Energy, power, delta functions and Fourier series.

Energy, power and delta functions

Energy and power

- The total energy content of a signal $f(t)$ is defined as.

$$\boxed{E_f = \lim_{T\to\infty} \int_{-T/2}^{T/2} |f(t)|^2 dt}$$

- The average power content of a signal $f(t)$ is defined as.

$$\boxed{P_f = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt}.$$

(this is the average over the time interval T).

* Usually only one of energy or power is defined.

Delta functions

- The delta function $\delta(t)$ can be defined in the following ways.

(i) $\delta(t) = \lim_{\varepsilon\to 0} f_1(t;\varepsilon)$, $f_1(t;\varepsilon) = \frac{1}{\varepsilon}$ for $t \in [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$. 0 o/w.

(ii). $\delta(t) = \lim_{\varepsilon\to 0} f_2(t;\varepsilon)$, $f_2(t;\varepsilon) = \frac{\varepsilon}{\varepsilon^2\pi^2 + t^2}$

(iii) $\delta(t) = \lim_{a\to\infty} f_3(t;a)$, $f_3(t;a) = \frac{\sin(at)}{\pi t}$

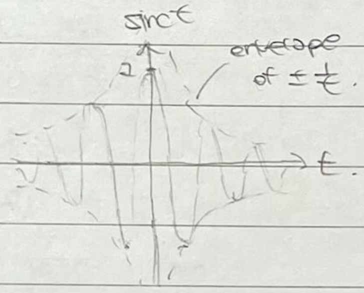
and have the following properties:

$$\delta(t) = 0 \quad \text{for} \quad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1.$$

$$\int_{-\infty}^{\infty} g(t)\delta(t-t_0)\, dt = g(t_0).$$

Sinc function.



- The sinc function is defined as

$$\boxed{\text{sinc}(x) = \frac{\sin x}{x}}$$

- Its integral from $-\infty$ to ∞ is

$$\boxed{\int_{-\infty}^{\infty} \text{sinc}(x)\, dx = \pi}.$$

Consider the function $I(b) = \int_0^\infty \frac{\sin x}{x} e^{-bx} dx$. $\rightarrow \int_{-\infty}^{\infty} \text{sinc}(x)\, dx = 2I(0)$.

$$\frac{dI}{db} = \int_0^\infty \frac{\partial}{\partial b} \frac{\sin x}{x} e^{-bx} dx$$

Feynman's technique of differentiating under the integral sign

$$= -\int_0^\infty \sin x\, e^{-bx} dx$$

$$= -\frac{1}{b^2+1}$$

$$\therefore I(b) = -\int \frac{1}{b^2+1} db$$

$$= -\tan^{-1}(b) + c.$$

As $b\to\infty$, $I(b) = \int_0^\infty \frac{\sin x}{x} e^{-bx} dx \to 0$, so $0 = -\frac{\pi}{2} + c \rightarrow c = \frac{\pi}{2}$.

$$\therefore I(b) = -\tan^{-1}(b) + \frac{\pi}{2}.$$

$$\int_{-\infty}^{\infty} \text{sinc}(x)\, dx = 2[-\tan^{-1}(0) + \tfrac{\pi}{2}] = \pi$$

A useful integral.

- Consider the integral I

$$I = \int_{-\infty}^{\infty} \exp(j\omega t)\,d\omega = \lim_{A\to\infty} \int_{-A}^{A} \exp(j\omega t)\,d\omega$$
$$= \lim_{A\to\infty} \left[\frac{e^{j\omega t}}{jt}\right]_{-A}^{A}$$
$$= \lim_{A\to\infty} \left(2\frac{\sin(At)}{t}\right)$$
$$= \lim_{A\to\infty} 2\pi f_3(t;A)$$
$$\therefore \boxed{\int_{-\infty}^{\infty} \exp(j\omega t)\,d\omega = 2\pi\delta(t)}$$

Fourier series.

Real and complex Fourier series.

- Any function, $g(t)$, which is periodic in the interval $[t_0, t_0+T]$ has a real or complex Fourier series representation.

Real:
$$g(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n\cos\left(\frac{2\pi n}{T}t\right) + b_n\sin\left(\frac{2\pi n}{T}t\right)\right\}$$
where $a_n = \frac{2}{T}\int_{t_0}^{t_0+T} g(t)\cos\left(\frac{2\pi n}{T}t\right)dt$, $\quad b_n = \frac{2}{T}\int_{t_0}^{t_0+T} g(t)\sin\left(\frac{2\pi n}{T}t\right)dt$

Complex:
$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}$$
where $c_n = \frac{1}{T}\int_{t_0}^{t_0+T} g(t)e^{-j\frac{2\pi n}{T}t}\,dt$.

- By considering the expression for c_n in real/imaginary parts,
$$c_n = \frac{1}{2T}\int_{t_0}^{t_0+T} g(t)\cos\left(\frac{2\pi n}{T}t\right)dt - \frac{j}{2T}\int_{t_0}^{t_0+T} g(t)\sin\left(\frac{2\pi n}{T}t\right)dt$$
$$= \tfrac{1}{2}a_n - \tfrac{1}{2}jb_n.$$

$\therefore$ For $n \geq 0$, and real $g(t)$,
$$\left[\begin{array}{l} \boxed{2c_n = a_n - jb_n} \\ \boxed{2c_n^* = a_n + jb_n} \end{array}\right. \quad \leftrightarrow \quad \left[\begin{array}{l} \boxed{a_n = c_n^* + c_n} \\ \boxed{jb_n = c_n^* - c_n} \end{array}\right.$$

- Also considering the expression for c_n,
$$c_n^* = \left[\frac{1}{T}\int_{t_0}^{t_0+T} g(t)e^{-j\frac{2\pi n}{T}t}dt\right]^* = \frac{1}{T}\int_{t_0}^{t_0+T} g(t)e^{j\frac{2\pi n}{T}t}dt = \frac{1}{T}\int_{t_0}^{t_0+T} g(t)e^{-j\frac{2\pi(-n)}{T}t}dt = c_{-n}$$
i.e. $\boxed{c_{-n} = c_n^*}$

- The modulus of c_n is easily obtained in terms of the real coefficients.
$$\boxed{2|c_n| = \sqrt{a_n^2 + b_n^2}}$$

Scaling, stretching and shifting.

- consider modifying $g(t)$ by scaling the amplitude by a factor of a, shifting it along the axis by b and changing the period to $T' = \beta T$. The modified signal $g_1(t)$ is.

$$g_1(t) = a\,g\left(\frac{t-b}{\beta}\right).$$

- Its Fourier series is hence

$$\begin{aligned} g_1(t) &= a \sum_{n=-\infty}^{\infty} c_n e^{j\omega_0 n\left(\frac{t-b}{\beta}\right)} \\ &= \sum_{n=-\infty}^{\infty} \left\{ a c_n e^{-\frac{j\omega_0 nb}{\beta}} \right\} e^{\frac{j\omega_0 nt}{\beta}} \\ &= \sum_{n=-\infty}^{\infty} c_n' e^{j\omega_0' nt} \end{aligned}$$

where the new coefficients and fundamental frequency are.

$$\boxed{c_n' = a c_n e^{-j\omega_0' nb}} \quad ; \quad \boxed{\omega_0' = \frac{\omega_0}{\beta}}$$

Differentiation and integration.

- suppose $g(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$, $h(t) = \frac{dg(t)}{dt}$, $f(t) = \int g(t)\,dt$.

(i) $h(t) = \frac{d}{dt} g(t) = \sum_{n=-\infty}^{\infty} \{ jn\omega_0 c_n \} e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} c_n' e^{jn\omega_0 t}$

where $\boxed{c_n' = jn\omega_0 c_n}$.

(ii) $f(t) = \int g(t)\,dt = c_0 t + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left\{ \frac{c_n}{jn\omega_0} \right\} e^{jn\omega_0 t} + K = c_0 t + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c_n' e^{jn\omega_0 t} + K.$

where $\boxed{c_n' = \frac{1}{jn\omega_0} c_n}$, $\boxed{K = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,dt}$

* (ii) can only be a Fourier series if $c_0 = 0 \rightarrow f(t) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c_n' e^{jn\omega_0 t} + K.$

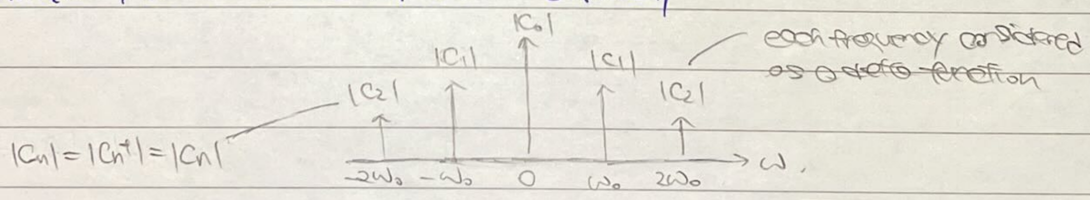
Interpretation of Fourier coefficients

- Consider the complex Fourier series for a periodic signal $g(t)$.

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

where each complex exponential and its conjugate together can be regarded as a pure frequency component of the signal, w/ frequency $n\omega_0$.

- This frequency content can be represented graphically



- The amplitudes r_i of the harmonics are defined by rewriting the real Fourier series as follows.

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \{ a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \} = r_0 + \sum_{n=1}^{\infty} r_n \sin(n\omega_0 t + \phi_n)$$

so $r_0 = a_0/2$ and $r_n = \sqrt{a_n^2 + b_n^2}$.

- Since $2|c_n| = \sqrt{a_n^2 + b_n^2}$, $\boxed{r_n = 2|c_n| = |c_n| + |c_{-n}|}$

For $n = 0$, $\boxed{r_0 = c_0}$.

Parseval's theorem for Fourier series.

- The magnitude of the complex coefficients can be related to the average power of the signal $g(t)$ over one period.

$$\boxed{P_g = \frac{1}{T}\int_0^T |g(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2}$$

* This is Parseval's Theorem for Fourier series.

The Fourier Transform and its properties.

The Fourier Transform

Mathematical formulation of the Fourier Transform

- Consider the complex Fourier series coefficient formula for the function $f(t)$

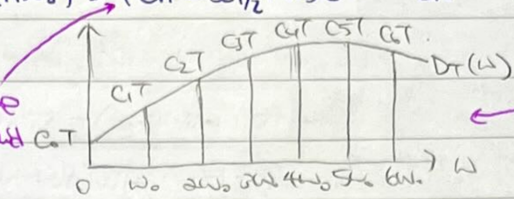
$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jw_0 n t}\,dt \qquad \text{where } w_0 = \frac{2\pi}{T}$$

(Note that c_n is the Fourier series component w/ frequency nw_0)

- Now we can define a function of frequency which equals Tc_n at the pts $w = nw_0$, i.e.

$$D_T(nw_0) = Tc_n = \int_{-T/2}^{T/2} f(t)e^{-jw_0 n t}\,dt \qquad n \in \mathbb{Z}$$

the factor T normalises the integral, o/w the integral would be zero for most functions



the scale factor T doesn't affect the shape of the plot

Another function that satisfies such condition can be obtained by replacing nw_0 w/ a continuous variable w,

$$D_T(w) = \int_{-T/2}^{T/2} f(t)e^{-jwt}\,dt$$

- As we increase the period T, w_0 decreases, and the discrete pts $D_T(nw_0)$ define the whole function $D_T(w)$ for all values of w.
- The Fourier Transform $F(w)$ is obtained as the limit of $D_T(w)$ as $T \to \infty$, i.e.

$$F(w) = \lim_{T\to\infty} D_T(w) = \lim_{T\to\infty}\int_{-T/2}^{T/2} f(t)e^{-jwt}\,dt$$

$$\boxed{F(w) = \int_{-\infty}^{\infty} f(t)\,e^{-jwt}\,dt}$$

The inverse Fourier Transform (IFT).

- It is possible to invert the Fourier Transform to determine uniquely the signal $f(t)$ from its Fourier Transform $F(w)$.
- Starting w/ the forward Fourier Transform formula,

$$F(w) = \int_{-\infty}^{\infty} f(t')e^{-jwt'}\,dt'$$

Now multiply $F(w)$ by e^{jwt} and integrate wrt w.

$$\begin{aligned}
I &= \int_{-\infty}^{\infty} F(w)\,e^{jwt}\,dw \\
&= \int_{w=-\infty}^{\infty}\int_{t'=-\infty}^{\infty} f(t')e^{-jwt'}\,dt'\,e^{jwt}\,dw \\
&= \int_{t'=-\infty}^{\infty} f(t')\int_{w=-\infty}^{\infty} e^{jw(t-t')}\,dw\,dt' && \text{[swap order of integration]} \\
&= \int_{-\infty}^{\infty} f(t')\cdot 2\pi\,\delta(t-t')\,dt' && \left[\int_{-\infty}^{\infty} e^{jw\tau}\,dw = 2\pi\delta(\tau)\right] \\
&= 2\pi f(t) && \text{[sifting theorem]}
\end{aligned}$$

$$\therefore\ f(t) = \frac{I}{2\pi} \quad \rightarrow \quad \boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)e^{jwt}\,dw}.$$

Important Fourier Transforms.

① Complex exponential.

$$f(t) = e^{j\omega_0 t}$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t} dt = 2\pi\,\delta(\omega_0 - \omega)$$

sifting thm.

$*\ \delta(\omega_0-\omega) = \delta(\omega-\omega_0)$

$$\therefore \boxed{F(\omega) = 2\pi\delta(\omega-\omega_0)}$$

② Cosine

$$f(t) = \cos(\omega_0 t) = \tfrac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

$$\boxed{F(\omega) = \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]}$$

③ Sine

linearity of Fourier transform

$$f(t) = \sin(\omega_0 t) = -\tfrac{j}{2}(e^{j\omega_0 t} - e^{-j\omega_0 t})$$

$$\boxed{F(\omega) = j\pi[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]}$$

④ Delta function

$$f(t) = a\delta(t)$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} a\delta(t) e^{-j\omega t} dt$$

$$\therefore \boxed{F(\omega) = a}$$

⑤ DC offset

$*\ \delta(\omega) = \delta(-\omega)$.

$$f(t) = a$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = a\int_{-\infty}^{\infty} e^{-j\omega t} dt = a \cdot 2\pi\delta(\omega)$$

$$\boxed{F(\omega) = 2\pi a\delta(\omega)}$$

⑥ Rectangular pulse

$\mathrm{rect}(t) = \begin{cases} 1 & -1/2 \le t \le 1/2 \\ 0 & \text{o/w.} \end{cases}$

$$f(t) = b\,\mathrm{rect}\left(\frac{t}{T}\right)$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \int_{-T/2}^{T/2} b e^{-j\omega t} dt = -\frac{1}{j\omega}\left[e^{-j\omega t}\right]_{-T/2}^{T/2}$$

$$\boxed{F(\omega) = bT\,\mathrm{sinc}\left(\frac{\omega T}{2}\right)}$$

⑦ Gaussian.

$$f(t) = e^{-a^2t^2}$$

$$\frac{df(t)}{dt} = -2a^2 t e^{-a^2t^2} = -2a^2 t f(t).$$

$FT\{tf(t)\} = j\dfrac{dF(\omega)}{d\omega}$.

$$j\omega F(\omega) = -2a^2 j\frac{dF(\omega)}{d\omega}$$

differentiation

$$\int_0^{\omega} \omega\, d\omega = -2a^2 \int_0^{\omega} \frac{1}{F(\omega)} dF(\omega)$$

$$\frac{\omega^2}{2} = -2a^2(\ln F(\omega) - \ln F(0))$$

$\dfrac{\sqrt{\pi}}{a}$.

$F(0) = \int_{-\infty}^{\infty} f(t) \cancel{e^{-j0t}}^{\,1} dt = \dfrac{\sqrt{\pi}}{a}$.

$$\boxed{F(\omega) = \frac{\sqrt{\pi}}{a} e^{-\frac{\omega^2}{4a^2}}}$$

the Fourier Transform of a Gaussian shape is itself a Gaussian shape

Properties of the Fourier Transform

Linearity of Fourier Transforms

- Consider the Fourier Transform of the linear combination of $f(t)$ and $g(t)$.

$$FT[\alpha f(t) + \beta g(t)] = \int_{-\infty}^{\infty} [\alpha f(t) + \beta g(t)]\, e^{-j\omega t} dt$$

$$= \alpha \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt + \beta \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt$$

$$\boxed{FT[\alpha f(t) + \beta g(t)] = \alpha FT[f(t)] + \beta FT[g(t)]}$$

i.e. the Fourier Transform is a linear operation

Derivatives.

- Let the Fourier Transform of $f(t)$ be $F(\omega)$, then $\frac{df}{dt}$ is given by

$$\frac{df}{dt} = \frac{d}{dt}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) \frac{d}{dt} e^{j\omega t} d\omega.$$

$$\therefore \frac{df}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega F(\omega) e^{j\omega t} d\omega$$

$$\rightarrow \boxed{FT\left[\frac{df}{dt}\right] = j\omega F(\omega)}$$

- In general, the Fourier Transform of $\frac{d^n f}{dt^n}$ is given by

$$\boxed{FT\left[\frac{d^n f}{dt^n}\right] = (j\omega)^n F(\omega)}$$

Multiplication by t.

- Consider the derivative of $F(\omega)$ wrt ω.

$$\frac{d}{d\omega} F(\omega) = \frac{d}{d\omega}\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} f(t) \frac{d}{d\omega} e^{-j\omega t} dt$$

$$= -j \int_{-\infty}^{\infty} t f(t) e^{-j\omega t} dt$$

$$= -j\, FT[t f(t)]$$

$$\therefore \boxed{FT[t f(t)] = j \frac{dF(\omega)}{d\omega}}$$

Similarity theorem (Time scaling).

- Let the Fourier Transform of $f(t)$ be $F(\omega)$, then .

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

Substitute αt for t,

$$f(\alpha t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega \alpha t} d\omega.$$

also true for $\alpha < 0$.

For $\alpha > 0$, substitute $\omega' = \alpha\omega$, so $d\omega = \frac{1}{\alpha} d\omega'$

$$f(\alpha t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\alpha} F\left(\frac{\omega'}{\alpha}\right) e^{j\omega' t} d\omega'$$

$$\therefore \boxed{FT[f(\alpha t)] = \frac{1}{\alpha} F\left(\frac{\omega'}{\alpha}\right)}$$

i.e. if we stretch in the time domain, we contract in the frequency domain and vice versa.

Heisenberg-Gabor principle.

- The Heisenberg-Gabor principle states if any function $f(t)$ has time duration T, and its Fourier Transform $F(\omega)$ has frequency bandwidth B, then.

time-bandwidth product. $\boxed{TB \geq 1}$

- This formalises the idea that stretching in the domain results in contracting in the frequency domain ($FT[f(\alpha t)] = \frac{1}{\alpha} F(\frac{\omega}{\alpha})$).

Frequency shift and modulation.

- Let the Fourier Transform of $f(t)$ be $F(\omega)$, then

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

substitute $\omega - \omega_0$ for ω,

$$F(\omega - \omega_0) = \int_{-\infty}^{\infty} f(t) e^{-j(\omega - \omega_0)t} dt = \int_{-\infty}^{\infty} e^{j\omega_0 t} f(t) e^{-j\omega t} dt$$

$$\therefore \boxed{FT[e^{j\omega_0 t} f(t)] = F(\omega - \omega_0)}$$

- since $f(t)\cos(\omega_0 t) = \frac{1}{2}[e^{j\omega_0 t} f(t) + e^{-j\omega_0 t} f(t)]$, by linearity,

$$\boxed{FT[\cos(\omega_0 t) f(t)] = \frac{1}{2} F(\omega - \omega_0) + \frac{1}{2} F(\omega + \omega_0)}$$

i.e. the spectrum of $f(t)$ is separated into two parts, each half of its original strength and shifted along the ω axis by $\pm\omega_0$.

Time shift

- Let the Fourier Transform of $f(t)$ be $F(\omega)$, then

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega.$$

Substitute $t - t_0$ for t,

$$f(t - t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega(t - t_0)} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j\omega t_0} F(\omega) e^{j\omega t} d\omega$$

$$\therefore \boxed{FT[f(t - t_0)] = e^{-j\omega t_0} F(\omega)}$$

Duality

- Let the Fourier Transform of $f(t)$ be $g(\omega)$, then.

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\omega) e^{j\omega t} d\omega$$

Substitute $-\omega'$ for t, and rename ω w/ t',

$$f(-\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(t') e^{jt'(-\omega')} dt'$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(t') e^{-j\omega t'} dt'$$

$$\therefore \boxed{FT[g(t)] = 2\pi f(-\omega)}$$

i.e. if we have one Fourier Transform pair $FT[f(t)] = g(\omega)$

then we automatically have the dual Fourier Transform pair. $FT[g(t)] = 2\pi f(-\omega)$

The multiplication theorem and Parseval's theorem

- consider two functions $f_1(t)$ and $f_2(t)$ w/ FTs $F_1(\omega)$ and $F_2(\omega)$. The integral of the product of f_1 and f_2^* is given by

$$\int_{-\infty}^{\infty} f_1(t) f_2^*(t)\,dt = \int_{-\infty}^{\infty} f_1(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F_2(\omega) e^{j\omega t} d\omega\right]^* dt$$
$$= \int_{-\infty}^{\infty} f_1(t) \frac{1}{2\pi}\int_{-\infty}^{\infty} F_2^*(\omega) e^{-j\omega t} d\omega\, dt$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F_2^*(\omega)\int_{-\infty}^{\infty} f_1(t) e^{-j\omega t} dt\, d\omega$$

$$\boxed{\int_{-\infty}^{\infty} f_1(t) f_2^*(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\omega) F_2^*(\omega)\,d\omega}$$ [Multiplication theorem]

- Putting $f_2(t) = f_1(t)$ in the above leads to Parseval's theorem.

$$\boxed{\int_{-\infty}^{\infty} |f_1(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F_1(\omega)|^2 d\omega}.$$ [Parseval's theorem]

i.e. the energy of of signal can be found either by integrating in the time domain or in the frequency domain, and that energy in time domain ∝ energy in frequency domain.

- The energy of a time function $f(t)$ between frequencies ω_1 and ω_2 is given by

$$\boxed{E = \frac{1}{2\pi}\int_{-\omega_2}^{-\omega_1} |F(\omega)|^2 d\omega + \frac{1}{2\pi}\int_{\omega_1}^{\omega_2} |F(\omega)|^2 d\omega}$$

For real valued functions $f(t)$,

$$\boxed{E = 2\cdot\frac{1}{2\pi}\int_{\omega_1}^{\omega_2} |F(\omega)|^2 d\omega}.$$

for real-valued $f(t)$, $F^*(\omega) = F(-\omega)$, so $|F(-\omega)|^2 = |F^*(\omega)|^2 = |F(\omega)|^2$

Convolution and multiplication.

- Let $h(t)$ be the convolution of $f(t)$ and $g(t)$, and their FTs be $H(\omega)$, $F(\omega)$, $G(\omega)$,

$$h(t) = f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau)\,d\tau$$

The Fourier transform of $h(t)$ is therefore.

$$H(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau) g(t-\tau)\,d\tau\, e^{-j\omega t} dt$$

using $u = t-\tau$, $dt = du$,

$$H(\omega) = \int_{-\infty}^{\infty} f(\tau)\int_{-\infty}^{\infty} g(u)\, e^{-j\omega(u+\tau)} du\, d\tau$$
$$= \int_{-\infty}^{\infty} f(\tau) e^{-j\omega\tau} d\tau \int_{-\infty}^{\infty} g(u) e^{-j\omega u} du$$
$$= F(\omega) G(\omega).$$

$$\therefore \boxed{FT[f(t) * g(t)] = FT[f(t)]\,FT[g(t)]}$$

- By duality, it can be shown that

$$\boxed{FT[f(t)\cdot g(t)] = \frac{1}{2\pi} FT[f(t)] * FT[g(t)]}$$

i.e. convolution in one domain is equivalent to multiplication in the other domain.

Symmetry.

- Consider the conjugate of the Fourier Transform of a real function $f(t)$, $F^*(\omega)$

$$F^*(\omega) = \left[\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt\right]^*$$
$$= \int_{-\infty}^{\infty} f^*(t) e^{j\omega t} dt$$
$$= \int_{-\infty}^{\infty} f(t) e^{j\omega t} dt.$$

Substituting $-\omega$ for ω in the definition of the forward Fourier Transform, we have.

$$F(-\omega) = \int_{-\infty}^{\infty} f(t) e^{-j(-\omega)t} dt$$
$$\therefore \boxed{F(-\omega) = F^*(\omega)}$$

- It can also be shown that
  - ↳ if $f(t)$ is real and even, then $F(\omega)$ is real and even
  - ↳ if $f(t)$ is real and odd, then $F(\omega)$ is imaginary and odd.

Fourier Transform and Laplace Transform.

- The definitions for the Fourier and Laplace Transforms are as follows:
  - ↳ Fourier: $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$
  - ↳ Laplace: $\int_0^{\infty} f(t) e^{-st} dt$
- The integrand is the same if we set $s = j\omega$, but the limits of integration are different.
- For a function $f(t)$ that is zero for $t<0$, we have that

$$\boxed{F(\omega) = \bar{f}(j\omega)}$$

provided that both integrals exist.

- For a causal LTI system, w/ impulse response $g(t)$, the frequency response is given by

$$\boxed{\text{Frequency response} = G(\omega) = \bar{g}(j\omega)}$$

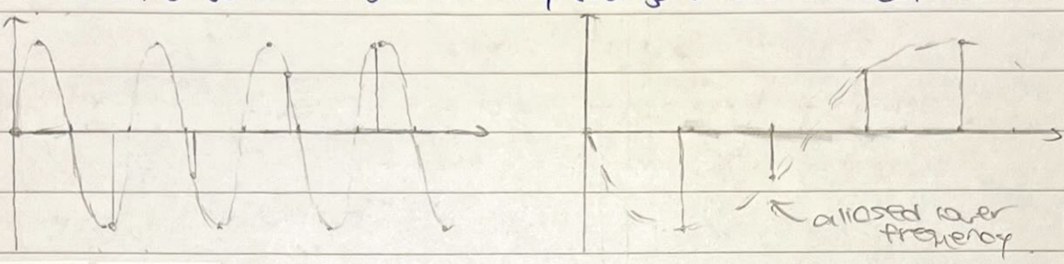
- DEs can be solved using either Fourier Transforms or Laplace Transforms.
  - ↳ Fourier is better suited to steady-state analysis
  - ↳ Laplace is better suited to problems w/ BC at $t=0$.

Sampling and the discrete Fourier Transform

Sampling theory

Digital sampling.

- Suppose a continuous time signal is given by $f(t)$, $-\infty < t < +\infty$.
  choose a sampling interval T and read off the value of $f(t)$ at $t = nT$, $n \in \mathbb{Z}$.
  The obtained values $f(nT)$ are the sampled version of $f(t)$.
- we need to choose the sampling interval T carefully, since
  ↳ Too large a value of T will mean loss of detail from $f(t)$ (aliasing).
  ↳ Too small a value means unnecessary storage of over-detailed / redundant data.



Mathematical representation of the sampled signal

- we can find a mathematical representation of the sampled signal using a train of delta functions.
- First, consider a single sample at $t = nT$. We multiply the sample $f(nT)$ and multiply it by a delta function centred at $t = nT$ to get

$$f(nT)\,\delta(t-nT).$$

since $\delta(t-nT) = 0$ except at $t = nT$, we can rewrite this as

$$f(t)\,\delta(t-nT).$$

The whole sampled signal is simply given by the sum of all such samples,

$$f_s(t) = \sum_{n=-\infty}^{\infty} f(t)\,\delta(t-nT)$$

$$= f(t)\sum_{n=-\infty}^{\infty} \delta(t-nT)$$

$$\boxed{f_s(t) = f(t)\,\delta_p(t)}$$

train of impulses, w/ period T.

As $\delta_p(t)$ is a periodic function, it can be represented as a Fourier series

$$\delta_p(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

where $\omega_0 = 2\pi/T$ and $c_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta_p(t) e^{-jn\omega_0 t}\,dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jn\omega_0 t} = \frac{1}{T}\ \forall n.$

sampling frequency

$$\therefore\ f_s(t) = f(t)\sum_{n=-\infty}^{\infty}\frac{1}{T} e^{jn\omega_0 t}$$

$$\boxed{f_s(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} f(t) e^{jn\omega_0 t}}$$

- $f_s(t)$ is a continuous-time signal which contains only the sampled data information $f(nT)$, and is zero elsewhere.
- $f_s(t)$ is only a conceptual version of the sampled signal — in no way are we implying that there are infinities / infinite energy in a real sampled signal.

Discrete-time Fourier Transform (DTFT).

- The Fourier Transform of the sampled signal $f_s(t)$, is the discrete-time Fourier Transform, (DTFT), $F_s(\omega)$. It can be written as follows:

$$F_s(\omega) = \int_{-\infty}^{\infty} f_s(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty}\left[\sum_{n=-\infty}^{\infty} f(t)\delta(t-nT)\right] e^{-j\omega t}\, dt$$

$$= \sum_{n=-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t) e^{-j\omega t}\delta(t-nT)\, dt\right]$$

$$\boxed{F_s(\omega) = \sum_{n=-\infty}^{\infty} f(nT)\, e^{-jn\omega T}}$$

- The DTFT gives the frequency content of the ideal sampled signal $f_s(t)$.

Nyquist frequency and reconstruction.

- Consider the ideal sampled signal $f_s(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} f(t) e^{jn\omega_0 t}$. The Fourier Transform of each term in the summation is given by

$$FT\left[f(t) e^{jn\omega_0 t}\right] = F(\omega - n\omega_0) \quad \text{[frequency shift]}$$

where $F(\omega)$ is the FT of $f(t)$.

By linearity of the FT, the FT of the whole ideal sampled signal $f_s(t)$, $F_s(\omega)$ is given by

$$\boxed{F_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} F(\omega - n\omega_0)}$$

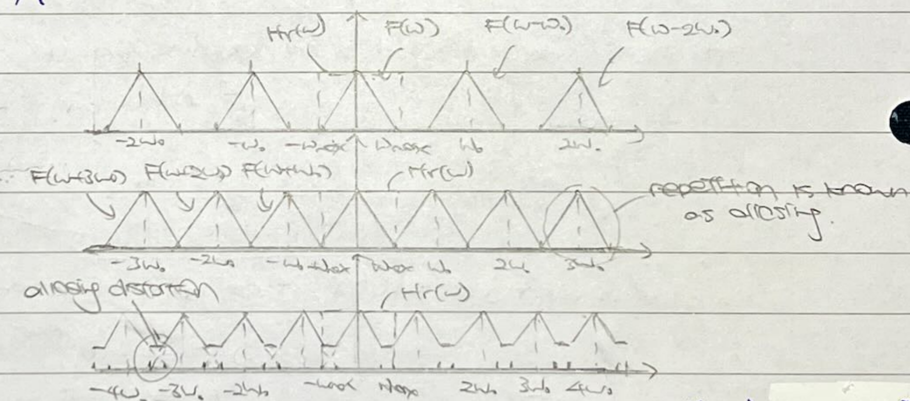
i.e. the FT of the idealised sampled signal is simply $\frac{1}{T}\times$ the FT of the continuous signal repeated every integer multiple of the sampling frequency and summed together

- Consider the sampled spectrum $F_s(\omega)$ of a signal w/ BW ω_{max} and for various sampling frequency $\omega_0 = 2\pi/T$:

(a) $\omega_0 = 3\omega_{max}$.

(b) $\omega_0 = 2\omega_{max}$.

(c) $\omega_0 = 1.6\omega_{max}$.



- For cases (a), (b), we can apply a filter w/ frequency response $H_r(\omega)$ to reconstruct the original spectrum $F(\omega)$ perfectly; for case (c), the original spectrum $F(\omega)$ is not properly constructed. (only possible when there is no overlap b/wn the periodic repetitions of $F(\omega)$).
- The Nyquist sampling theorem states that for a signal $f(t)$ that has a max. frequency content (BW ω_{max}, it is possible to reconstruct $f(t)$ perfectly from its sampled version $f_s(t)$, provided the sampling frequency is at least $\omega_0 = 2\omega_{max}$.
- The min. sampling frequency is the Nyquist frequency ω_{nyq}.

$$\boxed{\omega_{nyq} = 2\omega_{max}}$$

Ideal reconstruction filter.

- The ideal filter frequency response for perfect reconstruction is the rectangular pulse

$$H_r(\omega) = T \cdot rect\left(\frac{\omega}{2\omega_{max}}\right)$$

- The impulse response of the filter is then the IFT of $H_r(\omega)$, which is

$$h_r(\omega) = \frac{\omega_{max}T}{\pi} sinc(\omega_{max}t)$$

- If we are sampling at exactly the Nyquist frequency ($\omega_{max} = \frac{\omega_0}{2}$), then.

$$h_r(t) = \frac{\omega_0 T}{2\pi} sinc\left(\frac{\omega_0 t}{2}\right) = sinc\left(\frac{\omega_0 t}{2}\right)$$

Since multiplication in the frequency domain is equivalent to convolution in the time domain,

$$F(\omega) = H_r(\omega) F_s(\omega) \quad \overset{FT}{\leftrightarrow} \quad f(t) = h_r(t) * f_s(t)$$

$$\therefore f(t) = \int_{-\infty}^{\infty} f_s(\tau) sinc\left(\frac{\omega_0(t-\tau)}{2}\right) d\tau$$

$$= \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} f(nT)\delta(\tau - nT)\right] sinc\left(\frac{\omega_0(t-\tau)}{2}\right) d\tau$$

$$= \sum_{n=-\infty}^{\infty} f(nT) \int_{-\infty}^{\infty} \delta(\tau - nT) sinc\left(\frac{\omega_0(t-\tau)}{2}\right) d\tau$$

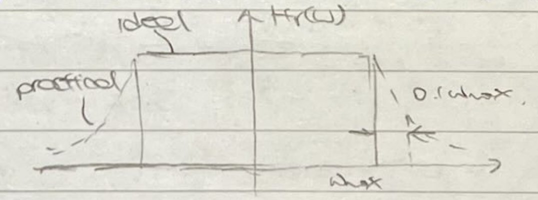
$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} f(nT) sinc\left(\frac{\pi}{T}(t-nT)\right)}$$

The above expression is an exact interpolation formula for determining $f(t)$ from its samples $f(nT)$.

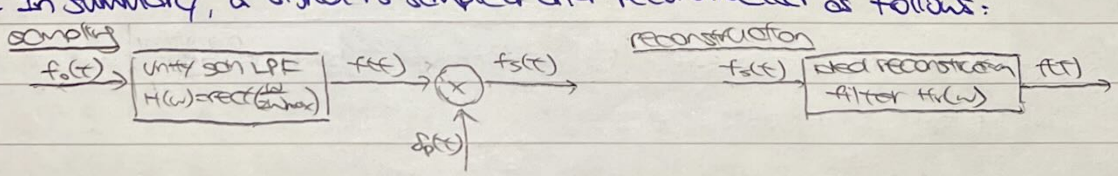
Practical considerations.

- we must determine the max. frequency component ω_{max} present in a signal before processing.
- To eliminate the aliasing effects of high frequency noise or unwanted high signal frequencies, we first filter the data w/ a unity gain LPF (frequency response $H(\omega) = rect(\frac{\omega}{2\omega_{max}})$)
- Then, perform sampling at the Nyquist rate $\omega_0 = 2\omega_{max}$ to get digital samples $f(nT)$.
- Reconstruct the signal by passing the sampled signal through the ideal reconstruction filter.

(In practice, we cannot exactly implement the ideal reconstruction filter $H_r(\omega)$ — we must allow ~10% extra signal BW for the transition band of the filter.)



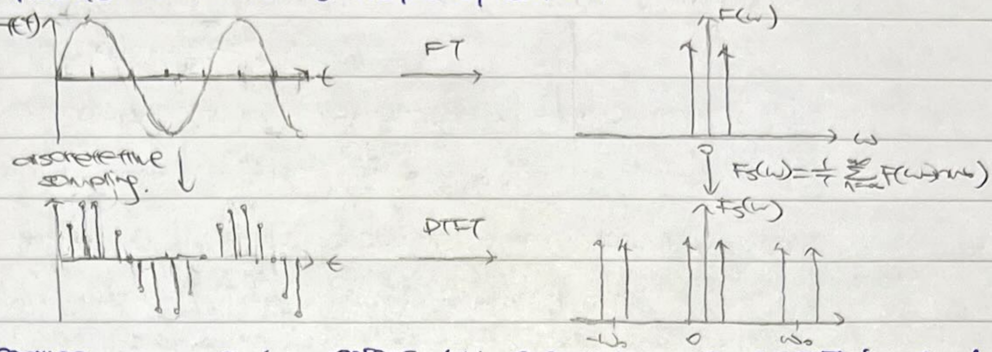
- In summary, a signal is sampled and reconstructed as follows:



The discrete Fourier transform

The discrete Fourier Transform (DFT).

- Given the sampled values of a signal $f(t)$, i.e. $\{f(-\infty), \dots f(-T), f(0), f(T) \dots f(\infty)\}$, the DTFT shows how to determine the frequency content.



- However in practice, there are two reasons why we can't compute the DTFT of a signal.
  - ↳ 1) We don't have access to all of the data pts from $-\infty$ to ∞ (even if we did, the computing time would be infinite)
  - ↳ 2) $F_S(\omega)$ is defined over the continuous range of frequencies from $-\infty$ to ∞. → cannot calculate and store $F_S(\omega)$ for all frequency values [uncountably infinite]
- The discrete Fourier transform (DFT) can bypass such practical difficulties.
  - ↳ 1) Only consider data pts within some finite range — say $t = 0, T, 2T \dots (N-1)T$.

    we assume $f(nT) = 0$ outside the range $n = 0, \dots, N-1$

    The DTFT is modified to.

$$F_S^N(\omega) = \sum_{n=0}^{N-1} f(nT) e^{-jn\omega T}$$

    Provided we have enough of the original continuous signal between $t=0$ and $t=nT$, the truncated DTFT will be similar to the full DTFT
  - ↳ 2) Calculate only over a finite grid of frequencies. As $F_S(\omega)$ is periodic w/ period ω_0, we don't need to calculate values beyond the Nyquist frequency. Evaluating $F_S(\omega)$ at N evenly spaced frequency values $\omega = 0, \frac{\omega_0}{N}, \frac{2\omega_0}{N} \dots \frac{(N-1)\omega_0}{N}$, we have.

$$F_S^N\left(\frac{m\omega_0}{N}\right) = \sum_{n=0}^{N-1} f(nT) e^{-jn\left(\frac{m\omega_0}{N}\right)T}$$
$$= \sum_{n=0}^{N-1} f(nT)^{-jnm2\pi/N}$$

- The above is the DFT. Defining $F_m = F_S^N\left(\frac{m\omega_0}{N}\right)$ and $f_n = f(nT)$, we have

$$\boxed{F_m = \sum_{n=0}^{N-1} f_n e^{-jnm2\pi/N}}$$

- We can interpret component F_k as the frequency content of the signal at frequency $k\frac{\omega_0}{N}$ / $k\omega$.
- The DFT is periodic, i.e.

$$\boxed{F_k = F_{k+N}}$$

- For real signals, using the defn for DFT, we can show that

$$\boxed{F_{-k} = F_k^*}$$

$$\left(F_{-k} = \sum_{n=0}^{N-1} f_n e^{-jn(-k)2\pi/N} = \sum_{n=0}^{N-1} f_n e^{jnk2\pi/N} \; ; \; F_k^* = \sum_{n=0}^{N-1} f_n^* \left(e^{-jnk2\pi/N}\right)^* = \sum_{n=0}^{N-1} f_n e^{jnk2\pi/N}\right)$$

Inverse discrete Fourier Transform (IDFT).

- Consider multiplying the forward DFT by $e^{jkm2\pi/N}$ and sum over $m=0$ to $N-1$.

$$\sum_{m=0}^{N-1} F_m e^{jkm2\pi/N} = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} f_n e^{-jnm2\pi/N} e^{jkm2\pi/N}$$
$$= \sum_{n=0}^{N-1} f_n \sum_{m=0}^{N-1} e^{j(k-n)m2\pi/N}$$

The inner summation over m is a GP w/ common ratio $r = e^{j(k-n)2\pi/N}$, so

$$\sum_{m=0}^{N-1} e^{j(k-n)2\pi/N} = \frac{1-e^{j(k-n)2\pi}}{1-e^{j(k-n)2\pi/N}} = \begin{cases} 0 & \text{if } k \neq n \\ N & \text{if } k = n \end{cases}$$

$$\therefore \sum_{m=0}^{N-1} F_m e^{jkm2\pi/N} = \sum_{n \neq k}^{N-1} f_n \cdot 0 + f_k \cdot N$$

- The expression for the inverse discrete Fourier Transform is therefore given by

$$\boxed{f_k = \frac{1}{N}\sum_{m=0}^{N-1} F_m e^{jkm2\pi/N}}.$$

Other considerations

- Fast algorithms – fast methods exist for computing the DFT, known generically as the Fast Fourier Transform (FFT).
- The effects of truncating a data sequence (when taking a finite set of samples) might introduce some discontinuities at the start and end of the data frame. → high frequency components. → this can be avoided by applying a window to the sampled data.

Summary of various Fourier Transforms.

① Fourier Transform (FT) and Inverse Fourier Transform (IFT)

$$\text{FT: } F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \qquad \text{IFT: } f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

② Discrete time Fourier Transform (DTFT)

$$\text{DTFT: } F_s(\omega) = \sum_{n=-\infty}^{\infty} f(nT) e^{-jn\omega T} = \frac{1}{T}\sum_{n=-\infty}^{\infty} F(\omega - n\omega_0)$$

③ Discrete Fourier Transform (DFT) and Inverse Discrete Fourier Transform (IDFT)

$$\text{DFT: } F_m = \sum_{n=0}^{N-1} f_n e^{-jnm2\pi/N} \quad [0 \le m \le N-1] \qquad \text{IDFT: } f_n = \frac{1}{N}\sum_{m=0}^{N-1} F_m e^{jnm2\pi/N} \quad [0 \le n \le N-1]$$