Signals and channels

Signals

Signal energy and power

- The energy E_x of a signal $x(t)$ is defined as

$$\boxed{E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt}$$

- If $X(\omega)$ is the FT of $x(t)$, using Parseval's thm,

$$\boxed{E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |X(f)|^2 df}$$

($|X(f)|^2$ is the energy spectral density — $|X(f)|^2 df$ represents the energy of a signal in the frequency band $[f, f+df]$)

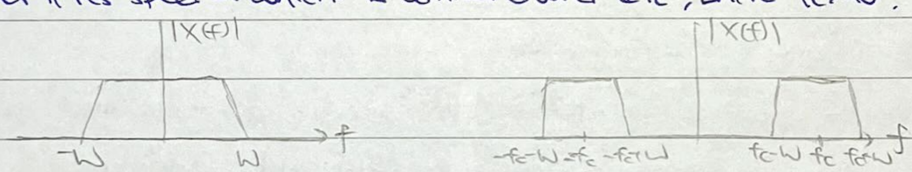
- For a signal $x(t)$ whose energy is infinite, the power P_x is defined as

$$\boxed{P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt}$$

- A lower transmit power implies a longer battery life for the device, but it also makes signal harder to detect at the receiver in the presence of noise.

Bandwidth

- Bandwidth (BW) is a measure of the extent of significant spectral content of the signal. It is roughly the range of frequencies over which its spectrum is nonzero.
- For real signals, BW is measured as the range of +ve frequencies as $|X(f)|$ is symmetric abt. 0.
- A signal is lowpass (baseband) if its spectral content is centred around $f=0$; A signal is passband if its spectral content is centred around $\pm f_c$, where $f_c \gg 0$.



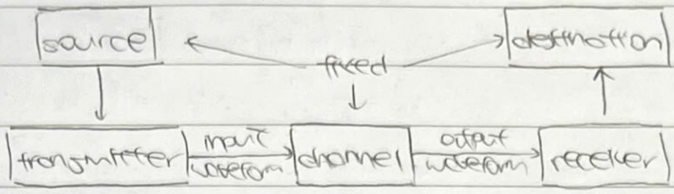
baseband signal w/ BW=W　　　passband signal w/ BW=2W.

- Most real-world signals are time-limited → not strictly limited in frequency → we have other definitions of BW:
  - ↳ 90% BW: The range of frequencies which contain 90% of the total energy of the spectrum
  - ↳ 3dB BW: The range of frequencies which contain 50% of the total energy of the spectrum
  - ↳ Null-to-null BW: The width of the "main lobe" of the spectrum (for $x(t)=\text{rect}(\frac{t}{T})$, $BW=\frac{1}{T}$).
- BW is a scarce resource, esp. in mobile (cellular) communication.
- Wired channels (telephone lines, USB cables) act as baseband filters (attenuates frequencies above W) so transmitted signals need to be bandlimited to W.

channels

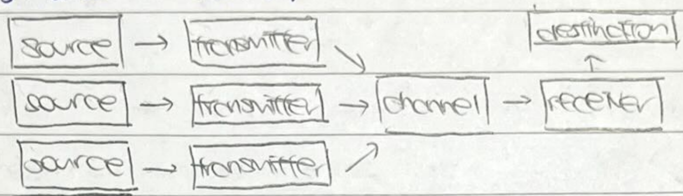
Communication

- communication is the process of delivering information source to a destination through a communication channel. The pt-to-pt communication channel is as follows:

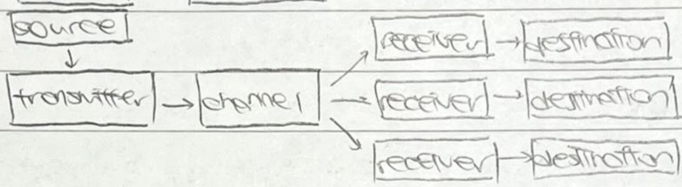


- More generally, we could have multiple sources delivering information to multiple destinations through a common channel.

Multiple access channel



broadcast channel



Communication channels

- A channel is the medium used to transmit the signal from transmitter to receiver.
- A channel introduces attenuation and noise → can cause errors at the receiver.
- Some real world channels include:
  - ↳ Mobile wireless channel (✗ has distortion due to multipath propagation and mobility)
  - ↳ Optical fibre channel (✓ large BW, cheap, low attenuation; ✗ dispersion of optical pulses)
  - ↳ Electrical wire channel (✓ cheap; ✗ limited BW, high attenuation)

Modelling a channel

- Channels are often modelled as linear systems w/ additive noise.
- Channel output $y(t)$ generated from input $x(t)$ for a channel w/ impulse response $h(t)$,

$$y(t) = h(t) * x(t) + n(t) \qquad \text{or} \qquad Y(f) = H(f)X(f) + N(f)$$

if the input is restricted to the band where the channel $H(f)$ is flat, then we have

$$y(t) = x(t) + n(t) \qquad \text{or} \qquad Y(f) = X(f) + N(f).$$

- $n(t)$ is the thermal noise at the Rx, which is generated by the thermal agitation of e^- inside an electrical conductor (happens regardless of applied voltage/type of Rx).
- Thermal noise $n(t)$ is modelled as a Gaussian random process (Gaussian additive noise)

Modulation

Modulation

- Modulation is the process by which some characteristic of a carrier wave is varied in accordance w/ an information bearing signal.
- A commonly used carrier is a sinusoidal wave $\cos(2\pi f_c t)$, f_c is the carrier frequency.
- Modulation is used instead of transmitting the original signal to reduce interference and antenna size ($l \approx \frac{\lambda}{4}$, so $\uparrow f_c \rightarrow \downarrow \lambda \rightarrow \downarrow l$).

Types of modulation.

- In analog modulation, a continuous information signal $x(t)$ is used to directly modulate the carrier wave
- In digital modulation, the continuous information signal $x(t)$ is first digitised into bits, then modulated to be transported across the channel.

Analog modulation

Amplitude modulation

Amplitude modulation (AM)

- For an information signal $x(t)$ and carrier $\cos(2\pi f_c t)$, the transmitted AM signal is.

$$\boxed{S_{AM}(t) = [a_0 + x(t)]\cos(2\pi f_c t)}$$

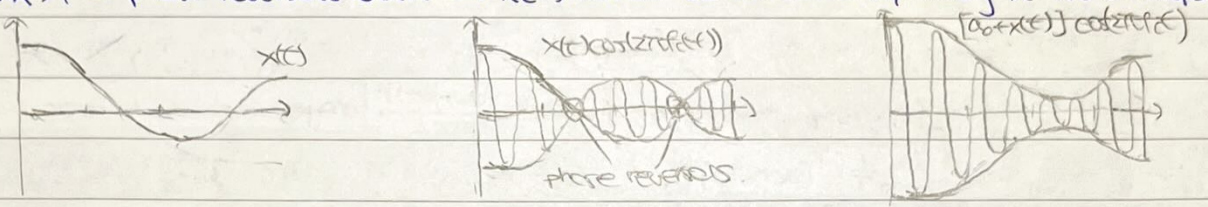
where a_0 is a +ve const. chosen s.t. $a_0 > \max_t |x(t)|$ (also the carrier amplitude).

- The modulation index M_A of the AM signal is defined as.

$$\boxed{M_A = \frac{\max_t |x(t)|}{a_0}}$$

- $M_A < 1$ is desirable because we can extract the information signal $x(t)$ from the modulated signal by envelope detection.

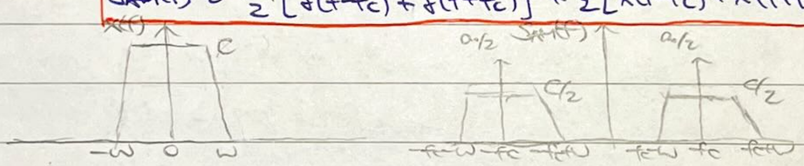
($M_A > 1$ → phase reversals occur → $x(t)$ cannot be detected by tracing the +ve envelope)



Spectrum of AM

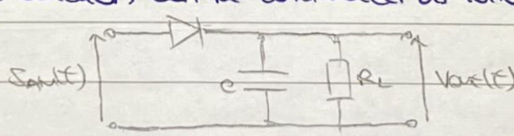
- The spectrum of the AM signal $S_{AM}(t) = [a_0 + x(t)]\cos(2\pi f_c t)$ is given by

$$S_{AM}(f) = FT[S_{AM}(t)]$$
$$= FT\left[[a_0 + x(t)]\frac{e^{j2\pi f_c t} + e^{-j2\pi f_c t}}{2}\right]$$
$$\boxed{S_{AM}(f) = \frac{a_0}{2}[\delta(f-f_c) + \delta(f+f_c)] + \frac{1}{2}[X(f-f_c) + X(f+f_c)]}$$



AM receiver

- The AM receiver (envelope detector) can be constructed as follows:



- On the +ve half-cycle of the input signal, capacitor C charges up rapidly to the peak value of input $S_{AM}(t)$. When the input falls below this peak, the diode becomes reverse-biased, and the capacitor discharges slowly through load R_L.
- In the next +ve half-cycle, when the input becomes greater than the voltage across capacitor C, the diode conducts again until its next peak value.
- The process repeats and the detector traces the +ve envelope, but has a jagged shape.
- The receiver is very inexpensive, but envelope detection req. $M_A < 1$.

properties of AM.

- The BW of AM is $\boxed{B_{AM} = 2W}$.

  ($x(t)$ is a baseband signal w/ one-sided BW W, $S_{AM}(t)$ is a passband signal w/ two sided BW $2W$)

- The power of AM is $\boxed{P_{AM} = \frac{a_0^2}{2} + \frac{P_x}{2}}$

  where P_x is the power of $x(t)$.

  ↳ Intuitively, we have $P_{AM} = 2\times\left[\left(\frac{a_0}{2}\right)^2 + \left(\frac{1}{2}\right)^2 P_x\right] = \frac{a_0^2}{2} + \frac{P_x}{2}$

  ↳ Formally, we have
$$P_{AM} = \lim_{T\to\infty}\frac{1}{T}\int_0^T [a_0 + x(t)]^2 \cos^2(2\pi f_c t)\,dt$$
$$= \lim_{T\to\infty}\frac{1}{T}\int_0^T [a_0 + x(t)]^2 \frac{1+\cos(4\pi f_c t)}{2}\,dt$$

x(t) has zero mean as any nonzero mean can be absorbed into a_0

$$= \frac{a_0^2}{2} + \lim_{T\to\infty} \frac{a_0}{T}\cancelto{0}{\int_0^T x(t)\,dt} + \frac{1}{T}\int_0^T x(t)^2\,dt + \frac{1}{T}\int_0^T [a_0+x(t)]^2 \frac{\cos(4\pi f_c t)}{2}\,dt$$
$$= \frac{a_0^2}{2} + \frac{P_x}{2} + \lim_{T\to\infty}\frac{1}{T}\int_0^T [a_0+x(t)]^2 \frac{\cos(4\pi f_c t)}{2}\,dt$$

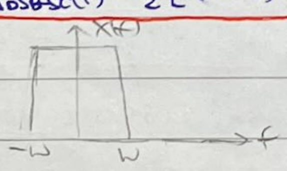
Consider the last term $\lim_{T\to\infty}\frac{1}{T}\int_0^T [a_0+x(t)]^2 \frac{\cos(4\pi f_c t)}{2}\,dt$. $\cos(4\pi f_c t)$ is a high frequency sinusoid w/ period $T_c = \frac{1}{2f_c}$. $g(t) = \frac{[a_0+x(t)]^2}{2}$ changes much slower than $\cos(4\pi f_c t)$,

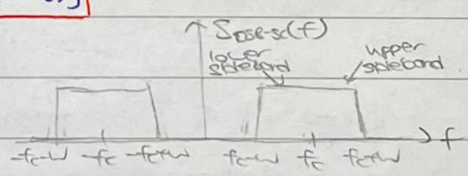
using $T = nT_c$, $\frac{1}{T}\int_0^T g(t)\cos(4\pi f_c t)\,dt \approx \frac{1}{nT_c}\left[\int_0^{T_c} g(0)\cos(4\pi f_c t)\,dt + \dots + \int_{(n-1)T_c}^{T_c} g((n-1)T_c)\cos(4\pi f_c t)\,dt\right]$
$$= \frac{1}{nT_c}\left[g(0)\int_0^{T_c}\cos(4\pi f_c t)\,dt + \dots + g((n-1)T_c)\int_{(n-1)T_c}^{T_c}\cos(4\pi f_c t)\,dt\right]$$
$$= 0.$$

$$\therefore P_{AM} = \frac{a_0^2}{2} + \frac{P_x}{2}$$

Double sideband suppressed carrier. (DSB-SC).

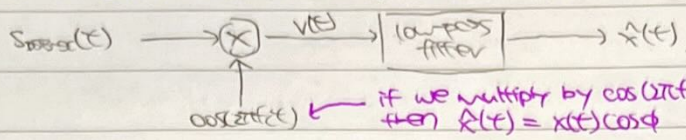
- In DSB-SC, we eliminate a_0 — transmit only the sidebands, suppress the carrier.
- For an information signal $x(t)$ and carrier $\cos(2\pi f_c t)$, the transmitted DSB-SC signal is
$$S_{DSB-SC}(t) = x(t)\cos(2\pi f_c t)$$
- The spectrum of the DSB-SC signal $S_{DSB-SC}(t) = x(t)\cos(2\pi f_c t)$ is given by
$$S_{DSB-SC}(f) = FT[S_{DSB-SC}(t)]$$
$$= FT\left[x(t)\,\frac{e^{j2\pi f_c t} + e^{-j2\pi f_c t}}{2}\right]$$
$$\boxed{S_{DSB-SC}(f) = \frac{1}{2}[X(f-f_c) + X(f+f_c)]}$$





DSB-SC receiver.

- The DSB-SC receiver can be constructed by a product modulator + LPF



if we multiply by $\cos(2\pi f_c t + \phi)$, then $\hat{x}(t) = x(t)\cos\phi$

Multiplying the received signal by $\cos(2\pi f_c t)$ gives.

$$v(t) = x(t)\cos^2(2\pi f_c t) = \underbrace{\frac{x(t)}{2}}_{\text{low freq.}} + \underbrace{\frac{x(t)\cos(4\pi f_c t)}{2}}_{\text{high freq.}}$$

The LPF eliminates the high freq. component. The ideal LPF has a transfer function $H(f)$

$$H(f) = 2\,\text{rect}\left(\frac{f}{2W}\right)$$

some constant to scale $\hat{x}(t)$ accordingly.

Properties of DSB-SC.

- The BW of DSB-SC is $\boxed{B_{DSB-SC} = 2W}$ (same as AM)
- The power of DSB-SC is $\boxed{P_{DSB-SC} = \frac{P_x}{2}}$ (same as AM, w/ $a_0 = 0$)

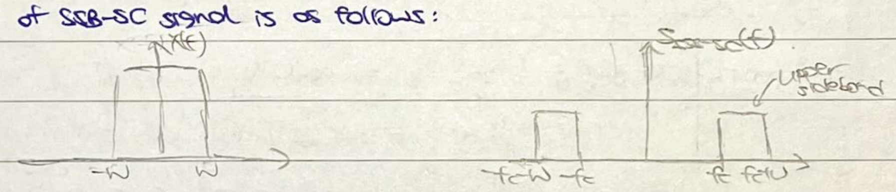
(DSB-SC req. less power than AM as the carrier is not transmitted but the DSB-SC receiver is more complex).

Single sideband suppressed carrier (SSB-SC).

- In SSB-SC, we only transmit the upper sideband.

($x(t)$ is real $\rightarrow X(-f) = X^*(f) \rightarrow$ we could obtain one sideband from the other).

- The spectrum of SSB-SC signal is as follows:



Properties of SSB-SC.

- The BW of SSB-SC is $B_{SSB-SC} = W$ (half of that of AM/DSB-SC)
- The power of SSB-SC is $P_{SSB-SC} = \frac{P_x}{4}$ (half of that of DSB-SC)

(SSB-SC req. less power than DSB-SC at the expense of a more complex transmitter).

Frequency modulation

Frequency modulation (FM)

- In FM, the information signal $x(t)$ modulates the instantaneous frequency of the carrier wave.
  The instantaneous frequency $f(t)$ is varied linearly w/ $x(t)$.

$$\boxed{f(t) = f_c + k_f x(t)}$$

  where k_f is the frequency-sensitivity factor.

  This translates to an instantaneous phase $\theta(t)$ given by

$$\boxed{\theta(t) = 2\pi\int_0^t f(u)\,du = 2\pi f_c t + 2\pi k_f\int_0^t x(u)\,du}$$

- For an information signal $x(t)$, the transmitted FM signal is

$$\boxed{S_{FM}(t) = A_c\cos(\theta(t)) = A_c\cos\left(2\pi f_c t + 2\pi k_f\int_0^t x(u)\,du\right)}$$

  where A_c is the carrier amplitude.

- Consider FM modulation of a tone $x(t) = a_x\cos(2\pi f_x t)$, we have

$$f(t) = f_c + k_f a_x\cos(2\pi f_x t) \xrightarrow{\theta(t) = 2\pi\int_0^t f(t)dt} \theta(t) = 2\pi f_c t + \frac{k_f a_x}{f_x}\sin(2\pi f_x t)$$

  $\Delta f = k_f a_x$ is the frequency deviation — max. deviation of carrier frequency $f(t)$ from f_c.

  $\beta = \frac{k_f a_x}{f_x} = \frac{\Delta f}{f_x}$ is the modulation index — max. deviation of carrier phase $\theta(t)$ from $2\pi f_c t$.

$$\therefore \boxed{S_{FM}(t) = A_c\cos(2\pi f_c t + \beta\sin(2\pi f_x t))}$$

Spectrum of FM

- The FM modulation of a tone $x(t) = a_x\cos(2\pi f_x t)$ can be rewritten as

$$\boxed{S_{FM}(t) = A_c\sum_{n=-\infty}^{\infty} J_n(\beta)\cos(2\pi(f_c + nf_x)t)}$$

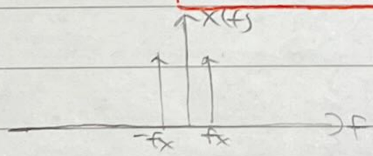
  where $J_n(\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta\sin u - nu)}\,du$ is the nth order Bessel function of the first kind.

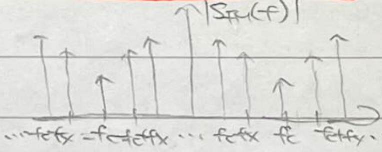
- The spectrum of the FM modulation of the tone is given by

$$S_{FM}(f) = FT[S_{FM}(t)]$$

$$= FT\left[A_c\sum_{n=-\infty}^{\infty} J_n(\beta)\,\frac{e^{j2\pi(f_c+nf_x)t} + e^{-j2\pi(f_c+nf_x)t}}{2}\right]$$

$$\boxed{S_{FM}(f) = \frac{A_c}{2}\sum_{n=-\infty}^{\infty} J_n(\beta)\left[\delta(f - f_c - nf_x) + \delta(f + f_c + nf_x)\right]}$$





* Consider $Re[\hat{s}(t)e^{j2\pi f_c t}]$, $\hat{s}(t) = A_c e^{j\beta\sin(2\pi f_x t)}$

$Re[\hat{s}(t)e^{j2\pi f_c t}] = Re[A_c e^{j[2\pi f_c t + \beta\sin(2\pi f_x t)]}] = A_c\cos(2\pi f_c t + \beta\sin(2\pi f_x t)) = S_{FM}(t)$.

$\hat{s}(t)$ is periodic w/ fundamental frequency f_x, so its Fourier expansion coefficient is

$$c_n = f_x\int_{-1/2f_x}^{1/2f_x}\hat{s}(t)e^{-j2\pi nf_x t}\,dt = f_x\int_{-1/2f_x}^{1/2f_x} A_c e^{j(\beta\sin(2\pi f_x t) - 2\pi nf_x t)}\,dt \overset{u = 2\pi f_x t}{=} \frac{A_c}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta\sin u - nu)}\,du = A_c J_n(\beta)$$

$$\therefore S_{FM}(t) = Re[\hat{s}(t)e^{j2\pi f_c t}] = Re\left[\sum_{n=-\infty}^{\infty} A_c J_n(\beta)e^{j2\pi nf_x t}e^{j2\pi f_c t}\right] = A_c\sum_{n=-\infty}^{\infty} J_n(\beta)\cos(2\pi(f_c + nf_x)t)$$

FM receiver.

- The FM receiver can be constructed by a differentiator + envelope detector.

$$s_{FM}(t) \longrightarrow \boxed{\text{differentiator}} \xrightarrow{v(t)} \boxed{\text{envelope detector}} \longrightarrow \hat{x}(t)$$

Differentiating the received signal gives

$$v(t) = \frac{d}{dt}\left(A_c \cos\left(2\pi f_c t + 2\pi k_f \int_0^t x(u)du\right)\right) = -2\pi A_c (f_c + k_f x(t)) \sin\left(2\pi f_c t + 2\pi k_f \int_0^t x(u)du\right)$$

The derivative is a passband signal w/ amplitude modulation by $(f_c + k_f x(t))$ → for large f_c, (s.t. $M_A = \frac{k_f}{f_c} < 1$), we can recover $x(t)$ by envelope detection.

Properties of FM.

absolute BW is infinite, but side components at $f_c \pm n f_x$ negligible for large n.

- The BW of FM is $\boxed{B_{FM} \approx 2\Delta f + 2W}$
  The BW of FM of a single tone is $\boxed{B_{FM} \approx 2\Delta f + 2f_x = 2\Delta f\left(1 + \frac{1}{\beta}\right)}$

Carson's rule for the effective BW of FM signals.

- The power of FM is $\boxed{P_{FM} = \frac{A_c^2}{2}}$
- FM is more robust to additive noise than AM, but at the cost of increased transmission BW.

Digitisation of analog signals.

Digitisation of analog signals.

Types of sources.

- There are two types of information sources:
  - ↳ Analog — continuous time, continuous-amplitude sources (e.g. speech, music).
  - ↳ Digital — can be represented into bits (e.g. email, png). In general, a digital source is a discrete-time sequence of symbols drawn from a finite alphabet. (e.g. $X_1, X_2, X_3 \dots$ where $X_i \in \underbrace{\{a, b, c, d\}}_{\text{alphabet}}$ for all i).

Digitisation

- The process by which an analog signal is converted into digital format (continuous in time and amplitude → discrete in time and amplitude) is known as digitisation.
- Digitisation consists of:
  - ↳ 1) Sampling (discretises the time axis)
  - ↳ 2) Quantisation (discretises the signal amplitude axis).
- There are many advantages of transmitting digital signals:
  - ↳ Robustness — digital signals are not attenuated over long distances → no need to amplify signal (which would also amplify noise) → can be transmitted over long distances.
  - ↳ Performance — powerful error-correcting codes can correct bit errors that may occur in the transmission of digital signals.
  - ↳ Encryption — digital communication systems can be made highly secure by exploiting powerful encryption algorithms.

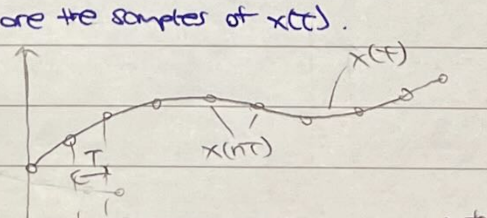
Sampling.

Sampling.

- Let $x(t)$ be a continuous-time signal for $-\infty < t < \infty$. We choose a sampling interval T, and read off the values of $x(t)$ at times

$$\dots, -3T, -2T, -T, 0, T, 2T, 3T \dots$$

The obtained values $x(nT)$ are the samples of $x(t)$.



Recovering original information from samples.

- Consider a signal $x(t)$ w/ BW W. We can recover $x(t)$ from its samples $\{x(nT)\}$ provided that the sampling frequency $f_s = \frac{1}{T}$ satisfies $\boxed{f_s > 2W}$ (Nyquist rate)
- A continuous-time representation of the sampled signal is,

$$x_s(t) = \sum_n x(nT)\,\delta(t-nT) = x(t)\sum_n \delta(t-nT)$$

$\sum_n \delta(t-nT)$ is periodic and can be expressed as a Fourier series.

$$\sum_n \delta(t-nT) = \frac{1}{T}\sum_n e^{jn\frac{2\pi}{T}t}$$

$$\therefore\ x_s(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} x(t)\,e^{jn\frac{2\pi}{T}t}$$

- The spectrum of the sampled signal $X_s(f)$ is therefore

$$X_s(f) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X\left(f-\frac{n}{T}\right)$$

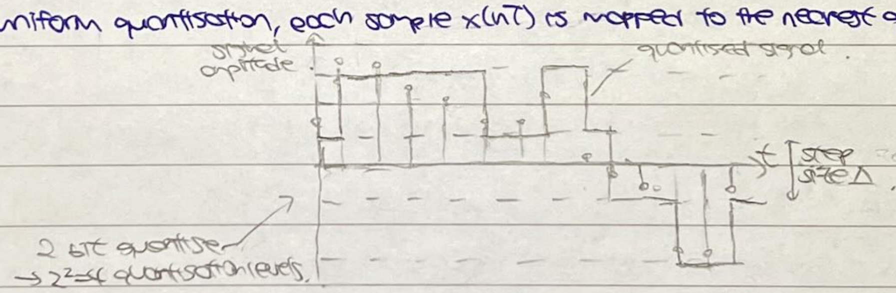
- The spectrum of the original signal $X(f)$ can be obtained from $X_s(f)$ using an ideal reconstruction / anti-aliasing filter. ($H(f) = \mathrm{rect}\left(\frac{f}{2W}\right)$)

* Sampling is a lossless procedure as long as the sampling rate is greater than the Nyquist rate.
→ $x(t)$ can be perfectly reconstructed from its samples $x(nT)$.

Quantisation

Uniform quantisation.

- The sampled signal can take continuous values. To convert it into digital, we first assign a discrete amplitude from a finite set of levels (w/ steps Δ) and assign bits to those amplitudes.
- For uniform quantisation, each sample $x(nT)$ is mapped to the nearest quantisation level.



* Quantisation is always lossy — you cannot recover $x(nT)$ from its quantised value

Quantisation noise

- If $Q(z)$ denotes the quantised value of a sample $z=x(n\tau)$, the quantisation noise $e_Q(z)$ is defined as.

$$\boxed{e_Q(z) = z - Q(z)}$$

- If the quantiser step size is Δ, then $e_Q(z)$ lies in the interval $[-\frac{\Delta}{2}, \frac{\Delta}{2}]$.
- We can model e_Q as a RV uniformly distributed in $[-\frac{\Delta}{2}, \frac{\Delta}{2}]$, i.e.

$$\boxed{e_Q(z) \sim \text{unif}\left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right]}.$$

The power of the noise is therefore

pdf of $e_Q(z)$.

$$\boxed{N_Q = E[e_Q^2] = \int_{-\Delta/2}^{\Delta/2} u^2 \cdot \frac{1}{\Delta} du = \frac{1}{\Delta}\left[\frac{u^3}{3}\right]_{-\Delta/2}^{\Delta/2} = \frac{\Delta^2}{12}}$$

and the corresponding RMS value of the noise is

$$\sqrt{N_Q} = \sqrt{E[e_Q^2]} = \frac{\Delta}{\sqrt{12}}$$

Signal to noise ratio.

- Consider a sinusoidal signal taking values btwn $-V$ and $+V$. The signal power is $\frac{V^2}{2}$ and the RMS signal is $\frac{V}{\sqrt{2}}$.
- The signal-to-noise ratio (SNR) is defined as.

$$\boxed{\text{SNR} = \frac{\text{signal power}}{\text{noise power}} = \frac{(\text{RMS signal})^2}{(\text{RMS noise})^2}}$$

- For the sinusoidal signal, the SNR is given by

$$\text{SNR} = \frac{V^2/2}{\Delta^2/12} = 6\frac{V^2}{\Delta^2}.$$

- An n-bit uniform quantiser has 2^n levels, and step size Δ is given by

$$\boxed{\Delta = \frac{V-(-V)}{2^n} = V \cdot 2^{1-n}}$$

Therefore the SNR can be written as.

$$\text{SNR} = 3 \cdot 2^{2n-1} = (1.76 + 6.02n)\text{dB}$$

- For a fixed signal amplitude $\pm V$, a larger n means a smaller step size, and lower quantisation noise (larger SNR), but more bits need to be transmitted.

Data rate of the digitised source

- If we sample a signal $x(t)$ having a BW W at the Nyquist rate $2W$, and we use an n-bit uniform quantiser, the digitised source will have a rate of

$$\boxed{R = n2W \text{ bits/s}}$$

($2W$ samples/s, each represented w/ n bits)

Non-uniform quantisation.

- Non-uniform quantisers can reduce the req. bit-rate by roughly a factor of 5, they use smaller step sizes for more frequently occurring symbol values and larger stepsizes for rarer symbol values.
- Non-uniform quantisation is sometimes also known as companding.

Digital modulation

Digital baseband modulation

Phase amplitude modulation (PAM).

- The PAM modulation scheme has two basic components:
  - ↳ 1) Mapping bits to real/complex no → constellation symbols
  - ↳ 2) Modulate the unit-energy baseband waveform $p(t)$ using the constellation symbols.
- The set of values the bits are mapped to is the constellation.
  - ↳ eg: binary symbols $0 \to -A, 1 \to A$ $\{-A, A\}$ binary constellation // 4-ary symbols $00 \to -3A, 01 \to -A, 10 \to A, 11 \to 3A$ $\{-3A, -A, A, 3A\}$ 4-ary constellation.
- Once we fix a constellation, a sequence of bits can be uniquely mapped to constellation symbols.
  - ↳ e.g: 01,01110010: using $\{-A, A\}$ $-A, A, -A, A, A, A, -A, -A, A, -A$.

    using $\{-3A, -A, A, 3A\}$ $-A, -A, 3A, -3A, A$
- In a constellation w/ M symbols, each symbol represents $\log_2 M$ bits.
- The unit-energy baseband waveform $p(t)$ used in PAM is the pulse shape. Common pulse shapes are the sinc pulse and the rectangular pulse.

$$\boxed{p(t) = \frac{1}{\sqrt{T}} \operatorname{sinc}\left(\frac{\pi t}{T}\right)} \qquad \boxed{p(t) = \frac{1}{\sqrt{T}} \operatorname{rect}(T)}$$

  where T is the symbol time of the pulse
- The sequence of constellation symbols $X_0, X_1 \ldots$ is used to modulate a series of pulses to generate a baseband signal as follows:

$$\boxed{x_b(t) = \sum_k X_k p(t - kT)}$$

- Every T seconds, a new symbol is introduced by shifting the pulse and modulating its amplitude w/ the symbol → $\boxed{\text{transmission rate is } \frac{1}{T} \text{ symbols/s, or } \frac{\log_2 M}{T} \text{ bits/s}}$

Desirable properties of the pulse shape $p(t)$.

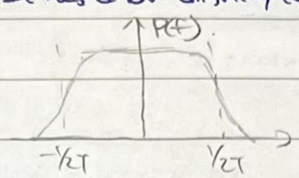
- The pulse shape $p(t)$ is chosen to satisfy the following important objectives:
  - ↳ $p(t)$ decays quickly in time — the effect of X_k should not start much before $t = kT$ or last much beyond $t = (k+1)T$.
  - ↳ $p(t)$ should be approx. band limited. For a fixed sequence of symbols $\{X_k\}$, the spectrum of $x_b(t)$ is.

$$X_b(f) = FT[x_b(t)] = FT\left[\sum_k X_k p(t - kT)\right] = P(f) \underbrace{\sum_k X_k e^{-j2\pi f kT}}_{\text{non-zero}}$$

    $\therefore$ BW of $x_b(t) \propto$ BW of $p(t)$ → bandlimited pulse gives bandlimited PAM signal.
  - ↳ The retrieval of the information sequence from the noisy received waveform $y(t) = x_b(t) + n(t)$ should be simple and rel reliable. In the absence of noise, the symbols $\{X_k\}_{k \in \mathbb{Z}}$ should be recovered perfectly at the receiver.

Time decay vs Bandwidth tradeoff.

- Ideally, we want both quick decay in time and approx. band-limited. However a faster decay in time means a larger BW.
- For the sinc pulse, $p(t)=\frac{1}{\sqrt{T}}\text{sinc}\left(\frac{\pi t}{T}\right)$, it's perfectly band-limited to $BW=\frac{1}{2T}$ but decays slowly in time $|p(t)| \sim \frac{1}{|t|}$.
- For the rectangular pulse, $p(t)=\frac{1}{\sqrt{T}}\text{rect}\left(\frac{t}{T}\right)$, it decays slowly in freq. $|P(f)| \sim \frac{1}{|f|}$, but perfectly time-limited to the interval $[-T/2, T/2]$.
- In practice, the root raised cosine pulse is used. It has a BW slightly larger than $\frac{1}{2T}$, but decays quicker in time, $|p(t)| \sim \frac{1}{|t|^2}$.



Orthonormality of pulse shifts.

- The orthonormal shift property of pulses make them simple and reliable to detect. It is as follows:

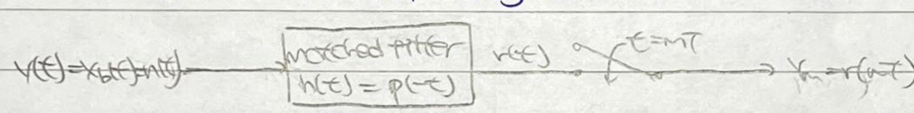
$$\boxed{\int_{-\infty}^{\infty} p(t-kT)\,p(t-mT)\,dt = \begin{cases} 1 & \text{if } k=m \\ 0 & \text{if } k\neq m \end{cases}}$$

← integral is 1 if the arguments of both functions are the same.

- Both the sinc and rectangular pulses satisfy the orthonormal shift property.

PAM receiver.

- The PAM receiver can be constructed by sampling the output of a matched filter.



- First consider the operation assuming no noise, i.e. $y(t)=x_b(t)$.

Passing $y(t)$ through the matched filter (impulse response $h(t)=p(-t)$), the filtered output is

$$\begin{aligned} r(t) = y(t)*h(t) &= x_b(t)*h(t). \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \\ &= \int_{-\infty}^{\infty} \sum_k x_k\, p(\tau-kT)\,p(\tau-t)\,d\tau \\ &= \sum_k x_k \int_{-\infty}^{\infty} p(\tau-kT)\,p(\tau-t)\,d\tau \\ &= \sum_k x_k \int_{-\infty}^{\infty} p(u+t-kT)\,p(u)\,du \qquad u=\tau-t. \\ &= \sum_k x_k\, g(t-kT) \end{aligned}$$

where $g(t)=\int_{-\infty}^{\infty} p(u+t)\,p(u)\,du$.

Sampling the filter output $r(t)$ at time $t=mT$, $m=0,1,2\ldots$, you get

$$r(mT) = \sum_k x_k\, g((m-k)T)$$

Applying the orthonormal shift property of $p(t)$, — integral is 1 if the arguments of both functions are the same

$$g((m-k)T) = \int_{-\infty}^{\infty} p(u+(m-k)T)\,p(u)\,du = \begin{cases} 1 & \text{if } k=m \\ 0 & \text{if } k\neq m \end{cases}$$

$$\therefore \boxed{r(mT) = \sum_k x_k\, g((m-k)T) = x_m}$$

PAM demodulation w/ noisy $y(t)$.

- Now consider the noisy case, i.e. $y(t) = x_b(t) + n(t)$.

Passing $y(t)$ through the matched filter (impulse response $h(t) = p(-t)$), the filtered output is.

$$r(t) = y(t) * h(t) = x_b(t) * h(t) + n(t) * h(t)$$
$$= \sum_k X_k g(t-kT) + \int_{-\infty}^{\infty} n(\tau) h(t-\tau)\, d\tau$$
$$= \sum_k X_k g(t-kT) + \int_{-\infty}^{\infty} n(\tau) p(\tau - t)\, d\tau$$

Sampling the filtered output $r(t)$ at time $t = mT$, $m = 0, 1, 2 \dots$, you get.

$$r(mT) = X_m + N_m$$

where $N_m = \int_{-\infty}^{\infty} n(\tau) p(\tau - mT)\, d\tau$.

- Denote $r(mT)$, the sampled output at time $t = mT$ by Y_m, so

$$\boxed{Y_m = X_m + N_m}, \quad m = 0, 1, 2 \dots$$

→ converted the continuous-time problem into a discrete time one of detecting the symbols X_m from the noisy outputs Y_m.

- If we model the noise $n(t)$ as a Gaussian process, the sequence of RVs $\{N_m\}_{m=0,1,2\dots}$ are iid. $N(0, \sigma^2)$, where σ^2 can be estimated empirically.

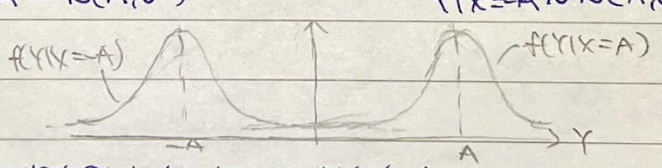
Detection for binary PAM.

- Consider a constellation where each $X_m \in \{-A, A\}$. This is binary PAM or binary phase shift keying (BPSK).
- As defined above, $Y = X + N$. So we observe that.

$$Y = A + N \text{ if } X = A \qquad Y = -A + N \text{ if } X = -A.$$

and since $N \sim N(0, \sigma^2)$, we can say.

$$Y \mid X = A \sim N(A, \sigma^2) \qquad Y \mid X = -A \sim N(-A, \sigma^2).$$



- Let $\hat{X}$ denote the decoded symbol. When the symbols A and $-A$ are a priori equally likely, the optimal detection rule is.

$$\hat{X} = A \text{ if } f(Y|X=A) \geq f(Y|X=-A) \qquad \hat{X} = -A \text{ if } f(Y|X=-A) \geq f(Y|X=A)$$

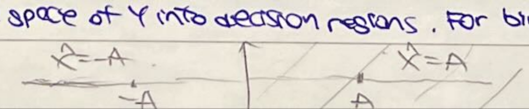
i.e. choose the symbol from which Y is most likely to have occurred (max-likelihood decoder).

- The detection rule can be more compactly written as,

$$\boxed{\hat{X} = \underset{x \in \{-A, A\}}{\arg\max}\, f(Y|X=x) = \underset{x \in \{-A, A\}}{\arg\max} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(Y-x)^2/2\sigma^2} = \underset{x \in \{-A, A\}}{\arg\min} (Y-x)^2}$$

i.e. choose the constellation symbol closest to the output Y.

- The decision rule partitions the space of Y into decision regions. For binary PAM,



Detection for general PAM constellations.

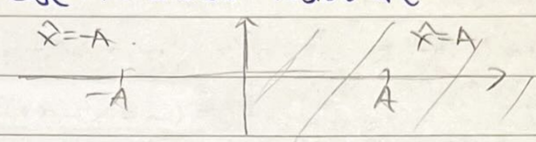
- The detection rule can be extended to a general constellation C.
- We still apply the max-likelihood principle – choose the constellation symbol from which Y is most likely to have occured.

$$\boxed{\hat{X} = \underset{x \in C}{\arg\max}\, f(Y \mid X = x) = \underset{x \in C}{\arg\max} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(Y-x)^2/2\sigma^2} = \underset{x \in C}{\arg\min} (Y-x)^2}$$

i.e. choose the constellation symbol closest to the output Y.

Probability of detection error

- Consider the binary PAM case. The decision regions are.



The detector makes an error when $X=A$ and $Y<0$ or when $X=-A$ and $Y>0$.

The probability of detection error is

$$P_e = P(\hat{X} \neq X) = P(\hat{X}=-A \mid X=A)P(X=A) + P(\hat{X}=A \mid X=-A)P(X=-A)$$

$$= \tfrac{1}{2} P(\hat{X}=-A \mid X=A) + \tfrac{1}{2} P(\hat{X}=A \mid X=-A)$$

symbols equally likely $\therefore P(X=-A)=P(X=A)=1/2$.

Consider $P(\hat{X}=A \mid X=-A)$.

$$P(\hat{X}=A \mid X=-A) = P(Y>0 \mid X=-A)$$

$$= P(-A+N>0 \mid X=-A)$$

$$= P(N>A \mid X=-A)$$

$$= P(N>A)$$

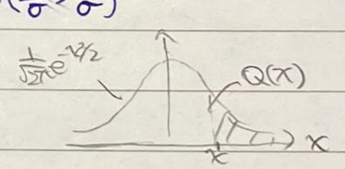
Similarly, and using symmetry,

$$P(\hat{X}=-A \mid X=A) = P(N<-A) = P(N>A).$$

standard Gaussian $\frac{N}{\sigma} \sim N(0,1)$

$$\therefore P_e = \tfrac{1}{2} P(\hat{X}=-A \mid X=A) + \tfrac{1}{2} P(\hat{X}=A \mid X=-A) = P(N>A) = P\left(\frac{N}{\sigma} > \frac{A}{\sigma}\right)$$

- We can express this using the Q function, defined as.



$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du = 1 - \Phi(x).$$

$$\therefore \boxed{P_e = P\left(\frac{N}{\sigma} > \frac{A}{\sigma}\right) = Q\left(\frac{A}{\sigma}\right)}$$

Power of PAM signal

- w/ any constellation, the power of the baseband PAM signal $x_b(t)$ is

$$\boxed{P_{PAM} = \frac{E_s}{T} = \frac{E_b \log_2 M}{T}}$$

where E_s is the average symbol energy of the constellation

and E_b is the average energy per bit.

- Intuitively, in each symbol period of length T, a symbol w/ average energy E_s modulates a unit energy pulse.

Probability of detection error and signal-to-noise ratio

- The average symbol energy E_s of the binary constellation is

$$\boxed{E_s = \frac{1}{2}(A^2 + (-A)^2) = A^2}$$

- For a binary constellation, each symbol corresponds to 1 bit, so the average energy per bit is

$$\boxed{E_b = \frac{E_s}{\log_2 M} = A^2}$$

- The probability of detection error P_e can be rewritten as

$$\boxed{P_e = Q\left(\frac{A}{\sigma}\right) = Q\left(\sqrt{\frac{E_b}{\sigma^2}}\right)}$$

where E_b/σ is the signal-to-noise ratio (SNR) of the transmission scheme.

- Since $Q(x) \sim e^{-x^2/2}$, $P_e = Q(\sqrt{SNR}) \sim e^{-SNR/2}$ which means P_e decays rapidly as SNR↑

- We cannot increase the SNR E_b/σ indefinitely by increasing E_b as the transmitted power would also increase ($P_{PAM} = \frac{E_b \log_2 M}{T}$), so ↑SNR → ↑$P_{PAM}$.

Digital passband modulation

Upconverted PAM.

- To upconvert from baseband to passband, we modulate the amplitude of a high frequency carrier using $x_b(t)$, so

$$\boxed{x(t) = x_b(t)\cos(2\pi f_c t) = \left[\sum_k X_k p(t-kT)\right]\cos(2\pi f_c t)}$$

- Upconverted PAM is the digital counterpart of DSB-SC, where $S_{DSB\text{-}SC}(t) = x(t)\cos(2\pi f_c t)$

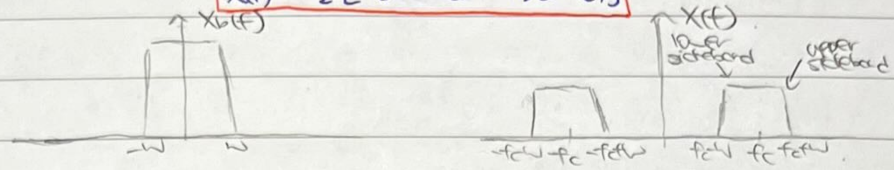
- Upconverted PAM w/ a rectangular pulse shape is known as amplitude shift keying (ASK)

(The rectangular pulse, and thus ASK is not BW efficient).

Spectrum of up-converted PAM

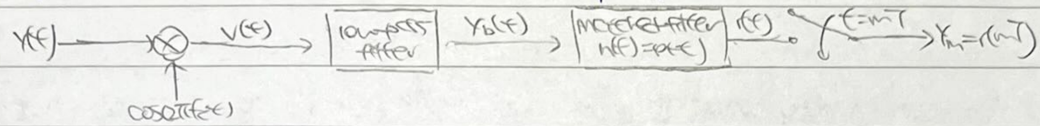
- The spectrum of the upconverted PAM signal $x(t) = x_b(t)\cos(2\pi f_c t)$ is given by.

$$X(f) = FT[x(t)]$$
$$= FT\left[x_b(t)\frac{e^{j2\pi f_c t} + e^{-j2\pi f_c t}}{2}\right]$$
$$\boxed{X(f) = \frac{1}{2}[X_b(f-f_c) + X_b(f+f_c)]}$$



Upconverted PAM receiver

- The upconverted PAM receiver can be constructed by a product modulator + LPF + PAM receiver.



- At the receiver, we have $y(t) = x_b\cos(2\pi f_c t) + n(t)$.

  Multiplying the received signal by $\cos(2\pi f_c t)$ gives.

$$v(t) = x_b\cos^2(2\pi f_c t) + n(t)\cos(2\pi f_c t) = \underbrace{\frac{x_b(t)}{2}}_{\text{low freq.}} + \underbrace{\frac{x_b(t)\cos(4\pi f_c t)}{2}}_{\text{high freq.}} + \underbrace{n(t)\cos(2\pi f_c t)}_{\text{noise}}$$

  The LPF eliminates the high freq. component. The ideal LPF has a transfer function $H(f)$.

$$H(f) = 2\,\mathrm{rect}\left(\frac{f}{2W}\right)$$

  After down-converting the received signal, we get the baseband waveform $y_b(t)$

$$y_b(t) = x_b(t) + \underbrace{n_b(t)}_{\text{filtered noise}}$$

  This waveform can then be demodulated and detected like the usual PAM signal.

Quadrature amplitude modulation (QAM)

- For PAM, $x_b(t) = \sum_k X_k\, p(t-kT)$ is a real signal since both the pulse $p(t)$ and the symbols $\{X_k\}$ are real-valued. $\rightarrow X_b(-f) = X_b^*(f)$. $\rightarrow$ we could obtain one sideband from the other.
- To save BW, we can transmit only one sideband (as in SSB-SC), but as we are transmitting digital information, we can simply make the information symbols $\{X_k\}$ complex valued.
- The baseband waveform is therefore still

$$\boxed{x_b(t) = \sum_k X_k\, p(t-kT)}$$

In PAM, $\mathrm{Im}(X_k) = 0$.

  but the constellation from which the symbols X_k are drawn is complex-valued.

- Although $x_b(t)$ is complex, the passband signal we transmit $x(t)$ must be real.

$$x(t) = \mathrm{Re}[x_b(t)e^{j2\pi f_c t}] = \mathrm{Re}(x_b(t))\cos(2\pi f_c t) - \mathrm{Im}(x_b(t))\sin(2\pi f_c t)$$
$$\boxed{x(t) = \sum_k p(t-kT)\left[\mathrm{Re}(X_k)\cos(2\pi f_c t) - \mathrm{Im}(X_k)\sin(2\pi f_c t)\right]}$$

QAM has two carriers, cosine carries $\mathrm{Re}(X_k)$, sine carries $\mathrm{Im}(X_k)$

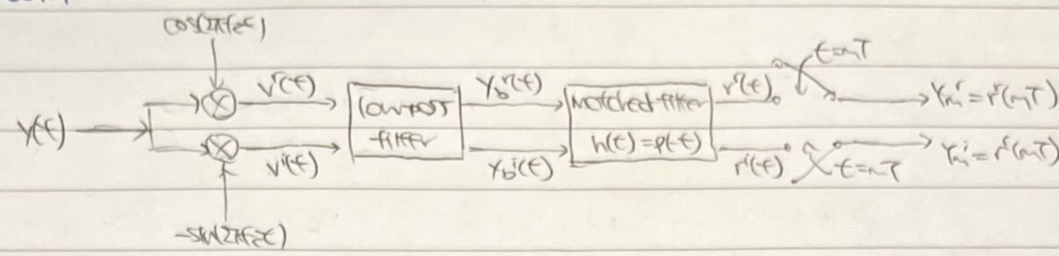
$$\boxed{x(t) = \sum_k p(t-kT)\,|X_k|\cos(2\pi f_c t + \phi)}$$

In QAM, the information symbol modulates both the amplitude + phase of the carrier.

where $|X_k|$ and ϕ are the magnitude and phase of the complex symbol X_k.

QAM receiver

- The QAM receiver can be constructed by product modulator x2 + LPF + PAM receiver



- At the receiver, we have $y(t) = \sum_k p(t-kT)\left[X_m^r \cos(2\pi f_c t) - X_m^i \sin(2\pi f_c t)\right] + n(t)$

Multiplying the received signal by $\cos(2\pi f_c t)$ / $-\sin(2\pi f_c t)$ gives.

$$v^r(t) = \sum_k p(t-kT)\left[X_k^r \cos^2(2\pi f_c t) - X_k^i \sin(2\pi f_c t)\cos(2\pi f_c t)\right] + n(t)\cos(2\pi f_c t)$$

$$= \sum_k p(t-kT)\Big[\overset{\text{low freq.}}{\frac{X_k^r}{2}} + \overset{\text{high freq.}}{\frac{X_k^r \cos(4\pi f_c t)}{2}} - \overset{\text{high freq.}}{\frac{X_k^i \sin(4\pi f_c t)}{2}}\Big] + \overset{\text{noise.}}{n(t)\cos(2\pi f_c t)}$$

$$v^i(t) = \sum_k p(t-kT)\left[-X_k^r \cos(2\pi f_c t)\sin(2\pi f_c t) + X_k^i \sin^2(2\pi f_c t)\right] - n(t)\sin(2\pi f_c t)$$

$$= \sum_k p(t-kT)\Big[-\overset{\text{high freq.}}{\frac{X_k^r \sin(4\pi f_c t)}{2}} + \overset{\text{low freq.}}{\frac{X_k^i}{2}} - \overset{\text{high freq.}}{\frac{X_k^i \cos(4\pi f_c t)}{2}}\Big] - \overset{\text{noise}}{n(t)\sin(2\pi f_c t)}$$

The LPF eliminates the high freq. component. The ideal LPF has a transfer function $H(f)$

$$H(f) = 2\,\text{rect}\left(\frac{f}{2W}\right)$$

After down-converting the received signal, we get the baseband waveforms $y_b^r(t)$ and $y_b^i(t)$.

$$y^r(t) = \sum_k X_k^r p(t-kT) + \overset{\text{filtered noise}}{n^r(t)}$$

$$y^i(t) = \sum_k X_k^i p(t-kT) + \overset{\text{filtered noise.}}{n^i(t)}$$

The waveform can then be demodulated like the usual PAM signal to give

$$\boxed{Y_m^r = X_m^r + N_m^r} \qquad \boxed{Y_m^i = X_m^i + N_m^i}$$

It can be shown that N_m^r and N_m^i are iid $N(0,\sigma^2)$ for each m.

Detection for QAM.

- consider a general constellation C. As defined above, $Y = X + N$, where $Y = \begin{bmatrix} Y_m^r \\ Y_m^i \end{bmatrix}$, $X = \begin{bmatrix} X_m^r \\ X_m^i \end{bmatrix}$, $N = \begin{bmatrix} N_m^r \\ N_m^i \end{bmatrix}$

- Assuming all constellation symbols are equally likely, the optimum detection rule is the max. likelihood detection rule,

$$\boxed{\hat{X} = \underset{x \in C}{\operatorname{argmax}}\ f(Y \mid X = x)}$$

i.e. choose the symbol from which y is most likely to have occured.

- The conditional distribution of $Y = \begin{bmatrix} Y_m^r \\ Y_m^i \end{bmatrix}$ given $X = \begin{bmatrix} x_m^r \\ x_m^i \end{bmatrix}$ is

$$f(Y \mid X = x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(Y^r - x^r)^2/2\sigma^2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(Y^i - x^i)^2/2\sigma^2} = \frac{1}{2\pi\sigma^2} e^{-\left[(Y^r - x^r)^2 + (Y^i - x^i)^2\right]/2\sigma^2}$$

So the optimal detector is therefore

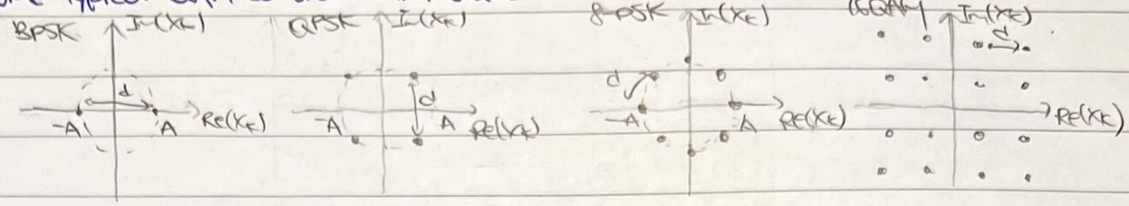
$$\boxed{\hat{X} = \underset{x \in C}{\operatorname{argmin}}\ (Y^r - x^r)^2 + (Y^i - x^i)^2 = \underset{x \in C}{\operatorname{argmin}}\ |Y - x|^2}$$

i.e. choose the constellation symbol closest to the output Y.

(same detection principle as PAM, but the symbols are complex in QAM)

Typical QAM constellations

- Some typical QAM constellations include



- In phase shift keying (PSK), the magnitude of X_k is const, and the information is in the phase of the symbol.

- For all the PSK constellations, the average symbol energy E_s is

$$\boxed{E_s = \frac{nA^2}{n} = A^2}$$

For 16QAM, the average symbol energy is

$$\boxed{E_s = 2.5d^2}$$

probability of detection error.

- It can be shown that the probability of detection error P_e for QAM is a Q-function depending on d/σ $\rightarrow$ P_e decays exponentially w/ d^2/σ^2 ($P_e \sim e^{-d^2/\sigma^2}$)
- Suppose we increase the no of constellation pts (e.g. 16 QAM $\rightarrow$ 256 QAM). while keeping E_s const. The transmission rate increases (4 bits/symbol $\rightarrow$ 8 bits/symbol), but to keep E_s const, d has to decrease $\rightarrow$ P_e increases.

Frequency shift keying. (FSK)

- Upconverted PAM and QAM are amplitude modulation for digital information.
- we can also transmit digital information by modulating the frequency/phase of the carrier.
- For binary FSK, in each symbol time $[kT, (k+1)T)$, transmit one bit X_k via

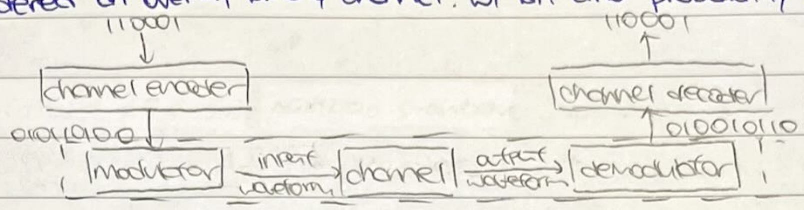
$$x(t) = \begin{cases} \cos(2\pi(f_c - \Delta f)t) & \text{if } X_k = 0 \\ \cos(2\pi(f_c + \Delta f)t) & \text{if } X_k = 1 \end{cases}$$

channel coding and multiple access

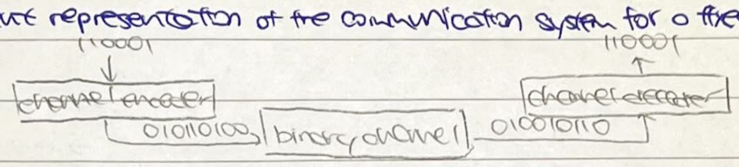
channel coding.

Binary channel

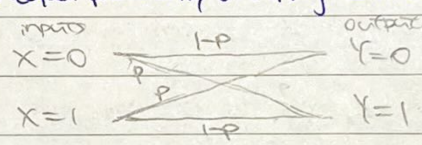
- Every modulation scheme has an associated probability of bit error, say p, that we can estimate theoretically or empirically.
- For a given modulation scheme, the part of the system enclosed by the dashed lines can thus be considered an overall binary channel. w/ bit error probability p.



Thus an equivalent representation of the communication system for a fixed modulation scheme is



* The binary channel is not the actual physical channel in the communication system, but the overall channel assuming that the modulation scheme is fixed and we estimate the bit error probability to be p.
- A modulation scheme w/ bit error probability p means a 0/1 is flipped by the binary channel to a 1/0 w/ probability p (equal probability of flipping both ways) → binary symmetric channel (BSC).



$P(Y=0|X=0)=1-p$ $\qquad$ $P(Y=1|X=1)=1-p$

$P(Y=1|X=0)=p$ $\qquad$ $P(Y=0|X=1)=p$

p is also known as the crossover probability, and the channel can be denoted as BSC(p).

Repetition code

- The simplest channel code for the BSC is a (n,1) repetition code:
  - ↳ encoding: repeat each source bit n times (n is odd)
  - ↳ decoding: by "majority vote" — declare 0 if greater than $n/2$ of the received bits are 0, o/w 1.
- The probability for decoding error for (n,1) repetition code is ← we want $P_e \to 0$

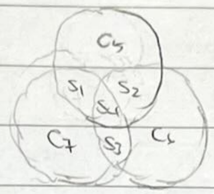
$$\boxed{P_e = \sum_{k=\frac{n+1}{2}}^{n} C^n_k p^k (1-p)^{n-k}} \quad (\text{flip } \tfrac{n}{2}+ \text{ bits})$$

The data rate for (n,1) repetition code is ← we want $r \to 1$.

$$\boxed{r = \frac{1}{n}}$$

Block codes.

- In a block code, every block of K source bits is represented by a sequence of N code bits. (To add redundancy, we have $N>K$).
- In a linear block code, the extra $N-K$ code bits are linear functions of the K source bits.
- An example is the $(N=7, K=4)$ Hamming code, where each 4-bit source block $s=(s_1,s_2,s_3,s_4)$ is encoded into 7-bit codeword $c=(c_1,c_2,c_3,c_4,c_5,c_6,c_7)$ as follows:
  $c_1=s_1$; $c_2=s_2$; $c_3=s_3$; $c_4=s_4$; $c_5=s_1\oplus s_2\oplus s_4$; $c_6=s_2\oplus s_3\oplus s_4$; $c_7=s_1\oplus s_3\oplus s_4$
  where $\oplus$ denotes the XOR operation.
- c_5, c_6, c_7 are parity check bits, and provide the redundancy.



- For any Hamming codeword, the parity of each circle is even (even no. of 1s in each circle).
- To encode a sequence of source bits w/ (K,N) block code, we split the sequence into blocks of K bits each, then transmit the N-bit codeword for each block over the BSC.
- The data rate for (N,K) block code is

$$\boxed{r=\frac{K}{N}}$$

- The $(7,4)$ Hamming code can correct any single bit error in a codeword — if any circle have odd parity, flip exactly one bit to make all of them have even parity.
- The probability for decoding error for $(7,4)$ Hamming code is

$$P_e=\sum_{k=2}^{7} p^k(1-p)^{7-k} \quad \text{(flip 2+ bits)}$$

The data rate for $(7,4)$ block code is

$$r=\frac{4}{7}.$$

Shannon's result

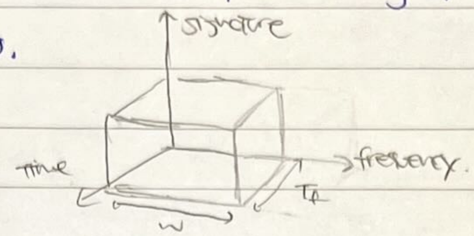
- Shannon showed that any communication channel has a capacity, which is the max rate of which the probability error can be made arbitrarily small.
- For BSC(0.1), there exist (N,K) block codes w/ rate $\boxed{\frac{K}{N}\approx 0.53}$ s.t. you can almost recover the correct codeword from the noisy output sequence of the BSC.

(But N has to be very large → large block lengths req.)

Multiple access

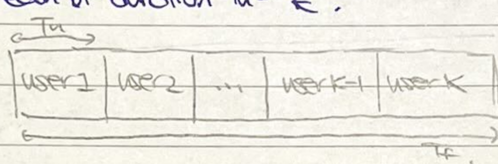
types of multiple access.

- The main techniques for multiple access include
  - ↳ Time division multiple access (TDMA)
  - ↳ Frequency division multiple access (FDMA)
  - ↳ Code division multiple access (CDMA)
- We can think of each multiple access technique as dividing up a "box" among the users, by cutting along diff. axes.



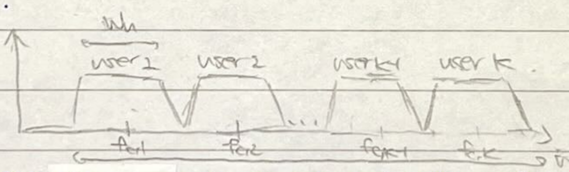
Time division multiple access (TDMA)

- In TDMA, multiple users are multiplexed in time, so they transmit one after the other, using the whole BW W.
- Each of K users gets one slot in a frame of duration T_f.
- K time slots in a frame, each of duration $T_u = \frac{T_f}{K}$.



Frequency division multiple access (FDMA).

- In FDMA, K users are multiplexed in the frequency domain by allocating a fraction of the total BW to each one.
- They communicate simultaneously on non-overlapping frequency bands of width $W_u < \frac{W}{K}$, so there is no interference.



- We can think of each user i using carrier $f_{c,i}$ to transmit a PAM signal $x_i(t)$, for $i = 1, \dots K$.

$$\boxed{s^{FDMA}(t) = \sum_{i=1}^{K} x_i(t)\cos(2\pi f_{c,i} t)}$$

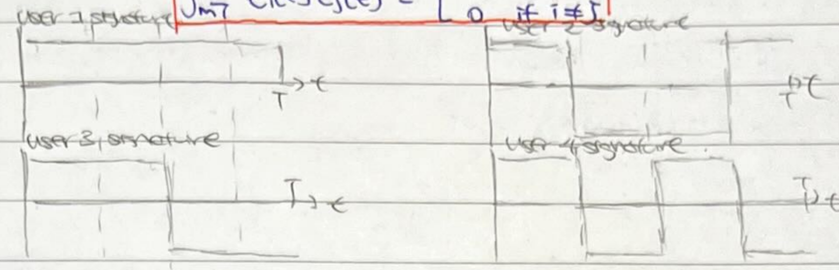
- At the receiver, we can separate $x_i(t)$ by multiplying $s^{FDMA}(t)$ by $\cos(2\pi f_{c,i} t)$ and pass through a LPF w/ pass band $[-\frac{B_u}{2}, \frac{B_u}{2}]$.

code division multiple access (CDMA)

- In CDMA, each user is given a unique signature function. The signatures are denoted $c_i(t)$, $i=1..K$ (for K users).
- These signatures are chosen to be orthogonal over each symbol period T, i.e. for $m=0,1,2\ldots$

$$\boxed{\int_{mT}^{(m+1)T} c_i(t)\,c_j(t) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}}$$

e.g. for K=4 users.



- Assume that user i wants to transmit a PAM signal $x_i(t)$, for $i=1\ldots K$, the signals of the K users are multiplexed as

$$\boxed{s^{CDMA}(t) = \left[\sum_{i=1}^{K} c_i(t)\,x_i(t)\right]\cos(2\pi f_c t)}$$

Thus each user i transmits their baseband signal $x_i(t)$ using the entire BW W over entire time frame of duration T_F.

- At the receiver, after downconverting using a product modulator + LPF, we get.

$$y(t) = \sum_{i=1}^{K} c_i(t)\,x_i(t) + n(t).$$

we can separate out the signal $x_j(t)$ by correlating w/ its signature $c_j(t)$.

- Assuming no noise, multiplying $y(t)$ by $c_j(t)$ and integrating, we get.

$$\int\left(\sum_{i=1}^{K} c_i(t)\,x_i(t)\right)c_j(t)\,dt = \sum_{i=1}^{K}\int x_i(t)\,c_i(t)\,c_j(t)\,dt$$

$$= \sum_{i=1}^{K}\sum_{m}\int_{m}^{(m+1)T} x_i(mT)\,c_i(t)\,c_j(t)\,dt$$

$$= x_j(t).$$

$x_i(t)$ is const. over each symbol.

- For large no. of users K, we may only be able to have approx. orthogonal signatures