

External forces

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Equilibrium of point forces, moments and couples

Forces as vectors

- Force is a vector \rightarrow has magnitude and direction.

$\hookrightarrow \underline{F} = F \underline{e}_F$, where the force \underline{F} of magnitude F is acting in dir. \underline{e}_F .

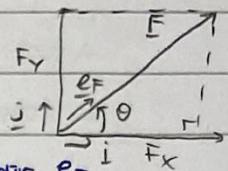
- In some problems, it is convenient to express a force \underline{F} in terms of its cartesian components.

$\hookrightarrow \underline{F} = F_x \underline{i} + F_y \underline{j} = \begin{pmatrix} F_x \\ F_y \end{pmatrix}$, $F_x = F \cos \theta$, $F_y = F \sin \theta$. // $F = \sqrt{F_x^2 + F_y^2}$, $\underline{e}_F = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$ [$\theta = \tan^{-1}\left(\frac{F_y}{F_x}\right)$]

* It is often helpful to think the forces as vectors instead of components, then solve using graphical methods.

\hookrightarrow we often know the dir. of a force, but not its magnitude \rightarrow 1 unknown.

\hookrightarrow using components, we lose the dir. information \rightarrow 2 unknowns.



Moments as vectors

- The turning effect of \underline{F} about a pt. O is measured by the moment \underline{M}_O , given by.

$$\underline{M}_O = \sum \underline{F}$$

moment arm d.

- The magnitude of \underline{M}_O is given by the magnitude of \underline{F} multiplied by the \perp distance from the line of action of \underline{F} to O ; the direction of \underline{M}_O is \perp to the plane containing \underline{F} and O .

$$|\underline{M}_O| = |\underline{F}| d$$

* The turning effect of \underline{F} about diff. pts. may not necessarily be the same.

Couples

- The moment produced by 2 equal and opposite non-collinear forces is a couple.



2 equal and opposite
non-collinear forces

An equivalent couple.

- Their combined moment abt. a pt. O in the plane is the couple \underline{M} , which has magnitude

$$|\underline{M}| = F(a+d) - F(a) = Fd$$

\rightarrow As the magnitude of \underline{M} is independent of a , the moment of a couple is the same abt. any pt.

Moment vs couple.

- Moments are the turning effect created by 1 resultant force around some particular pt.

- Couples depend on a pair of separated but equal and opposite forces and are not associated w/ a specific pt. of application.

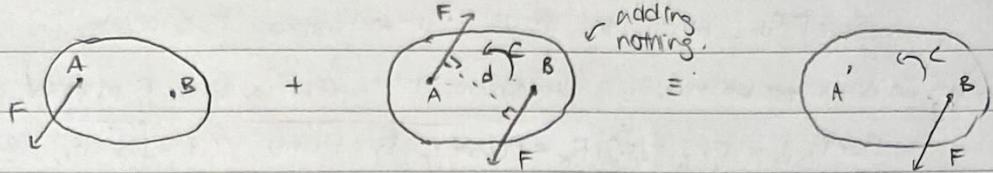
* Force (LoA through com) \rightarrow translation only Couple \rightarrow rotation only.

Force (LoA not through com) \rightarrow translation + rotation [moment]

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Replacement of a force by a force-and-couple.

- A force \underline{F} applied at pt. A can be replaced w/ a force \parallel to \underline{F} and of equal magnitude to \underline{F} by introducing a couple of suitable magnitude.



where $C = Fd$ (M is anticlockwise as the moment of original \underline{F} about B is anticlockwise).

- The 2 systems are statically equivalent but may lead to diff. int. forces.

Resultants

- The resultant of a system of forces and couples acting on a body is the simplest force combination that can replace the original system w/o altering the effect of the system on the body.

① Resultant of intersecting forces (concurrent forces)

- The resultant force \underline{F} of any no. of forces is the vector sum of all the individual forces.

Graphically \underline{F} can be found using a force polygon.

- Varigny's thm for concurrent forces states that if a body acted on by only no. of concurrent forces, the sum of the moments of all forces about any pt. O is equal to the moment of the resultant force \underline{F}_R about O. Proof is as follows:

↳ As all the forces intersect at P. The moment of the ith force about O is

$$M_i = \vec{OP} \times \underline{F}_i$$

Summing all these moments, the total moment about O is

$$M_R = \sum M_i = \sum \vec{OP} \times \underline{F}_i = \vec{OP} \times \sum \underline{F}_i = \vec{OP} \times \underline{F}_R$$

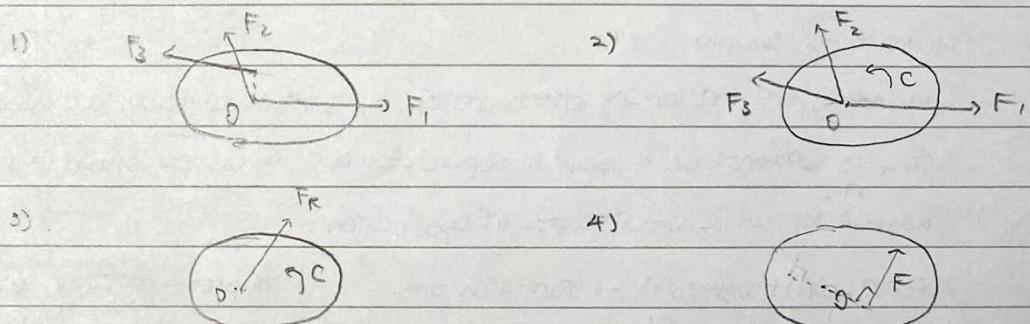
② Resultant of non-intersecting forces.

- A system of non-intersecting forces can be replaced by a single force \underline{F} , w/ a \perp distance d to O.

↳ Shift the position of forces so the forces become concurrent (by adding a couple).

↳ Combine the forces to get a resultant force.

↳ Shift the position of the resultant force to cancel out any existing couple.



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③ Resultant of a system of forces and couples (general case).

- Given a system of 2D forces F_i and couples C_i , we can replace the system by a single force E .

$$E = \sum F_i$$

and the \perp distance d of the line of action of E to an arbitrary pt. O is given by.

$$Fd = \sum F_i d_i + \sum C_i$$

where d_i are the \perp distances from the lines of action of F_i to O .

- * ① and ② are special cases of ③ where $\sum C_i = 0$.

Newton's laws

① When all ext. influences on a particle are removed, the particle moves w/ constant velocity.

This velocity may be 0 in which case the particle remains at rest.

② When a force E acts on a particle of mass m , the particle moves w/ instantaneous acceleration a given by the formula : $\sum E = ma$.

③ When 2 particles exert forces upon each other, these forces are equal + opposite and // to the straight line joining the 2 particles.

Equilibrium expressed in vector resultant

- For static equilibrium:

↳ The resultant force on the body must be 0 i.e. $\sum F_i = 0$

↳ The resultant moment about any pt. must be 0 i.e. $\sum M_i = 0$

- In 2D, the 2 eqn. can be stated as 3 scalar eqn. (Graphical method is more convenient as we know more about the dir. of applied forces rather than their magnitude)

- In 3D, the 2 eqn. can be stated as 6 scalar eqn. (Graphical method is not convenient since it is very difficult to draw in 3D).

Forces in eqm. form a closed polygon.

Accuracy in structural mechanics.

- In reality, we do not know accurate information on the loads \rightarrow there are few situations in which you need to be more accurate than 1 sig.fig.

- We need to know the order of magnitude of the deflections more than the detailed stiffness of the structure.

- The answer should never be more precise.

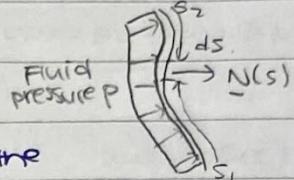
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Distributed loads

Constant fluid pressure

- According to Pascal's law, the pressure p at any given pt. in a stationary fluid is the same in all directions.
- where the fluid meets a constraining surface, over any infinitesimal area, it imparts a force $\delta F = p \delta A$ normally to the surface.
- The resultant force per unit width (PLW) of pressure acting on some surface that varies in just 1D is given by

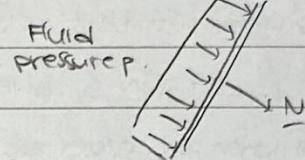
$$F = \int_{S_1}^{S_2} p \cdot \underline{N}(s) ds.$$



where F is the equivalent force per unit width of the pressure acting on the surface b/w s_1 and s_2 w/ surface normal $N(s)$ at s .

- For pressure acting on a flat surface, the dir. of \underline{N} is constant for the whole surface, so the integral is solved as

$$F = p L N$$



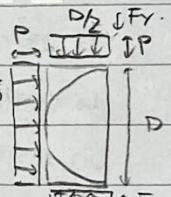
where L is the length of the surface and F is the force per unit width.

- The magnitude of the component of pressure force on an object in a given dir. is equal to the pressure multiplied by the projected area of the object in that dir.

e.g.: find the resultant force of a uniform pressure acting on the block.

$$\text{Force PLW } \times \text{dir: } F_x = p D$$

$$y\text{-dir: } F_y = p \frac{D}{2}$$



The distributed pressure can be replaced by a pt. force F per unit width acting through the COM of the area representation of the load. Refer to trapezoidal distributed load example.

The resultant force on any body subject to a uniform pressure over its entire surface is 0.

Hydrostatic pressure

f remember both vertical and horizontal hydrostatic forces

- The hydrostatic pressure $p(h)$ of a depth h below the surface of a liquid of uniform density ρ is given by

$$p(h) = \rho g h.$$

- Consider the column of liquid above a horizontal plane at depth h .

By NIL I:

$$W = p(h) A.$$

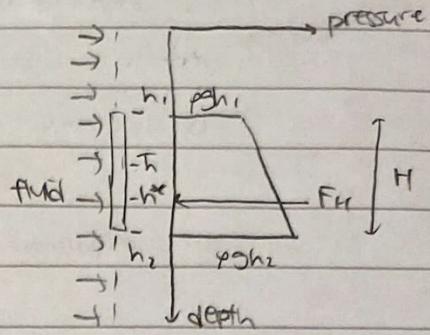
$$\rho g h = p(h) A$$

$$p(h) = \rho g h.$$

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- consider the horizontal loading on the plate shown below due to the hydrostatic pressure in the fluid which varies w/ depth h .

$$\begin{aligned}
 F_H &= \int_{h_1}^{h_2} p(h) dh \\
 &= \int_{h_1}^{h_2} \rho g h dh \\
 &= \frac{\rho g}{2} (h_2^2 - h_1^2) \\
 &= \rho g \frac{h_1 + h_2}{2} \cdot (h_2 - h_1) \\
 &= \rho g \bar{h} H \quad [\bar{h} = \frac{h_1 + h_2}{2}, H = h_2 - h_1].
 \end{aligned}$$



(M_{h⁺}): $F_H h^* = \int_{h_1}^{h_2} p(h) \cdot h dh \quad [F(h) = p(h) \cdot h, d = h]$,

$$\begin{aligned}
 &= \int_{h_1}^{h_2} \rho g h^2 dh \\
 &= \frac{\rho g}{3} (h_2^3 - h_1^3).
 \end{aligned}$$

$$\begin{aligned}
 \therefore h^* &= \frac{F_H h^*}{F_H} \\
 &= \frac{\frac{\rho g}{3} (h_2^3 - h_1^3)}{\frac{\rho g}{2} (h_1 + h_2) (h_2 - h_1)} \\
 &= \frac{2}{3} \frac{h_1^2 + h_1 h_2 + h_2^2}{h_1 + h_2}
 \end{aligned}$$

- In the special case that the top of the body is at the surface of the fluid, i.e. $h_1 = 0$.

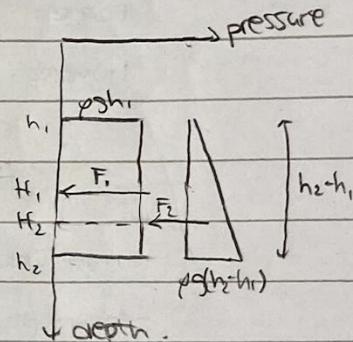
$$F_H = \frac{\rho g}{2} h_2^2 : F_H h^* = \frac{\rho g}{3} h_2^3 ; h^* = \frac{2}{3} h_2.$$

- * we can split the trapezoidal distributed load into a rectangular distributed load and a triangular distributed load:

$$\begin{aligned}
 F_1 &= \rho L \\
 &= \rho g h_1 (h_2 - h_1) \\
 H_1 &= h_1 + \frac{h_2 - h_1}{2} \\
 &= \frac{h_1 + h_2}{2}
 \end{aligned}$$

$$\begin{aligned}
 F_2 &= \frac{\rho g}{2} H^2 \\
 &= \frac{\rho g}{2} (h_2 - h_1)^2 \\
 H_2 &= h_1 + \frac{2}{3} (h_2 - h_1) \\
 &= \frac{h_1 + 2h_2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \sum F_i &= F_1 + F_2 = \rho g h_1 (h_2 - h_1) + \frac{1}{2} \rho g (h_2 - h_1)^2 \\
 &= \frac{1}{2} \rho g (h_2 - h_1) (2h_1 + h_2 - h_1) \\
 &= \frac{1}{2} \rho g (h_1 + h_2) (h_2 - h_1) \quad \text{as expected.}
 \end{aligned}$$



$$\begin{aligned}
 \sum F_i H_i &= F_1 H_1 + F_2 H_2 = \rho g h_1 (h_2 - h_1) \cdot \frac{h_1 + h_2}{2} + \frac{1}{2} \rho g (h_2 - h_1)^2 \cdot \frac{h_1 + 2h_2}{3} \\
 &= \frac{1}{6} \rho g (h_2 - h_1) [3h_1(h_1 + h_2) + (h_2 - h_1)(h_1 + 2h_2)] \\
 &= \frac{1}{6} \rho g (h_2 - h_1) (3h_1^2 + 3h_1 h_2 + h_1 h_2 + 2h_2^2 - h_1^2 - 2h_1 h_2) \\
 &= \frac{1}{6} \rho g (h_2 - h_1) (2h_1^2 + 2h_1 h_2 + h_2^2) \\
 &= \frac{1}{3} \rho g (h_2 - h_1) (h_1^2 + h_1 h_2 + h_2^2) \quad \text{as expected.}
 \end{aligned}$$

Note that h^* is equivalent to the CDM of the area representation of the load.

$$M\bar{Y} = M_1 Y_1 + M_2 Y_2$$

$$\begin{aligned}
 &\cancel{[\rho g h_1 (h_2 - h_1) + \frac{1}{2} \rho g (h_2 - h_1) (h_1 + h_2)]} \bar{Y} = \cancel{\rho g h_1 (h_2 - h_1) \cdot H_1} + \cancel{\rho g (h_2 - h_1) \cdot H_2} \\
 &\bar{Y} = \frac{2}{3} \frac{h_1^2 + h_1 h_2 + h_2^2}{h_1 + h_2} \quad \text{as expected.}
 \end{aligned}$$

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Gravity

- The weight loading / self-weight of a structure is always distributed, but we can treat it as a pt. force through the COM of the system. (Also the centroid for uniform laminae)

- For discretely distributed masses,

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i} = \frac{\sum A_i x_i}{\sum A_i}$$

$$\bar{y} = \frac{\sum m_i y_i}{\sum m_i} = \frac{\sum A_i y_i}{\sum A_i}$$

\bar{x} has the symbol  on diagrams.

- For continuously distributed masses,

$$\bar{x} = \frac{\int x dm}{\int dm} = \frac{\int x dA}{\int dA}$$

$$\bar{y} = \frac{\int y dm}{\int dm} = \frac{\int y dA}{\int dA}$$

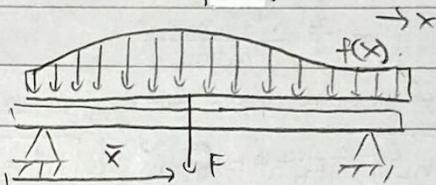
- For a continuously distributed lamina w/ height $h(x)$,

$$\bar{x} = \frac{\int x h(x) dx}{\int h(x) dx}$$

$$\bar{y} = \frac{\int y h(x) dx}{\int h(x) dx}$$

General distributed force.

- Consider a general distributed force per unit length $f(x)$.



We can replace the distributed force w/ a force F at position $x = \bar{x}$.

Force : $F = \int f(x) dx$

Moment : $F\bar{x} = \int f(x)x dx$

$$\bar{x} = \frac{F\bar{x}}{F} = \frac{\int f(x)x dx}{\int f(x) dx}$$

Contact forces ✓ contact forces are reaction forces \rightarrow we can only find the contact force as the reaction to other applied forces.

Kuhn-Tucker conditions.

- At any pt. on the interface of 2 contacting bodies, 1 of 2 Kuhn-Tucker conditions must apply:

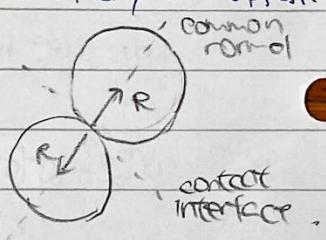
① The distance between the bodies is 0 \rightarrow force between them is true.

② The distance between the bodies is true \rightarrow force between them is 0.

\rightarrow either separation or contact force is always 0, and the other is always true.

Fictionless contact.

- In the frictionless contact of 2 bodies, a distributed force acts equally and oppositely on both bodies, normally to the contact interface
- It is often convenient to replace the distributed force w/ a pt. force anywhere in the region of contact. (provided that the force is along the common normal).



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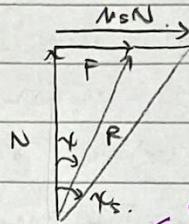
Contact forces with friction.

- Friction varies w/ the materials of the 2 contacting bodies, the detailed geometry of their surfaces (rough or smooth, textured or not) and the presence or not of another material (usually a lubricant) at the contact surface.
 - A good approx. of dry friction - where there is no lubricant between 2 relatively rough bodies, the limiting value of friction F is prop. to the normal force N :
 - For static contact (no sliding) : $|F| \leq M_s N$
 - For dynamic contact (sliding) : $|F| = M_d N$
- where M_s and M_d are the coefficient of static and dynamic friction respectively.
- Friction always opposes motion : if the dir. of red/incipient motion changes, so does the dir. of the frictional force.
 - * $M_s < M_d$. (which causes a ruler to "judder" when sliding 2 fingers underneath it).

Angle of friction

very useful for graphical methods

- It is often convenient to replace F and N by a single resultant force R .



It is good practice to always draw the common normal before friction.

The force R is inclined to the common normal at an angle γ , given by

$$\gamma = \tan^{-1}\left(\frac{F}{N}\right)$$

- For static contact (no sliding) : $|\gamma| \leq \gamma_s$ * $\tan \gamma_s = M_s$
 - For dynamic contact (sliding) : $|\gamma| = \gamma_d$ * $\tan \gamma_d = M_d$.
- where γ_s and γ_d are the angle of static and dynamic friction respectively.

Types of friction problems.

① static contact w/o impeding motion.

$|F| \leq M_s N$, $|\gamma| < \gamma_s$. F found by considering eqm. of the body.

② static contact, at the pt. of slipping.

$|F| = M_s N$, $|\gamma| = \gamma_s$.

③ dynamic contact, surfaces slipping.

$|F| = M_d N$, $|\gamma| = \gamma_d$.

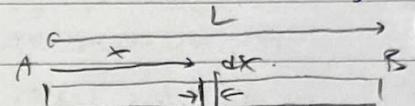
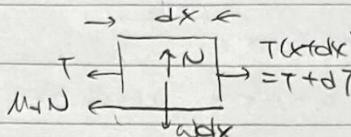
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Distributed friction

- If a body is sliding over another, the coefficient of dry friction is constant, but the forces within the sliding body will work continuously, due to the integration ("accumulation") of friction along the interface. → may lead to nonuniform distribution of pressure between the bodies along their mutual interface.
- Generally, we solve problems of distributed friction by considering the eqn. of a small (infinitesimal) element and then integrate to find the force distribution.

e.g.: Sliding carpet. Find the force T_b req. to pull a carpet of weight per unit length w at steady speed over a floor w/ coefficient of dynamic friction M_d . ($T_f = 0$).

FBD of small element



N2L I:

$$N = wdx$$

\hookrightarrow :

$$T + dT = T + M_d N$$

$$\text{so } dT = M_d wdx$$

$$\int_0^{T(x)} dT = M_d w \int_0^x dx$$

$$T(x) = M_d wx + C.$$

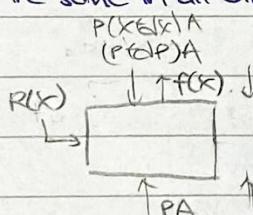
boundary cond: when $x=0$, $T(x)=0$ so $C=0 \rightarrow T(x) = M_d wx$.

$$\therefore T_b = T(L) = M_d w L.$$

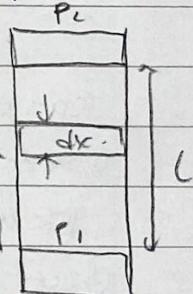
e.g.: Sand in a cylinder. A column of dry sand of length L is pushed at constant speed through a cylindrical tube of radius R b/w 2 smooth left pistons applying pressures p_1 & p_2 ($p_2 > p_1$). Ignoring the self weight of the sand and assuming that at any pt in the sand the pressure is the same in all dir. and that the coefficient of dynamic friction is M_d .

Find p_2/p_1 ,

FBD of small element



$$\begin{aligned} \text{Around the slice, } \\ R(x) = 2\pi R P(x) \Delta x \\ f(x) = M_d R(x) \\ = M_d \cdot 2\pi R P(x) \Delta x \\ A = \pi R^2. \end{aligned}$$



N2L I:

$$PA + f(x) = (P + dP)A$$

$$2M_d \pi R P dx = dP \cdot \pi R^2$$

$$2M_d dx = \frac{R}{P} dP$$

$$R \int_{p_1}^{p_2} \frac{1}{P} dP = 2M_d \int_0^L dx$$

$$\ln \left| \frac{p_2}{p_1} \right| = \frac{2M_d}{R} L$$

$$\frac{p_2}{p_1} = e^{\frac{2M_d}{R} L}$$

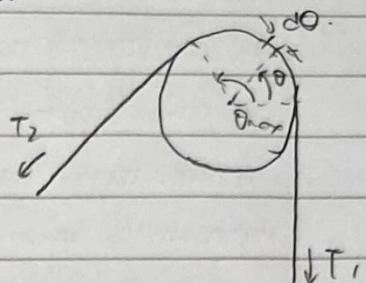
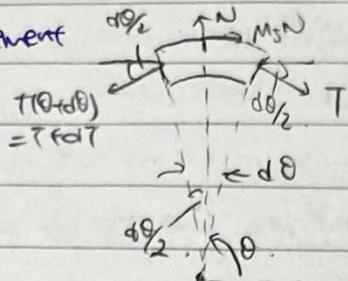
Note $R(x)$ is the total force acting on the slice so we integrate $P dA$ for the entire surface.

$$\begin{aligned} \text{As } P \text{ is constant, } R(x) &= \int P dA \\ &= P \int dA \\ &= P \cdot 2\pi R dx. \end{aligned}$$

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e.g.: Flexible belt on a pulley. A flexible belt is passed over a pulley. If the coefficient of static friction between the belt and pulley is μ_s , and $T_2 > T_1$. Find the max. ratio of T_2/T_1 before slipping occurs.

FBD of small element



By N2L J :

$$\begin{aligned} N &= T \sin\left(\frac{d\theta}{2}\right) + (T + dT) \sin\left(\frac{d\theta}{2}\right) \\ &= 2T \sin\left(\frac{d\theta}{2}\right) + dT \sin\left(\frac{d\theta}{2}\right) \quad \text{2nd order form} \\ &= T d\theta. \end{aligned}$$

for small θ

$$\rightarrow : (T + dT) \cos\left(\frac{d\theta}{2}\right) = T \cos\left(\frac{d\theta}{2}\right) + \mu_s N$$

$$dT \cos\left(\frac{d\theta}{2}\right) = \mu_s N \quad \text{for small } \theta$$

$$dT = \mu_s T d\theta$$

$$\int_{T_1}^{T_2} \frac{dT}{T} = \mu_s \int_{0}^{\theta_{max}} d\theta$$

$$\ln\left|\frac{T_2}{T_1}\right| = \mu_s \theta_{max}.$$

DB. $T_2 = T_1 e^{\mu_s \theta_{max}}$ * $T_2 > T_1$

supports and free body diagrams

Applied forces and reaction forces.

- Forces can be applied to a structure by an ext. agency in which case the force is known and the displacement of the structure under the action of the force must be found.
- Forces can be the reactions of the structure at a geometric constraint or support, in which case the displacement is 0 and the force is found by overall eqn.

Pin-joints

- A frictionless pin joint can only provide a reaction force, but not apply a moment to the structure.
- The reaction force at a pin joint can act in any direction.
- The pin-joint support restricts 2 degrees of freedom (vertical & horizontal translation)
- In analysing problems w/ pin-joints, it is assumed that they are frictionless and apply a pt. force to the structure.

direction can be found
easily via graphical methods.

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Roller supports

- Roller supports provide a vertical restrain on displacement but does not inhibit horizontal movement (or rotations).
- As w/ frictionless contact, the reaction force of a roller support is in the dir. of the common normal (\perp to the motion of the rollers).
- A more formal representation of the roller support is  which restricts translations up and down. However, for simplicity  is assumed to constrain motion in both the vertical directions.

Built-in / encastre supports.

- Encastre supports constrains all 3 deg. of freedom - linear displacement in the vertical / horizontal dir. and rotation.
 - The reaction forces provided by an encastre support include 2 components of a force and a moment in reaction to the constraint on motion.
- * DO NOT confuse this w/ a string/cable, which can only transmit a force along its dir.

The direction of reaction forces,

- In general, a body subject to 2 forces (and no moments) can only be in eqm. if the forces are co-linear.
 - In general, a body subject to 3 forces (and no moments) can only be in eqm. if the forces are concurrent (their lines of action intersect at a single pt.).
 - * If the 3 forces are \parallel , the intersection pt. is effectively at infinity \rightarrow the body can still be in eqm.
 - For a pin-joint support,
 - \hookrightarrow 1 applied force \rightarrow reaction force collinear w/ applied force.
 - \hookrightarrow 2 applied forces \rightarrow reaction force's line of action passes through the intersection pt. of the applied forces' lines of action.
 - For a roller support, the reaction force is \perp to the motion of the rollers.
 - For an encastre support (of a concentrated beam), there is a component of force along the length of the beam (tensile/compressive force), a component of force \perp to the length of the beam (shear force) and a moment (bending moment).
 - For a cable, the dir. of force is always along the dir. of the cable.
- * We can only find the reaction forces of a structure that is neither under or over constrained. (by ext. constraints / supports)

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Free body diagrams (FBD)

- When a structure comprises of many components/bodies connected at contact interfaces or w/ joints, \rightarrow we may want to find the forces between the bodies.
- These forces are internal to the overall structure but external rel. to the bodies once they have been separated.
- We can separate the different bodies and draw FBDs to solve for the (external) int. forces.
- To construct FBDs for a 2D structure:
 - \hookrightarrow cut along a closed curve and draw each part separated from each other.
 - \hookrightarrow Add the original applied & reaction forces at the pt. where they apply and add force/moment pairs (by N3L) at the cut-through interface.

problem solving strategy (graphical method).

- 3 Forces : We know the pt. the force is acting on \rightarrow find intersection pt. of the lines of action to find the dir. (if force triangle to find the magnitude).
We don't know the pt. the force is acting on \rightarrow force triangle to find the magnitude.
- 4 Forces: Take moments abt the intersection of the lines of action of 2+ forces \rightarrow find the other forces.
- If 2+ forces act at the same pt. \rightarrow merge them into 1 force.
After obtaining the resultant, use a force polygon to solve for the individual forces.
- In general, draw a rough sketch + think of the solving process (steps before drawing an accurate sketch).

Problem solving strategy (algebraic method).

- Use N3L in 2 directions to get 2 scalar eqn. of eqm.
- Use moments to find the magnitude of the reaction forces.
 - \hookrightarrow Take moments abt. one support to find the (vertical) contact force at the other support.
 - \hookrightarrow Take moments abt. a pt. in the structure (above ground) to find the (horizontal) frictional force of the contacts. [using a section created by cutting at a pin joint]

Pin-jointed trusses.

Pin-jointed plane trusses.

- A truss is a framework of members joined at their ends.
- For a plane truss, all the members lie in 1 plane.
- In general, a truss may be joined at their ends by welding, rivets or bolts and gusset plates, pins or adhesives.
- For simplicity, we will only consider the idealised frictionless pin as the type of connection \rightarrow pin-jointed plane trusses.

(For any truss w/ slender members, only type of joint may be modelled as a pin joint)

- for now, we will consider only pin-jointed trusses subject to pt. loads applied at the joints.
- As these loads are usually large in comparison to the weight of the members, we will also neglect the self-weight of the members.
- Each member in the truss must be an axial force member — it is in a state of pure tension or pure compression.
- This is because each member is only loaded at the 2 nodes, which each has a resultant force \rightarrow the member is acted on by only 2 forces \rightarrow the reactions must be coplanar.

Statically determinate frames

- If the int. components of a structure are under-connected will be a mechanism and may collapse w/o the application of any load ; if they are over-connected, this may be a sign of inefficiency and the analysis is difficult as the structure is statically-indeterminate. (the int. forces depend on the interaction of components which depend on their material prop.).
- statically determinate structures are the easiest to analyse — the int. forces can be found directly by the method of joints/sections from eqn. w/ the ext. forces.
- A characteristic of statically determinate trusses is that the bars can be assembled w/o forcing, even if they may not be all exactly of the correct length.
- It follows that changes of length of the members (eg due to change in temp/humidity) do not produce additional stresses in the structure.
- A formal approach to characterising a pin-jointed truss structure is provided by counting the no. of joints (J), bars (b), restraints (r) and dimensions (D) in the problem.

$D_J > b+r$: mechanism , $D_J = b+r$: statically determinate , $D_J < b+r$: statically indeterminate.

* Do not blindly rely on the rule. $D_J = b+r$ can mean mechanism+statically indeterminate.

* In general, a statically determinate truss is made up of triangles and has 3 deg.

of freedom restrained at the joints (usually $\frac{J}{2} + \frac{r}{2}$)

$= 1$

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Method of joints

- The method of joints req. that we create a FBD for each joint of the truss. It provides a systematic way of calculating all the bar forces in a structure, by considering the conditions of force eqn. at each joint.
- By defn., the forces meet at the joint \rightarrow moment eqn. satisfied; we req. the vector sum of all forces acting on the joint to be 0.
- We can obtain 2 scalar eqns of eqn (from NLL) applying at each joint \rightarrow we can only find 2-unknown bar forces (reactions per joint). [start from the supports].
- * For trusses whose members align simply w/ a coord system \rightarrow algebraic methods. For trusses w/ more complex geometries \rightarrow graphical methods.
- * Method of joints is the same as method of sections where the section is the joint.

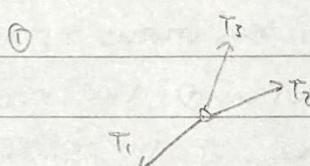
\hookrightarrow both methods req. us to first find all free ext.
 \hookrightarrow forces (reaction forces at the supports).

Method of sections

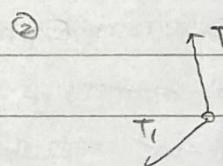
- The method of sections can help us find the axial tension in 1 specific bar \rightarrow better than method of joints when the members are far from the supports.
- To find a useful section:
 - \hookrightarrow Cut through the member of interest. Then include any known ext. forces (applied or reaction forces). Always choose a section w/ less ext. forces \rightarrow less work.
 - \hookrightarrow The section will be in planar eqn \rightarrow we can obtain 3 scalar eqns of eqn. (from NLL and moments) \rightarrow we can only find 3-unknown bar forces/reactions per section.
- To use the section to find axial forces:
 - \hookrightarrow Use NLL to find diagonal bars.
 - \hookrightarrow Take moments at the intersection pt. of lines of action to find vertical/horizontal bars.
- * The method of section often leads to an eqn. of moment eqn \rightarrow algebraic methods. If we need to use NLL and the geometry is inconvenient \rightarrow graphical methods.

Some simplifications in analysing planar pin-jointed trusses.

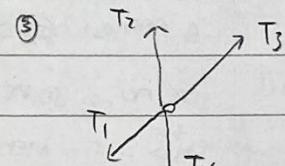
- The calculation of bar forces can be simplified by noting the special cases.



$$T_1 \parallel T_2 \rightarrow T_3 = 0$$

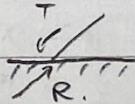


$$T_1 = T_2 = 0$$



$$T_1 \parallel T_3, T_2 \parallel T_4 \rightarrow T_1 = T_3, T_2 = T_4$$

- At a support, if there is only 1 bar, the reaction force must be in the dir. of the bar since there are only 2 forces acting at the node.



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Superposition

- When there are many forces acting on a truss, w_1, w_2, w_3 etc. and the axial force in a particular member due to w_1 alone is $\lambda_1 w_1$, then the force T in the member is given by

$$\text{Principle of Superposition: } T = \lambda_1 w_1 + \lambda_2 w_2 + \lambda_3 w_3 + \dots$$

where the coefficients λ_i depend on the truss geometry.

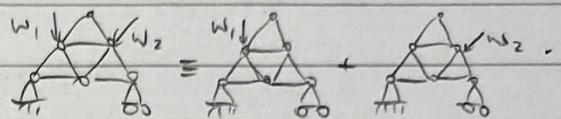
- The principle of superposition is not restricted to bar forces in trusses; it applies to all linear systems.

- The behaviour of a truss is linear if:

↳ The material of the truss remains in the linear elastic range.

↳ The geometry does not change much, i.e. the distortion of the structure caused by the loads are small and hence the eqn. written in the undeformed configuration are also valid after the structure has deformed.

(These conditions are true for most structures made of metal).



Symmetry

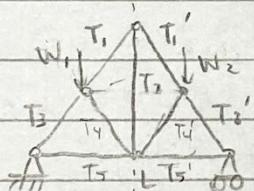
- Consider a truss that has symmetry about a line.

↳ If the loads are symmetric → the bar tensions are symmetric (about that line)

↳ If the loads are anti-symmetric → the bar tensions are anti-symmetric (about that line).

* Note the supports are not symmetric, but their effect is to apply symmetric reactions, so both applied and reaction ext-forces are symmetric.

(This works as there is no net horizontal applied force \Rightarrow defn of (anti-)symmetry about the line).

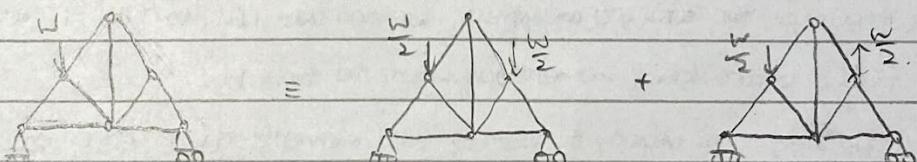


$$\text{Symmetric load } (w_1 = w_2) : T_i = T'_i$$

$$\text{Anti-symmetric load } (w_1 = -w_2) : T_i = -T'_i$$

* symmetric about mirror line L.

- For a symmetric structure, it is always possible to represent any set of applied loads by superposing a symmetric and an anti-symmetric set of loads.



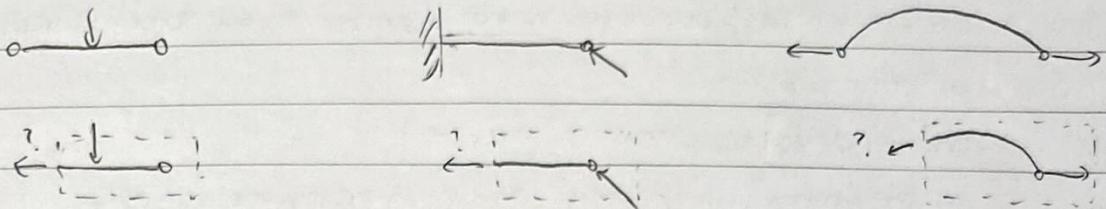
- In general for a single load w , it can be replaced a symmetric+antisymmetric loading of $\frac{w}{2}$. Use the principle of superposition for multiple loads.

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Shear force and bending moments

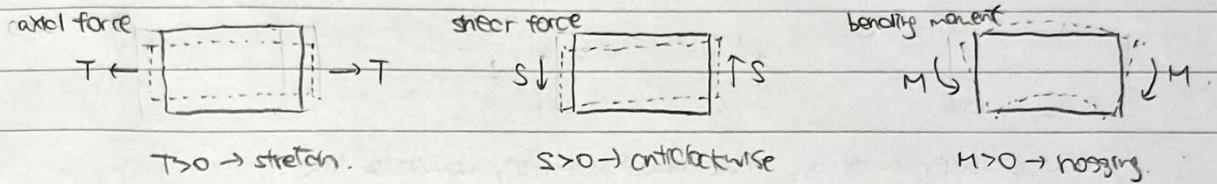
shear forces and bending moments.

- Considering beams w/ transverse loading (load not applied at node or an unpinned one joint encastre) and non-straight members, we cannot reach eqn. w/ a single axial force at a cut (when using method of sections). \rightarrow we need shear force + bending moments



Sign conventions:

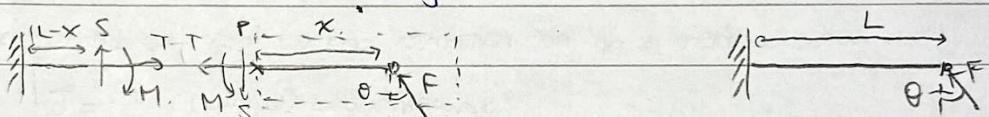
- deformations arising from the axial forces (T), shear forces (S) and bending moments (M) are defined as follows: (for this convention)



* Using this convention, shear forces (S) and bending moments (M) have opposite signs.

Beams with transverse loadings.

- Consider a beam w/ transverse loading - an unpinned one joint encastre,



* The dir. of T , S , and M are chosen so they are fve (using the convention above).

$$NIL \Rightarrow T + F \sin \theta = 0 \rightarrow T = -F \sin \theta$$

$$\int: S = F \cos \theta \rightarrow S = F \cos \theta$$

$$MP: M + F \cos \theta x = 0 \rightarrow M = -F \cos \theta x.$$

* In general, for straight members, we can use NIL to find T , then use NIL again to find S before taking moments anywhere to find M .

(However, it is usually faster to take moments about either end, whichever end has more (unknown) forces.)

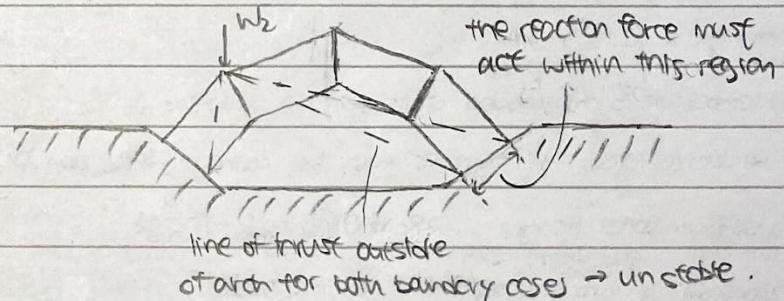
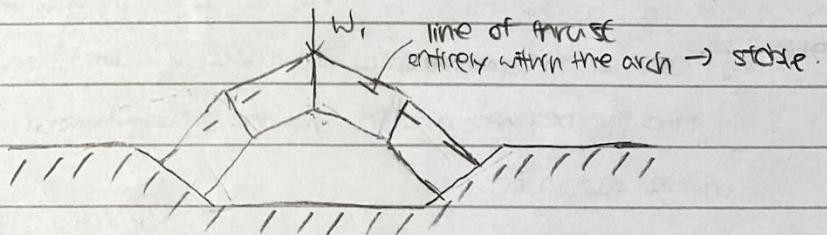
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Arches

- The structural benefit of arches is that their shape can be chosen s.t. the internal forces in the arch are mainly in compression. \rightarrow useful for:
 - \hookrightarrow Stone masonry, joints b/wn stone blocks are strong in compression but weak in tension
 - \hookrightarrow Metal structures, reduced bending moment allows the use of lighter members.
- Arches rely upon their supports, called abutments to resist lateral movement. If abutment movement is significant, the arch may deform or even collapse.

Segmented arches

- Segmented arches, those made of blocks, (stone arches in bridges and churches) generally cannot carry tensile loads at joints. We make the following assumptions:
 - \hookrightarrow The blocks are infinitely strong (no compression failure).
 - \hookrightarrow The block interfaces are unable to carry bending moments (no tensile capacity, i.e. the block cannot be in tension \rightarrow the arch just opens up).
 - \hookrightarrow Block interfaces have infinite friction (no lateral sliding along the interface).
- Based on these assumptions, global eqm. rather than material failure determines the arch capacity \rightarrow if the line of thrust can remain completely within the structure, the arch is in eqm. and stable. [use geometric thickness to resist loading].
- If we consider a stone arch subjected to a large pt. load, we can ignore the self-weight of the stones that make up the arch. In this case, the line of thrust comprises of 2 straight lines that connect the pt. load to meet the abutments either side of the arch.
- If we can draw a line of thrust from each abutment to the pt. load that remains entirely within the arch, the arch can support the load.



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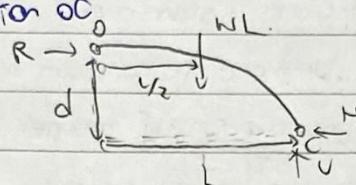
Continuous arches

- Continuous (flat) arches (made of steel or reinforced concrete) use bending moment capacity to resist loading.
- In general, we work out the reaction forces at the abutment, then we use the method of sections to cut through the pt. of interest before taking moments abt. that pt. to find the bending moment.
- * For non-straight members, we must take moments abt. the cut pt. to find the bending moment.

- e.g.: Consider an arch w/ uniform load w per unit horizontal length, and has the parabolic shape $y = d \frac{x^2}{L^2}$. Find the abutment reactions, and the bending moments in the arch.

Consider section OC₀

FBD :



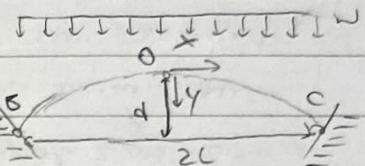
NIL I :

$$V = wL$$

(M_p) :

$$NL \cdot \frac{L}{2} + Hd = VL$$

$$H = \frac{wL^2}{2d}$$

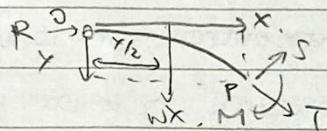


V can be different for different sections of the arch as the ext. load changes.

H is the same no matter what section of the arch we take as all the loads applied are vertical.

Consider a general section OP.

FBD.



(M_p) :

$$Nx \cdot \frac{x}{2} = M + R \cdot y$$

$$\frac{1}{2}N x^2 = M + \frac{wL^2}{2d} \cdot d \frac{x^2}{L^2}$$

$$M = 0$$

choose a section that starts from the origin (so we can use the org. easily).

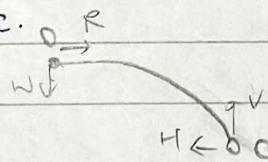
→ special result: the bending moment throughout a parabolic arch under this loading condition is 0.

↳ replacing a distributed force w/ a single pt. force does not help as analyse the internal forces. → different bending moments.

e.g.: Consider the same arch as above but w/ a central pt. load W.

Find free abutment reactions and the bending moments in the arch.

Consider section OC.

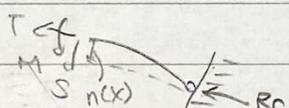


Member OC is only loaded at the pins → subject to 2

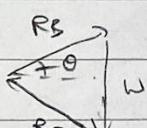
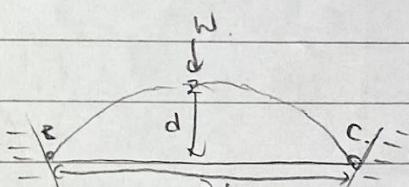
resultant forces → resultant must be center → RC day OC.

Using a force triangle, $W = 2R \sin \theta = 2R_c \cdot \sqrt{\frac{d}{L^2 + d^2}}$

Consider an arbitrary section



$M(x) = R_c \cdot n(x)$, (non-zero bending moment).



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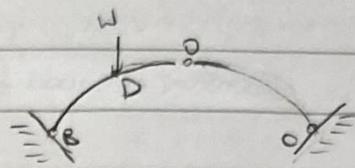
e.g.: Consider the same arch as above but w/ a non-central pt. load w .

Explain why the max. bending moment in OC is where the tangent is \parallel OC.

Explain why the max. bending moment in OB is at D.

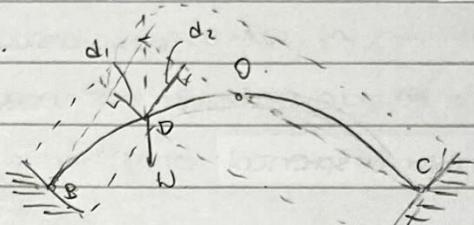
Consider member OC.

2 resultant forces \rightarrow collinear $\rightarrow R_c$ in the dir. OC.



Bending moment is given by R_c times the \perp distance from the line of action of R_c (along OC) and the arch \rightarrow max. when the tangent is \parallel OC. \rightarrow max. M.

Consider the entire arch.



By considering the lines of action of the load and R_c , we can find the dir. of R_s .

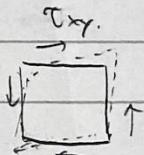
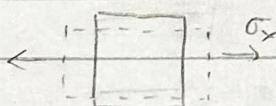
Consider a section that cuts through D \rightarrow we can ignore W. (as we take moments abt. the cut)

The \perp distance from the lines of action of R_s , R_c to the arch is: max at D, \rightarrow max. M.

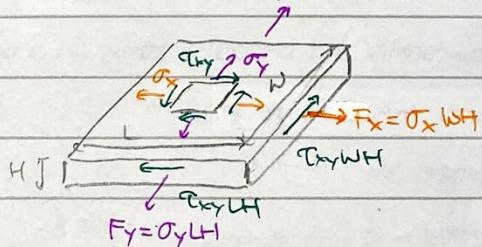
STRESS

2D plane stress in thin walled shells.

- We can use stress (w/ sufficient components) to represent the influence of all really deformed interatomic bonds.
- In 2D, the state of stress has 3 components \rightarrow 2 direct stresses σ_x, σ_y (tend to stretch/compress the material along an axis) and the shear stress τ_{xy} (tend to shear the material from a shell square to a rhomboid form).



- relationship between ext. forces and the int. state of stress..



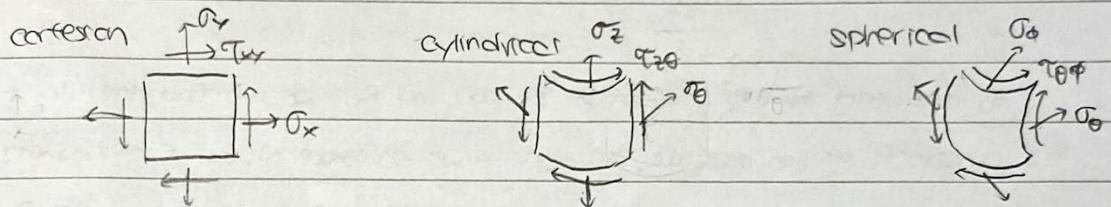
Stress has units of force/area so the int. forces (stresses) can be related to ext. loads by integrating (stress) wrt area over the boundary influenced by the load.

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- The units of stress are the same as pressure, but pressure has no dir. - at any pt. in a fluid, it acts identically in all dir. In contrast, the components of the state of stress have specific dir. (indicated by their subscripts).
- The state of stress must always be stated wrt a set of axes, as a given component of stress under a constant ext. load varies as the axes rotate.

Thin walled shells w/ uniform stress.

- For thin-walled pressure vessels, we can assume the stress distribution is uniform through the thickness.
- We will only consider vessels w/ axis-symmetric shape (spheres or cylinders), for which the stress is uniform along any circumferential or longitudinal section.
- The components of the state of stress for these vessels do not fit the Cartesian coords system \rightarrow use cylindrical-polar or spherical-polar rotation.



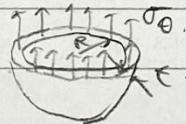
- We can find the stress in the pressure vessel using the method of sections:

- After making a plane cut through the vessel, asserting eqn. across this plane, the ext. force (pressure \times proj. area of the vessel's interior) must equal to the int. force (stress in the wall normal to the plane \times area of the cut surface of the vessel).

$$\frac{\text{external force}}{p \times A_{\text{projected}}} = \frac{\text{internal force}}{\sigma \times A_{\text{cut material}}}$$

- e.g.: Find the stress distribution in a sphere w/ thickness t , radius R and int. pressure p .

Make a cut.



$$p \times A_{\text{projected}} = \sigma \times A_{\text{cut material}} : p \cdot \pi R^2 = \sigma_0 \cdot 2\pi R t \rightarrow \sigma_0 = \frac{pR}{2t}$$

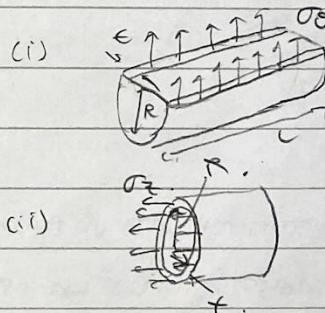
- e.g.: find the circumferential and longitudinal stress in a cylindrical pressure vessel w/ thickness t , radius R , length L and int. pressure P

Make an appropriate cut.

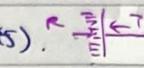
$$p \times A_{\text{projected}} = \sigma \times A_{\text{cut material}}$$

$$(i) \quad p \cdot \pi R^2 = \sigma_0 \cdot \pi R L \rightarrow \sigma_0 = \frac{pR}{L}$$

$$(ii) \quad p \cdot \pi R^2 = \sigma_z \cdot 2\pi R t \rightarrow \sigma_z = \frac{pR}{2t}$$

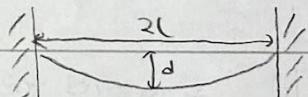


Cables and bar extensions**Cables.**

- Cables, strings, ropes and chains are able to carry unidirectional tension but they are unable to carry compression or bending as they have negligible bending stiffness.
- It is a good assumption that the tension T of any pt. acts in the dir. of the tangent to the cable at the pt. (They look like encastre supports). 
- In many applications, it can be assumed that cable length will remain approx constant throughout, and hence the cable can be treated as inextensible.
- The deformed geometry of inextensible cables can be predicted purely by considering the eqn. of ext. and int. forces.
- Typically, for a cable of known length and subject to specified loads and boundary conditions (fixed ends) we try to find:
 - ↳ The support reactions
 - ↳ The shape adopted by the cable.
 - ↳ The tension distribution throughout the cable.

Cables subjected to concentrated loads.

- If the cable is subjected only to concentrated loads, then it will adopt a straight line between loading pts.
- The overall shape of the cable is thus a broken line, consisting of straight segments each of which carry constant tension (the tension in each segment can be different if the load is fixed).
- If the ext. loads are known, we can work out the reaction forces at the ends of the cable and the cable tension using the method of joints (sections).
- If it is assumed the ext. supports provide no moment \rightarrow the cable can leave the support at any angle (although they are drawn like encastre supports).

**Cables subjected to distributed loads.**

- Cables subject to distributed loads adopt the shape of a smooth curve
- For a cable with constant weight w per unit length hanging between level supports,
 - ↳ If the cable is shallow ($d \ll L$) \rightarrow assume self weight is w per unit horizontal length of cable \rightarrow parabolic shape
 - ↳ If the cable is not shallow (d not $\ll L$) \rightarrow cannot assume self weight is w per unit horizontal length of cable \rightarrow catenary (cosh curve shape).

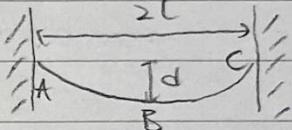
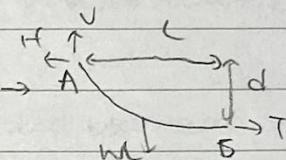
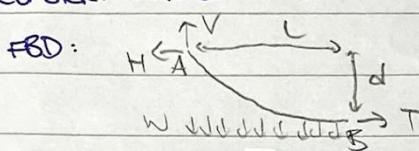
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- The standard procedure for cable questions :
 - ↳ Generally, we can find the vertical comp. of the 2 reaction forces by eqn only ($N2L+M$)
 - ↳ To find the horizontal comp. of the reaction forces, we need 1 additional piece of information about the shape of the cable
 - ↳ After finding the full reaction force, we can take a FBD diagram at any location in the cable to find the deflected shape, angle and its tension.
- An important result is that a cable subjected to a uniform vertical load distribution has a parabolic shape, which is independent of the magnitude of the loading.
- An approximate formula for finding the length of a parabolic cable

$$S = 2L \left(1 + \frac{wL^2}{8d} \right)$$

e.g.: Find the shape of a cable under uniform vertical load w per unit length.

Consider half of the cable.



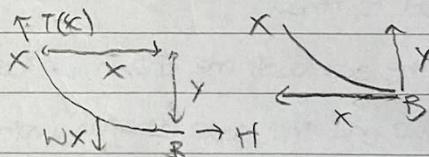
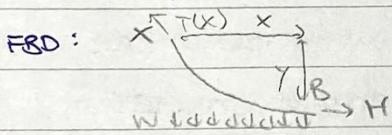
$$\text{By NCL } \int: V = WL$$

$$(M_0^{\infty}): WL \cdot \frac{L}{2} + Hd = VL$$

$$Hd = WL \cdot L - \frac{1}{2}WL^2$$

$$H = \frac{WL^2}{2d}$$

Consider a general section.



$$(M_k^{\infty}): Hx = WX \cdot \frac{x}{\Sigma}$$

$$\frac{Wx^2}{2d} Y = W \frac{x^2}{2}$$

$$Y = d \left(\frac{x}{\Sigma} \right)^2$$

* To find the angle and tension.

use eqn. ($N2L+M$) to find T_x and T_y

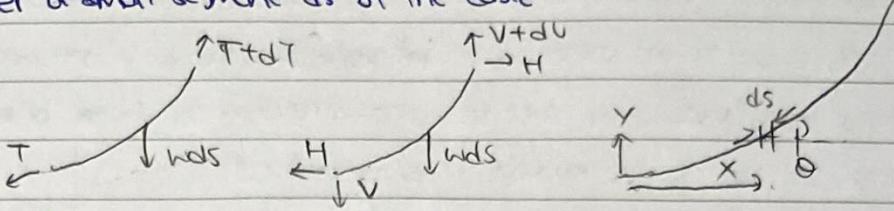
As the tension must act in the tangential direction of the cable, $\tan \theta = \frac{T_y}{T_x}$

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e.g.: Find the shape of a cable with constant weight w per unit length

Consider a small segment ds of the cable

FBD:



$$\text{By N2L } \sum F_y: V + dV = V + wds$$

$$\frac{dV}{ds} = w.$$

$$\text{At the general pt. } x, \quad \tan\theta = \frac{dy}{dx} = \frac{V}{H} = \frac{dy}{dx}.$$

Differentiating w.r.t x

$$\frac{dy}{dx} = \frac{1}{H} \left(\frac{V}{H} \right)$$

* $\frac{dy}{dx}$ term is useful!

$$\frac{d^2y}{dx^2} = \frac{1}{H^2} \frac{dV}{dx} \quad \text{as } H \text{ is constant}$$

For a small segment ds ,

$$ds^2 = dx^2 + dy^2$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx \rightarrow \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx} \right)^2}.$$

$$\therefore \frac{ds}{dx} = \frac{1}{H} \frac{dV}{dx}$$

$$= \frac{1}{H} \frac{dV}{ds} \cdot \frac{ds}{dx}$$

$$= \frac{w}{H} \sqrt{1 + \left(\frac{dy}{dx} \right)^2}$$

$$\text{Let } z = \frac{dy}{dx}, \text{ then } \frac{dz}{dx} = \frac{d^2y}{dx^2}.$$

$$\frac{dz}{dx} = \frac{w}{H} \sqrt{1 + z^2}$$

$$\int \frac{1}{\sqrt{1+z^2}} dz = \frac{w}{H} \int dx$$

$$\operatorname{arcsinh} z = \frac{w}{H} x + C$$

$$\text{At } x=0, \frac{dy}{dx} = z = 0$$

$$\operatorname{arcsinh} 0 = \frac{w}{H} (0) + C \rightarrow C = 0.$$

$$\therefore \operatorname{arcsinh} z = \frac{w}{H} x$$

$$z = \sinh \left(\frac{w}{H} x \right)$$

$$\frac{dy}{dx} = \sinh \left(\frac{w}{H} x \right)$$

$$y = \int \sinh \left(\frac{w}{H} x \right) dx$$

$$= \frac{w}{H} \cosh \left(\frac{w}{H} x \right) + C_1$$

$$\text{At } x=0, y=0$$

$$0 = \frac{w}{H} \cosh \left(\frac{w}{H} \cdot 0 \right) + C_1 \rightarrow C_1 = -1.$$

$$\therefore y = \frac{w}{H} \cosh \left(\frac{w}{H} x \right) - 1.$$

* Generally, to find the shape of the cable via integration,

draw a FBD at a general segment w/ x-coord x , in terms of V and H .

use $\frac{dy}{dx} = \frac{V}{H}$ and differentiate w.r.t x

combine the 2 expressions and solve to find $y(x)$.

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shows Hooke's law over bar extension

- For typical metals, the extension of the wire ϵ increases linearly w/ weight. Fundamentally, this is due to the stretching of the interatomic bonds in the metal.

- For many metals, the limit to elastic stretching is of the order of 0.3%.

- In 1D, the Young's modulus E is defined as

$$E = \frac{\sigma}{\epsilon} = \frac{F/A}{\Delta L/L}$$

interatomic force roughly prop.
to atomic separation near
the equilibrium spacing.

where σ is the tensile stress and ϵ is the tensile strain.

- In the context of a truss member carrying an axial force T ($T > 0$ indicates tension),

$$E = \frac{TL}{Ae} \rightarrow e = \frac{TL}{AE}$$

* only valid for stiff materials (metal/concrete), up till the elastic limit and constant stress (good approximation for slender members, i.e. $A \ll L$)

Thermal expansion

- Experiments have shown that the (1D) thermal strain ϵ_T in a (long straight slender) member due to a uniform temp. change $\Delta\theta$ can be predicted by

$$\epsilon_T = \alpha \Delta\theta$$

where α is the coefficient of thermal expansion.

- For a member subject to an axial force T and temp. change $\Delta\theta$, the total strain ϵ can be found by superposition as

$$\epsilon = \frac{TL}{AE} + \alpha \Delta\theta$$

- therefore the elongation is given by

$$e = \epsilon L_0 = \frac{TL_0}{AE} + \alpha L_0 \Delta\theta$$

* For structurally indeterminate (over-constrained) structures, uneven thermal expansion cannot be compensated by nodal movement \rightarrow structure has int. loads w/o ext. loads.

Displacement diagrams and red/virtual work method.

Compatibility

- The individual elements of the whole structure can only deform in a way that is compatible w/ the overall deformation of the structure.

- To solve for the overall structure after all the members in the structure change length, we can use:

↳ displacement diagrams

[graphical]

↳ red. work / virtual work method

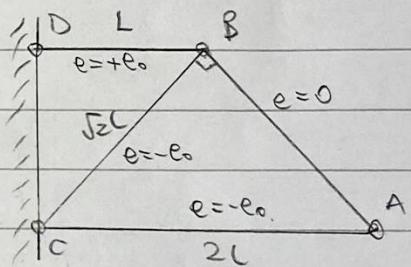
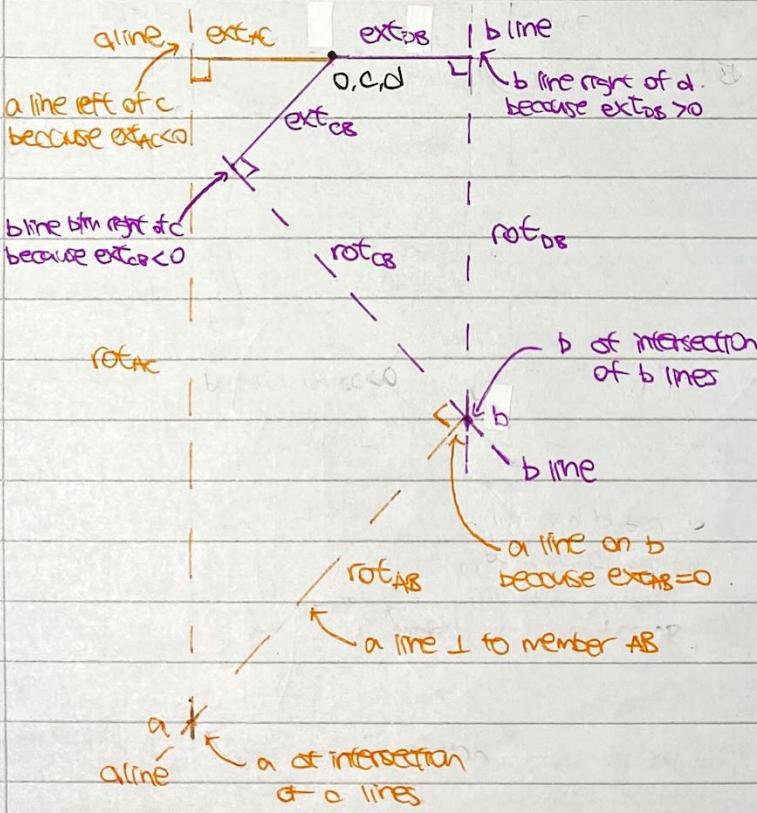
[algebraic].

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Displacement diagrams

- When using displacement diagrams, we assume the extensions are small compared to the member lengths \rightarrow replace circular arcs w/ tangents.
- In a displacement diagram, we draw the extension \parallel to the bar and the displacements \perp to the bar. (displacements \perp to bar for $e \ll L$).
- In a displacement diagram, the pt. O is the origin/pole, where all the displacements of all other pts on the diagram can be measured rel. to.
- In general, to find the displacement of a node, it must be connected to 2 other nodes w/ known displacements (eg fixed nodes, from symmetry).
- We first draw the extension from the known node (some "order" of nodes) for the +ve, reverse "order" of nodes for -ve e) then draw a dotted rot. line \perp to it.
- Repeat for the other known node and the intersection of the 2 dotted rot. lines give the position of the node on the displacement diagram. The displacement of the node is given by the position of the node rel. to O on the displacement diagram.
- * Displacement diagrams are to scale \rightarrow more accurate if the diagram fills on ALL sheet.
Draw a rough sketch to get a feel for the sdin \rightarrow choose a good place for O.
If the lines are drawn to $\pm 1\text{mm}$ in length, $\pm 1^\circ$ in angles \rightarrow roughly 5% error.

- ej: Draw a displacement diagram of the structure
given the extensions are $e_0, -e_0, e_0, 0$ for BD, AC, BC, AB.



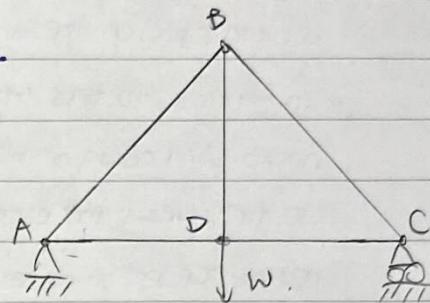
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Very small rigid-body rotation of the whole structure

- If we made a wrong assumption (e.g. a slanted member assumed to be horizontal), as observed when the predicted displacements do not make sense, we can fix this by applying a small rigid-body rotation to the whole structure.

- e.g.: Draw a displacement diagram of the structure given

Bars	Length	T	$e (\times \frac{WL}{AE})$
BD	L	$+w$	1
AB	$\sqrt{2}L$	$-\frac{w}{\sqrt{2}}$	-1
BC	$\sqrt{2}L$	$-\frac{w}{\sqrt{2}}$	-1
AD	L	$+\frac{w}{2}$	$\frac{1}{2}$
DC	L	$+\frac{w}{2}$	$\frac{1}{2}$

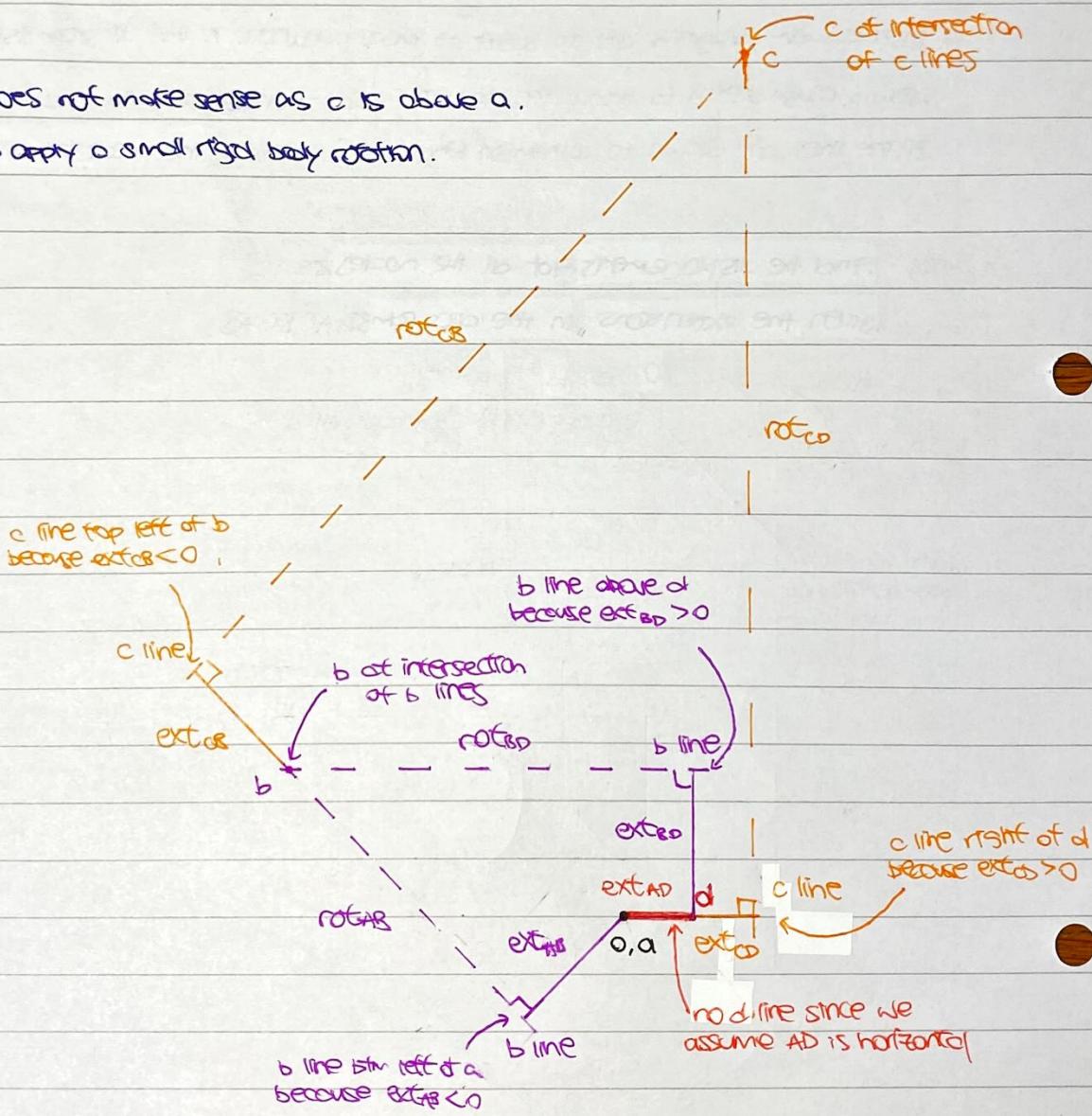


and all the bars have cross-sectional area A and Young's modulus E .

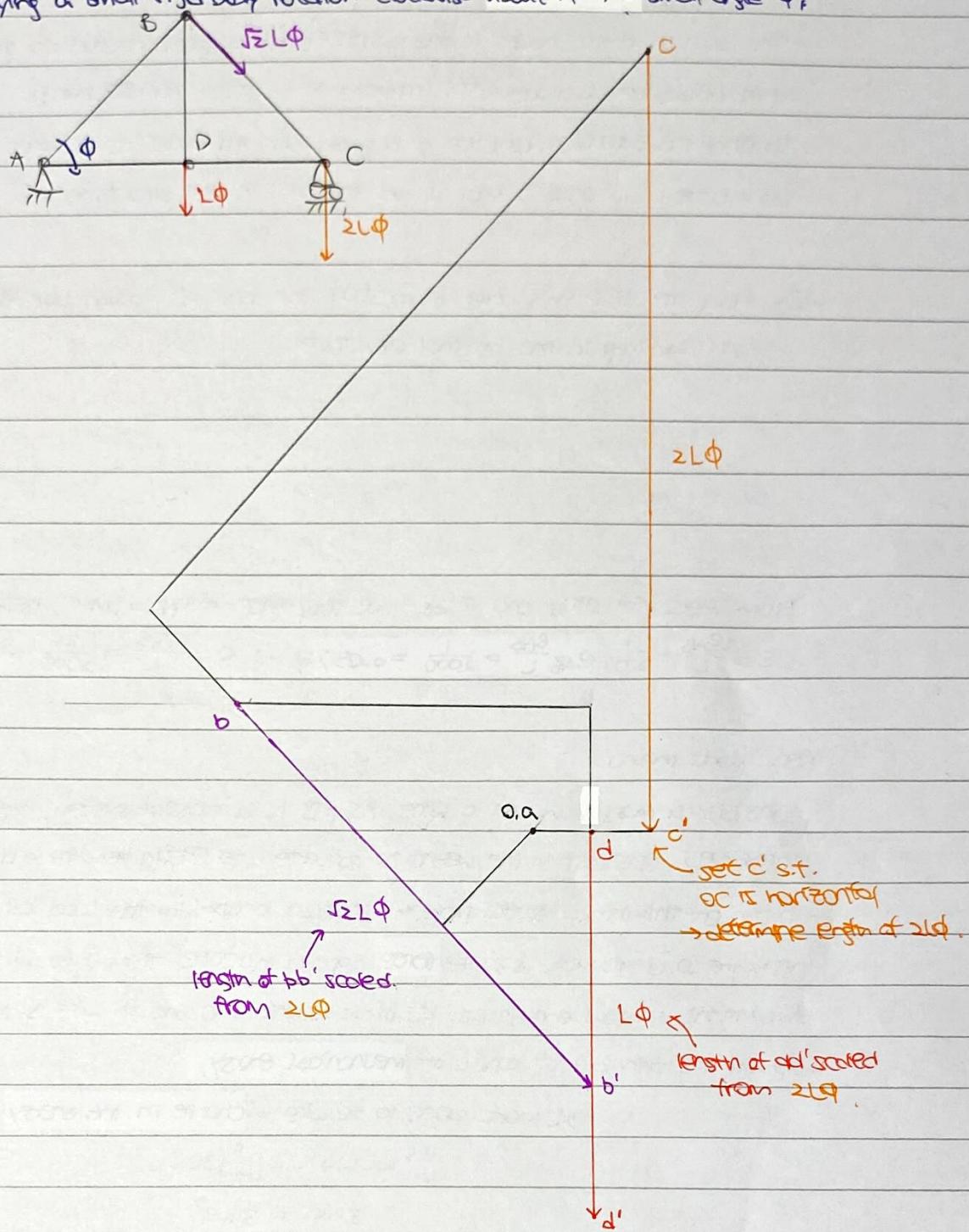
Assume member AD remains horizontal (wrong assumption!).

DOES not make sense as c is above a.

→ apply a small rigid body rotation.



Applying a small rigid-body rotation clockwise about A by a small angle ϕ ,



For a joint X, the displacement when rotating about A by a small angle ϕ , has a magnitude $|AX|\phi$ in the direction \perp to AX .

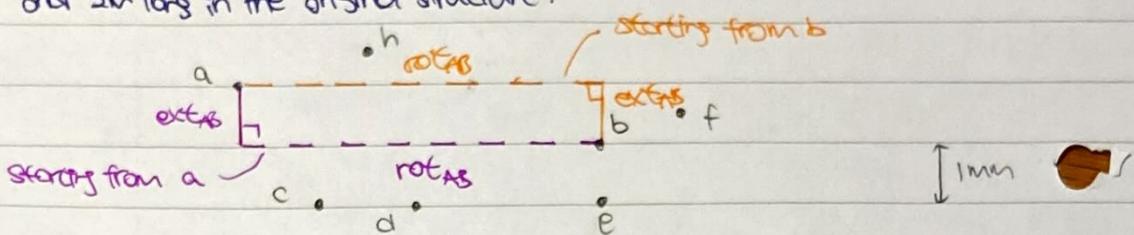
* In this question, we should have used symmetry and deduced that b must be above d and draw the displacement diagram starting from d.

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Interpreting displacement diagrams.

- The location of the nodes in the displacement diagram relative to the origin shows their absolute displacement. (remember to state the direction!)
- To find the extension/rotation of a given bar, the nodes at its ends must be connected by lines // and \perp to the bar in the structure.

e.g.: Find the strain and angle of rotation for bar AB, given bar AB was vertical and 2m long in the original structure.



From measuring ext_{AB} and rot_{AB} , we find that $ext_{AB} = 1\text{mm}$, $rot_{AB} = 11.6\text{mm}$

$$E = \frac{ext_{AB}}{L} = \frac{1}{2000} = 0.05\% \quad // \quad \Delta\phi = rot_{AB} \rightarrow \phi = \frac{rot_{AB}}{L} = \frac{11.6}{2000} = 5.8 \times 10^{-3} \text{ rad.}$$

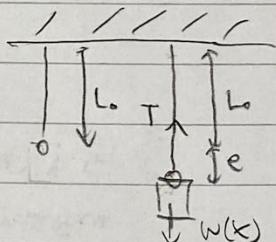
Real work method

- Consider a mass hung on a wire. As this is a closed system, the total energy is conserved - GPE lost in the weight is converted to EPE in the wire and KE of the weight.
- To convert this into a static problem, instead of applying the load all at once (SFM), we apply the load gradually s.t. the load applied increases from 0 to W so $W(x) = \frac{x}{L}W$. (assume this is slow enough s.t. KE of the weight is 0 and the wire is in eqm).
- Applying the principle of consv. of energy,

$$\text{Ext. work done to the wire} = \text{Int. energy stored}$$

$$\int_0^L w(x)dx = \int_0^L kx dx$$

$$\frac{1}{2}We = \frac{1}{2}ke^2$$



When the load has been fully applied, $T = ke$, therefore

$$\frac{1}{2}We = \frac{1}{2}Te.$$

- Generalising this for a pin-jointed planar truss, to include a no. of externally applied forces F_i , each of which is applied to some node that displaces by d_i . Meanwhile each straight member in the truss experiences an axial force T_j leading to extension e_j .

$$\Sigma \text{ext. work applied to truss} = \Sigma \text{changes in int. stored energy}$$

$$\frac{1}{2} \sum_i F_i \cdot d_i = \frac{1}{2} \sum_j T_j e_j$$

ext. forces F_i in eqm. with int. bar tensions T_j

$$\sum_i F_i \cdot d_i = \sum_j T_j e_j$$

nodal deflections d_i compatible with bar extensions e_j

* Reaction forces act in directions where displacement is constrained $\rightarrow WD = 0$, T_j and e_j always have the same sign.

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Derivation of virtual work method for pin-jointed frames.

- Now if we remove the original set of ext. forces E_i and apply a different set of ext. forces E_i^* . Using the eqn. for real work, we now have

$$\sum E_i \cdot d_i = \sum T_j e_j \quad \text{and} \quad \sum E_i^* d_i^* = \sum T_j^* e_j^*$$

- Then we return to the original set of ext. forces E_i , however, this time we will add the new set of ext. forces E_i^* while the original set of ext. forces E_i are in place. (For $E_i^* \ll E_i$, deflections are small \rightarrow system is locally linear \rightarrow principle of superposition applies).

- When adding the new set of ext. forces E_i^* , both sets of ext. forces do additional ext. work. add therefore additional int. energy is stored.

$$\sum (E_i + E_i^*) \cdot d_i^* = \sum (T_j + T_j^*) e_j^*$$

$$\sum E_i \cdot d_i + \sum E_i^* \cdot d_i^* = \sum T_j e_j + \sum T_j^* e_j^*$$

ext. forces E_i in eqn. with
int. bar tensions T_j

$$\boxed{\sum E_i^* \cdot d_i^* = \sum T_j^* e_j^*}$$

nodal deflections d_i^* compatible
with bar extensions e_j^*

* The additional ext. work is not $\sum (E_i + E_i^*) (d_i + d_i^*) - \sum E_i \cdot d_i$ because we are not applying both sets of ext. forces all at one instant $\rightarrow E_i^*$ only does work during displacement d_i^* (but not d_i) .

- If we applied the new set of ext. forces E_i^* first, then superimpose the original set of ext. forces E_i , using the same logic, we would get

ext. forces E_i^* in eqn. with
int. bar tensions T_j^*

$$\boxed{\sum E_i^* \cdot d_i = \sum T_j^* e_j}$$

nodal deflections d_i compatible
with bar extensions e_j

Using virtual work to find displacements,

- If we want to find nodal displacements, then these must be real (along w/ bar extensions), so the forces and bar tensions must be virtual

$$\text{Virtual } \boxed{\sum E_i^* \cdot d_i = \sum T_j^* e_j} \quad \text{Real}$$

- To find a nodal displacement, we add a unit virtual force $E_i^* = 1$ at the node of interest in the direction of interest \rightarrow it becomes $\sum E_i^* \cdot d_i = E_i^* \cdot d = d$

- From the virtual applied force E_i^* , we can find the virtual reaction forces R_i^* and thus the virtual bar-tensions T_j^* .

- From the real applied forces E_i , we can find the real reaction forces R_i and thus the real bar-tensions T_j , from which we can find the real bar extensions e_j .

- Summing all the products of virtual bar-tensions T_j^* and real bar extensions e_j ,

$$d = \sum T_j^* e_j$$

* Don't bother finding T^* for bars w/ 0-extension ($e=0$),

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Using virtual work to find forces or bar tensions.

- If we want to find forces or bar tensions, then these must be real, so the nodal displacements and bar extensions must be virtual

$$\text{Real } \sum_i F_i \cdot d_i^* = \sum_j T_j e_j^* \quad \text{Virtual}$$

- To find a force, we apply a virtual rigid-body rotation about a node (s.t. the forces acting on the other nodes are known / the req. one) by a small angle ϕ . \rightarrow The virtual extensions are 0 ($e^* = 0$) \rightarrow RHS becomes 0.
- The virtual displacement of node X , d_i^* when rotated about node A by a small angle ϕ has a magnitude $|AX|\phi$ and is in the direction \perp to AX .
- Summing all the dot products of the virtual displacements d_i^* and the real forces F_i and equate this to 0 to solve for the req. force.

$$\sum_i F_i \cdot d_i^* = 0,$$

- To find a bar tension, we extend the member of interest by a unit virtual extension $\epsilon^* = 1$. \rightarrow RHS becomes $\sum_j T_j e_j^* = T \epsilon^* = T$
- This virtual extension ϵ^* results in a rigid-body rotation in part of the structure about a particular node by a small angle ϕ . Using geometry, we can find the virtual displacements d_i^* at the nodes that have a real applied force F_i .
- Summing all the dot products of the virtual displacements d_i^* and the real forces F_i to get

$$T = \sum_i F_i \cdot d_i^*.$$

real work and virtual work

- In general, for both real work and virtual work,

$$\text{Set I } \sum_i F_i \cdot d_i = \sum_j T_j e_j \quad \text{Set II}$$

where both sets are real for real work

and one set is real & one set is virtual for virtual work

- We can now determine 4 features for a truss' state:

Ext. forces	F_i	N2L + moments
	F_i	rigid-body rotation abt. a node by $\phi \rightarrow e^* = 0$ so $\sum_i F_i \cdot d_i^* = 0$
Bar tensions	T_j	$F_i \rightarrow R_i \rightarrow T_j$
	T_j	partial rigid-body rotation s.t. $\epsilon^* = 1$ and find d_i^* by geometry, $T = \sum_i F_i \cdot d_i^*$
Nodal displacement	d_i	if load in some dir. as req. $d_i \rightarrow$ real work, $F_d = \sum_j T_j e_j$
	d_i	apply F_i^* of req. node in req. dir. find $(F_i \rightarrow R_i \rightarrow T_j \rightarrow e_j)$; find T_j^* ($F_i^* \rightarrow R_i^* \rightarrow T_j^* \rightarrow e_j^*$) $\rightarrow d = \sum_j T_j^* e_j$
Bar extensions	e_j	$F_i \rightarrow R_i \rightarrow T_j \rightarrow e_j$

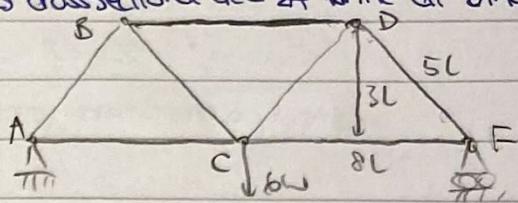
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-eg: find the vertical displacement of node C and the horizontal displacement of node D.

All bars have Young's modulus E, bar BD has cross-sectional area 2A while all other bars have cross-sectional area A.

We want real displacements:

$$\sum F_i \cdot d_i = \sum T_j e_j \quad \text{or} \quad \sum F_i^* \cdot d_i = \sum T_j^* e_j$$



In both cases, we need to find the real bar extensions.

Bar	T	Length	Area	$e \left(\times \frac{WL}{AE} \right)$	$T_e \left(\times \frac{h^2}{AE} \right)$	
AB	-5W	5L	A	-25	125	always true.
DF	-5W	5L	A	-25	125	
BC	+5W	5L	A	+25	125	
CD	+5W	5L	A	+25	125	
AC	+4W	4L	A	+32	128	
CF	+4W	4L	A	+32	128	
BD	-8W	8L	2A	-32	256	

$$\sum 1012.$$

Vertical load + vertical displacement @ node C \rightarrow real work.

$$\sum F_i \cdot d_i = \sum T_j e_j$$

$$Fd = 1012 \frac{WL}{AE}$$

$$6Wd = 1012 \frac{WL}{AE} \rightarrow d = \frac{170WL}{AE}.$$

No horizontal load + horizontal displacement @ node D \rightarrow virtual work δF^* .

Apply a unit virtual horizontal force $F^* = 1$ @ node D.

After finding R_A^* and R_F^* ,

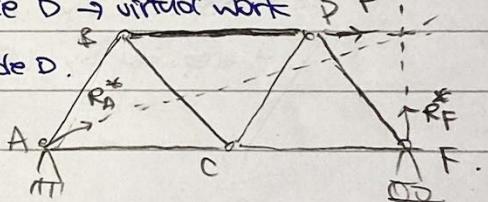
We find the virtual bar tensions.

Bar	T^*	$e \left(\times \frac{WL}{AE} \right)$	$T^*e \left(\times \frac{WL}{AE} \right)$
AB	$+\frac{5}{16}$	-25	-125
DF	$-\frac{5}{16}$	-25	+125
BC	$-\frac{5}{16}$	+25	-125
CD	$+\frac{5}{16}$	+25	+125
AC	$+\frac{12}{16}$	+32	+384
CF	$+\frac{4}{16}$	+32	+128
BD	$+\frac{8}{16}$	-32	-256

$$\sum +256.$$

$$\sum F_i^* d_i = \sum T_j^* e_j$$

$$F^* d = 256 \frac{WL}{AE} \rightarrow d = \frac{16WL}{AE}$$



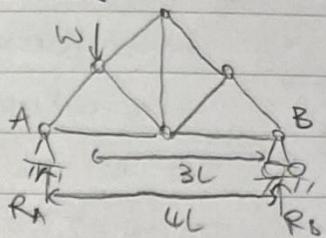
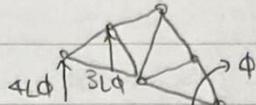
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-ej: Find the reaction force at support A, using virtual work.

We want real forces:

$$\sum \underline{F}_i \cdot \underline{d}_i^* = \sum T_j e_j^*$$

Apply a virtual rigid-body rotation about B by ϕ .



As this is a pure rotation, $e_i^* = 0 \rightarrow \sum T_j e_j^* = 0$.

$$\sum \underline{F}_i \cdot \underline{d}_i^* = 0$$

$$R_A \cdot 4L\phi - W \cdot 3L\phi = 0$$

$$R_A = \frac{3}{4}W$$

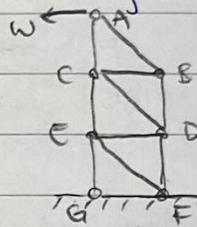
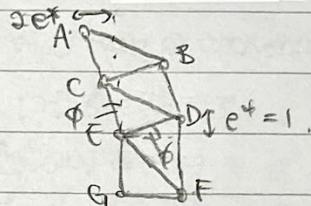
- sign for $-W \cdot 3L\phi$ term because W acts downwards (opp. $3L\phi$).

-ej: Find the axial force in member DF caused by the load W using virtual work.

We want real axial forces:

$$\sum \underline{F}_i \cdot \underline{d}_i^* = \sum T_j e_j^*$$

Apply a unit virtual extension e_j^* in member DF.



The upper part of the structure rotates by a small angle ϕ anticlockwise. (about E).

$$\rightarrow D \text{ displaces by } |AE|\phi = e^* = 1 \rightarrow \phi = \frac{1}{L}$$

$$\therefore A \text{ displaces by } |AE|\phi = 2L \cdot \frac{1}{L} = 2.$$

$$\sum \underline{F}_i \cdot \underline{d}_i^* = \sum T_j e_j^*$$

$$W \cdot 2 = T \cdot 1$$

$$T = 2W.$$

Displacement diagrams vs real/virtual work.

- Displacement diagrams are easier to check and usually quicker if the displacements of many nodes are req, however, displacement diagrams are complex in some cases.

- When seeking the displacement of a single joint, real/virtual work is often easier, given the geometry is simple.

- Regardless of method, look for opportunities to exploit symmetry and simplifications where bar forces or nodal displacements are zero or known.

* Remember to state the direction of any displacements.

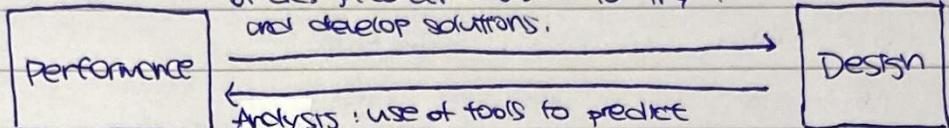
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Structural design

Iterative design

- structural design involves synthesis and analysis.

Synthesis: evolving understanding of clients needs and accumulated evolution of design concepts used to try and develop solutions.



Analysis: use of tools to predict how structures respond to specific loads (i.e. find the deflections).

- design is fundamentally iterative - partially specified set of req. → early ideas
→ greater clarity of requirements → reduce no. of design options.
- As the design process progresses, the designer can focus more on the details than concepts.
(use paper-calculations for concepts; computer calculations for details).

Structural optimization

- optimize cross-sectional area to minimize deflection

- Using the method of virtual work, we find the deflection of a given node is given by.

$$d = \sum T_j f_{ej} = \sum \left(T_j \frac{I_j L_j}{A_j E} \right)$$

- If we want to reduce the deflection under the same loading by adding some mass (volume) to one of the bars, we can increase the cross-sectional area. (change length → expensive, assume same material → same Young's modulus and density).

- Expressing the deflection in terms of the volume of each bar and taking the derivative wrt volume for each bar j in turn,

$$d = \sum T_j \frac{I_j L_j}{A_j E} = \sum T_j \frac{I_j L_j^2}{V_j E}$$

$$[V_j = A_j L_j]$$

$$\frac{\partial d}{\partial V_j} = - \frac{T_j^2 I_j L_j^2}{V_j^2 E} = - \frac{T_j^2 T_j}{A_j^2 E}$$

$$[V_j = A_j L_j]$$

- If $d > 0$, we want the most $\frac{\partial d}{\partial V_j}$; If $d < 0$, we want the most $\frac{\partial d}{\partial V_j}$, in order to have d closer to 0 (i.e. minimize $|d|$).

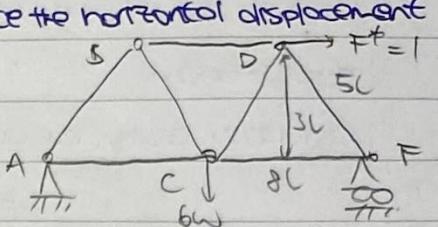
- After adding volume (increase cross-sectional area) to the member, the partial derivative for that member can be updated and we can add volume to the next priority member. We repeat this until all the partial derivatives are equal.

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-e.g.: Which bar should we add mass to to reduce the horizontal displacement of node D most effectively.

Using virtual work method, we find that

$$d = \frac{16WL}{AE}$$



As $d > 0$, we want to choose the bar w/ the most -ve $\frac{\partial d}{\partial T}$ to minimize $|d|$.

Bar	Area	T	T^*	$\frac{\partial d}{\partial T} (\times \frac{W}{16AE})$
AB	A	-5W	$+\frac{5}{16}$	+25
DF	A	-5W	$-\frac{5}{16}$	-25
BC	A	+5W	$-\frac{5}{16}$	+25
CD	A	+5W	$+\frac{5}{16}$	-25
AC	A	+4W	$+\frac{12}{16}$	-48 most -ve.
CF	A	+4W	$+\frac{4}{16}$	-16
BD	2A	-8W	$+\frac{8}{16}$	+16.

→ Adding mass to bar AC will be the most effective in reducing horizontal displacement of D.

check the analysis by doubling the cross-sectional area of bar AC:

Bar	Area	$e(\times \frac{WL}{AE})$	T^*	$T^* e (\times \frac{WL}{16AE})$
AB	A	-25	$+\frac{5}{16}$	-125
DF	A	-25	$-\frac{5}{16}$	+125
BC	A	+25	$-\frac{5}{16}$	-125
CD	A	+25	$+\frac{5}{16}$	+125
AC	2A	($\frac{1}{2}$) ($\frac{1}{2}2$) +16	$+\frac{12}{16}$	($\frac{1}{2}24$) +192
CF	A	+32	$+\frac{4}{16}$	+128
BD	2A	-32	$+\frac{8}{16}$	-256 <small>(+256)</small> $\Sigma +64$.

$$\therefore d = \frac{64WL}{16AE} = \frac{4WL}{AE}.$$

→ The horizontal deflection of node D decreased from $\frac{16WL}{AE}$ to $\frac{4WL}{AE}$.

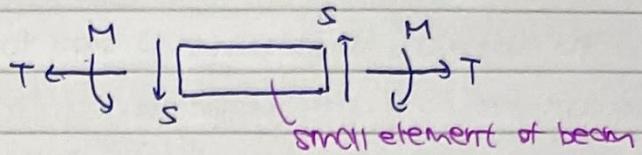
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Equilibrium of elastic beams

conventions and assumptions

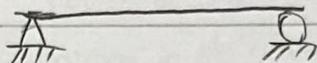
sign conventions.

- The positive sense of M, S, and T are defined as follows:



Drawing conventions of supports.

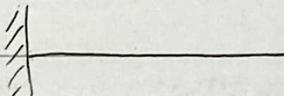
- ① simply supported beam.



- The end supports can:

- ↳ prevent vertical movement // provide vertical forces
- ↳ allow rotation // not transmit moments
- ↳ The intrasular support prevents horizontal movement // provide horizontal forces

- ② cantilever beam



- The end support can:

- ↳ prevent vertical movement // provide vertical forces
- ↳ prevent rotation // transmit moments
- ↳ prevent horizontal movement // provide horizontal forces

Assumptions

- the bending moments are too small to break or permanently bend the beam.
- Only 2D situations are considered.
- The structure does not deflect significantly under load so we can consider eqm. in the original configuration.

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Shear force and bending moment diagrams

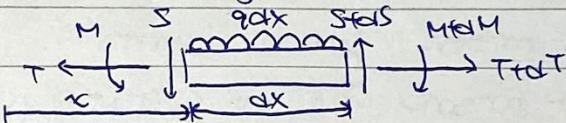
Calculation of M, S and T by analysis of free bodies

- First, find only support reactions, then make a cut at a variable distance x to find expressions for axial force $T(x)$, shear force $S(x)$ and bending moment $M(x)$.
- (We don't need to find the support reactions if we start from a free end, say in a cantilever w/ known load).
- Here are several facts useful for checking shear force / bending moment diagrams:
 - ↳ Under a conc. force, S is discontinuous. There is a jump equal to the conc. force at the pt. the conc. force is applied.
 - ↳ Under a conc. force, M is continuous but its slope is discontinuous.
 - ↳ Given an applied couple, M is discontinuous. There is a jump equal to the couple at the pt. the couple is applied.
 - ↳ When S is zero, M is at a max./min.
 - ↳ M is zero at the simply supported ends / free end.
- * If we start from the RFS, $\uparrow F$ results in $\uparrow S$; $\curvearrowright C$ results in $\uparrow M$.

If we draw a FBD, there are no ext. forces \rightarrow no int. forces needed to balance
 $\therefore T=0, S=0, M=0$.

Differential relationships between q , S and M .

- Consider the eqn. of a small element of beam w/ infinitesimal length under a distributed force of intensity q , acting downwards.



* We implicitly assume that the beam is straight.

- NCL \hookrightarrow : $T+dT=T \rightarrow dT=0$, so $\frac{dT}{dx}=0$
- $\int : S+qdx = S+qdx \rightarrow dS=qdx$ so $\frac{dS}{dx}=q$.

$\curvearrowleft M_{\text{centre}} : M+dM-M = S \cdot \frac{dx}{2} + (S+dS) \frac{dx}{2} \rightarrow dM = Sdx + \frac{1}{2}qdx^2$ so $\frac{dM}{dx} = S$.

$\therefore \text{using this sign convention, } q = \frac{dS}{dx}, S = \frac{dM}{dx}$

- When integrating to find S/M , we need a boundary condition to find the constant of integration. \rightarrow Find BC via inspection ($M=0$ or S at end supports) or via FBD calc.
- This method does not work well for conc. loads or distributed loads w/ abrupt change of intensity. \rightarrow We would need to have different DE for each section, w/ BC used to ensure continuity between sections.

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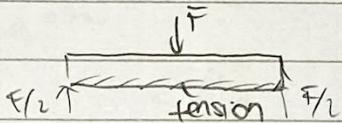
shortcut for beams under point forces.

- When a straight beam is loaded exclusively by pt. forces and reactions, then between only 2 consecutive pts. of application of force,
 - ↳ 1) The shearing force is constant
 - ↳ 2) The bending moment varies linearly.
- As the beam is unloaded between pts. of application of force, $q=0$.
so $S = \int q dx = A$ and $M = \int S dx = Ax + B$, where A, B are constants.
- To find the bending moment diagram for beams under pt. loads quickly,
 - ↳ 1) calculate M at all pts. of application of force
 - ↳ 2) Joint up consecutive pts. on the plot $\rightarrow M(x)$ is piecewise linear.

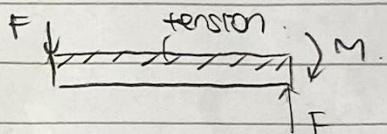
Bending moment diagrams for simple frames and floating structures.

- Find / write down T, S, M for each beam and calculate M of all pts. of application of force, then join the consecutive pts. (as above).
- By convention, we draw the bending moment diagram on the "tension side" of the member,

↳ force applied in between



↳ force applied at ends



recognising statically determinate structures

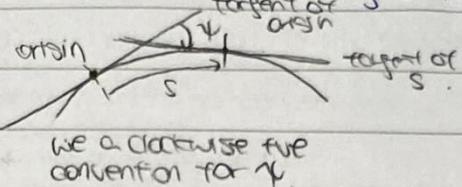
- To find the internal forces T, S, M , we need to find the reaction forces via eqm., which is only possible if the structure is statically determinate.
- For statically indeterminate structures, we must invoke compatibility (how the structure deforms) to find the unknown reaction force.
- In general, any "free structure", i.e. any series of cantilever beams connected to one another which does not form any closed loops is statically determinate.

Deflection of straight elastic beams

curvature, rotation and deflection

curvature and change of curvature

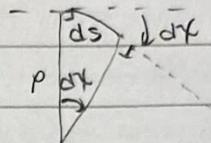
- The curvature of a plane curve is the rate of turning of the tangent to the curve



- Using the intrinsic coordinate system (s, τ) as above, the curvature K at a pt. of the curve is

$$K = \frac{ds}{d\tau} = \frac{1}{r}$$

where r is the radius of curvature.



$$\text{so } r \frac{ds}{dr} = \frac{ds}{d\tau} \quad \frac{ds}{dr} = \frac{1}{r}$$

curvature, rotation and deflection.

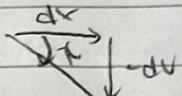
valid for $|r| = |\frac{dv}{dx}| < 0$

- For small rotations τ , we can approximate $x \approx s$, so curvature K is given by

$$K \approx \frac{dr}{dx}$$

- The deflection v can be related to the rotation via

$$\tau = -\frac{dv}{dx}$$

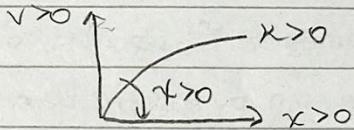


- Combining the 2 eqns. above, we get

$$K \approx \frac{dr}{dx} = \frac{1}{x} \left(-\frac{dv}{dx} \right)$$

$$K \approx -\frac{d^2v}{dx^2}$$

The eqns. above use the following sign convention:



- Given a curvature function $K(x)$, we can integrate and apply the boundary conditions on slope and/or deflections to fix the constants of integration \rightarrow find $v(x)$ or $r(x)$. (usually 2 simple supports $\rightarrow v=0$ at ends; cantilever $\rightarrow v=t=0$ at end; symmetry $\rightarrow \tau=0$ at centre)
- For situations that have the same curvature function but different boundary conditions, they are related via a rigid body rotation.

Distortion produced by internal forces

Distortion produced by internal forces.

- consider a short element of beam subjected to internal forces T, S, M .

\hookrightarrow Tension T causes extension of the element [ignore]

valid for slender beams

\hookrightarrow Shearing force S causes distortion of the element [ignore]

as distortions due to S is negligible compared to distortions due to M .

\hookrightarrow Bending moment M causes change of curvature of the element.

- Ext forces set up int-forces \rightarrow produce distortion in beam elements \rightarrow deflection of structure

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Deflection of elastic beams

linear-elastic beam

- The resultant curvature of a beam is directly proportional to the applied bending moment.

$$M = BK \quad [\text{Elastic moment/curvature relationship}]$$

where B is the bending stiffness / flexural stiffness / flexural rigidity of the beam.

- Note the eqn. is only true for an initially straight beam. When $M=0$. More generally,

$$M = B \Delta K$$

where ΔK is the elastic change of curvature of a beam w/ initial curvature K_0 .

- Unless stated otherwise, we assume the beams are

↳ Linear-elastic and uniform, w/ bending stiffness B

↳ Have zero self-weight

↳ Initially straight when $M=0$.

- Moreover, we assume the beam deflections and rotations are small so $K = -\frac{d^2v}{dx^2}$

Calculation of deflections by integration

- To find deflections, the method is as follows:

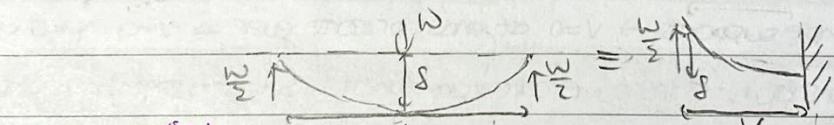
↳ Given the loading, find $M(x)$ by eqn. (Graphical or algebraic)

↳ Use the elastic moment/curvature relationship $M=BK$ to find $K(x)$ - ignore effects of T.S.

↳ Use $K = -\frac{d^2v}{dx^2}$ and apply boundary conditions to find $v(x)$.

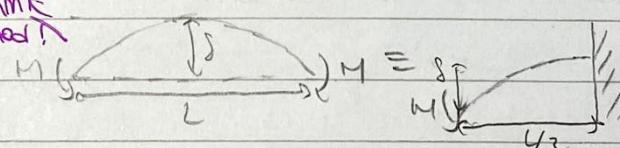
- In cases where $\frac{d^2v}{dx^2}$ (i.e. $M(x)$) is discontinuous, we cannot integrate a single, standard function of x → integrate $\frac{d^2v}{dx^2}$ separately over intervals where it is continuous and match the separate sol'n by suitable boundary conditions.

- We can replace each half of a symmetrically loaded beam supported at 2 ends w/ a concrete pier:



see this applied moment configuration, think of cantilever method!

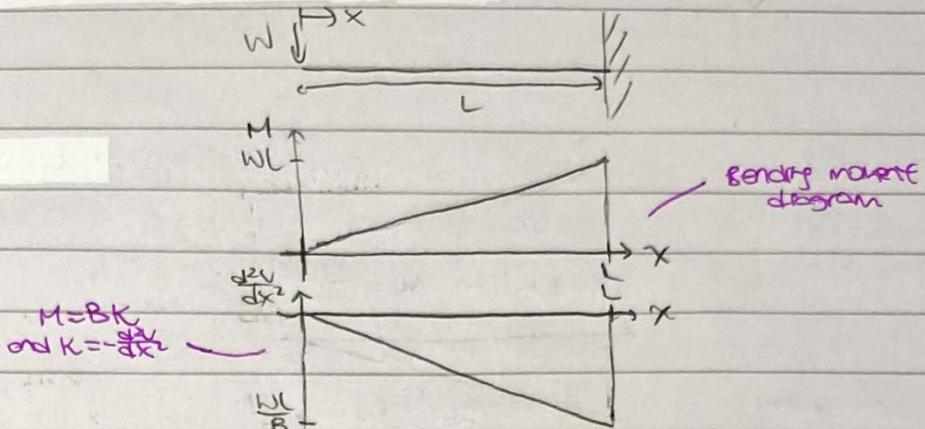
Symmetric loading
→ zero rotation at the centre (end of cantilever support)



→ This method doesn't work for distributed loads.

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Deflection of a cantilever loaded w/ pt. load.



From the graph, we see $\frac{d^2v}{dx^2} = -\frac{WL}{BL}x = -\frac{W}{B}x$

$$\int \frac{dv}{dx} = -\frac{W}{2B}x^2 + C_1$$

$$v = -\frac{W}{6B}x^3 + C_1x + C_2$$

$$v(L) = -v'(0) = 0 : \quad C_1 = \frac{WL^2}{2B}$$

$$v(L) = 0 : \quad 0 = -\frac{WL^3}{6B} + \frac{WL^3}{2B} + C_2 \rightarrow C_2 = -\frac{WL^3}{3B}$$

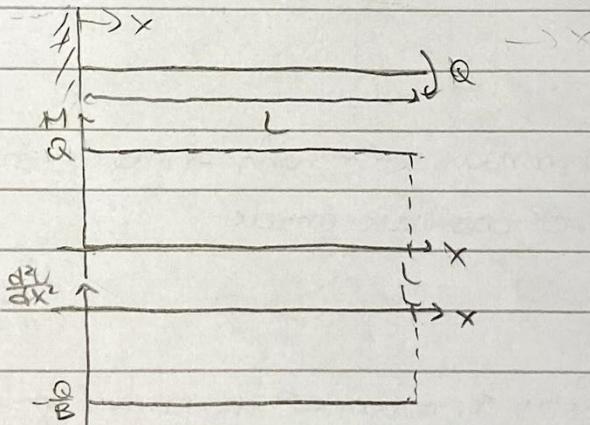
$$\therefore v = \frac{WL^3}{B} \left[-\frac{1}{6} \left(\frac{x}{L} \right)^3 + \frac{1}{2} \left(\frac{x}{L} \right) - \frac{1}{3} \right]$$

$$\frac{dv}{dx} = \frac{WL^2}{B} \left[-\frac{1}{2} \left(\frac{x}{L} \right)^2 + \frac{1}{2} \right]$$

end deflection: $v(0) = -\frac{WL^3}{3B}$

end rotation: $-v'(0) = -\frac{WL^2}{2B}$

Deflection of a cantilever loaded w/ couple



From the graph, we see $\frac{dv}{dx^2} = -\frac{Q}{B}$.

$$\int \frac{dv}{dx} = -\frac{Q}{B}x + C_1$$

$$v = -\frac{Q}{2B}x^2 + C_1x + C_2$$

$$v(0) = -v'(0) = 0 : \quad C_1 = 0$$

$$v(0) = 0 : \quad C_2 = 0$$

$$\therefore v = -\frac{Q}{2B}x^2$$

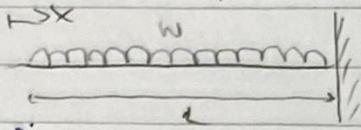
$$\frac{dv}{dx} = -\frac{Q}{B}x$$

end deflection: $v(L) = -\frac{QL^2}{2B}$

end rotation: $-v'(L) = -\frac{QL}{B}$

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Deflection of a cantilever under uniformly distributed load



$$q = \frac{dS}{dx} \rightarrow S = \int q dx = wx + C_1 ; S(0) = 0 \rightarrow C_1 = 0 \therefore S = wx$$

$$S = \frac{dM}{dx} \rightarrow M = \int S dx = \frac{1}{2}wx^2 + C_2 ; M(0) = 0 \rightarrow C_2 = 0 \therefore M = \frac{1}{2}wx^2$$

$$M = BK \rightarrow K = \frac{w}{2B}x^2$$

$$K = -\frac{dU}{dx} \rightarrow \frac{d^2U}{dx^2} = -\frac{w}{2B}x^2$$

Integrating,

$$\frac{dU}{dx} = -\frac{w}{6B}x^3 + C_3$$

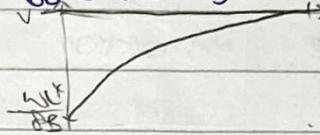
$$U = -\frac{w}{24B}x^4 + C_3x + C_4$$

$$U(L) = -V'(L) = 0 : \quad C_3 = \frac{wL^3}{6B}$$

$$V(L) = 0 : \quad 0 = -\frac{wL^4}{24B} + \frac{wL^4}{6B} + C_4 \rightarrow C_4 = -\frac{wL^4}{8B}$$

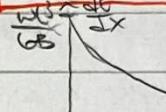
$$\therefore V = \frac{wL^4}{8} \left[-\frac{1}{24} \left(\frac{x}{L}\right)^4 + \frac{1}{6} \left(\frac{x}{L}\right) - \frac{1}{8} \right]$$

$$\frac{dU}{dx} = \frac{wL^3}{6B} \left[-\left(\frac{x}{L}\right)^3 + 1 \right]$$

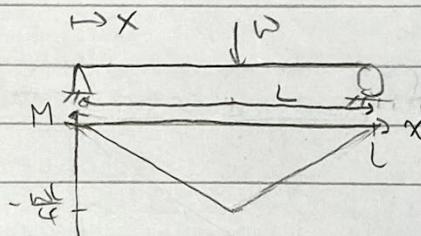


$$\boxed{\text{end deflection: } V(0) = -\frac{wL^4}{24B}}$$

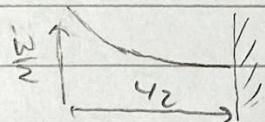
$$\boxed{\text{end rotation: } \gamma(0) = -V'(0) = \frac{wL^3}{6B}}$$



Deflection of a simply supported beam loaded w/ pt. load at midspan



$M(x)$ is discontinuous at midspan \rightarrow integrate each half separately or notice that each half acts like a cantilever.



We only consider the reaction forces to find the equivalent cantilever

Using the results for a cantilever loaded w/ a pt. load at mid, ($W = \frac{w}{2}$, $L = \frac{\Sigma}{2}$)

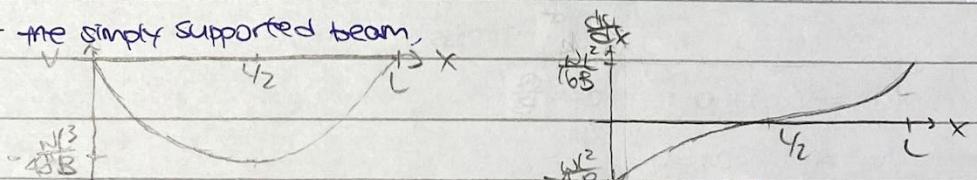
$$V = \frac{wL^3}{16B} \left[\frac{4}{3} \left(\frac{x}{L}\right)^3 - \left(\frac{x}{L}\right) + \frac{1}{3} \right] \quad \parallel \text{for } x \in [0, \frac{\Sigma}{2}]$$

$$\frac{dU}{dx} = \frac{wL^2}{8B} \left[2 \left(\frac{x}{L}\right)^2 - \frac{1}{2} \right]$$

$$\text{end deflection: } V(0) = \frac{wL^3}{48B}$$

$$\text{end rotation: } -V'(0) = \frac{wL^2}{16B}$$

\therefore For the simply supported beam,

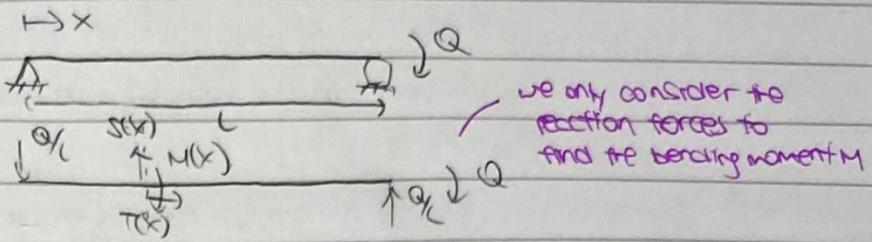


$$\boxed{\text{central deflection: } V\left(\frac{\Sigma}{2}\right) = -\frac{wL^3}{48B}}$$

$$\boxed{\text{end rotations: } -V'(0) = \frac{wL^2}{16B}; -V'(\Sigma) = \frac{wL^2}{16B}}$$

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Deflection of simply supported beam loaded w/ a couple.



consider the left part of the beam at the cut x (variable distance x from origin).

$$M(x) = \frac{Q}{L}x$$

$$M = BK \rightarrow K = \frac{Q}{BL}x$$

$$K = -\frac{d^2v}{dx^2} \rightarrow \frac{d^2v}{dx^2} = -\frac{Q}{BL}x$$

Integrating,

$$\frac{dv}{dx} = -\frac{Q}{2BL}x^2 + C_1$$

$$v = -\frac{Q}{6BL}x^3 + C_1x + C_2$$

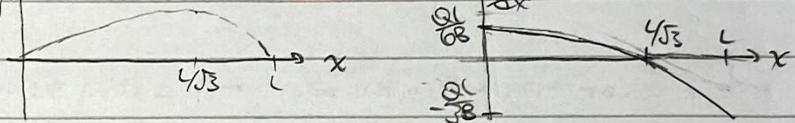
$$v(0) = 0 \rightarrow C_2 = 0$$

$$v(L) = 0 \rightarrow 0 = -\frac{QL^2}{6B} + C_1L \rightarrow C_1 = \frac{QL}{6B}$$

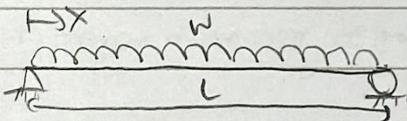
$$\therefore v = \frac{QL^2}{6B} \left[-\left(\frac{x}{L}\right)^3 + \left(\frac{x}{L}\right) \right]$$

$$\frac{dv}{dx} = \frac{QL}{2B} \left[-\left(\frac{x}{L}\right)^2 + \frac{1}{3} \right]$$

end rotations: $v(0) = -\frac{QL}{6B}$, $v'(L) = \frac{QL}{3B}$



Deflection of a simply supported beam under uniformly distributed load.



$$q = \frac{dw}{dx} \rightarrow s = \int q dx = wx + C_1 ; s(0) = -\frac{wL}{2} \rightarrow C_1 = -\frac{wL}{2} \therefore s = wx - \frac{wL}{2}$$

$$s = \frac{dw}{dx} \rightarrow M = \int s dx = \frac{1}{2}wx^2 - \frac{wL}{2}x + C_2 ; M(0) = 0 \rightarrow C_2 = 0 \therefore M = \frac{1}{2}wx^2 - \frac{wL}{2}x$$

$$M = BK \rightarrow K = \frac{w}{2B}x^2 - \frac{wL}{2B}x$$

$$K = -\frac{d^2v}{dx^2} \rightarrow \frac{d^2v}{dx^2} = -\frac{w}{2B}x^2 + \frac{wL}{2B}x$$

Integrating,

$$\frac{dv}{dx} = -\frac{w}{6B}x^3 + \frac{wL}{4B}x^2 + C_3$$

$$v = -\frac{w}{24B}x^4 + \frac{wL}{12B}x^3 + C_3x + C_4$$

$$v(0) = 0 \rightarrow C_4 = 0$$

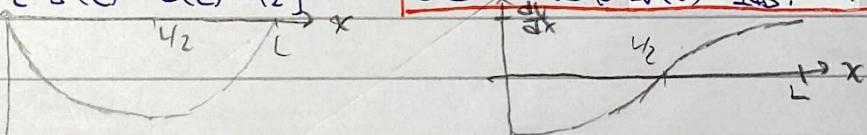
$$v(L) = 0 \rightarrow 0 = -\frac{wL^4}{24B} + \frac{wL^3}{12B} + C_3L \rightarrow C_3 = -\frac{wL^3}{24}$$

$$\therefore v = \frac{wL^3}{12B} \left[-\frac{1}{2}\left(\frac{x}{L}\right)^4 + \left(\frac{x}{L}\right)^3 + \frac{1}{2}\left(\frac{x}{L}\right) \right]$$

$$\frac{dv}{dx} = \frac{wL^3}{2B} \left[-\frac{1}{3}\left(\frac{x}{L}\right)^3 + \frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{1}{12} \right]$$

central deflection: $v(\frac{L}{2}) = -\frac{5wL^4}{96B}$

end rotations: $v(0) = \frac{wL^3}{24B}$, $v'(L) = -\frac{wL^3}{24B}$



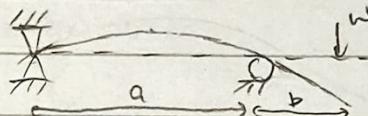
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calculation of deflections by superposition

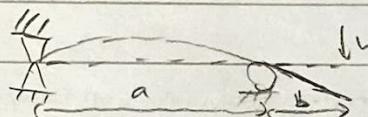
- As long as the deflections of the structure are small, the principle of superposition applies.
- We can find the deflection of a beam by superposing a series of previously known standard solutions.
- In superposing these results, we need to ensure that (i) the curvature in the solns that are superposed matches the curvature of the beam, and (ii) the boundary conditions are met, or any mismatches are corrected for.
- In general, we find the deflection, assuming a part of the beam (between 2 supports or overhang) has been put in a rigid sleeve. Then we repeat by assuming other parts of the beam are put in a rigid sleeve.
- The sum of these deflections give the total deflection (what we req.)

e.g.: overhanging beam (Find tip deflection)

Consider the overhanging beam as shown below: find the tip deflection.

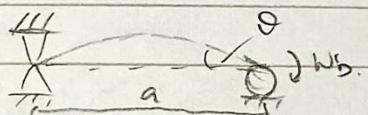


First, we assume the overhang has been put in a rigid sleeve.



W at distance b transmits moment Wb .
 Q at any distance transmits moment Q .

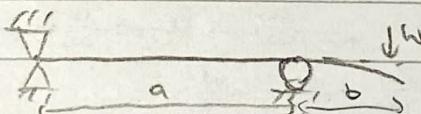
Therefore the overhang can transmit a moment Wb to the end of the simply-supported beam.



Wb/a at overhang $\rightarrow \theta$ in beam
 Wb^2/a of beam \rightarrow nothing in overhang.

$$\text{From the textbook, } \theta = \frac{Wb \cdot a}{3B} \rightarrow V_1 = b\theta = \frac{Wb^2 a}{3B}$$

Second, we assume the simply-supported beam has been put in a rigid sleeve.



part where F/Q is not applied directly find deflection.

$$\text{From the textbook, } V_2 = \frac{Wb^3}{3B}.$$

$$\therefore \text{Total deflection: } V = V_1 + V_2 = \frac{Wb^2}{3B} (atb)$$

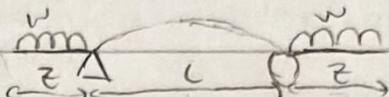
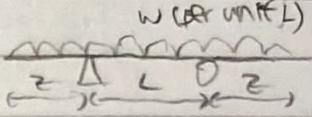
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e.g. beam w/ unknown overhang length (Find central deflection)

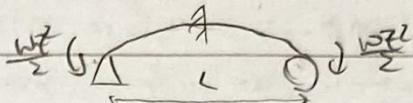
The applied loading is uniform along the full length of the beam. However, the length of overhang, z , has not yet been decided upon.

What z is req. for the central deflection to be zero?

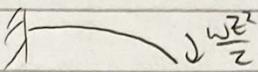
First, we assume the overhang has been put in a rigid sleeve.



Therefore the overhang can transmit a moment $\frac{wz^2}{2}$ to the end of the simply supported beam.



By symmetry, the centre of the beam acts like a conference support.



$$\text{From the databook } V_1 = -\left(\frac{wz^2}{2}\right)\left(\frac{1}{2}\right) = -\frac{wz^2}{16B}$$

Second, consider the deflection due to the distributed load in the simply supported beam



$$\text{From the databook } V_2 = \frac{5wL^4}{384B}$$

$$\therefore \text{Total deflection } V = V_1 + V_2 = -\frac{wz^2L^2}{16B} + \frac{5wL^4}{384B} = 0$$

$$z = \sqrt{\frac{5}{34}}L$$

strategy for using superposition.

- To find the deflection of the req. pt, consider the deflection of that section of the beam due to applied loads/moment from each section of the whole beam.
- For tip deflections, total deflection = deflection due to load/moment in overhang + rotation in support (overhang + simply supported beam) \times length of overhang.
- For central deflections, total deflection = deflection due to load in simply supported beam + deflection due to moment (overhang) in simply supported beam

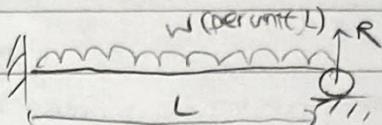
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Finding the reaction forces of statically indeterminate structures.

- we superpose the deflections due to different forces (including unknown reaction forces) and set them to meet the boundary condition.

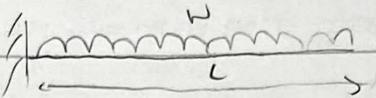
e.g. propped cantilever.

Consider a propped cantilever as shown below.



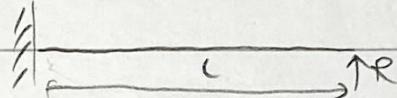
Find the tip reaction R s.t. there is no deflection.

Consider the deflection due to the distributed load alone



$$\text{From the textbook, } v_1 = \frac{wL^4}{8B}$$

Consider the deflection due to the tip reaction alone



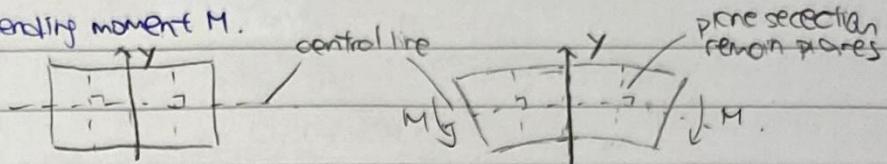
$$\text{From the textbook, } v_2 = -\frac{RL^3}{3B}$$

$$\therefore \text{Total deflection } v = v_1 + v_2 = \frac{wL^4}{8B} - \frac{RL^3}{3B} = 0$$

$$R = \frac{3wL}{8}$$

Bending of beamsGeometric relationships

- Consider a side view of the beam that is initially straight, but curved due to the application of a bending moment M .



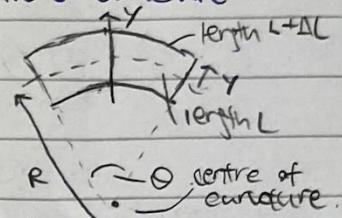
We make 2 assumptions:

- The central line doesn't change in length. This line traces the position of the neutral axis.
- Plane sections remain planes (Euler-Bernoulli hypothesis).
- consider a beam element originally of length L . After bending, the top/bottom edges change in length by $\pm \Delta L$ and the beam element subtends an angle θ at the centre of curvature.

$$L = \theta R \quad ; \quad L + \Delta L = \theta(R + y)$$

$$\epsilon = \frac{\Delta L}{L} = \frac{\theta(R+y) - \theta R}{\theta R} = \frac{\theta y}{\theta R}$$

$$\epsilon = \frac{y}{R} = Ky$$



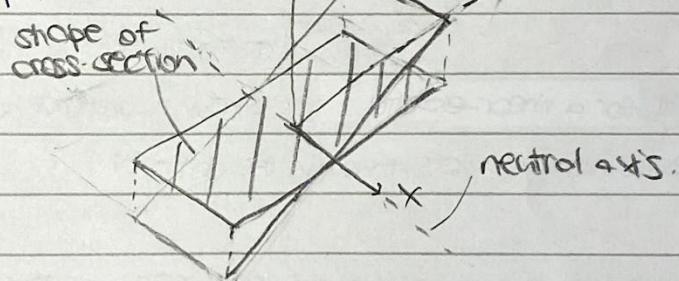
- For a beam subjected to pure bending, the Euler-Bernoulli hypothesis assumption implies that the strain ϵ is prop. to the curvature k and the distance from neutral axis y .

Stress-strain law.

- Assuming that the material is linear elastic,

$$\sigma = E\epsilon = EKy$$

- Plotting this on a graph,

Stress resultant from bending.① Tension T .

- On an elemental area dA of the cross-section, the force is $dT = \sigma dA$. Integrating over the whole area,

$$T = \int_A dT = \int_A \sigma dA \\ = EK \int_A y dA = 0$$

the centroid defines the neutral axis for pure bending.

- * The resultant tensile force T is 0 (because how we chose the neutral axis).

✓ makes sense as we have pure bending

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Bending moment.

- On an elemental area dA of the cross section, the bending moment is $\delta M = y \delta T = y \sigma dA$.

Integrating over the whole area,

$$M = \int_A dM = \int_A y dT = \int_A y \sigma dA \\ = EK \int_A y^2 dA$$

$$\boxed{M = EKI}$$

where I is the 2nd moment of area of the cross section about the neutral axis.

- Recall for an elastic beam, $M = BK$, so

$$\boxed{B = EI}$$

Summary of results

- Since $\sigma = Eky$ and $M = EKI$, these relationships can be collected:

$$\boxed{\frac{\sigma}{y} = \frac{M}{I} = EK}$$

- This relationship is useful for finding resultant internal stress from the applied moment and vice versa.
- The neutral axis is defined as the line in the cross section where $\sigma=0$ under a pure bending moment. (Neutral axis is \perp to plane of bending, ie the 'centre line').
consequently, neutral axis is at centroidal level
the only plane that contains the deformed beam

Combined bending moment & axial force.

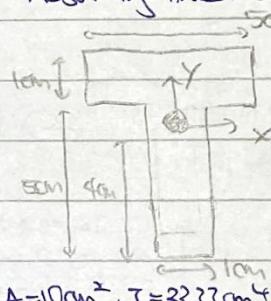
- Often a beam may have to carry both a bending moment M and an axial force T . The stress distribution can be obtained by superposing the stresses due to T and M :

$$\text{Stresses due to } T \quad \boxed{\sigma = \frac{T}{A} + \frac{My}{I}} \quad \text{Stresses due to } M$$

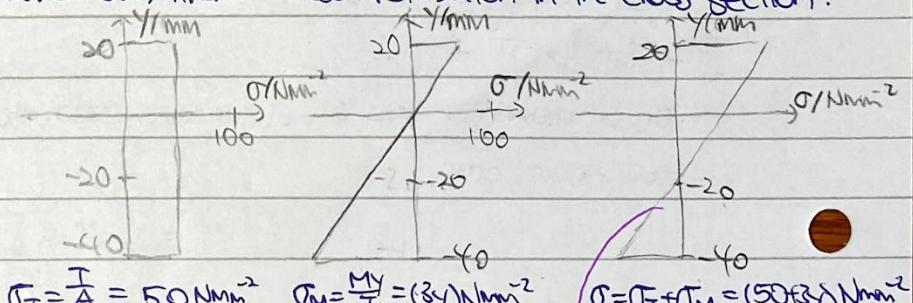
* Note under uniaxial tension, the stress is uniform across the cross section, and the resultant force acts through the centroid.

- e.g.: A member w/o T-shaped cross section carries $M_x = 1 \text{ kNm}$ and $T = 50 \text{ kN}$.

Assuming linear elastic behaviour, find the stress distribution in the cross section.



$$A = 10 \text{ cm}^2, I = 3333 \text{ cm}^4$$



$$\sigma_T = \frac{T}{A} = 50 \text{ N/mm}^2 \quad \sigma_M = \frac{My}{I} = (3y) \text{ N/mm}^2$$

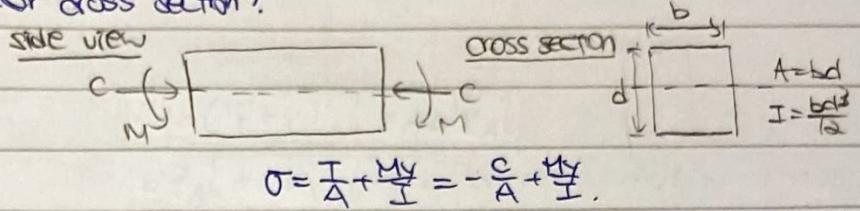
$\sigma = \sigma_T + \sigma_M = (50 + 3y) \text{ N/mm}^2$
At $y=0$, $\sigma \neq 0$
as it is not pure bending.

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Axial compression to avoid tensile stress

- In masonry structures, the material will crack in tension \rightarrow we have an axial compression C provided by the self weight to avoid tensile stress

e.g. What is the value of axial compression C req. to avoid tensile stresses in a rectangular cross section?



Max. tensile stress σ_{max} at the top, $y = \frac{d}{2}$,

$$\sigma_{max} = -\frac{C}{A} + \frac{M y_{max}}{I}$$

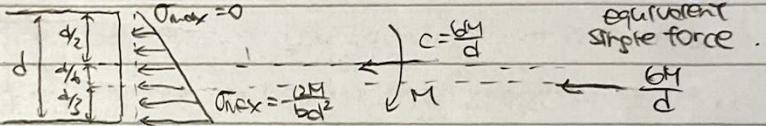
$$= -\frac{C}{bd} + \frac{M \cdot d/2}{bd^3/12}$$

To avoid tensile stresses, we req. $\sigma_{max} < 0$

$$\frac{M}{bd^2} - \frac{C}{bd} < 0$$

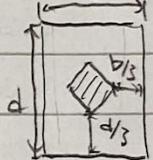
$$C > \frac{6M}{d}$$

For the particular case $C = \frac{6M}{d}$, it is useful to consider the stresses acting on the cross section.



\rightarrow If the resultant thrust (single force resultant of C and M) acts within the middle third of the section, there will be no tensile stress anywhere in the cross section.

*Also considering bending moments about the y-axis, the thrust must lie within a zone w/ the shape of a rhombus to avoid tensile stress.



Calculation of the 2nd moment of area I .

Centroid of the cross section.

- The centroid defines the neutral axis for pure bending.

Assuming plane sections remain planes, $\epsilon = k_y \rightarrow \sigma = E k_y$

As we have pure bending, $T=0$

$$\therefore T = \int_A \sigma dA = E k \int_A k dA = 0$$

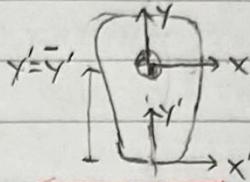
\rightarrow We must locate the origin at a place s.t. the 1st moment of area is zero (centroid)

$$\int_A y dA = 0$$

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Finding the centroid of the cross section

- For a doubly symmetric shape, the centroid is at the centre.
- For a singly symmetric shape, we set up some temporary axes x, y' at a pt. on the axis of symmetry to define the position of the centroid.



(Refer to mechanics notes for more details)

Finding the 2nd moment of area I of the cross-section

- Once the centroid has been located, the 2nd moment of area abt. the neutral axis is

$$I = \int_A y^2 dA$$

where y is the distance from the neutral axis.

- We can find I using the parallel axis theorem

$$I_o = I_g + Ad^2$$

where d is the distance between the given axis and the centroid.

(Refer to mechanics notes (moment of inertia) for more details).

we can find 2nd moment of area of standard shapes using the moment of inertia of that shape by converting m to A.

- * 2nd moment of area $I = \int_A y^2 dA$ and moment of inertia $I = \int r^2 dm$ have similar eqns but have no physical relation w/ one another. However techniques for finding I are the same.
- Values of I and other section properties for various standard cross-sections are tabulated in the data book (section tables for universal columns/beams).

Elastic section modulus

- If we want to find the magnitude of the max. moment that a beam could carry $|M_{al}|$, before the magnitude of the stress in the material reached some max. value, (σ_{al}) , we use.

$$\frac{|M_{al}|}{(\sigma_{al})} = \frac{|M_{al}|}{I} \rightarrow |M_{al}| = \frac{I}{(\sigma_{al})}$$

- $\frac{I}{(\sigma_{al})}$ is such a useful property that it is given a name, the elastic section modulus Z/Z_e .

$$Z = \frac{I}{(\sigma_{al})}$$

so $|M_{al}| = Z (\sigma_{al})$

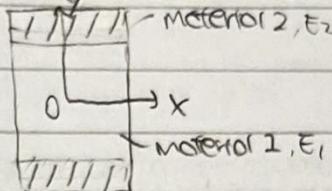
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Bending stresses in composite beams

Bending stresses in composite beams

- A composite beam is made from 2 or more different materials.

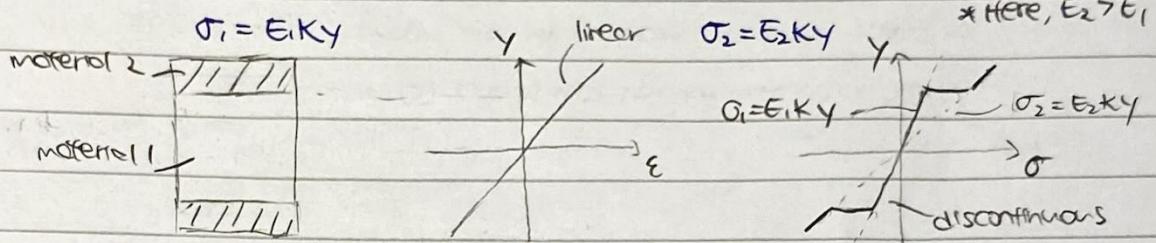
- Consider a composite made from 2 materials as shown below:



Assuming plane sections remain planes, from geometry

$$\epsilon = Ky$$

For a linear-elastic material, we can find the corresponding stresses.



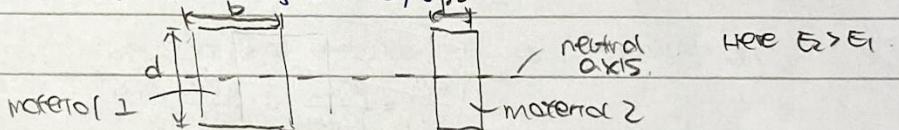
From this, we can find the bending moment M .

$$M = \int A y \sigma dA$$

where the integral would be divided into 2 parts, 1 for each material.

Method of transformed section

- The method is based on the following analogy:



Given $E_2 > E_1$, for the 2 sections to have equal bending stiffness B , the section made of material 2 would have a reduced width of βb .

$$B = EI = E_1 \frac{bd^3}{12} = E_2 \frac{\beta b d^3}{12}$$

$$\therefore \boxed{\beta = \frac{E_1}{E_2}}$$

for material 1 \rightarrow material 2

- Using this method, we can transform a composite beam into a beam entirely made of material 1/2 (They would still have the same bending stiffness: $B = E_1 I_1 = E_2 I_2$)

- Under an applied bending moment, the curvatures and strains will be correctly calculated from either of the transformed sections.

$$\epsilon = \frac{My}{B} = \frac{My}{E_1 I_1} = \frac{My}{E_2 I_2}$$

$$K = \frac{M}{B} = \frac{M}{E_1 I_1} = \frac{M}{E_2 I_2} \quad / \text{use correction factor } E/E_2.$$

- However the stresses calculated are only correct for the untransformed part of the section.

$$\sigma_1 = \frac{My}{I_1}$$

Correct stresses in material 1

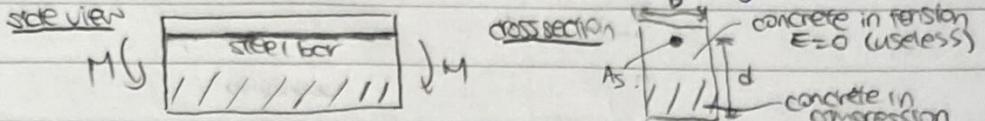
$$\sigma_2 = \frac{My}{I_2}$$

Correct stresses in material 2

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Pure bending of reinforced concrete beams

- Concrete is strong in compression but weak in tension; steel is equally strong in compression and tension \rightarrow in reinforced concrete, use slender steel bars to withstand tension and concrete to withstand compression due to the bending moment.

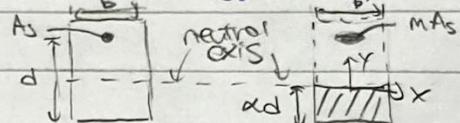


- Concrete is so weak in tension that it is normally treated as a no-tension material. This means it has no strength/stiffness in tension. $E=0$.
- By contrast, concrete is a linear elastic material in compression. $E=E_c$
- We transform the section to one entirely made of material w/ Young's modulus E_c :

\hookrightarrow Steel has to be widened by the modular ratio $M = \frac{E_s}{E_c}$

\hookrightarrow The concrete in tension disappears (\Rightarrow width).

α : reinforcement means $A_s = \alpha / . bd$



- we set the centroid to be ad from the bottom and solve for α to find the centroid.

$$\int_A y da = (-\sum ad)(bd) + (1-ad)ad(mA_s)$$

$$0 = \frac{1}{2} bd \alpha^2 + m A_s \alpha - m A_s$$

$$\alpha = \frac{-m A_s \pm \sqrt{m^2 A_s^2 + 2bd m A_s}}{bd} \quad / \text{take the true solution}$$

- From this, we can work out the 2nd moment of area,

$$I = m A_s (1-\alpha)^2 d^2 + \frac{1}{2} b (ad)^2 + bd (\frac{ad}{2})^2$$

- the strain ratio between concrete and steel is

$$\left| \frac{E_s}{E_c} \right| = \frac{\epsilon_{Ymax}}{\epsilon_{Ymaxc}} = \frac{1-\alpha}{\alpha}$$

and hence the stress ratio between concrete and steel is

$$\left| \frac{\sigma_s}{\sigma_c} \right| = \frac{E_s}{E_c} \left| \frac{\epsilon_s}{\epsilon_c} \right| = m \frac{1-\alpha}{\alpha}$$

Shear stresses in beams

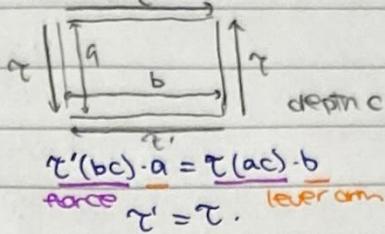
shear stresses.

- In a beam, both T and M are carried by distribution of longitudinal stresses σ that are normal to the cut face.
- By contrast S is carried by shear stresses τ that are parallel to the cut surface.
- These shear stresses are not uniform across the surface. Simply dividing the shear force by the area gives the average shear stress. (no information on peak shear stress).

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Principle of complementary shear.

- A state of simple shear req. shear stress on 4 faces of an arbitrary small block (to maintain static eqm. in translation and rotation)

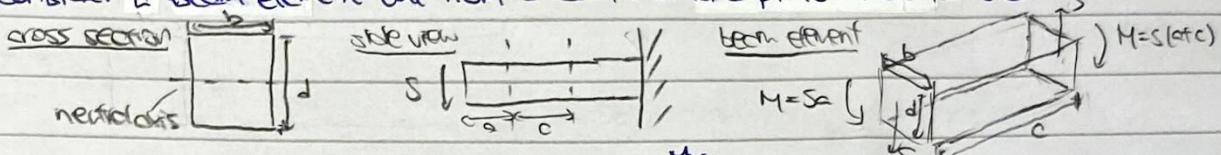


- Principle of complementary shear states if τ acts on a plane, at a pt. in a stressed body, then τ acts also on the orthogonal plane.

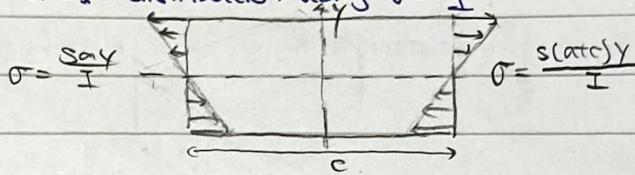
Shear stresses at various depths

if s is not constant across the beam,
set $c = dx$ and let $dx \rightarrow 0$.

- Consider a beam element cut from a cantilever w/o pt. load at the end.



- We can find the stress distribution using $\sigma = \frac{My}{I}$



shear force on section

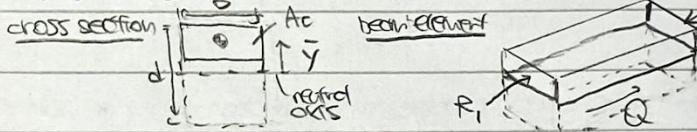
variation of M along beam
as $S = \frac{I}{y}$, $s = 0$.

difference in longitudinal stress
along the 2 cut-out sections of the beam

longitudinal shear force/stress

comp. shear stress in section

- Making an arbitrary cut as shown below.



The longitudinal force at a and ccc are

$$R_1 = \int_{Ac} \sigma dA = \frac{S_a}{I} \int_{Ac} y dA \quad R_2 = \int_{Ac} \sigma dA = \frac{S_{ac}}{I} \int_{Ac} y dA$$

so the shear force Q req. to maintain eqm. is

$$Q = R_2 - R_1 = \frac{S_c}{I} \int_{Ac} y dA$$

useful for finding the no. of bolts req. to support a longitudinal force.

and hence the shear flow q is (shear force per unit length)

$$q = \frac{Q}{c} = \frac{S}{I} \int_{Ac} y dA$$

whereas the shear stress τ is (shear force per unit area).

$$\tau = \frac{Q}{bc} = \frac{S}{bI} \int_{Ac} y dA$$

Note $\tau = \frac{q}{b}$

where b is the width of the cut section.

- We can write the 2nd moment of area as

$$\int_{Ac} y^2 dA = Ac \bar{y}^2$$

useful for checking whether it is within the materials limit

where \bar{y} is the distance from the centroid of the entire cross-section to the centroid of the cut section.

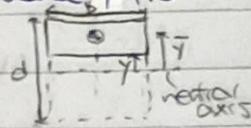
* I used in the formulae is the 2nd moment of area of the whole section.

* THIS method is restricted to cross sections that have axis of symmetry in plane of bending.

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Shear stress distribution in a rectangular section

- consider a cut at an arbitrary distance, y from the centroid.



$$\begin{aligned}\tau &= \frac{S}{Ib} \int_A c dy = \frac{SA_c y}{Ib} \\ &= \frac{6S}{bd^3} \cdot b \left(\frac{d}{2} - y\right) \cdot \left(y + \frac{1}{2} \left(\frac{d}{2} - y\right)\right) \\ &= \frac{12S}{bd^3} \left(\frac{d}{2} - y\right) \cdot \frac{1}{2} \left(y + \frac{d}{2}\right) \\ &= \frac{6S}{bd^3} \left(\frac{d^2}{4} - y^2\right)\end{aligned}$$

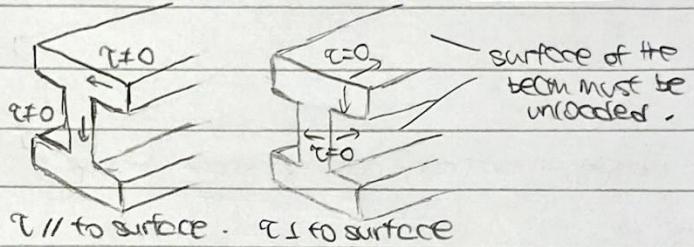
$$\text{At mid-depth, } y=0, \quad \tau = \frac{6S}{bd^3} \left(\frac{d^2}{4}\right) = \frac{3S}{2bd} = \frac{3}{2} S_{avg}$$

At top/bottom, $y = \pm \frac{d}{2}$, $\tau = 0$ in accordance w/ the principle of complementary shear, (as the top/bottom surfaces are stress free).

→ The variation of τ on the cross section is parabolic: max. at mid-depth, zero at top/bottom.

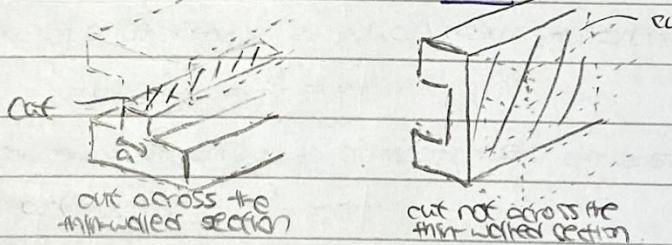
Shear stress in thin-walled beams

- The key idea is to realise that the shear stress at a pt. of a cross section of a thin-walled beam must be // to the surface of the section. (i.e. along the length of thin wall)



- This is because shear stresses in a direction ⊥ to the surface of the section (i.e. along width of thin wall) would be accompanied by complementary shear stresses acting on the surface of the beam, which is unloaded.

- The eqn. derived for the shear stress/flow in a beam can also be used to find the shear stresses in thin-walled beams - but the cut must be across the thin-walled section.



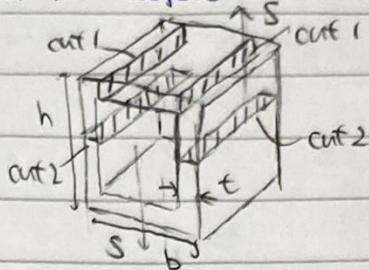
If the cut we have made is of length a ,

$$\tau_a = \frac{S A_c y}{I}$$

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Shear stresses in a box beam

- Consider a rectangular box beam subject to shear force S .



For the entire section, the 2nd moment of area is

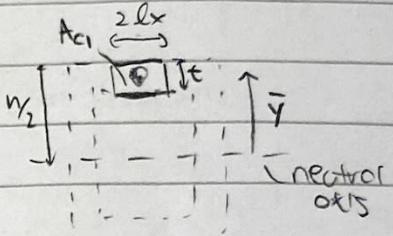
$$I = \frac{bh^3}{12} - \frac{(b-2t)(h-2t)^3}{12} \approx \frac{ht}{6}(2bt+h)$$

Consider making the cuts 1. The 1st moment of area of the cut section is,

$$A_{c1}\bar{y} = 2bx \cdot \left(\frac{h}{2} - \frac{t}{2}\right)$$

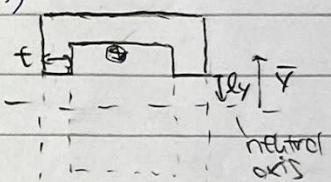
The shear stress is therefore given by

$$\begin{aligned} \tau_1 &= \frac{S}{Ia} A_{c1} \bar{y} \\ &= \frac{S}{I(a)} 2bx \left(\frac{h}{2} - \frac{t}{2}\right) \\ &= \frac{S}{2It} (h-t) bx \quad \text{linear variation in flange} \end{aligned}$$



Consider making the cuts 2. The 1st moment of area of the cut section is,

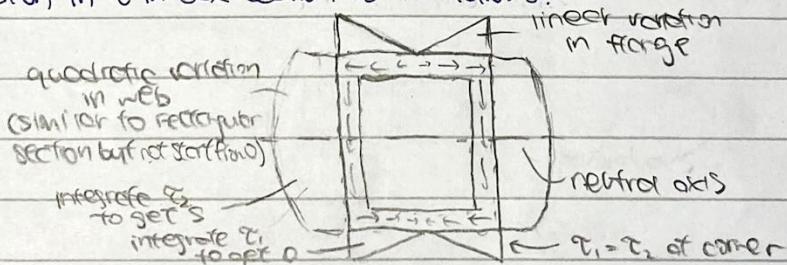
$$\begin{aligned} A_{c2}\bar{y} &= (bt)\left(\frac{h}{2} - \frac{t}{2}\right) + 2\left[\left(\frac{h}{2} - t - b_x\right)t\right]\left[\frac{1}{2}(h-t+b_x)\right] \\ &= \frac{bt}{2}(h-t) + \left[\left(\frac{h}{2} - t\right)^2 - b_x^2\right]t \end{aligned}$$



The shear stress is therefore given by

$$\begin{aligned} \tau_2 &= \frac{S}{Ia} A_{c2} \bar{y} \\ &= \frac{S}{I(a)} \left[bt \left(\frac{h}{2} - t \right) + \left[\left(\frac{h}{2} - t \right)^2 - b_x^2 \right] t \right] \\ &= \frac{S}{2I} \left[\frac{bt(h-t)}{2} + \left(\frac{h}{2} - t \right)^2 - b_x^2 \right] t \quad \text{quadratic variation in web.} \end{aligned}$$

- The variation in τ in box section is as follows.

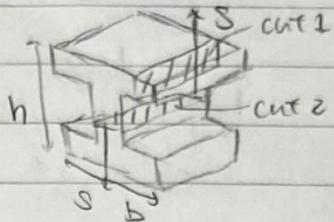


- The entire shear force (vertical) is carried by the web

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shear stresses in an I-beam

- consider an I-beam subject to shear force S .



For the entire section, the 2nd moment of area is

$$I = \frac{bh^3}{12} - \frac{(b-t)(h-t)}{12} \approx \frac{ht}{12}(6b+h^3)$$

Consider making the cut 1. The 1st moment of area of the cut section is

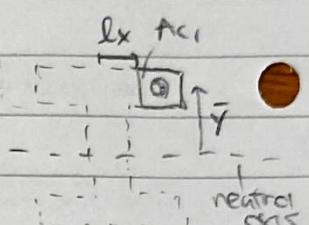
$$A_{c1}\bar{y} = \left(\frac{b}{2} - l_x\right)t \cdot \left(\frac{h}{2} - \frac{t}{2}\right)$$

The shear stress is therefore given by

$$\tau_1 = \frac{S}{Ia} A_{c1}\bar{y}$$

$$= \frac{S}{It} \left(\frac{b}{2} - l_x\right)t \left(\frac{h}{2} - \frac{t}{2}\right)$$

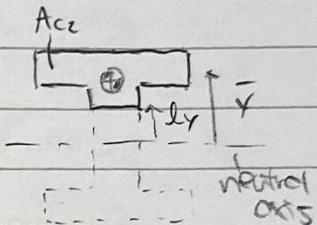
$$= \frac{S}{2t} (h-t) \left(\frac{b}{2} - l_x\right) \quad \text{linear variation in flange}$$



Consider making the cut 2. The 1st moment of area of the cut section is

$$A_{c2}\bar{y} = bt\left(\frac{h}{2} - \frac{t}{2}\right) + t\left(\frac{b}{2} - t - l_y\right) \cdot \left[\frac{1}{2}\left(\frac{h}{2} - t + l_y\right)\right]$$

$$= \frac{bt}{2}(h-t) + \frac{t}{2}\left[\left(\frac{b}{2}-t\right)^2 - l_y^2\right]$$



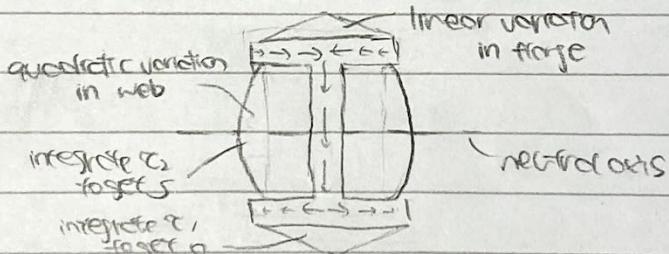
The shear stress is therefore given by

$$\tau_2 = \frac{S}{Ia} A_{c2}\bar{y}$$

$$= \frac{S}{It} \left[\frac{bt}{2}(h-t) + \frac{t}{2}\left[\left(\frac{b}{2}-t\right)^2 - l_y^2\right] \right]$$

$$= \frac{S}{I} \left[\frac{b}{2}(h-t) + \frac{1}{2}\left[\left(\frac{b}{2}-t\right)^2 - l_y^2\right] \right] \quad \text{quadratic variation in web}$$

- The variation in τ I-section is as follows:



- The entire shear force (vertical) is carried by the web.

- In the web of an I-beam, τ varies linearly from zero in a beam of solid rectangular cross-section.

- The shear stress in the web is approximately $\tau \approx \frac{S}{A_{web}}$ (A_{web} not A)

- I-beams are used w/ their webs vertical, so that the bending moment due to vertical loads is carried by the beam in its stiff (strong) direction.

Buckling of columns

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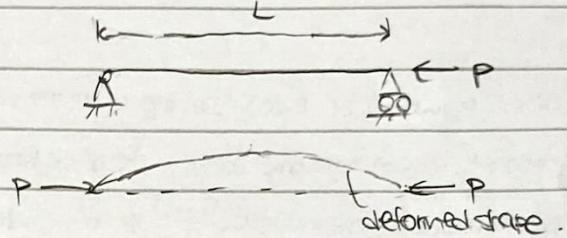
Buckling of (perfect) columns

Buckling.

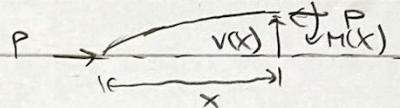
- Buckling is fundamentally nonlinear. We need to consider eqm. of the deformed shape.
- Looking at the deformed shape that is in eqm. w/ the applied load gives us a bifurcation load.
- To consider the buckling of a compression member, we initially make a no. of key assumptions:
 - ↪ Pin ended bar ← we will remove these assumptions → Linear elastic
 - ↪ Initially perfectly straight ✓ later ↪ All deflections are assumed to be small
 - ↪ Initially stress free.

Euler column

- A simple pinned ended bar in compression is known as an Euler column



- The column buckles when a non-straight eqm. shape becomes possible. → we analyse the deformed shape v(x) to determine when it will be in eqm.



Eqm.

$$PV(x) = M(x)$$

Elastic law

$$M(x) = EI\Delta K(x)$$

$$= EI K(x) \quad \text{here } K_0 = 0$$

Geometry

$$K(x) = -\frac{d^2v(x)}{dx^2}$$

$$\therefore PV(x) = -EI \frac{d^2v(x)}{dx^2}$$

$$\frac{EI}{P} \frac{d^2v}{dx^2} + v = 0 \quad [\text{Governing DE}]$$

Rearranging,

$$\text{Define } \alpha^2 = \frac{P}{EI},$$

$$\frac{1}{\alpha^2} \frac{dv}{dx^2} + v = 0 \quad \leftarrow \text{homogeneous DE}$$

$$v = A \sin(\alpha x) + B \cos(\alpha x)$$

Boundary conditions

$$v(0) = 0 \rightarrow B = 0$$

$$v(L) = 0 \rightarrow A \sin(\alpha L) = 0$$

so either $A = 0$ (no deformation)

or $\sin(\alpha L) = 0 \rightarrow \alpha L = n\pi, n = 1, 2, 3, \dots$

since $\alpha^2 = \frac{P}{EI}$, the corresponding values of P are

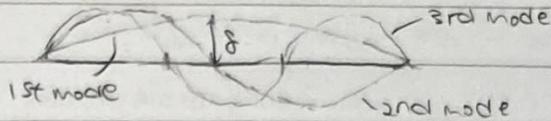
$$P = \alpha^2 EI = \frac{\pi^2}{L^2} EI, \frac{4\pi^2}{L^2} EI, \dots \frac{n^2 \pi^2}{L^2} EI \dots$$

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- We can substitute the values of a buck into the GS.

$$V = A \sin\left(\frac{\pi x}{L}\right), A \sin\left(\frac{2\pi x}{L}\right) \dots A \sin\left(\frac{n\pi x}{L}\right) \dots$$

The sides have fixed mode shapes but the amplitudes A are indeterminate.



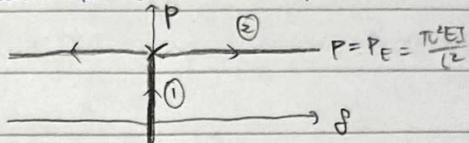
- Therefore, the bar remains straight, $A=0$, except at certain critical values of P . At these critical values, there is a fixed mode shape but the size of the deflection is indeterminate.

- The lowest critical value of P for a pin-jointed column is the Euler load P_E .

$$P_E = \frac{\pi^2 EI}{L^2}$$

choose smallest I from I_{xx}, I_{yy} (weakest axis)

- If we plot P vs the central deflection of the bar δ , we have.



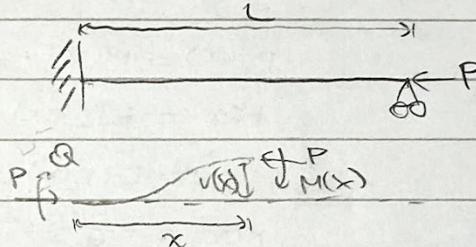
①: $\delta = 0$ for all P , ($P > 0$ or $P < 0$). In this case the bar doesn't buckle and remains straight.

②: At $P = P_E = \text{const.}$, δ can have any value : the axial load remains constant while the buckle grows.

- The 2 points in the P, δ space intersect at a pt. of bifurcation.

- In practice, as the compressive load is increased, we go up ① to $P=P_E$, then we go either to the left or to the right on ②. We never go beyond $P=P_E$ for ① as the eqm. is unstable.

fixed end column,
no longer assume
pin-jointed bar



Eqm.

$$Pv(x) = M(x) + Q$$

Elastic law

$$M(x) = EI\Delta K(x)$$

$$= EI K(x) \quad \text{here } K_0 = 0$$

Geometry

$$K(x) = -\frac{d^2 V(x)}{dx^2}$$

$$\therefore Pv(x) = -EI \frac{d^2 V(x)}{dx^2} + Q \quad [\text{Governing DE}]$$

Rearranging.

$$\frac{EI}{P} \frac{d^2 V}{dx^2} + Pv = \frac{Q}{P} \quad \text{non-homogeneous DE}$$

Define $\alpha^2 = \frac{P}{EI}$.

$$\frac{1}{\alpha^2} \frac{d^2 V}{dx^2} + V = \frac{Q}{P}$$

$$V = A \sin \alpha x + B \cos \alpha x + \frac{Q}{P}$$

Boundary Conditions

$$V(0) = 0 \rightarrow \frac{Q}{P} = -B$$

we implied symmetry

fixed end

$$V'(0) = 0 \rightarrow A = 0$$

so $V'(L) = 0$ is not another boundary condition

$$V(L) = 0 \rightarrow B(\cos \alpha L - 1) = 0$$

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so either $\theta = 0$ (no deformation)

or $\cos(\alpha L) = 1 \rightarrow \alpha L = 0 \text{ (trivial)}, \pi, 2\pi, \dots, 2n\pi \dots$

Since $\alpha^2 = \frac{P}{EI}$, the corresponding values of P are

$$P = \alpha^2 EI = \frac{\pi^2 E I}{L^2}, \frac{4\pi^2 E I}{L^2}, \dots, \frac{4n^2 \pi^2 E I}{L^2} \dots$$

a times larger than
for the pinned-jointed case.

- We can substitute the values of α back into the GS

$$v = B \left(\cos\left(\frac{\pi x}{L}\right) - 1 \right), B \left(\cos\left(\frac{2\pi x}{L}\right) - 1 \right) \dots$$

The soils have fixed node shapes but the amplitudes B are indeterminate.

Effective lengths. / no longer assume pinned-end bar

/ may have to reflect about the cantilever support for 2nd Pf.

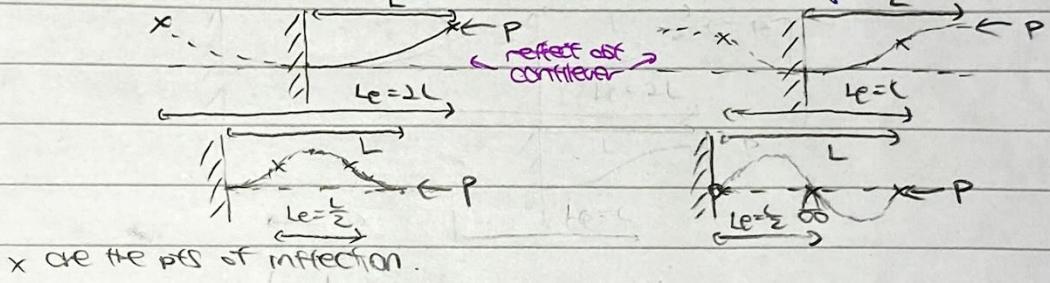
- For a non-pinned-jointed column, we look for the 2 pts. of inflection, where the curvature is zero \rightarrow so moment is zero (as a pin-joint would be)

- The section between 2 pts. of inflection is equivalent to a pinned-end column. The length of this section is the effective length of the column, L_e .

- Now, the buckling load is given by

$$P_E = \frac{\pi^2 EI}{L_e^2}$$

- Here are examples where we can consider the effective length:



Critical stress of (perfect) columns

- For a perfect strut, it could fail in compression by general yielding ($\sigma_{cr} = \sigma_y$) or by buckling ($\sigma_{cr} = \sigma_E$). The critical stress at failure σ_{cr} is the smaller of σ_y and σ_E .

- The buckling stress σ_E can be obtained from P_E

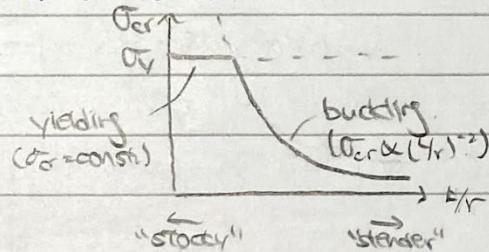
$$\sigma_E = \frac{P_E}{A} = \frac{\pi^2 E}{L_e^2} \left(\frac{I}{A} \right) = \frac{\pi^2 E}{(4r)^2}$$

where r is the radius of gyration, defined as

$$r = \sqrt{\frac{I}{A}}$$

- The ratio $4r/L$ is the slenderness of the column.

- Plotting a graph of σ_{cr} against $4r/L$:

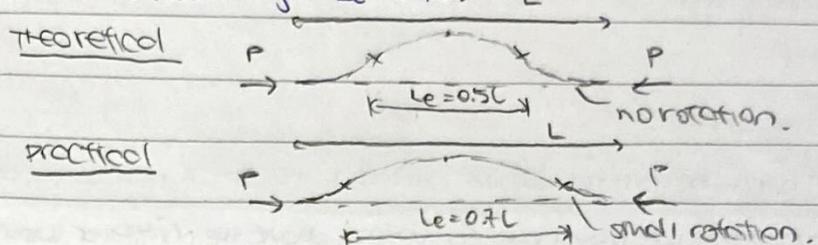


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Buckling of imperfect columns

Practical effective lengths

- In practice, restraints on columns are never perfectly rigid \rightarrow some design codes specify a larger effective length than the theoretical value (lower Euler buckling load).
- For example, the effective length for the fixed-end column is theoretically $l_e = 0.5L$. However, since full fixity is nearly impossible to achieve, all design codes recommend use of a larger value of effective length $l_e = 0.7L$.

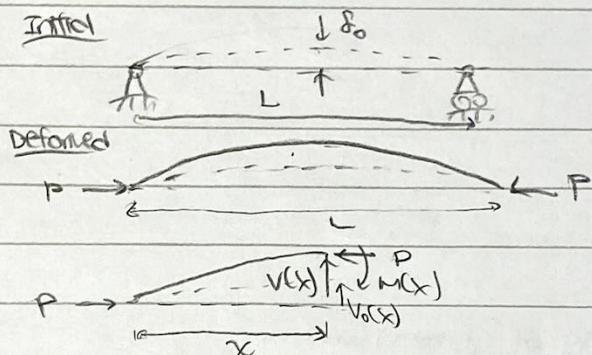


Imperfections

- In practice, bars are never absolutely straight – consider that the bar has an initial bow, defined by

$$v_0 = \delta_0 \sin\left(\frac{\pi x}{L}\right)$$

we typically carry out Fourier analysis on any initial shape and use the most significant component



E.g.m.

$$Pv(x) = M(x)$$

Elastoic (law)

$$M(x) = EI \Delta K(x)$$

$$= EI K(x) - EI k_o(x)$$

Geometry

$$K(x) = -\frac{d^2v(x)}{dx^2} ; \quad K_o(x) = -\frac{d^2v_o(x)}{dx^2}$$

$$\therefore Pv(x) = -EI \frac{d^2v(x)}{dx^2} + EI \frac{d^2v_o(x)}{dx^2} \quad [\text{Governing DE}]$$

Rearranging,

$$\frac{EI}{P} \frac{d^2v}{dx^2} + v = -\frac{EI}{P} \cdot \left(\frac{\pi}{L}\right)^2 \delta_0 \sin\left(\frac{\pi x}{L}\right)$$

$$\frac{EI}{P} \frac{d^2v}{dx^2} + v = -\left(\frac{PE}{P}\right) \delta_0 \sin\left(\frac{\pi x}{L}\right)$$

Define $\alpha = \frac{P}{EI}$,

$$\frac{1}{\alpha^2} \frac{d^2v}{dx^2} + v = -\left(\frac{PE}{P}\right) \delta_0 \sin\left(\frac{\pi x}{L}\right)$$

$$v = A \sin(\alpha x) + B \cos(\alpha x) + \frac{\delta_0}{1 - P/E} \sin\left(\frac{\pi x}{L}\right)$$

Boundary conditions

$$v(0) = 0 \rightarrow B = 0$$

$$v(L) = 0 \rightarrow A \sin(\alpha L) = 0$$

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so either $A=0$

$$\text{or } \sin(\alpha L) = 0 \rightarrow \alpha L = 0, \pi, 2\pi, \dots$$

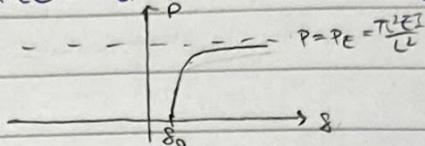
For loads less than the Euler buckling load, $A=0$, so

$$V = \frac{\delta_0}{1-P/P_E} \sin\left(\frac{\pi x}{L}\right)$$

The central deflection is given by

$$\delta = V\left(\frac{L}{2}\right) = \frac{\delta_0}{1-P/P_E}$$

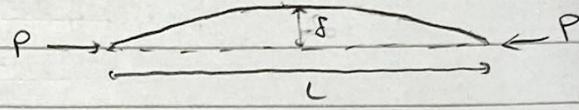
- If we plot P vs the central deflection of the bar δ , we have



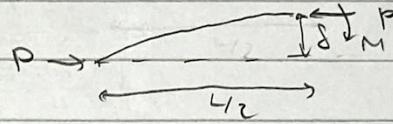
- There is no longer a bifurcation in the eqn. paths, but the initially cracked bar tends asymptotically towards the buckling load of the perfectly straight bar.

Critical stress of imperfect columns

- Consider an initially imperfect strut, carrying some axial load.



- Cut at midspan to find the max stress



The max stress is given by (Here compressive stress is true)

$$\sigma_{max} = \frac{P}{A} + \frac{M y_{max}}{I} = \frac{P}{A} + \frac{P \delta y_{max}}{I}$$

Defining the average stress $\sigma_{avg} = P/A$, we obtain

$$\sigma_{max} = \sigma_{avg} + \frac{P \delta y_{max}}{A r^2} = \sigma_{avg} \left(1 + \frac{\delta y_{max}}{r^2}\right)$$

σ_{avg} is the stress we expect for a perfect strut below the Euler buckling load

Perry defined the critical stress σ_{cr} to be σ_{avg} when the max. stress is σ_y .

$$\sigma_y = \sigma_{cr} \left(1 + \frac{\delta y_{max}}{r^2}\right)$$

Rearranging, we find that as the load is increased, δ increases until first yield occurs

$$\delta = \frac{r^2}{y_{max}} \left(\frac{\sigma_y}{\sigma_{cr}} - 1 \right)$$

- Recall for an imperfect strut, the central deflection δ is given by

$$\delta = \frac{\delta_0}{1-P/P_E} = \frac{\delta_0}{1-\sigma_y/\sigma_{crE}} = \frac{\delta_0}{1-\sigma_y/\sigma_E}$$

- Equating the 2 expressions for the central deflection δ

$$\frac{\delta_0}{1-\sigma_y/\sigma_E} = \frac{r^2}{y_{max}} \left(\frac{\sigma_y}{\sigma_{cr}} - 1 \right)$$

$$(\sigma_y - \sigma_{cr})(\sigma_E - \sigma_{cr}) = \frac{y_{max} \delta_0}{r^2} \sigma_{cr} \sigma_E$$

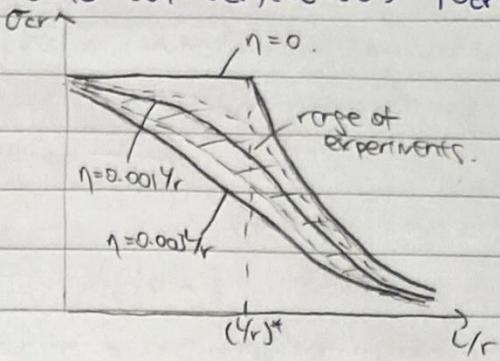
Defining $\eta = \frac{\delta_0 y_{max}}{r^2}$ as a dimensionless measure of initial imperfection,

$$(\sigma_y - \sigma_{cr})(\sigma_E - \sigma_{cr}) = \eta \sigma_{cr} \sigma_E$$

[Perry's formula]

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- Plotting the graph of solns to $(\sigma_y - \sigma_{cr})(\sigma_E - \sigma_{cr}) = \eta \sigma_{cr} \sigma_E$, for various η ,



- The greatest reduction in strength occurs at the pt. of intersection of the 2 original curves (i.e. $Y_r = (Y_r)^*$) \rightarrow critical stress of columns w/ slenderness of def. $(Y_r)^*$ is very imperfection sensitive (small imperfections result in large reduction in strength).
- At the pt. of intersection ($\sigma_y = \sigma_E = \sigma^*$), Perry's formula becomes,

$$(\sigma^* - \sigma_{cr})^2 = \eta \sigma^* \sigma_{cr}$$

$$\sigma_{cr}^2 - (z + \eta) \sigma^* \sigma_{cr} + \sigma^*^2 = 0$$

$$\left(\frac{\sigma_{cr}}{\sigma^*}\right)^2 - (z + \eta) \frac{\sigma_{cr}}{\sigma^*} + 1 = 0$$

$$\frac{\sigma_{cr}}{\sigma^*} = \frac{z + \eta \pm \sqrt{z^2 + 4\eta z - 4}}{2}$$

$$= \frac{z + \eta - \sqrt{z^2 + 4\eta z - 4}}{2}$$

$$= 1 + \frac{\eta}{2} - \frac{1}{2}(\eta^2 + 4\eta)^{1/2}$$

$$= 1 + \frac{\eta}{2} - \frac{1}{2}\sqrt{\eta}(\eta + 4)^{1/2}$$

$$\approx 1 + \frac{\eta}{2} - \frac{\sqrt{\eta}}{2}(2 + \frac{\eta}{4})^{1/2}$$

$$\approx 1 - \sqrt{\eta}$$

$\frac{\eta}{2} \ll \frac{\sqrt{\eta}}{2}$ for small η ($\eta < 1$)

$$\therefore \boxed{\sigma_{cr} \approx (1 - \sqrt{\eta}) \sigma^*}$$

- An analysis of a large no. of experiments on hot-rolled sections showed that
 - ↪ $\eta = 0.001(Y_r)$ provides a good average value to fit the experiments
 - ↪ $\eta = 0.003(Y_r)$ provides a good lower bound for the experiments \rightarrow good choice for design