Network Creation with Selfish Agents

Nuo Xu

Technical University of Munich

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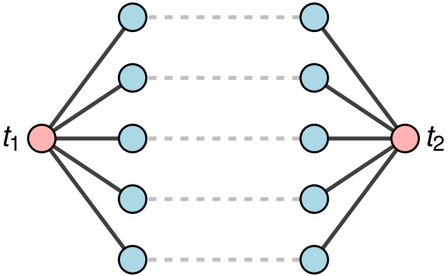
Classic Steiner Tree Problem

Given an undirected graph $G = (V, E)$ with non-negative edge cost $c_e \geq 0$ for every edge $c_e \in E$.

In the Steiner Tree problem, we are given a set of terminals $R \subseteq V$. The goal is to compute the minimum-cost subgraph that spans all terminals.

Steiner Tree with Two Terminals

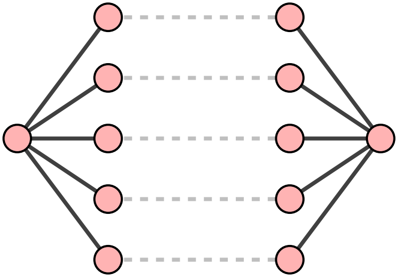
If $|R| = 2$, then it is equal to find the shortest path between two terminals.



Dijkstra's algorithm can find the shortest path polynomial time.

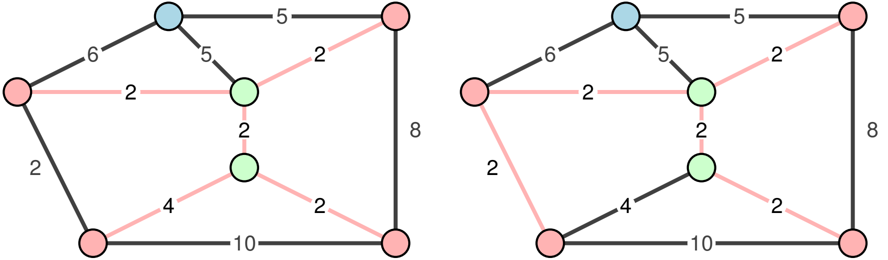
Steiner Tree with N Terminals

If $|R| = |V|$, then it is equal to find the MST.



Prim's algorithm can find the minimum spanning tree polynomial time.

Steiner Tree Example



Steiner Forest Problem

Given a collection of disjoint subsets of $V : V_1, V_2, V_3, ... V_n$. The goal is to compute a subgraph that any two vertices that belong to the same subset V_i are connected.

Introduction of Selfish Agents

What if we don't have a director in change of the whole network, instead there exist many agents acting selfishly?

For example:

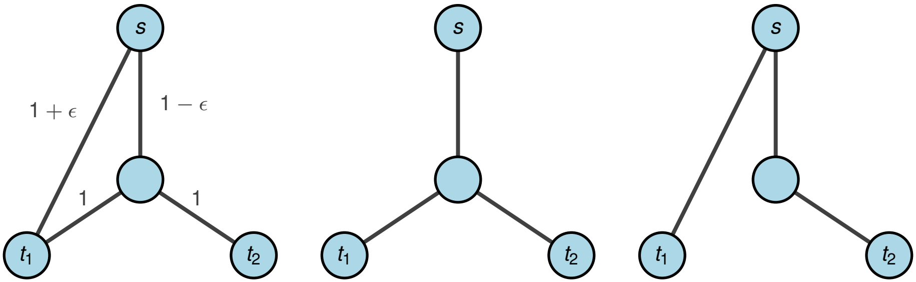
- The railway network across European
- Internet
- ...

Introduction of Selfish Agents

Given an undirected graph G with non-negative edge costs and N players, each player is interested in connecting a set of terminals (nodes in G) via buying a subgraph of G. Players offer each edge in G certain payment $p_i(e)$

When players start to behave selfish, i.e. they would like to seek the minimum total payment, some players might deviate best centralized optimum.

How Selfish Agents Cause Deviation from Central OPT



Formal Definition of Connection Game

Definition (Connection Game)

- An undirected graph $G = (V, E)$.
- Non-negative edge cost $c_e \geq 0$ for every edge $c_e \in E$.
- A subset of V for each player that they must connect to.
- A payment function p_i indicates that player's payment strategy. $p_i(e)$ is the contribution that player i would like to offer for edge e.

Rules of Connection Game

Definition (Connection Game)

- If the sum of payment on certain edge e is larger than the cost on that edge c_e, this edge is considered as bought.
- If an edge is bought, it can be used by all players no matter they contribute to it or not.
- The goal of all players is to connect all of their terminals. If in the end, a player's terminates are not fully connected, they will face an infinite penalty.

Nash's Theorem

Theorem (Nash's theorem)

With randomization, any game with finite number of players and actions has a mixed-strategy of Nash equilibrium.

Nash's theorem states that any finite game has at least one mixed strategies Nash equilibrium but no grantee on pure strategy equilibrium.

In the context of large-scale network creation, allowing players choosing their strategies randomly does not make much sense. We are also not interested in the expected payoff but a certain result.

We are only interested in the pure Nash equilibrium in connection game.

Nash Equilibrium in Connection Game

Definition (Nash Equilibrium in Connection Game)

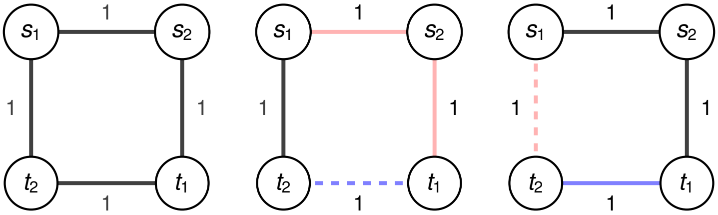
A Nash equilibrium of the connection game is a payment function p such that, if players offer payments p, no payer has an incentive to deviate from their payment.

Some Properties of Nash Equilibrium

- G_p is a forest.
- Let T^i be the smallest tree in G_p connecting all terminals of player i, then player i only contributes to edges in T^i.
- Each edge is either bought or not at all.

No Existence of Nash Equilibrium

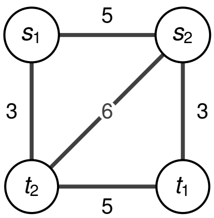
Pure Nash equilibrium may not exist in the connection game if we only players to pay for an edge or not at all.



Fractional Nash Equilibrium

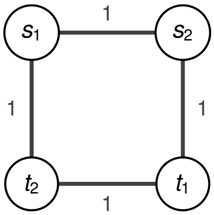
Definition (Fractional Nash Equilibrium)

Game instances where the only Nash equilibrium in existence require players to split the cost of an edge.



No Equilibrium Even with Shared Costs

In fact, even if we allow players to only pay part of an edge, there still exists no equilibrium in this graph.



Price of Anarchy

As shown before, the introduction of selfish agents can lead to worse equilibrium than the best centralized optimum. The question is how bad an equilibrium can be.

Definition (Price of Anarchy)

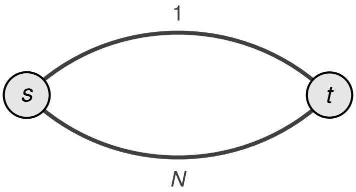
The price of anarchy of connection game is defined as the ratio of the cost of worst Nash equilibrium over the best centralized design.

$$P_A = \frac{\text{Cost of Worst Equilibrium}}{OPT}$$

Price of Anarchy

Lemma

The Price of Anarchy in connection game can be as bad as N.



Price of Stability

Price of stability is a complementary concept of price of anarchy which evaluate how good the best equilibrium can be.

Definition (Price of Stability)

The price of anarchy of connection game is defined as the ratio of the cost of best Nash equilibrium over the best centralized design.

$$P_A = \frac{\text{Cost of Best Equilibrium}}{OPT}$$

Single Source Game

In the single source game, we only allow every player i has one unique terminal t_i that they all would like to connect a common terminal s. This can be considered as a special version of Steiner tree problem where $R = \{s, t_1, t_2, ..., t_n\}$.

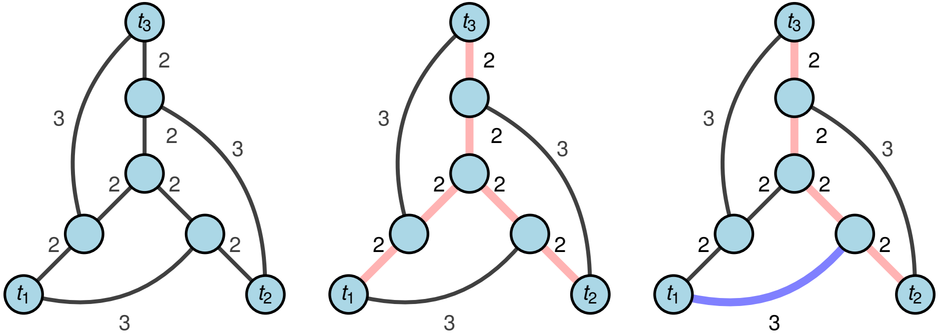
Definition (Single source Game)

A single source game is a game in which all players share a common terminal s and in addition, each player i has exactly one other terminal t_i.

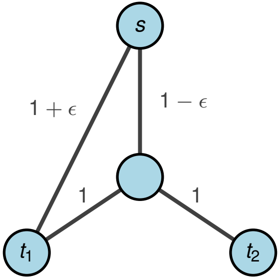
Possible Strategies

1. Best Response Dynamics
2. Shapely Value
3. Utilizing minimum Steiner Tree

Best Response Dynamics



Shapely Value



Suppose We Have T^* in Our Hands . . .

Now suppose the minimum cost Steiner tree T^* over all players' terminals is given on a sliver plate, one natural thought is can we achieve an equilibrium that equals to the OPT?

The answer is YES!

It is trivial to see that every leaf node in T^* is a terminal as if not it is possible to simply discard this node and corresponding edge without affecting the connection of any terminals.

Fact

Every leaf node in T^* is a terminal.

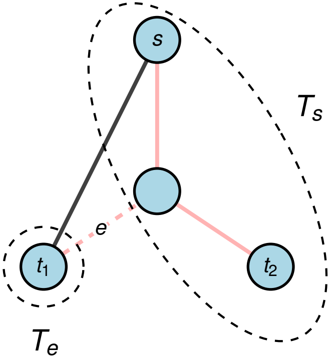
How Should We Traverse T^*?

First, let T^* be rooted from s.

For every edge e, suppose e is removed from T^*, T^* will be divided into two subtrees.

Let the subtree that does not contain s be T_e and the one contains s be T_s.

We will visit T^* in reverse breadth-first-search order.



How to Assign $p_i(e)$?

In every iteration, we modify the cost of every edge according to the following rules:

- let costs of edges $e \in T_e$ be $p_i(e)$
- let costs of edges $e \in T_s$ be 0
- costs of edges $e \notin T^*$ stay the same

How to Assign $p_i(e)$?

We can find the cheapest alternative path A_i from t_i to s from $G \setminus \{e\}$ under modified cost in polynomial time.

Find the minimum value of:

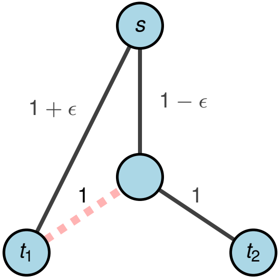
- $c(e)-$ contribution from other players to edge e
- cost of A_i- sum of all contribution of player i to T^*

Algorithm for Assigning $p_i(e)$

Algorithm 1 pseudocode for assigning $p_i(e)$

$p_i(e) \leftarrow 0, \forall t_i \in R, \forall e \in E$
while $e \in ReverseBFS(T^*)$ **do**
    if e is a cut **then**
        $p_i(e) \leftarrow c(e)$
    else
        $c(e) \leftarrow p_i(f) \qquad \forall e \in T_e$
        $c(e) \leftarrow 0 \qquad \forall e \in T_S$
        $A_i \leftarrow$ the cheapest alternative path from s to t_i in $G \setminus \{e\}$
        $p_i(e) \leftarrow min\{c(A_i) - \sum_{e \in T^*} p_i(e), c(e) - \sum_j p_j(e)\}$
    end if
end while

Example



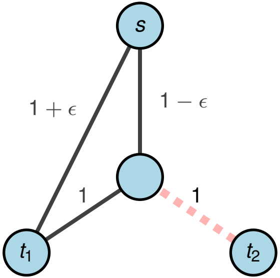
e is not a cut.

consider $c(A_1) = 1 + \epsilon, \quad p_1(T^*) = 0$

$c(e) = 1, \quad \sum_j p_j(e) = 0$

set $p_1(e) = 1$

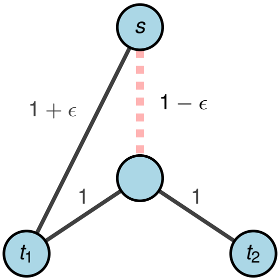
Example



e is a cut.

set $p_2(e) = 1$

Example



e is not a cut.

for t_1

$c(A_1) = 1+\epsilon, \quad p_1(T^*) = 1$

$c(e) = 1-\epsilon, \quad \sum_j p_j(e) = 0$

set $p_1(e) = \epsilon$

for t_2

$c(A_2) = 3+\epsilon, \quad p_2(T^*) = 1$

$c(e) = 1-\epsilon, \quad \sum_j p_j(e) = \epsilon$

set $p_2(e) = 1-2\epsilon$

Proof of p Being Nash Equilibrium

Corollary

If T^* is fully paid in the end of the game, the final payment function produced by the above algorithm is indeed a Nash equilibrium.

By assigning $p_i(e) = min\{c(A_i) - \sum_{e \in T^*} p_i(T^*), c(e) - \sum_{j \in T_i, j \neq i} p_j(e)\}$, we ensure that player i will never contribute to e more than the cost of deviation.

In another word, it is never player i's interests to deviate from buying e. And this holds true for every player in every edge in T^*. Therefore, p is indeed a Nash equilibrium.

An Assumption

First, assume that the following lemma is true.

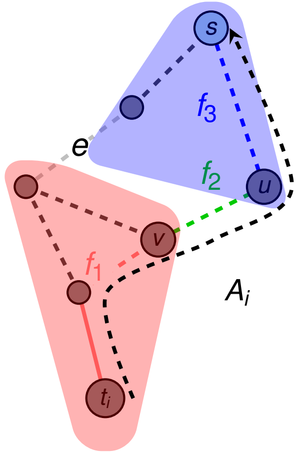
Suppose A_i is i's alternative path at some stage of the algorithm. Then suppose there are two nodes v and w on A_i that divides this path into three parts. Let edges on A_i from t_i to v be f_1, from v to w be f_2, from w to s be f_3.

Lemma

There must exist a pair of $\{v, w\}$ such that $f_1 \in T_e$, $f_2 \notin T^*$, $f_3 \in T_s$.

Prove T^* is fully paid via contradiction.

An Assumption



Proof via Contradiction

Lemma

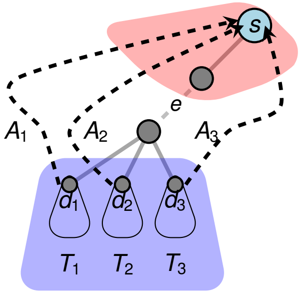
All edges in T^* are bought, i.e. $\sum p_i(e) = c(e)$ for any $e \in T^*$.

Suppose there exists some edge e such that after all players in T_e have contributed to that edge, $p_i(e) < c(e)$, we can prove that T_e costs more than $\bigcup_{i \in T_e} A_i$ which is a contradiction of T^* is OPT.

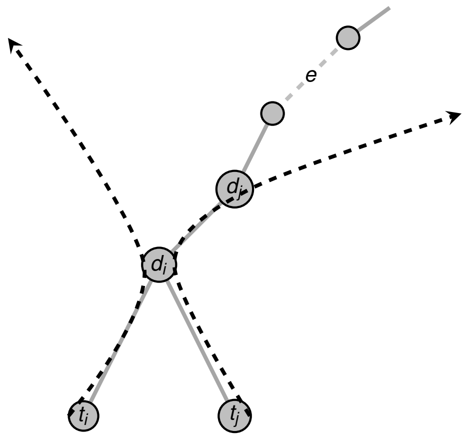
Deviation Point

Definition

For each player i, name the highest ancestor of t_i in A_i that is also in T_e, player i's deviation point, denoted d_i. Let T_i be the subtree rooted from d_i.



d_is Are Siblings in $\bigcup_{i \in T_e} A_i$



A_i Can Be Fully Paid Without Increasing Any Payment

- Recall in each step when we decide a payment $p_i(e)$ for player i on e, we choose $p_i(e) \leftarrow min\{c(A_i) - \sum_{e \in T^*} p_i(e), c(e) - \sum_j p_j(e)\}$
- If we decide player i needs to pay $c(e) - \sum_j p_j(e)$, then e is bought.
- If e is not bought, then we set all players in T_e, their payment function on e to be $c(A_i) - \sum_{e \in T^*} p_i(e)$

T^* Is Not Unique

- Modify T^* by replacing T_e with $\bigcup_{i \in T_e} A_i$.
- All other players leave their payments unchanged.
- A_i will be fully paid.
- By the lemma, we know that all players in T_e can stay connect to s after the modification without increasing their expenditures.

This is a contradiction to T^* being unique.

Proof of Lemma

Lemma

$f_1 \in T_e$, $f_2 \notin T^*$, $f_3 \in T_s$.

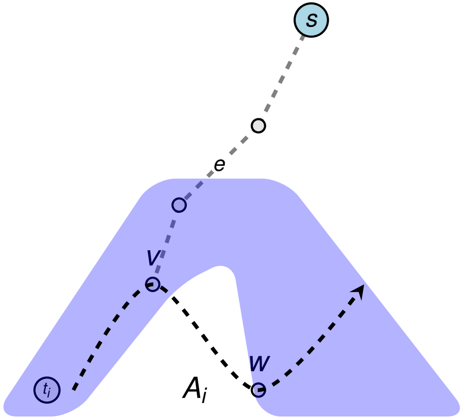
Suppose once A_i reaches a node in T_s, then all subsequent edges will be in T_s as edges in T_s cost 0 under modified cost.

A_i will always reach a node in T_s since $s \in T_s$.

So we only need to prove that only A_i leaves T_e it will only be in $G \setminus T_e$, i.e. A_i **does not go back to T_e anymore.** We show this by contradiction.

Proof of Lemma

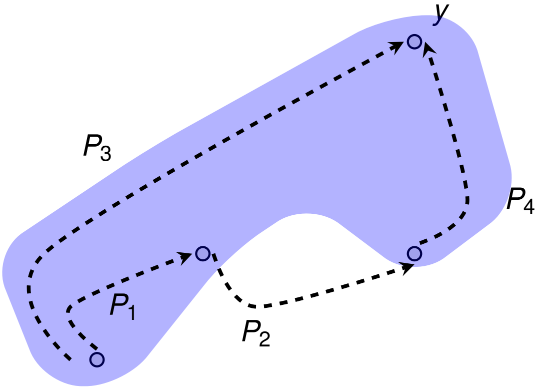
Suppose there are two nodes v and w on A_i that divides the part of A_i before reaching to T_s into three parts such that $P_1 \in T_e$, $P_2 \notin T^*$, $P_4 \in T_e$.



Proof of Lemma

Let y be the lowest common ancestor of t_i and w in T_e.

We will show that by replacing $P_1 \bigcup P_2$ with $P_3 \bigcup P_4$, player i would obtain a better deviation than A_i.



P1 is Strictly Below y

Claim

$c'(P_4) = p_i(P_4) = 0$ for i.

- None of edges in P_4 are on player i's path from t_i to s.

Claim

P_1 is restrictly below y, i.e. P_1 is a subpath of P_3.

- If not, P_3 is just a subpath of P_1.
- $c'(P_3) \leq c'(P_1)$ and $c'(P_4) = 0$.
- $c'(P_3 \bigcup P_4) \leq c'(P_1 \bigcup P_2)$.

$P_3 \bigcup P_4$ is at least as cheap as $P_1 \bigcup P_2$

- When deciding the payment of each edge in P_3, $p_i(e) \leq c(A_i)$
- At any time, player *i*'s payments are upper bounded by the modified cost of his alternate path, which is in turn upper bounded by the modified cost of any path from t_i to s.
- $c^{'}(P_3 \bigcup P_4) = c^{'}(P_3) \leq c^{'}(P_1 \bigcup P_2)$

Best Equilibrium

Theorem

For every single source game, there always exists a Nash equilibrium with price of stability being 1.

Hardness of Steiner Tree

Although we proved that the determined existence of Nash equilibrium in single source game, **the algorithm presented is not feasible** as the finding the minimum cost Steiner tree is NP-hard by itself.

Fact

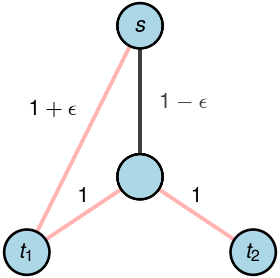
Steiner Tree problem is APX-hard.

$$c(T) = \alpha c(T^*)$$

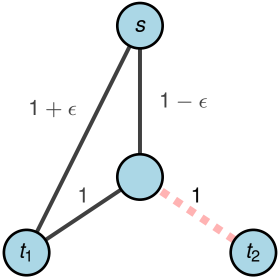
It has been proven that it is NP-hard to approximate Steiner tree problem within ratio $96/95$.

Approximate Steiner Tree Will Fail at Archiving Nash Equilibrium

If we are given an approximated Steiner tree T and try to use the same algorithm to assign $p_i(e)$ in T, since T is not optimal, there will be some edge e that players are unwilling to pay for.



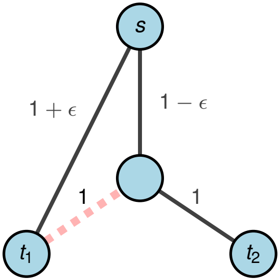
Example



e is a cut.

set $p_2(e) = 1$

Example



e is not a cut.

$c(A_2) = 2 - \epsilon, \quad p_2(T^*) = 1$

$c(e) = 1, \quad \sum_j p_j(e) = 0$

set $p_2(e) = 1 - \epsilon$

In the end of the game, e is not fully paid!

Player 2 will deviate.

Approximate Nash Equilibrium in Connection Game

Definition (Approximate Equilibrium)

A $(1+\epsilon)$-approximate Nash Equilibrium is a payment function p such that player i would not deviate their payment by a factor of $1+\epsilon$.

Let $p_i^{'}(T)$ be player's final payment

$$p_i^{'}(e) \leq (1+\epsilon)p_i(e)$$

Approximate Nash Equilibrium in Connection Game

Theorem

Given a single source game and α approximation of minimum-cost Steiner tree T, $\forall \epsilon > 0$, there is a polynomial algorithm which return a $(1+\epsilon)$-approximation Nash equilibrium on Steiner Tree $T^{'}$, where $c(T^{'}) \leq c(T)$.

γ Reduction

Instead of trying to fully pay every edge:

- We try to pay T with $c(e) = c(e) - \gamma, \quad \gamma = \frac{\epsilon c(T)}{(1+\epsilon)n\alpha}$
- If all players agree to pay for T with γ reduction, we can increase their payment on every edge in T by $\gamma \cdot \frac{p_i(T)}{P(T)}$
- T will be full paid, and it is a $(1 + \epsilon)$ Nash Equilibrium
- Forms a cheaper tree whenever a Nash equilibrium cannot be found even with γ reduction.

How Do We Get The Value of γ

If all players ends of buying T with γ reduction, $p_i'(e) \leftarrow p_i(e) + \gamma \frac{p_i(T)}{P(T)}$.
Suppose there are m edges in T,

$$p_i'(T) - p_i(T) \leq \epsilon p_i(T)$$

$$p_i'(T) - p_i(T) = \gamma \frac{p_i(T)}{P(T)} m = \gamma \frac{p_i(T)}{c(T) - m\gamma} m \leq \epsilon p_i(T)$$

$$\frac{m\gamma}{c(T) - m\gamma} \leq \epsilon \Rightarrow \gamma \leq \frac{\epsilon c(T)}{(1+\epsilon)m}$$

Since we also need to bound the value after reconstruction, we define
$\gamma = \frac{\epsilon c(T)}{(1+\epsilon)n\alpha}, \quad n = |V|$

Algorithm for Approximate Nash Equilibrium

Algorithm 2 Modify T

```
c'(e) ← c(e) − γ   ∀e ∈ T
Run Algorithm 1 to attempt to pay for on T under modified cost
while e ∈ T do
    if e is not fully paid then
        Adjust T by replacing T_e with ⋃_{i∈T_e} A_i to get T'
        BREAK
    end if
end while
Modify(T')
```

$P(T') \leftarrow \sum_i p_i(T')$

$p_i'(e) \leftarrow p_i(e) + \gamma \frac{p_i(T')}{P(T')}$ for all players and every $e \in T'$

$T^{'}$ Is Fully Paid under $P^{'}$ and Algorithm 2 Runs in Polynomial Time

$T^{'}$ is fully paid as $\sum_i p^{'}_i(e) = \sum_i p_i(e) + \gamma$

Whenever $e \in T$ is not fully paid, we form a new tree $T^{'}$, and $c(T^{'}) \leq c(T) - \gamma$. Therefore, we need to reconstruct our Steiner tree most $\frac{c(T)}{\gamma} = \frac{(1+\epsilon)n\alpha}{\epsilon}$ times.

$P^{'}(T^{'})$ Is A $(1+\epsilon)$ Nash Equilibrium

$$p_i^{'}(e) = p_i(e) + \gamma \frac{p_i(T^{'})}{P(T^{'})}$$

Suppose $T^{'}$ has $m^{'}$ edges:

$$p_i^{'}(T^{'}) = p_i(T^{'}) + \gamma \frac{p_i(T^{'})}{P(T^{'})} m^{'} = p_i(T^{'}) + \gamma \frac{p_i(T^{'})}{c(T^{'}) - m^{'}\gamma}$$

$P'(T')$ Is A $(1+\epsilon)$ Nash Equilibrium

$$
\begin{aligned}
p_i'(T') - p_i(T') &= \gamma \frac{p_i(T')}{c(T') - m'\gamma} m' \\
&= \frac{\epsilon c(T) p_i(T') m'}{(1+\epsilon) n\alpha (c(T') - m'\gamma)} \\
&= \frac{\epsilon c(T) p_i(T')}{(1+\epsilon)\alpha n (\frac{c(T')}{m'} - \gamma)} \\
&= \frac{\epsilon c(T) p_i(T')}{(1+\epsilon)\alpha (\frac{n}{m'} - \frac{n\gamma}{c(T')}) c(T')}
\end{aligned}
$$

$P'(T')$ Is A $(1+\epsilon)$ Nash Equilibrium

$$\frac{n\gamma}{c(T')} = \frac{\epsilon c(T)n}{(1+\epsilon)n\alpha c(T')} = \frac{\epsilon c(T)}{(1+\epsilon)\alpha c(T')}$$

$$c(T) = \alpha c(T^*)$$

$$\frac{n\gamma}{c(T')} = \frac{\epsilon c(T^*)}{(1+\epsilon)c(T')} < \epsilon$$

$$p_i'(T') - p_i(T') \leq \frac{\epsilon c(T)p_i(T')}{(1+\epsilon)\alpha(1-\epsilon)c(T')} = \frac{\epsilon p_i(T')}{(1+\epsilon)(1-\epsilon)} \leq \epsilon p_i(T')$$

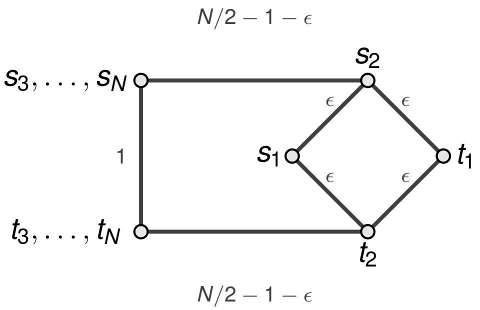
General Game

In the general case, players can have different numbers of terminals and do not necessarily share the same source.

Lemma

The price of stability in general cases can be bad as $\Theta(N)$.

Proof of Lemma



$$OPT = 1 + 3\epsilon$$

Any equilibrium ends up paying $N - 2$.

Lower Bounds on Nash Equilibrium in General Case

Since the price of stability can be as bad as $\Theta(N)$ and pure Nash equilibrium may not exist at all, we cannot hope to be able to provide cheap Nash equilibrium for multi-source games.

Therefore, we instead hope that we can get a cheap $(1+\epsilon)$ approximate equilibrium.

Theorem

There exists such a graph that any equilibrium that purchases the optimal Steiner forest is at least a $(3/2-\epsilon)$ approximate equilibrium for any $\epsilon > 0$.

Graph Construction

- Start with a cycle with $2N$ vertices from v_1 to v_{2N} clockwise.
- For v_i, add an edge from vertex i to vertex $(i + N - 1) mod(2N)$ and an edge from vertex i to $(i + N + 1) mod(2N)$.
- All edges have cost 1.
- Add N players and each of them have a source s_i and a terminal t_i.
- Let vertices from v_1 to v_N be the s for each player and vertices v_{N+1} to v_{2N} be the terminals.

$OPT = 2N - 1$ with all the edges in the outer cycle with edge from s_1 to t_N.

Proof

Lemma

For any equilibrium than is better than 3/2, player 1 and player N won't pay more than 3.

- Suppose player 1 pays x to $s_1 \to t_1$, y to $t_1 \to t_N$.
- Player 1 has the choice to pay for only x or $1 + y$.

$$\frac{x+y}{x} \leq \frac{3}{2} \quad \frac{x+y}{y+1} \leq \frac{3}{2}$$

$$x + y \leq 3$$

Proof

Lemma

Given there at least exists a player i needs to pay at least $\frac{2N-7}{N-2}$, we find that player i will at least have the incentive $\frac{6N-21}{4N-11}$ to deviate.

- Suppose player i pays x to $s_i \rightarrow t_i$, y to $s_{i-1} \rightarrow s_i$, z to $t_i \rightarrow t_{i+1}$.
- Player 1 has the choice to pay for only x, $1 + y$,or $1 + z$.

Player i has $max\{\frac{x+y+z}{x}, \frac{x+y+z}{1+y}, \frac{x+y+z}{1+z}\}$ incentive to deviate.

Proof

$max\{\frac{x+y+z}{x}, \frac{x+y+z}{1+y}, \frac{x+y+z}{1+z}\}$ is minimized when $x = 1 + y = 1 + z$.

$$x + y + z \geq \frac{2N-7}{N-2}$$

$$x \geq \frac{4N-11}{3N-6}$$

$$\frac{x+y+z}{x} \geq \frac{3x-2}{x} = 3 - \frac{2}{x} \geq \frac{6N-21}{4N-11}$$

Bicriteria Approximation

	Single Source	Multi-Source
Exists Nash	(1,1)	(3,1)
Can find Nash in poly-time	$(1+\epsilon, 1.55)$	$(4.65+\epsilon, 2)$
Lower Bounds on Existence	(1,1)	(1.5,1)

The closest approximation ratio has been obtained so far is 1.55 by k-restricted Steiner Tree.
A better approximation ratio, 1.39, was archived 2 years after this paper by modeling Steiner tree into linear programming relaxtion and itertative randomized rounding.