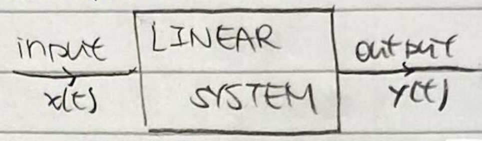
Linear systems

Linear systems

Linear system.

- Schematic of a linear system:

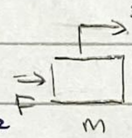


- For a linear system, the output $y(t)$ is related to the input $x(t)$ via a linear differential equation. $\left[a_n\frac{d^n y}{dt^n}+\ldots+a_1\frac{dy}{dt}+a_0 y = b_n\frac{d^n x}{dt^n}+\ldots+b_1\frac{dx}{dt}+b_0 x\right]$.
- Linearity implies that superposition applies, i.e. if input $x_1(t)$ gives output $y_1(t)$ and input $x_2(t)$ gives output $y_2(t)$, input $\alpha x_1(t)+\beta x_2(t)$ gives output $\alpha y_1(t)+\beta y_2(t)$.

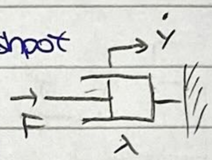
System elements.

① Mechanical system (Translational)

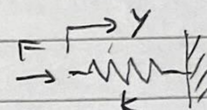
- Mass.
  - ↳ $F = m\ddot{y}$
  - ↳ $KE = \frac{1}{2}m\dot{y}^2$



- Damper/dashpot
  - ↳ $F = \lambda\dot{y}$
  - ↳ $P = \lambda\dot{y}^2$

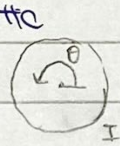


- Spring.
  - ↳ $F = ky$
  - ↳ $PE = \frac{1}{2}ky^2$



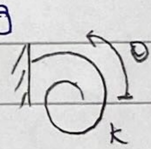
② Mechanical system (Rotational)

- Moment of inertia
  - ↳ $\tau = I\ddot{\theta}$
  - ↳ $KE = \frac{1}{2}$



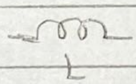
- Torsional damper
  - ↳ $\tau = \lambda\dot{\theta}$
  - ↳ $P = \lambda\dot{\theta}^2$

- Torsional spring
  - ↳ $\tau = k\theta$
  - ↳ $PE = \frac{1}{2}k\theta^2$

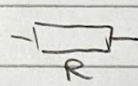


③ Electrical system

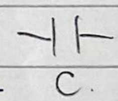
- Inductance
  - ↳ $V = L\ddot{Q}$
  - ↳ $PE = \frac{1}{2}L\dot{Q}^2$



- Resistance
  - ↳ $V = R\dot{Q}$
  - ↳ $P = R\dot{Q}^2$



- Capacitance
  - ↳ $V = \frac{1}{C}Q$
  - ↳ $P = \frac{1}{2}\left(\frac{1}{C}\right)Q^2$



* $W = \int F\,dy$, $P = F\dot{y}$, $\ddot{y} = \dot{y}\frac{d\dot{y}}{dy}$

Obtaining differential equations.

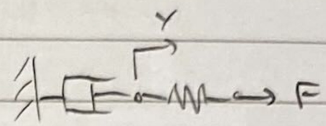
- To find the governing DE of a system, we use:
  - ↳ Sum of forces / torques (N2L) // Sum of voltages (KVL)
  - ↳ Compatibility (if applicable).

First order systems

Standard form

- For a general first-order system,

$$\lambda \dot{y} + ky = F.$$



Rearranging to set the coefficient of y to be 1,

$$\boxed{T\dot{y} + y = x} \quad \text{[Standard form].}$$

- T is the time constant. $\boxed{T = \frac{\lambda}{k}}$ [units: s]

General response

- We can solve the standard form DE directly using integrating factor:

$$\dot{y} + \frac{1}{T}y = \frac{1}{T}x$$

$$\frac{d}{dt}\left(y e^{\frac{t}{T}}\right) = \frac{1}{T} x e^{\frac{t}{T}}$$

$$y e^{\frac{t}{T}} = \frac{1}{T}\int_0^t x e^{\frac{t}{T}} dt + A.$$

depends on boundary conditions

$$\therefore\ y = \underbrace{Ae^{-t/T}}_{CF} + \underbrace{\frac{1}{T}e^{-t/T}\int_0^t x e^{t/T} dt}_{PI}$$

depends on the input x.

- The total response can be split into the complementary function (CF) and the particular integral (PI), i.e. $y = y_{CF} + y_{PI}$.

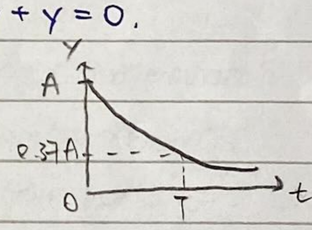
* In practice, we often don't evaluate the integral as we can find the PI via alt. means.

Free response

Solving for CF directly.

- Consider the homogeneous case, $x(t) = 0$, i.e. $T\dot{y} + y = 0$.
- Solving the DE gives

$$\boxed{y = Ae^{-t/T}}$$



- At $t = T$, $y = y(T) = A \cdot \frac{1}{e} \approx 0.37A$.

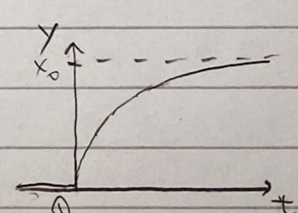
Step response

Solving for PI directly.

- Consider the input $x(t) = x_0 H(t)$, i.e. $T\dot{y} + y = x_0 H(t)$
- For $t \geq 0$, $x(t) = x_0$. → Try $y_{PI} = k$, so $\dot{y}_{PI} = 0$.

$$T(0) + k = x_0 \quad \rightarrow \quad y_{PI} = x_0$$

(or by inspection, we can see $y_{PI} = x_0$)



- For $t \geq 0$, the total response is $y = Ae^{-t/T} + x_0$

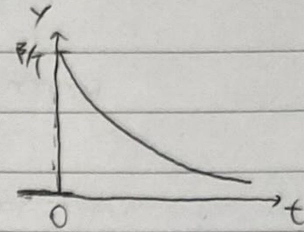
Continuity → $y(0) = 0$ $\quad \therefore \boxed{y = x_0(1 - e^{-t/T})}$

* Note for $t < 0$, $y = 0$.

Impulse response

Solving for PI directly

- Consider the input $x(t) = \beta\,\delta(t)$, i.e. $T\dot{y} + y = \beta\,\delta(t)$.
- For $t>0$, $x(t)=0$. By inspection, $y_{PI}=0$.
- For $t>0$, the total response is $y = Ae^{-t/T}$



continuity → $y(0^-)=0$.

$$\int_{0^-}^{0^+} (T\dot{y}+y)\,dt = \int_{0^-}^{0^+} \beta\,\delta(t)$$

$$T\int_{t=0^-}^{t=0^+} dy + 0 = \beta$$

$$y(0^+) - \cancelto{0}{y(0^-)} = \beta/T \qquad \text{so } A = y(0^+) = \beta/T.$$

$$\therefore \boxed{y = \frac{\beta}{T}e^{-t/T}}$$

Differentiate the step response

- The impulse function $\delta(t)$ is the derivative of the step function $H(t)$ $\frac{d}{dt}[H(t)] = \delta(t)$.
  We can differentiate the step response to get the impulse response.
- The impulse $x_I(t) = \beta\delta(t)$ corresponds to the step $x_S(t) = \beta H(t)$. $\left(\frac{d}{dt}[\beta H(t)] = \beta\delta(t)\right)$
  The step response is $y_S = \beta(1-e^{-t/T})$

$$\therefore\ y_I = \frac{d}{dt}\left[\beta(1-e^{-t/T})\right]$$

$$= \frac{\beta}{T}e^{-t/T}$$

* Note differentiation is a linear operator so for a linear system, if input $x(t)$ gives output $y(t)$, then input $\dot{x}(t)$ gives output $\dot{y}(t)$.

Ramp response

Solving for PI directly

- Consider the input $x(t) = \alpha t\, H(t)$ i.e. $T\dot{y} + y = \alpha t \cdot H(t)$
- For $t>0$, $x(t) = \alpha t$. → Try $y_{PI} = at+b$, so $\dot{y}_{PI} = a$,

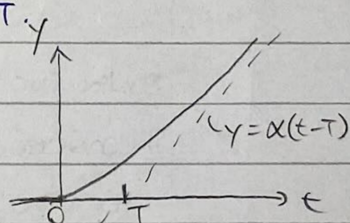
$$Ta + at + b = \alpha t$$

Comparing coefficients, $a=\alpha$, $b=-\alpha T$. → $y_{PI} = \alpha t - \alpha T$.

- For $t>0$, the total response is $y = Ae^{-t/T} + \alpha(t-T)$.

continuity → $y(0)=0$. $\qquad 0 = A - \alpha T$ → $A = \alpha T$

$$\therefore \boxed{y = \alpha T e^{-t/T} + \alpha(t-T)}$$



Integrate the step response.

- The linear function t is the integral of the step function. $\int_0^t H(t)\,dt = t$.

  We can integrate the step response to get the ramp response.
- The linear function $x_R(t) = \alpha t$ corresponds to the step $x_S(t) = \alpha H(t)$. $\left(\int_0^t \alpha H(t)\,dt = \alpha t\right)$.

  The step response is $y_S = \alpha(1 - e^{-t/T})$.

$$\therefore\ y_R = \int_0^t \alpha(1 - e^{-t/T})\,dt$$
$$= \left[\alpha t + \alpha T e^{-t/T}\right]_0^t$$
$$= \alpha T e^{-t/T} + \alpha(t - T).$$

* Note integration is a linear operator so for a linear system, if input $x(t)$ gives output $y(t)$, then input $\int_0^t x(t)\,dt$ gives output $\int_0^t y(t)\,dt$.

Harmonic response.

Solving for PI directly.

- Consider the input $x(t) = X\cos(\omega t)H(t)$. i.e. $T\dot{y} + y = X\cos(\omega t)\cdot H(t)$.
- For $t \geq 0$, $x(t) = X\cos(\omega t)$ $\rightarrow$ Try $y_{PI} = A\sin(\omega t) + B\cos(\omega t)$, so $\dot{y}_{PI} = \omega(A\cos(\omega t) - B\sin(\omega t))$

$$\omega T(A\cos(\omega t) - B\sin(\omega t)) + A\sin(\omega t) + B\cos(\omega t) = X\cos(\omega t).$$

Comparing coefficients, $A = B\omega T$, $B + \omega T A = X$ $\rightarrow$ $B + \omega T(B\omega T) = X$

so $B = \frac{1}{1+\omega^2T^2}X$, $A = \frac{\omega T}{(1+\omega^2T^2)}X$ $\rightarrow$ $y_{PI} = \frac{X}{1+\omega^2T^2}(\omega T\sin(\omega T) + \cos(\omega T))$.

Alternatively, $y_{PI} = \frac{X}{1+\omega^2T^2}\left[\sqrt{1+\omega^2T^2}\cos(\omega T - \arctan(\omega T))\right] = \frac{X}{\sqrt{1+\omega^2T^2}}\cos(\omega T - \arctan(\omega T))$.

* NB $A\sin x + B\cos x = R\cos(x - \phi_1) = R\cos(x + \phi_2)$, $R = \sqrt{A^2+B^2}$, $\tan\phi_1 = \frac{A}{B}$, $\tan\phi_2 = \frac{B}{A}$.

- For $t \geq 0$, the total response is $y = Ae^{-t/T} + \frac{X}{\sqrt{1+\omega^2T^2}}\cos(\omega T - \arctan(\omega T))$.

  Continuity $\rightarrow y(0) = 0$ so we can find A. However, we are more interested in the steady state response rather than the transient response.

  y lags x.

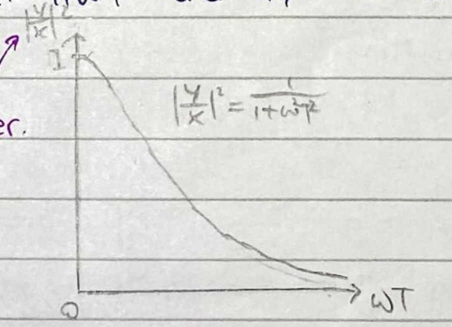
  Steady state (large t), $y_{CF} = Ae^{-t/T} \rightarrow 0$ so $\boxed{y \rightarrow y_{PI} = \frac{X}{\sqrt{1+\omega^2T^2}}\cos(\omega T - \arctan(\omega T))}$

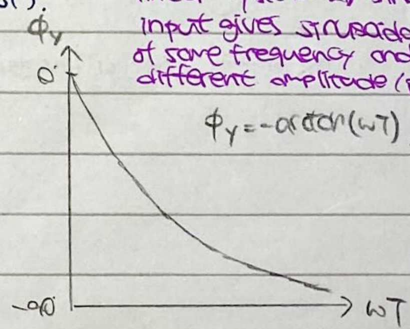
- Just considering the steady state response, $y = \frac{X}{\sqrt{1+\omega^2T^2}}\cos(\omega T + \phi_y)$.

  where $\left|\frac{Y}{X}\right|^2 = \frac{1}{1+\omega^2T^2}$ and $\phi_y = -\arctan(\omega T)$.

  linear system w/ sinusoidal input gives sinusoidal output of same frequency and possibly different amplitude/phase

$\left|\frac{Y}{X}\right|^2$ related to power.





* For large ω, $\dot{y}$ is in phase w/ x (easier to see using the $e^{i\omega t}$ method)

$e^{i\omega t}$ method.

- Assume x and y have the form

$$x = \mathrm{Re}\{Xe^{i\omega t}\} \qquad y = \mathrm{Re}\{Ye^{i\omega t}\}.$$

where X, Y are complex amplitudes (amplitude + phase).

- Drop the "Real" specification and substitute into the eqn. of motion,

$$x = Xe^{i\omega t} \qquad y = Ye^{i\omega t}$$

$$T(i\omega Y\cancel{e^{i\omega t}}) + Y\cancel{e^{i\omega t}} = X\cancel{e^{i\omega t}}$$

$$Y(1+i\omega T) = X$$

$$\boxed{Y = \frac{X}{1+i\omega T}}.$$

here we assume the phase of y is measured rel. to x. so $\phi_x = 0$, i.e. $X \in \mathbb{R}$.

- Take $\mathrm{Re}\{Ye^{i\omega t}\}$ to give the PI.

$$y = \mathrm{Re}\{Ye^{i\omega t}\} = \mathrm{Re}\{\frac{X}{1+i\omega T}e^{i\omega t}\}.$$

$$= \mathrm{Re}\{\frac{X}{1+\omega^2T^2}(1-i\omega T)(\cos\omega T + i\sin\omega T)\}$$

$$= \frac{X}{1+\omega^2T^2}[\omega T\sin(\omega T) + \cos(\omega T)] = \frac{X}{\sqrt{1+\omega^2T^2}}\cos(\omega T - \arctan(\omega T))$$

* NB. Real and imaginary parts are orthogonal to each other → they do not interfere w/ each other when passing a linear system.

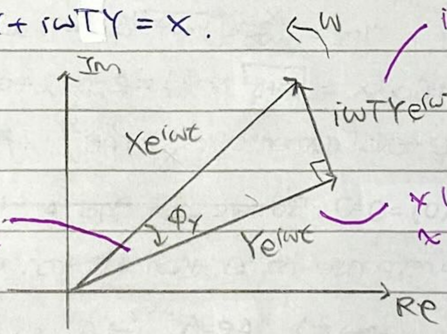
Phasor diagrams. — graphical approach for $e^{i\omega t}$ method

- From the $e^{i\omega t}$ method, we found that $Y(1+i\omega T) = X$.

i.e. $Y + i\omega TY = X$.

$i\omega TYe^{i\omega t} = e^{\frac{\pi}{2}}\omega TYe^{i\omega t} = \omega TYe^{i(\omega t+\frac{\pi}{2})}$.

- On a phasor diagram,



note ϕ_Y is the phase of Y measured relative to x.

y lags x or x leads y.

we can see $|X|^2 = |Y|^2 + |\omega TY|^2 = |Y|^2(1+\omega^2T^2) \rightarrow |\frac{Y}{X}| = \frac{1}{1+\omega^2T^2}$

and $\tan\phi_Y = -\frac{\omega TY}{Y} \rightarrow \phi_Y = \arctan(-\omega T) = -\arctan(\omega T)$.

$\therefore\ Y = \frac{|X|}{1+\omega^2T^2}\cos(\omega T - \arctan(\omega T))$.

Second order systems

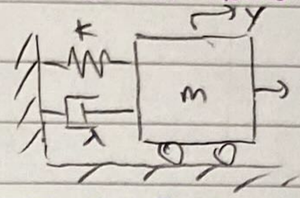
Second order systems

Standard form

- For a general second-order system,

$$m\ddot{y} + \lambda\dot{y} + ky = F$$

Power balance:
$m\ddot{y}\dot{y} + ky\dot{y} = F\dot{y} - \lambda\dot{y}^2$
$\frac{d}{dt}\left(\frac{1}{2}m\dot{y}^2 + \frac{1}{2}ky^2\right) = P_{in} - P_{losses}$.



- Rearranging to set the coefficient of y to be 1,

$$\frac{m}{k}\ddot{y} + \frac{\lambda}{k}\dot{y} + y = \frac{F}{k}.$$

$$\boxed{\frac{1}{\omega_n^2}\ddot{y} + \frac{2\zeta}{\omega_n}\dot{y} + y = x}$$ [standard form]

- ω_n is the undamped natural frequency, $\boxed{\omega_n = \sqrt{\frac{k}{m}}}$ [units: rad s^{-1}]
- ζ is the damping ratio, $\boxed{\zeta = \frac{\lambda}{2\sqrt{mk}}}$ [units: –]
- x is the displacement input, $\boxed{x = \frac{F}{k}}$ [units: m]

We can avoid any static forces (like weight) by measuring y from the static eqm position.

Free response

Solving for CF directly

- Consider the homogeneous case, $x(t) = 0$, i.e. $\frac{1}{\omega_n^2}\ddot{y} + \frac{2\zeta}{\omega_n}\dot{y} + y = 0$.

AE: $\frac{1}{\omega_n^2}\alpha^2 + \frac{2\zeta}{\omega_n}\alpha + 1 = 0$

$\alpha^2 + 2\zeta\omega_n\alpha + \omega_n^2 = 0$

$\alpha = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

CF: $\boxed{y = Ae^{\alpha_1 t} + Be^{\alpha_2 t}, \quad \alpha_1, \alpha_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}}$

- There are 3 different cases:
  - ↳ $\zeta > 1$ (overdamped) → $\alpha_{1,2}$ are real and distinct
  - ↳ $\zeta = 1$ (critically damped) → $\alpha_{1,2}$ are real and repeated

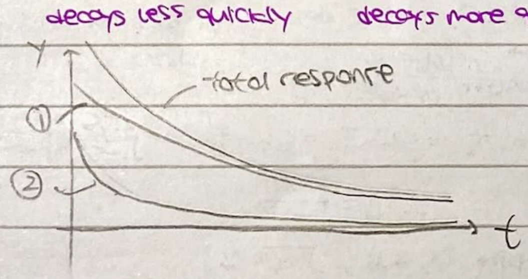
  ] no vibration

  - ↳ $\zeta < 1$ (underdamped) → $\alpha_{1,2}$ are complex – vibration.

Case 1: $\zeta > 1$ (overdamped)

- α_1 and α_2 are both real (and negative). $\zeta > \sqrt{\zeta^2 - 1}$

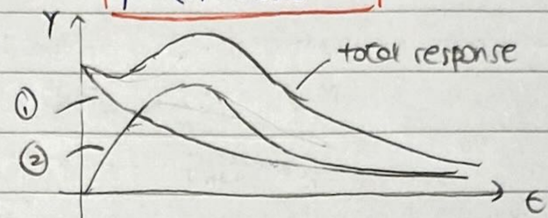
$$\boxed{y = \underbrace{A\exp\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}\right)t}_{\text{① decays less quickly}} + \underbrace{B\exp\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}\right)t}_{\text{② decays more quickly}}}$$



case 2 : $\zeta = 1$ (critically damped)

- $\alpha_1 = \alpha_2 = -\omega_n$. special case → an additional term must be added to maintain 2 independent components of response.

$$\boxed{y = (\overset{①}{A} + \overset{②}{Bt})e^{-\omega_n t}}$$



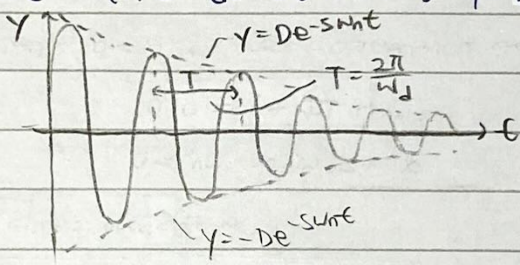
case 3 : $\zeta < 1$ (underdamped)

- $\alpha_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$.

  Define the damped natural frequency ω_d, $\boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}}$

$$\boxed{y = e^{-\zeta\omega_n}(Ae^{i\omega_d t} + Be^{i\omega_d t})}$$

or equivalently, $y = e^{-\zeta\omega_n}(A'\cos\omega_d t + B'\sin\omega_d t)$; $y = De^{-\zeta\omega_n t}\cos(\omega_d t - \psi)$



- Damping reduces the vibration frequency from undamped natural frequency ω_n to the damped natural frequency ω_d. (due to energy lost)

* when $\zeta = 0$, the curve is SHM at the undamped natural frequency ω_n.

Logarithmic decrement (Log-dec)

- Logarithmic decrement is a method of measuring the damping ratio from the transient response of a system (✓ for free response, step response, impulse response).
- Logarithmic decrement Δ is defined as

$$\boxed{\Delta = \ln\left(\frac{y_1}{y_2}\right)}$$

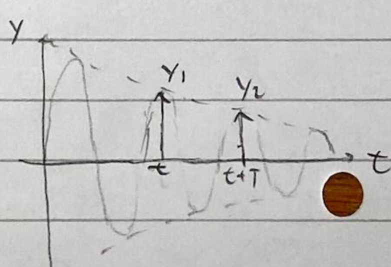
ratio of amplitudes of only 2 successive cycles

- For free response, (same result for step response and impulse response)

$$\frac{y_1}{y_2} = \frac{De^{-\zeta\omega_n t}\cos(\omega_d t - \psi)}{De^{-\zeta\omega_n (t+T)}\cos(\omega_d(t+T) - \psi)} = e^{\zeta\omega_n T}$$

$$\rightarrow \Delta = \ln\left(\frac{y_1}{y_2}\right) = \zeta\omega_n T = \zeta\omega_n \frac{2\pi}{\omega_d} = \zeta\omega_n \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$\therefore \boxed{\Delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}}$$



- For light damping ($\zeta \ll 1$), $\sqrt{1-\zeta^2} \approx 1$,

$$\boxed{\Delta = 2\pi\zeta}$$

* $\Delta_N = \ln\left(\frac{y_1}{y_{1+N}}\right) = \ln\left(\frac{y_1}{y_2}\right) + \ln\left(\frac{y_2}{y_3}\right) \ldots \ln\left(\frac{y_N}{y_{1+N}}\right) = N\Delta$

Step response

Solving for PI directly

- Consider the input $x(t) = x_o H(t)$, i.e. $\frac{1}{\omega_n^2}\ddot{y} + \frac{2\zeta}{\omega_n}\dot{y} + y = x_o H(t)$
- For $t \geq 0$, $x(t) = x_o$ → Try $y_{PI} = k$ so $\dot{y}_{PI} = 0$, $\ddot{y}_{PI} = 0$

$$\frac{1}{\omega_n^2}(0) + \frac{2\zeta}{\omega_n}(0) + k = x_o \rightarrow y_{PI} = x_o.$$

(or by inspection, we can see $y_{PI} = x_o$)

- We only consider the case $\zeta < 1$ (as there are vibrations)

For $t \geq 0$, the total response is $y = De^{-\zeta\omega_n t}\cos(\omega_d t - \psi) + x_o$

continuity → $y(0) = \dot{y}(0) = 0$. Note $\dot{y} = De^{-\zeta\omega_n t}(-\zeta\omega_n\cos(\omega_d t - \psi) - \omega_d\sin(\omega_d t - \psi))$

$y(0) = 0$: $0 = D\cos(-\psi) + x_o \rightarrow D = -\frac{x_o}{\cos\psi}$

$\dot{y}(0) = 0$: $0 = \cancel{D}(-\zeta\omega_n\cos(-\psi) - \omega_d\sin(-\psi)) \rightarrow \tan\psi = \frac{\zeta\omega_n}{\omega_d} = \frac{\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}} = \frac{\zeta}{\sqrt{1-\zeta^2}}$

Since $\tan\psi = \frac{\zeta}{\sqrt{1-\zeta^2}}$, $\sin\psi = \zeta$ and $\cos\psi = \sqrt{1-\zeta^2}$.

$$\therefore y = -\frac{x_o}{\cos\psi}e^{-\zeta\omega_n t}\cos(\omega_d t - \psi) + x_o$$

$$\boxed{y = x_o\left[1 - \frac{1}{\cos\psi}e^{-\zeta\omega_n t}\cos(\omega_d t - \psi)\right]}, \text{ where } \sin\psi = \zeta$$

- For light damping ($\zeta \ll 1$), $\sqrt{1-\zeta^2} \approx 1 \rightarrow \cos\psi \approx 1$, $\psi \approx 0$, $\omega_d \approx \omega_n$.

$$\therefore \boxed{y = x_o\left[1 - e^{-\zeta\omega_n t}\cos(\omega_n t)\right]}$$

Impulse response

Differentiate the step response

- The impulse function $\delta(t)$ is the derivative of the step function $H(t)$, $\frac{d}{dt}[H(t)] = \delta(t)$.

we can differentiate the step response to get the impulse response

- The impulse $x_I(t) = \beta\delta(t)$ corresponds to the step $x_S(t) = \beta H(t)$ ($\frac{d}{dt}[\beta H(t)] = \beta\delta(t)$)

The step response is $y_S = \beta\left[1 - \frac{1}{\cos\psi}e^{-\zeta\omega_n t}\cos(\omega_d t - \psi)\right]$, $\sin\psi = \zeta$ / $\cos\psi = \sqrt{1-\zeta^2}$

$$\therefore y_I = \frac{\beta}{\cos\psi}e^{-\zeta\omega_n t}(\zeta\omega_n\cos(\omega_d t - \psi) + \omega_d\sin(\omega_d t - \psi))$$

$$= \frac{\beta}{\cos\psi}e^{-\zeta\omega_n t}\left[\zeta\omega_n\cos(\omega_d t - \psi) + \omega_n\sqrt{1-\zeta^2}\sin(\omega_d t - \psi)\right]$$

$$= \frac{\beta}{\cos\psi}e^{-\zeta\omega_n t}\omega_n\left[\sin\psi\cos(\omega_d t - \psi) + \cos\psi\sin(\omega_d t - \psi)\right]$$

$$= \frac{\beta}{\cos\psi}e^{-\zeta\omega_n t}\omega_n\sin(\cancel{\psi} + \omega_d t - \cancel{\psi})$$

$$\boxed{y = \frac{\beta\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_d t)}$$

- For light damping, ($\zeta \ll 1$), $\sqrt{1-\zeta^2} \approx 1$.

$$\therefore \boxed{y = \beta\omega_n e^{-\zeta\omega_n t}\sin(\omega_d t)}$$

* Note $\dot{y}(0) \neq 0$, $\dot{y}(0) = \frac{\beta\omega_n\omega_d}{\sqrt{1-\zeta^2}} = \beta\omega_n^2$

For an impulsive force $F = I\delta(t)$, we have a corresponding displacement x, where $F = kx$.

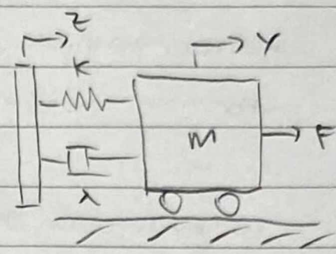
→ $\frac{F}{k} = \frac{I}{k}\delta(t) = \beta\delta(t) = x$, i.e. $\frac{I}{k} = \beta$. impulse = initial change in momentum

$$\therefore \dot{y}(0) = \beta\omega_n^2 = \frac{I}{k}\cdot\frac{k}{m} = \frac{I}{m} \rightarrow I = m\dot{y}(0)$$

Harmonic response

Equation of motion.

- For a general second-order system w/ base motion



$$m\ddot{y} + \lambda\dot{y} + ky = F + \lambda\dot{z} + kz$$

- Rearranging to set the coefficient of y to be 1,

$$\frac{m}{k}\ddot{y} + \frac{\lambda}{k}\dot{y} + y = \frac{F}{k} + \frac{\lambda}{k}\dot{z} + z$$

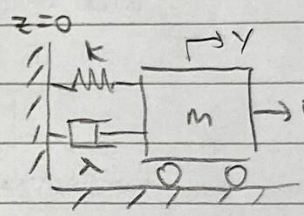
$$\frac{1}{\omega_n^2}\ddot{y} + \frac{2\zeta}{\omega_n}\dot{y} + y = x + \frac{2\zeta}{\omega_n}\dot{z} + z \quad \text{[standard form]}$$

where ω_n, ζ and x are defined as before.

Case (a) – Response to an applied force

- Here, we put the base motion to zero, $z(t) = \dot{z}(t) = 0$, so

$$\boxed{\frac{1}{\omega_n^2}\ddot{y} + \frac{2\zeta}{\omega_n}\dot{y} + y = x}$$ — not a true displacement



- Using the $e^{i\omega t}$ method.

$$x = Xe^{i\omega t} \qquad y = Ye^{i\omega t}$$

$$-\frac{\omega^2}{\omega_n^2}Y\cancel{e^{i\omega t}} + \frac{2\zeta\omega}{\omega_n}iY\cancel{e^{i\omega t}} + Y\cancel{e^{i\omega t}} = X\cancel{e^{i\omega t}}$$

$$Y\left(1 - \frac{\omega^2}{\omega_n^2} + \frac{2\zeta\omega}{\omega_n}i\right) = X$$

$$Y = \frac{X}{1 - \frac{\omega^2}{\omega_n^2} + \frac{2\zeta\omega}{\omega_n}i}$$

$\phi = 0$ stiffness
$\phi = -90$ damping forces
$\phi = -180$ inertia forces

→ Response ratio : $\boxed{\frac{Y}{X} = \frac{1}{1 - \omega^2/\omega_n^2 + i2\zeta\omega/\omega_n}}$

Amplitude : $\boxed{\left|\frac{Y}{X}\right| = \frac{1}{\sqrt{(1-\omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2}}}$, phase : $\boxed{\phi = -\arctan\left(\frac{2\zeta\omega/\omega_n}{1-\omega^2/\omega_n^2}\right)}$

$$\therefore y_{PI} = \frac{X_1}{\sqrt{(1-\omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2}}\cos\left(\omega t - \arctan\left(\frac{2\zeta\omega/\omega_n}{1-\omega^2/\omega_n^2}\right)\right)$$

- We can find the damped resonant frequency ω_r (the angular frequency where $|Y/X|$ is max) by differentiating $|Y/X|$ or $\frac{1}{|Y/X|^2}$ (mathematically easier).

For convenience, set $\Omega = \omega/\omega_n$,

$$\frac{1}{|Y/X|^2} = (1-\Omega^2)^2 + (2\zeta\Omega)^2$$

$$\frac{d}{d\Omega}\left[\frac{1}{|Y/X|^2}\right] = 2(1-\Omega^2)\cdot(-2\Omega) + 8\zeta^2\Omega = 0$$

$$\therefore \Omega = \sqrt{1-2\zeta^2} \quad \rightarrow \quad \boxed{\omega_r = \omega_n\sqrt{1-2\zeta^2}}$$

- The max. response $\left|\frac{Y}{X}\right|_{max}$ is

$$\left|\frac{Y}{X}\right|_{max} = \frac{1}{\sqrt{(1-\Omega^2)^2 + (2\zeta\Omega)^2}}$$

$$= \frac{1}{\sqrt{4\zeta^4 + 4\zeta^2(1-2\zeta^2)}}$$

$$\boxed{\left|\frac{Y}{X}\right|_{max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}} \quad \text{[For } \zeta < 1/\sqrt{2}\text{, o/w there is no local max.]}$$

- For light damping, ($\zeta \ll 1$), $\sqrt{1-\zeta^2} \approx 1$

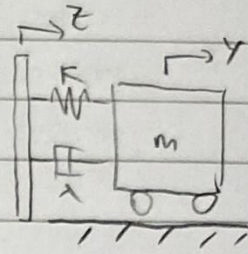
$$\boxed{\left|\frac{Y}{X}\right|_{max} = \frac{1}{2\zeta}}$$

- The Quality factor is defined as $\boxed{Q = \frac{1}{2\zeta}}$, so $Q = \left|\frac{Y}{X}\right|_{max}$ for $\zeta \ll 1$

Case (c) — Absolute response to base movement

- Here, we put the applied force to zero, $F(t) = x(t) = 0$, so

$$\boxed{\frac{1}{\omega_n^2}\ddot{y} + \frac{2\zeta}{\omega_n}\dot{y} + y = \frac{2\zeta}{\omega_n}\dot{z} + z}$$



- Using the $e^{i\omega t}$ method,

$$z = Ze^{i\omega t} \qquad y = Ye^{i\omega t}$$

$$-\frac{\omega^2}{\omega_n^2}Y\cancel{e^{i\omega t}} + \frac{2\zeta\omega}{\omega_n}iY\cancel{e^{i\omega t}} + Y\cancel{e^{i\omega t}} = \frac{2\zeta\omega}{\omega_n}iZ\cancel{e^{i\omega t}} + Z\cancel{e^{i\omega t}}$$

$$Y\left(1 - \frac{\omega^2}{\omega_n^2} + \frac{2\zeta\omega}{\omega_n}i\right) = Z\left(\frac{2\zeta\omega}{\omega_n}i + 1\right)$$

$$Y = \frac{Z\left(\frac{2\zeta\omega}{\omega_n}i + 1\right)}{1 - \frac{\omega^2}{\omega_n^2} + \frac{2\zeta\omega}{\omega_n}i}$$

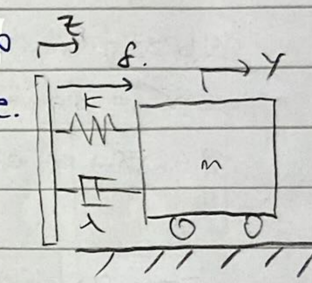
→ Response ratio : $\boxed{\frac{Y}{Z} = \frac{1 + i2\zeta\omega/\omega_n}{1 - \omega^2/\omega_n^2 + i2\zeta\omega/\omega_n}}$

$\arctan(u) + \arctan(v) = \arctan\left(\frac{u+v}{1-uv}\right)$

Amplitude : $\boxed{\left|\frac{Y}{Z}\right| = \sqrt{\frac{1 + (2\zeta\omega/\omega_n)^2}{(1 - \omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2}}}$; phase : $\boxed{\phi = \arctan(2\zeta\omega/\omega_n) - \arctan\left(\frac{2\zeta\omega/\omega_n}{1 - \omega^2/\omega_n^2}\right)}$

Case (b) — Relative response to base displacement

- Here, we put the applied force to zero, $F(t) = x(t) = 0$, also we are interested in the relative motion btwn the mass and the base.



Set $\delta = y - z$, → $\dot{\delta} = \dot{y} - \dot{z}$; $\ddot{\delta} = \ddot{y} - \ddot{z}$

From case (c), $\frac{1}{\omega_n^2}\ddot{y} + \frac{2\zeta}{\omega_n}\dot{y} + y = \frac{2\zeta}{\omega_n}\dot{z} + z$

$$\frac{1}{\omega_n^2}(\ddot{\delta} + \ddot{z}) + \frac{2\zeta}{\omega_n}(\dot{\delta} + \cancel{\dot{z}}) + \delta + \cancel{z} = \cancel{\frac{2\zeta}{\omega_n}\dot{z}} + \cancel{z}$$

$$\boxed{\frac{1}{\omega_n^2}\ddot{\delta} + \frac{2\zeta}{\omega_n}\dot{\delta} + \delta = -\frac{1}{\omega_n^2}\ddot{z}}$$

- Using the $e^{i\omega t}$ method,

$$z = Ze^{i\omega t} \qquad \delta = \Delta e^{i\omega t}$$

$$-\frac{\omega^2}{\omega_n^2}\Delta\cancel{e^{i\omega t}} + \frac{2\zeta\omega}{\omega_n}i\Delta\cancel{e^{i\omega t}} + \Delta\cancel{e^{i\omega t}} = \frac{\omega^2}{\omega_n^2}Z\cancel{e^{i\omega t}}$$

$$\Delta\left(1 - \frac{\omega^2}{\omega_n^2} + \frac{2\zeta\omega}{\omega_n}i\right) = Z\cdot\left(\frac{\omega^2}{\omega_n^2}\right)$$

$$\Delta = \frac{Z\left(\frac{\omega^2}{\omega_n^2}\right)}{1 - \frac{\omega^2}{\omega_n^2} + \frac{2\zeta\omega}{\omega_n}i}$$

→ Response ratio : $\boxed{\frac{\Delta}{Z} = \frac{\omega^2/\omega_n^2}{1 - \omega^2/\omega_n^2 - i2\zeta\omega/\omega_n}}$

Amplitude : $\boxed{\left|\frac{\Delta}{Z}\right| = \frac{\omega^2/\omega_n^2}{\sqrt{(1 - \omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2}}}$; phase : $\boxed{\phi = \arctan\left(\frac{2\zeta\omega/\omega_n}{1 - \omega^2/\omega_n^2}\right)}$

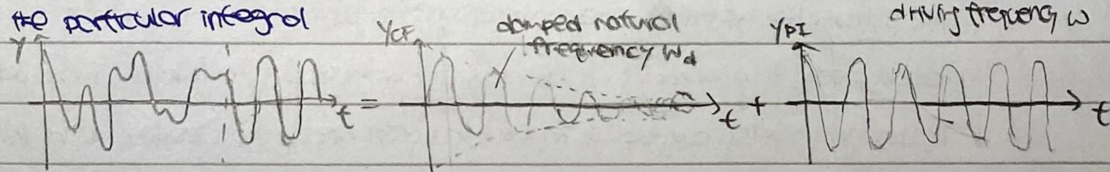
same expression as case (a)

The effect of the complementary function

- The total response is the sum of the complementary function and the particular integral

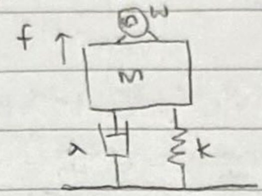
$$y = De^{-\zeta\omega_n t}\cos(\omega_d t - \psi) + y_{PI}$$

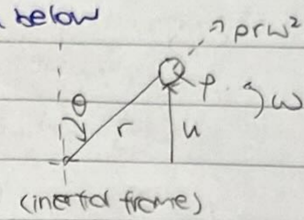
- The CF represents an initial transient motion at the damped natural frequency, which adds to the particular integral



Excitation by an off-balance rotor.

- Acceleration excitation (case (b)) can also arise from an off-balance rotor (turbine)
- there is an outward-acting inertia force associated w/ the centripetal acceleration.
- consider the off-balance rotor as shown below





As $u = r\cos\omega t$, $\dot{u} = -r\omega\sin\omega t$ and $\ddot{u} = -r\omega^2\cos\omega t$ $\Big]\ f = -p\ddot{u}$

N2L ↑ :

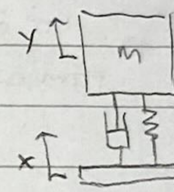
$$f = pr\omega^2\cos\theta = pr\omega^2\cos\omega t.$$

$$\therefore \frac{1}{\omega_n^2}\ddot{y} + \frac{2\zeta}{\omega_n}\dot{y} + y = \frac{f}{k}$$

$$\frac{1}{\omega_n^2}\ddot{y} + \frac{2\zeta}{\omega_n}\dot{y} + y = -p\frac{\ddot{u}}{k} \qquad \text{i.e. case (b)}$$

Vibration isolation and transmission

- If we want to isolate a mass from ground vibrations (decoupling), we use case (c) to find an expression for transmissibility $T(\omega)$, defined as



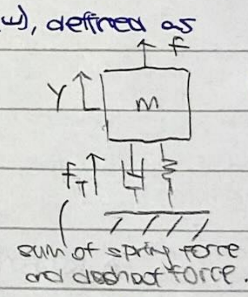
$$T(\omega) = \frac{Y}{X}.$$

- using the expression for $\frac{Y}{X}$ from case (c),

$$T(\omega) = \frac{1 + i2\zeta\omega/\omega_n}{1 - (\omega^2/\omega_n^2) + i2\zeta\omega/\omega_n}$$

- Given an applied force on a mass and we wish to minimise the force transmitted to the ground, we can find an expression for transmissibility $T(\omega)$, defined as

$$T(\omega) = \frac{F_T}{F}$$



- considering the forces,

$$\frac{1}{\omega_n^2}\ddot{y} + \frac{2\zeta}{\omega_n}\dot{y} + y = \frac{f}{k}$$

$$f_T = ky + \lambda\dot{y} \quad \rightarrow \quad \frac{f_T}{k} = y + \frac{2\zeta}{\omega_n}\dot{y}$$

- using the $e^{i\omega t}$ method, we can show that [$f = Re\{Fe^{i\omega t}\}$, $f_T = Re\{F_Te^{i\omega t}\}$, $y = Re\{Ye^{i\omega t}\}$]

$$T(\omega) = \frac{F_T}{F} = \frac{1 + i2\zeta(\omega/\omega_n)}{1 - (\omega^2/\omega_n^2) + i2\zeta\omega/\omega_n}$$

- Looking at the graph for curve (c), it is clear that

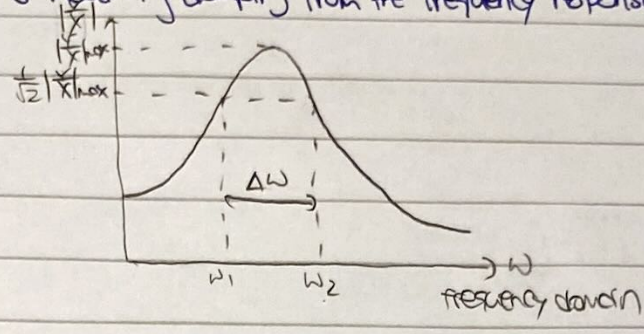
$$|T(\omega)| < 1 \text{ for } \omega/\omega_n > \sqrt{2}.$$

thus we need to design the support so the excitation frequency ω is greater than $\sqrt{2}\times$ the natural frequency ω_n (i.e. we want a low ω_n → soft support).

- Also from the graph, the response decreases w/ decreasing damping, so we would ideally req. low damping. (Not too low damping as the system may pass through resonance during run-up → make sure response isn't excessive during this stage).

Damping measurement.

- There are 2 ways of measuring damping from the frequency response function:



① Q-factor measurement

$$\boxed{Q = \left|\frac{Y}{X}\right|_{max} = \frac{1}{2\zeta}}.$$ — valid for all 3 cases.

given that ζ is small (light damping).

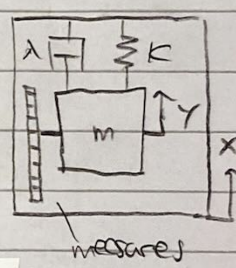
② Half-power bandwidth method.

At ω_1 and ω_2, we have $\left|\frac{Y}{X}\right| = \frac{1}{\sqrt{2}}\left|\frac{Y}{X}\right|_{max}$. (3dB pt).

$$\Delta\omega = \omega_2 - \omega_1 = 2\zeta\omega_n \rightarrow \boxed{\Delta\omega/\omega_n = 2\zeta}$$

Vibration measurement

① Seismic transducers — displacement measurement — we want $\left|\frac{Z}{X}\right| \approx 1$



measures $z = y - x$.

- The device measures the relative motion between the mass and the outer casing (attached to the object to be measured). [case (b)]
- The measured output is equal to the case motion (what we want to measure) for $\omega \gg \omega_n$
- In practice, reasonable accuracy is obtained for $(\omega/\omega_n) > 0.85$ if $\zeta \approx 0.5$

② The geophone — velocity measurement — we want $\left|\frac{Z}{X}\right| \approx 1$

- The geophone is very similar to the seismic transducer, except the relative motion between the mass and the outer casing is used to move a magnet inside a coil.
- The resultant voltage depends on the relative velocity between the mass and the outer casing.
- The measured output is equal to the case motion (what we want to measure) for $\omega \gg \omega_n$
- In practice, reasonable accuracy is obtained for $(\omega/\omega_n) > 0.85$ if $\zeta \approx 0.5$.

② The accelerometer — acceleration measurement — we want $\left|\frac{Z}{A/\omega_n^2}\right| \approx 1$.

- The piezoelectric crystal produces a charge prop. to the relative displacement z.

$$\left|\frac{Z}{X}\right| = \frac{\omega^2/\omega_n^2}{\sqrt{(1-\omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2}} \qquad \text{[case (b)]}$$

- Given that ground acceleration $A = -\omega^2 X$, so

$$\left|\frac{Z}{A/\omega_n^2}\right| = \frac{1}{\sqrt{(1-\omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2}} \qquad \text{[case (a)]}$$

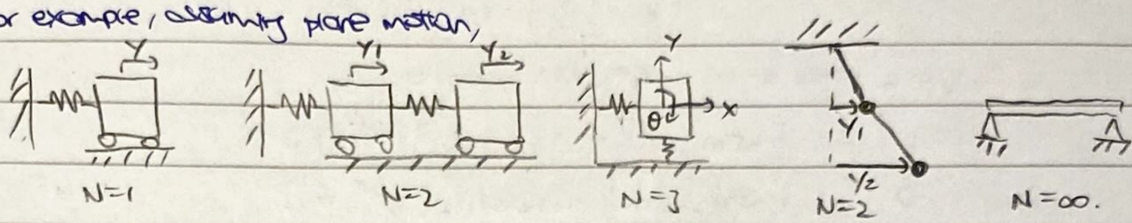
- The measured output is equal to the base motion (what we want to measure) for $\omega \ll \omega_n$.

Multi-degree of freedom systems

Multi-degree of freedom systems

Degrees of freedom.

- The no. of degrees of freedom N is the no. of coordinates needed to describe the configuration of a system.
- For example, assuming plane motion,



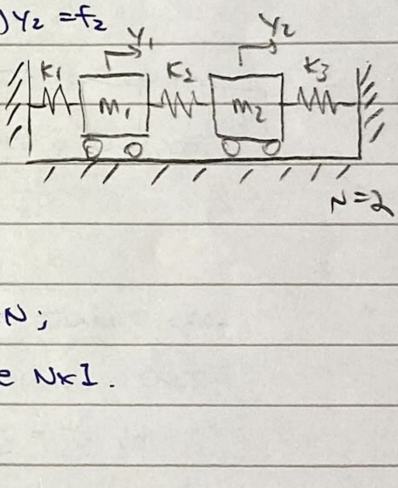
* The continuous beam has an infinite no. of d.o.f (i.e. to fully describe the position of the beam, we would need to describe every one of the infinite no. of pts. along the beam).
In practice, we approximate the deformation by using the displacement at a finite no. of pts.

Equations of motion.

- Consider a 2 degree of freedom system:

① $m_1\ddot{y}_1 = f_1 + k_2(y_2-y_1) - k_1y_1 \rightarrow m_1\ddot{y}_1 + (k_1+k_2)y_1 - k_2y_2 = f_1$

② $m_2\ddot{y}_2 = f_2 - k_2(y_2-y_1) - k_3y_1 \rightarrow m_2\ddot{y}_2 - k_2y_1 + (k_2+k_3)y_2 = f_2$



Using matrix notation,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

- For a general N degree of freedom system,

$$\boxed{\underline{\underline{M}}\,\ddot{\underline{y}} + \underline{\underline{K}}\,\underline{y} = \underline{f}}$$

where the mass matrix $\underline{\underline{M}}$ and stiffness matrix $\underline{\underline{K}}$ are $N \times N$;
the displacement vector $\underline{y}$ and force vector $\underline{f}$ are $N \times 1$.

Free response

Natural frequencies and mode shapes

- We can expect a system w/ N degrees of freedom to have N natural frequencies.
- At each natural frequency, the system will vibrate in a particular shape, known as the mode shape.
- The natural frequencies and mode shapes can be found using
  - ↳ First principles (Doable for 2 dof systems).
  - ↳ Inspection (Doable for 2dof systems w/ symmetry)
  - ↳ Eigen approach (General)

Using first principles. (2 dof system)

- Assume $\underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is of the form

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathrm{Re}\left\{ \begin{bmatrix} Y_1 e^{i\omega_1 t} \\ Y_2 e^{i\omega_2 t} \end{bmatrix} \right\}$$

also assume that $\omega_1 = \omega_2 = \omega$. so

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathrm{Re}\left\{ \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} e^{i\omega t} \right\}$$

- The 2 rows of the matrix eqn. becomes

① $$[-\omega^2 m_1 Y_1 + (k_1+k_2)Y_1 - k_2 Y_2]\cancel{e^{i\omega t}} = 0$$

② $$[-\omega^2 m_2 Y_2 - k_2 Y_1 + (k_2+k_3)Y_2]\cancel{e^{i\omega t}} = 0$$

The trivial sol'n is $Y_1 = Y_2 = 0$, o/w,

$$\frac{Y_1}{Y_2} = \frac{k_2}{k_1+k_2-\omega^2 m_1} = \frac{k_2+k_3-\omega^2 m_2}{k_3}.$$

- Multiply out to get an eqn. for ω,

$$a\omega^4 + b\omega^2 + c = 0$$

where $a = m_1 m_2$

$b = -[m_1(k_2+k_3) + m_2(k_1+k_2)]$

$c = k_1k_2 + k_1k_3 + k_2k_3$.

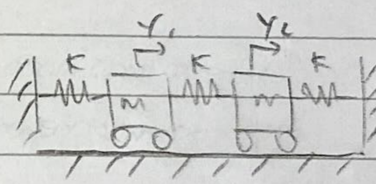
$\rightarrow \omega_1^2, \omega_2^2 = \dfrac{-b \mp \sqrt{b^2-4ac}}{2a}$, both real and positive.

ω_1 and ω_2 are the natural frequencies of the system.

- For each natural frequency, there is an associated mode shape.

$$\frac{Y_1^{(1)}}{Y_2^{(1)}} = \frac{k_2}{k_1+k_2-\omega_1^2 m_1} \quad ; \quad \frac{Y_1^{(2)}}{Y_2^{(2)}} = \frac{k_2}{k_1+k_2-\omega_2^2 m_1}$$

- e.g.: symmetric 2 mass problem.



Same as above, but $m_1 = m_2 = m$; $k_1 = k_2 = k_3 = k$.

so $\dfrac{Y_1}{Y_2} = \dfrac{k}{2k-\omega^2 m} = \dfrac{2k-\omega^2 m}{k}$.

$a = m^2$; $b = -4mk$; $c = 3k^2$ $\rightarrow$ $\omega_1 = \sqrt{\frac{k}{m}}$; $\omega_2 = \sqrt{\frac{3k}{m}}$

At $\omega = \omega_1$, $\dfrac{Y_1^{(1)}}{Y_2^{(1)}} = \dfrac{k}{2k-\omega_1^2 m} = \dfrac{1}{2-1} = 1$ $\rightarrow$ $\underline{Y}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

At $\omega = \omega_2$, $\dfrac{Y_1^{(2)}}{Y_2^{(2)}} = \dfrac{k}{2k-\omega_2^2 m} = \dfrac{1}{2-3} = -1$ $\rightarrow$ $\underline{Y}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

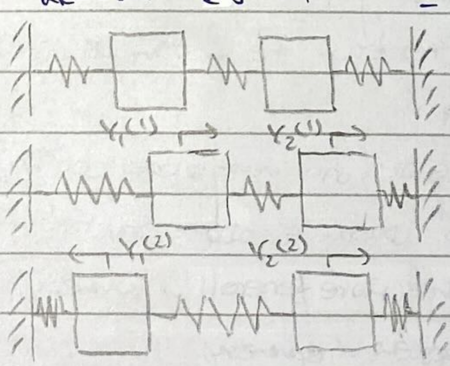
At rest

Mode ①

$\omega = \omega_1 = \sqrt{\frac{k}{m}}$

Mode ②

$\omega = \omega_2 = \sqrt{\frac{3k}{m}}$



$\underline{y} = \mathrm{Re}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i\omega t} \right\}$

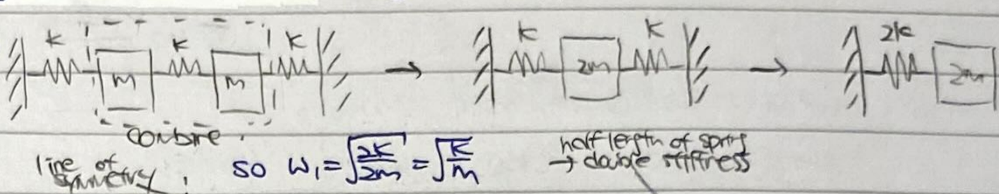
$\underline{y} = \mathrm{Re}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i\omega t} \right\}$.

By inspection (2 dof system w/ symmetry)

- We convert the system to a physically equivalent system w/ 1 degree of freedom.
- Consider the symmetric 2 mass problem:

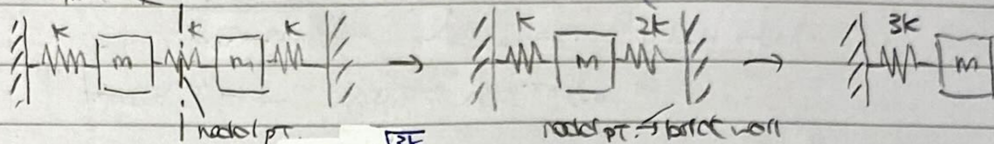
↳ We can guess the 2 mode shapes using symmetry.

Mode ①



so $\omega_1 = \sqrt{\frac{2k}{2m}} = \sqrt{\frac{k}{m}}$

Mode ②



so $\omega_2 = \sqrt{\frac{3k}{m}}$

* This method could only be used rarely.

Eigen approach (General)

- Assume $\underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is of the form

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathrm{Re}\left\{ \begin{bmatrix} Y_1 e^{i\omega_1 t} \\ Y_2 e^{i\omega_2 t} \end{bmatrix} \right\}$$

also assume that $\omega_1 = \omega_2 = \omega$, so

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathrm{Re}\left\{ \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} e^{i\omega t} \right\}.$$

- The matrix eqn. becomes.

$$[-\omega^2 \underline{\underline{M}} + \underline{\underline{K}}]\underline{Y}\cancel{e^{i\omega t}} = \underline{0}$$

$$\boxed{[-\omega^2 \underline{\underline{M}} + \underline{\underline{K}}]\underline{Y} = \underline{0}}$$

$\underline{\underline{K}}\underline{Y} = \omega^2 \underline{\underline{M}}\underline{Y}$ generalised eigenvalue problem

or

$\underline{\underline{M}}^{-1}\underline{\underline{K}}\underline{Y} = \omega^2 \underline{Y}$ standard eigenvalue problem.

- The natural frequencies are the eigenvalues ω_j^2 and the mode shapes are the eigenvectors $\underline{Y}_j$
- For a non-trivial sol'n to the matrix eqn. above,

$$\boxed{|-\omega^2 \underline{\underline{M}} + \underline{\underline{K}}| = 0}$$

This yields an algebraic eqn. in ω^2 which yields N sol'ns corresponding to the natural frequencies for an N degree of freedom system.

$$\omega_1^2, \omega_2^2, \ldots, \omega_N^2$$

- Having found ω_j^2, the jth eigenvector $\underline{Y}^{(j)}$ must satisfy the eqn.

$$\boxed{[-\omega_j^2 \underline{\underline{M}} + \underline{\underline{K}}]\underline{Y}^{(j)} = \underline{0}}$$

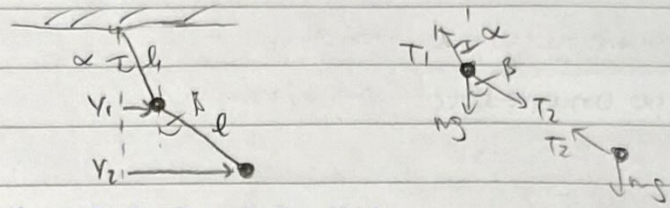
(use N-1 rows to find $\frac{Y_2^{(j)}}{Y_1^{(j)}}, \frac{Y_3^{(j)}}{Y_1^{(j)}}, \ldots \frac{Y_N^{(j)}}{Y_1^{(j)}}$)

Eigenvector $\underline{Y}_j$ has an arbitrary amplitude → one of the entries of the eigenvector can be placed to unity, say $Y_1^{(j)}$

- The eigenvectors $\underline{Y}^{(j)}$ are independent → any motion of the system can be expressed as a linear combination of the eigenvectors
- If $\omega_j = 0$ (i.e. $T = \infty$), then the eigenvector corresponds to a rigid body mode.

-eg: A double pendulum.

Consider the double pendulum below.



* small oscillations so $\alpha \approx 0$, $\beta \approx 0$.

For small oscillations, $\alpha \approx 0$, $\beta \approx 0$, so

vertical accelerations ≈ 0 ; $\cos\alpha \approx 1$, $\cos\beta \approx 1$.

→ N2L ↕: $mg \approx T_2 \cos\beta \quad \rightarrow \quad T_2 \approx mg$

$mg + T_2\cos\beta \approx T_1 \cos\alpha \quad \rightarrow \quad T_1 \approx 2mg$

→ N2L ↔: $m\ddot{y}_1 = -T_1 \sin\alpha + T_2 \sin\beta$

$$= -2mg\cdot\frac{y_1}{l} + mg\cdot\frac{y_2 - y_1}{l} \quad \rightarrow \quad m\ddot{y}_1 = -\frac{3mg}{l}y_1 + \frac{mg}{l}y_2$$

$m\ddot{y}_2 = -T_2 \sin\beta$

$$= -mg\cdot\frac{y_2 - y_1}{l} \quad \rightarrow \quad m\ddot{y}_2 = \frac{mg}{l}y_1 - \frac{mg}{l}y_2$$

$$\therefore \; m\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}\ddot{y}_1\\ \ddot{y}_2\end{bmatrix} + \frac{mg}{l}\begin{bmatrix}3&-1\\-1&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Assume $y = \mathrm{Re}\{Y e^{i\omega t}\}$

$$-\omega^2\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}Y_1\\Y_2\end{bmatrix} + \frac{g}{l}\begin{bmatrix}3&-1\\-1&1\end{bmatrix}\begin{bmatrix}Y_1\\Y_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Eigenvalues: $$\left| -\omega^2\begin{bmatrix}1&0\\0&1\end{bmatrix} + \frac{g}{l}\begin{bmatrix}3&-1\\-1&1\end{bmatrix}\right| = 0$$

$$\begin{vmatrix}-\omega^2 + \frac{3g}{l} & -\frac{g}{l}\\ -\frac{g}{l} & -\omega^2 + \frac{g}{l}\end{vmatrix} = 0$$

$$\omega^4 - 4\omega^2(g/L) + 2(g/L)^2 = 0$$

$$\omega_1^2, \omega_2^2 = (2g/L) \mp (\sqrt{2}g/L)$$

i.e. $\omega_1 = \sqrt{\frac{(2-\sqrt{2})g}{L}}$; $\omega_2 = \sqrt{\frac{(2+\sqrt{2})g}{L}}$

NB $\omega = \sqrt{\frac{g}{L}}$ for a single pendulum

Eigenvectors: $$\begin{bmatrix}-\omega^2 + \frac{3g}{l} & -\frac{g}{l}\\ -\frac{g}{l} & -\omega^2 + \frac{g}{l}\end{bmatrix}\begin{bmatrix}Y_1\\Y_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

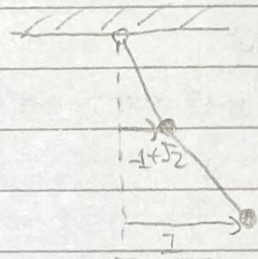
using the 1st row, $\left(-\omega^2 + \frac{3g}{l}\right)Y_1 - \frac{g}{l}\cdot Y_2 = 0$

$$\frac{Y_1}{Y_2} = \frac{g/L}{3g/L - \omega^2}$$

For $\omega = \omega_1$, $\frac{Y_1^{(1)}}{Y_2^{(1)}} = \frac{g/L}{3g/L - \omega_1^2} = \frac{1}{3-(2-\sqrt{2})} = -1+\sqrt{2} \quad \rightarrow \quad \underline{Y}^{(1)} = \begin{bmatrix}-1+\sqrt{2}\\1\end{bmatrix}$

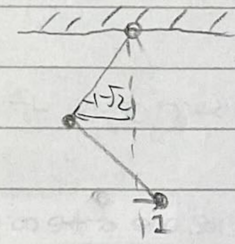
For $\omega = \omega_2$, $\frac{Y_1^{(2)}}{Y_2^{(2)}} = \frac{g/L}{3g/L - \omega_2^2} = \frac{1}{3-(2+\sqrt{2})} = -1-\sqrt{2} \quad \rightarrow \quad \underline{Y}^{(2)} = \begin{bmatrix}-1-\sqrt{2}\\1\end{bmatrix}$

Mode ①



$\omega = \omega_1 = \sqrt{\frac{(2-\sqrt{2})g}{L}}$

Mode ②



$\omega = \omega_2 = \sqrt{\frac{(2+\sqrt{2})g}{L}}$

Undamped free vibration.

- The total complementary function can be written as a linear combination of vibration in the normal modes.

$$\boxed{\underline{y} = \sum_{j=1}^{N} \underbrace{\underline{Y}^{(j)}}_{\text{mode shape } \underline{Y}^{(j)}} \overbrace{A_j' \cos(\omega_j t + \phi_j)}^{\text{generic sinusoidal vibration at freq. } \omega_j}}$$

or equivalently,

$$\boxed{\underline{y} = \sum_{j=1}^{N} \underline{Y}^{(j)} (A_j \cos(\omega_j t) + B_j \sin(\omega_j t))}$$

- We have $2N$ constants $[(A_j', \phi_j)$ or $(A_j, B_j)]$ can be found using the initial conditions.

- e.g. consider the double pendulum example.

Find the CF given the initial conditions $\underline{y}(0) = \begin{bmatrix}1\\1\end{bmatrix}$ and $\dot{\underline{y}}(0) = \begin{bmatrix}0\\0\end{bmatrix}$

Given $\omega_1 = \sqrt{\frac{(2-\sqrt{2})g}{L}}$, $\omega_2 = \sqrt{\frac{(2+\sqrt{2})g}{L}}$; $\underline{Y}^{(1)} = \begin{bmatrix}-1+\sqrt{2}\\1\end{bmatrix}$, $\underline{Y}^{(2)} = \begin{bmatrix}-1-\sqrt{2}\\1\end{bmatrix}$.

$$\underline{y} = \underline{Y}^{(1)}(A_1\cos(\omega_1 t) + B_1 \cos(\omega_1 t)) + \underline{Y}^{(2)}(A_2\cos(\omega_2 t) + B_2 \sin(\omega_2 t))$$

Using $\dot{\underline{y}}(0) = \begin{bmatrix}0\\0\end{bmatrix}$, $B_1 = B_2 = 0$

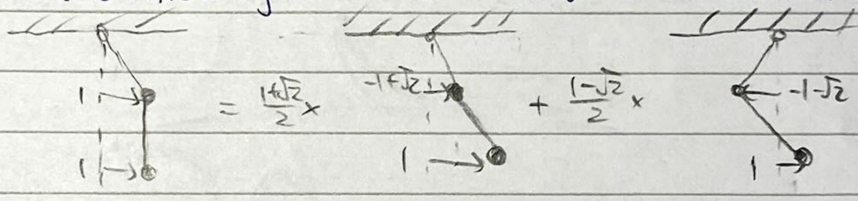
Using $y_1(0) = 1$, $(-1+\sqrt{2})A_1 + (-1-\sqrt{2})A_2 = 1$

Using $y_2(0) = 1$, $1 \cdot A_1 + 1 \cdot A_2 = 1$

$\Big]$ $A_1 = \frac{1+\sqrt{2}}{2}$, $A_2 = \frac{1-\sqrt{2}}{2}$

$$\therefore \underline{y} = \begin{bmatrix}-1+\sqrt{2}\\1\end{bmatrix}\left(\frac{1+\sqrt{2}}{2}\right)\cos\left(\sqrt{\frac{(2-\sqrt{2})g}{L}}\,t\right) + \begin{bmatrix}-1-\sqrt{2}\\1\end{bmatrix}\left(\frac{1-\sqrt{2}}{2}\right)\cos\left(\sqrt{\frac{(2+\sqrt{2})g}{L}}\,t\right)$$

* This is equivalent to representing the initial displacement as a sum of the normal modes.



Harmonic response

Harmonic response.

- If we have forces acting on the system, then the eqn. of motion becomes

$$\underline{\underline{M}}\ddot{\underline{y}} + \underline{\underline{K}}\underline{y} = \underline{f}$$

- Using the $e^{i\omega t}$ method,

$$\underline{y} = \mathrm{Re}\{\underline{Y}e^{i\omega t}\} \qquad \underline{f} = \mathrm{Re}\{\underline{F}e^{i\omega t}\}.$$

$$-\omega^2\underline{\underline{M}}\,\underline{Y}e^{i\omega t} + \underline{\underline{K}}\,\underline{Y}e^{i\omega t} = \underline{F}e^{i\omega t}$$

$$[-\omega^2\underline{\underline{M}} + \underline{\underline{K}}]\underline{Y} = \underline{F}$$

We define the dynamic stiffness matrix $\underline{\underline{D}}$. $\boxed{\underline{\underline{D}} = [-\omega^2\underline{\underline{M}} + \underline{\underline{K}}]}$, so

$$\underline{\underline{D}}\,\underline{Y} = \underline{F}$$

$$\boxed{\underline{Y} = \underline{\underline{D}}^{-1}\underline{F}}$$

* Note the response must go to infinity when $\omega = \omega_j$ since $|\underline{\underline{D}}|$ would be zero and.

$\underline{\underline{D}}^{-1} = \frac{1}{|\underline{\underline{D}}|}\mathrm{adj}(\underline{\underline{D}})$ $\quad [\text{As } |\underline{\underline{D}}| \to 0, \underline{\underline{D}}^{-1} \to \infty]$

- eg: Consider the symmetric 2 mass problem

Now a sinusoidally varying force f_2 is applied to the RH mass.

The eqn. of motion is

$$\begin{bmatrix} M & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ f_2 \end{bmatrix}$$

using the $e^{i\omega t}$ method,

$$\underline{y} = \mathrm{Re}\{\underline{Y} e^{i\omega t}\} \qquad f_2 = \mathrm{Re}\{F_2 e^{i\omega t}\}.$$

$$-\omega^2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\underline{Y}\cancel{e^{i\omega t}} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\underline{Y}\cancel{e^{i\omega t}} = \begin{bmatrix} 0 \\ F_2 \end{bmatrix}\cancel{e^{i\omega t}}$$

$$\begin{bmatrix} -\omega^2 m+2k & -k \\ -k & -\omega^2 m+2k \end{bmatrix}\underline{Y} = \begin{bmatrix} 0 \\ F_2 \end{bmatrix}$$

$$\underline{Y} = \begin{bmatrix} -\omega^2 m+2k & -k \\ -k & -\omega^2 m+2k \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ F_2 \end{bmatrix}$$

$$= \frac{1}{(2k-\omega^2 m)^2 - k^2}\begin{bmatrix} -\omega^2 m+2k & k \\ k & -\omega^2 m+2k \end{bmatrix}\begin{bmatrix} 0 \\ F_2 \end{bmatrix}$$

The determinant $|\underline{D}|$ can be written as

$$(2k-\omega^2 m)^2 - k^2 = (k-\omega^2 m)(3k-\omega^2 m)$$
$$= 3k^2[1-\omega^2(m/k)][1-\omega^2(m/3k)]$$
$$= 3k^2(1-\omega^2/\omega_1^2)(1-\omega^2/\omega_2^2)$$

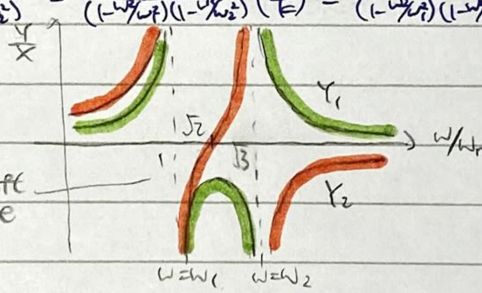
For ANY problem, the determinant can always be written in the form "const $\times (1-\omega^2/\omega_1^2)(1-\omega^2/\omega_2^2)$" because it must be zero at each natural freq. (except $\omega_j = 0$).

Expanding the matrix eqn.

$$Y_1 = \frac{kF_2}{3k^2(1-\omega^2/\omega_1^2)(1-\omega^2/\omega_2^2)} = \frac{1/3}{(1-\omega^2/\omega_1^2)(1-\omega^2/\omega_2^2)}\left(\frac{F_2}{k}\right) = \frac{1/3}{(1-\omega^2/\omega_1^2)(1-\omega^2/\omega_2^2)}X$$

$$Y_2 = \frac{(-\omega^2 m+2k)F_2}{3k^2(1-\omega^2/\omega_1^2)(1-\omega^2/\omega_2^2)} = \frac{1/3(2-\omega^2/\omega_1^2)}{(1-\omega^2/\omega_1^2)(1-\omega^2/\omega_2^2)}\left(\frac{F_2}{k}\right) = \frac{1/3(2-\omega^2/\omega_1^2)}{(1-\omega^2/\omega_1^2)(1-\omega^2/\omega_2^2)}X$$

passes through 0 at $\omega/\omega_n = \sqrt{2}$



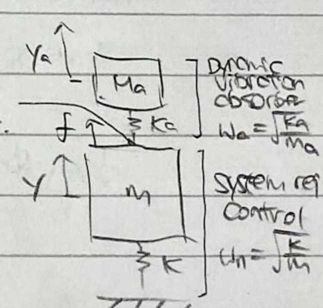
The dynamic vibration absorber (Tuned mass damper)

- consider the dynamic vibration absorber as shown below:

- The eqns. of motion:

$$\begin{bmatrix} M_a & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{y}_a \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} k_a & -k_a \\ -k_a & k+k_a \end{bmatrix}\begin{bmatrix} y_a \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

it constrains the coupling pt. $\omega = \omega_n \rightarrow k_d = \infty$



- The determinant of the dynamic stiffness matrix $|\underline{D}|$

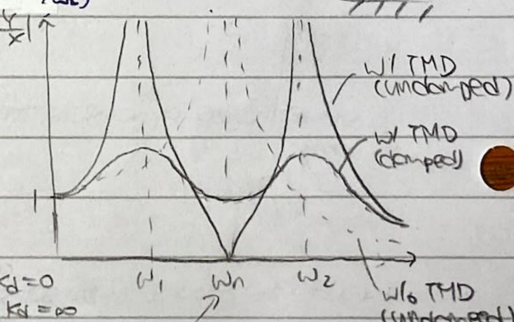
$$|\underline{D}| = (-M_a\omega^2+k_a)(-m\omega^2+k+k_a) - k_a^2 = k_a k(1-\omega^2/\omega_1^2)(1-\omega^2/\omega_2^2)$$

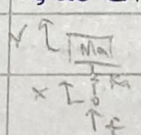
- The final result for the response:

$$\frac{Y}{F/k} = \frac{1-\omega^2/\omega_a^2}{(1-\omega^2/\omega_1^2)(1-\omega^2/\omega_2^2)}$$ At $\omega = \omega_a$, $Y = 0$

$$\frac{Y_a}{F/k} = \frac{1}{(1-\omega^2/\omega_1^2)(1-\omega^2/\omega_2^2)}$$

→ we choose M_a, k_a s.t. $\omega_a = \omega_n$





$f = m\ddot{y} \rightarrow F = -m\omega^2 Y$

$f = k_a(X-Y) \rightarrow F = k_a(X-Y)$

Eliminate Y. $k_d = F/X = \frac{k_a}{1-k_a/m\omega^2}$

$\omega = 0 \rightarrow k_d = 0$

$\omega = \omega_n \rightarrow k_d = \infty$

$\omega = \infty \rightarrow k_d = k_a$