conduction

steady heat conduction

conductive heat transfer.

- when a temp. gradient exists in a body, there is an energy transfer from the hotter to the cooler part. The rate of energy transfer is prop. to the temp. gradient.

$$\boxed{\dot{q} = \frac{\dot{Q}}{A} = -\lambda \frac{\partial T}{\partial x}}$$ [Fourier's law] ← In 3D, we have $\dot{q} = -\lambda \nabla T$

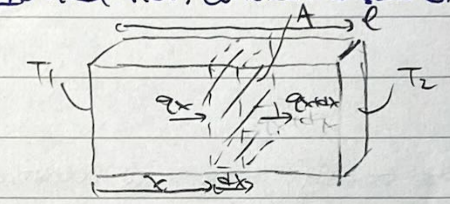
where λ is the thermal conductivity (const. of proportionality)

$\dot{q}$ is the heat flux (rate of heat flow per unit area).

- Note that Fourier's law of heat conduction is an empirical relationship. we can generally neglect any variation of λ w/ temp.

steady heat conduction

- consider a steady state, 1D heat flow. consider a slab of material as shown below



For the small element, the heat transferred into the face at x and the heat conducted out of the face at $x+dx$, are given, per unit area, by

$$\dot{q}_x = -\lambda \frac{dT}{dx} \qquad \dot{q}_{x+dx} = \dot{q}_x + \frac{d}{dx}(\dot{q}_x)dx$$

At steady state, there can be no accumulation of energy in the element, so

$$\dot{q}_x = \dot{q}_{x+dx}$$

$$\cancel{\dot{q}_x} = \cancel{\dot{q}_x} + \frac{d\dot{q}_x}{dx}dx$$

$$\frac{d}{dx}\left(-\lambda\frac{dT}{dx}\right) = 0$$

$$\therefore -\lambda \frac{dT}{dx} = \text{const}$$

- If the thermal conductivity λ is constant, the temp. gradient $\frac{dT}{dx}$ is constant.

$$-\lambda \int dT = k \int dx$$

$$-\lambda T = kx + c$$

$x = 0,\ T = T_1$: $\quad -\lambda T_1 = c$

$x = l,\ T = T_2$: $\quad -\lambda T_2 = kl - \lambda T_1 \quad \rightarrow \quad k = \frac{\lambda(T_1 - T_2)}{l}$

$$\therefore \quad -\lambda T = \frac{\lambda(T_1-T_2)}{l}x - \lambda T_1$$

$$\boxed{T = T_1 - \frac{x}{l}(T_1 - T_2)}$$

$\frac{dT}{dx}$ is const. → $\frac{dT}{dx} = \frac{T_2 - T_1}{l}$ so $\boxed{\dot{Q} = -A\lambda\left(\frac{T_2-T_1}{l}\right)}$

$$\dot{Q} = \frac{T_1 - T_2}{R_{th}}$$

where the thermal resistance R_{th} is given by

$$\boxed{R_{th} = \frac{l}{\lambda A}}$$

Convective heat losses at fluid interface

- Often, heat from a surface is transferred to a flowing fluid (convective BC).
  The heat flux out of the solid $\dot{q}_x$ is proportional to the temp. diff. $T_\infty - T_w$

freestream temp. — wall temp.

$$\boxed{\dot{Q} = -hA(T_\infty - T_w)}$$
$$= \frac{T_w - T_\infty}{R_{th}}$$

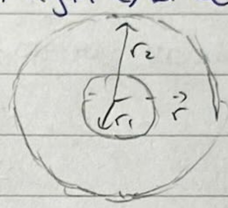
where the thermal resistance R_{th} is given by

$$\boxed{R_{th} = \frac{1}{hA}}$$

and h is the heat transfer coefficient.

Radial heat transfer.

- consider a circular cylinder of length ℓ, w/ BC $T(r_1) = T_1$, $T(r_2) = T_2$.



consider the rate of energy transfer $\dot{Q}$ (const. over r, whereas $\dot{q}$ not const. over r)

$$\dot{Q} = -(2\pi r \ell)\lambda \frac{dT}{dr}$$
$$\int_{r_1}^{r_2} \frac{\dot{Q}}{r}\,dr = -2\pi\ell\lambda \int_{T_1}^{T_2} dT$$
$$\boxed{\dot{Q} = \frac{2\pi\ell\lambda(T_1 - T_2)}{\ln(r_2/r_1)}}$$
$$= \frac{T_1 - T_2}{R_{th}}$$

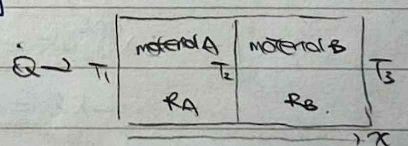
work w/ total heat flow $\dot{Q}$ instead of heat flux $\dot{q}$ for systems w/ varying CSA.

where the thermal resistance R_{th} is given by

$$\boxed{R_{th} = \frac{\ln(r_2/r_1)}{2\pi\ell\lambda}}$$

Heat transfer resistance in series.

- consider heat flowing through slabs of different materials



For this 1D, steady conduction problem, the heat flow $\dot{Q}$ (and heat flux $\dot{q}$) is not a function of x.

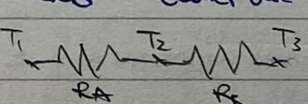
Material A: $\dot{Q} = \frac{T_1 - T_2}{R_A} \rightarrow R_A\dot{Q} = T_1 - T_2$

Material B: $\dot{Q} = \frac{T_2 - T_3}{R_B} \rightarrow R_B\dot{Q} = T_2 - T_3$.

so $\dot{Q}(R_A + R_B) = T_1 - T_3$.

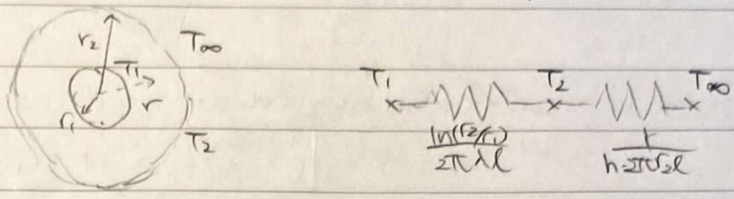
$\therefore R_{total} = R_A + R_B$.

i.e. heat transfer resistances in series add → could use ohm's law analogy



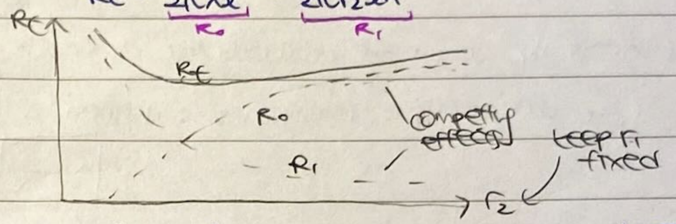
Insulation on a pipe

- Consider a hot pipe, wrapped in insulating material of thermal conductivity λ. The inner and outer radii of the insulating material are r_1 and r_2 respectively. The temp. at $r = r_1$ is T_1. The outer surface of the insulator is in contact w/ the environment, w/ convective thermal resistance $\frac{1}{hA}$.



The total thermal resistance is thus

$$R_t = \underbrace{\frac{\ln(r_2/r_1)}{2\pi\lambda l}}_{R_0} + \underbrace{\frac{1}{2\pi r_2 l h}}_{R_1}$$



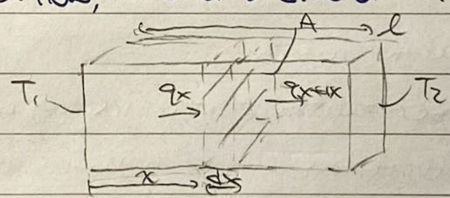
The first term R_0 increases w/ insulator thickness but the second term R_1 decreases w/ insulator thickness → competing effects → a min. value of R_t exists for some non-zero thickness

* For small pipes, (small r_1), $r_2 \gg r_1$ for R_t to increase compared to its initial value, so generally it is better not to insulate small pipes.

Unsteady heat conduction

Unsteady heat conduction

- Consider a general 1D heat flow, consider a slab of material as shown below.



Assume that the internal heat generation is G Wm^{-3}, and that the temp. change of the element in time δt is δT. The net energy input to element = change in int. energy of element, so

$$\underbrace{(\rho A dx) c \frac{\delta T}{\delta t}}_{\text{change in int. energy}} = \underbrace{\dot{q}_x A}_{\text{in}} - \underbrace{\left(\dot{q}_x + \frac{\partial \dot{q}_x}{\partial x} dx\right) A}_{\text{out}} + \underbrace{G A dx}_{\text{generation}}.$$

where c is the specific heat capacity of the material.

- Rearranging the eqn, we get

$$\rho c \frac{\partial T}{\partial t} = -\frac{\partial \dot{q}_x}{\partial x} + G$$

If the thermal conductivity λ is const, $-\frac{\partial}{\partial x}\left[-\lambda \frac{\partial T}{\partial x}\right] = \lambda \frac{\partial^2 T}{\partial x^2}$, so

$$\boxed{\frac{\partial T}{\partial t} = \left(\frac{\lambda}{\rho c}\right)\frac{\partial^2 T}{\partial x^2} + \frac{G}{\rho c}} \quad \text{[transient heat transport eqn.]}$$

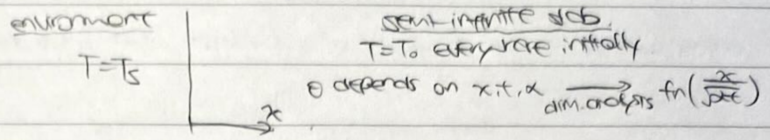
where $\frac{\lambda}{\rho c} = \alpha$ is the thermal diffusivity

- In the absence of heat generation ($G = 0$), we have the 1D transient heat diffusion eqn.

$$\boxed{\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}}$$

Conduction in a semi-infinite slab

- conduction in a semi-infinite slab is one case where it is possible to derive an analytical sol'n to the 1D transient heat diffusion eqn. The setup is as follows.



It can be shown that the analytical sol'n is as follows.

$$\boxed{\theta = \frac{T-T_0}{T_s-T_0} = 1 - \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)}.$$

- say the characteristic time τ for heat to penetrate a distance x is when $\frac{x}{\sqrt{\alpha\tau}} = 1$, so

$$\tau = \frac{x^2}{\alpha}$$

generally, we use the characteristic dimension s of the body as the characteristic length x.

(the characteristic dimension s is defined as $\boxed{s = \frac{\text{volume}}{\text{surface area}}}$)

$$\boxed{\tau = \frac{s^2}{\alpha}}$$

- This can also be derived by applying scaling analysis on the 1D transient heat diffusion eqn.

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

Applying scaling analysis,

$$\frac{\partial T}{\partial t} \sim \frac{\theta}{\tau}, \quad \alpha\frac{\partial^2 T}{\partial x^2} \sim \frac{\alpha\theta}{s^2}$$

$$\therefore \frac{1}{\tau} \sim \frac{\alpha}{s^2}.$$

- The Fourier no. Fo is a NDG useful in transient heat conduction problems, defined as

$$\boxed{Fo = \frac{\text{time } t}{\text{characteristic time } \tau} = \frac{\alpha t}{s^2}}$$

A body can be considered to have heated up when $Fo > 1$. ($t > \tau$)

Lumped heat capacity analysis.

- If a transient heat transfer situation involves a solid w/ rel. low "int". thermal resistance and a solid/liquid interface w/ a rel. high thermal resistance, we can apply lumped heat capacity analysis — we assume the solid body is at a uniform temp.
- consider a lump of material w/ specific heat capacity c, density ρ, volume V, SA A w/ convective heat transfer coefficient h. If we assume uniform temp throughout the body, we can write

heat in

$$\dot{Q} = (\rho V)c\frac{dT}{dt} = hA(T_\infty - T)$$

For an initial condition of $T=T_0$ at $t=0$, the sol'n is

analogous to a capacitor

$$\theta = \frac{T-T_\infty}{T_0-T_\infty} = \exp\left(-\frac{t}{\tau_c}\right) \quad \text{where} \quad \boxed{\tau_c = \frac{c\rho V}{hA}}$$

- The Biot no. Bi is defined as

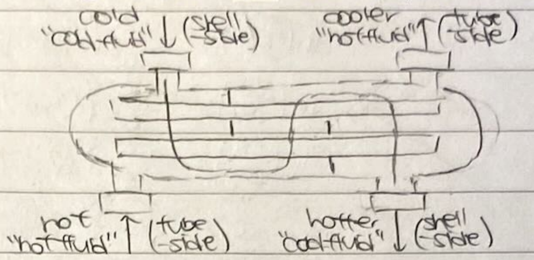
$$\boxed{Bi = \frac{\text{internal thermal resistance}}{\text{surface thermal resistance}} = \frac{s/\lambda A}{1/hA} = \frac{hs}{\lambda}}$$

In general, $Bi < 0.1$ is often used as an acceptable condition for applying lumped heat capacity analysis.

Heat exchangers

Heat exchangers (HX)

- The most common form of HX is the shell and tube HX, illustrated below.



The overall heat transfer coefficient

- Previously, an overall thermal resistance R_{total} is calculated s.t.

$$\dot{Q} = \frac{\Delta T}{R_{total}}$$

However, common usage is to express the above relationship as

$$\boxed{\dot{Q} = UA\Delta T}$$

where U is the overall heat transfer coefficient, and $UA = 1/R_{total}$.

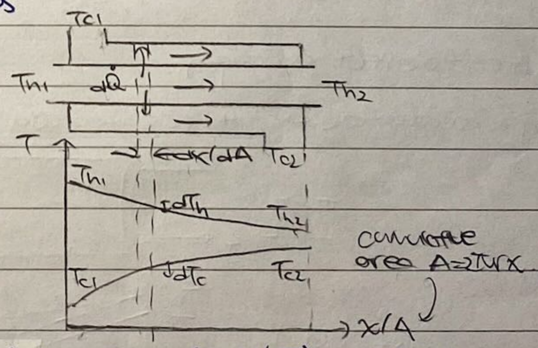
* For problems w/ radial conduction, we need to specify which diameter/area the overall heat transfer coefficient U is based on

Log-mean temperature

- In a HX, we have heat transfer taking place due to a continuously changing temp. diff. so we can define a log-mean temp. diff. ΔT_{lm} s.t.

$$\boxed{\dot{Q} = UA\Delta T_{lm}}$$

- consider a co-flow HX as follows



Local analysis:

$$d\dot{Q} = (\dot{m}c)_h(-dT_h) = (\dot{m}c)_c(dT_c) = U\Delta T\,dA$$

$$\therefore \quad dT_h = -\frac{d\dot{Q}}{(\dot{m}c)_h}, \quad dT_c = \frac{d\dot{Q}}{(\dot{m}c)_c}, \quad \frac{d\dot{Q}}{\Delta T} = U\,dA$$

$$d(\Delta T) = d(T_h - T_c) = dT_h - dT_c = -d\dot{Q}\left(\frac{1}{(\dot{m}c)_h} + \frac{1}{(\dot{m}c)_c}\right)$$

$$\frac{d(\Delta T)}{\Delta T} = -U\,dA\left(\frac{1}{(\dot{m}c)_h} + \frac{1}{(\dot{m}c)_c}\right)$$

$$\int_{\Delta T_1}^{\Delta T_2} \frac{1}{\Delta T}\,d(\Delta T) = -\int_0^A U\left(\frac{1}{(\dot{m}c)_h} + \frac{1}{(\dot{m}c)_c}\right) dA$$

$$\ln\left(\frac{\Delta T_2}{\Delta T_1}\right) = -UA\left(\frac{1}{(\dot{m}c)_h} + \frac{1}{(\dot{m}c)_c}\right)$$

Global analysis: $\dot{Q} = (\dot{m}c)_h(-(T_{h2}-T_{h1})) = (\dot{m}c)_c(T_{c2}-T_{c1})$

$$\therefore \frac{1}{(\dot{m}c)_h} = \frac{T_{h1}-T_{h2}}{\dot{Q}}, \quad \frac{1}{(\dot{m}c)_c} = \frac{T_{c2}-T_{c1}}{\dot{Q}}$$

Combining the two eqns above, we get

$$\ln\left(\frac{\Delta T_2}{\Delta T_1}\right) = -\frac{UA}{\dot{Q}}(T_{h1}-T_{h2}+T_{c2}-T_{c1})$$

$$\ln\left(\frac{\Delta T_2}{\Delta T_1}\right) = -\frac{UA}{\dot{Q}}[(T_{h1}-T_{c1})-(T_{h2}-T_{c2})]$$

$$\ln\left(\frac{\Delta T_2}{\Delta T_1}\right) = -\frac{UA}{\dot{Q}}(\Delta T_1-\Delta T_2)$$

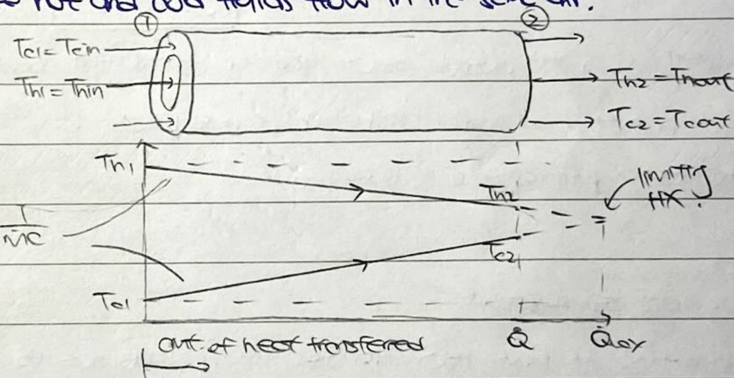
$$\dot{Q} = UA\frac{\Delta T_2-\Delta T_1}{\ln(\Delta T_2/\Delta T_1)}$$

$$\therefore \boxed{\Delta T_M = \frac{\Delta T_2-\Delta T_1}{\ln(\Delta T_2/\Delta T_1)}}$$

- we can use the log-mean temp. diff. ΔT_{lm} to calculate the total heat exchanged $\dot{Q}$ for both the co-flow HXs and the counter-flow HXs. For cross-flow HXs, we use $\dot{Q} = UA\Delta T_{lm}F$.

Co-flow heat exchangers.

- In a co-flow HX, the "hot" and "cold" fluids flow in the same dir.



$d\dot{Q} = \dot{m}c\,dT$

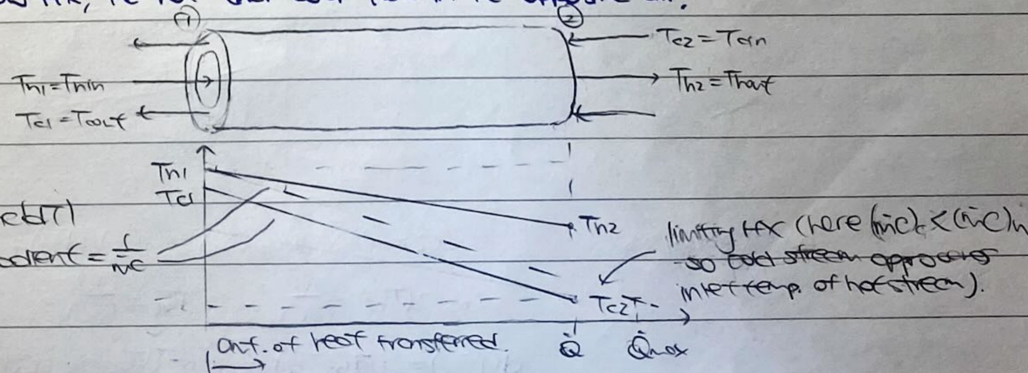
$\therefore \left|\frac{dT}{d\dot{Q}}\right| = \text{gradient} = \frac{1}{\dot{m}c}$

- For max heat transfer, the exit temp. must become equal ($T_{h2} = T_{c2}$). This would req. an infinite heat transfer area since $\Delta T_{lm} = 0$

for HX w/ a fluid held at const. temp (change of state), its c is effectively infinite

Counter-flow heat exchangers

- In a counter-flow HX, the "hot" and "cold" flow in the opposite dir.



$d\dot{Q} = \dot{m}c\,dT$

$\therefore \left|\frac{dT}{d\dot{Q}}\right| = \text{gradient} = \frac{1}{\dot{m}c}$

limiting HX (here $(\dot{m}c)_c < (\dot{m}c)_h$ so cold stream approaches inlet temp. of hot stream).

- For max. heat transfer, one of the end temps must become equal ($T_{h1} = T_{c1}$ or $T_{h2} = T_{c2}$), (depending on which fluid has the lower heat capacity flow rate $\dot{m}c$) This would req. an infinite heat transfer area since $\Delta T_{lm} = 0$.

* fluid w/ lower $\dot{m}c$ approaches inlet temp. of fluid w/ higher $\dot{m}c$ for max. heat transfer.

effectiveness of heat exchangers

- The max. possible heat exchange is achieved in the counter-flow case, where

$$\boxed{\dot{Q} = |\dot{m}c|_{min}(T_{hin} - T_{cin})}$$

(counter-flow HX allows for more heat to be transferred as smaller ΔT are involved, so there are less irreversibilities)

- since max. possible heat exchange req. an infinite area for heat transfer, it cannot practically be achieved, but it can be a reference.

- The effectiveness of a HX ε, is defined as

$$\boxed{\varepsilon = \frac{\dot{Q}}{\dot{Q}_{max}}}.$$

Convection

Convection

Convective heat transfer

- For a convective BC, the local heat transfer coefficient h is defined s.t.

$$\dot{q} = -h\Delta T$$

since h may vary position, we often use the averaged heat transfer coefficient.

$$\boxed{\dot{q} = -h_{av}\Delta T_{mean}}$$

(h and h_{av} are often used interchangeably → use h_{av} since it is easier to find)

- To estimate the value of h, there are two distinct situations to consider:
  - ↳ 1) Forced convection — when the fluid is driven over the surface by a fan/pump.
  - ↳ 2) Free/natural convection — when density differences due to buoyancy drive the flow.

(both cases consider the flow of fluid over the surface).

viscous dissipation
forced convection
mechanical energy field (p, V) → thermal energy field (T)
only if compressible flow

Heat transfer and fluid mechanics

- Mechanical fields can always affect the thermal field by viscous dissipation / move a convecting hot fluid through space.
- The thermal field can only affect the mechanical field if the flow is compressible (i.e. density cannot be assumed to be constant.
  variation of fluid properties due to temp.
  variation only affects the mech. field slightly.
- For the eqn. of state of ideal gases,

$$\rho = \frac{P}{RT} \quad \rightarrow \quad \frac{d\rho}{\rho} = -\frac{dT}{T}$$

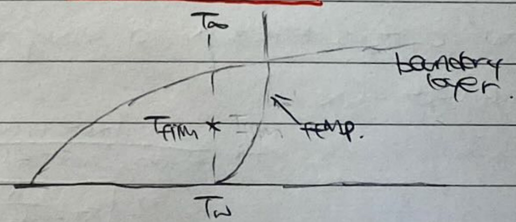
we can assume incompressibility when temp. fluctuations are less than 0.1 of background temp.

- Incompressibility also req. the mach no. M to be less than 0.3, so

$$\underbrace{c_pT}_{\text{thermal}} >> \underbrace{\tfrac{1}{2}V^2}_{\text{mechanical}} \quad \rightarrow \quad \dot{Q} = \dot{m}c_p\Delta T$$

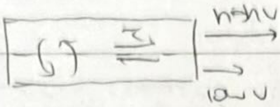
- Fluid properties (μ, λ, c) depend on temp. For small temp. variations, the variations in these properties are small → take the value of the mean temp, the film temp.

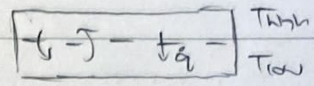
$$\boxed{T_{film} = \tfrac{1}{2}(T_w + T_\infty)}.$$



The analogy between heat and momentum transfer

- The random swapping of places of molecules due to a velocity / temp. gradient results in momentum/thermal energy exchange and thus a shear stress / heat flux.





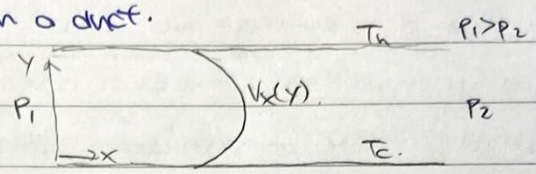
molecules swap places → momentum exchange → shear stress

$$\tau = \mu \frac{\partial V_x}{\partial y}$$

molecules swap places → thermal energy exchange → heat flux.

$$\dot{q} = -\lambda \frac{\partial T}{\partial y}$$

- consider a flow down a duct.



momentum conservation:

$$\rho \frac{DV_x}{Dt} = \nabla \cdot \tau_x - \frac{\partial P}{\partial x}$$

thermal energy conservation:

$$\rho c_p \frac{DT}{Dt} = -\nabla \dot{q} + S_h$$

- Combining the above eqns, we get

$$\frac{DV_x}{Dt} = \frac{\mu}{\rho}\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2}\right) - \frac{1}{\rho}\frac{dp}{dx}$$

$$\frac{DT}{Dt} = \frac{\lambda}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \frac{1}{\rho c_p} S_h$$

where $\nu = \frac{\mu}{\rho}$ is the momentum diffusivity / kinematic viscosity

and $\alpha = \frac{\lambda}{\rho c_p}$ is the thermal diffusivity.

- The Prandtl no. Pr is defined as.

$$\boxed{Pr = \frac{\nu}{\alpha} = \frac{\mu c_p}{\lambda}}$$

Non-dimensional heat transfer coefficients

- The Nusselt no. Nu is defined by normalising the convective heat flux by a conductive heat flux.

$$Nu = \frac{\text{heat flux}}{\text{heat flux w/o flow}} = \frac{h\Delta T}{\lambda(\Delta T/D)}$$

$$\therefore \boxed{Nu = \frac{hD}{\lambda}}$$

where D is some characteristic length scale

- The Stanton no St is defined by normalising the convective heat flux by a characteristic convective heat flux

$$St = \frac{\text{heat flux}}{\text{charac. convective heat flux}} = \frac{h\Delta T}{\rho V c_p \Delta T}$$

$$\therefore \boxed{St = \frac{h}{\rho V c_p}}$$

- The Nusselt no. Nu and Stanton no. St are related by

$$\boxed{Nu = Re \cdot Pr \cdot St}$$

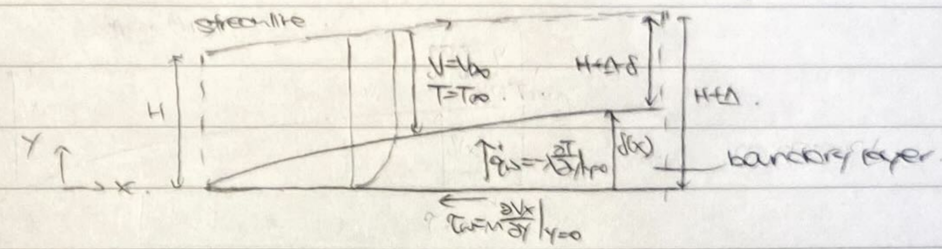
↳ LHS = $Nu = \frac{hD}{\lambda}$

↳ RHS = $Re \cdot Pr \cdot St = \frac{\cancel{\rho V} D}{\cancel{\mu}} \cdot \frac{\cancel{\mu c_p}}{\lambda} \cdot \frac{h}{\cancel{\rho V c_p}} = \frac{hD}{\lambda}$

Forced convection

Reynold's analogy ($Pr = 1$)

- Consider a laminar (momentum/thermal) boundary layer over a flat plate.



Conservation of mass

$$\underbrace{\rho V H}_{\text{mass in}} = \underbrace{\rho V(H+\Delta-\delta) + \int_0^\delta \rho V_x \, dy}_{\text{mass out}}$$

Conservation of x-momentum

$$\underbrace{-\int_0^x \tau_w \, dx}_{\text{force}} = \underbrace{\rho V(H+\Delta-\delta)V + \int_0^\delta (\rho V_x) V_x \, dy}_{\text{momentum out}} - \underbrace{(\rho V) H V}_{\text{momentum in}}$$

$\Rightarrow$ using $V' = \frac{V_x}{V}$, we get

$$-\mu \int_0^x \frac{\partial V'}{\partial y}\bigg|_{wall} dx = \rho V(H+\Delta-\delta) + \int_0^\delta (\rho V_x) V' \, dy - (\rho V) H$$

Conservation of energy

$$\underbrace{\int_0^x \dot{q}_w \, dx}_{\text{heat flow}} = \underbrace{\rho V(H+\Delta-\delta) c_p (T_\infty - T_w) + \int_0^\delta (\rho V_x) c_p (T - T_w) \, dy}_{\text{enthalpy out}} - \underbrace{(\rho V) H c_p (T_\infty - T_w)}_{\text{enthalpy in}}$$

$\Rightarrow$ using $\theta = \frac{T - T_w}{T_\infty - T_w}$, we get

$$-\frac{\lambda}{c_p} \int_0^x \frac{\partial \theta}{\partial y}\bigg|_{wall} dx = \rho V(H+\Delta-\delta) + \int_0^\delta (\rho V_x) \theta \, dy - (\rho V) H$$

$\therefore$ $\boxed{V' = \frac{V_x}{V} \text{ and } \theta = \frac{T-T_w}{T_\infty - T_w} \text{ satisfy the same eqn when } Pr = \frac{\mu c_p}{\lambda} = 1}$

↳ check BC, for $y = 0$, $V_x = 0 \rightarrow V' = 0$; for $y \geq \delta$, $V_x = V \rightarrow V' = 1$

$T = T_w \rightarrow \theta = 0$; $T = T_\infty \rightarrow \theta = 1$.

- Using the relationship btwn V_x and T (V' and θ), we can relate h to τ_w (c_f).

$$\theta' = \frac{T - T_w}{T_\infty - T_w} = \frac{V_x}{V} = V'$$

Differentiate wrt y (eval at $y=0$)

$$\frac{1}{T_\infty - T_w} \frac{\partial T}{\partial y}\bigg|_{y=0} = \frac{1}{V} \frac{\partial V_x}{\partial y}\bigg|_{y=0}$$

$\dot{q}_w = -\lambda \frac{\partial T}{\partial y}\big|_{y=0}$, $\tau_w = \mu \frac{\partial V_x}{\partial y}\big|_{y=0}$

$$\frac{-\dot{q}_w/\lambda}{T_\infty - T_w} = \frac{\tau_w/\mu}{V}$$

$\dot{q}_w = -h(T_\infty - T_w)$

$$\frac{h\cancel{(T_\infty - T_w)}}{\lambda\cancel{(T_\infty - T_w)}} = \frac{\tau_w}{\mu V}$$

$$\boxed{h = \frac{\lambda \tau_w}{\mu V}}$$

- Since $\tau_w = \frac{1}{2}\rho V^2 c_f$,

$$h = \frac{\lambda}{\mu V} \frac{1}{2} \rho V^2 c_f$$

$$\frac{h}{\rho c_p V} = \frac{1}{2} c_f \frac{\lambda}{\mu c_p}$$

$$St = \frac{1}{2} c_f \frac{1}{Pr}\,{}^{=1}$$

Reynold's analogy $Pr = 1$.

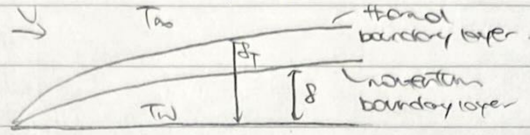
$$\boxed{St = \frac{1}{2} c_f}$$

- This result indicates that we cannot have high heat transfer w/o having high wall friction.
- This is expected as both heat/momentum transfer have the same underlying mechanism of molecular motion (laminar) or eddy motion (turbulent).

as in momentum transfer, the additional transport in thermal energy due to turbulent eddies greatly increase the heat transfer.

Heat transfer for $Pr \neq 1$.

- If the Prandtl no. Pr is not equal to 1, the transport eqns are no longer identical and we have two separate boundary layers (rel. thickness determined by Pr).



$Pr \ll 1 \rightarrow \delta \ll \delta_T$

$Pr = 1 \rightarrow \delta \approx \delta_T$

$Pr \gg 1 \rightarrow \delta \gg \delta_T$

From experiments, it is found that

$$\boxed{St = \tfrac{1}{2} C_f Pr^{-2/3}}$$

(For gases, $Pr \sim 0.7$, so the correction is small and usually ignored).

Evaluation of St and Nu for a flat plate boundary layer.

- The velocity profile for the laminar boundary layer is approximately

$$\frac{V_x}{V} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

The corresponding shear stress at the wall is

$$\tau_w = \mu \frac{\partial V_x}{\partial y}\bigg|_{y=0} = \mu \frac{3V}{2\delta}.$$

The local friction factor C_{fx} is thus

$$C_{fx} = \frac{\tau_w}{\frac{1}{2}\rho V^2} = \mu \frac{3}{\rho \delta V}$$

substituting the above into $St_x = \frac{1}{2} C_{fx} Pr^{-2/3}$, we get

$$St_x = \frac{3\mu}{2\rho V \delta} Pr^{-2/3}$$

The thickness of the boundary layer δ is given by $\delta = \frac{4.64x}{\sqrt{Re_x}}$, so

$$\boxed{St_x = \frac{3}{2}\left(\frac{\mu}{\rho x V}\right)\frac{\sqrt{Re_x}}{4.64} Pr^{-2/3} = 0.323\, Re_x^{-1/2} Pr^{-2/3}}$$

or using the Nusselt no. Nu, $Nu = Re\,Pr\,St$

$$\boxed{Nu_x = 0.323\, Re_x^{1/2} Pr^{1/3}}$$

- The expressions above give a value for the local value of Nu, St (and h) as a function of x. Suitable averages need to be taken to give the avg heat transfer coefficient h_{av} over the plate.

Evaluation of St and Nu for fully developed flow in circular pipes of diameter d.

- Overall heat transfer for laminar flow w/ const. wall temp. [$Re_d < 2300$, $Pr > 0.1$]

$$St_d = 3.66\, Re_d^{-1} Pr^{-1} \qquad Nu_d = 3.66$$

- Overall heat transfer for turbulent flow w/ const. wall temp. [$2300 < Re_d < 10^7$, $0.5 < Pr < 120$]

$$St_d = 0.023\, Re_d^{-0.2} Pr^{-0.6} \qquad Nu_d = 0.023\, Re_d^{0.8} Pr^{0.4}$$

(In databook).

The forced convection experiment

- Often the flow field is too complicated to calculate by hand, too expensive to use CFD
  → perform an experiment on a geometrically-identical situation and scale up using NDG.
- For the forced convection experiment, applying dim. analysis gives.

$$Nu = fn(Re, Pr)$$

- When we plot Nu as a function of Re (given that Pr is fixed), the results collapse to a single line → useful for calculating heat transfer (Nu) for a geometrically similar situation of any size, btwn the range of Re studied.

Natural convection

Characteristics of natural convection

- When heated, fluids expand and are driven upward by buoyancy forces. Such flows have no natural reference velocity → no natural Reynolds no. Re associated w/ the flow.
- We will assume that pressure variations due to convection are small and that the motion is driven primarily driven by changes in density.

Changes in pressure

- To equally easily deal w/ liquids and gases, we use the coefficient of expansion β, defined as

$$\boxed{\beta = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p}$$

where v is the specific volume.

- Since $\rho = \frac{1}{v}$, $\frac{d\rho}{\rho} = -\frac{dv}{v}$, so

$$\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p$$

- For small changes in density (at const. pressure),

$$\boxed{\delta\rho = \left(\frac{\partial \rho}{\partial T}\right)_p \delta T = -\bar{\rho}\beta\delta T}$$

where $\bar{\rho}$ is the mean value of the density.

- For liquids, β must be obtained from tables.
- For gases, β can be derived from the eqn. of state for ideal gases, $\rho = \frac{p}{RT}$.

$$\left(\frac{\partial \rho}{\partial T}\right)_p = -\frac{p}{RT^2} = -\frac{\rho}{T}$$

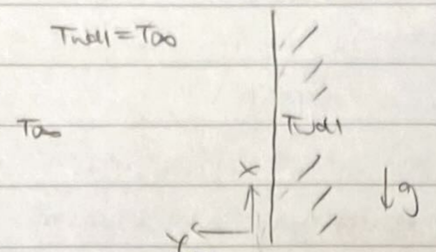
$$\therefore \boxed{\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p = \frac{1}{T}}$$

which is evaluated at the mean temp. $\bar{T}$.

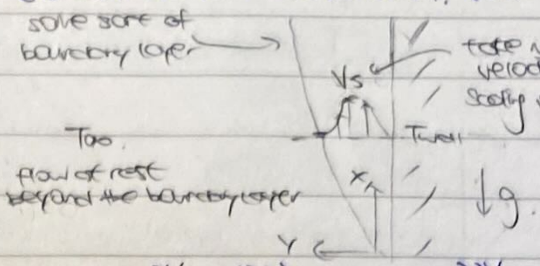
The Boussinesq equation.

- Consider a convective flow near a hot plate.

(a) base case – no flow, $V=0$, $T=T_\infty$



(b) flow due to temp. variation, $T=T$



Hydrostatic: $0=-\left(\frac{\partial p}{\partial x}\right)_A-\rho_A g$ N-S: $\rho V\frac{\partial V}{\partial x}=-\left(\frac{\partial p}{\partial x}\right)_B-\rho_B g+\mu\frac{\partial^2 V}{\partial y^2}$

- Subtract the base case (A) from the flow case (B) gives.

$$\rho V\frac{\partial V}{\partial x}=-\left(\frac{\partial p}{\partial x}\right)_B-\rho_B g+\mu\frac{\partial^2 V}{\partial y^2}-\left[-\left(\frac{\partial p}{\partial x}\right)_A-\rho_A g\right]$$

Since we assume convective flow does not result in significant pressure change, we can say the pressure distribution is the same in both cases. i.e. $\left(\frac{\partial p}{\partial x}\right)_A=\left(\frac{\partial p}{\partial x}\right)_B$.

$$\rho V\frac{\partial V}{\partial x}=-(\rho_B-\rho_A)g+\mu\frac{\partial^2 V}{\partial y^2}$$

Assuming the density diff. in the two cases is small, we can say $\delta\rho=\rho_B-\rho_A=-\bar{\rho}\beta(T_B-T_A)$

$$\boxed{\rho V\frac{\partial V}{\partial x}=\bar{\rho}\beta(T-T_\infty)g+\mu\frac{\partial^2 V}{\partial y^2}}\quad\text{[Boussinesq eqn.]}$$

- If the momentum term $\left(\rho V\frac{\partial V}{\partial x}\right)$ is much smaller than the buoyancy $\left(\bar{\rho}\beta(T-T_\infty)g\right)$ and viscous $\left(\mu\frac{\partial^2 V}{\partial y^2}\right)$ terms, then we can write

$$\mu\frac{\partial^2 V}{\partial y^2}=-\bar{\rho}\beta(T-T_\infty)g$$

Applying scaling analysis, $\mu\frac{\partial^2 V}{\partial y^2}\sim\mu\frac{V_s}{\delta^2}$ where δ is the boundary layer thickness.

$$\therefore\ \boxed{V_s=\frac{\bar{\rho}\beta g\Delta T\delta^2}{\mu}}$$

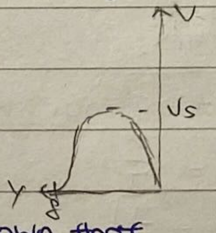
- The BC for the problem is:

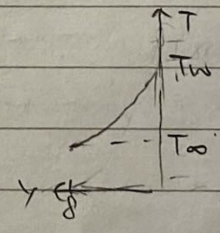
↳ $y=0$: $V=0$; $T=T_w$

↳ $y=\delta$: $V=0,\ \mu\frac{\partial V}{\partial y}=0$; $T=T_\infty,\ \frac{dT}{dy}=0$

The simplest polynomials that satisfies the above BC are

$$V=V_s\frac{y}{\delta}\left(1-\frac{y}{\delta}\right)^2 \qquad T-T_\infty=(T-T_\infty)\left(1-\frac{y}{\delta}\right)^2$$





- It can be shown that

$$\delta\propto x^{1/4} \qquad V_s\propto x^{1/2}$$

- For scaling general problems, we replace δ w/ the characteristic length D, so

$$V_s=\frac{\bar{\rho}\beta g\Delta T D^2}{\mu}$$

large Gr – turbulent
small Gr – laminar.

- For natural convection problems, the Reynolds no. Re equivalent is the Grashof no. Gr, defined as

$$\boxed{Gr=\frac{\bar{\rho}V_s D}{\mu}=\frac{\bar{\rho}^2\beta g\Delta T D^3}{\mu^2}=\frac{\beta g\Delta T D^3}{\nu^2}}$$

Evaluation of Nu for a vertical isothermal plate of length L.

- Overall heat transfer [$Pr \approx 1$],

$$Nu_L = 0.52(GrPr)^{0.25}$$

(In Databook)

Dimensional analysis for natural convection problems.

- For natural convection problems, $\beta \Delta T g$ must appear together, i.e. they represent one independent dimensional quantity, not three.
- For example, $h_{overall} = fn(D, \bar{\rho}, \mu, \lambda, c_p, \beta \Delta T g)$.

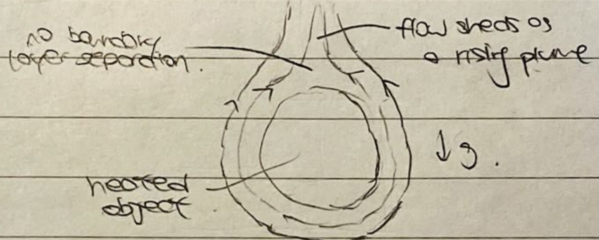
Applying dim. analysis, we find that

$$\frac{h_{overall} D}{\lambda} = fn\left(\frac{\bar{\rho}^2 \beta g \Delta T D^3}{\mu^2}, \frac{\mu c_p}{\lambda}\right)$$

$$Nu_{overall} = fn(Gr, Pr).$$

Typical free convection

- In general, for natural convection,
  - ↳ The flows are laminar as the velocities involved are small.
  - ↳ The boundary layer does not separate since there are no (adverse) pressure gradients.



Radiation

Radiation

Radiative heat transfer

- Heat can be transferred by EM waves emitted by a body due to its temperature.
- It does not req. a medium b/twn the bodies and can occur through a medium which is colder than either body.
- Thermal radiation lies roughly in the wavelength range 0.1 μm to 100 μm.

The Black body.

- A Black body is an idealised object, which will
  - ↳ completely absorb all radiation incident upon it.
  - ↳ emit the largest amt. of energy per unit area possible for a given temp.
- The max. rate at which a body can radiate energy is given by the Stefan-Boltzmann Law.

E_b is power per unit area → $\boxed{E_b = \sigma T^4}$

where $\sigma = 5.67 \times 10^{-8}\, W m^{-2} K^{-4}$ is the Stefan-Boltzmann constant and E_b is the black body emissive power.

- The monochromatic emissive power of a black body $\dot{E}_{b\lambda}$ (power emitted b/twn wavelengths λ and $\lambda + d\lambda$) is given by

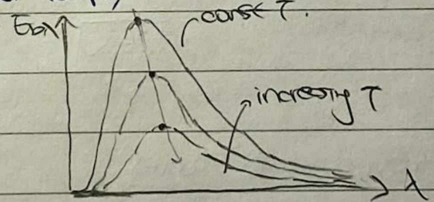
$$E_{b\lambda} = \frac{C_1 \lambda^{-5}}{e^{C_2/\lambda T} - 1}$$

where $C_1 = 3.743 \times 10^8\, W \mu m\, m^{-2}$, $C_2 = 1.4387 \times 10^4\, \mu m K$

- Integrating the monochromatic emissive power over all wavelengths, we get

$$\int_0^\infty E_{b\lambda}\, d\lambda = \sigma T^4 = E_b$$

- Plotting $E_{b\lambda}$ for a range of temps,



The peak value of $E_{b\lambda}$ is at a wavelength

$\boxed{\lambda_{max} = \frac{2897.6}{T}\, \mu m}$ [Wien displacement law].

Real surfaces.

Real surfaces.

- Real surfaces emit less radiation than a black body. The monochromatic emissivity ε_λ characterises the surface, and is defined as.

$$\varepsilon_\lambda = \frac{\text{radiation emitted per unit area btwn } \lambda \text{ \& } \lambda + d\lambda}{E_{b\lambda}}.$$

which is a function of the wavelength of radiation λ.

- The (total) emissivity is given by

$$\boxed{\varepsilon = \frac{E}{E_b} = \frac{\int_0^\infty \varepsilon_\lambda E_{b\lambda} d\lambda}{\int_0^\infty E_{b\lambda} d\lambda} = \frac{\int_0^\infty \varepsilon_\lambda E_{b\lambda} d\lambda}{\sigma T^4}}$$

which is in general a function of temp.

- A Grey body is defined s.t. the monochromatic emissivity ε_λ is independent of wavelength λ, i.e. ε_λ is constant $\rightarrow$ $\boxed{\varepsilon_\lambda = \varepsilon}$. (reasonably good approximation).

Absorption, transmission and reflection.

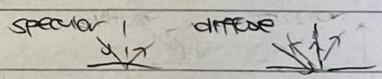
- When radiation strikes a surface, a fraction ρ (reflectivity) will be reflected, a fraction α (absorbtivity) will be absorbed, and a fraction τ (transmittivity) will be transmitted.

$$\boxed{\rho + \alpha + \tau = 1}.$$

- Most solid bodies (opaque bodies) do not transmit thermal radiation, so $\tau = 0$,

$$\rho + \alpha = 1.$$

- The reflected part of the radiation may be specular or diffuse (or a combination of both). Most surfaces are predominantly diffuse reflectors



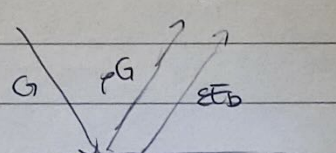
- Due to quantum effects, the prob. of a photon being absorbed on hitting the surface is equal to the prob. of a photon being emitted. This gives rise to Kirchoff's identity.

$$\boxed{\varepsilon = \alpha}$$

Radiation to and from non-black bodies.

- The irradiation G is defined as the total radiation incident on a surface per unit time and area. The radiosity J is defined as the total radiation leaving a surface per unit time and area.
- we will assume these quantities are uniform over each surface.
- The radiosity J of a surface is given by

$$\boxed{J = \varepsilon E_b + \rho G}$$



- Since $\rho = 1 - \alpha = 1 - \varepsilon$, (for non-transmissive surfaces), the expression for J can be rewritten.

$$J = \varepsilon E_b + (1-\varepsilon)G$$

$$G = \frac{J - \varepsilon E_b}{1-\varepsilon}$$

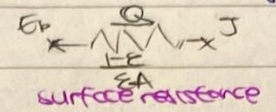
- The net radiation leaving a surface per unit area and time $\dot{q}$ is the diff. btwn radiosity J and the irradiation G, so

$$\dot{q} = J - G = J - \frac{J - \varepsilon E_b}{1-\varepsilon}$$

$$\dot{Q} = \frac{A(\cancel{J} - \varepsilon J - \cancel{J} + \varepsilon E_b)}{1-\varepsilon}$$

$$\boxed{\dot{Q} = \frac{E_b - J}{(1-\varepsilon)/\varepsilon A}}$$

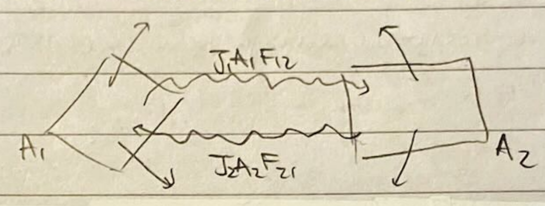
This can be interpreted using an ohm's law analogy



Radiative exchange between surfaces

Radiative exchange between surfaces.

- Consider the heat transfer btwn two surfaces A_1 and A_2 (both diffuse reflectors)



we define the shape factor F_{ij} as the fraction of energy leaving surface i that arrives at surface j, which is purely geometric and depends on how much of surface j can be seen by i

- Since EM rays travel in straight lines, if radiation can travel from 1 to 2 along a path, it can also travel from 2 to 1 along the same path, so we have the reciprocity relationship.

$$\boxed{A_1F_{12} = A_2F_{21}}$$

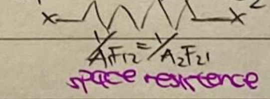
- of the total radiation that leaves surface 1 (J_1A_1), a quantity $J_1A_1F_{12}$ reaches surface 2. Likewise from 2 to 1, a quantity $J_2A_2F_{21}$ reaches surface 1 from surface 2. The net exchange is

$$\dot{Q}_{12} = J_1A_1F_{12} - J_2A_2F_{21}$$

Applying the reciprocity relationship $A_1F_{12} = A_2F_{21}$, we get

$$\boxed{\dot{Q}_{12} = \frac{J_1 - J_2}{1/A_1F_{12}} = \frac{J_1 - J_2}{1/A_2F_{21}}}$$

This can be interpreted using an ohm's law analogy



Shape factor algebra

- By defn, the shape factor is the fraction of energy leaving one surface and hitting another, so shape factors must sum up to unity.

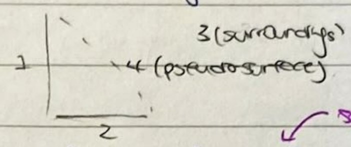
$$\boxed{\sum_i F_{1i} = 1}$$

* Note we need to include the self view factor F_{11} since a surface may be able to see itself.

- Convex surfaces cannot see themselves, so their self view factor is 0. If surface 1 is convex,

$$F_{11} = 0.$$

- Sometimes, it is useful to replace the surroundings w/ a pseudo surface.

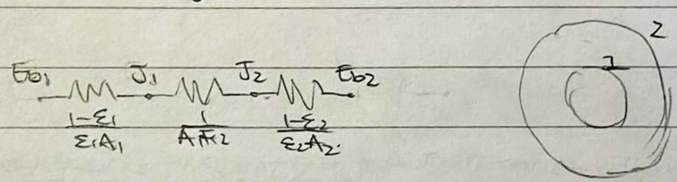


so $\dot{Q}$ through $\frac{1-\varepsilon}{\varepsilon A}$ is 0.

- An insulated / refractory surface has infinite surface resistance, and thus acts like a black body (although it isn't a black body). It reflects any incoming radiation entirely.

Radiation exchange between concentric cylinders

- Consider the radiation exchange btwn concentric cylinders 1, 2.



since 1 is convex, $F_{11} = 0$

sum of shape factors is unity $F_{11} + F_{12} = 1 \rightarrow F_{12} = 1$

The overall resistance $R = \frac{1-\varepsilon_1}{\varepsilon_1 A_1} + \frac{1}{A_1 F_{12}} + \frac{1-\varepsilon_2}{\varepsilon_2 A_2}$

Total heat exchanged

$$\dot{Q} = \frac{\sigma T_1^4 - \sigma T_2^4}{R} = \frac{\sigma (T_1^4 - T_2^4)}{1-\varepsilon_1/\varepsilon_1 A_1 + \frac{1}{A_1 F_{12}} + 1-\varepsilon_2/\varepsilon_2 A_2}$$

$$= \frac{A_1 \sigma (T_1^4 - T_2^4)}{1-\varepsilon_1/\varepsilon_1 + 1 + (A_1/A_2)\,1-\varepsilon_2/\varepsilon_2}$$

* The above expression is valid as long as the inner surface is convex ($F_{11} = 0$)

- When $A_2 \rightarrow \infty$ (or $\varepsilon_2 \rightarrow 1$), we get

$$\boxed{\dot{Q} = A_1 \varepsilon_1 \sigma (T_1^4 - T_2^4)}$$

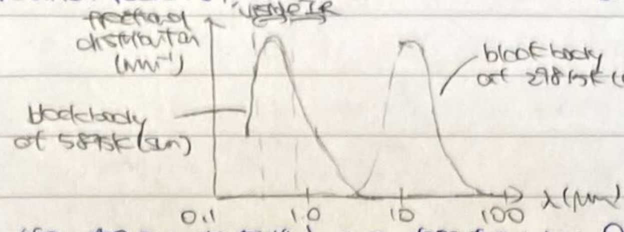
which shows that when an object is in a large enclosure, the net radiation exchange is independent of the emissivity of the enclosure, and it is as if the enclosure were a black body.

pf note last part.

Radiation in the environment

Radiation in the environment

- compared to terrestrial radiation, solar radiation is of much shorter wavelengths



- Incoming UV rays (from the sun to Earth) are absorbed by O_2 and O_3.

outgoing IR radiation (from the Earth) is absorbed by CO_2, H_2O, CH_4 etc (absorb $\lambda \sim 10\,\mu m$).

A simple model of the greenhouse effect.

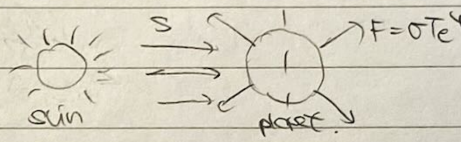
- In this model, the absorbance α, reflectivity ρ and emissivity ε are now functions of wavelength.

(The assumption that they are independent of wavelength is no longer valid).

- The albedo A is defined as the fraction of the incident radiation which is reflected.

- First consider a planet, radius R, w/ solar flux S (short-wave radiation), and albedo A.

For long wave radiation, the planet can be taken to be black.



The overall energy absorbed is

$$\dot{Q}_S = S\underbrace{\pi R^2}_{\text{projected area}}(1-A)$$

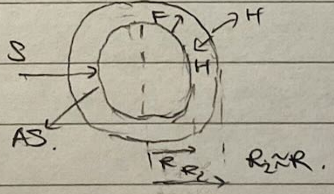
If the planet's surface is at T_e, then the (mainly long wave) energy emitted is

$$\dot{Q}_P = 4\pi R^2 \sigma T_e^4$$

At steady state, $\dot{Q}_S = \dot{Q}_P$, so

$$\sigma T_e^4 = \frac{S(1-A)}{4}.$$

- Now, also consider IR absorbing gases that form a layer of some thickness above the surface, spread equally over the entire planet.



The overall energy absorbed is

$$\dot{Q}_S = S\underbrace{\pi R^2}_{\text{projected area}}(1-A)$$

The gas layer absorbs F and itself emit H both inwards and outwards (so overall the layer emits a total of 2H flux).

In this new situation the outer surface of the gas layer becomes the outer surface of the planet.

The amt. of energy radiated by the planet is therefore

$$\dot{Q}_P = 4\pi R^2 H$$

At steady state, $\dot{Q}_S = \dot{Q}_P$, so

$$H = \frac{(1-A)S}{4}$$

At the planet's surface, the energy balance is

$$\overset{\text{energy in}}{4\pi R^2 F} = \overset{\text{energy out}}{H \cdot 4\pi R^2 + (1-A) S\pi R^2}$$

$$F = H + \frac{(1-A)S}{4}$$

$$\sigma T_e^4 = \frac{(1-A)S}{2}.$$

- substituting numbers, $S = 1000\,Wm^{-2}$, $A = 0.1$,

w/o gas layer: $T_e = 250K$.

w/ gas layer: $T_e = 298.5K$

→ w/o the greenhouse effect, life on Earth wouldn't exist.