Econ, Problem Set #2, Dynamic Programming

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Set 1, Ex 2) Proof.

$|f(x) - f(x')|$
$= |c(1-\beta) + \beta \sum_{k=1}^{K} max\{W_k, x\} P_k - c(1-\beta) - \beta \sum_{k=1}^{K} max\{W_k, x'\} P_k|$
$= |\beta|| \sum_{k=1}^{K} max\{W_k, x\} P_k - max\{W_k, x'\} P_k|$
$= |\beta|| \sum_{k=1}^{K} P_k(max\{W_k, x\} - max\{W_k, x'\})|$
$\leq |\beta| \sum_{k=1}^{K} |P_k(max\{W_k, x\} - max\{W_k, x'\}|)$
$\leq |\beta| \sum_{k=1}^{K} |P_k||x - x'|$
$= |\beta|| \sum_{k=1}^{K} P_k|x - x'|$
$\leq |\beta||x - x'|.$

Therefore, by Banach's Fixed Point Theorem, there exists a unique solution to the equation. As such, we can make use of this to iterate values on the equation to solve for the fixed point in Exercise 3.

Set 2, Ex 1) Proof.

$|Uw - Uw'|$
$= |U(\sigma(y)) + \beta \int w(f(y - \sigma(y))z)\phi(dz) - U(\sigma(y)) - \beta \int w'(f(y - \sigma(y))z)\phi(dz)|$
$= |\beta|| \int w(f(y - \sigma(y))z) - w'(f(y - \sigma(y))z)\phi(dz)|$
$\leq |\beta| sup_y | \int w(f(y - \sigma(y))z) - w'(f(y - \sigma(y))z)\phi(dz)|$
$= |\beta| \|w(f(y - \sigma(y))z) - w'(f(y - \sigma(y))z)\|_\infty$
$= |\beta| \|w - w'\|_\infty$

Therefore, U is a contraction mapping with respect to the supremum distance.

Since U is a contraction mapping, it follows from Banach's Fixed Point Theorem that U has only one fixed point w^* in C. Now note that for a given sequence of incomes y, $v_\sigma(y)$, which is the expected lifetime utility over infinite periods does not change from period to period. This is because if it did change, it would mean that the utility function and therefore preferences have changed, which contradicts the assumption of unchanging preferences in our model. Therefore, v_σ is a fixed point. Since there is only one fixed point, it follows that v_σ is the unique fixed point of U in C.