

Elastic properties and constitutive response**Stress and strain**

- Normal stress σ is defined as

$$\sigma = \frac{F}{A}$$

where F is the force carried \perp to the plane within the material.

- An applied normal stress σ results in deformation - length increases in tension; decreases in compression.

- Normal strain ϵ is defined as

$$\epsilon = \frac{\Delta L}{L_0}$$

- By convention, tensile stress/strain is +ve, compressive stress/strain is -ve.

- Uniaxial stress produces triaxial strain (Poisson's ratio).

Stiffness and strength

- The stiffness S of a component/structure is defined as

$$S = \frac{F}{\Delta L}$$

- Stiffness S depends on the material, geometrical shape and mode of bending.

(Axial stiffness > bending stiffness \rightarrow easier to bend a beam than to stretch it)

- The strength σ of a material can either be defined as the nominal stress at the elastic limit (σ_{el}/σ_y) or the nominal stress at failure (σ_{fs})
- Strength can be thought of as the resistance to permanent distortion (σ_y) or total failure (σ_{fs})

Young's modulus

- Young's modulus E is a material property and is defined as

$$E = \frac{\sigma}{\epsilon}$$

- The Young's modulus E characterizes the intrinsic stiffness of the material

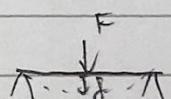
Measuring Young's modulus**① Bending stiffness of a beam**

- A beam of uniform GFA is loaded in 3 pt. bending.

- In this configuration, $f = \frac{FL^3}{4EI}$

- For a square section of side length D , $I = \frac{D^4}{12}$, so

$$E = \frac{(f/D)D^3}{4D^4} = \frac{(f/D)L^3}{4D^2}$$



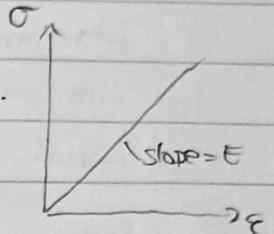
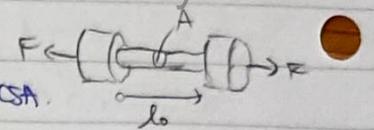
- Bending gives more deflection for a given load than tension

- E is sensitive to L and $D \rightarrow$ we req. accurate measurement of dimensions

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④ Tensile testing.

- A uniaxial tension is applied to sample of standard, uniform CSA.
- We measure the extension of gage length from machine extension or directly from the sample using a strain gauge.
- Convert the load-extension graph to stress-strain graph. The Young's modulus E is given by the slope.
- As the extensions are small ($\epsilon < 0.1\%$), it is hard to measure precisely.
- Measurement from machine must allow for flexure of machine
- The strain gauge has a much higher precision.



⑤ Natural frequency of vibration (beams and plates).

- The natural frequency of vibration in bending depends on flexural rigidity $B=EI$.
- A beam is supported at nodal pts and set over a loudspeaker.
- A plate is supported on soft mounts over a loudspeaker (Chladni patterns).
- The frequency of the sound is adjusted until the beam/plate reaches its natural frequency (max. amplitude vibration, stable pattern).
- Measuring frequency is more accurate than deflection.
- The calculated E is sensitive to dimensions (via I and L).

⑥ Speed of sound in the material

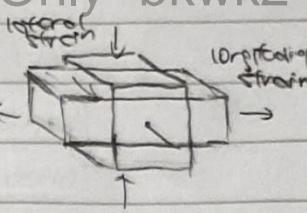
- Both the Young's modulus and speed of sound in a solid depend on the stiffness of interatomic bonds.
- To measure the speed of sound, we strike a bar of material on the end and time the wave reflected from the far end of the bar.
- $E = \rho V^2$, where ρ is the density of the material.
- The accuracy depends on precise time measurement, which is relatively easy w/ piezoelectric transducers.

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Poisson's ratio and 2D Hooke's law

- Poisson's ratio ν is a material property and is defined as

$$\nu = -\frac{\text{lateral strain}}{\text{longitudinal strain}}$$



- The lateral strain is not due to volume conservation (density change) but reflects the way atomic bonds deform under load.

metals, ceramics: $\nu = 0.2 - 0.33$

- The Poisson's ratio ν usually ranges from -1 to 0.5

↳ Materials w/ $\nu = 0.5$ are incompressible materials — volume conserved, $\Delta = 0$.

↳ Materials w/ $\nu = 0$ have a constant CSA even when deformed longitudinally.

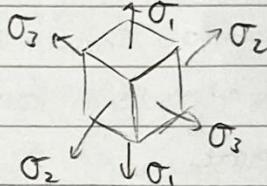
↳ Materials w/ $\nu < 0$ are auxetic materials.

rubber
foam, cork

- Some engineering components are loaded uniaxially (cables, struts), but often stresses are multiaxial (beams, arches, pressure vessels)

- Strain is always 2D (even for uniaxial stress due to Poisson's ratio).

- Consider a unit cube of material under a general set of normal stresses $(\sigma_1, \sigma_2, \sigma_3)$



To find the resulting strains, we superpose the strains due to each stress.

$$\epsilon_1 = \frac{1}{E} (\sigma_1 - \nu \sigma_2 - \nu \sigma_3)$$

in matrix form

$$\epsilon = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \sigma$$

$$\epsilon_2 = \frac{1}{E} (-\nu \sigma_1 + \sigma_2 - \nu \sigma_3)$$

$$\epsilon_3 = \frac{1}{E} (-\nu \sigma_1 - \nu \sigma_2 + \sigma_3)$$

[Generalised Hooke's law]

i.e. isotropic elastic constitutive response

Dilatation and bulk modulus.

- Volumetric strain/dilatation Δ is a material property and is defined as

$$\Delta = \frac{\Delta V}{V_0}$$

- Consider a unit cube of material. For a general strain state $(\epsilon_1, \epsilon_2, \epsilon_3)$:

Initial volume $V_0 = 1$

Final volume $V = (1 + \epsilon_1)(1 + \epsilon_2)(1 + \epsilon_3) = 1 + (\epsilon_1 + \epsilon_2 + \epsilon_3) + O(\epsilon^2)$

→ for small strains ($\epsilon \ll 1$), $\Delta = \frac{V - V_0}{V_0} = \epsilon_1 + \epsilon_2 + \epsilon_3$ — valid for metal
not valid for rubber

- Bulk modulus K is a material property and is defined as

$$K = \frac{\sigma}{\Delta}$$

where σ is isotropic volumetric stress, i.e. hydrostatic stress.

$$-\Delta = \epsilon_1 + \epsilon_2 + \epsilon_3 = \frac{1}{E} [\sigma_x - \nu(\sigma_y + \sigma_z)] + \dots = \frac{1-2\nu}{E} (\sigma_x + \sigma_y + \sigma_z) \quad [\text{for small strains}]$$

$$\text{Isotropic so } \sigma_x = \sigma_y = \sigma_z = \sigma \rightarrow \Delta = \frac{3\sigma(1-2\nu)}{E} \quad \therefore K = \frac{\sigma}{\Delta} = \frac{E}{3(1-2\nu)}$$

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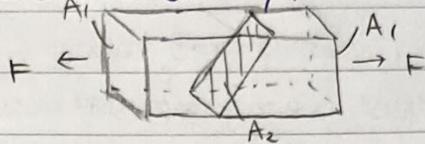
shear stress, strain and modulus.

- shear stress τ is defined as

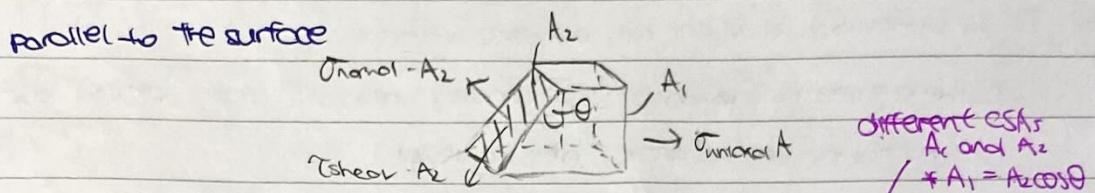
$$\tau = \frac{F}{A}$$

where F is the force carried // to the plane within the material

- For uniaxial tension, consider an arbitrary plane which is inclined to the axis.



For estm, the shaded surface must carry components of force both normal and parallel to the surface



different cosines
 A_1 and A_2
 $/ A_1 = A_2 \cos \theta$

- Forces may be related by resolving but this is not true for stresses

- The description of the stress state depends on the orientation of the surface.

($\sigma_{uniaxial}$ is equivalent to $\sigma_{normal} + \tau_{shear}$) $\sigma_{normal} = \sigma_{uniaxial} \cos \theta$
 $\tau_{shear} = \sigma_{uniaxial} \sin \theta$ $\rightarrow K = \frac{\theta}{2}$ ($\theta = 45^\circ$)

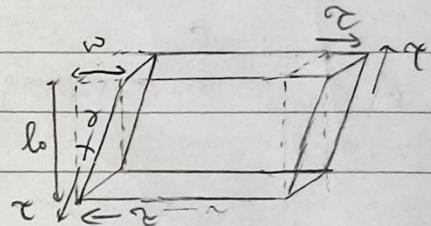
- An applied shear stress τ results in distortion rather than change in axial dimensions.

- Shear strain γ is defined as

$$\tan \gamma = \frac{\tau}{l_0}$$

For small strains, $\tan \gamma \approx \gamma$, so

$$\gamma = \frac{\tau}{l_0}$$

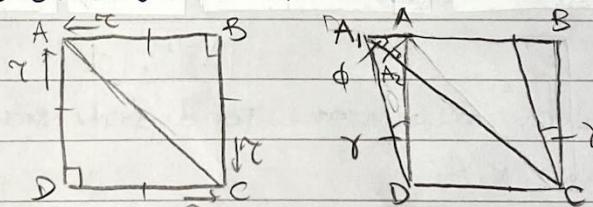


- Shear modulus G is a material property and is defined as

$$G = \frac{\tau}{\gamma}$$

- The shear modulus G characterizes the intrinsic stiffness in shear of the material.

- Consider the 2D element under applied shear stress τ .



$$\epsilon_{AC} = \frac{AC - AC'}{AC} = \frac{AC - A_1 C}{AC} = \frac{A_1 A_2}{AC}$$

small deformation $\rightarrow A_1 C \approx A_2 C$

Consider $\Delta A A_1 A_2$ $A_1 A_2 = A A_1 \cos \phi = A A_1 \cos 45^\circ$ small deformation $\rightarrow \phi \approx 45^\circ$

$$AC^2 = AD^2 + CD^2 \rightarrow AC = \sqrt{2} AD$$

$$\therefore \epsilon_{AC} = \frac{AA_1 \cos 45^\circ}{\sqrt{2}AD} = \frac{AA'}{2AD} = \frac{\gamma}{2} = \frac{\tau}{2G} \quad [\epsilon = \frac{\gamma}{2}]$$

$$\epsilon_{AC} = \epsilon_{AC1} (\text{due to } \tau_{AC}) + \epsilon_{AC2} (\text{due to } \tau_{BD}) = \frac{\tau}{E} + \frac{\tau}{E} \cdot v = \frac{\tau}{E} (1+v)$$

$$\rightarrow \epsilon_{AC} = \frac{\tau}{2G} = \frac{\tau}{E} (1+v)$$

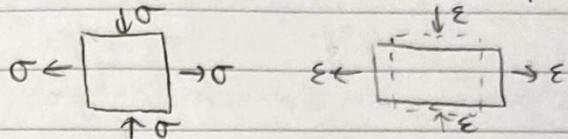
$$G = \frac{E}{2(1+v)}$$

for crystalline materials,
 $v = 1/3$ so $G = 3/5 E$.

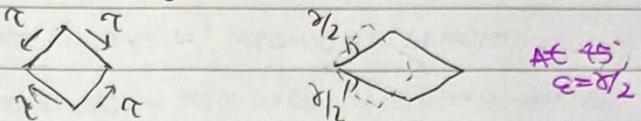
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Relation between normal vs shear stress/strain

- Both stress and strain stresses depend on the axes chosen.
- consider the 2D element loaded w/ biaxial normal tension/compression.



consider the plane inclined at a 45° angle;



There is pure shear stress and pure shear strain.

- * For biaxial/triaxial loading, find strains by superposing strains due to loading in each axis.

constrained deformation

- A constraint induces a transverse stress: nominally uniaxial loads generate biaxial or triaxial stresses in the material. [As we req. $\sum \epsilon = 0$]
- consider a cube of material loaded w/ a compressive stress σ_1 . Deformation is constrained in direction 3:

deformation constrained in direction 3, i.e. $\epsilon_3 = 0 \rightarrow$ there must be σ_3 s.t.

$$\sum \epsilon_i = \frac{\sigma_i}{E} - v \frac{\sigma_i}{E} = 0 \quad \rightarrow \quad \sigma_3 = v \sigma_1 \quad \begin{matrix} \text{biaxial stress} \\ \text{effective modulus} \end{matrix}$$

$$\therefore \epsilon_1 = \frac{\sigma_1}{E} - v \frac{\sigma_1}{E} = \frac{\sigma_1}{E} - v^2 \frac{\sigma_1}{E} = \frac{\sigma_1}{E} (1 - v^2) \rightarrow \boxed{\frac{\sigma_1}{\epsilon_1} = \frac{E}{1-v^2}}$$

- consider a cube of material loaded w/ a compressive stress σ_1 . Deformation is constrained in directions 2,3:

deformation constrained in directions 2,3, i.e. $\epsilon_2 = \epsilon_3 = 0 \rightarrow$ there must be σ_2, σ_3 s.t.

$$\sum \epsilon_2 = \frac{\sigma_2}{E} - v \left(\frac{\sigma_1}{E} + \frac{\sigma_3}{E} \right) = 0 \quad \left[\sigma_2 = \frac{v \sigma_1}{1-v} \right]$$

$$\sum \epsilon_3 = \frac{\sigma_3}{E} - v \left(\frac{\sigma_1}{E} + \frac{\sigma_2}{E} \right) = 0 \quad \left[\sigma_3 = \frac{v \sigma_1}{1-v} \right]$$

$$\therefore \epsilon_1 = \frac{\sigma_1}{E} - v \left(\frac{\sigma_2 + \sigma_3}{E} \right) = \frac{\sigma_1}{E} - v \left(\frac{2v \sigma_1}{(1-v)E} \right) = \frac{\sigma_1}{E} \left(\frac{1-v-2v^2}{1-v} \right) \rightarrow \boxed{\frac{\sigma_1}{\epsilon_1} = \frac{E(1-v)}{1-v-2v^2}}$$

- The apparent material stiffness is increased by a factor $\frac{1}{1-v^2}$ or $\frac{1-v}{1-v-2v^2}$.

$$\frac{1}{1-v^2} \quad \frac{1-v}{1-v-2v^2}$$

crystalline materials $v=1/3$

$$1.12$$

$$1.5$$

rubber $v=1/2$

$$1.23$$

tends to ∞

- * Rubber has a low Young's modulus E but its effective modulus is very high if constrained in both directions (since rubber is incompressible)

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Thermal stress and strain

- All materials expand as temp. rises. To a good approximation, the thermal strain ϵ_T caused by a temp. change $\Delta\theta$ is given by

$$\epsilon_T = \alpha \Delta\theta$$

where α is the thermal expansion coefficient.

α is of order $\sim 10^{-4} - 10^{-6} K^{-1}$

- Thermal expansion is important in design when:

↳ Constrained deformation \rightarrow thermal stresses

$$\sum \epsilon = \epsilon_T + \epsilon_{elastic} + \epsilon_{plastic} = 0$$

$$u \epsilon_T = \alpha \Delta\theta$$

$$\epsilon_{elastic} = \frac{\sigma}{E} - \nu \left(\frac{\sigma_x + \sigma_y}{E} \right)$$

↳ Temperature gradient exist across a product \rightarrow thermal stresses.

↳ Dissimilar materials joined and subjected to temp. change \rightarrow differential thermal expansion \rightarrow thermal stresses

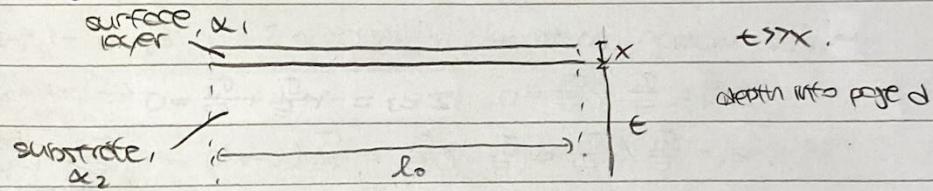
↳ For a bimetallic strip, curvature K is

$$K \propto \frac{(d_1 - d_2) \Delta T}{(E_{d_1} + E_{d_2} + V)}$$

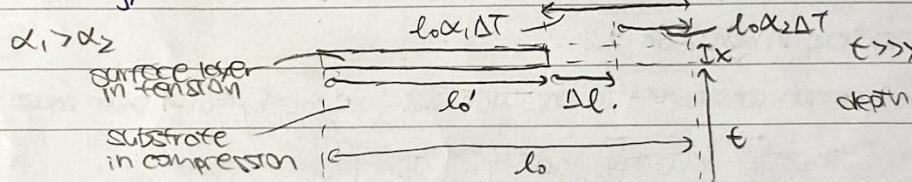
Constrained surface layers

- Many coating technologies (for wear/corrosion resistance) deposit a thin surface layer of a different material on a component at high temp.

- Consider a thin film on a component of thickness t \rightarrow film thickness x and exclude a length l_0 , away from the edges. \rightarrow edge effects can be ignored



- On cooling, differential contraction of substrate and surface layer induces thermal stress.



- For $\alpha_1 > \alpha_2$, surface layer is restrained from contracting by the substrate \rightarrow tension; tension in surface layer balanced by compression in substrate. \checkmark analysis still valid for $\alpha_2 > \alpha_1$.

- As the surface layer is thin compared to the substrate ($t \gg x$). \checkmark stress negligible

↳ Compressive stress in substrate is negligible (large area to provide force)

↳ Final length of surface layer is that of substrate, $l_0(1 - \alpha_2 \Delta T)$. \checkmark strain negligible

- After cooling, the surface layer effectively expands from l_0' to $l_0' + \Delta l$:

$$\Delta l = l_0(1 - \alpha_2 \Delta T) - l_0'(1 - \alpha_1 \Delta T) = l_0(\alpha_1 - \alpha_2) \Delta T$$

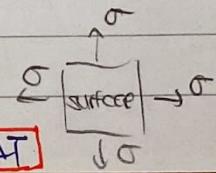
$$\therefore \epsilon = \frac{\Delta l}{l_0'} = \frac{l_0(\alpha_1 - \alpha_2) \Delta T}{l_0(1 - \alpha_1 \Delta T)} = \frac{(\alpha_1 - \alpha_2) \Delta T}{1 - \alpha_1 \Delta T}$$

since $\alpha_1 \Delta T \ll 1$, $1 - \alpha_1 \Delta T \approx 1$ so

$$\epsilon = (\alpha_1 - \alpha_2) \Delta T$$

- The surface stress is bi-axial and equal in all directions

$$\sigma = \frac{\epsilon}{E} - \nu \frac{\sigma}{E} = \frac{\epsilon}{E}(1 - \nu) \rightarrow \sigma = \frac{\epsilon E}{1 - \nu} = \frac{\epsilon}{1 - \nu} (\alpha_1 - \alpha_2) \Delta T$$



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Plastic properties and constitutive response

shear yield stress K
is given by $K = \frac{\sigma_y}{2}$

Yield stress, strength and ductility.

the yield strain ϵ_y is the corresponding strain

- the yield strength σ_y is the stress beyond which the material deforms plastically.
- ↳ for metals, the onset of plasticity is not always distinct → we identify σ_y w/ the 0.2% proof stress $\sigma_{0.2}$ (σ when $\epsilon = 0.2\%$).
- ↳ for polymers, σ_y is identified as the stress at which the stress-strain curve becomes markedly nonlinear – typically $\epsilon = 1\%$.
- the tensile strength σ_t is the max tensile stress that can be applied w/o the material failing.
- plastic strain ϵ_{pl} is the permanent strain resulting from plasticity, thus it is the total strain ϵ_{tot} minus the recoverable elastic strain ϵ_{el}

$$\epsilon_{pl} = \epsilon_{tot} - \epsilon_{el} = \epsilon_{tot} - \frac{\sigma}{E}$$

- the elongation/strain-to-failure ϵ_f is the tensile strain at failure
- ductility is a measure of how much plastic strain a material can tolerate, measured in standard tensile tests by elongation ϵ_f (we use standard test geometry).

Tensile testing and compressive testing. → destructive tests

- In tensile testing, standard specimens are loaded uniaxially in tension. The load-extension response is then recorded.
- For a typical ductile metal, the extension increases linearly w/ the applied force up until the elastic limit.
- The material then starts to yield and experiences permanent plastic strain.
- The yield load increases as work hardening occurs. This stabilises yielding so the whole cross section reduces uniformly. volume conserved
- At the neck, local, the material's ability to work harden in tension has reduced. The plastic strain localises and necking occurs.
- In compressive testing, the specimens are loaded uniaxially in compression. The load-extension response is then recorded volume conserved
- For a ductile material, the area increases during test. Localisation and necking do not occur – the test is geometrically stable. (we can reach higher strains).
- However, deformation can be non-uniform due to friction on the ends.
- For metals, compressive strength \approx tensile strength; for ceramics (brittle), compressive strength \gg tensile strength.

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Nominal Stress and strain

- The nominal stress σ_n is defined as

$$\sigma_n = \frac{F}{A_0}$$

where A_0 is the initial CSA. (constant)

- The nominal strain ϵ_n is defined as

$$\epsilon_n = \frac{l - l_0}{l_0}$$

where l_0 is the initial gauge length. (constant)

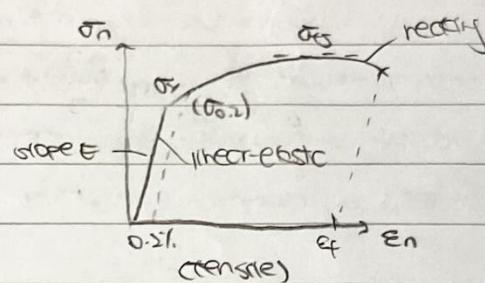
- Nominal stress-strain curve is identical in shape to the load-extension curve

(Nominal) stress-strain curves.

① Metals / alloys

Work hardening may give non-linear response

→ use proof stress to define yield (e.g. $\sigma_{0.2}$)



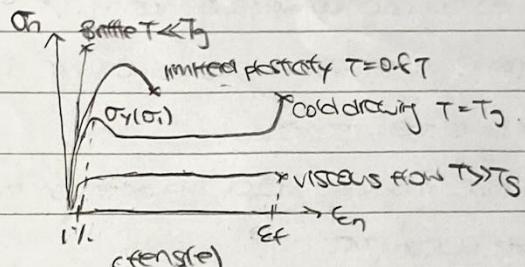
terminology for alloys

- Annealed: softened by heat treatment (σ_y, E_f)

- Drawn: previously hardened by stretching.

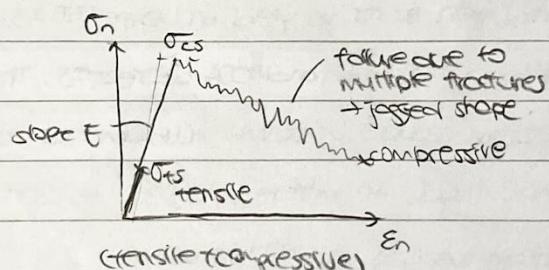
② Polymers

some polymers are brittle (PMMA, perspex) while others are very ductile (PE) or RTP



③ Ceramics

tensile strength controlled by growth of waist flaw; compressive strength controlled by crushing.



- Elastic energy stored per unit volume is the area under the linear-elastic region

$$\epsilon \leq \epsilon_y$$

$$\frac{U}{V} = \int_0^{\epsilon_y} \sigma d\epsilon = \frac{1}{2} \sigma_y \epsilon_y$$

$$[\frac{U_{max}}{V} = \frac{1}{2} \sigma_y \epsilon_y]$$

- Plastic energy stored per unit volume is the area under the entire curve minus the final elastic contribution

$$\epsilon > \epsilon_y$$

$$\frac{U}{V} = \int_0^{\epsilon} \sigma d\epsilon - \frac{1}{2} \sigma_y \epsilon_y$$

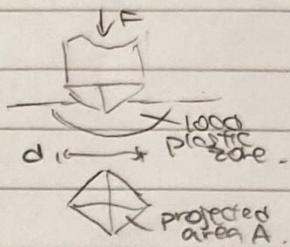
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Hardness test → non-destructive test

- The hardness test is a standard low-cost NDT technique for measuring strength. The component to be tested could be returned to service after testing.
- In the Vickers hardness test, a diamond pyramid w/ shallow tip angle is pressed into the surface under a constant load.
- Local plastic deformation occurs until load is supported. The load is then removed and the resulting indent size, d , is measured.
- Hardness is defined as

$$H = \frac{\text{Load}}{\text{projected area of indent}} = \frac{F}{A}$$

where $A = (\frac{d}{2})^2 = \frac{d^2}{2}$



- Under the indenter, a steep strain gradient exists - the deformation and work hardening is greatest near the indenter. Plastic analysis indicates that:
 - Average strain is $\approx 8\%$. → Hardness is related to yield stress at $\epsilon = 8\%$.
 - Hardness is approximately equal to $3 \times$ yield stress, i.e.

$$H \propto 3\sigma_y$$

- e.g.: For low-carbon steel, $HV = 120 \text{ kg/mm}^2 \rightarrow H = 1200 \text{ MPa} \rightarrow \sigma_y = 400 \text{ MPa}$.

True stress and strain

- To describe the constitutive response, we need to account for the changes in geometry, using true stress and true strain.
- The true stress σ_t is defined as

$$\sigma_t = \frac{F}{A}$$

- For constant volume, (plastic deformation): $V = V_0 \rightarrow A_l = A_{0l}$

$$\sigma_t = \frac{F}{A_0} \left(\frac{A_0}{A} \right) = \sigma_n \left(\frac{l}{l_0} \right)$$

$$\therefore \sigma_t = \sigma_n (1 + \epsilon_n)$$

i.e. true stress > nominal stress for $\epsilon_n > 0$.

plastic deformation only, $\sigma_t > \sigma_n$

continuation of the true stress-strain curve found in compression

- True stress σ_t can easily be calculated from nominal stress-strain data, but only as far as the onset of yielding in tension (as deformation is no longer uniform)

- An incremental true strain $\delta\epsilon_t$ is defined as

$$\delta\epsilon_t = \frac{dl}{l}$$

we req. uniform deformation to measure the characteristics of the whole specimen.

So the total true strain is found by integrating between original and final lengths.

$$\epsilon_t = \int_{l_0}^l \delta\epsilon_t = \int_{l_0}^l \frac{1}{l} dl = \ln \left(\frac{l}{l_0} \right)$$

$$\therefore \epsilon_t = \ln (1 + \epsilon_n)$$

true strain is additive
nominal strain is not additive

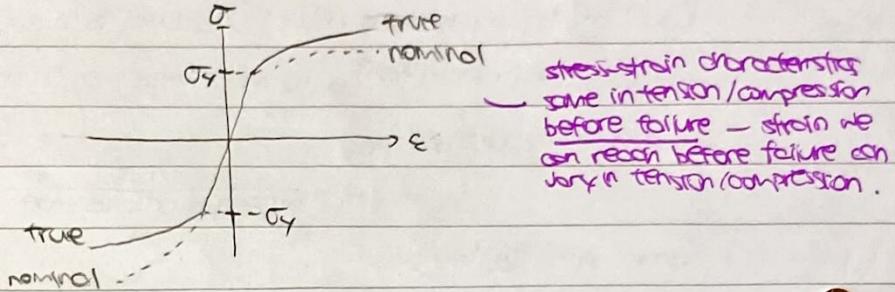
i.e. true strain < nominal strain for $\epsilon_n > 0$

- True strain ϵ_t can easily be calculated from the nominal strain ϵ_n

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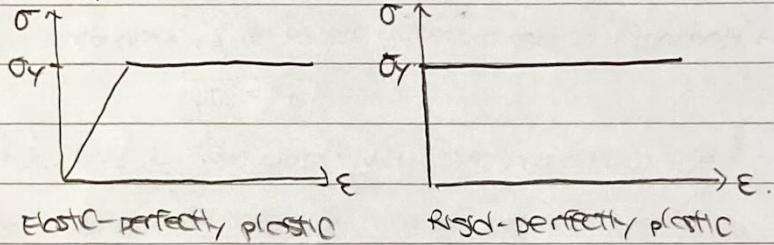
(True) stress-strain curves.

- True stress-strain \approx nominal stress-strain for small strains, $\epsilon_T \approx \epsilon_n$ for $\epsilon_n \ll 1$.
- True stress and strain take account of geometry change, so true stress-strain curves for tension and compression are identical.
- At the yield pt, $\epsilon_T = \epsilon_n$, and $\sigma_T \neq \sigma_n \rightarrow$ yield stress σ_y is the same in tension and in compression.

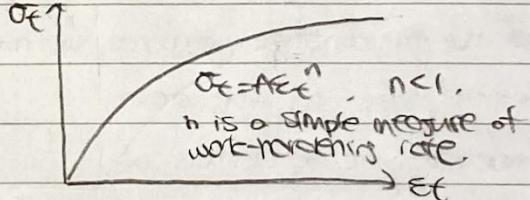


Modelling uniaxial elastic-plastic stress-strain response

- For numerical computation, empirical fits to stress-strain response are req.



- Another common form is a power-fit law for materials w/o a distinct yield pt.



* The eqn. can be converted into the corresponding nominal stress-strain response

$$\text{if req.: } \sigma_n = A\epsilon_n^{\frac{n}{n-1}} \rightarrow \sigma_n(1+\epsilon_n) = A[\ln(1+\epsilon_n)]^{\frac{n}{n-1}}$$

Atoms and bonding

Atoms, solutions and compounds.

- Atoms consist of a nucleus of protons and neutrons, w/ electrons orbiting the nucleus in quantised energy levels
- The 2 most important quantities of an atom is its atomic no. Z and atomic mass A.
 - ↳ Atomic no. Z is the no. of protons
 - ↳ Atomic mass A is the no. of protons + no. of neutrons.
- All atoms are similar size (atomic radii $\approx 0.1\text{--}0.2\text{ nm}$) while the atomic mass spans a factor of over 200.
- Mixture of atoms of different elements can pack together efficiently into crystal lattice structures, forming solid salts or compounds.
- Most solid salts are substitutional - atoms of similar size replace one another in the lattice.
- only small atoms (H, C) form interstitial solid salts - atoms fit into the gaps between atoms.
- Compounds can form readily w/ lattices that satisfy the req. stoichiometry.

Interatomic/intermolecular bonding.

- Interatomic/intermolecular bonding is determined by the interaction between the outermost shell e^- in atoms.
- There are 2 types of bonds:
 - ↳ Primary (interatomic) : Metals, ceramics, along polymer chains
 - ↳ Secondary (intermolecular) : Between polymer chains, molecules
- Primary bonds are 100x stronger than secondary bonds.

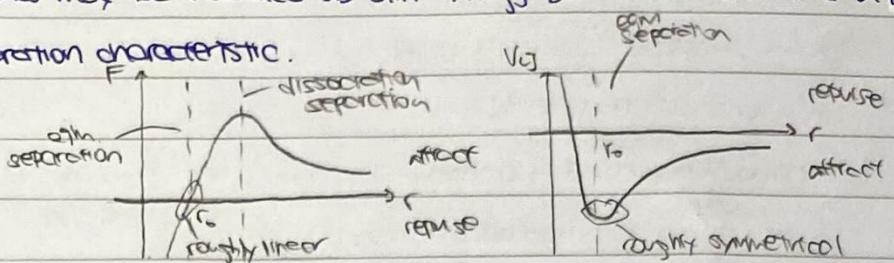
Primary bonding.

- Metallic bonding : the strong non-directional electrostatic attraction between the metal cations and the sea of delocalised e^- .
- Ionic bonding : the strong non-directional electrostatic attraction between oppositely charged ions
- Covalent bonding : the strong directional electrostatic attraction between the bonding nuclei and bond pair e^- .

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Modelling primary bonds

- primary bonds may be modelled as stiff springs between atoms/ions, w/ a non-linear force-separation characteristic.



- The atoms/ions have an eqm. separation r_0 , governed by the balance of attractive and repulsive forces. At dissociation separation, the atoms/ions can be separated completely.

Secondary bonding.

- secondary bonds operate at much larger atomic separation than primary bonds, and are much weaker. They are associated w/ dipoles.
- hydrogen bonds are the strongest type of secondary bonding (strongest dipole)

Microstructure of metals and ceramics

Unit cells

crystalline materials: atoms pack in a regular repeating lattice structure

- A unit cell is the smallest unit which can be replicated by translation in all directions to build up the 3D crystal structure
- The unit cell dimensions are lattice constants.
- Unit cells are drawn w/ the atoms reduced in size for clarity (in reality, the atoms touch in close-packed directions)

FCC or CCP structures

close-packed structure [Metals]

- close-packed structures have the max. theoretical packing fraction (74%). The basic building block of such structures is the close-packed plane (hexagonal packing)
- The close-packed directions are the lines through the centre of atoms (3 in each close-packed plane)

non-close-packed structures / can have close-packed dir. (dir. where atoms touch)

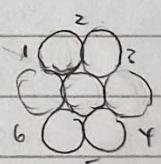
- A 3D lattice can be built by stacking these close-packed planes in 2 ways.

↳ Around each atom, there are 6 locations in which atoms can sit but only 3 of these can be occupied at once (odd/even)

↳ The layers above/below the reference layer could be put in the same location (ABA) or alternating locations (ABC)

- FCC is ABC stacking; CCP is ABA stacking.

- The minor packing difference doesn't affect the elastic properties much, but significantly influences plastic deformation.



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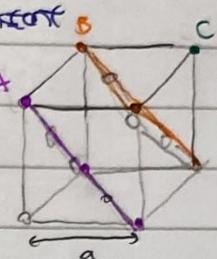
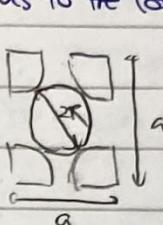
Face-centred cubic structure (FCC)

- The FCC structure is described by a cubic unit cell w/ an atom at each corner and at the centre of each face.
- The close-packed planes are \perp to the cube diagonals, and are packed in ABC sequence.
- Since the FCC stacking has a high degree of symmetry, and generates close packing in multiple orientations (only diagonal of any face is a close-packed direction)
- We can use geometry to relate the atomic radius to the lattice constant

$$a^2 + a^2 = (2R + R + R)^2$$

$$2a^2 = (6R)^2$$

$$\boxed{a = 2\sqrt{2}R}$$



- FCC metals have the following characteristics:

- ↳ They are very ductile when pure, work hardening rapidly but softening again when annealed \rightarrow able to be shaped by deformation processing.
- ↳ They are generally tough (resistant to crack propagation)
- ↳ They return their ductility and toughness to absolute zero

Close-packed hexagonal structure (CPH)

- The CPH structure is described by a prismatic hexagonal unit cell w/ an atom at each corner, one at the centre of the hexagonal faces and three in the middle.
- The close-packed planes are \perp to the axes of the prism and are packed in ASA sequence.
- Contrary to FCC, this stacking does not generate close-packed planes / directions in any other orientations

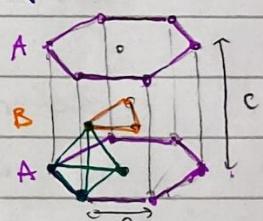
- We can use geometry to relate the atomic radius to the lattice constant

Consider the green tetrahedron:

$$\boxed{a = 2R}$$

$$c = \frac{2\sqrt{2}}{\sqrt{3}}a$$

$$\therefore \boxed{c = \frac{4\sqrt{6}}{3}R}$$



- CPH metals have the following characteristics:

- ↳ They are reasonably ductile (at least when hot) \rightarrow able to be shaped by deformation processing, but in a more limited way than FCC metals.
- ↳ Their structure makes them more anisotropic than FCC metals.

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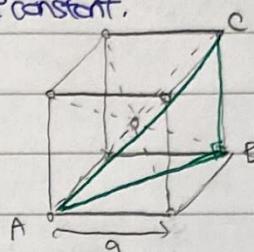
body-centred cubic structure (BCC)

- The BCC structure is described by a cubic unit cell w/ an atom at each corner and in the middle of the cube.
- The structure is not close-packed - it is made by stacking planes of atoms in a square array (not hexagonal)
- Atoms touch along the cube diagonals - these are the close-packed directions.
- We can use geometry to relate the atomic radius to the lattice constant.

Consider $\triangle ABC$: $(AB)^2 = a^2 + a^2 = 2a^2$

$$(AC)^2 = (AB)^2 + (BC)^2 = 2a^2 + a^2 = (4R)^2$$

$$\therefore a = \frac{4\sqrt{3}R}{3}$$



- BCC metals have the following characteristics:

- ↳ They are ductile (esp. when hot) → able to be shaped by deformation processing.
- ↳ They are seismically tough (resistant to crack propagation)
- ↳ They become brittle at low temp. (At the ductile-brittle transition temp.)

Grain structure

- Metal components are commonly manufactured by casting - solidification of a liquid poured into a shaped mould.
- The solidification mechanism involves the formation of many small crystalline nuclei, which grow by attachment of atoms to the crystal at the liquid-solid interface.
- Solidification is completed when adjacent crystals impinge on one other.
- Since orientation of the packing in each crystal is random, there is a misfit in the atomic packing at the interface between 2 crystals - the grain boundary.

Ceramic crystals / compounds of metals / ST w/ nonmetals.

- The structures of ceramic crystals often look complicated, but can mostly be interpreted as atoms of one type, arranged on a simple FCC/CPH/RCC lattice, w/ the atoms of the second type inserted into the interstitial holes of the first lattice.
- Diamond cubic structure has one type of atom arranged in FCC lattice. e.g. diamond, SiC
- Corundum structure has one type of atom arranged in CPH lattice. e.g. oxides, Mg_2O_3
- ↳ Glasses are a type of ceramic material but they do not have a crystalline structure - they are amorphous.

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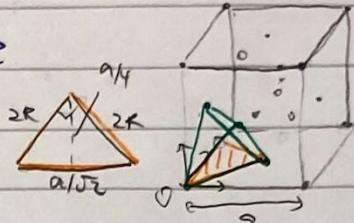
Diamond cubic structure (DC)

- The DC structure is an FCC lattice w/ additional atoms in half of the tetrahedral interstitial spaces
- As the tetrahedral hole is far too small to accommodate a full-sized atom, the others are pushed further apart, lowering the density.
- We can use geometry to relate the atomic radius to the lattice constant

Consider the regular tetrahedron, the Δ formed by the centre connected to an edge

$$(2R)^2 = \left(\frac{a}{4}\right)^2 + \left(\frac{a\sqrt{3}}{2}\right)^2$$

$$a = \frac{8\sqrt{3}}{3} R$$



+ The centre of the tetrahedral interstitial space is $(\frac{a}{4}, \frac{a}{4}, \frac{a}{4})$

Packing fraction

- packing fraction is the fraction of space occupied by atoms.

- Face-centred cubic structure (FCC) : $\boxed{\text{PF} = 0.74}$

↳ No. of atoms per unit cell : $8 \times \frac{1}{8} + 6 \times \frac{1}{2} = 4$

$$\text{ATOMIC PF} = \frac{4 \cdot \frac{4}{3}\pi R^3}{a^3} = \frac{16/3\pi R^3}{(2\sqrt{2}R)^3} = \frac{\sqrt{2}}{6}\pi = 0.74$$

- Close-packed hexagonal structure (CPH) : $\boxed{\text{PF} = 0.74}$

↳ No. of atoms per unit cell : $12 \times \frac{1}{6} + 2 \times \frac{1}{2} + 3 \times 1 = 6$

$$\text{ATOMIC PF} = \frac{6 \cdot \frac{4}{3}\pi R^3}{3\sqrt{2}/2 a^2 c} = \frac{24/3\pi R^3}{24\sqrt{2}R^3} = \frac{\sqrt{2}}{8}\pi = 0.74$$

- Body-centred cubic structure (BCC) : $\boxed{\text{PF} = 0.68}$

↳ No. of atoms per unit cell : $8 \times \frac{1}{8} + 1 \times 1 = 2$

$$\text{ATOMIC PF} = \frac{2 \cdot \frac{4}{3}\pi R^3}{a^3} = \frac{8/3\pi R^3}{(4\sqrt{3}/3 R)^3} = \frac{\sqrt{3}}{8}\pi = 0.68$$

- Diamond cubic structure (DC) : $\boxed{\text{PF} = 0.34}$

↳ No. of atoms per unit cell : $8 \times \frac{1}{8} + 6 \times \frac{1}{2} + 4 = 8$

$$\text{ATOMIC PF} = \frac{8 \cdot \frac{4}{3}\pi R^3}{a^3} = \frac{32/3\pi R^3}{(8\sqrt{3}/3 R)^3} = \frac{\sqrt{3}}{16}\pi = 0.34$$

Theoretical density

- The theoretical density is the mass per unit volume for the unit cell, given by

$$\rho = \frac{m_c}{V_c} = \frac{N A / N_A}{V_c}$$

where N is the no. of atoms in a unit cell, A is the atomic mass, N_A is Avogadro's constant

- since the atomic mass and the crystal packing are both physically well-defined,

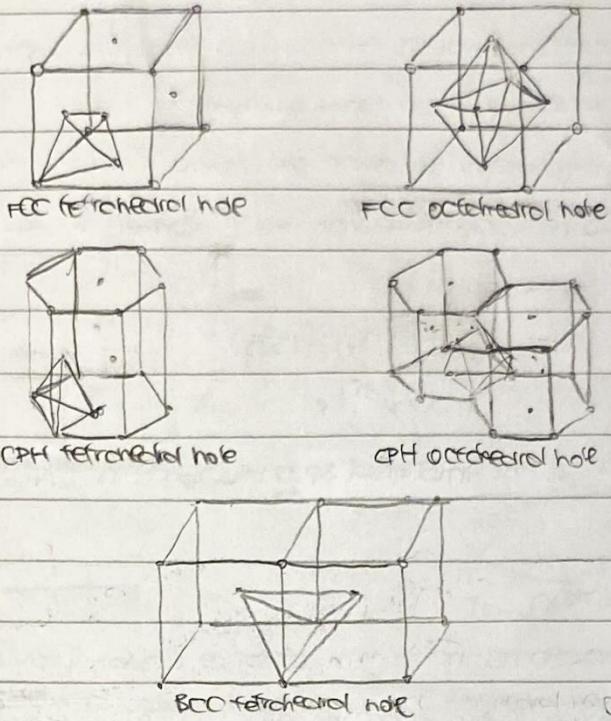
↳ densities of metals and ceramics have narrow ranges

↳ there is no scope to modify density (e.g. by processing a metal differently)

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Interstitial space.

- An interstitial space is the space between the atoms / molecules.
- The FCC, CPH and BCC structures contain interstitial spaces - tetrahedral or octahedral.



- Interstitial holes are important because small foreign atoms can fit into them
- For FCC/CPH structures, the tetrahedral hole can accommodate, w/o strain, a sphere w/ a radius of 0.22 of that of the host
- For BCC structures, the tetrahedral hole can accommodate, w/o strain a sphere w/ a radius of 0.29 of that of the host
- For FCC(CPH) structures, the octahedral hole can accommodate, w/o strain a sphere w/ a radius of 0.41 of that of the host ($\sqrt{2}/4$)
- Interstitial solute atoms are important for carbon steel - Fe w/ C in some of the interstitial holes.
 - At room temp., C atoms are too large to fit in the BCC tetrahedral holes → distort the crystal structure → increases strength.
 - At high temp. ($> 800^\circ\text{C}$), Fe transforms to FCC → max. hole size increases → more C can be "dissolved" → increases strength.

Microstructure of glasses and polymers

Glasses

- When crystalline materials melt, the atoms lose their regular packing but are still loosely held together → on solidification, crystals form readily / amorphous - no long range order
- Glasses are all based on silica SiO_2 , for which crystallisation is difficult. → In the solid state, silica usually has an amorphous structure and only crystallises if cooled very slowly. amorphous structures give transparency

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Polymer

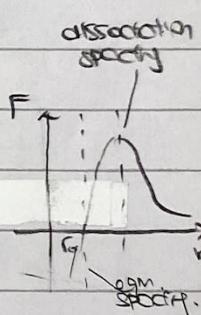
/ polymer "alloys" inc. copolymers - multiple types of monomers
and polymer blends - molecular scale mixtures of polymers.

- There are 3 main classes of polymers - thermoplastics, thermosets and elastomers
- Thermoplastics contain no crosslinks, but are divided into 2 sub-groups - amorphous and semi-crystalline
- In amorphous thermoplastics, the long-chain molecules are arranged entirely at random, w/ occasional entanglements b/w chains.
- At the entanglements there is no additional bonding, but they do restrain the deformation and sliding of the molecules.
- Semi-crystalline thermoplastics are partly amorphous and partly ordered (in crystalline regions).
- In the crystalline regions, the long chains line up and pack to give an ordered, repeating structure. The low symmetry of the individual molecules means that the unit cell is defined by 3 dimensions.
- Few polymers are fully crystalline, but many have as much as 90% crystallinity.
- Since the crystalline regions are more tightly packed than amorphous regions, there is significant shrinkage on crystallisation \rightarrow low dimensional precision in manufacturing
- The ease w/ which a thermoplastic crystallises is determined by the complexity of side groups:
 - \hookrightarrow simple thermoplastics : semi-crystalline (e.g.: Nylon, polyethylene)
 - \hookrightarrow complex thermoplastics : fully amorphous (e.g.: Perspex, polycarbonate)
- In pure form, amorphous thermoplastics are transparent and semi-crystalline thermoplastics are translucent. (Crystallinity, \downarrow transparency)
- Elastomers contain a small no. of cross-links between chains. (e.g.: Rubber) /
- Thermosets have extensive cross-links between chains. They burn before they melt.

Elasticity of metals and ceramics

Atomic basis of elasticity of metals and ceramics

- Primary bonds behave as stiff elastic springs, w/ large displacement of the term
- The gradient of the F-r response at eqm. spacing r_0 is the bond stiffness S .
- For small displacement from eqm. u,



$$\text{Restoring force per atom } F = S \cdot u ; \text{ Area per bond } A \propto r_0^2$$

$$\therefore \text{Atomic scale stress } \sigma \propto \frac{S}{r_0^2}$$

$$\text{Atomic scale strain } \epsilon = \frac{u}{r_0}$$

$$\rightarrow \text{Young's modulus } E = \frac{S/r_0^2}{u/r_0} = \frac{S}{u} \quad \begin{matrix} S \text{ depends on element} \\ u \text{ depends on element + structure} \end{matrix}$$

- Young's modulus of a component directly reflects the bond stiffness so atomic F-r is approx. linear near eqm. spacing r_0 means linear elasticity at macro scale.
- S and r_0 depend on atomic packing and atomic bonding (physically prescribed)

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Property of alloys

- From the hard sphere model, solid solns and compounds will form densities between those of pure elements. (approx. rule of mixtures)
- Density depends on atomic mass and atomic packing (physically prescribed)
- Solid solns contain a mixture of different bond stiffnesses → Young's modulus of A-B solns is somewhere between pure A and pure B (approx. rule of mixtures)
- Compounds have stiffer bonds and higher moduli (strong chemical bonds are geometrically favorable)
- ρ and E are well-defined by the composition and crystal structure → they are microstructure-insensitive properties. → little scope for manipulating ρ / E by material processing.

Amorphous metals

- Metals crystallise easily but can be forced to retain an amorphous structure if cooled very quickly.
- Bulk amorphous metals have been produced which solidify at conventional rates, using very unusual compositions (widely different sized atoms → regular crystal packing is difficult)
- Amorphous metals have unusual properties:
 - ↳ Mechanically hard
 - ↳ Magnetically either soft or hard
 - ↳ Very low damping (little energy lost in collisions)
as there is no dislocations that can move, i.e. no plastic deformation,

Elasticity of polymers

Glass transition temperature

- In polymers, the weaker secondary bonds are overcome by thermal energy at the glass transition temp. T_g (lower than the melting pt. T_m , where strong primary bonds are broken)
- Above T_g , the behaviour differs between the polymer classes:
 - ↳ Amorphous thermoplastics : melt to a viscous liquid (entangled molecules slide over one another)
 - ↳ Semicrystalline thermoplastics : amorphous regions melt, crystalline regions survive to the higher melting pt. T_m (usually $T_m \approx 1.5T_g$ in K), above which a viscous liquid forms.
 - ↳ Elastomers/thermosets : secondary bonds melt at T_g but crosslinks do not → on heating the polymer does not melt, but decomposes (burns).
- Therefore, thermoplastics are moulded at 150°C above T_g → easy to remould, weld and recycle; elastomers/thermosets can be moulded once only → cannot be recycled / limited reuse only
- * T_g depends on the rate of cooling → T_g must be defined at a reference cooling rate. (why T_g is meaningless)

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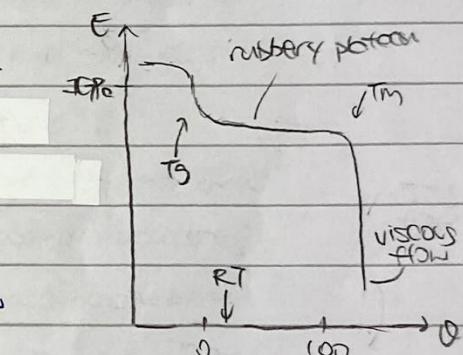
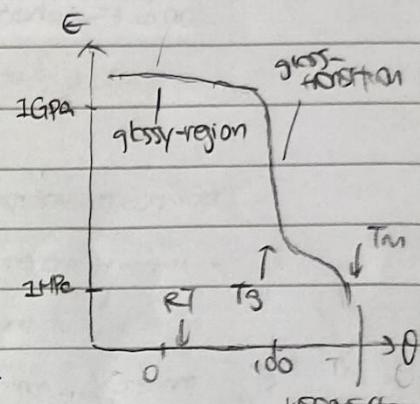
Elastic response of polymers

- elasticity of polymers is sensitive to temp. (rel. to T_g) and rate of loading.
- deformation around and above T_g relies on molecular sliding, which is sensitive to the rate of deformation. $\rightarrow E$ and T_g depend on the loading rate.
- elasticity also differs between polymer classes, due to crystallinity and cross-linking.

we use a reference rate of loading,
usually quasi-static loading

due to
molecular structure

- (1a) Amorphous thermoplastics / usually $RT < T_g$,
→ used in glassy state.
- In the glassy region ($T < T_g$),
 - ↳ loading stretches low stiffness secondary bonds.
 - ↳ E of order $1-2 \text{ GPa}$.
 - Around the glass transition temperature ($T \approx T_g$)
 - ↳ segments of the chain melt $\rightarrow E$ falls steeply.
 - ↳ reptation forces place — molecules slide past one another.
 - Between glass transition to melting pt. ($T_g < T < T_m$)
 - ↳ very low residual stiffness ($\sim 10^3 \times$ lower)
 - ↳ residual elasticity is due to molecular chain entanglement
 - ↳ above T_g , the chains all slip \rightarrow melting by viscous flow
- ↳ usually $RT > T_g$
→ used in rubbery state



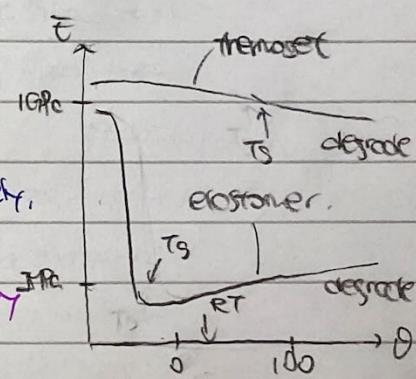
(1c) Elastomers, thermosets — effect of cross-linking

- Elastomers:

- ↳ below T_g (well below RT), glassy
- ↳ above T_g (pronounced rubbery region), there are large recoverable strains of low stresses as chains unravel extensively, pulled back by cross-links when load released \rightarrow low E ("memory" from crosslinks and entropy)

- Thermosets:

- ↳ stiffer than thermoplastics in glassy region
- ↳ Highly cross-linked \rightarrow no effect of T_g .
- * Elastomers and thermosets degrade before melting.



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Manipulating elastic properties

Foams

- Foams are porous solids
- Commercial foams use polymers of various stiffnesses, w/ open or closed cells. (Recently, ceramic and metallic foams have been developed)
- Foams are described by their relative density, defined as,

$$\text{relative density} = \frac{\rho_{\text{foam}}}{\rho_{\text{solid}}}$$

- The elasticity of foams is dominated by bending of the solid segments. The key result is

$$\frac{\nu_{\text{foam}}}{\nu_{\text{solid}}} = \left(\frac{\rho_{\text{foam}}}{\rho_{\text{solid}}} \right)^2$$

- This means lines of slope 2 lead from solid to foam on the E- ν chart, (enables us to predict E- ν of new foams)

Composite materials

- Composite materials combine 2 materials to produce new property profiles, exploiting separate qualities of the individual components.
- There are 3 main composite geometries, dependent on component architecture
 - ↳ Particulate, metal-ceramic, polymer-ceramic ("filled-polymers")
 - ↳ Fibres : carbon/sbs/kevlar fibre - polymer
 - ↳ Laminates : plywood.

Composite processing.

- Particulate composites

↳ Add micron-scale particles to melt before casting or molding.

- Fibre composites,

↳ short chopped fibres - mix w/ resin, shape in a mould.

↳ long continuous fibres - lay out mat in mould and infiltrate w/ resin, then combine fibre + resin in "prepreg" plies, then laminate.

- Laminates

↳ Wood - stack and glue thin layers.

↳ Long-fibre composites - stack multiple layers of prepreg plies w/ different orientations, not form to shape, curing the resin.

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Estimates for composite properties

- Composites are defined by the volume fraction V_f of one of the components.

① Density.

- For all composites, density follows a simple rule of mixtures between the densities of the 2 components (mass conservation).

$$\rho_c = V_f \rho_f + (1-V_f) \rho_m$$

* c: composite
f: reinforcement
m: matrix

② Young's modulus.

- The Voigt-Reuss eqns are upper and lower bounds for $E \rightarrow E_c$ must lie between or on these limits for all composites.

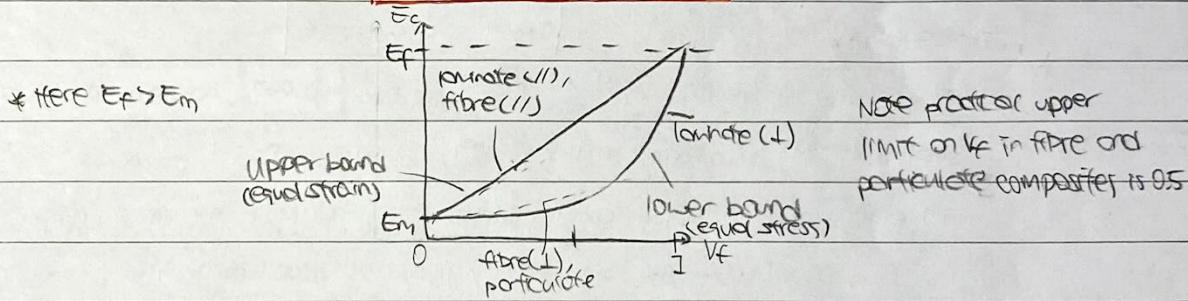
- The upper and lower bounds of E corresponds to axial and transverse loading of a fibre composite / laminate. — these composites are ANISOTROPIC

- The upper bound of E is when we have axial loading - equal strain (Voigt model)

$$E_c = V_f E_f + (1-V_f) E_m \quad \text{rule of mixtures for } E$$

- The lower bound of E is when we have transverse loading - equal stress (Reuss model)

$$1/E_c = V_f / E_f + (1-V_f) / E_m \quad \text{rule of mixtures for } 1/E$$



- Particulate composites are isotropic, close to lower bound (equal stress)

- Fibre composites / laminates are anisotropic (stiffer when fibre / laminate // to force).

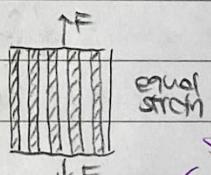
↪ fibre / laminate (i) // to force : exact on upper bound (equal strain)

(ii) ⊥ to force : close to / exact on lower bound (equal stress).

axial loading.

Derivation of the upper bound for E_c .

Total force F : $F = A_f \sigma_f + A_m \sigma_m$.



Stresses follow a rule of mixtures.

Average stress σ_c : $\sigma_c = \frac{F}{A} = \frac{A_f}{A} \sigma_f + \frac{A_m}{A} \sigma_m = V_f \sigma_f + (1-V_f) \sigma_m$

$$* \frac{A_f}{A} = V_f$$

Hooke's law $\sigma = E\epsilon$: $\sigma_c = V_f E_f \epsilon_f + (1-V_f) E_m \epsilon_m = V_f E_f \epsilon_c + (1-V_f) E_m \epsilon_c$.

$$\therefore E_c = \frac{\sigma_c}{\epsilon_c} = V_f E_f + (1-V_f) E_m$$

equal strain
 $\therefore \epsilon_c = \epsilon_f = \epsilon_m$

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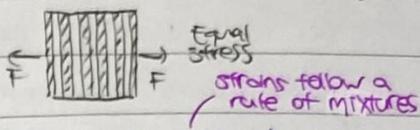
Derivation of the lower bound for $\bar{\epsilon}_c$:

$$\text{Total extension } \Delta L = \Delta L_f + \Delta L_m$$

$$\text{Average strain } \bar{\epsilon}_c = \frac{\Delta L}{L} = V_f \frac{\epsilon_f}{E_f} + \frac{V_m}{L} \epsilon_m = V_f \frac{\sigma_f}{E_f} + (1-V_f) \frac{\sigma_m}{E_m}$$

$$\text{Hooke's law } \epsilon = \frac{\sigma}{E}: \quad \bar{\epsilon}_c = V_f \frac{\sigma_f}{E_f} + (1-V_f) \frac{\sigma_m}{E_m} = V_f \frac{\sigma_f}{E_f} + (1-V_f) \frac{\sigma_f}{E_m}$$

$$\therefore \bar{\epsilon}_c = \frac{\sigma_f}{E_f} = V_f \frac{1}{E_f} + (1-V_f) \frac{1}{E_m}$$



$$* \frac{1}{E_f} = V_f$$

$$\downarrow \text{equal stress} \\ \therefore \sigma_f = \sigma_m$$

Plasticity of crystalline materials

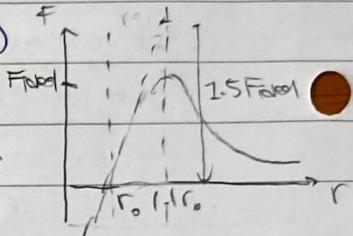
Ideal strength of crystalline materials

- We can estimate the upper limit of strength from the atomic force-distance curve:

↳ bonds rupture at the dissociation separation, $\sim 1.1 r_0$ ($E=0.1$)

↳ force-distance curve is linear near $r=r_0$. \rightarrow extrapolating, the

linear-elastic force of separation of $1.1 r_0$ is higher than the peak force by a factor of $\sim 1.5 \times$.



- Therefore, the tensile stress needed to break all the bonds simultaneously is of order $1/5$ of nominal linear-elastic stress at a strain of $\sim 10\%$. (Estimate of ideal strength)

$$\sigma_{\text{ideal}} = \frac{E}{1.5} \cdot 0.1 = \frac{E}{15}$$

- If we consider the ratio of $\frac{\sigma_{\text{el}}}{E}$, $(\frac{\sigma_{\text{el}}}{E})_{\text{ideal}} = \frac{1}{15} \cdot 10^{-1}$. However, real materials have smaller failure stresses, $\sim 10-1000 \times$ smaller $(\frac{\sigma_{\text{el}}}{E})_{\text{real}} \sim 10^{-4}-10^{-2}$; $(\frac{\sigma_{\text{el}}}{E})_{\text{atomic}} \sim 10^{-3}$

↳ Metals: plastic yielding, enabling deformation that keeps the material intact at stresses well below the ideal strength. (movement of dislocations).

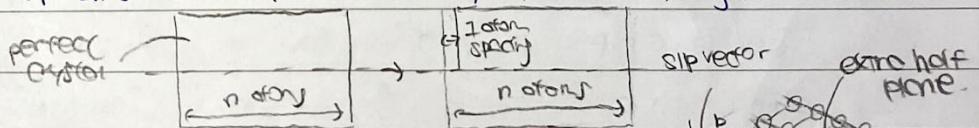
↳ Ceramics: fracture, from inherent flaws and cracks.

Dislocations

- Dislocations are line defects in a crystal.

- Consider a block of perfect material, w/ the atoms in a cubic lattice.

- Displace the top half of the block, on one side only, by 1 atomic spacing rel to the lower half

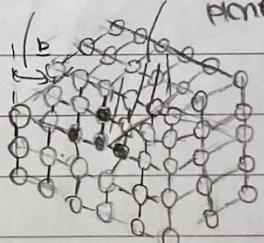


- To accommodate this displacement,

↳ part of the interface between the blocks have slipped.

↳ the top block contains an extra half plane of atoms

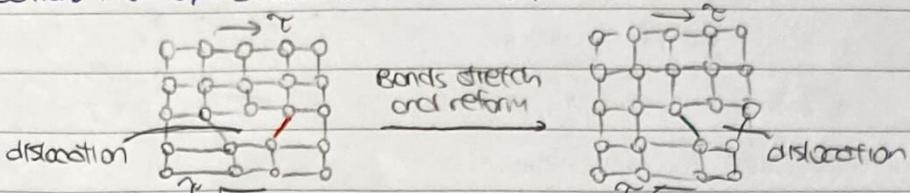
- The extra half-plane is found at the boundary between slipped and unslipped regions — the crystal defect at this pt. is a dislocation.



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Dislocation motion

- Dislocations move by the action of shear stress parallel to the slip plane.

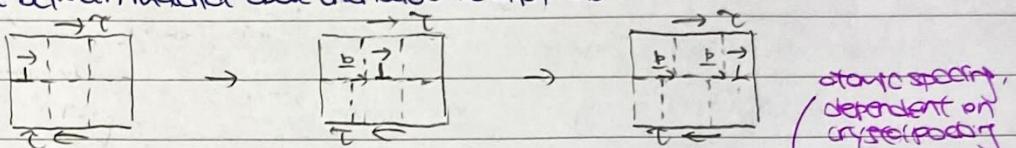


- When a dislocation moves,

↳ no atom moves more than a fraction of the atomic spacing

↳ the adjacent set of atoms become the new plane.

- Consider a dislocation moving right through a block of material. This would give a net displacement between material above and below the slip plane.



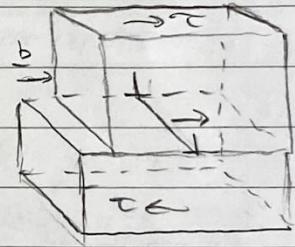
The dip step produced by the passage of 1 dislocation is the Burgers vector b .

- Since dislocations enable incremental slip by extending a few bonds of a time, the yield stress is much less than the rated strength (all bonds broken simultaneously)

Types of dislocations.

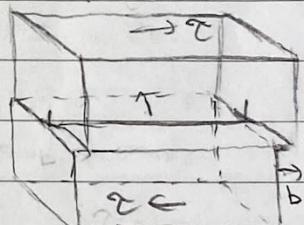
- In an edge dislocation,

↳ shear stress and Burgers vector both at right angles to the dislocation



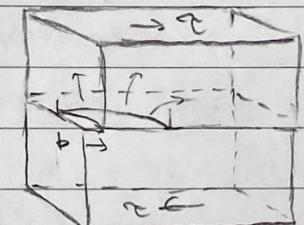
- In a screw dislocation,

↳ shear stress and Burgers vector both parallel to the dislocation



- Generally, dislocations are mixed,

↳ curved, and varying between pure edge and pure screw.



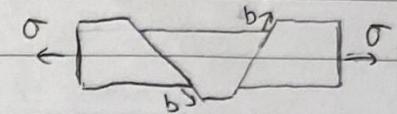
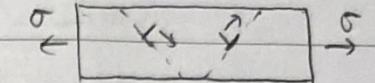
* For all 3 types of dislocations, the net effect is a slip step in the direction of the shear stress.

* The Burgers vector is always in the same direction as the shear stress.

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Macroscopic plastic strain

- Although a dislocation crossing a lattice leads to an incremented slip step of the order of 1 atomic spacing, we see plastic strains of the order 0.1% - 10%. This is because
 - ↳ crystals contain very many dislocations, w/ many different planes they can glide on.
 - ↳ in any stress state, shear stresses exist to move dislocations
- Consider a crystal loaded in tension, w/ 2 dislocations crossing at 45°.
 - ↳ The net effect is that the crystal becomes longer and thinner by a small increment.
 - ↳ Replicating this increment $\times 1000$ s of dislocations on multiple slip planes produces continuum bulk plasticity
- Since the blocks of material slip past one another w/o affecting crystal packing, volume is conserved during plastic deformation.



Forces on dislocations

Looking at a 2D cross section, per unit depth, into the page.

- The shear stress τ applies a shear force (per unit length) to the dislocation.
- The crystal resists dislocation motion w/ a resistance per unit length f .
- The dislocation moves when the shear force is equal to the resistance.
- By considering the work done by τ , as the dislocation moves,

$$\tau b = f$$

- The intrinsic lattice resistance to dislocation motion T , comes from additional bond stretching as the dislocation moves each Burgers vector step.
- This intrinsic resistance depends on the type of bonding:
 - ↳ technical ceramics, diamond: ionic/covalent bonds \rightarrow high T .
 - ↳ (unreduced pure) metals: metallic bonds \rightarrow low T .

Dislocation energy per unit length — line tension.

dislocations try to be as short as possible, as if they were in "tension"

- Atoms around a dislocation are displaced from their eqm. spacing and thus have a higher energy.
- The energy per unit length T can be calculated from the elastic stress-strain field around the dislocation core.

$$T \propto G b^2$$

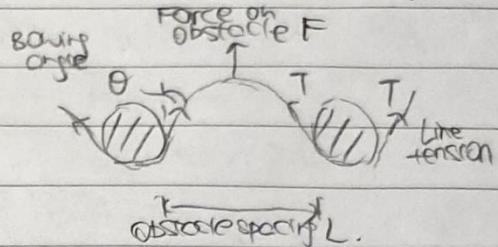
where G is the shear modulus and b is the Burgers vector.

- Line tension governs how dislocations interact w/ obstacles

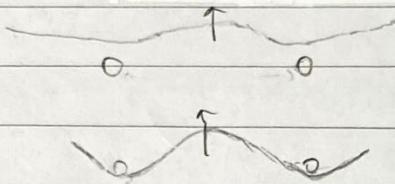
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Dislocation pinning

- When a gliding dislocation meets obstacles in its slip plane, it is pinned by the obstacles, and is forced to bow out between them \rightarrow resistance per unit length increases.
- This means an additional shear stress $\Delta\tau$ is needed to overcome this increased resistance.
- As the dislocation bows out, it applies a force to the obstacle (via the line tension T)
 - \hookrightarrow Force on obstacle F , $F = 2T \cos\theta$
 - \hookrightarrow Dislocation escapes when either
 - force $>$ obstacle strength ($\theta > 0^\circ$)
 - dislocation forms a semi-circle ($\theta = 0^\circ$)

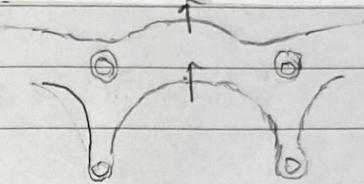


* Weak obstacles: $\theta > 0^\circ \rightarrow F < T$



Dislocations cut through/past the obstacle

Strong obstacles: $\theta = 0^\circ \rightarrow F = 2T$



Dislocations escape by leaving a loop of dislocation round the obstacle.

Shear stress to overcome obstacles.

- For projected length L of dislocation between obstacles, additional force F due to shear stress $\Delta\tau$ is given by

$$F = \frac{\Delta\tau}{b} b \cdot L$$

- As the force required to overcome obstacles F is given by

$$F = k_0 T, \text{ where } k_0 \leq 2.$$

$$\therefore \Delta\tau = \frac{k_0 T}{b L}$$

$$\text{so } \Delta\tau_{\text{min}} = \frac{G b}{L}$$

- Since $T \approx \frac{1}{2} G b^2$, we get

$$\Delta\tau \approx \alpha \frac{G b}{L}, \text{ where } \alpha \leq 1.$$

we can only manipulate α and L .
(G and b are fixed)

where α is a measure of the obstacle strength ($\alpha = 1$: strong, $\alpha \leq 1$: weak).

- Metals and alloys use several methods to pin/disrupt dislocations:

\hookrightarrow other dislocations : work hardening / dislocation hardening ($\Delta\tau_{\text{wh}}$)

\hookrightarrow solute atoms (foreign atoms) : solid soln hardening ($\Delta\tau_{\text{ss}}$)

\hookrightarrow particles of another solid (foreign compound) : precipitation hardening ($\Delta\tau_{\text{pf}}$).

- Therefore the shear stress // to the slip plane τ_y needed to move dislocations is

$$\boxed{\tau_y = \tau_0 + (\Delta\tau)_i}$$

\nearrow INTRINSIC RESISTANCE

\nearrow $(\Delta\tau)_{\text{wh}}, (\Delta\tau)_{\text{ss}}$ or $(\Delta\tau)_{\text{pf}}$.

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Yield stress in polycrystals.

Costly, recrystallisation

- Metals are polycrystalline, made up of grains as a result of processing
- Under a remote shear stress τ , the slip planes in different grains will vary in their alignment w/ the shear stress. As a result,
 - ↳ Dislocations move first in grains which are favourably oriented (A)
 - ↳ Yield occurs progressively throughout all the grains (B,C), at higher remote shear stress τ .
- As the shear stress τ to a slip plane τ_y is needed to move dislocations, we need a corresponding remote shear stress τ given by

$$\boxed{\tau \approx \frac{3}{2} \tau_y} \quad \text{→ } \frac{3}{2} \text{ from statistics}$$

which is known as the shear yield stress K .

- Recall that $K = \frac{\sigma_y}{2}$, so combining these results,

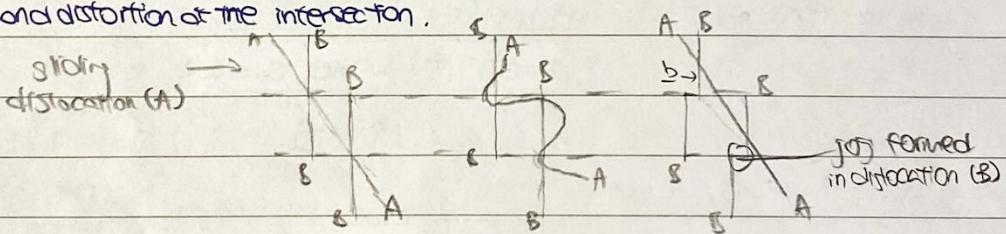
$$\boxed{\sigma_y = 2K = 3\tau_y} \quad \begin{matrix} K \text{ for polycrystal} \\ \tau_y \text{ for single crystal} \end{matrix}$$

- Therefore the primary contributes to σ_y , i.e. ($\Delta\sigma_y$); translate directly into σ_y , ($\Delta\sigma_y$) by a factor of 3.

Manipulating plastic properties (metals)

Work hardening (Dislocation hardening)

- Gliding dislocations on different slip planes interact — pinning occurs due to the additional bond distortion at the intersection.



- The gliding dislocation (A) bows out until the (weak) pinning pt. gives way, creating a jog in the pinned dislocation (B). This new imperfection reduces the mobility of dislocation (B).

- The additional strength contribution comes from dislocation pinning (using other dislocations), $\Delta\sigma_y \propto \frac{1}{L}$. For work hardening, the spacing L depends on the dislocation density ρ_d .

(Dislocation density ρ_d is the total dislocation length per unit volume, units: m/m^3).

- When strain increases, new dislocations form (from existing dislocations) → increase in ρ_d , → reduce $L \rightarrow$ increase resistance.

- Assume dislocations form a // array on a square grid, $L \times L$:

For unit length of dislocation, $A_{\text{per dislocation}} = L^2$; $V_{\text{per dislocation}} = L^2 \cdot L$

$$\text{so } \rho_d = \frac{1}{L^2 \cdot L} \quad \rightarrow \quad L = \frac{1}{\rho_d}$$

∴ Yield stress contribution from work hardening

$$(\Delta\sigma_y)_{\text{wh}} = 3\alpha \frac{Gb}{L} = 3\alpha Gb \rho_d$$

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- Alloys are work hardened by deformation processing (e.g. rolling, wire drawing) to increase the dislocation density and while shaping the product.
- * The dislocations store elastic energy in the lattice \rightarrow drive microstructural change during heat treatments (annealing) of deformed metals.
- Typically, the dislocation density ρ_d is of the order $\sim 10^{14} \text{ m}^{-2}$ for annealed alloys and $\sim 10^{15} \text{ m}^{-2}$ for work-hardened alloys. \rightarrow Dislocation spacing $L \sim 30 \text{ nm}$.
- Yield stress data for work hardened alloys:
 - \hookrightarrow pure annealed Cu, $\sigma_y = 50-60 \text{ MPa}$
 - \hookrightarrow cold-drawn Cu, $\sigma_y = 180-350 \text{ MPa}$.

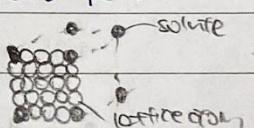
Solid solution hardening.

- Most mixtures of metal + other elements form solid solutions, sometimes over wide ranges of composition.
- Substitutional solid solns provide hardening as the solute atoms have a diff. size and local bonding to host atoms in the lattice \rightarrow roughening the slip plane.
- Interstitial solid solns provide hardening by displacing host atoms from their eqm. positions \rightarrow roughening the slip plane.
- Both types of solid solns provide a weak obstacle to dislocations,
- Since spacing L of solute atoms scales w/ solute conc. C as $L \propto 1/C^{1/2}$,
 \therefore yield stress contribution from solid soln hardening $(\Delta\sigma)_ss = 3\alpha \frac{G}{L} \propto 3\alpha G \sqrt{C}$
- Casting is used to mix elements together in the liquid state, enabling solid solns to be manufactured.
- Typically, the atomic fraction / solute conc. is $\sim 1.6\%$. \rightarrow spacing of solute atoms $L \sim 1 \text{ nm}$
- \hookrightarrow Consider a cubic lattice, w/ solute atoms regularly spaced & atoms apart:

$$\text{Atomic fraction of solute } C = \frac{V_{\text{solute}}}{V_{\text{lattice}}} = 1.6\%$$

$$\text{Spacing of solute atoms } L = 4 \times 0.2 = 0.8 \text{ nm}$$

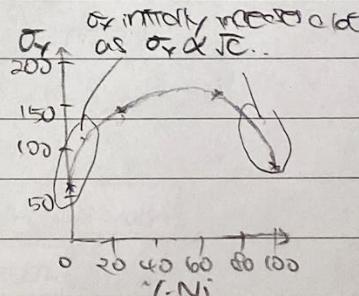
$$C = \frac{V_{\text{solute}}}{V_{\text{lattice}}} = 1.6\%$$



- Yield stress data for solid soln hardened alloys:

\hookrightarrow Cu- γ -Ni	0	10	30	70	100
$\sigma_y (\text{MPa})$	60	115	145	170	80

i.e. not a rule of mixtures



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Precipitation hardening / most significant contain Mo⁺-O²⁻ atoms.

- Alloying elements also form compounds. When distributed as small particles within a lattice, they provide pinning pts for dislocations.
- Particles of foreign compounds provide strong obstacles. → max. shear stress req. to pass strong particles.

∴ Yield stress contribution from precipitation hardening $(\Delta\sigma_y)_{\text{ppt}} = 3 \frac{Gb}{L}$

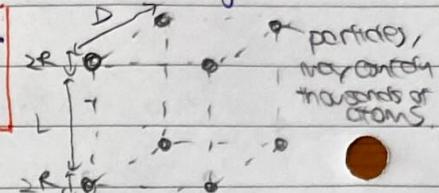
- The particle spacing is determined by the particles size R and volume fraction, f.

↳ Assume a cubic array of particles of radius R and centre-to-centre spacing D.

$$\text{Volume fraction } f \text{ of particles} : f = \frac{1/8 \times 8 \times \pi/6 \times R^3}{D^3}$$

$$\text{The min. sep between particles} : L = D - 2R.$$

→ Given R and f, we eliminate D to find L



- The main practical manufacturing route is to use heat treatment in each stage, forming fine precipitates from a solid solution (hard to manufacture such small particles for mixing)
- Controlled temp-time histories offers a route to control ppt. structure, size and vol. fraction.
- Typically, the solute spacing $L \sim 55\text{nm}$

- Yield stress data for precipitation hardened alloys :

↳ Pure Al, $\sigma_y = 25\text{MPa}$

↳ High strength aerospace Al alloy, $\sigma_y = 500\text{MPa}$

↳ Pure Fe, $\sigma_y = 120\text{MPa}$

↳ Quenched/tempered high alloy steel, $\sigma_y = 2000\text{MPa}$

Grain boundary hardening.

- Grain boundary hardening is a relatively weak hardening mechanism → it is only useful for pure metals or dilute alloys (little effect from other hardening mechanisms)
- Since the lattice orientation changes at a grain boundary, dislocations cannot slip directly from grain to grain. → dislocation pileups occur at the boundary → additional stress from pileups nucleates dislocations in the adjoining grain.
- The finer the grain size D, the more often boundaries obstruct dislocations → grain boundary hardening given by the Hall-Petch relationship: $(\Delta\sigma_y)_{gb} \propto \frac{1}{D}$.

Overview: alloy processing for strength

- Alloy composition and process route determine microstructure and thus σ_y .

PRIMARY PROCESSES

shape casting

in-situ casting → deformation processing (rolling)

Axial comp.
form initial grain size

SECONDARY PROCESSES

CAST

wrought

heat treatment

Welding

Surface engineering

Joining

pt hardening
annelling (reduce pd
and change grain size)

harden surface

heat may
alter the
microstructure

Plasticity of polymers

Strength of polymers

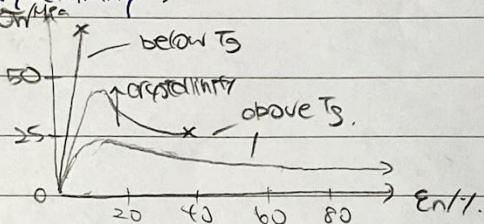
- Polymer strength (and stress-strain response) are determined by:

- ↳ the ability of the chain molecules to unravel and slide
- ↳ temperature (rel. to T_g) and the strain rate
- ↳ crystallinity (thermoplastics) and cross-linking (elastomers, thermosets)
- If we consider the ratio $\frac{\sigma_{EL}}{E}$, $(\frac{\sigma_{EL}}{E})_{EL} \sim 0.1 - 1.0$; $(\frac{\sigma_{EL}}{E})_{TP/TS} \sim 10^2 - 10^4$.
- Polymers are considerably stronger than metals/ceramics, rel. to their Young's modulus.
- ↳ stiffness is dominated by secondary bonds between the molecular chains, and/or re-aligning the molecules
- ↳ strength usually involves the breaking of primary bonds along the chains

Thermoplastics > effect of glass-transition, crystallinity

- For thermoplastics, below T_g : elastic-brittle; above T_g : elastic-plastic

- The peak stress rises w/ crystallinity.

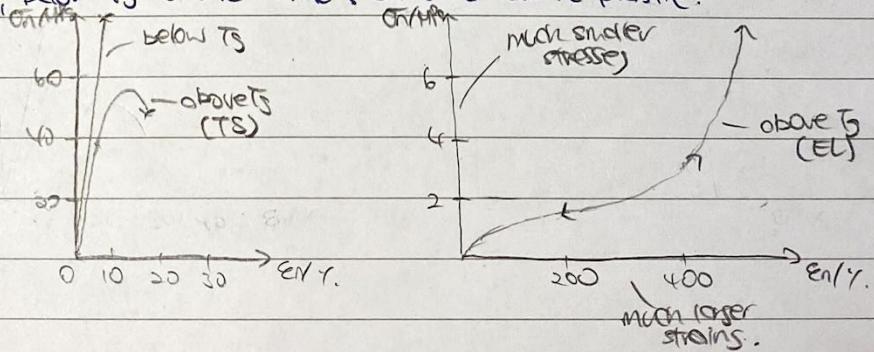


Elastomers and thermosets: effect of glass transition, cross-linking

- For elastomers, below T_g : elastic-brittle; above T_g : non-linear elastic

* When above T_g , elastomers have very large strains to failure. However the elastic strain is recovered → little/no ductility; i.e. brittle (elongation to failure is zero).

- For thermosets, below T_g : elastic-brittle; above T_g : elastic-plastic.



Engineering materials.

Classes of engineering materials

- Engineering materials can be grouped into 5 generic classes: metals, polymers, elastomers, glasses, ceramics, based on similarities in microstructure and properties.
- Hybrids combine materials from the 5 classes (composites), or have internal architecture (foams), or both (natural materials: wood, bone).

Structural vs functional materials

- Structural materials are used to resist mechanical loading (sometimes thermal and electrical loads), and is not limited to "civil engineering" materials.
- Functional materials include optical, electronic, magnetic materials.

Material property charts

Material property charts

- Material property charts have log scales of properties spanning orders of magnitude.
- The ranges of the 2 properties define a "bubble" on the chart.
- Materials in a given class cluster together (indicating similarity in underlying physical basis of the properties)
- The databook contains 4 material-property charts: E vs ρ , σ_y vs ρ , E vs σ_y , K_{ic} vs σ_y .

Young's modulus.

- Young's modulus for solid materials spans over 6 orders of magnitude.
- Metals, ceramics and composites are stiff; Polymers, natural materials and foams are generally less stiff and cover a very wide range of stiffness.
- Individual classes of metals have well-defined values of Young's modulus

Density

Generally, Young's modulus increases w/
density for ceramics and metals
(except Pb alloys)

- Density for solid materials spans a factor of 20.
- Polymer foams and porous natural materials (e.g. wood) extend the range to much lower values.
- Most ceramics and composites have similar densities to the "light metals" (Mg, Al, Ti); other metals and steel have higher densities.
- Individual classes of metals have well-defined values of density; density of polymers all around $\sim 1 \text{ Mg/m}^3$.
- Only porous materials and the lightest polymers float in water ($\rho < 1 \text{ Mg/m}^3$).

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Strength.

- Strength for solid materials spans over 3 orders of magnitude.
- Metals, ceramics and composites are the strongest materials; natural materials and polymer foams are less strong and extend the range to much lower strengths.
- Individual classes of metals cover much wider range of strength than is observed for Young's modulus or density, as strength is microstructure sensitive, and thus can be manipulated widely by composition and processing.
- * Any envelopes in strength-related material property charts are shown dotted for technical ceramics, silicones and porous ceramics as their data is for compression only.

Fracture toughness

- Fracture toughness for solid materials spans over 4 orders of magnitude.
- Metals are the toughest materials - followed by ceramics, then polymers and finally foams.

Material selection.

functionally, we have
(i) stiffness-limited design
(ii) strength-limited design.

Property trade-off

- The material choice is determined by any constraint placed on the design (stiffness, strength, size) and the overall objective (min. mass / cost / environmental impact)
- The mass/cost/environmental impact depends on multiple properties → we need to trade-off material properties against one another → use material property charts.

Performance indices. (merit indices)

sometimes, we can skip steps
for more straight-forward cases.

- To find the performance index,
 - ↳ 1) Identify the objective (minimise mass / cost / embodied energy etc. / maximise length, strength etc.)
 - ↳ 2) Identify functional constraints (specified stiffness, strength)
 - ↳ 3) Identify geometrical constraints (identify fixed and variable dimensions)
 - ↳ 4) Eliminate free variables to find the performance index. *take reciprocal if we want to minimise the quantity.*
- After finding the performance index, we can plot it on the material property chart:
If the performance index is $\frac{a^{ln}}{b}$, for all materials that have the index value of C,

$$\log a - \log b = \log C$$

$$\log a = n \log b + c'$$

→ they lie on the same line w/ slope = n, on the a vs b material property chart.

- Increasing the performance index is equal to shifting the line towards the top left.

* Best materials can be identified w/ numerical values for the design constants.

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Secondary constraints

- After short-listing the materials w/ the greater performance index, we need to refine the selection using secondary constraints.
- We can either use simple property limits, qualitative measures of performance and suitability.
- We may consider the toughness, corrosion, cost, manufacturing limits, size limits, shape etc.

Toughness.

- We can set a lower limit for the fracture toughness K_{IC} .
- For now, we exclude all ceramics/glass from any structural application, particularly where impact is expected. (since they are brittle and would fracture)

Corrosion

- Materials must resist chemical attack by the service environment (e.g. water, salt water)
 - To account for corrosion resistance, we use a 5-pt. scale grading of typical performance in different environments (found in databook).
- * Poor resistance may not exclude a material, but implies the need for protection → extra costs.

Cost.

- Product cost includes material and manufacturing cost.
 - As a first indicator, we can consider the material cost only.
- * Cost can be an object (min. cost) or a constraint (upper limit for cost).

Manufacturing limits: size/shape/joining.

- Not all materials may be available in the req. geometry due to manufacturing limits:
 - ↳ Metals/polymers: easy to form into wide range of shapes and sizes — casting/moulding as a liquid, deformation processing as a solid (powder), easy to join (welding).
 - ↳ Ceramics: much more limited, processed by powder compaction.
 - ↳ Fibre composites: difficult to shape, must usually joined by adhesives.
 - ↳ Wood: some shape limitations, well-suited to mechanical joining and adhesives.

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Size limits

- sometimes, one of the dimensions (e.g. length, CSA) is a free variable. In reality, there may be upper/lower limits in the practical size.
- This size limit results in an upper/lower limit on E/σ_y
- * Fixing the value of this limit requires the values of the design parameter.

shape.

- shape is significant in material selection for bending applications.
- If area and shape can both be varied, we can
 - i) use performance indices inc. shape factor
 - ii) use performance indices w/o shape factor, but comment on likely effect of shape.

(most shapable metals > composites > wood (least shapable))

Shape factor

- define shape factor, for stiffness in bending ϕ_e as

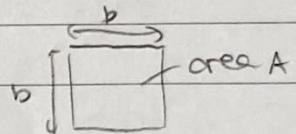
$$\phi_e = \frac{I_{\text{for shaped section}}}{I_{\text{for reference shape}}} = \frac{I}{I_0}$$

where the reference shape is a solid square section,

$$A = b^2 \quad ; \quad I_0 = \frac{b^4}{12} = \frac{A^2}{3}$$

$$\text{Here, } \phi_e = \frac{I}{I_0} = \frac{12I}{A^2} \quad [\text{max } A \propto 1/\phi_e]$$

$$I > \int y^2 dA, \quad \delta = k \frac{WL^3}{EI}$$



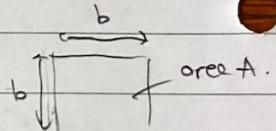
- define shape factor, for strength in bending ϕ_f as

$$\phi_f = \frac{Z_e \text{ for shaped section}}{Z_e \text{ for reference shape}} = \frac{Z_e}{Z_{e0}}$$

where the reference shape is a solid square section

$$A = b^2 \quad ; \quad I_0 = \frac{b^4}{12} \quad ; \quad Z_{e0} = \frac{\frac{b^3}{4}}{\frac{b}{2}} = \frac{b^2}{6}$$

$$\text{Here, } \phi_f = \frac{Z_e}{Z_{e0}} = \frac{6Z_e}{A^{3/2}} \quad [\text{max } A \propto Y_{f43}]$$



- The max. shape factor depends on the physical limits of thickness due to the capabilities of manufacturing processes and buckling failure of thin-walled sections.

- Different materials can be shaped to a different extent:

Material	Typical max. shape factor ϕ_e	Typical mass ratio by shaping, $1/\phi_e^2$	Typical max. shape factor ϕ_f	Typical mass ratio by shaping, $1/\phi_f^{2/3}$
most shapable steels	64	$1/8$	1.3	0.18
Al alloys	49	$1/7$	1.0	0.22
Fibre composite	86	$1/6$	9	0.23
Wood (least shapable)	9	$1/3$	3	0.48

* To find the performance index w/ shape factor, replace I w/ $\phi_e I_0$ and $Z_e = \frac{I}{Y_{max}}$ w/ $\phi_f Z_{e0}$.

* Shaping has a smaller influence on strength than on stiffness (as Y_{max} increases less than I)

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Selection for minimum cost / embodied energy.

- Material cost C , $C = mC_m \rightarrow$ To minimize cost, replace m w/ mC_m
- Embodied energy H , $H = mH_m \rightarrow$ To minimize embodied energy, replace m w/ mH_m .
- * When considering lightweight design (minimize m), we simply replace p w/ pC_m or pH_m

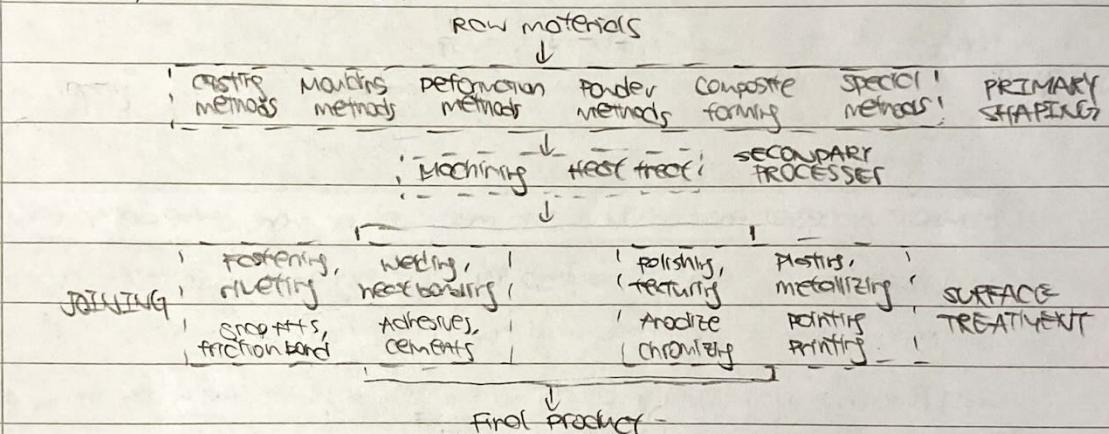
in step ① at finding the performance index

Material selection w/ multiple constraints.

- If we have both functional constraints — a required stiffness and avoiding failure, we need to find the free variable in both cases and take the larger one (st. both req. are met).
- From this, we can then plug in the free variable into the objective eqn. and repeat for each material to rank them. (can't use performance indices).
- * Since we need to calculate the free variable for each material, we need design constants.

Manufacturing processes and process selection

Hierarchy of manufacturing processes



Top level hierarchy of processing functions in design

↳ Primary shaping : turn raw materials into components

↳ Secondary processes : add features to components, modify bulk properties

↳ Joining : assemble components into products

↳ Surface treatment : modify surface properties

Physical basis of different process families

↳ Casting: pour liquid (metal), solidify and cool, remove mould.

↳ Moulding: viscous flow of molten polymer or glass

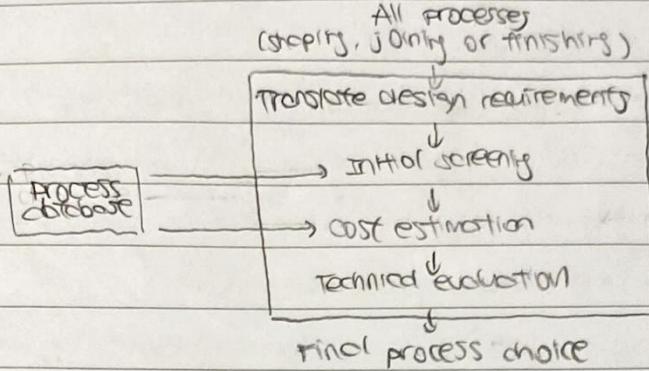
↳ Forming: plastically deform solid metal to shape

↳ Powder: fill die w/ powder (ceramic/metal) and hot press

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Process selection

- Material processing achieves 2 simultaneous outcomes:
 - ↳ shaping components and assembling them into products
 - ↳ Manipulating the microstructure and properties to give the req. performance in service.
- Process selection consists of the following stages:



- Process selection applies separately to the 3 process classes : shaping, joining, surface treatment.
- These do not compete w/ each other — they offer different functions and each has its own design requirements.
- * For now, we only consider primary shaping.

Initial screening

- The procedure for initial screening is as follows:
 - ↳ 1) Assemble info about the design req: material class, shape class, approx mass, section thickness, tolerances and req. surface finish.
 - ↳ 2) Plot on process attribute charts to identify processes that have the req. attributes.
 - ↳ 3) Consider "stacking" of processes to bypass problems (e.g. if shaping processes fail on tolerance or roughness, consider shaping then machining).
- * Note the charts show the "normal" viable ranges for each process — operating outside these ranges may be feasible, but probably only at a cost penalty

Material-process compatibility

- Material-process compatibility depends on the physical nature of the process, and whether the material has suitable properties
- Process attribute chart in database. Extra notes as follows:
 - ↳ some process limits w/ high Tm metals
 - ↳ ceramics, glasses and composites are difficult to join ; Natural materials are easy to join
 - ↳ Thermoplastics can be softened → can hot-form, weld ; Thermosets must be moulded to re-shape.

adhesive or
mechanical (screws)

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Shape-process compatibility

- choice of shaping process is strongly influenced by geometric characteristics of the components being shaped → prismatic, sheet or 3D.
 - ↳ Prismatic : rolling, extrusion
 - ↳ Sheet : sheet metal forming
 - ↳ 3D : casting, forging, powder, moulding, machining
- * Not in databook.

Mass and section thickness

- Mass and thickness only discriminating at the extremes
- Wide range of mass and thickness can be handled by almost all processes
- Machining used for shaping at all length scales → can also be used as a secondary process for "tidy-up" processes.

Tolerance and roughness

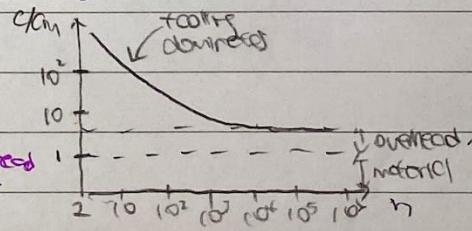
- Polymers give a smooth finish, but poor dimensional accuracy
- Tolerance and roughness more discriminating between processes
- Machining after shaping used in metals to reach target processes
- Expensive to over-specify precision and finish

Cost-based ranking

- After short-listing possible processes from the initial screening stage, we make a final selection based on cost.
- Manufactured cost can be estimated approximately for mass-produced, net-shaped products.
- The total cost of a component depends on 3 contributions:
 - ↳ Material cost, C_m - product and consumables
 - ↳ Tooling cost, C_c - dedicated tooling (dies, moulds etc.)
 - ↳ Overhead cost, C_L - labour, energy, share of capital
- The relative importance of these depend on
 - ↳ Batch size, n - total no. being made
 - ↳ Production rate, r - no./hour which can be made
- The general cost eqn. is as follows

$$C = C_m + \frac{C_c}{n} + \frac{C_L}{r}$$

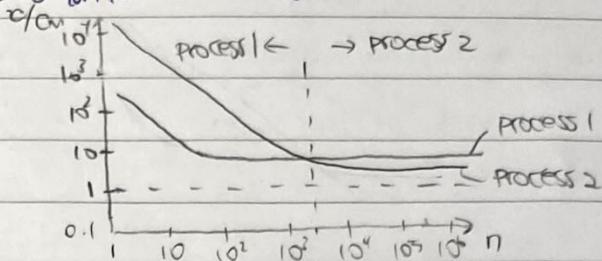
shares of tooling per part shares of overhead per part



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Economic batch size.

- The cost eqn. allows competing processes to be ranked approximately in order of increasing cost. The ranking depends on the batch size.



- Experience shows that each process has a characteristic range of batch sizes for which it is usually competitive. (Economic batch size)
- A preliminary cost assessment can be made on the basis of this range of economic batch size, before plotting the cost eqns for detailed cost analysis.

Technical evaluation.

- After the initial screening and cost-based analysis, technical evaluation is still needed to
 - ↳ determine process operating conditions
 - ↳ refine part geometry / material choice
 - ↳ ensure defect-free part that provides target microstructure / properties

Environmental impact of materials

Exponential growth in consumption

- The production of materials contribute to a significant proportion of CO₂ emission and energy consumption, which is increasing exponentially
- If at time $t=t_0$, the consumption rate is $C=C_0$, and the exponential growth rate is $r\%$ per year, i.e. $\frac{dC}{dt} = \frac{r}{100} C$. Integrating,

$$C = C_0 \exp \left[\frac{r(t-t_0)}{100} \right]$$

- The time for rate to double, t_D is given by

$$t_D = \frac{100}{r} \ln 2$$

- The amount of "stuff" consumed, Q from $t=t_0$ to $t=t$ is given by

$$Q = \int_{t_0}^t C(t) dt$$

$$= \frac{C_0}{r} \left[\exp \left[\frac{(t-t_0)}{100} \right] - \exp \left[\frac{-t_0}{100} \right] \right]$$

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Life cycle assessment (LCA)

- Full LCA outputs the resource consumption, emissions inventory and impact assessment, which is time consuming, expensive and subjective (ref. weighting of impacts)
- Moreover, the data can only be obtained in retrospect, but impact should be assessed early in design
- We req. a simple approach - choose a single measure of impact: energy or CO₂.
- Using eco-audit, we can have a quick, approximate overview of impact of products and identify the life phase that consumes the most energy / releases the most CO₂.
- After identifying the dominant life phase, we can modify the design accordingly:
 - ↳ Material - minimise material in the part, embodied energy and CO₂/kg.
 - ↳ Manufacture - minimise process energy and CO₂/kg.
 - ↳ Transport - minimise distance moved, mass and energy made of transport
 - ↳ USE - minimize mass, thermal loss and electrical loss
 - ↳ DISPOSAL - select non-toxic and recyclable materials

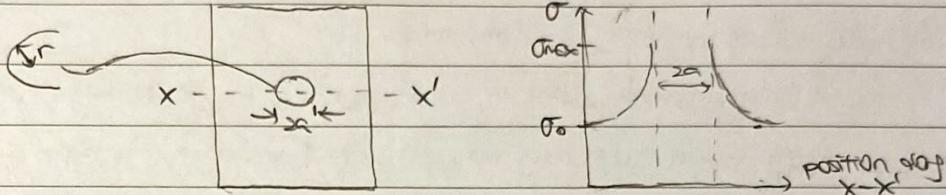
Brittle fracture

stress concentration factor. (blunt features)

- The stress concentrated at the edge (tip) of a flaw is larger than the remote applied stress σ_0 . (usually occur at changes in section (threads)).
- For blunt features, we can define the stress concentration factor (SCF) as:

$$\text{dimensions} \quad \text{SCF} = \frac{\sigma_{\max}}{\sigma_0} = [1 + 2(\gamma r)^{1/2}] \geq 1$$

where σ_{\max} is the peak stress, $2a$ is the flaw length, r is the radius of curvature of the tip.



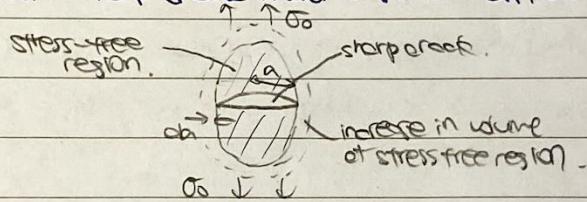
Energy-based approach

strain energy release rate (sharp cracks)

- The driving force for crack advance is the elastic strain energy stored in the stressed material.

$$\text{The elastic strain energy per unit volume is } U = \frac{1}{2} \sigma_0 \epsilon_0 = \frac{1}{2} \frac{\sigma_0^2}{E}$$

- consider a material w/ a sharp crack under a remote applied stress σ_0 .



- For a plate w/ thickness t , the energy released dW when the crack extends (at both ends) by da is given by

$$dW = \frac{\sigma_0^2}{2E} (2\pi a + da) = \frac{\sigma_0^2 t da}{E}$$

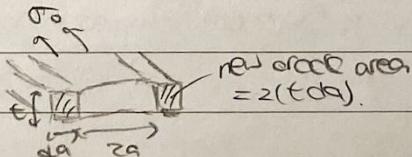
- The eqn. above assumes the stress-free region in the shadow of the crack is circular. A more accurate expression would be

$$dW \approx \frac{2\sigma_0 t a^2 da}{E}$$

- For a plate of thickness t , the strain energy release rate G , defined as the rate of energy released per unit of new crack area, is given by

$$G = \frac{dW}{\text{new crack area}} = \frac{2\sigma_0^2 t a da / E}{t^2 da} = \frac{\sigma_0^2 a}{E}$$

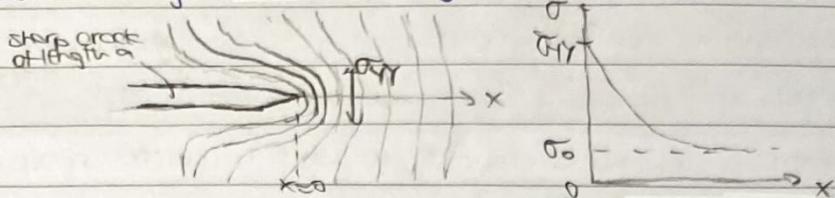
- * The new crack area is given by $2 \cdot (t da)$ — consider the cross-section through the thickness t .



stress-based approach

stress intensity factor (sharp cracks)

- consider loading a material containing an infinitely sharp crack of length a .



- Analysis of the stress field ahead of a sharp crack shows that the stress σ_{YY} is given by

$$\sigma_{YY} = \frac{K}{2\pi x}$$

where K is the stress intensity factor.

- The stress intensity factor K is a measure of the crack tip loading, and depends on the applied stress and specimen/crack geometry, given by

$$K = Y \sigma_0 \sqrt{\pi a}$$

where a is some measure of the crack length, Y is a dimensionless constant.

- For a centre crack, we take a as half the total crack length, whereas for an edge crack, we take a as the total crack length.
- Stress intensity handbooks give Y for many geometries:
 - ↪ Centre crack w/ semi-length a in infinite plate : $Y=1$
 - ↪ Edge crack w/ crack length a in semi-infinite plate : $Y=1.12$

uniting the stress and energy approaches.

- Fracture is expected when G reaches a critical value, G_{IC} , the critical strain energy release rate (fracture energy),

$$G \geq G_{IC}$$

- Fracture is also expected when K reaches a critical value, K_{IC} , the critical stress intensity factor (fracture toughness).

$$K \geq K_{IC}$$

both G_{IC} and K_{IC} are measures of the loading at the tip of a sharp crack.

- G_{IC} and K_{IC} are material properties and they are related by

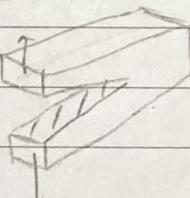
$$K = \sqrt{EG}$$

and

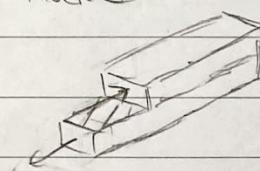
$$K_{IC} = \sqrt{EG_{IC}}$$

- The I subscript for K_{IC} denotes that this critical value of K is for mode I crack loading. There are 3 basic modes of loading.

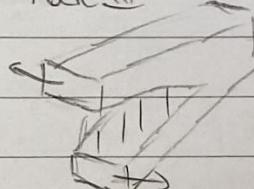
Mode I



Mode II



Mode III

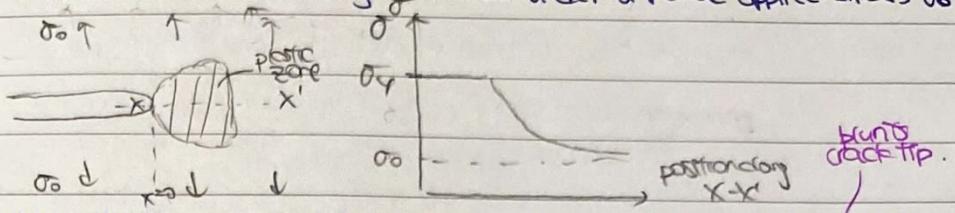


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Ductile fracture

Crack tip plasticity

- consider loading a ductile material containing a crack under a remote applied stress σ_0 .



- Under the remote applied stress σ_0 , plastic flow occurs at the crack tip, which raises the radius of curvature at the crack tip and reduces the stress concentration effect.
- This involves extensive energy absorption and results in the peak stress being lowered to yield stress σ_y (or higher slightly if work hardening occurs).
- In general, as the yield stress σ_y increases, the fracture toughness K_{IC} decreases due to the reduced plasticity at the crack tip.

The crack tip process zone

- the size of the process zone (plastic zone) r_p is given by

$$r_p = \frac{K_{IC}^2}{\sigma_y E}$$

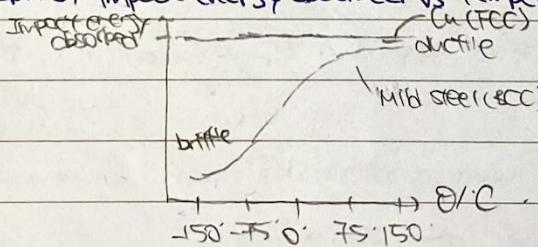
the diameter of the process zone

although the process zone is a region of plastic deformation, it locally affects the elastic stress field → we can use linear elastic fracture mechanics to model it

- K can be used to predict fracture if the process zone size r_p is small. When r_p becomes of the order of the crack length/specimen size, then the stresses in the process zone are no longer characterised only by K . → check validity of fracture mechanics if
fracture may be due to extensive yielding.

The ductile-brittle transition temperature.

- As temperature decreases, the dislocations are less mobile, so there is less plastic flow, therefore fracture toughness K_{IC} decreases, and vice versa.
- Metals w/ FCC crystal structure often display ductile-brittle transition properties. - this can be seen on a graph of impact energy absorbed vs temperature.



- Fracture toughness K_{IC} is often measured by the energy absorbed during an impact test (Izod impact strength test)

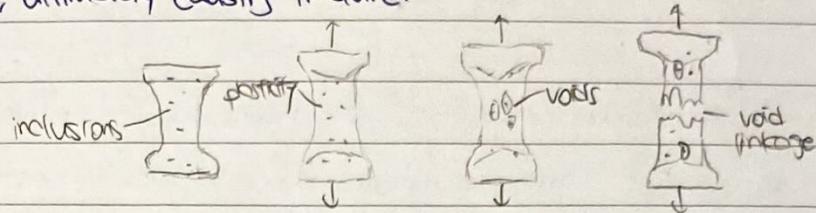
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Micromechanism of fracture

Metals

① Ductile fracture

- Plasticity concentrates stress on inclusions/impurities, nucleating voids that grow and coalesce, ultimately causing fracture.



- From a micrograph, you could see evidence of extensive plastic flow and regions in which voids formed on the fracture surface.

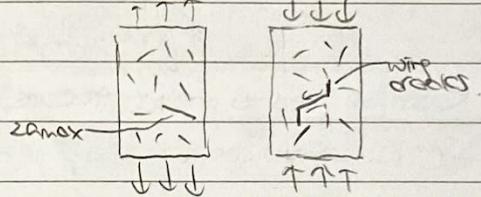
- For a pure metal (no inclusions), ductile rupture occurs instead

② Brittle fracture.

- At a low temperature, below the ductile-brittle transition temperature, there is little plastic flow and the energy absorbed during fracture is rel. low.

- From a micrograph, there is little evidence of plastic flow and no voids on the fracture surface

Ceramics



- Fracture of ceramics is dominated by their brittle nature. Failure starts from flaws or pores which are introduced during their processing.

- Fracture may occur through grains (transgranular) or along the grain boundaries (intergranular)

① Tension

worst when crack length a is \perp to σ_0 .

- Tension failure occurs at the worst flaw — in terms of size and orientation to σ_0 .

$$\sigma_t = \frac{K_{Ic}}{\gamma J I C_{max}}$$

where a is the half-length of a central crack or the length of an edge crack.

- As the tensile stress depends on the worst flaw, it is not well defined \rightarrow Weibull statistics.

② Compression

- Compressive failure is initiated in mode II, which will occur on the planes of max. shear stress (45° to the loading axis)

- Once propagation starts, the whole loading situation changes, cracks are likely to follow a mode I path (i.e. parallel to the loading axis \rightarrow wavy cracks).

- Failure under compressive load req. a much higher driving force than in tension — mainly due to frictional work being done behind the crack tip as the asperities slide over each other

- Therefore, for a brittle material, compressive strength σ_c is much greater than tensile strength σ_t

$$\sigma_c \approx 10-15 \sigma_t$$

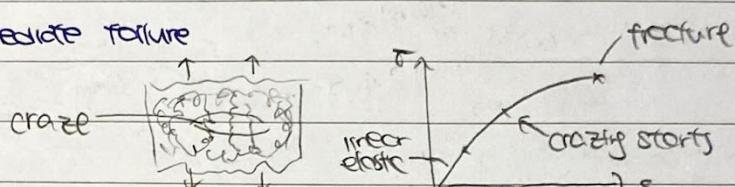
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Polymers (Thermoplastics)

- Below the glass transition temp. T_g ($T < T_g$), thermoplastics are elastic-brittle.
 - ↳ chain sliding is limited \rightarrow little or no ductility
 - ↳ brittle fracture occurs from inherent flaws of the material.
- Above the glass transition temp. T_g ($T > T_g$), thermoplastics are elastic-plastic.
 - ↳ chain mobility increases around T_g as the van der Waals bonds melt.
 - ↳ yielding can take place by crazing, shear yielding or cold drawing.

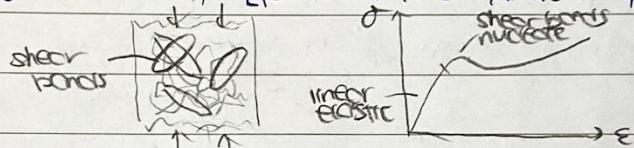
① crazing

- Microcracks open in tension, bridged by stiff fibres of material w/ aligned molecules, preventing immediate failure



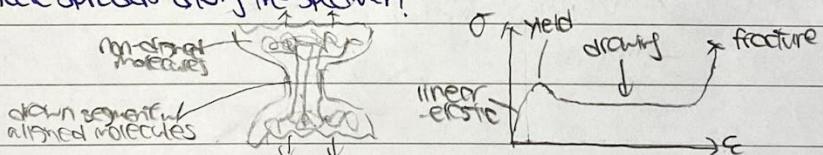
② shear yielding

- shear bonds form and are stabilised by alignment of molecules. Multiple shear bonds form \rightarrow greater ductility [possible in tension, more likely in compression]



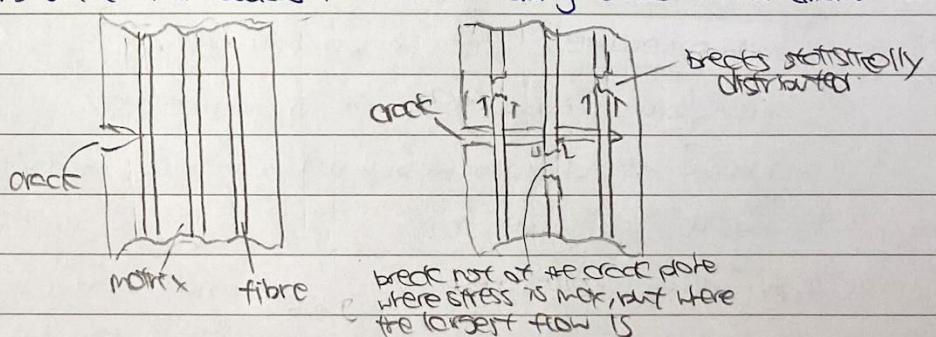
③ cold drawing

- polymers which do not craze can often be cold drawn.
- necking occurs but the neck is stable - the molecules align in the neck and strengthen it, so the neck spreads along the specimen.



Composites

- The main energy-absorbing mechanism raising the toughness of fibre composites is the pulling of fibres out of their sockets in the matrix during crack advance (frictional sliding).



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Weibull Statistics

Weibull statistics

- The strength of which brittle materials fail depends on the presence of flaws, which can be treated on a statistical basis
- The Weibull approach considers a chain of links — the survival of the chain under load req. that ALL the links survive, i.e. failure occurs at the weakest link. [Weakest link theory]
- Design w/ brittle materials is about ensuring an acceptably low risk of failure choosing the appropriate failure probability P_f .
- The survival probability P_s is related to the failure probability by $P_s = 1 - P_f$.
- The Weibull eqn. gives the survival probability P_s as

$$P_s(V) = \exp \left[\int_0^V \left(\frac{\sigma}{\sigma_0} \right)^m dV \right] \quad \text{varying } \sigma$$

where m is the Weibull modulus

- For constant applied stress σ , the Weibull eqn. becomes

$$P_s(V) = \exp \left[-\frac{V}{V_0} \left(\frac{\sigma}{\sigma_0} \right)^m \right] \quad \text{constant } \sigma$$

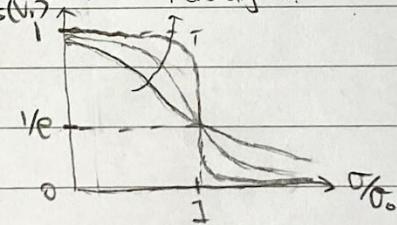
* V_0 and σ_0 are the volume and failure stress of an arbitrary test sample (analogous to a link in the chain).

Variation of strength for constant applied stress.

- Consider a specimen w/ $V = V_0$ under a constant applied stress σ . The survival probability P_s is

$$P_s(V_0) = \exp \left[-\left(\frac{\sigma}{\sigma_0} \right)^m \right]$$

- Plotting $P_s(V_0)$ vs (σ/σ_0) on a graph, increasing m .



$$\frac{dP_s(V_0)}{d(\sigma/\sigma_0)} = -m \left(\frac{\sigma}{\sigma_0} \right)^{m-1} \exp \left[-\left(\frac{\sigma}{\sigma_0} \right)^m \right] ; \text{ At } \sigma = \sigma_0, \frac{dP_s(V_0)}{d(\sigma/\sigma_0)} = -m (1)^{m-1} \exp(-1) = -\frac{m}{e}$$

\therefore Bigger m means longer slope at $\sigma = \sigma_0$.

- The Weibull modulus m is a measure of the variability of failure stress — a low m means wide spread of failure stresses and vice versa.
- When a material has a Weibull modulus $m > 100$, we say it has a single well-defined failure stress.
- Engineering ceramics have $m \approx 10$; steels have $m \approx 100$.

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Weakest link theory

- Consider a specimen w/ volume V under a constant applied stress σ . The specimen can be regarded as n samples of volume V_0 , each w/ a survival probability $P_S(V_0)$.
- The weakest link theory states that

$$P_S(V) = [P_S(V_0)]^n$$

$$= \exp\left[-\left(\frac{\sigma}{\sigma_0}\right)^m\right]^{V/V_0}$$

$$= \exp\left[-\frac{V}{V_0} \left(\frac{\sigma}{\sigma_0}\right)^m\right]$$

note more
 $V=nV_0$

which satisfies the Weibull eqn.

Effect of specimen volume

- On average, a larger sample is more likely to contain one of the larger flaws and therefore fail at a lower stress.
- The statistics established from smaller lab specimens thus may not extrapolate to large components.
- Consider 2 specimens w/ volume V_1 and V_2 , under a constant applied stress σ . If the 2 specimens have the same survival probability P_S ,

$$P_S(V_1) = P_S(V_2) \rightarrow \exp\left[-\frac{V_1}{V_0} \left(\frac{\sigma}{\sigma_0}\right)^m\right] = \exp\left[-\frac{V_2}{V_0} \left(\frac{\sigma}{\sigma_0}\right)^m\right] \rightarrow V_1 \sigma_1^m = V_2 \sigma_2^m \rightarrow \frac{\sigma_2}{\sigma_1} = \left(\frac{V_1}{V_2}\right)^{1/m}$$

\therefore If $V_1 > V_2$, then $\sigma_2 > \sigma_1$, i.e. the failure stress decreases w/ specimen volume.

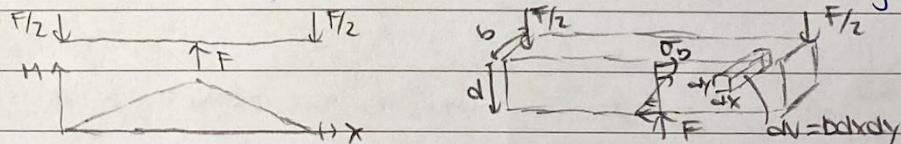
Non-uniform stress.

- Typically, components will be subject to varying stresses (e.g. bending).

* Any part of the specimen in compression will be OMITTED from the integral as its strength will be much greater than the tensile part

- e.g.: Ceramic beam in 3-pt. bend loading and uniform tension.

- Consider a beam of length L , thickness d and width b subject to 3-pt. bending.



M varies linearly w/ the position along the beam x , w/ M_{max} in the middle. ($M_{max} = \frac{FL}{4}$)

since $\sigma = \frac{M y}{I}$, the stress varies linearly w/ the distance from the neutral axis y , w/

the σ_{max} at $x=0, y=d/2$. ($\sigma_{max} = \sigma_b = \frac{F L d}{8 I}$).

In general, $M(x) = \frac{FL}{2} \left(\frac{1}{2} - |x| \right)$

$$\frac{\sigma}{\sigma_b} = \frac{M(x) \cdot y/I}{M(0) \cdot d/2/I} = \frac{y}{d/2} \frac{L/2 - |x|}{L/2}$$

As the stress varies w/ both x and y , we have a 2D integral!

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We integrate over the upper half of the beam only (tensile side)

$$P_{sb} = \exp \left[\int_V -\left(\frac{\sigma}{\sigma_0} \right)^m dV / V_0 \right] = \exp \left[-\frac{b}{V_0 \sigma_0^m} \int_{-L/2}^{L/2} \int_{-H/2}^{H/2} \sigma^m dx dy \right]$$

Substituting the expression for σ and taking logs,

$$-\ln P_{sb} = \frac{b \sigma_0^m}{V_0 \sigma_0^m} \int_{-L/2}^{L/2} \int_{-H/2}^{H/2} \left(\frac{y - Lx}{2L/2} \frac{H/2 - Lx}{H/2} \right)^m dx dy = \frac{Ldb}{2V_0} \left(\frac{\sigma_0}{\sigma_e} \right)^m \frac{1}{(m+1)^2}$$

- Consider a beam of length L , thickness t and width b under uniform tension σ_e .

Under uniform tension, the stresses are uniform, so

$$P_{st} = \exp \left[-\frac{V}{V_0} \left(\frac{\sigma_e}{\sigma_0} \right)^m \right] = \exp \left[-\frac{Ltb}{V_0} \left(\frac{\sigma_e}{\sigma_0} \right)^m \right]$$

- For the same survival probability for bending and tension,

$$P_{sb} = P_{st} \rightarrow -\frac{Ldb}{2V_0} \left(\frac{\sigma_b}{\sigma_0} \right)^m \frac{1}{(m+1)^2} = -\frac{Ltb}{V_0} \left(\frac{\sigma_e}{\sigma_0} \right)^m \rightarrow 2\sigma_e^m = \frac{\sigma_b^m}{(m+1)^2} \rightarrow \frac{\sigma_b}{\sigma_e} = (2m+1)^{1/m} \geq 1.$$

i.e. $\sigma_b > \sigma_e$ which makes sense because there is a larger volume of high stress in the tension specimen \rightarrow weaker

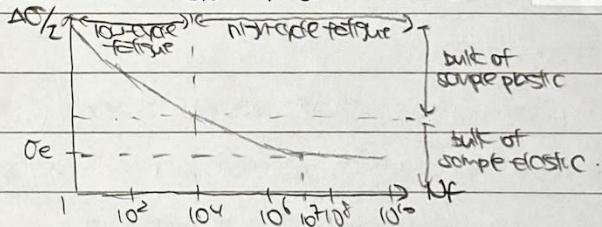
* Note uniform tension is a difficult test to apply to brittle materials due to the stress concentrations at the grips \rightarrow bend tests / Brazilian test

Fatigue — causes 90% of mechanical failures

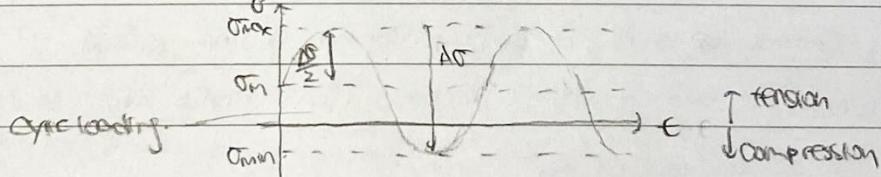
DEFINITIONS

- Fatigue is the damage or failure of materials due to cyclic loading.

- Fatigue data are presented as $\Delta\sigma-N_f$ (stress-life) curves



where $\Delta\sigma$ is the stress range, $\frac{\Delta\sigma}{2}$ is the stress amplitude, N_f is the no. of fatigue cycles to failure.



here, $\Delta\sigma = \sigma_{max} - \sigma_{min}$ and the mean stress σ_m is $\sigma_m = \frac{\sigma_{max} + \sigma_{min}}{2}$

- Fatigue data are often reported in terms of the stress ratio R , defined as

$$R = \frac{\sigma_{min}}{\sigma_{max}}$$

$R=1$ for a static load.

$R=0$ means that the stress cycles from 0 to σ_{max} (zero-tension fatigue)

$R=-1$ means that the mean stress σ_m is zero (fully reversed loading)

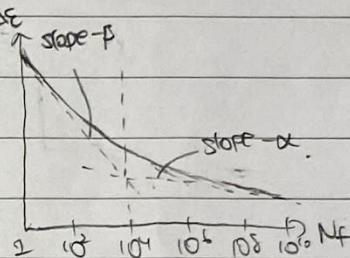
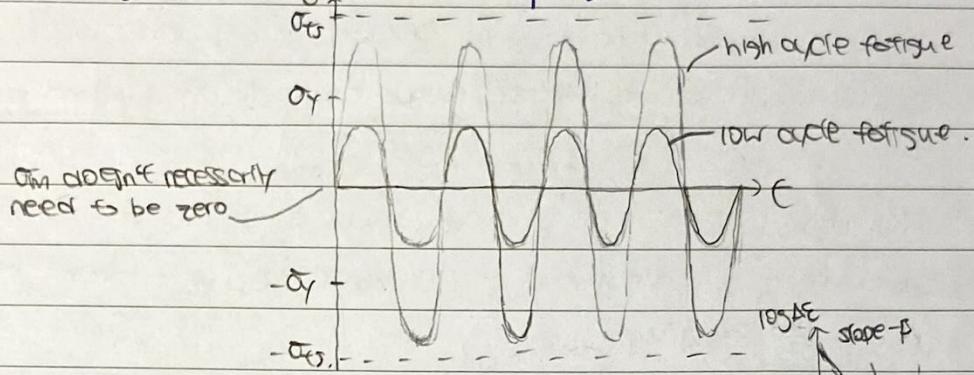
- The endurance limit of a naturally defect-free material is the applied stress amplitude σ_e about zero σ_m below which fracture does not occur at all, or occurs only after $N_f > 10^7$.

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Types of fatigue

- There is low cycle fatigue and high cycle fatigue

Low cycle fatigue	High cycle fatigue
- High-amplitude cyclic stresses ($\sigma_{\text{peak}} > \sigma_y$)	- Low-amplitude cyclic stresses ($\sigma_{\text{peak}} < \sigma_y$)
- Plastic deformation	- Primarily elastic deformation.
- $N_f < 10^4$ cycles	- $N_f > 10^4$ cycles
- e.g.: breaking an out-of-shape credit card	- e.g.: Aircraft gas turbine engine.



Empirical description of fatigue for uncracked components

① Low cycle fatigue - Coffin-Manson law.

- The low cycle part of the S-N curve can be expressed using Coffin-Manson law.

$$\Delta \epsilon_{pl} N_f^\beta = C_2$$

where C_2 and β are constants, w/ β typically being 0.5-0.7.

- Taking logs,

$$\log(\Delta \epsilon_{pl}) = \log(C_2) - \beta \log(N_f) \quad \text{slope of } -\beta$$

* The plastic strain $\Delta \epsilon_{pl}$ is given by the total strain $\Delta \epsilon_{tot}$ minus the elastic strain $\Delta \epsilon_{el}$.

However since $\Delta \epsilon_{el}$ is very small, we can say $\Delta \epsilon_{pl} \approx \Delta \epsilon_{tot}$.

② High cycle fatigue - Basquin's law.

- The high cycle part of the S-N curve can be expressed using Basquin's law.

$$\Delta \epsilon_{tot} N_f^\alpha = C_1$$

where C_1 and α are constants, w/ α typically being 0.07-0.13.

- Dividing the stress range $\Delta \sigma$ by the modulus E gives the strain range.

$$\Delta \epsilon_{tot} N_f^\alpha = \frac{C_1}{E} \quad [\Delta \epsilon_{tot} = \frac{\Delta \sigma}{E}]$$

- Taking logs,

$$\log(\Delta \epsilon_{tot}) = \log(C_1/E) - \alpha \log(N_f) \quad \text{slope of } -\alpha$$

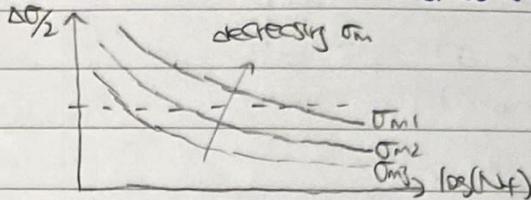
* Note both laws describe fatigue failure of constant amplitude about a zero mean stress σ_m , of uncracked components.

* $|\alpha| \ll |\beta|$

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EFFECT OF MEAN STRESS EFFECTS ON FATIGUE LIFE (Unnotched)

- Consider the S-N plot of a material w/ different mean stress levels $\sigma_m < \sigma_{m0}$.



- If we consider a fixed stress amplitude, we can see that the fatigue life (N_f) increases w/ decreasing mean stress levels σ_m .

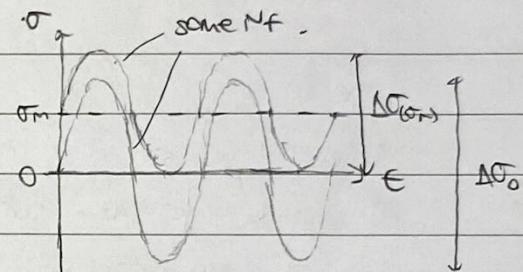
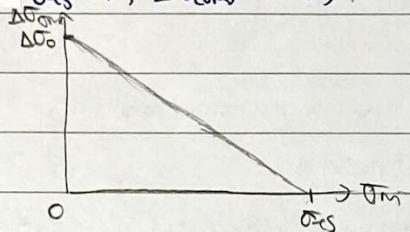
- Goodman's empirical rule allows for the effect of mean stress on S-N data.

- Goodman's rule assumes that the no. of cycles to failure N_f is conserved and states

$$\Delta\sigma_{(m)} = \Delta\sigma_0 \left(1 - \frac{\sigma_m}{\sigma_{m0}}\right)$$

- This means a stress range of $\Delta\sigma_{(m)}$ about a non-zero mean stress σ_m is as damaging as a stress range of $\Delta\sigma_0$ about zero mean stress.

$$(AS 0 < 1 - \frac{\sigma_m}{\sigma_{m0}} < 1, \Delta\sigma_{(m)} < \Delta\sigma_0)$$



CUMULATIVE DAMAGE.

- In practice, most engineering components are subjected to varying cyclic stress ranges.
- We can use Miner's empirical rule to predict the fatigue life in a loading sequence consisting of various blocks of diff. stress ranges about zero mean.
- Miner's rule states that the specimen fails when the proportion of the life-time used up by each block add up to 1.

$$\sum_i \frac{N_i}{N_f} = 1$$

where N_i is the no. of cycles corresponding to the i th block of const. stress range $\Delta\sigma_i$, and N_f is the no. of cycles to failure at that stress range.

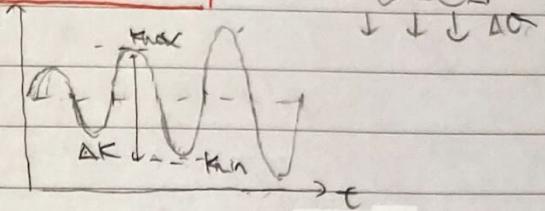
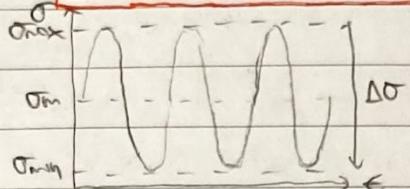
- Miner's rule ignores the load sequence events and is only applicable for high cycle fatigue (elastic deformation).

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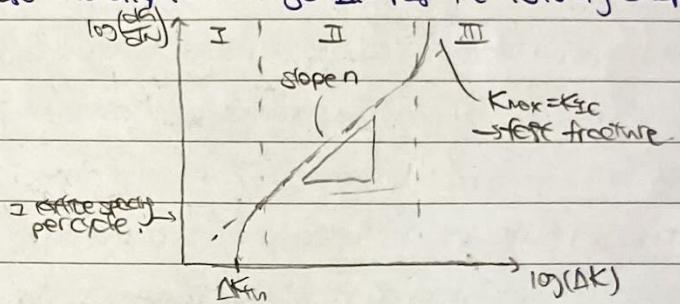
Characterisation of fatigue crack propagation.

- We define the cyclic stress intensity factor range ΔK as

$$\boxed{\Delta K = K_{max} - K_{min} = Y \Delta \sigma \sqrt{\rho} = Y (\sigma_{max} - \sigma_{min}) \sqrt{\rho}}$$



- For a constant stress range $\Delta\sigma$, α increases, so ΔK also increases over time.
- For most engineering alloys, a plot of the log of crack growth per cycle $\frac{da}{dN}$ vs the log of the stress intensity factor range ΔK has the following shape.



- Region I : crack initiation

- ↳ crack growth per cycle is zero below a threshold cyclic stress intensity factor range ΔK_{th} .
- ↳ Above ΔK_{th} , crack growth per cycle is v. small (less than I (office specify)).

- Region II : steady-state crack propagation

- ↳ this region can be described by Paris law:

$$\boxed{\frac{da}{dN} = A \cdot \Delta K^n}$$

$$\log\left(\frac{da}{dN}\right) = \log A + n \log(\Delta K)$$

where A and n are constants.

- Region III : fast fracture

- ↳ At high ΔK , crack growth rate increases rapidly

- ↳ As K_{max} approaches K_{IC} , fast fracture occurs.

- The fatigue life N_f , the no. of cycles to failure can be taken as the sum of the no. of cycles for crack initiation N_i and crack propagation N_p ($N_f = N_i + N_p$).

(the contribution of the final failure step to the total fatigue life is negligible).

- For low cycle fatigue, stress levels are high ($\sigma_{max} > \sigma_y$) so initiation is easy → crack propagation takes up most of the cycle life ($N_p > N_i$)

- For high cycle fatigue, stress levels are low ($\sigma_{max} < \sigma_y$) so initiation is hard → crack initiation takes up most of the cycle life ($N_i > N_p$)

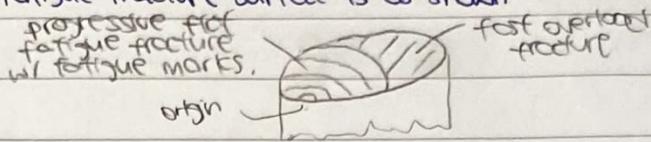
- The fatigue life is influenced by stress concentrations such as geometric variations in surfaces and material inhomogeneities (stress conc → local stress levels → faster crack initiation).

- Crack initiation can be delayed via shot peening or surface hardening. compressively stressed → no crack propagation

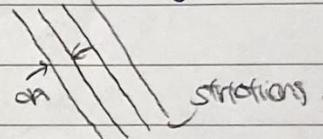
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Surface Fatigue

- A typical fatigue fracture surface is as shown:

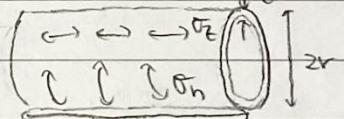


- Fracture surfaces formed during fatigue crack propagation may be characterised by 2 types of markings - clamshell marks and striations
- Both these features indicate the position of the crack tip at some pt. in time and appears as concentric ridges that expand away from the crack initiation site(s), usually in a circular or semi-circular pattern.
- Clamshell marks are of macroscopic dimensions and may be observed w/ the unaided eye.
- They are produced by a change in crack growth conditions, such as a drop in stress level or a pause in stress during growth rate.
- Fatigue striations are microscopic in size and can be observed under high magnification
- The striations are produced as the crack advances over 1 cycle, i.e. each striation corresponds to Δt . → slower crack growth rate means smaller striation spacing.
- * Clamshell marks may contain thousands of striations.



Pressurized thin wall cylinders

- consider a thin-walled cylinder ($t \ll r$) w/ internal pressure P



The hoop stress σ_h and the longitudinal stress σ_z is given by.

$$\sigma_h(2\pi L) = (2\pi L)P \rightarrow \sigma_h = \frac{P r}{t}$$

$$\sigma_z(2\pi r^2) = (\pi r^2)P \rightarrow \sigma_z = \frac{P r}{2t}$$

As $\sigma_h > \sigma_z$, the cracks are always // to the length (normal to σ_h).

- In pressurised cylinders, the main concern is to avoid catastrophic failure, particularly when the pressurised fluid is a gas, in which case an explosion is likely to result.
- Such rupture often occurs in the form of fast crack propagation → use the "leak before break" criterion as a guideline when designing high pressure systems.
- If $A_{crit} \text{ (critical flow area)} < \epsilon \text{ (vessel wall thickness)}$, then fast fracture will occur.
- If $A_{crit} \text{ (critical flow area)} > \epsilon \text{ (vessel wall thickness)}$, then gas will leak out through the crack before the crack is large enough to propagate under fast fracture conditions.
- * A_{crit} is the critical flow area for fast fracture to occur.

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- The leak-before-break criterion can be written by setting the crack length a to the vessel wall thickness t

$$K = Y \sigma \sqrt{\pi t} < K_{IC}$$

- Using the expression for the higher hoop stress to substitute t ,

$$Y \sqrt{\pi} \frac{P}{\sigma_y} = Y \sqrt{\pi P r} < K_{IC}$$

- If r is also req. that plastic deformation should not occur, we set the stress to the yield stress σ_y to obtain an expression for the max. operating pressure P_{max} .

$$P_{max} = \frac{K_{IC}}{Y^2 \sigma_y \pi r} = \frac{1}{Y \pi r} \left(\frac{K_{IC}^2}{\sigma_y} \right)$$

- The ratio K_{IC}/σ_y thus represents a material performance index for being able to operate safely under high pressure.

Proof testing.

- We can proof test a pressure vessel via a hydrostatic test - fill the vessel w/ water and pressuring it to a level above the planned working pressure P_w .
(Water is used as the stored energy in water is small (\rightarrow okay if it explodes))
- If the vessel survives, an upper bound on cracks present a_0 can be obtained (i.e. all cracks must have crack length $a \leq a_0$). This max. (initial) crack length can be found via,

$$K_{IC} = Y P_w \sqrt{\pi a_0}$$

- We can then calculate the min. no. of cycles N_f the vessel can last before cracks of length a_0 would grow to a length a_f , where fast fracture or leakage occurs.

↳ For fast fracture, a_f can be found using $K_{IC} = Y \sigma_{max} \sqrt{\pi a_f}$

↳ For leakage, a_f can be found using $a_f = t$.

- For a working stress range $\Delta \sigma_w$,

$$\frac{da}{dN} = A \cdot \Delta \sigma^{\alpha} \quad \text{and} \quad \Delta \sigma = Y \Delta \sigma_w \sqrt{\pi a}$$

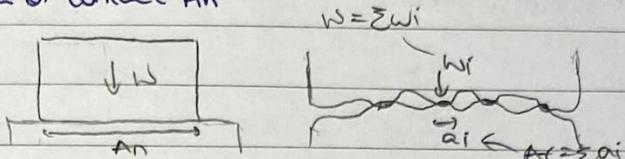
$$\therefore \int_{a_0}^{a_f} \frac{1}{A \cdot (Y \Delta \sigma_w \sqrt{\pi a})^{\alpha}} da = \int_0^{N_f} dN$$

Tribology, friction and wear

Contact between surface

Contact between rough metallic surfaces. (ductile, linear-elastic)

- consider 2 rough metallic surfaces in contact w/ an apparent/nominal area of contact A_n
- Due to the roughness of the surface, the true area of contact A_t is much smaller than the nominal area of contact A_n



- For metals, we can use the plastic model to relate the area of contact A_t at each asperity to the load W_i . Each contact region is like a small hardness test.

$$W_i \approx H_a i$$

- This leads to the macroscopic relationship between the load W and the true area of contact A_t .

$$W = \sum W_i = \sum H_a i = H_a A_t$$

* The hardness of the metal H_a is a constant.

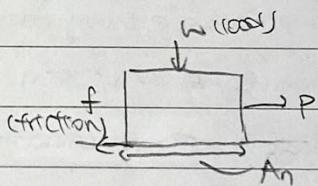
↳ only valid for ductile materials w/ linear elasticity

Contact between other types of surface.

- Elastomer (rubber) — these materials are non-linear elastic and compliant. At small loads, the true area of contact A_t increases non-linearly w/ the load. (At high loads, $A_t \approx A_n$)
- Ceramic — these materials are brittle. Asperities would fracture and the proportionality between the load and the contact area is not valid.

Friction

Amontons-Coulomb's laws of friction



- ① The force of friction is directly proportional to the applied load (Amontons 1st law)
- ② the force of friction is independent of the apparent area of contact A_n (Amontons 2nd law)
- ③ kinetic friction is independent of the sliding velocity (Coulomb's law).

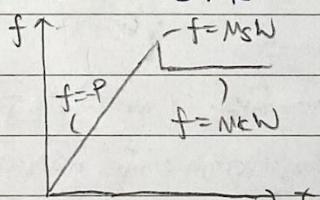
- The force of friction f is related to the applied load W by

$$f \leq M_s W$$

or

$$f = M_k W$$

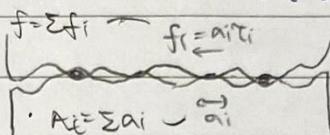
where M_s is the coefficient of static friction; M_k is the coefficient of kinetic friction,



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Adhesive contact

- In adhesive contact, we have 2 surfaces of similar hardness brought in contact w/ each other → adhesion as the contributing factor to frictional effect.
- For ductile, linear-elastic materials, the surface will start sliding if the shear stress τ_i on each contact reaches the material shear strength τ_s .



- Considering that $\tau_s \approx k \approx \frac{\sigma_y}{2}$ and $H \approx 3\sigma_y$, (k : shear yield stress, σ_y : unyielded stress)

$$F = \sum f_i = \sum \tau_i a_i = \tau_s \sum a_i = \tau_s A_c = \frac{\tau_s}{H} W \approx \frac{\sigma_y/2}{3\sigma_y} W$$

- We define the coefficient of friction μ as

$$\mu = \frac{\tau_s}{H} \quad \text{so} \quad \mu \approx 1/6.$$

- Here, we recover Amontons' law, and explain why the coefficient of friction is often in the range 0.2-0.3.

Friction in metals.

- FOR metals w/ an oxide layer, the oxide layer is responsible for the shear resistance

$$\mu = \frac{\tau_y(\text{oxide})}{H(\text{metal})}$$

- As indentation is a bulk property, we use the hardness of metal for H .
- For noble metals or metals kept protected from oxygen, there is a lack of oxide layer.
- After some sliding occurs, the asperities plastically deform so the true area of contact A_c becomes larger than $W/H \rightarrow$ strong adhesion/frictional force.
- If the true area of contact A_c approaches the nominal area of contact A_n , the frictional shear stress tends to the shear yield stress $k \rightarrow$ sticking friction.

Application – friction welding.

- In friction welding, mechanical friction is used to increase the temp. of the interface between 2 objects to be welded. (only interface heats up)
- Combined w/ the imposition of large normal loading at the interface, the 2 parts plastically yield and fuse together.
- Spin welding is mostly used w/ metals. A piece is rapidly rotated and pressed against a static part. Firstly, Coulomb friction causes heat and removes the oxide layer. Then, sticking friction takes place and remodels the interface until they fully fuse (good if melting damages the metal microstructure)
- Linear vibration welding is used w/ thermoplastics. A piece is rapidly vibrating and rubbed against a static part → the thermoplastics fuse.

Application - cold welding.

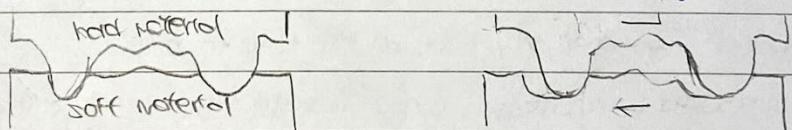
- In cold welding, 2 flat surfaces of similar metals are welded w/o heating by exploiting the natural adhesion of metallic surfaces
- Metals like Cu and Al could be welded using large pressures to break the oxide layers and bring non-oxidised material into contact \rightarrow electronic connectors/join long cables.
- Noble metals like Au can be welded w/o excessive pressure \rightarrow weld conductors at nano-scale.

Friction in rubber metals

- In elastomers (rubber), deformations are elastic and the true area of contact A_c tend to be large at typical loads.
- Energy is dissipated due to work of adhesion and also hysteresis in the loading/unloading cycles in each asperity \rightarrow Amonton's laws are usually not followed.
- The coefficient of friction μ tends to be 0.1–0.5.
- In ceramics, energy is dissipated due to brittle fracture of the asperities.

Abrasive contact

- In abrasive contact, we have 2 surfaces of very different hardness brought in contact w/ each other.
- The properties of the hard material will penetrate the soft one \rightarrow the true area of contact A_c will be mostly controlled by the load and the hardness of the softer material.
- During sliding, hard asperities will plough through the soft material, causing large dissipation of energy.
- The coefficient of friction of abrasive contacts tends to be large



Wear

Wear and wear rate

- The wear rate Q is defined as the volume worn away per unit sliding distance. It is found to be proportional to the true area of contact $A_c = W/H$.
- The wear coefficient K is defined as

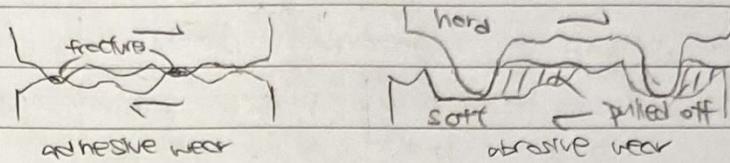
$$Q = K A_c = K \frac{W}{H}$$

- The value of K depends on how the material is pulled off at the contacts.

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Adhesive and abrasive wear

- In adhesive wear, small bits are pulled off every time yield occurs at the asperity tips
- In abrasive wear, materials come off as hard asperities plough through the soft material.
- The rate of abrasive wear is usually much larger than the rate of adhesive wear.



Friction and wear in polymers

- sometimes, when a surface slides on a polymer material, a transfer layer of polymer chains form on the surface → reduce friction and hence wear rate.
- Wear in brittle materials is hard to describe and sensitive to environmental factors (humidity)

Positive aspects of wear

- Although it is a pity to lose material, there are some positive aspects of wear
 - ↳ shiny metallic surfaces are obtained by polishing the metal until the asperities are smaller than the wavelength of light
 - ↳ Pencil/chalk only work due to the abrasive wear between the tool and paper (useful for the transfer of material(s))

Lubrication

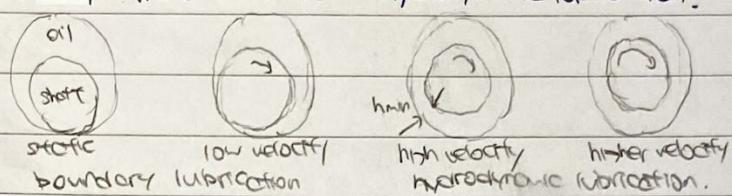
Lubrication

- Lubrication consists of interposing a substance (lubricant) between 2 surfaces in order to reduce wear when they move relative to each other.
- The interposed lubricant film can be a solid (e.g. graphite), a liquid (e.g oil) or a combination of both.
- The addition of lubricants can reduce the coefficient of friction between surfaces.
- The true area of contact A_c is still largely determined by the harder surface but the shear strength of the junction is reduced to that of the lubricant.
- Lubrication also ensures a decrease in the energy dissipation. In high speed motors involving pistons/bearings, lubrication avoids excessive heat generation.

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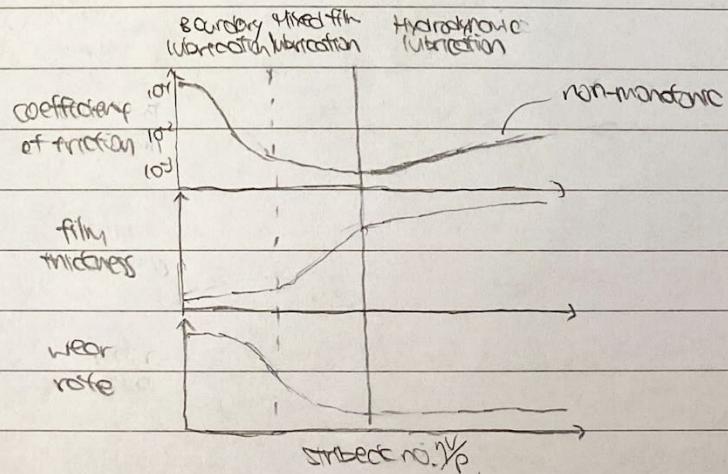
Journal bearing lubrication

- In journal bearing, the regime of lubrication depends on the shaft rotation.
- ↳ In static or low velocity situations, there is contact between the shaft and the bush
→ boundary lubrication
- ↳ As velocity increases, the shaft becomes increasingly supported by the oil phase. The thickness of the oil layer increases, as well as the oil pressure. Eventually, the pressure is sufficient to fully support the shaft → hydrodynamic lubrication.



Optimal lubrication

- The Stribeck no. is a good way to characterise the different regimes of lubrication.
- It is defined as $\frac{\eta V}{P}$ where η is the oil viscosity, V is the sliding speed at the interface and P is the pressure due to shaft loading.
- Friction decreases w/ increasing Stribeck no. as the oil layer is getting thicker and thicker, taking progressively a larger proportion of the shaft's load.
- When the surfaces are fully separated, the dissipation is mostly due to oil viscosity. Oil increases w/ velocity → friction increases.
- In a plot of friction against Stribeck no., the min. of the curve defines the optimum at which the device should operate. (Here the wear rate goes to nearly zero)
- At low velocities, the wear rate and friction are high → starting and stopping the machine would req. additional protection.



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The wheel

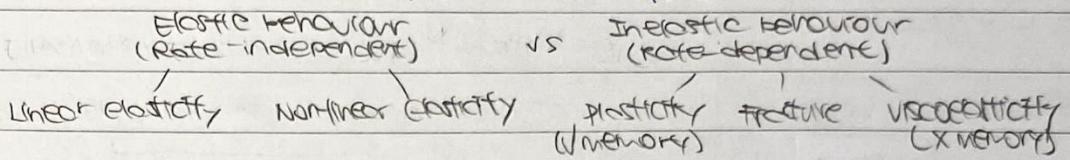
- wear can also be reduced by achieving the zone relative motion w/o sliding interfaces.
- ball bearings deliver very low friction even at low rotation speeds. However, they cannot take high loads as the contact forces are very localised
- the shape of the teeth of gears are optimised s.t. the teeth roll against each other rather than slide → reduce wear significantly.

Constitutive modelling

Constitutive modelling

Constitutive model

- A constitutive model relates stress and strain for a particular model.
- We determine the material's operating regime, then establish the related constitutive model for the material's macroscopic mechanical response.



Linear elasticity

The linear elastic regime

- For small deformations, the relationship between stress and strain is often linear and defines the material's elastic modulus.
 - Different measures of material stiffness are measured for different types of deformation:
 - ↳ uniaxial tensile test → Young's modulus E
 - ↳ shear test → shear modulus G
 - ↳ volumetric changes → bulk modulus K .
- ↳ related via Poisson's ratio ν for isotropic materials.

3D linear elasticity

- The general representation of stress and strain in 3D consists of 6 components - 3 direct components and 3 shear components.

$$\underline{\sigma} = [\sigma_1, \sigma_2, \sigma_3, \tau_{23}, \tau_{13}, \tau_{12}]^T \quad ; \quad \underline{\epsilon} = [\epsilon_1, \epsilon_2, \epsilon_3, \gamma_{23}, \gamma_{13}, \gamma_{12}]^T$$

- The stress and shear vectors are related by

$$\underline{\sigma} = \underline{C} \underline{\epsilon} \quad \text{or} \quad \underline{\epsilon} = \underline{C}^{-1} \underline{\sigma}$$

- For isotropic materials, we can write

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix}$$

- For orthotropic materials, we can write (i.e. 3 planes of symmetry)

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} 1/E_1 & -G_{12}/E_2 & -G_{13}/E_3 & 0 & 0 & 0 \\ -G_{12}/E_1 & 1/E_2 & -G_{23}/E_3 & 0 & 0 & 0 \\ -G_{13}/E_1 & -G_{23}/E_2 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{12} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix}$$

* In both cases, the compliance matrix \underline{C} is symmetric

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Physical origin of the elastic regime.

- Metals: atomic interactions
- Foams: beam bending in the microstructure
- Polymers below T_g : chains interact w/ each other via weaker interactions \rightarrow compliant.
- Elastomers above T_g : entropic forces. (Non-linear elastic)

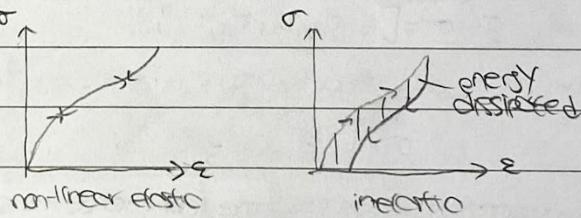
Non-linear elasticity

Definition of elasticity.

- Elasticity is the ability of a body to resist a distorting influence and to return to its original size and shape when that influence is removed.
- This implies that the mechanical work done on the material is stored, not dissipated.
- Linear elasticity is an approximation and works usually well at small deformations
 - ↳ FOR materials like metals/ceramics, which plastically deform/fraction at low strain, linear elasticity captures their elastic regime well.
 - ↳ Many materials like rubber can undergo large deformations w/o significant plastic deformation or permanent damage. \rightarrow linear elasticity fails to capture their response properly.

Non-linear elasticity.

- We can identify non-linear elasticity by checking the reversibility of the stress-strain curve.



- If the loading and unloading curves overlap, the behaviour is elastic. Otherwise, some energy has been dissipated (i.e. not elastic).
- Most compliant materials have a nonlinear response at large deformations as the internal microstructure of the material evolves w/ the deformation.
 - ↳ In rubber, polymer chains align at large deformation \rightarrow material stiffens
 - ↳ Most fabrics and fibrous materials are nonlinear \rightarrow fibres bend at first even stretch.

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Modelling nonlinear materials

- For elastic materials, we can define an energy function from which stress and strain can be derived.

- Consider an elastic material in 1D. We can introduce a strain energy function as.

$$U = \int \sigma d\varepsilon (= \frac{1}{2} E \varepsilon^2)$$

To relate stress and strain, we use. linear-elastic

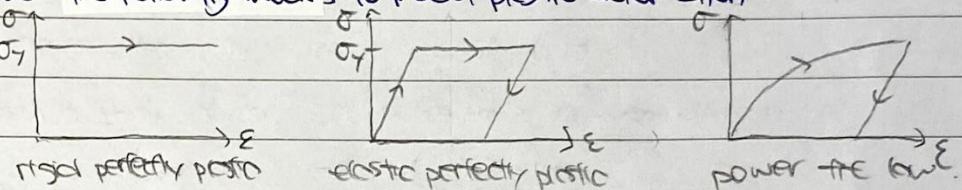
$$\sigma = \frac{dU}{d\varepsilon} (= E\varepsilon)$$

Time dependent behaviour

characterising inelastic behaviours

- Beyond the elastic regime, materials start to dissipate energy \rightarrow the stress-strain curve will be different on loading and unloading (energy dissipated \propto area under curve)

- We can use the following models to model plastic deformation

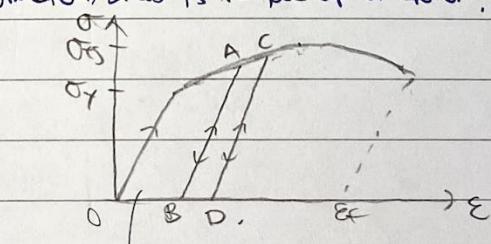


Physical origin of plasticity

- Metals: movement of dislocations
- Foams: cells may get crushed or fractured
- Polymers: chains start to slide past each other.

Time dependence of inelastic response

- In general, the state of an inelastic material depends on
 - ↳ The particular sequence of deformation (stress imposed to the material (memory effects))
 - ↳ How fast the deformation/stress is ramped up or down.



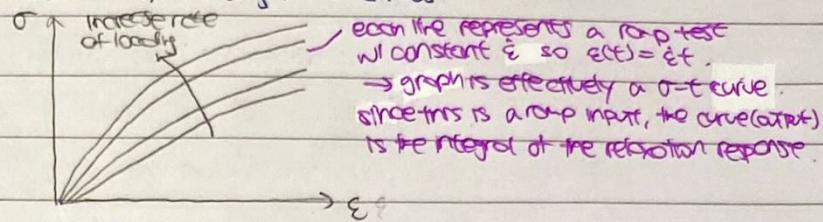
loading curves OA, BA are different \rightarrow memory effects.

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Time dependent stress-strain graph

- For an inelastic material, the stress-strain curve depends on the rate of loading.

The faster the material is deformed, the larger the stress



- We can capture the time-dependent response of such materials by considering their creep and relaxation response (decouple time and strain effects)

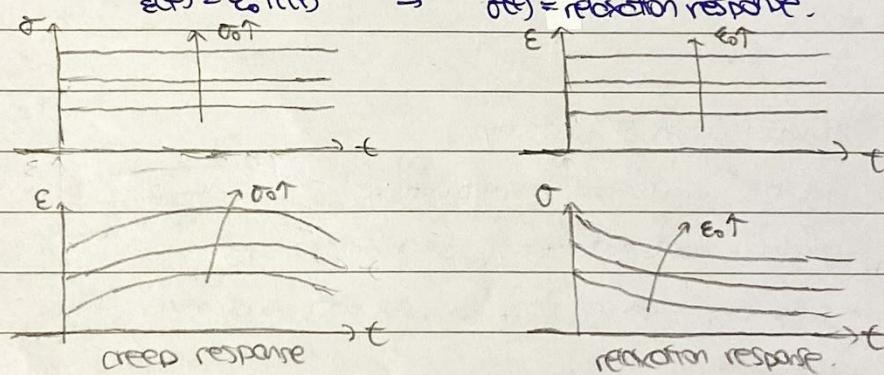
Creep and relaxation responses

- It is useful to record the temporal evolution of stress/strain as a result of an applied step function of the other variable.
- The creep response is the evolution of strain as a result of imposing a step in stress

$$\sigma(t) = \sigma_0 H(t) \rightarrow \epsilon(t) = \text{creep response}$$

- The relaxation response is the evolution of stress as a result of imposing a step in strain.

$$\epsilon(t) = \epsilon_0 H(t) \rightarrow \sigma(t) = \text{relaxation response.}$$



- The form of the response functions is described by the material constitutive model

Newtonian fluids as linear materials

Shear resistance in fluids

- By defn, fluids cannot sustain a shear stress at rest. However, they can generate a shear stress during flow (i.e. they are viscous)
- For Newtonian fluids, the shear rate $\dot{\gamma}$ and shear stress τ are related by

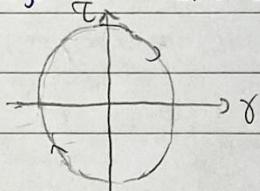
$$\tau = \eta \dot{\gamma}$$

where η is the viscosity of the fluid.

- Newtonian fluids exhibit time-dependent and linear behaviour.

Cycles of deformation on a fluid sample

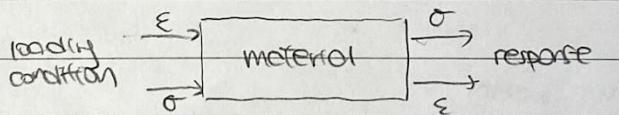
- If we consider cycles of deformation - e.g. $\gamma = \gamma_0 \sin(\omega t)$, then $\tau = \eta \omega \gamma_0 \cos(\omega t)$
- The resulting cycles enclose a large area on stress-strain curves \rightarrow they are dissipative



Linear systems

Constitutive equation as a linear system

- Considering ϵ and σ as input or output of a dynamical system, we assume that their relationship takes the form of a linear system



$$a_0\epsilon + a_1 \frac{d\epsilon}{dt} + \dots + a_m \frac{d^m\epsilon}{dt^m} = b_0\sigma + b_1 \frac{d\sigma}{dt} + \dots + b_n \frac{d^n\sigma}{dt^n}$$

* Here we use $\epsilon, \sigma, \dot{\epsilon}, \ddot{\epsilon}$, but we can apply this to shear - $\gamma, \tau, \dot{\gamma}$.

- This is valid if the material's response is linear - if inputs $\epsilon_1(t), \epsilon_2(t)$ give outputs $\sigma_1(t), \sigma_2(t)$ respectively, then the input $a\epsilon_1(t) + b\epsilon_2(t)$ should give an output $a\sigma_1(t) + b\sigma_2(t)$.
- In practice, the system is also time-invariant, i.e. the material properties do not evolve significantly over time
- The material also needs to be causal, i.e. present input cannot influence past response

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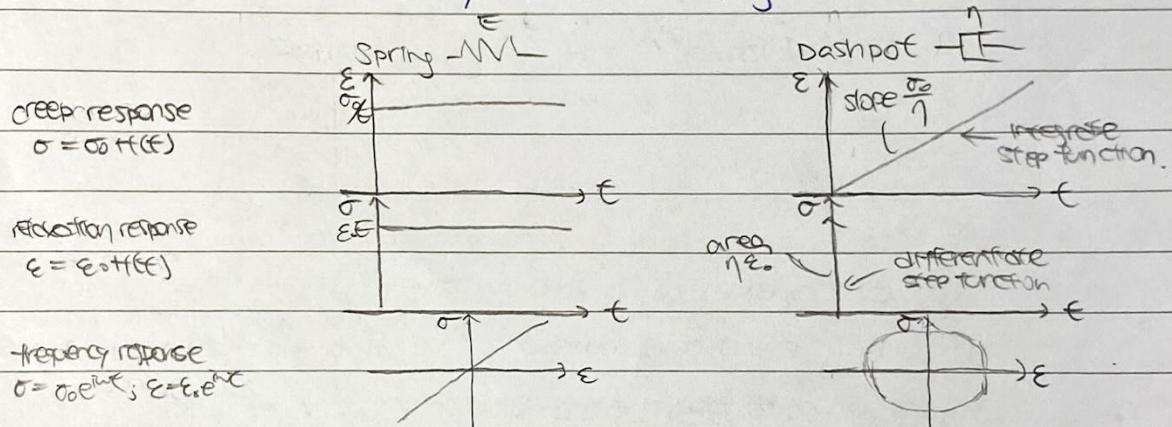
Working with linear-viscoelastic models

- Start from an abstract representation of a material in terms of springs/dashpots \rightarrow DE
- To predict the response to a controlled $\sigma(t)/\epsilon(t)$, solve the DE for $\epsilon(t)$ by substituting $\sigma(t)/\epsilon(t)$.
- For harmonic inputs, solve using the $e^{j\omega t}$ method.
- For periodic inputs, use Fourier series and the harmonic responses.
- For arbitrary inputs, use convolution

Common linear viscoelastic models

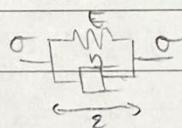
Linear viscoelastic models

- In a viscoelastic model, we combine parameters characterizing elastic components (spring) and viscous components (dashpot) to account for the relationship between stress and strain
- The creep, relaxation and frequency responses for a spring and dashpot are as follows:



Kelvin-Voigt model.

- The Kelvin-Voigt model (KV) is as follows.



$$\sigma = E\epsilon + \eta \dot{\epsilon}$$

$$\sigma = E(\epsilon + \tau \dot{\epsilon})$$

$$\tau = \frac{1}{E}$$

- Creep response - $\sigma(t) = \sigma_0 H(t)$

$$\tau \dot{\epsilon} + \epsilon = \frac{\sigma_0}{E} H(t)$$

$$\therefore \epsilon(t) = \frac{\sigma_0}{E} (1 - \exp(-t/\tau))$$

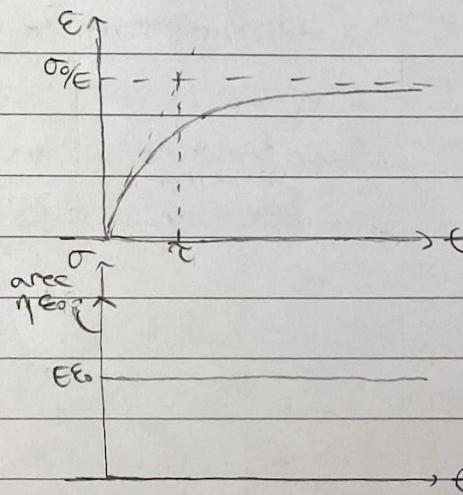
$$\text{and } \tau(t) = \frac{\sigma(t)}{\sigma_0} = \frac{1}{E} (1 - \exp(-t/\tau))$$

- Relaxation response - $\epsilon(t) = \epsilon_0 H(t)$

$$\sigma(t) = E(\epsilon_0 H(t) + \tau \frac{d}{dt} \epsilon_0 H(t))$$

$$\therefore \sigma(t) = E \epsilon_0 (H(t) + \tau f(t))$$

$$\text{and } E f(t) = \frac{\sigma(t)}{\epsilon_0} = E(H(t) + \tau f(t))$$



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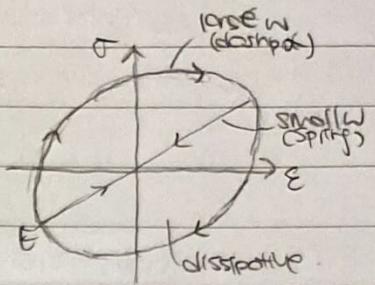
- Frequency response — $\sigma(t) = \sigma_0 e^{i\omega t}$; $\epsilon(t) = \epsilon_0 e^{i\omega t}$

$$\sigma_0 e^{i\omega t} = E \epsilon_0 e^{i\omega t} + i\eta \epsilon_0 e^{i\omega t}$$

$$\frac{\sigma}{\epsilon} = E + i\eta$$

small ω (slow oscillations / long timescale) : $\frac{\sigma}{\epsilon} \propto E$

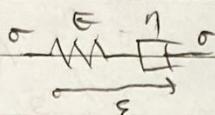
large ω (fast oscillations / short timescale) : $\frac{\sigma}{\epsilon} \approx i\eta$



- The Kelvin model is good for modelling materials that behave like liquids when deformed quickly but behave like solids at long timescales (e.g. jelly, hydrogels)

Maxwell model

- The Maxwell model is as follows



$$\sigma = E\epsilon_1 + \eta \dot{\epsilon}_2 \rightarrow \dot{\epsilon}_1 = \frac{\sigma}{E}; \dot{\epsilon}_2 = \frac{\sigma}{\eta}$$

$$\dot{\epsilon} = \frac{\dot{\epsilon}_1}{E} + \frac{\sigma}{\eta} \rightarrow \eta \dot{\epsilon} = \tau \dot{\sigma} + \sigma$$

$$\tau = \frac{1}{\eta}$$

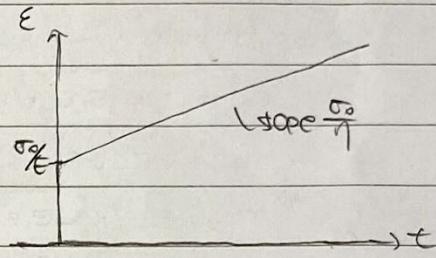
- Creep response — $\sigma(t) = \sigma_0 H(t)$

$$\eta \dot{\epsilon} = \tau \dot{\sigma} (\sigma_0 H(t)) + \sigma_0 H(t)$$

$$\dot{\epsilon}(t) = \frac{\sigma_0}{E} \dot{\sigma}(t) + \frac{\sigma_0}{\eta} H(t)$$

$$\therefore \epsilon(t) = \frac{\sigma_0}{E} H(t) + \frac{\sigma_0}{\eta} t$$

$$\text{and } J(t) = \frac{\epsilon(t)}{\sigma_0} = \frac{1}{E} H(t) + \frac{1}{\eta} t$$



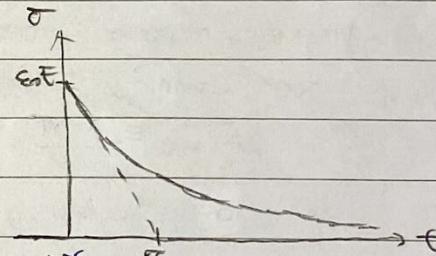
- Relaxation response — $\epsilon(t) = \epsilon_0 H(t)$

$$\frac{d}{dt}(\epsilon_0 H(t)) = \tau \dot{\sigma} + \sigma$$

$$\epsilon_0 \delta(t) = \tau \dot{\sigma} + \sigma$$

$$\therefore \sigma(t) = \epsilon_0 E \exp(-\frac{t}{\tau})$$

$$\text{and } E(t) = \frac{\sigma(t)}{\epsilon_0} = E \exp(-\frac{t}{\tau})$$



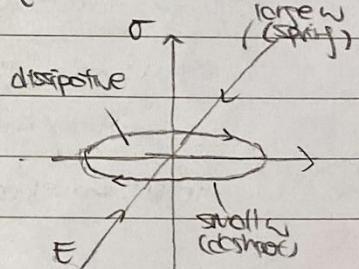
- Frequency response — $\sigma(t) = \sigma_0 e^{i\omega t}$; $\epsilon(t) = \epsilon_0 e^{i\omega t}$

$$i\omega \eta \epsilon_0 e^{i\omega t} = i\omega \tau \sigma_0 e^{i\omega t} + \sigma_0 \dot{\sigma} e^{i\omega t}$$

$$\frac{\sigma}{\epsilon} = \frac{i\omega \eta}{1 + i\omega \tau} = \frac{\omega^2 \eta \tau}{1 + \omega^2 \tau^2} + i \frac{\omega \eta}{1 + \omega^2 \tau^2}$$

small ω (slow oscillations / long timescale) : $\frac{\sigma}{\epsilon} \approx i\omega \eta$

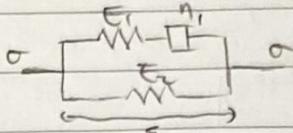
large ω (fast oscillations / short timescale) : $\frac{\sigma}{\epsilon} \approx \frac{\eta}{\tau} = E$



- The Maxwell model is good for modelling materials that behave like solids when deformed quickly but behave like liquids at long timescales (e.g. silly putty)

Standard linear solid model (SLS)

- The standard linear solid model is as follows:



$$\sigma = \sigma_1 + \sigma_2 \quad [**]$$

$$\sigma_1 + \eta_1 \dot{\sigma} = \eta_1 \dot{\varepsilon} \text{ (top)} ; \quad \sigma_2 = E_2 \varepsilon \text{ (bottom)}$$

$$[**] \leftarrow \tau_1 \dot{\varepsilon} \dot{\sigma} [**] : \quad \sigma + \tau_1 \dot{\sigma} = \sigma_1 + \tau_1 \dot{\sigma}_1 + \sigma_2 + \tau_1 \dot{\sigma}_2$$

$$= \eta_1 \dot{\varepsilon} + E_2 \varepsilon + \tau_1 \dot{\varepsilon}_2 \dot{\varepsilon}$$

$$= E_2 \left[\varepsilon + \left(\frac{\eta_1}{E_2} + \tau_1 \right) \dot{\varepsilon} \right]$$

$$\boxed{\sigma + \tau_1 \dot{\sigma} = E_2 \left(\varepsilon + \frac{\eta_1}{E_2} \dot{\varepsilon} \right)} \quad \rightarrow \tau_1^* = \frac{\eta_1}{E_2} + \tau_1 = \tau_1 \left(\frac{E_1}{E_2} + 1 \right), \quad \tau_1 = \frac{\eta_1}{E_1}$$

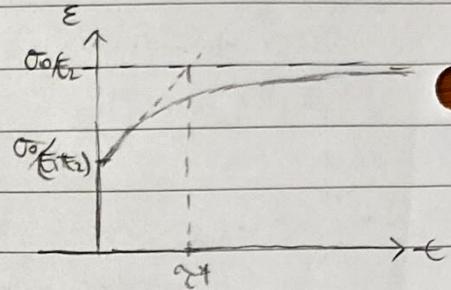
- creep response — $\sigma(t) = \sigma_0 H(t)$

$$\tau_1^* \dot{\varepsilon} + \varepsilon = \frac{\sigma_0}{E_2} H(t) + \frac{\sigma_0 \tau_1^*}{E_2} f(t)$$

$$\therefore \sigma(t) = \frac{\sigma_0}{E_2} (1 - e^{-\frac{t}{\tau_1^*}}) + \frac{\sigma_0 \tau_1^*}{E_2} \cdot \frac{1}{\tau_1^*} e^{-\frac{t}{\tau_1^*}}$$

$$\boxed{\varepsilon(t) = \sigma_0 \left(\frac{1}{E_2} - \frac{E_1}{E_2(E_1+E_2)} e^{-\frac{t}{\tau_1^*}} \right)}$$

$$\therefore J(t) = \frac{1}{E_2} - \frac{E_1}{E_2(E_1+E_2)} e^{-\frac{t}{\tau_1^*}}$$



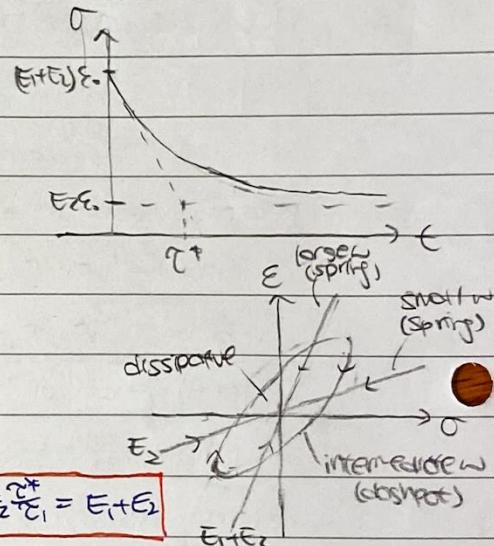
- relaxation response — $\varepsilon(t) = \varepsilon_0 H(t)$,

$$\tau_1 \dot{\sigma} + \sigma = \varepsilon_0 E_2 H(t) + \varepsilon_0 E_2 \tau_1^* f(t)$$

$$\therefore \sigma(t) = \varepsilon_0 E_2 (1 - e^{-\frac{t}{\tau_1}}) + \varepsilon_0 E_2 \tau_1^* \cdot \frac{1}{\tau_1} e^{-\frac{t}{\tau_1}}$$

$$\boxed{\sigma(t) = \varepsilon_0 (E_2 + E_1 e^{-\frac{t}{\tau_1}})}$$

$$\therefore \varepsilon(t) = E_2 + E_1 e^{-\frac{t}{\tau_1}}$$



- frequency response — $\sigma(t) = \sigma_0 e^{i\omega t}, \quad \varepsilon(t) = \varepsilon_0 e^{i\omega t}$

$$\sigma_0 e^{i\omega t} + i\omega \tau_1 \sigma_0 e^{i\omega t} = E_2 (\varepsilon_0 e^{i\omega t} + i\omega \tau_1^* \varepsilon_0 e^{i\omega t})$$

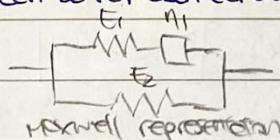
$$\boxed{\frac{\sigma}{\varepsilon} = E_2 \frac{1 + i\omega \tau_1}{1 + i\omega \tau_1^*}}$$

$$\text{small } \omega \text{ (slow oscillations / long timescale)} : \quad \boxed{\frac{\sigma}{\varepsilon} \approx E_2}$$

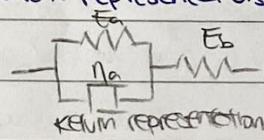
$$\text{large } \omega \text{ (fast oscillations / short timescale)} : \quad \boxed{\frac{\sigma}{\varepsilon} \approx E_2 \tau_1^* = E_1 + E_2}$$

- The SLS model is the simplest model that describes both creep and stress relaxation. It is good for modelling materials that exhibit both phenomena (e.g., thermoplastics below T_g).

- The SLS model can be represented using 2 equivalent representations



Maxwell representation



Kelvin representation

By equating the creep responses, we find the 2 representations are related by

$$J(t) = \frac{1}{E_2} - \frac{E_1}{E_2(E_1+E_2)} e^{-\frac{t}{\tau_1}} \quad J(t) = \frac{1}{E_1} + \frac{1}{E_1} (1 - e^{-\frac{t}{\tau_1}})$$

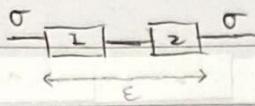
$$\rightarrow E_a = \frac{E_2}{E_1} (E_1 + E_2); \quad E_b = E_1 + E_2; \quad \eta_a = \eta_1 \left(\frac{E_1 + E_2}{E_1} \right)^2$$

* To prove 2 representations are equivalent, we consider their DEs, creep responses $J(t)$ or relaxation responses $E(t)$.

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Graphical approach for deducing response of a mechanical circuit.

- Components in series



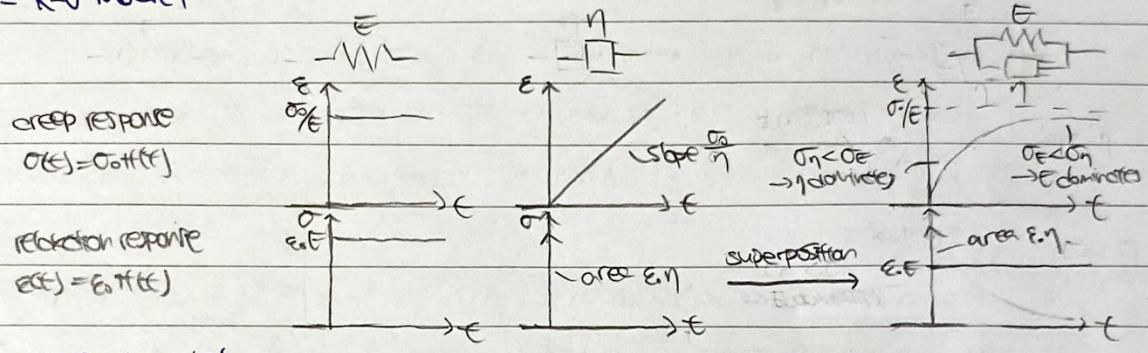
$$\hookrightarrow \text{strains add } \epsilon = \epsilon_1 + \epsilon_2$$

$$\hookrightarrow \text{equal stresses } \sigma = \sigma_1 = \sigma_2$$

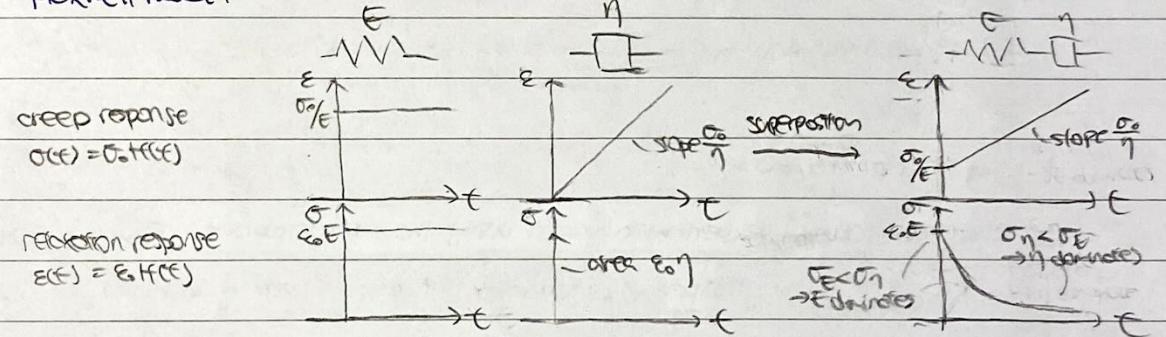
- We can superpose the responses if the stresses/strains add.

- If we have equal stress/strain, we need to consider the component that dominates the response (the component w/ a smaller value of stress/strain dominates)

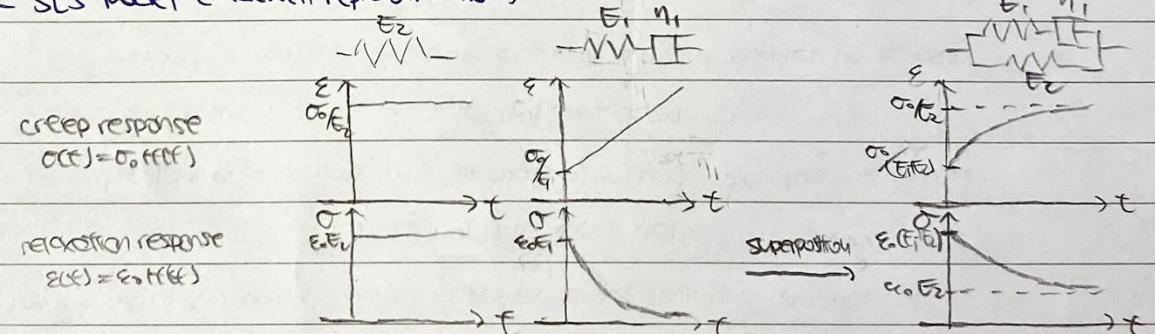
- K-U model



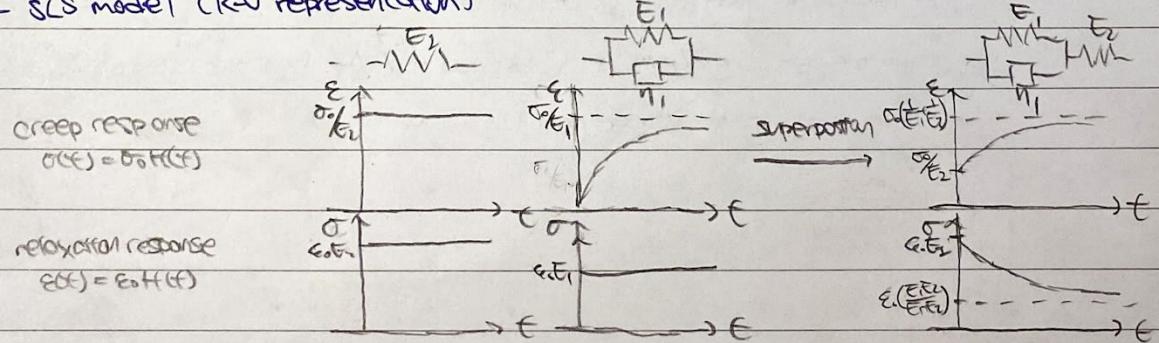
- Maxwell model



- SLS model (Maxwell representation)



- SLS model (K-U representation)



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Measuring and exploiting data

Common methods

① Tensile testing machines

- When using a tensile testing machine, the material is clamped to two grips, one of which is connected to a position controlled arm able to impose set displacement & record load.
- 3-pt. bending tests can be performed by using appropriate grips and sample geometry.
- This instrument is used to study the low-frequency response, through ramps or by holding deformation or stress.

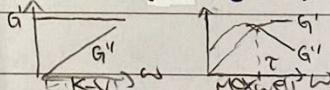
② Dynamical mechanical analyser (DMA)

- The DMA can impose cyclic deformations to small samples, and study their response as a function of frequency and temperature. (The DMA outputs the frequency response of the material at a given temp. and strain amplitude)
- Using different sample geometries, tensile, bending and shear deformations can be studied.

③ Rheometer

- The rheometer imposes an oscillating shear deformation on a liquid sample confined between two surfaces so the frequency response can be measured.
- The measurements are limited to shear deformations/stresses only, and not solids.

From data to model

- We can model a material from its creep, relaxation or frequency response.
- Creep response : Jump in deformation at $t=0 \rightarrow$ spring in series
steady strain rate after transient \rightarrow dashpot in series
- Relaxation response : Jump in stress at $t=0 \rightarrow$ spring in parallel
defo function at $t=0 \rightarrow$ dashpot in parallel.
- Frequency response : Consider separately the storage modulus ($G = \text{Re}\{\frac{\sigma}{\epsilon}(w)\}$) and the loss modulus ($G'' = \text{Im}\{\frac{\sigma}{\epsilon}(w)\}$).
- In general, we can find the parameters of the components in the model by considering what happens at small/large values of t/w .
- For the frequency response, we could check if the parameters are consistent by considering the intersection of the storage and loss moduli