Notes on Calculus

Calculus I, II & III

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I. Limits

A. Definitions

1) *Precise Definition*
We say $\lim_{x\to a} f(x) = L$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

2) *"Working" Definition*
We say $\lim_{x\to a} f(x) = L$ if we can make $f(x)$ as close to L as we want by taking x sufficiently close to a (on either side of a) without letting $x = a$.

3) *Right hand limit*
$\lim_{x\to a^+} f(x) = L$ This has the same definition as the limit except it requires $x > a$.

4) *Left hand limit*
$\lim_{x\to a^-} f(x) = L$ This has the same definition as the limit except it requires $x < a$.

5) *Limit at Infinity*
We say $\lim_{x\to\infty} f(x) = L$ if we can make $f(x)$ as close to L as we want by taking x large enough and positive. There is a similar definition for $\lim_{x\to-\infty} f(x) = L$ except we require x large and negative.

6) *Infinite Limit*
We say $\lim_{x\to a} f(x) = \infty$ if we can make $f(x)$ arbitrarily large (and positive) by taking x sufficiently close to a (on either side of a) without letting $x = a$. There is a similar definition for $\lim_{x\to a} f(x) = -\infty$ except we make $f(x)$ arbitrarily large and negative.

B. Relationship between the limit and one-sided limits

$$\begin{aligned}
\lim_{x\to a} f(x) = L &\implies \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L \\
\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L &\implies \lim_{x\to a} f(x) = L \\
\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x) &\implies \lim_{x\to a} f(x) \text{ Does not exist.}
\end{aligned}$$

C. Properties

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and c is any number, then,

$$\begin{aligned}
\lim_{x\to a} [cf(x)] &= c \lim_{x\to a} f(x) \\
\lim_{x\to a} [f(x) \pm g(x)] &= \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) \\
\lim_{x\to a} [f(x)g(x)] &= \lim_{x\to a} f(x) \lim_{x\to a} g(x) \\
\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] &= \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \iff \lim_{x\to a} g(x) \neq 0 \\
\lim_{x\to a} [f(x)]^n &= \left[\lim_{x\to a} f(x)\right]^n \\
\lim_{x\to a} \left[\sqrt[n]{f(x)}\right] &= \sqrt[n]{\lim_{x\to a} f(x)}
\end{aligned}$$

D. Basic Limit Evaluations

Note: $sgn(a) = 1$ if $a > 0$ and $sgn(a) = 1$ if $a < 0$.

$$\begin{aligned}
\lim_{x\to\infty} e^x &= \infty \\
\lim_{x\to-\infty} e^x &= 0 \\
\lim_{x\to\infty} \ln(x) &= \infty \\
\lim_{x\to 0^+} \ln(x) &= -\infty \\
\lim_{x\to\infty} \frac{b}{x^r} &= 0, r > 0 \\
\lim_{x\to-\infty} \frac{b}{x^r} &= 0, r > 0, x^r \in \mathbb{R} \text{ for } -x
\end{aligned}$$

Odd and even powers:

$$\begin{aligned}
n \text{ even}: &\quad \lim_{x\to\pm\infty} x^n = \infty \\
n \text{ odd}: &\quad \lim_{x\to\infty} x^n = \infty, \ \& \lim_{x\to-\infty} x^n = -\infty \\
n \text{ even}: &\quad \lim_{x\to\pm\infty} ax^n + \cdots + bx + c = sgn(a)\infty \\
n \text{ odd}: &\quad \lim_{x\to\infty} ax^n + \cdots + bx + c = sgn(a)\infty \\
n \text{ odd}: &\quad \lim_{x\to-\infty} ax^n + \cdots + bx + c = -sgn(a)\infty
\end{aligned}$$

E. Evaluation Techniques

1) *Continuous Functions*
If $f(x)$ is continuous at a then $\lim_{x\to a} f(x) = f(a)$.
2) *Continuous Functions and Composition*
If $f(x)$ is continuous at b and $\lim_{x\to a} g(x) = b$ then $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(b)$.

3) *Factor and Cancel*
Example:

$$\begin{aligned}\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x} &= \lim_{x\to 2}\frac{(x-2)(x+6)}{x(x-2)}\\ &= \lim_{x\to 2}\frac{x+6}{x}\\ &= \frac{8}{2}=4\end{aligned}$$

4) *Rationalize Numerator/Denominator*
Example:

$$\begin{aligned}\lim_{x\to 9}\frac{3-\sqrt{x}}{x^2-81} &= \lim_{x\to 9}\frac{3-\sqrt{x}}{x^2-81}\cdot\frac{3+\sqrt{x}}{3+\sqrt{x}}\\ &= \lim_{x\to 9}\frac{9-x}{(x^2-81)(3+\sqrt{x})}\\ &= \lim_{x\to 9}\frac{-1}{(x+9)(3+\sqrt{x})}\\ &= \frac{-1}{(18)(6)}=\frac{-1}{108}\end{aligned}$$

5) *Combine Rational Expressions*
Example:

$$\begin{aligned}\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{x+h}-\frac{1}{x}\right) &= \lim_{h\to 0}\frac{1}{h}\left(\frac{x-(x+h)}{x(x+h)}\right)\\ &= \lim_{h\to 0}\frac{1}{h}\left(\frac{-h}{x(x+h)}\right)\\ &= \lim_{h\to 0}\frac{-1}{x(x+h)}\\ &= -\frac{1}{x^2}\end{aligned}$$

6) *L'Hospital's Rule*
If $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$ or $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\pm\infty}{\pm\infty}$ then, $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}a$ is a number, ∞ or $-\infty$.

7) *Polynomials at Infinity*
Define $p(x)$ and $q(x)$ polynomials. To compute $\lim\limits_{x\to\pm\infty}\dfrac{p(x)}{q(x)}$, then factor the largest power of x in $q(x)$ out of both $p(x)$ and $q(x)$ then compute limit. Example:

$$\begin{aligned}\lim_{x\to -\infty}\frac{3x^2-4}{5x-2x^2} &= \lim_{x\to -\infty}\frac{x^2(3-\frac{4}{x^2})}{x^2(\frac{5}{x}-2)}\\ &= \lim_{x\to -\infty}\frac{3-\frac{4}{x^2}}{\frac{5}{x}-2}\\ &= -\frac{3}{2}\end{aligned}$$

8) *Piecewise Function*
Example:

$\lim\limits_{x\to -2} g(x)$ where $g(x)\begin{cases}x^2+5 \text{ if } x<-2,\\ 1-3x \text{ if } x\geq -2\end{cases}$

Compute two, one-sided limits,

$$\lim_{x\to -2^-} g(x)=\lim_{x\to -2^-} x^2+5=9$$

$$\lim_{x\to -2^+} g(x)=\lim_{x\to -2^+} 1-3x=7$$

One-sided limits are different so $\lim\limits_{x\to -2} g(x)$ doesn't exist. If the two one-sided limits had been equal, then $\lim\limits_{x\to -2} g(x)$ would have existed and had the same value.

F. Some Continuous Functions

Partial list of continuous functions and the values of x for which they are continuous.

- Polynomials for all x.
- Rational function, except for xs that give division by zero.
- $\sqrt[n]{x}$ (n odd) for all x.
- $\sqrt[n]{x}$ (n even) for all $x\geq 0$.
- e^x for all x.
- $\ln x$ for $x>0$.
- $\cos(x)$ and $\sin(x)$ for all x.
- $\tan(x)$ and $\sec(x)$ provided $x\neq\cdots,\frac{-3\pi}{2},\frac{-\pi}{2},\frac{\pi}{2},\frac{3\pi}{2},\cdots$.
- $\cot(x)$ and $\csc(x)$ provided $x\neq\cdots,-2\pi,-\pi,0,\pi,2\pi,\cdots$.

G. Intermediate Value Theorem

Suppose that $f(x)$ is continuous on $[a,b]$ and let M be any number between $f(a)$ and $f(b)$. Then there exists a number c such that $a<c<b$ and $f(c)=M$.

II. Derivatives

A. Definition and Notation

If $y=f(x)$ then the derivative is defined to be

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

If $y=f(x)$ then all of the following are equivalent notations for the derivative.

$$f'(x)=y'=\frac{df}{dx}=\frac{dy}{dx}=\frac{d}{dx}\left(f(x)\right)=Df(x).$$

If $y=f(x)$ all of the following are equivalent notations for derivative evaluated at $x=a$.

$$f'(a)=y'\big|_{x=a}=\frac{df}{dx}\Big|_{x=a}=\frac{dy}{dx}\Big|_{x=a}=Df(a).$$

B. Interpretation of the Derivative

If y = f (x) then,

1) $m=f(a)$ is the slope of the tangent line to $y=f(x)$ at $x=a$ and the equation of the tangent line at $x=a$ is given by $y=f(a)+f(a)(xa)$.
2) $f(a)$ is the instantaneous rate of change of $f(x)atx=a$.
3) If $f(x)$ is the position of an object at time x then $f(a)$ is the velocity of the object at $x=a$.

C. Basic Properties and Formulas

If $f(x)$ and $g(x)$ are differentiable functions (the derivative exists), c and n are any real numbers,

1) $(cf)' = cf'(x)$.
2) $(f \pm g)' = f'(x) \pm g'(x)$.
3) $(f\ g)' = f'\ g + f\ g'$ – Product Rule.
4) $\left(\frac{f}{g}\right)' = \frac{f' g_f\ g'}{g^2}$ – Quotient Rule.
5) $\frac{d}{dx}(c) = 0$.
6) $\frac{d}{dx}(x^n) = n\ x^{n-1}$ – Power Rule.
7) $\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))g'(x)$ – Chain Rule.

This is stated more completely in the following subsections:

Derivative of a Scalar Multiple A constant factor c can be factored out from the differential sign:

$$(cu)' = cu' \quad \equiv \quad d(cu) = c\ du.$$

Derivative of a Sum If the functions u, v, w, etc. are differentiable one by one, ther sum and difference is also differentiable, and equal to the sim or difference of the derivatives:

$$(u+v-w)' = u'+v'-w' \quad \equiv \quad d(u+v-w) = du+dv-dw.$$

It is possible that the summands are not differentiable separately, but their sum or difference is. Then we calculate the derivative by a definition.

Derivative of a Product of Two Functions

$$(uv)' = u'v + uv' \quad \equiv \quad d(uv) = v\ du + u\ dv.$$

It is possible that the terms are not differentiable separately, but their product is. Then we calculate the derivative by a definition.

Derivative of a Quotient If both u and v are differentiable, and $v(x) \neq 0$, their ratio is also differentiable:

$$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2} \quad \equiv \quad d\left(\frac{u}{v}\right) = \frac{v\ du - u\ dv}{v^2}.$$

Chain Rule The composite function $y = u(v(x))$ has the derivative

$$\frac{dy}{dx} = u'(v)v'(x) = \frac{du}{dv}\frac{dv}{dx},$$

where the functions $u = u(v)$ and $v = v(x)$ are differentiable functions with respect to their own variables. $u(v)$ is called the exterior function, and $v(x)$ is called the interior function. According to this, $\frac{du}{dv}$ is the *exterior derivative*, and $\frac{dv}{dx}$ is the *interior derivative*. It is possible that the functions u and v are not differentiable separately, but the composite function is. Then we calculate the derivative by a definition.

The similar method is applied to the function $y = u(v(w(x)))$:

$$y' = \frac{dy}{dx} = \frac{du}{dv}\frac{dv}{dw}\frac{dw}{dx}$$

Partial Derivatives Partial Derivatives are a derivative of part of a function while holding all other variables constant (and thus act like constants for the derivative

Given $z = f(x,y)$, the partial derivative of z with respect to x is

$$f_x(x,y) = z_x = \frac{\partial z}{\partial x} = \frac{\partial f(x,y)}{\partial x}$$

The partial derivative with respect to y is

$$f_y(x,y) = z_y = \frac{\partial z}{\partial y} = \frac{\partial f(x,y)}{\partial y}$$

Given f_{xyy}, work "inside to outside", e.g. f_x, then f_{xy}, then f_{xyy}. The result is $f_{xyy} = \frac{\partial^3 f}{\partial^2 y \partial x}$.

Given $\frac{\partial^3 f}{\partial^2 y \partial x}$, work right to left in the denominator.

U-Substitution Let $u = f(x)$ (can be more than one variable). Determine: $du = \frac{f(x)}{dx} dx$ and solve for dx.

D. Frequently used derivatives

Derivatives of common functions.

$$\begin{aligned}
\frac{d}{dx}(C(\text{constant})) &= 0 \\
\frac{d}{dx}(x) &= 1 \\
\frac{d}{dx}(x^n) &= nx^{n-1},\ (n \in \mathbb{R}) \\
\frac{d}{dx}(\frac{1}{x}) &= -\frac{1}{x^n} \\
\frac{d}{dx}(\frac{1}{x^n}) &= -n\frac{1}{x^{n+1}} \\
\frac{d}{dx}(\sqrt{x}) &= \frac{1}{\sqrt{x}} \\
\frac{d}{dx}(\sqrt[n]{x}) &= \frac{1}{n\sqrt[n]{x^{n-1}}},\ (n \in \mathbb{R}, n \neq 0, x > 0) \\
\frac{d}{dx}(e^x) &= e^x \\
\frac{d}{dx}(e^{bx}) &= be^{bx},\ (b \in \mathbb{R}) \\
\frac{d}{dx}(a^x) &= a^x \ln a,\ (a > 0) \\
\frac{d}{dx}(a^{bx}) &= ba^{bx} \ln a,\ (b \in \mathbb{R}, a > 0) \\
\frac{d}{dx}(\ln(x)) &= \frac{1}{x} \\
\frac{d}{dx}(\log_a(x)) &= \frac{1}{x}\log_a e \\
&= \frac{1}{x \ln a},\ (a > 0, a \neq 1, x > 0)
\end{aligned}$$

Derivatives of common Trigonometry functions.

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$
$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$
$$\frac{d}{dx}(\tan(x)) = \sec^2(x)$$
$$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$$
$$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$$
$$\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$$
$$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}, x \in [-1,1]$$
$$\frac{d}{dx}(\cos^{-1}(x)) = \frac{-1}{\sqrt{1-x^2}}, x \in [-1,1]$$
$$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}, \frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$$
$$\frac{d}{dx}(\sec^{-1}(x)) = \frac{1}{\mid x \mid \sqrt{x^2-1}}, |x| > 1$$
$$\frac{d}{dx}(\sinh(x)) = \cosh(x)$$
$$\frac{d}{dx}(\cosh(x)) = \sinh(x)$$
$$\frac{d}{dx}(\tanh(x)) = \mathrm{sech}^2(x)$$
$$\frac{d}{dx}(\coth(x)) = -\mathrm{csch}^2(x)$$
$$\frac{d}{dx}(\mathrm{sech}(x)) = -\mathrm{sech}(x)\tanh(x)$$
$$\frac{d}{dx}(\mathrm{csch}(x)) = -\mathrm{csch}(x)\coth(x)$$
$$\frac{d}{dx}(\sinh^{-1}) = \frac{1}{\sqrt{x^2+1}}$$
$$\frac{d}{dx}(\cosh^{-1}) = \frac{-1}{\sqrt{x^2-1}}, x > 1$$
$$\frac{d}{dx}(\tanh^{-1}) = \frac{1}{1-x^2} - 1 < x < 1$$
$$\frac{d}{dx}(\mathrm{sech}^{-1}) = \frac{1}{x\sqrt{1-x^2}}, 0 < x < 1$$

Derivatives with Chain Rule variants.

$$\frac{d}{dx}\left([f(x)]^n\right) = n[f(x)]^{n-1}f'(x)$$
$$\frac{d}{dx}\left(e^{f(x)}\right) = f'(x)e^{f(x)}$$
$$\frac{d}{dx}\left(\ln[f(x)]\right) = \frac{f'(x)}{f(x)}$$
$$\frac{d}{dx}\left(\sin[f(x)]\right) = f'(x)\cos[f(x)]$$
$$\frac{d}{dx}\left(\cos[f(x)]\right) = -f'(x)\sin[f(x)]$$
$$\frac{d}{dx}\left(\tan[f(x)]\right) = f'(x)\sec^2[f(x)]$$
$$\frac{d}{dx}\left(\sec[f(x)]\right) = f'(x)\sec[f(x)]\tan[f(x)]$$
$$\frac{d}{dx}\left(\tan^{-1}[f(x)]\right) = \frac{f'(x)}{1+[f(x)]^2}$$

E. *Higher Order Derivatives*

The Second Derivative is denoted as

$$f''(x) = f^{(2)}(x) = \frac{d^2f}{dx^2}$$

and is defined as

$$f''(x) = (f'(x))'$$

i.e. the derivative of the first derivative, $f'(x)$.

The n^{th} derivative is denoted as

$$f^{(n)}(x) = \frac{d^nf}{dx^n}$$

and is defined as

$$f^{(n)}(x) = (f^{(n-1)}(x))'$$

i.e. the derivative of the $(n-1)^{st}$ derivative, $f^{(n-1)}(x)$.

F. *Implicit Differentiation*

Example: Find y if $e^{2x-9y} + x^3y^2 = sin(y) + 11x$. Remember $y = y(x)$ here, so products/quotients of x and y will use the product/quotient rule and derivatives of y will use the chain rule. The "trick" is to differentiate as normal and every time you differentiate a y you tack on a y (from the chain rule). After differentiating solve for y.

$$\begin{aligned} e^{2x-9y} + x^3y^2 &= \sin(y) + 11x \\ e^{2x-9y}(2-9y') + 3x^2y^2 + 2x^3y\ y' &= \cos(y)y' + 11 \\ 2e^{2x-9y} - 9y'e^{2x-9y} + 3x^2y^2 + 2x^3y\ y' &= \cos(y)y' + 11 \\ \left(2x^3y - 9e^{2x-9y} - \cos(y)\right)y' &= 11 - 2e^{2x-9y} - 3x^2y^2 \\ \implies y' = \frac{11 - 2e^{2x-9y} - 3x^2y^2}{2x^3y - 9e^{2x-9y} - \cos(y)} \end{aligned}$$

G. *Increasing/Decreasing Concave Up/Concave Down*

Critical Points $x = c$ is a critical point of $f(x)$ provided either

1) $f(c) = 0$,
2) $f(c)$ doesnt exist.

Increasing/Decreasing

1) If $f(x) > 0$ for all x in an interval I then $f(x)$ is increasing on the interval I.
2) If $f(x) < 0$ for all x in an interval I then $f(x)$ is decreasing on the interval I.
3) If $f(x) = 0$ for all x in an interval I then $f(x)$ is constant on the interval I.

Concave Up/Concave Down

1) If $f(x) > 0$ for all x in an interval I then $f(x)$ is concave up on the interval I.
2) If $f(x) < 0$ for all x in an interval I then $f(x)$ is concave down on the interval I.

Inflection Points $x = c$ is a inflection point of $f(x)$ if the concavity changes at $x = c$.

H. Extrema

Absolute Extrema

1) $x = c$ is an absolute maximum of $f(x)$ if $f(c)f(x)$ for all x in the domain.
2) $x = c$ is an absolute minimum of $f(x)$ if $f(c)f(x)$ for all x in the domain.

Fermats Theorem If $f(x)$ has a relative (or local) extrema at $x = c$, then $x = c$ is a critical point of $f(x)$.

Extreme Value Theorem If $f(x)$ is continuous on the closed interval $[a, b]$ then there exist numbers c and d so that,

1) ac, db,
2) $f(c)$ is the abs. max. in $[a, b]$,
3) $f(d)$ is the abs. min. in $[a, b]$.

Finding Absolute Extrema To find the absolute extrema of the continuous function $f(x)$ on the interval $[a, b]$ use the following process.

1) Find all critical points of $f(x)$ in $[a, b]$.
2) Evaluate $f(x)$ at all points found in Step 1.
3) Evaluate $f(a)$ and $f(b)$.
4) Identify the abs. max. (largest function value) and the abs. min.(smallest function value) from the evaluations in Steps 2 & 3.

Relative (local) Extrema

1) $x = c$ is a relative (or local) maximum of $f(x)$ if $f(c)f(x)$ for all x near c.
2) $x = c$ is a relative (or local) minimum of $f(x)$ if $f(c)f(x)$ for all x near c.

1st Derivative Test If $x = c$ is a critical point of $f(x)$ then $x = c$ is

1) a rel. max. of $f(x)$ if $f(x) > 0$ to the left of $x = c$ and $f(x) < 0$ to the right of $x = c$.
2) a rel. min. of $f(x)$ if $f(x) < 0$ to the left of $x = c$ and $f(x) > 0$ to the right of $x = c$.
3) not a relative extrema of $f(x)$ if $f(x)$ is the same sign on both sides of $x = c$.

2nd Derivative Test If $x = c$ is a critical point of $f(x)$ such that $f(c) = 0$ then $x = c$

1) is a relative maximum of $f(x)$ if $f(c) < 0$.
2) is a relative minimum of $f(x)$ if $f(c) > 0$.
3) may be a relative maximum, relative minimum, or neither if $f(c) = 0$.

Finding Relative Extrema and/or Classify Critical Points

1) Find all critical points of $f(x)$.
2) Use the 1^{st} derivative test or the 2^{nd} derivative test on each critical point.

Mean Value Theorem If $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) then there is a number $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Newtons Method If x_n is the n^{th} guess for the root/solution of $f(x) = 0$ then $(n + 1)^{st}$ guess is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ provided $f'(x_n)$ exists.

III. Integrals

A. Definitions

Definite Integral Suppose $f(x)$ is continuous on $[a, b]$. Divide $[a, b]$ into n subintervals of width Δx and choose x_i^* from each interval. Then

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Anti-Derivative An anti-derivative of $f(x)$ is a function, $F(x)$, such that $F(x) = f(x)$.

Indefinite Integral Given $F(x)$ is an anti-derivative of $f(x)$, then

$$\int f(x)dx = F(x) + c.$$

B. Fundamental Theorem of Calculus

Part I If $f(x)$ is continuous on $[a, b]$ then $g(x) = \int_a^x f(t)dt$ is also continuous on $[a, b]$ and

$$g'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x).$$

Part II $f(x)$ is continuous on $[a, b]$, $F(x)$ is an anti-derivative of $f(x)$ (i.e. $F(x) = \int f(x)dx$) then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Variants of Part I

$$\frac{d}{dx}\int_a^{u(x)} f(t)dt = u' f[u(x)]$$

$$\frac{d}{dx}\int_{v(x)}^{b} f(t)dt = -v' f[v(x)]$$

$$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)dt = u' f[u(x)] - v' f[v(x)]$$

C. Properties

$$\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$$
$$\int_a^b f(x) \pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$
$$\int_a^a f(x)dx = 0$$
$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \forall c$$
$$\int cf(x)dx = c\int f(x)dx, c \text{ is a constant}$$
$$\int_a^b cf(x)dx = c\int_a^b f(x)dx, c \text{ is a constant}$$
$$\int_a^b cdx = c(b-a)$$
$$\Big|\int_a^b f(x)dx\Big| \leq \int_a^b \Big|f(x)\Big|dx$$

If $f(x) \geq g(x)$ on $a \leq x \leq b$, then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx.$$

If $f(x) \geq 0$ on $a \leq x \leq b$, then

$$\int_a^b f(x)dx \geq 0.$$

If $m \leq f(x) \leq M$ on $a \leq x \leq b$, then

$$m(b-a)\int_a^b f(x)dx \leq M(b-a).$$

D. Frequently used Integrals

Integrals of common functions.

$$\int \frac{1}{x}dx = \ln|x| + c$$
$$\int e^x dx = e^x + c$$
$$\int a^x dx = \frac{1}{\ln a}a^x + c$$
$$\int e^{ax}dx = \frac{1}{a}e^{ax} + c$$
$$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}(x) + c$$
$$\int \frac{1}{1+x^2}dx = \tan^{-1}(x) + c$$
$$\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1}(x) + c$$

Sine and cosine properties.

$$\int \tan(x)dx = -\ln|\cos(x)| + c$$
$$\int \cot(x)dx = \ln|\sin(x)| + c$$
$$\int \cos(x)dx = \sin(x) + c$$
$$\int \sin(x)dx = -\cos(x) + c$$

Inverse properties.

$$\int \frac{1}{\sqrt{a^2-u^2}}dx = \sin^{-1}\left(\frac{u}{a}\right) + c$$
$$\int \frac{1}{a^2+u^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + c$$
$$\int \ln(x)dx = (x\ln(x)) - x + c$$

Hyperbolic properties.

$$\int \sinh(x)dx = \cosh(x) + c$$
$$\int \cosh(x)dx = \sinh(x) + c$$
$$\int \tanh(x)dx = \ln|\cosh(x)| + c$$
$$\int \tanh(x)\text{sech}(x)dx = -\text{sech}(x) + c$$
$$\int \text{sech}^2(x)dx = \tanh(x) + c$$
$$\int \text{csch}(x)\coth(x)dx = -\text{csch}(x) + c$$

E. Standard Integration Techniques

U-Substitution Let $u = g(x)$ (can be more than one variable), then using $du = g'(x)dx$

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

For indefinite integrals drop the limits of integration.
Integration by Parts

$$\int udv = uv - \int vdu$$

and

$$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu.$$

Choose u and dv from integral and compute du by differentiating u and compute v using $v = \int dv$.
Trig Substitutions If the integral contains the following root use the given substitution and formula to convert into an integral involving trig functions.

$$\sqrt{a^2-b^2x^2} \mapsto x = \frac{a}{b}\sin\theta, \text{ where } \cos^2\theta = 1-\sin^2\theta$$

$$\sqrt{b^2x^2-a^2} \mapsto x = \frac{a}{b}\sec\theta, \text{ where } \tan^2\theta = \sec^2\theta - 1$$

$$\sqrt{a^2+b^2x^2} \mapsto x = \frac{a}{b}\tan\theta, \text{ where } \sec^2\theta = 1+\tan^2\theta$$

Partial Fractions If integrating $\int \frac{P(x)}{Q(x)}dx$ where the degree of $P(x)$ is smaller than the degree of $Q(x)$. Factor the denominator as completely as possible and find the partial fraction decomposition of the rational expression. Integrate the partial fraction decomposition (P.F.D.). For each factor in the denominator we get term(s) in the decomposition according to the following table.

Factor in $Q(x)$	Term in P.F.D.
$ax+b$	$\frac{A}{ax+b}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
$(ax+b)^k$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}$
$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Double Integrals With Respect to the xy-axis, if taking an integral

- $\int\int dydx$ cuts in vertical rectangles
- $\int\int dxdy$ cuts in horizontal rectangles

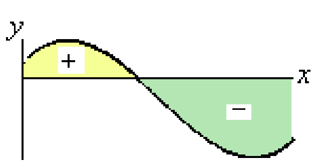
Triple Integrals Given

$$\int\int\int_s f(x,y,z)dv = \int_{a_1}^{a_2}\int_{\phi_1(x)}^{\phi_2(x)}\int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dzdydx$$

Note: dv can be exchanged for $dx\ dy\ dz$ in any order, but you must then choose your limits of integration according to that order.

F. Applications of Integrals

Net Area $\int_a^b f(x)dx$ represents the net area between $f(x)$ and the x-axis with area above x-axis positive and area below x-axis negative.



Area Between Curves The general formulas for the two main cases for each are,

1) $y = f(x)$
$\implies A = \int_a^b[\text{upper function}] - [\text{lower function}]\ dx,$
2) $x = f(y)$
$\implies A = \int_c^d[\text{right function}] - [\text{left function}]\ dy.$

If the curves intersect then the area of each portion must be found individually. Here are some sketches (see Figs. 1-3) of possible situations and formulas for these cases.

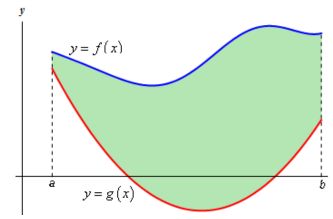


Fig. 1. $A = \int_a^b f(x) - g(x)dx$

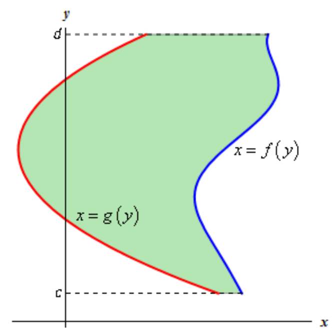


Fig. 2. $A = \int_c^d f(y) - g(y)dy$

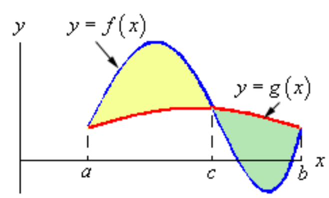


Fig. 3. $A = \int_a^c f(x) - g(x)dx + \int_c^b g(x) - f(x)dx$

Work If a force of $F(x)$ moves an object in $a \le x \le b$, the work done is $W = \int_a^b F(x)dx$.
Average Function Value The average value of $f(x)$ on $a \le x \le b$ is $f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$

G. Improper Integral

An improper integral is an integral with one or more infinite limits and/or discontinuous integrands. Integral is called convergent if the limit exists and has a finite value and divergent if the limit doesnt exist or has infinite value.
Infinite Limit

1) $\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$
2) $\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty}\int_t^b f(x)dx$
3) $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$, provided BOTH integrals are convergent.

Discontinuous Integrand

1) Discontinuity at a:

$$\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx$$

2) Discontinuity at b:

$$\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx$$

3) Discontinuity at $a < c < b$:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

provided both are convergent.

Comparison Test for Improper Integrals If $f(x) \geq g(x) \geq 0$ on $[a,\infty)$ then,

1) If $\int_a^\infty f(x)dx$ converges, then $\int_a^\infty g(x)dx$ converges.

2) If $\int_a^\infty g(x)dx$ diverges, then $\int_a^\infty f(x)dx$ diverges.

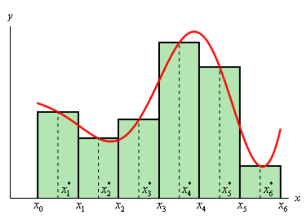
Note: If $a > 0$ then $\int_a^\infty \frac{1}{x^p}dx$ converges if $p > 1$ and diverges for $p \leq 1$.

H. Approximating Definite Integrals

For given integral $\int_a^b f(x)dx$ and n (must be even for Simpson's Rule), define $\Delta x = \frac{b-a}{n}$ and divide $[a,b]$ into n subintervals $[x_0,x_1],[x_1,x_2],\ldots,[x_{n-1},x_n]$ with $x_0 = a$ and $x_n = b$, then the following rules apply:

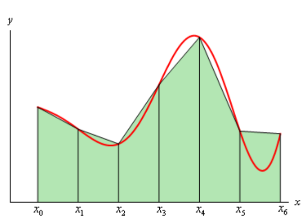
Midpoint Rule Define x_i^* as midpoint $[x_{n-1},x_n]$, then

$$\int_a^b f(x)dx \approx \Delta x[f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)].$$



Trapezoid Rule

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)].$$



Simpsons Rule

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_1) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)].$$

IV. Trigonometry Functions & Identities

Inverse properties.

$$\sin(\cos^{-1}(x)) = \sqrt{1-x^2}$$
$$\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$$
$$\sec(\tan^{-1}(x)) = \sqrt{1+x^2}$$
$$\tan(\sec^{-1}(x)) = (\sqrt{x^2-1} \text{ if } x \geq 1)$$
$$= (-\sqrt{x^2-1} \text{ if } x \leq -1)\sinh^{-1}(x)$$
$$= \ln x + \sqrt{x^2+1}$$

Inverse and hyperbolic properties.

$$\sinh^{-1}(x) = \ln x + \sqrt{x^2-1},\ x \geq -1$$
$$\tanh^{-1}(x) = \frac{1}{2}\ln x + \frac{1+x}{1-x},\ 1 < x < -1$$
$$\text{sech}^{-1}(x) = \ln[\frac{1+\sqrt{1-x^2}}{x}],\ 0 < x \leq -1$$
$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$
$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

Sine and cosine properties.

$$\sin^2(x) + \cos^2(x) = 1$$
$$1 + \tan^2(x) = \sec^2(x)$$
$$1 + \cot^2(x) = \csc^2(x)$$
$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$$
$$\cos(x \pm y) = \cos(x)\cos(y) \pm \sin(x)\sin(y)$$
$$\tan(x \pm y) = \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)}$$
$$\sin(2x) = 2\sin(x)\cos(x)$$
$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

Hyperbolic properties.

$$\cosh(n^2x) - \sinh^2 x = 1$$
$$1 + \tan^2(x) = \sec^2(x)$$
$$1 + \cot^2(x) = \csc^2(x)$$
$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$
$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$
$$\tan^2(x) = \frac{1-\cos(2x)}{1+\cos(2x)}$$
$$\sin(-x) = -\sin(x)$$
$$\cos(-x) = \cos(x)$$
$$\tan(-x) = -\tan(x)$$

V. Multivariate Calculus

A. Cartesian coords in 3D

Given two points (x_1, y_1, z_1) and (x_2, y_2, z_2), the distance between them is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

The midpoint is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

Sphere with center (h, k, l) and radius r

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2.$$

B. Vectors

Define a vector as $\vec{u}$, magnitude of a vector $||\vec{u}|| = \sqrt{u_1^2 + u_2^2 + u_3^2}$, and a unit vector $\hat{u} = \frac{\vec{u}}{||\vec{u}||}$.

1) Dot Product: The dot product $\vec{u} \cdot \vec{v}$ produces a scalar. Geometrically, the dot product is a vector projection. Given $\vec{u} =< u_1, u_2, u_3 >$ and $\vec{v} =< v_1, v_2, v_3 >$ the dot product $\vec{u} \cdot \vec{v} = \vec{0}$ implies the two vectors are perpendicular.

Given the angle between them θ, define

$$\vec{u} \cdot \vec{v} = ||\vec{u}|| \, ||\vec{v}|| \cos(\theta),$$

and

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

The following are useful relations

$$\begin{aligned} \hat{u} \cdot \hat{v} &= \cos(\theta) \\ ||\vec{u}||^2 &= \vec{u} \cdot \vec{u} \\ \vec{u} \cdot \vec{v} &= 0 \text{ when } \perp \end{aligned}$$

Angle Between $\vec{u}$ and $\vec{v}$ is

$$\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| \, ||\vec{v}||}\right)$$

The projection of $\vec{u}$ onto $\vec{v}$ is

$$pr_{\vec{v}} \ \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2}\right)\vec{v}$$

2) Cross Product: The cross product $\vec{u} \times \vec{v}$ produces a vector. Geometrically, the cross product is the area of a parallelogram with sides $||\vec{u}||$ and $||\vec{v}||$, where $\vec{u} =< u_1, u_2, u_3 >$ and $\vec{v} =< v_1, v_2, v_3 >$ and

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

If $\vec{u} \times \vec{v} = \vec{0}$, the vectors are parallel.

C. Lines and Planes

1) Equation of a line: A line requires a Direction Vector $\vec{u} =< u_1, u_2, u_3 >$ and a point (x_1, y_1, z_1). A parameterization of a line could be:

$$\begin{aligned} x &= u_1 t + x_1 \\ y &= u_2 t + y_1 \\ z &= u_3 t + z_1 \end{aligned}$$

2) Equation of a Plane: (x_0, y_0, z_0) is a point on the plane, and $\langle a, b, c \rangle$ is a normal vector. Then

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

and

$$< a, b, c > \cdot < x - x_0, y - y_0, z - z_0 >= 0$$

and

$$ax + by + cz = d$$

where

$$d = ax_0 + by_0 + cz_0.$$

3) Distance from a Point to a Plane: The distance from a point (x_0, y_0, z_0) to the plane $ax + by + cz = d$ is

$$d = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

D. Coordinate System Conversion

1) Cylindrical to Rectangular Coordinates:

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \\ z &= z \end{aligned}$$

2) Rectangular to Cylindrical Coordinates:

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \tan(\theta) &= \frac{y}{x} \\ z &= z \end{aligned}$$

3) Spherical to Rectangular Coordinates:

$$\begin{aligned} x &= \rho\sin(\phi)\cos(\theta) \\ y &= \rho\sin(\phi)\sin(\theta) \\ z &= \rho\cos(\phi) \end{aligned}$$

4) Rectangular to Spherical Coordinates:

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2 + z^2} \\ \tan(\theta) &= \frac{y}{x} \\ \cos(\phi) &= \frac{z}{\sqrt{x^2 + y^2 + z^2}} \end{aligned}$$

5) Spherical to Cylindrical Coordinates:

$$\begin{aligned} r &= \rho\sin(\phi) \\ \theta &= \theta \\ z &= \rho\cos(\phi) \end{aligned}$$

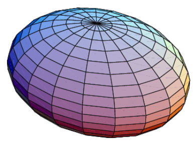
6) Cylindrical to Spherical Coordinates:

$$\begin{aligned} \rho &= \sqrt{r^2 + z^2} \\ \theta &= \theta \\ \cos(\phi) &= \frac{z}{\sqrt{r^2 + z^2}} \end{aligned}$$

E. Surfaces

1) Ellipsoid:

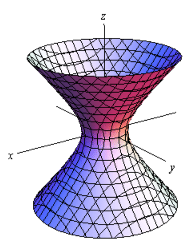
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



2) Hyperboloid of One Sheet:

(Major Axis: z because it follows -)

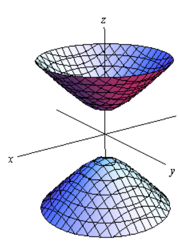
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$



3) Hyperboloid of Two Sheets:

(Major Axis: z because it is the one not subtracted)

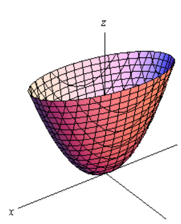
$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



4) Elliptic Paraboloid:

(Major Axis: z because it is the variable NOT squared)

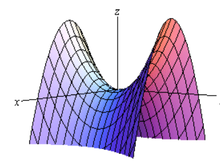
$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$



5) Hyperbolic Paraboloid:

(Major Axis: z axis because it is not squared)

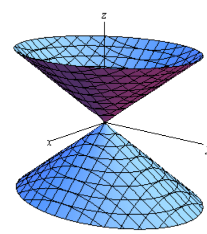
$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$



6) Elliptic Cone:

(Major Axis: z axis because it's the only one being subtracted)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$



7) Cylinder:

One of the variables is missing or

$$(x - a)^2 + (y - b^2) = c$$

(Major Axis is missing variable)

VI. Rules

A. Gradients

The Gradient of a function in 2 variables is $\nabla f = \langle f_x, f_y \rangle$. The Gradient of a function in 3 variables is $\nabla f = \langle f_x, f_y, f_z \rangle$.

B. Directional Derivatives

Let $z = f(x, y)$ be a function, (a, b) a point in the domain (a valid input point) and $\hat{u}$ a unit vector (2D).

The Directional Derivative is then the derivative at the point (a, b) in the direction of $\hat{u}$ or

$$D_{\vec{u}} f(a, b) = \hat{u} \cdot \nabla f(a, b),$$

which returns a *scalar*.

The 4-D version is

$$D_{\vec{u}} f(a, b, c) = \hat{u} \cdot \nabla f(a, b, c)$$

C. Tangent Planes

Let $F(x, y, z) = k$ be a surface and $P = (x_0, y_0, z_0)$ be a point on that surface. The equation of a Tangent Plane is

$$\nabla F(x_0, y_0, z_0) \cdot < x - x_0, y - y_0, z - z_0 >$$

D. Approximations

Let $z = f(x, y)$ be a differentiable function total differential of $f = dz$ then

$$dz = \nabla f \cdot < dx, dy >$$

This is the *approximate* change in z. The actual change in z is the difference in z values: $\Delta z = z - z_1$

E. Lagrange Multipliers

Given a function $f(x, y)$ with a constraint $g(x, y)$, solve the following system of equations to find the max and min points on the constraint (NOTE: may need to also find internal points.):

$$\nabla f = \lambda \nabla g$$

$$g(x, y) = 0 \text{ (or } k \text{ if given)}$$

F. Surface Area of a Curve

let $z = f(x, y)$ be continuous over S (a closed Region in 2D domain). Then the surface area of $z = f(x, y)$ over S is

$$SA = \int\int_S \sqrt{f_x^2 + f_y^2 + 1} dA$$

G. Jacobian Method

Given

$$\int\int_G f(g(u, v), h(u, v))|J(u, v)|dudv = \int\int_R f(x, y)dxdy$$

where

$$J(u, v) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

Common Jacobians are

- Rectangular to Cylindrical: r.
- Rectangular to Spherical: $\rho^2 \sin(\phi)$.

H. Vector Fields

Let $f(x, y, z)$ be a scalar field and

$$\vec{F}(x, y, z) = M(x, y, z)\hat{i} + N(x, y, z)\hat{j} + P(x, y, z)\hat{k}$$

be a vector field. The **Gradient** of f is

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle.$$

The **Divergence** of $\vec{F}$ is

$$\nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$

The **Curl** of $\vec{F}$ is

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}.$$

I. Line Integrals

The integral over C given by $x = x(t), y = y(t), t \in [a, b]$ is

$$\int_c f(x, y)ds = \int_a^b f(x(t), y(t))ds$$

where

$$ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt,$$

or

$$\sqrt{1 + (\frac{dy}{dx})^2} dx,$$

or

$$\sqrt{1 + (\frac{dx}{dy})^2} dy.$$

To evaluate a Line Integral

- Get a parameterized version of the line (usually in terms of t, though in exclusive terms of x or y is ok)
- Evaluate for the derivatives needed (usually dy, dx, and/or dt)
- Plug in to original equation to get in terms of the independent variable
- Solve integral

VII. Theorems

A. Independence of Path

Fundamental Theorem of Line Integrals
If C is curve given by $\vec{r}(t), t \in [a, b]$ then $\vec{r}\prime(t)$ exists. If $f(\vec{r})$ is continuously differentiable on an open set containing C, then

$$\int_c \nabla f(\vec{r}) \cdot d\vec{r} = f(\vec{b}) - f(\vec{a}).$$

Equivalent Conditions
$\vec{F}(\vec{r})$ continuous on open connected set D. Then, $(a)\vec{F} = \nabla f$ for some function f (if $\vec{F}$ is conservative).

$$\Leftrightarrow (b) \int_c \vec{F}(\vec{r}) \cdot d\vec{r},$$

is independent of path in D

$$\Leftrightarrow (c) \int_c \vec{F}(\vec{r}) \cdot d\vec{r} = 0,$$

for all closed paths in D.

Conservation Theorem
Define

$$\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$$

as continuously differentiable on open, and a simply connected set D. Then $\vec{F}$ conservative $\Leftrightarrow \nabla \times \vec{F} = \vec{0}$ (in 2D $\nabla \times \vec{F} = \vec{0}$ iff $M_y = N_x$).

B. Green's Theorem

The method of changing line integral for double integral. Use for Flux and Circulation across 2D curve and line integrals over a closed boundary. Let

- R be a region in the xy-plane.
- C is simple, closed curve enclosing R (w/ parametrization $\vec{r}(t)$).
- $\vec{F}(x,y) = M(x,y)\hat{i} + N(x,y)\hat{j}$ be continuously differentiable over $R \cup C$.

Then the integrals are defined as

$$\oint Mdy - Ndx = \int\int_R (M_x + N_y)dxdy,$$

$$\oint Mdx + Ndy = \int\int_R (N_x - M_y)dxdy.$$

Form 1: Flux Across Boundary
Let $\vec{n}$ = unit normal vector to C, then

$$\oint_c \vec{F} \cdot \vec{n} = \int\int_R \nabla \cdot \vec{F} dA$$

$$\Leftrightarrow \oint Mdy - Ndx = \int\int_R (M_x + N_y)dxdy.$$

Form 2: Circulation Along Boundary
Let

$$\oint_c \vec{F} \cdot d\vec{r} = \int\int_R \nabla \times \vec{F} \cdot \hat{u} dA$$

$$\Leftrightarrow \oint Mdx + Ndy = \int\int_R (N_x - M_y)dxdy.$$

Area of R
Define

$$A = \oint (\frac{-1}{2}ydx + \frac{1}{2}xdy).$$

C. Gauss' Divergence Theorem

This is a 3D Analog of Green's Theorem, used for Flux over a 3D surface. Let

- $\vec{F}(x,y,z)$ be vector field continuously differentiable in solid S.
- S is a 3D solid.
- ∂S boundary of S (a Surface).
- $\hat{n}$ unit outer normal to ∂S.

Then,

$$\int\int_{\partial S} \vec{F}(x,y,z) \cdot \hat{n} dS = \int\int\int_S \nabla \cdot \vec{F} dV,$$

where $dV = dxdydz$.

D. Surface Integrals

Let

- R be closed, bounded region in xy-plane.
- f be a fn with first order partial derivatives on R.
- G be a surface over R given by $z = f(x,y)$.
- $g(x,y,z) = g(x,y,f(x,y))$ is continuous on R.

Then,

$$\int\int_G g(x,y,z)dS = \int\int_R g(x,y,f(x,y))dS,$$

where

$$dS = \sqrt{f_x^2 + f_y^2 + 1}dydx.$$

To compute the **flux of $\vec{F}$ across** G, define

$$\int\int_G \vec{F} \cdot ndS = \int\int_R [-Mf_x - Nf_y + P]dxdy,$$

where

- $\vec{F}(x,y,z) = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$.
- G is surface $f(x,y) = z$.
- $\vec{n}$ is upward unit normal on G.
- $f(x,y)$ has continuous 1^{st} order partial derivatives.

E. Stokes Theorem

Let

- S be a 3D surface.
- $\vec{F}(x,y,z) = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{l}$.
- M, N, P have continuous 1^{st} order partial derivatives.
- C is piece-wise smooth, simple, closed, curve, positively oriented.
- $\hat{T}$ is unit tangent vector to C.

Then,

$$\oint \vec{F}_c \cdot \hat{T} dS = \int\int_s (\nabla \times \vec{F}) \cdot \hat{n} dS = \int\int_R (\nabla \times \vec{F}) \cdot \vec{n} dxdy.$$

Remember:

$$\oint \vec{F} \cdot \vec{T} ds = \int_c (Mdx + Ndy + Pdz).$$