Problem Chosen	2024 MCM/ICM Summary Sheet	Team Control Number
A		2528508

Old Stairs Are An Open Book

January, 27th 2025

We will be addressing Problem A. This problem asks our group to assist a team of archaeologists in determining the answer to several questions in regards to the history of a set of stairs.

First, we are tasked with determining a set of measurements to request from the archaeologists that will assist in the development of our model. The measurements that we have settled on require the following tools: A level, Level-able reference edges, a machined straight edge, a digital micrometer that can measure depth, and optionally a portable LIDAR scanner. The LIDAR scanner will save time and increase the accuracy of the model drastically. The measurements themselves consist of measuring the length and width of each tread, along with measuring depths across a $5cm \times 5cm$ grid covering each tread. This grid will then be converted into a matrix with the $(0,0)$ entry being the back left corner of a tread. The bottom of the matrix will be the front edge of each step. If the archaeologists have LIDAR available to them then this process is made quite easy with one LIDAR scan of the staircase. Note, LIDAR is readily available to most people through iPhones.

Using the LIDAR scan we can obtain these results:

- A 3D model to help us visualize the stairs
- Volume lost to wear
- A map of where low spots are

Utilizing these results we can determine:

- The frequency of stair use
- The favored direction of travel
- A prediction of the amount of people utilizing a stair case at once
- A prediction on the age of a stairwell
- The density of use on the stair

After producing these results, we applied our model onto a set of wild stairs on our college campus. We then compared our prediction of how many steps per day to the information that we had available regarding the history of the building. We then concluded that the prediction was accurate in relation to the history of the building and our university.

Contents

1 Problem

Archaeologists want to be able to estimate how much foot traffic occurred on a set of staircases during a time window. By measuring the volume lost to wear and where the wearing occurred we claim we can answer the following:

1. How often were the stairs used?
2. Was a certain direction of travel favored?
3. How many people used the stairs simultaneously?

We will also attempt to address the following questions assuming more information is available including an estimate of the age of the stairs exists, the way the stairwell was used, and an estimate of the daily patterns of life in the structure are available.

4. Is the wear consistent with the additional information?
5. What is the age of the stairwell and how reliable is the given estimate?
6. How do repairs effect the results?
7. Can the materials source be corroborated?
8. Can determine the foot traffic volume per estimated time frame?

2 Development of the Model

2.1 Assumptions

Assumptions

- Anyone who used the staircase traversed the entire staircase.
- Treads are assumed to start as flat surfaces.
- Weathering from climate is uniform across the treads.
- The softest material in the person-grit-stair system will be the tread of the stair.
- We assume the grit to be of quartz sand.
- We can assign a shoe material based on the era and location associated with a staircase. The era and location could also indicate bare feet.
- The use of archaeologists implies only old structures, so we will only consider archaic materials for stair construction being that of stone, aggregate, or wood.
- The force on the stair per step is approximated to the weight of the average person in the era which the staircase was active.
- The average shoulder width of a human has not changed drastically over our history.
- People will not normally crowd on stairs.

2.2 Definitions and Variables

Below are a few definitions and variables that will be used throughout the paper. First, is a figure showing the different elements of a staircase that will be referenced throughout.
Definitions:

- Ball of foot: The area on the bottom of the foot starting after the toe and ending before the arch of a foot
- Wear: The removal of material via various processes, such as: abrasion, erosion, corrosion, adhesion, etc.

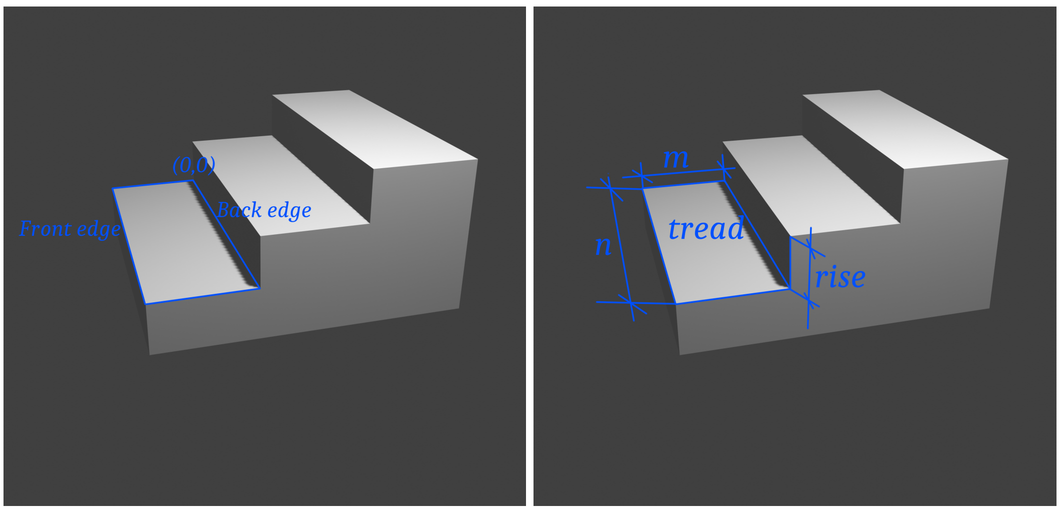


Figure 1: Elements of a stair

Variables:

- V_{tot}: total volume of tread lost to abrasion mm^3
- δ: side length of a cell in the grid cm
- V_0: volume of tread lost to one footstep mm^3
- k: coefficient of abrasion (for the stair) *dimensionless*
- S: distance a foot slips per step mm
- F: weight of a person on average N
- H: hardness of the softest material in the system *Mohs*
- m: number of rows in tread matrix corresponding to its position from the back of the tread.
- n: number of columns in the tread matrix corresponding to its position to the right of the tread.
- T: tread matrix
- $d_{m,n}$: depth at a point m, n
- D: diffusion constant
- ρ_{air}: density of carbon as carbon dioxide in the air $\frac{g}{cm^3}$
- ρ_{lichen}: density of carbon as carbon dioxide in the lichen $\frac{g}{cm^3}$
- H: height of the lichen mm
- R: radius of the lichenmm

2.3 Model Motivation

We began the development of our model with some field research. We used a staircase that we knew was built in 1915, made of hard sandstone, uncovered from weather, and had minimal repair. By direct observation, we could conclude:

- The ball of the foot comes into contact first with the tread going along either directions
- When walking down stairs, the ball of the foot lands around the upper third of the tread down from the front edge.
- When walking up stairs, the ball of the foot lands around a third of the tread from the back edge.
- From the pattern of wear along the m axis, a direction could be determined.
- From the pattern of wear along the n axis, the most used side of the stair could be determined

From these observations we hypothesized that the wear marks location along the n axis could determine from what direction people were walking towards the stair. For example, if the wear marks are in the middle of the treads and are indicative of people going up the stairs, along with moving further from one side towards the middle of the stairs. Then, we know that people likely went up the stairs starting on one side and then moved towards the center to enter a passage.

This justifies the need to make accurate measurements of the treads topography, which we will cover in the next section.

2.4 Measurements required

Tools required:

- Level
- level-able reference edges
- machined straight edge
- Digital Micrometer that can measure depth
- Portable LIDAR scanner (optional but will save time and increase accuracy)

The archaeologists will be asked for length and widths of each of the treads. A basic heat map generated from the wear patterns of each tread is denoted as a **depth map**. The archaeologists will need to bring multiple reference flats (whose ends heights can be adjusted and tuned to be level) to span the length of the treads long axis. Once all flats are made level to each other, we can measure depths in a $5cm \times 5cm$ grid covering each tread with reference to our levels and the back edge. The data recorded by this method will be arranged as by facing the front edge closest to you and the back edge furthest, the corner on the far left on the back edge is $0,0$ and every coordinate on the tread corresponds to the indices of matrix T whose entries are the depths in mm. However, LIDAR, which is readily available to most people through iPhones, can make short work of the topography measurements of the stairs. Moreover, if the stairs are in a non-extreme environment, lichen growth is rather likely, especially upon any stone or aggregate stairs. We ask the archaeologists measure the diameter and radius of the largest lichen growing on the structure.

3 Overall Model

3.1 Frequency of Stairs Use

Using our matrix of recorded depths, T can be converted into a rough depth map by importing the matrix into Excel, followed by applying a color scale to the data. This adds a color to each depth value, allowing us to accurately visualize the wear pattern on the stair.

If the archaeologists have access to LIDAR, like Polycam [1],then they can capture a scan of the stairs in a matter of 30 minutes depending on the size of the stair case. We can import the scan into blender3D [3] to which:

1. Make a duplicate of the model and trim away unnecessary geometry (walls, geomerty that isn't in the top layer of the tread)
2. cut out the stairs treads and move them to be as planar to each other as possible
3. Apply a material that takes the z value and assigns it a brightness from black to white
4. adjust scaling until you have at least one spot fully black and one fully white

Once we have our data, be it from LIDAR or manual measurements, we can calculate the volume via two methods:

1. Directly through the 3D software
2. Using the manual data collected in matrix T, the volume can be approximated by taking each $d_{i,j}, d_{i+1,j}, d_{i,j+1}, d_{i+1,j+1} \in T$, then finding

$$V_{tot} = \sum_{i=0}^{n} \sum_{j=0}^{m} \delta * \frac{d_{i,j} + d_{i+1,j} + d_{i,j+1} + d_{i+1,j+1}}{4}$$

Now with v_{tot} estimated, an archaeologist can identity the material type by reasonable means of its look, hardness, and grain structure, which will help us find matching material samples. Using Archard's Law $V = \frac{kSF}{H}$, [5] we can find how many steps were taken in total on a tread by:

1. determining the coefficient of abrasion k via a matched material
2. determining the average weight F of a person in the area and era
3. measuring how much a foot slips while walking given a person of weight F
4. knowing the hardness H of the tread.

Then our formula for total steps taken is,

$$\text{Steps} = \frac{V_{tot} * H}{kSF}$$

This method addresses problem number 1 and provides us with a rough estimate for the amount of steps ever taken on a given step in the staircase, telling us how often the stairs were used.

3.2 Favored direction of Travel

To determine the direction of travel favored, we first have to consider where the stairs lead to. If the stairs in question are the only entrance to a location, then wear should be equal on both sides since a person heading into a dead end will need to return the same way. We can determine if a particular direction of travel was favored by analyzing the depth map of our staircase. Hence, if we split the volume of the stair in half to have a right and left hand side, then if the left hand side has more wear (i.e. more volume lost) with a wear pattern indicative of going up, then its clear more people went up than down on this staircase. This is assuming people in this culture keep to the right hand side while walking forward as opposed to the left hand side. In general, to determine a direction favored of up versus down, regardless of cultural norms, one only has to study the depth map. If wear is more severe along the front end of the tread, more people were traveling down. If the wear is more severe along a lower value on the m-axis, it is evident that more people were traveling up the staircase.

3.3 A Prediction of Tread Occupancy

To figure out how many people may have used these stairs simultaneously, we will first look at the lengths of the treads on the staircase being observed and take the smallest length. Let this length be n. According to CDC survey data [6] from 1988 to 1994, the average male shoulder width is about 41.1 cm. We can take this measurement instead of the female shoulder width, because if 2 males can walk side by side on a staircase, it is fair to assume that so can 2 females or 1 female and 1 male. Another assumption is that most people enjoy a bit of personal space while using the stairs. Specifically, we believe that most people prefer around 20cm between themselves and another person while walking on a set of stairs. The last assumption for this problem we can make is that at least one person will be close to the edge of the stairwell, presumably near a railing. This person will be around 20cm from the wall.

With these assumptions, we have derived this formula that determines an estimate of how many people could comfortably use the stairs at once.

$$X = \left\lceil \frac{n - 100cm}{50cm} \right\rceil$$

It is important to note that this is just a rough estimate of how many people could use a tread simultaneously. Pairing a value for X with an analysis of the depth map, one can determine if the stair tread was one person wide or two people wide based on if there was one uniform wear pattern on the stair tread or if there are two or more separate wear patterns on the stair tread.

While determining what guidance can be provided to the archaeologists for the following 5 problems, it is important to remember we have access to an estimated age of the staircase, the way the stairwell was used, and estimates on daily patterns of life in the structure.

3.4 Is Our Model Consistent With Background Info?

For question number 4, archaeologists would need to construct their own depth map of their staircase of interest taking into account all information available. They should also take into consideration their climate and whether the stairs are indoors or outdoors. A staircase that is outside in a rainy part of the world will incur much more weathering than any staircase inside. The archaeologists could simulate an entire tread matrix to convert into a depth map, or they could just create a rudimentary one in Microsoft Paint. Either one will serve our purpose. Then, the archaeologists should render the actual depth map of the staircase and compare the model with the actual. This will allow them to find out if the wear is consistent with the available information. While a perfect replica is not expected, a depth map that preserves direction and wear severity would be enough to conclude that the wear is consistent with the information available.

3.5 Estimated Age of Stairwell

Determining an exact age of the staircase may be beyond the scope of what is feasible, but there are ways to determine another estimation. In the case of wooden steps, it is very likely that it will be rotted too much to glean any useful information with methods such as dendrochronolgy. With the method described in subsection 3.1, we are able to deduce the total amount of steps in a stair case over its life time. This is assuming little to no renovations. It is important to remember for this problem we know the estimated age of the staircase, the way the stairwell was used along with estimates on daily patterns of life. With this additional information we can deduce an estimated steps per year that makes sense for how the stair well was used. Now that we have a total amount of steps in the stairs lifetime along with an estimated steps per year we have:

$$\text{Age} = \frac{\text{Total Steps On Stair}}{\text{Estimated Steps Per Year}}$$

With this new estimate of the age of the stairwell we can then compare it to the estimated age and conclude if the first estimated age is likely to be accurate. Note, there are other ways to date materials including radio carbon dating. However, these methods may be damaging to the structure itself and require a lab.

3.6 Determining Stair Renovations

Guidance for what repairs have been done is relatively straightforward. We will operate under the assumption that any wooden staircases will have rotted, and obtaining further information about repairs is futile. The only evidence that may have survived is extra nails dated younger than the estimated age of the nails in the staircase, if any. For stone and aggregate stairs, we will consider three main types of repairs. That is, shaving, filling, and replacement. Shaving would be sanding down the tread when it is no longer flat to make it flat again. This would be evident if the age and material of the step is consistent through the whole step and staircase, but it appears to have significantly less wear than anticipated given its estimated age and patterns of use. Filling would be if material is added to the tread to fill in any cracks, grooves, holes, or indentations. These repairs would be evident by parts of the tread having either a completely different material, or a relatively younger age than the rest of the step. Different material will be easy to identify, but as we've covered, assigning exact age can be tricky. Something to look for could be areas of very different discoloration, inconsistent with discoloration from wear. Shallower or fewer cracks in a particular section of the tread can also indicate younger material used to fill gaps. Finally, the archaeologists can identify replacement stairs by analyzing how a step appears compared to the staircase as a whole. This should be significantly easier than fillings. The most obvious indicator would be if a step was made with a completely different material. This would be easy to spot and strongly implies a damaged step was replaced. Another indicator that a step was replaced is if the age of the step is younger than the estimated age of the staircase. Similar to determining this with fillings, analyzing the cracks and discoloration can give great insight to the age of a particular step. Further techniques in age analysis can be found in Section 3.5 and Section 3.8.

3.7 Discovering the Source

As far as we are aware, the source of material can not be determined by wear patterns on steps alone. However, it is understood that archaeologists likely have the ability to determine what a material is made up of. So for the case of stone stairs, it should be more than reasonable to assume they can find matching materials, whether that be a nearby quarry or a location further away. For the case of wood, finding a matching material will be quite difficult. If the wood is severely decayed, they can apply several techniques including radio carbon dating to find the period the wood is from, however this would require a lab. They could also look into the native tree species around the area during the estimated age the stairs were built. With these species in mind they should be able to deduce which species the wood is from based on the grain density and hardness. If the stairs are made with a cement aggregate mixture, there are only a few systems that are known to exist. With historical and geographical context the archaeologist should be able to understand which system they are dealing with.

3.8 Calculating Density of Stair Use

Luckily there is a handy equation to help us with this problem in Section 3.1. Our archaeologists can use the formula

$$\text{Steps} = \frac{V_{tot} * H}{kSF}$$

to find the total number of steps taken on one step of our staircase. We can then use the estimated age of the last repair and use the simple equation

$$\text{Steps per day} = \frac{\text{Total Steps}}{\text{Total Days Since Last Repair}}$$

The formula calculating steps will automatically calculate the steps since the last repair, since it works from the point when the tread was last level. Thus, we need to use the length of time since the last repair, assuming we can accurately estimate it. Tackling the problem of establishing if there was a lot of people using the stairs over a short period of time or a small number of people using the stairs over a long period of time is tricky. This requires establishing a time frame for when the staircase started being used to when it stopped. We can then determine the time frame of use with a technique called Lichenometry [4]. It is a common practice in archaeological dating since lichen grows at a rather predictable rate. The archaeologists will use the diameter of the largest lichen growing on the staircase to calculate for how long the lichen has been growing. It would help make more accurate predictions if they are able to figure out what kind of lichen they are dealing with. Since one of the limiting factors of lichen growth is foot traffic, the lichen will only have started flourishing when the staircase became abandoned. Thus, subtracting the estimated age of the lichen from the estimated age of the staircase will give you a prediction for the amount of time the staircase was in use. To find the estimated age of the lichen, the archaeologists can use the equation [2],

$$\frac{dR}{dt} = \frac{2D\rho_{air}}{\pi H \rho_{lichen}}$$

This equation will provide an estimated growth rate of the lichen in units of $\frac{mm}{year}$. The constants of ρ_{air} and ρ_{lichen} can be looked up based on the location of the staircase and the species of lichen. The archaeologists can then use

$$\text{Lichen Age} = \frac{\text{Radius of Lichen}}{\text{Radius Growth per Year}}$$

to obtain a rough estimate about the age of the lichen. An estimated age of the lichen implies an estimated date in which the stair case was no longer being used, since lichen will start to grow once there is no longer foot traffic. Subtracting the age of the lichen from the estimated ago of the lichen will provide an estimated span of time in which the stairs were used. Using our algorithm from Section 3.1, the archaeologists can calculate how many steps were taken on the staircase. Then, using the equation

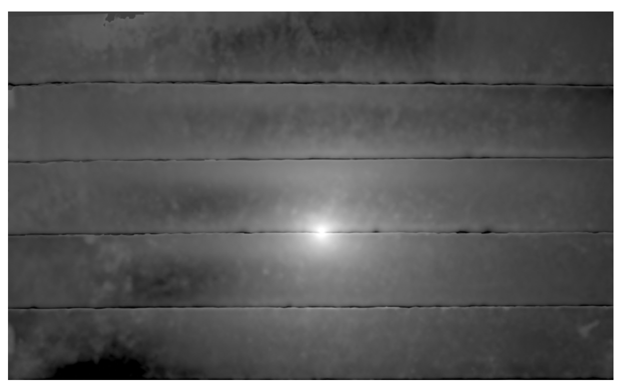
$$\text{Steps per Year} = \frac{\text{Total Steps}}{\text{Estimated Age}}$$

a prediction for the density of stair use can be obtained in units of $\frac{\text{steps}}{\text{year}}$. The archaeologists can determine how many people per year is a large number or a small number, and this algorithm will allow them to get a rough estimate as to whether a large number of people used the stairs in a short period of time, or a small number of people in a long period of time.

3.9 Applied Model on Wild Stairs

During our field research period, we identified a set of stairs on our college campus that was located in the entrance of one of the original buildings on campus. This resulted in an old set of stairs that was perfect to apply our model onto. Below is the LIDAR scan of these stairs that we identified. We were able to produce a LIDAR scan through the app Polycam.

(a) Image of the example stairs as a raw scan



(b) LIDAR scan rendered depth map

Figure 2: LIDAR Scan of Wild Stairs

Looking at this processed LIDAR scan, overlaying a height map, we can observe a few things about the wear on the steps. First, on the right side of the stairs, it seems as if there is no wear. This could be the result of a low quality scan. However, we came to a different conclusion. We realized that during most of this building's lifetime, the vast majority of other buildings on campus have been to the

left of this building. Thus, any people coming or going, would be coming from the same direction, resulting in wearing on only one side of the staircase. Thus, we believe that the wear pattern on this wear map shows us both people leaving and entering the building along this direct path to class, library, or cafeteria at the time.
By using the scan, we can compute the volume of the wear void space with blender3d. For this stair case example,(assuming limestone) the average volume per step was $V_{tot} = 6 * 10^{-4} m^3$, $S \approx 0.001m$, $k = 0.00164$ [5], $F \approx 686N$, $H = 5 * 10^6 Pa$

$$\begin{aligned} \text{Steps} &= \frac{V_{tot} * H}{kSF} \\ &= 2670000 \text{ steps} \end{aligned}$$

The building was built in 1915, so this number amounts to roughly 66 steps per day. Taking into account all of the breaks in school, this number looks more like 132 steps per day since we are only in school for half of the calendar year. Currently our student enrollment is around 3000 students, however 110 years ago the student enrollment was roughly 500 students. Further, since this building is one of two currently used for traditional arts and this staircase leads to one of three entrances to the building, the average steps per day seems to be a fair estimate.

4 Conclusions

A LIDAR scan can deduce a good amount of information about a set of stairs. The depth map produced is key to estimating with better certainty to the amount of foot traffic upon a set of stairs along with additional visual analysis. Further, our model is adaptable to be used without the need for LIDAR and can still be insightful with more basic tools. When provided with additional information such as the estimated age of the stairs, the way the stairwell was used, and estimates about daily patterns of life, it is easier to obtain much more information, and in some cases, the model does not even need to be used. However, that information combined with the model can be very powerful and can tackle some difficult questions that are hard to quantify.

5 Strengths and Weaknesses

5.1 Strengths

- Relatively easy and straight forwards to use.
- Requires minimal tools and measurements.
- Model produces pretty pictures
- Observation of results does not require a background in mathematics
- Results in accurate measuremetns in the case that LIDAR is used.

5.2 Weaknesses

- Cannot determine absolute age
- Model does not work well with wooden stairs
- If the tread is skinny it is hard to determine if people utilize the stairs to go up more vs down more as the wear pattern will be overlapping.
- Lichenometry is inexact and requires additional information that is not measurable on site.
- Our methods does not explicitly take into account severe climate effects.

References

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