Connection Game

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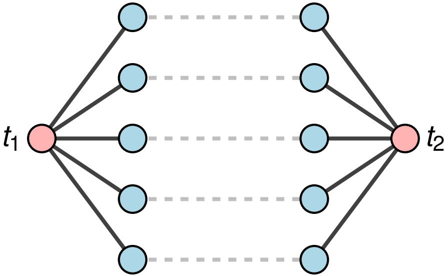
Classic Steiner Tree Problem

Given an undirected graph $G = (V, E)$ with non-negative edge cost $c_e \geq 0$ for every edge $c_e \in E$.

In the Steiner Tree problem, we are given a set of terminals $R \subseteq V$. The goal is to compute the minimum-cost subgraph that spans all terminals.

Steiner Tree with Two Terminals

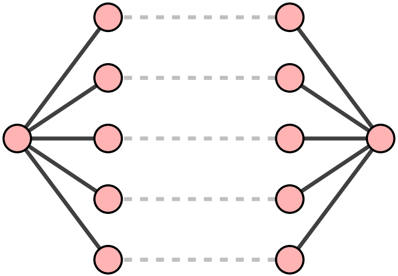
If $|R| = 2$, then it is equal to find the shortest path between two terminals.



Dijkstra's algorithm can find the shortest path polynomial time.

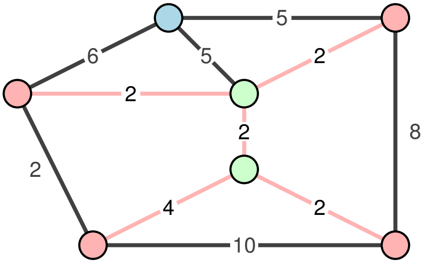
Steiner Tree with N Terminals

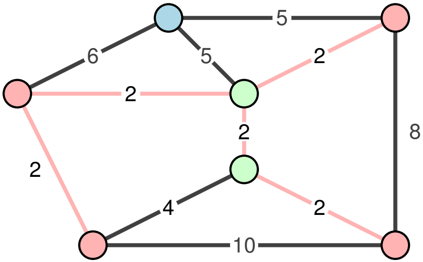
If $|R| = |V|$, then it is equal to find the MST.



Prim's algorithm can find the minimum spanning tree polynomial time.

Steiner Tree Example





Steiner Forest Problem

Given a collection of disjoint subsets of $V : V_1, V_2, V_3, ...V_n$. The goal is to compute a subgraph that any two vertices that belong to the same subset V_i are connected.

Introduction of Selfish Agents

What if we don't have a director in change of the whole network, instead there exist many agents acting selfishly?

For example:

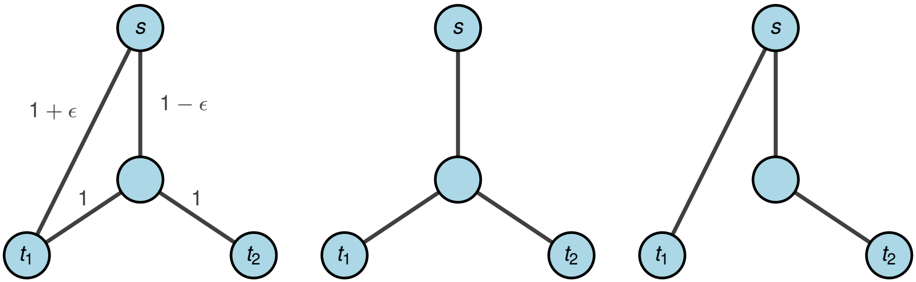
- The railway network across European
- Internet
- ...

Introduction of Selfish Agents

Given an undirected graph G with non-negative edge costs and N players, each player is interested in connecting a set of terminals (nodes in G) via buying a subgraph of G. Players offer each edge in G certain payment $p_i(e)$

When players start to behave selfish, i.e. they would like to seek the minimum total payment, some players might deviate best centralized optimum.

How Selfish Agents Cause Deviation from Central OPT



Formal Definition of Connection Game

Definition (Connection Game)

- An undirected graph $G = (V, E)$.
- Non-negative edge cost $c_e \geq 0$ for every edge $c_e \in E$.
- A subset of V for each player that they must connect to.
- A payment function p_i indicates that player's payment strategy. $p_i(e)$ is the contribution that player i would like to offer for edge e.

Rules of Connection Game

Definition (Connection Game)

- If the sum of payment on certain edge e is larger than the cost on that edge c_e, this edge is considered as bought.
- If an edge is bought, it can be used by all players no matter they contribute to it or not.
- The goal of all players is to connect all of their terminals. If in the end, a player's terminates are not fully connected, they will face an infinite penalty.

Nash Equilibrium in Connection Game

Definition (Nash Equilibrium in Connection Game)

A Nash equilibrium of the connection game is a payment function p such that, if players offer payments p, no payer has an incentive to deviate from their payment.

Nash's Theorem

Theorem (Nash's theorem)

With randomization, any game with finite number of players and actions has a mixed-strategy of Nash equilibrium.

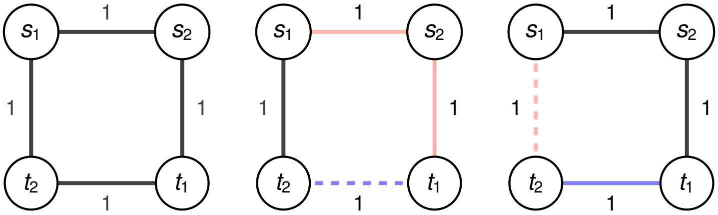
Nash's theorem states that any finite game has at least one mixed strategies Nash equilibrium but no grantee on pure strategy equilibrium.

In the context of large-scale network creation, allowing players choosing their strategies randomly does not make much sense. We are also not interested in the expected payoff but a certain result.

We are only interested in the pure Nash equilibrium in connection game.

No Existence of Nash Equilibrium

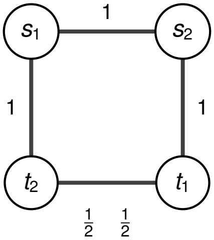
Pure Nash equilibrium may not exist in the connection game if we only allow players to pay for the whole edge or not at all.



Fractional Nash Equilibrium

Definition (Fractional Nash Equilibrium)

Game instances where the only Nash equilibrium in existence require players to split the cost of an edge.



Some Properties of Nash Equilibrium

- G_p is a forest.
- Let T^i be the smallest tree in G_p connecting all terminals of player i, then player i only contributes to edges in T^i.
- Each edge is either bought or not at all.

Price of Anarchy

As shown before, the introduction of selfish agents can lead to worse equilibrium than the best centralized optimum. The question is how bad an equilibrium can be.

Definition (Price of Anarchy)

The price of anarchy of connection game is defined as the ratio of the cost of worst Nash equilibrium over the best centralized design.

$$P_A = \frac{\sum_1^N p_i(e)}{OPT}$$

Price of Anarchy

Lemma

The Price of Anarchy in connection game can be as bad as N.

Price of Stability

Price of stability is a complementary concept of price of anarchy which evaluate how good the best equilibrium can be.

Definition (Price of Stability)

The price of anarchy of connection game is defined as the ratio of the cost of best Nash equilibrium over the best centralized design.

$$P_A = \frac{\sum_1^N p_i(e)}{OPT}$$

Single Source Game

In the single source game, we only allow every player i has one unique terminal t_i that they all would like to connect a common terminal s. This can be considered as a special version of Steiner tree problem where $R = \{s, t_1, t_2, ..., t_n\}$.

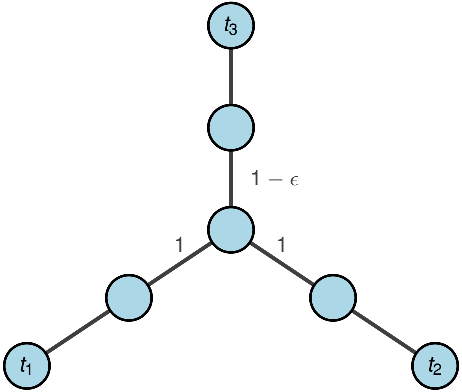
Definition (Single source Game)

A single source game is a game in which all players share a common terminal s and in addition, each player i has exactly one other terminal t_i.

Possible Strategies

1. Best Response Dynamics
2. Marginal Cost mechanisms
3. Shapely Value
4. Utilizing minimum Steiner Tree

Best Response Dynamics



Suppose We have T^* in our hands . . .

Now suppose the minimum cost Steiner tree T^* over all players' terminals is given on a sliver plate, one natural thought is can we achieve an equilibrium that equals to the OPT?

The answer is YES!

A Not Important Property of T^*

It is trivial to see that every leaf node in T^* is a terminal as if not it is possible to simply discard this node and corresponding edge without affecting the connection of any terminals.

Fact

Every leaf node in T^* is a terminal.

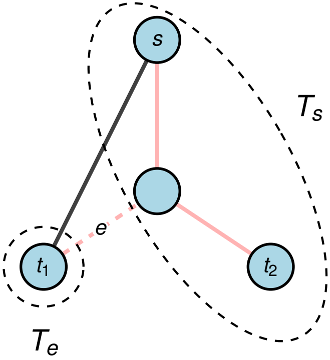
How Should We Traverse T^*?

First, let T^* be rooted from s.

For every edge e, suppose e is removed from T^*, T^* will be divided into two subtrees.

Let the subtree that does not contain s be T_e and the one contains s be T_s.

We will visit T^* in reverse breadth-first-search order.



How to Assign $p_i(e)$?

In every iteration, we modify the cost of every edge according to the following rules:

- let costs of edges $e \in T_e$ be $p_i(e)$
- let costs of edges $e \in T_s$ be 0
- costs of edges $e \notin T^*$ stay the same

How to Assign $p_i(e)$?

We can find the cheapest alternative path A_i from t_i to s from $G \setminus \{e\}$ under modified cost in polynomial time.

Find the minimum value of:

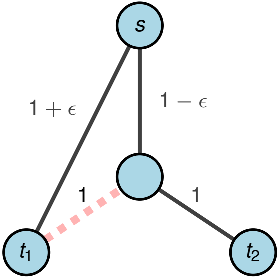
- $c(e)-$ contribution from other players to edge e
- cost of A_i- sum of all contribution of player i to T^*

Algorithm for Assigning $p_i(e)$

Algorithm 1 pseudocode for assigning $p_i(e)$

$p_i(e) \leftarrow 0, \forall t_i \in R, \forall e \in E$
while $e \in \mathit{ReverseBFS}(T^*)$ **do**
  if e is a cut **then**
    $p_i(e) \leftarrow c(e)$
  else
    $c(e) \leftarrow p_i(f) \qquad \forall e \in T_e$
    $c(e) \leftarrow 0 \qquad \forall e \in T_S$
    $A_i \leftarrow$ the cheapest alternative path from s to t_i in $G \setminus \{e\}$
    $p_i(e) \leftarrow \min\{c(A_i) - \sum_{e \in T^*} p_i(e), c(e) - \sum_j p_j(e)\}$
  end if
end while

Example



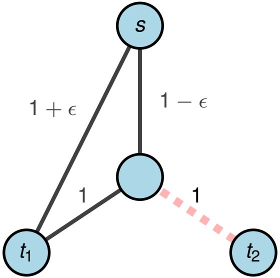
e is not a cut.

consider $c(A_1) = 1+\epsilon, \quad p_1(T^*) = 0$

$c(e) = 1, \quad \sum_j p_j(e) = 0$

set $p_1(e) = 1$

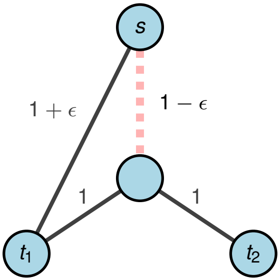
Example



e is a cut.

set $p_2(e) = 1$

Example



e is not a cut.

for t_1

$c(A_1) = 1+\epsilon, \quad p_1(T^*) = 1$

$c(e) = 1-\epsilon, \quad \sum_j p_j(e) = 0$

set $p_1(e) = \epsilon$

for t_2

$c(A_2) = 3+\epsilon, \quad p_2(T^*) = 1$

$c(e) = 1-\epsilon, \quad \sum_j p_j(e) = \epsilon$

set $p_2(e) = 1-2\epsilon$

Proof of p Being Nash Equilibrium

Corollary

If T^* is fully paid in the end of the game, the final payment function produced by the above algorithm is indeed a Nash equilibrium.

By assigning $p_i(e) = min\{c(A_i) - \sum_{e \in T^*} p_i(T^*), c(e) - \sum_{j \in T_i, j \neq i} p_j(e)\}$, we ensure that player i will never contribute to e more than the cost of deviation.

In another word, it is never player i's interests to deviate from buying e. And this holds true for every player in every edge in T^*. Therefore, p is indeed a Nash equilibrium.

An Assumption

First, assume that the following lemma is true.

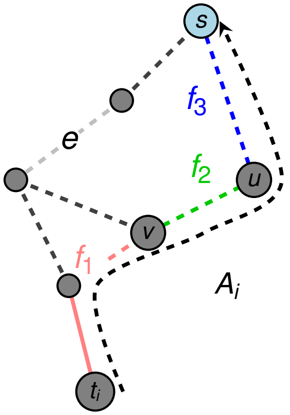
Suppose A_i is i's alternative path at some stage of the algorithm. Then suppose there are two nodes v and w on A_i that divides this path into three parts. Let edges on A_i from t_i to v be f_1, from v to w be f_2, from w to s be f_3.

Lemma

There must exist a pair of $\{v, w\}$ such that $f_1 \in T_e$, $f_2 \notin T^*$, $f_3 \in T_s$.

Prove T^* is fully paid via contradiction.

An Assumption



Proof via Contradiction

Lemma

All edges in T^* are bought, i.e. $\sum p_i(e) = c(e)$ for any $e \in T^*$.

Suppose there exists some edge e such that after all players in T_e have contributed to that edge, $p_i(e) < c(e)$, we can prove that T_e costs more than $\bigcup_{i \in T_e} A_i$ which is a contradiction of T^* is OPT.

Deviation Point

Definition

For each player i, name the highest ancestor of t_i in A_i that is also in T_e, player i's deviation point, denoted d_i.

T_i is fully paid

Let T_i be the subtree rooted from d_i

Fact

T_i is fully paid.

T^* is fully paid

We modify T^* by replacing T_e as follows: those players i whose alternative paths A_i are associated with nodes in D deviate to A_i.

All other players leave their payments unchanged.

Note that no player has increased their expenditures.

If we can show that all terminals in T_e are connected to $T^* \setminus T_e$ after this modification, we're done.

Proof of Lemma

Lemma

$f_1 \in T_e$, $f_2 \notin T^*$, $f_3 \in T_s$.

Suppose once A_i reaches a node in T_s, then all subsequent edges will be in T_s as edges in T_s cost 0 under modified cost.

A_i will always reach a node in T_s since $s \in T_s$.

So we only need to prove that only A_i leaves T_e it will only be in $G \setminus T_e$, i.e. A_i **does not go back to T_e anymore.** We show this by contradiction.

Proof of Lemma

Suppose there are two nodes v and w on A_i that divides the part of A_i before reaching to T_s into three parts such that $f_1 \in T_e$, $f_2 \notin T^*$, $f_3 \in T_e$.

Proof of Lemma

Let y be the lowest common ancestor of t_i and u in T_e

Best Equilibrium

Theorem

For every single source game, there always exists a Nash equilibrium with price of stability being 1.

Hardness of Steiner Tree

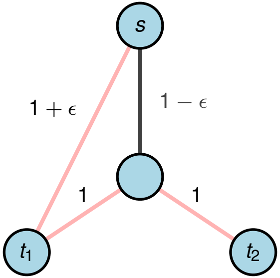
Although we proved that the determined existence of Nash equilibrium in single source game, **the algorithm presented is not feasible** as the finding the minimum cost Steiner tree is NP-hard by itself.

It has been proven that it is NP-hard to approximate Steiner tree problem within ratio $96/95$.

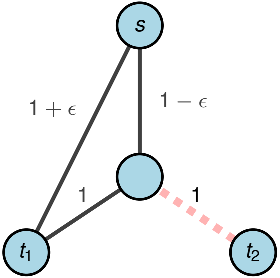
$$c(T) = \alpha c(T^*)$$

Approximate Steiner Tree Will Fail at Archiving Nash Equilibrium

If we are given an approximated Steiner tree T and try to use the same algorithm to assign $p_i(e)$ in T, since T is not optimal, there will be some edge e that players are unwilling to pay for.



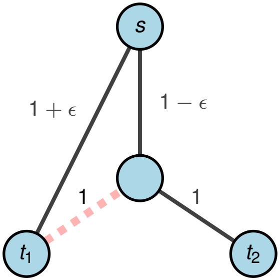
Example



e is a cut.

set $p_2(e) = 1$

Example



e is not a cut.

$c(A_2) = 2 - \epsilon, \quad p_2(T^*) = 1$

$c(e) = 1, \quad \sum_j p_j(e) = 0$

set $p_2(e) = 1 - \epsilon$

In the end of the game, e is not fully paid!

Player 2 will deviate.

Approximate Nash Equilibrium in Connection Game

Definition (Approximate Equilibrium)

A $(1+\epsilon)$-approximate Nash Equilibrium is a payment function p such that player i would not deviate their payment by a factor of $1+\epsilon$.

Let $p_i'(T)$ be player's final payment

$$p_i'(e) \leq (1+\epsilon)p_i(e)$$

Approximate Nash Equilibrium in Connection Game

Theorem

Given a single source game and α approximation of minimum-cost Steiner tree T, $\forall\epsilon > 0$, there is a polynomial algorithm which return a $(1+\epsilon)$-approximation Nash equilibrium on Steiner Tree $T^{'}$, where $c(T^{'}) \leq c(T)$.

Proof Sketch

Propose an algorithm that forms a cheaper tree whenever a Nash equilibrium cannot be found.

$$\gamma = \frac{\epsilon c(T)}{(1+\epsilon)n\alpha}$$

In every iteration of previous algorithm, we pay every edge γ

Algorithm for Approximate Nash Equilibrium

Algorithm 2 pseudocode for approximate Nash Equilibrium

$c(e) \leftarrow c(e) - \gamma \quad \forall e \in T$
Run Algorithm 1 to attempt to pay for on T under modified cost
while $e \in T$ **do**
    if e is not fully paid **then**
        Adjust T by replacing T_e with $\bigcup_{i \in T_e} A_i$ to get $T^{'}$
        $c(e) \leftarrow c(e) - \gamma \quad \forall e \in T^{'}$
        Run Algorithm 1 to pay for $T^{'}$ under modified cost
        $P(T^{'}) \leftarrow \sum_i p_i(T^{'})$
        $p_i^{'}(e) \leftarrow p_i(e) + \gamma \frac{p_i(T^{'})}{P(T^{'})}$ for all players and every $e \in T^{'}$
    end if
end while

$T^{'}$ Is Fully Paid under $P^{'}$ and Algorithm 2 Runs in Polynomial Time

$T^{'}$ is fully paid as $\sum_i p_i^{'}(e) = \sum_i p_i(e) + \gamma$

Whenever $e \in T$ is not fully paid, we form a new tree $T^{'}$, and $c(T^{'}) \leq c(T) - \gamma$.

Therefore, we need to reconstruct our Steiner tree most $\frac{c(T)}{\gamma} = \frac{(1+\epsilon)n\alpha}{\epsilon}$ times.

$P'(T')$ Is A $(1+\epsilon)$ Nash Equilibrium

$$p_i'(e) = p_i(e) + \gamma \frac{p_i(T')}{P(T')}$$

Suppose T' has m edges:

$$p_i'(T') = p_i(T') + \gamma \frac{p_i(T')}{P(T')} m = p_i(T') + \gamma \frac{p_i(T')}{c(T') - m\gamma}$$

$P'(T')$ Is A $(1+\epsilon)$ Nash Equilibrium

$$
\begin{aligned}
p_i'(T') - p_i(T') &= \gamma \frac{p_i(T')}{c(T') - m\gamma} m \\
&= \frac{\epsilon c(T) p_i(T') m}{(1+\epsilon) n\alpha (c(T') - m\gamma)} \\
&= \frac{\epsilon c(T) p_i(T')}{(1+\epsilon)\alpha n (\frac{c(T')}{m} - \gamma)} \\
&= \frac{\epsilon c(T) p_i(T')}{(1+\epsilon)\alpha n (\frac{1}{m} - \frac{\gamma}{c(T')}) c(T')}
\end{aligned}
$$

Lower Bound of Approximate Nash Equilibrium

The closest approximation ratio has been obtained so far is 1.55 by modeling this problem into linear programming and randomized rounding.

Frame Title

For any $\epsilon > 0$, there is a game such that any equilibrium which purchases the optimal network is at least a $(3/2 - \epsilon)$-approximate Nash equilibrium.