallel wires



D due to an infinitely long wire : $D = \frac{\rho}{2\pi r}$



$$\Sigma D = D_{left} + D_{right}$$

$$= \frac{\rho}{2\pi(d+x)} + \frac{\rho}{2\pi(d-x)}$$

Find E $\qquad \varepsilon_0 E = D$

$$E = \frac{\rho}{2\pi\varepsilon_0(d+x)} + \frac{\rho}{2\pi\varepsilon_0(d-x)}$$

EMN $\qquad V_2 - V_1 = -\int_1^2 E\, dl$

$$0 - V = -\int_{-d+a}^{d-a} \frac{\rho}{2\pi\varepsilon_0}\left[\frac{1}{d+x} + \frac{1}{d-x}\right] dx$$

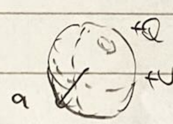
$$V = \frac{\rho}{\pi\varepsilon_0}\ln\left(\frac{2d-a}{a}\right)$$

Defn of C $\qquad C_l = \frac{Q}{Vl} = \frac{\rho l}{Vl}$

$$= \frac{\pi\varepsilon_0}{\ln\left(\frac{2d-a}{a}\right)}$$

since $a \ll d$, $2d-a \approx 2d$ $\qquad \therefore \boxed{C_l \approx \frac{\pi\varepsilon_0}{\ln(2d/a)}}$

capacitance of a sphere.



GL $\qquad Q_{enc} = \oint\!\!\oint_S \underline{D}\cdot d\underline{S}$

$$Q = D\cdot(4\pi r^2)$$

$$D = \frac{Q}{4\pi r^2}$$

Find E $\qquad \varepsilon_0 E = D$

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

EMN $\qquad V_2 - V_1 = -\int_1^2 E\, dl$

$$0 - V = -\int_a^\infty \frac{Q}{4\pi\varepsilon_0 r^2} dr$$

$$V = \frac{Q}{4\pi\varepsilon_0 a}$$

Defn of C $\qquad C = \frac{Q}{V}$

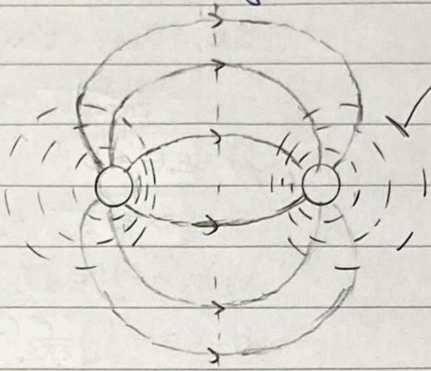
$$\boxed{C = 4\pi\varepsilon_0 a}$$

Equipotentials

Equipotentials.

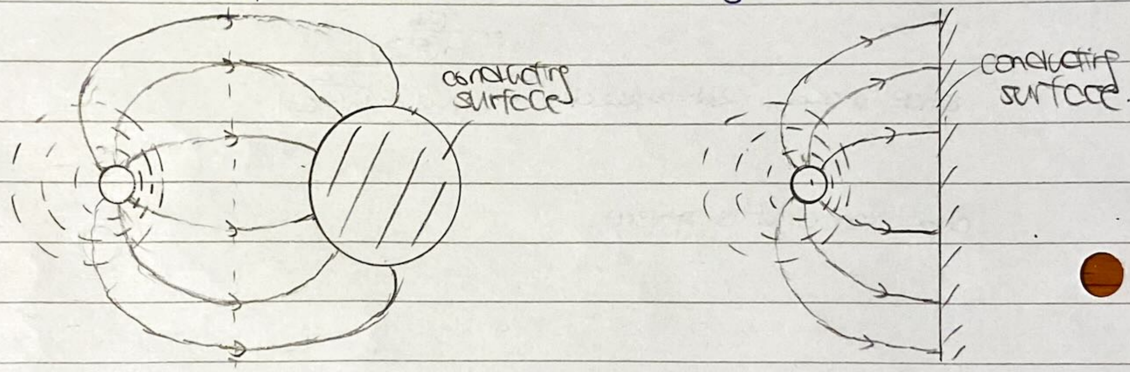
- An equipotential is a line/surface of constant potential.
- We can move a pt. charge along an equipotential w/o doing work against electric fields.
- Equipotential lines are ⊥ to electric field lines — if a charge travels ⊥ to an electric field, no work is done by/against the field.
- Consider two equal and opposite charges +Q and −Q.

  The equipotentials are circles of increasing radius.



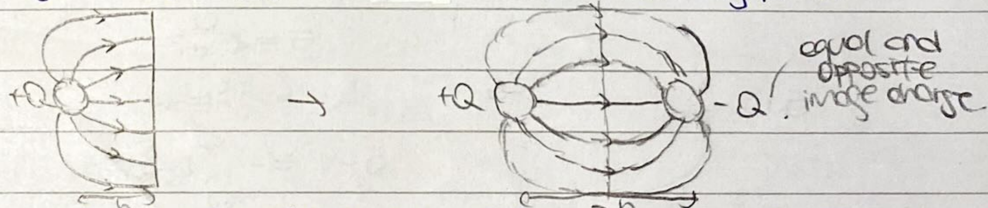
The equipotential midway between the conductors is a line (circle w/ infinite radius)

- We can replace any equipotential w/ a conducting surface.

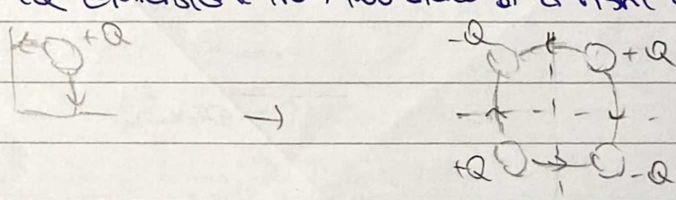


Method of images.

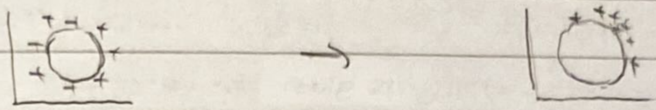
- We can replace a conducting surface w/ an image charge (that gives same equipotentials) ← usually equal and opposite charge.
- Consider a charge +Q at a height h next to a metal conducting plate.



- Consider a charge +Q equidistant from two sides of a right angle conductor.



- This method is only an approximation as we assume the charged sphere to behave as if it were uniformly charged.
- In practice, the metal plane will distort the charge distribution of the sphere.

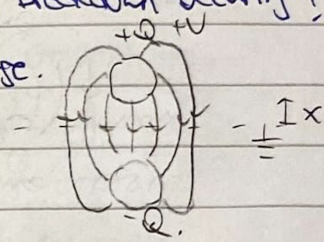


- An exact sol'n requires finite element and finite difference methods.

- e.g: Voltage of sphere above a conducting plane.

A metal sphere of radius $a = 5mm$ is centred at a height $h = 10mm$ from an earthed conducting sheet. The electrical breakdown strength of air $|E_{max}|$ is $3 \times 10^6 Vm^{-1}$. What is the max. voltage / charge applied to the sphere w/o breakdown occurring?

First, we replace the conducting sheet w/ an image charge.



For a sphere, $E = \frac{Q}{4\pi\varepsilon_0 r^2}$, so by superposition,

$$E = -\left[\frac{Q}{4\pi\varepsilon_0 (h-x)^2} + \frac{Q}{4\pi\varepsilon_0 (h+x)^2}\right]$$

E field dir. is opposite to x dir. / due to real charge at $x = h$ / due to image charge at $x = -h$.

As E field is max at the edge of a charged sphere, sub $x = h - a$.

$$|E_{max}| = \frac{Q}{4\pi\varepsilon_0 a^2} + \frac{Q}{4\pi\varepsilon_0 (2h-a)^2}$$

Substituting $a = 5mm$, $h = 10mm$, $|E_{max}| = 3 \times 10^6 Vm^{-1}$, $Q = 7.5nC$.

$$V_2 - V_1 = -\int_1^2 E\,dl.$$

$$V - 0 = -\int_0^{h-a} -\left[\frac{Q}{4\pi\varepsilon_0 (h-x)^2} + \frac{Q}{4\pi\varepsilon_0 (h+x)^2}\right] dx$$

$$V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{2h-a}\right).$$

Substituting $a = 5mm$, $h = 10mm$, $Q = 7.5nC$. $V = 9000V$.

* Note, w/o the metal plane, $E = \frac{Q}{4\pi\varepsilon_0 r^2}$ so $|E_{max}| = \frac{Q}{4\pi\varepsilon_0 a^2}$ and $V = \frac{Q}{4\pi\varepsilon_0 a}$

substituting $a = 5mm$, $|E_{max}| = 3 \times 10^6 Vm^{-1}$, $Q = 8.34nC$, $V = 15000V$.

i.e. as we move a conductor closer to a charged sphere, the voltage req. for breakdown drops.

Electrostatic energy and forces

Energy stored in an electric field.

- Consider the work done δW in adding a charge δq to a capacitor at voltage V.

$I = \frac{\delta q}{\delta t}$, $V = \frac{q}{C}$, $P = IV$ // $\delta W = P\delta t = \frac{\delta q}{\delta t}\frac{q}{C}\delta t = \frac{1}{C} q\,\delta q$.

$$\therefore W = \int_0^Q \frac{1}{C} q\,dq = \frac{1}{2C} Q^2$$

As $Q = CV$, $\boxed{W = \frac{Q^2}{2C} = \frac{1}{2}QV = \frac{1}{2}CV^2}$ → E field can store energy.

* The expression above applies to any capacitor of any geometry.

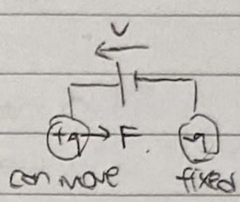
Energy stored in a // plate capacitor.

- Consider a // plate capacitor w/ area A and plate separation x. We neglect any fringe effects at the edges of the plate.

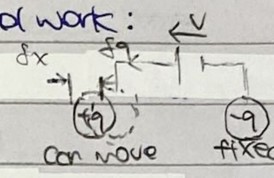
$$V = Ex, \quad C = \frac{\varepsilon_r \varepsilon_0 A}{x}, \quad W = \tfrac{1}{2}CV^2 \rightarrow W = \tfrac{1}{2}\frac{\varepsilon_0 \varepsilon_r A}{x}E^2x^2 = \tfrac{1}{2}\varepsilon_0\varepsilon_r E^2(Ax).$$

$\therefore$ The energy density stored in the electric field is given by $\boxed{\frac{W}{Ax} = \tfrac{1}{2}\varepsilon_0\varepsilon_r E^2}$

Electrostatic force (virtual work method).



- Consider a system consisting of two arbitrary charged conductors:
  - ↳ one of the conductors is fixed (cannot move)
  - ↳ the conductors have a charge of $+q$ and $-q$ → they experience a Coulomb force F
  - ↳ The battery between the conductors keep the potential V constant
- To find the Coulomb force F, we use the method of virtual work:
  - ↳ Apply a virtual force F^* equal and opposite to F so the conductors are held in static eqm.



  - ↳ Imagine the moveable conductor moves a small distance δx towards the other.
- Consider the energy changes,
  - ↳ Mechanical work is done on the system $-F^*\delta x$.
  - ↳ The battery does work on the system $V\delta q$ by providing an extra charge δq (so V is const)
  - ↳ There is a net change in electrostatic energy of $\tfrac{1}{2}V\delta q$. ($W = \tfrac{1}{2}QV$, so $\delta W = \tfrac{1}{2}V\delta q$).

δx results in δC.
As $q = CV$, $\delta q = V\delta C$.
so δx results in δq.

- Using consv. of mechanical energy, (considering the system).

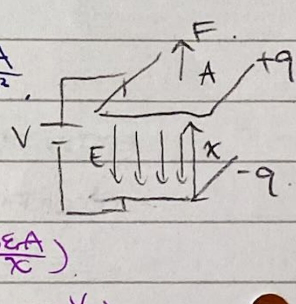
$$-F\delta x + V\delta q = -\tfrac{1}{2}V\delta q$$
$$F\delta x = \tfrac{1}{2}V\delta q$$
$$F = \tfrac{1}{2}V\frac{\delta q}{\delta x}.$$

As $q = CV$, $\frac{\delta q}{\delta x} = V\frac{\partial C}{\partial x}$, so $\boxed{F = \tfrac{1}{2}V^2\frac{\partial C}{\partial x}}$

useful as we just need to know V and $\frac{\partial C}{\partial x}$.
(no need to know q, r)
* To find $\frac{\partial C}{\partial x}$, we need $C = fn(x)$.

Force between capacitor plates.

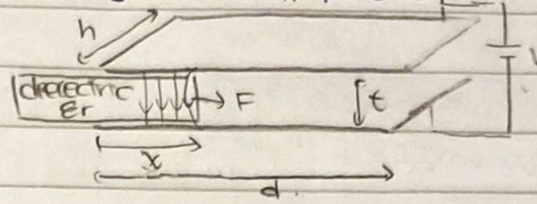
- For a // plate capacitor, $C = \frac{\varepsilon_0\varepsilon_r A}{x}$, so $\frac{\partial C}{\partial x} = -\frac{\varepsilon_0\varepsilon_r A}{x^2}$.



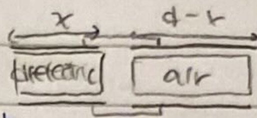
$$F = \tfrac{1}{2}V^2\frac{\partial C}{\partial x}$$
$$= \tfrac{1}{2}V^2\left(-\frac{\varepsilon_r\varepsilon_0 A}{x^2}\right).$$
$$= -\tfrac{1}{2}V^2\frac{C}{x}. \qquad \left(C = \frac{\varepsilon_0\varepsilon_r A}{x}\right)$$
$$\boxed{F = -\tfrac{1}{2}qE} \qquad \left(q = CV,\ E = \frac{V}{x}\right).$$

* Note the electrostatic force F is not qE.
* The -ve sign means the force is attractive (opposite the dir. of +ve x).

Force on dielectric slab inserted into // plate capacitor.



- The system is essentially two capacitors connected in //



$$\therefore C_{eq} = \frac{\varepsilon_0 \varepsilon_r x h}{t} + \frac{\varepsilon_0 (d-x) h}{t} \rightarrow \frac{\partial C}{\partial x} = \frac{\varepsilon_0 \varepsilon_r h}{t} - \frac{\varepsilon_0 h}{t} = \frac{\varepsilon_0 h}{t}(\varepsilon_r - 1)$$

$$F = \frac{1}{2}V^2 \frac{\partial C}{\partial x}$$

$$\boxed{F = \frac{1}{2}V^2 \frac{\varepsilon_0 h}{t}(\varepsilon_r - 1)}$$

Electric current

Current.

- Any conductor has delocalised e^- that can move freely about the conductor.
- If an ext. E field is applied to the conductor, the delocalised e^- will accelerate under the applied force and drift in a single direction, creating a current.
- Current I is defined as the rate of change of charge through a fixed area wrt. time, i.e. $\boxed{I = \frac{dQ}{dt}}$

Current density.

- current density $\underline{J}$ is defined as.

$$dI = \underline{J} \cdot d\underline{S} \quad , \quad \text{or} \quad \boxed{I = \iint_S \underline{J} \cdot d\underline{S}} \qquad \left(J = \frac{\Delta I}{\Delta S}\right)$$

- If the conductor has a uniform cross section, $\boxed{J = \frac{I}{S}}$
- Considering a conducting volume w/ charge density ρ_v that moves at a drift velocity $\underline{u}$, then the current density J is given by

$$\boxed{J = \rho_v \underline{u}}$$

Resistance.

- Considering conduction currents, e^- experience a force $\underline{F}$ when an E field $\underline{E}$ is applied, $\underline{F} = -e\underline{E}$.
- This force move the e^- along the wire but the many collisions slow them down to an average drift velocity $\underline{u}$. If the e^- have mass m and the average time between collisions is τ, the force is given by

$$\underline{F} = \frac{m\underline{u}}{\tau} = -e\underline{E}$$

$$\therefore \underline{u} = -\frac{e\tau}{m}\underline{E}.$$

- If there are n e^- in a unit volume, then charge density ρ_v is simply

$$\rho_v = -ne.$$

- Using the defn of current density $\underline{J}$,

$$\boxed{\underline{J} = \rho_v \underline{u} = \frac{ne^2\tau}{m}\underline{E} = \sigma \underline{E}}$$ [Raw form of ohm's law].

where σ is the conductivity of the conductor.

- For a non-perfect conductor, we can define the imperfections in the conductor as resistance R, where $R = \frac{V}{I}$.

- For a non-perfect conductor w/ uniform cross-sectional area A and length ℓ,

$$\boxed{R = \frac{V}{I} = \frac{V}{JA} = \frac{E\ell}{\sigma EA} = \frac{\ell}{\sigma A}}$$

- We can define resistivity ρ as $\rho = \frac{1}{\sigma}$, so

$$\boxed{R = \rho\frac{\ell}{A}}$$

Magnetism

Magnetic flux density

Magnetic flux density

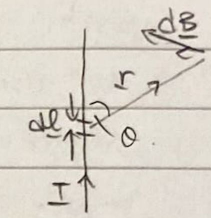
- Magnetic flux density is a vector denoted as $\underline{B}$
- Magnetic flux density $\underline{B}$ has units T or Wbm^{-2}.

Biot-Savart Law.

- Biot-Savart Law states that a short element $d\underline{l}$ of wire carrying a current I produces a magnetic flux density $d\underline{B}$ at a position described by the vector $\underline{r}$ is

$$\boxed{d\underline{B} = \frac{\mu_0 I\, d\underline{l} \times \underline{r}}{4\pi r^3} = \frac{\mu_0 I\, d\underline{l} \times \hat{\underline{r}}}{4\pi r^2}}.$$

use RH rule to determine the dir. of $\underline{B}$



- In scalar form,

$$\boxed{dB = \frac{\mu_0 I \sin\theta\, dl}{4\pi r^2}}$$

where θ is the angle between the vectors $\underline{r}$ and $d\underline{l}$.

* μ_0 is the permeability of free space. $\mu_0 = 4\pi \times 10^{-7}\ Hm^{-1}$.

For a moving charge,
$$\underline{B} = \frac{\mu_0}{4\pi} \frac{q\, \underline{v} \times \hat{\underline{c}}}{r^2}$$

Ampère's Law.

- Ampère's Law states that for any closed loop C (Amperian loop.) (just for air/free space)

$$\boxed{\oint_C \underline{B} \cdot d\underline{l} = \mu_0 I_{enc}} \qquad (I_{enc} = NI)$$

where $\oint_C \underline{B} \cdot d\underline{l}$ is the circulation of $\underline{B}$ and I_{enc} is the current enclosed in the closed loop.

- Alternatively, using magnetic field intensity $\underline{H}$ (general, inc. nonlinear materials)

$$\boxed{\oint_C \underline{H} \cdot d\underline{l} = NI}$$

where $\oint_C \underline{H} \cdot d\underline{l}$ is the circulation of $\underline{H}$ and NI is the magnetomotive force (mmf).

Magnetic flux density around a straight wire.

(i) Biot-Savart Law.

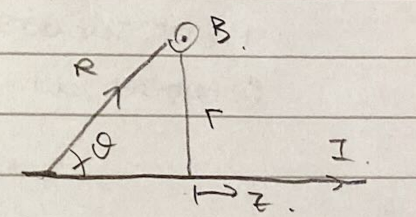
$$dB = \frac{\mu_0 I \sin\theta\, dz}{4\pi R^2}$$

$$B = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{r/\sqrt{r^2+z^2}}{r^2+z^2}\, dz$$

$$= \frac{\mu_0 I}{4\pi} \int_{-\pi/2}^{\pi/2} \frac{r}{(r^2 \sec^2\alpha)^{3/2}} (r\sec^2\alpha\, d\alpha)$$

$$= \frac{\mu_0 I}{4\pi} \int_{-\pi/2}^{\pi/2} \frac{\cos\alpha}{r}\, d\alpha.$$

$$\boxed{B = \frac{\mu_0 I}{2\pi r}}$$



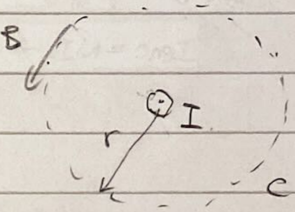
$R^2 = z^2 + r^2$, $z = r\tan\alpha$, $dz = r\sec^2\alpha\, d\alpha$.

(ii) Ampère's Law.

$$\oint_C \underline{B} \cdot d\underline{l} = \mu_0 I_{enc}$$

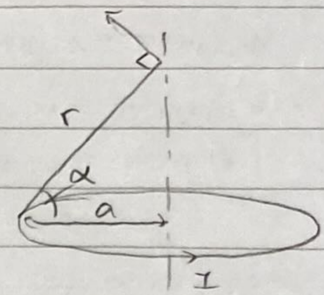
$$B(2\pi r) = \mu_0 I$$

$$\boxed{B = \frac{\mu_0 I}{2\pi r}}$$



Magnetic flux density on axis of a wire loop.

(i) Biot-Savart Law



Note $\underline{B}$ is axial only (horizontal components cancel)

$$dB = \frac{\mu_0 I \sin\theta \, dl}{4\pi r^2}$$

axial B only.

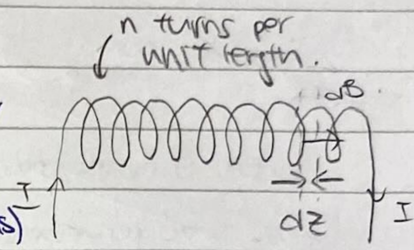
$$B = \frac{\mu_0 I \sin 90^\circ}{4\pi r^2} \int dl \cdot \cos\alpha$$

$$= \frac{\mu_0 I}{4\pi(a^2+z^2)} (2\pi a) \left(\frac{a}{\sqrt{a^2+z^2}}\right)$$

$$\boxed{B = \frac{\mu_0 I a^2}{2(a^2+z^2)^{3/2}}}$$

Magnetic flux density inside a solenoid.

(i) Biot-Savart Law



Consider the magnetic flux density $d\underline{B}$ produced by the short section dz.

Using superposition (valid for infinitesimally small sections)

$$dB = \frac{\mu_0 I a^2}{2(a^2+z^2)^{3/2}} \cdot n\,dz.$$

(derived using Biot-Savart Law)

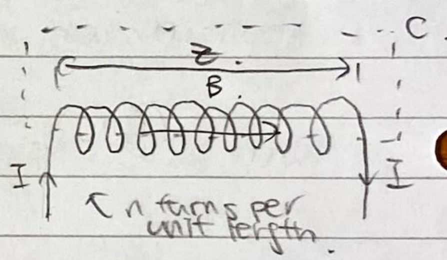
$$B = \frac{\mu_0 I a^2 n}{2} \int_{-\infty}^{\infty} \frac{1}{(a^2+z^2)^{3/2}}\, dz \qquad z = a\tan\alpha$$

$$= \frac{\mu_0 I a^2 n}{2} \int_{-\pi/2}^{\pi/2} \frac{1}{(a^2\sec^2\alpha)^{3/2}} (a\sec^2\alpha\, d\alpha) \qquad dz = a\sec^2\alpha$$

$$= \frac{\mu_0 I n}{2} \int_{-\pi/2}^{\pi/2} \cos\alpha\, d\alpha$$

$$\boxed{B = \mu_0 n I}$$

(ii) Ampère's Law.



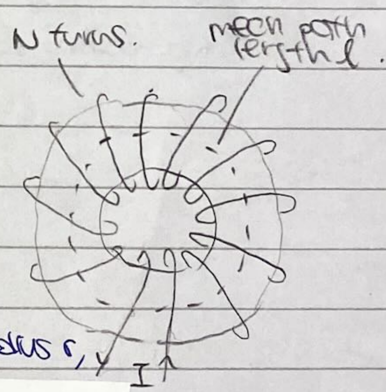
$$\oint_C \underline{B} \cdot d\underline{l} = \mu_0 I_{enc}$$

$$B\cancel{z} = \mu_0 (n\cancel{z}) I$$

$$\boxed{B = \mu_0 n I}$$

Magnetic flux density inside a toroidal coil.

(i) Ampère's Law



$$\oint \underline{B} \cdot d\underline{l} = \mu_0 I_{enc}$$

$$B(2\pi r) = \mu_0 N I$$

$$\boxed{B = \frac{\mu_0 N I}{2\pi r}}$$

*. If we consider the mean path length l, at a certain radius r,

$2\pi r = l$ so $B = \frac{\mu_0 N I}{l} = \mu_0 n I$.

Magnetic force

- The magnetic force $\underline{F}$ due to a charge q moving at velocity $\underline{v}$ in magnetic field is
$$\boxed{\underline{F} = q(\underline{v} \times \underline{B})}$$
- The magnetic force $\underline{F}$ of a wire w/ length $\underline{l}$ carrying current I in magnetic field is
$$\boxed{\underline{F} = I(\underline{l} \times \underline{B})}.$$

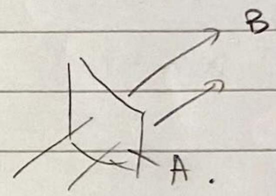
* Note $I = \frac{q}{V}Av = \frac{q}{Al}Av = \frac{qv}{l}$. → both eqns are equivalent.

- The magnetic force $\underline{F}$ between two // current carrying wires is given by.
$$F = B_2 I_1 l$$
$$= \frac{\mu_0 I_2}{2\pi r} I_1 l$$
$$\boxed{F = \frac{\mu_0 I_1 I_2 l}{2\pi r}}$$

Magnetic flux and self inductance.

Magnetic flux

- Flux is the idea of evaluating the amt. of a vector field that passes through a surface of area A.
- Magnetic flux Φ_B is given by
$$\boxed{\Phi_B = \iint_S \underline{B} \cdot d\underline{A}}$$



- If the field is always ⊥ to the surface, $\Phi_B = BA$.

Flux linkage.

- In solenoids and inductors, we have many turns and each turn is effectively another area, so we sum the flux for each turn to get the flux linkage Φ_B'.
$$\boxed{\Phi_B' = N\Phi_B}$$
- For a uniform flux density B passing through N turns each of area A, the flux linkage Φ_B' is given by $\Phi_B' = N\Phi_B = NBA$.

Faraday's Law of electromagnetic induction.

- Faraday's Law of electromagnetic induction states that the emf induced $\mathcal{E}$ is prop. to the rate of change of flux linking the circuit.
$$\mathcal{E} = \frac{d\Phi'}{dt}$$
- Lenz's law states that the emf induced by the change in flux is in the opposite dir.
$$\boxed{\mathcal{E} = -\frac{d\Phi'}{dt}}$$
(i.e. the induced emf opposes the flux that generated it).

* Practically, we ignore the − sign and figure out the dir. of $\mathcal{E}$ in the end.

self inductance.

- self inductance L is the ratio of flux linkage to the current generating the flux. (units: H).

$$L = \frac{\Phi_B'}{I}$$

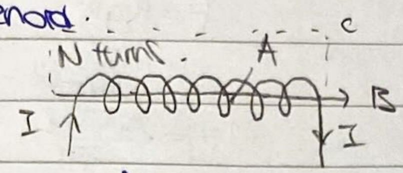
- strategy to find self-inductance:

use superposition for combinations of wires, loops.

↳ ① specify currents ↳ ② calculate B (using Ampère's Law)

↳ ③ calculate Φ_B'. (using $\Phi_B' = N\iint_S \underline{B}\cdot d\underline{A}$). ↳ ④ calculate L (using $L = \frac{\Phi'}{I}$).

self inductance of a solenoid.



A/L $\oint_c \underline{B}\cdot d\underline{l} = \mu_0 I_{enc}$

$$Bl = \mu_0 N I$$

$$B = \mu_0 \frac{NI}{l}.$$

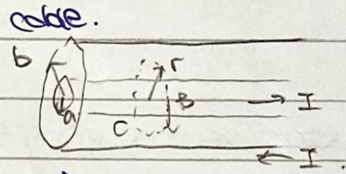
Flux Φ' $\Phi_B' = NBA$

$$= \mu_0 \frac{N^2}{l} IA$$

Defn of L $L = \frac{\Phi'}{I}$

$$\boxed{L = \frac{\mu_0 N^2 A}{l}.}$$

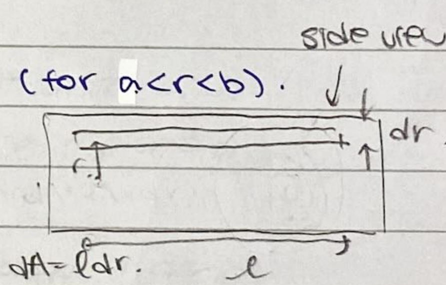
self inductance of a coaxial cable.



A/L $\oint_c \underline{B}\cdot d\underline{l} = \mu_0 I_{enc}$

$$B(2\pi r) = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi r}. \quad (\text{for } a<r<b).$$



Flux Φ' $\Phi_B' = \cancel{N}\iint_S \underline{B}\cdot d\underline{A}$

$$= \int_a^b \frac{\mu_0 I}{2\pi r} l\,dr$$

$$= \frac{\mu_0 I l}{2\pi}\ln\left(\frac{b}{a}\right).$$

Defn of L $L = \frac{\Phi'}{I}$

$$\boxed{L = \frac{\mu_0 l}{2\pi}\ln\left(\frac{b}{a}\right)}$$

* Note self inductance per unit length L_l is given by $L_l = \frac{L}{l}$.

so for a coaxial cable, $L_l = \frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right)$.

recall that for a coaxial cable, $C_l = \frac{2\pi\varepsilon_0}{\ln\left(\frac{b}{a}\right)}$.

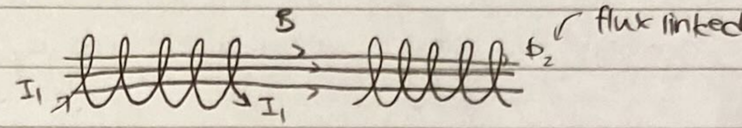
so $\sqrt{\frac{L_l}{C_l}} \propto \sqrt{\frac{\mu_0}{\varepsilon_0}} = Z_0 = \frac{E}{H}$. where Z_0 is the characteristic impedance of free space.

and $\frac{1}{\sqrt{L_l C_l}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = c$, where c is the speed of light in free space.

Mutual inductance.

Mutual inductance.

- when two electrical circuits are close to one another, a current I_1 in one may cause magnetic flux Φ_2 to link the other.

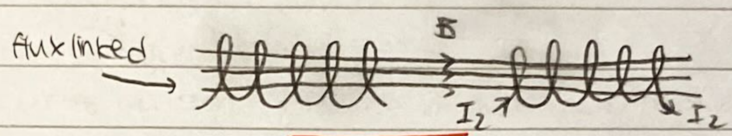


- Assuming linear behaviour, flux Φ_2 must be prop. to current I_1

$$\boxed{\Phi_2 = M_{12} I_1}$$

where the const. of prop. M_{12} is the mutual inductance between the circuits.

- similarly, w/ the roles of the above circuits reversed, a current I_2 may cause magnetic flux Φ_1 to link the other.



$$\boxed{\Phi_1 = M_{21} I_2}$$

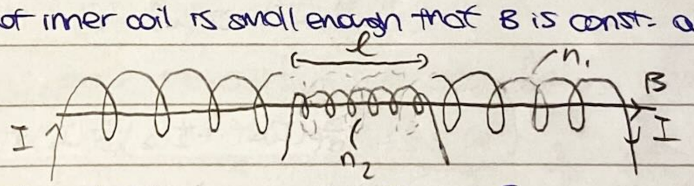
- It can be shown that the const. of prop. M_{12} and M_{21} are equal.

$$\boxed{M_{12} = M_{21} = M}$$

The quantity M is mutual inductance (units: H).

Mutual inductance of concentric solenoids

- consider a pair of concentric coils. the outer coil has n_1 turns per unit length; the inner coil has n_2 turns per unit length, CSA A and length ℓ.
* Assume diameter of inner coil is small enough that B is const. over its CSA.



Magnetic flux density B due to outer coil $B = \mu_0 n_1 I$

Magnetic flux Φ_2 through 1 turn of inner coil $\Phi_2 = BA = \mu_0 n_1 IA$.

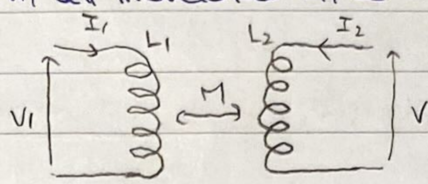
Magnetic flux linked Φ_2' through $N_2\ell$ turns of inner coil $\Phi_2' = n_2 \ell \Phi_2 = \mu_0 n_1 n_2 \ell IA$.

Mutual inductance $\boxed{M = \frac{\Phi_2'}{I} = \mu_0 n_1 n_2 \ell A}$

(similar strategy to finding the self inductance but we want to find Φ_2' rather than Φ_1' for a current I_1).

Magnetic energy in current-carrying circuits

- When currents I are established in an electrical circuit (change in I), there is a corresponding change in magnetic flux (density) B/ϕ, hence an induced emf ε. This means the voltage source must do work against this emf → energy stored in circuit.
- Consider two circuits w/ self inductances L_1, L_2 and mutual inductance M.



- The flux linking each circuit is a combination of the flux generated by its own current and the flux generated by the other current.

$$\boxed{\begin{aligned}\phi_1 &= \underbrace{L_1 I_1}_{\text{self}} + \underbrace{M I_2}_{\text{mutual}} \\ \phi_2 &= L_2 I_2 + M I_1\end{aligned}}$$

- As the flux in each circuit is changing, there will be induced emfs V_1 and V_2 in each circuit, w/ the magnitude given by

$$V_1 = \frac{d\phi_1}{dt} = \frac{d}{dt}(L_1 I_1 + M I_2) = L_1\frac{dI_1}{dt} + M\frac{dI_2}{dt}$$

$$V_2 = \frac{d\phi_2}{dt} = \frac{d}{dt}(L_2 I_2 + M I_1) = L_2\frac{dI_2}{dt} + M\frac{dI_1}{dt}$$

- The rate of change electrical energy $\frac{dW}{dt}$ is given by.

$$\begin{aligned}\frac{dW}{dt} = P &= I_1 V_1 + I_2 V_2 \\ &= I_1\left(L_1\frac{dI_1}{dt} + M\frac{dI_2}{dt}\right) + I_2\left(L_2\frac{dI_2}{dt} + M\frac{dI_1}{dt}\right) \\ &= L_1 I_1\frac{dI_1}{dt} + L_2 I_2\frac{dI_2}{dt} + M\left(I_1\frac{dI_2}{dt} + I_2\frac{dI_1}{dt}\right) \\ &= \frac{1}{2}L_1\frac{d}{dt}(I_1^2) + \frac{1}{2}L_2\frac{d}{dt}(I_2^2) + M\frac{d}{dt}(I_1 I_2)\end{aligned}$$

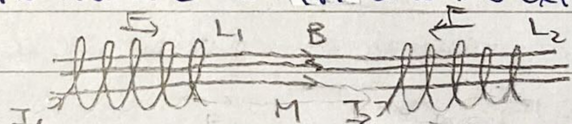
- Integrating wrt time gives the total energy stored

$$W = \frac{1}{2}L_1\int d(I_1^2) + \frac{1}{2}L_2\int d(I_2^2) + M\int d(I_1 I_2)$$

$$\boxed{W = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M I_1 I_2}$$

i.e. the total energy in the circuit is the sum of the individual inductances $\left(\frac{1}{2}L_i I_i^2\right)$ plus the energy of interaction btwn the circuits from the mutual inductance $(M I_1 I_2)$.

- The method of virtual work can be applied to this expression to find the force F.



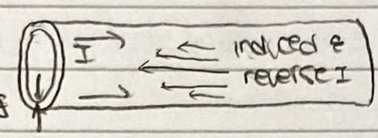
$$\boxed{F = I_1 I_2 \frac{\delta M}{\delta x}}$$

Force F symmetric wrt. interchanging currents I_1, I_2

* In practice, this is a difficult expression to use as it is usually to define the mutual inductance $M(x)$.

The skin effect

- Consider a cylindrical conducting wire along which a high frequency AC current ($f > 100$ kHz)
- Inside the wire, there will be circular lines of magnetic flux, and these will also be oscillating at high frequency
- As a result of Faraday's law, an emf will be induced along the wire ; As a result of Lenz's law, this emf opposes the original change in flux, setting up a reverse current.
- At the centre of the wire, magnetic flux ϕ is max, therefore a larger reverse current is produced → current tend to flow near the surface of the wire.



- In this course, we consider that all currents pass through the thin layer of thickness δ (skin depth) near the surface of the wire. Actually, at skin depth δ, the current drops to $1/e$ of the max. current.
- The higher the frequency, the larger the $\frac{d\Phi}{dt}$, so the larger the emf and cancellation of currents, thus the smaller the skin depth.
- The skin depth δ is given by the expression

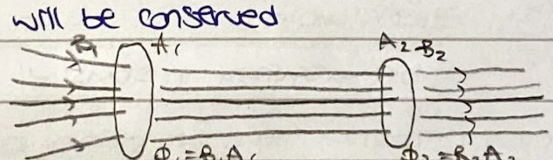
$$\boxed{\delta = \frac{1}{\sqrt{\pi \mu_0 \sigma f}}}$$

- A smaller skin depth δ means a higher resistance in the wire R since the current has a smaller cross section A through which to flow ($A = \pi[r^2 - (r-\delta)^2]$, $R = \rho \frac{l}{A}$)

Magnetic materials

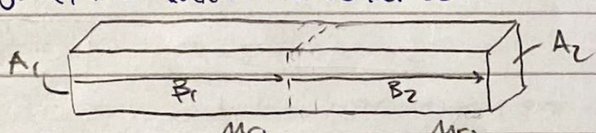
Conservation of magnetic flux

- Lines of magnetic flux density B always form closed loops. i.e. $\nabla \cdot \underline{B} = 0$ (In contrast to E-fields, which always begin/end on charges → no magnetic charges (monopoles))
- If we have a region of space where the B lines are contained / confined, then the flux of B across the area will be conserved



$$\boxed{\iint_{A_1} \underline{B} \cdot d\underline{S} = \Phi_1 = \Phi_2 = \iint_{A_2} \underline{B} \cdot d\underline{S}}$$

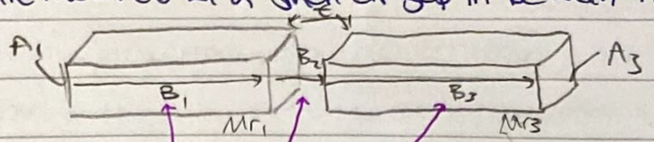
- This means given that flux travels from a magnetic material into another magnetic material w/o significant divergence, then flux will be conserved.



If $A_1 = A_2 = A$, $\quad \Phi_1 = \Phi_2 \rightarrow B_1 \cancel{A} = B_2 \cancel{A} \rightarrow B_1 = B_2$

i.e. cons. of B as long as it passes through the same CSA A.

- Consider two magnetic materials w/ a small air gap in between them



$$\phi_1 = \phi_2 = \phi_3$$

$$B_1 A_1 = B_2 A_2 = B_3 A_3$$

very useful

If $A_1 = A_2 = A_3 = A$, then $B_1 = B_2 = B_3$ → we can create a magnetic field inside an air gap

* If only given $A_1 = A_3$, we can only assume $A_1 = A_2 = A_3 = A$ when the air gap is small enough to prevent any fringing magnetic fields (A_2 would be larger for large air gaps)

- Due to the principle of flux conservation, we can treat magnetically susceptible materials like wires in a magnetic circuit.

(we assume high μr materials effectively contain the magnetic flux). $R_m = \frac{l}{\mu_0 \mu_r A}$

- This also assumes that fringing effects at sharp corners can be neglected.

Magnetic materials

- There is a limit to the magnetic flux density B we can generate using air-filled coils
  - ↳ ↑I → resistive heating → wires become hot → ↑R in coil
    - → exert forces (as $F = BIl$) → wires become in tension
  - ↳ ↑n → ↓d → ↓A → ↑R in coil
    - → ↓d → mechanically weaker
  - → Filling the coil filled w/ magnetic material can enhance the magnetic flux density B.
- The behaviour of materials when placed in a magnetic field is a consequence of three types of behaviour:
  - ↳ Paramagnetism (Linear): Internal magnetic fields line up and reinforce the external field. Atoms that have a net dipole moment. (Unpaired e^-)
  - ↳ Diamagnetism (Linear): Internal magnetic fields add up in a way that opposes the external field. Atoms that have no net dipole moment. (No unpaired e^-)
  - ↳ Ferromagnetism (Non-linear): Internal fields are strongly interacting w/ each other lining up into magnetic domains, even in the absence of an external field.
- When subjected to an ext. magnetic flux density B_0, there is an internally generated magnetic flux density I (intensity of magnetization) that add to the ext. field, given by

$$\boxed{\underline{I} = \mu_0 \underline{M}} \qquad \boxed{\underline{M} = \frac{M_{total}}{V}}$$

where M is the magnetization and M_{total} is the total magnetic moment of the material.

- The resultant magnetic flux density B is therefore given by:

$$\underline{B} = \underline{B_0} + \underline{I}$$

$$\boxed{\underline{B} = \mu_0 (\underline{H} + \underline{M})}$$

Linear magnetic material. (Paramagnetic / Diamagnetic).

- For a linear magnetic material, the magnetization M is directly proportional to the magnetic field intensity of the ext. field H

$$\boxed{\underline{M} = \chi_m \underline{H}}$$

where χ_m is the magnetic susceptibility of the material.

- The resultant magnetic flux density B is therefore given by

$$\underline{B} = \mu_0(\underline{H} + \underline{M})$$
$$= \mu_0(\underline{H} + \chi_m \underline{H})$$
$$= \mu_0 \underline{H}(1 + \chi_m)$$
$$\boxed{\underline{B} = \mu_0 \mu_r \underline{H}}.$$

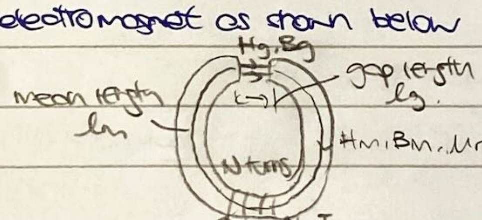
- The relative permeability μ_r and magnetic susceptibility χ_m are related by

$$\boxed{\mu_r = 1 + \chi_m}$$

- Paramagnetic and diamagnetic materials are soft (temporary) and weak ($\mu_r \approx 1$).
- Paramagnetic materials have a +ve magnetic susceptibility χ_m; Diamagnetic materials have a -ve magnetic susceptibility χ_m. Curie's Law; material property.
- Paramagnetism is temperature dependent ($\chi_m = \frac{C}{T}$, where C is the Curie constant); Diamagnetism is independent of temperature.

Magnetic flux in a linear electromagnet.

- Consider a toroidal electromagnet as shown below



A/L
$$\oint_C \underline{H} \cdot d\underline{l} = NI$$
$$H_m l_m + H_g l_g = NI$$

Use B
$$\frac{B_m}{\mu_0 \mu_r} l_m + \frac{B_g}{\mu_0} l_g = NI.$$

Consv. of flux
$$\Phi_m = \Phi_g$$
assume fringing effects are negligible.
$$B_m \cancel{A} = B_g \cancel{A} \rightarrow B_m = B_g.$$
$$\therefore \frac{B_g}{\mu_0 \mu_r} l_m + \frac{B_g}{\mu_0} l_g = NI.$$

For large μ_r, $\frac{B_g}{\mu_0\mu_r} l_m \ll \frac{B_g}{\mu_0} l_g$ so $\frac{B_g}{\mu_0 \mu_r} l_m + \frac{B_g}{\mu_0} l_g \approx \frac{B_g}{\mu_0} l_g$.

i.e. $B_g = \frac{NI\mu_0}{l_g}$.

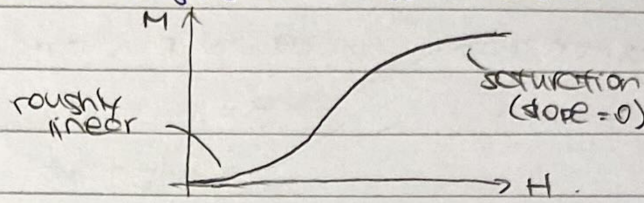
- This setup is used in tape recorder heads, where magnetization in magnetic tape (B_g) is related to the audio signal strength (I) $B_g \propto I$.

Non-linear magnetic material (Ferromagnetic). only Fe, Co, Ni (above RTP)

- For a non-linear magnetic material, the magnetization M is related to the magnetic field intensity of the ext. field H through a non-linear function fn.

$$M = fn(H).$$

- The function fn follows the rough general shape as follows:



- Initially, M increases w/ H at an increasing rate, but it eventually flattens off as all of the magnetic domains available are aligned (saturation).
- The resultant magnetic flux density B is still given by

$$\underline{B} = \mu_0(\underline{H} + \underline{M})$$
$$= \mu_0 \mu_r \underline{H}$$

← linear magnetic materials can also experience saturation at very large applied H but is practically always linear

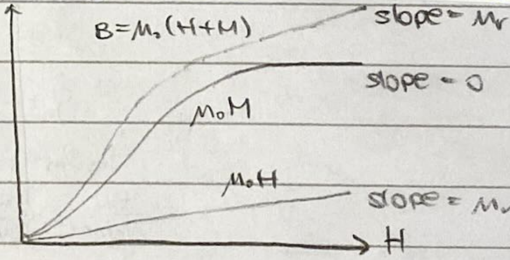
but now that $\underline{M}$ varies non-linearly w/ $\underline{H}$, μ_r is no longer constant.

(we use differential susceptibility $\chi_m^d = \frac{\partial M}{\partial H}$ or apparent susceptibility).

- Ferromagnetic materials can be hard (permanent) and are strong ($\mu_r \approx 10^4$).
- When heated above the Curie temperature T_c, ferromagnetic materials become paramagnetic (degree of alignment of atomic magnetic moments decreases). Above T_c, susceptibility χ_m is given by $\chi_m = \frac{C}{T - T_c}$ where C is the Curie constant
← Curie-Weiss Law

Relationship between B and M.

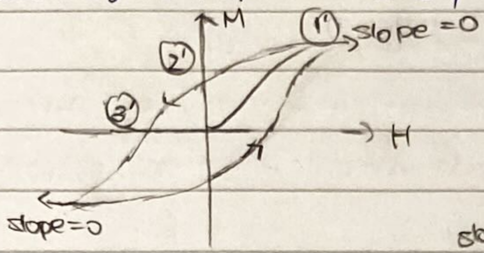
- The relationship between B and M. $\underline{B} = \mu_0(\underline{H} + \underline{M})$ can be shown graphically:

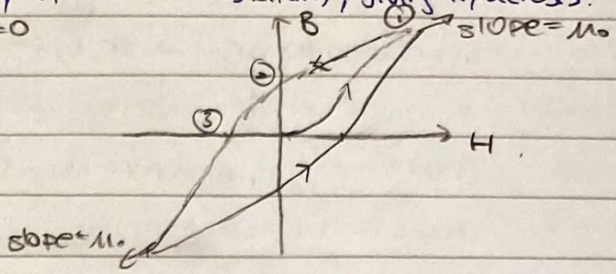


- At infinitely large H, B tends to infinity but M stays a constant value. (so B does not saturate!)
- The M-H curve is an intrinsic characteristic of the magnetic material.

Magnetic hysteresis

- Non-linear magnets may have memory (permanent magnetism), giving hysteresis.

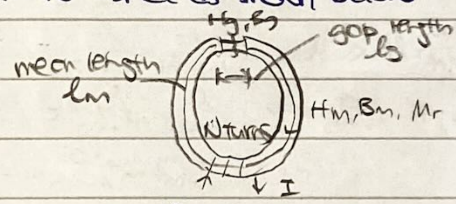




- The magnetic material is saturated at ①.
- At ②, H is zero but there is a remenant flux density B_r (y-intercept) — or retentivity/remanence
- At ③, a -ve H is req. to reach zero B, this H is the coercivity H_c (x-intercept)
- the H at ③ and ③' are different. They are the coercivity H_c and intrinsic coercivity H_{ci} respectively (for soft magnets, $H_c \approx H_{ci}$)
- M-H and B-H curves are not necessarily symmetrical. It may be easier/harder to reach saturation depending on the crystallographic direction.
- The portion from ② to ③ is the demagnetization curve.

Magnetic flux in a non-linear electromagnet

- consider a toroidal electromagnet as shown below

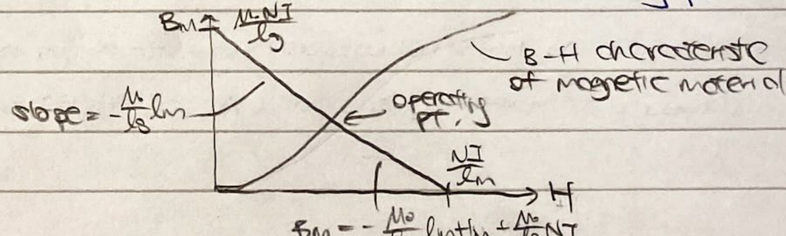


A/L
$$\oint_C \underline{H}\cdot d\underline{l} = NI$$
$$H_m l_m + H_g l_g = NI$$

Use B
$$H_m l_m + \frac{B_g}{\mu_0} l_g = NI$$
assume fringing effects are negligible.

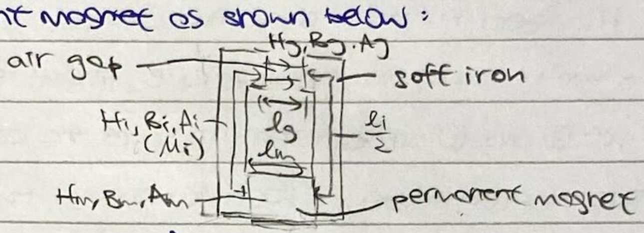
Consv. of flux
$$\Phi_m = \Phi_g$$
$$B_m \cancel{A} = B_g \cancel{A} \rightarrow B_m = B_g$$
$$\therefore H_m l_m + \frac{B_m}{\mu_0} l_g = NI$$
$$B_m = -\frac{\mu_0}{l_g} l_m H_m + \frac{\mu_0}{l_g} NI$$
load line.

Plotting the load line on the B-H characteristic to find the operating pt.



Magnetic flux in a permanent magnet

- A permanent magnet produces a magnetic flux even after any ext. magnetic field have been removed, w/ the max. remenant flux B_r.
- permanent magnetic materials are expensive → rarely used alone → we usually combine a small piece of permanent magnetic material w/ soft iron pieces to channel the magnetic flux to the req. pt.
- By reducing the CSA of the air gap, the flux becomes more conc. (flux consv.).
- Consider a permanent magnet as shown below:



A/C
$$\oint_C \underline{H} \cdot d\underline{l} = NI$$

$$H_m l_m + H_i l_i + H_g l_g = 0$$ ← $NI = 0$ for permanent magnets

Use B
$$H_m l_m + \frac{B_i}{\mu_0 \mu_i} l_i + \frac{B_g}{\mu_0} l_g = 0 .$$

Consv. of flux
$$\phi_m = \phi_i = \phi_g$$ assume fringing effects are negligible

$$B_m A_m = B_i A_i = B_g A_g \rightarrow B_i = \frac{B_m A_m}{A_i} ; \; B_g = \frac{B_m A_m}{A_g}$$

$$\therefore H_m l_m + \frac{B_m A_m}{\mu_0 \mu_i A_i} l_i + \frac{B_m A_m}{\mu_0 A_g} l_g = 0$$

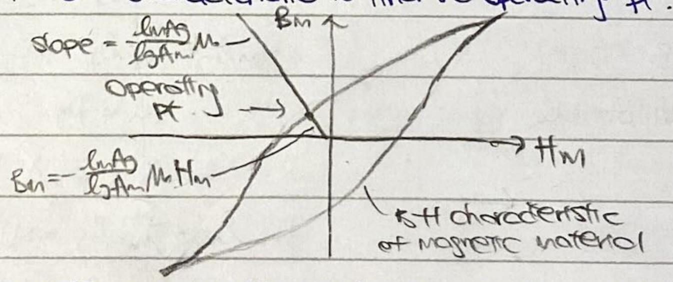
$$B_m A_m \left(\frac{l_i}{\mu_0 \mu_i A_i} + \frac{l_g}{\mu_0 A_g} \right) = -H_m l_m$$

For large μ_i, $\frac{l_i}{\mu_0 \mu_i A_i} \ll \frac{l_g}{\mu_0 A_g}$, so $\frac{l_i}{\mu_0 \mu_i A_i} + \frac{l_g}{\mu_0 A_g} \approx \frac{l_g}{\mu_0 A_g}$

i.e. $$B_m A_m \frac{l_g}{\mu_0 A_g} = -H_m l_m$$

$$B_m = -\frac{l_m A_g}{l_g A_m} \mu_0 H$$ ← load line

Plotting the load line on the B-H characteristic to find the operating pt.



* The permanent magnet operates on the demagnetisation section of the hysteresis curve
* We have assumed that B_i is prop. to H_i for soft iron. However, this is not strictly true for iron as μ_i is not constant. However as μ_i is very large $\sim 10^3$, the term has disappeared from the final answer (we can neglect any terms w/ $\mu_i \gg 1$).

Magnetostatic energy and forces

Energy stored in a magnetic field.

- Consider the work done δW in increasing the flux by $\delta\Phi'$ against the induced emf V.

$V = \frac{\delta\Phi'}{\delta t}$, $\delta\Phi' = NA\delta B$, $NI = Hl$. $P = IV$ // $\delta W = P\delta t = I\frac{\delta\Phi'}{\delta t}\delta t = INA\delta B = HlA\delta B = VH\delta B$

$$\therefore\ W = V\int_0^B H\,dB$$

work done per unit volume

$$\boxed{\frac{W}{V} = \int_0^B H\,dB}$$

— area under the H-B curve.

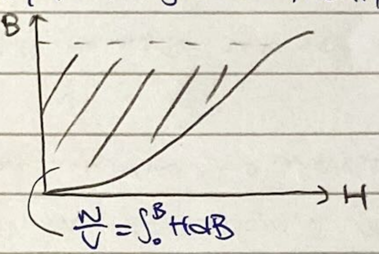
- For linear magnetic materials, B is prop. to H, $B = \mu_0\mu_r H$, so

$$\frac{W}{V} = \int_0^B \frac{B}{\mu_0\mu_r}\,dB$$

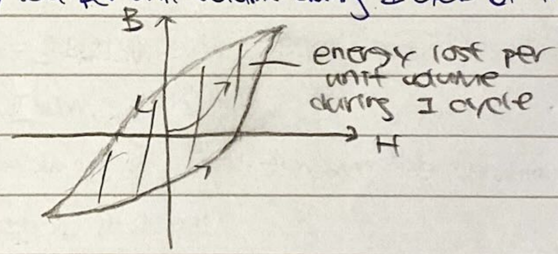
$$\boxed{\frac{W}{V} = \frac{B^2}{2\mu_0\mu_r}}$$

— can be v. large as $\mu_0\mu_r$ is v. small

- For nonlinear magnetic materials, B and H are related by the B-H characteristic of the material, so we cannot simplify the integral analytically.

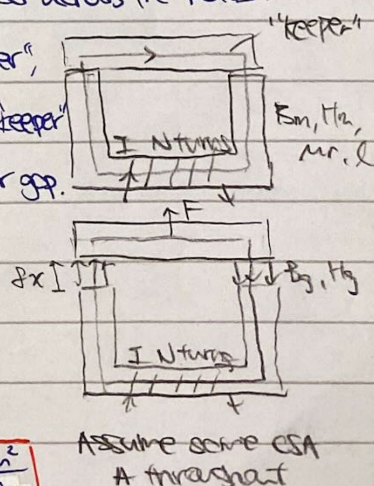


- If the material has hysteresis, the area inside the B-H loop is not accessible and this area represents the energy lost per unit volume during 1 cycle of the B-H loop.



Force from an electromagnet

- Consider an electromagnet w/ a core of permeability $\mu_0\mu_r$ wound w/ a coil of N turns carrying a current I. A "keeper" of the same material is placed across the poles.
- To calculate the force F needed to just remove the "keeper", we equate the mechanical work done by the force F moving the "keeper" by a short distance δx to the magnetostatic energy stored in the new air gap.



$$F\delta x = V\int_0^{B_g} H_g\,dB$$

$V = 2A\delta x$ as A is the area of 1 gap, we have 2 gaps —

$$F\cancel{\delta x} = 2A\cancel{\delta x}\int_0^{B_g}\frac{B_g}{\mu_0}\,dB$$

$$F = \cancel{2}A\frac{B_g^2}{\cancel{2}\mu_0}$$

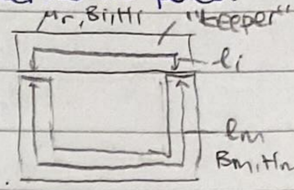
$$\frac{F}{A} = \frac{B_g^2}{\mu_0}$$

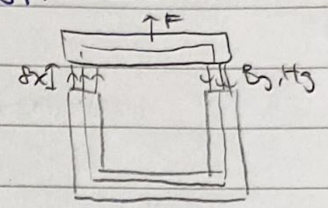
cons. of flux $\Phi_m = \Phi_g \rightarrow B_m\cancel{A} = B_g\cancel{A} \rightarrow B_m = B_g$ $\quad\therefore \boxed{\frac{F}{A} = \frac{B_m^2}{\mu_0}}$

* Assume δx is small so no fringing effects $\rightarrow \delta\Phi = 0 \rightarrow$ induced emf $V = 0 \rightarrow$ no need to account for the work done against the induced emf $\delta W = IV\delta t$.

Forces from a permanent magnet

- Consider a permanent magnet w/ a piece of soft iron (keeper) placed across the poles.
- To calculate the force F needed to just remove the "keeper", we equate the mechanical work done by the force F in moving the "keeper" by a short distance δx to the magnetostatic energy stored in the new air gap.





Assume constant CSA A throughout

$$F\delta x = V\int_0^{B_g} H_g \, dB$$

$V = 2A\delta x$ as A is the area of 1 gap, we have 2 gaps

$$F\cancel{\delta x} = 2A\cancel{\delta x}\int_0^{B_g} \frac{B_g}{\mu_0} dB$$

$$F = 2A\frac{B_g^2}{2\mu_0}$$

$$\frac{F}{A} = \frac{B_g^2}{\mu_0}$$

consv. of flux $\phi_m = \phi_g \rightarrow B_m \cancel{A} = B_g \cancel{A} \rightarrow B_m = B_g \quad \therefore \boxed{\frac{F}{A} = \frac{B_m^2}{\mu_0}}$

* Assume δx is small so no fringing effects $\rightarrow \delta\phi = 0 \rightarrow$ induced emf $V = 0 \rightarrow$ no need to account for the work done against the induced emf $\delta W = IV\delta t$.

Forces from electromagnets vs permanent magnets.

- The expression for the magnetostatic force (per unit area) is the same for both electromagnets and permanent magnets: $\frac{F}{A} = \frac{B_m^2}{\mu_0}$
- The only difference is how we obtain an expression/value for B_m / using techniques mentioned before.
  - ↳ Linear electromagnet: Use $\oint_C \underline{H}_m \cdot d\underline{l} = NI$ to find H_m
    Use $\underline{B}_m = \mu_0\mu_r\underline{H}_m$ to find B_m
  - ↳ Nonlinear electromagnet: Use $\oint_C \underline{H}_m \cdot d\underline{l} = NI$ to find H_m
    Use B-H characteristic to find B_m
  - ↳ Permanent magnet: Use $\oint_C \underline{H}\cdot d\underline{l} = NI$ to find an expression w/ H_m, H_i
    Use $\underline{B}_i = \mu_0\mu_i\underline{H}_i$ to express in terms of B_i
    Use consv. of flux to express in terms of B_m, H_m only $\rightarrow$ load line
    Use load line + B-H characteristic to find the operating pt. (B_m)