Z-Transform

Discrete time signals and systems

Discrete time signals

- Discrete time: Discrete in time only (sampled but not quantised)
  Digital: Discrete in time and amplitude (sampled and quantised)
- Discrete time signals $\underline{x} = (\ldots x_{-1}, x_0, x_1 \ldots)$ typically have real/complex elements, and could be right-sided, left-sided, two-sided or finite-length
  - ↳ Right-sided: starts at x_m and $x_k = 0$ for $k < m$ [start at x_0: semi-infinite]
  - ↳ Left-sided: ends at x_m and $x_k = 0$ for $k > m$
  - ↳ Two-sided: no highest/lowest index for non-zero values
  - ↳ Finite-length: starts at x_m and ends at x_{m+l-1} (length l).
- A discrete time signal may be a sampled continuous signal, in which case, w/ reference to the original continuous (sample $x_k = x(kT)$), we can denote the discrete time signal as

$$\underline{x} = (x(0), x(kT), x(2kT) \ldots)$$

where T is the sampling interval

- We could treat a discrete time signal as a sequence of no./ has a normalised sampling freq. of 1Hz.

Discrete time systems

- A discrete time system takes discrete time signal i/ps and produces discrete time signal o/ps.
- The defn of linear time-invariant (LTI) system L extends from continuous to discrete time
  If for i/p $\{u_k\}$ we get o/p $\{y_k\}$ and for i/p $\{u'_k\}$ we get o/p $\{y'_k\}$, then
  - ↳ Linear: $L(\alpha\{u_k\} + \beta\{u'_k\}) = \alpha L(\{u_k\}) + \beta L(\{u'_k\}) = \alpha\{y_k\} + \beta\{y'_k\}$
  - ↳ Time-invariant: $L(\{u_{k+d}\}) = \{y_{k+d}\}$
- Discrete LTI systems are governed by difference eqns w/ const. coeffs.

Discrete convolution

- The convolution of the sequences $\{a_k\}$ and $\{b_k\}$ is the sequence $\{y_k\}$.

$$y_k = \sum_{n=0}^{\infty} a_n b_{k-n}$$

- The summation limits can be simplified for the special cases:
  - ↳ Semi-infinite sequence $\{a_k\}_{k\geq 0}$ and $\{b_k\}_{k\geq 0}$ $\quad y_k = \sum_{n=0}^{k} a_n b_{k-n}$
  - ↳ Finite-length sequence $\{a_k\}_{k=0}^{l-1}$ $\quad y_k = \sum_{n=0}^{\min(l,k)} a_n b_{k-n}$

Convolution and polynomial multiplication

- We can map semi-infinite sequences to power series, i.e. $\{a_k\} \to a(D)$

$$a(D) = a_0 + a_1 D^1 + a_2 D^2 + \dots$$

The convolution of semi-infinite sequences is equivalent to multiplication of power series.

$$a(D)b(D) = \left(\sum_{i=0}^{\infty} a_i D^i\right)\left(\sum_{j=0}^{\infty} b_j D^j\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_i b_j D^{i+j} = \sum_{i=0}^{\infty}\sum_{k=i}^{\infty} a_i b_{k-i} D^k \quad (\text{sub } k=i+j)$$

$$= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} a_i b_{k-i}\right) D^k = \sum_{k=0}^{\infty} y_k D^k = y(D) \quad \to \boxed{a(D)b(D) = y(D)}$$

where we used $\sum_{i=0}^{\infty}\sum_{k=i}^{\infty} \leftrightarrow \sum_{k=0}^{\infty}\sum_{i=0}^{k}$

i \ j	0	1	2
0	(0,0)	(0,1)	(0,2) ←
1	↑	(1,1)	(1,2) ←

- D is a dummy variable — we never plan to assign a value to the variable → formal power series.
- As we do not assign values to the dummy variable, we do not need to worry about the radius of convergence (sufficient for mapping convolutions to multiplications)

* Z-transform is not a formal power series as we do assign a complex value to the variable z.

Z-transform

Z-transform

- For a semi-infinite sequence $\{a_k\}_{k\geq 0}$, the one-sided Z-transform is defined as

$$\boxed{\mathcal{Z}[\{a_k\}] = A(z) = \sum_{k=0}^{\infty} a_k z^{-k}}$$

For a two-sided sequence $\{a_k\}_{k=-\infty}^{\infty}$, the two sided Z-transform is defined as

$$\mathcal{Z}[\{a_k\}] = A(z) = \sum_{k=-\infty}^{\infty} a_k z^{-k}$$

- For the two-sided z-transform, the sequence $\{a_k\}$ is not uniquely determined by $A(z)$, so we must also specify the radius of convergence (not an issue for one-sided transforms)

Properties of Z-transform

① Linearity: $\boxed{\mathcal{Z}[\alpha\{a_k\} + \beta\{b_k\}] = \alpha\mathcal{Z}[\{a_k\}] + \beta\mathcal{Z}[\{b_k\}]}$, $\boxed{\mathcal{Z}[\frac{d}{d\alpha}\{a_k\}] = \frac{d}{d\alpha}\mathcal{Z}[\{a_k\}]}$

- The z-transform only involves doing linear operations on the sequence values,

$$\mathcal{Z}[\alpha\{a_k\} + \beta\{b_k\}] = \sum_{k=0}^{\infty}(\alpha a_k + \beta b_k) z^{-k} = \alpha\sum_{k=0}^{\infty} a_k z^{-k} + \beta\sum_{k=0}^{\infty} b_k z^{-k} = \alpha\mathcal{Z}[\{a_k\}] + \beta\mathcal{Z}[\{b_k\}]$$

It also follows that if the sequence data involves only a parameter α, i.e. $a_k = f_k(\alpha)$,

$$\mathcal{Z}[\tfrac{d}{d\alpha}\{a_k\}] = \sum_{k=0}^{\infty}(\tfrac{d}{d\alpha}a_k) z^{-k} = \tfrac{d}{d\alpha}\sum_{k=0}^{\infty} a_k z^{-k} = \tfrac{d}{d\alpha}\mathcal{Z}[\{a_k\}]$$

② Conjugate symmetry: $\boxed{A(\bar{z}) = \overline{A(z)}}$

- The z-transform has conjugate symmetry for real-valued signals $\{a_k\}$.

$$A(\bar{z}) = \sum_{k=0}^{\infty} a_k \bar{z}^{-k} = \overline{\left(\sum_{k=0}^{\infty} a_k z^{-k}\right)} = \overline{A(z)}$$

- For the two-sided transform, there is the reverse property that for $\{a_k\}$ s.t. $a_{-k} = \bar{a}_k$, the z-transform of $\{a_k\}$, $A(z)$ is real on the unit circle.

(e.g. $a_k = a + bj$, $a_{-k} = a - bj$ and $z = e^{j\theta}$, $A(e^{j\theta}) = a_0 + \sum_{k=1}^{\infty}(a+bj)e^{jk\theta} + (a-bj)e^{-jk\theta} = a_0 + 2\sum_{k=1}^{\infty}(a\cos(k\theta) + b\sin(k\theta))$)

③ Time delay / time advance: $\mathcal{Z}[\{a_{k-1}\}] = z^{-1}A(z) + a_{-1}$; $\mathcal{Z}[\{a_{k+1}\}] = zA(z) - za_0$

- Defining the time delay operation: $\{a_k\} \to \{a_{k-1}\}$.

$$\mathcal{Z}[\{a_{k-1}\}] = \sum_{k=0}^{\infty} a_{k-1} z^{-k} = z^{-1} \sum_{k=0}^{\infty} a_{k-1} z^{-(k-1)} = z^{-1} \sum_{k=0}^{\infty} a_k z^{-k} + a_{-1} = z^{-1}A(z) + a_{-1}$$

Assuming $a_{-1} = 0$, then $\mathcal{Z}[\{a_{k-1}\}] = z^{-1}A(z)$.

- Defining the time advance operation: $\{a_k\} \to \{a_{k+1}\}$

$$\mathcal{Z}[\{a_{k+1}\}] = \sum_{k=0}^{\infty} a_{k+1} z^{-k} = z \sum_{k=0}^{\infty} a_{k+1} z^{-(k+1)} = z \sum_{k=0}^{\infty} a_k z^{-k} - za_0 = zA(z) - za_0$$

* The time-shift properties can be applied repeatedly for larger shifts.

④ Convolution: $\mathcal{Z}[\{a_k\} * \{b_k\}] = A(z)B(z)$

- Consider the z-transform of the convolution btwn $\{a_k\}, \{b_k\}$.

$$\mathcal{Z}[\{a_k\} * \{b_k\}] = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} a_i b_{k-i} \right) z^{-k} = \sum_{k=0}^{\infty} \sum_{i=0}^{k} a_i b_{k-i} z^{-k} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_i b_j z^{-(i+j)} = \left(\sum_{i=0}^{\infty} a_i z^{-i} \right) \left(\sum_{j=0}^{\infty} b_j z^{-j} \right) = A(z)B(z)$$

where we used $\sum_{k=0}^{\infty} \sum_{i=0}^{k} \leftrightarrow \sum_{i=0}^{\infty} \sum_{k=i}^{\infty}$ as before

Common z-transforms

① Geometric: $\mathcal{Z}[q^k] = \frac{1}{1-qz^{-1}}$; $\mathcal{Z}[1] = \frac{1}{1-z^{-1}}$; $\mathcal{Z}[e^{ak}] = \frac{1}{1-e^{a}z^{-1}}$

- Consider the geometric sequence $a_k = q^k$.

$$\mathcal{Z}[q^k] = \sum_{k=0}^{\infty} q^k z^{-k} = \sum_{k=0}^{\infty} (qz^{-1})^k = \frac{1}{1-qz^{-1}}$$

$q = 1 \to$ step sequence, $a_k = 1$ $\quad \mathcal{Z}[1] = \frac{1}{1-z^{-1}}$

$q = e^{a} \to$ exponential sequence, $a_k = e^{ak}$ $\quad \mathcal{Z}[e^{ak}] = \frac{1}{1-e^{a}z^{-1}}$

② Sine/cosine: $\mathcal{Z}[\sin(ak)] = \frac{z \sin a}{1 - 2z\cos a + z^{-2}}$; $\mathcal{Z}[\cos(ak)] = \frac{1 - z\cos a}{1 - 2z\cos a + z^{-2}}$

- Applying linearity, we can get the z-transform for the sine/cosine sequences.

$a_k = \sin(ak)$: $\mathcal{Z}[\sin(ak)] = \mathcal{Z}\left[\frac{1}{2j}(e^{jak} - e^{-jak})\right] = \frac{1}{2j}\left(\frac{1}{1-e^{ja}z^{-1}} - \frac{1}{1-e^{-ja}z^{-1}}\right) = \frac{z^{-1}\sin a}{1 - 2z\cos a + z^{-2}}$

$a_k = \cos(ak)$: $\mathcal{Z}[\cos(ak)] = \mathcal{Z}\left[\frac{1}{2}(e^{jak} + e^{-jak})\right] = \frac{1}{2}\left(\frac{1}{1-e^{ja}z^{-1}} + \frac{1}{1-e^{-ja}z^{-1}}\right) = \frac{1 - z^{-1}\cos a}{1 - 2z\cos a + z^{-2}}$

③ Scale by r^k: $\mathcal{Z}[r^k g_k] = G(r^{-1}z)$

- Scaling a sequence $\{g_k\}$ w/ a geometric sequence r^k, i.e. $a_k = r^k g_k$.

$$\mathcal{Z}[r^k g_k] = \sum_{k=0}^{\infty} r^k g_k z^{-k} = \sum_{k=0}^{\infty} g_k (r^{-1}z)^{-k} = G(r^{-1}z)$$

④ Ramp: $\mathcal{Z}[k] = \frac{z^{-1}}{(1-z^{-1})^2}$

- Differentiating the geometric sequence,

$$\mathcal{Z}[kq^{k-1}] = \mathcal{Z}\left[\frac{d}{dq} q^k\right] = \frac{d}{dq} \mathcal{Z}[q^k] = \frac{d}{dq} \frac{1}{1-qz^{-1}} = \frac{z^{-1}}{(1-qz^{-1})^2}$$

$q = 1 \to$ ramp sequence, $a_k = k$ $\quad \mathcal{Z}[k] = \frac{z^{-1}}{(1-z^{-1})^2}$

Initial value theorem (IVT) and final value theorem (FVT)

- The IVT states that

$$\lim_{z\to\infty} A(z) = a_0$$

↳ $\lim_{z\to\infty} A(z) = \lim_{z\to\infty}\left(\sum_{k=0}^{\infty} a_k z^{-k}\right) = \lim_{z\to\infty}\left(a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots\right) = a_0$

- The FVT states if all the poles of $(z-1)A(z)$ lie strictly within the unit circle,

$$\lim_{z\to 1}(z-1)A(z) = \lim_{k\to\infty} a_k$$

↳ All poles of $A(z)$ are in the unit circle, except possibly a pole at $z=1$.

Assuming distinct poles, $A(z) = \frac{\alpha_0}{1-z^{-1}} + \sum_i \frac{\alpha_i}{1-p_i z^{-1}}$ where $|p_i|<1$

By inverse z-transform, $a_k = \alpha_0 + \sum_i \alpha_i p_i^k$

Taking limits $k\to\infty$, $\lim_{k\to\infty} a_k = \lim_{k\to\infty}\left(\alpha_0 + \sum_i \alpha_i p_i^k\right) = \alpha_0$

Consider $(z-1)A(z) = z(1-z^{-1})A(z) = z\alpha_0 + (z-1)\sum_i \frac{\alpha_i}{1-p_i z^{-1}}$

Taking limits $z\to 1$. $\lim_{z\to 1}(z-1)A(z) = \lim_{z\to 1}\left(z\alpha_0 + (z-1)\sum_i \frac{\alpha_i}{1-p_i z^{-1}}\right) = \alpha_0$

$$\therefore \alpha_0 = \lim_{k\to\infty} a_k = \lim_{z\to 1}(z-1)A(z)$$

Z-transforms and other transforms

Z-transform and Laplace transform

- For continuous time signals $x(t)$, the Laplace transform is defined as

$$\bar{x}(s) = \int_{0^+}^{\infty} x(t) e^{-st}\, dt$$

Analogously, we can define a transform for a discrete time signal $\{x(kT)\}_{k\to 0}$, as

$$\sum_{k=0}^{\infty} x(kT) e^{-skT}$$

If we define $x_k = x(kT)$ and $z = e^{sT}$, we get the z-transform

$$X(z) = \sum_{k=0}^{\infty} x_k z^{-k}$$

- Alternatively, we can obtain the z-transform by taking the Laplace transform of a continuous time representation of a discrete time signal, $x^*(t)$.

$$x^*(t) = \sum_{k=0}^{\infty} x(kT)\,\delta(t-kT)$$

$$\therefore L[x^*(t)] = \int_0^{\infty} \sum_{k=0}^{\infty} x(kT)\,\delta(t-kT)\, e^{-st}\, dt = \sum_{k=0}^{\infty} x(kT) e^{-skT} = \sum_{k=0}^{\infty} x_k z^{-k} = Z[\{x_k\}].$$

Z-transform and DTFT

- consider a band limited signal $x(t)$ where $x(t)=0$ for $t<0$.

The DTFT of the sampled signal is defined as

$$X_\delta(\theta) = \sum_{k=0}^{\infty} x(kT) e^{-jk\theta}$$

where $\theta = \Omega T$ is the normalised frequency.

If we define $x_k = x(kT)$ and $z = e^{j\theta}$, we get the z-transform

$$X(z) = \sum_{k=0}^{\infty} x_k z^{-k}$$

i. $X(z=e^{j\theta}) = X_\delta(\theta)$, i.e. z-transform on the unit circle is the DTFT.

- since the z-transform on the unit circle is the DTFT, we can express the IDTFT as

$$x_k = x(kT) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_\delta(\theta) e^{jk\theta} d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta}) e^{jk\theta} d\theta$$

i.e. we can do an inverse z-transform using $\boxed{x_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta}) e^{jk\theta} d\theta}$

- The z-transform on the unit circle shows the normalised Fourier spectrum

(i.e. normalised frequency $\theta \in [-\pi, \pi]$ and actual frequency $\omega \in [-f_s/2, f_s/2]$.

- For real-valued signals, the conjugate symmetric property states $X(e^{j\theta}) = \overline{X(e^{j\theta})}$

(i) $|X(e^{j\theta})| = |X(e^{j\theta})|$ and (ii) $\angle X(e^{j\theta}) = -\angle X(e^{j\theta})$

Inverse Z-transform

Inverse Z-transform

- The inverse z-transform can be found using

↳ partial fractions ↳ polynomial long division

↳ Taylor expansion. ↳ IDTFT formula.

① Partial fractions

- decompose the rational function into partial fractions and use standard z-transforms

- eg: $A(z) = \frac{z^{-3}}{z^2(z-1)(z-2)}$ Find $\{a_k\}_{k\geq 0}$.

$$A(z) = \frac{z^{-3}}{z^2(z-1)(z-2)} = z^{-2}\left[\frac{2}{z-1} - \frac{1}{z-2}\right] = z^{-3}\left[2\frac{1}{1-z^{-1}} - \frac{1}{1-2z^{-2}}\right] \rightarrow a_k = \begin{cases} 0 & k<3 \\ 2(1)^{k-3} - 2^{k-3} & k\geq 3 \end{cases}$$

② Polynomial long division

- Explicitly compute the coefficients of $z^0, z^{-1}, \ldots$ → first few terms of $\{x_k\}$. [not analytic]

- eg: $A(z) = \frac{z^{-3}}{z^2(z-1)(z-2)}$ Find $\{a_k\}_{k\geq 0}$.

$$A(z) = \frac{z^{-3}}{z^2(z-1)(z-2)} = \frac{1}{z^2}\frac{z^{-3}}{z^2-3z+2}$$

$$z^2-3z+2 \;\overline{)\; z^{-3}} \quad = z^{-1} + 0z^{-2} + 2z^{-3} + 6z^{-4} + \ldots$$

$$-\frac{z^{-3}-3z^{-4}+2z^{-5}}{-2z^{-1}}$$

$$-\frac{-2z^{-1}+6z^{-2}-4z^{-3}}{-6z^{-2}+4z^{-3}}$$

$$\rightarrow A(z) = z^{-2}(z^{-1} - 2z^{-3} - 6z^{-4} + \ldots)$$

$$= z^{-3}(1 - 2z^{-2} - 6z^{-3} + \ldots)$$

$\therefore \{a_k\}_{k\geq 0} = \{0, 0, 0, 1, 0, -2, -6, \ldots\}$.

③ Taylor expansion

- Use the taylor expansion to get a power series → read off the coefficients.

- eg. $A(z)=\ln(1+z^{-1})$ Find $\{a_k\}_{k\geq 0}$.

$$A(z)=\ln(1+z^{-1})=z^{-1}-\tfrac{1}{2}z^{-2}+\tfrac{1}{3}z^{-3}-\tfrac{1}{4}z^{-4}+\ldots \rightarrow \{a_k\}_{k\geq 0}=\{0,1,-1/2,1/3,-1/4\ldots\}$$

④ IDTFT formula.

- Given the z-transform on the unit circle, we can use $a_k=\frac{1}{2\pi}\int_{-\pi}^{\pi}A(e^{j\theta})e^{jk\theta}d\theta$.

- eg: $A(z=e^{j\theta})=\begin{cases}1 & \text{if } \theta\in[-\pi/4,\pi/4]\\ 0 & \text{o/w.}\end{cases}$

$$a_k=\frac{1}{2\pi}\int_{-\pi}^{\pi}A(e^{j\theta})e^{jk\theta}d\theta=\frac{1}{2\pi}\int_{-\pi/4}^{\pi/4}1\cdot e^{jk\theta}d\theta=\frac{1}{\pi k}\frac{1}{2j}\left(e^{jk\pi/4}-e^{-jk\pi/4}\right)=\frac{\sin(k\pi/4)}{\pi k}=\frac{1}{4}\operatorname{sinc}\left(\frac{k}{4}\right).$$

System transfer function

System transfer function

Pulse response of LTI systems

- We define the unit pulse (Kronecker delta function) δ_k in discrete time as

$$\delta_k = \begin{cases} 1 & k=0 \\ 0 & k \neq 0 \end{cases}$$

- If the i/p to a system is the unit pulse input $\{x_k\}_{k\geq 0} = \delta_k = (1, 0, 0, \ldots)$, then the o/p to a system $\{h_k\}_{k\geq 0} = (h_0, h_1, h_2 \ldots)$ is the pulse response of the system.

- For a LTI system w/ pulse response $\{h_k\}_{k\geq 0}$, we can write the general i/p $\{x_k\}_{k\geq 0}$ in the form

$$x_k = \sum_{n=0}^{k} x_n \delta_{k-n}$$

Using the superposition principle, the o/p of the system is thus

$$y_k = \sum_{n=0}^{k} x_n h_{k-n}$$

Z-transfer function.

- The system described by the following difference eqn, subject to zero initial conditions is LTI.

$$y_k + a_1 y_{k-1} + \ldots + a_n y_{k-n} = b_0 x_k + b_1 x_{k-1} + \ldots + b_m x_{k-m} \quad , \quad x_k = y_k = 0 \text{ for } k<0.$$

Taking the Z-transform,

$$Y(z)\left[1 + a_1 z^{-1} + \ldots + a_n z^{-n}\right] = X(z)\left[b_0 + b_1 z^{-1} + \ldots + b_m z^{-m}\right]$$

We define the transfer function $G(z)$ to be

$$G(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_n z^{-n}}$$

Pulse response and Z-transfer function.

- The o/p $\{y_k\}_{k\geq 0}$ of a LTI system can be found by convolving the i/p $\{x_k\}_{k\geq 0}$ w/ the pulse response $\{h_k\}_{k\geq 0}$ of the system

$$y_k = \sum_{n=0}^{k} x_n h_{k-n} \qquad \text{or} \qquad \{y_k\} = \{x_k\} * \{h_k\}.$$

Taking the Z-transform gives

$$Y(z) = H(z) X(z)$$

where $H(z) = \frac{Y(z)}{X(z)}$ is the system transfer.

→ The system transfer function $H(z)$ is the Z-transform of the system pulse response $\{h_k\}_{k\geq 0}$.

$$H(z) = \mathcal{Z}[\{h_k\}]$$

Solution of linear difference equations.

- Take the Z-transform on the linear difference eqn, applying the linearity and time shift property.
- Substitute in any initial conditions and or inputs
- Solve for $Y(z)$ and take the inverse Z-transform to find $\{y_k\}_{k\geq 0}$.

Finite impulse response (FIR) and infinite impulse response (IIR)

- FIR filters have pulse responses that terminate after a finite no. of time steps.

$$\{h_k\}_{k\geq 0} = \{h_0, h_1 \ldots h_n, 0, 0 \ldots\}.$$

IIR filters have pulse responses that do not terminate after a finite no. of time steps

$$\{h_k\}_{k\geq 0} = \{h_0, h_1, h_2 \ldots\}.$$

- FIR systems have transfer functions of the special form

$$G(z) = b_0 + b_1 z^{-1} + \ldots + b_m z^{-m} = \frac{b_0 z^m + b_1 z^{m-1} + b_m}{z^m}.$$

where $a_k = 0$ (no feedback), and all the poles are at $z = 0$.

IIR systems have transfer functions in the general form

$$G(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_m z^{-m}}{1 + a_1 z^{-1} + \ldots + a_n z^{-n}}.$$

where the poles are at arbitrary locations.

Causality

- Discrete time systems whose pulse response $\{h_k\}$ where $h_k = 0$ for $k < 0$ are causal.
- Consider a non-causal pulse response $\{g_k\} = \{\ldots, g_{-1}, g_0, g_1 \ldots\}$.

By convolution, the o/p of the system is

$$y_k = \sum_{n=-\infty}^{\infty} g_n x_{k-n} = \ldots \underbrace{g_{-2} x_{k+2} + g_{-1} x_{k+1}}_{\text{future}} + \underbrace{g_0 x_k + g_1 x_{k-1}}_{\text{past}} + \ldots$$

i.e. the current value of the o/p y_k depends on future values of the i/p $x_{k+1}, x_{k+2} \ldots$

- In causal systems, only the past of an i/p sequence $x_k, x_{k-1} \ldots$ can influence the current o/p y_k.
- The transfer function of causal systems have the form

$$H(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + \ldots$$

For a transfer function w/ the following form to be causal

$$H(z) = \frac{b_0 z^m + b_1 z^{m-1} + \ldots + b_m}{z^n + a_1 z^{n-1} + \ldots + a_n}$$

we require $\boxed{m \leq n}$, o/w we can rewrite the transfer function as

$$H(z) = c_1 z^{m-n} + \ldots + c_{m-n} z + \frac{b_0' z^n + b_1 z^{n-1} + \ldots b_n'}{z^n + a_1' z^{n-1} + \ldots + a_n'}$$

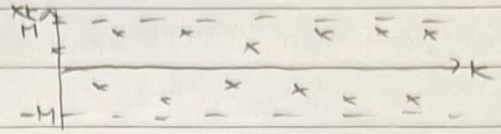
which includes non-causal terms $z, z^2 \ldots z^{m-n}$

+ Note for causal systems, $\lim_{z \to \infty} G(z) = g_0 < \infty$.

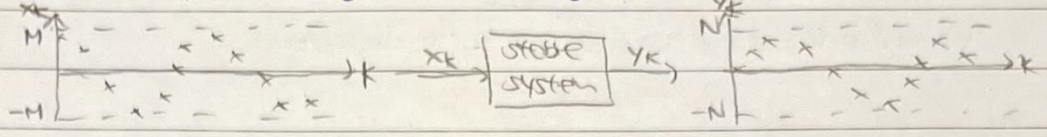
- Systems that operate in function of time and in real time are always causal.
- Examples of non-causal "real world systems" include
  - ↳ Image processing (function of location)
  - ↳ Signal processing system w/ stored/buffered data (not in real time).

Stability

- A signal $\{x_k\}$ is bounded if there exists a +ve const. M s.t. $|x_k| < M \ \forall k$



- A discrete time system is stable if bounded i/ps give bounded o/ps (BIBO stability)



- Let G be a discrete time system w/ a rational transfer function

causal system

$$G(z) = \frac{b(z)}{a(z)} = \frac{b_0 z^m + \dots + b_m}{z^n + a_1 z^{n-1} + \dots + a_n}$$

where $m \le n$ and there are no common factors btwn $n(z)$ and $d(z)$.

- Let the pulse response of G be $\{g_k\}_{k \ge 0}$, then the following are equivalent: (i) G is stable, (ii) all of the roots of $a(z)$, p_i, satisfy $|p_i| < 1$, and (iii) $\sum_{k=0}^{\infty} |g_k|$ is finite.

↳ Proof of (i) → (ii) (using contrapositive, ¬(ii) → ¬(i), i.e. ¬(|pi|<1 ∀i) → G not stable)

- Suppose $p_1, p_2 \dots$ are distinct, then we can decompose G using partial fractions

$$G(z) = \frac{\alpha_1}{1-p_1 z^{-1}} + \dots + \frac{\alpha_n}{1-p_n z^{-1}} \quad \rightarrow \quad g_k = \alpha_1 p_1^k + \dots + \alpha_n p_n^k$$

- Suppose $|p_i| > 1$ for some i, then g_k is unbounded.

∴ A pulse i/p (bounded) gives an unbounded o/p, so G is unstable.

- Suppose we have a pole on the unit circle, i.e. $p_i = e^{j\theta}$ for some i. G can rewritten as

$$G(z) = \frac{\alpha_i}{1-e^{j\theta}z^{-1}} + \tilde{G}(z) \quad \rightarrow \quad g_k = \alpha_i e^{j\theta k} + \dots$$

The term that corresponds to the pole on the unit circle $g_k' = \alpha_i e^{j\theta k}$ → bounded ($|g_k'|$ is const.)

- Now consider the i/p $U(z) = \frac{\alpha_i z^{-1}}{1-e^{j\theta}z^{-1}}$, which corresponds to the sequence $g_{k-1}' \rightarrow$ bounded.

The corresponding o/p is $Y(z) = G(z)U(z) = \left(\frac{\alpha_i}{1-e^{j\theta}z^{-1}} + \tilde{G}(z)\right)\left(\frac{\alpha_i z^{-1}}{1-e^{j\theta}z^{-1}}\right) = \frac{\alpha_i^2 z^{-1}}{(1-e^{j\theta}z^{-1})^2} + \tilde{G}(z)U(z)$

where the first term corresponds to the sequence $y_k' = \alpha_i^2 k e^{j\theta(k-1)} \rightarrow$ unbounded ($|y_k'|$ grows w/ k.)

∴ A bounded i/p gives an unbounded o/p, so G is unstable

⇒ If G is stable, then all of the $a(z)$, p_i, satisfy $|p_i| < 1$.

↳ Proof of (ii) → (iii) (i.e. $|p_i|<1 \ \forall i \rightarrow \sum_{k=0}^{\infty} |g_k| < \infty$)

- Suppose $p_1, p_2 \dots$ are distinct. Then we can decompose G using partial fractions.

$$G(z) = \frac{\alpha_1}{1-p_1 z^{-1}} + \dots + \frac{\alpha_n}{1-p_n z^{-1}} \quad \rightarrow \quad g_k = \alpha_1 p_1^k + \dots + \alpha_n p_n^k$$

$$\therefore \sum_{k=0}^{\infty} |g_k| \le \alpha_1 \sum_{k=0}^{\infty} |p_1|^k + \dots + \alpha_n \sum_{k=0}^{\infty} |p_n|^k$$ where RHS is finite if $|p_i|<1 \ \forall i$

↳ Proof of (iii) → (i) (i.e. $\sum_{k=0}^{\infty} |g_k| < \infty \rightarrow G$ is stable)

- Let $\{x_k\}$ be an arbitrary bounded i/p, i.e. $|x_k| < M$ for $k > 0$. Then the o/p is given by

$$y_k = \sum_{n=0}^{\infty} g_n x_{k-n} \quad \rightarrow \quad |y_k| = \left|\sum_{n=0}^{\infty} g_n x_{k-n}\right| \le \sum_{n=0}^{\infty} |g_n||x_{k-n}| < \sum_{n=0}^{\infty} |g_n| M = M \sum_{n=0}^{\infty} |g_n|$$

∴ Given $\sum_{n=0}^{\infty} |g_n| < \infty$, we have $|y_k| = M \sum_{n=0}^{\infty} |g_n| < \infty$ (bounded o/p).

(We can also prove (i) → (iii) using contrapositive, w/ $u_{N-k} = \text{sign}(g_k) \rightarrow y_N = \sum_{k=0}^{N} g_k u_{N-k} = \sum_{k=0}^{N} |g_k| > M$)

Transient and steady state response

Transient and steady state for discrete-time systems.

- The general soln of a linear difference eqn. is

$$\{y_k\}_{k\geq 0} = \{t_k\}_{k\geq 0} + \{s_k\}_{k\geq 0}$$

where $\{t_k\}$ is the transient response and $\{s_k\}$ is the steady state response.

- The transient response $\{t_k\}$ decays for stable systems.

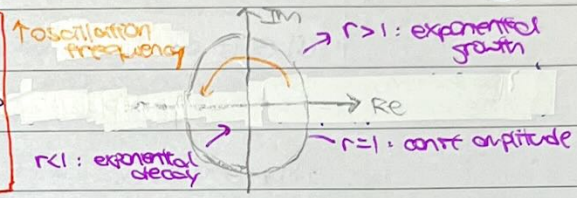
Pulse response and poles.

- Consider a LTI system $G(z)$ subjected to a pulse i/p $\{\delta_k\} = \{1,0,0\ldots\}$, so $\Delta(z) = 1$.

The o/p $\{y_k\}$ is given by $y_k = \mathcal{Z}^{-1}[Y(z)] = \mathcal{Z}^{-1}[G(z)\Delta(z)] = \mathcal{Z}^{-1}[G(z)]$

i.e. the poles of $G(z)$ define the response to a pulse.

- In general, for a pole of $z = re^{j\theta}$,
  - ↳ the magnitude r gives the amplitude decay rate
  - ↳ the argument θ gives the oscillation rate



- Recall all the poles inside the unit circle $|p_i| < 1\ \forall i$ → stable system.

- consider the following pole locations
  - ↳ Real poles $[p = \lambda]$: $g_k = \lambda^k$

$$G(z) = \frac{1}{1-\lambda z^{-1}} \rightarrow g_k = \lambda^k$$

  - ↳ Conjugate symmetric poles $[p_1 = \lambda e^{j\theta}, p_2 = \lambda e^{-j\theta}]$: $g_k = 2\lambda^k \cos(k\theta)$

$$G(z) = \frac{1}{1-\lambda e^{j\theta} z^{-1}} + \frac{1}{1-\lambda e^{-j\theta} z^{-1}} \rightarrow g_k = \lambda^k (e^{jk\theta} + e^{-jk\theta}) = 2\lambda^k \cos(k\theta)$$

  - * Real-valued impulse responses have complex conjugate pair poles.
  - ↳ Repeated poles $[p_1 = \lambda, p_2 = \lambda]$: $g_k = (1+k)p^k$

$$G(z) = \frac{1}{(1-pz^{-1})^2} = \frac{z^{-1}}{(1-pz^{-1})^2} \cdot z \rightarrow g_k = (1+k)p^k$$

Stable filters and steady state response.

- In the z-domain, any linear filter can be written as

$$A(z)Y(z) = B(z)X(z) + C(z, y_i)$$

where $C(z, y_i)$ takes into account initial conditions of the system

- Rearranging for $Y(z)$ and taking the inverse z-transform,

$$y_k = \mathcal{Z}^{-1}\left[\frac{B(z)}{A(z)}X(z) + \frac{C(z,y_i)}{A(z)}\right]$$

As $k \to \infty$, $\lim_{k\to\infty} y_k = \lim_{k\to\infty} \mathcal{Z}^{-1}\left[\frac{B(z)}{A(z)}X(z) + \frac{C(z,y_i)}{A(z)}\right] = \lim_{k\to\infty} \mathcal{Z}^{-1}\left[\frac{B(z)}{A(z)}X(z)\right]$

- For a stable system, stable roots in $A(z)$ enforce the exponential decay of $\mathcal{Z}^{-1}\left[\frac{C(z,y_i)}{A(z)}\right]$,

whereas $\frac{B(z)}{A(z)}X(z)$ may not necessarily decay exponentially → stable filter 'forgets' initial conditions.

Steady state response to a step input

- Provided all the poles of $(z-1)Y(z) = (z-1)G(z)U(z)$ lie strictly within the unit circle, we can use FVT,

$$\lim_{k\to\infty} y_k = \lim_{z\to 1}(z-1)G(z)$$

- For a step i/p, $U(z) = \frac{1}{1-z^{-1}}$, so the steady-state value of the o/p is given by $\lim_{k\to\infty} y_k = G(1)$

$\lim_{k\to\infty} y_k = \lim_{z\to 1}(z-1)Y(z) = \lim_{z\to 1}(z-1)G(z)U(z) = \lim_{z\to 1}(z-1)G(z)\frac{z}{z-1} = \lim_{z\to 1} zG(z) = G(1)$

(For the continuous case, FVT: $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sG(s)$, $U(s) = \frac{1}{s}$, so $\lim_{t\to\infty} y(t) = G(0)$ for step i/p).

Steady state response to a sinusoidal input

- Consider the complex sinusoidal i/p $x_k = e^{j\theta k}$ to a system $G(z) = \frac{b(z)}{(z-p_1)(z-p_2)\cdots(z-p_n)}$

$$Y(z) = G(z)X(z) = \frac{b(z)}{(z-p_1)(z-p_2)\cdots(z-p_n)}\frac{1}{1-e^{j\theta}z^{-1}} = \frac{G(e^{j\theta})}{(-e^{j\theta}z^{-1})} + \text{terms of the form } \frac{\beta_i}{1-p_i z^{-1}}$$

(In general, $Y(z) = \frac{\beta_0}{1-e^{j\theta}z^{-1}}$ + terms of the form $\frac{\beta_i}{(-p_i z^{-1})}$, consider $[(1-e^{j\theta}z^{-1})Y(z)]\big|_{z=e^{j\theta}}$,

$[(1-e^{j\theta}z^{-1})Y(z)]\big|_{z=e^{j\theta}} = \beta_0$, $[(1-e^{j\theta}z^{-1})G(z)X(z)]\big|_{z=e^{j\theta}} = G(e^{j\theta}) \rightarrow \beta_0 = G(e^{j\theta})$)

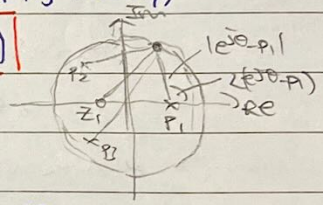
- Taking the inverse z-transform,

$$y_k = G(e^{j\theta})e^{j\theta k} + \text{terms decaying to 0 as } k\to\infty.$$

$\therefore$ The stationary o/p is $\tilde{y}_k = G(e^{j\theta})e^{j\theta k} = |G(e^{j\theta})|\, e^{j(\theta k + \angle G(e^{j\theta}))}$

- Now consider the real sinusoidal i/p $x_k = \cos(\theta k) = \frac{1}{2}(e^{j\theta k} + e^{-j\theta k})$. Applying linearity,

$$\tilde{y}_k = \frac{|G(e^{j\theta})|}{2}\left(e^{j(\theta k + \angle G(e^{j\theta}))} + e^{-j(\theta k + \angle G(e^{j\theta}))}\right) = |G(e^{j\theta})|\cos(k\theta + \angle G(e^{j\theta}))$$



Bode diagrams

- A bode diagram plots the filter amplification $|G(e^{j\theta})|$ and phase shift $\angle G(e^{j\theta})$ at each frequency θ.

- Consider a transfer function of the form $G(z) = c\frac{\prod_{k=1}^{m}(z-z_k)}{\prod_{k=1}^{n}(z-p_k)}$

↳ $|G(e^{j\theta})| = |c|\frac{\prod_{k=1}^{m}|e^{j\theta}-z_k|}{\prod_{k=1}^{n}|e^{j\theta}-p_k|}$ ⇝ $|G(e^{j\theta})|_{dB} = 20\log(|G(e^{j\theta})|) = 20(\log|c| + \sum_{k=1}^{m}\log|e^{j\theta}-z_k| - \sum_{k=1}^{n}\log|e^{j\theta}-p_k|)$

i.e. $|G(e^{j\theta})|$ = product of distances from zeroes to $e^{j\theta}$ divided by product of distances from poles to $e^{j\theta}$

(in dB, the products/divisions become sums/subtractions)

↳ $\angle G(e^{j\theta}) = \angle c + \sum_{k=1}^{m}\angle(e^{j\theta}-z_k) - \sum_{k=1}^{n}\angle(e^{j\theta}-p_k)$

i.e. $\angle G(e^{j\theta})$ = sum of angles from zeroes to $e^{j\theta}$ minus sum of angles from poles to $e^{j\theta}$.

- When $e^{j\theta}$ is close to a pole, the magnitude of the response rises (resonance)

When $e^{j\theta}$ is close to a zero, the magnitude of the response falls (null)

- Since $e^{jk\theta} = e^{jk(\theta+2\pi)}$, going more than a complete revolution is redundant

- For real-valued signals, the conjugate symmetric property of z-transform means $G(\rho e^{j\theta}) = \overline{G(\rho e^{j\theta})}$,

i.e. $|G(e^{j\theta})| = |G(e^{-j\theta})|$ and $\angle G(e^{j\theta}) = -\angle G(e^{-j\theta})$, so -ve frequencies mirror +ve frequencies.

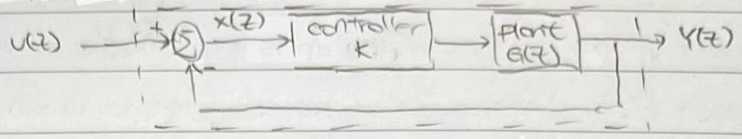
- Therefore, Bode diagrams are generally limited to $\theta\in[0,\pi]$ and $\omega\in[0, f_s/2]$.

- The gain and phase margins can be read off the Bode diagram as in the continuous time case

Nyquist stability criterion

closed-loop control systems

- consider the closed-loop control system below.



where we will assume the open loop system is causal (but not necessarily assume it is stable)

$$X(z) = U(z) - Y(z), \quad Y(z) = KG(z)X(z) \longrightarrow Y(z) = \frac{U(z)\cdot KG(z)}{1+KG(z)}$$

$$\therefore H(z) = \frac{Y(z)}{U(z)} = \frac{KG(z)}{1+KG(z)}$$

* same form as for continuous time signals.

Nyquist diagrams

- The Nyquist diagram of G is $G(e^{j\theta})$ plotted on the complex plane for $\theta \in [-\pi, \pi]$. To draw the Nyquist plot,
  - ↳ Read from the Bode diagram, obtain the real/imaginary parts using the gain/phase.
  - ↳ Substitute various value of θ into $G(e^{j\theta})$ and plot directly (try values like $\theta = 0, \theta = \pi/2, \theta = \pi$).
- If there are open-loop poles on the unit circle, the Nyquist trajectory, $G(e^{j\theta})$ will go to infinity as it passes through the pole on the unit circle
- For real-valued systems, the conjugate symmetry property of Z-transforms gives $G(e^{j\theta}) = \overline{G(e^{j\theta})}$, thus to find a complete Nyquist diagram, we only need to plot $G(e^{j\theta})$ for $0 \le \theta \le \pi$, then draw in the complex conjugate locus.

Nyquist stability criterion.

- The Nyquist stability criterion allows us to predict the stability of the closed loop $H(z) = \frac{KG(z)}{1+KG(z)}$ from the Nyquist diagram of the open loop G
- The Nyquist stability criterion states that the closed loop system is stable iff the no. of encirclements (CCW encirclements) of $-1/K$ by $G(e^{j\theta})$ as θ increases from $-\pi$ to π equals the no. of open loop unstable poles $N_{OP,U}$.
- Our goal is to vary K to shift the pt $-1/K$ on the real axis s.t. we get a stable closed loop system.
- The interpretation is the same as the continuous-time Nyquist diagram — gain and phase margins can be read off it the same way and have the same meanings.

Encirclements of the origin by $F(e^{j\theta})$.

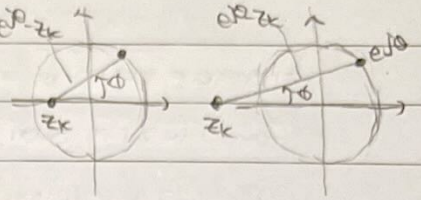
- consider the rational function $F(z) = A\frac{\prod_{k=1}^{m}(z-z_k)}{\prod_{k=1}^{n}(z-p_k)}$. The phase of $F(e^{j\theta})$ is given by.

$$\angle F(e^{j\theta}) = \angle A + \sum_{k=1}^{m}\angle(e^{j\theta}-z_k) - \sum_{k=1}^{n}\angle(e^{j\theta}-p_k)$$

- As θ increases from 0 to 2π (or equivalently $-\pi$ to π),

(i) $|z_i|<1$, then the increase in $\angle(e^{j\theta}-z_k)$ is 2π

(ii) $|z_i|>1$, then the increase in $\angle(e^{j\theta}-p_k)$ is 0.



+ similar for poles but -ve effect

- Let $N_{z,i}/N_{p,i}$ be the no. of zeros/poles of $F(z)$ inside the unit circle

- The overall increase α of $\angle F(e^{j\theta})$ as θ increases from 0 to 2π is

$$\alpha = 2\pi(N_{z,i}-N_{p,i})$$

The no. of encirclements C of the origin by the Nyquist trajectory $F(e^{j\theta})$ as θ increases from 0 to 2π is

$$C = N_{z,i}-N_{p,i}$$

Encirclements of the origin by $1+kG(z)$.

- The open loop transfer function is $H_o(z) = G(z) = \frac{b(z)}{a(z)}$ → open loop poles are soln's to $a(z)=0$

The closed loop transfer function is $H_c(z) = \frac{kG(z)}{1+kG(z)}$ → closed loop poles are soln's to $1+kG(z) = \frac{a(z)+kb(z)}{a(z)} = 0$.

- Consider $1+kG(z) = \frac{a(z)+kb(z)}{a(z)}$.

↳ The zeros of $1+kG(z)$ (soln to $a(z)+kb(z)=0$) are the closed loop poles

↳ The poles of $1+kG(z)$ (soln to $a(z)=0$) are the open loop poles.

- The no. of zeros/poles of $1+kG(z)$ inside the unit circle, $N_{z,i}/N_{p,i}$ is equivalent to the no. of closed/open loop poles inside the unit circle $N_{cp,i}/N_{op,i}$, i.e.

$$N_{z,i} = N_{cp,i} \qquad N_{p,i} = N_{op,i}$$

- The no. of encirclements C of the origin by the Nyquist trajectory $1+kG(e^{j\theta})$ as θ increases from 0 to 2π is

$$C = N_{z,i}-N_{p,i} = N_{cp,i}-N_{op,i}$$

Pole arithmetic

- Let m/n be the degree of $b(z)/a(z)$. Since the open-loop system $G(z)=\frac{b(z)}{a(z)}$ is causal, $m \le n$.

Note that $\deg(a(z)+kb(z)) = \max(\deg(a(z)), \deg(kb(z))) = \max(n,m) = n$.

- Therefore, $1+kG(z) = \frac{a(z)+kb(z)}{a(z)}$ is a fraction of two polynomials of degree n → both have n roots → no. of zeros = no. of poles = n.

- Let the subscript cp/op denote closed/open loop poles, z/p denote zeros/poles and i/o denote inside/outside the unit circle. [As before, $N_{cp}=N_z$, $N_{op}=N_p$]

$$n = \underbrace{N_{cp,i}+N_{cp,o}}_{N_{cp}} = \underbrace{N_{op,i}+N_{op,o}}_{N_{op}}$$

∴ The no. of encirclements C satisfies $C = N_{cp,i}-N_{op,i} = (n-N_{cp,o})-(n-N_{op,o}) = N_{op,o}-N_{cp,o}$

Encirclements of the point $-1/k$ by $G(e^{j\theta})$.

- When $1+KG(e^{j\theta})$ encircles the origin, equivalently,
  - ↳ $KG(e^{j\theta})$ encircles the pt. -1
  - ↳ $G(e^{j\theta})$ encircles the pt. $-1/k$.
- The no. of encirclements C of the pt. $-1/k$ by the Nyquist trajectory $G(e^{j\theta})$ as θ increases from 0 to 2π is

$$C = N_{op,o} - N_{cp,o}$$

- The closed loop system is stable if $N_{cp,o} = 0$, i.e. no closed loop poles outside the unit circle.

Therefore, for closed loop stability, we req.

$$\boxed{C = N_{op,o} = N_{p,o}}$$

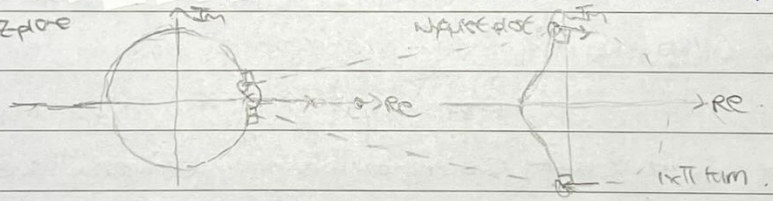
as stated in the Nyquist stability criterion.

Open loop poles on the unit circle

- The Nyquist trajectory $G(e^{j\theta})$ goes to infinity as it passes an open-loop pole on the unit circle, so to correctly count the no. of encirclements, we need to define a closed curve for the Nyquist locus.
- To close the Nyquist trajectory, we indent the path of z around the poles on the unit circle w/ a small semi-circular excursion outside the unit circle.
- This means open loop poles on the unit circle are counted as stable in the stability criterion.

i.e. $N_{op,o} = N_{p,o}$ is the no. of open loop poles <u>strictly outside</u> the unit circle.

- The "right turn" in the z-plane then gives a "right turn" in the Nyquist plot, and a circular arc of large radius is produced which continues for $m\pi$ rad, where m is the multiplicity of the unit circle pole.



- If $m=1$, the Nyquist locus will be asymptotic to a straight line as it tends to infinity.
  - ↳ Suppose that $G(z)$ has a pole at $z=1$, i.e. $G(z) = \frac{F(z)}{z-1}$, where $F(z)$ has no poles/zeros at $z=1$.

Taylor expanding about $z=1$, $G(z) = \frac{F(z)}{z-1} = \frac{1}{z-1}[F(1) + F'(1)(z-1) + \dots] \approx \frac{F(1)}{z-1} + F'(1)$

Note that $\frac{1}{e^{j\theta}-1} = \frac{1}{e^{j\theta/2}(e^{j\theta/2}-e^{-j\theta/2})} = \frac{e^{-j\theta/2}}{e^{j\theta/2}-e^{-j\theta/2}} = \frac{\cos(\theta/2) - j\sin(\theta/2)}{2j\sin(\theta/2)} = -\frac{1}{2} - \frac{j}{2\tan(\theta/2)}$

$\therefore G(e^{j\theta}) = F'(1) + \frac{F(1)}{e^{j\theta}-1} = F'(1) - \frac{1}{2}F(1) - \frac{jF(1)}{2\tan(\theta/2)} \to \boxed{G(e^{j\theta}) \to F'(1) - \frac{1}{2}F(1)}$ as $\theta \to 0$

- ↳ Consider $G(e^{j\theta})$ directly, then Taylor expand the $e^{j\theta}$ terms, $e^{j\theta} = 1 + j\theta + \frac{(j\theta)^2}{2!} + \dots$

Take out a factor of $j\theta$ from the $(e^{j\theta}-1)$ term, then expand to get the form $\frac{1}{j\theta}(a + bj\theta + c(\theta^2))$

The x-coord of the asymptote is b.

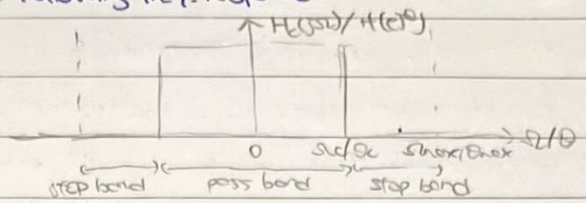
* May need to use the binomial expansion $(1+k\theta)^{-1} = 1 - k\theta + O(\theta^2)$

Digital filters

Design of filters

Ideal filters

- The ideal LPF has the following freq. response.

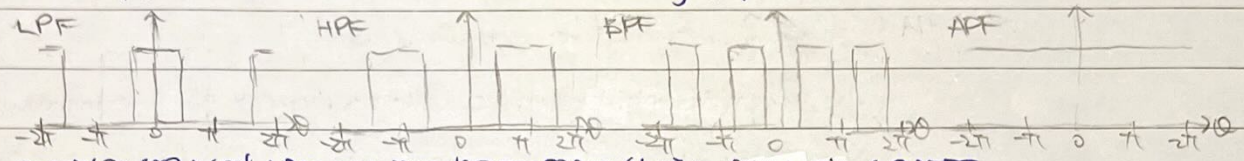


Analog: $H_c(j\Omega) = \begin{cases} 1 & \text{if } |\Omega| \le \Omega_c \\ 0 & \text{if } |\Omega| > \Omega_c \end{cases}$ Digital: $H(e^{j\theta}) = \begin{cases} 1 & \text{if } |\theta| \le \theta_c \\ 0 & \text{if } |\theta| > \theta_c \end{cases}$

where $\theta = \omega T$ is the normalised freq. and $T = \frac{1}{f_s} = \frac{2\pi}{\omega_s}$ is the sampling freq.

and $\theta_c \le \theta_{max} = \pi$ / $\Omega_c \le \Omega_{max} = \frac{\omega_s}{2}$

- similarly, the ideal HPF/BPF/APF have the following freq. response.



- Consider the ideal LPF. Its impulse response $\{h_k\}$ is found using the DTFT.

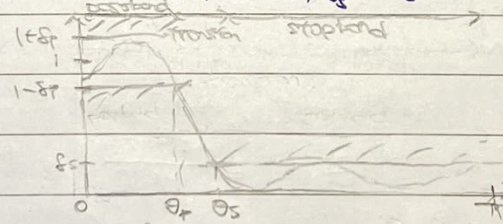
$$h_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\theta}) e^{j\theta k} d\theta = \frac{1}{2\pi}\int_{-\theta_c}^{\theta_c} e^{j\theta k} d\theta = \frac{1}{2\pi}\frac{e^{j\theta_c k} - e^{-j\theta_c k}}{jk} = \frac{\sin(\theta_c k)}{\pi k} = \frac{\theta_c}{\pi}\sin(\theta_c k)$$

i.e. it is non-causal and has a two-sided infinite impulse response → not realisable.

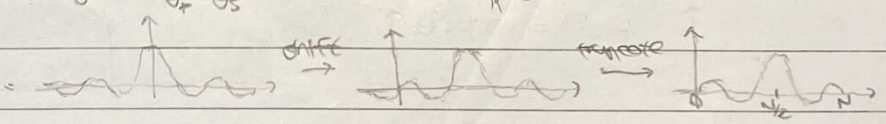
Realisable filter specifications

- A typical filter specification must specify the max. permissible deviations from the ideal; including (i) band edge freq. (θ_p, θ_s), (ii) max. passband ripple $(1-\delta_p, 1+\delta_p)$, (iii) min. stopband attenuation $(0, \delta_s)$.
- Using the dB scale, we need to specify (i) passband ripple $(20\log_{10}(1+\delta_p)\,dB)$, (ii) peak-to-peak passband ripple $(20\log_{10}(1+2\delta_p)\,dB)$, (iii) min. stopband attenuation $(-20\log_{10}(\delta_s)\,dB)$.

($\delta_p = 0.06$ gives peak-to-peak passband ripple = 2dB, $\delta_s = 0.01$ gives min. stopband attenuation = 40dB).



FIR filters.



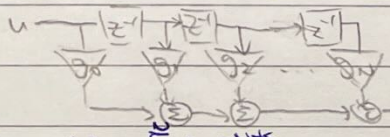
- The FIR filter G is derived by shifting + truncating at N+1 samples of the ideal response H, i.e.

$$G(z) = \sum_{k=0}^{N} g_k z^{-k} \quad \text{where } g_k = h_{k-N/2}$$

and h_k is the non-causal impulse response of the ideal filter.

- FIR filters are feed-forward / non-recursive → efficiently realised in hardware (FFT).

$$y_n = \sum_{k=0}^{N} g_k u_{n-k}$$



- FIR filters are inherently stable. $G(z) = \frac{\sum_{k=0}^{N} g_k z^{N-k}}{z^N}$ → all poles at $z=0$ (inside unit circle)

IIR filters

- IIR filters have a finite polynomial representation but infinite impulse response.
- IIR filters has feedback — the o/p "goes back" to the filter.

$$y_k = \underbrace{-a_{n-1}y_{k-1} + \dots - a_0 y_{k-n}}_{\text{feedback}} + b_n u_{k-n} + b_{n-1}u_{k-n+1} + \dots + b_0 u_{k-n}$$

- IIR filters typically meet a given set of specifications w/ a much lower filter order than their FIR filter counterpart → used for digital implementation of analog filters.

FIR vs IIR filters

- FIR filters: ✓ simple, stable, feedforward, fast implementation (FFT)
  - ✓ can have exactly linear phase $G(e^{j\theta}) = |G(e^{j\theta})| e^{j\theta}$
  - ✓ design methods are generally linear
  - × Higher filter order than IIR filter for same level of performance
  - × often greater delay than IIR filter w/ equal performance.
- IIR filters: ✓ Lower filter order than FIR filter for same level of performance
  - ✓ Shorter delay than FIR filter w/ equal performance.
  - × Can be unstable, uses feedback.

Design of FIR filters

Frequency distortion of truncation/windowing.

- To get the truncated unit sample response g_k from the shifted desired impulse, we effectively multiply by a rectangular window.

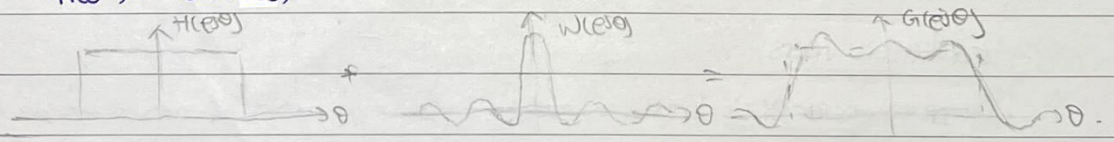
$$g_k = h_k \quad 0 \le k \le N \quad \rightarrow \quad g_k = h_k w_k, \quad w_k = \begin{cases} 1 & 0 \le k < N \\ 0 & \text{o/w.} \end{cases}$$

- Using duality of multiplication/convolution for Fourier transforms

$$G(e^{j\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\sigma}) W(e^{j(\theta-\sigma)})\, d\sigma$$

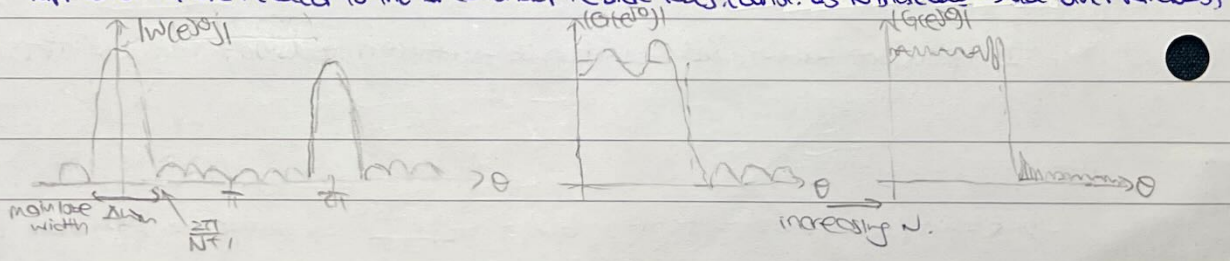
where $W(e^{j\theta}) = \sum_{k=0}^{N-1} w_k e^{j\theta k} = \sum_{k=0}^{N-1} \theta^{j\theta k} = \frac{1-e^{j\theta N}}{1-e^{j\theta}} = e^{-j\frac{\theta}{2}(N-1)} \frac{\sin(\frac{\theta N}{2})}{\sin(\theta/2)}$

$H(e^{j\theta}) = \text{rect}(\frac{\theta}{2\theta_c})$



- The transition band is related to the main lobe, (reduces as $N \to \infty$)

Ripples of G are related to the area under the side lobes. (const. as N increases → use other windows)



Design by window method

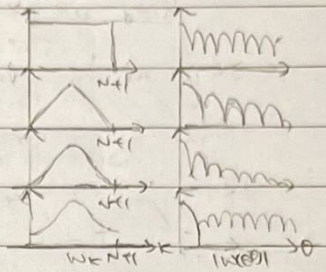
- We can reduce freq. distortion by adopting diff windows, common windows include.

↳ Rectangular $w_k = \begin{cases} 1 & 0 \le k \le N \\ 0 & o/w \end{cases}$

↳ Triangular (Bartlett) $w_k = \begin{cases} 2k/N & 0 \le k \le N/2 \\ 2-2k/N & N/2 < k \le N \\ 0 & o/w \end{cases}$

↳ Hanning $w_k = \begin{cases} 0.5 - 0.5\cos(2\pi k/N) & 0 \le k \le N \\ 0 & o/w \end{cases}$

↳ Hamming $w_k = \begin{cases} 0.54 - 0.46\cos(2\pi k/N) & 0 \le k \le N \\ 0 & o/w \end{cases}$



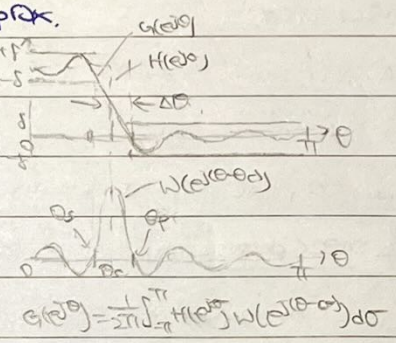
- The shape and duration of the window to control the resulting filter properties:

↳ Smaller side lobes yield better approx. of the ideal response

↳ Narrow transition bandwidth for increasing N, symmetric at cut-off θ_c.

↳ Save δ for passband error and stopband approx.

	Peak side lobe amplitude (relative)	Approx width of main lobe	Peak approx error (dB)
Rectangular	-13	$\frac{4\pi}{N+1}$	-21
Bartlett	-25	$\frac{8\pi}{N}$	-25
Hanning	-31	$\frac{8\pi}{N}$	-44
Hamming	-41	$\frac{8\pi}{N}$	-53



- The window method can be used to quickly design filters to approx. a given target response.

- It does not impose amplitude response constraints (passband ripple, stopband attenuation) ↳ need to be used iteratively to produce designs which meet such specs.

- To design a filter using the window method:

↳ 1) Select a suitable window function w_k, ie. Hanning/Bartlett etc.

↳ 2) Specify an ideal freq. response H, ie. LPF/BPF etc.

↳ 3) Compute the coeff. for the ideal filter, h_k

↳ 4) Delay the ideal filter coeff h_k by $N/2$, $h_{k-N/2}$, then multiply by the window function w_k

↳ 5) Evaluate the freq. response of the resulting filter and repeat the above steps iteratively if needed.

Low pass filter design (FIR)

-ex: Design a LPF w passband $\theta_p = 0.2\pi$, stopband $\theta_s = 0.3\pi$ and approx. error $\delta = 0.01$

① Max. allowed approx error in dB $= 20\log_{10}\delta = -40\text{dB} > -44\text{dB}$ → use Hanning window.

② cut-off freq. $\theta_c = \frac{\theta_s + \theta_p}{2} = 0.25\pi$ mainlobe width $\Delta\theta = \theta_p - \theta_s = 0.1\pi = \frac{8\pi}{N}$ → $N = 80$

③ Ideal filter $h_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\theta}) e^{j\theta k} d\theta = \frac{1}{2\pi}\int_{-\theta_c}^{\theta_c} e^{j\theta k} d\theta = \frac{1}{2\pi}\frac{e^{j\theta_c k} - e^{-j\theta_c k}}{jk} = \frac{\sin(\theta_c k)}{\pi k} = \frac{\theta_c}{\pi}\text{sinc}(\theta_c k)$.

④ Filter coeff $g_k = h_k - w_k/2 = \frac{1}{4}\text{sinc}\left(\frac{\pi}{4}(k-40)\right)\frac{1}{2}\left(1-\cos\left(\frac{\pi k}{40}\right)\right) 1_{k\in[0,80]}$

Multiband FIR design.

- We can design HPF/BPF using composition of LPF

$$H_{HPF[\theta_c,\pi]}(e^{j\theta}) = H_{LPF[0,\pi]} - H_{LPF[0,\theta_c]} \qquad H_{BPF[\theta_1,\theta_2]}(e^{j\theta}) = H_{LPF[0,\theta_2]} - H_{LPF[0,\theta_1]}$$

where $H_{LPF[0,\theta_c]} = \begin{cases} 1 & 0 \le \theta \le \theta_c \\ 0 & \text{o/w} \end{cases}$ and its corresponding $h_{k,LPF} = \frac{\theta_c}{\pi}\text{sinc}(\theta_c k)$

- The ideal impulse response for HPF/BPF is found by linearity of the antifourier,

$$h_{k,HPF} = \text{sinc}(\pi k) - \frac{\theta_c}{\pi}\text{sinc}(\theta_c k) \qquad h_{k,BPF} = \frac{\theta_2}{\pi}\text{sinc}(\theta_2 k) - \frac{\theta_1}{\pi}\text{sinc}(\theta_1 k)$$

- The final filter is thus obtained as usual, using,

$$g_k = h_{k-N/2} w_k .$$

Linear phase

- A linear phase filter has freq. response $G(e^{j\theta})$ that satisfies

$$\boxed{G(e^{j\theta}) = |G(e^{j\theta})| e^{-j\theta\frac{N}{2}}}$$

i.e the phase response is a linear function of freq. ($\phi(\theta) = \frac{N}{2}\theta$).

$g_k \to G(z)$, $g_{k-\frac{N}{2}} \to z^{-\frac{N}{2}} G(z)$

$\therefore$ For $z = e^{j\theta}$, we have $G(z) e^{-j\theta\frac{N}{2}}$

- For a linear phase filter, all the freq. components of the i/p signal are shifted in time by the same amt → no phase distortion in the o/p signal (i/p and o/p signals have the same shape)

- Linear phase is achieved if the filter coeff g_k is symmetric about $g_{N/2}$, i.e.

$$g_k = g_{N-k}$$

$$G(e^{j\theta}) = \sum_{k=0}^{N} g_k e^{-j\theta k} = g_0 e^{-j\theta 0} + g_1 e^{-j\theta N} + g_2 e^{-j\theta 1} + g_{N-1} e^{-j\theta(N-1)} + \dots$$

$$= e^{-j\theta\frac{N}{2}}\left(g_0 e^{j\theta\frac{N}{2}} + g_1 e^{j\theta(\frac{N}{2}-1)} + g_2 e^{j\theta(\frac{N}{2}-1)} + \dots\right)$$

$$= e^{-j\theta\frac{N}{2}}\left(\underbrace{2g_0\cos(\theta\tfrac{N}{2}) + 2g_1\cos(\theta(\tfrac{N}{2}-1)) + \dots}_{\text{real}}\right).$$

- Note that the window method gives linear phase filters ($w_k = w_{N-k}$ by defn, so if the desired h_k is symmetric, i.e. $h_k = h_{N-k}$, then $g_k = h_{k-N/2} w_k$ is also symmetric → linear phase).

Design by optimisation.

- Given the ideal filter H and the weighting function W, we can find the optimal filter G (and its coefficients g_k) of length N by optimisation.

- We define a cost/error function $E(\theta)$

$$E(\theta) = W(\theta)[H(\theta) - G(\theta)]$$

- G can be found by minimising

(i) least squares error $\int_{-\pi}^{\pi} E^2(\theta)\, d\theta$ i.e $G^*(\theta) = \underset{G(\theta)}{\text{argmin}} \int_{-\pi}^{\pi} E^2(\theta)\, d\theta$

(ii) max. error $\max_{-\pi \le \theta \le \pi} |E(\theta)|$ i.e. $G^*(\theta) = \underset{G(\theta)}{\text{argmin}} \max_{-\pi \le \theta \le \pi} |E(\theta)|$

↑ equiripple filter.

Design of IIR filters

Discretization by response matching.

- We can convert a continuous time filter $G_c(s)$ into an equivalent discrete time filter $G(z)$ by matching their response to a certain i/p (impulse, step, ramp)
- For a certain i/p, the response is matched if the o/p y satisfies

$$\boxed{\mathcal{L}^{-1}[Y(s)]\big|_{t=kT} = y(kT) = y_k = \mathcal{Z}^{-1}[Y(z)]}$$

① Impulse invariance.

$$G_c(s) \xrightarrow{\mathcal{L}^{-1}} g(t) \xrightarrow{\text{sample}} g(kT) = g_k \xrightarrow{\mathcal{Z}} G(z)$$

$$\therefore \boxed{G(z) = \mathcal{Z}[\mathcal{L}^{-1}[G_c(s)]\big|_{t=kT}]}$$

* For bandlimited filters, the discrete filter freq. response closely approx the continuous time freq. response
* Impulse invariance preserves stability, i.e. $\text{Re}[p_k] < 0 \rightarrow |e^{p_k T}| < 1$

② Step invariance

$$G_c(s) \xrightarrow{\text{step}} \frac{G_c(s)}{s} \xrightarrow{\mathcal{L}^{-1}} y(t) \xrightarrow{\text{sample}} y(kT) = y_k \xrightarrow{\mathcal{Z}} Y(z) \xrightarrow{\text{step}} \frac{z-1}{z} Y(z)$$

$$\therefore \boxed{G(z) = \frac{z-1}{z} \mathcal{Z}\left[\mathcal{L}^{-1}\left(\frac{G_c(s)}{s}\right)\Big|_{t=kT}\right]}$$

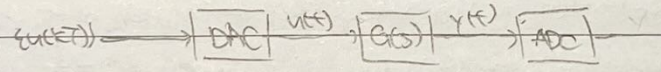
③ Ramp invariance

$$G_c(s) \xrightarrow{\text{ramp}} \frac{G_c(s)}{s^2} \xrightarrow{\mathcal{L}^{-1}} y(t) \xrightarrow{\text{sample}} y(kT) = y_k \xrightarrow{\mathcal{Z}} Y(z) \xrightarrow{\text{ramp}} \frac{(z-1)^2}{Tz} Y(z)$$

$$\therefore \boxed{G(z) = \frac{(z-1)^2}{Tz} \mathcal{Z}\left[\mathcal{L}^{-1}\left(\frac{G_c(s)}{s^2}\right)\Big|_{t=kT}\right]}$$

Choice of waveform for response matching.

- The choice of waveform to achieve response invariance depends on the hybrid analog/digital system.



- Within the hybrid system, it is the properties of the DAC and ADC that influence the choice of waveform for response invariance.
  - ↳ Sample + hold DAC → step invariance.
  - ↳ Moving average DAC → ramp invariance

Discretization by algebraic transformation

- Algebraic transformations are used to map systems that are not necessarily constrained at the cost of some distortion btwn the freq. response of the continuous/discrete time system.

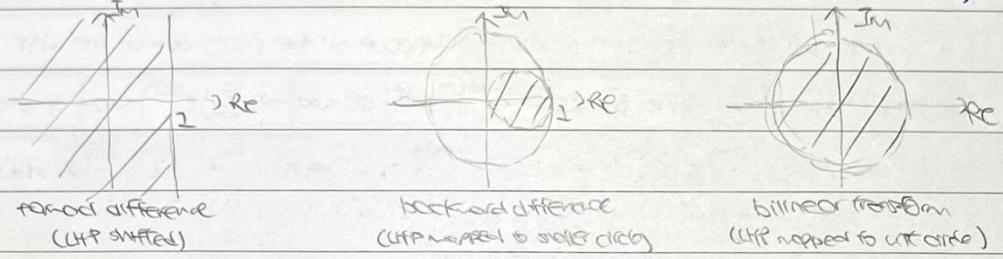
$$H(z) = H_c(s)\,|_{s=\gamma(z)}$$

where $\gamma(z)$ is given by

Forward difference (Euler)	$s=\frac{z-1}{T}$	$\dot{x} = \frac{x(t+T)-x(t)}{T} \rightarrow sX(s) = \frac{z-1}{T}X(z)$
Backward difference	$s=\frac{z-1}{zT}$	$\dot{x} = \frac{x(t)-x(t-T)}{T} \rightarrow sX(s) = \frac{1-z^{-1}}{T}X(z)$
Bilinear transform (Tustin)	$s=\frac{2}{T}\frac{z-1}{z+1}$	$z = e^{sT} = \frac{e^{sT/2}}{e^{-sT/2}} = \frac{1+sT/2}{1-sT/2} = \frac{2+sT}{2-sT} \rightarrow s=\frac{2}{T}\frac{z-1}{z+1}$

- Each of these transformations corresponds to a certain mapping btwn splane and zplane.

The shaded region below in the zplane corresponds to the LHP of the splane (stable region)



forward difference (LHP shifted) | backward difference (LHP mapped to stable circle) | bilinear transform (LHP mapped to unit circle)

- Applying algebraic transformation on a stable continuous system results in a stable discrete time system for backward diff/ bilinear transform, but is not necessarily true for forward diff.

Bilinear transform

- Consider the bilinear transform $s=\gamma(z)=\frac{z-1}{z+1}$
- Analog filter $G_c(s)$ w/ freq. resp. $G_c(j\Omega)$; Digital filter $G(z)$ w/ freq. resp. $G(e^{j\theta})$.

Using the algebraic transformation $G(z)=G_c(\gamma(z))$, the freq. resp. of the digital filter for $|\theta|<\pi$ is

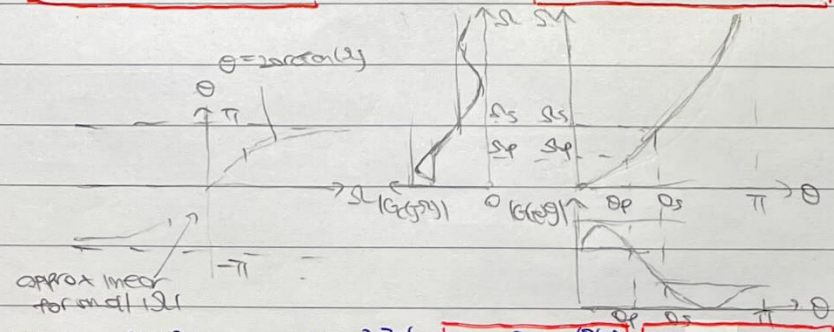
$$G(e^{j\theta}) = G_c(\gamma(e^{j\theta})) = G_c(j\tan(\theta/2)) = G_c(j\Omega) \quad \text{since } \gamma(e^{j\theta}) = \frac{e^{j\theta}-1}{e^{j\theta}+1} = \frac{e^{j\theta/2}(e^{j\theta/2}-e^{-j\theta/2})}{e^{j\theta/2}(e^{j\theta/2}+e^{-j\theta/2})} = j\tan(\theta/2)$$

- This means the freq. resp. of the analog filter at Ω is mapped to the freq. resp. of the digital filter at θ, where Ω and θ are related by

$$\Omega_c = \tan(\theta/2) \qquad \theta = 2\arctan(\Omega)$$

- Therefore, we can map btwn the discrete/continuous time filters using.

$$G(e^{j\theta}) = G_c(j\tan(\theta/2)) \qquad G_c(j\Omega) = G(e^{j2\arctan(\Omega)})$$



* Using the bilinear transform $s=\gamma(z)=\frac{z-1}{z+1}$, $\Omega_c=\frac{2}{T}\tan(\theta/2)$, $\theta=2\arctan(\frac{\Omega T}{2})$, and the mapping is

$$G(e^{j\theta}) = G_c(j\tfrac{2}{T}\tan(\theta/2)) \qquad G_c(j\Omega) = G(e^{j2\arctan(\Omega T/2)})$$

- consider the bilinear transform $s = Y(z) = \frac{z-1}{z+1} \rightarrow z = Y^{-1}(s) = \frac{1+s}{1-s}$

For $s = \lambda + j\omega$, $|z|^2 = zz^* = Y^{-1}(s)Y^{-1}(s)^* = \frac{1+\lambda+j\omega}{1-\lambda-j\omega} \cdot \frac{1+\lambda-j\omega}{1-\lambda+j\omega} = \frac{(1+\lambda)^2+\omega^2}{(1-\lambda)^2+\omega^2}$

If $\lambda = 0$, then $|z|^2 = \frac{1+\omega^2}{1+\omega^2} = 1$, i.e. on unit circle

If $\lambda < 0$, then $|z|^2 = \frac{(1+\lambda)^2+\omega^2}{(1-\lambda)^2+\omega^2} < 1$, i.e. inside the unit circle.

→ the LHP in the s-plane is mapped exactly to the unit circle in the z-plane → stability preserved.

Low pass filter design (IIR)

- Consider the first order analog LPF $G_c(s) = \frac{1}{1+s/\Omega_c}$, which has gain 1 at $s = j0$ and gain $= 0.5$ at $s = j\Omega_c$. Applying the bilinear transform $s = Y(z) = \frac{z-1}{z+1}$,

$$G(z) = G_c(Y(z)) = \frac{1}{1+Y(z)/\Omega_c} = \frac{1}{1+\frac{z-1}{(z+1)\Omega_c}} = \frac{(z+1)\Omega_c}{(z+1)\Omega_c + z - 1} = \frac{(z+1)\Omega_c}{(1+\Omega_c)z + (\Omega_c - 1)} = \frac{(z+1)\frac{\Omega_c}{1+\Omega_c}}{z + \frac{\Omega_c - 1}{\Omega_c + 1}}$$

- Given a desired cutoff freq. for the digital filter Ω_c, we can find the equivalent pre-warped analog filter cut-off freq. Ω_c using $\Omega_c = \tan(\theta_c/2)$.

- e.g. Design a first order digital LPF w/ -3dB freq. of 2kHz and a sampling freq. of 8kHz.

We are given $f_c = 1\text{kHz}$, $f_s = 8\text{kHz}$ → $\omega_c = 2\pi f_c$, $T = \frac{1}{f_s}$ → $\theta_c = \omega_c T = \frac{2\pi f_c}{f_s} = \frac{\pi}{4}$ rad/sample

$$\therefore \Omega_c = \tan\left(\frac{\pi/4}{2}\right) = \tan\left(\frac{\pi}{8}\right) = 0.414 \text{ rad/s} \rightarrow G(z) = \frac{(z+1)\frac{\Omega_c}{1+\Omega_c}}{z + \frac{\Omega_c - 1}{\Omega_c + 1}} = \frac{0.293(z+1)}{z - 0.414}$$

since $G(z) = \frac{Y(z)}{U(z)}$, we have $Y(z)(1 - 0.414z^{-1}) = U(z)(0.293(1+z^{-1}))$, so the difference eqn. is

$$y_k = 0.414 y_{k-1} + 0.293(u_k + u_{k-1})$$

* This is an IIR filter as the transfer function $G(z)$ has a pole at $z \neq 0$.

Band transformations

- Analog prototypes are typically LPF. Standard transformations can be used to achieve HPF/BPF/BSF.

- Assuming we have a LPF w/ cutoff at $\omega = 1$ rad/s.

↳ LPF → LPF: $s \rightarrow \frac{s}{\omega_c}$

↳ LPF → HPF: $s \rightarrow \frac{\omega_c}{s}$

} ω_c is the cutoff freq.

↳ LPF → BPF: $s \rightarrow \frac{s^2 + \omega_l\omega_u}{s(\omega_u - \omega_l)}$

↳ LPF → BSF: $s \rightarrow \frac{s(\omega_u - \omega_l)}{s^2 + \omega_l\omega_u}$

} ω_l is the lower cutoff freq.
ω_u is the upper cutoff freq.

Discrete Fourier Transform

Discrete Fourier Transform

Discrete Fourier Transform (DFT)

- The DFT is given by

$$X_n = \sum_{k=0}^{N-1} x_k e^{-j\frac{2\pi}{N}nk}, \quad \text{for } 0 \le n \le N-1$$

- since the transform maps a finite length sequence $\{x_k\}_{k=0}^{N-1}$ to a finite length sequence $\{X_n\}_{n=0}^{N-1}$, we can rewrite the DFT as a matrix-vector operation.

$$\begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ \vdots \\ X_{N-1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix}}_{\text{DFT Vandermonde matrix } \underline{\underline{F}}} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{bmatrix}$$

where $W_N = e^{-j\frac{2\pi}{N}}$ is a primitive Nth root of unity (i.e. $W_N^N = 1$ and $W_N^m \neq 1$ for $m = 1, 2, \ldots, N-1$)

- The N freq. readings obtained from the DFT are "scaled" freq. btwn 0 and 2π in normalized freq. (2π corresponds to sampling freq. f_s).
  - ↳ X_0 ↪ freq. 0 ; $X_{N/2}$ ↪ freq. π (i.e. $f_s/2$)
  - ↳ $X_1 \ldots X_{\pi N/2 - 1}$ ↪ equally spaced +ve freq. pts btwn 0 and π (i.e. $f_s/2$)
  - ↳ $X_{N/2} \ldots X_{\lfloor N/2 \rfloor + 1}$ ↪ equally spaced -ve freq. pts btwn 0 and $-\pi$ (i.e. $-f_s/2$)

Cyclic properties of the powers of W_N

- Any integer k can be written as

$$k = qN + r, \quad \text{w/ } 0 \le r < N$$

where q is the quotient and $r = R_N(k)$ is the remainder

- We can therefore write W_N^k as

$$W_N^k = W_N^{qN+r} = W_N^{qr} \cdot W_N^r = 1 \cdot W_N^r = W_N^r \rightarrow \boxed{W_N^k = W_N^{R_N(k)}}$$

So the DFT Vandermonde matrix $\underline{\underline{F}}$ can be rewritten s.t. powers of W_N higher than $N-1$ can be expressed as powers btwn 0 and $N-1$.

Constructing the DFT Vandermonde matrix $\underline{\underline{F}}$.

- The DFT Vandermonde matrix $\underline{\underline{F}}$ can be constructed as follows:

$$\underline{\underline{F}} = \begin{bmatrix} 1 & 1 & \cdots & \text{(row 0: N ones)} \\ 1 & W_N & \cdots & \text{(row 1: powers 0 to N-1 of } W_N\text{)} \\ 1 & W_N^2 & \cdots & \text{(row 2: every 2nd element of row 1)} \\ 1 & W_N^3 & \cdots & \text{(row 3: every 3rd element of row 1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & \cdots & \text{(row N-1: row 1 cyclically backward)} \end{bmatrix}$$

where "every kth element" is understood cyclically.

↳ Note the last row is "every (N-1)th element", which is the same as going backwards.

Properties of the DFT

① periodicity: $X_n = X_{n+N}$

$$X_{n+N} = \sum_{k=0}^{N-1} x_k e^{j\frac{2\pi}{N}k(n+N)} = \sum_{k=0}^{N-1} x_k e^{j\frac{2\pi}{N}kn} e^{j2\pi k} = X_n$$

② Linearity: $DFT[x_k + y_k] = DFT[x_k] + DFT[y_k]$

$$DFT[x_k+y_k] = \sum_{k=0}^{N-1}(x_k+y_k)e^{-j\frac{2\pi}{N}kn} = \sum_{k=0}^{N-1} x_k e^{-j\frac{2\pi}{N}kn} + \sum_{k=0}^{N-1} y_k e^{-j\frac{2\pi}{N}kn} = DFT[x_k] + DFT[y_k]$$

③ Conjugate symmetry: $X_{-n} = X_n^*$ for real x_k; $x_k = x_k^*$ for real X_n

$$X_{-n} = \sum_{k=0}^{N-1} x_k e^{j\frac{2\pi}{N}k(-n)} = \left(\sum_{k=0}^{N-1} x_k e^{-j\frac{2\pi}{N}kn}\right)^* = X_n^*$$

$$x_{-k} = \frac{1}{N}\sum_{n=0}^{N-1} X_n e^{j\frac{2\pi}{N}n(-k)} = \left(\frac{1}{N}\sum_{n=0}^{N-1} X_n e^{j\frac{2\pi}{N}nk}\right)^* = x_k^*$$

i.e. real for time/freq. ⇔ time has CS; complex for time/freq. ⇔ freq./time does not have CS

④ Time/frequency shift: $DFT[x_{R_N(k-l)}] = W_N^{nl} X_n$; $IDFT[X_{R_N(n-l)}] = W_N^{kl} x_k$

$$DFT[x_{R_N(k-l)}] = \sum_{k=0}^{N-1} x_{R_N(k-l)} W_N^{nk} = \sum_{p=-l}^{N-1-l} x_p W_N^{n(p+l)} = \sum_{p=-l}^{N-1-l} x_p W_N^{np} W_N^{nl} = W_N^{nl} X_n$$

$$IDFT[X_{R_N(n-l)}] = \frac{1}{N}\sum_{n=0}^{N-1} X_{R_N(n-l)} W_N^{-nk} = \frac{1}{N}\sum_{p=-l}^{N-1-l} X_p W_N^{-k(p+l)} = W_N^{kl}\frac{1}{N}\sum_{p=-l}^{N-1-l} X_p W_N^{-kp} = W_N^{kl} x_k$$

⑤ cyclic convolution: $DFT[\{x_k\} \circledast \{y_k\}] = X_n Y_n$ $\left[\{z_k\} = \{x_k\} \circledast \{y_k\},\ Z_n = \sum_{k=0}^{N-1} x_k y_{R_N(k-n)}\right]$

$$DFT[\{x_k\} \circledast \{y_k\}] = \sum_{n=0}^{N-1}(x_k \circledast y_k) W_N^{nk} = \sum_{n=0}^{N-1}\left(\sum_{m=0}^{N-1} x_m y_{R_N(k-m)}\right) W_N^{nk} = \sum_{m=0}^{N-1} x_m \sum_{n=0}^{N-1} y_{R_N(k-m)} W_N^{nk}$$

$$= \sum_{m=0}^{N-1} x_m \sum_{k=0}^{N-1} y_{R_N(k-m)} W_N^{nk} = \sum_{m=0}^{N-1} x_m \sum_{k=0}^{N-1} y_k W_N^{n(k+m)} = \sum_{m=0}^{N-1} x_m W_N^{nm} \sum_{k=0}^{N-1} y_k W_N^{nk} = X_n Y_n$$

i.e. circular convolution in time domain is equivalent to pointwise multiplication in freq. domain.

The DFT Vandermonde matrix W.

- The sum of column n of the DFT Vandermonde matrix W is given by $\sum_{k=0}^{N-1} W_N^{nk}$

$n=0$: $\sum_{k=0}^{N-1} W_N^{0k} = \sum_{k=0}^{N-1} 1^k = N$

$n \neq 0$: $\sum_{k=0}^{N-1} W_N^{nk} = \frac{1-(W_N^n)^N}{1-W_N^n} = \frac{1-W_N^{n(N)}}{1-W_N^n} = \frac{1-W_N^{nN}}{1-W_N^n} = 0$

i.e. the sum of any column of W except column 0 is zero.

- Let f_l be the lth column of the DFT Vandermonde matrix W. For $l \geq m$,

complex dot product → $f_l \cdot f_m = \sum_{k=0}^{N-1} W_N^{lk}\left(W_N^{mk}\right)^* = \sum_{k=0}^{N-1} W_N^{(l-m)k}$

i.e. the sum of the $(l-m)$th column of W, and hence 0 when $l > m$ / N when $l = m$

- This means the columns of W are orthogonal, and their magnitude is $\sqrt{N}$.

$$f_i \cdot f_j = \begin{cases} N & i=j \\ 0 & i \neq j \end{cases}$$

∴ The DFT is an orthogonal basis transform (w/ scaling factor N).

- If $X = Wx$ and $Y = Wy$, the scaling factor of $\sqrt{N}$ on the basis vectors give

$$X \cdot Y = \left(\frac{1}{\sqrt{N}}X\right)\cdot\left(\frac{1}{\sqrt{N}}Y\right) = \frac{1}{N} X \cdot Y \qquad \|x\|^2 = \left(\frac{1}{\sqrt{N}}X\right)\cdot\left(\frac{1}{\sqrt{N}}X\right) = \frac{1}{N}\|X\|^2$$

Inverse Discrete Fourier Transform (IDFT)

- The IDFT is given by

$$x_k = \frac{1}{N}\sum_{n=0}^{N-1} X_n e^{j\frac{2\pi}{N}kn} \quad \text{for } 0 \le k \le N-1$$

- Since the transform maps a finite length sequence $\{x_n\}_{n=0}^{N-1}$ to a finite length sequence $\{X_k\}_{k=0}^{N-1}$, we can rewrite the IDFT as a matrix-vector operation.

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{bmatrix} = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)^2} \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ \vdots \\ X_{N-1} \end{bmatrix}$$

where $W_N = e^{-j\frac{2\pi}{N}}$ is a primitive Nth root of unity (i.e. $W_N^N = 1$ and $W_N^m \neq 1$ for $m = 1, 2, \ldots N-1$)

- By inspection, the IDFT is simply DFT but $W_N \to \frac{1}{N} W_N^{-1}$!

Alternatively, we can obtain the IDFT by i/p conjugation → DFT → o/p conjugation → divide by N.

$$\frac{1}{N}\left[\mathrm{DFT}[X_n^*]\right]^* = \frac{1}{N}\left[\sum_{n=0}^{N-1} X_n^* e^{j\frac{2\pi}{N}kn}\right]^* = \frac{1}{N}\sum_{n=0}^{N-1} X_n e^{+j\frac{2\pi}{N}kn} = \mathrm{IDFT}[X_n]$$

- Considering the IDFT matrix, it is simply the inverse of the DFT Vandermonde matrix $\underline{\underline{F}}$, $\underline{\underline{F}}^{-1}$. Since $\underline{\underline{F}}$ is orthogonal, it is easy to determine its inverse.

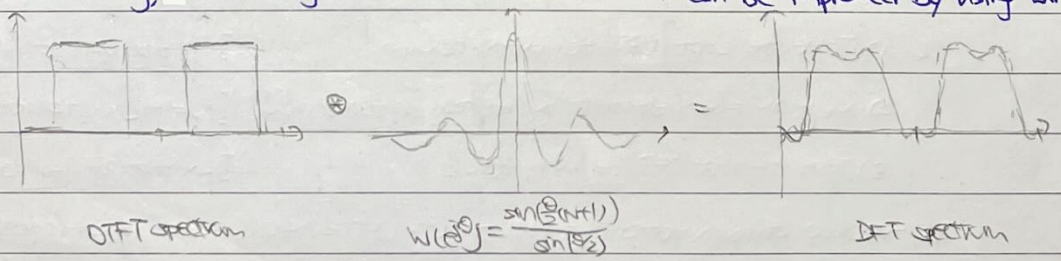
$$\underline{\underline{F}}\,\underline{\underline{F}}^H = N\underline{\underline{I}}_N \quad \to \quad \underline{\underline{F}}^{-1} = \frac{1}{N}\underline{\underline{F}}^H.$$

Note that $\underline{\underline{F}}$ is symmetric, so we just need to conjugate the entries of $\underline{\underline{F}}$, i.e. $\underline{\underline{F}}^{-1} = \frac{1}{N}\underline{\underline{F}}^*$

- As $W_N^{-nk} = (W_N^{nk})^*$, the inverse matrix $\underline{\underline{E}} = \underline{\underline{F}}^*$ has the same row 0 as $\underline{\underline{F}}$ and its rows 1 to N-1 are the corresponding rows of $\underline{\underline{F}}$ in reverse order (counting down from N-1 to 1).

DFT and DTFT frequency comparison.

- The DFT can be interpreted as an approx. of the continuous DTFT spectrum by truncation + sampling.
- Truncation of the sequence $\{x_k\}_{k=0}^{\infty}$ (DTFT → DFT) is equivalent to a multiplication of the sequence by a rectangular window of length N. / convolution of the spectrum w/ $W(e^{j\theta}) = \frac{\sin(\frac{\theta}{2}(N+1))}{\sin(\frac{\theta}{2})}$.
- Therefore the value of X_n depends on the whole DTFT spectrum by convolution (instead of only over the ideal interval $[\frac{2\pi}{N}(n-\frac{1}{2}), \frac{2\pi}{N}(n+\frac{1}{2})]$) → leakage from neighbouring freq. bins.
- As for filtering, the matching of DFT to DTFT spectrum can be improved by using windowing.



Fast Fourier Transform

Fast Fourier Transform (FFT)

- FFT algorithms exploit properties of the DFT Vandermonde matrix E to perform the DFT in $O(N \log N)$ operations instead of $O(N^2)$ normally needed for matrix-vector multiplication.
  - ↳ FFT relies on redundancy in the calculation of the basic DFT → reuse past computations.
  - ↳ FFT is recursive → logarithmic complexity.
- There are many FFT algorithms that work for length N w/ specific properties.
  - ↳ (General) Cooley Tukey: $N = LM$, where $L, M \in \mathbb{Z}^+$.
  - ↳ Radix-2: $N = 2^m$, where $m \in \mathbb{Z}^+$.

Cooley Tukey algorithm.

- The Cooley Tukey algorithm works for composite numbers $N = LM$, where $L, M \in \mathbb{Z}^+$.
  If further factorisation of L/M is possible, the method can be applied recursively.
- Re-consider the N-length i/p and o/p vectors $\underline{x}, \underline{X}$ as a $L \times M$ array w/ L rows and M columns, i.e.

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix} \rightarrow \begin{matrix} x_0 & x_1 & x_2 & \cdots & x_{M-1} \\ x_M & x_{M+1} & x_{M+2} & \cdots & x_{2M-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{(L-1)M} & \cdots & & \cdots & x_{LM-1} \end{matrix} \quad \left.\vphantom{\begin{matrix} a \\ b \\ c \\ d \end{matrix}}\right\} L$$

$\underbrace{\qquad\qquad}_{M}$

This is row major form — consecutive entries in a row are consecutive in $\underline{x}$; consecutive entries in a column are M apart in $\underline{x}$.

- Reconsidering the time/freq. domain index k/n using a 2D index, we have

$$k = \ell M + s \qquad\qquad n = mL + r$$

where ℓ, r indicate the row no.; m, s indicate the column no.

- Using the 2D index, we can re-write the DFT as.

$$X_n = X_{mL+r} = \sum_{k=0}^{N-1} x_k W_N^{kn} = \sum_{s=0}^{M-1} \sum_{\ell=0}^{L-1} x_{\ell M+s} W_N^{(\ell M+s)(mL+r)} = \sum_{s=0}^{M-1} \sum_{\ell=0}^{L-1} x_{\ell M+s} W_N^{\ell mLM} W_N^{\ell Mr} W_N^{smL} W_N^{sr}$$

Noting that $W_N^{\ell mLM} = e^{-j\frac{2\pi}{N}\ell mLM} = 1$, $W_N^L = e^{-j\frac{2\pi}{N}L} = e^{-j\frac{2\pi}{M}} = W_M$, $W_N^M = e^{-j\frac{2\pi}{N}M} = e^{-j\frac{2\pi}{L}} = W_L$,

$$X_{mL+r} = \sum_{s=0}^{M-1} W_M^{sm} W_N^{sr} \sum_{\ell=0}^{L-1} x_{\ell M+s} W_L^{\ell r}$$

twiddle factor · L-pt DFT · M-pt DFT.

i.e. 1) Take the L-pt DFT for every column [$M \times L$-pt DFT → ML^2 computations]

2) Multiply every element by its twiddle factor W_N^{sr} [$N = ML$ multiplications → N computations]

3) Take the M-pt DFT of every row. [$L \times M$-pt DFT → LM^2 computations].

- The total no. of computations is $ML + ML^2 + LM^2 = ML(1 + L + M) \approx N(1 + L + M)$

The total no. of computations for the N-pt DFT is N^2.

Radix-2 algorithm

- The Radix-2 algorithm works for powers of 2, $N=2^m$, where $m \in \mathbb{Z}^+$.

- We can split the summation of the N-pt DFT into two parts.

$$X_n = \sum_{k=0}^{N-1} x_k e^{-j\frac{2\pi}{N}nk} = \sum_{k=0}^{\frac{N}{2}-1} x_{2k} e^{-j\frac{2\pi}{N}(2k)n} + \sum_{k=0}^{\frac{N}{2}-1} x_{2k+1} e^{-j\frac{2\pi}{N}(2k+1)n} = \underbrace{\sum_{k=0}^{\frac{N}{2}-1} x_{2k} e^{-j\frac{2\pi}{\frac{N}{2}}nk}}_{A_n} + \underbrace{e^{-j\frac{2\pi}{N}n}}_{W_N^n} \underbrace{\sum_{k=0}^{\frac{N}{2}-1} x_{2k+1} e^{-j\frac{2\pi}{\frac{N}{2}}nk}}_{B_n}$$

DFT of even index; DFT of odd index

$$\therefore \boxed{X_n = A_n + W_N^n B_n}, \quad \text{where } A_n, B_n \text{ are } \tfrac{N}{2}\text{-pt DFTs.}$$

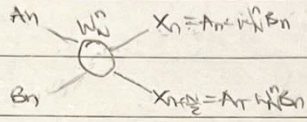
- Consider $X_{n+N/2}$

$$X_{n+N/2} = \sum_{k=0}^{\frac{N}{2}-1} x_{2k} e^{-j\frac{2\pi}{\frac{N}{2}}(n+\frac{N}{2})k} + e^{-j\frac{2\pi}{N}(n+\frac{N}{2})} \sum_{k=0}^{\frac{N}{2}-1} x_{2k+1} e^{-j\frac{2\pi}{\frac{N}{2}}(n+\frac{N}{2})k} = \underbrace{\sum_{k=0}^{\frac{N}{2}-1} x_{2k} e^{-j\frac{2\pi}{\frac{N}{2}}nk}}_{A_n} - \underbrace{e^{-j\frac{2\pi}{N}n}}_{W_N^n} \underbrace{\sum_{k=0}^{\frac{N}{2}-1} x_{2k+1} e^{-j\frac{2\pi}{\frac{N}{2}}nk}}_{B_n}$$

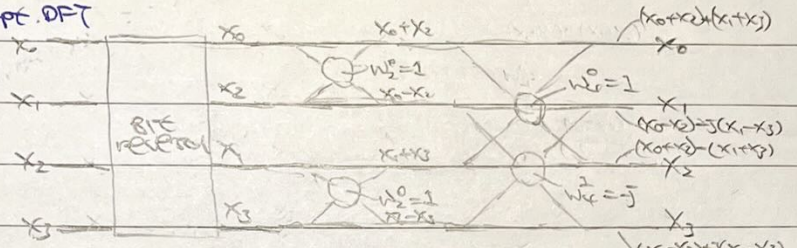
part of even index; DFT of odd index.

$$\therefore \boxed{X_{n+\frac{N}{2}} = A_n - W_N^n B_n}, \quad \text{where } A_n, B_n \text{ are } \tfrac{N}{2}\text{-pt DFTs.}$$

- We have recursion + redundancy by decomposing a N-pt DFT into $2 \times \frac{N}{2}$-pt DFTs + resulting post computations.



- eg: Consider the 4-pt DFT



Bit reversal: convert index from decimal to binary → reverse bits → convert from binary to decimal.

Applications of the DFT

Cyclic convolution

- Consider independent RVs $X_1, X_2 \ldots X_n$ defined over an alphabet $\mathcal{X} = \{0, 1 \ldots p-1\}$, where $p \in \mathbb{Z}^+$.

Consider the RV Z

$$Z = X_1 \oplus X_2 \oplus \ldots \oplus X_n, \quad \text{where } \oplus \text{ is mod-}p \text{ addition } (x \oplus y = R_p(x+y))$$

For all n and p, the pmf of Z, P_Z, is given by the cyclic convolution of the pmfs of X_i, P_{X_i}.

$$\underline{P}_Z = \underline{P}_{X_1} \circledast \underline{P}_{X_2} \circledast \ldots \circledast \underline{P}_{X_n}$$

- For $p=2$, $X_1, X_2 \ldots X_n$ are binary RVs w/ pmf $\underline{P}_{X_i} = [1-p_i, p_i]$

1) 2-pt DFT of $\underline{P}_{X_i}$: $\underline{Q}_{X_i} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1-p_i \\ p_i \end{bmatrix} = \begin{bmatrix} 1 \\ 1-2p_i \end{bmatrix}$

2) Component wise multiplication: $\underline{Q}_Z = \begin{bmatrix} \prod_{i=1}^n 1 \\ \prod_{i=1}^n (1-2p_i) \end{bmatrix} = \begin{bmatrix} 1 \\ \prod_{i=1}^n (1-2p_i) \end{bmatrix}$

3) Inverse 2-pt DFT: $\underline{P}_Z = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ \prod_{i=1}^n (1-2p_i) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}\prod_{i=1}^n (1-2p_i) \\ \frac{1}{2} - \frac{1}{2}\prod_{i=1}^n (1-2p_i) \end{bmatrix}$

- In general, we calculate the circular convolution using FFT → component-wise multiplication → IFFT.

OFDM (Fixing a convolution channel in doing cyclic convolution)

(Refer to DFT notes)

Linear convolution

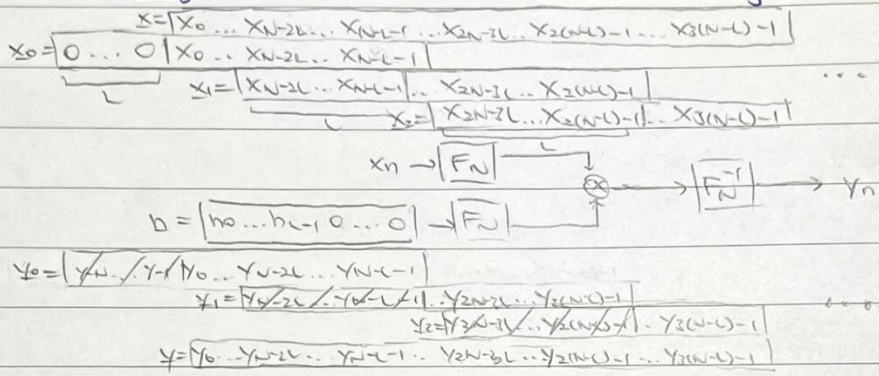
- Linear convolutions of arbitrary sequences cannot in general be computed via an FFT.

However, consider the case of an FIR filter of length L+1.

semi-infinite finite semi-infinite

$$\{y_k\}_{k\geq 0} = \{h_k\}_{k=0}^{L} * \{x_k\}_{k\geq 0}, \quad \text{i.e. } y_k = \sum_{m=0}^{L} h_m x_{k-m}$$

- If we take a block $\underline{x} = [x_0 \dots x_{N-1}]$ and calculate the circular convolution $\underline{y_c} = \underline{h} \circledast \underline{x}$, the first L o/ps differ from linear convolution because they depend on the last L entries of the i/p block, instead of $x_L \dots x_{-1} = 0$ for linear convolution.
- For $N \gg L$, the o/p beyond the first L will be identical to linear convolution.
- We can use length N FFT to do linear convolutions w/ FIR of length L+1 $\ll N$ w/ overlap-save method.



Interpolation and decimation.

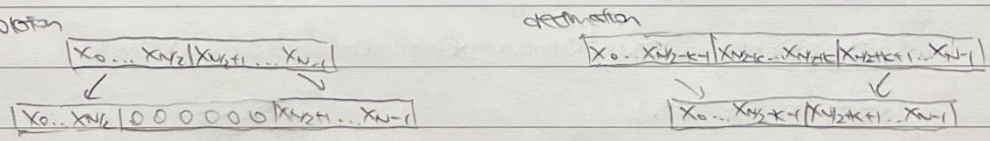
- Signal processing systems often req. a change of sampling freq. f_s so processing can be done at a higher f_s (better accuracy) or lower f_s (reduce computations)
- Interpolation (upping to a higher freq.) is done by first zero stuffing: insert K zeros btwn every two samples, then filtering: low-pass) and scale the resulting signal.
- Decimation (reducing to a lower freq.) is done by dropping samples: keep only one in every K samples. (To avoid aliasing, this can only be done when $BW < \frac{f_s^{new}}{2}$)
- Interpolating / decimating via the FFT gives a graceful transition btwn freq. and no longer req. f_s^{new} to be a multiple or divisor of f_s^{old}.
- To interpolate/decimate, we pick N and N' s.t. $\boxed{N'/N = f_s^{new}/f_s^{old}}$.

1) Take the N-pt FFT of the i/ps $\underline{x}$.

2) Interpolation: Insert zeros btwn $X_{\lfloor N/2 \rfloor}$ and $X_{\lfloor N/2+1 \rfloor}$ (odd N) / either side of $X_{N/2}$ (even N)

   Decimation: Delete components symmetrically around $X_{N/2}$.

3) Take the N'-pt IFFT of the modified spectrum



- The FFT method has the usual effect of a cut in the spectrum being equivalent to multiplying by a rectangular window → can be improved using window-based approaches.

Continuous time random signals and system response

Continuous time random signals

Random process

- A random signal is a time-dependent RV, $X(t)$, which is an instance drawn randomly from a set of possible signal waveforms. (ensemble)
- The ensemble of random signals + the associated probability distributions is known as a random process.
- We denote the random process as $\{X(t,\alpha)\}$ and an instance of the random process, the random signal as $X(t,\alpha)$, where α is a draw from some set A.

Correlation and covariance functions

- The autocorrelation function of a random process $\{X(t,\alpha)\}$ is

$$r_{XX}[t_1,t_2] = E_A[X(t_1,\alpha)X(t_2,\alpha)] = \int_A x(t_1,\alpha)\,x(t_2,\alpha)\,f_A(\alpha)\,d\alpha$$

The autocovariance function of a random process $\{X(t,\alpha)\}$ is

$$c_{XX}[t_1,t_2] = E_A[(X(t_1,\alpha)-M_{t_1})(X(t_2,\alpha)-M_{t_2})] = \int_A (x(t_1,\alpha)-M_{t_1})(x(t_2,\alpha)-M_{t_2})\,f_A(\alpha)\,d\alpha$$

- The crosscorrelation function btwn random processes $\{X(t,\alpha)\}$ and $\{Y(t,\alpha)\}$ is

$$r_{XY}[t_1,t_2] = E_A[X(t_1,\alpha)Y(t_2,\alpha)] = \int_A x(t_1,\alpha)\,y(t_2,\alpha)\,f_A(\alpha)\,d\alpha$$

The crosscovariance function btwn random processes $\{X(t,\alpha)\}$ and $\{Y(t,\alpha)\}$ is

$$c_{XY}[t_1,t_2] = E_A[(x(t_1,\alpha)-M_{X,t_1})(X(t_2,\alpha)-M_{Y,t_2})] = \int_A (x(t_2,\alpha)-M_{X,t_1})(Y(t_2,\alpha)-M_{Y,t_2})\,f_A(\alpha)\,d\alpha$$

* Note the expectations $E_A[\,]$ are over the whole ensemble.

Stationarity

- stationarity means the statistical characteristics of a signal do not change over time.
- A random process is strict sense stationary (SSS) if its probability distribution does not change over time.

Formally, for all finite N and all sets of time pts $\{t_1,\dots t_N\}$, the Nth order joint pdf is invariant for any time shift T.

$$f_{X(t_1)\dots X(t_N)}(x_1,\dots x_N) = f_{X(t_1+T)\dots X(t_N+T)}(x_1,\dots x_N)$$

- A random process is wide sense stationary (WSS) iff

(i) $E[X(t)] = M \;\forall t$ (const. mean)

(ii) $\mathrm{Var}[X(t)] < \infty$ / $E[X(t)^2] < \infty \;\forall t$ (finite variance/power) [ignored in 3F1]

(iii) $r_{XX}[t_1,t_2] \to r_{XX}[t_2-t_1]$ $\quad (r_{XX} = r_{XX}(t_2-t_1))$

- For WSS $\{X(t,\alpha)\}$, $r_{XX}[\tau] = r_{XX}[-\tau]$, i.e. $r_{XX}[\tau]$ is even.

$$r_{XX}[\tau] = E[X(t)X(t+\tau)] = E[X(t+\tau)X(t)] = r_{XX}[-\tau]$$

Moreover, as τ becomes large, $X(t)$ and $X(t+\tau)$ tends to become decorrelated. ($r_{XX}[\tau] \to E[X(t)]E[X(t+\tau)] = M^2$)

Therefore, $\max|r_{XX}[\tau]| = r_{XX}[0]$ (Formal proof in 3F3 notes)

The width of the autocorrelation function $r_{XX}[\tau]$ tells us how slowly the signal fluctuates in time (width $\propto \frac{1}{\text{freq}}$)

Power spectral density (PSD)

- For WSS $\{X(t)\}$ the PSD function is the FT of the autocorrelation function $r_{XX}[\tau]$

$$S_X(\omega) = \int_{-\infty}^{\infty} r_{XX}[\tau] e^{-j\omega\tau} d\tau \qquad r_{XX}[\tau] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega) e^{j\omega\tau} d\omega$$

- For jointly WSS $\{X(t,\alpha)\}, \{Y(t,\alpha)\}$, the cross PSD function is the FT of the crosscorrelation function $r_{XY}[\tau]$

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} r_{XY}[\tau] e^{-j\omega\tau} d\tau \qquad r_{XY}[\tau] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega) e^{j\omega\tau} d\omega$$

Ergodicity

- If we can exchange ensemble averages $E_A[\,]$ for time averages $\langle\,\rangle_T$, the random process is ergodic.
- This means the ensemble of the random process simply composed of all possible time shifts of a single random signal.
- Ensemble averages $E_A[\,]$ and time averages $\langle\,\rangle_T$ are defined as

$$E_A[X(t,\alpha)] = \int_A x(t,\alpha) f_A(\alpha) d\alpha \qquad \langle X(t,\alpha)\rangle_T = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t,\alpha) dt$$

- A WSS $\{X(t,\alpha)\}$ could be mean ergodic/correlation ergodic, defined as
  - ↳ mean ergodic $\mu = E_A[X(t,\alpha)] = \langle X(t,\alpha)\rangle_T$
  - ↳ correlation ergodic $r_{XX}[\tau] = E_A[X(t,\alpha)X(t+\tau,\alpha)] = \langle X(t,\alpha)X(t+\tau,\alpha)\rangle_T$
- In reality, no process is truly stationary (therefore cannot be ergodic), but many noise (white noise) processes are approx stationary → use ergodicity to simplify measurement of WSS process.
- We may estimate quantities from a finite period $[T_1, T_2]$ of a ergodic process using.

$$\mu \approx \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} X(t,\alpha) dt \qquad r_{XX}(t) \approx \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} X(t,\alpha) X(t+\tau,\alpha) dt$$

System response to random signals

Linear systems and random processes

- If the i/p $\{X(t)\}$ of a LTI system w/ impulse resp $h(t)$ is WSS, then the o/p $\{Y(t)\}$ is also WSS.

(i) Consider the mean $E[Y(t)]$

$$E[Y(t)] = E\left[\int h(k)X(t-k)dk\right] = \int h(k) E[X(t-k)]dk = \mu_X \int h(k)dk \to \text{const. over all } t.$$

(ii) consider the autocorrelation $r_{XX}[\tau]$

$$r_{YY}[\tau] = E\left[\int h(k_1)X(t-k_1)dk_1 \int h(k_2) X(t+\tau-k_2)dk_2\right] = E\left[\iint h(k_1)h(k_2)X(t-k_1)X(t+\tau-k_2)dk_1 dk_2\right]$$

$$= \iint h(k_1)h(k_2)E\left[X(t-k_1)X(t+\tau-k_2)\right)dk_1 dk_2 = \int h(k_1)\left(\int h(k_2) r_{XX}[\tau+k_1-k_2]dk_2\right)dk_1$$

$$= \int h(k_1)\left(h(\tau+k_1) * r_{XX}[\tau+k_1]\right)dk_1 = h(-\tau) * h(\tau) * r_{XX}[\tau] \to \text{only a function of } \tau.$$

- The i/p $\{X(t)\}$ and o/p $\{Y(t)\}$ of the LTI system is jointly WSS. We already know $\{X(t)\}, \{Y(t)\}$ are WSS.

$$r_{XY}[\tau] = E[X(t)Y(t+\tau)] = E\left[X(t)\int h(k)Y(t+\tau-k)dk\right] = \int h(k)E[X(t)X(t+\tau-k)]dk = \int h(k) r_{XX}[\tau-k]dk = h(\tau) * r_{XX}[\tau]$$

- Notice that the correlation functions can be expressed as convolutions

$$r_{YY}[\tau] = h(\tau) * h(-\tau) * r_{XX}[\tau] \qquad r_{XY}[\tau] = h(\tau) * r_{XX}[\tau]$$

- Taking the FT of the correlation functions expressed as convolutions,

$$S_Y(\omega) = H(\omega)H^*(\omega)S_X(\omega) = |H(\omega)|^2 S_X(\omega) \qquad S_{XY}(\omega) = H(\omega)S_X(\omega).$$

Rearranging for $H(\omega)$,

only magnitude: $$|H(\omega)| = \sqrt{\frac{S_Y(\omega)}{S_X(\omega)}}$$

both magnitude and phase: $$H(\omega) = \frac{S_{XY}(\omega)}{S_X(\omega)}$$

- If an important system is subject to measurable random fluctuations at the i/p, we can measure the system freq. response w/o taking the plant offline. (estimate $r_{XX}[\tau], r_{XY}[\tau], r_{YY}[\tau]$ w/ time averages $\rightarrow S_X(\omega), S_Y(\omega), S_{XY}(\omega) \rightarrow H(\omega)$)

Physical interpretation of PSD.

- For a deterministic signal $x(t)$, the instantaneous power is $x(t)^2$ and the average power is $\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)^2 dt$
- For a random process $\{X(t)\}$, the expected instantaneous power is $E[X(t)^2]$ and the expected average power is $E\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} X(t)^2 dt\right] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} E[X(t)^2]dt$
- If $\{X(t)\}$ is WSS, then $E[X(t)^2] = r_{XX}[0]$ for all t, so the expected average power is $r_{XX}[0]$.

$$r_{XX}[0] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)e^{j\omega(0)}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega$$

i.e. the area under $S_X(\omega)$ gives the total power of $\{X(t)\}$.

- To find the power of a WSS process $\{X(t)\}$, in a particular freq. band, say $[\omega_1, \omega_2]$ s.t. $0<\omega_1<\omega_2$, consider passing $\{X(t)\}$ through an ideal band-pass filter w/ TF $H(\omega) = \begin{cases} 1 & \text{if } \omega_1 \le |\omega| \le \omega_2 \\ 0 & \text{o/w} \end{cases}$, then find the o/p power $r_{YY}[0]$.

$$r_{YY}[0] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 S_X(\omega)d\omega = \frac{1}{2\pi}\left[\int_{-\omega_2}^{-\omega_1} S_X(\omega)d\omega + \int_{\omega_1}^{\omega_2} S_X(\omega)d\omega\right] = \frac{1}{\pi}\int_{\omega_1}^{\omega_2} S_X(\omega)d\omega$$

→ The power of $\{X(t)\}$ in a particular freq. band $\omega_1 \le \omega \le \omega_2$ is $\frac{1}{\pi}\int_{\omega_1}^{\omega_2} S_X(\omega)d\omega$.

White noise process.

- A WSS process $\{X(t)\}$ is termed white noise if

$$C_{XX}[\tau] = \sigma^2\delta(\tau)$$

where $\delta(\tau)$ is the delta function.

- The PSD $S_X(\omega)$ for white noise is given by

$$S_X(\omega) = \int_{-\infty}^{\infty} r_{XX}[\tau]e^{-j\omega\tau}d\tau = \int_{-\infty}^{\infty}(C_{XX}[\tau]+\mu^2)e^{-j\omega\tau}d\tau = \int_{-\infty}^{\infty}\sigma^2\delta(\tau)e^{-j\omega\tau}d\tau + \mu^2\int_{-\infty}^{\infty}e^{-j\omega\tau}d\tau = \sigma^2 + \mu^2 2\pi\delta(\omega)$$

For zero-mean white noise, $\mu = 0$, so $S_X(\omega) = \sigma^2$, i.e. flat across all freq.

- Note that power for white noise is $E[X(t)^2] = r_{XX}[0] = \sigma^2 + \mu^2 2\pi\delta(0) \rightarrow \infty$, so pure white noise is unrealizable in practice. (However, we can have "nearly white" processes).

From discrete time to continuous time.

- consider a discrete time i/p x_k that is zero for $k \notin [-\frac{N}{2}+1, \frac{N}{2}]$ and each x_k is sampled independently from a distribution w/ $\mu=0$, $\sigma^2=1$.

- Consider $|X(e^{j\theta})|^2 = X(e^{j\theta})X^*(e^{j\theta})$

$$|X(e^{j\theta})|^2 = \left(\sum_k x_k e^{-jk\theta}\right)\left(\sum_\ell x_\ell e^{j\ell\theta}\right) = \sum_k w_k^2 + \sum_{k\neq\ell} w_k w_\ell e^{j\theta(\ell-k)}$$

Taking expectations, noting that $E[x^2]=1$ and $E[x_k x_\ell]=E[x_k]E[x_\ell]=0$ for $k\neq\ell$,

$$E[|X(e^{j\theta})|^2] = N$$

$$\therefore \frac{1}{N}E[|Y(e^{j\theta})|^2] = \frac{1}{N}E[|H(e^{j\theta})|^2|X(e^{j\theta})|^2] = \frac{1}{N}|H(e^{j\theta})|^2 \underbrace{E[|X(e^{j\theta})|^2]}_{N} = |H(e^{j\theta})|^2$$

- we can derive the continuous case by taking the limit as sampling time tends to zero.

- consider a continuous time i/p $x_h(t)$ btwn $-T/2$ and $T/2$, given by

$$x_h(t) = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}+1} p(t-kh)\, w_k$$

where $h=T/N$ is the sampling time, and each w_k is sampled independently from a distribution w/ $\mu=0$, $\sigma^2=\frac{1}{h}$.

$p(t) = H(t)-H(t-h)$ is the pulse function.

- The FT of $x_h(t)$, $X_h(j\omega)$ is given by

FT[H(t)] FT[H(t-h)] time shift by kh

$$X_h(j\omega) = FT\left[\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}+1} p(t-kh)w_k\right] = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}+1} FT[p(t-kh)]w_k = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}+1}\left(\frac{1}{j\omega}-\frac{1}{j\omega}e^{-j\omega h}\right)e^{-j\omega kh}w_k$$

Consider $|X_h(j\omega)|^2 = X_h(j\omega)X_h^*(j\omega)$

$$|X_h(j\omega)|^2 = \left(\sum_k w_k\left(\frac{1-e^{-j\omega h}}{j\omega}\right)e^{-j\omega kh}\right)\left(\sum_\ell w_\ell\left(\frac{1-e^{j\omega h}}{-j\omega}\right)e^{j\omega\ell h}\right) = \sum_k w_k^2\left(\frac{1-e^{-j\omega h}}{j\omega}\right)\left(\frac{1-e^{j\omega h}}{-j\omega}\right) + \sum_{k\neq\ell} w_k w_\ell \left(\frac{1-e^{-j\omega h}}{j\omega}\right)\left(\frac{1-e^{j\omega h}}{-j\omega}\right)e^{j\omega(\ell-k)h}$$

Taking expectations, noting that $E[w_k^2]=\frac{1}{h}$ and $E[w_k w_\ell]=E[w_k]E[w_\ell]=0$ for $k\neq\ell$, $h=T/N \rightarrow T=Nh$

$$E[|X_h(j\omega)|^2] = \sum_k E[w_k^2]\left(\frac{(1-e^{-j\omega h})(1-e^{j\omega h})}{\omega^2}\right) = \sum_k\left(\frac{1}{h}\right)\left(\frac{(1-(1-j\omega h))(1-((1+j\omega h)))}{\omega^2}\right) = \sum_k h = T$$

$\therefore$ Define $x(t) = \lim_{h\to 0} x_h(t)$, i.e. white noise, then we have.

$$\frac{1}{T}E[|Y(j\omega)|^2] = \frac{1}{T}E[|H(j\omega)|^2|X(j\omega)|^2] = \frac{1}{T}|H(j\omega)|^2 \underbrace{E[|X(j\omega)|^2]}_{T} = |H(j\omega)|^2$$