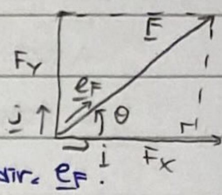
External forces

Equilibrium of point forces, moments and couples



Forces as vectors

- Force is a vector → has magnitude and direction.
  ↳ $\underline{F} = F\underline{e}_F$, where the force $\underline{F}$ of magnitude F is acting in dir. $\underline{e}_F$.
- In some problems, it is convenient to express a force $\underline{F}$ in terms of its cartesian components.
  ↳ $\underline{F} = F_x\underline{i} + F_y\underline{j} = \begin{pmatrix} F_x \\ F_y \end{pmatrix}$, $F_x = F\cos\theta$, $F_y = F\sin\theta$. // $F = \sqrt{F_x^2 + F_y^2}$. $e_F = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ $[\theta = \tan^{-1}(\frac{F_y}{F_x})]$

* It is often helpful to think the forces as vectors instead of components, then solve using graphical methods.
  ↳ we often know the dir. of a force, but not its magnitude → 1 unknown.
  ↳ using components, we lose the dir. information → 2 unknowns.

Moments as vectors

- The turning effect of $\underline{F}$ about a pt. $\underline{O}$ is measured by the moment $\underline{M}_O$, given by.

$$\boxed{\underline{M}_O = \underline{r} \times \underline{F}}$$

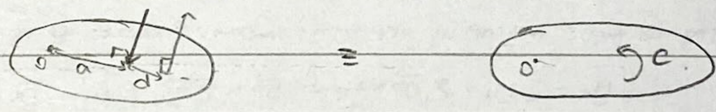
- The magnitude of $\underline{M}_O$ is given by the magnitude of F multiplied by the ⊥ distance from the line of action of $\underline{F}$ to $\underline{O}$; the direction of M_O is ⊥ to the plane containing $\underline{F}$ and $\underline{O}$.
  moment arm d.

$$\boxed{|\underline{M}_O| = |\underline{F}|\, d}$$

* The turning effect of $\underline{F}$ about diff. pts. may not necessarily be the same.

Couples.

- The moment produced by 2 equal and opposite non-collinear forces is a couple.



2 equal and opposite non-collinear forces

An equivalent couple.

- Their combined moment abt. a pt. O in the plane is the couple $\underline{M}$, which has magnitude

$$\boxed{|\underline{M}| = F(a+d) - F(a) = Fd}$$

→ As the magnitude of $\underline{M}$ is independent of a, the moment of a couple is the same abt any pt.

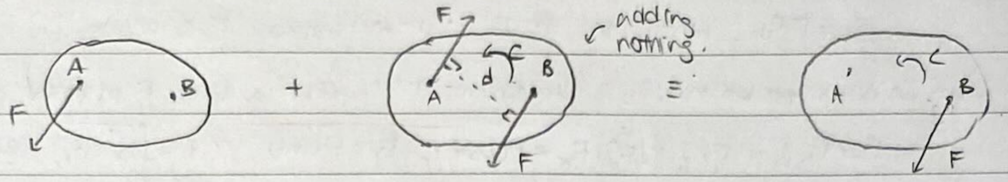
Moment vs couple.

- Moments are the turning effect created by 1 resultant force around some particular pt.
- Couples depend on a pair of separated but equal and opposite forces and are not associated w/ a specific pt. of application.

* Force (LOA through CoM) → translation only Couple → rotation only.
  Force (LOA not through CoM) → translation + rotation [moment]

Replacement of a force by a force-and-couple.

- A force $\underline{F}$ applied at pt. A can be replaced w/ a force // to $\underline{F}$ and of equal magnitude to $\underline{F}$ by introducing a couple of suitable magnitude.



where $C = Fd$ (M is anticlockwise as the moment of original $\underline{F}$ about B is anticlockwise)

- The 2 systems are statically equivalent but may lead to diff. int. forces.

Resultants

- The resultant of a system of forces and couples acting on a body is the simplest force combination that can replace the original system w/o altering the effect of the system on the body.

① Resultant of intersecting forces. (concurrent forces)

- The resultant force $\underline{F}$ of any no. of forces is the vector sum of all the individual forces. Graphically $\underline{F}$ can be found using a force polygon.
- Varignon's thm for concurrent forces states that the for a body acted on by any no. of concurrent forces, the sum of the moments of all forces about any pt. $\underline{O}$ is equal to the moment of the resultant force $\underline{F}_R$ about $\underline{O}$. Proof is as follows:

↳ As all the forces intersect at $\underline{P}$. The moment of the ith force about O is.

$$\underline{M}_i = \vec{OP} \times \underline{F}_i .$$

summing all these moments, the total moment about $\underline{O}$ is .

$$\boxed{\underline{M}_R = \sum \underline{M}_i = \sum \vec{OP} \times \underline{F}_i = \vec{OP} \times \sum \underline{F}_i = \vec{OP} \times \underline{F}_R}$$

② Resultant of non-intersecting forces.

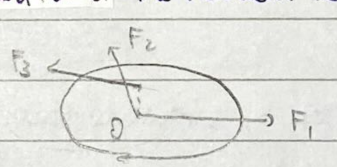
- A system of non-intersecting forces can be replaced by a single force $\underline{F}$, w/ a ⊥ distance d to $\underline{O}$.

↳ shift the position of forces so the forces become concurrent (by adding a couple).

↳ Combine the forces to get a resultant force.

↳ shift the position of the resultant force to cancel out any existing couple.

1)



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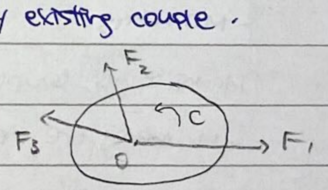

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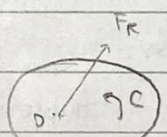

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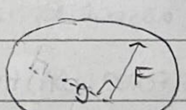


③ Resultant of a system of forces and couples (general case).

- Given a system of 2D forces $\underline{F}_i$ and couples C_i, we can replace the system by a single force $\underline{F}$.

$$\boxed{\underline{F} = \Sigma \underline{F}_i}$$

and the ⊥ distance d of the line of action of $\underline{F}$ to an arbitrary pt. O is given by.

$$\boxed{Fd = \Sigma F_i d_i + \Sigma C}$$

where d_i are the ⊥ distances from the lines of action of $\underline{F}_i$ to O.

* ① and ② are special cases of ③ where $\Sigma C = 0$.

Newton's laws

① When all ext. influences on a particle are removed, the particle moves w/ constant velocity. This velocity may be 0 in which case the particle remains at rest

② When a force $\underline{F}$ acts on a particle of mass m, the particle moves w/ instantaneous acceleration $\underline{a}$ given by the formula : $\boxed{\Sigma \underline{F} = m\underline{a}}$.

③ When 2 particles exert forces upon each other, these forces are equal + opposite and // to the straight line joining the 2 particles.

Equilibrium expressed in vector resultants

- For static equilibrium:

Forces in eqm. form a closed polygon.

↳ The resultant force on the body must be 0 i.e. $\boxed{\underline{F}_R = \Sigma \underline{F}_i = 0}$

↳ the resultant moment about any pt. must be 0 i.e. $\boxed{\underline{M}_R = \Sigma \underline{M}_i = 0}$

- In 2D, the 2 eqn. can be stated as 3 scalar eqn. (Graphical method is more convenient as we know more about the dir. of applied forces rather than their magnitude)
- In 3D, the 2 eqn. can be stated as 6 scalar eqn. (Graphical method is not convenient since it is very difficult to draw in 3D).

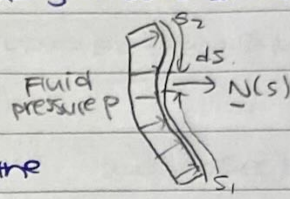
Accuracy in structural mechanics.

- In reality, we do not know accurate information on the loads → there are few situations in which you need to be more accurate than 1 sig. fig.
- We need to know the order of magnitude of the deflections more than the detailed stiffness of the structure.
- The answer should never be more precise.

Distributed loads

Constant fluid pressure

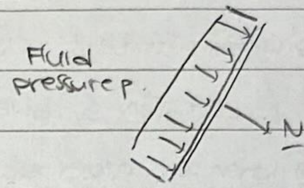
- According to Pascal's law, the pressure p at any given pt. in a stationary fluid is the same in all directions.
- Where the fluid meets a constraining surface, over any infinitesimal area, it imparts a force $\delta F = p\,\delta A$ normally to the surface.
- The resultant force per unit width (PUW) of pressure acting on some surface that varies in just 1D is given by



$$\boxed{\underline{F} = \int_{s_1}^{s_2} p.\underline{N}(s)\,ds}$$

where $\underline{F}$ is the equivalent force per unit width of the pressure acting on the surface btwn s_1 and s_2 w/ surface normal $\underline{N}(s)$ at s.

- For pressure acting on a flat surface, the dir. of $\underline{N}$ is constant for the whole surface, so the integral is solved as

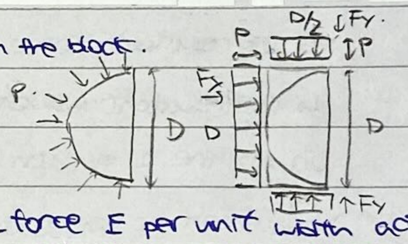


$$\boxed{\underline{F} = pL\underline{N}}$$

where L is the length of the surface and $\underline{F}$ is the force per unit width.

- The magnitude of the component of pressure force on an object in a given dir. is equal to the pressure multiplied by the projected area of the object in that dir.

eg: Find the resultant force of a uniform pressure acting on the block.



Force PUW x-dir: $F_x = pD$

y-dir: $F_y = p\frac{D}{2}$

* The distributed pressure can be replaced by a pt. force $\underline{F}$ per unit width acting through the CoM of the area representation of the load. Refer to trapezoidal distributed load example.
* The resultant force on any body subject to a uniform pressure over its entire surface is 0.

Hydrostatic pressure

* remember both vertical and horizontal hydrostatic forces

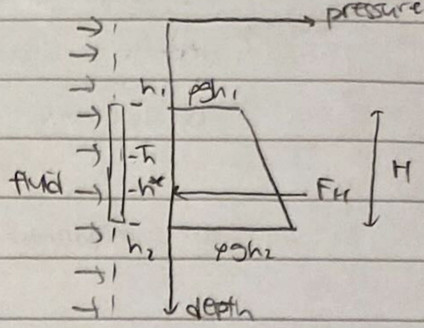
- The hydrostatic pressure $p(h)$ at a depth h below the surface of a liquid of uniform density ρ is given by

$$\boxed{p(h) = \rho g h}.$$

- Consider the column of liquid above a horizontal plane at depth h.

By NII I:
$$W = p(h)A$$
$$\rho \cancel{A} g h = p(h)\cancel{A}$$
$$p(h) = \rho g h.$$

- consider the horizontal loading on the plate shown below due to the hydrostatic pressure in the fluid which varies w/ depth h.



$$F_H = \int_{h_1}^{h_2} p(h)\,dh.$$
$$= \int_{h_1}^{h_2} \rho g h\,dh$$
$$= \frac{\rho g}{2}(h_2^2 - h_1^2)$$
$$= \rho g \frac{h_1+h_2}{2}\cdot(h_2-h_1)$$
$$= \rho g \bar{h} H \quad [\bar{h} = \frac{h_1+h_2}{2},\ H = h_2-h_1].$$

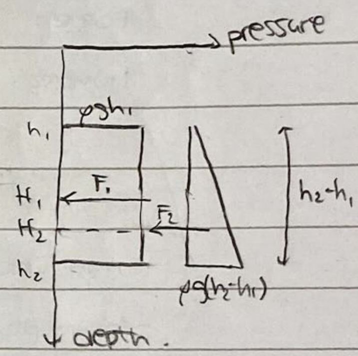
$M_{h=0}$:
$$F_H h^* = \int_{h_1}^{h_2} p(h)\cdot h\,dh \quad [F(h) = p(h)\,dh,\ d = h].$$
$$= \int_{h_1}^{h_2} \rho g h^2\,dh$$
$$= \frac{\rho g}{3}(h_2^3 - h_1^3).$$

$$\therefore h^* = \frac{F_H h^*}{F_H}$$
$$= \frac{\rho g}{3}(h_2^3 - h_1^3) \div \frac{\rho g}{2}(h_1+h_2)(h_1-h_2)$$
$$= \frac{2}{3}\frac{h_1^2 + h_1h_2 + h_2^2}{h_1+h_2}$$

- In the special case that the top of the body is at the surface of the fluid, i.e. $h_1 = 0$.

$$\boxed{F_H = \frac{\rho g}{2}h_2^2}\ ;\ F_H h^* = \frac{\rho g}{3}h_2^3\ ;\ \boxed{h^* = \frac{2}{3}h_2}$$

* we can split the trapezoidal-distributed load into a rectangular distributed load and a triangular distributed load:



$$F_1 = pL \qquad\qquad F_2 = \frac{\rho g}{2}H^2$$
$$= \rho g h_1(h_2-h_1) \qquad\qquad = \frac{\rho g}{2}(h_2-h_1)^2$$
$$H_1 = h_1 + \frac{h_2-h_1}{2} \qquad\qquad H_2 = h_1 + \frac{2}{3}(h_2-h_1)$$
$$= \frac{h_1+h_2}{2} \qquad\qquad = \frac{h_1+2h_2}{3}$$

$$\Sigma F_i = F_1 + F_2 = \rho g h_1(h_2-h_1) + \frac{1}{2}\rho g(h_2-h_1)^2$$
$$= \frac{1}{2}\rho g(h_2-h_1)(2h_1 + h_2 - h_1)$$
$$= \frac{1}{2}\rho g(h_1+h_2)(h_2-h_1) \quad \text{as expected.}$$

$$\Sigma F_i H_i = F_1H_1 + F_2H_2 = \rho g h_1(h_2-h_1)\cdot\frac{h_1+h_2}{2} + \frac{1}{2}\rho g(h_2-h_1)^2\cdot\frac{h_1+2h_2}{3}.$$
$$= \frac{1}{6}\rho g(h_2-h_1)[3h_1(h_1+h_2) + (h_2-h_1)(h_1+2h_2)]$$
$$= \frac{1}{6}\rho g(h_2-h_1)(3h_1^2 + 3h_1h_2 + h_1h_2 + 2h_2^2 - h_1^2 - 2h_1h_2).$$
$$= \frac{1}{6}\rho g(h_2-h_1)(2h_1^2 + 2h_1h_2 + 2h_2^2)$$
$$= \frac{1}{3}\rho g(h_2-h_1)(h_1^2 + h_1h_2 + h_2^2) \quad \text{as expected.}$$

Note that h^* is equivalent to the CoM of the area representation of the load.

$$M\bar{y} = m_1y_1 + m_2y_2$$

$$\cancel{\rho}[\cancel{\rho g}h_1\cdot\cancel{(h_2-h_1)} + \tfrac{1}{2}\cancel{\rho g}(h_2-h_1)\cdot\cancel{(h_2-h_1)}]\bar{y} = \cancel{\rho}\cancel{\rho g}h_1\cdot\cancel{(h_2-h_1)}\cdot H_1 + \cancel{\rho}\cdot\tfrac{1}{2}\cancel{\rho g}(h_2-h_1)\cdot\cancel{(h_2-h_1)}\cdot H_2$$

$$\bar{y} = \frac{2}{3}\frac{h_1^2 + h_1h_2 + h_2^2}{h_1+h_2} \quad \text{as expected.}$$

Gravity

- The weight loading / self-weight of a structure is always distributed, but we can treat it as a pt. force through the CoM of the system. (Also the centroid for uniform lamina)
has the symbol ⊕ on diagrams.
- For discretely distributed masses,

$$\boxed{\bar{x} = \frac{\Sigma m_i x_i}{\Sigma m_i} = \frac{\Sigma A_i x_i}{\Sigma A_i}} \qquad \boxed{\bar{y} = \frac{\Sigma m_i y_i}{\Sigma m_i} = \frac{\Sigma A_i y_i}{\Sigma A_i}}$$

- For continuously distributed masses,

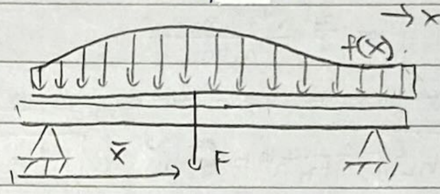
$$\boxed{\bar{x} = \frac{\int x\,dm}{\int dm} = \frac{\int x\,dA}{\int dA}} \qquad \boxed{\bar{y} = \frac{\int y\,dm}{\int dm} = \frac{\int y\,dA}{\int dA}}$$

- For a continuously distributed lamina w/ height $h(x)$,

$$\boxed{\bar{x} = \frac{\int x\,h(x)\,dx}{\int h(x)\,dx}} \qquad \boxed{\bar{y} = \frac{\int y\,h(x)\,dx}{\int h(x)\,dx}}$$

General distributed force.

- Consider a general distributed force per unit length $f(x)$.



We can replace the distributed force w/ a force F at position $x = \bar{x}$.

Force: $\boxed{F = \int f(x)\,dx}$

Moment: $F\bar{x} = \int f(x)\,x\,dx$ $\rightarrow$ $\boxed{\bar{x} = \frac{F\bar{x}}{F} = \frac{\int f(x)\,x\,dx}{\int f(x)\,dx}}$

Contact forces — contact forces are reaction forces → we can only find the contact force as the reaction to other applied forces.

Kuhn-Tucker conditions.

- At any pt. on the interface of 2 contacting bodies, 1 of 2 Kuhn-Tucker conditions must apply:

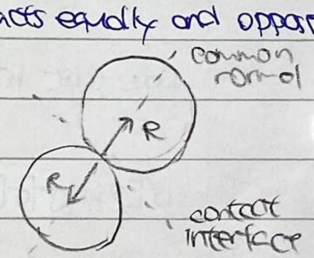
① The distance between the bodies is 0 → force between them is +ve.

② The distance between the bodies is +ve → force between them is 0.

→ either separation or contact force is always 0, and the other is always +ve.

Frictionless contact.

- In the frictionless contact of 2 bodies, a distributed force acts equally and oppositely on both bodies, normally to the contact interface
- It is often convenient to replace the distributed force w/ a pt. force anywhere in the region of contact. (provided that the force is along the common normal).



Contact forces with friction.

- Friction varies w/ the materials of the 2 contacting bodies, the detailed geometry of their surfaces (rough or smooth, textured or not) and the presence or not of another material (usually a lubricant) at the contact surface.
- A good approx. of dry friction – where there is no lubricant between 2 relatively rough bodies, the limiting value of friction F is prop. to the normal force N:
- For static contact (no sliding) : $\boxed{|F| \le \mu_s N}$

  For dynamic contact (sliding) : $\boxed{|F| = \mu_d N}$

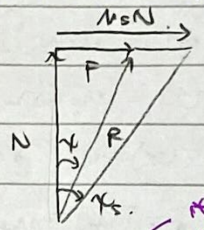
  where μ_s and μ_d are the coefficient of static and dynamic friction respectively.
- Friction always opposes motion : if the dir. of real/incipient motion changes, so does the dir. of the frictional force.

* $\mu_s < \mu_d$. (which causes a ruler to "judder" when sliding 2 fingers underneath it).

Angle of friction

very useful for graphical methods.

- It is often convenient to replace F and N by a single resultant force R.



It is good practice to always draw the common normal before friction.

The force R is inclined to the common normal at an angle φ, given by

$$\boxed{\varphi = \tan^{-1}\left(\frac{F}{N}\right)}$$

- For static contact (no sliding) : $|\varphi| \le \varphi_s$ * $\tan\varphi_s = \mu_s$

  For dynamic contact (sliding) : $|\varphi| = \varphi_d$ * $\tan\varphi_d = \mu_d$.

  where φ_s and φ_d are the angle of static and dynamic friction respectively.

Types of friction problems.

① Static contact w/o impeding motion.

$|F| < \mu_s N$, $|\varphi| < \varphi_s$. F found by considering eqm. of the body.

② Static contact, at the pt. of slipping.

$|F| = \mu_s N$, $|\varphi| = \varphi_s$.

③ Dynamic contact, surfaces slipping.

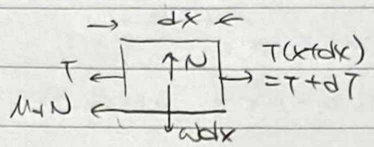
$|F| = \mu_d N$, $|\varphi| = \varphi_d$.

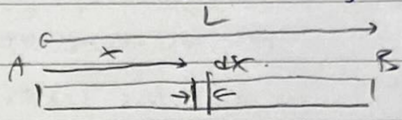
Distributed friction

- If a body is sliding over another, the coefficient of dry friction is constant, but the forces within the sliding body will vary continuously, due to the integration ("accumulation") of friction along the interface. → may lead to nonuniform distribution of pressure between the bodies along their mutual interface.
- Generally, we solve problems of distributed friction by considering the eqm. of a small (infinitesimal) element and then integrate to find the force distribution.

- eg: Sliding carpet. Find the force T_B req. to pull a carpet of weight per unit length w at steady speed over a floor w/ coefficient of dynamic friction μ_d. ($T_A = 0$).

FBD of small element





N2L ↕ :

$$N = w\,dx$$

→ :

$$T + dT = T + \mu_d N$$

so

$$dT = \mu_d w\,dx$$

$$\int_0^{T(x)} dT = \mu_d w \int_0^x dx$$

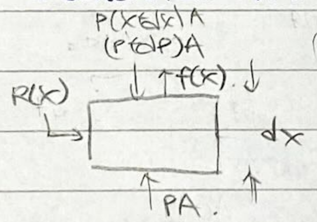
$$T(x) = \mu_d w x + C.$$

Boundary cond : when $x = 0$, $T(x) = 0$ so $C = 0$ → $T(x) = \mu_d w x$.

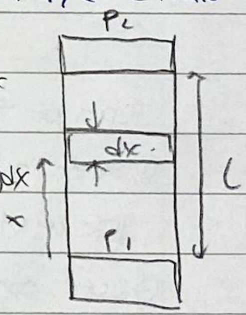
$$\therefore T_B = T(L) = \mu_d w L.$$

- eg: Sand in a cylinder. A column of dry sand of length L is pushed at constant speed through a cylindrical tube of radius R btwn 2 smooth light pistons applying pressures p_1 & p_2 ($p_2 > p_1$). Ignoring the self weight of the sand and assuming that at any pt in the sand the pressure is the same in all dir. and that the coefficient of dynamic friction is μ_d. Find p_2/p_1.

FBD of small element



(Around the slice,
$R(x) = 2\pi R\,p(x)\,dx$
$f(x) = \mu_d R(x)$
$= \mu_d \cdot 2\pi R\,p(x)\,dx$
$A = \pi R^2$



N2L ↕ :

$$PA + f(x) = (P + dP)A$$

$$2\mu_d \cancel{\pi} \cancel{R}\, P\,dx = dP \cdot \cancel{\pi} R^{\cancel{2}}$$

$$2\mu_d\,dx = \frac{R}{P}\,dP$$

$$R\int_{P_1}^{P_2} \frac{1}{P}\,dP = 2\mu_d \int_0^L dx$$

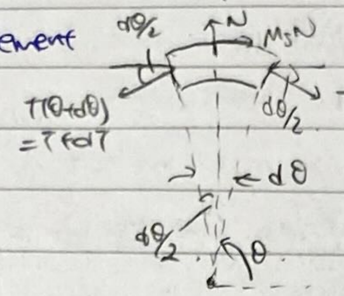
$$\ln\left|\frac{P_2}{P_1}\right| = \frac{2\mu_d}{R} L$$

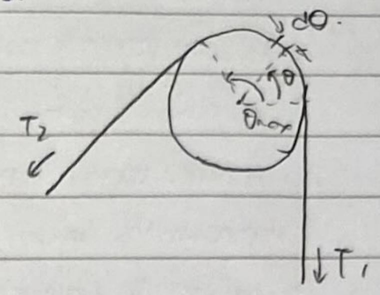
$$P_2/P_1 = e^{\frac{2\mu_d}{R}L}.$$

Note $R(x)$ is the total force acting on the slice so we integrate $p\,dA$ for the entire surface. As p is constant, $R(x) = \int p\,dA$
$= p\int dA$
$= p \cdot 2\pi R\,dx$

- e.g. Flexible belt on a pulley. A flexible belt is passed over a pulley. If the coefficient of static friction b/wn the belt and pulley is μ_s, and $T_2 > T_1$. Find the max. ratio of T_2/T_1 before sliding occurs.

FBD of small element





By N2L ↑ :

$$N = T\sin\left(\frac{d\theta}{2}\right) + (T+dT)\sin\left(\frac{d\theta}{2}\right)$$

$$= 2T\sin\left(\frac{d\theta}{2}\right) + dT\sin\left(\frac{d\theta}{2}\right)$$ 2nd order term — For small θ

$$= T\,d\theta.$$

→ :

$$(T+dT)\cos\left(\frac{d\theta}{2}\right) = T\cos\left(\frac{d\theta}{2}\right) + \mu_s N$$

$$dT\cos\left(\frac{d\theta}{2}\right) = \mu_s N$$ For small θ

$$dT = \mu_s T\,d\theta$$

$$\int_{T_1}^{T_2}\frac{1}{T}\,dT = \mu_s\int_0^{\theta_{max}} d\theta$$

$$\ln\left|\frac{T_2}{T_1}\right| = \mu_s\theta_{max}.$$

DB. $$\boxed{T_2 = T_1 e^{\mu_s\theta_{max}}}$$ * $T_2 > T_1$

Supports and free body diagrams.

Applied forces and reaction forces.

- Forces can be applied to a structure by an ext. agency in which case the force is known and the displacement of the structure under the action of the force must be found.
- Forces can be the reactions of the structure at a geometric constraint or support, in which case the displacement is 0 and the force is found by overall eqm.

Pin-joints

- A frictionless pin joint can only provide a reaction force, but not apply a moment to the structure.
- The reaction force at a pin-joint can act in any direction. (direction can be found easily via graphical methods.)
- The pin-joint support restricts 2 degrees of freedom (vertical + horizontal translation)
- In analysing problems w/ pin-joints, it is assumed that they are frictionless and apply a pt. force to the structure.

Roller supports

- Roller supports provide a vertical restrain on displacement but does not inhibit horizontal movement (or rotations).
- As w/ frictionless contact, the reaction force of a roller support is in the dir. of the common normal ($\perp$ to the motion of the rollers).
- A more formal representation of the roller support is , which restricts translations up and down. However, for simplicity is assumed to constrain motions in both the vertical directions.

Built in / encastré supports.

- Encastré supports constrains all 3 deg. of freedom - linear displacement in the vertical / horizontal dir. and rotation.
- The reaction forces provided by an encastré support include 2 components of a force and a moment in reaction to the constraint on motion.

* Do not confuse this w/ a string / cable, which can only transmit a force along its dir.

The direction of reaction forces.

- In general, a body subject to 2 forces (and no moments) can only be in eqm. if the forces are co-linear.
- In general, a body subject to 3 forces (and no moments) can only be in eqm. if the forces are concurrent (their lines of action intersect at a single pt.).

* If the 3 forces are //, the intersection pt. is effectively at infinity → the body can still be in eqm.

- For a pin-joint support,
  - ↳ 1 applied force → reaction force colinear w/ applied force.
  - ↳ 2 applied forces → reaction force's line of action passes through the intersection pt. of the applied forces' lines of action.
- For a roller support, the reaction force is $\perp$ to the motion of the rollers.
- For an encastré support (of a cantilevered beam), there is a component of force along the length of the beam (tensile / compressive force), a component of force $\perp$ to the length of the beam (shear force) and a moment (bending moment).
- For a cable, the dir. of force is always along the dir. of the cable.

* We can only find the reaction forces of a structure that is neither under or over constrained (by ext. constraints / supports).

Free body diagrams (FBD)

- When a structure comprises of many components/bodies connected at contact interfaces or w/ joints, → we may want to find the forces between the bodies.
- These forces are internal to the overall structure but external rel. to the bodies once they have been separated.
- We can separate the different bodies and draw FBDs to solve for the (overall) int. forces.
- To construct FBDs for a 2D structure:
  ↳ Cut along a closed curve and draw each part separated from each other.
  ↳ Add the original applied + reaction forces at the pt. where they apply and add force/moment pairs (by N3L) at the cut-through interface.

Problem solving strategy (graphical method).

- 3 Forces: We know the pt. the force is acting on → find intersection pt. of the lines of action to find the dir, (+ force triangle to find the magnitude).
  We don't know the pt. the force is acting on → force triangle to find the magnitude.
- 4 Forces: Take moments abt the intersection of the lines of action of 2 + forces → find the other forces.
- If 2+ forces act at the same pt. → merge them into 1 force.
  After obtaining the resultant, use a force polygon to solve for the individual forces.
- In general, draw a rough sketch + think of the solving process/steps before drawing an accurate sketch.

Problem solving strategy (algebraic method).

- Use N1L in 2 directions to get 2 scalar eqn. of eqm.
- Use moments to find the magnitude of the reaction forces.
  ↳ Take moments abt. one support to find the (vertical) contact force at the other support.
  ↳ Take moments abt. a pt. in the structure (above ground) to find the (horizontal) frictional force at the contacts. [using a section created by cutting at a pin joint]

Internal forces

Pin-jointed trusses

Pin-jointed plane trusses.

- A truss is a framework of members joined at their ends.
- For a plane truss, all the members lie in 1 plane.
- In general, a truss may be joined at their ends by welding, rivets or bolts and gusset plates, pins or adhesives.
- For simplicity, we will only consider the idealised frictionless pin as the type of connection → pin-jointed plane trusses.

(For any truss w/ slender members, any type of joint may be modelled as a pin joint)

- For now, we will consider only pin-jointed trusses subject to pt. loads applied at the joints.
- As these loads are usually large in comparison to the weight of the members, we will also neglect the self-weight of the members.
- Each member in the truss must be an axial force member — it is in a state of pure tension or pure compression.
- This is because each member is only loaded at the 2 nodes, which each has a resultant force → the member is acted on by only 2 forces → the resultants must be colinear.

Statically determinate frames

- If the int. components of a structure are under-connected will be a mechanism and may collapse w/o the application of any load; if they are over-connected, this may be a sign of inefficiency and the analysis is difficult as the structure is statically-indeterminate. (the int. forces depend on the interaction of components which depend on their material prop.).
- Statically determinate structures are the easiest to analyse — the int. forces can be found directly by the method of joints/sections from eqm. w/ the ext. forces.
- A characteristic of statically determinate trusses is that the bars can be assembled w/o forcing, even if they may not be all exactly of the correct length.
- It follows that changes of length of the members (eg due to change in temp/humidity) do not produce additional stresses in the structure.
- A formal approach to characterising a pin-jointed truss structure is provided by counting the no. of joints (j), bars (b), restraints (r) and dimensions (D) in the problem.

$Dj > b+r$: mechanism, $Dj = b+r$: statically determinate, $Dj < b+r$: statically indeterminate.

* Do not blindly rely on the rule. $Dj = b+r$ can mean mechanism + statically indeterminate.

* In general, a statically determinate truss is made up of triangles and has 3 dofs of freedom restrained at the joints (usually +)
2 + 1

Method of joints

- The method of joints req. that we create a FBD for each pin joint of the truss. It provides a systematic way of calculating all the bar forces in a structure, by considering the conditions of force eqm. at each joint.
- By defn, the forces meet at the joint → moment eqm. satisfied ; we req. the vector sum of all forces acting on the joint to be 0.
- We can obtain 2 scalar eqns of eqm (from N1L) applying at each joint → we can only find 2- unknown bar forces / reactions per joint. [start from the supports].

* For trusses whose members align simply w/ a co-ord system → algebraic methods
  For trusses w/ more complex geometries → graphical methods.
* Method of joints is the same as method of sections where the section is the joint.
  ↳ both methods req. us to first find all the ext. forces (reaction forces at the supports).

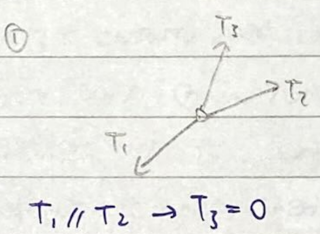
Method of sections

- The method of sections can help us find the axial tension in 1 specific bar → better than method of joints when the members are far from the supports.
- To find a useful section:
  ↳ Cut through the member of interest. Then include any known ext. forces (applied or reaction forces). Always choose a section w/ less ext. forces → less work.
  ↳ The section will be in planar eqm → we can obtain 3 scalar eqns of eqm (from N1L and moments) → we can only find 3- unknown bar forces / reactions per section.
- To use the section to find axial forces :
  ↳ Use N1L to find diagonal bars.
  ↳ Take moments at the intersection pt. of lines of action to find vertical / horizontal bars.

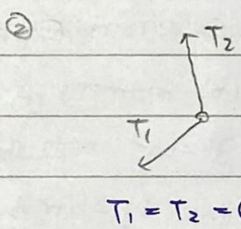
* The method of section often leads to an eqn. of moment eqm → algebraic methods
  If we need to use N1L and the geometry is inconvenient → graphical methods.

Some simplifications in analysing planar pin-jointed trusses.

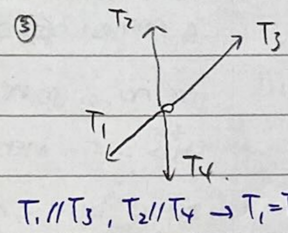
- The calculation of bar forces can be simplified by noting the special cases.

① 

$T_1 \parallel T_2 \rightarrow T_3 = 0$

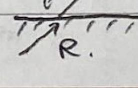
② 

$T_1 = T_2 = 0$

③ 

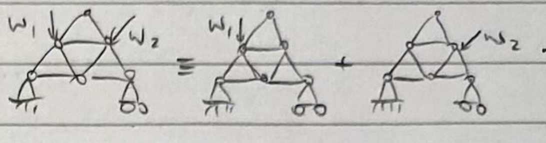
$T_1 \parallel T_3$, $T_2 \parallel T_4 \rightarrow T_1 = T_3$, $T_2 = T_4$.

- At a support, if there is only 1 bar, the reaction force must be in the dir. of the bar since there are only 2 forces acting at the node.



Superposition

- When there are many forces acting on a truss, W_1, W_2, W_3 etc. and the axial force in a particular member due to W_1 alone is $\lambda_1 W_1$, then the force T in the member is given by principle of superposition: $T = \lambda_1 W_1 + \lambda_2 W_2 + \lambda_3 W_3 + \ldots$
  where the coefficients λ_i depend on the truss geometry.
- The principle of superposition is not restricted to bar forces in trusses; it applies to all linear systems.



- The behaviour of a truss is linear if:
  - ↳ The material of the truss remains in the linear elastic range.
  - ↳ The geometry does not change much, i.e. the distortion of the structure caused by the loads are small and hence the eqn. written in the undeformed configuration are also valid after the structure has deformed.

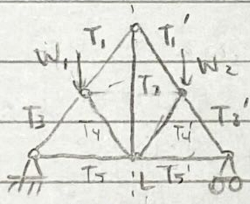
(These conditions are true for most structures made of metal).

Symmetry

- Consider a truss that has symmetry about a line:
  - ↳ If the loads are symmetric → the bar tensions are symmetric (about that line)
  - ↳ If the loads are anti-symmetric → the bar tensions are antisymmetric (about that line).

* Note the supports are not symmetric, but their effect is to apply symmetric reactions, so both applied and reaction ext. forces are symmetric.

(This works as there is no net horizontal applied force by defn of (anti-)symmetry abt the line).

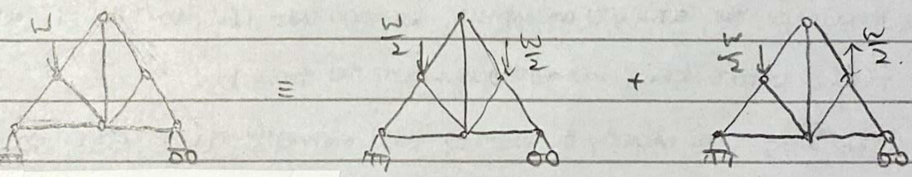


Symmetric load ($W_1 = W_2$): $T_i = T_i'$

Anti-symmetric load ($W_1 = -W_2$): $T_i = -T_i'$

* symmetric about mirror line L.

- For a symmetric structure, it is always possible to represent any set of applied loads by superposing a symmetric and an antisymmetric set of loads.

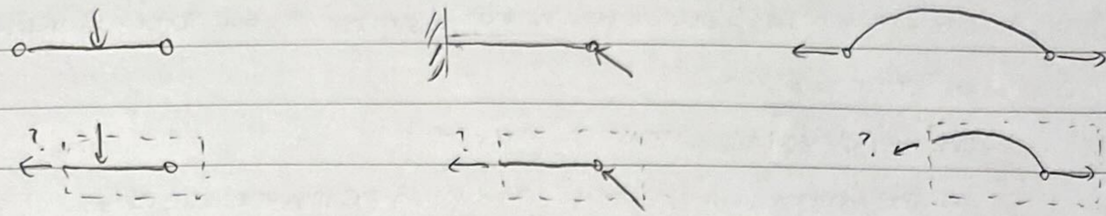


- In general for a single load W, it can be replaced a symmetric + antisymmetric loading of $\frac{W}{2}$. Use the principle of superposition for multiple loads.

Shear force and bending moments

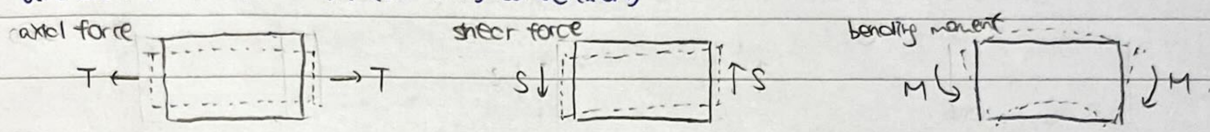
Shear forces and bending moments.

- Considering beams w/ transverse loading (load not applied at node or an unpinned one joint encastré) and non-straight members, we cannot reach eqm. w/ a single axial force at a cut (when using method of sections). → we need shear force + bending moments



Sign conventions.

- Deformations arising from +ve axial forces (T), shear forces (S) and bending moments (M) are defined as follows: (for this convention)

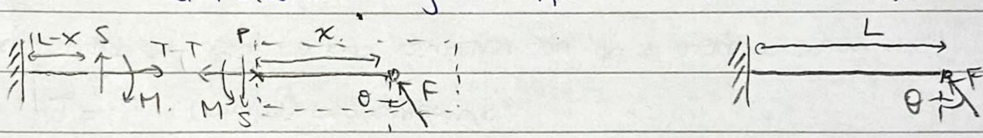


$T>0 \rightarrow$ stretch. $S>0 \rightarrow$ anticlockwise $M>0 \rightarrow$ hogging.

* Using this convention, shear forces (S) and bending moments (M) have opposite signs.

Beams with transverse loadings.

- Consider a beam w/ transverse loading — an unpinned one joint encastré.



* The dir. of T, S and M are chosen so they are +ve (using the convention above).

N1L (↔): $T + F\sin\theta = 0 \rightarrow T = -F\sin\theta$

(↕): $S = F\cos\theta \rightarrow S = F\cos\theta$

M^P: $M + F\cos\theta\, x = 0 \rightarrow M = -F\cos\theta\, x$.

* In general, for straight members, we can use N1L to find T, then use N1L again to find S before taking moments anywhere to find M.

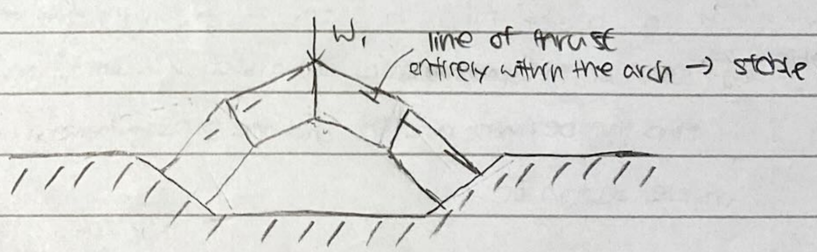
(However, it is usually faster to take moments about either end, whichever end has more (unknown forces))

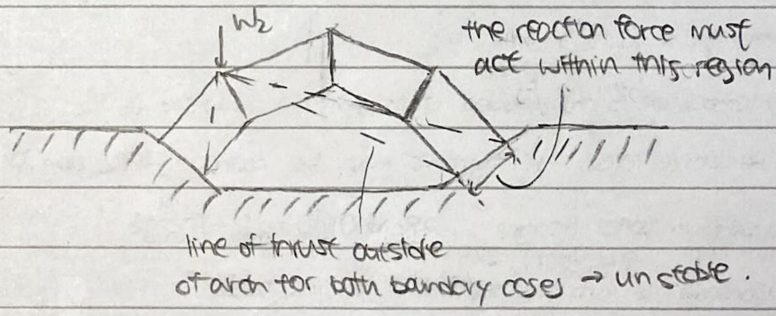
Arches

- The structural benefit of arches is that their shape can be chosen s.t. the internal forces in the arch are mainly in compression. → useful for:
  - ↳ Stone masonry, joints btwn stone blocks are strong in compression but weak in tension
  - ↳ Metal structures, reduced bending moment allows the use of lighter members.
- Arches rely upon their supports, called abutments to resist lateral movement. If abutment movement is significant, the arch may deform or even collapse.

Segmented arches

- Segmented arches, those made of blocks, (stone arches in bridges and churches) generally cannot carry tensile loads at joints. We make the following assumptions:
  - ↳ The blocks are infinitely strong (no compression failure).
  - ↳ The block interfaces are unable to carry bending moments (no tensile capacity, i.e. the block cannot be in tension → the arch just opens up).
  - ↳ Block interfaces have infinite friction (no lateral sliding along the interfaces).
- Based on these assumptions, global eqm. rather than material failure determines the arch capacity → if the line of thrust can remain completely within the structure, the arch is in eqm. and stable. [use geometric thickness to resist loading].
- If we consider a stone arch subjected to a large pt. load, we can ignore the selfweight of the stones that make up the arch. In this case, the line of thrust comprises of 2 straight lines that connect the pt. load to meet the abutments either side of the arch.
- If we can draw a line of thrust from each abutment to the pt. load that remains entirely within the arch, the arch can support the load.





Continuous arches

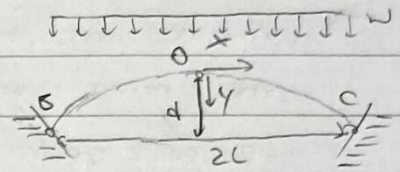
- Continuous (twin) arches (made of steel or reinforced concrete) use bending moment capacity to resist loading.

N1L + moments or intersecting lines of action

- In general, we work out the reaction forces at the abutment, then we use the method of sections to cut through the pt. of interest before taking moments abt. that pt. to find the bending moment.

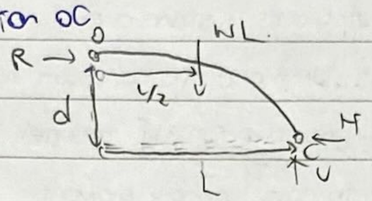
* For non-straight members, we must take moments abt. the cut pt. to find the bending moment.

- eg: Consider an arch w/ uniform load w per unit horizontal length, and has the parabolic shape $y = d\frac{x^2}{L^2}$. Find the abutment reactions and the bending moments in the arch.



Consider section OC

FBD :



N1L ↕ : $V = wL$

V can be different for different sections of the arch as the ext. load changes.

$\circlearrowleft M_O$: $wL \cdot \frac{L}{2} + Hd = VL$

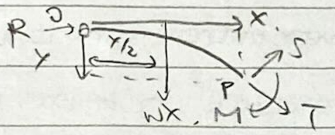
$H = \frac{wL^2}{2d}$.

H is the same no matter what section of the arch we take as all the loads applied are vertical.

Consider a general section OP.

Choose a section that starts from the origin (so we can use the eqn. easily).

FBD.



$\circlearrowleft M_P$: $wx \cdot \frac{x}{2} = M + R \cdot y$

$\frac{1}{2}wx^2 = M + \frac{wL^2}{2d} \cdot d\frac{x^2}{L^2}$

$M = 0$

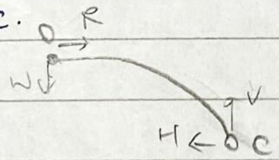
→ special result: the bending moment throughout a parabolic arch under this loading condition is 0.

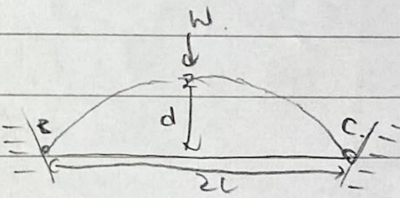
replacing a distributed force w/ a single pt. force does not help us analyse the internal forces. → different bending moments.

- eg: Consider the same arch as above but w/ a central pt. load W.

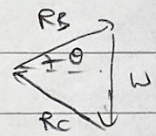
Find the abutment reactions and the bending moments in the arch

Consider section OC.



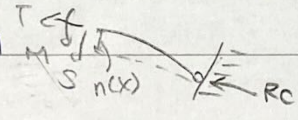


Member OC is only loaded at the pins → subject to 2 resultant forces → resultants must be colinear → R_C along OC.



Using a force triangle, $W = 2R_C\sin\theta = 2R_C \cdot \frac{d}{\sqrt{L^2+d^2}}$

Consider an arbitrary section



$M(x) = R_C \cdot n(x)$.

(non-zero bending moment).

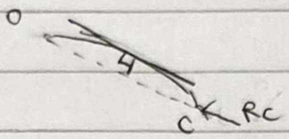
eg: Consider the same arch as above but w/ a non-central pt. load W.

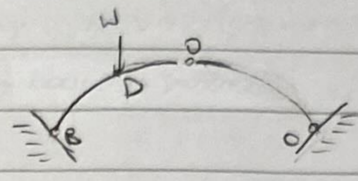
Explain why the max. bending moment in OC is where the tangent is // OC.

Explain why the max. bending moment in OB is at D.

Consider member OC.

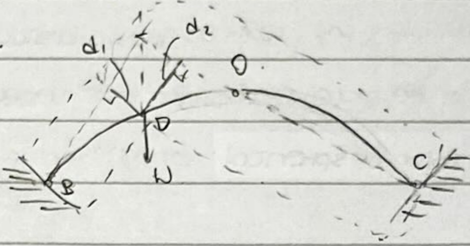
2 resultant forces → colinear → R_C in the dir. OC.





Bending moment is given by R_C times the ⊥ distance from the line of action of R_C (along OC) and the arch → max. when the tangent is // OC. → max. M.

Consider the entire arch.



By considering the lines of action of the load and R_C, we can find the dir. of R_B.

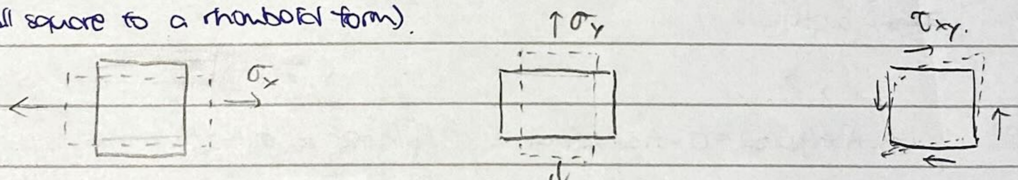
Consider a section that cuts through D → we can ignore W. (as we take moments abt. the cut)

The ⊥ distance from the lines of action of R_B, R_C to the arch is max at D. → max. M.

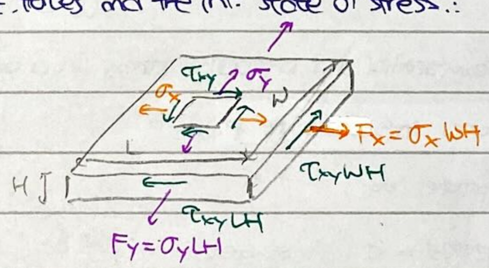
Stress

2D plane stress in thin walled shells.

- We can use stress (w/ sufficient components) to represent the influence of all locally deformed inter-atomic bonds.
- In 2D, the state of stress has 3 components — 2 direct stresses σ_x, σ_y (tend to stretch/compress the material along an axis) and the shear stress τ_{xy} (tend to shear the material from a small square to a rhomboid form).



- relationship between ext. forces and the int. state of stress:

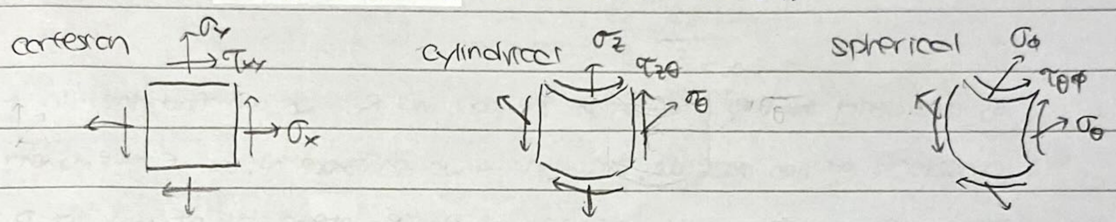


Stress has units of force/area so the int. forces (stress) can be related to ext. loads by integrating (stress) wrt area over the boundary influenced by the load.

- The units of stress are the same as pressure, but pressure has no dir. — at any pt. in a fluid, it acts identically in all dir. In contrast, the components of the state of stress have specific dir. (indicated by their subscripts).
- The state of stress must always be stated wrt a set of axes, as a given component of stress under a constant ext. load varies as the axes rotate.

Thin walled shells w/ uniform stress.

- For thin-walled pressure vessels, we can assume the stress distribution is uniform through the thickness.
- We will only consider vessels w/ axi-symmetric shape (spheres or cylinders), for which the stress is uniform along any circumferential or longitudinal section.
- The components of the state of stress for these vessels do not fit the Cartesian coords system → use cylindrical-polar or spherical-polar notation.

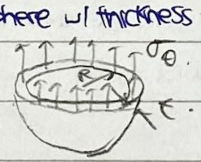


- We can find the stress in the pressure vessel using the method of sections:
- After making a plane cut through the vessel, asserting eqm. across this plane, the ext. force (pressure × proj. area of the vessel's interior) must equal to the int. force (stress in the wall normal to the plane × area of the cut surface of the vessel).

$$\boxed{\underbrace{p \times A_{projected}}_{\text{external force}} = \underbrace{\sigma \times A_{cut\ material}}_{\text{internal force}}}$$

- e.g.: Find the stress distribution in a sphere w/ thickness t, radius R and int. pressure p.

Make a cut.

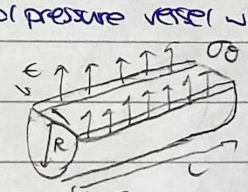


$p \times A_{projected} = \sigma \times A_{cut\ material}$: $p \cdot \cancel{\pi} R^{\cancel{2}} = \sigma_\theta \cdot 2\cancel{\pi}\cancel{R}t \rightarrow \boxed{\sigma_\theta = \frac{pR}{2t}}$

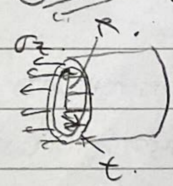
- e.g.: Find the circumferential and longitudinal stress in a cylindrical pressure vessel w/ thickness t, radius R, length L and int. pressure p

Make an appropriate cut.

(i)



(ii)



$p \times A_{projected} = \sigma \times A_{cut\ material}$

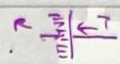
(i). $p \cdot 2R\cancel{L} = \sigma_\theta \cdot 2t\cancel{L} \rightarrow \boxed{\sigma_\theta = \frac{pR}{t}}$ circumferential stress

(ii) $p \cdot \cancel{\pi} R^{\cancel{2}} = \sigma_z \cdot 2\cancel{\pi}\cancel{R}t \rightarrow \boxed{\sigma_z = \frac{pR}{2t}}$ longitudinal stress.

Deflections

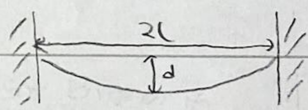
Cables and bar extensions

Cables.

- Cables, strings, ropes and chains are able to carry uniaxial tension but they are unable to carry compression or bending as they have negligible bending stiffness.
- It is a good assumption that the tension T at any pt. acts in the dir. of the tangent to the cable at the pt. (They look like encastré supports). 
- In many applications, it can be assumed that cable length will remain approx constant throughout, and hence the cable can be treated as inextensional.
- The deformed geometry of inextensional cables can be predicted purely by considering the eqm. of ext. and int. forces.
- Typically, for a cable of known length and subject to specified loads and boundary conditions (fixed ends) we try to find:
  - ↳ The support reactions
  - ↳ The shape adopted by the cable.
  - ↳ the tension distribution throughout the cable.

Cables subjected to concentrated loads.

- If the cable is subjected only to concentrated loads, then it will adopt a straight line between loading pts.
- The overall shape of the cable is thus a broken line, consisting of straight segments each of which carry constant tension (the tension in each segment can be different if the load is fixed).
- If the ext. loads are known, we can work out the reaction forces at the ends of the cable and the cable tension using the method of joints / sections.
- It is assumed the ext. supports provide no moment → the cable can leave the support at any angle (although they are drawn like encastré supports).



Cables subjected to distributed loads.

- Cables subject to distributed loads adopt the shape of a smooth curve
- For a cable with constant weight w per unit length hanging between level supports,
  - ↳ If the cable is shallow ($d \ll L$) → assume self weight is w per unit horizontal length of cable → parabolic shape
  - ↳ If the cable is not shallow (d not $\ll L$) → cannot assume self weight is w per unit horizontal length of cable → catenary / cosh curve shape.

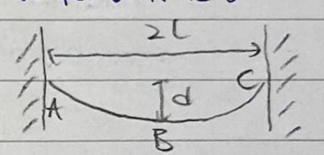
- The standard procedure for cable questions:
  - ↳ Generally, we can find the vertical comp. of the 2 reaction forces by eqm only (N2L + M)
  - ↳ To find the horizontal comp. of the reaction forces, we need 1 additional piece of information about the shape of the cable
  - ↳ After finding the full reaction force, we can take a FBD diagram of any location in the cable to find the deflected shape, angle and its tension.
- An important result is that a cable subjected to a uniform vertical load distribution has a parabolic shape, which is independent of the magnitude of the loading.

$$\boxed{y(x) = \frac{w}{2H}x^2 = d\left(\frac{x}{L}\right)^2} \quad [y = kx^2]$$

- An approximate formula for finding the length of a parabolic cable

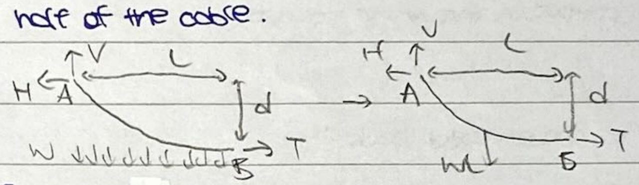
$$\boxed{S = 2L\left(1 + \frac{2d^2}{3L^2}\right)}$$

- e.g.: Find the shape of a cable under uniform vertical load w per unit length.



Consider half of the cable.

FBD:



By N2L ↕: $V = wL$

$\overset{\curvearrowright}{M_B}$:

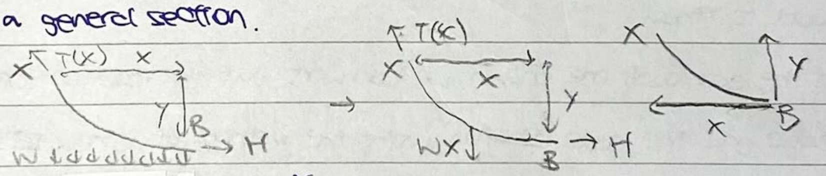
$$wL \cdot \frac{L}{2} + Hd = VL$$

$$Hd = wL \cdot L - \frac{1}{2}wL^2$$

$$H = \frac{wL^2}{2d}.$$

Consider a general section.

FBD:



$\overset{\curvearrowright}{M_X}$:

$$Hy = wx \cdot \frac{x}{2}$$

$$\frac{\cancel{w}L^2}{2d} y = \cancel{w}\frac{x^2}{2}$$

$$y = d\left(\frac{x}{L}\right)^2$$

* To find the angle and tension,

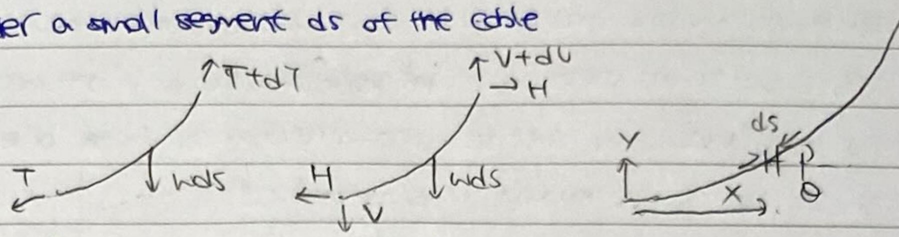
use eqm. (N2L + M) to find T_x and T_y

As the tension must act in the tangential direction of the cable, $\tan\theta = \frac{T_y}{T_x}$

-e.g: Find the shape of a cable with constant weight w per unit length

Consider a small segment ds of the cable

FBD:



By N2L ↕: $$V + dV = V + w\,ds$$
$$\frac{dV}{ds} = w.$$

At the general pt. x, $$\tan\theta = \frac{T\sin\theta}{T\cos\theta} = \frac{V}{H} = \frac{dy}{dx}.$$

Differentiating wrt x $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{V}{H}\right)$$ * $\frac{d^2y}{dx^2}$ term is useful!
$$\frac{d^2y}{dx^2} = \frac{1}{H}\frac{dV}{dx}$$ as H is constant

For a small segment ds, $$ds^2 = dx^2 + dy^2$$
$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \rightarrow \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$
$$\therefore \frac{d^2y}{dx^2} = \frac{1}{H}\frac{dV}{dx}$$
$$= \frac{1}{H}\frac{dV}{ds}\cdot\frac{ds}{dx}$$
$$= \frac{w}{H}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Let $z = \frac{dy}{dx}$, then $\frac{dz}{dx} = \frac{d^2y}{dx^2}$: $$\frac{dz}{dx} = \frac{w}{H}\sqrt{1+z^2}$$
$$\int \frac{1}{\sqrt{1+z^2}}\,dz = \frac{w}{H}\int dx$$
$$\operatorname{arcsinh} z = \frac{w}{H}x + C$$

At $x=0$, $\frac{dy}{dx} = z = 0$ $$\operatorname{arcsinh} 0 = \frac{w}{H}(0) + C \rightarrow C = 0.$$
$$\therefore \operatorname{arcsinh} z = \frac{w}{H}x$$
$$z = \sinh\left(\frac{w}{H}x\right)$$

substituting back $z = \frac{dy}{dx}$, $$\frac{dy}{dx} = \sinh\left(\frac{w}{H}x\right)$$
$$y = \int \sinh\left(\frac{w}{H}x\right)dx$$
$$= \frac{H}{w}\cosh\left(\frac{w}{H}x\right) + C_1$$

At $x=0$, $y=0$ $$0 = \frac{H}{w}\cosh\left(\frac{w}{H}\cdot 0\right) + C_1 \rightarrow C_1 = -1.$$
$$\therefore \boxed{y = \frac{H}{w}\cosh\left(\frac{w}{H}x\right) - 1}$$

* Generally, to find the shape of the cable via integration,

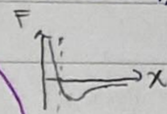
Draw a FBD at a general segment w/ x-coord x, in terms of V and H.

Use $\frac{dy}{dx} = \frac{V}{H}$ and differentiate wrt x

Combine the 2 expressions and solve to find y(x).

Strains, Hooke's law and bar extension

- For typical metals, the extension of the wire e increases linearly w/ weight. Fundamentally, this is due to the stretching of the interatomic bonds in the metal.



- For many metals, the limit to elastic stretching is of the order of 0.3%.

interatomic force roughly prop. to atomic separation near the equilibrium spacing.

- In 1D, the Young's modulus E is defined as

$$\boxed{E = \frac{\sigma}{\varepsilon} = \frac{F/A}{\Delta L / L}}$$

where σ is the tensile stress and ε is the tensile strain.

- In the context of a truss member carrying an axial force T ($T>0$ indicates tension),

$$\varepsilon = \frac{TL}{Ae} \rightarrow \boxed{e = \frac{TL}{AE}}$$

* only valid for stiff materials (metal/concrete), up till the elastic limit and constant stress (good approximation for slender members, i.e. $A \ll L$)

Thermal expansion

- Experiments have shown that the (1D) thermal strain ε_T in a (long straight slender) member due to a uniform temp. change $\Delta\theta$ can be predicted by.

$$\boxed{\varepsilon_T = \alpha \Delta\theta}$$

where α is the coefficient of thermal expansion.

- For a member subject to an axial force T and temp. change $\Delta\theta$, the total strain ε can be found by superposition as

$$\boxed{\varepsilon = \frac{T}{AE} + \alpha\Delta\theta}$$

- Therefore the elongation is given by

$$\boxed{e = \varepsilon L_0 = \frac{TL_0}{AE} + \alpha L_0 \Delta\theta}$$

* For structurally indeterminate (over-constrained) structures, uneven thermal expansion cannot be compensated by nodal movement → structure has int. loads w/o ext. loads.

Displacement diagrams and real/virtual work method.

Compatibility

- The individual elements of the whole structure can only deform in a way that is compatible w/ the overall deformation of the structure.
- To solve for the overall structure after all the members in the structure change length, we can use:

↳ displacement diagrams [graphical]

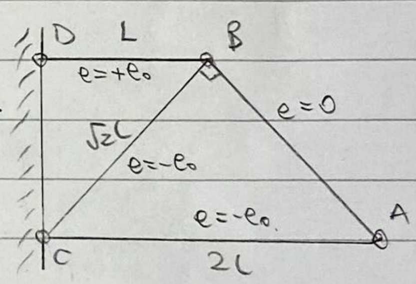
↳ real work/virtual work method [algebraic].

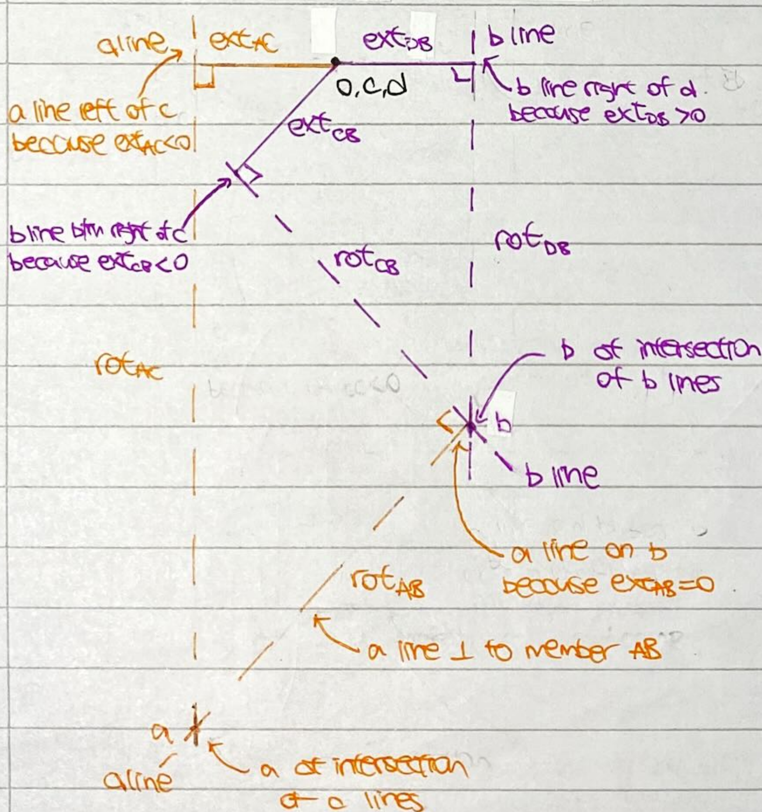
Displacement diagrams

- When using displacement diagrams, we assume the extensions are small compared to the member lengths → replace circular arcs w/ tangents.
- In a displacement diagram, we draw the extension // to the bar and the displacements ⊥ to the bar. (displacements ⊥ to bar for e<<L).
- In a displacement diagram, the pt. O is the origin/pole, where all the displacements of all other pts on the diagram can be measured rel. to.
- In general, to find the displacement of a node, it must be connected to 2 other nodes w/ known displacements (eg fixed nodes, from symmetry).
- We first draw the extension from the known node (same "order" of nodes for +ve e, reverse "order" of nodes for -ve e) then draw a dotted rot. line ⊥ to it.
- Repeat for the other known node and the intersection of the 2 dotted rot. lines give the position of the node on the displacement diagram. The displacement of the node is given by the position of the node rel. to O on the displacement diagram.

* Displacement diagrams are to scale → more accurate if the diagram fills an A4 sheet. Draw a rough sketch to get a feel for the sol^n → choose a good place for O. If the lines are drawn to ±1mm in length, ±1° in angles → roughly 5% error.

- e.g.: Draw a displacement diagram of the structure given the extensions are $e_0, -e_0, -e_0, 0$ for BD, AC, BC, AB.



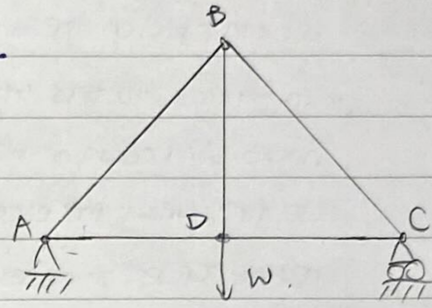


Using (small) rigid body rotation of the whole structure

- If we made a wrong assumption (eg a slanted member assumed to be horizontal), as observed when the predicted displacements do not make sense, we can fix this by applying a small rigid body rotation to the whole structure.

- e.g: Draw a displacement diagram of the structure given

Bars	Length	T	$e\left(\times \frac{WL}{AE}\right)$
BD	L	$+W$	1
AB	$\sqrt{2}L$	$-\frac{W}{\sqrt{2}}$	-1
BC	$\sqrt{2}L$	$-\frac{W}{\sqrt{2}}$	-1
AD	L	$+\frac{W}{2}$	$\frac{1}{2}$
DC	L	$+\frac{W}{2}$	$\frac{1}{2}$

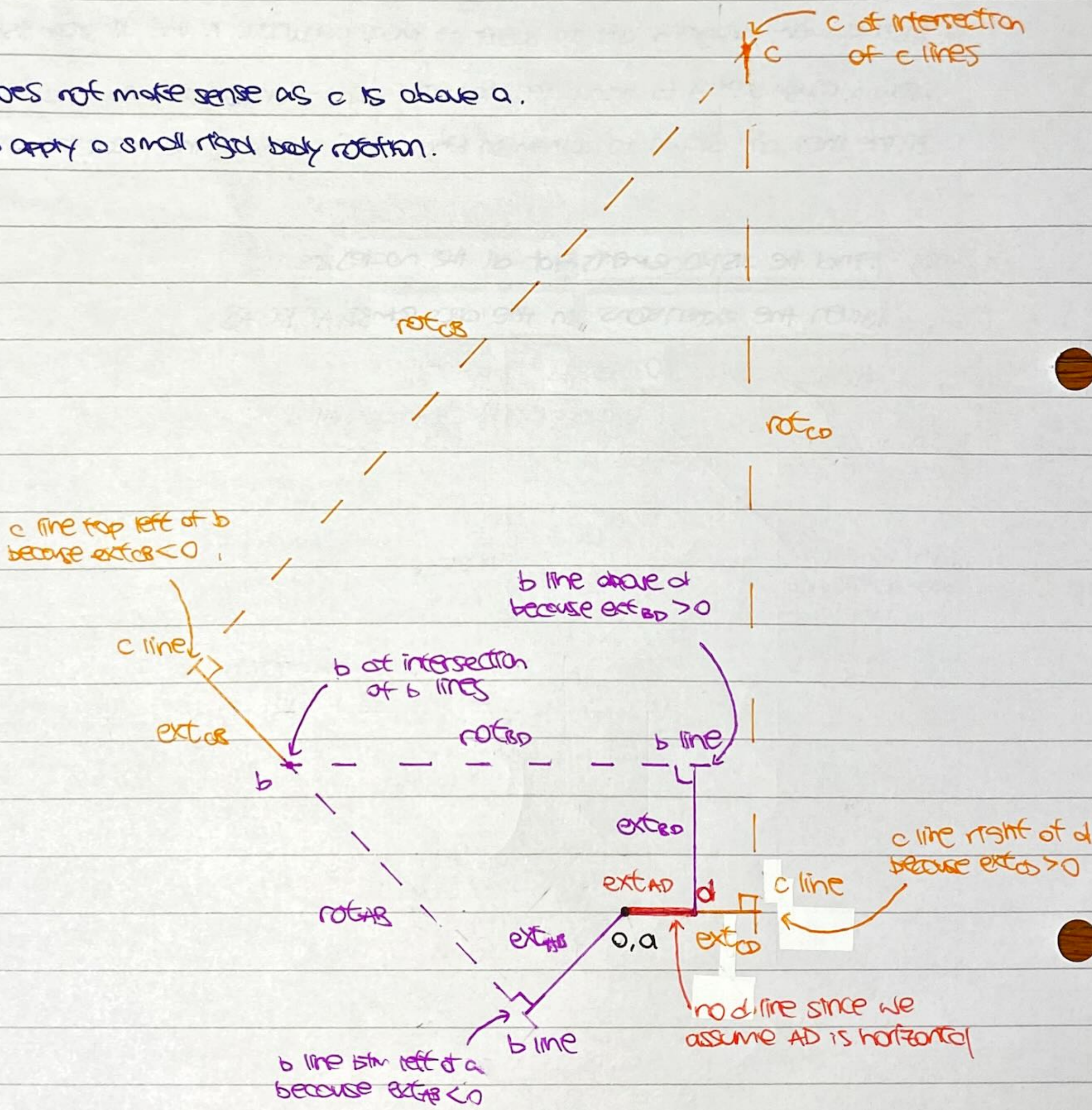


and all the bars have cross-sectional area A and Young's modulus E.

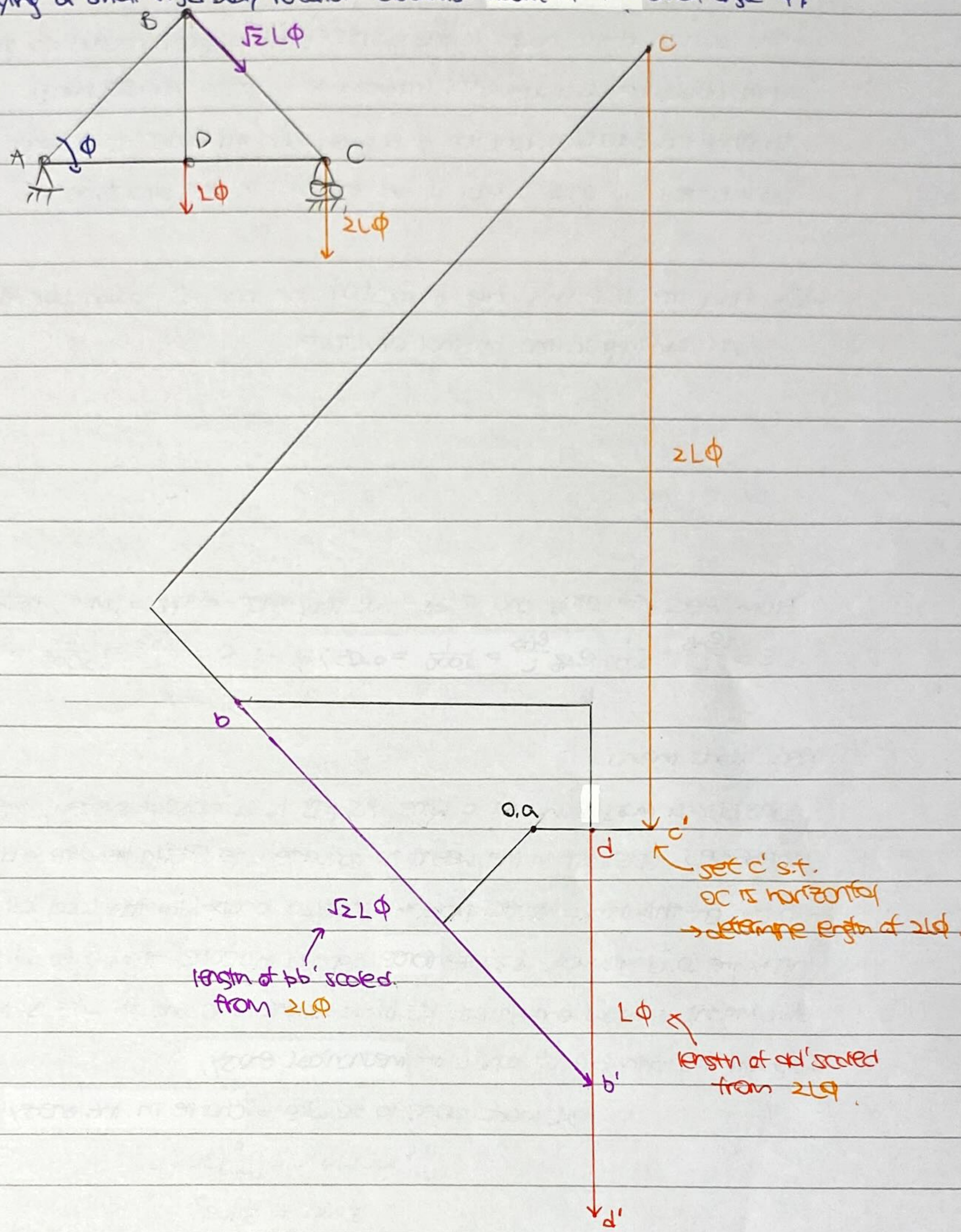
Assume member AD remains horizontal (wrong assumption!).

Does not make sense as c is above a.

→ apply a small rigid body rotation.



Applying a small rigid-body rotation clockwise about A by a small angle ϕ,



For a joint X, the displacement when rotating about A by a small angle ϕ, has a magnitude $|AX|\phi$ in the direction $\perp$ to AX.

* In this question, we should have used symmetry and deduced that b must be above d and draw the displacement diagram starting from d.

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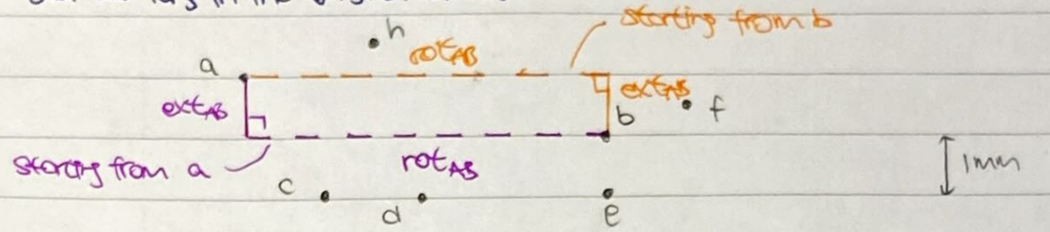
Interpreting displacement diagrams.

- The location of the nodes in the displacement diagram relative to the origin shows their absolute displacement. (remember to state the direction!)
- To find the extension/rotation of a given bar, the nodes at its ends must be connected by lines // and ⊥ to the bar in the structure.

- eg: Find the strain and angle of rotation for bar AB, given bar AB was vertical and 2m long in the original structure.



From measuring ext$_{AB}$ and rot$_{AB}$, we find that ext$_{AB}$ = 1mm, rot$_{AB}$ = 11.6mm

$\varepsilon = \frac{ext_{AB}}{L} = \frac{1}{2000} = 0.05\%$ // $L\phi = rot_{AB} \rightarrow \phi = \frac{rot_{AB}}{L} = \frac{11.6}{2000} = 5.8\times10^{-3}$ rad.

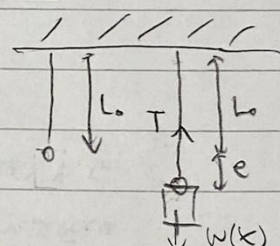
Real work method

- Consider a mass hung on a wire. As this is a closed system, the total energy is conserved - GPE lost in the weight is converted to EPE in the wire and KE of the weight.
- To convert this into a statics problem, instead of applying the load all at once (SHM), we apply the load gradually s.t. the load applied increases from 0 to W so $W(x) = \frac{x}{e}W$. (assume this is slow enough s.t. KE of the weight is 0 and the wire is in eqm.)
- Applying the principle of consv. of energy,

Ext work done to the wire = Int. energy stored

$$\int_0^e W(x)dx = \int_0^e kx\,dx$$

$$\tfrac{1}{2}We = \tfrac{1}{2}ke^2$$



When the load has been fully applied, T=ke, therefore

$$\tfrac{1}{2}We = \tfrac{1}{2}Te.$$

- Generalising this for a pin-jointed plane truss, to include a no. of externally applied forces F_i, each of which is applied to some node that displaces by d_i. Meanwhile each straight member in the truss experiences an axial force T_j leading to extension e_j.

Σext. work applied to truss = Σ changes in int. stored energy

$$\tfrac{1}{2}\sum_i \underline{F}_i \cdot \underline{d}_i = \tfrac{1}{2}\sum_j T_j e_j$$

ext. forces F_i in eqm. with int. bar tensions T_j — $\boxed{\sum_i \underline{F}_i \cdot \underline{d}_i = \sum_j T_j e_j}$ — nodal deflections d_i compatible with bar extensions e_j

* Reaction forces act in directions where displacement is constrained → WD = 0.
T_j and e_j always have the same sign.

Derivation of virtual work method for pin-jointed frames.

- Now if we remove the original set of ext. forces $\underline{F}_i$ and apply a different set of ext. forces $\underline{F}_i^*$. Using the eqn. for real work, we now have

$$\sum_i \underline{F}_i \cdot \underline{d}_i = \sum_j T_j e_j \quad \text{and} \quad \sum_i \underline{F}_i^* \cdot \underline{d}_i^* = \sum_j T_j^* e_j^*$$

- Then we return to the original set of ext. forces $\underline{F}_i$, however, this time we will add the new set of ext. forces $\underline{F}_i^*$ while the original set of ext. forces $\underline{F}_i$ are in place. (For $\underline{F}_i^* \ll \underline{F}_i$, deflections are small → system is locally linear → principle of superposition applies).
- When adding the new set of ext. forces $\underline{F}_i^*$, both sets of ext. forces do additional ext. work and therefore additional int. energy is stored.

$$\sum_i (\underline{F}_i + \underline{F}_i^*) \cdot \underline{d}_i^* = \sum_j (T_j + T_j^*) e_j^*$$

$$\sum_i \underline{F}_i \cdot \underline{d}_i^* + \cancel{\sum_i \underline{F}_i^* \cdot \underline{d}_i^*} = \sum_j T_j e_j^* + \cancel{\sum T_j^* e_j^*}$$

ext. forces $\underline{F}_i$ in eqm. with int. bar tensions T_j
$$\boxed{\sum_i \underline{F}_i \cdot \underline{d}_i^* = \sum_j T_j e_j^*}$$
nodal deflections $\underline{d}_i^*$ compatible with bar extensions e_j^*

* The additional ext. work is not $\Sigma(\underline{F}_i + \underline{F}_i^*)(\underline{d}_i + \underline{d}_i^*) - \Sigma \underline{F}_i \cdot \underline{d}_i$ because we are not applying both sets of ext. forces all at an instant → $\underline{F}_i^*$ only does work during displacement $\underline{d}_i^*$ (but not $\underline{d}_i$).

- If we applied the new set of ext. forces $\underline{F}_i^*$ first, then superimpose the original set of ext. forces $\underline{F}_i$, using the same logic, we would get

ext. forces $\underline{F}_i^*$ in eqm. with int. bar tensions T_j^*
$$\boxed{\sum_i \underline{F}_i^* \cdot \underline{d}_i = \sum T_j^* e_j}$$
nodal deflections $\underline{d}_i$ compatible with bar extensions e_j

Using virtual work to find displacements.

- If we want to find nodal displacements, then these must be real (along w/ bar extensions), so the forces and bar tensions must be virtual

$$\text{Virtual} \quad \sum_i \underline{F}_i^* \cdot \underline{d}_i = \sum T_j^* e_j \quad \text{Real}$$

- To find a nodal displacement, we add a unit virtual force $\underline{F}^* = 1$ at the node of interest in the direction of interest → LHS becomes $\sum_i \underline{F}_i^* \cdot \underline{d}_i = \underline{F}^* \cdot \underline{d} = d$
- From the virtual applied force $\underline{F}^*$, we can find the virtual reaction forces $\underline{R}_i^*$ and thus the virtual bar tensions T_j^*.
- From the real applied forces $\underline{F}_i$, we can find the real reaction forces $\underline{R}_i$ and thus the real bar tensions T_j, from which we can find the real bar extensions e_j.
- Summing all the products of virtual bar tensions T_j^* and real bar extensions e_j,

$$d = \sum_j T_j^* e_j$$

* Don't bother finding T^* for bars w/ 0 extension ($e=0$).

Using virtual work to find forces or bar tensions.

- If we want to find forces or bar tensions, then these must be real, so the nodal displacements and bar extensions must be virtual

$$\text{Real} \quad \sum_i \underline{F}_i \cdot \underline{d}_i^* = \sum_j T_j e_j^* \quad \text{Virtual}$$

- To find a force, we apply a virtual rigid-body rotation about a node (s.t. the forces acting on the other nodes are known (the req. one) by a small angle ϕ. → The virtual extensions are 0 ($e^* = 0$) → RHS becomes 0.
- The virtual displacement of node X, $\underline{d}_i^*$ when rotated about node A by a small angle ϕ has a magnitude $|AX|\phi$ and is in the direction $\perp$ to AX.
- Summing all the dot products of the virtual displacements $\underline{d}_i^*$ and the real forces $\underline{F}_i$ and equate this to 0 to solve for the req. force.

$$\sum_i \underline{F}_i \cdot \underline{d}_i^* = 0.$$

- To find a bar tension, we extend the member of interest by a unit virtual extension $e^* = 1$.
  → RHS becomes $\sum_j T_j e_j^* = Te^* = T$
- This virtual extension e^* results in a rigid-body rotation in part of the structure about a particular node by a small angle ϕ. Using geometry, we can find the virtual displacements $\underline{d}_i^*$ of the nodes that has a real applied force $\underline{F}_i$.
- Summing all the dot products of the virtual displacements $\underline{d}_i^*$ and the real forces $\underline{F}_i$ to get

$$T = \sum_i \underline{F}_i \cdot \underline{d}_i^*.$$

Real work and virtual work

- In general, for both real work and virtual work,

$$\text{set I} \quad \sum_i \underline{F}_i \cdot \underline{d}_i = \sum_j T_j e_j \quad \text{set II}$$

where both sets are real for real work

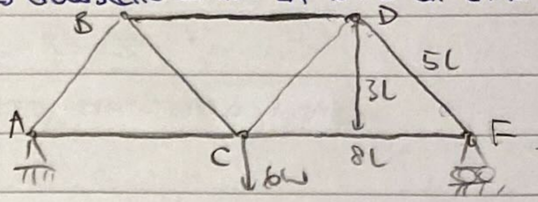
and one set is real & one set is virtual for virtual work

- We can now determine 4 features for a truss' state:

Ext. forces	$\underline{F}_i$	N2L + moments
		rigid-body rotation abt. a node by ϕ → $e^* = 0$ so $\sum \underline{F}_i \cdot \underline{d}_i^* = 0$
Bar tensions	T_j	$F_i \to R_i \to T_j$
		partial rigid-body rotation s.t. $e^* = 1$ and find $\underline{d}_i^*$ by geometry, $T = \sum \underline{F}_i \cdot \underline{d}_i^*$
Nodal displacement	$\underline{d}_i$	if load in same dir. as req. $\underline{d}_i$ → real work, $Fd = \sum T_j e_j$
		apply $\underline{F}_i^*$ at req. node in req. dir. find e_j ($\underline{F}_i \to \underline{R}_i \to T_j \to e_j$); find T_j^* ($\underline{F}_j^* \to \underline{R}_j^* \to T_j^*$) → $d = \sum T_j^* e_j$
Bar extensions	e_j	$\underline{F}_i \to \underline{R}_i \to T_j \to e_j$

-eg: Find the vertical displacement of node C and the horizontal displacement of node D.

All bars have Young's modulus E, bar BD has cross-sectional area 2A while all other bars have cross-sectional area A.



We want real displacements:

$\Sigma \underline{F}_i \cdot \underline{d}_i = \Sigma T_j e_j$ or $\Sigma \underline{F}_i^* \cdot \underline{d}_i = \Sigma T_j^* e_j$

In both cases, we need to find the real bar extensions.

Bar	T	Length	Area	$e\,(\times \frac{WL}{AE})$	$Te\,(\times \frac{W^2L}{AE})$
AB	-5W	5L	A	-25	125 ← always +ve.
DF	-5W	5L	A	-25	125
BC	+5W	5L	A	+25	125
CD	+5W	5L	A	+25	125
AC	+4W	8L	A	+32	128
CF	+4W	8L	A	+32	128
BD	-8W	8L	2A	-32	256
					Σ 1012

Vertical load + vertical displacement @ node C → real work.

$$\Sigma \underline{F}_i \cdot \underline{d}_i = \Sigma T_j e_j$$

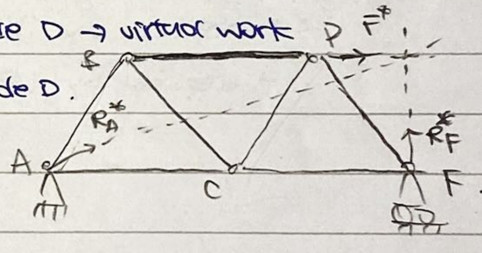
$$Fd = 1012 \frac{WL}{AE}$$

$$6W\,d = 1012 \frac{W^2L}{AE} \rightarrow d = \frac{170WL}{AE}.$$

No horizontal load + horizontal displacement @ node D → virtual work.

Apply a unit virtual horizontal force $F^* = 1$ @ node D.

After finding R_A^* and R_F^*, we find the virtual bar tensions.

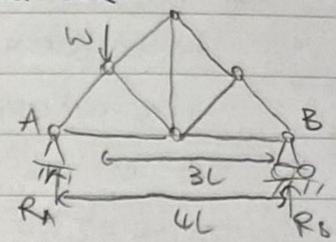


Bar	T^*	$e\,(\times \frac{WL}{AE})$	$T^*e\,(\times \frac{WL}{16AE})$
AB	$+\frac{5}{16}$	-25	-125
DF	$-\frac{5}{16}$	-25	+125
BC	$-\frac{5}{16}$	+25	-125
CD	$+\frac{5}{16}$	+25	+125
AC	$+\frac{12}{16}$	+32	+384
CF	$+\frac{4}{16}$	+32	+128
BD	$+\frac{8}{16}$	-32	-256
			Σ +256.

$$\Sigma \underline{F}_i^* \cdot \underline{d}_i = \Sigma T_j^* e_j$$

$$\underline{F}^* \cdot \underline{d} = 256 \frac{WL}{16AE} \rightarrow d = \frac{16WL}{AE}.$$

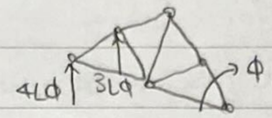
- e.g.: Find the reaction force of support A using virtual work.



We want real forces:

$\Sigma \underline{F}_i \cdot \underline{d}_i^* = \Sigma T_j e_j^*$.

Apply a virtual rigid-body rotation about B by ϕ.



As this is a pure rotation, $e_j^* = 0 \rightarrow \Sigma T_j e_j^* = 0$.

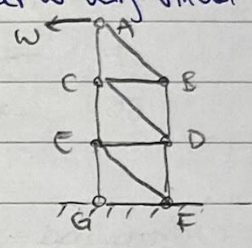
$$\Sigma \underline{F}_i \cdot \underline{d}_i = 0.$$

$$R_A \cdot 4\cancel{L}\cancel{\phi} - W \cdot 3\cancel{L}\cancel{\phi} = 0$$

- sign for $-W \cdot 3L\phi$ term because W acts downwards (opp. $3L\phi$).

$$R_A = \frac{3}{4} W$$

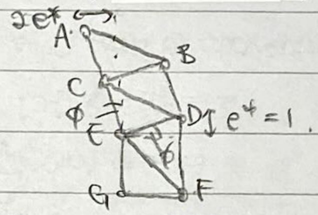
- e.g.: Find the axial force in member DF caused by the load W using virtual work.



We want real axial forces:

$\Sigma \underline{F}_i \cdot \underline{d}_i^* = \Sigma T_j e_j^*$

Apply a unit virtual extension e^* in member DF.



The upper part of the structure rotates by a small angle ϕ anticlockwise (about E).

$\rightarrow$ D displaces by $|DE|\phi = e^* = 1 \rightarrow \phi = \frac{1}{L}$.

$\therefore$ A displaces by $|AE|\phi = 2L \cdot \frac{1}{L} = 2$.

$$\Sigma \underline{F}_i \cdot \underline{d}_i^* = \Sigma T_j e_j^*$$

$$W \cdot 2 = T \cdot 1$$

$$T = 2W.$$

Displacement diagrams vs real/virtual work.

- Displacement diagrams are easier to check and usually quicker if the displacements of many nodes are req. However, displacement diagrams are complex in some cases.
- When seeking the displacement of a single joint, real/virtual work is often easier, given the geometry is simple.
- Regardless of method, look for opportunities to exploit symmetry and simplifications where bar forces or nodal displacements are zero or known.

★ Remember to state the direction of any displacements.

Structural design

Iterative design

- structural design involves synthesis and analysis:

Synthesis: evolving understanding of client's needs and accumulated evaluation of design concepts used to try and develop solutions.

Performance → Design

Performance ← Design

Analysis: use of tools to predict how structures respond to specific loads (i.e. find the deflections).

- Design is fundamentally iterative — partially specified set of req. → early ideas → greater clarity of requirements → reduce no. of design options.
- As the design process progresses, the designer can focus more on the details than concepts. (Use paper-calculations for concepts; computer calculations for details).

Structural optimization. — optimize cross-sectional area to minimise deflection

- Using the method of virtual work, we find the deflection of a given node is given by.

$$d = \sum T_j^* e_j = \sum\left(T_j^* \frac{T_j L_j}{A_j E}\right)$$

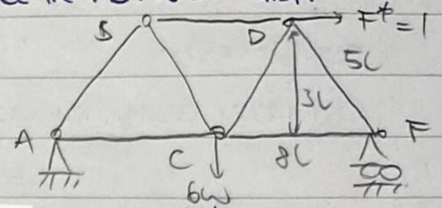
- If we want to reduce the deflection under the same loading by adding some mass (volume) to one of the bars, we can increase the cross-sectional area. (change length → expensive, assume same material → same Young's modulus and density).
- Expressing the deflection in terms of the volume of each bar and taking the derivative wrt volume for each bar $\tilde{j}$ in turn,

$$d = \sum T_j^* \frac{T_j L_j}{A_j E} = \sum T_j^* \frac{T_j L_j^2}{V_j E} \qquad [V_j = A_j L_j]$$

$$\boxed{\frac{\partial d}{\partial V_{\tilde{j}}} = -\frac{T_{\tilde{j}}^* T_{\tilde{j}} L_{\tilde{j}}^2}{V_{\tilde{j}}^2 E} = -\frac{T_{\tilde{j}}^* T_{\tilde{j}}}{A_{\tilde{j}}^2 E}} \qquad [V_{\tilde{j}} = A_{\tilde{j}} L_{\tilde{j}}]$$

- If $d > 0$, we want the most -ve $\frac{\partial d}{\partial V_{\tilde{j}}}$; If $d < 0$, we want the most +ve $\frac{\partial d}{\partial V_{\tilde{j}}}$, in order to have d closer to 0 (i.e. minimize $|d|$).
- After adding volume (increase cross-sectional area) to the member, the partial derivative for that member can be updated and we can add volume to the next priority member. We repeat this until all the partial derivatives are equal.

-e.g.: Which bar should we add mass to to reduce the horizontal displacement of node D most effectively.



Using virtual work method, we find that

$$d = \frac{16WL}{AE}$$

As $d > 0$, we want to choose the bar w/ the most -ve $\frac{\partial d}{\partial V_j}$ to minimize $|d|$.

Bar	Area	T	T^*	$\frac{\partial d}{\partial V_j}\left(\times \frac{W}{16AE}\right)$	
AB	A	$-5W$	$+\frac{5}{16}$	$+25$	
DF	A	$-5W$	$-\frac{5}{16}$	-25	
BC	A	$+5W$	$-\frac{5}{16}$	$+25$	
CD	A	$+5W$	$+\frac{5}{16}$	-25	
AC	A	$+4W$	$+\frac{12}{16}$	$\boxed{-48}$	most -ve.
CF	A	$+4W$	$+\frac{4}{16}$	-16	
BD	2A	$-8W$	$+\frac{8}{16}$	$+16$.	

→ Adding mass to bar AC will be the most effective in reducing horizontal displacement of D.

Check the analysis by doubling the cross-sectional area of bar AC:

Bar	Area	$e\left(\times \frac{WL}{AE}\right)$	T^*	$T^* e\left(\times \frac{WL}{16AE}\right)$
AB	A	-25	$+\frac{5}{16}$	-125
DF	A	-25	$-\frac{5}{16}$	$+125$
BC	A	$+25$	$-\frac{5}{16}$	-125
CD	A	$+25$	$+\frac{5}{16}$	$+125$
AC	(A) 2A	(+32) +16	$+\frac{12}{16}$	(+384) +192
CF	A	$+32$	$+\frac{4}{16}$	$+128$
BD	2A	-32	$+\frac{8}{16}$	-256 (+256)

Σ +64.

$$\therefore d = \frac{64WL}{16AE} = \frac{4WL}{AE}.$$

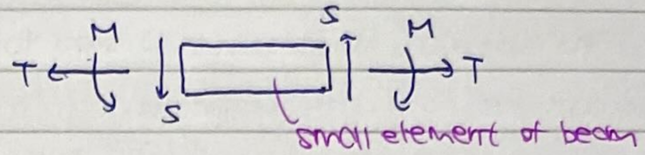
→ The horizontal deflection of node D decreased from $\frac{16WL}{AE}$ to $\frac{4WL}{AE}$.

Equilibrium of elastic beams

Conventions and assumptions

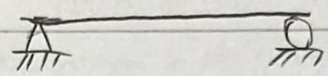
Sign conventions.

- The positive sense of M, S, and T are defined as follows:



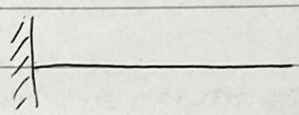
Drawing conventions of supports.

① simply supported beam.



- The end supports can:
  - ↳ prevent vertical movement // provide vertical forces
  - ↳ Allow rotation // not transmit moments ;
  - ↳ The triangular support prevents horizontal movement // provide horizontal forces.

② Cantilever beam



- The end support can:
  - ↳ Prevent vertical movement // provide vertical forces.
  - ↳ Prevent rotation // transmit moments
  - ↳ prevent horizontal movement // provide horizontal forces

Assumptions

- the bending moments are too small to break or permanently bend the beam.
- Only 2D situations are considered.
- The structure does not deflect significantly under load so we can consider eqm. in the original configuration.

Shear force and bending moment diagrams

Calculation of M, S and T by analysis of free bodies

- First, find any support reactions, then make a cut at a variable distance x to find expressions for axial force $T(x)$, shear force $S(x)$ and bending moment $M(x)$.

(We don't need to find the support reactions if we start from a free end, say in a cantilever w/ known load).

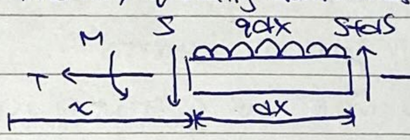
- Here are several facts useful for checking shear force / bending moment diagrams:
  - ↳ Under a conc. force, S is discontinuous. There is a jump equal to the conc. force at the pt the conc. force is applied.
  - ↳ under a conc. force, M is continuous but its slope is discontinuous.
  - ↳ Given an applied couple, M is discontinuous. There is a jump equal to the couple at the pt the couple is applied.
  - ↳ When S is zero, M is at a max./min.
  - ↳ M is zero at the simply supported ends / free end.

If we draw a FBD, there are no ext. forces → no int. forces needed to balance ∴ $T=0, S=0, M=0$.

* If we start from the RHS, ↑F results in ↑S; ↻ C results in ↑M.

Differential relationships between q, S and M.

- Consider the eqm. of a small element of beam w/ infinitesimal length under a distributed force of intensity q, acting downwards.



* We implicitly assume that the beam is straight.

- N2L ↔: $T+dT=T \rightarrow dT=0$ so $\frac{dT}{dx}=0$

  ↕ : $S+dS=S+q\,dx \rightarrow dS=q\,dx$ so $\frac{dS}{dx}=q$.

  M_{centre} : $M+dM-M = S\frac{dx}{2}+(S+dS)\frac{dx}{2} \rightarrow dM = S\,dx + \cancel{\tfrac{1}{2}dS\,dx}^{\,0}$ so $\frac{dM}{dx}=S$.

∴ Using this sign convention, $\boxed{q=\frac{dS}{dx},\ S=\frac{dM}{dx}}$

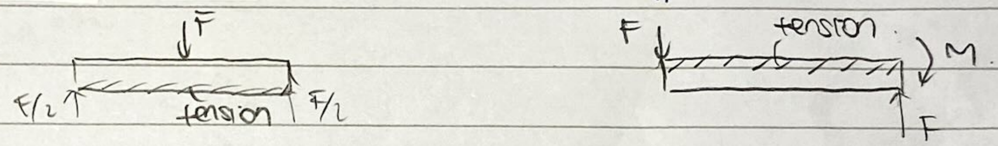
- When integrating to find S/M, we need a boundary condition to find the constant of integration. → Find BC via inspection ($M=0$ or S at end supports) or via FBD calc.
- This method does not work well for conc. loads or distributed loads w/ abrupt change of intensity. → we would need to have different DE for each section, w/ BC used to ensure continuity between sections.

Shortcut for beams under point forces.

- When a straight beam is loaded exclusively by pt. forces and reactions, then between any 2 consecutive pts. of application of force,
  - ↳ 1) The shearing force is constant
  - ↳ 2) The bending moment varies linearly.
- As the beam is unloaded between pts of application of force, $q=0$.
  so $S=\int q\,dx = \underbrace{A}_{\text{constant}}$ and $M=\int S\,dx = \underbrace{Ax+B}_{\text{linearly varying}}$, where A, B are constants.
- To find the bending moment diagram for beams under pt. loads quickly,
  - ↳ 1) Calculate M at all pts of application of force
  - ↳ 2) Join up consecutive pts. on the plot → $M(x)$ is piecewise linear.

Bending moment diagrams for simple frames and floating structures.

- Find / write down T, S, M for each beam and calculate M at all pts. of application of force, then join the consecutive pts (as above).
- By convention, we draw the bending moment diagram on the "tensile side" of the member.
  - ↳ Force applied in between
  - ↳ Force applied at ends



Recognising statically determinate structures

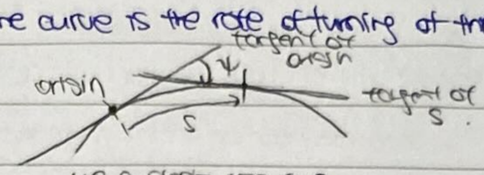
- To find the internal forces T, S, M, we need to find the reaction forces via eqm., which is only possible if the structure is statically determinate.
- For statically indeterminate structures, we must involve compatibility (how the structure deforms) to find the unknown reaction force.
- In general, any "tree structure", i.e. any series of cantilever beams connected to one another which does not form any closed loops is statically determinate.

Deflection of straight elastic beams

curvature, rotation and deflection

curvature and change of curvature

- The curvature of a plane curve is the rate of turning of the tangent to the curve

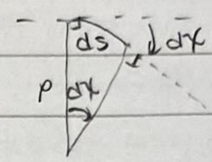


we a clockwise +ve convention for ψ.

- Using the intrinsic coord system (s, ψ) as above, the curvature κ at a pt. of the curve is

$$\boxed{\kappa = \frac{d\psi}{ds} = \frac{1}{\rho}}$$

where ρ is the radius of curvature.



so $\rho d\psi = ds$
$\frac{d\psi}{ds} = \frac{1}{\rho}$.

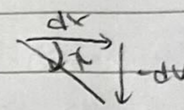
curvature, rotation and deflection.

valid for $|\psi| = |\frac{dv}{dx}| < 0.1$

- For small rotations ψ, we can approximate $x \approx s$, so curvature κ is given by

$$\boxed{\kappa \approx \frac{d\psi}{dx}}$$

- The deflection v can be related to the rotation via.



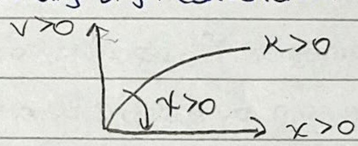
$$\boxed{\psi = -\frac{dv}{dx}}$$

- combining the 2 eqns above, we get

$$\kappa \approx \frac{d\psi}{dx} = \frac{d}{dx}\left(-\frac{dv}{dx}\right)$$

$$\boxed{\kappa \approx -\frac{d^2v}{dx^2}}$$

* The eqns. above use the following sign convention:



- Given a curvature function $\kappa(x)$, we can integrate and apply the boundary conditions on slope and/or deflections to fix the constants of integration → find $\psi(x)$ or $v(x)$. (usually 2 simple supports → $v=0$ at ends; cantilever → $v=\psi=0$ at end; symmetry → $\psi=0$ at centre)
- For situations that have the same curvature function but different boundary conditions, they are related via a rigid body rotation.

Distortion produced by internal forces

Distortion produced by internal forces.

- consider a short element of beam subjected to internal forces T, S, M.
  - ↳ Tension T causes extension of the element [ignore]
  - ↳ shearing force S causes distortion of the element [ignore]
  - ↳ Bending moment M causes change of curvature of the element.

valid for slender beams as distortions due to S negligible compared to distortions due to M.

- Ext forces set up int. forces → produce distortion in beam elements → deflection of structure

Deflection of elastic beams

- The resultant curvature of a beam (linear-elastic beam) is directly proportional to the applied bending moment.

$$\boxed{M = B\kappa}$$ [elastic moment/curvature relationship].

where B is the bending stiffness / flexural stiffness / flexural rigidity of the beam.

- Note the eqn. is only true for an initially straight beam when M=0. More generally,

$$\boxed{M = B\Delta\kappa}$$

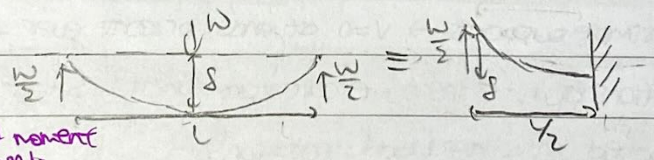
where $\Delta\kappa$ is the elastic change of curvature of a beam w/ initial curvature κ_0

- Unless stated otherwise, we assume the beams are
  - ↳ Linear-elastic and uniform, w/ bending stiffness B .
  - ↳ Have zero self-weight
  - ↳ Initially straight when M=0.
- Moreover, we assume the beam deflections and rotations are small so $\kappa = -\frac{d^2v}{dx^2}$

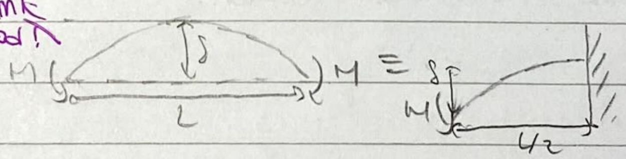
Calculation of deflections by integration

- To find deflections, the method is as follows:
  - ↳ Given the loading, find M(x) by eqm. (Graphical or algebraic)
  - ↳ Use the elastic moment/curvature relationship $M = B\kappa$ to find $\kappa(x)$ — ignore effects of T,S.
  - ↳ Use $\kappa = -\frac{d^2v}{dx^2}$ and apply boundary conditions to find $v(x)$.
- In cases where $\frac{d^2v}{dx^2}$ (i.e. M(x)) is discontinuous, we cannot integrate a single, standard function of x → integrate $\frac{d^2v}{dx^2}$ seperately over intervals where it is continuous and match the separate sol'n by suitable boundary conditions.
- We can replace each half of a symmetrically loaded beam supported at 2 ends w/ a cantilever:

symmetrical loading → zero rotation at the centre (end of cantilever support)

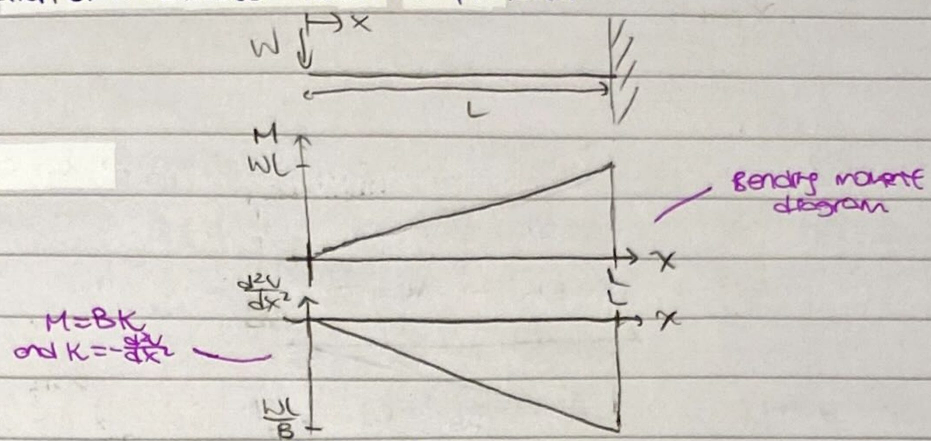


see this applied moment configuration, think of cantilever method!



*This method doesn't work for distributed loads.

Deflection of a cantilever loaded w/ pt. load.



From the graph, we see $\frac{d^2v}{dx^2} = -\frac{WL}{BL}x = -\frac{W}{B}x$

Integrating, $\frac{dv}{dx} = -\frac{W}{2B}x^2 + C_1$

$$v = -\frac{W}{6B}x^3 + C_1x + C_2$$

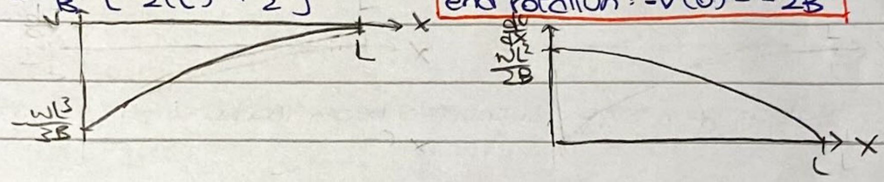
$\psi(L) = -v'(0) = 0$: $C_1 = \frac{WL^2}{2B}$

$v(L) = 0$: $0 = -\frac{WL^3}{6B} + \frac{WL^3}{2B} + C_2 \rightarrow C_2 = -\frac{WL^3}{3B}$

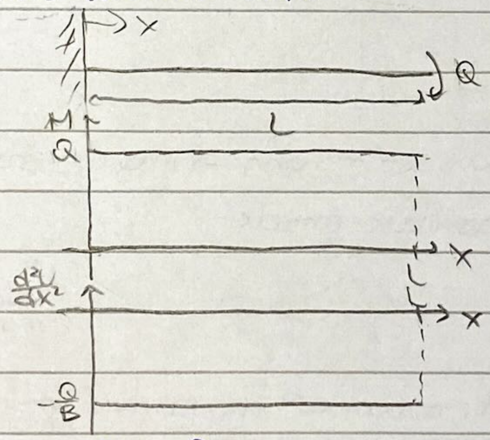
$\therefore\ v = \frac{WL^3}{B}\left[-\frac{1}{6}\left(\frac{x}{L}\right)^3 + \frac{1}{2}\left(\frac{x}{L}\right) - \frac{1}{3}\right]$

$\frac{dv}{dx} = \frac{WL^2}{B}\left[-\frac{1}{2}\left(\frac{x}{L}\right)^2 + \frac{1}{2}\right]$

$\boxed{\begin{array}{l}\text{end deflection}: v(0) = -\frac{WL^3}{3B} \\ \text{end rotation}: -v'(0) = -\frac{WL^2}{2B}\end{array}}$



Deflection of a cantilever loaded w/ couple



From the graph, we see $\frac{d^2v}{dx^2} = -\frac{Q}{B}$.

Integrating, $\frac{dv}{dx} = -\frac{Q}{B}x + C_1$

$$v = -\frac{Q}{2B}x^2 + C_1x + C_2$$

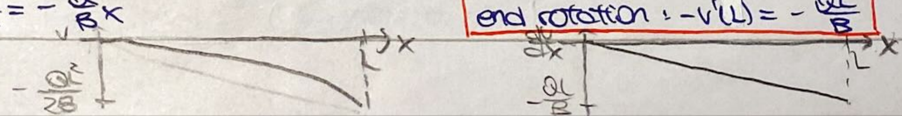
$\psi(0) = -v'(0) = 0$: $C_1 = 0$

$v(0) = 0$: $C_2 = 0$.

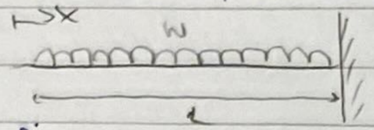
$\therefore\ v = -\frac{Q}{2B}x^2$

$\frac{dv}{dx} = -\frac{Q}{B}x$

$\boxed{\begin{array}{l}\text{end deflection}: v(L) = -\frac{QL^2}{2B} \\ \text{end rotation}: -v'(L) = -\frac{QL}{B}\end{array}}$



Deflection of a cantilever under uniformly distributed load



$q = \frac{dS}{dx} \rightarrow \quad S = \int q\,dx = wx + C_1$; $S(0) = 0 \rightarrow C_1 = 0 \quad \therefore S = wx$

$S = \frac{dM}{dx} \rightarrow \quad M = \int S\,dx = \frac{1}{2}wx^2 + C_2$; $M(0) = 0 \rightarrow C_2 = 0 \quad \therefore M = \frac{1}{2}wx^2$.

$$M = B\kappa \rightarrow \kappa = \frac{w}{2B}x^2$$

$$\kappa = -\frac{d^2v}{dx^2} \rightarrow \frac{d^2v}{dx^2} = -\frac{w}{2B}x^2$$

Integrating,
$$\frac{dv}{dx} = -\frac{w}{6B}x^3 + C_3$$
$$v = -\frac{w}{24B}x^4 + C_3x + C_4$$

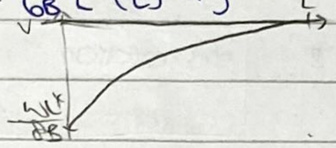
$\psi(L) = -v'(L) = 0$: $\quad C_3 = \frac{wL^3}{6B}$

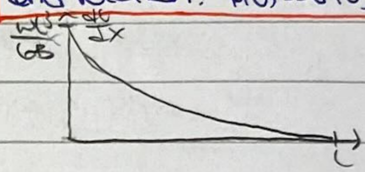
$v(L) = 0$: $\quad 0 = -\frac{wL^4}{24B} + \frac{wL^4}{6B} + C_4 \rightarrow C_4 = -\frac{wL^4}{8B}$

$\therefore v = \frac{wL^4}{8}\left[-\frac{1}{24}\left(\frac{x}{L}\right)^4 + \frac{1}{6}\left(\frac{x}{L}\right) - \frac{1}{8}\right]$

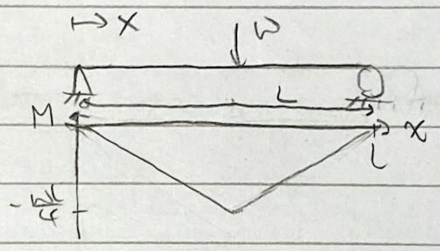
$\frac{dv}{dx} = \frac{wL^3}{6B}\left[-\left(\frac{x}{L}\right)^3 + 1\right]$

$\boxed{\text{end deflection: } v(0) = -\frac{wL^4}{8B} \qquad \text{end rotation: } \psi(0) = -v'(0) = \frac{wL^3}{6B}}$

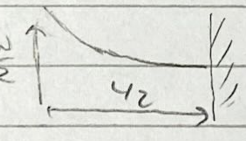




Deflection of a simply supported beam loaded w/ pt. load at midspan



M(x) is discontinuous at mid-span → integrate each half seperately or notice that each half acts like a cantilever,



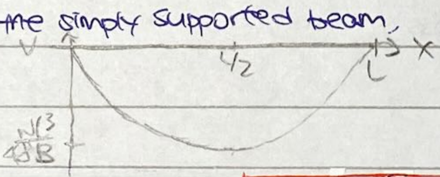
we only consider the reaction forces to find the equivalent cantilever

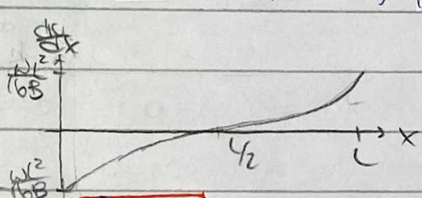
Using the results for a cantilever loaded w/ a pt. load, ($W = -\frac{W}{2}$, $L = \frac{L}{2}$)

$v = \frac{WL^3}{16B}\left[\frac{4}{3}\left(\frac{x}{L}\right)^3 - \left(\frac{x}{L}\right) + \frac{1}{3}\right]$ || for $x \in [0, L/2]$ — end deflection: $v(0) = \frac{WL^3}{48B}$

$\frac{dv}{dx} = \frac{WL^2}{8B}\left[2\left(\frac{x}{L}\right)^2 - \frac{1}{2}\right]$ — end rotation: $-v'(0) = \frac{WL^2}{16B}$

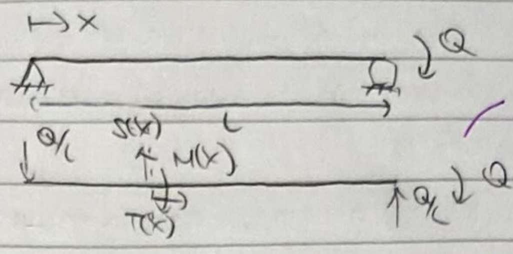
$\therefore$ For the simply supported beam,





$\boxed{\text{central deflection: } v\left(\frac{L}{2}\right) = -\frac{WL^3}{48B} \qquad \text{end rotations: } -v'(0) = \frac{WL^2}{16B};\ -v'(L) = -\frac{WL^2}{16B}}$

Deflection of simply supported beam loaded w/ a couple.



we only consider the reaction forces to find the bending moment M

consider the left part of the beam at the cut x (variable distance x from origin).

$$M(x) = \frac{Q}{L}x$$

$$M = BK \rightarrow K = \frac{Q}{BL}x$$

$$K = -\frac{d^2v}{dx^2} \rightarrow \frac{d^2v}{dx^2} = -\frac{Q}{BL}x$$

Integrating, $$\frac{dv}{dx} = -\frac{Q}{2BL}x^2 + C_1$$

$$v = -\frac{Q}{6BL}x^3 + C_1x + C_2$$

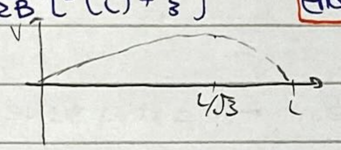
$v(0)=0$: $C_2 = 0$.

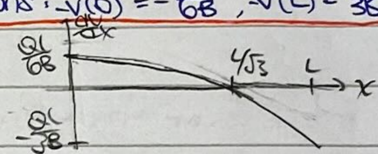
$v(L)=0$: $0 = -\frac{QL^2}{6B} + C_1L \rightarrow C_1 = \frac{QL}{6B}$

$$\therefore v = \frac{QL^2}{6B}\left[-\left(\frac{x}{L}\right)^3 + \left(\frac{x}{L}\right)\right]$$

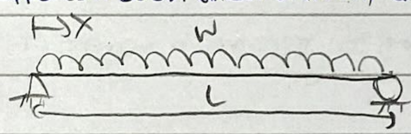
$$\frac{dv}{dx} = \frac{QL}{2B}\left[-\left(\frac{x}{L}\right)^2 + \frac{1}{3}\right]$$

$\boxed{\text{end rotations: } -v'(0) = -\frac{QL}{6B},\ -v'(L) = \frac{QL}{3B}}$





Deflection of a simply supported beam under uniformly distributed load.



$q = \frac{dS}{dx} \rightarrow S = \int q\,dx = wx + C_1$; $S(0) = -\frac{wL}{2} \rightarrow C_1 = -\frac{wL}{2}$ $\therefore S = wx - \frac{wL}{2}$

$S = \frac{dM}{dx} \rightarrow M = \int S\,dx = \frac{1}{2}wx^2 - \frac{wL}{2}x + C_2$; $M(0) = 0 \rightarrow C_2 = 0$ $\therefore M = \frac{1}{2}wx^2 - \frac{wL}{2}x$

$$M = BK \rightarrow K = \frac{w}{2B}x^2 - \frac{wL}{2B}x$$

$$K = -\frac{d^2v}{dx^2} \rightarrow \frac{d^2v}{dx^2} = -\frac{w}{2B}x^2 + \frac{wL}{2B}x$$

Integrating, $$\frac{dv}{dx} = -\frac{w}{6B}x^3 + \frac{wL}{4B}x^2 + C_3$$

$$v = -\frac{w}{24B}x^4 + \frac{wL}{12B}x^3 + C_3x + C_4$$

$v(0)=0$: $C_4 = 0$

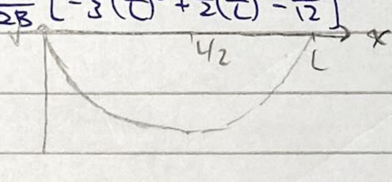
$v(L)=0$: $0 = -\frac{wL^4}{24B} + \frac{wL^4}{12B} + C_3L \rightarrow C_3 = -\frac{wL^3}{24}$

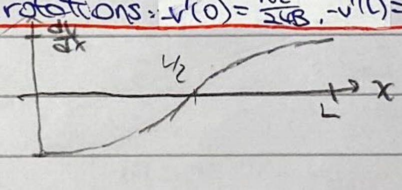
$$\therefore v = \frac{wL^4}{12B}\left[-\frac{1}{2}\left(\frac{x}{L}\right)^4 + \left(\frac{x}{L}\right)^3 - \frac{1}{2}\left(\frac{x}{L}\right)\right]$$

$$\frac{dv}{dx} = \frac{wL^3}{2B}\left[-\frac{1}{3}\left(\frac{x}{L}\right)^3 + \frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{1}{12}\right]$$

$\boxed{\text{central deflection: } v\left(\frac{L}{2}\right) = -\frac{5wL^4}{384B}}$

$\boxed{\text{end rotations: } -v'(0) = \frac{wL^3}{24B},\ -v'(L) = -\frac{wL^3}{24B}}$





Calculation of deflections by superposition

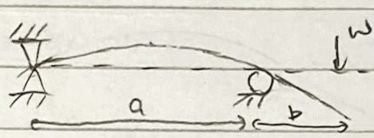
- As long as the deflections of the structure are small, the principle of superposition applies.
- We can find the deflection of a beam by superposing a series of previously known standard solutions.
- In superposing these results, we need to ensure that (i) the curvature in the sol'ns that are superposed matches the curvature of the beam, and (ii) the boundary conditions are met, or any mismatches are corrected for.

  equivalently, the BM in the sol'ns that are superposed matches the BM of the beam
- In general, we find the deflection, assuming a part of the beam (between 2 supports or 1 support and free end) has been put in a rigid sleeve. Then we repeat by assuming other parts of the beam are put in a rigid sleeve.

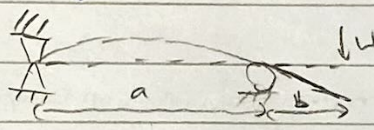
  simply-supported beam; overhang
- The sum of these deflections give the total deflection (what we req.)

- e.g.: overhanging beam (find tip deflection)

  · consider the overhanging beam as shown below. Find the tip deflection.

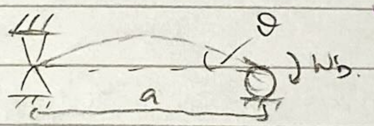


First, we assume the overhang has been put in a rigid sleeve.



W at distance b transmits moment Wb.
Q at any distance transmits moment Q.

Therefore the overhang can transmit a moment Wb to the end of the simply-supported beam.

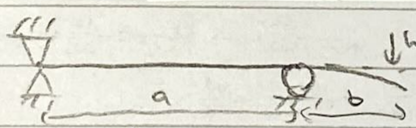


for rigid overhang,
W/Q at overhang → θ in beam
W/Q at beam → nothing in overhang.

From the databook, $\theta = \frac{Wb \cdot a}{3B}$ → $v_1 = b\theta = \frac{Wb^2 a}{3B}$

Second, we assume the simply supported beam has been put in a rigid sleeve.

part where F/Q is not applied → directly find deflection.

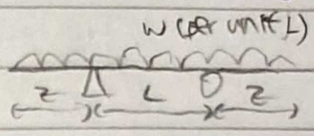


From the databook, $v_2 = \frac{Wb^3}{3B}$.

∴ Total deflection: $v = v_1 + v_2 = \frac{Wb^2}{3B}(a+b)$

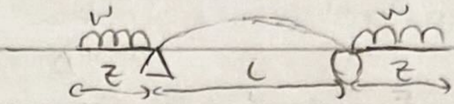
- eg: Beam w/ unknown overhang length (Find central deflection)

The applied loading is uniform along the full length of the beam. However, the length of overhang, z, has not yet been decided upon.

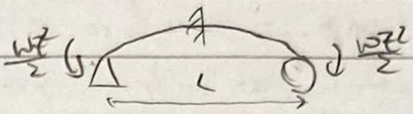


What z is req. for the central deflection to be zero?

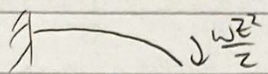
First, we assume the overhang has been put in a rigid sleeve.



Therefore the overhang can transmit a moment $\frac{wz^2}{2}$ to the end of the simply-supported beam.



By symmetry, the centre of the beam acts like a cantilever support.



From the databook $\quad v_1 = -\left(\frac{wz^2}{2}\right)\left(\frac{L}{2}\right)^2\left(\frac{1}{2B}\right) = -\frac{wz^2L^2}{16B}$

Second, consider the deflection due to the distributed load in the simply supported beam

From the databook $\quad v_2 = \frac{5wL^4}{384B}$

$\therefore$ Total deflection $\quad v = v_1 + v_2 = -\frac{wz^2L^2}{16B} + \frac{5wL^4}{384B} = 0$

$$z = \sqrt{\frac{5}{24}}\,L$$

Strategy for using superposition.

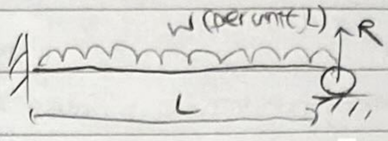
- To find the deflection of the req. pt, consider the deflection of each section of the beam due to applied loads/moments from each section of the whole beam.
- For tip deflections, total deflection = deflection due to load/moment in overhang + rotation in support (overhang + simply supported beam) × length of overhang.
- For central deflections, total deflection = deflection due to load in simply supported beam + deflection due to moment (overhang) in simply supported beam

Finding the reaction forces of statically indeterminate structures.

- we superpose the deflections due to different forces (including unknown reaction forces) and set them to meet the boundary condition.

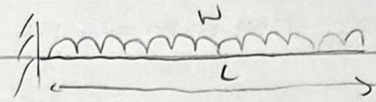
-eg: Propped cantilever.

Consider a propped cantilever as shown below.



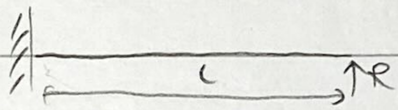
Find the tip reaction R s.t. there is no deflection .

Consider the deflection due to the distributed load alone



From the databook, $v_1 = \frac{wL^4}{8B}$

Consider the deflection due to the tip reaction alone .



From the databook, $v_2 = -\frac{RL^3}{3B}$.

$\therefore$ Total deflection $v = v_1 + v_2 = \frac{wL^4}{8B} - \frac{RL^3}{3B} = 0$

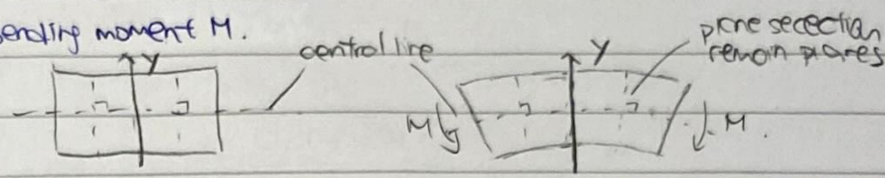
$$R = \frac{3wL}{8}.$$

Stresses in elastic beams

Bending of beams

Geometric relationships

- Consider a side view of the beam that is initially straight, but curved due to the application of a bending moment M.

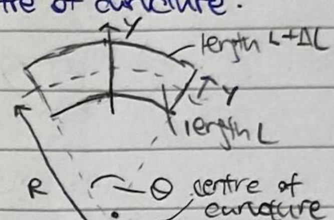


We make 2 assumptions:

↳ The central line doesn't change in length. This line traces the position of the neutral axis.

↳ Plane sections remain planes (Euler-Bernoulli hypothesis).

- Consider a beam element originally of length L. After bending, the top/bottom edges change in length by $\pm\Delta L$ and the beam element subtends an angle θ at the centre of curvature.



$$L = \theta R \quad ; \quad L+\Delta L = \theta(R+y)$$

$$\varepsilon = \frac{\Delta L}{L} = \frac{\theta(R+y)-\theta R}{\theta R} = \frac{\cancel{\theta}y}{\cancel{\theta}R}$$

$$\boxed{\varepsilon = \frac{y}{R} = \kappa y}$$

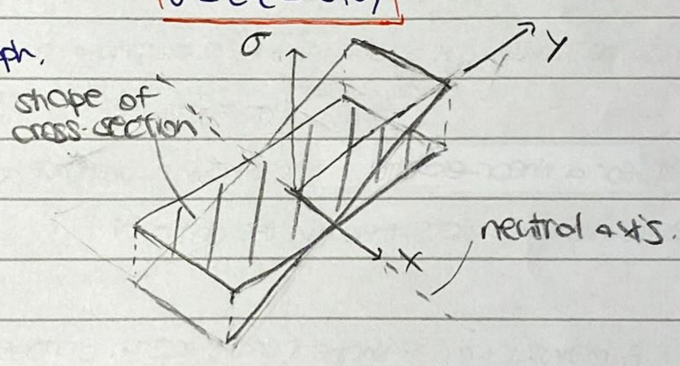
→ For a beam subjected to pure bending, the Euler-Bernoulli hypothesis assumption implies that the strain ε is prop. to the curvature κ and the distance from neutral axis y.

Stress-strain law.

- Assuming that the material is linear elastic.

$$\boxed{\sigma = E\varepsilon = E\kappa y}$$

- Plotting this on a graph,



Stress resultant from bending.

① Tension T.

- On an elemental area δA of the cross-section, the force is $\delta T = \sigma\delta A$. Integrating over the whole area,

$$T = \int_A dT = \int_A \sigma\, dA$$
$$= E\kappa\int_A y\, dA = 0$$

the centroid defines the neutral axis for pure bending.

* The resultant tensile force T is 0 (because how we chose the neutral axis).

makes sense as we have pure bending

① Bending moment.

- On an elemental area δA of the cross section, the bending moment is $\delta M = y\,\delta T = y\sigma\,\delta A$.

Integrating over the whole area,

$$M = \int_A dM = \int_A y\,dT = \int_A y\sigma\,dA$$
$$= EK\int_A y^2\,dA$$
$$\boxed{M = EKI}$$

where I is the 2nd moment of area of the cross section about the neutral axis.

- Recall for an elastic beam, $M = BK$, so

$$\boxed{B = EI}$$

Summary of results

- Since $\sigma = EKy$ and $M = EKI$, these relationships can be collected:

$$\boxed{\frac{\sigma}{y} = \frac{M}{I} = EK}$$

- This relationship is useful for finding resultant internal stress from the applied moment and vice versa.
- The neutral axis is defined as the <u>line in the cross section</u> where $\sigma = 0$ under a pure bending moment. (Neutral axis is $\perp$ to plane of bending, ie the "centre line".)

consequently, neutral axis is at centroidal level

the only plane that contains the deformed beam

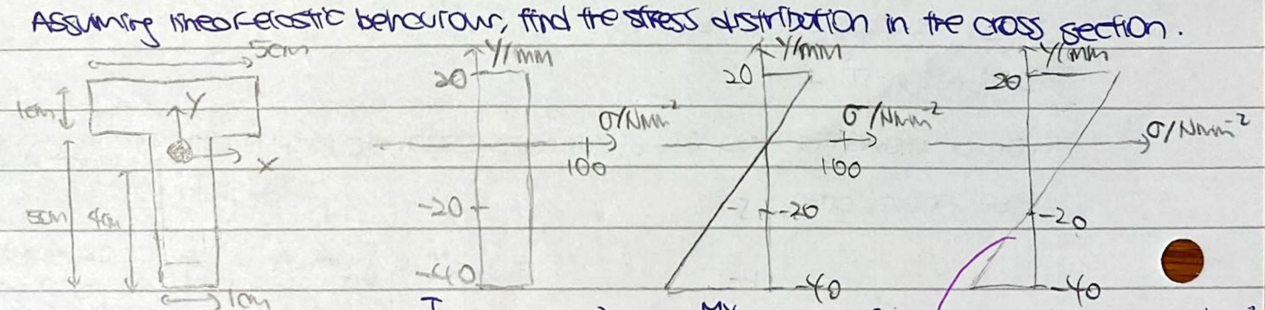
Combined bending moment an axial force.

- Often a beam may have to carry both a bending moment M and an axial force T. The stress distribution can be obtained by superposing the stresses due to T and M:

$$\boxed{\sigma = \underbrace{\frac{T}{A}}_{\text{stresses due to } T} + \underbrace{\frac{My}{I}}_{\text{stresses due to } M}}$$

*Note under uniaxial tension, the stress is uniform across the cross-section, and the resultant force acts through the centroid.

- e.g.: A member w/ a T-shaped cross section carries $M_x = 1\text{kNm}$ and $T = 50\text{kN}$. Assuming linear-elastic behaviour, find the stress distribution in the cross section.



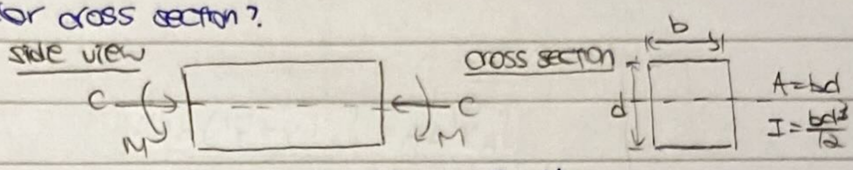
$A = 10\text{cm}^2$, $I = 33.33\text{cm}^4$ $\quad \sigma_T = \frac{T}{A} = 50\text{Nmm}^{-2}$ $\quad \sigma_M = \frac{My}{I} = (3y)\text{Nmm}^{-2}$ $\quad \sigma = \sigma_T + \sigma_M = (50+3y)\text{Nmm}^{-2}$

At $\sigma = 0$, $y \neq 0$ as it is not pure bending.

Axial compression to avoid tensile stress

- In masonry structures, the material will crack in tension → we have an axial compression C provided by the self weight to avoid tensile stress

- e.g. What is the value of axial compression C req. to avoid tensile stresses in a rectangular cross section?



$$\sigma = \frac{T}{A} + \frac{My}{I} = -\frac{C}{A} + \frac{My}{I}.$$

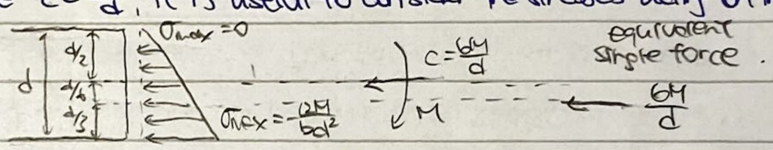
Max. tensile stress σ_{max} at the top, $y = \frac{d}{2}$.

$$\sigma_{max} = -\frac{C}{A} + \frac{My_{max}}{I}$$
$$= -\frac{C}{bd} + \frac{M \cdot d/2}{bd^3/12}$$

To avoid tensile stresses, we req. $\sigma_{max} < 0$

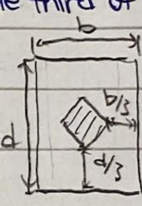
$$\frac{M}{6bd^2} - \frac{C}{bd} < 0$$
$$C > \frac{6M}{d}$$

For the particular case $C = \frac{6M}{d}$, it is useful to consider the stresses acting on the cross section.



→ If the resultant thrust (single force resultant of C and M) acts within the middle third of the section, there will be no tensile stress anywhere in the cross section.

*Also considering bending moments about the y-axis, the thrust must lie within a zone w/ the shape of a rhombus to avoid tensile stress.



Calculation of the 2nd moment of area I.

Centroid of the cross section.

- The centroid defines the neutral axis for pure bending:

Assuming plane sections remain planes, $\varepsilon = ky$ → $\sigma = Eky$

As we have pure bending, $T = 0$

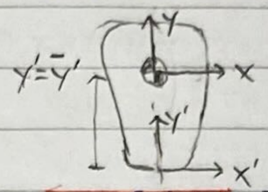
$$\text{so} \quad T = \int_A \sigma dA = Ek\int_A y dA = 0$$

→ We must locate the origin at a place s.t. the 1st moment of area is zero (centroid)

$$\int_A y dA = 0$$

Finding the centroid of the cross section

- For a doubly symmetric shape, the centroid is at the centre.
- For a singly symmetric shape, we set up some temporary axes x', y' at a pt. on the axis of symmetry to define the position of the centroid.



$$\boxed{\bar{y}' = \frac{\int_A y' dA}{\int_A dA}}$$

(Refer to mechanics notes for more details)

Finding the 2nd moment of area I of the cross-section

- Once the centroid has been located, the 2nd moment of area abt. the neutral axis is

$$\boxed{I = \int_A y^2 dA}$$

where y is the distance from the neutral axis.

- We can find I using the parallel axis theorem

$$\boxed{I_o = I_G + Ad^2}$$

where d is the distance between the given axis and the centroid.

(Refer to mechanics notes (moment of inertia) for more details).

We can find 2nd moment of area of standard shapes using the moment of inertia of that shape by converting m to A.

* 2nd moment of area $I = \int_A y^2 dA$ and moment of inertia $I = \int r^2 dm$ have similar eqns but have no physical relation w/ one another. However techniques for finding I are the same.

- Values of I and other section properties for various standard cross-sections are tabulated in the data book. (Section tables for universal columns/beams).

Elastic section modulus

- If we want to find the magnitude of the max. moment that a beam could carry $|M_a|$, before the magnitude of the stress in the material reached some max. value, $|\sigma_a|$, we use

$$\frac{|\sigma_a|}{|y_{max}|} = \frac{|M_a|}{I} \rightarrow |M_a| = \frac{I}{|y_{max}|}|\sigma_a|$$

- $\frac{I}{|y_{max}|}$ is such a useful property that is is given a name, the elastic section modulus Z/Z_e.

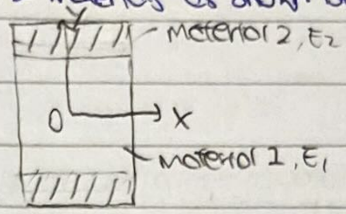
$$\boxed{Z = \frac{I}{|y_{max}|}}$$

so $\boxed{|M_a| = Z|\sigma_a|}$

Bending stresses in composite beams

Bending stresses in composite beams

- A composite beam is made from 2 or more different materials.
- Consider a composite made from 2 materials as shown below:

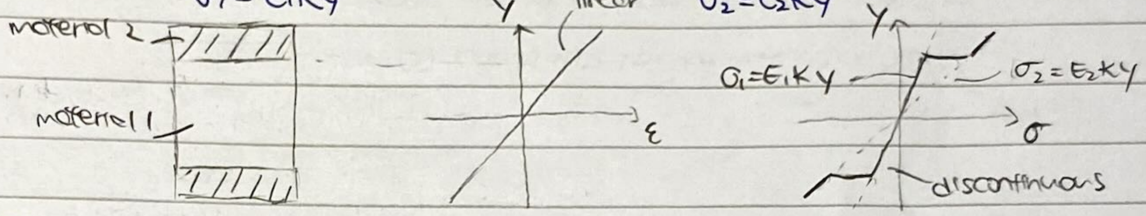


Assuming plane sections remain planes, from geometry

$$\varepsilon = \kappa y$$

For a linear-elastic material, we can find the corresponding stresses.

$\sigma_1 = E_1 \kappa y$ $\quad\quad$ $\sigma_2 = E_2 \kappa y$ $\quad\quad$ * Here, $E_2 > E_1$



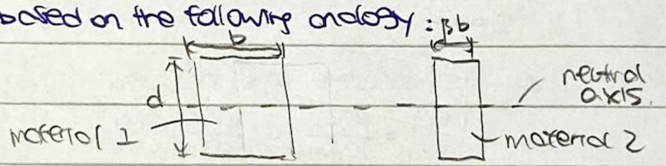
From this, we can find the bending moment M.

$$M = \int_A y \sigma \, dA$$

where the integral would be divided into 2 parts, 1 for each material.

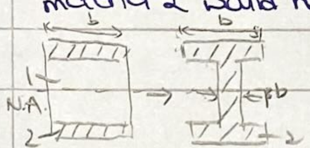
Method of transformed section

- The method is based on the following analogy:



Here $E_2 > E_1$

Given $E_2 > E_1$, for the 2 sections to have equal bending stiffness B, the section made of material 2 would have a reduced width of βb.



$$B = EI = E_1 \frac{bd^3}{12} = E_2 \frac{\beta b d^3}{12}$$

$$\therefore \boxed{\beta = \frac{E_1}{E_2}} \quad \text{for material 1} \rightarrow \text{material 2}$$

- Using this method, we can transform a composite beam into a beam entirely made of material 1/2 (They would still have the same bending stiffness: $B = E_1 I_1 = E_2 I_2$)
- Under an applied bending moment, the curvatures and strains will be correctly calculated from either of the transformed sections.

$$\varepsilon = \frac{My}{B} = \frac{My}{E_1 I_1} = \frac{My}{E_2 I_2} \quad\quad \kappa = \frac{M}{B} = \frac{M}{E_1 I_1} = \frac{M}{E_2 I_2}$$

use correction factor E_1/E_2.

- However the stresses calculated are only correct for the untransformed part of the section.

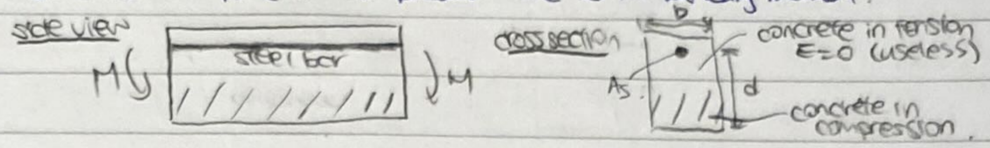
$$\sigma_1 = \frac{My}{I_1} \quad\quad \sigma_2 = \frac{My}{I_2} \quad\quad * \ I_1 = I_2 \cdot \frac{E_2}{E_1}$$

correct stresses in material 1

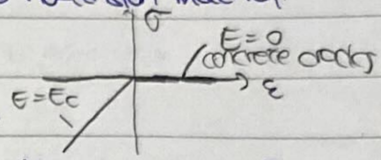
correct stresses in material 2

Pure bending of reinforced concrete beams

- Concrete is strong in compression but weak in tension; steel is equally strong in compression and tension → in reinforced concrete, use slender steel bars to withstand tension and concrete to withstand compression due to the bending moment.

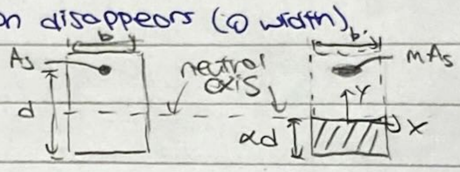


- Concrete is so weak in tension that it is normally treated as a no-tension material. This means it has no strength/stiffness in tension. $E=0$.



- By contrast, concrete is a linear-elastic material in compression.
- We transform the section to one entirely made of material w/ Young's modulus E_c:
  - ↳ steel has to be widened by the modular ratio $m = \frac{E_s}{E_c}$
  - ↳ The concrete in tension disappears (0 width).

α% reinforcement means $A_s = \alpha\% \, bd$



- We set the centroid to be αd from the bottom and solve for α to find the centroid.

$$\int_A y\,dA = \left(-\tfrac{1}{2}\alpha d\right)(b\alpha d) + (1-\alpha)d(mA_s)$$

$$0 = \tfrac{1}{2}bd\alpha^2 + mA_s\alpha - mA_s$$

$$\alpha = \frac{-mA_s \pm \sqrt{m^2A_s^2 + 2bdmA_s}}{bd}$$ take the +ve solution

- From this, we can work out the 2nd moment of area,

$$I = \underbrace{mA_s(1-\alpha)^2d^2}_{\text{steel (pt. mass)}} + \underbrace{\tfrac{1}{12}b(\alpha d)^3 + b\alpha d\left(\tfrac{\alpha d}{2}\right)^2}_{\text{concrete + ll axis thm.}}$$

- The strain ratio between concrete and steel is

$$\left|\frac{\varepsilon_s}{\varepsilon_c}\right| = \frac{\cancel{\kappa} y_{max\,s}}{\cancel{\kappa} y_{max\,c}} = \frac{1-\alpha}{\alpha}$$

and hence the stress ratio between concrete and steel is

$$\left|\frac{\sigma_s}{\sigma_c}\right| = \frac{E_s}{E_c}\left|\frac{\varepsilon_s}{\varepsilon_c}\right| = m\frac{1-\alpha}{\alpha}$$

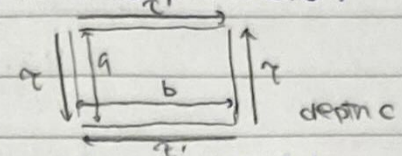
Shear stresses in beams

Shear stresses.

- In a beam, both T and M are carried by distribution of longitudinal stresses σ that are normal to the cut face.
- By contrast S is carried by shear stresses τ that are parallel to the cut surface.
- These shear stresses are not uniform across the surface. Simply dividing the shear force by the area gives the average shear stress. (no information on peak shear stress).

Principle of complementary shear.

- A state of simple shear req. shear stress on 4 faces of an arbitrary small block (to maintain static eqm. in translation and rotation)



$\underbrace{\tau'(bc)}_{\text{force}} \cdot \underbrace{a}_{\text{lever arm}} = \tau(ac) \cdot b$

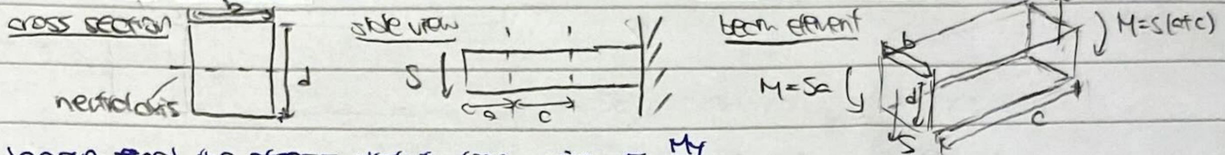
$\tau' = \tau.$

- Principle of complementary shear states if τ acts on a plane, at a pt. in a stressed body, then τ acts also on the orthogonal plane.

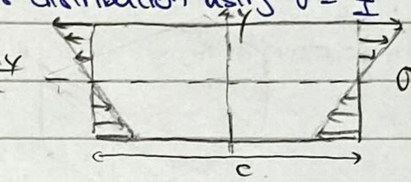
Shear stresses at various depths

If S is not constant across the beam, set $c = dx$ and let $dx \to 0$.

- Consider a beam element cut from a cantilever w/ a pt. load at the end.



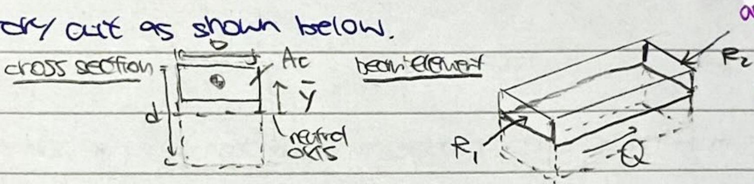
- We can find the stress distribution using $\sigma = \frac{My}{I}$



$\sigma = \frac{Say}{I}$ $\qquad$ $\sigma = \frac{S(a+c)y}{I}$

shear force on section
↓
variation of M along beam as $S = \frac{dM}{dx}$, $S \neq 0$.
↓
difference in longitudinal stress along the 2 cut-out sections of the beam
↓
longitudinal shear force / stress
↓ comp. shear
shear stress in section

- Making an arbitrary cut as shown below.



The longitudinal force at a and a+c are

$R_1 = \int_{A_c} \sigma \, dA = \frac{Sa}{I}\int_{A_c} y \, dA \qquad R_2 = \int_{A_c} \sigma \, dA = \frac{S(a+c)}{I}\int_{A_c} y \, dA$

so the shear force Q req. to maintain eqm is

$$\boxed{Q = R_2 - R_1 = \frac{Sc}{I}\int_{A_c} y \, dA}$$

useful for finding the no. of bolts req. to support a longitudinal force.

and hence the shear flow q is (shear force per unit length)

$$\boxed{q = \frac{Q}{c} = \frac{S}{I}\int_{A_c} y \, dA}$$

Note $\boxed{\tau = \frac{q}{b}}$ where b is the width of the cut section.

whereas the shear stress τ is (shear force per unit area)

$$\boxed{\tau = \frac{Q}{bc} = \frac{S}{bI}\int_{A_c} y \, dA}$$

useful for checking whether it is within the material's limit

- We can write the 1st moment of area as

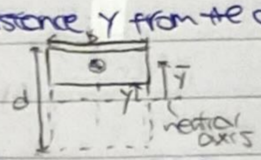
$$\boxed{\int_{A_c} y \, dA = A_c \bar{y}}$$

where $\bar{y}$ is the distance from the centroid of the entire cross-section to the centroid of the cut section.

* I used in the formulae is the 2nd moment of area of the <u>whole</u> section.
* This method is restricted to cross sections that have axis of symmetry in plane of bending.

Shear stress distribution in a rectangular section

- consider a cut at an arbitrary distance, y from the centroid.



$$\tau = \frac{S}{Ib}\int_{A_c} y\,dA = \frac{SA_c\bar{y}}{Ib}$$

$$= \frac{S}{\frac{bd^3}{12}\cdot \cancel{b}}\,\cancel{b}\left(\frac{d}{2}-y\right)\cdot\left(y+\frac{1}{2}\left(\frac{d}{2}-y\right)\right)$$

$$= \frac{12S}{bd^3}\left(\frac{d}{2}-y\right)\cdot\frac{1}{2}\left(y+\frac{d}{2}\right)$$

$$= \frac{6S}{bd^3}\left(\frac{d^2}{4}-y^2\right)$$

At mid-depth, $y=0$, $\tau = \frac{6S}{bd^3}\left(\frac{d^2}{4}\right) = \frac{3S}{2bd} = \frac{3}{2}S_{avg}$

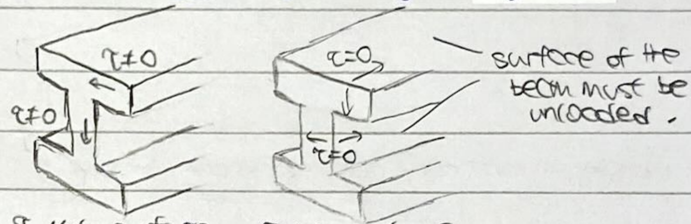
At top/bottom, $y=\pm\frac{d}{2}$, $\tau=0$ in accordance w/ the principle of complementary shear.

(as the top/bottom surfaces are stress free).

→ the variation of τ on the cross section is parabolic: max. at mid-depth, zero at top/bottom.

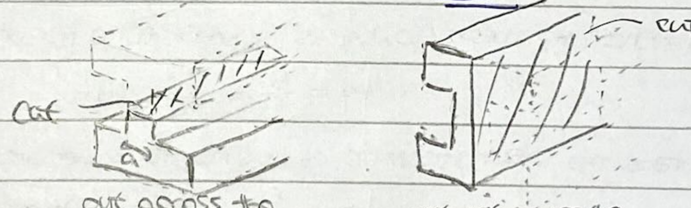
Shear stress in thin-walled beams

- The key idea is to realise that the shear stress at a pt. of a cross section of a thin-walled beam must be // to the surface of the section. (i.e. along the length of thin wall)



τ // to surface. $\tau \perp$ to surface

- This is because shear stresses in a direction ⊥ to the surface of the section (i.e. along width of thin wall) would be accompanied by complementary shear stresses acting on the surface of the beam, which is unloaded.
- The eqn. derived for the shear stress/flow in a beam can also be used to find the shear stresses in thin-walled beams — but the cut must be across the thin-walled section.



cut across the thin-walled section

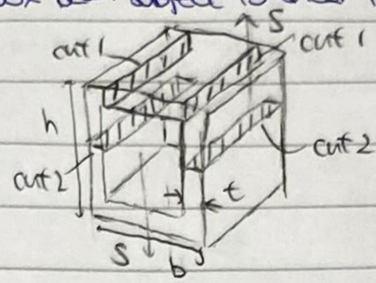
cut not across the thin-walled section

If the cut we have made is of length a,

$$\tau a = \frac{SA_c\bar{y}}{I}$$

Shear stresses in a box beam

- Consider a rectangular box beam subject to shear force S.

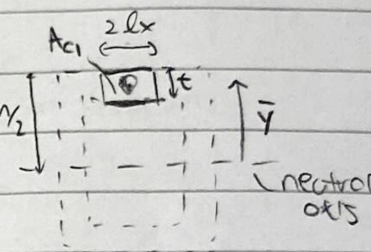


For the entire section, the 2nd moment of area is

$$I = \frac{bh^3}{12} - \frac{(b-2t)(h-2t)^3}{12} \approx \frac{h^2t}{6}(3b+h)$$

Consider making the cuts 1. The 1st moment of area of the cut section is,

$$A_{c1}\bar{y} = 2l_x t \cdot \left(\frac{h}{2} - \frac{t}{2}\right)$$



The shear stress is therefore given by

$$\tau_1 = \frac{S}{Ia} A_{c1}\bar{y}$$

*we have cut in 2 places to give a free body ∴ $a = 2t$

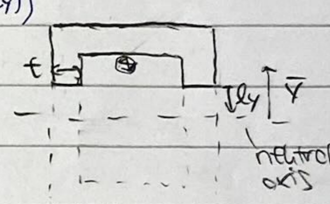
$$= \frac{S}{I(2t)} 2l_x t \left(\frac{h}{2} - \frac{t}{2}\right)$$

$$= \frac{S}{2I}(h-t) l_x$$ linear variation in flange

Consider making the cuts 2. The 1st moment of area of the cut section is,

$$A_{c2}\bar{y} = (bt)\left(\frac{h}{2} - \frac{t}{2}\right) + 2\left[\left(\frac{h}{2} - t - l_y\right)t\right]\left[\frac{1}{2}\left(\frac{h}{2} - t + l_y\right)\right]$$

$$= \frac{bt}{2}(h-t) + \left[\left(\frac{h}{2} - t\right)^2 - l_y^2\right]t$$



The shear stress is therefore given by

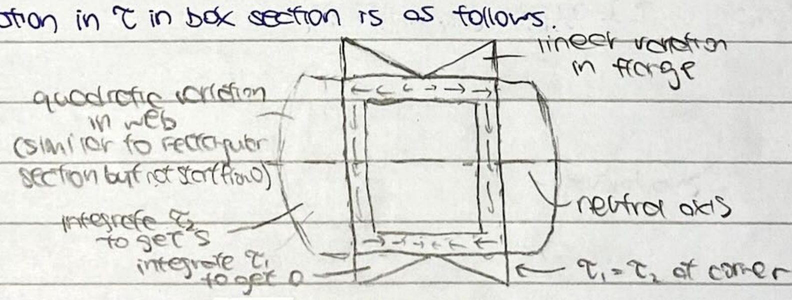
$$\tau_2 = \frac{S}{Ia} A_{c2}\bar{y}$$

*we have cut in 2 places to give a free body ∴ $a = 2t$

$$= \frac{S}{I(2t)}\left[\frac{bt}{2}(h-t) + \left[\left(\frac{h}{2} - t^2\right) - l_y^2\right]t\right]$$

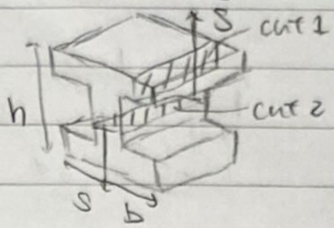
$$= \frac{S}{2I}\left[\frac{b(h-t)}{2} + \left(\frac{h}{2} - t\right)^2 - l_y^2\right]$$ quadratic variation in web.

- The variation in τ in box section is as follows.



- The entire shear force (vertical) is carried by the web shear stresses in an I-beam

- consider an I-beam subject to shear force S.

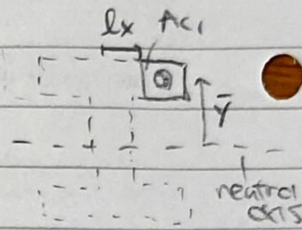


For the entire section, the 2nd moment of area is

$$I = \frac{bh^3}{12} - \frac{(b-t)(h-2t)^3}{12} \approx \frac{h^2 t}{12}(6b+h^3)$$

Consider making the cut 1. The 1st moment of area of the cut section is

$$A_{c1}\bar{y} = \left(\frac{b}{2}-l_x\right)t\cdot\left(\frac{h}{2}-\frac{t}{2}\right)$$

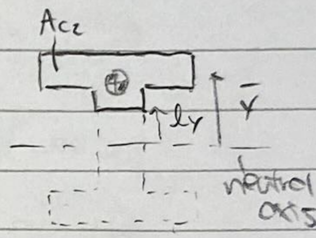


The shear stress is therefore given by

$$\tau_1 = \frac{S}{Ia}A_{c1}\bar{y}$$
$$= \frac{S}{It}\left(\frac{b}{2}-l_x\right)t\left(\frac{h}{2}-\frac{t}{2}\right)$$
$$= \frac{S}{2I}(h-t)\left(\frac{b}{2}-l_x\right)$$ linear variation in flange

Consider making the cut 2. The 1st moment of area of the cut section is

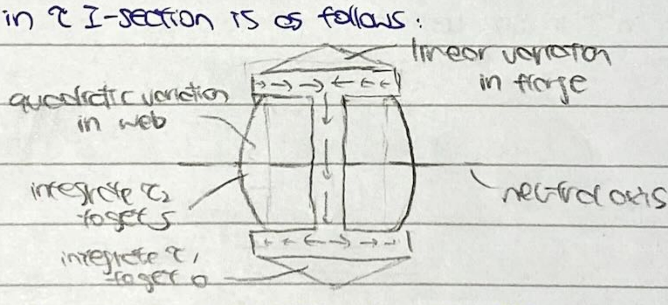
$$A_{c2}\bar{y} = bt\left(\frac{h}{2}-\frac{t}{2}\right)+t\left(\frac{h}{2}-t-l_y\right)\cdot\left[\frac{1}{2}\left(\frac{h}{2}-t+l_y\right)\right]$$
$$= \frac{bt}{2}(h-t) + \frac{t}{2}\left[\left(\frac{h}{2}-t\right)^2 - l_y^2\right]$$



The shear stress is therefore given by

$$\tau_2 = \frac{S}{Ia}A_{c2}\bar{y}$$
$$= \frac{S}{It}\left[\frac{bt}{2}(h-t) + \frac{t}{2}\left[\left(\frac{h}{2}-t\right)^2 - l_y^2\right]\right]$$
$$= \frac{S}{I}\left[\frac{b}{2}(h-t) + \frac{1}{2}\left[\left(\frac{h}{2}-t\right)^2 - l_y^2\right]\right]$$ quadratic variation in web

- The variation in τ I-section is as follows:



- The entire shear force (vertical) is carried by the web.
- In the web of an I-beam, τ varies far less than in a beam of solid rectangular cross-section.
- The shear stress in the web is approximately $\tau \approx \frac{S}{A_{web}}$ (A_{web} not A)
- I-beams are used w/ their webs vertical, so that the bending moment due to vertical loads is carried by the beam in its stiff/strong direction.

Buckling of columns

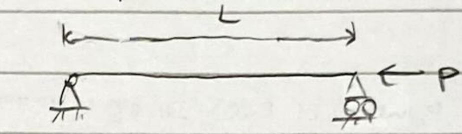
Buckling of (perfect) columns

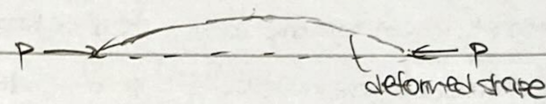
Buckling.

- Buckling is fundamentally non-linear. We need to consider eqm. of the deformed shape.
- Looking at the deformed shape that is in eqm. w/ the applied load give us a bifurcation load.
- To consider the buckling of a compression member, we initially make a no. of key assumptions:
  - ↳ Pin ended bar ← we will remove these assumptions later → ↳ Linear elastic
  - ↳ Initially perfectly straight ↳ All deflections are assumed to be small
  - ↳ Initially stress free.

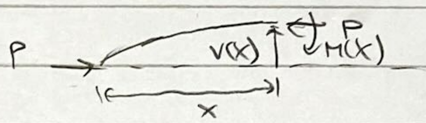
Euler column

- A simple pin-ended bar in compression is known as an Euler column





- The column buckles when a non-straight eqm. shape becomes possible. → we analyse the deformed shape $v(x)$ to determine when it will be in eqm.



Eqm. $\quad Pv(x) = M(x)$

Elastic law $\quad M(x) = EI\Delta K(x)$

$= EIK(x)$ here $K_0 = 0$

Geometry $\quad K(x) = -\frac{d^2v(x)}{dx^2}$

$\therefore Pv(x) = -EI\frac{d^2v(x)}{dx^2}$

Rearranging, $\quad \frac{EI}{P}\frac{d^2v}{dx^2} + v = 0$ [Governing DE]

Define $\alpha^2 = \frac{P}{EI}$, $\quad \frac{1}{\alpha^2}\frac{d^2v}{dx^2} + v = 0$ ← homogeneous DE

$v = A\sin\alpha x + B\cos\alpha x$

Boundary conditions $\quad v(0) = 0 \rightarrow B = 0$

$v(L) = 0 \rightarrow A\sin(\alpha L) = 0$

so either $A = 0$ (no deformation)

or $\sin(\alpha L) = 0 \rightarrow \alpha L = 0$ (trivial), $\pi, 2\pi \dots n\pi \dots$

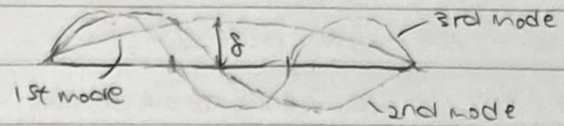
since $\alpha^2 = \frac{P}{EI}$, the corresponding values of P are

$$P = \alpha^2 EI = \frac{\pi^2}{L^2}EI, \frac{4\pi^2}{L^2}EI \dots \frac{n^2\pi^2}{L^2}EI \dots$$

- We can substitute the values of α back into the GS.

$$\boxed{v = A\sin\left(\frac{\pi x}{L}\right),\ A\sin\left(\frac{2\pi x}{L}\right) \ldots A\sin\left(\frac{n\pi x}{L}\right) \ldots}$$

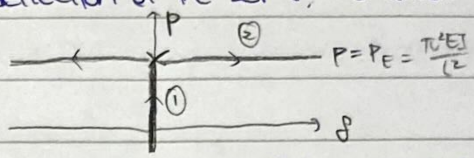
The solns have fixed mode shapes but the amplitudes A are indeterminate.



- Therefore, the bar remains straight, A=0, except at certain critical values of P. At these critical values, there is a fixed mode shape but the size of the deflection is indeterminate.
- The lowest critical value of P for a pin-jointed column is the Euler load P_E.

$$\boxed{P_E = \frac{\pi^2 EI}{L^2}}$$ — choose smallest I from I_{xx}, I_{yy} (weakest axis)

- If we plot P vs the central deflection of the bar δ, we have.

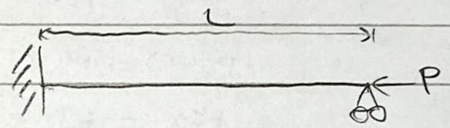


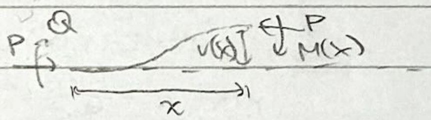
① : $\delta = 0$ for all P, (P>0 or P<0). In this case the bar doesn't buckle and remains straight.

② : At $P = P_E$ = const., δ can have any value : the axial load remains constant while the buckle grows.

- The 2 paths in the P, δ space intersect at a pt. of bifurcation.
- In practice, as the compressive load is increased, we go up ① to $P = P_E$, then we go either to the left or to the right on ②. We never go beyond $P = P_E$ for ① as the eqm. is unstable.

Fixed end column. — no longer assume pin-ended bar





Eqm. $\quad Pv(x) = M(x) + Q$

Elastic law $\quad M(x) = EI\Delta K(x)$

$\quad = EIK(x)$ here $K_0 = 0$

Geometry $\quad K(x) = -\frac{d^2v(x)}{dx^2}$

$\therefore Pv(x) = -EI\frac{d^2v(x)}{dx^2} + Q$ [Governing DE]

Rearranging. $\quad \frac{EI}{P}\frac{d^2v}{dx^2} + Pv = \frac{Q}{P}$ ← non-homogeneous DE

Define $\alpha^2 = \frac{P}{EI}$. $\quad \frac{1}{\alpha^2}\frac{d^2v}{dx^2} + v = \frac{Q}{P}$

$$v = A\sin\alpha x + B\cos\alpha x + \frac{Q}{P}$$

Boundary conditions $\quad v(0) = 0 \rightarrow \frac{Q}{P} = -B$

fixed end → $\quad v'(0) = 0 \rightarrow A = 0$ — we implied symmetry so $v'(L) = 0$ is not another boundary condition

$\quad v(L) = 0 \rightarrow B(\cos\alpha L - 1) = 0$

so either $B=0$ (no deformation)

or $\cos(\alpha L)=1 \rightarrow \alpha L = 0$ (trivial), $2\pi, 4\pi \ldots 2n\pi \ldots$

Since $\alpha^2 = \frac{P}{EI}$, the corresponding values of P are

$$P = \alpha^2 EI = \frac{4\pi^2}{L^2}EI,\ \frac{16\pi^2}{L^2}EI \ldots \frac{4n^2\pi^2}{L^2}EI \ldots$$

4 times larger than for the pinjointed case.

- We can substitute the values of α back into the GS

$$\boxed{v = B\left(\cos\left(\frac{2\pi x}{L}\right)-1\right),\ B\left(\cos\left(\frac{4\pi x}{L}\right)-1\right)\ldots}$$

The solns have fixed mode shapes but the amplitudes B are indeterminate.

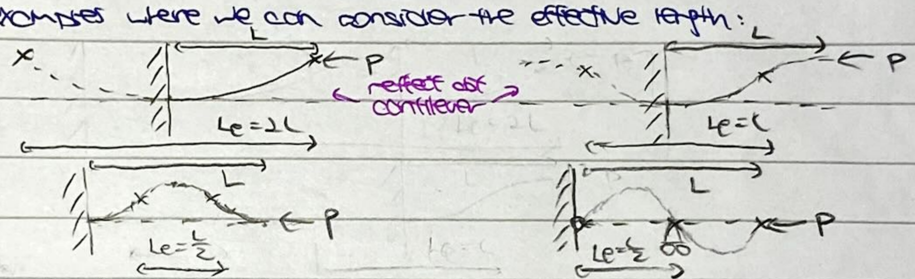
Effective lengths. no longer assume pin-ended bar

may have to reflect about the cantilever support for 2nd pt.

- For a non-pinjointed column, we look for the 2 pts. of inflection, where the curvature is zero → so moment is zero (as a pinjoint would be)
- The section between 2 pts. of inflection is equivalent to a pinended column. The length of this section is the effective length of the column, Le.
- Now, the buckling load is given by

$$\boxed{P_E = \frac{\pi^2 EI}{L_e^2}}$$

- Here are examples where we can consider the effective length:



x are the pts of inflection.

Critical stress of (perfect) columns

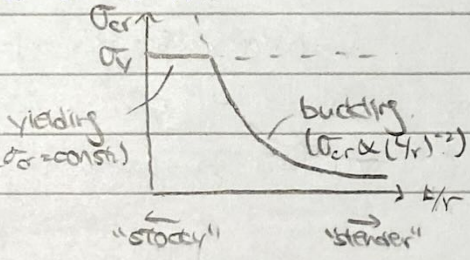
- For a perfect strut, it could fail in compression by general yielding ($\sigma_{cr} = \sigma_y$) or by buckling ($\sigma_{cr} = \sigma_E$). The critical stress at failure σ_{cr} is the smaller of σ_y and σ_E.
- The buckling stress σ_E can be obtained from P_E

$$\boxed{\sigma_E = \frac{P_E}{A} = \frac{\pi^2 E}{L_e^2}\left(\frac{I}{A}\right) = \frac{\pi^2 E}{(L_e/r)^2}}$$

where r is the radius of gyration, defined as

$$\boxed{r = \sqrt{\frac{I}{A}}}$$

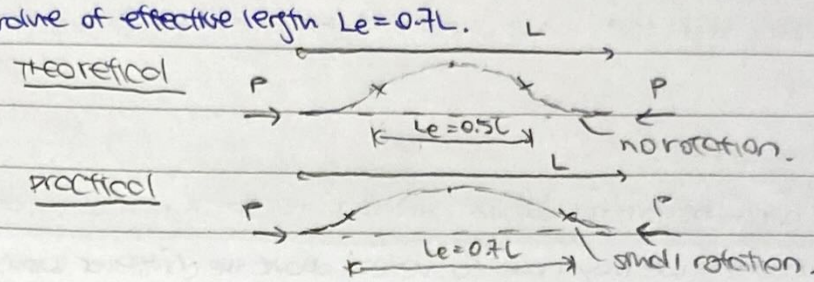
- The ratio L/r is the slenderness of the column.
- Plotting a graph of σ_{cr} against L/r:



Buckling of imperfect columns

Practical effective lengths

- In practice, restraints on columns are never perfectly rigid → some design codes specify a larger effective length than the theoretical value (lower Euler buckling load).
- For example, the effective length for the fixed-end column is theoretically $L_e = 0.5L$. However, since full fixity is nearly impossible to achieve, old design codes recommend use of a larger value of effective length $L_e = 0.7L$.



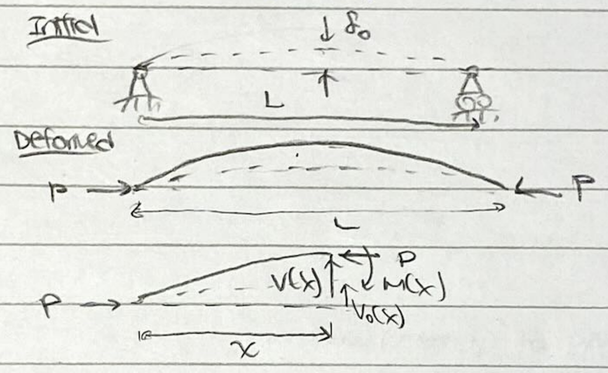
Imperfections

no longer assume initially perfectly straight

- In practice, bars are never absolutely straight — consider that the bar has an initial bow, defined by

$$v_0 = \delta_0 \sin\left(\frac{\pi x}{L}\right)$$

we typically carry out Fourier analysis on any initial shape and use the most significant component



Eqm. $\quad Pv(x) = M(x)$

Elastic law $\quad M(x) = EI\Delta K(x)$

$$= EIK(x) - EIK_0(x)$$

Geometry $\quad K(x) = -\frac{d^2v(x)}{dx^2} \quad ; \quad K_0(x) = -\frac{d^2v_0(x)}{dx^2}$

$$\therefore Pv(x) = -EI\frac{d^2v(x)}{dx^2} + EI\frac{d^2v_0(x)}{dx^2} \quad \text{[Governing DE]}$$

Rearranging,

$$\frac{EI}{P}\frac{d^2v}{dx^2} + v = -\frac{EI}{P}\left(\frac{\pi}{L}\right)^2 \delta_0 \sin\left(\frac{\pi x}{L}\right)$$

$$\frac{EI}{P}\frac{d^2v}{dx^2} + v = -\left(\frac{P_E}{P}\right)\delta_0 \sin\left(\frac{\pi x}{L}\right)$$

Define $\alpha = \frac{P}{EI}$,

$$\frac{1}{\alpha^2}\frac{d^2v}{dx^2} + v = -\left(\frac{P_E}{P}\right)\delta_0 \sin\left(\frac{\pi x}{L}\right)$$

$$v = A\sin\alpha x + B\cos\alpha x + \frac{\delta_0}{1 - P/P_E}\sin\left(\frac{\pi x}{L}\right)$$

Boundary conditions $\quad v(0) = 0 \rightarrow B = 0$

$$v(L) = 0 \rightarrow A\sin(\alpha L) = 0$$

So either $A = 0$

or $\sin(\alpha L) = 0 \rightarrow \alpha L = 0, \pi, 2\pi \ldots$

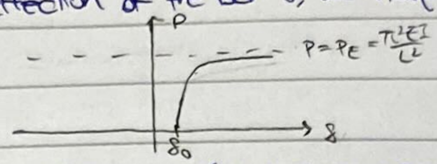
For loads less than the Euler buckling load, $A = 0$, so

$$\boxed{v = \frac{\delta_0}{1 - P/P_E} \sin\left(\frac{\pi x}{L}\right)}$$

The central deflection is given by

$$\boxed{\delta = v\left(\frac{L}{2}\right) = \frac{\delta_0}{1 - P/P_E}}$$

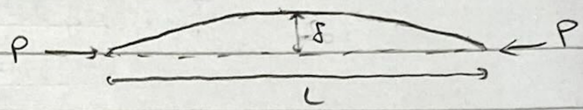
- If we plot P vs the central deflection of the bar δ, we have



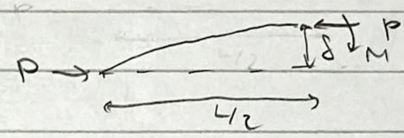
- There is no longer a bifurcation in the eqm. paths, but the initially crooked bar tends asymptotically towards the buckling load of the perfectly straight bar.

Critical stress of imperfect columns

- Consider an initially imperfect strut, carrying some axial load.



- Cut at midspan to find the max. stress



The max stress is given by (Here compressive stress is +ve)

$$\sigma_{max} = \frac{P}{A} + \frac{M y_{max}}{I} = \frac{P}{A} + \frac{P \delta y_{max}}{I}$$

Defining the average stress $\sigma_{avg} = P/A$, we obtain

σ_{avg} is the stress we expect for a perfect strut below the Euler buckling load

$$\sigma_{max} = \frac{P}{A} + \frac{P \delta y_{max}}{A r^2} = \sigma_{avg}\left(1 + \frac{\delta y_{max}}{r^2}\right).$$

Perry defined the critical stress σ_{cr} to be σ_{avg} when the max. stress is σ_y.

$$\sigma_y = \sigma_{cr}\left(1 + \frac{\delta y_{max}}{r^2}\right)$$

Rearranging, we find that as the load is increased, δ increases until first yield occurs

$$\delta = \frac{r^2}{y_{max}}\left(\frac{\sigma_y}{\sigma_{cr}} - 1\right)$$

- Recall for an imperfect strut, the central deflection δ is given by

$$\delta = \frac{\delta_0}{1 - P/P_E} = \frac{\delta_0}{1 - \sigma A/\sigma_E A} = \frac{\delta_0}{1 - \sigma/\sigma_E}$$

- Equating the 2 expressions for the central deflection δ

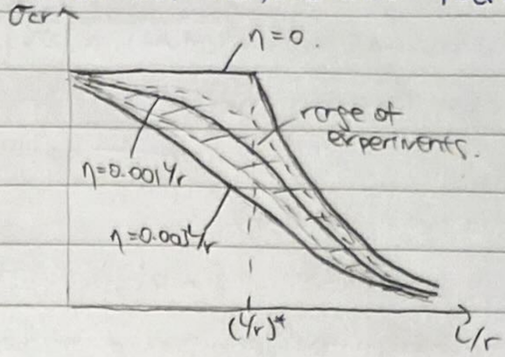
$$\frac{\delta_0}{1 - \sigma_{cr}/\sigma_E} = \frac{r^2}{y_{max}}\left(\frac{\sigma_y}{\sigma_{cr}} - 1\right)$$

$$(\sigma_y - \sigma_{cr})(\sigma_E - \sigma_{cr}) = \frac{y_{max}\delta_0}{r^2}\sigma_{cr}\sigma_E$$

Defining $\boxed{\eta = \frac{\delta_0 y_{max}}{r^2}}$ as a dimensionless measure of initial imperfection,

$$\boxed{(\sigma_y - \sigma_{cr})(\sigma_E - \sigma_{cr}) = \eta\, \sigma_{cr}\sigma_E}$$ [Perry's formula]

- Plotting the graph of sol'ns to $(\sigma_Y - \sigma_{cr})(\sigma_E - \sigma_{cr}) = \eta\,\sigma_{cr}\sigma_E$, for various η,



- The greatest reduction in strength occurs at the pt. of intersection of the 2 original curves (i.e. $L/r = (L/r)^*$) → critical stress of columns w/ slenderness of abt. $(L/r)^*$ is very imperfection sensitive (small imperfections result in large reduction in strength).
- At the pt. of intersection ($\sigma_Y = \sigma_E = \sigma^*$), Perry's formula becomes

$$(\sigma^* - \sigma_{cr})^2 = \eta\,\sigma^*\sigma_{cr}$$

$$\sigma_{cr}^2 - (2+\eta)\sigma^*\sigma_{cr} + \sigma^{*2} = 0$$

$$\left(\frac{\sigma_{cr}}{\sigma^*}\right)^2 - (2+\eta)\frac{\sigma_{cr}}{\sigma^*} + 1 = 0$$

$$\frac{\sigma_{cr}}{\sigma^*} = \frac{2+\eta \pm \sqrt{(2+\eta)^2 - 4}}{2}$$

$$= \frac{2+\eta - \sqrt{\cancel{4} + 4\eta + \eta^2 - \cancel{4}}}{2}$$ lower root as we reach this critical stress first

$$= 1 + \frac{\eta}{2} - \frac{1}{2}(\eta^2 + 4\eta)^{1/2}$$

$$= 1 + \frac{\eta}{2} - \frac{1}{2}\sqrt{\eta}\,(\eta + 4)^{1/2}$$

$$\approx 1 + \frac{\eta}{2} - \frac{\sqrt{\eta}}{2}\left(2 + \cancel{\frac{\eta}{4}}^{\text{HOT}}\right)$$

$$\approx 1 - \sqrt{\eta}$$ $\frac{\eta}{2} \ll \frac{\sqrt{\eta}}{2}$ for small η ($\eta < 1$).

$$\therefore \boxed{\sigma_{cr} \approx (1 - \sqrt{\eta})\,\sigma^*}$$

- An analysis of a large no. of experiments on hot-rolled sections showed that
  - ↳ $\eta = 0.001(L/r)$ provides a good average value to fit the experiments
  - ↳ $\eta = 0.003(L/r)$ provides a good lower bound for the experiments → good choice for design