Math, Problem Set #1, Probability Theory

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1. **Exercises from chapter.** Do the following exercises in Chapter 3 of **?**: 3.6, 3.8, 3.11, 3.12 (watch this movie clip), 3.16, 3.33, 3.36

3.6)

The probability of event A maybe rewritten as:

$= P(\Omega \cap A)$

$= P(\bigcup_{i \in I} B_i \cap A)$ because $B_{i i \in I}$ is a partition of Ω.

$= \sum_{i \in I} P(A \cap B_i)$ by finite additivity.

3.8)

$P(\bigcup_{k=1}^{n} E_k)$

$= 1 - P((\bigcup_{k=1}^{n} E_k)^c)$

$= 1 - P(E_1^c \cap E_2^c \cap ... \cap E_n^c)$ by De Morgan's Law

$= 1 - P(E_1^c)P(E_2^c)...P(E_n^c)$ because if $E_k's$ are independent, so are $E_k^{c}{}'s$

$= 1 - \prod_{k=1}^{n} 1 - P(E_k)$

3.11)

$P(s = crime \mid s\ tested+)$

$= \frac{P(s\ =\ crime,\ s\ tested\ +)}{P(s\ tested\ +)}$

$= \frac{P(s\ tested\ +|\ s\ =\ crime)P(s\ =\ crime)}{P(s\ tested\ +)}$

$= \frac{P(s\ in\ sample|\ s\ =\ crime)P(s\ =\ crime)}{P(s\ tested\ +)}$

$= \frac{\frac{1}{250}\frac{1}{250{,}000{,}000}}{\frac{1}{3{,}000{,}000}}$

$= \frac{3}{250^2}$

$= \frac{3}{62{,}500}$

3.12)

We name the three doors as door 1, door 2, and door 3. Without loss of generality, suppose the contestant chose door 1, and Monty Hall opened door 2.

$$P(prize\ 1 \mid\ opened\ 2)$$
$$= \frac{P(opened\ 2 \mid\ prize\ 1)P(prize\ 1)}{P(opened\ 2)}$$
$$= \frac{(\frac{1}{2})(\frac{1}{3})}{\frac{1}{2}}$$
$$= \frac{1}{3}$$

On the other hand, $P(prize\ 3 \mid opened\ 2) = \frac{P(opened\ 2|prize\ 3)P(prize\ 3)}{P(opened\ 2)} = \frac{2}{3}$

Therefore, there is always a higher probability of winning a car if one switches doors, once one door has been opened by Monty Hall.

For the case with 10 doors, without loss of generality, suppose the contestant chose door 1, and Monty Hall opened doors 2 to 9.

$$P(prize\ 1 \mid opened\ 2\ to\ 9)$$
$$= \frac{P(opened\ 2\ to\ 9|prize\ 1)P(prize\ 1)}{P(opened\ 2\ to\ 9)}$$
$$= \frac{\frac{1}{\binom{9}{8}}\frac{1}{10}}{\frac{1}{\binom{9}{8}}}$$
$$= \frac{1}{10}$$

On the other hand, $P(prize\ 10 \mid opened\ 2\ to\ 9) = \frac{P(opened\ 2\ to\ 9|prize\ 10)P(prize\ 10)}{opened\ 2\ to\ 9}$
$$= \frac{\frac{1}{10}}{\frac{1}{\binom{9}{8}}}$$
$$= \frac{\binom{9}{8}}{10}$$
$$= \frac{9}{10}$$

3.16)

$$Var(X)$$
$$= E((X-\mu)^2)$$

$= E(X^2) - 2E(\mu X) + E(\mu^2)$
$= E(X^2) + \mu^2 - 2\mu E(X)$
$= E(X^2) + \mu^2 - 2\mu^2$
$= E(X^2) - \mu^2$

3.33)

First note that $E(B) = np$ and $Var(B) = np(1-p)$. Then,
$P(|\frac{B}{n} - p)| \geq \epsilon)$
$= P(|B - np)| \geq n\epsilon)$ because n is positive
$\leq \frac{np(1-p)}{n\epsilon^2}$ by Chebyshev's Inequality
$= \frac{p(1-p)}{n\epsilon^2}$

3.36)

Let X = the total number of enrolled students. Let the Bernoulli random variable $X_i = 1$ if student i enrolls, 0 otherwise.
Let $S_{6242} = X_1 + X_2 + ... + X_{6242}$.
We know $\mu = 0.801 = p$ where p is the probability of enrollment.
By the Central Limit Theorem, we know $\frac{S_{6242}-6242(0.801)}{\sqrt{0.801(1-0.801)}\sqrt{6242}} \sim N(0,1)$.
Therefore, $P(X\,5500) = 1 - P(X \leq 5500) = 1 - \Phi(\frac{5500-6242(0.801)}{\sqrt{0.801(1-0.801)}\sqrt{6242}}) = 0$

2. Construct examples of events A, B, and C, each of probability strictly between 0 and 1, such that

(a) $P(A \cap B) = P(A)P(B)$, $P(A \cap C) = P(A)P(C)$, $P(B \cap C) = P(B)P(C)$, but $P(A \cap B \cap C) \neq P(A)P(B)P(C)$.

Consider the simultaneous rolling of two fair dice, where the numbers on each die are independent of each other. Then define the events A, B, and C as follows:

A: Sum of points on the dice is 7
B: 1 die came up as 3
C: 1 die came up as 4

Then, we know the following:
$P(A) = \frac{6}{36} = \frac{1}{6}, P(B) = \frac{1}{6}, P(C) = \frac{1}{6}$
$P(A \cap B \cap C) = P(\{3,4\}) = (\frac{1}{6})^2 = \frac{1}{36} \neq P(A)P(B)P(C) = \frac{1}{216}$
$P(A \cap B) = P(\{3,4\}) = \frac{1}{36} = P(A)P(B)$
$P(A \cap C) = P(\{3,4\}) = \frac{1}{36} = P(A)P(C)$
$P(B \cap C) = P(\{3,4\}) = \frac{1}{36} = P(B)P(C)$

(b) $P(A \cap B) = P(A)P(B)$, $P(A \cap C) = P(A)P(C)$, $P(A \cap B \cap C) = P(A)P(B)P(C)$, but $P(B \cap C) \neq P(B)P(C)$. (Hint: You can let Ω be a set of eight equally likely points.)

Let Ω = {1,2,3,4,5,6,7,8}, where each event is equally likely, each occurring with probability $\frac{1}{8}$. Then let:

A = {1,2,3,4}
B = {3,4,5,6}
C = {1,3,7,8}

Then, we know that:
$P(A \cap B) = P(\{3,4\}) = \frac{2}{8} = \frac{1}{4} = (\frac{1}{2})^2 = P(A)P(B)$
$P(A \cap C) = P(\{1,3\}) = \frac{2}{8} = \frac{1}{4} = (\frac{1}{2})^2 = P(A)P(C)$
$P(A \cap B \cap C) = P(\{3\}) = \frac{1}{8} = (\frac{1}{2})^3 = P(A)P(B)P(C)$

However, we know that $P(\{B \cap C\}) = P(\{3\}) = \frac{1}{8} \neq (\frac{1}{2})^2 = P(B)P(C)$

3. Prove that Benford's Law is, in fact, a well-defined discrete probability distribution.

Let D={1,2,3,4,5,6,7,8,9}. To be a well-defined probability space, we want to show that the following three conditions hold:

1) $0 \leq P(d) \leq 1 \; for \; d \in D$

Since $log_{10}(x)$ is monotone in x, we know $log_{10}(1+\frac{1}{d})$ is bounded above by $log_{10}(1+1) = log_{10}(2) \leq 1$ and bounded below by $log_{10}(1) = 0$. Therefore, $log_{10}(x)$ is bounded between 0 and 1 for all $d \in D$.

2) $\sum_{d \in D} P(d) = 1$

Computing, we find that $log_{10}(1+1) + log_{10}(1+\frac{1}{2}) + ... + log_{10}(1+\frac{1}{9}) = 1$, as desired.

3) Finite additivity: For any collection of pairwise disjoint events $P(\bigcup\limits_{d \in D} E_d)$ $= \sum_{d \in D} P(E_d)$

This last condition holds if we assume the axiom of countable additivity, which states that $P(\bigcup\limits_{d=1}^{\infty} E_d) = \sum_{d=1}^{\infty} P(E_d)$. Finite additivity follows from countable additivity because:

$P(A_1 \cup A_2 \cup ... \cup A_n)$
$= P(A_1 \cup A_2 \cup ... \cup A_n \cup \emptyset \cup \emptyset ...)$
$=$ P(A_1) $+ ... + P(A+n) + P(\emptyset)$ + P($\emptyset$) + ...
$=$ P(A_1) $+ ... + P(A+n)$
$=\sum_{d=1}^{n} P(A_d)$

4. A person tosses a fair coin until a tail appears for the first time. If the tail appears on the nth flip, the person wins 2^n dollars. Let the random variable X denote the player's winnings.

(a) (St. Petersburg paradox) Show that $E[X] = +\infty$.

E(X) $= (\frac{1}{2})^{1-1}\frac{1}{2}2 + (\frac{1}{2})^{2-1}\frac{1}{2}2^2 + ...$
$= \sum_{n=1}^{\infty} (\frac{1}{2})^n 2^n$
$= \sum_{n=1}^{\infty} (1)^n$
$= +\ \infty$

(b) Suppose the agent has log utility. Calculate $E[\ln X]$.

$\sum_{n=1}^{\infty} (\frac{1}{2})^n \ln(2^n)$
$= (\ln 2) \sum_{n=1}^{\infty} \frac{n}{2^n}$
$= (\ln 2) S$ where S $= \sum_{n=1}^{\infty} \frac{n}{2^n}$
$= (\ln 2) \frac{\frac{1}{2}}{(\frac{1}{2}-1)^2}$
$= 2(\ln 2)$

5. (Siegel's paradox) Suppose the exchange rate between USD and CHF is 1:1. Both a U.S. investor and a Swiss investor believe that a year from now the exchange rate will be either 1.25 : 1 or 1 : 1.25, with each scenario having a probability of 0.5. Both investors want to maximize their wealth in their respective home currency (a year from now) by investing in a risk-free asset; the risk-free interest rates in the U.S. and in Switzerland are the same. Where should the two investors invest?

   For the US investor:
   If she only holds USD, then she bears no currency risk and has an expected value of 1.
   If she only holds CHF, then her expected value = 0.5*1.24 + 0.5*0.80 = 1.025

   For the Swiss investor:
   If she only holds CHF, then she bears no currency risk and has an expected value of 1.
   If she only holds USD, then her expected value = 0.5*1.24 + 0.5*0.80 = 1.025

   Therefore, each investor should invest all her capital in the currency foreign to where they are from to get a higher expected return.

6. Consider a probability measure space with $\Omega = [0, 1]$.

   (a) Construct a random variable X such that $E[X] < \infty$ but $E[X^2] = \infty$.

   Consider the Pareto distribution with parameter $\alpha = \frac{3}{2}$, which has the probability mass function $f_x(x) = \frac{3}{2}\frac{x_0^{\frac{3}{2}}}{x^{\frac{3}{2}+1}}$ if x $\geq x_0$ and 0 otherwise. Then notice that $E(X) = \frac{\frac{3}{2}x_m}{\frac{1}{2}}$ and $E(X^2) = \infty$. This is because the Pareto distribution has the following mean and variance functions:

E(X) $= \infty$ if $\alpha \leq 1$ and E(X) $= \frac{\alpha x_m}{\alpha - 1}$ else.
Var(X) $= \infty$ if $\alpha \in (0, 2]$ and Var(X) $= \frac{x_m^2 \alpha}{(\alpha-1)^2(\alpha-2)}$ else.

(b) Construct random variables X and Y such that $P(X > Y) > \frac{1}{2}$ but $E[X] < E[Y]$.

Consider the joint density function $f_{x,y}(x, y) = 15yx^2$ if $0 \leq x \leq 1$ and $0 \leq y \leq x$ and 0 otherwise. The marginal density functions f_x and f_y are as follows:

$f_y(y) = 5y^4$ if $0 \leq y \leq 1$ and $f_y(y) = 0$ else.
$f_x(x) = \frac{15}{2}x^2(1 - x^2)$ if $0 \leq x \leq 1$ and $f_x(x) = 0$ else.

E(X) $= \frac{15}{2} \int_0^1 x(x^2 - x^4)dx = \frac{5}{8}$
E(Y) $= 5 \int_0^1 y(y^4)dy = \frac{5}{6}$
Therefore, E(X) $\leq$ E(Y).

Also, $P(X > Y) = 15 \int_0^1 \int_0^y yx^2 dxdy = 15 \int_0^1 \frac{x^3}{3} dy = 1 > \frac{1}{2}$

(c) Construct random variables X, Y, and Z such that $P(X > Y)P(Y > Z)P(X > Z) > 0$ and $E(X) = E(Y) = E(Z) = 0$.

Let X, Y, and Z be normally distributed random variables with mean 0 and variance 1. Then, $P(X > Y)P(Y > Z)P(X > Z) = (\frac{1}{2})^3 = \frac{1}{8} > 0$ by the symmetry of the normal distribution. Also, $E(X) = E(Y) = E(Z) = 0$ by the assumption we made.

7. Let the random variables X and Z be independent with $X \sim N(0, 1)$ and $P(Z = 1) = P(Z = -1) = \frac{1}{2}$. Define $Y = XZ$ as the product of X and Z. Prove or disprove each of the following statements.

(a) $Y \sim N(0, 1)$.

Ans: True

$E(XZ) = E(X)E(Z) = 0$

$Var(Y) = Var(XZ) = E(X^2Z^2) - (E(XZ))^2$
$= Var(X)Var(Z) + Var(X)(E(Z))^2 + Var(Z)(E(X))^2$
$= 1 + (E(Z))^2 + ((E(X))^2$

$= 1 + 0 + 0$
$= 1$

CDF of Y $= P(Y < y)$
$= P(Z = 1)P(X < y) + P(Z = -1)P(X > -y)$
$= 0.5P(X < y) + 0.5P(X > -y)$
$= 2(0.5)P(X < y)$
$= P(X < y) =$ CDF of X
Therefore, Y is normally distributed with mean 0 and variance 1.

(b) $P(|X| = |Y|) = 1$.

Ans: True
Consider the following cases:
Case 1: Z = 1, so Y = X. Then $P(|X| = |Y|) = P(|X| = |X|) = 1$
Case 2: Z = -1, so Y = -X. Then $P(|X| = |Y|) = P(|X| = |-X|) = P(|X| = |-X|) = 1$

(c) X and Y are not independent.

Ans: True.
X and Y are not independent because $P(X = 2 \mid Y = 2) = \frac{1}{2} \neq P(X = 2) = 0$ because the normal distribution is continuous.

(d) $Cov[X, Y] = 0$.

Ans: True.
Cov[X,Y]
$= Cov[X, XZ]$
$= E(X^2Z) - E(X)Cov(X, Z) - (E(X))^2E(Z)$
$= E(XXZ)$
$= E(XY)$
$= 0.5X^2 - 0.5X^2 = 0$

(e) If X and Y are normally distributed random variables with $Cov[X, Y] = 0$, then X and Y must be independent.

Ans: False.

The example in question 7 through parts a) and d) provide a counterexample.

8. Let the random variables X_i, $i = 1, 2, \ldots, n$, be i.i.d. having the uniform distribution on $[0,1]$, denoted $X_i \sim U[0,1]$. Consider the random variables $m = \min\{X_1, X_2, \ldots, X_n\}$ and $M = \max\{X_1, X_2, \ldots, X_n\}$. For both random variables m and M, derive their respective cumulative distribution (cdf), probability density function (pdf), and expected value.

   CDF of m $= P(m \leq x)$
   $= P(\text{at least one of } \{X_1, ..., X_n\} \leq x)$
   $= 1 - P(all\ of\ \{X_1, ..., X_n\} > x)$
   $= 1 - [1 - P(X \leq x)]^n$
   $= 1 - (1 - F(x))^n$
   $= 1 - (1 - x)^n$
   CDF of M $= P(all\ of\ \{X_1, ..., X_n\} \leq x)$
   $= (P(X \leq x))^n$
   $= (F(x))^n$
   $= x^n$

   PDF of m $= P(m = x) = n(1-x)^{n-1}$
   PDF of M $= P(M = x) = nx^{n-1}$

   E(m) $= \int_0^1 xn(1-x)^{n-1}dx = \frac{1}{n+1}$
   E(M) $= \int_0^1 xnx^{n-1}dx = \frac{n}{n+1}$

9. You want to simulate a dynamic economy (e.g., an OLG model) with two possible states in each period, a "good" state and a "bad" state. In each period, the probability of both shocks is $\frac{1}{2}$. Across periods the shocks are independent. Answer the following questions using the Central Limit Theorem and the Chebyshev Inequality.

   (a) What is the probability that the number of good states over 1000 periods differs from 500 by at most 2%?

Let the Bernoulli random variable $X_i = 1$ if the state is good and $X_i = 0$ if the state is bad. P(no. of good states over 1000 periods differs from 500 by at most 2%)

$= P(|S_{1000} - 500| \leq 10)$ *where* $S_{1000} = X1 + ... + X_{1000}$

$= P(\frac{|S_{1000}-500|}{0.5\sqrt{1000}} \leq \frac{10}{0.5\sqrt{1000}})$ by the Central Limit Theorem

$= P(\frac{10}{0.5\sqrt{1000}} \leq \frac{S_{1000}-500}{0.5\sqrt{1000}} \leq \frac{10}{0.5\sqrt{1000}})$

$= \Phi(\frac{10}{0.5\sqrt{1000}}) - \Phi(\frac{-10}{0.5\sqrt{1000}})$

$= 0.473$

(b) Over how many periods do you need to simulate the economy to have a probability of at least 0.99 that the proportion of good states differs from $\frac{1}{2}$ by less than 1%?

$P(|\frac{X_1+...+X_n}{n} - \frac{1}{2} < (0.5)(0.01)) \geq 0.99$

$= 1$ - $P(|\frac{X_1+...+X_n}{n} - \frac{1}{2} \geq 0.005) \geq 0.99$

$= P(|\frac{X_1+...+X_n}{n} - \frac{1}{2} \geq 0.005) \leq 0.01$

$= \frac{\sigma^2}{n\epsilon^2}$ by the Weak Law of Large Numbers

$= \frac{0.25}{n(0.005)^2}$

Solving for n, we get n $= \frac{0.25}{0.01(0.005)^2} =$ 1,000,000

10. If $E[X] < 0$ and $\theta \neq 0$ is such that $E[e^{\theta X}] = 1$, prove that $\theta > 0$.

Proof by Contradiction

Let $f(x) = e^{\theta x}$. Assume that $E(X) < 0$ and $\theta \neq 0$ s.t. $E(e^{\theta x}) = 1$. Suppose to the contrary that $\theta < 0$. Because $f''(x) = (\theta)^2 e^{\theta x} > 0$, we know $f(x)$ is a convex function. Therefore, by Jensen's Inequality, $E(e^{\theta x}) \geq e^{\theta E(X)}$ implies that $e^{\theta E(X)} \leq 1$. Since $E(X) < 0$ and $\theta \neq 0$, we know that $\theta E(X) > 0$. Therefore, $e^a \leq 1$ where $a = \theta E(X)$. However, this contradicts the fact that $e^a \geq 1 \ \forall \ a > 0$. Therefore, it follows that $\theta > 0$.