# Connection Game

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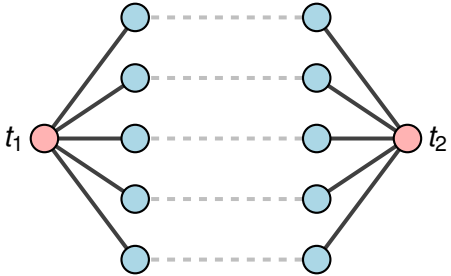
# Classic Steiner Tree Problem

Given an undirected graph $G = (V, E)$ with non-negative edge cost $c_e \geq 0$ for every edge $c_e \in E$.

In the Steiner Tree problem, we are given a set of terminals $R \subseteq V$. The goal is to compute the minimum-cost subgraph that spans all terminals.

# Steiner Tree with Two Terminals

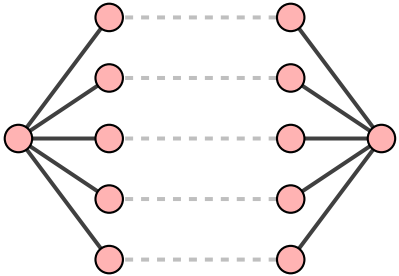
If $|R| = 2$, then it is equal to find the shortest path between two terminals.




Dijkstra's algorithm can find the shortest path polynomial time.

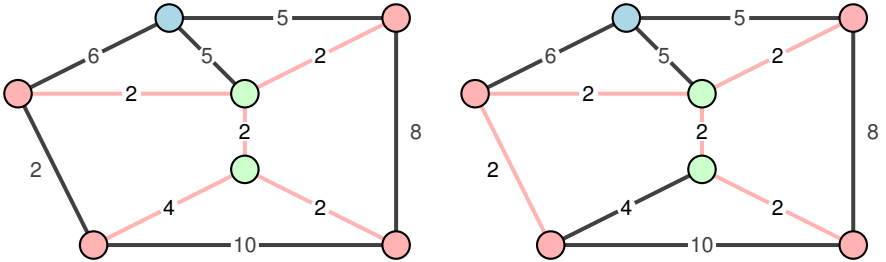
# Steiner Tree with N Terminals

If $|R| = |V|$, then it is equal to find the MST.



Prim's algorithm can find the minimum spanning tree polynomial time.

# Steiner Tree Example



# Steiner Forest Problem

Given a collection of disjoint subsets of $V : V_1, V_2, V_3, ... V_n$. The goal is to compute a subgraph that any two vertices that belong to the same subset $V_i$ are connected.

# Introduction of Selfish Agents

**What if we don't have a director in change of the whole network, instead there exist many agents acting selfishly?**

For example:

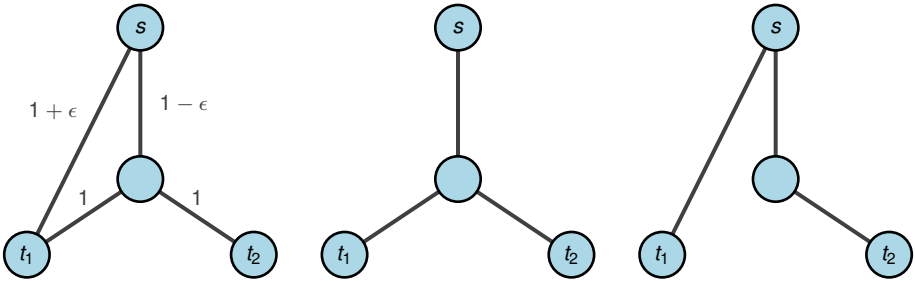
- The railway network across European
- Internet
- ...

# Introduction of Selfish Agents

Given an undirected graph $G$ with non-negative edge costs and $N$ players, each player is interested in connecting a set of terminals (nodes in $G$) via buying a subgraph of $G$. Players offer each edge in $G$ certain payment $p_i(e)$

When players start to behave selfish, i.e. they would like to seek the minimum total payment, some players might deviate best centralized optimum.

# How Selfish Agents Cause Deviation from Central OPT



# Formal Definition of Connection Game

**Definition (Connection Game)**

- An undirected graph $G = (V, E)$.
- Non-negative edge cost $c_e \geq 0$ for every edge $c_e \in E$.
- A subset of $V$ for each player that they must connect to.
- A payment function $p_i$ indicates that player's payment strategy. $p_i(e)$ is the contribution that player $i$ would like to offer for edge $e$.

# Rules of Connection Game

**Definition (Connection Game)**

- If the sum of payment on certain edge $e$ is larger than the cost on that edge $c_e$, this edge is considered as bought.
- If an edge is bought, it can be used by all players no matter they contribute to it or not.
- The goal of all players is to connect all of their terminals. If in the end, a player's terminates are not fully connected, they will face an infinite penalty.

# Nash's Theorem

**Theorem (Nash's theorem)**

*With randomization, any game with finite number of players and actions has a mixed-strategy of Nash equilibrium.*

Nash's theorem states that any finite game has at least one mixed strategies Nash equilibrium but no grantee on pure strategy equilibrium.

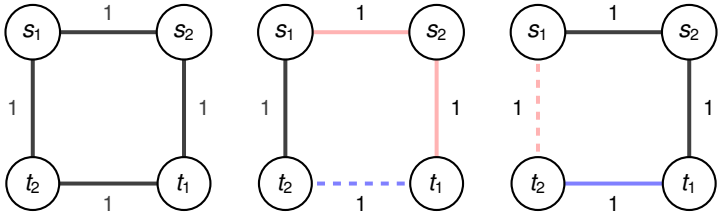
# Nash Equilibrium in Connection Game

**Definition (Nash Equilibrium in Connection Game)**

A Nash equilibrium of the connection game is a payment function p such that, if players offer payments $p$, no payer has an incentive to deviate from their payment.

**Pure Nash equilibrium may not exist in the connection game.**

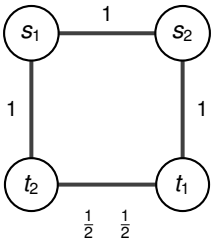
# No Existence of Nash Equilibrium



# Fractional Nash Equilibrium

**Definition (Fractional Nash Equilibrium)**

Game instances where the only Nash equilibrium in existence require players to split the cost of an edge.



# Some Properties of Nash Equilibrium

- $G_p$ is a forest.
- Let $T^i$ be the smallest tree in $G_p$ connecting all terminals of player $i$, then player $i$ only contributes to edges in $T^i$.
- Each edge is either bought or not at all.

# Price of Anarchy

As shown before, the introduction of selfish agents can lead to worse equilibrium than the best centralized optimum. The question is how bad an equilibrium can be.

**Definition (Price of Anarchy)**

The price of anarchy of connection game is defined as the ratio of the cost of worst Nash equilibrium over the best centralized design.

$$P_A = \frac{\sum_1^N p_i(e)}{OPT}$$

# Price of Anarchy

**Lemma**

*The Price of Anarchy in connection game can be as bad as N.*

# Price of Stability

Price of stability is a complementary concept of price of anarchy which evaluate how good the best equilibrium can be.

**Definition (Price of Stability)**

The price of anarchy of connection game is defined as the ratio of the cost of best Nash equilibrium over the best centralized design.

$$P_A = \frac{\sum_1^N p_i(e)}{OPT}$$

# Single Source Game

In the single source game, we only allow every player $i$ has one unique terminal $t_i$ that they all would like to connect a common terminal $s$. This can be considered as a special version of Steiner tree problem where $R = \{s, t_1, t_2, ..., t_n\}$.

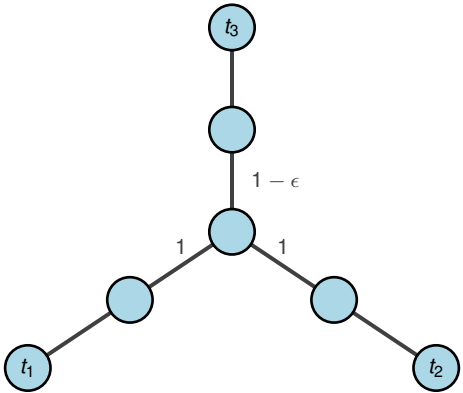
**Definition (Single source Game)**

A single source game is a game in which all players share a common terminal $s$ and in addition, each player $i$ has exactly one other terminal $t_i$.

# How Can We Assign $P$

1. Best Dynamics
2. Shapely Value
3. Utilizing minimum Steiner Tree

# Best Dynamics



# Suppose We have $T^*$ in our hands ...

Now suppose the minimum cost Steiner tree $T^*$ over all players' terminals is given on a sliver plate, one natural thought is can we achieve an equilibrium that equals to the $OPT$?

The answer is YES!

# A Not Important Property of $T^*$

It is trivial to see that every leaf node in $T^*$ is a terminal as if not it is possible to simply discard this node and corresponding edge without affecting the connection of any terminals.

**Fact**

*Every leaf node in $T^*$ is a terminal.*

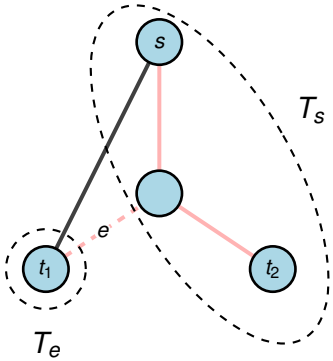
# How Should We Traverse $T^*$?

First, let $T^*$ be rooted from $s$.

For every edge $e$, suppose $e$ is removed from $T^*$, $T^*$ will be divided into two subtrees.

Let the subtree that does not contain $s$ be $T_e$ and the one contains $s$ be $T_s$.

**We will visit $T^*$ in reverse breadth-first-search order.**



# How to Assign $p_i(e)$?

In every iteration, we modify the cost of every edge according to the following rules:

- let costs of edges $e \in T_e$ be $p_i(e)$
- let costs of edges $e \in T_s$ be 0
- costs of edges $e \notin T^*$ stay the same

# How to Assign $p_i(e)$?

We can find the cheapest alternative path $A_i$ from $t_i$ to $s$ from $G \setminus \{e\}$ under modified cost in polynomial time.

Find the minimum value of:

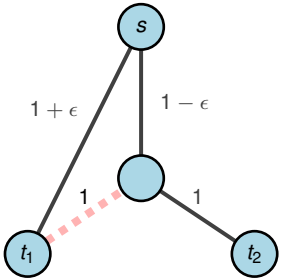
- $c(e)-$ contribution from other players to edge $e$
- cost of $A_i-$ sum of all contribution of player $i$ to $T^*$

# Algorithm

**Algorithm 1** pseudocode for assigning $p_i(e)$

$p_i(e) \leftarrow 0, \forall t_i \in R, \forall e \in E$
**while** $e \in ReverseBFS(T^*)$ **do**
  **if** $e$ is a cut **then**
    $p_i(e) \leftarrow c(e)$
  **else**
    $c(e) \leftarrow p_i(f) \qquad \forall e \in T_e$
    $c(e) \leftarrow 0 \qquad \forall e \in T_S$
    $A_i \leftarrow$ the cheapest alternative path from $s$ to $t_i$ in $G \setminus \{e\}$
    $p_i(e) \leftarrow min\{c(A_i) - \sum_{e \in T^*} p_i(e), c(e) - \sum_j p_j(e)\}$
  **end if**
**end while**

# Example



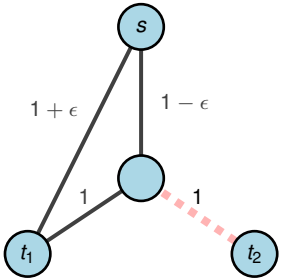

$e$ is not a cut.

consider $c(A_1) = 1 + \epsilon, \quad p_1(T^*) = 0$

$c(e) = 1, \quad \sum_j p_j(e) = 0$

set $p_1(e) = 1$

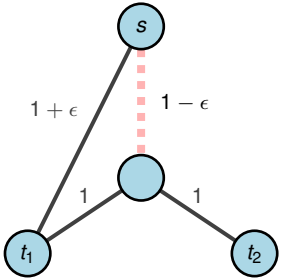
# Example




$e$ is a cut.

set $p_2(e) = 1$

# Example




$e$ is not a cut.

**for** $t_1$

$c(A_1) = 1 + \epsilon, \quad p_1(T^*) = 1$

$c(e) = 1 - \epsilon, \quad \sum_j p_j(e) = 0$

set $p_1(e) = \epsilon$

**for** $t_2$

$c(A_2) = 3 + \epsilon, \quad p_2(T^*) = 1$

$c(e) = 1 - \epsilon, \quad \sum_j p_j(e) = \epsilon$

set $p_2(e) = 1 - 2\epsilon$

# Proof of $p$ Being Nash Equilibrium

**Corollary**

*If $T^*$ is fully paid in the end of the game, the final payment function produced by the above algorithm is indeed a Nash equilibrium.*

By assigning $p_i(e) = min\{c(A_i) - \sum_{e \in T^*} p_i(T^*), c(e) - \sum_{j \in T_i, j \neq i} p_j(e)\}$, we ensure that player $i$ will never contribute to $e$ more than the cost of deviation.

In another word, it is never player $i$'s interests to deviate from buying $e$. And this holds true for every player in every edge in $T^*$. Therefore, $p$ is indeed a Nash equilibrium.

# Proof Agenda

First, assume that the following lemma is true.

Suppose $A_i$ is $i$'s alternative path at some stage of the algorithm. Then suppose there are two nodes $v$ and $w$ on $A_i$ that divides this path into three parts. Let edges on $A_i$ from $t_i$ to $v$ be $f_1$, from $v$ to $w$ be $f_2$, from $w$ to $s$ be $f_3$.

**Lemma**

*There must exist a pair of* $\{v, w\}$ *such that* $f_1 \in T_e$, $f_2 \notin T^*$, $f_3 \in T_s$.

**Prove $T^*$ is fully paid via contradiction.**

# Proof via Contradiction

**Lemma**

*All edges in $T^*$ are bought, i.e. $\sum p_i(e) = c(e)$ for any $e \in T^*$.*

Suppose there exists some edge $e$ such that after all players in $T_e$ have contributed to that edge, $p_i(e) < c(e)$, we can prove that $T_e$ costs more than $\bigcup_{i \in T_e} A_i$ which is a contradiction of $T^*$ is $OPT$.

# Deviation Point

**Definition**

For each player $i$, name the highest ancestor of $t_i$ in $A_i$ that is also in $T_e$, player $i$'s deviation point, denoted $d_i$.

# $T_i$ is fully paid

Let $T_i$ be the subtree rooted from $d_i$

**Fact**

*$T_i$ is fully paid.*

# $T^*$ is fully paid

We modify $T^*$ by replacing $T_e$ as follows: those players $i$ whose alternative paths $A_i$ are associated with nodes in D deviate to $A_i$.

All other players leave their payments unchanged.

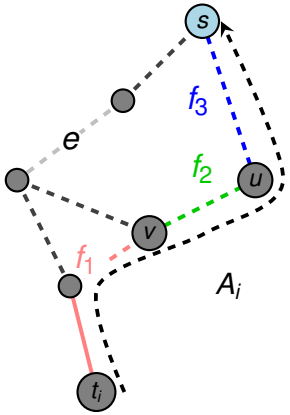
Note that no player has increased their expenditures.

If we can show that all terminals in $T_e$ are connected to $T^* \setminus T_e$ after this modification, we're done.

# Proof of Lemma

**Lemma**

$f_1 \in T_e$, $f_2 \notin T^*$, $f_3 \in T_s$.



# Best Equilibrium

**Theorem**

*For every single source game, there always exists a Nash equilibrium with price of stability being 1.*

# Approximate Nash Equilibrium

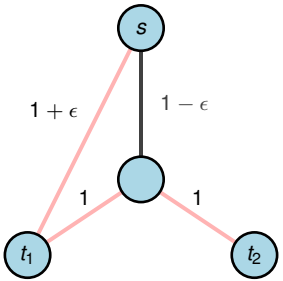
Although we proved that the determined existence of Nash equilibrium in single source game, **the algorithm presented is not feasible** as the finding the minimum cost Steiner tree is NP-hard by itself.

**Definition (Approximate Equilibrium)**

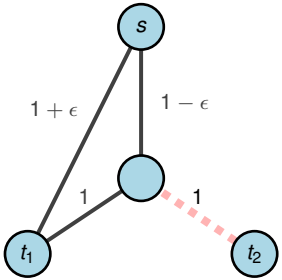
A $(1+\epsilon)$-approximate Nash Equilibrium is a payment function $p$ such that player $i$ would not deviate their payment by a factor of $1+\epsilon$.

# Approximate Nash Equilibrium in Connection Game

If we are given an approximated Steiner tree $T$ and try to use the same algorithm to assign $p_i(e)$ in $T$, since $T$ is not optimal, there will be some edge $e$ that players are unwilling to pay for.



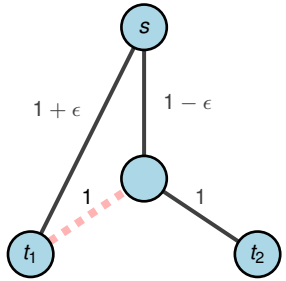
# Example




$e$ is a cut.

set $p_2(e) = 1$

# Example




$e$ is not a cut.

$c(A_2) = 2 - \epsilon, \quad p_2(T^*) = 1$

$c(e) = 1, \quad \sum_j p_j(e) = 0$

set $p_2(e) = 1 - \epsilon$

In the end of the game, $e$ is not fully paid!

Player 2 will deviate.

# Approximate Nash Equilibrium in Connection Game

**Theorem**

*Given a single source game and $\alpha$ approximation of minimum-cost Steiner tree $T$, $\forall \epsilon > 0$ , there is a polynomial algorithm which return a $(1+\epsilon)$-approximation Nash equilibrium on Steiner Tree $T'$, where $c(T') \leq c(T)$.*

# Lower Bound of Approximate Nash Equilibrium

It has been proven that it is NP-hard to approximate Steiner tree problem within ratio $96/95$.

The closest approximation ratio has been obtained so far is 1.55 by modeling this problem into linear programming and randomized rounding.

# Frame Title

For any $\epsilon > 0$, there is a game such that any equilibrium which purchases the optimal network is at least a $(3/2 - \epsilon)$-approximate Nash equilibrium.