Mechanics summary

Kinematics

Differentiating unit vectors

$$\dot{\underline{e}} = \frac{d\underline{e}}{dt} = \underline{\omega} \times \underline{e}$$

velocity and acceleration in polar and intrinsic coordinates

polar $\quad \dot{\underline{r}} = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta \quad \ddot{\underline{r}} = (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta$

Intrinsic $\quad \dot{\underline{r}} = \dot{s}\underline{e}_t \quad \ddot{\underline{r}} = \ddot{s}\underline{e}_t + \frac{\dot{s}^2}{R}\underline{e}_n$

Relating position, velocity and acceleration

$$\underline{a} = \frac{d^2x}{dt^2} = \underline{v}\frac{dv}{dx} \qquad \underline{v} = \frac{dx}{dt}$$

General relative motion

$$\underline{r}_{B/O} = \underline{r}_{B/A} + \underline{r}_{A/O}$$

$$\underline{v}_{B/O} = \underline{v}_{B/A} + \underline{v}_{A/O}$$

$$\underline{a}_{B/O} = \underline{a}_{B/A} + \underline{a}_{A/O}$$

Relative motion using rotating reference frames

$$\underline{r}_B = \underline{r}_A + \underline{r}_{B/A}$$

$$\underline{v}_B = \underline{v}_A + [\dot{\underline{r}}_{B/A}]_R + \underline{\omega} \times \underline{r}_{B/A}$$

$$\underline{a}_B = \underline{a}_A + [\ddot{\underline{r}}_{B/A}]_R + \dot{\underline{\omega}} \times \underline{r}_{B/A} + 2\underline{\omega} \times [\dot{\underline{r}}_{B/A}]_R + \underline{\omega} \times (\underline{\omega} \times \underline{r}_{B/A})$$

Relative motion of points within a rigid body

$$\underline{v}_A \cdot \underline{e} = \underline{v}_B \cdot \underline{e}$$

$$\underline{r}_{B/A} = r\underline{e}$$

$$\dot{\underline{r}}_{B/A} = \underline{\omega} \times \underline{r}_{B/A}$$

$$\ddot{\underline{r}}_{B/A} = \dot{\underline{\omega}} \times \underline{r}_{B/A} + \underline{\omega} \times (\underline{\omega} \times \underline{r}_{B/A})$$

Relative motion of points within a rigid body in planar motion

$$\underline{r}_{B/A} = r\underline{e}_1$$

$$\dot{\underline{r}}_{B/A} = r\dot{\theta}\underline{e}_2$$

$$\ddot{\underline{r}}_{B/A} = -r\dot{\theta}^2\underline{e}_1 + r\ddot{\theta}\underline{e}_2$$

Instantaneous centres. (rigid body in planar motion)

$$\omega = \frac{v_i}{r_i}$$

Elliptical orbits

$$\frac{r_P}{r_A} = \frac{(1-e)a}{(1+e)a} \qquad \frac{l}{r} = 1 + e\cos\theta$$

Vis-viva equation

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

Dynamics

Moment of a force about O / an axis

$$\underline{q}_o = \underline{r} \times \underline{F} \qquad Q_o = \underline{q}_o \cdot \underline{e}_n$$

Moment of a force about other points

$$\underline{q}_o = \underline{q}_G + \underline{r}_{G/O} \times \underline{F}$$

Moment of momentum about O / an axis

$$\underline{h}_o = \underline{r} \times m\underline{v} \qquad H_o = \underline{h}_o \cdot \underline{e}_n$$

Moment of momentum about other points

$$\underline{h}_o = \underline{h}_G + \underline{r}_{G/O} \times m\underline{v}$$

Centre of mass of a general body

$$\underline{r}_G = \frac{1}{M}\int \underline{r}\, dm$$

Moment of inertia for a general rigid body rotating about a fixed axis.

$$I_o = \int r_o^2\, dm$$

Moment of inertia for a planar body in general planar motion

$$I_G = \int r_G^2\, dm$$

Parallel axis theorem (general rigid body rotating about a fixed axis).

$$I_o = I_G + m r_{G/O}^2$$

Perpendicular axis theorem (planar body rotating about a fixed axis)

$$I_{zz} = I_{xx} + I_{yy}$$

Moment of momentum of a planar body rotating about a fixed axis about G/O

$$\underline{h}_o = I_o \underline{\omega}_{body} \qquad \underline{h}_o = \underline{h}_G + \underline{r}_{G/O} \times m\underline{v} = (I_G + m r_{G/O}^2)\,\underline{\omega}_{body}$$

Moment of momentum of a planar body in general planar motion about G/O

$$\underline{h}_G = I_G \underline{\omega}_{body} \qquad \underline{h}_o = \underline{h}_G + \underline{r}_{G/O} \times m\underline{v}$$

Newton's 2nd law for a particle / general body

$$\underline{F} = m\underline{a} \qquad \underline{F} = m\ddot{\underline{r}}_G$$

Newton's 2nd law for rotation for a general rigid body rotating about a fixed axis.

$$\underline{q}_o = I_o \dot{\underline{\omega}}_{body}$$

Newton's 2nd law for rotation for a planar body in general planar motion.

$$\underline{q}_G = I_G \dot{\underline{\omega}}_{body}$$

D'Alembert for a particle / rigid body.

Particle, general motion $\quad \underline{F} - m\underline{a} = 0$

General body, general motion $\quad \underline{F} - m\ddot{\underline{r}}_G = 0$

Planar body, planar motion $\quad \underline{q}_G - I_G \dot{\underline{\omega}}_{body} = 0$

Impulse-momentum principle for a particle / general body

$$\underline{F} = \frac{d\underline{p}}{dt} \qquad \underline{J} = \underline{p}_B - \underline{p}_A = \int_{t_A}^{t_B} \underline{F}\,dt$$

Angular impulse-angular momentum principle for a particle about O

$$\underline{q}_o = \frac{d\underline{h}_o}{dt} \qquad \underline{r} \times \underline{J} = \underline{h}_{oB} - \underline{h}_{oA} = \int_{t_A}^{t_B} \underline{q}_o\,dt$$

Angular impulse-angular momentum principle for a general rigid body rotating about a fixed axis

$$\underline{q}_o = \frac{d\underline{h}_o}{dt} \qquad \underline{r} \times \underline{J} = \underline{h}_{oB} - \underline{h}_{oA} = \int_{t_A}^{t_B} \underline{q}_o\,dt$$

Angular impulse-angular momentum principle for a general rigid body in general motion.

$$\underline{q}_G = \frac{d\underline{h}_G}{dt} \qquad \underline{r} \times \underline{J} = \underline{h}_{GB} - \underline{h}_{GA} = \int_{t_A}^{t_B} \underline{q}_G\,dt$$

Kinetic energy of a particle / rigid body

$$T = \frac{1}{2} m\underline{v}\cdot\underline{v} = \frac{1}{2} m|\underline{v}|^2 \qquad T = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega_{body}^2$$

Work-energy principle

$$\underline{F}\cdot\underline{v} = \frac{dT}{dt} \qquad T_B - T_A = \int_{t_A}^{t_B} \underline{F}\cdot d\underline{r}$$

Potential energy (in a conservative force field)

$$\underline{F}\cdot d\underline{r} = -d\underline{V} \qquad V_B - V_A = -\int_{t_A}^{t_B} \underline{F}_{consv}\cdot d\underline{r}$$

Conservation of mechanical energy

$$T + V = E \text{ (constant)}$$

Conservation of energy

$$\Delta T + \Delta V = WD$$

Newton's law of impact

$$e = -\frac{v_2 - v_1}{u_2 - u_1}$$

Variable mass systems

$$F_{body} = \dot{m}\,\underline{V}_{P/B}$$

Kinematics of particles.

Force, mass and acceleration

Newton's laws.

① When all ext. influences on a particle are removed, the particle moves w/ constant velocity (velocity can be 0 → the particle remains at rest).

② When a force $\underline{F}$ acts on a particle of mass m, the particle moves w/ instantaneous acceleration $\underline{a}$ given by $\Sigma\underline{F} = m\underline{a}$.

③ When 2 particles exert forces upon each other, these forces are (i) equal in magnitude, (ii) opposite in direction and (iii) // to the straight line joining the 2 particles.

Definitions

- A particle has all its mass concentrated at a single pt → no need to worry about its physical size or the effects of rotation.
- Mass is the reluctance of a body to being accelerated

Forces.

- There are 4 fundamental forces — gravitation, electromagnetic, strong and weak (subatomic scale only). However it is more useful to have "force models", i.e. working at a larger scale.
- Common forces include
  - ↳ Gravity : $F = \frac{GMm}{r^2}$ (planet-scale) ; $F = mg$, $g = 9.81\,ms^{-2}$ (surface-scale)
  - ↳ Contact : $N = mg$ if $a = 0$.
  - ↳ Friction : $f = \mu N$.
  - ↳ Drag : $D = \frac{1}{2} C_D \rho A V^2$

Free body diagram (FBD)

- In a FBD, what matters is
  - ↳ The forces that act on the body
  - ↳ The magnitude and direction of each force applied.
  - ↳ The line of action of each force (i.e. where the force is applied).
- To apply N2L, sum the force vectors to obtain $\Sigma\underline{F}$ by using the tip-to-tail method
  - ↳ If $\Sigma\underline{F} = 0$: closed polygon → N1L
  - ↳ If $\Sigma\underline{F} \neq 0$: not closed polygon → N2L (the body will accelerate).

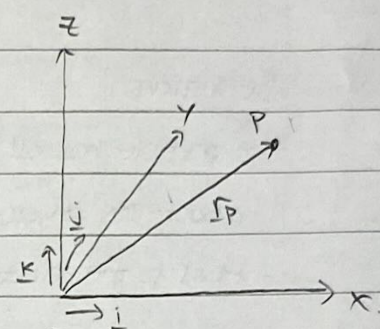
Coordinate systems.

Frames of reference

- To define position, velocity and acceleration as vectors, we need to define a frame of reference to measure them relative to. / choose anything that is the most convenient!
- It may be useful to choose a reference frame moving w/ the object of interest. However, Newton's laws may not be valid in that frame.
- A frame of reference is inertial when the frame itself has no acceleration → we can apply Newton's laws.

Cartesian coordinates.

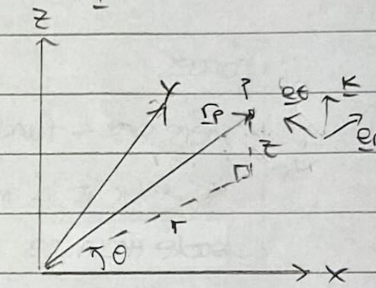
- Coordinates : (x, y, z).
- Unit vectors : $\underline{i}, \underline{j}, \underline{k}$ or $\underline{e}_x, \underline{e}_y, \underline{e}_z$
- Position vector : $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$



(Cylindrical) polar coordinates.

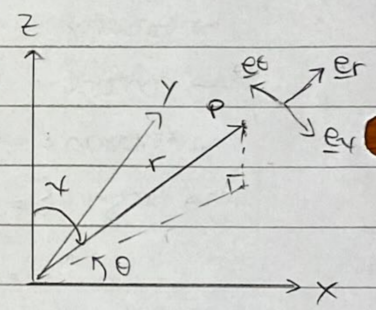
- Coordinates : (r, θ, z)
- unit vectors : $\underline{e}_r, \underline{e}_\theta, \underline{k}$ or $\underline{e}_r, \underline{e}_\theta, \underline{e}_z$
- Position vector : $\underline{r} = r\underline{e}_r + z\underline{k}$ *

distinction not useful since we are doing 2D problems. → cylindrical w/o z.



(Spherical) polar coordinates.

- Coordinates : (r, θ, ψ)
- Unit vectors : $\underline{e}_r, \underline{e}_\theta, \underline{e}_\psi$
- Position vector : $\underline{r} = r\underline{e}_r$ *

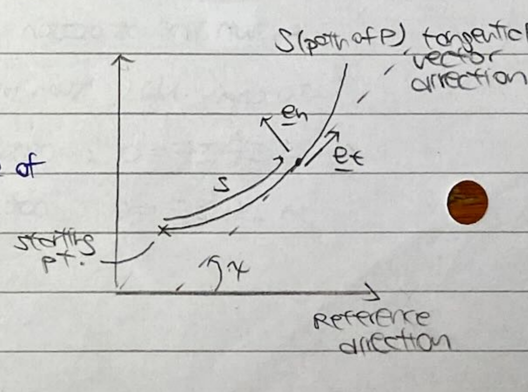


* Note for both cylindrical/spherical polar coordinates, we don't need $\underline{e}_\theta$ to reach an arbitrary position since $\underline{e}_r$ is defined to be aligned s.t. $\underline{e}_r$ points radially towards the position. For a 2nd particle, we just define a new $\underline{e}_r'$

Intrinsic coordinates.

- Coordinates : (s, ψ)
- Unit vectors : $\underline{e}_t, \underline{e}_n$ / points towards the centre of curvature of the path.
- Position vector : N/A (in general)
- Velocity vector : $\underline{v} = \dot{s}\underline{e}_t$

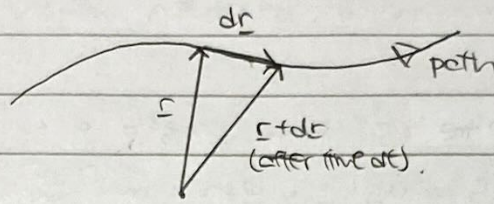
* The path of P, S must be pre-defined.



velocity and acceleration

Definitions.

- Consider a particle moving along some path, w/ instantaneous position vector $\underline{r}$:



- The average velocity is defined as the change in displacement divided by the time interval of the change: $\bar{v} = \frac{\Delta r}{\Delta t}$.
- The instantaneous velocity is the instantaneous rate of change of position as a vector: $\underline{v} = \frac{d\underline{r}}{dt} = \dot{\underline{r}}$ ← tangential to path
- The acceleration (generally not tangential to path) is the rate of change of velocity as a vector:
  $\underline{a} = \frac{d\underline{v}}{dt} = \frac{d^2\underline{r}}{dt^2} = \ddot{\underline{r}}$
- To express velocity and acceleration as vectors, we need to be able to differentiate the position vector (in different coordinate systems).

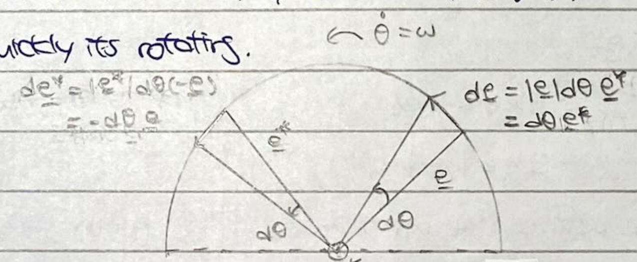
Cartesian coordinates.

because $\underline{i}, \underline{j}, \underline{k}$ are constant vectors in cartesian coordinates.

- Position: $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$
- Velocity: $\dot{\underline{r}} = \frac{d\underline{r}}{dt} = \frac{dx}{dt}\underline{i} + x\cancel{\frac{d\underline{i}}{dt}}^{\,0} + \frac{dy}{dt}\underline{j} + y\cancel{\frac{d\underline{j}}{dt}}^{\,0} + \frac{dz}{dt}\underline{k} + z\cancel{\frac{d\underline{k}}{dt}}^{\,0}$.
  $= \dot{x}\underline{i} + \dot{y}\underline{j} + \dot{z}\underline{k}$
- Acceleration: $\ddot{\underline{r}} = \frac{d\dot{\underline{r}}}{dt} = \ddot{x}\underline{i} + \ddot{y}\underline{j} + \ddot{z}\underline{k}$ [since $\frac{d\underline{e}}{dt} = 0$].

Differentiating rotating unit vectors.

- For a particle travelling along its path, the unit vectors in polar coords change direction (although they remain unit length) → differentiation depends on how quickly its rotating.



* $|\underline{e}| = |\underline{e}^*| = 1$.

$\dot{\theta}/\omega$ is the angular velocity

* $\underline{e}^*$ is defined s.t. it is rotated $+90°$ from $\underline{e}$ (in this case +ve angle is anticlockwise).

- we can see that $d\underline{e} = d\theta\,\underline{e}^*$ so $\frac{d\underline{e}}{dt} = \frac{d\theta}{dt}\underline{e}^* = \dot{\theta}\underline{e}^*$
  and $d\underline{e}^* = -d\theta\,\underline{e}$ so $\frac{d\underline{e}^*}{dt} = -\frac{d\theta}{dt}\underline{e} = -\dot{\theta}\underline{e}$

Angular velocity.

- The angular velocity can be represented by a vector that has:
  - ↳ magnitude: equal to the rotation rate $\dot{\theta}$
  - ↳ direction: // to the axis of rotation, using the RH grip rule $\underline{k}$ $\quad \omega \circlearrowleft \odot \underline{k}$
  - → $\underline{\omega} = \dot{\theta}\underline{k}$
- Considering, $\underline{k} \times \underline{e} = \underline{e}^*$ and $\underline{k} \times \underline{e}^* = -\underline{e}$,

  then $\frac{d\underline{e}}{dt} = \dot{\theta}\underline{e}^* = \dot{\theta}\underline{k} \times \underline{e} = \underline{\omega} \times \underline{e}$ and $\frac{d\underline{e}^*}{dt} = -\dot{\theta}\underline{e} = \dot{\theta}\underline{k} \times \underline{e}^* = \underline{\omega} \times \underline{e}^*$.
- In general, the result for differentiating unit vectors is: $\boxed{\frac{d\underline{e}}{dt} = \underline{\omega} \times \underline{e}}$
- Physically, when pre-multiplying by $\underline{\omega} = \dot{\theta}\underline{k}$, we rotate +90°.
  - ↳ so $\underline{e} \to \underline{e}^* \to -\underline{e} \to -\underline{e}^* \to \underline{e}$

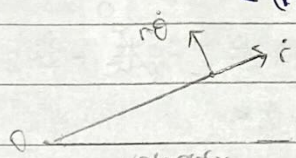
Polar coordinates. [Same as cylindrical but ignore $z(\underline{k})$].

- Position: $\underline{r} = r\underline{e}_r$
- Velocity: $\dot{\underline{r}} = \frac{d\underline{r}}{dt} = \frac{dr}{dt}\underline{e}_r + r\frac{d\underline{e}_r}{dt}$ $\quad * \frac{d\underline{e}_r}{dt} = \underline{\omega} \times \underline{e}_r = \dot{\theta}\underline{e}_\theta$

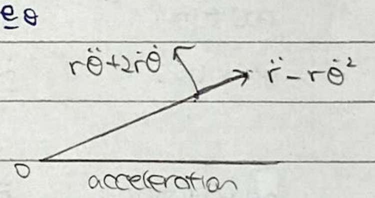
  $= \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta$ $\quad$ triple product rule
- Acceleration: $\ddot{\underline{r}} = \frac{d\dot{\underline{r}}}{dt} = \ddot{r}\underline{e}_r + \dot{r}\dot{\underline{e}}_r + \dot{r}\dot{\theta}\underline{e}_\theta + r\ddot{\theta}\underline{e}_\theta + r\dot{\theta}\dot{\underline{e}}_\theta$

  $= \ddot{r}\underline{e}_r + \dot{r}\dot{\theta}\underline{e}_\theta + \dot{r}\dot{\theta}\underline{e}_\theta + r\ddot{\theta}\underline{e}_\theta - r\dot{\theta}^2\underline{e}_r$

  $= (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta$



velocity

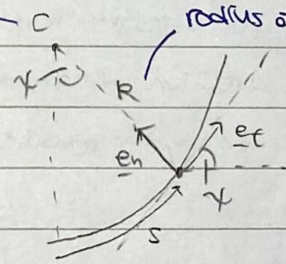


acceleration

DB $\boxed{\dot{\underline{r}} = \underbrace{\overbrace{\dot{r}\underline{e}_r}^{\text{radial}} + r\dot{\theta}\underline{e}_\theta}_{\text{circumferential}}}$

DB $\boxed{\ddot{\underline{r}} = \overbrace{(\ddot{r} - r\dot{\theta}^2)\underline{e}_r}^{\text{radial}} + \overbrace{(r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta}^{\text{circumferential}}}$ $\quad$ centripetal $\quad$ coriolis.

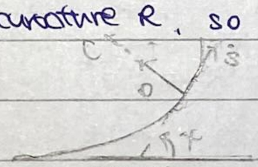
Intrinsic coordinates.



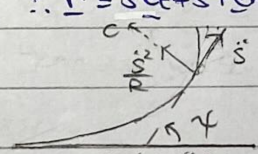
- Position: N/A.
- Velocity: $\dot{\underline{r}} = \dot{s}\underline{e}_t$
- Acceleration: $\ddot{\underline{r}} = \frac{d\dot{\underline{r}}}{dt} = \ddot{s}\underline{e}_t + \dot{s}\dot{\underline{e}}_t$

  $= \ddot{s}\underline{e}_t + \dot{s}\dot{\psi}\underline{e}_n$

- A short section of the path can be approximated as a section of a circle w/ radius of curvature R, so $\dot{s} = R\dot{\psi} \to \dot{\psi} = \frac{\dot{s}}{R}$. $\therefore \ddot{\underline{r}} = \ddot{s}\underline{e}_t + \dot{s}\dot{\psi}\underline{e}_n = \ddot{s}\underline{e}_t + \frac{\dot{s}^2}{R}\underline{e}_n$



velocity



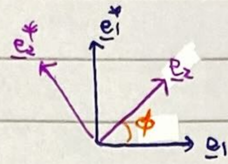
acceleration

DB $\boxed{\dot{\underline{r}} = \dot{s}\underline{e}_t}$ $\quad$ tangential

DB $\boxed{\ddot{\underline{r}} = \overbrace{\ddot{s}\underline{e}_t}^{\text{tangential}} + \underbrace{\frac{\dot{s}^2}{R}\underline{e}_n}_{\text{radial}}}$

Conversion between unit vectors.

- In general, we can convert the unit vectors by overlaying them on top of each other. Then we need to find the angle between them.



- Converting from $\underline{e}_1, \underline{e}_1^*$ to $\underline{e}_2, \underline{e}_2^*$: $\underline{e}_2 = \cos\phi\, \underline{e}_1 + \sin\phi\, \underline{e}_1^*$

  $\underline{e}_2^* = -\sin\phi\, \underline{e}_1 + \cos\phi\, \underline{e}_1^*$

$$\begin{pmatrix} e_2 \\ e_2^* \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} e_1 \\ e_1^* \end{pmatrix}$$

rotation matrix

- Converting from $\underline{e}_2, \underline{e}_2^*$ to $\underline{e}_1, \underline{e}_1^*$: $\underline{e}_1 = \cos(-\phi)\, \underline{e}_2 + \sin(-\phi)\, \underline{e}_2^*$

  $\underline{e}_1^* = -\sin(-\phi)\, \underline{e}_2 + \cos(-\phi)\, \underline{e}_2^*$

$$\begin{pmatrix} e_1 \\ e_1^* \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} e_2 \\ e_2^* \end{pmatrix}$$

* For polar $\Leftrightarrow$ intrinsic, $\phi = |\theta - \chi|$

Common tricks for solving questions.

- $\dot{s} = |\dot{\underline{r}}_P|$
- $\underline{e}_t = \frac{1}{\dot{s}} \dot{\underline{r}}_P$
- $a_t = \ddot{\underline{r}}_P \cdot \underline{e}_t$
- $\underline{a}_n = \ddot{\underline{r}}_P - \underline{a}_t$ so $|\underline{a}_n|^2 = |\ddot{\underline{r}}_P|^2 - |\underline{a}_t|^2$

we found $\dot{\underline{r}}_P$ / $\ddot{\underline{r}}_P$ using cartesian/polar

→ we want to use intrinsic coords (so we can find inst. rad. of curvature)

Dynamics of particles

Equations of motion (Newton's laws and D'Alembert forces)

Deriving the equation of motion

- The eqn. of motion of a system is the eqn. that determines the motion of the particle.
- The eqn. defines the rules that the motion is governed by but does not tell us what the actual motion is. → solve numerically/analytically to find the actual motion.
- In general, we can find the eqn. of motion:
  - ↳ 1) Define the coordinate system (type + orientation)
  - ↳ 2) Identify the deg of freedom of the body (if useful)
  - ↳ 3) Draw a BIG free body diagram, only inc. the forces acting on the body.
  - ↳ 4) Apply N2L to find the eqn. of motion.

D'Alembert forces

- N2L for a particle is : $\Sigma F = ma$.

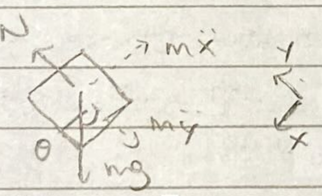
  Rearranging gives : $\boxed{\Sigma F - ma = 0}$

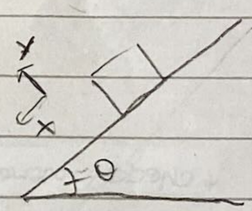
- We can think of mass × acceleration as an inertial force, then the new resultant force $\Sigma F' = \Sigma F - ma = 0$ → we can apply the principles of static eqm.
- To do this, we add the inertial force onto the FBD. (i.e. add a force of magnitude ma in the $-a$ direction)

* If the direction of a is unknown, set them in the -ve direction of the unit vectors.

-e.g: Block on slope.

FBD:





N2L: x-dir $mg\sin\theta - m\ddot{x} = 0$ → $\ddot{x} = g\sin\theta$

y-dir $N - \cancel{m\ddot{y}}^{\,0} - mg\cos\theta = 0$ → $N = mg\cos\theta$

since the block must stay on the plane.

Solving equations of motion

- For simple systems, we can find analytical sol'n to the DE (eqn. of motion)
- Sometimes, we can integrate directly wrt time or displacement :

  ↳ $\boxed{a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = v\frac{dv}{dx}}$ ↳ $v = \frac{dx}{dt}$

- For a nth order DE, we need n boundary conditions to solve for the arbitrary constants. (we can put the limits straight into the integral expression to do so)

Integrating N2L.

- The 2 approaches to integrating the eqn. of motion – displacement and time integrals lead to the fundamental principles of energy and momentum.

N2L

$F = mv\frac{dv}{dx} = ma$

displacement integral: $\int \underline{F} \cdot d\underline{x} = \int mv\,dv$

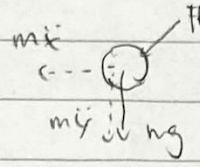
time integral: $\int F\,dt = \int m\,a\,dt$

Energy: $\int \underline{F} \cdot d\underline{x} = \frac{1}{2}mv^2$

Momentum: $\int F\,dt = \Delta mv$

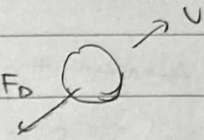
- eg: Projectile motion w/ air resistance., $F_D = -k\underline{v}$. At $t=0$, $\underline{x} = 0$, $\dot{\underline{x}} = \underline{v}_0$

FBD:



$F_D = -k\underline{v}$

$= -k(\dot{x}\underline{i} + \dot{y}\underline{j})$



N2L: x-dir. $-k\dot{x} - m\ddot{x} = 0 \rightarrow m\ddot{x} + k\dot{x} = 0$ [1]

y-dir. $-k\dot{y} - mg - m\ddot{y} = 0 \rightarrow m\ddot{y} + k\dot{y} + mg = 0$ [2].

Consider x-dir. Integrate [1] wrt time $m\dot{x} + kx = c$

Solving the DE gives $x = Ae^{-\frac{k}{m}t} + B$.

Using the initial conditions $t=0$, $x=0$, $\dot{x} = V_0\cos\theta$.

$A = -\frac{mV_0\cos\theta}{k}$, $B = \frac{mV_0\cos\theta}{k}$ $\therefore x = \frac{mV_0\cos\theta}{k}(1 - e^{-\frac{k}{m}t})$

Consider y-dir Integrate [2] wrt time $m\dot{y} + ky + mgt = C_1$

Solving the DE gives $y = A_1e^{-\frac{k}{m}t} + A_2t + A_3$

Using the initial conditions $t=0$, $y=0$, $\dot{y} = V_0\sin\theta$

$A_1 = -(\frac{mg}{k} + V_0\sin\theta)\frac{m}{k}$, $A_2 = -\frac{mg}{k}$, $A_3 = (\frac{mg}{k} + V_0\sin\theta)\frac{m}{k}$ $\therefore y = \frac{m}{k}(\frac{mg}{k} + V_0\sin\theta)(1 - e^{-\frac{k}{m}t}) - \frac{mg}{k}t$

As $t \to \infty$, $x \to \frac{mV_0\cos\theta}{k}$ so $x_{max} = \frac{mV_0\cos\theta}{k}$.

Energy

For a particle, over a given interval, the increase in KE is equal to the work done by the applied forces.

Deriving the energy principle

$* \frac{d}{dt}(\underline{a} \cdot \underline{b}) = \underline{a} \cdot \frac{d\underline{b}}{dt} + \underline{b} \cdot \frac{d\underline{a}}{dt}$

so $\frac{d}{dt}(\underline{v} \cdot \underline{v}) = 2\underline{v} \cdot \frac{d\underline{v}}{dt}$.

$* \underline{v}\,dt = \frac{d\underline{r}}{dt}dt = d\underline{r}$

- From N2L, $F = m\underline{a} = m\frac{d\underline{v}}{dt}$

Dotting both sides by $\underline{v}$, $\underline{F} \cdot \underline{v} = m\frac{d\underline{v}}{dt} \cdot \underline{v}$

using $\underline{v} \cdot \frac{d\underline{v}}{dt} = \frac{d}{dt}(\frac{1}{2}\underline{v} \cdot \underline{v})$ $= \frac{d}{dt}(\frac{1}{2}m\underline{v} \cdot \underline{v})$

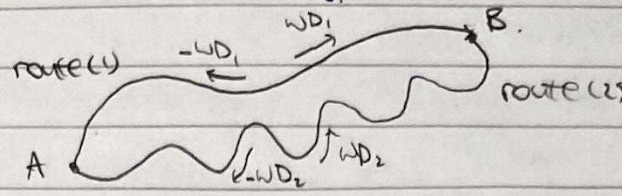
- We define the kinetic energy of a particle $\boxed{T = \frac{1}{2}m\underline{v} \cdot \underline{v} = \frac{1}{2}m|\underline{v}|^2}$

so $\boxed{\underline{F} \cdot \underline{v} = \frac{dT}{dt}}$ $\rightarrow \int_{t_A}^{t_B} \underline{F} \cdot \underline{v}\,dt = \int_{T_A}^{T_B} dT$

inc. all forces except D'Alembert

$\boxed{\underbrace{T_B - T_A}_{\text{change in KE of the system}} = \int_{t_A}^{t_B} \underline{F} \cdot \underline{v}\,dt = \underbrace{\int_{t_A}^{t_B} \underline{F} \cdot d\underline{r}}_{\text{WD on the system going from A to B.}}}$ [Work-energy principle]

Conservative forces and potential energy



- Consider moving a particle from A to B through a force field $\underline{F}(x,y,z)$ that depends on position.
- If the work done in moving from A to B along a particular route is WD, the work done in moving from B to A along the same route is −WD. This is because the forces are the same but the direction of travel is opposite.

* If the force field depends on velocity, $WD_{AB} \neq -WD_{BA}$ since the force changes direction when the route is reversed, so $WD_{AB} = WD_{BA}$.

- If the work done in moving from A to B along any route is the same (path independent) and $WD_{AB} = -WD_{BA}$, then the force field $\underline{F}$ is conservative.
- For a conservative force field, the WD from A to B is always the same, so we can associate A and B w/ a particular energy V_A and V_B respectively.
- As the energy expended in moving from A to B has the potential to be recovered when moving from B to A (regardless of route taken), the energy associated w/ a given position is the potential energy $V(x,y,z)$.
- The work req. to move from A to B by an ext. force is equal to the increase in PE.

$$\boxed{\int_A^B (-F)\cdot dx = V_B - V_A = \Delta U}$$

F must be a conservative force
Ext. force F_{ext} opposes force field so $F_{ext} = -F$.

As this is true for all A and B, $\quad F dx = -dU \quad , \quad F = -\frac{dU}{dx}$ [scalar]

WD by a conservative force field is equal to the loss in potential energy

- In the 3D case, $\boxed{\underline{F}\cdot d\underline{r} = -dU \quad , \quad \underline{F} = -\frac{dU}{d\underline{r}}}$ [vector]

Conservation of mechanical energy.

- As $\int_A^B \underline{F}\cdot d\underline{r} = T_B - T_A$

and for a conservative force $\underline{F}$, $\int_A^B \underline{F}\cdot d\underline{r} = -(V_B - V_A)$.

$\therefore \int_A^B \underline{F}\cdot d\underline{r} = T_B - T_A = -(V_B - V_A)$

$$T_A + V_A = T_B + V_B.$$

- If we choose to define a reference starting energy $E = T_A + V_A$, this leads to the principle of conservation of mechanical energy — when a particle moves in a conservative force field, the mechanical energy (KE+PE) remains constant.

$$\boxed{T + V = E \text{ (constant)}}$$

Common forces and their potential energies.

① Gravity (local)

$\underline{F} = -mg\underline{k}$ — as gravity always vertically

$\underline{F} \cdot d\underline{r} = -dV$

$\begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = -dV$ — the horizontal motion is orthogonal to the force

$-mg\,dz = -dV$ — → does not contribute to the work done.

$V = mgz$.

② Gravity (inverse square law)

$\underline{F} = -\frac{GMm}{r^2}\underline{e}_r$ — Newton's law of gravitation.

$\underline{F} \cdot d\underline{r} = -dV$

$-\frac{GMm}{r^2}dr = -dV$ — only the $\underline{e}_r$ component of displacements contribute to the work done.

$V = -\frac{GMm}{r}$

③ Springs (w/o hysteresis)

$F = -kx$ — Hooke's law (linear springs).

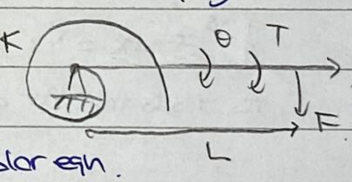
$F\,dx = -dV$ — considering the scalar eqn.

$-kx\,dx = -dV$

$V = \frac{1}{2}kx^2$

④ Torsional springs

$T = -k\theta$ — assuming a linear torsional spring.



$FL = -k\left(\frac{x}{L}\right)$

$F = -\frac{k}{L^2}x$

$F\,dx = -dV$ — considering the scalar eqn.

$-\frac{k}{L^2}x\,dx = -dV$

$V = \frac{1}{2}\frac{k}{L^2}x^2 = \frac{1}{2}\frac{k}{L^2}(L\theta)^2 = \frac{1}{2}k\theta^2$ — * similar form as normal spring.

- other common conservative force models inc. magnetic / electrostatic force.
- Common non-conservative force models inc. springs w/ hysteresis, friction, drag force.

(The non-conservative forces depend on velocity or past motion in some way as the force cannot be independent of the path btwn 2 pts → the specific motion affects the force).

Conservation of energy

- In general, there may be some forces acting on a particle that are conservative and others that are not.

$$\Delta T = \int_A^B \underline{F} \cdot d\underline{r}$$
$$= \int_A^B (\underline{F}_{consv} + \underline{F}_{nonconsv}) \cdot d\underline{r}$$
$$= -\Delta V + WD$$

work done on the system

* $\int_A^B \underline{F}_{consv} \cdot d\underline{r} = -\Delta U$
$\int_A^B \underline{F}_{non\text{-}consv.} \cdot d\underline{r} = WD.$

- Rearranging, the increase in mechanical energy of a particle is equal to the work done by non-conservative forces acting on the particle.

$$\boxed{\Delta T + \Delta V = WD}$$

* WD is -ve for energy loss (e.g. friction)

* For the work done by the system, the work done on the system is -ve so WD term on the RHS would be -ve.

Potential energy, equilibrium and stability

- A particle at rest and not accelerating (i.e. $\underline{a} = \underline{v} = 0$) is said to be in eqm, which can only happen if the net force acting on the particle is 0. $\Sigma \underline{F} = 0$.
- For a particle in a conservative force field, the force $F(x)$ is the derivative of the potential energy $V(x)$. $F = -\frac{dU}{dx}$.

So at eqm, the PE is a stationary pt. $\frac{dU}{dx} = 0$. → PE constant at an eqm. pt.

- There are 3 types of eqm:
  - ↳ Stable eqm when $V(x)$ is a minima (i.e. $U'' > 0$)
  - ↳ Unstable eqm when $V(x)$ is a maxima (i.e. $U'' < 0$)
  - ↳ Neutral eqm when $V(x)$ is a pt. of inflection.
- For conservative forces, $T + U = E$ (constant) so

$$\frac{1}{2}mv^2 + V(x) = E = \text{constant}.$$

As v is non-negative, ($U \geq 0$), $V(x)$ is always less than or equal to E and the particle remains in the region $V(x) \leq E$

Also, when KE is max, PE is min. and vice versa.

Linear momentum

Linear momentum and impulse

- The linear momentum $\underline{p}$ of a particle of mass m travelling at velocity $\underline{v}$ is defined to be.

$$\underline{p} = m\underline{v}$$

- We can rewrite N2L in terms of linear momentum

$$\boxed{\underline{F} = m\underline{a} = m\frac{d\underline{v}}{dt} = \frac{d}{dt}(m\underline{v}) = \frac{d\underline{p}}{dt}}$$

- The resultant force applied to a particle is equal to the rate of change of momentum, and if no resultant force is applied then linear momentum is conserved — this is the principle of conservation of linear momentum. $(\underline{F} = \frac{d\underline{p}}{dt} = 0)$
- Integrating wrt time gives

$$\boxed{\int_{t_A}^{t_B} \underline{F}\,dt = \int_A^B d\underline{p} = \underline{p}_B - \underline{p}_A} \quad \text{[Impulse-momentum principle]}$$

- The integral of force over time is the impulse and gives the net change of momentum in that time window.
- Consider a collection of particles. The total linear momentum is the sum of each particle's momentum.

$$\underline{P}_{total} = \sum_{\forall i} \underline{p}_i = \sum_{\forall i} m\underline{v}_i$$

The ith particle has mass m_i and is acted on by a net ext. force $\underline{F}_i$ and int. forces $\underline{G}_{ij}$ (exerted by other particles).

Consider the internal forces $\underline{G}_{ij}$.

↳ N3L tells us $\underline{G}_{ij} = -\underline{G}_{ji}$ so $\Delta p_i = -\Delta p_j$ → the int. forces cancel in pairs

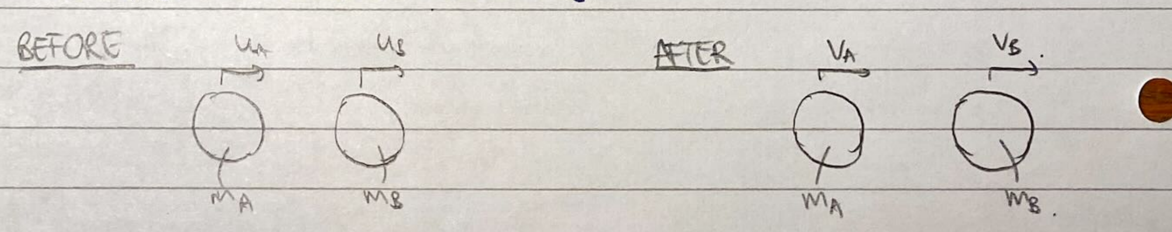
→ The int. forces make no difference to the total linear momentum.

Therefore, $\sum_{\forall i} \underline{F}_i = \underline{F} = \frac{d\underline{p}}{dt}$, $\int_{t_A}^{t_B} \underline{F}\,dt = \underline{p}_B - \underline{p}_A$

even for a collection of particles. (i.e. only the ext. forces matter).

- This is the linear momentum principle — the rate of increase of linear momentum is equal to the total ext. force applied. Equivalently, the net increase of linear momentum is equal to the total ext. impulse applied.

- eg: Colliding particles. What are the final velocities if 2 particles of mass m_A and m_B collide, given initial speeds u_A and u_B.



- The int. forces and duration of the collision makes no difference to the total linear momentum so we can say

$$\underline{P}_{before} = \underline{P}_{after}$$

$$m_A u_A + m_B u_B = m_A v_A + m_B v_B.$$

- To solve for the separate motion of A and B, we need more info (eg the amt of energy lost / they stick together after impact). This is because the int. forces can do work so they can effect the total change of KE of the pairs of particles.
- An important special case is when the collisions are elastic — the interaction forces during a collision are conservative and no energy is lost.

Energy: $\frac{1}{2}m_A u_A^2 + \frac{1}{2}m_B u_B^2 = \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2$ Momentum: $m_A u_A + m_B u_B = m_A v_A + m_B v_B$.

$$m_A(u_A^2 - v_A^2) = m_B(v_B^2 - u_B^2)$$ $$m_A(u_A - v_A) = m_B(v_B - v_A) \quad [2].$$

$$m_A(u_A + v_A)(u_A - v_A) = m_B(v_B + u_B)(v_B - u_B) \quad [1]$$

sub [2] into [1], $\cancel{m_B}\cancel{(v_B - u_B)}(u_A + v_A) = \cancel{m_B}(v_B + u_B)\cancel{(v_B - u_B)}$

$$u_A + v_A = u_B + v_B.$$

Coefficient of restitution.

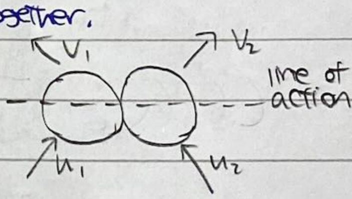
- We can quantify the energy loss of a collision using the coefficient of restitution e that relates the ratio of relative velocities before and after the impact.

$$\boxed{e = -\frac{v_2 - v_1}{u_2 - u_1}} = \frac{\text{relative velocity after}}{\text{relative velocity before}} = \frac{\text{separation velocity}}{\text{approach velocity}}.$$

- No energy lost when $e = 1$ (elastic) and max. energy lost when $e = 0$ (completely inelastic)

* when $e = 0$, separation velocity $= 0$ so the particles stick together.

- For oblique impacts, we use the velocity components along the line of impact (orthogonal to the contact surface)



Variable mass systems   only use this if $\underline{v}_{p/b}$ is constant.

- The general expression for both expelling and collecting mass scenarios is:

Force acted on the body $\boxed{\underline{F}_{body} = \dot{m}\,\underline{v}_{p/b}}$ Velocity of particle relative to the body.

- Treat $\dot{m}$ as $\pm|\dot{m}|$. +ve if the body gains mass; -ve if the body expells mass.
- Treat $\underline{v}_{p/b}$ as $\pm|\underline{v}_{p/b}|$. +ve if $\underline{v}_{p/b}$ in +ve dir; -ve if $\underline{v}_{p/b}$ in -ve dir.
- To avoid confusion, always draw $\underline{F}_{body}$ in the +ve dir. in the FBD. $\underline{F}_{body}$ would simply be -ve if the direction is incorrect.

* Body gain mass = approach; body expel mass = separation.

* If confused, draw a table and the corresponding diagrams.

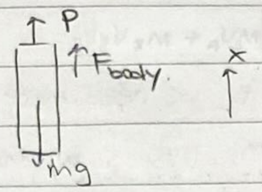
	$+v_{p/b}$	$-v_{p/b}$
$+\dot{m}$	$+F_{body}$	$-F_{body}$
$-\dot{m}$	$-F_{body}$	$+F_{body}$

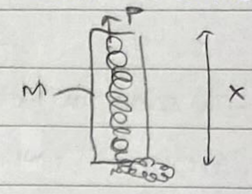
-eg: A chain of total length L and mass M is resting on a horizontal flat surface.

It is pulled up vertically at a speed of v by a force P.

Determine an expression for how P varies w/ the height above the surface x.

FBD:





* F_{body} is in the +ve x direction in the FBD.

$m = M \cdot \frac{x}{L}$ so $|\dot{m}| = \frac{M}{L}|\dot{x}| = \frac{M}{L}|v|$.

$\underline{F_{body}} = \dot{m}\, \underline{v_{P/b}}$ $\dot{m}$ +ve since the body is gaining mass.

$= (+\frac{M}{L}|v|)(-v) = -\frac{M}{L}v^2$ $\underline{v_{P/b}} = \underbrace{\underline{v_{P/O}}}_{0} - \underbrace{\underline{v_{b/O}}}_{+ve} \rightarrow \underline{v_{P/b}}$ -ve.

N2L: $P + F_{body} - mg = 0$

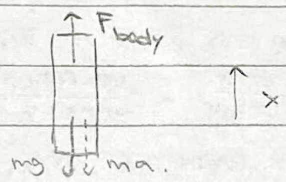
$$P = (M \cdot \tfrac{x}{L})g - (-\tfrac{M}{L}v^2)$$

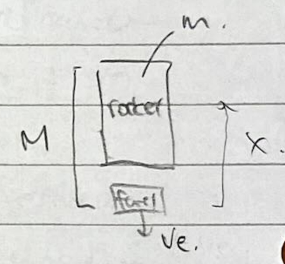
$$= M\tfrac{x}{L}g + \tfrac{M}{L}v^2$$

-eg: A rocket of mass M is launched from Earth. Fuel is expelled at a rate of $|\dot{m}|$ and at a speed of $|v_e|$ relative to the rocket. The rocket is initially at rest.

What is the speed of the rocket at time t?

FBD:





* F_{body} is in the +ve x direction in the FBD.

$m = M - |\dot{m}|t$.

$\underline{F_{body}} = \dot{m}\, \underline{v_{P/b}}$ $\dot{m}$ -ve since the body is losing mass

$= (-|\dot{m}|)(-|v_e|) = |\dot{m}|v_e$. $\underline{v_{P/b}} = \underbrace{\underline{v_{P/O}}}_{-ve} - \underbrace{\underline{v_{b/O}}}_{+ve} \rightarrow \underline{v_{P/b}} < 0$.

N2L: $F_{body} - mg - ma = 0$.

$$|\dot{m}|v_e = (M - |\dot{m}|t)(g + a)$$

$$|\dot{m}|v_e \frac{1}{M - |\dot{m}|t} = g + \frac{dv}{dt}$$

$$|\dot{m}|v_e \int_0^t \frac{1}{M - |\dot{m}|t}\,dt = \int_0^t g\,dt + \int_0^v dv$$

$$-\frac{|\dot{m}|v_e}{|\dot{m}|}\left[\ln|M - |\dot{m}|t|\right]_0^t = gt + v$$

$$v = v_e \ln\left|\frac{M}{M - \dot{m}t}\right| - gt.$$

If we set $m_f = M - |\dot{m}|t$, $v = v_e \ln\left|\frac{M}{m_f}\right| - gt$.

Ignoring gravity (i.e. set $g = 0$), we get the Tsiolkovsky rocket equation.

$$\Delta v = v_e \ln\left|\frac{M}{m_f}\right|$$

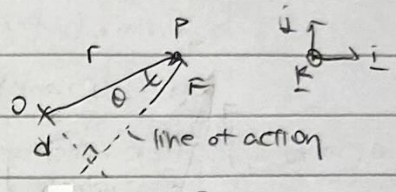
Angular momentum

Moment of a force (Torque)

- Consider applying a force $\underline{F}$ to a particle P, whose position vector is $\underline{r}$ relative to the origin O. The moment $\underline{g}_o$ of the force $\underline{F}$ about the pt. O is defined as

$$\boxed{\underline{g}_o = \underline{r} \times \underline{F}}$$

- Alternatively, the moment about a pt. is the force × ⊥ distance to the line of action of the force about the pt.



$$\underline{g}_o = \underline{r} \times \underline{F} = |\underline{r}||\underline{F}|\sin\theta\, \underline{k} = F|r|\sin\theta\, \underline{k} = Fd\, \underline{k}$$

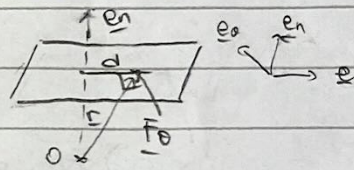
and the direction is defined by the RH grip rule.

- The moment Q_o of the force $\underline{F}$ about the axis in the direction $\underline{e}_n$ passing through the origin O is the component of the torque $\underline{g}_o$ about pt. O in the $\underline{e}_n$ direction.

$$\boxed{Q_o = \underline{g}_o \cdot \underline{e}_n = (\underline{r} \times \underline{F}) \cdot \underline{e}_n}$$

- Alternatively, the moment about an axis is the shortest distance d from the axis to the particle P × the component of the force that is ⊥ to both the axis and the shortest-distance line.

$\underline{e}_r$ chosen s.t. $\underline{e}_r$ // d line
∴ $\underline{r}$ has no $\underline{e}_\theta$ component



$$\underline{F} = F_r \underline{e}_r + F_\theta \underline{e}_\theta + F_n \underline{e}_n \; ; \; \underline{r} = d\underline{e}_r + z\underline{e}_n$$

$$Q_o = (\underline{r} \times \underline{F}) \cdot \underline{e}_n = (d\underline{e}_r \times F_\theta \underline{e}_\theta + \ldots) \cdot \underline{e}_n = dF_\theta \underline{e}_n$$

Moment of momentum (Angular momentum)

- The moment of momentum of a particle about a pt. O is defined as

$$\boxed{\underline{h}_o = \underline{r} \times \underline{p} = \underline{r} \times m\underline{v}}$$

- If we differentiate wrt time, we can see its relation w/ moment of a force (about a pt)

$$\begin{aligned}
\frac{d\underline{h}_o}{dt} &= \frac{d}{dt}(\underline{r} \times \underline{p}) \\
&= \frac{d\underline{r}}{dt} \times m\underline{v} + \underline{r} \times \frac{d\underline{p}}{dt} \\
&= \underbrace{\underline{v} \times m\underline{v}}_{0} + \underline{r} \times \frac{d\underline{p}}{dt} \\
&= \underline{r} \times \underline{F}
\end{aligned}$$

$* \frac{d}{dt}(\underline{a} \times \underline{b}) = \frac{d\underline{a}}{dt} \times \underline{b} + \underline{a} \times \frac{d\underline{b}}{dt}$

$\underline{v} \times m\underline{v} = m(\underbrace{\underline{v} \times \underline{v}}_{0}) = 0$.

- The rate of increase of angular momentum of a particle P about a pt. O is equal to the moment of force about O.

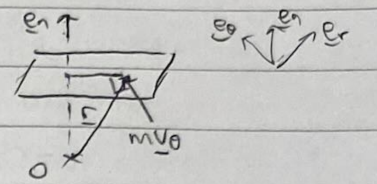
$$\boxed{\frac{d\underline{h}_o}{dt} = \underline{r} \times \underline{F} = \underline{g}_o}$$

- This extends to a collection of particles (like linear momentum) to give the angular momentum principle about a fixed pt — the total rate of increase of moment of momentum is equal to the total ext. moment of force about the same pt. (only the ext. forces matter, the int. forces cancel out)

- The angular momentum H_o of a particle about an axis in the direction $\underline{e_n}$ that goes through the origin, is defined as

$$\boxed{H_o = \underline{h_o} \cdot \underline{e_n} = (\underline{r} \times m\underline{v}) \cdot \underline{e_n}}$$

- Alternatively, the angular momentum about an axis is the shortest distance d from the axis to the particle P × the component of momentum (velocity) that is ⊥ to both the axis and the shortest-distance line.



$m\underline{v} = m(V_r\underline{e_r} + V_\theta\underline{e_\theta} + V_n\underline{e_n})$; $\underline{r} = d\underline{e_r} + z\underline{e_n}$.

$H_o = (\underline{r} \times m\underline{v}) \cdot \underline{e_n} = m(d\underline{e_r} \times V_\theta\underline{e_\theta} + \ldots) \cdot \underline{e_n} = mV_\theta d$

- If we differentiate wrt time, we can see its relation w/ moment of a force (about an axis)

$$\frac{dH_o}{dt} = \frac{d}{dt}[(\underline{r} \times \underline{p}) \cdot \underline{e_n}]$$

$$= \frac{d}{dt}(\underline{r} \times \underline{p}) \cdot \underline{e_n} \qquad * \underline{e_n} \text{ is a constant vector}$$

$$= (\underline{r} \times \underline{F}) \cdot \underline{e_n}$$

- The rate of increase of angular momentum about an axis is equal to the moment of force about that axis.

$$\boxed{\frac{dH_o}{dt} = (\underline{r} \times \underline{F}) \cdot \underline{e_n} = \underline{q_o} \cdot \underline{n} = Q_o}$$

Conservation of angular momentum

- The expression for the rate of change of angular momentum about a pt. O is

$$\frac{d\underline{h_o}}{dt} = \underline{q_o} = \underline{r} \times \underline{F}$$

so angular momentum is conserved about pt. if the moment of force about that pt. is 0.

* This is true if the line of action of the resultant force passes through O.

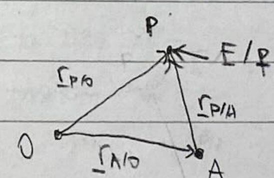
- The expression for the rate of change of angular momentum about an axis is

$$\frac{dH_o}{dt} = Q_o = \underline{q_o} \cdot \underline{n} = (\underline{r} \times \underline{F}) \cdot \underline{n}$$

so angular momentum is conserved about an axis if the moment of force about that axis is 0.

* This is true if the line of action of the resultant force passes through or // to the axis.

(More general principle – there are systems where angular momentum about a pt. is not conserved, but about an axis it is).



Moment of a force / momentum about other points.

- The moment of a force / momentum depends on which pt. are taking about:

$$\underline{q_A} = \underline{r_{P/A}} \times \underline{F} \qquad \underline{h_A} = \underline{r_{P/A}} \times \underline{p}$$

$$= (\underline{r_{P/O}} - \underline{r_{A/O}}) \times \underline{F} \qquad = (\underline{r_{P/O}} - \underline{r_{A/O}}) \times \underline{p}$$

$$\boxed{\underline{q_A} = \underline{q_o} - \underline{r_{A/O}} \times \underline{F}} \quad \underline{q_o} = \underline{r_{P/O}} \times \underline{F} \qquad \boxed{\underline{h_A} = \underline{h_o} - \underline{r_{A/O}} \times \underline{p}} \quad \underline{h_o} = \underline{r_{P/O}} \times \underline{p}$$

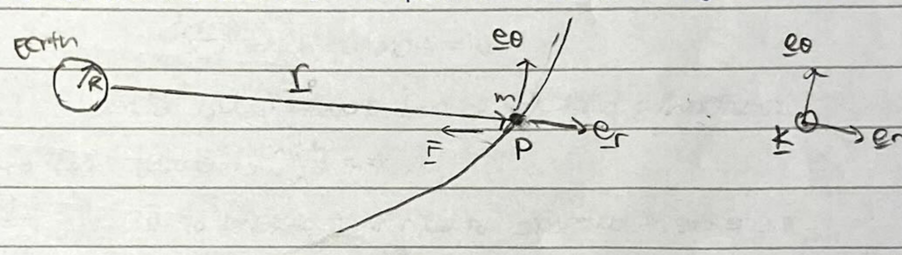
Satellite dynamics

Four key principles of classical mechanics.

- N2L $\qquad \sum \underline{F} = m\underline{a}$
- Energy $\qquad \frac{dT}{dt} = \underline{F}\cdot\underline{v} \qquad \int_A^B \underline{F}\cdot d\underline{r} = T_B - T_A$
- Linear momentum $\qquad \frac{d\underline{p}}{dt} = \underline{F} \qquad \int_A^B \underline{F}\cdot dt = \underline{p}_B - \underline{p}_A$
- Angular momentum $\qquad \frac{d\underline{h}_o}{dt} = \underline{r}\times\underline{F} \qquad \int_A^B (\underline{r}\times\underline{F})dt = \underline{h}_B - \underline{h}_A$
- Combining these key principles, we can find the eqn. of motion solving that would give the shape of the orbit.

Satellite dynamics.

- There are 2 special features of the gravitational force exerted by a planet on a satellite
  - ↳ It is a conservative force → consv. of energy
  - ↳ It always acts towards the centre of the planet → consv. of angular momentum abt. planet. centre field



- Consv. of (mechanical) energy.:

$$T + V = E \text{ (constant)}.$$

$$\frac{1}{2}mv^2 + \left(-\frac{GMm}{r}\right) = E$$

$$\frac{1}{2}m[\dot{r}^2 + (r\dot{\theta})^2] - \frac{GMm}{r} = E. \quad [1] \qquad \underline{v} = \dot{r}\underline{e_r} + r\dot{\theta}\underline{e_\theta}$$

Consv. of angular momentum:

$$\underline{r}\times(m\underline{v}) = \underline{h_o} \text{ (constant)}$$

$$\underline{h_o} = r\underline{e_r}\times m(\dot{r}\underline{e_r} + r\dot{\theta}\underline{e_\theta})$$

$$= mr^2\dot{\theta}\,\underline{k}.$$

Rearranging. $\dot{\theta} = \frac{h_o}{mr^2}$ [2], where $h_o = |\underline{h_o}|$

Sub [2] into [1] : $\frac{1}{2}m\left[\dot{r}^2 + \frac{h_o^2}{m^2r^2}\right] - \frac{GMm}{r} = E$

Differentiate both sides wrt time to get rid of the unknown constant E.

$$\frac{1}{2}\cancel{m}\left[\cancel{2}\cancel{\dot{r}}\ddot{r} - \frac{\cancel{2}h_o^2}{m^2r^3}\cancel{\dot{r}}\right] + \frac{GM\cancel{m}}{r^2}\cancel{\dot{r}} = 0. \qquad \text{[use chain rule!].}$$

$$\ddot{r} - \frac{h_o^2}{m^2r^3} + \frac{GM}{r^2} = 0.$$

- This can be solved by using N2L only (not recommended).

N2L:

$$\underline{F} = m\underline{a}$$

$$-\frac{GMm}{r^2}\underline{e_r} = m[(\ddot{r} - r\dot{\theta}^2)\underline{e_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e_\theta}].$$

Radial dir: $-\frac{GMm}{r^2} = m(\ddot{r} - r\dot{\theta}^2)$ $\qquad$ Tangential dir: $0 = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$.

→ Combining the eqns give the eqn. of motion. (eliminate θ) related to consv. of angular momentum.

Shape of the orbit.

- To find an eqn. for the shape of the orbit, we need an expression that relates r directly to θ.
- The eqn. of motion : $\ddot{r} - \frac{h_0^2}{m^2 r^3} + \frac{GM}{r^2} = 0$.

The sol'n would give us r as a function of t → we need to eliminate t from this 2nd-order non-linear DE.

- Defining a new variable u $u(r) = \frac{1}{r}$, so $r = \frac{1}{u}$.

$\dot{r} = \frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) = \frac{d}{du}\left(\frac{1}{u}\right)\frac{du}{d\theta}\frac{d\theta}{dt} = -\frac{1}{u^2}\dot{\theta}\frac{du}{d\theta} = -(r^2\dot{\theta})\frac{du}{d\theta} = -\frac{h_0}{m}\frac{du}{d\theta}$.

$\ddot{r} = -\frac{h_0}{m}\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -\frac{h_0}{m}\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -\frac{h_0}{m}\dot{\theta}\frac{d^2u}{d\theta^2} = -\frac{h_0}{m}\left(\frac{h_0}{mr^2}\right)\frac{d^2u}{d\theta^2} = -\frac{h_0^2}{m^2}u^2\frac{d^2u}{d\theta^2}$.

Substituting this and u(r) into the eqn. of motion,

$$-\frac{h_0^2}{m^2}\cancel{u^2}\frac{d^2u}{d\theta^2} - \frac{h_0^2}{m^2}u^{\cancel{3}} + GM\cancel{u^2} = 0$$

$$\frac{h_0^2}{m^2}\frac{d^2u}{d\theta^2} + \frac{h_0^2}{m^2}u = GM$$

This is now a 2nd order linear DE w/ constant coefficients. Solving for u,

$$u = A\cos(\theta + \alpha) + \frac{GMm^2}{h_0^2}$$

Transforming back to r and choosing $\alpha = 0$ gives

$$\frac{1}{r} = \frac{GMm^2}{h_0^2}(1 + e\cos\theta) \quad \rightarrow \text{eqn. of a conic section.}$$

- There are 4 categories of sol'n that depend on e. $\left(\frac{l}{r} = 1 + e\cos\theta,\ \frac{1}{l} = \frac{GM}{h^2},\ l = \frac{b^2}{a}\right)$

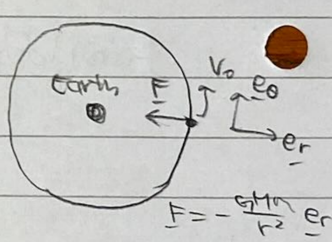
↳ $e = 0$: circle] periodic orbit.
↳ $0 < e < 1$: ellipse

↳ $e = 1$: parabola
↳ $e > 1$: hyperbola.

① Circular orbit ($e=0$) [no need to use general sol'n].

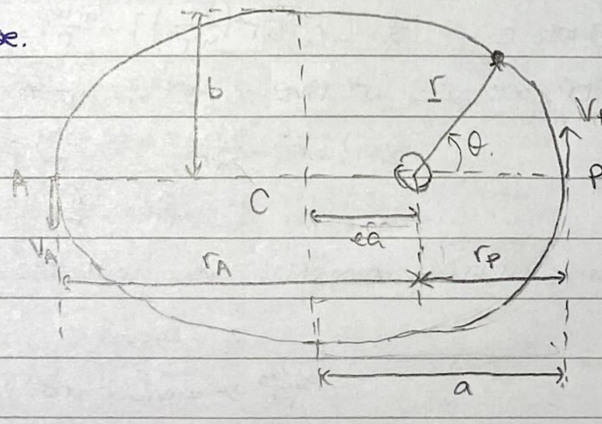
- By N2L: $\frac{GMm}{R^2} = m\frac{V_0^2}{R}$.

$$R = \frac{GM}{V_0^2}$$



② Elliptical orbit ($0<e<1$)

- Geometry of an ellipse.



The ellipse can be defined in terms of its major semi-axis a and its minor semi-axis b.

Point P is the perigee of the orbit : the closest pt. on the path to the Earth.

$$\theta = 0, \quad r_P = (1-e)a$$

Point A is the apogee of the orbit : the furthest pt. on the path to the Earth.

a for away

$$\theta = \pi, \quad r_A = (1+e)a.$$

* $r_P + r_A = 2a$.

- At perigee and apogee, the satellite's angular momentum is easy to find ($\underline{r} \perp \underline{v}$).

$h_P = r_P m v_P$ $\qquad\qquad$ $h_A = r_A m v_A$.

* Note $\dot{\theta} = \frac{h_0}{mr^2}$ so when $r\downarrow$, $\dot{\theta}\uparrow$, $\rightarrow$ $\dot{\theta}_{max}$ at P, $\dot{\theta}_{min}$ at A.

Since angular momentum is consv. about the Earth (F_G always pts towards the Earth $\rightarrow \underline{q}=0$).

$$m v_P r_P = m v_A r_A \qquad [h_P = h_A].$$

$$\rightarrow \boxed{\frac{v_A}{v_P} = \frac{r_P}{r_A} = \frac{1-e}{1+e}\frac{\cancel{a}}{\cancel{a}}}$$

From the general soln, we can rearrange to find an expression for a.

$$\frac{1}{r} = \frac{GMm^2}{h_0^2}(1+e\cos\theta)$$

At $\theta = 0$, $r = r_P$.

$$\frac{1}{r_P} = \frac{GMm^2}{h_0^2}(1+e) = \frac{1}{(1-e)a}$$

$$a = \frac{h_0^2}{GMm^2}\frac{1}{(1+e)(1-e)} = \frac{h_0^2}{GMm^2(1-e^2)}$$

Then we can find an expression for b using $\boxed{\frac{b}{a} = \sqrt{1-e^2}}$ $\quad [e = \sqrt{1-(\frac{b}{a})^2}]$.

Vis-viva equation

- For any Keplerian orbit, the vis-viva eqn. is as follows,

$$\boxed{v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)}$$

* a is the major semi-axis ($a>0$ for ellipses, $a=\infty$ for parabolas, $a<0$ for hyperboles).

- Derivation of vis-viva eqn. for ellipses ($0<e<1$).

Consv. of (mechanical) energy : $\frac{1}{2}mv_A^2 - \frac{GMm}{r_A} = \frac{1}{2}mv_P^2 - \frac{GMm}{r_P}$

$$\frac{1}{2}v_A^2 - \frac{GM}{r_A} = \frac{1}{2}v_P^2 - \frac{GM}{r_P} \qquad [1]$$

Consv. of angular momentum : $m v_A r_A = m v_P r_P$

$$v_P = v_A\frac{r_A}{r_P}. \qquad [2]$$

Sub [2] into [1] : $\frac{1}{2}v_A^2\left(1 - \frac{r_A^2}{r_P^2}\right) = GMm\left(\frac{1}{r_A} - \frac{1}{r_P}\right)$

$$\frac{1}{2}v_A^2 = GMm\left(\frac{r_P - r_A}{r_A r_P}\right)\left(\frac{r_P^2}{r_P^2 - r_A^2}\right)$$

$$= GMm\left(\frac{r_P}{r_A(r_P + r_A)}\right)$$

For an ellipse, $r_P + r_A = 2a$: $\frac{1}{2}v_A^2 = GMm\left(\frac{2a - r_A}{r_A(2a)}\right) = GMm\left(\frac{1}{r_A} - \frac{1}{2a}\right)$

Consv. of (mechanical) energy : $\frac{1}{2}mv_A^2 - \frac{GMm}{r_A} = \frac{1}{2}mv^2 - \frac{GMm}{r}$

$$\cancel{\frac{GMm}{r_A}} - \frac{GMm}{2a} - \cancel{\frac{GMm}{r_A}} = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

$$\frac{1}{2}\cancel{m}v^2 = GM\cancel{m}\left(\frac{1}{r} - \frac{1}{2a}\right)$$

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

Other central forces

- similar methodology can be applied to conservative central forces other than the gravitational inverse square law. (periodic orbit but not elliptical orbit)

↳ conservative force → conservation of mechanical energy.

↳ central force → conservation of angular momentum about the centre.

- we can use these 2 principles to quickly calculate the behaviour at specific pts in the path — where the velocities are orthogonal to the position vector from the centre.

* At these specific pts. ($\underline{v} \perp \underline{r}$), we can use the same velocity for both consv. of ME and consv. of angular momentum about the centre / central axis. [since $|v| = v_{tangential}$].

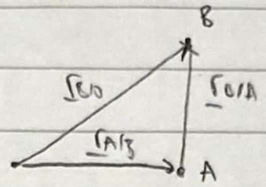
* The converse is true, given at a pt P, $KE_P = \frac{1}{2}mv^2$ and $h_P = mrv$, then $r_P \perp v_P$.

Kinematics of rigid bodies

Rigid body kinematics

Relative motion (Also applies to particles).

- Consider pts O, A, B.



The vector expression for position, velocity and acceleration all take the same form

$$\boxed{\begin{aligned} \underline{r}_{B/O} &= \underline{r}_{B/A} + \underline{r}_{A/O} \\ \underline{v}_{B/O} &= \underline{v}_{B/A} + \underline{v}_{A/O} \\ \underline{a}_{B/O} &= \underline{a}_{B/A} + \underline{a}_{A/O} \end{aligned}}$$

similar to fractions : $\frac{B}{O} = \frac{B}{A} \times \frac{A}{O}$

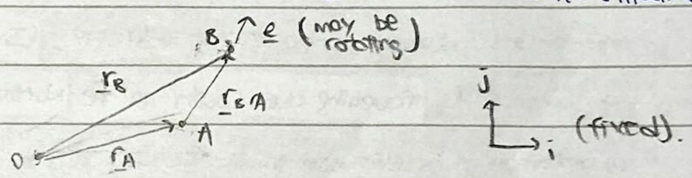
These pt-relative-to-pt. expressions are completely general.

Relative motion using rotating reference frames.

- In the case above, the reference frames are fixed / translating → unit vectors are fixed.

  If the reference frame is rotating, then so are the corresponding unit vectors. ↖ Take care when differentiating.

- Consider 2 pts A, B, and a unit vector $\underline{e}$ in the dir $\underline{r}_{B/A}$



Position: $\boxed{\underline{r}_B = \underline{r}_A + \underline{r}_{B/A}} = \underline{r}_A + r\underline{e}$

fixed frame, rotating frame. (fixed on A).

Velocity: $\underline{v}_B = \frac{d}{dt}(\underline{r}_A + r\underline{e})$

$= \underline{v}_A + \dot{r}\underline{e} + r(\underline{\omega} \times \underline{e})$.

$$\boxed{\underline{v}_B = \underline{v}_A + [\dot{\underline{r}}_{B/A}]_R + \underbrace{\underline{\omega} \times \underline{r}_{B/A}}_{\text{correction term}}}$$

Acceleration: $\underline{a}_B = \frac{d}{dt}(\underline{v}_A + \dot{r}\underline{e} + r(\underline{\omega} \times \underline{e}))$

$= \underline{a}_A + \ddot{r}\underline{e} + \dot{r}(\underline{\omega} \times \underline{e}) + \dot{r}(\underline{\omega} \times \underline{e}) + r(\dot{\underline{\omega}} \times \underline{e}) + r[\underline{\omega} \times (\underline{\omega} \times \underline{e})]$.

$$\boxed{\underline{a}_B = \underline{a}_A + [\ddot{\underline{r}}_{B/A}]_R + \underbrace{\dot{\underline{\omega}} \times \underline{r}_{B/A} + 2\underline{\omega} \times [\dot{\underline{r}}_{B/A}]_R + \underline{\omega} \times (\underline{\omega} \times \underline{r}_{B/A})}_{\text{correction term}}}$$

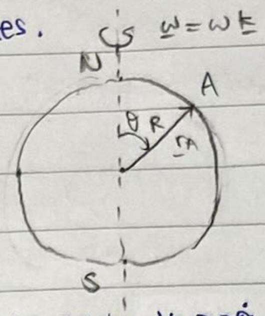
* In the rotating reference frame (fixed on A),

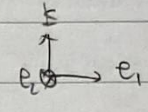
$\underline{r}_{B/A} = r\underline{e}$; $[\dot{\underline{r}}_{B/A}]_R = \dot{r}\underline{e}$; $[\ddot{\underline{r}}_{B/A}]_R = \ddot{r}\underline{e}$

Fixed frame	Rotating frame
$\underline{r}_{B/A}$	$\underline{r}_{B/A}$
$\dot{\underline{r}}_{B/A}$	$[\dot{\underline{r}}_{B/A}]_R + \underline{\omega} \times \underline{r}_{B/A}$
$\ddot{\underline{r}}_{B/A}$	$[\ddot{\underline{r}}_{B/A}]_R + \dot{\underline{\omega}} \times \underline{r}_{B/A} + 2\underline{\omega} \times [\dot{\underline{r}}_{B/A}]_R + \underline{\omega} \times (\underline{\omega} \times \underline{r}_{B/A})$

* The vector $\underline{\omega}$ is the angular velocity of the chosen reference frame / unit vector $\underline{e}$.

- e.g.: Coriolis effect in hurricanes.





Assumptions: Constant speed due south, $V_0 = R\dot{\theta} = \text{constant}$. (i.e. $\ddot{\theta} = 0$).

Constant self rotation, $\dot{\omega} = 0$

$$\underline{r}_A = R\sin\theta\,\underline{e}_1 + R\cos\theta\,\underline{k}$$

$$\dot{\underline{r}}_A = R\cos\theta\,\dot{\theta}\,\underline{e}_1 + R\sin\theta(\underline{\omega}\times\underline{e}_1) + R(-\sin\theta)\dot{\theta}\,\underline{k} + R\cos\theta\,\overset{0}{\cancel{(\underline{\omega}\times\underline{k})}}$$

$$= R\dot{\theta}\cos\theta\,\underline{e}_1 + R\omega\sin\theta\,\underline{e}_2 - R\dot{\theta}\sin\theta\,\underline{k}$$

$$\ddot{\underline{r}}_A = R\overset{0}{\cancel{\ddot{\theta}}}\cos\theta\,\underline{e}_1 + R\dot{\theta}(-\sin\theta)\dot{\theta}\,\underline{e}_1 + R\dot{\theta}\cos\theta(\underline{\omega}\times\underline{e}_1) + R\overset{0}{\cancel{\dot{\omega}}}\sin\theta\,\underline{e}_2 + R\omega\cos\theta\dot{\theta}\,\underline{e}_2$$
$$+ R\omega\sin\theta(\underline{\omega}\times\underline{e}_2) - R\overset{0}{\cancel{\ddot{\theta}}}\sin\theta\,\underline{k} - R\dot{\theta}\cos\theta\dot{\theta}\,\underline{k} - R\dot{\theta}\sin\theta\,\overset{0}{\cancel{(\underline{\omega}\times\underline{k})}}$$

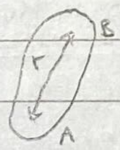
$$= -R(\dot{\theta}^2+\omega^2)\sin\theta\,\underline{e}_1 + \overset{\text{coriolis accel.}}{2R\dot{\theta}\omega\cos\theta\,\underline{e}_2} - R\dot{\theta}^2\cos\theta\,\underline{k}$$

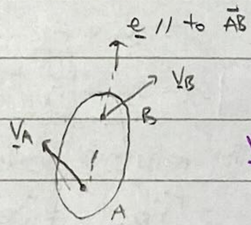
- A coriolis acceleration occurs when a body has a component of velocity inwards/outwards from an axis of spin, giving a Coriolis acceleration that is in a dir. $\perp$ to both the axis and the inward/outward component of velocity. ($\underline{v}$ has $\underline{e}_1$ component so $\underline{a}$ has $\underline{e}_2$ component).
- For a particle constrained to travel due South in the Northern Hemisphere ($\dot{\theta} > 0$, $\cos\theta > 0$), it would experience an acceleration in the $\underline{e}_2$ dir. This is because it moves further from the Earth's axis of rotation so it would need to accelerate to keep the same angular speed.
- However, in reality, there is no constraint so the particle appears to accelerate in the $-\underline{e}_2$ dir.

Relative motion of pts. within a rigid body.

- A rigid body is an object w/ continuously distributed mass that cannot deform — it can be thought of as a collection of particles w/ fixed spacing between them.
- Consider 2 pts A, B on a rigid body:

~~r is fixed~~
$\rightarrow \dot{r} = \ddot{r} = 0$.





$\underline{V}_A \cdot \underline{e} = \underline{V}_B \cdot \underline{e}$

By defn, the distance r btwn A and B is fixed because they are pts within the rigid body. They can move and rotate but the distance between the pts must stay the same.

- Therefore, the velocity v_A and v_B must be the same in the $\underline{r}_{B/A}$ dir. (but can be different in the perpendicular direction). Formally,

$$\boxed{\underline{V}_A \cdot \underline{e} = \underline{V}_B \cdot \underline{e}}$$

- Rearranging,

$$\underline{v}_B \cdot \underline{e} - \underline{v}_A \cdot \underline{e} = 0$$
$$(\underline{v}_B - \underline{v}_A) \cdot \underline{e} = 0.$$
$$\underline{v}_{B/A} \cdot \underline{e} = 0.$$

From this, we can see that the relative velocities on a rigid body are orthogonal to their relative displacements.

- In other words, the motion of B rel to A must be due to rotation alone. (no translation).
- Within a rigid body, the general expressions for velocity and acceleration become much simpler because distance r is fixed.

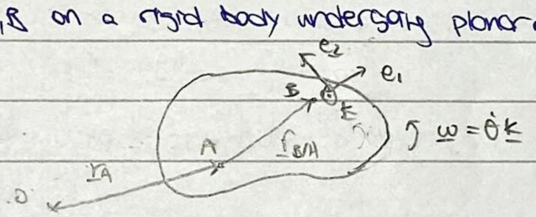
$$\boxed{\begin{aligned} \underline{r}_{B/A} &= r\underline{e} \\ \dot{\underline{r}}_{B/A} &= r(\underline{\omega} \times \underline{e}) = \underline{\omega} \times \underline{r}_{B/A} \\ \ddot{\underline{r}}_{B/A} &= r(\dot{\underline{\omega}} \times \underline{e}) + r[\underline{\omega} \times (\underline{\omega} \times \underline{e})] = \dot{\underline{\omega}} \times \underline{r}_{B/A} + \underline{\omega} \times (\underline{\omega} \times \underline{r}_{B/A}) \end{aligned}}$$

(Same as rel. motion w/ rotating reference frames but the $\dot{r}$ and $\ddot{r}$ terms are 0)

* The vector $\underline{\omega}$ is the angular velocity of the rigid body, and therefore is also the angular velocity of the vectors $\underline{r}_{B/A}$ / $\underline{e}$, which are fixed within the rigid body.

Relative motion of points within a rigid body : 2D planar motion.

- Consider 2 pts. A, B on a rigid body undergoing planar motion



Position : $\underline{r}_B = \underline{r}_A + \underline{r}_{B/A}$

$$\boxed{\underline{r}_B = \underline{r}_A + \boxed{r\underline{e}_1}}$$

velocity : $\underline{v}_B = \frac{d\underline{r}_B}{dt} = \underline{v}_A + r(\underline{\omega} \times \underline{e}_1)$

$$\boxed{\underline{v}_B = \underline{v}_A + \boxed{r\dot{\theta}\underline{e}_2}}$$

polar expressions for $\underline{r}, \dot{\underline{r}}, \ddot{\underline{r}}$ can be found in the databook (just set the $\dot{r}, \ddot{r}$ terms to 0).

Acceleration : $\underline{a}_B = \frac{d\underline{v}_B}{dt} = \underline{a}_A + r\ddot{\theta}\underline{e}_2 + r\dot{\theta}(\underline{\omega} \times \underline{e}_2)$

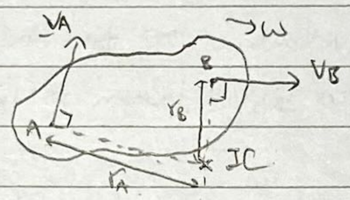
$$\boxed{\underline{a}_B = \underline{a}_A + \boxed{r\ddot{\theta}\underline{e}_2 - r\dot{\theta}^2\underline{e}_1}}$$

Degrees of freedom

- For a general 3D rigid body, there are 6 degrees of freedom : v_x, v_y, v_z translation of a pt. and $\omega_x, \omega_y, \omega_z$ components of an angular velocity vector through the same pt.
- For a general rigid body undergoing planar motion, there are 3 degrees of freedom: v_x, v_y translation of a pt. and angular velocity $\underline{\omega} = \omega\underline{k}$.

Instantaneous centres.

- For 2D planar motion, there is a pt. somewhere in the plane of a lamina that has zero velocity (i.e. instantaneously at rest). This pt. is the instantaneous centre.
- At a given instant, the lamina behaves as if it were rotating w/ angular velocity ω about the instantaneous centre.
- A fixed pt. is automatically an instantaneous centre.
- In general, the instantaneous centre is not at the centre of curvature of the paths of the lamina.
- Instantaneous centres move from one moment to the next.
- Instantaneous centres need not be on the lamina itself. In fact, the instantaneous centre is at ∞ when the lamina is travelling in a straight line.
- We can find the instantaneous centre if we know either
  ↳ the direction of velocities at 2 pts. of the lamina.

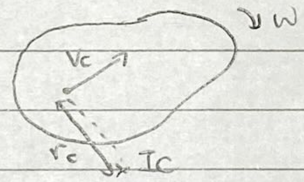


The instantaneous centre is the intersection pt. of the 2 perpendiculars to the velocities.

If we know the magnitude of 1 velocity, we can find the others' magnitude:

$$\boxed{\omega = \frac{V_A}{r_A} = \frac{V_B}{r_B}}$$

↳ The direction and magnitude of velocity at 1 pt. on the lamina and the lamina's angular velocity.



We can work out the position of IC by calculating r_c using $r_c = \frac{V_c}{\omega}$

$\underline{r_c}$ is $\perp$ to $\underline{V_c}$.

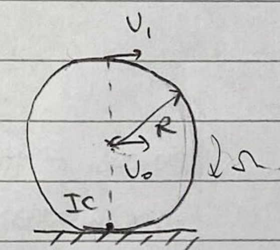
- e.g.: Rolling wheel w/o slip.

At the contact pt, the instantaneous velocity is 0 → IC.

Given R and Ω,

$$V_0 = R\Omega$$

$$V_1 = (2R)\Omega = 2R\Omega$$

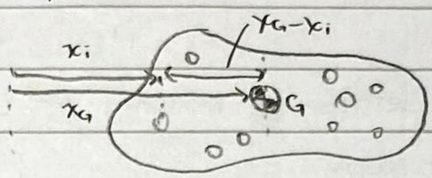


Dynamics of rigid bodies

Rigid body properties I

Centre of mass (CoM)

- The CoM represents a balance pt. where if a rigid body were simply supported at this pt, it would not rotate.
- Considering a rigid body as a collection of particles,



The x-coordinate of the CoM must be where the total moment due to the weight of each mass is 0,

$$\Sigma (x_G - x_i) m_i g = 0$$

Similarly for y, z-coordinates (simply rotated),

$$\Sigma (y_G - y_i) m_i g = 0 \qquad \Sigma (z_G - z_i) m_i g = 0$$

In vector notation,

$$\Sigma (\underline{r}_G - \underline{r}_i) m_i = 0$$

Since a rigid body is a collection of infinitesimally small masses, taking the limit,

$(m_i \to dm)$ $$\int (\underline{r}_G - \underline{r})\, dm = 0$$

Rearranging, $$\int \underline{r}_G\, dm = \int \underline{r}\, dm$$

$\underline{r}_G$ is a constant vector $$\underline{r}_G \int dm = \int \underline{r}\, dm$$

$$\underline{r}_G M = \int \underline{r}\, dm$$

$$\boxed{\underline{r}_G = \frac{1}{M} \int \underline{r}\, dm}$$

where $M = \int dm$ is the total mass.

- The CoM can often be seen by inspection using symmetry.
- For composite bodies, the combined CoM can be found by treating the sections as particles at their CoMs.
- If the body has a hole, then the hole can be treated as -ve mass.
- The CoM need not be within the body itself.

Newton's 2nd law and centre of mass

- Applying N2L to each particle within the rigid body,

$$m_i \underline{a}_i = \underline{F}_{i\,total}$$

$$= \underbrace{\underline{F}_{i\,ext}}_{\text{ext. force}} + \underbrace{\sum_{j \neq i} \underline{G}_{ij}}_{\text{int. force}}$$

Adding up all the particles, $$\Sigma m_i \underline{a}_i = \Sigma \underline{F}_{i\,ext} + \sum_i \sum_{j \neq i} \underline{G}_{ij}$$

By N3L, the net internal force must be 0, i.e. $\sum_i \sum_{j \neq i} \underline{G}_{ij} = 0$, so

$$\sum m_i \underline{a}_i = \sum \underline{F}_i$$

For infinitesimally small masses,

$$\int \underline{a}\, dm = \underline{F} \quad \text{where } \underline{F} \text{ is the total ext. force.}$$

We also know $\int \underline{a}\, dm = \int \underline{\ddot{r}}\, dm = \int \underline{\ddot{r}}_G\, dm = \underline{\ddot{r}}_G \int dm = \underline{\ddot{r}}_G M$, so

$$\boxed{\underline{F} = M \underline{\ddot{r}}_G} \quad \text{where } \underline{\ddot{r}}_G \text{ is the acceleration of CoM.}$$

- N2L can be used to describe the motion of the CoM of any rigid body (or more generally any kind of body).

Mass moment of inertia.

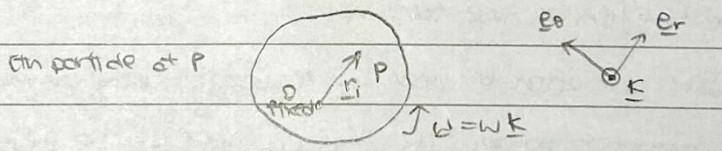
- The general expression relating torque to angular acceleration of a 3D rigid body is:

$$\begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} \qquad \boxed{\underline{q} = \underline{\underline{I}}\, \underline{\dot{\omega}}}$$

- To describe general 3D rotational dynamics req. a 3x3 matrix that characterises the rotational inertia of a rigid body → complicated eqn. of motion.
- If we consider planar motion — where all the particles move in a plane that is // to a single plane of motion, the eqn. of motion simplifies considerably.
  - ↳ A general rigid body rotating about a fixed axis.
  - ↳ General motion of a lamina within its own plane.

① A general rigid body rotating about a fixed axis

- Consider a general rigid body that can rotate about a fixed axis.



As the body has some angular velocity and acceleration ω and $\dot{\omega}$, all the particles rotate about the axis O, w/ velocity and acceleration $\underline{v}_P$ and $\underline{a}_P$, given by.

$$\underline{v}_P = r\omega \underline{e}_\theta \ ; \ \underline{a}_P = -r\omega^2 \underline{e}_r + r\dot{\omega}\underline{e}_\theta \quad [\underline{r}_P = r\underline{e}_r + \text{diff / find in DB, then set } \dot{r} = \ddot{r} = 0]$$

So the tangential force req. to accelerate each particle within the rigid body is:

$$\underline{F}_{\tan i} = m_i r_i \dot{\omega}\, \underline{e}_\theta$$

The moment of this force for each particle about the axis O is:

$$\underline{Q}_i = \underline{r}_i \times \underline{F}_i = \underline{r}_i \times (\underline{F}_{\tan i} + \ldots) = m_i r_i \dot{\omega} (\underline{r}_i \times \underline{e}_\theta) = m_i r_i^2 \dot{\omega}\, \underline{k}$$

The total moment is found by integrating over all particles.

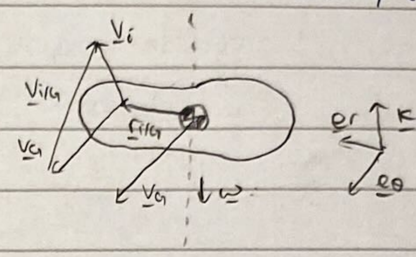
abt. axis of rot.

$$\underline{Q}_0 = \left(\int r^2 dm\right) \dot{\omega}\, \underline{k} \qquad \left[\text{Discrete: } \underline{Q} = \sum_i (m_i r_i^2)\, \dot{\omega}\, \underline{k}\right]$$

- The total torque is equal to the quantity $\boxed{I_0 = \int r^2 dm}$ times the angular acceleration for a general rigid body rotating about a fixed axis.

② General motion of a lamina within its own plane

- consider a lamina made up of a collection of particles, w/ CoM at G. The velocity and acceleration of any given particle is.



$$\underline{v}_i = \underline{v}_G + \underline{v}_{i/G} = \underline{v}_G + \underline{\omega} \times \underline{r}_{i/G}$$

$$\underline{a}_i = \underline{a}_G + \underline{a}_{i/G} = \underline{a}_G + \dot{\underline{\omega}} \times \underline{r}_{i/G} + \underline{\omega} \times (\underline{\omega} \times \underline{r}_{i/G})$$

so applying N2L to each particle gives.

$$\underline{F}_i = m_i a_i$$
$$= m_i [\underline{a}_G + \dot{\underline{\omega}} \times \underline{r}_{i/G} + \underline{\omega} \times (\underline{\omega} \times \underline{r}_{i/G})].$$

The moment of each force is

$$q_i = \underline{r}_{i/G} \times \underline{F}_i$$
$$= \underline{r}_{i/G} \times m_i [\underline{a}_G + \dot{\underline{\omega}} \times \underline{r}_{i/G} + \underline{\omega} \times (\underline{\omega} \times \underline{r}_{i/G})].$$

In the case for a lamina (undergoing planar motion), the angular velocity and acceleration vectors $\underline{\omega}$ and $\dot{\underline{\omega}}$ are orthogonal to the plane of the rigid body. (i.e. they are orthogonal to all the $\underline{r}_i$ vectors).

Expanding and simplifying,

$$q_i = m_i [\underline{r}_{i/G} \times \underline{a}_G + \underline{r}_{i/G} \times (\dot{\underline{\omega}} \times \underline{r}_{i/G}) + \underline{r}_{i/G} \times [\underline{\omega} \times (\underline{\omega} \times \underline{r}_{i/G})]]$$
$$= m_i [\underline{r}_{i/G} \times \underline{a}_G + \dot{\underline{\omega}} (\underline{r}_{i/G} \cdot \underline{r}_{i/G}) - \underline{r}_{i/G} \overset{0}{\cancel{(\underline{r}_{i/G} \cdot \dot{\underline{\omega}})}} + C \underline{e}_r \times [\underline{k} \times (\underline{k} \times \underline{e}_r)]]$$
$$= m_i [\underline{r}_{i/G} \times \underline{a}_G + \dot{\underline{\omega}} |\underline{r}_{i/G}|^2 + C \underline{e}_r \times (\underline{k} \times \underline{e}_\theta)]$$
$$= m_i [\underline{r}_{i/G} \times \underline{a}_G + \dot{\underline{\omega}} |\underline{r}_{i/G}|^2 + \overset{0}{\cancel{C \underline{e}_r \times (-\underline{e}_r)}}].$$
$$= m_i (\underline{r}_{i/G} \times \underline{a}_G + \dot{\underline{\omega}} (|\underline{r}_{i/G}|^2).$$

The total moment of all the forces about the CoM is found by integrating over the whole lamina.

$$q_G = \overset{0}{\cancel{\left(\int \underline{r}\, dm\right)}} \times \underline{a}_G + \left(\int |\underline{r}|^2 dm\right) \dot{\underline{\omega}}$$
$$= \left(\int |\underline{r}|^2 dm\right) \dot{\underline{\omega}}$$

$\int \underline{r}\, dm = 0$ by defn of G
($\underline{r}$ is the distance from CoM).

- For a planar rigid body (lamina) undergoing planar motion, the total torque about the CoM is equal to the quantity $\boxed{I_G = \int |\underline{r}|^2 dm}$ times the angular acceleration.

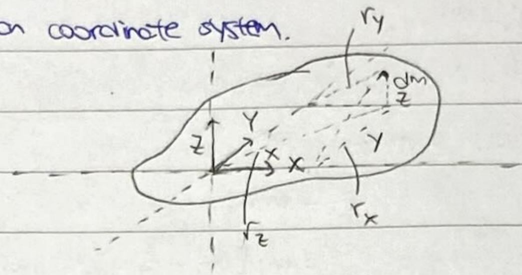
- The quantity I_O / I_G is called the mass moment of inertia about the axis O/ the CoM G. It is analogous to mass as it represents the resistance of a rigid body to changing angular velocity.

Translational dynamics	$\underline{F} = m\dot{\underline{v}}$	
Rotational dynamics	$q_O = I_O \dot{\underline{\omega}}$	[general rigid body, fixed axis of rotation]
	$q_G = I_G \dot{\underline{\omega}}$	[planar rigid body, general planar motion]

Rigid body properties II

perpendicular axis theorem

- consider a small element of mass dm within a general 3D rigid body defined wrt a Cartesian coordinate system.



The mass moment of inertia about each axis is given by:

$$I_{xx} = \int r_x^2 dm = \int (y^2+z^2) dm \qquad r_x: \text{distance from x-axis}$$

$$I_{yy} = \int r_y^2 dm = \int (x^2+z^2) dm \qquad r_y: \text{distance from y-axis}$$

$$I_{zz} = \int r_z^2 dm = \int (x^2+y^2) dm \qquad r_z: \text{distance from z-axis.}$$

- When the system is a lamina, then it has 0 depth → z^2 term can be neglected.

therefore for a lamina,

$$I_{xx} = \int r_x^2 dm = \int y^2 dm$$

$$I_{yy} = \int r_y^2 dm = \int x^2 dm$$

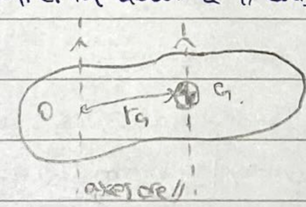
$$I_{zz} = \int r_z^2 dm = \int (x^2+y^2) dm$$

→ $\boxed{I_{zz} = I_{xx} + I_{yy} = \int (x^2+y^2) dm}$ (Perpendicular axis thm)

* The perpendicular axis thm is only valid for lamina where thickness z is negligible.

Parallel axis theorem.

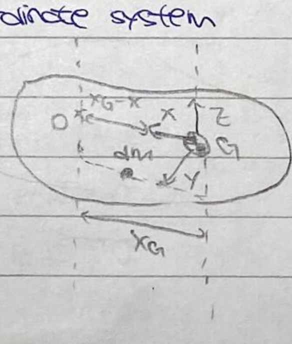
- The parallel axis thm. states that the moment of inertia about an axis passing through O is equal to the moment of inertia about a // axis passing through G plus a correction term.



always in relation to CoM!

$\boxed{I_O = I_G + M r_G^2}$. where r_G is the shortest distance between the 2 axes.

- For simplicity, consider a small element of mass dm within a lamina defined wrt a Cartesian coordinate system



In this coord system, $I_G = I_{zz} = \int (x^2+y^2)\,dm$

and $I_o = \int [(x_G - x)^2 + y^2]\,dm$

Therefore $I_o = \int (x_G^2 - 2x_Gx + x^2 + y^2)\,dm$

$= \int (x^2+y^2)\,dm + x_G^2 \int dm - 2x_G \int x\,dm$ $\int x\,dm = 0$ by defn of G.

$= I_G + M x_G^2$

Note x_G is equivalent to r_G since they are both the shortest distance btwn the // axes.

Note that for a differently chosen dm, it is possible that $I_o = \int [(x_G+x)^2 + y^2]\,dm$.

After expanding, the only difference is the $\int 2x_Gx\,dm$ term, which evaluates to 0.

* The parallel axis thm is valid for general 3D bodies as well.

Using the mechanics databook.

think about what m is the mass of → annulus

- The mechanics databook lists $k_\#^2$ of various common shapes.
- $k_\#$ is the radius of gyration. For a uniformly distributed mass, $k_\#^2 = \frac{1}{A}\int r_\#^2\,dA$.
- This can be used to find the moment of inertia provided mass is uniformly distributed.

$$\boxed{\begin{aligned} I_{xx} &= \int r_x^2\,dm = mk_x^2 \\ I_{yy} &= \int r_y^2\,dm = mk_y^2 \\ I_{zz} &= \int r_z^2\,dm = mk_z^2 \end{aligned}}$$

I_{zz} is the polar moment of inertia, sometimes denoted by J

Composite bodies.

- The moment of inertia of a composite body about an axis is the sum of the moments of inertia of the individual components about the same axis.
- Holes can be thought of as components w/ -ve mass.
- The process for calculating moments of inertia of composite bodies is:
  - ↳ 1) Find I_G for each component
  - ↳ 2) Use the parallel axis thm to find I_o for each component
  - ↳ 3) Sum the contributions from each component.

Equations of motion and D'Alembert

D'Alembert forces and moments

- For a planar rigid body, $\sum \underline{F}_i = m\underline{a}_G$

  and $\sum \underline{Q}_i = I_G \underline{\dot{\omega}}_{body}$

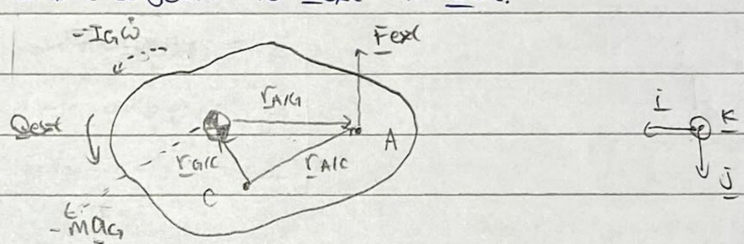
- Moving them to the LHS and adding them to the FBD as D'Alembert forces / moments,

  For a planar rigid body $\boxed{\sum \underline{F}_i - m\underline{a}_G = 0}$

  and $\boxed{\sum \underline{Q}_i - I_G \underline{\dot{\omega}}_{body} = 0}$

- Nothing is fundamentally changed, but it can help drawing everything on the FBD and keep track of all the terms.
* Be careful w/ the sign of the inertial forces/moments → draw them in the +ve direction of the unit vectors to be safe.
* Remember to multiply by mass m or moment of inertia I_G.
- After adding the inertial force/moment, the problem turns into a statics problem.
  → we can take moments about any pt. (We can take moments about pts w/ unknown forces to eliminate them).
- w/o D'Alembert force/moment the eqn. for angular planar motion $\underline{Q} = I_G \dot{\underline{\omega}}_{body}$ only applies to moments about the CoM so we need to combine it w/ the translational eqn. $\underline{F} = m\underline{a}_G$. W/ D'Alembert approach, the inertial force "$-m\underline{a}_G$" is automatically included.
- In other words, after using D'Alembert's approach, taking moments about a general pt. C is equivalent to applying $\sum \underline{F}_i = m\underline{a}_G$ and $\sum \underline{Q}_{Gi} = I_G \dot{\underline{\omega}}_{body}$. Proof as follows:

Consider a general lamina subjected to $\underline{F}_{ext}$ and $\underline{Q}_{ext}$.



Moments about C.: $\underline{Q}_{ext} + \underline{r}_{A/C} \times \underline{F}_{ext} + \underline{r}_{G/C} \times (-m\underline{a}_G) - I_G\dot{\underline{\omega}} = 0$.

$$\underline{Q}_{ext} + \underline{r}_{A/C} \times \underline{F}_{ext} + \underline{r}_{G/C} \times (-\underline{F}_{ext}) - I_G\dot{\underline{\omega}} = 0$$

$$\underline{Q}_{ext} + (\underline{r}_{A/C} - \underline{r}_{G/C}) \times \underline{F}_{ext} - I_G\dot{\underline{\omega}} = 0.$$

$$\underline{Q}_{ext} + \underline{r}_{A/G} \times \underline{F}_{ext} - I_G\dot{\underline{\omega}} = 0.$$

all torques about G. $\sum \underline{Q}_G - I_G\dot{\underline{\omega}} = 0$

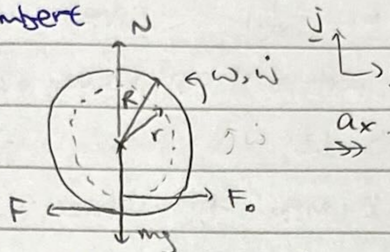
→ The underlying eqns are the same no matter where we take moments about.
(i.e. w/ D'Alembert's approach, ✓ moments about any pt ; w/o D'Alembert's approach, ✓ moments about G).

- For rotations about a fixed axis, we can also find the reaction forces at the pivot of rotation by taking moments about the CoM using D'Alembert's approach:
  ↳ Take moments about the pivot to find angular acceleration $\dot{\omega}$
  ↳ Take moments about the CoM and use N2L in $\underline{e}_r$ dir. to find the reaction forces.
- We can only solve for angular acceleration if we find the moment of inertia about the pivot of rotation I_O and use $\sum \underline{Q} = I_O \dot{\underline{\omega}}_{body}$ → it is more general to use D'Alembert's approach.
* Keep your "I" on G. → we can add D'Alembert force/moment → statics problem!

- e.g.: A reel of cable is wound around a hollow cylinder of mass $m/3$ and radius r. A disc also of mass $m/3$ but radius R caps each end of the cylinder. It is proposed to unroll the cable by pulling on the loose end w/ force F_0. What is the angular acceleration of the reel assuming no slip?

w/o D'Alembert

FBD:



No slip: $V_x = -R\omega \rightarrow a_x = -R\dot{\omega}$

N2L $\rightarrow$: $F_0 - F = ma_x$

$$F = F_0 + mR\dot{\omega}$$

$\overset{\curvearrowright}{M_G}$: $F_0 r - FR = I_G\dot{\omega}$

$$F_0 r - F_0 R - mR^2\dot{\omega} = I_G\dot{\omega}$$

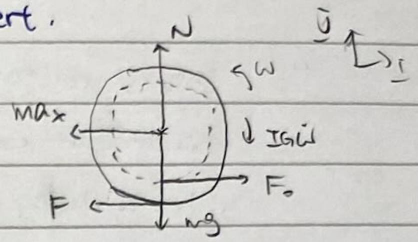
$$-F_0(R-r) = \dot{\omega}(I_G + mR^2)$$

$$\dot{\omega} = -\frac{F_0(R+r)}{I_G + mR^2}$$

(Fixed axis of rotation method also works since the contact pt. is the IC).

w/ D'Alembert.

FBD + D'A:



No slip: $V_x = -R\omega \rightarrow a_x = -R\dot{\omega}$

$\overset{\curvearrowright}{M_C}$: $F_0(R-r) - ma_x R + I_G\dot{\omega} = 0$.

$$F_0 R - F_0 r + mR^2\dot{\omega} + I_G\dot{\omega} = 0$$

$$-F_0(R-r) = \dot{\omega}(I_G + mR^2)$$

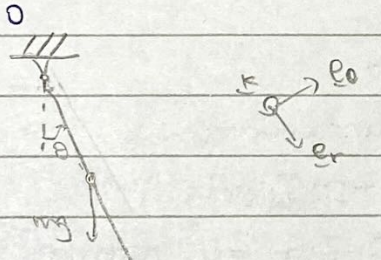
$$\dot{\omega} = -\frac{F_0(R+r)}{I_G + mR^2}.$$

combining 2 steps into 1.

- e.g.: What is the eqn. of motion / reaction force of the pivot of a pendulum of a clock, assuming that the pendulum is a rigid bar of length L and mass m.

Taking I at O

FBD



$I_0 = \frac{1}{3}mL^2$

$\overset{\curvearrowright}{M_0}$: $\Sigma Q = I_0\ddot{\theta}$

$$-\cancel{m}g\frac{\cancel{L}}{2}\sin\theta = \frac{1}{3}\cancel{m}L^{\cancel{2}}\ddot{\theta}$$

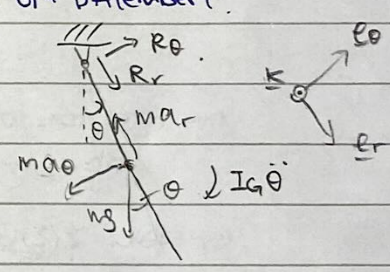
$$\ddot{\theta} = \frac{3g}{2L}\sin\theta$$

Cannot find the reaction forces

Faster in finding $\ddot{\theta}$ but not general.

Taking I at G + D'Alembert.

FBD + D'A



$I_G = \frac{1}{12}mL^2$, $\underline{a} = -r\dot{\theta}^2\underline{e_r} + r\ddot{\theta}\underline{e_\theta}$

$\overset{\curvearrowright}{M_0}$: $ma_\theta \cdot \frac{L}{2} + mg\sin\theta\frac{L}{2} + I_G\ddot{\theta} = 0$.

$$m\frac{L}{2}\ddot{\theta}\cdot\frac{L}{2} + \frac{1}{12}mL^2\ddot{\theta} + mg\sin\theta\frac{L}{2} = 0$$

$$\frac{1}{3}\cancel{m}L^{\cancel{2}}\ddot{\theta} = -\frac{1}{2}\cancel{m}g\cancel{L}\sin\theta$$

$$\ddot{\theta} = -\frac{3g}{2L}\sin\theta.$$

$\overset{\curvearrowright}{M_G}$: $R_\theta\cdot\frac{L}{2} + I_G\ddot{\theta} = 0$

$$\frac{1}{\cancel{2}}R_\theta \cancel{L} = \frac{1}{12}mL^{\cancel{2}}\cdot\frac{3g}{2\cancel{L}}\sin\theta$$

$$R_\theta = \frac{1}{4}mg\sin\theta.$$

N2L $\searrow$: $R_r + mg\cos\theta = ma_r$

$$R_r = -m\frac{L}{2}\dot{\theta}^2 - mg\cos\theta$$

Momentum of rigid bodies

Linear momentum

- For a group of particles, the rate of increase of linear momentum $\underline{P}$ is equal to the total ext force $\underline{F}$,

$$\boxed{\underline{F} = \frac{d\underline{P}}{dt}} \qquad \text{[linear momentum principle]}$$

- This is true for any collection of particles (rigid bodies, elastic bodies or disconnected collections of many particles)

* This is closely related to N2L $\underline{F} = m\underline{a}_G$ which applies to the CoM of any collection of particles.

Angular momentum

① Rotation about a fixed pt.

- Consider a collection of particles: the ith particle has mass m_i and a net ext. force $\underline{F}_i$ + int. forces $\underline{G}_{ij}$ (exerted by other particles) acting on it.

The rate of increase of angular momentum of the ith particle is:

$$\frac{d\underline{h}_{oi}}{dt} = \frac{d}{dt}(\underline{r}_i \times m_i \underline{v}_i)$$

$$= \underbrace{\dot{\underline{r}}_i \times m_i \underline{v}_i}_{0} + \underline{r}_i \times m_i \dot{\underline{v}}_i$$

$$= \underline{r}_i \times m\underline{a}_i$$

$$= \underline{r}_i \times \left(\underline{F}_i + \sum_{j \neq i} \underline{G}_{ij}\right)$$

$$= \underline{r}_i \times \underline{F}_i + \underline{r}_i \times \sum_{j \neq i} \underline{G}_{ij}$$

The total rate of increase of angular momentum is:

$$\frac{d\underline{h}_o}{dt} = \sum_i \frac{d\underline{h}_{oi}}{dt} = \sum_i \left[\underline{r}_i \times \underline{F}_i + \underline{r}_i \times \sum_{j \neq i} \underline{G}_{ij}\right].$$

By N3L, $\sum_i \sum_{j \neq i} \underline{G}_{ij} = 0$ so $\sum_i \underline{r}_i \times \sum_{j \neq i} \underline{G}_{ij} = 0$, therefore.

$$\frac{d\underline{h}_o}{dt} = \sum(\underline{r}_i \times \underline{F}_i) = \underline{q}_o$$

- This gives the angular momentum principle about a fixed pt. — for any system, the rate of increase of the angular momentum about a fixed pt. is equal to the total external moment applied to the system about that pt.

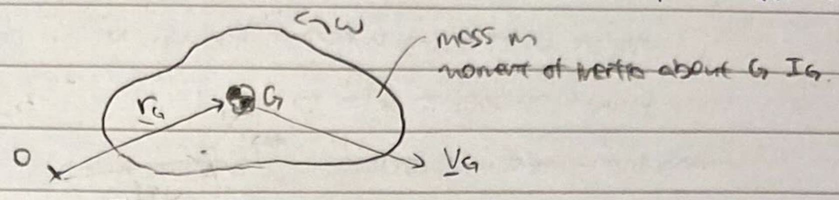
$$\boxed{\underline{q}_o = \frac{d\underline{h}_o}{dt} \qquad [\underline{h}_o = \underline{h}_G + \underline{r}_G \times m\underline{v}_G]}$$

- This applies to general 3D motion about a fixed pt. Although the result is simple, it is hard to solve the 3D case because $\underline{h}_o$ is difficult to find (we req. $\underline{I}_o$).

* The angular momentum principle above does not normally hold if the fixed pt. O moves, except when the angular momentum and moments are taken about the CoM G instead.

② General motion

- Consider a general system (collection of particles) w/ fixed reference pt. O and CoM G.



Recall that $\underline{h}_A = \underline{h}_O - \underline{r}_{A/O} \times \underline{p}_A$ $\qquad$ $\underline{q}_A = \underline{q}_O - \underline{r}_{A/O} \times \underline{F}_A$

$\underline{h}_O = \underline{h}_A + \underline{r}_{A/O} \times \underline{p}_A$ $\qquad$ $\underline{q}_O = \underline{q}_A + \underline{r}_{A/O} \times \underline{F}_A$

In this context, $\underline{h}_O = \underline{h}_G + \underline{r}_G \times m\underline{V}_G$ and $\underline{q}_O = \underline{q}_G + \underline{r}_G \times \underline{F}_G$.

↳ intuitive meaning: $\underline{r}_G \times m\underline{V}_G$ is the contribution from translation of CoM.

$\underline{h}_G$ is the contribution due to the body's angular momentum about G.

Applying the principle of angular momentum about a fixed pt. gives

$$\underline{q}_O = \frac{d\underline{h}_O}{dt} = \frac{d}{dt}(\underline{h}_G + \underline{r}_G \times m\underline{V}_G)$$

$$= \frac{d\underline{h}_G}{dt} + m\frac{d}{dt}(\underline{r}_G \times \underline{V}_G).$$

Using $\underline{q}_O = \underline{q}_G + \underline{r}_G \times \underline{F}_G$ and rearranging,

$$\underline{q}_G = \underline{q}_O - \underline{r}_G \times \underline{F}_G$$

$$= \frac{d\underline{h}_G}{dt} + m(\underline{r}_G \times \underline{a}_G) - \underline{r}_G \times \underline{F}_G$$

$$= \frac{d\underline{h}_G}{dt} + \underline{r}_G \times \underbrace{(m\underline{a}_G - \underline{F}_G)}_{0}$$

$$= \frac{d\underline{h}_G}{dt}.$$

- This gives the angular momentum principle about G — the rate of increase of angular momentum of any group of particles about their CoM is equal to the total ext. torque applied about their CoM.

$$\boxed{\underline{q}_G = \frac{d\underline{h}_G}{dt}. \qquad [\underline{h}_G = \underline{\underline{I}}_G \underline{\omega}_{body}]}$$

- This applies to 3D motion in general. However, it is hard to solve the 3D case because $\underline{h}_G$ is difficult to find (we req. $\underline{\underline{I}}_G$).

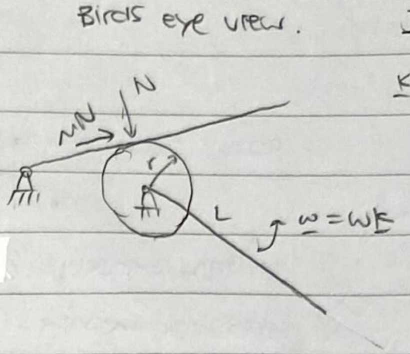
Linear and angular momentum principles.

- The 2 fundamental results governing the rate of change of momentum of any collection of particles (rigid bodies, flexible bodies or disconnected collections of particles) are:
  - ↳ Linear momentum principle: $\underline{F} = \frac{d\underline{p}}{dt}$ $\iff$ $\int_{t_A}^{t_B} \underline{F}\,dt = \underline{p}_B - \underline{p}_A$
  - ↳ Angular momentum principle: $\underline{q}_G = \frac{d\underline{h}_G}{dt}$ $\iff$ $\int_{t_A}^{t_B} \underline{q}_G\,dt = \underline{h}_{GB} - \underline{h}_{GA}$
- If the ext. force applied to a system is 0 → linear momentum is conserved.
- If the ext. torque applied to a system about a given pt is 0 → angular momentum is conserved about that pt.

* Note linear / angular momentum are vector quantity so both the magnitude and direction are conserved.

- eg: (fixed axis). A rod of length L and mass m is attached at one end to a light shaft of radius r and is spinning at a rate of ω. A friction brake is applied to the shaft using a normal preload N. The coefficient of friction is μ. How long does it take to stop?

Birds eye view.



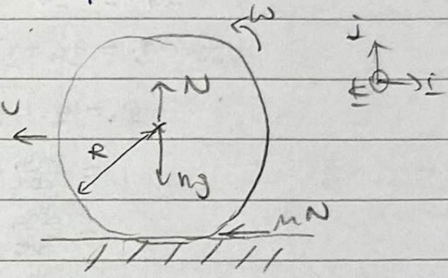
Angular momentum principle, $\int_0^{T_0} \underline{q}_o \, dt = \underline{h}_T - \underline{h}_o$

$$\int_0^{T_0} (-\mu N r) \, dt = 0 - I\omega$$

$$-\mu N r T_0 = -\tfrac{1}{3} m L^2 \omega$$

$$T_0 = \frac{m L^2 \omega}{3 \mu N r}$$

- eg: (moving axis) A solid ball of mass m and radius R is given an initial angular velocity $\Omega_1 \underline{k}$ rad s⁻¹ before being dropped vertically onto a rough surface. The coefficient of friction is μ and it can be assumed that the ball does not bounce upon landing. How long does it take before the ball starts rolling w/o slip.



Rolling w/o slip : $v = \Omega_2 R$

Linear momentum principle : $\int_0^{T_0} \underline{F} \, dt = \underline{P}_{T_0} - \underline{P}_0$

$$\int_0^{T} (+\mu N) \, dt = m \Omega_2 R - 0$$

$$\mu N T_0 = m \Omega_2 R \quad [1]$$

Angular momentum principle : $\int_0^{T_0} \underline{q}_G \, dt = \underline{h}_{G T_0} - \underline{h}_{G 0}$

$$\int_0^{T} (-\mu N R) \, dt = I_G \Omega_2 - I_G \Omega_1$$

$$\mu N R T_0 = \tfrac{2}{5} m R^2 (\Omega_1 - \Omega_2) \quad [2].$$

From [1], $\mu N T_0 = m \Omega_2 R \rightarrow \Omega_2 = \frac{\mu N T_0}{m R}$.

$$\therefore \mu N R T_0 = \tfrac{2}{5} m R^2 \left(\Omega_1 - \frac{\mu N T_0}{m R}\right)$$

$$T_0 \left(\mu N R + \tfrac{2}{5} \cancel{m} R^{\cancel{2}} \cdot \frac{\mu N}{\cancel{m R}}\right) = \tfrac{2}{5} m R^2 \Omega_1$$

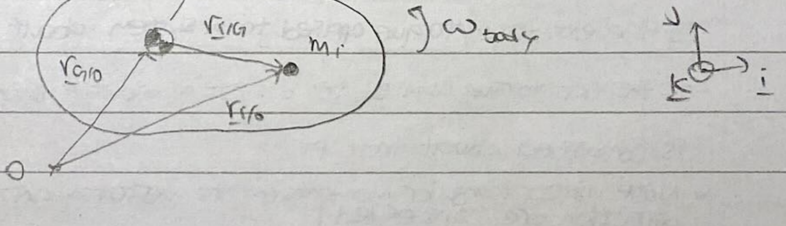
$$\tfrac{7}{5} \mu \cancel{m} g R T_0 = \tfrac{2}{5} \cancel{m} R^{\cancel{2}} \Omega_1$$

$$T_0 = \frac{2 R \Omega_1}{7 \mu g}.$$

Energy of rigid bodies.

Energy of rigid bodies in planar motion

- Consider a rigid body that is translating and rotating.



At a given instant, the velocity of the ith particle is.

$$\underline{v}_i = \underline{v}_G + \underline{\omega} \times \underline{r}_{i/G}$$

and therefore its kinetic energy is.

$$T_i = \tfrac{1}{2}m_i|\underline{v}_i|^2 = \tfrac{1}{2}m_i|\underline{v}_G + \underline{\omega}\times\underline{r}_{i/G}|^2$$

The total kinetic energy is

$$\begin{aligned}T = \sum T_i &= \sum \tfrac{1}{2}m_i|\underline{v}_G + \underline{\omega}\times\underline{r}_{i/G}|^2 \\ &= \tfrac{1}{2}\sum m_i(\underline{v}_G + \underline{\omega}\times\underline{r}_{i/G})\cdot(\underline{v}_G + \underline{\omega}\times\underline{r}_{i/G}) \\ &= \tfrac{1}{2}\sum m_i[\underline{v}_G\cdot\underline{v}_G + (\underline{\omega}\times\underline{r}_{i/G})\cdot(\underline{\omega}\times\underline{r}_{i/G}) + 2\underline{v}_G\cdot(\underline{\omega}\times\underline{r}_{i/G})] \\ &= \tfrac{1}{2}\sum m_i(|\underline{v}_G|^2 + |\underline{r}_{i/G}|^2\omega^2) + \sum m_i\underline{v}_G\cdot(\underline{\omega}\times\underline{r}_{i/G}) \\ &= \tfrac{1}{2}Mv_G^2 + \tfrac{1}{2}I_G\omega^2 + \underline{v}_G\cdot(\underline{\omega}\times\overset{0}{\cancel{\sum m_i\underline{r}_{i/G}}}) \\ &= \tfrac{1}{2}Mv_G^2 + \tfrac{1}{2}I_G\omega^2\end{aligned}$$

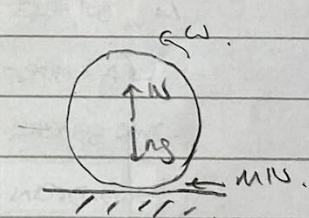
→ The total kinetic energy of a rigid body that can translate and rotate is.

$$\boxed{T = \overset{\text{translational}}{\tfrac{1}{2}Mv_G^2} + \overset{\text{rotational}}{\tfrac{1}{2}I_G\omega_{body}^2}}$$

i.e. the sum of the translational KE of the CoM and the rotational KE about the CoM.

* A force can affect both the body's translational and rotational KE if it is not applied through the CoM → we can separate the 2 effects by shifting the force to the CoM (by adding a moment).

e.g: A solid ball of mass m and radius R is given an initial angular velocity $\Omega_1\underline{k}$ rad s^{-1} before being dropped vertically onto a rough surface. The coefficient of friction is μ and it can be assumed that the ball does not bounce when it lands. How far does the ball travel before rolling w/o slip.

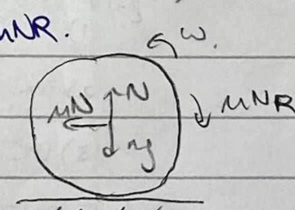


The frictional force is not applied through the CoM.

→ affects both translational + rotational KE.

We can shift the force to separate the 2 effects:

After shifting the force to the CoM, we add a torque $|\underline{\tau}_G| = \mu NR$.



Rolling w/o slip: $v = \Omega_2 R$.

Recalling it takes $T = \frac{2\Omega_1 R}{7\mu g}$ to roll w/o slip,

and $\mu NT = m\Omega_2 R$

$$\Omega_2 = \frac{\mu g}{R}\cdot\frac{2\Omega_1 R}{7\mu g} = \frac{2}{7}\Omega_1$$

Work-energy principle: $WD_{total} = \Delta KE_{total}$

$$\underbrace{\mu mgd} + (-\mu NR)\theta = \underbrace{\Delta KE_{linear}} + \Delta KE_{rotational}$$

Using the linear part, $\mu \cancel{m} gd = \tfrac{1}{2}\cancel{m}[(\Omega_2 R)^2 - (\Omega_1 R)^2]$

$$d = \frac{2\Omega_1^2 R^2}{49\mu g}$$

Impacts

Impacts.

- The impulse $\underline{J}$ is equal to the net change in linear momentum.

$$\boxed{\underline{J} = \int_{t_A}^{t_B} \underline{F}\,dt = \underline{P}_B - \underline{P}_A}$$

where $\underline{F}$ is the total ext force.

- The moment of impulse $\underline{r} \times \underline{J}$ is equal to the net change in angular momentum about the fixed pt. O / the moving CoM G.

$$\boxed{\begin{aligned} \underline{r}_{J/O} \times \underline{J} &= \int_{t_A}^{t_B} \underline{\tau}_O\,dt = \underline{h}_{OB} - \underline{h}_{OA} \\ \underline{r}_{J/G} \times \underline{J} &= \int_{t_A}^{t_B} \underline{\tau}_G\,dt = \underline{h}_{GB} - \underline{h}_{GA} \end{aligned}}$$

[about the fixed pt. O]

[about the moving CoM G].

where τ_O / τ_G is the total ext. torque about the fixed pt. O / the moving CoM G.

- The term impulse can mean forces over an arbitrarily long time, but a collision event will imply a short duration impulse (impact).
- During a collision, the contact time is short and so there is a large force but a finite momentum change.
- In order to predict what happens during a collision event, it is helpful to make the following set of assumptions:
  1. The impulse is ideal, i.e.
     - ↳ It is infinitesimally short w/ an infinite force.
     - ↳ The direction of $\underline{J}$ is the same as the force $\underline{F}$ and does not change during impact.
     - ↳ The magnitude of $\underline{J}$ is the area under the F-t curve i.e. $\underline{J} = \int \underline{F}\,dt$.
  2. During the impact the bodies do not change position significantly (as the contact time is short) → allow us to use the geometry at impact to find moments.
  3. All other impulsive forces (eg self weight) are negligible (as the impulsive forces are infinite).
- The general method is as follows:
  1. Draw 3 diagrams for before, during and after the impact.
     - ↳ Each diagram should show the system in the same configuration (moment of impact).
     - ↳ The before and after diagrams should define before and after velocities
     - ↳ The during diagram is like the FBD but showing impulses and ignoring non-impulsive forces.
  2. Equate linear momentum changes to impulsive forces (2 components)
  3. Equate angular momentum changes to impulsive moments.
  4. The coefficient of restitution e is defined as before – the ratio of relative separation to approach velocities (along the line of impact!).

* when finding the velocity of the contact pt, remember to add the rotational component

Fixed axis of rotation vs general planar motion.

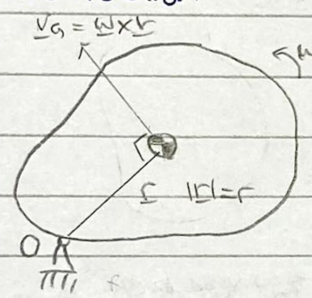
- For a fixed axis of rotation (the body is attached to a pivot pt), we can use either I_O or I_G. usually use I_O to avoid unknown reactions at pivot first. ~~(take moments about fixed axis first)~~.
  - ↳ If we use I_O, we must only take moments about the fixed axis.
  - ↳ If we use I_G, we can take moments about the fixed axis or a moving axis. ↖ then use I_G to find reactions at pivot
- For general planar motion, we must use I_G.
  - ↳ we can take moments about a fixed axis or a moving axis.
- When taking moments about a fixed axis, use

$$\underline{r}_{J/O} \times \underline{J} = \int_{t_A}^{t_B} \underline{\tau}_O \, dt = \underline{h}_{OB} - \underline{h}_{OA} \qquad * \ \underline{h}_O = \underline{h}_G + \underline{r}_G \times m\underline{v}_G$$

- When taking moments about a moving axis, use

$$\underline{r}_{J/G} \times \underline{J} = \int_{t_A}^{t_B} \underline{\tau}_G \, dt = \underline{h}_{GB} - \underline{h}_{GA}$$

Fixed axis of rotation



Rigid body pivots about pt. O

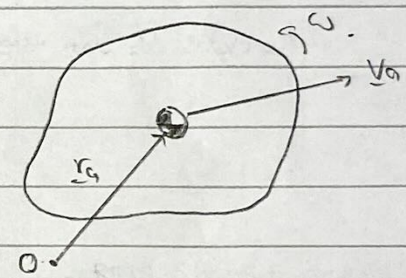
$$\underline{h}_O = I_O \underline{\omega}$$
$$= (I_G + mr^2)\underline{\omega} \quad (\text{// axis thm.})$$

or using the general expression,

$$\underline{h}_O = \underline{h}_G + \underline{r}_G \times m\underline{v}_G$$
$$= I_G\underline{\omega} + mr^2\underline{\omega}$$
$$= (I_G + mr^2)\underline{\omega}$$

Works because $\underline{v} = \underline{\omega} \times \underline{r}$ and $\underline{r} \perp \underline{v}$.

General planar motion



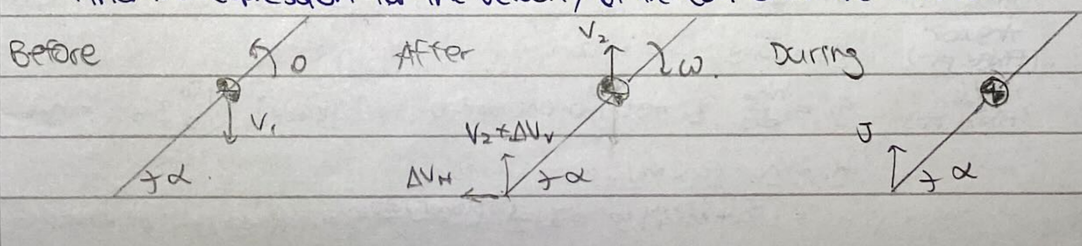
Rigid body not constrained at pt. O.

$$\underline{h}_O = \underline{h}_G + \underline{r}_G \times m\underline{v}_G$$
$$= I_G\underline{\omega} + \underline{r}_G \times m\underline{v}_G$$

* More general to keep your I on G (i.e. use I_G).

- e.g.: A phone of length L and mass m is dropped from a height h onto a smooth floor, starting w/ 0 angular velocity and at an angle α to the horizontal. The coefficient of restitution is e=1.

  Find the expression for the velocity of the CoM after the bounce.



During the fall $\frac{1}{2}mv_1^2 = mgh \rightarrow v_1 = \sqrt{2gh}$

$\Delta V_V = \omega_2 \frac{L}{2}\cos\alpha$; $\Delta V_H = \omega_2 \frac{L}{2}\sin\alpha$. * $\Delta \underline{V} = \underline{\omega} \times \underline{r}$

① Linear momentum ↕ $J = m(v_2 - (-v_1))$

Linear momentum ↔ : $0 = \omega_2 \frac{L}{2}\sin\alpha$ ($J \neq 0$ if the floor is not smooth)

② Angular momentum Abt. contact: $0 = (I_G\omega_2 - mv_2\frac{L}{2}\cos\alpha) - (+mv_1\frac{L}{2}\cos\alpha)$

Abt. G: $J\frac{L}{2}\cos\alpha = I_G\omega_2$ } equivalent.

③ Coeff of restitution: $v_2 + \omega_2\frac{L}{2}\cos\alpha = v_1$

$$\omega_2 = \frac{2(v_1 - v_2)}{L\cos\alpha}$$

sub into ②, $mv_1\frac{L}{2}\cos\alpha = I_G\frac{2(v_2 - v_1)}{L\cos\alpha} - mv_2\frac{L}{2}\cos\alpha$.

$$\cancel{m}v_1\frac{\cancel{L}}{2}\cos\alpha = \frac{1}{12}\cancel{m}L^{\cancel{2}}\cdot\frac{2v_2}{\cancel{L}\cos\alpha} - \frac{1}{12}\cancel{m}L^{\cancel{2}}\cdot\frac{2v_1}{\cancel{L}\cos\alpha} - \cancel{m}v_2\frac{\cancel{L}}{2}\cos\alpha$$

$$v_1\left(3\cos\alpha + \frac{1}{\cos\alpha}\right) = v_2\left(\frac{1}{\cos\alpha} - 3\cos\alpha\right)$$

$$v_2 = v_1\left(\frac{3\cos^2\alpha + 1}{1 - 3\cos^2\alpha}\right)$$

$$= \sqrt{2gh}\left(\frac{1 + 3\cos^2\alpha}{1 - 3\cos^2\alpha}\right).$$

* Note. V_2 only true if $1 - 3\cos^2\alpha > 0$

$$\cos^2\alpha < 1/3$$

$$\cos\alpha < 1/\sqrt{3}$$

$$\alpha > 55^\circ$$

- Angular momentum is conserved about the contact pt. as no ext. moment abt. that pt.
- The contact pt. is fixed, but it is not a fixed axis of rotation. (the rotation of the phone is not constrained). → must calculate angular momentum $\underline{h_o}$ using $\underline{h_o} = \underline{h_G} + \underline{r_G} \times m\underline{v_G}$. (cannot use $\underline{h_o} = I_o\underline{\omega} = (I_G + mr_G^2)\underline{\omega}$).
- Alternatively, we can consider angular momentum abt. the CoM but angular momentum is not conserved → we need another eqn. to eliminate the contact impulse $\underline{J}$.
- In this case we must use I_G because the phone is not attached to a pivot.

Rigid body key equations

	Momentum (differential form)	Impulse (integral form)
Use	Eqn. of motion, full soln	Before/after calculations.
Linear	$\underline{F} = \frac{d\underline{p}}{dt}$, $\underline{p} = m\underline{v_G}$	$\underline{J} = \int_{t_A}^{t_B} \underline{F} dt = \underline{p_B} - \underline{p_A}$
Angular (About G)	$\underline{q_G} = \frac{d\underline{h_G}}{dt}$, $\underline{q_G}$ = ext. moment abt. G.	$\underline{r_{J/G}} \times \underline{J} = \int_{t_A}^{t_B} \underline{q_G} dt = \underline{h_{GB}} - \underline{h_{GA}}$
	$\underline{h_G} = I_G\underline{\omega}$ planar motion	
Angular (Fixed pt.)	$\underline{q_o} = \frac{d\underline{h_o}}{dt}$. $\underline{q_o}$ = ext. moment abt. O.	$\underline{r_{J/o}} \times \underline{J} = \int_{t_A}^{t_B} \underline{q_o} dt = \underline{h_{oB}} - \underline{h_{oA}}$
	$\underline{h_o} = \underline{h_G} + \underline{r_G} \times m\underline{v_G}$ planar motion	
	$= (I_G + mr_G^2)\underline{\omega} = I_o\underline{\omega}$ planar motion, fixed axis of rotation.	

Numerical methods

Numerical methods

Numerical methods.

- When analytical soln cannot be found, then soln can be found numerically by considering small but discrete time steps Δt. (discretisation).
- There are 2 main steps for systems consisting of particles or rigid bodies:
  - ↳ Approximate the derivatives in the eqn. of motion using discrete time steps.
  - ↳ Solve the discrete eqn. at every time step
- The smaller the time step Δt, the more accurate the numerical approximation.

Numerical differentiation.

- From the defn of the derivative, $\dot{x}(t) = \lim_{\Delta t \to 0} \dfrac{x(t+\Delta t) - x(t)}{\Delta t}$.

If we consider the nth time interval, dividing time into discrete intervals Δt,

$$\dot{x}(n\Delta t) = \frac{x(n\Delta t + \Delta t) - x(n\Delta t)}{\Delta t}.$$

$$\dot{x}_n = \frac{x_{n+1} - x_n}{\Delta t}.$$

- We can estimate the velocity at $t = n\Delta t$, $\dot{x}_n$ using

① Single-sided estimate $\quad \dot{x}_n = \dfrac{x_n - x_{n-1}}{\Delta t}$

② Central difference $\quad \dot{x}_n = \frac{1}{2}\left(\dfrac{x_{n+1} - x_n}{\Delta t} + \dfrac{x_n - x_{n-1}}{\Delta t}\right) = \frac{1}{2}\left(\dfrac{x_{n+1} - x_{n-1}}{\Delta t}\right)$

Similarly, we can estimate the acceleration at $t = n\Delta t$, $\ddot{x}_n$ using

① Single-sided estimate $\quad \ddot{x}_n = \dfrac{\dot{x}_n - \dot{x}_{n-1}}{\Delta t}$

② Central difference $\quad \ddot{x}_n = \frac{1}{2}\left(\dfrac{\dot{x}_{n+1} - \dot{x}_n}{\Delta t} + \dfrac{\dot{x}_n - \dot{x}_{n-1}}{\Delta t}\right) = \frac{1}{2}\left(\dfrac{\dot{x}_{n+1} - \dot{x}_{n-1}}{\Delta t}\right)$

* Central difference method works for middle values (that have a $k+1$, $k-1$ term) but we have to use the single-sided estimate at the ends. (unless we extrapolate values).

Numerical integration.

- Using the Taylor series (single variable), $x(t+\Delta t) = x(t) + \dot{x}(t)\Delta t + \frac{1}{2}\ddot{x}(t)\Delta t^2 + \ldots$

Taking the first order approximation, $x(t+\Delta t) = x(t) + \dot{x}(t)\Delta t$.

If we consider the nth time interval, dividing time into discrete periods Δt.

$$x(n\Delta t + \Delta t) = x(n\Delta t) + \dot{x}(n\Delta t)\Delta t$$

$$x_{n+1} = x_n + \dot{x}_n \Delta t.$$

- We can estimate the velocity at $t = n\Delta t$, $\dot{x}_n$, using

① Euler's method. $\quad \dot{x}_n = \dot{x}_{n-1} + \ddot{x}_{n-1}\Delta t$

② Semi-implicit method $\quad \dot{x}_n = \dot{x}_{n-1} + \ddot{x}_{n-1}\Delta t$

where $\ddot{x}_{n-1}$ and $\ddot{x}_n$ are found from the equation of motion. $\left(\ddot{x} = \dfrac{F}{m}\right)$

- Similarly, we can estimate the position at $t=n\Delta t$, x_n, using:

  ① Euler's method $\quad x_n = x_{n-1} + \dot{x}_{n-1}\Delta t$

  ② semi-implicit method $\quad x_n = x_{n-1} + \dot{x}_n \Delta t$.

- To check whether the answers are accurate:

  ↳ check a known case/aspect of the simulation.

  ↳ check for convergence when increasing the no. of time steps, N.

- For 1st order methods, error $\propto \Delta t \propto \frac{1}{N}$. (error $= |d_{num} - d_{analytical}|$).