Connection Game

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Classic Generalized Steiner Tree and Steiner Forest problem

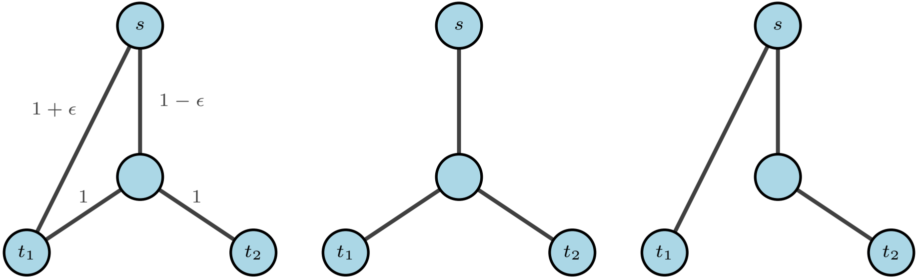
Given an undirected graph $G = (V, E)$ with non-negative edge cost $c_e \geq 0$ for every edge $c_e \in E$.

- In the Steiner Tree problem, we are given a set of terminals $R \subseteq V$. The goal is to compute the minimum-cost subgraph that spans all terminals.
- Given a collection of disjoint subsets of $V : V_1, V_2, V_3, ... V_n$. The goal is to compute a subgraph that any two vertices that belong to the same subset V_i are connected.
- The Steiner Tree problem is a Steiner Forest problem with a single subset of V.
- Denote the optimal solution as OPT. Finding OPT is NP-hard.

Introduction of Selfish Agents

However, when players start to have self-interests, some players might need to pay more if they choose to discard self-interests to archive best centralized optimum. Given an undirected graph G with non-negative edge costs and N players, each player is interested in connecting a set of terminals (nodes in G) via buying a subgraph of G. Players offer each edge in G certain amount of money, and they would like to pay a little as possible.

How Selfish Agents Cause Deviation from Central OPT



Formal Definition of Connection Game

- An undirected graph $G = (V, E)$.
- Non-negative edge cost $c_e \geq 0$ for every edge $c_e \in E$.
- A subset of V for each player that they must connect to.
- A payment function p_i indicates that player's payment strategy. $p_i(e)$ is the contribution that player i would like to offer for edge e.

If the sum of payment on certain edge e is larger than the cost on that edge c_e, this edge is considered as bought and can be used by all players no matter they contribute to it or not. The goal of all players is to connect all of their terminals. If in the end, a player's terminates are not fully connected, they will face an infinite penalty.

Nash Equilibrium in Connection Game

Definition (Nash Equilibrium in Connection Game)

A Nash equilibrium of the connection game is a payment function p such that, if players offer payments p, no payer has an incentive to deviate from their payment.

Theorem (Nash's theorem)

With randomization, any game with finite number of players and actions has a mixed-strategy of Nash equilibrium.

Some Properties of Nash Equilibrium

G_p is a forest.

Let T^i be the smallest tree in G_p connecting all terminals of player i, then player i only contributes to edges in T^i.

Each edge is either bought or not at all.

Price of Anarchy

As shown before, the introduction of selfish agents can lead to worse equilibrium than the best centralized optimum. The question is how bad an equilibrium can be.

Definition (Price of Anarchy)

The price of anarchy of connection game is defined as the ratio of the cost of worst Nash equilibrium over the best centralized design.

$$P_A = \frac{\sum_1^N p_i(e)}{OPT}$$

Price of Anarchy

Lemma

The Price of Anarchy in connection game can be as bad as N.

Price of Stability

Price of stability is a complementary concept of price of anarchy which evaluate how good the best equilibrium can be.

Definition (Price of Stability)

The price of anarchy of connection game is defined as the ratio of the cost of best Nash equilibrium over the best centralized design.

$$P_A = \frac{\sum_1^N p_i(e)}{OPT}$$

Single Source Game

In the single source game, we only allow every player i has one unique terminal $t_i$that they all would like to connect a common terminal s. This can be considered as a special version of Steiner tree problem where $R = \{s, t_1, t_2, ..., t_n\}$.

Definition (Single source Game)

A single source game is a game in which all players share a common terminal s and in addition, each player i has exactly one other terminal t_i.

Algorithm of Assigning Payment Strategies

Proof of p Being Nash Equilibrium

Proof of Price of Stability Being 1

Approximate Nash Equilibrium