

vectors and coordinate systems

vectors

- A vector has both magnitude and direction. The magnitude of \underline{E} is a scalar quantity written as $|E|$. A unit vector \underline{e}_E has the same direction as \underline{E} .

$$\underline{E} = E \underline{e}_E.$$

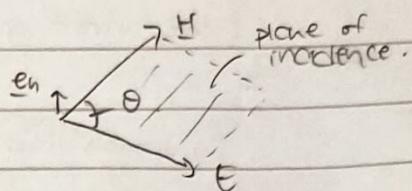
- A vector can be represented in any suitable coord system as a series of component vectors. eg in Cartesian coords, $\underline{E} = E_x \underline{e}_x + E_y \underline{e}_y + E_z \underline{e}_z$.

- The dot product of \underline{E} and \underline{H} is given by

$$\underline{E} \cdot \underline{H} = |E| |H| \cos \theta.$$

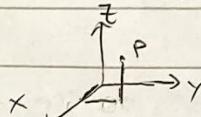
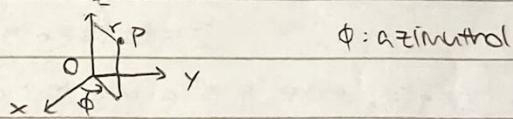
- The cross product of \underline{E} and \underline{H} is given by

$$\underline{E} \times \underline{H} = |E| |H| \sin \theta \underline{e}_n$$

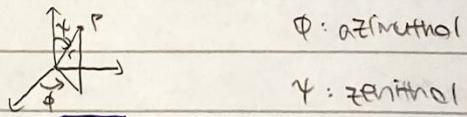


Coordinate systems.

- The key to solving any electromagnetic problem is to use the correct symmetry to simplify the maths. → use the correct coordinate system.

① Cartesian/rectangular coordinates. (x, y, z)② Circular cylindrical coordinates (r, ϕ, z) [wires]

$$r = \sqrt{x^2 + y^2}, \quad \phi = \arctan(\frac{y}{x}), \quad z = z. \quad // \quad x = r \cos \phi, \quad y = r \sin \phi, \quad z = z.$$

③ Spherical coordinates (r, ϕ, γ) [balls, pts, spheres].

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \phi = \arctan(\frac{y}{x}), \quad \gamma = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right) \quad // \quad x = r \sin \gamma \cos \phi, \quad y = r \sin \gamma \sin \phi, \quad z = r \cos \gamma$$

Electric force and electric field.

Electric charge.

- The charge of an e^- is $-1.6 \times 10^{-19} C$. → a unit charge $e = 1.6 \times 10^{-19} C$.

- The interaction of charges generates force and force fields.

↳ charges of the same sign - repel

↳ charges of the opposite sign - attract.

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Electrical conductors and insulators.

- In insulators (i.e. plastics, ceramics, dielectrics), charge motion is restricted within the material \rightarrow charge is assumed to be distributed throughout the material (unless told otherwise).
- However, in some cases, there may still be an imbalance of e^- throughout the molecules in the material \rightarrow net overall charge. (dielectrics).
- In conductors, (i.e metals), charges are free to move throughout the material \rightarrow charge is assumed to be uniformly distributed on the surface of the material (since charges are furthest apart in this configuration).

Coulomb's law

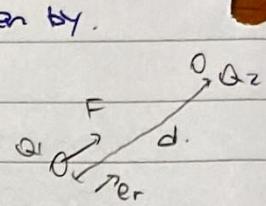
- Coulomb's law states that the electrostatic force F is given by.

$$F = k_e \frac{Q_1 Q_2}{r^2} \epsilon_r = \frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{r^2} \epsilon_r$$

where r is the distance between the 2 charges Q_1, Q_2 .

k_e is Coulomb's constant, $k_e = 8.988 \times 10^9 \text{ Nm}^2 \text{ C}^{-2}$.

ϵ_0 is the dielectric permittivity of free space, $\epsilon_0 = 8.854 \times 10^{-12} \text{ Fm}^{-1}$.

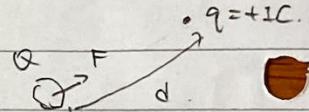


Electric field

- The electric field E represents the lines of force surrounding charges in an electrostatic system. It is the electrostatic force experienced by a unit charge.

$$E = k_e \frac{Q}{r^2} \epsilon_r = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \epsilon_r$$

and $E = qE$, where q is a unit test charge.



- The density of electric field lines represents the electric field strength, ϵ_r .

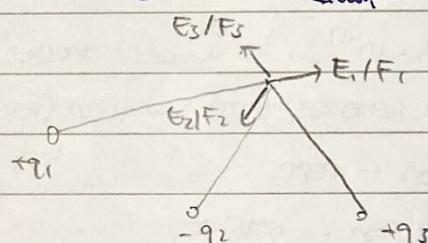
- Electric field lines always begin and end at a charged particle/part. or surface (infinity).

Superposition

- The electrostatic force F and electric field E are vector quantities \rightarrow the net force and electric field can be found by the vector sum of the contributions of each charge.

$$F = \sum F_i = \sum \frac{Q_i Q_i}{4\pi\epsilon_0 r_i^2} \epsilon_r$$

$$E = \sum E_i = \sum \frac{Q_i}{4\pi\epsilon_0 r_i^2} \epsilon_r$$

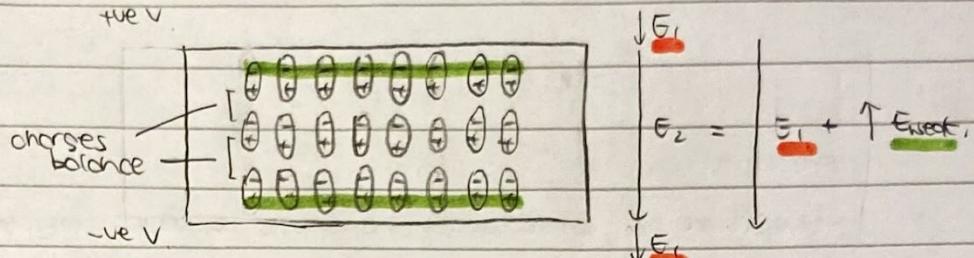


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Dielectrics and electric flux density.

Dielectrics

- When charges within a material are subjected to an electric field,
- ↳ metal : charges are free to move \rightarrow current flow
- ↳ dielectric : charges bound inside the molecules but can move within each molecule.
- \rightarrow charge imbalance (polarisation).
- Consider a dielectric material subject to an electric field E_1 .



- The electric field polarises the molecules within the dielectric, slightly separating the +ve and -ve charges.
- In the bulk of the dielectric, the +ve/-ve charges cancel out, but on the surface, a net +ve and -ve charge appears. $E_{\text{weak}} = -\chi_e E_2$, χ_e is the electrical susceptibility.
- The net charge at the edge sets up a weak internal E field E_{weak} , which is the opposite dir. of the ext. Efield E_1 .
- The net electric field E_2 in the dielectric material is therefore given by

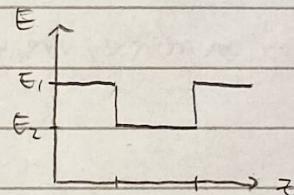
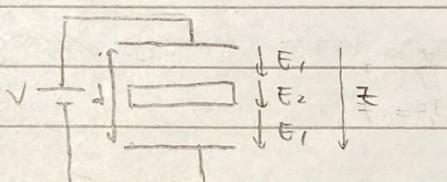
$$(E_2 = E_1 + E_{\text{weak}} = E_1 - \chi_e E_2) \quad E_2 = E_1 - E_{\text{weak}}. \quad \text{so } |E_2| < |E_1|$$

- For a linear material, the Efield inside the dielectric material is proportional to the ext. Efield E_1 , so E_1 across air \downarrow across dielectric \downarrow

$$E_1 = \epsilon_r E_2$$

where ϵ_r is the rel. permittivity / dielectric constant of the material.

- * $\epsilon_r \geq 1$. It is assumed that $\epsilon_r = 1$ for vacuum/air. [permittivity $\epsilon = \epsilon_0 \epsilon_r$],
- This assumes that materials are perfect dielectrics. However in reality, when the E-field is large, the e⁻ will be pulled out of the molecules \rightarrow material starts conducting. (dielectric breakdown). The dielectric strength E_{max} of a material is defined as the max. E-field it can tolerate before breakdown occurs. ($E_{\text{max air}} = 3 \times 10^6 \text{ Vm}^{-1}$),
- Consider a // plate capacitor w/ a dielectric.



\rightarrow E-z graph is discontinuous \rightarrow define a new quantity : Electric flux density D.

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Electric flux density

- It is more convenient to use electric flux density D rather than E-field E to analyse dielectrics.
- The electric flux density D is defined as.

$$D = \epsilon E = \epsilon_0 \epsilon_r E$$

D is continuous

E is directly related to voltage

- The normal component of electric flux density D takes the same value on passing from one dielectric to another \rightarrow flux conservation. (always continuous!)

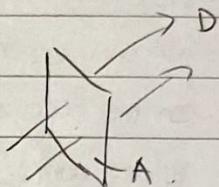
Gauss law and electric potential.

Electric flux.

- Flux is the idea of evaluating the out. of a vector field that passes through a surface of area A .
- Electric flux ϕ_D is given by.

$$\phi_D = \iint_S D \cdot dA$$

- If the field is always \perp to the surface, $\phi_D = DA$.



Gauss's law.

Gaussian surface.

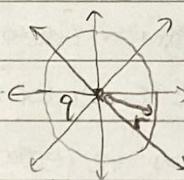
- Gauss's law states that the total electric flux of D through any closed surface S equals the charge enclosed by the surface.

$$Q_{\text{enc}} = \iint_S D \cdot dS = \phi_D$$

- Gauss's law is very useful because if we know the charge Q , we can calculate D .

- ① A point charge (or sphere of charge).

Consider a pt. charge, $+q$



G/L

$$Q_{\text{enc}} = \iint_S D \cdot dS$$

$$q = D \cdot 4\pi r^2$$

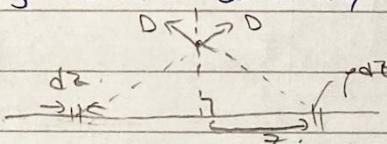
D always $\perp S$.

$$D = \frac{q}{4\pi r^2}$$

consistent w/ Coulomb's law.

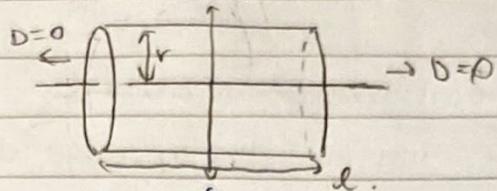
- ② A line of charge.

Consider an infinitely long wire w/ charge density (charge per unit length) ρ .



Horizontal component
of D cancels out.

By symmetry, the resultant D must act radially outwards at every pt. along the line.



G/L

$$Q_{enc} = \oint_S D \cdot d\vec{S}$$

$$\rho f = D \cdot (2\pi r l)$$

$$D = \frac{\rho}{2\pi r}$$

the ends of the cylinder are ignored as flux through them is 0.

Alternatively, this could be solved using Coulomb's law.

$$R = \sqrt{r^2 + z^2}, \cos\theta = \frac{r}{R} = \frac{r}{\sqrt{r^2 + z^2}}$$

Consider 2 elements either side a distance z apart.

$$dq = \rho dz,$$

Coulomb's law.

$$\frac{dV}{dz} = \frac{1}{4\pi\epsilon R^2} \frac{dq}{R^2} \cos\theta \times 2$$

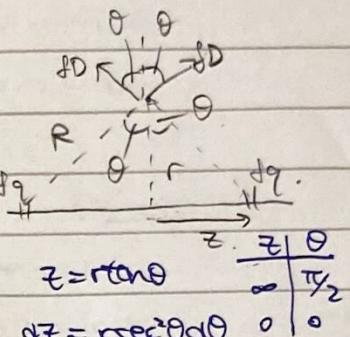
$$\int dV = \frac{1}{2\pi} \frac{\rho r}{(r^2 + z^2)^{3/2}} dz$$

$$D = \frac{\rho r}{2\pi} \int_0^\infty \frac{1}{(r^2 + z^2)^{3/2}} dz$$

$$= \frac{\rho r}{2\pi} \int_0^{\pi/2} \frac{1}{(r \sec^2\theta)^{3/2}} r \sec^2\theta d\theta$$

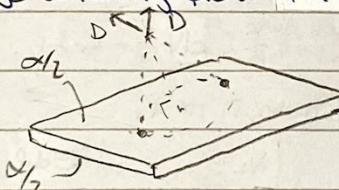
$$= \frac{\rho r}{2\pi} \int_0^{\pi/2} \frac{\cos\theta}{r^2} d\theta$$

$$= \frac{\rho}{2\pi r}$$



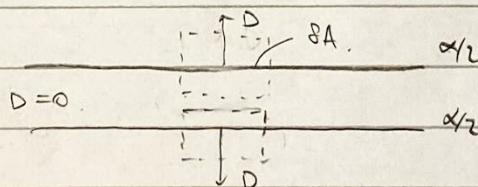
(3) Thin uniform metal plane.

Consider an infinitely large conducting plane w/ a charge density (charge per unit area) α .



component of D // to plane cancels out.

By symmetry, the resultant D must be \perp to the planes surface



By symmetry, $D_{top} = D_{bottom}$. Consider the top surface.

G/L

$$Q_{enc} = \oint_S D \cdot d\vec{S}$$

$$\frac{\alpha}{2} \cdot A = D \cdot A$$

$$D = \frac{\alpha}{2}$$

Key results: Point/Sphere:

$$D = \frac{q}{4\pi\epsilon r^2}$$

$$E = \frac{q}{4\pi\epsilon r^2}$$

Line:

$$D = \frac{\rho}{2\pi r}$$

$$E = \frac{\rho}{2\pi\epsilon r}$$

Plane:

$$D = \frac{\alpha}{2}$$

$$E = \frac{\alpha}{2\epsilon}$$

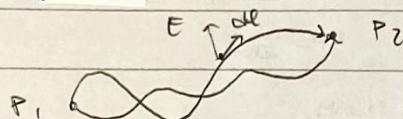
The Faraday cage.

- A simple but important consequence of Gauss's law is the absence of an electric field inside a hollow, charged conducting (thin-walled) shell.
- When considering a conducting object where charges are free to move, we always assume the charge resides on the surface.
- A Gaussian surface S inside must have 0 flux through it, since it encloses 0 charge.
- Given 0 flux for a finite surface area \rightarrow E-field is 0 everywhere inside the hollow conducting shell.



Potential difference.

- Consider the work req. to move a unit charge from pt. P_1 to P_2 in the presence of an E-field E . At any pt. along the path, we must exert a force $-E$ to hold the charge in eqm.
- The electrostatic potential difference b/w 2 pts. is the work done when a unit +ve charge is moved from one pt. to another.



some work done for both paths
as the force field is conservative.

- Work done in moving distance dl is:

$$\text{work done on the system} \curvearrowright dV = -\underline{E} \cdot \underline{dl}$$

So total work to move from P_1 to P_2 :

$$V_2 - V_1 = - \int_{P_1}^{P_2} \underline{E} \cdot \underline{dl} \quad [\text{Eqn. w/o name}]$$

- Electrostatic potential is defined to be 0 at infinity.

Capacitance

Capacitance

- Capacitance exists between any pair of conductors which are electrically insulated from each other (Actually everything has a capacitance but it may be extremely small!).
- Capacitance C is the ratio of charge to potential difference applied (units $F(m^2/m^2)$)

$$Q = CV$$

point

$$P = C \Delta V$$

line

$$\alpha = C \Delta V$$

plane

- Strategy to find capacitance:

\hookrightarrow ① Specify voltages / charges

\hookrightarrow ② Calculate E (using $D = \epsilon_0 \epsilon_r E$)

\hookrightarrow ③ Calculate C (using $C = \frac{Q}{V}$)

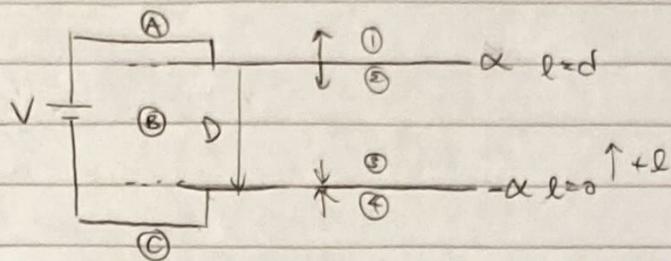
use superposition for
combinations of pts./lines/planes.

\hookrightarrow ④ Calculate V (using EV)

\hookrightarrow ⑤ Calculate D (using Gauss' law)

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Capacitance of a pair of // conducting plates.



$$\text{At } \textcircled{A} \quad \sum D = D_1 + D_3$$

$$= \frac{\alpha}{2} \uparrow + \frac{\alpha}{2} \downarrow = 0$$

$$\text{At } \textcircled{D} \quad \sum D = D_2 + D_4$$

$$= \frac{\alpha}{2} \downarrow + \frac{\alpha}{2} \uparrow = 0.$$

$$\text{At } \textcircled{B} \quad \sum D = D_2 + D_3$$

$$= \frac{\alpha}{2} \downarrow + \frac{\alpha}{2} \downarrow = \alpha \downarrow.$$

$\therefore D = 0$ above top plate / below bot. plate ; $D = \alpha \downarrow$ between the plates.

Find E

$$\epsilon_0 E = D$$

$$E = \frac{\alpha}{\epsilon_0}$$

ENN.

$$V_2 - V_1 = - \int_{t_1}^{t_2} E \cdot dL$$

$$V - 0 = - \int_{t_1}^{t_2} (-E) \cdot dL$$

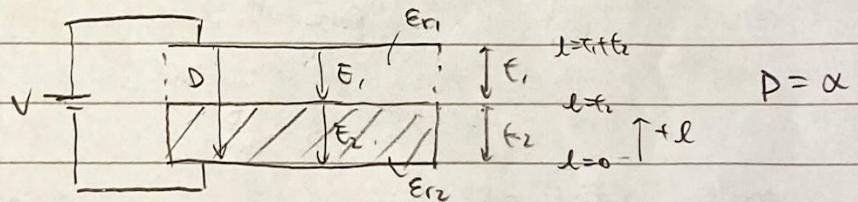
$$V = Ed = \frac{\alpha}{\epsilon_0} d = \frac{Qd}{A\epsilon_0}$$

Defn of C

$$C = \frac{Q}{V}$$

$$C = \frac{\epsilon_0 A}{d}$$

Capacitance of a pair of // conducting plates w/ dielectric.



Find E

$$\epsilon_0 \frac{1}{\epsilon_r} E_1 = D$$

$$E_1 = \frac{D}{\epsilon_0 \epsilon_r}$$

$$\epsilon_0 \epsilon_r E_2 = D$$

$$E_2 = \frac{D}{\epsilon_0 \epsilon_r} = \frac{E_1}{\epsilon_r}$$

ENN

$$V_2 - V_1 = - \int_{t_1}^{t_2} E \cdot dL$$

$$V - 0 = - \int_{t_1}^{t_2} (E_1) \cdot dL - \int_{t_2}^{t_1} (E_2) \cdot dL$$

$$V = E_1 t_2 + E_1 t_1 + E_2 t_2 - E_2 t_1$$

$$= \frac{E_1}{\epsilon_0 \epsilon_r} t_2 + E_1 t_1$$

$$= \frac{D}{\epsilon_0} \left(\frac{t_2}{\epsilon_0 \epsilon_r} + t_1 \right)$$

$$= \frac{\alpha}{\epsilon_0} \left(\frac{t_2}{\epsilon_0 \epsilon_r} + t_1 \right) = \frac{Q}{A \epsilon_0} \left(\frac{t_2}{\epsilon_0 \epsilon_r} + t_1 \right)$$

Defn of C .

$$C = \frac{Q}{V}$$

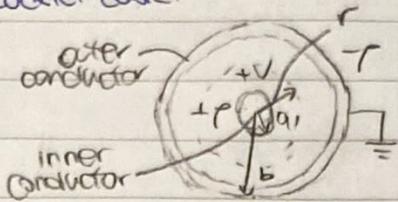
$$C = \frac{\epsilon_0 A}{\epsilon_0 + \frac{A}{\epsilon_0}} = \frac{\epsilon_0 \epsilon_r A}{\epsilon_0 \epsilon_r + A}$$

$$\text{check } t_1 = 0, C = \frac{\epsilon_0 \epsilon_r A}{\epsilon_0 + \frac{A}{\epsilon_0}}$$

$$t_2 = 0, C = \frac{\epsilon_0 A}{\epsilon_0 + \frac{A}{\epsilon_0}}$$

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Capacitance of air-filled coaxial cable.



G/L

$$Q_{enc} = \oint_S D \cdot d\vec{S}$$

$$\rho l = D \cdot 2\pi r dr$$

$$D = \frac{\rho}{2\pi r}$$

Find E

$$\epsilon_0 E = D$$

$$E = \frac{\rho}{2\pi\epsilon_0 r}$$

ENN

$$V_2 - V_1 = - \int_1^2 E dl$$

$$0 - V = - \int_a^b \frac{\rho}{2\pi\epsilon_0 r} dr$$

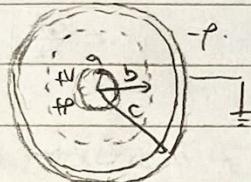
$$V = \frac{\rho}{2\pi\epsilon_0} \ln\left(\frac{b}{a}\right)$$

Defn of C.

$$C_L = \frac{Q}{V_L} = \frac{\rho l}{V_L}$$

$$C_L = \frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}$$

Capacitance of coaxial cable w/ two dielectrics



G/L

$$Q_{enc} = \oint_S D \cdot d\vec{S}$$

$$\rho l = D \cdot 2\pi r dr$$

$$D = \frac{\rho}{2\pi r}$$

Find E

$$\epsilon_0 \epsilon_{r1} E_1 = D$$

$$E_1 = \frac{D}{2\pi\epsilon_0\epsilon_{r1} r}$$

$$\epsilon_0 \epsilon_{r2} E_2 = D$$

$$E_2 = \frac{D}{2\pi\epsilon_0\epsilon_{r2} r}$$

ENN

$$V_2 - V_1 = - \int_1^2 E dl$$

$$0 - V = - \left[\int_a^b \frac{D}{2\pi\epsilon_0\epsilon_{r1} r} dr + \int_b^c \frac{D}{2\pi\epsilon_0\epsilon_{r2} r} dr \right]$$

$$V = \frac{\rho}{2\pi\epsilon_0} \left[\frac{1}{\epsilon_{r1}} \ln\left(\frac{b}{a}\right) + \frac{1}{\epsilon_{r2}} \ln\left(\frac{c}{b}\right) \right]$$

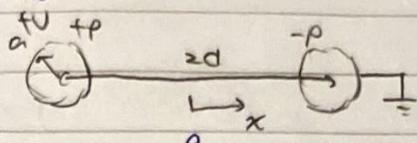
Defn of C

$$C_L = \frac{Q}{V_L} = \frac{\rho l}{V_L}$$

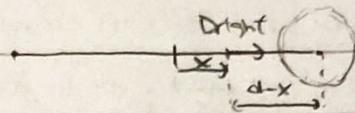
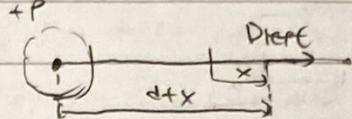
$$C_L = \frac{2\pi\epsilon_0}{\epsilon_{r1} \ln\left(\frac{b}{a}\right) + \epsilon_{r2} \ln\left(\frac{c}{b}\right)}$$

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Capacitance of two parallel wires



$$D \text{ due to an infinitely long wire} : D = \frac{\rho}{2\pi\epsilon_0 r}$$



$$\sum D = D_{\text{left}} + D_{\text{right}}$$

$$= \frac{\rho}{2\pi\epsilon_0(d+x)} + \frac{\rho}{2\pi\epsilon_0(d-x)}$$

Find E

$$\epsilon_0 E = D$$

$$E = \frac{\rho}{2\pi\epsilon_0(d+x)} + \frac{\rho}{2\pi\epsilon_0(d-x)}$$

ENR

$$V_2 - V_1 = - \int_1^2 E dx$$

$$0 - V = - \int_{d-x}^{d+x} \frac{\rho}{2\pi\epsilon_0} \left[\frac{1}{d+x} + \frac{1}{d-x} \right] dx$$

$$V = \frac{\rho}{4\pi\epsilon_0} \ln\left(\frac{2d}{a}\right)$$

Depth of C

$$C_L = \frac{Q}{V_L} = \frac{\rho L}{V_L}$$

$$= \frac{\pi\epsilon_0}{\ln\left(\frac{2d}{a}\right)}$$

since $a \ll d$, $2d-a \approx 2d$

$$\therefore C_L \approx \frac{\pi\epsilon_0}{\ln(2d/a)}$$

Capacitance of a sphere.



GIC

$$Q_{enc} = \iint_D D \cdot dS$$

$$Q = D \cdot (4\pi a^2)$$

$$D = \frac{Q}{4\pi a^2}$$

Find E

$$\epsilon_0 E = D$$

$$E = \frac{Q}{4\pi\epsilon_0 a^2}$$

ENR

$$V_2 - V_1 = - \int_1^2 E dx$$

$$0 - V = - \int_a^\infty \frac{Q}{4\pi\epsilon_0 r^2} dr$$

$$V = \frac{Q}{4\pi\epsilon_0 a}$$

Defn of C

$$C = \frac{Q}{V}$$

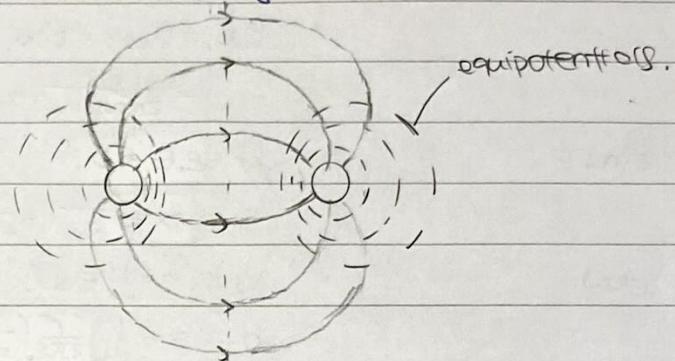
$$C = 4\pi\epsilon_0 a$$

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Equipotentials

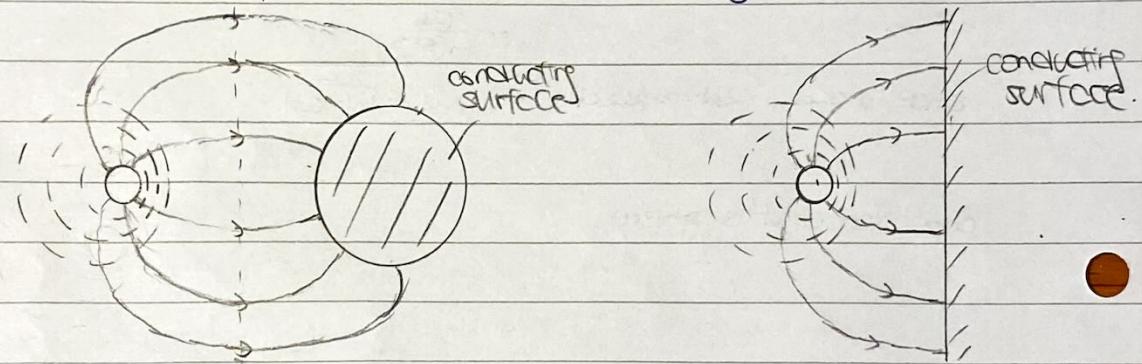
Equipotentials.

- An equipotential is a line/surface of constant potential.
- We can move a pt. charge along an equipotential w/o doing work against electric fields.
- Equipotential lines are \perp to electric field lines — if a charge moves \perp to an electric field, no work is done by/against the field.
- Consider two equal and opposite charges $+Q$ and $-Q$.
The equipotentials are circles of increasing radius.



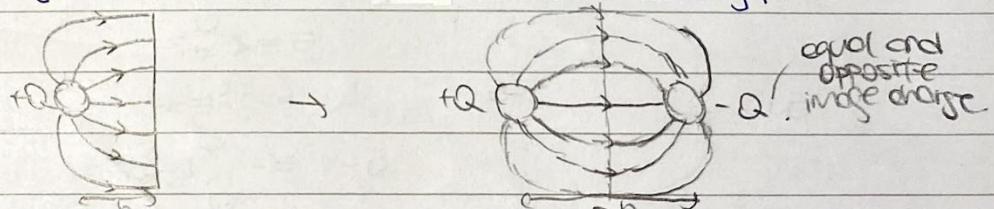
The equipotential midway between the conductors is a line (circle w/ infinite radius)

- We can replace any equipotential w/ a conducting surface.

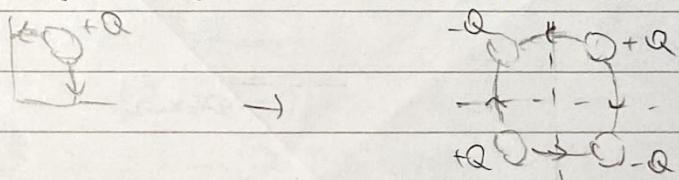


Method of images.

- We can replace a conducting surface w/ an image charge (that gives same equipotential).
- Consider a charge $+Q$ at a height h next to a metal conducting plate.

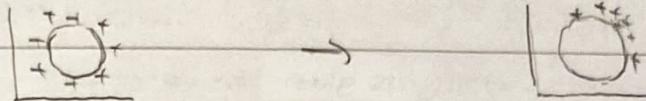


- Consider a charge $+Q$ equidistant from two sides of a right angle conductor.



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- This method is only an approximation as we assume the charged sphere to behave as if it were uniformly charged.
- In practice, the metal plane will distort the charge distribution of the sphere.



- An exact sol'n requires finite element and finite difference methods.

- e.g.: Voltage of sphere above a conducting plane.

A metal sphere of radius $a=5\text{mm}$ is centred at a height $h=10\text{mm}$ from an earthed conducting sheet. The electrical breakdown strength of air (E_{break}) is $3 \times 10^6 \text{Vm}^{-1}$. What is the max. voltage / charge applied to the sphere w/o breakdown occurring?

First, we replace the conductive sheet w/ an image charge.

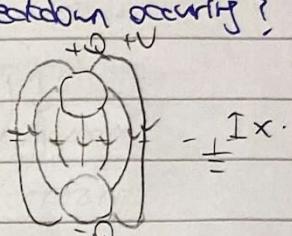
For a sphere, $E = \frac{Q}{4\pi\epsilon_0 r^2}$, so by superposition,

$$E = -\left[\frac{Q}{4\pi\epsilon_0 (h+x)^2} + \frac{Q}{4\pi\epsilon_0 (h-x)^2} \right]$$

Effect dir. is
opposite to x dir.
due to real
charge at $x=h$.

due to real
charge at $x=h$.

due to image
charge at $x=-h$.



As E field is max at the edge of a charged sphere, sub $x = h-a$.

$$|E_{\text{max}}| = \frac{Q}{4\pi\epsilon_0 a^2} + \frac{Q}{4\pi\epsilon_0 (2a)^2}$$

Substituting $a=5\text{mm}$, $h=0\text{mm}$, $|E_{\text{max}}|=3 \times 10^6 \text{Vm}^{-1}$, $Q=7.5\text{nC}$.

$$V = V_1 = - \int_0^a E dx$$

$$V = 0 = - \int_0^{h-a} - \left[\frac{Q}{4\pi\epsilon_0 (h+x)^2} + \frac{Q}{4\pi\epsilon_0 (h-x)^2} \right] dx$$

$$V = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{2a} \right)$$

Substituting $a=5\text{mm}$, $h=10\text{mm}$, $Q=7.5\text{nC}$, $V=7000\text{V}$.

* Note, w/o the metal plane, $E = \frac{Q}{4\pi\epsilon_0 r^2}$ so $|E_{\text{max}}| = \frac{Q}{4\pi\epsilon_0 a^2}$ and $V = \frac{Q}{4\pi\epsilon_0 a}$.

Substituting $a=5\text{mm}$, $|E_{\text{max}}|=3 \times 10^6 \text{Vm}^{-1}$, $Q=8.34\text{nC}$, $V=15000\text{V}$.

i.e. as we move a conductor closer to a charged sphere, the voltage req. for breakdown drops.

Electrostatic energy and forces

Energy stored in an electric field.

- Consider the work done δW in adding a charge q_1 to a capacitor at voltage V .

$$I = \frac{q}{t}, \quad V = \frac{I}{C}, \quad P = IV \quad // \quad \delta W = P \delta t = \frac{q_1}{t} \frac{q_2}{C} \delta t = \frac{1}{C} q_1 q_2$$

$$\therefore W = \int_0^Q \frac{1}{C} \frac{1}{2} q_1 q_2 = \frac{1}{2C} Q^2$$

$$\text{As } Q=CV, \quad W = \frac{Q^2}{2C} = \frac{1}{2} QV = \frac{1}{2} CV^2 \rightarrow \text{E field can store energy.}$$

* The expression above applies to any capacitor of any geometry.

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Energy stored in σ // plate capacitor.

- Consider a // plate capacitor w/ area A and plate separation x . We neglect only fringe effects at the edges of the plate.

$$V = Ex, \quad C = \frac{\epsilon_0 \sigma A}{x}, \quad W = \frac{1}{2} CV^2 \rightarrow W = \frac{1}{2} \frac{\epsilon_0 \sigma A}{x} E^2 x^2 = \frac{1}{2} \epsilon_0 \sigma E^2 (Ax).$$

\therefore The energy density stored in the electric field is given by $\boxed{\frac{W}{Ax} = \frac{1}{2} \epsilon_0 \sigma E^2}$

Electrostatic force (virtual work method).

- Consider a system consisting of two arbitrary charged conductors:
- ↳ One of the conductors is fixed (cannot move)
- ↳ The conductors have a charge of $+q$ and $-q \rightarrow$ They experience a Coulomb force F
- ↳ The battery between the conductors keep the potential V constant
- To find the Coulomb force F , we use the method of virtual work:
- ↳ Apply a virtual force F^* equal and opposite to F so the conductors are held in static eqm.
- ↳ Imagine the movable conductor moves a small distance δx towards the other.
- Consider the energy changes,
- ↳ Mechanical work is done on the system $-F^* \delta x$.
- ↳ The battery does work on the system $\delta V q$ by providing an extra charge δq (so V is const).
- ↳ There is a net change in electrostatic energy of $\frac{1}{2} \delta V \delta q$. ($W = \frac{1}{2} QUV$, so $\delta W = \frac{1}{2} V \delta q$).
- Using cons. of mechanical energy, (considering the system).

$$-F\delta x + V\delta q = -\frac{1}{2} V\delta q$$

$$F\delta x = \frac{1}{2} V\delta q$$

$$F = \frac{1}{2} V \frac{\delta q}{\delta x}$$

$$\text{As } q = CV, \quad \frac{\delta q}{\delta x} = V \frac{\delta C}{\delta x}, \quad \text{so} \quad \boxed{F = \frac{1}{2} V \frac{\delta C}{\delta x}}$$

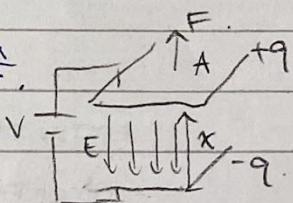
useful as we just need to know V and $\frac{\delta C}{\delta x}$.

(no need to know q, r)

* To find $\frac{\delta C}{\delta x}$, we need $C = f(x)$.

Force between capacitor plates.

$$\begin{aligned} \text{- For a } \sigma \text{ // plate capacitor, } \quad C &= \frac{\epsilon_0 \sigma A}{x}, \quad \text{so} \quad \frac{\delta C}{\delta x} = -\frac{\epsilon_0 \sigma A}{x^2}, \\ F &= \frac{1}{2} V^2 \frac{\delta C}{\delta x} \\ &= \frac{1}{2} V^2 \left(-\frac{\epsilon_0 \sigma A}{x^2} \right), \\ &= -\frac{1}{2} V^2 \frac{C}{x}. \quad (C = \frac{\epsilon_0 \sigma A}{x}) \\ \boxed{F = -\frac{1}{2} qE} \quad (q = CV, E = \frac{V}{x}). \end{aligned}$$

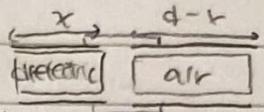
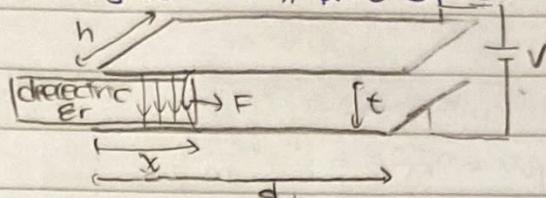


* Note the electrostatic force F is not qE .

* The -ve sign means the force is attractive (opposite the dir. of the x).

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Force on dielectric slab inserted into // plate capacitor.



- The system is essentially two capacitors connected in //

$$\therefore C_{eq} = \frac{\epsilon_0 \epsilon_r x h}{t} + \frac{\epsilon_0 (d-x) h}{t} \rightarrow \frac{\partial C}{\partial x} = \frac{\epsilon_0 \epsilon_r h}{t} - \frac{\epsilon_0 h}{t} = \frac{\epsilon_0 h}{t} (\epsilon_r - 1)$$

$$F = \frac{1}{2} V \frac{\partial C}{\partial x}$$

$$\boxed{F = \frac{1}{2} V^2 \frac{\epsilon_0 h}{t} (\epsilon_r - 1)}$$

Electric current

Current.

- Any conductor has delocalised e⁻ that can move freely about the conductor.
- If an ext. E field is applied to the conductor, the delocalised e⁻ will accelerate under the applied force and drift in a single direction, creating a current.
- Current I is defined as the rate of change of charge through a fixed area wrt. time. i.e. $\boxed{I = \frac{dQ}{dt}}$

Current density.

- Current density J is defined as.

$$dI = J \cdot dS, \quad \text{or} \quad \boxed{I = \iint_S J \cdot dS} \quad (J = \frac{dI}{dS})$$

- If the conductor has a uniform cross section, $\boxed{J = \frac{I}{S}}$

- Considering a conducting volume w/ charge density ρ, that moves at a drift velocity u, then the current density J is given by

$$\boxed{J = \rho u}$$

Resistance.

- Considering conduction currents, e⁻ experience a force F when an E field E is applied, $F = -eE$.

- This force move the e⁻ along the wire but the many collisions slow them down to an average drift velocity u. If the e⁻ have mass m and the average time between collisions is τ, the force is given by

$$F = \frac{mu}{\tau} = -eE$$

$$\therefore u = -\frac{e\tau}{m} E$$

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- If there are $n e^-$ in a unit volume, then charge density ρ_v is simply

$$\rho_v = -ne.$$

- Using the defn of current density J ,

$$J = \rho_v v = \frac{n e^2 c}{m} E = \sigma E \quad [\text{rown form of ohms law}].$$

where σ is the conductivity of the conductor.

- For a nonperfect conductor, we can define the imperfections in the conductor as resistance R , where $R = \frac{V}{I}$.

- For a nonperfect conductor w/ uniform cross-sectional area A and length l ,

$$R = \frac{V}{I} = \frac{V}{JA} = \frac{El}{\sigma EA} = \frac{l}{\sigma A}$$

- We can define resistivity ρ as $\rho = \frac{l}{\sigma A}$, so

$$R = \rho \frac{l}{A}$$

Magnetic flux density**Magnetic flux density**

- Magnetic flux density is a vector denoted as \underline{B}
- Magnetic flux density \underline{B} has units T or Wb m^{-2} .

Biot-Savart Law.

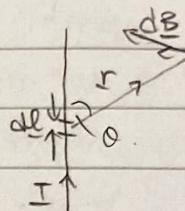
- BIOT-SAVART LAW states that a short element $d\underline{l}$ of wire carrying a current I produces a magnetic flux density $d\underline{B}$ at a position described by the vector \underline{r} is

$$d\underline{B} = \frac{\mu_0 I d\underline{l} \times \underline{r}}{4\pi r^3} = \frac{\mu_0 I d\underline{l} \times \hat{\underline{r}}}{4\pi r^2}$$

- In vector form,

use RH rule to determine the dir. of \underline{B}

$$d\underline{B} = \frac{\mu_0 I \sin\theta d\underline{l}}{4\pi r^2}$$

where θ is the angle between the vectors \underline{r} and $d\underline{l}$.+ μ_0 is the permeability of free space, $\mu_0 = 4\pi \times 10^{-7} \text{ Tm/A}$.

For a moving charge,

$$\underline{B} = \frac{\mu_0 q v \times \hat{\underline{r}}}{4\pi r^2}$$
Ampere's Law.**Ampere loop.**

- Ampere's law states that for any closed loop C (just for air/free space)

$$\oint \underline{B} \cdot d\underline{l} = \mu_0 I_{\text{enc}} \quad (I_{\text{enc}} = NI)$$

where $\oint \underline{B} \cdot d\underline{l}$ is the circulation of \underline{B} and I_{enc} is the current enclosed in the closed loop.

- Alternatively, using magnetic field intensity \underline{H} (general, inc. nonlinear materials)

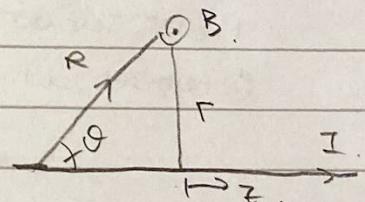
$$\oint \underline{H} \cdot d\underline{l} = NI$$

where $\oint \underline{H} \cdot d\underline{l}$ is the circulation of \underline{H} and NI is the magnetomotive force (mmf).**Magnetic flux density around a straight wire.****i) BIOT-SAVART law.**

$$\begin{aligned} d\underline{B} &= \frac{\mu_0 I \sin\theta d\underline{l}}{4\pi r^2} \\ \underline{B} &= \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{d\underline{l}}{r^2 + z^2} \cdot \hat{\underline{z}} \\ &= \frac{\mu_0 I}{4\pi} \int_{\pi/2}^{r'} \frac{r'}{(r'^2 \sec^2 \alpha)^{1/2}} (r \sec \alpha d\alpha) \\ &= \frac{\mu_0 I}{4\pi} \int_{\pi/2}^{r'} \frac{r \cos \alpha}{r} d\alpha. \end{aligned}$$

$r^2 = z^2 + r^2$, $dz = r \sec^2 \alpha d\alpha$.

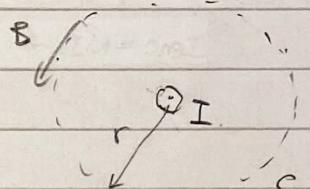
$$\boxed{\underline{B} = \frac{\mu_0 I}{2\pi r} \hat{\underline{z}}}$$

**ii) Ampere's Law.**

$$\oint \underline{B} \cdot d\underline{l} = \mu_0 I_{\text{enc}}$$

$$B(2\pi r) = \mu_0 I$$

$$\boxed{B = \frac{\mu_0 I}{2\pi r}}$$



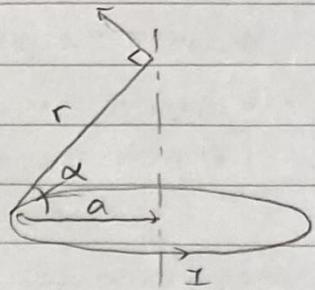
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Magnetic flux density on axis of a wire loop.

① Biot-Savart Law

Note B is axial only (horizontal components cancel)

$$\begin{aligned} dB &= \frac{\mu_0 I \sin \theta dl}{4\pi r^2} \quad \text{axial } B \\ B &= \frac{\mu_0 I \sin \theta}{4\pi r^2} \int dl \cdot \cos \alpha \\ &= \frac{\mu_0 I}{4\pi (a^2 + r^2)^{3/2}} \sin \theta \left(\frac{a}{\sqrt{a^2 + r^2}} \right) \\ B &= \frac{\mu_0 I a}{2(a^2 + r^2)^{3/2}} \end{aligned}$$

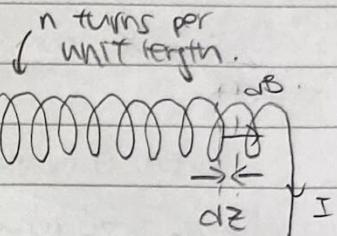


Magnetic flux density inside a solenoid.

① Biot-Savart Law

Consider the magnetic flux density dB produced by the short section dz .

Using superposition (valid for infinitesimally small sections)



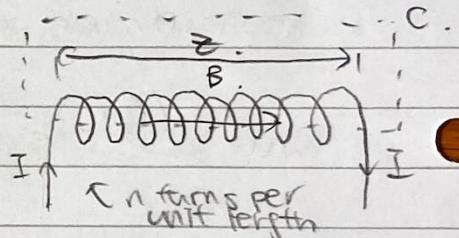
$$\begin{aligned} dB &= \frac{\mu_0 I a^2}{2(a^2 + z^2)^{3/2}} \cdot ndz. \quad (\text{derived using BIOT-SAVART LAW}) \\ B &= \frac{\mu_0 I a^2 n}{2} \int_{-\infty}^{\infty} \frac{1}{(a^2 + z^2)^{3/2}} dz \\ &= \frac{\mu_0 I a^2 n}{2} \int_{-\pi/2}^{\pi/2} \frac{1}{(a \sec^2 \alpha)^{3/2}} (a \sec^2 \alpha d\alpha) \quad dz = a \sec^2 \alpha \\ &= \frac{\mu_0 I n}{2} \int_{-\pi/2}^{\pi/2} \cos \alpha d\alpha \\ B &= \mu_0 n I \end{aligned}$$

② Ampere's Law

$$\oint_C \underline{B} \cdot d\underline{l} = \mu_0 I_{enc}$$

$$B \neq \mu_0 (n\pi) I$$

$$\boxed{B = \mu_0 n I}$$



Magnetic flux density inside a toroidal coil.

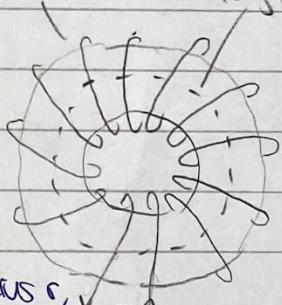
N turns. mean path length ℓ .

② Ampere's Law

$$\oint \underline{B} \cdot d\underline{l} = \mu_0 I_{enc}$$

$$B(2\pi r) = \mu_0 N I$$

$$\boxed{B = \frac{\mu_0 N I}{2\pi r}}$$



* If we consider the mean path length ℓ , at a certain radius r ,

$$2\pi r = \ell \text{ so } B = \frac{\mu_0 N I}{\ell} = \mu_0 n I.$$

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Magnetic force

- The magnetic force F due to a charge q moving at velocity v in magnetic field is

$$F = q(v \times B)$$

- The magnetic force F of a wire w/ length l carrying current I in magnetic field is

$$F = I(l \times B)$$

* Note $I = \frac{q}{t} A v = \frac{1}{A t} A v = \frac{q v}{l}$, \rightarrow both eqns are equivalent.

- The magnetic force F between two // current carrying wires is given by,

$$F = B I l$$

$$= \frac{\mu_0 I_1}{2\pi r} I_2 l$$

$$F = \frac{\mu_0 I_1 I_2 l}{2\pi r}$$

Magnetic flux and self inductance.

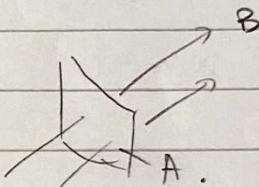
Magnetic flux

- Flux is the idea of evaluating the out. of a vector field that passes through a surface of area A .

- Magnetic flux Φ_B is given by

$$\Phi_B = \iint_S B \cdot dA$$

- If the field is always \perp to the surface, $\Phi_B = BA$.



Flux linkage

- In solenoids and inductors, we have many turns and each turn is effectively another area, so we sum the flux for each turn to get the flux linkage Φ'_B .

$$\Phi'_B = N\Phi_B$$

- For a uniform flux density B passing through N turns each of area A, the flux linkage Φ'_B is given by $\Phi'_B = N\Phi_B = NBA$.

Faraday's law of electromagnetic induction.

- Faraday's Law of Electromagnetic Induction states that the emf induced \mathcal{E} is prop. to the rate of change of flux linking the circuit.

$$\mathcal{E} = \frac{d\Phi'}{dt}$$

- Lenz's Law states that the emf induced by the change in flux is in the opposite dir.

$$\mathcal{E} = - \frac{d\Phi'}{dt}$$

(i.e. the induced emf opposes the flux that generated it).

* Practically, we ignore the - sign and figure out the dir. of \mathcal{E} in the end.

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Self Inductance.

- self inductance L is the ratio of flux linkage to the current generating the flux. (units: H).

$$L = \frac{\Phi}{I}$$

- strategy to find self inductance :

use superposition for
combinations of wires, loops.

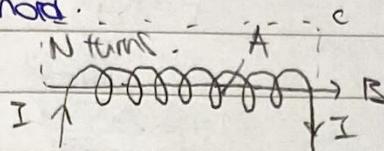
↳ ① Specify currents

↳ ② calculate B (using Ampere's Law)

↳ ③ calculate Φ_B . (using $\Phi_B = N \int_S B \cdot dA$)

↳ ④ Calculate L (using $L = \frac{\Phi}{I}$)

Self inductance of a solenoid.



A/L

$$\oint_C B \cdot d\ell = M_0 I_{enc}$$

$$B \cdot l = M_0 N I$$

$$B = M_0 \frac{N}{l} I$$

Flux Φ'

$$\Phi' = NBA$$

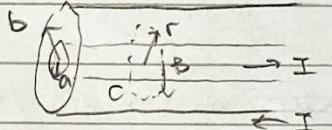
$$= M_0 \frac{N^2}{l} IA$$

Defn of L

$$L = \frac{\Phi'}{I}$$

$$L = \frac{M_0 N^2 A}{l}$$

Self inductance of a coaxial cable.



A/L

$$\oint_C B \cdot d\ell = M_0 I_{enc}$$

$$B(2\pi r) = M_0 I$$

$$B = \frac{M_0 I}{2\pi r}$$

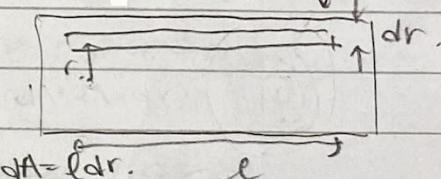
Flux Φ'

$$\Phi' = \oint_C B \cdot dA$$

$$= \int_a^b \frac{M_0 I}{2\pi r} l dr$$

$$= \frac{M_0 I l}{2\pi} \ln\left(\frac{b}{a}\right)$$

side view
(for $a < r < b$). ✓



Defn of L

$$L = \frac{\Phi'}{I}$$

$$L = \frac{M_0 l}{2\pi} \ln\left(\frac{b}{a}\right)$$

* Note self inductance per unit length L_L is given by $L_L = \frac{L}{l}$.

so for a coaxial cable, $L_L = \frac{M_0}{2\pi} \ln\left(\frac{b}{a}\right)$.

Recall that for a coaxial cable, $C_L = \frac{2\pi \epsilon_0}{\ln(b/a)}$.

so $\frac{L_L}{C_L} \propto \sqrt{\frac{M_0}{\epsilon_0}} = Z_0 = \frac{E}{H}$, where Z_0 is the characteristic impedance of free space.

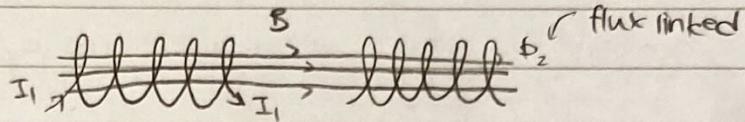
and $\frac{1}{Z_0 C_L} = \frac{1}{\sqrt{M_0 \epsilon_0}} = c$, where c is the speed of light in free space.

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Mutual inductance.

Mutual inductance.

- when two electrical circuits are close to one another, a current I_1 in one may cause magnetic flux ϕ_2 to link the other.

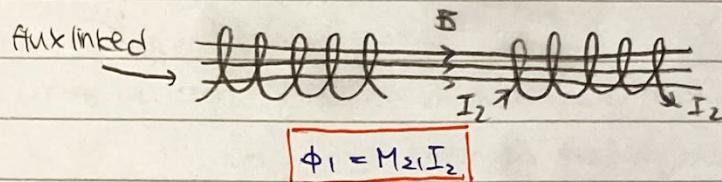


- Assuming linear behaviour, flux ϕ_2 must be prop. to current I_1 ,

$$\boxed{\phi_2 = M_{12} I_1}$$

where the const. of prop. M_{12} is the mutual inductance between the circuits.

- similarly, w/ the roles of the above circuits reversed, a current I_2 may cause magnetic flux ϕ_1 to link the other.



$$\boxed{\phi_1 = M_{21} I_2}$$

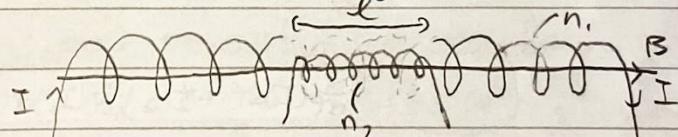
- It can be shown that the const. of prop. M_{12} and M_{21} are equal.

$$\boxed{M_{12} = M_{21} = M}$$

The quantity M is mutual inductance (units: H).

Mutual inductance of concentric solenoids

- consider a pair of concentric coils. the outer coil has n_1 turns per unit length; the inner coil has n_2 turns per unit length, CSA A and length l.
- Assume diameter of inner coil is small enough that B is const. over its CSA.



Magnetic flux density B due to outer coil $B = \mu_0 n_1 I$

Magnetic flux ϕ_2 through 1 turn of inner coil $\phi_2 = BA = \mu_0 n_1 I A$.

Magnetic flux linked ϕ'_2 through $N_2 l$ turns of inner coil $\phi'_2 = n_2 l \phi_2 = \mu_0 n_1 n_2 l I A$.

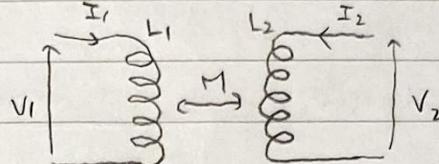
Mutual inductance $\boxed{M = \frac{\phi'_2}{I} = \mu_0 n_1 n_2 l A}$.

(similar strategy to finding the self inductance but we want to find ϕ'_2 rather than ϕ_2 for a current I_2).

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Magnetic energy in current-carrying circuits

- When currents I are established in an electrical circuit (charge in I), there is a corresponding change in magnetic flux (density) B/d , hence an induced emf ϵ . This means the voltage source must do work against this emf \rightarrow energy stored in circuit.
 - Consider two circuits w/ self inductances L_1, L_2 and mutual inductance M .



- The flux linking each circuit is a combination of the flux generated by its own current and the flux generated by the other current.

$$\Phi_1 = L_1 I_1 + M I_2$$

self mutual

$$\Phi_2 = L_2 I_2 + M I_1$$

- As the flux in each circuit is changing, there will be induced emfs V_1 and V_2 in each circuit, w/ the magnitude given by

$$\begin{aligned}
 V_1 &= \frac{d\Phi_1}{dt} & V_2 &= \frac{d\Phi_2}{dt} \\
 &= \frac{d}{dt}(L_1 I_1 + M I_2) & &= \frac{d}{dt}(L_2 I_2 + M I_1) \\
 &= L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt} & &= L_2 \frac{dI_2}{dt} + M \frac{dI_1}{dt}
 \end{aligned}$$

- The rate of change electrical energy $\frac{dW}{dt}$ is given by .

$$\begin{aligned}
 \frac{d\mathbf{v}}{dt} &= \mathbf{P} = \mathbf{I}_1 \mathbf{V}_1 + \mathbf{I}_2 \mathbf{V}_2 \\
 &= \mathbf{I}_1 \left(L_1 \frac{d\mathbf{I}_1}{dt} + M \frac{d\mathbf{q}_1}{dt} \right) + \mathbf{I}_2 \left(L_2 \frac{d\mathbf{I}_2}{dt} + M \frac{d\mathbf{q}_2}{dt} \right) \\
 &= L_1 \mathbf{I}_1 \frac{d\mathbf{I}_1}{dt} + L_2 \mathbf{I}_2 \frac{d\mathbf{I}_2}{dt} + M \left(\mathbf{I}_1 \frac{d\mathbf{q}_1}{dt} + \mathbf{I}_2 \frac{d\mathbf{q}_2}{dt} \right) \\
 &= \frac{1}{2} L_1 \frac{d}{dt} (\mathbf{I}_1^2) + \frac{1}{2} L_2 \frac{d}{dt} (\mathbf{I}_2^2) + M \frac{d}{dt} (\mathbf{I}_1 \mathbf{I}_2)
 \end{aligned}$$

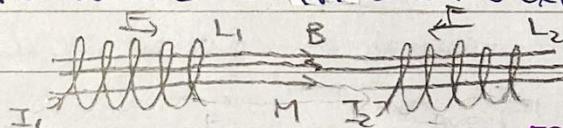
- Integrating w.r.t time gives the total energy stored

$$W = \frac{1}{2} L_1 \int d(I_x^2) + \frac{1}{2} L_2 \int d(I_z^2) + M \int d(I_x I_z)$$

$$N = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M I_1 I_2$$

i.e. the total energy in the circuit is the sum of the individual inductances ($\sum L_i I_i^2$) plus the energy of interaction between the circuits from the mutual inductance ($M_1 M_2$).

- The method of virtual work can be applied to this expression to find the force F.



$$F = I_1 I_2 \frac{\delta y}{\delta x}$$

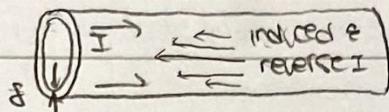
\leftarrow Force F symmetric
w.r.t. interchanging currents I_1, I_2

* In practice, this is a difficult expression to use as it is usually to define the mutual inductance $M(x)$.

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The skin effect

- Consider a cylindrical conducting wire along which a high frequency AC current (I) flows.
- Inside the wire, there will be circular lines of magnetic flux, and these will also be oscillating at high frequency.
- As a result of Faraday's law, an emf will be induced along the wire; as a result of Lenz's law, this emf opposes the original change in flux, setting up a reverse current.
- At the centre of the wire, magnetic flux Φ is max, therefore a larger reverse current is produced \rightarrow current tends to flow near the surface of the wire.



- In this course, we consider that all currents pass through the thin layer of thickness δ (skin depth) near the surface of the wire. Actually, at skin depth δ , the current drops to $1/e$ of the max. current.
- The higher the frequency, the larger the $\frac{d\Phi}{dt}$, so the larger the emf and cancellation of currents, thus the smaller the skin depth.
- The skin depth δ is given by the expression

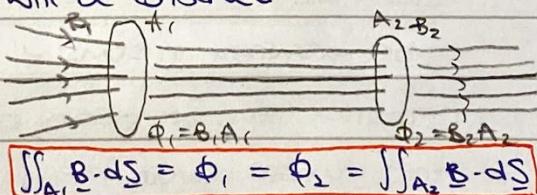
$$\delta = \sqrt{\frac{1}{\mu_0 \sigma f}}$$

- A smaller skin depth δ means a higher resistance in the wire R since the current has a smaller cross section A through which to flow ($A = \pi[r^2 - (r-\delta)^2]$, $R = \rho \frac{l}{A}$)

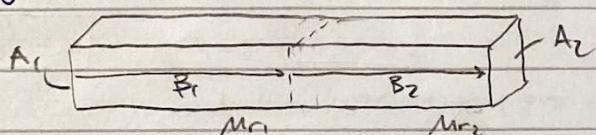
Magnetic materials

Conservation of magnetic flux

- Lines of magnetic flux density B always form closed loops. i.e. $\nabla \cdot B = 0$
(In contrast to E-fields, which always begin/end on charges \rightarrow no magnetic charges (monopoles))
- If we have a region of space where the B lines are contained / confined, then the flux of B across the area will be conserved



- This means given that flux travels from a magnetic material into another magnetic material w/o significant divergence, then flux will be conserved.

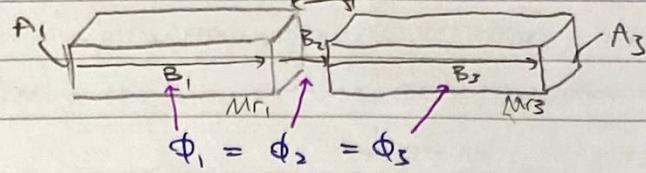


$$\text{If } A_1 = A_2 = A, \quad \Phi_1 = \Phi_2 \rightarrow B_1 A_1 = B_2 A_2 \rightarrow B_1 = B_2$$

i.e. const. of B as long as it passes through the same CSA A .

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- Consider two magnetic materials w/ a small air gap in between them



$$B_1 A_1 = B_2 A_2 = B_3 A_3$$

very useful!

If $A_1 = A_2 = A_3 = A$, then $B_1 = B_2 = B_3$ → we can create a magnetic field inside an air gap

- * If only given $A_1 = A_3$, we can only assume $A_1 = A_2 = A_3 = A$ when the air gap is small enough to prevent any fringing magnetic fields (A_2 would be larger for large air gaps)

- Due to the principle of flux conservation, we can treat magnetically susceptible materials like wires in a magnetic circuit.

(we assume high M_r materials effectively conform the magnetic flux). $R_m = \frac{l}{\mu_0 M_r A}$

- This also assumes that fringing effects at sharp corners can be neglected.

Magnetic materials

- There is a limit to the magnetic flux density B we can generate using air-filled coils

↪ ↑ I → resistive heating → wires become hot → ↑ R in coil

→ exert force ($F = B I l$) → wires become in tension

↪ ↑ n → ↓ d → ↓ A → ↑ R in coil

→ ↓ d → mechanically weaker

→ Filling the coil filled w/ magnetic material can enhance the magnetic flux density B .

- The behaviour of materials when placed in a magnetic field is a consequence of three types of behaviour:

↪ paramagnetism: Internal magnetic fields line up and reinforce the external field
(Linear)

Atoms that have a net dipole moment. (Unpaired e^-)

↪ diamagnetism: Internal magnetic fields add up in a way that opposes the external field.
(Linear)

Atoms that have no net dipole moment. (No unpaired e^-)

↪ ferromagnetism: Internal fields are strongly interacting w/ each other lining up into magnetic domains, even in the absence of an external field.

- When subjected to an ext. magnetic flux density B_0 , there is an internally generated magnetic flux density I (intensity of magnetization) that adds to the ext. field, given by

$$I = M_0 M$$

$$M = \frac{M_{tot}}{\sqrt{V}}$$

where M is the magnetization and M_{tot} is the total magnetic moment of the material.

- The resultant magnetic flux density B is therefore given by,

$$B = B_0 + I$$

$$B = M_0 (H_0 + M)$$

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Linear magnetic material. (Paramagnetic / Diamagnetic).

- For a linear magnetic material, the magnetization M is directly proportional to the magnetic field intensity of the ext. field H .

$$M = \chi_m H$$

where χ_m is the magnetic susceptibility of the material.

- The resultant magnetic flux density B is therefore given by

$$\begin{aligned} B &= M_0 (H + M) \\ &= M_0 (H + \chi_m H) \\ &= M_0 H (1 + \chi_m) \\ B &= M_0 M_r H \end{aligned}$$

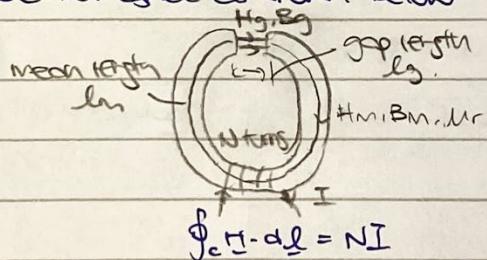
- The relative permeability M_r and magnetic susceptibility χ_m are related by

$$M_r = 1 + \chi_m$$

- Paramagnetic and diamagnetic materials are soft (temporary) and weak ($M_r \approx 1$).
- Paramagnetic materials have a +ve magnetic susceptibility χ_m ; Diamagnetic materials have a -ve magnetic susceptibility χ_m , Curie's Law Material property.
- Paramagnetism is temperature dependent ($\chi_m = \frac{C}{T}$, where C is the Curie constant); Diamagnetism is independent of temperature.

Magnetic flux in a linear electromagnet

- Consider a toroidal electromagnet as shown below



A/I L

$$H_m l_m + H_g l_g = NI$$

use B

$$\frac{B_m}{M_r M_r l_m} l_m + \frac{B_g}{M_r} l_g = NI$$

conserv. of flux

$$B_m = B_g$$

$$B_m H_m = B_g H_g \rightarrow B_m = B_g$$

$$\therefore \frac{B_g}{M_r M_r l_m} l_m + \frac{B_g}{M_r} l_g = NI$$

For large M_r , $\frac{B_g}{M_r M_r l_m} l_m \ll \frac{B_g}{M_r} l_g$ so $\frac{B_g}{M_r M_r l_m} l_m + \frac{B_g}{M_r} l_g \approx \frac{B_g}{M_r} l_g$.

$$\text{i.e. } B_g = \frac{NI l_m}{l_g}$$

- This setup is used in tape recorder heads, where magnetization in magnetic tape (B_g) is related to the audio signal strength (I) $B_g \propto I$.

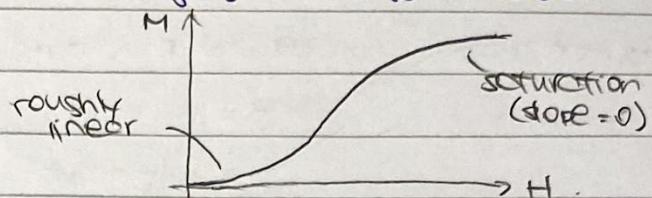
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non-linear magnetic material (ferromagnetic), only Fe, Co, Ni above RTP

- For a non-linear magnetic material, the magnetization M is related to the magnetic field intensity of the ext. field H through a non-linear function f_H .

$$M = f_H(H)$$

- The function f_H follows the rough general shape as follows:



- Initially, M increases w/ H at an increasing rate, but it eventually flattens off as all of the magnetic domains available are aligned (saturation).

- The resultant magnetic flux density B is still given by

$$\begin{aligned} B &= M_0(H+M) \\ &= M_0 M_r + H \end{aligned}$$

Linear magnetic materials can also experience saturation at very large applied H , but is practically always linear.

but now that M varies non-linearly w/ H , M_r is no longer constant,

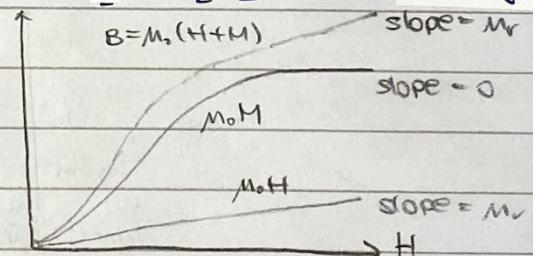
(we use differential susceptibility $\chi_m^d = \frac{\partial M}{\partial H}$ or apparent susceptibility).

- Ferromagnetic materials can be hard (paramagnet) and are strong ($M_r \approx 10^4$).
- When heated above the Curie temperature T_c , ferromagnetic materials become paramagnetic (degree of alignment of atomic magnetic moments decreases). Above T_c , susceptibility χ_m is given by $\chi_m = \frac{C}{T-T_c}$ where C is the Curie constant

← Curie-Weiss Law

Relationship between B and M .

- The relationship between B and M , $B = M_0(H+M)$ can be shown graphically:

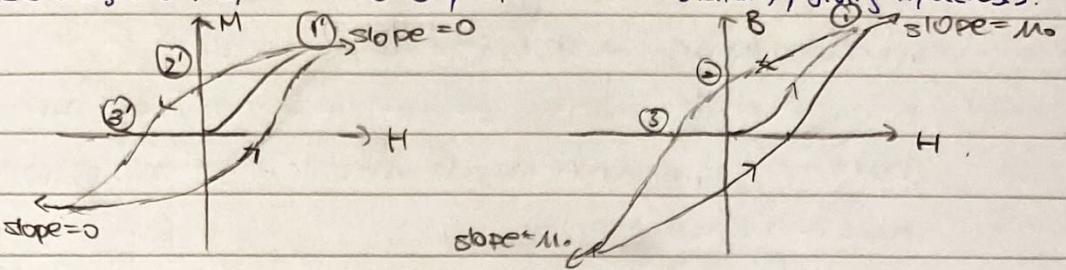


- At infinitely large H , B tends to infinity but M stays a constant value.
(so B does not saturate!)
- The M - H curve is an intrinsic characteristic of the magnetic material.

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Magnetic hysteresis

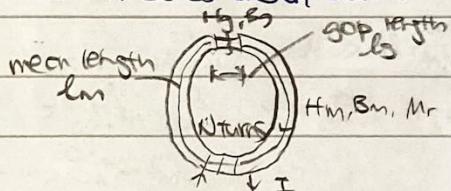
- Non-linear magnets may have memory (permanent magnetism), giving hysteresis.



- The magnetic material is saturated at ①. or retentivity/remance
- At ②, H is zero but there is a remanent flux density B_r (y -intercept)
- At ③, a -ve H is req. to reach zero B, this H is the coercivity H_c (x -intercept)
- The H at ② and ③ are different. They are the coercivity H_c and intrinsic coercivity H_{ci} , respectively (for soft magnets, $H_c \approx H_{ci}$)
- M-H and B-H curves are not necessarily symmetrical. It may be easier/harder to reach saturation depending on the crystallographic direction.
- The portion from ② to ③ is the demagnetization curve.

Magnetic flux in a nonlinear electromagnet

- consider a toroidal electromagnet as shown below



A/L

$$\oint_C H \cdot dL = NI$$

$$H_m l_m + H_g l_g = NI$$

use B

$$H_m l_m + \frac{B_m}{M_o} l_g = NI$$

cons. of flux

$$\Phi_m = \Phi_g$$

assume fringing effects are negligible.

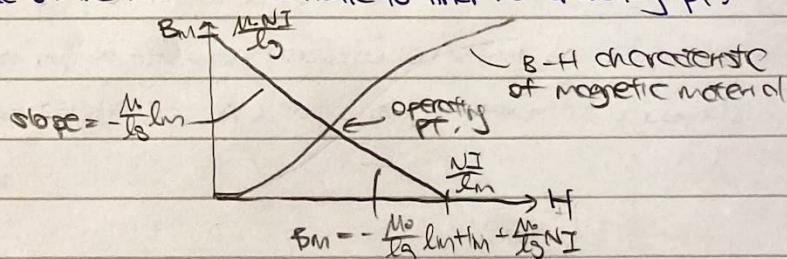
$$B_m A = B_g A \rightarrow B_m = B_g$$

$$\therefore H_m l_m + \frac{B_m}{M_o} l_g = NI$$

$$B_m = -\frac{M_o}{l_g} l_m H_m + \frac{M_o}{l_g} NI$$

load line.

Plotting the load line on the B-H characteristic to find the operating pt.

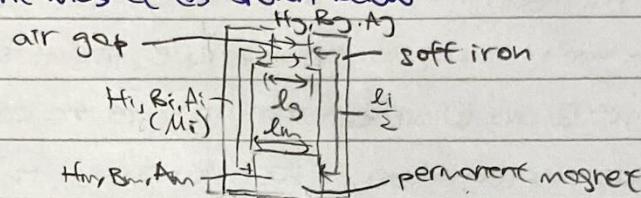


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Magnetic flux in a permanent magnet

- A permanent magnet produces a magnetic flux even after any ext. magnetic field have been removed, w/ the max. remanent flux B_r .
- Permanent magnetic materials are expensive \rightarrow rarely used alone \rightarrow we usually combine a small piece of permanent magnetic material w/ soft iron pieces to channel the magnetic flux to the req. pt.
- By reducing the CSA of the air gap, the flux becomes more conc. (flux consu.).

- Consider a permanent magnet as shown below:



A/C

$$\oint \mathbf{H} \cdot d\mathbf{l} = NI$$

$$H_m l_m + H_i l_i + H_g l_g = 0 \quad \leftarrow NI = 0 \text{ for permanent magnet}$$

use B

$$H_m l_m + \frac{B_i}{\mu_0 M_i} l_i + \frac{B_g}{\mu_0 M_g} l_g = 0.$$

Consrv. of flux

$$\phi_m = \phi_i = \phi_g \quad / \text{assume tracing effects are negligible}$$

$$B_m A_m = B_i A_i = B_g A_g \rightarrow B_i = \frac{B_m A_m}{A_i} \quad ; \quad B_g = \frac{B_m A_m}{A_g}$$

$$\therefore H_m l_m + \frac{B_m A_m}{\mu_0 M_i A_i} l_i + \frac{B_m A_m}{\mu_0 M_g A_g} l_g = 0$$

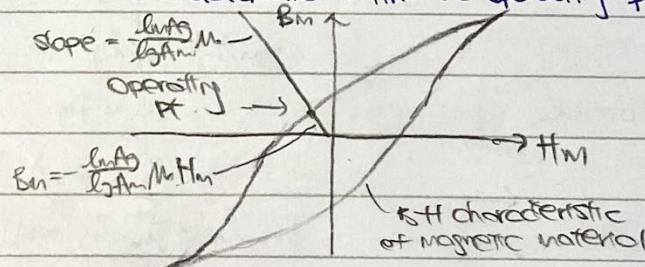
$$B_m A_m \left(\frac{l_i}{\mu_0 M_i A_i} + \frac{l_g}{\mu_0 M_g A_g} \right) = -H_m l_m$$

for large M_i , $\frac{l_i}{\mu_0 M_i A_i} \ll \frac{l_g}{\mu_0 M_g A_g}$. so $\frac{l_i}{\mu_0 M_i A_i} + \frac{l_g}{\mu_0 M_g A_g} \approx \frac{l_g}{\mu_0 M_g A_g}$

$$\text{r.e. } B_m A_m \frac{l_g}{\mu_0 M_g A_g} = -H_m l_m$$

$$B_m = -\frac{l_g A_g}{l_m A_m \mu_0 M_g} H \quad \text{load line}$$

Plotting the load line on the B-H characteristic to find the operating pt.



* The permanent magnet operates on the demagnetisation section of the hysteresis curve

* We have assumed that B_i is prop. to H_i for soft iron. However, this is not strictly true for iron as M_i is not constant. However as M_i is very large $\approx 10^3$, the term has disappeared from the final answer (We can neglect any terms w/ $M_i \gg 1$).

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Magnetostatic energy and forces

Energy stored in a magnetic field.

- Consider the work done δW in increasing the flux by $\delta\Phi'$ against the induced emf V .
 $V = \frac{\delta\Phi'}{\delta t}$, $\delta\Phi' = NAB$, $N=nl$, $P=IV$ // $\delta W = P\delta t = I \frac{\delta\Phi}{\delta t} \delta t = INAB = HlA\delta B = VH\delta B$

$$\therefore W = V \int_0^B H dB$$

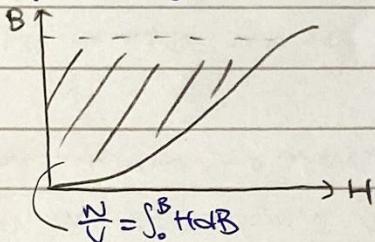
Work done per unit volume $\boxed{\frac{W}{V} = \int_0^B H dB}$ — area under the $H-B$ curve.

- For linear magnetic materials, B is prop. to H , $B = M_r M_0 H$, so

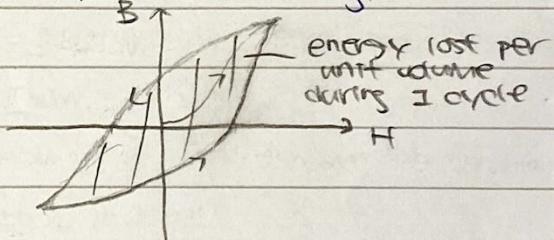
$$\frac{W}{V} = \int_0^B \frac{B}{M_r M_0} dH$$

$$\boxed{\frac{W}{V} = \frac{B^2}{2M_r M_0}} \quad \text{can be } V \text{-large as } M_r M_0 \text{ is } V\text{-small}$$

- For nonlinear magnetic materials, B and H are related by the $B-H$ characteristic of the material, so we cannot simplify the integral analytically.



- If the material has hysteresis, the area inside the $B-H$ loop is not accessible and this area represents the energy lost per unit volume during 1 cycle of the $B-H$ loop.

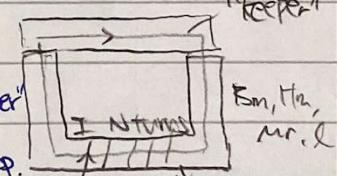


Force from an electromagnet

- Consider an electromagnet w/ a core of permeability $M_r M_0$ wound w/ a coil of N turns carrying a current I . A "keeper" of the same material is placed across the poles.

- To calculate the force F needed to just remove the "keeper",

we equate the mechanical work done by the force F moving the "keeper" by a short distance δx to the magnetostatic energy stored in the new air gap.



$$F \delta x = V \int_0^B H dB$$

$V = 2A\delta x$ as
A is the area of 1 —
gap, we have 2 gaps

$$F \delta x = 2A \delta x \int_0^B \frac{B^2}{M_r M_0} dB$$

$$F = 2A \frac{B^2}{2M_r M_0}$$

$$\frac{F}{A} = \frac{B^2}{M_r M_0}$$

cons. of flux $\Phi_m = \Phi_g \rightarrow B_m A = B_g A \rightarrow B_m = B_g \quad \therefore \frac{F}{A} = \frac{B_m^2}{M_r M_0}$ Assume same CSA

+ throughout

* Assume δx is small so no fringing effects $\rightarrow \delta\Phi = 0 \rightarrow$ induced emf $V = 0 \rightarrow$ no need to account for the work done against the induced emf $\delta W = IV \delta x$.

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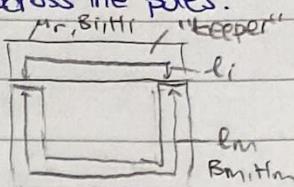
Forces from a permanent magnet

- Consider a permanent magnet w/ a piece of soft iron (keeper) placed across the poles.

- To calculate the force F needed to just remove the "keeper", we

equate the mechanical work done by the force F in moving the "keeper" by

a short distance δx to the magnetostatic energy stored in the new air gap.



$$F\delta x = V \int_0^{B_g} H d\delta B$$

$$V = 2A\delta x \text{ as}$$

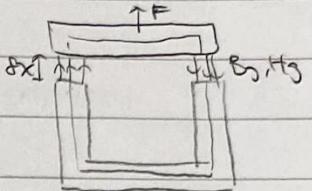
A is the area of 1 gap, we have 2 gaps

$$F\delta x = 2A\delta x \int_0^{B_g} \frac{B_g}{M_0} dB$$

$$F = 2A \frac{B_g^2}{2M_0}$$

$$\frac{F}{A} = \frac{B_g^2}{M_0}$$

$$\text{cons. of flux } \Phi_m = B_g \rightarrow B_m \delta x = B_g \delta x \rightarrow B_m > B_g \quad \therefore \boxed{\frac{F}{A} = \frac{B_m^2}{M_0}}$$



Assume constant CSA
A throughout

* Assume δx is small so no fringing effects $\rightarrow \delta\Phi = 0 \rightarrow$ induced emf $V = 0 \rightarrow$ no need to account for the work done against the induced emf $\Delta W = IV\delta t$.

Forces from electromagnets vs permanent magnets.

- the expression for the magnetostatic force (per unit core) is the same for both electromagnets and permanent magnets: $\frac{F}{A} = \frac{B_m^2}{M_0}$

- the only difference is how we obtain an expression/value for B_m .

↳ linear electromagnet: Use $\oint C H dl = NI$ to find H_m

$$\text{use } B_m = M_r H_m \text{ to find } B_m$$

↳ nonlinear electromagnet: Use $\oint C H_m dl = NI$ to find H_m

use B-H characteristic to find B_m

↳ permanent magnet: Use $\oint C H dl = NI$ to find an expression w/ H_m, H_i

$$\text{use } B_i = M_r H_i \text{ to express in terms of } B_i$$

use cons. of flux to express in terms of B_m, H_m only \rightarrow load line

use load line + B-H characteristic to find the operating pt. (B_m)