Notes on Trigonometry

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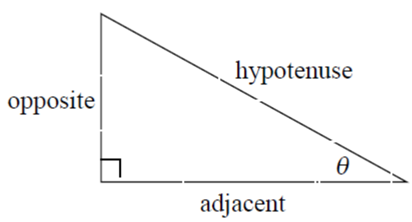
I. Definitions

Right triangle definition
For this definition we assume that $0 < \theta < \frac{\pi}{2}$ or $0^\circ < \theta < 90^\circ$.

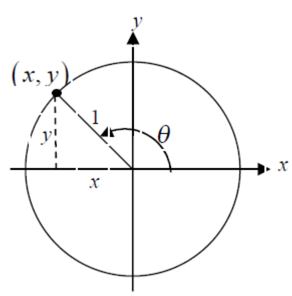


$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

Unit circle definition
For this definition θ is any angle.



$$\sin\theta = \frac{y}{1} = y \qquad \csc\theta = \frac{1}{y}$$

$$\cos\theta = \frac{x}{1} = x \qquad \sec\theta = \frac{1}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

II. Facts and Properties

Domain
The domain is all the values of θ that can be plugged into the function.

$$\begin{aligned}
&\sin\theta,\ \theta \text{ can be any angle.}\\
&\cos\theta,\ \theta \text{ can be any angle.}\\
&\tan\theta,\ \theta \neq (n + \frac{1}{2})\pi, n = 0, \pm1, \pm2, \ldots\\
&\csc\theta,\ \theta \neq n\pi, n = 0, \pm1, \pm2, \ldots\\
&\sec\theta,\ \theta \neq (n + \frac{1}{2})\pi, n = 0, \pm1, \pm2, \ldots\\
&\cot\theta,\ \theta \neq n\pi, n = 0, \pm1, \pm2, \ldots
\end{aligned}$$

Range
The range is all possible values to get out of the function.

$$\begin{gathered}
-1 \leq \sin \leq 1\\
-1 \leq \cos \leq 1\\
-\infty \leq \tan \leq \infty\\
\csc\theta \geq 1 \text{ and } \csc\theta \leq -1\\
\sec\theta \geq 1 \text{ and } \sec\theta \leq -1\\
-\infty \leq \cot \leq \infty
\end{gathered}$$

Period
The period of a function is the number T such that $f(\theta + T) = f(\theta)$. If ω is a fixed number and θ is any angle, the following periods exist.

$$\sin(\omega\theta) \implies T = \frac{2\pi}{\omega} \qquad \csc(\omega\theta) \implies T = \frac{2\pi}{\omega}$$

$$\cos(\omega\theta) \implies T = \frac{2\pi}{\omega} \qquad \sec(\omega\theta) \implies T = \frac{2\pi}{\omega}$$

$$\tan(\omega\theta) \implies T = \frac{\pi}{\omega} \qquad \cot(\omega\theta) \implies T = \frac{\pi}{\omega}$$

III. Formulas and Identities

Tangent and Cotangent Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot = \frac{\cos\theta}{\sin\theta}$$

Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Pythagorean Identities

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \tan^2\theta + 1 &= \sec^2\theta \\ 1 + \cot^2\theta &= \csc^2\theta \end{aligned}$$

Even/Odd Formulas

$$\begin{aligned} \sin(-\theta) &= -\sin\theta & \csc(-\theta) &= -\csc\theta \\ \cos(-\theta) &= \cos\theta & \sec(-\theta) &= \sec\theta \\ \tan(-\theta) &= -\tan\theta & \cot(-\theta) &= -\cot\theta \end{aligned}$$

Periodic Formulas

For $n \in \mathbb{Z}$

$$\begin{aligned} \sin(\theta + 2\pi n) &= \sin\theta & \csc(\theta + 2\pi n) &= \csc\theta \\ \cos(\theta + 2\pi n) &= \cos\theta & \sec(\theta + 2\pi n) &= \sec\theta \\ \tan(\theta + \pi n) &= \tan\theta & \cot(\theta + \pi n) &= \cot\theta \end{aligned}$$

Double Angle Formulas

$$\begin{aligned} \sin(2\theta) &= 2\sin\theta\cos\theta \\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta \\ \tan(2\theta) &= \frac{2\tan\theta}{1-\tan^2\theta} \end{aligned}$$

Degrees to Radians Formulas

If x is an angle in degrees and t is an angle in radians then

$$\frac{\pi}{180} = \frac{t}{x} \quad \Longrightarrow \quad t = \frac{\pi x}{180}, \text{ and } x = \frac{180t}{\pi}$$

Half Angle Formulas

$$\begin{aligned} \sin\frac{\theta}{2} &= \pm\sqrt{\frac{1-\cos\theta}{2}} & \sin^2\theta &= \frac{1}{2}(1-\cos(2\theta)) \\ \cos\frac{\theta}{2} &= \pm\sqrt{\frac{1+\cos\theta}{2}} & \cos^2\theta &= \frac{1}{2}(1+\cos(2\theta)) \\ \tan\frac{\theta}{2} &= \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} & \tan^2\theta &= \frac{1-\cos(2\theta)}{1+\cos(2\theta)} \end{aligned}$$

Sum and Difference Formulas

$$\begin{aligned} \sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\ \cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \\ \tan(\alpha \pm \beta) &= \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} \end{aligned}$$

Product to Sum Formulas

$$\begin{aligned} \sin\alpha\sin\beta &= \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] \\ \cos\alpha\cos\beta &= \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)] \\ \sin\alpha\cos\beta &= \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)] \\ \cos\alpha\sin\beta &= \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)] \end{aligned}$$

Sum to Product Formulas

$$\begin{aligned} \sin\alpha + \sin\beta &= 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \\ \sin\alpha - \sin\beta &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \\ \cos\alpha + \cos\beta &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \\ \cos\alpha - \cos\beta &= -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \end{aligned}$$

Cofunction Formulas

$$\begin{aligned} \sin\left(\frac{\pi}{2}-\theta\right) &= \cos\theta & \csc\left(\frac{\pi}{2}-\theta\right) &= \sec\theta \\ \cos\left(\frac{\pi}{2}-\theta\right) &= \sin\theta & \sec\left(\frac{\pi}{2}-\theta\right) &= \csc\theta \\ \tan\left(\frac{\pi}{2}-\theta\right) &= \cot\theta & \cot\left(\frac{\pi}{2}-\theta\right) &= \tan\theta \end{aligned}$$

IV. Inverse Trig Functions

Definition

$$\begin{aligned} y = \sin^{-1}x &\quad\equiv\quad x = \sin y \\ y = \cos^{-1}x &\quad\equiv\quad x = \cos y \\ y = \tan^{-1}x &\quad\equiv\quad x = \tan y \end{aligned}$$

Domain and Range

Function	Domain	Range
$y = \sin^{-1}x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \cos^{-1}x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \tan^{-1}x$	$-\infty \le x \le \infty$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

Inverse Properties

$$\sin(\sin^{-1}(x)) = x \qquad \sin^{-1}(\sin(\theta)) = \theta$$
$$\cos(\cos^{-1}(x)) = x \qquad \cos^{-1}(\cos(\theta)) = \theta$$
$$\tan(\tan^{-1}(x)) = x \qquad \tan^{-1}(\tan(\theta)) = \theta$$

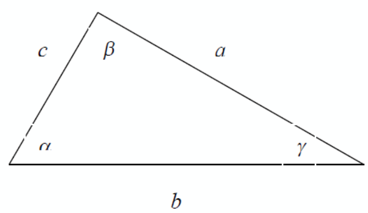
Alternate Notation

$$\sin^{-1} x = \arcsin x$$
$$\cos^{-1} x = \arccos x$$
$$\tan^{-1} x = \arctan x$$

V. Law of Sines, Cosines and Tangents

Law of Sines

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$



Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

Mollweides Formula

$$\frac{a+b}{c} = \frac{\cos\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}\gamma}$$

Law of Tangents

$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(\alpha-\beta)}{\tan\frac{1}{2}(\alpha+\beta)}$$
$$\frac{b-c}{b+c} = \frac{\tan\frac{1}{2}(\beta-\gamma)}{\tan\frac{1}{2}(\beta+\gamma)}$$
$$\frac{a-c}{a+c} = \frac{\tan\frac{1}{2}(\alpha-\gamma)}{\tan\frac{1}{2}(\alpha+\gamma)}$$

VI. Unit Circle

For any ordered pair on the unit circle (x, y) : $\cos\theta = x$ and $\sin\theta = y$. Example: $\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}$, and $\sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$.