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Symmetric Ternary Switching Functions: Their Detection and Realization with Threshold Logic

ROY D. MERRILL, JR., MEMBER, IEEE

Abstract—**A simple, systematic procedure for detecting symmetric ternary switching functions is developed. The detection procedure is essentially based on a derived counting theorem. Other important properties and close bounds on the number of ternary symmetric functions are developed. A design method is given whereby symmetric functions can be readily synthesized with switching networks which are economical, and, in certain instances, minimal in the number of ternary threshold devices required. The greatest lower bound on the number of devices required for a given symmetric function is developed.**

Index Terms—**Counting theorem, design method, detection algorithm for symmetric functions, logical design, multivalued logic, switching networks, symmetric ternary switching functions, ternary logic, ternary threshold devices, threshold logic.**

I. Introduction

IN RECENT YEARS, there has been a growing awareness of the potential usefulness of ternary logic and 3-state switching networks in engineering applications.[1] This paper attempts to broaden the knowledge of ternary switching theory in considering a class of functions called "symmetric ternary functions." These functions, like their binary counterpart, arise in arithmetic networks, code checking and correcting circuits, counting circuits, and certain coding-decoding networks, to name a few. The class of symmetric functions considered numbers between 3^{2n+1} and $(n!)(2^n)(3^{3n+1})$ of the 3^{3n} ternary functions of n variables.

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The author is with the Lockheed Palo Alto Research Laboratory, Lockheed Missiles & Space Company, Palo Alto, Calif.

[1] The reader is directed to the references[1]–[3] for a fuller coverage of the applications and theory of ternary switching functions.

This paper presents properties of symmetric ternary functions and develops a simple, systematic procedure for detecting these functions. Also, a design method is given whereby symmetric functions can be synthesized with switching networks which are economical, and, in certain instances, minimal in the number of ternary threshold devices required. These results are somewhat general in that an arbitrary ternary switching function $f(x_1, x_2, \cdots, x_n)$ can be expressed as the symmetric function in $(3^n-1)/2$ variables: $g(x_1, x_2, x_2, x_2, \cdots, x_n)$ where each x_i appears with multiplicity 3^{i-1}.

II. Definitions

Let the vector space V^n be the set of 3^n n-triples (or vectors) $V_i=(v_{i1}, v_{i2}, \cdots, v_{in})$, $i=1, 2, \cdots, 3^n$, representing all assignments of the truth values -1, 0, $+1$ to the variables $x_1, x_2, \cdots, x_n$. A *ternary switching function* $f(x_1, x_2, \cdots, x_n)$, or simply $f(X)$, defines a partition of V^n into three sets of n-tuples f^-, f^0, and f^+, called *truth sets*, which are mapped by f into -1, 0, and $+1$, respectively. The truth table representation for f will be

$$\begin{array}{c} x_1 \quad x_2 \cdots x_n \\ \left[\begin{array}{c} f^+ \\ \hline f^- \end{array}\right] \end{array}$$

called the *principal matrix* for f where f^+ and f^- are the *principal submatrices* for f.

Two parameters of importance are 1) the number of -1, 0, and $+1$ truth values in each column of the truth table matrix and its submatrices; these three counts for the column of the ith variable are the ordered elements of the 3-tuples c_i, c_i^+, and c_i^- called the *column counts* of the truth table matrix, the f^+ submatrix, and the f^- submatrix, respectively; and 2) the *row weight*

$$r_i = \sum_{j=1}^{n} v_{ij} \quad \text{for the } n\text{-tuple } V_i.$$

Two columns, i and j, are said to have *matrix (submatrix) equivalent counts* if c_i (c_i^- or c_i^+) can be made equal to $c_j(c_j^-$ or $c_j^+)$ by a permutation of the counts. All such possible transformations are the permutations of degree three, $\phi_k(x)$, $k=0, 1, 2, 3, 4, 5$ defined in Table I.[2] The term $\phi_k(x)$ will sometimes be written as x^k.

TABLE I
PERMUTATIONS OF DEGREE THREE

i	$\phi_i(x)$					
x	0	1	2	3	4	5
-1	-1	0	-1	1	0	1
0	0	-1	1	-1	1	0
1	1	1	0	0	-1	-1

The function $f(X)$ is said to have a *standard matrix* if the column counts of the principal matrix, c_i, c_i^-, and c_i^+, are equivalent to c_j, c_j^-, and c_j^+, respectively, for all $i, j=1, 2, \cdots, n$. If $f(X)$ has a standard matrix, then it is obtained from the principal matrix by transforming the x_i (and, identically, the truth values of column i) using appropriate ϕ_k, $k=0, 1, \cdots, 5$, for $i=1, 2, \cdots, n$, such that all columns of the resulting matrix have the same set of matrix and submatrix column counts. $f(X)$ has at least six standard matrices, since if one standard matrix is represented in terms of $z_1, z_2, \cdots, z_n$, then five more can be generated, each in terms of $z_1^k, z_2^k, \cdots, z_n^k$, $k=1, 2, 3, 4, 5$, respectively. Later, however, it will be shown that there can be many more than six standard matrices for certain functions.

$f(X)$ is said to be a *symmetric ternary function* if there exist two sets of numbers A and B, called *a- and b-numbers*, and a set of transformations $\phi_{k_i}(x_i)=y_i$, $i=1, 2, \cdots, n$, sometimes written $Y=\Phi(X)$, such that

$$f(X) = \begin{cases} +1 & \text{iff there exists an } a_i \in A \text{ such that } \sum_{j=1}^{n} y_j = a_i \\ -1 & \text{iff there exists a } b_i \in B \text{ such that } \sum_{j=1}^{n} y_j = b_i \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

$f(X)$ will be written in Shannon's notation: $S_{A,B}(Y)$.

Clearly, if $f(X)$ is symmetric on $y_1, y_2, \cdots, y_n$, then it defines a partition of the set of numbers $(-n, -n+1, \cdots, n)$ into the sets A, B, and an additional set C, called *c-numbers*. Thus $A \cup B \cup C$ is the set of $2n+1$ numbers which are all possible distinct sums

$$\sum_{j=1}^{n} y_j,$$

$y_j = -1, 0,$ or $+1$.

III. Symmetric Function Properties

This section develops the theory used in the algorithm for detecting symmetric functions presented in Section IV. In Section V, a procedure is given whereby symmetric functions can be realized quite economically with threshold logic networks. Since this is not generally possible for arbitrary ternary switching functions, there is an added incentive in being able to determine whether a function is symmetric.

Clearly, by (1), symmetric functions can be detected by complete enumeration; that is, by examining all possible transformations y_i of the x_i, $i=1, 2, \cdots, n$, for $f(X)$. Then for each of these up to 6^n possible forms of the $f(X)$, compute the row weights for f^+, f^-, and f^0 and check to see if an A and B that satisfy (1) exist. This procedure is obviously costly; however, a much simpler, systematic method exists, as will be proven in the following development. The detection method which is developed is analogous to one used for binary functions.[5] Theorem proofs are given in the Appendix.

Theorem 1 is the fundamental theorem for symmetric functions.

Theorem 1:[3]

$$S_{A,B}(Y) = S_{A,B}(y_1, y_2, \cdots, y_{i-1}, y_j, y_{i+1}, \cdots, y_{j-1}, y_i, y_{j+1}, \cdots, y_n).$$

The two forms of $S_{(1),(2,3)}(x_1, x_2^4, x_3)$ in Table II, columns (d) and (e), illustrate this identity.

Theorem 2: If $f(X)$ is a symmetric function, then it has standard matrices.

To determine if $f(X)$ has standard matrices, first calculate the principal matrix and submatrix column counts: c_i, c_i^-, and c_i^+, $i=1, 2, \cdots, n$, as illustrated in Table II, column (a). Next, examine the columns by pairs. If every pair i and j has equivalent counts for c_i, c_i^-, c_i^+ and c_j, c_j^-, c_j^+, then $f(X)$ has standard matrices. Example 1 has such matrices, one form of which is shown in Table II, column (d).

Corollary 1: If $f(X)$ is a symmetric function, it will be

[2] These six transformations form a group under the operation defined by the mappings in Table I.[4]

[3] It should be noted that there can exist $f(X)$ such that for some $Y=\Phi(X)$, $f(y_1, y_2, \cdots, y_{i-1}, y_j, y_{i+1}, \cdots, y_{j-1}, y_i, y_{j+1}, \cdots, y_n) = f(Y)$ for any $1 \le i, j \le n$ which are not symmetric by the Section II definition. The class of such functions is exemplified by the $n=3$ residue functions having only one principal submatrix containing the subset, either $\{(1, 1, -1), (1, -1, 1), (-1, 1, 1)\}$, or $\{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}$.

TABLE II
EXAMPLE 1

	(a)				(b)				(c)				(d)				(e)			
	x_1	x_2	x_3	r	x_3	x_2	x_1	r	x_2	x_1	x_3	r	x_1	x_2^4	x_3	r	x_2^4	x_1	x_3	r
	1	−1	0	0	0	−1	1	0	−1	1	0	0	1	0	0	1	0	1	0	1
	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	0	0	1
+truth	0	−1	1	0	1	−1	0	0	−1	0	1	0	0	0	1	1	0	0	1	1
set:	1	0	−1	0	−1	0	1	0	0	1	−1	0	1	1	−1	1	1	1	−1	1
	1	1	1	3	1	1	1	3	1	1	1	3	1	−1	1	1	−1	1	1	1
	−1	0	1	0	1	0	−1	0	0	−1	1	0	−1	1	1	1	1	−1	1	1
Submatrix																				
−1 count:	1	2	1		1	2	1		2	1	1		1	1	1		1	1	1	
0 count:	2	3	2		2	3	2		3	2	2		2	2	2		2	2	2	
+1 count:	3	1	3		3	1	3		1	3	3		3	3	3		3	3	3	
	1	0	1	2	1	0	1	2	0	1	1	2	1	1	1	3	1	1	1	3
−truth	1	0	0	1	0	0	1	1	0	1	0	1	0	1	1	2	1	0	1	2
set:	1	−1	1	1	1	−1	1	1	−1	1	1	1	1	0	1	2	0	1	1	2
	0	0	1	1	1	0	0	1	0	0	1	1	1	1	0	2	1	1	0	2
Submatrix																				
−1 count:	0	1	0		0	1	0		1	0	0		0	0	0		0	0	0	
0 count:	1	3	1		1	3	1		3	1	1		1	1	1		1	1	1	
+1 count:	3	0	3		3	0	3		0	3	3		3	3	3		3	3	3	
Matrix																				
−1 count:	1	3	1		1	3	1		3	1	1		1	1	1		1	1	1	
0 count:	3	6	3		3	6	3		6	3	3		3	3	3		3	3	3	
+1 count:	6	6	6		6	6	6		6	6	6		6	6	6		6	6	6	

symmetric on the variables of at least one of its standard matrices.

Theorem 2 provides a necessary, but not sufficient, condition for a function to be symmetric. By the Corollary, given that $f(X)$ has standard matrices, one can determine symmetry by examining only the standard matrices. In testing each of the standard matrices, the basic question arises. Given that a standard submatrix vector (row) has a row weight r, is every vector in V^n with a row weight r also in the submatrix? If the answer is yes for each row of both submatrices, then the standard matrix is said to have a *proper row weight distribution.* Clearly, f is symmetric on $z_1, z_2, \cdots, z_n$ only if the standard matrix for f has a proper row weight distribution. This is true, since if two rows exist with the same row weight which are not both in the same standard submatrix, then f will not map the vectors corresponding to these rows into the same truth set. But by (1), if f is symmetric on $z_1, z_2, \cdots, z_n$, the vectors with equal row weights are mapped by f into the same truth set. This result is formally stated in Theorem 3.

Theorem 3: $f(X)$ is a symmetric function if and only if at least one standard matrix of the set of all standard matrices for $f(X)$ has a proper row weight distribution.

In the next theorem a simple, systematic method is given for determining whether a standard matrix has a proper row weight distribution.

Theorem 4: The number of vectors (rows) in V^n with a row weight r is[4]

$$N(r) = \sum_{i=0}^{[(n-|r|)/2]} \binom{n}{(i)\quad(|r|+i)}. \tag{2}$$

An obvious consequence of Theorem 4 is that

$$\sum_{r=-n}^{n} N(r) = 3^n$$

which results from the fact that r can take any one of the $2n+1$ values in the set $(-n, -n+1, \cdots, n)$.

Example 1 in Table II illustrates Theorem 4. Starting with the first row of the standard matrix (d):

$$N(1) = \sum_{i=0}^{1} \binom{3}{(i)\quad(1+i)} = 3 + 3 = 6$$

$$N(3) = \sum_{i=0}^{0} \binom{0}{(i)\quad(3+i)} = \binom{3}{3} = 1$$

$$N(2) = \sum_{i=0}^{0} \binom{3}{(i)\quad(2+i)} = \binom{3}{0\quad 2} = 3.$$

[4] $[q]$ is defined as the largest integer not greater than q.

$$\binom{n}{(i)\quad(j)}$$

is the trinomial coefficient.

TABLE III
COUNTER EXAMPLE

	$f(y_1, y_2)$	$f(y_1^1, y_2^1)$	$f(y_1^2, y_2^2)$	$f(y_1^3, y_2^3)$	$f(y_1^4, y_2^4)$
+truth	0 0	− −	+ +	− −	+ +
set:	− +	0 +	− 0	+ 0	0 −
	+ −	+ 0	0 −	0 +	− 0
	+ +	+ +	0 0	0 0	− −
	– – –	– – –	– – –	– – –	– – –
−truth	+ 0	+ −	0 +	0 −	− +
set:	0 +	− +	+ 0	− 0	+ −

TABLE IV
EXAMPLE 3

z_1	z_2	r	z_1^1	z_2^1	r	z_1^2	z_2^2	r
−	−	−2	0	0	0	−	−	−2
0	+	+1	−	+	0	+	0	+1
+	0	+1	+	−	0	0	+	+1
0	0	0	−	−	−2	+	+	+2
–	–		–	–		–	–	
+	−	0	+	0	+1	0	−	−1
−	+	0	0	+	+1	−	0	−1
+	+	+2	+	+	+2	0	0	0

Hence, this standard matrix has the proper row weight distribution which by Theorem 3 proves Example 1 to be the function $S_{(1),(2,3)}(x_1, x_2^4, x_3)$.

Although Theorem 3 gives the necessary and sufficient conditions for a switching function to be a symmetric function, in some instances there can be many standard matrices; more, in fact, than six. This will occur if, in the standard matrices examined, two or three column counts are equal in each c_i, c_i^-, and c_i^+. In the worst case it can be shown that there can be as many as 6^n standard matrices. The next three theorems provide a basis for a method of substantially reducing the number of standard matrices which must be examined in verifying that a function is or is not symmetric.

Consider the cases where at least one standard submatrix or the standard matrix itself has distinct column counts. It is easily shown that in all such cases there can be only six distinct standard matrices per function. Theorems 5 and 6 show that at most three of these need be examined in determining whether a function is symmetric.

Theorem 5: $S_{A,B}(Y) = S_{A^-,B^-}(y_1^5, y_2^5, \cdots, y_n^5)$ where A^- and B^- are A and B, respectively, except that the elements of each set A and B are multiplied by -1.

Relationships similar to Theorem 5 do not generally exist for ϕ_1, ϕ_2, ϕ_3, and ϕ_4 as demonstrated by the counter examples in Table III where $f(y_1, y_2) = S_{(0,2),(1)}(y_1, y_2)$. However, $f(y_1^i, y_2^i)$, $i = 1, 2, 3$ and 4, are not symmetric on their variables, since each has $y_1^i + y_2^i = 0$ in two different truth sets.

Theorem 6: If $f(X)$ is a symmetric function with a standard matrix in terms of $z_1, z_2, \cdots, z_n$ where at least one submatrix or the matrix itself has distinct -1, 0, and 1 column counts, then $f(X)$ is symmetric on either z_i, z_i^1, or z_i^2, $i = 1, 2, \cdots, n$ (or, alternatively, either z_i^5, z_i^4, or z_i^3, $i = 1, 2, \cdots, n$).

Theorem 6 is illustrated in Table IV. Note that $f(X)$ is symmetric on z_1^1, z_2^1 but not z_1, z_2, or z_1^2, z_2^2.

Bounds on the Number of Symmetric Ternary Functions: Consider the number of symmetric functions there are for a given n. The a-, b-, and c-numbers for a specific symmetric function together comprise the set of $2n+1$ numbers $(-n, \cdots, n)$. Conversely, each symmetric function defines a partition of $(-n, \cdots, n)$ into the a-, b-, and c-numbers dependent on the variables $y_1, y_2, \cdots, y_n$. Therefore, given $y_1, y_2, \cdots, y_n$, there are exactly 3^{2n+1} ways of partitioning $(-n, -n+1, \cdots, n)$ into a-, b-, and c-number sets. It follows then that there are at least 3^{2n+1} unique symmetric functions for a given n.

It is possible, however, for y_i to be any one of six transformations of x_i. Further, there are $n!$ ways of ordering the y_i. Consequently, there can be no more than $n! \cdot 6^n \cdot 3^{2n+1} = n! \cdot 2^n \cdot 3^{3n+1}$ symmetric functions for a given n. It is known by Theorem 5 that this is not the lowest upper bound, however.

From Theorem 6, at most three standard matrices for a given function need be examined in determining symmetry if at least one submatrix or the matrix itself has distinct column counts. As previously noted, if this is not the case, there are many more than six standard matrices. A similar condition can arise for binary functions. McCluskey developed a technique for simplifying the binary detection procedure whereby the function is expanded about one of its variables and the residues are tested for symmetry.[5] It will now be shown that an analogous procedure can be used for ternary functions.

Theorem 7: $S_{A,B}(Y)$ can be expanded about any variable, say y_1, to obtain the following three residues:

$$S_{A,B}(Y) = \begin{cases} S_{A_-,B_-}(y_2, y_3, \cdots, y_n) & \text{if } y_1 = +1 \\ S_{A_0,B_0}(y_2, y_3, \cdots, y_n) & \text{if } y_1 = 0 \\ S_{A_+,B_+}(y_2, y_3, \cdots, y_n) & \text{if } y_1 = -1 \end{cases}$$

where the new a-, b-, and c-numbers are defined as follows: for every a_i in A there corresponds an $(a_i - 1)$ in A_-, a_i in A_0 and (a_i+1) in A_+ except that A_-, A_0, and A_+ are constrained from containing elements greater than $n-1$ or less than $-n+1$. B_-, B_0, B_+, C_-, C_0, and C_+ are similarly defined.

Corollary 2: Given three arbitrary functions $S_{A_1,B_1}(y_2, y_3, \cdots, y_n)$, $S_{A_2,B_2}(y_2, y_3, \cdots, y_n)$, and $S_{A_3,A_3}(y_2, y_3, \cdots, y_n)$, if there exist two sets A and B where $A \cap B = \Phi$, the null set, and $A, B \subset (-n, -n+1, \cdots, n)$ such that $A_- = A_1$, $A_0 = A_2$, $A_+ = A_3$, $B_- = B_1$, $B_0 = B_2$, and $B_+ = B_3$, then

TABLE V

a- AND b-NUMBER COMBINATIONS*

Row	a-Number Combinations			b-Number Combinations			k_1	k_2	k_3	y_1
0	D_+	G	K_-	E_+	H	L_-	$+1$	0	-1	x_1^0
1	D_+	K	G_-	E_+	L	H_-	$+1$	-1	0	x_1^1
2	G_+	D	K_-	H_+	E	L_-	0	$+1$	-1	x_1^2
3	G_+	K	D_-	H_+	L	E_-	0	-1	$+1$	x_1^3
4	K_+	D	G_-	L_+	E	H_-	-1	$+1$	0	x_1^4
5	K_+	G	D_-	L_+	H	E_-	-1	0	$+1$	x_1^5

* Parentheses have been dispensed with for convenience. Thus, for example, $D_+=(D)_+$ and $D=(D)$.

$$S_{A,B}(Y) = \begin{cases} S_{A_1,B_1}(y_2, y_3, \cdots, y_n) & \text{if } y_1 = +1 \\ S_{A_2,B_2}(y_2, y_3, \cdots, y_n) & \text{if } y_1 = 0 \\ S_{A_3,B_3}(y_2, y_3, \cdots, y_n) & \text{if } y_1 = -1. \end{cases}$$

It follows from Theorem 7 and its Corollary that a given switching function is not symmetric if any of the following cases arise for the residues of $f(X)$:

1) at least one of the three residues is not a symmetric function;
2) all three residues are not symmetric on the same set of variables;
3) there exists no pair of sets A and B for which the a- and b-numbers of the residues can be formed as in Corollary 2.

The first two cases can be checked out easily on the basis of the development thus far. Case 3) requires further consideration, however. The following procedure provides a convenient means of determining whether sets A and B exist, given that the residues of $f(X)$ are all symmetric on the same set of variables [say $(y_2, y_3, \cdots, y_n)$ for the case of expanding $f(X)$ about x_1] with the form

$$\begin{aligned} f(x_1 = k_1) &= S_{D,E}(y_2, y_3, \cdots, y_n) \\ f(x_1 = k_2) &= S_{G,H}(y_2, y_3, \cdots, y_n) \\ f(x_2 = k_3) &= S_{K,L}(y_2, y_3, \cdots, y_n) \end{aligned} \tag{3}$$

where (k_1, k_2, k_3) equals $(-1, 0, +1)$ up to a permutation of the truth values.

Suppose $f(X) = S_{A,B}(Y)$ and let $A = (a_1, a_2, \cdots, a_r)$ where $a_1 < a_2 < \cdots < a_r$. Define A_-, A_0, and A_+ as in Theorem 7. Further, define $(A_-)_+$, (A_0), and $(A_+)_-$ as follows: for every element of an arbitrary set Q, there is a corresponding element q_i+1 in $(Q)_+$, q_i-1 in $(Q)_-$, and q_i in (Q). Now it can be stated that

$$(A_-)_+ \cup (A_0) \cup (A_+)_- = A. \tag{4}$$

Equation (4) can be proved by considering the following conditions of A:

a) $$\begin{aligned} a_1 &= -n \\ a_2 &= -n+1 \\ a_{r-1} &= n-1 \\ a_r &= n \end{aligned}$$

b) $$\begin{aligned} a_1 &\geq -n+2 \\ a_r &\leq n-2. \end{aligned}$$

Note that for a),

$$\begin{aligned} (A_-)_+ &= (a_3, a_4, \cdots, a_r) \\ (A_0) &= (a_2, a_3, \cdots, a_{r-1}) \\ (A_+)_- &= (a_1, a_2, \cdots, a_{r-2}) \end{aligned}$$

and for b),

$$(A_-)_+ = (A_0) = (A_+)_- = A.$$

It follows that (4) holds for both conditions. Since a) and b) represent the extremes in the values taken by the a_i, (4) holds for all A. Similarly, the equality

$$(B_-)_+ \cup (B_0) \cup (B_+)_- = B \tag{5}$$

is true for all B. As mentioned earlier, A and B are distinct sets with elements in the range $(-n, -n+1, \cdots, n)$. Therefore, since $A \cap B = \Phi$ from (4) and (5),

$$\{(A_-)_+ \cup (A_0) \cup (A_+)_-\} \cap \{(B_-)_+ \cup (B_0) \cup (B_+)_-\} = \Phi. \tag{6}$$

Equations (4), (5), and (6) then provide a basis for determining if, given the sets D, E, G, H, K, and L of (3), there exist sets A and B such that $f(X) = S_{A,B}(Y)$. The first step in the procedure is to construct a table of the form in Table V. Note that the six combinations of sets in this table are exhaustive under the condition that all three residues of $f(X)$ are symmetric on the same set of variables.

The second step in determining if A and B exist is to find all rows of Table V which satisfy (6), that is, all rows for which the union of a-number combinations, $\hat{A}$, intersected with the union of b-number combinations, $\hat{B}$, produce the null or empty set. The third step is to consider each row that satisfies (6) and again take $\hat{A}$ and $\hat{B}$. Then form $(\hat{A}_-, \hat{A}_0, \hat{A}_+)$ and $(\hat{B}_-, \hat{B}_0, \hat{B}_+)$ and compare with the a-number combinations and b-number combinations, respectively, of that row. If in comparing these combinations the corresponding sets are equal, say in the ith row of the table, then $f(X)$ is symmetric on $x_1^i, y_2, \cdots, y_n$ with the a- and b-numbers $\hat{A}$ and $\hat{B}$. Since the table constructed as in Table V gives all possible combinations whereby one might reconstruct the A and B sets given the a- and b-numbers of the residues, it follows that if no row of the table satisfies the conditions of the third step, then $f(X)$ is not symmetric.

In carrying out the expansion of $f(X)$, it is possible that for each of the residues neither the standard submatrix nor the standard matrix itself has distinct -1, 0, and $+1$ column counts. In this case, it will be necessary to expand one of the residues and test the resulting standard matrices for distinct column counts. If the above problem arises again, the expansion and distinct column count test process is repeated. We are always assured of finding a residue which can easily be tested for symmetry properties, even if it requires n expansions, since the residues of the nth expansion are constants. Generally, however, one expansion provides the desired matrix. Having once found a residue which has distinct column counts in its matrix or at least one of its submatrices, then it is possible to determine if the function is symmetric on the basis of Theorems 2, 3, 4, and 6. By Theorem 7, if any of the three residues is not symmetric, then the original function is not symmetric. If all three residues are symmetric on the same set of variables, then the above procedure is used to determine if the function expanded to get the residues is symmetric. If this function is itself a residue and also symmetric, the above process is repeated, etc., until either a residue is found not to be symmetric or the original function is found to be symmetric. In the next section an example is given which illustrates the procedure under one expansion.

IV. An Algorithm for Detecting Symmetric Ternary Functions

In Section III, seven properties of symmetric ternary functions were presented. Of these seven, Theorems 2, 3, 4, 6, and 7 can be used directly as a basis for a symmetric function detection procedure. By Theorem 2 a function must have a standard matrix to be symmetric. Theorem 3 states that for a function to be symmetric on the variables of its standard matrix, the standard matrix must have the proper row weight distributions as determined by the counting technique given in Theorem 4. Theorems 6 and 7, together with the construction technique delineated in Table V, provide the techniques for finding all standard matrices of a function which must be examined in determining whether the function is symmetric.

The following algorithm utilizes these five theorems to provide a systematic procedure for detecting symmetric ternary functions. The mechanics of implementing the algorithm are illustrated with an example. The illustrative function used exhibits all the basic difficulties which might be encountered in the typical logical design problem.

Consider the function given in truth table form as in Table VI(a).[5]

Step 1: Form the column counts for the principal matrix and both submatrices. If the function has a standard matrix, then by Theorem 2 it may be a symmetric function; otherwise, it is not. Example 4 has a standard matrix.

Step 2: If the principal matrix or at least one submatrix has distinct column counts, then transform the principal matrix into a standard matrix; otherwise, go to Step 4. $f(X)$ in the example does not pass this test, but its residue $f_+ = f(x_1 = +1)$ does, Table VI(b). A standard matrix for f_+ is given in Table VI(c).

Step 3: Form the row weights for the standard matrix obtained in Step 2. For each distinct row weight, calculate by Theorem 4 the number of vectors or rows which must have this weight. If the standard matrix in terms of $z_1, z_2, \cdots, z_n$ has the proper row weight distribution, then by Theorem 3 its function is symmetric. Form the standard matrix in terms of $z_1^1, z_2^1, \cdots, z_n^1$ and again perform the above test if the first standard matrix does not have the proper row weight distribution. If neither satisfies this condition, then form the standard matrix in terms of $z_1^2, z_2^2, \cdots, z_n^2$. If one of the three standard matrices formed has the proper row weight distributions, then the function is symmetric; otherwise, it is not a symmetric function. In Example 4 the standard matrix for f_+ in terms of x_2^3, x_3 does not have the proper row weight distribution, Table VI(c). However, the standard matrix in Table VI(d) does. Hence, f_+ is symmetric. Using the same variables, the standard matrices for f_- and f_0, Table VI(e) and (f), also have the proper row weight distributions. Their symmetric function representations are given in Table VI(g).

Step 4: Proceeding from Step 2, expand the function about one variable. If at least one residue has distinct column counts in one or more columns, carry out Steps 1, 2, and 3 for that residue; otherwise, repeat Step 4 on either of the residues. In Example 4, f_+ was found to have distinct column counts as shown in Table VI(b). If all three residues are found to be symmetric on the same set of variables, carry out Step 5; otherwise stop, since by Theorem 7 it is known that the original function is not symmetric.

Step 5: Form the a- and b-number combinations for the three residues as stipulated in Table V. Then perform the tests prescribed with relationship to (4), (5), and (6) to determine the A and B sets if they exist. For the illustrative example, Table VI(g), (h), and (i) demonstrates this process. If the function whose residues were just used is symmetric and if it is a residue, repeat the above procedure for each expansion, stopping only if a residue or the original function is found not to be symmetric.

[5] It is assumed that the function has no degenerate variables. A systematic procedure for determining all such variables in the original function is given in Merrill.[8]

TABLE VI
EXAMPLE 4

	(a) $f(x_1, x_2, x_3)$ x_1	x_2	x_3	Σy	(b) $f_+(x_2, x_3)$ x_2	x_3		(c) $f_+(x_2^3, x_3)$ x_2^3	x_3	Σy	(d) $f_+(x_2^5, x_3^1)$ $(x_2^3)^1$	x_3^1	Σy	(e) $f_0(x_2^5, x_3^1)$ x_2^5	x_3^1	Σy	(f) $f_-(x_2^5, x_3^1)$ x_2^5	x_3^1	Σy
	0	1	−1	0	0	−1		−1	−1	−2	0	0	0	−1	0	−1	−1	−1	−2
	−1	1	0	0 +	−1	0	+	1	0	1 +	1	−1	0 +	0	−1	−1 +	1	0	1
	0	0	0	0	1	1		0	1	1	−1	1	0	1	1	2	0	1	1
	1	0	−1	0	1	1		1	1		1	1		1			1	1	
+truth set	−1	−1	−1	−3	1	1		1	1		1	1		1	1		1	1	
	−1	0	1	0	1	1		1	1		1	1		1	1		1	1	
	1	−1	1	0	−1	1		1	1	2	1	1	2	−1	−1	−2 −	1	1	2
	1	1	1	+3 −	−1	−1	−	1	−1	0 −	1	0	1	1	1		0	0	
	0	−1	1	0	0	1		−1	1	0	0	1	1	0	0		0	0	
−1 count:	3	3	3		2	1		1	1		0	0		0	0		1	1	
0 count:	3	3	3		1	0		0	0		1	1							
+1 count:	3	3	3		0	2		2	2		2	2							
	1	−1	1	1															
−truth set:	1	−1	−1	−1															
	1	0	1	2															
	−1	−1	1	−1															
	0	1	0	1															
−1 count:	1	3	1																
0 count:	1	1	1																
+1 count:	3	1	3																
−1 count:	4	6	4																
0 count:	4	4	4																
+1 count:	6	4	6																

(g)

$f_+=f_{k_1}=S_{(0),(1,2)}(x_2^5, x_3^1)$, $D=(0)$, $E=(1, 2)$

$f_0=f_{k_2}=S_{(-1,2),(-2)}(x_2^5, x_3^1)$, $G=(-1, 2)$ $H=(-2)$

$=f_{k_3}=S_{(-2,1)(2)}(x_2^5, x_3^1)$, $K=(-2, 1)$, $L=(2)$

(h)

Row	a-number Combinations	b-number Combinations	Intersection
0	(1), (−1, 2), (−3, 0)	(2, 3), (−2), (1)	(1)
1	(1), (−2, 1), (−2, 1)	(2, 3), (2), (−3)	ϕ
2	(0, 3), (−2, 1), (−3, 0)	(−1), (1, 2), (1)	(1)
3	(0, 3), (−2, 1), (−1)	(−1), (2), (0, 1)	(0, 1)
4	(−2, 2), (0), (−2, 1)	(3), (1, 2), (−3)	(2)
5	(−2, 2), (−1, 2), (−1)	(3), (−2), (0, 1)	(−2)

(i)

$A=(-2, 1)$, $B=(-3, 2, 3)$

$A_-=(0)$ $B_-=(1, 2)$

$A_0=(-2, 1)$ $B_0=(2)$

$A_+=(-1, 2)$ $B_+=(-2)$

$\therefore f(x_1, x_2, x_3,)=S_{A,B}\ (x_1^1, x_2^5, x_3^1)$

V. Threshold Logic Design Procedure for Symmetric Functions

This section deals with the realization of symmetric ternary functions with ternary threshold networks. A procedure is developed which can be systematically applied to obtain economical, and in some cases, minimal threshold network realizations of symmetric functions. The results obtained have application to the larger class of arbitrary ternary switching functions in the nature of the approach taken as well as the fact that any function $f(X)$ can be expressed as the symmetric function in $(3^n-1)/2$ variables: $g(x_1, x_2, x_2, x_2, \cdots, x_n)$ where the x_i appear with multiplicity 3^{i-1}.

Properties of Threshold Devices: Ternary threshold functions appear promising as a source of gating functions for logical design because of their algebraic and switching theoretic properties.[3],[6]–[8] Moreover, for a reasonable number of inputs, threshold functions can be feasibly mechanized with a range of electronic devices; for example, parametron circuits,[9] nonlinear Hall-effect circuits,[10] and analog summer and quantizer circuits.[8],[11]

The author's definition of threshold functions is used here[7]: $f(X)$ is a ternary threshold function if there exist sets of integral constants $(w_1, w_2, \cdots, w_n)$ called *weights* and (t_+, t_-) called *thresholds* such that

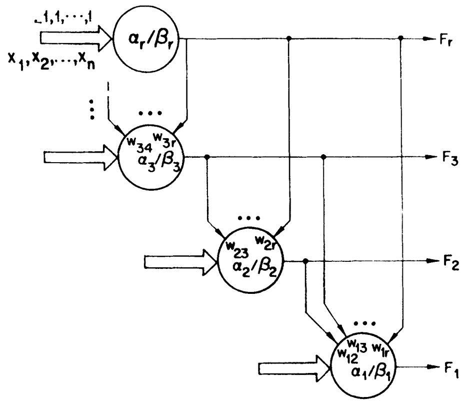

Fig. 1. Feed-forward threshold network.

TABLE VII

THRESHOLD REALIZATIONS FOR THE PERMUTATIONS OF DEGREE THREE*

$\phi_0(x) = x,$	$\phi_5(x) = -x$
$\phi_1(x) = T_{-1,2;1/-2,}^{(x,T^x_{1;1/0})}$	$\phi_4(x) = \phi_5(x^1)$
$\phi_2(x) = T_{-1,2;2/-1}^{(x,T^x_{1;0/-1})},$	$\phi_3(x) = \phi_5(x^2)$

* $\phi_5(x)$ is obtained by modifying the receiving threshold device according to (8). $\phi_5(x^i)$ can be obtained from the realization for $\phi_i(x)$ using (9).

$$f(X) = \begin{cases} +1 & \leftrightarrow \sum_{j=1}^{n} x_j w_j \geq t_+ \\ -1 & \leftrightarrow \sum_{j=1}^{n} x_j w_j \leq t_- \\ 0 & \text{otherwise} \end{cases}$$

which is written

$$T_{w_1,w_2,\cdots,w_n;t_+/t_-}^{x_1,x_2,\cdots,x_n}. \tag{7}$$

Two identities of use which have been treated previously[7] are as follows:

$$T_{w_1,w_2,\cdots,w_n;t_+/t_-}^{\phi_5(x_1),x_2,\cdots,x_n} = T_{-w_1,w_2,\cdots,w_n;t_+/t_-}^{x_1,x_2,\cdots,x_n} \tag{8}$$

$$\phi_5\left(T_{w_1,w_2,\cdots,w_n;t_+/t_-}^{x_1,x_2,\cdots,x_n}\right) = T_{-w_1,-w_2,\cdots,-w_n;-t_-/-t_+}^{x_1,x_2,\cdots,x_n}. \tag{9}$$

Logical Design with Threshold Devices: This subsection develops the theory by which it will be possible to realize arbitrary symmetric ternary functions with the feed-forward threshold device network of Fig. 1. (As shown, each device is represented by a circle; thresholds t_+/t_- appear in this form adjacent to the device output; primary inputs x_i and associated unit weights are presented as double arrows; and the weights of the secondary inputs F_i appear within the circle.) The basic design approach is first to derive the network for $f(X) = S_{A,B}(Y)$ with inputs $y_1, y_2, \cdots, y_n$, then augment this result with the realizations for $\phi_i(x)$, $i = 1, 2, 3, 4, 5$ given in Table VII to obtain the final form with inputs $x_1, x_2, \cdots, x_n$.

Consider an arbitrary $S_{A,B}(X)$ and partition A into subsets $A_1, A_2, \cdots, A_m$, called *grouped sets*, such that the a-numbers in A_k completely fill the range of integers between the maximum and minimum a-numbers in A_k, $k = 1, 2, \cdots, m$. The grouped sets of B and C are similarly formed. Two sets have the ordered relation $A_i < B_j$ if $a_k < b_l$ for each $a_k \in A_i$ and every $b_l \in B_j$.

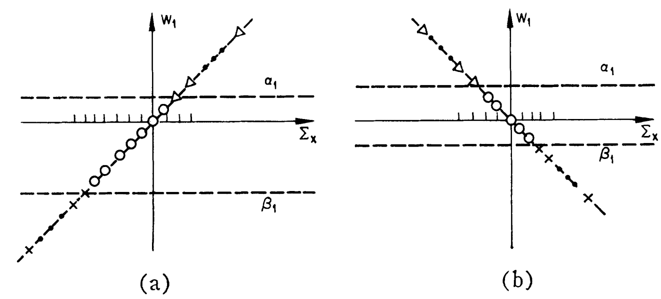

Fig. 2. Weight-line representation for symmetric functions that are threshold functions. (a) Positive primary weights. (b) Negative primary weights.

Theorem 8: $S_{A,B}(X)$ is a threshold function if A, B, and C are each grouped sets and either $A > C > B$ or $A < C < B$.

The proof of this theorem is presented in the Appendix.

$S_{A,B}(X)$ is uniquely defined in terms of

$$\sum_{i=1}^{n} x_i, \text{ hereafter written } \sum x,$$

and the a- and b-numbers. Hence, $S_{A,B}(X)$ can be represented graphically in a useful way by adopting the weight-line plots introduced by Sheng for symmetric binary functions.[12] The weight line for each F_i of Fig. 1 is specified by W_i vs. Σx where

$$W_i = \pm \sum x + \sum_{j=i+1}^{r} w_{ij} F_j.$$

If $S_{A,B}(X)$ is a threshold function, it will have a straight continuous weight line $W_1 = \pm \Sigma x$ as illustrated in Fig. 2 where the function truth values $+1$, 0, and -1 are denoted by Δ, $\bigcirc$, and $\times$, respectively. The thresholds are represented by horizontal dashed lines. Fig. 2(a) depicts the situation where $A > C > B$, and Fig. 2(b) where $A < C < B$.

It is now of interest to show that every symmetric ternary function is realizable with a feed-forward network and, to establish bounds on r, the number of devices required in the realization.

Consider an arbitrary $S_{A,B}(X)$ and partition A, B, and C exhaustively into their grouped sets. Order these sets, monotonically increasing from the set containing $-n$ to the set containing $+n$. As Σx is increased from $-n$ to $+n$, each unit increment $\Sigma x = k-1$ to k will result in a transition of the truth value $S_{A,B}(X) = S^{k-1}$ to S^k, respectively. As Σx increments from the maximum number in a grouped set to the minimum number in a monotonically ordered adjacent grouped set, the transition in $S_{A,B}(X)$ is said to be *positive* if $S^{k-1} < S^k$, and *negative* if $S^{k-1} > S^k$. A transition within a grouped set is

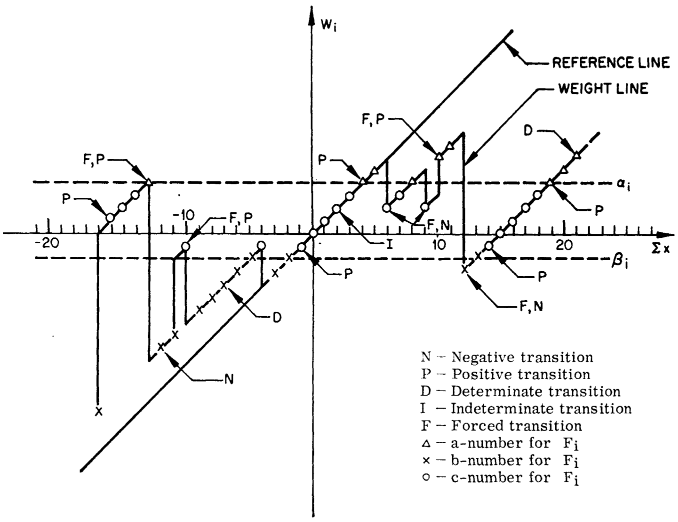

Fig. 3. Weight line with typical transitions.

said to be *determinate* with the exception that if $S_{A,B}(X)$ is two-valued, $S^{k-1}=S^k=0$ is called an *indeterminate transition*.

The numbers of positive, negative, determinate, and indeterminate transitions on the F_i function weight line are denoted as T_{Pi}, T_{Ni}, T_{Di}, T_{Ii}. The corresponding nomenclature for $S_{A,B}(X)$ is T_{NS}, T_{PS}, T_{DS}, T_{IS}, respectively. The example in Fig. 3 shows a weight line and the transitions for the symmetric function with

$$A = (-13, 4, 5, 8, 10, 11, 19, 20, 21)$$

and

$$B = (-21, -20, -19, -18, -17, -12, -11, -9, -8, -7, -6, -5, -3, -2, 12, 13).$$

From Figs. 2 and 3, it can be seen that the value of F_i at $\Sigma x = k$ is determined by the expression

$$F_i^k = \begin{cases} +1 & \text{if } \mathfrak{F}_i^k + k \geq \alpha_i \\ -1 & \text{if } \mathfrak{F}_i^k + k \leq \beta_i, \qquad i = 1, 2, \cdots, r \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

where

$$\mathfrak{F}_i^k \triangleq \sum_{j=i+1}^{r} w_{ij} F_j^k. \tag{11}$$

It can be shown that for an arbitrary F_i, as Σx increments from $k-1$ to k,

$$\mathfrak{F}_i^k - \mathfrak{F}_i^{k-1} < -1 \quad \text{for all negative transitions} \tag{12a}$$

$$\mathfrak{F}_i^k - \mathfrak{F}_i^{k-1} > -1 \quad \text{for all positive transitions} \tag{12b}$$

$$-\alpha_i + \beta_i - 1 < \mathfrak{F}_i^k - \mathfrak{F}_i^{k-1} < \alpha_i - \beta_i - 1 \quad \text{for all } 0 \to 0 \text{ transitions} \tag{12c}$$

$$\mathfrak{F}_i^{k-1} \leq \beta_i - k + 1 \text{ and } \mathfrak{F}_i^k \leq \beta_i - k \quad \text{for all } -1 \to -1 \text{ transitions} \tag{12d}$$

$$\mathfrak{F}_i^{k-1} \geq \alpha_i - k + 1 \text{ and } \mathfrak{F}_i^k \geq \alpha_i - k \quad \text{for all } +1 \to +1 \text{ transitions.} \tag{12e}$$

If $\mathfrak{F}_i^k \neq \mathfrak{F}_i^{k-1}$, then the change in value for F_i as Σx increments from $k-1$ to k is called a *forced transition*. It will be noted in Fig. 3 that a weight line having forced transitions is constructed to exhibit these transitions where W_i actually changes as Σx increases for $\Sigma x \geq 0$ and decreases for $\Sigma x < 0$. It will become clear later that this convention is helpful in the network synthesis procedure.

From (12), a negative transition is a forced transition (i.e., it always requires a change in $\mathfrak{F}_i$); a positive or indeterminate transition may require a change in $\mathfrak{F}_i$; but a determinate transition does not require a change in $\mathfrak{F}_i$. It should be observed, however, that more economical realizations are obtained for some functions if $\mathfrak{F}_i$ is allowed to change for certain determinate transitions. Nonetheless, it is clear that in the worst case $T_{Ni}+T_{Pi}+T_{Ii}$ changes in $\mathfrak{F}_i$ can be required as Σx increments from $-n$ to $+n$. It follows that since a change in $\mathfrak{F}_i$ during a transition comes about because one or more F_j, $j>i$ change value, any symmetric function F_i can be realized with the feed-forward network in Fig. 2. One such realization is as follows.

1) Select α_i and β_i so that each effects a positive transition. (Note that if F_i is not 3-valued, at most one positive transition can be effected by threshold selection.)
2) Select a F_j and its w_{ij} to effect at least one negative or indeterminate transition.

These results can be stated formally as follows:

Theorem 9: An arbitrary $S_{A,B}(X)$ is realizable with the feed-forward network of Fig. 1.

Corollary 4: Every $S_{A,B}(X)$ has a feed-forward realization with $r \leq (T_{NS}+T_{PS}+T_{IS})$.

Two examples with minimum realizations requiring the number of devices stipulated in Corollary 4 are $S_{(-2,-1,+1,+2),(0)}(x_1, x_2)$ and $S_{(-1,+1,+2),(-2)}(x_1, x_2)$ with $T_{NS}=T_{PS}=1$, $r=2$ and $T_{NS}=1$, $T_{NS}=2$, $r=2$, respectively.

In developing an economical feed-forward realization, it is sometimes necessary to examine $\phi_5(S_{A,B}(X))$ rather than $S_{A,B}(X)$. This usually comes about when $T_{NS}>T_{PS}$, the principal objective being to minimize the number of forced transitions. It is easily shown that T_{NS} and T_{PS} are equal to the number of positive and negative transitions, respectively, for $\phi_5(S_{A,B}(X))$. Hence, by designing a network for $\phi_5(S_{A,B}(X))$ where $T_{NS}>T_{PS}$, one is generally confronted with fewer forced transitions. The network obtained for $\phi_5(S_{A,B}(X))$ can be easily transformed to realize $S_{A,B}(X)$ using (8) and (9). Clearly, this general consideration is equally applicable to the treatment of the F_j selected to obtain a suitable weight line for F_1.

In network design, it is important to know whether one has achieved the ultimate realization. The following theorem provides such information on the minimal number of devices possible in a feed-forward network realization.

Theorem 10: Every $S_{A,B}(X)$ has a feed-forward realization with an $r \geq 1+\log_3 (T_{N1}+1)$, the greatest lower bound or r, where

$$S_{A,B}(X) = \begin{cases} F_1 & \text{if } T_{NS} < T_{PS} \\ \phi_5(F_1) & \text{if } T_{PS} < T_{NS} \\ F_1 \text{ or } \phi_5(F_1) & \text{otherwise.} \end{cases}$$

The ternary "parity" or modulo 3 addition functions comprise a class of symmetric functions which can be realized with the greatest lower bound (or minimum number) of devices in the feed-forward network. (This has been proved using an approach differing from that in the Theorem 10 proof contained in the Appendix.[7]) Specifically, consider $f = x_1 \oplus x_2 \oplus \cdots \oplus x_{13}$ which is the symmetric function $S_{(-11,-8,-5,-2,1,4,7,10,13),(-13,-10,-7,-4,-1,+2,+5,+8,+11)}(X)$. By using $\triangle$, $\times$, and $\bigcirc$ to denote a-, b-, and c-numbers on the real number line, the grouped sets can be easily ordered monotonically. There are eight negative and eight positive transitions as shown:

$$\begin{array}{ccccccccccccc} -12 & -10 & -8 & -6 & -4 & -2 & 0 & 2 & 4 & 6 & 8 & 10 & 12 \end{array}$$

$$\times \bigcirc \triangle \times \bigcirc \triangle \times \bigcirc \triangle \times \bigcirc \triangle \times \bigcirc \triangle \times \bigcirc \triangle \times \bigcirc \triangle \times \bigcirc \triangle \times \bigcirc \triangle$$

Hence, letting $F_1 = f$ gives $r \geq 1+\log_3 (8+1) = 3$. Fig. 4 shows the weight lines and feed-forward realization for f. It has been shown[7] that, in general, if $n \leq (3^r - 1)/2$, the parity function of n variables can be realized with a feed-forward threshold network in which $\alpha_i = -\beta = (3^{i-1}+1)/2$, $i = 1, 2, \cdots, r$, and $w_{ij} = w_{kj} = -3^{i-1}$, $i, k < j = 2, 3, \cdots, r$.

Referring again to Fig. 4, it can be seen that weight line graphs afford a convenient means of determining the parameters w_{ij}, α_i, and β_i, $i < j = 2, 3, \cdots, r$, of the network. This is done by dissecting into rhomboids the area between a reference line (passing through the origin at 45 deg to the Σx axis) and the weight line. These rhomboids have vertical sides and 45-deg sloped tops and bottoms parallel to the weight and reference lines. Further, each side of a rhomboid is at a Σx for which a forced transition (negative for this example) has occurred, with the possible exception of $\Sigma x = \pm n$. The height of rhomboid R_{ij} is denoted by h_{ij}. The h_{ij} determine the secondary weights w_{ij}. For this example, $w_{12} = -h_{12} = -3$, $w_{13} = -h_{13} = -9$, and $w_{23} = h_{23} = -9$. A rhomboid R_{ij} is said to be produced by F_j for F_i. In general, there are two types of rhomboids: single and compound. The compound rhomboid is produced by more than one F_j, for example, F_k and F_l, such that the height of the rhomboid is the difference between the heights of the single-type rhomboids produced separately by F_k and F_l. In Fig. 4, $R_{13} - R_{12}$ is such a rhomboid.

The Design Procedure: The truth-value transition characteristics of symmetric functions afford a convenient means of deriving economical feed-forward threshold network realizations. The design procedure utilizes the weight line graph as a means of relating the transition characteristics to the feed-forward network. In accomplishing a design, it is convenient to take the following approach.

1) Draw a reference line.
2) Draw a weight line $W_1 = \Sigma x + \mathfrak{F}_1$ which satisfies F_1. The steps in this operation are as follows.
   a) Represent $S_{A,B}(X)$ on the abscissa by labeling integral values of Σx with $\times$, $\bigcirc$, and $\triangle$ where the function truth value is -1, 0, and 1, respectively. This representation quickly reveals the grouped sets and transitions.
   b) Select (α_1, β_1) for $S_{A,B}(X)$, then for $\phi_5(S_{A,B}(X))$ to minimize the number of forced transitions on the weight line in each case. Accomplishing this step usually requires a cut-and-try approach.
   c) From b), select F_1 as the function having the least number of forced transitions.
3) Decompose the area between the weight line and reference line into rhomboids. Label the rhomboids to be produced by an upper stage F_j of the network as R_{1j}. All R_{1j} are: a) of the same height, b) present only once for any given Σx, and c) said to be either negative or positive. Most often, $w_{1j} = -h_{1j}$, the height of R_{1j}. This means of selecting w_{1j} usually results in an F_j with a weight line having fewer forced transitions than the one for $\phi_5(F_j)$. Further, a negative transition on the F_1 weight line will always result from positive transitions on the weight lines of one or more upper stages F_j. A negative R_{1j} occurs when $F_j = -1$, and a positive R_{1j} occurs when $F_j = +1$. Thus, F_j as a function of Σx is determined directly from the appearance of R_{1j} in the F_1 graph. In decomposing the F_1 graph, one strives to minimize the number of distinct rhomboids. This has the effect of minimizing r.
4) Repeat Step 3) for F_2. Again we strive to minimize the number of rhomboids. Note that if F_j produces an R_{2j}, it does so in the same interval of Σx as it produces R_{1j}, and conversely.
5) Repeat Step 3) for $F_3, F_4, \cdots, F_r$.

In general, Steps 3), 4), and 5) will be accomplished by an iterative cut-and-try process. The potential of the design procedure can best be illustrated by an example. Consider the function

$$S_{(-11,-10,-9,-4,-3,-2,-1,1,5,6,9),(-8,-7,-6,2,3,4,7,10,11)}(x_1, x_2, \cdots, x_{11}).$$

This function has $T_{NS} = 5$, $T_{PS} = 6$, $T_{IS} = 0$, and $T_{FS} = 6$. From Theorems 9 and 10: $3 \leq r \leq 11$. Carrying out the design procedure results in the weight line graphs shown in Fig. 5(a), (b), and (c) and a realization, Fig. 5(d), which is minimal.

This graphic procedure has been found to work well

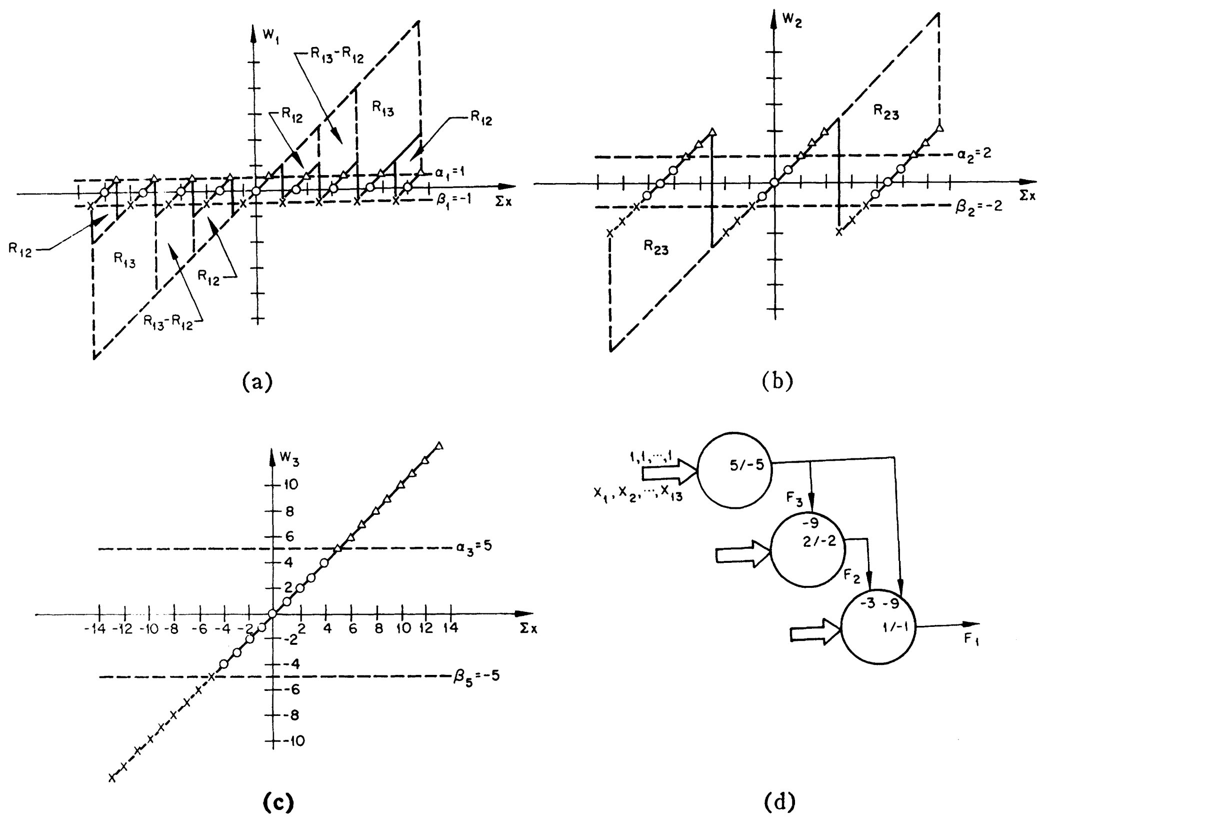

Fig. 4. Realization for $f=x_1 \oplus x_2 \oplus \cdots \oplus x_{13}$: (a) f. (b) $S_{(-7,-6,-5,2,3,4,11,12,13),\ (-13,-11,-12,-4,-3,-2,5,6,7)}(x_1, x_2, \cdots, x_{13})$. (c) $S_{(5,6,\cdots,13),\ (-13,-12,\cdots,-5)}(x_1, x_2, \cdots, x_{13})$. (d) The feed-forward network.

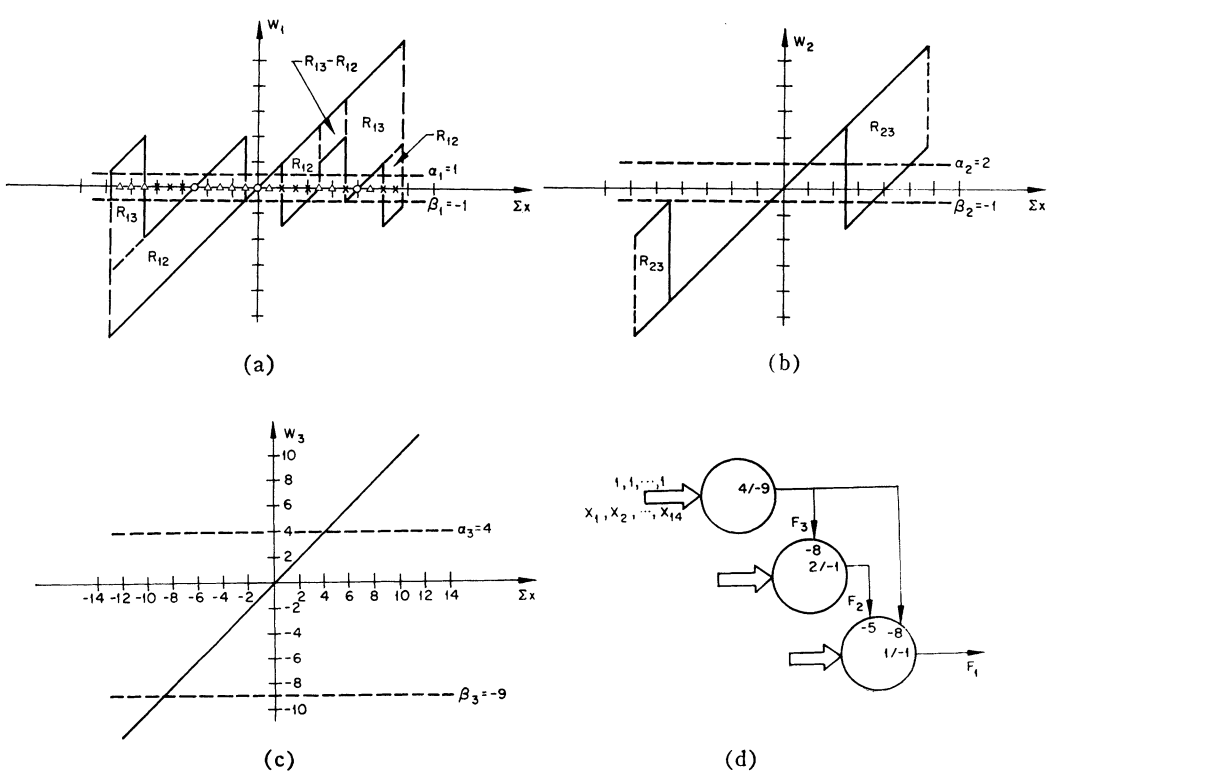

Fig. 5. Realization for a typical symmetric function:
(a) $S_{(-11,-10,-9,-4,-3,-2,-1,1,5,6,9),\ (-8,-7,-6,2,3,4,7,10,11)}(x_1, x_2, \cdots, x_{11})$.
(b) $S_{(2,3,4,10,11),\ (-11,-10,\cdots,-1,5,6,7)}(x_1, x_2, \cdots, x_{11})$.
(c) $S_{(5,6,\cdots,11),\ (-11,-10,-9)}(x_1, x_2, \cdots, x_{11})$.
(d) The realization.

for functions of up to 10 or 20 variables. Like so many cases in logical design, the realization obtained depends on the skill and thoroughness of the designer. Because of the simplicity of the approach, one can be expected to develop the skill after working through several typical problems.

Appendix

Theorem Proofs

Theorem 1 Proof: The truth value of a symmetric function for any $Y=V_i$ is dependent solely on

$$\sum_{j=1}^{n} v_{ij}$$

by definition, (1). It follows directly that the function is independent of the order of its variables $y_1, y_2, \cdots, y_n$.

Theorem 2 Proof: Suppose permuting two columns of a truth table matrix does not alter the matrix, aside from rearranging the rows within each submatrix. Clearly, then, for each submatrix the two columns which were interchanged must both have had the same set of $+1$, 0, and -1 counts, since rearranging rows of a submatrix does not change the $+1$, 0, and -1 counts in any column of the submatrix or its parent matrix.

By Theorem 1, a symmetric function $f(X)$ is invariant to a permutation of the y_i variables, which implies that a standard matrix for $f(X)$ in terms of the y_i is invariant to a column permutation in the sense that only rows within submatrices will be interchanged. It is easily shown by example that some nonsymmetric functions have standard matrices. Hence, the theorem provides a necessary but not sufficient condition for symmetry.

Theorem 3 Proof: This proof is contained in text preceding the statement of Theorem 3.

Theorem 4 Proof: Consider the case $r \geq 0$ and require that for some V_k,

$$\sum_{j=1}^{n} v_{kj} = r.$$

If no $v_{kj}=-1$, then r of the $v_{kj}=+1$ and $(n-r)$ of the $v_{kj}=0$. If one of the $v_{kj}=-1$, then $(r+1)$ of the $v_{kj}=+1$ and $(n-r-2)$ of the $v_{kj}=0$. If p of the $v_{kj}=-1$, then $(r+p)$ of the $v_{kj}=+1$ and $(n-r-2p)$ of the $v_{kj}=0$ provided $p \leq (n-r)/2$. Clearly then, for each integer in the closed interval $[0, [(n-r)/2]]$ one can construct a unique nonordered n-tuple with element truth values of -1, 0, or $+1$ and whose sum of elements is r. Further, the set of $[(n-r)/2]+1$ such nonordered n-tuples is exhaustive for the condition

$$\sum_{j=1}^{n} v_{kj} = r.$$

Finally, for each nonordered n-tuple with i of the $v_{kj}=-1$, $(r+i)$ of the $v_{kj}=+1$ and $(n-r-2i)$ of $v_{kj}=0$ in the set of v_{kj}, $j=1, 2, \cdots, n$, there will be

$$\binom{n}{(i) \quad (r+i)} = \frac{n!}{(i)!(r+i)!(n-r-2i)!}$$

unique ordered n-tuples V_k for which

$$\sum_{j=1}^{n} v_{kj} = r.$$

Because of the symmetry about 0 in the set of real numbers $\{-n, -n+1, \cdots, n\}$ which comprise all possible values for r, it is obvious that the above arguments also hold for $r<0$. Specifically, there are $[(n+r)/2]+1$ nonordered n-tuples whose sum of elements is r, and there are

$$\binom{n}{(i) \quad (-r+i)}$$

ordered n-tuples V_k for which

$$\sum_{j=1}^{n} v_{kj} = r$$

and in which i of the v_{kj} are $+1$, $-p+1$ of the v_{kj} are -1, and $(n+p-2i)$ of the v_{kj} are 0. Q.E.D.

Theorem 5 Proof: For an arbitrary $Y=V_i$,

$$\sum_{j=1}^{n} v_{ij} = a_k, b_k, \text{ or } c_k$$

implies that

$$\sum_{j=1}^{n} v_{ij}{}^5 = -a_k, -b_k, \text{ or } -c_k,$$

respectively.

Hence, by the definitions for A^- and B^-,

$$S_{A,B}(V_i) = S_{A^-,B^-}(v_{i1}{}^5, v_{i2}{}^5, \cdots, v_{in}{}^5).$$

Since this is true for an arbitrary V_i, it is true for all $V_i \in V^n$.

Theorem 6 Proof: Suppose $f(X)$ is a symmetric function with at least one submatrix or the parent matrix itself having distinct -1, 0, and $+1$ column counts. Clearly, at most six distinct standard matrices can be formed: one each in terms of $z_1{}^i, z_2{}^i, \cdots, z_n{}^i$, $i=0, 1, 2, 3, 4, 5$. It follows directly from Theorem 5 that $f(X)$ is symmetric on $z_1{}^i, z_2{}^i, \cdots, z_n{}^i$ if and only if it is symmetric on $\phi_5(z_1{}^i), \phi_5(z_2{}^i), \cdots, \phi_5(z_n{}^i)$ for $i=0, 1$, and 2.

Theorem 7 Proof: Suppose $Y=V_i$: $v_i{}^i=+1$ implies

$$\sum_{j=1}^{n} v_{ij} = a_k, b_k, \text{ or } c_k$$

if and only if

$$\sum_{k=1}^{n} v_{ij} = a_{k-1},\ b_{k-1},\ \text{or}\ c_{k-1}$$

respectively. Similarly, if $v_{i1}=0$, then

$$\sum_{j=1}^{n} v_{ij} = \sum_{j=2}^{n} v_{ij}.$$

Finally, $v_{i1}=-1$ implies

$$\sum_{j=1}^{n} v_{ij} = a_k,\ b_k,\ \text{or}\ c_k$$

if and only if

$$\sum_{j=2}^{n} v_{ij} = a_{k+1},\ b_{k+1},\ \text{or}\ c_{k+1}$$

respectively. Finally, note that $v_{i1}=+1$, 0, or -1 implies $-n<a_k, b_k, c_k \leq n$. Thus,

$$-n < \sum_{j=2}^{n} v_{ij} < n \quad \text{for all } V_i.$$

It follows, therefore, from the definitions for A_-, B_-, A_0, B_0, A_+, and B_+ that for all $Y=V_i$ in which $v_{i1}=+1$

$$S_{A,B}(Y) = S_{A_-,B_-}(y_2, y_3, \cdots, y_n).$$

For all $Y=V_i$ in which $v_{i1}=0$,

$$S_{A,B}(Y) = A_{A_0,B_0}(y_2, y_3, \cdots, y_n).$$

And for all $Y=V_i$ in which $v_{i1}=-1$,

$$S_{A,B}(Y) = S_{A_+,B_+}(y_2, y_3, \cdots, y_n).$$

Theorem 8 Proof: Suppose A, B, and C are grouped sets where a_1 and b_1 are the smallest elements and a_s and b_t are the largest elements in A and B, respectively. It follows that if $A>C>B$,

$$S_{A,B}(X) = \begin{cases} +1 & \text{if } \sum x \geq a_1 \\ -1 & \text{if } \sum x \leq b_t \\ 0 & \text{otherwise} \end{cases}$$

and by (7) it is a threshold function realizable with unit weights and thresholds $(t_+, t_-)=(a_1, b_t)$.

Similarly, if $B>C>A$, then

$$S_{A,B}(X) = \begin{cases} -1 & \text{if } \sum x \geq b_1 \\ +1 & \text{if } \sum x \leq a_s \\ 0 & \text{otherwise} \end{cases}$$

or

$$\phi_5(S_{A,B}(X)) = \begin{cases} +1 & \text{if } \sum x \geq b_1 \\ -1 & \text{if } \sum x \leq a_s \\ 0 & \text{otherwise.} \end{cases}$$

$S_{A,B}(X)$ is obviously a threshold function which by (9) is realizable with negative unit weights and $(t_+, t_-) = (-a_s, -b_1)$.

Theorem 9 Proof: This proof is contained in the text preceding the statement of Theorem 9.

Theorem 10 Proof: Every negative transition is a force transition. Therefore, the least number of forced transitions on the F_1 weight line is T_{N1}. Each such forced transition requires a change in $\mathfrak{F}_1$. It follows, therefore, that T_{N1} is a measure of the least number of F_j, $j>1$, or devices in the feed-forward network required to realize F_1. Since $T_{N1}=T_{NS}$ if $F_1=S_{A,B}(X)$ and $T_{N1}=T_{PS}$ if $F_1=\psi_{-8}S_{A,B}(X)$, to establish the greatest lower bound on r we select whichever case results in the smallest T_{N1}.

Each forced transition on the F_1 weight line requires at least one F_j weight line, $j>1$, to exhibit a positive transition. The following inductive proof is developed on the basis of this fact.

If $r=1$, then $T_{N1}=0$, and $T_{P1}\leq 2$ as can be easily verified in reference to Fig. 3. If $r=2$, again $T_{N2}=0$ and $T_{P2}\leq 2$, but for every positive transition on the F_2 weight line, it is possible to have a corresponding negative transition on the F_1 weight line. Moreover, for every negative transition on the F_1 weight line, it is possible to have two subsequent positive transitions as Σx increases from the negative transition (again, see Fig. 3). Also, without any negative transitions, it is possible to have two positive transitions. Therefore, $T_{P1}\leq 2(T_{N1}+1)$ and $T_{N1}\leq 2$ and $T_{P1}\leq 6$.

Suppose with p devices in the feed-forward network $T_{N1}\leq 3^{p-1}-1$. Then, for the reasons stated above, $T_{P1}\leq 2\cdot 3^{p-1}$. Consider the network with $p+1$ devices. Clearly, $T_{N2}\leq 3^{p-1}-1$ and $T_{P2}\leq 2\cdot 3^{p-1}$. It is possible for the outputs of devices $p+1$, p, $\cdots$, 2 each to separately affect the F_1 weight line. That is, for every positive transition on each F_i weight line, $i=2, 3, \cdots, p+1$, it is possible to have a corresponding negative transition on the F_1 weight line. Therefore, it follows that

$$T_{N1} \leq 2\sum_{i=1}^{p} 3^{i-1} = 3^{(p+1)-1} - 1.$$

Hence, by inductive reasoning an r-device network can have as many as $3^{r-1}-1$ negative or forced transitions. Thus,

$$T_{N1} \leq 3^{r-1} - 1$$

or

$$r \geq 1 + \log_3 (T_{N1} + 1). \qquad \text{Q.E.D.}$$

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On the Synthesis of Signal Switching Networks with Transient Blocking

LARRY JOEL GOLDSTEIN AND STEPHEN W. LEIBHOLZ, MEMBER, IEEE

Abstract—**Signal switching networks with transient blocking are defined. The number of 2×2 crossbars necessary to synthesize a signal switching network with transient blocking capable of performing all one-to-one connections of N inputs to N outputs is shown to be at least $N\log_2 N - N\log_2 e + (1/2)\log_2 N + \log_2 2 + 0(1)$ as $N\to\infty$. It is shown that this lower bound can never be attained for $N>2$. An algorithm for building a network using at most $2N\log_2 N$ 2×2 crossbars is described. If N is a power of 2, $N=2^m$, then the algorithm described requires $N\log_2 N - N + 1$ 2×2 crossbars, which is close to the theoretical minimum. Generalizations of this work to networks performing an arbitrary permutation group of connections of inputs to outputs are indicated. Explicit results are obtained in the case of Abelian groups.**

Index Terms—**Group theory, permutation network, switching network.**

I. Introduction

A SIGNAL switching network is defined as an ordered collection of four objects: a black box, a set of input wires, a set of output wires, and a control mechanism [Fig. 1(a)]. In response to settings of the control mechanism, the network can connect specified inputs to specified outputs. A signal switching network can possibly perform all or only some of the possible connections of inputs to outputs, but is usually restricted to one-to-one connections with possibly some inputs and outputs not being connected at a given time. Such networks, which form the heart of telephone and other switching systems, have been extensively studied. (The bibliography at the end of the paper is representative.)

Early studies of signal switching networks concentrated on designing optimal networks by means of simplification of Boolean functions.[14] The computations associated with such simplification are, in general, very difficult. Recent work by Beizer,[1],[2] Benes,[5] Clos,[7] Paull,[16] and Shannon[17] has been devoted to replacing the manipulation of logical functions by a more arithmetically oriented technique. The basic theme of these papers has been to build signal switching networks using a given type of atomic switch (e.g., $m\times m$ crossbar) and to minimize the number of atomic switches used. This paper represents an extension of the work cited.

A signal switching network is called *rearrangeable* (Benes[5]) if, given any connection of inputs to outputs in which one input and one output are unconnected, it is possible to reconnect the existing connections in such a way that the idle input is connected to the idle output. If such a connection can always be made without any reconnection of the existing connections, then the network is said to be *without transient blocking*. Otherwise, the network is called a *network with transient blocking*. Transient blocking is distinguished from conventional static blocking, wherein specified connections cannot be made at all because of prior existing connections. Thus, conventional blocking, by definition, can occur only in networks which are not rearrangeable.

Rearrangeable signal switching networks in which

Manuscript received March 29, 1967.
L. J. Goldstein is with the Dept. of Mathematics, Yale University, New Haven, Conn.
S. W. Leibholz is with Computer Command and Control Company, Philadelphia, Pa.
The authors were formerly with Auerbach Corporation, Philadelphia, Pa., and are now consultants to this company.