Math, Problem Set #4, Optimization

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Solutions

6.1 minimize $-e^{-w^T x}$ subject to:

$w^T Aw - w^T Ay - w^T x \leq -a$

$y^T w - w^T x = b$

6.5 Let B be the number of bottles, and K be the number of the knobs. We want to minimize $-f(B, K) = -0.07B + 0.05K$, subject to:

$7(B + K) \leq 240000$

$3(B + K) \leq 6000$

$-B \leq 0$

$-K \leq 0$

6.6 Given $f(x, y) = 3x^2y + 4x * y^2 + xy$, we know the partial derivatives of f are given by:

$f_x(x, y) = 6xy + 4y^2 + y = 0$

$f_y(x, y) = 3x^2 + 8xy + x = 0$

Simplifying, we get:

$x(3x + 8y + 1) = 0$ and $y(6x + 4y + 1) = 0$. Therefore, one pair of roots is $x = 0$ and $y = 0$. Substituting $4y = -6x - 1$, we get $3x - 12x - 2 + 1 = 0$. Solving, we get $x = -\frac{1}{9}$ and $y = -\frac{1}{12}$. The Hessian is given by: $D^2 f(x, y) = \begin{bmatrix} 6y & 6x + 8y + 1 \\ 6x + 8y + 1 & 8x \end{bmatrix}$.

$D^2 f(-\frac{1}{9}, -\frac{1}{12}) = \begin{bmatrix} -\frac{1}{2} & -\frac{6}{9} - \frac{8}{12} + 1 \\ -\frac{6}{9} - \frac{8}{12} + 1 & -\frac{8}{9} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{8}{9} \end{bmatrix}$, which is negative definite because the first and second leading principal minors are negative and positive respectively. This means $(-\frac{1}{9}, -\frac{1}{12})$ is a local maximum. On the other hand,

$D^2 f(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is an indefinite matrix because its eigenvalues are both positive and negative.

6.11 Claim: For any initial real-valued guess, one iteration of Newton's method lands at the unique minimizer of f. To see why, note that $f'(x) = 2ax + b$ and $f''(x) = 2a$. Using Newton's method, we obtain $x_1 = x_0 - \frac{2ax_0+b}{2a} = -\frac{b}{2a}$. Then, $x_2 = -\frac{b}{2a} - \frac{2a(-b/2a)+b}{2a} = -\frac{b}{2a} - (-\frac{b}{2a} + \frac{b}{2a}) = -\frac{b}{2a}$. By induction, one can see that $x_k = -\frac{b}{2a}$ for all $k \geq 2$.