Lecture 15: Elastic Airplane Dynamics, Structural-Mode Control, and Nutation Control

Textbook Sections 10.4, 11.3, & 11.4

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Introduction

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  - Do not include vibration dynamics
- Structural-Mode Control
  - Additional loop-shaping possible for targeting structural modes in attitude control systems
- Spinning satellites marginally stable or unstable
  - Stabilize with nutation control systems

Static-Elastic Effects

- Recall: elastic flight vehicle EOMs

$$\begin{aligned}\dot{\vec{x}}_{rig} &= f_{rig}(\vec{x}_{rig}, \phi, \theta) + \mathcal{A}_{rig\leftarrow rig}\vec{x}_{rig} + \begin{bmatrix}\mathcal{A}_{rig\leftarrow\eta} & \mathcal{A}_{rig\leftarrow\dot{\eta}}\end{bmatrix}\vec{x}_{vib} + \mathcal{B}_{rig}\vec{u} \\ \dot{\vec{x}}_{vib} &= \begin{bmatrix}0_{n\times 6} \\ A_{vib\leftarrow rig}\end{bmatrix}\vec{x}_{rig} + \begin{bmatrix}0_{n\times n} & I_{n\times n} \\ A_{vib\leftarrow\eta} & A_{vib\leftarrow\dot{\eta}}\end{bmatrix}\vec{x}_{vib} + \begin{bmatrix}0_{n\times 4} \\ B_{vib}\end{bmatrix}\vec{u}\end{aligned} \tag{1}$$

Static-Elastic Effects

- Recall: elastic flight vehicle EOMs

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- Deformation equilibrium, i.e. static-elastic effects: $\ddot{\eta}_i = \dot{\eta}_i = 0 \; \forall i$

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- Deformation equilibrium, i.e. static-elastic effects: $\ddot{\eta}_i = \dot{\eta}_i = 0\ \forall i$
- Solve for **static-elastic modal coordinates**:

$$\bar{\eta} = \begin{bmatrix}\bar{\eta}_1 & \cdots & \bar{\eta}_n\end{bmatrix} \tag{2}$$

- In terms of rigid state and control inputs

Static-Elastic EOMs

- Vibration EOM:

$$\vec{0} = \begin{bmatrix} 0_{n\times 6} \\ A_{vib\leftarrow rig} \end{bmatrix} \vec{x}_{rig} + \begin{bmatrix} 0_{n\times n} & I_{n\times n} \\ A_{vib\leftarrow \eta} & A_{vib\leftarrow \dot{\eta}} \end{bmatrix} \begin{bmatrix} \bar{\eta} \\ \vec{0} \end{bmatrix} + \begin{bmatrix} 0_{n\times 4} \\ B_{vib} \end{bmatrix} \vec{u} \tag{3}$$

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- Non-trivial portion:

$$0 = A_{vib\leftarrow rig}\vec{x}_{rig} + A_{vib\leftarrow \eta}\bar{\eta} + B_{vib}\vec{u} \tag{4}$$

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- Non-trivial portion:

$$0 = A_{vib\leftarrow rig}\vec{x}_{rig} + A_{vib\leftarrow \eta}\bar{\eta} + B_{vib}\vec{u} \tag{4}$$

- **Static-elastic constraint**:

$$\bar{\eta} = A_{vib\leftarrow \eta}^{-1}\left(A_{vib\leftarrow rig}\vec{x}_{rig} + B_{vib}\vec{u}\right) \tag{5}$$

Static-Elastic Effects (continued)

- Rigid-body EOM:

$$\dot{\vec{x}}_{rig} = f_{rig}(\vec{x}_{rig}, \phi, \theta) + \mathcal{A}_{rig\leftarrow rig}\vec{x}_{rig} + \begin{bmatrix} \mathcal{A}_{rig\leftarrow\eta} & \mathcal{A}_{rig\leftarrow\dot{\eta}} \end{bmatrix} \begin{bmatrix} \bar{\eta} \\ \vec{0} \end{bmatrix} + \mathcal{B}_{rig}\vec{u} \tag{6}$$

Static-Elastic Effects (continued)

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- By back-substitution:

$$\begin{aligned}\dot{\vec{x}}_{rig} =& f_{rig}(\vec{x}_{rig}, \phi, \theta) + \mathcal{A}_{rig\leftarrow rig}\vec{x}_{rig} \\ &+ \mathcal{A}_{rig\leftarrow\eta}A_{vib\leftarrow\eta}^{-1}\left(A_{vib\leftarrow rig}\vec{x}_{rig} + B_{vib}\vec{u}\right) + B_{rig}\vec{u}\end{aligned} \tag{7}$$

Static-Elastic Effects (continued)

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- **Static-elastic rigid vehicle EOMs**

$$\begin{aligned}\dot{\vec{x}}_{rig} =& f_{rig}(\vec{x}_{rig}, \phi, \theta) + \left(\mathcal{A}_{rig \leftarrow rig} - \mathcal{A}_{rig \leftarrow \eta}A_{vib \leftarrow \eta}^{-1}A_{vib \leftarrow rig}\right)\vec{x}_{rig} \\ &+ \left(B_{rig} - \mathcal{A}_{rig \leftarrow \eta}\mathcal{A}_{vib \leftarrow \eta}^{-1}B_{vib}\right)\vec{u}\end{aligned} \tag{8}$$

Residualization

- Process called **residualization** of vibration degrees-of-freedom into new matrices of static-elastic stability and control derivatives/coefficients
  - Residualized static-elastic derivatives: elements of $\left(\mathcal{A}_{rig\leftarrow rig} - \mathcal{A}_{rig\leftarrow\eta} A_{vib\leftarrow\eta}^{-1} A_{vib\leftarrow rig}\right)$ & $\left(B_{rig} - \mathcal{A}_{rig\leftarrow\eta} A_{vib\leftarrow\eta}^{-1} B_{vib}\right)$

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- In general depend on flight conditions
  - Directly affect loads on vehicle's structure, especially dynamic pressure

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- In general depend on flight conditions
  - Directly affect loads on vehicle's structure, especially dynamic pressure
- If aerodynamic forces and moments not truly linear $\rightarrow$ numerical techniques to find static-elastic modal coordinates

Rigid Body Modeling

- Linearized EOMs for elastic airplanes typically use fuselage body frame (subscript *F*) instead of stability body frame (subscript *S*) for developing vibration and dynamic-elastic coefficients

Rigid Body Modeling

- Linearized EOMs for elastic airplanes typically use fuselage body frame (subscript F) instead of stability body frame (subscript S) for developing vibration and dynamic-elastic coefficients
- Linearized rigid airplane dynamics in stability frame $\rightarrow$ transform perturbed rigid body aerodynamic and propulsive forces and moments

$$\begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix}_F = \begin{bmatrix} \cos\bar{\alpha} & 0 & -\sin\bar{\alpha} \\ 0 & 1 & 0 \\ \sin\bar{\alpha} & 0 & \cos\bar{\alpha} \end{bmatrix} \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix}_S \tag{9}$$

$$\begin{bmatrix} \Delta L \\ \Delta M \\ \Delta N \end{bmatrix}_F = \begin{bmatrix} \cos\bar{\alpha} & 0 & -\sin\bar{\alpha} \\ 0 & 1 & 0 \\ \sin\bar{\alpha} & 0 & \cos\bar{\alpha} \end{bmatrix} \begin{bmatrix} \Delta L \\ \Delta M \\ \Delta N \end{bmatrix}_S \tag{10}$$

F Frame Linearized EOMs

- Note: $\bar{\alpha}$ may not equal 0

$$\begin{bmatrix}\Delta X\\ \Delta Y\\ \Delta Z\end{bmatrix} + g\begin{bmatrix}-\cos\bar{\theta} & 0\\ -\sin\bar{\theta}\sin\bar{\phi} & \cos\bar{\theta}\cos\bar{\phi}\\ \sin\bar{\theta}\cos\bar{\phi} & \cos\bar{\theta}\sin\bar{\phi}\end{bmatrix}\begin{bmatrix}\theta\\ \phi\end{bmatrix} = \begin{bmatrix}\Delta\dot{u}\\ \Delta\dot{v}\\ \Delta\dot{w}\end{bmatrix}$$
$$+\begin{bmatrix}0 & -\bar{r} & \bar{q}\\ \bar{r} & 0 & -\bar{p}\\ -\bar{q} & \bar{p} & 0\end{bmatrix}\begin{bmatrix}\Delta u\\ \Delta v\\ \Delta w\end{bmatrix} + \begin{bmatrix}0 & \bar{w} & -\bar{v}\\ -\bar{w} & 0 & \bar{u}\\ \bar{v} & -\bar{u} & 0\end{bmatrix}\begin{bmatrix}\Delta p\\ \Delta q\\ \Delta r\end{bmatrix} \tag{11}$$

F Frame Linearized EOMs (continued)

$$\begin{aligned}\begin{bmatrix}\Delta L\\ \Delta M\\ \Delta N\end{bmatrix} &= \begin{bmatrix}1 & 0 & -\frac{I_{xz}}{I_{xx}}\\ 0 & 1 & 0\\ -\frac{I_{xz}}{I_{zz}} & 0 & 1\end{bmatrix}\begin{bmatrix}\Delta\dot{p}\\ \Delta\dot{q}\\ \Delta\dot{r}\end{bmatrix}\\ &+ \begin{bmatrix}-\frac{I_{xz}}{I_{xx}}\bar{q} & -\frac{I_{xz}}{I_{xx}}\bar{p}+\frac{I_{zz}-I_{yy}}{I_{xx}}\bar{r} & \frac{I_{zz}-I_{yy}}{I_{xx}}\bar{q}\\ \frac{I_{xx}-I_{zz}}{I_{yy}}\bar{r}+2\frac{I_{xz}}{I_{yy}}\bar{p} & 1 & \frac{I_{xx}-I_{zz}}{I_{yy}}\bar{p}-2\frac{I_{xz}}{I_{yy}}\bar{r}\\ \frac{I_{yy}-I_{xx}}{I_{zz}}\bar{q} & \frac{I_{xz}}{I_{zz}}\bar{r}+\frac{I_{yy}-I_{xx}}{I_{zz}}\bar{p} & \frac{I_{xz}}{I_{zz}}\bar{q}\end{bmatrix}\begin{bmatrix}\Delta p\\ \Delta q\\ \Delta r\end{bmatrix}\end{aligned} \tag{12}$$

Perturbed Forces/Moments

$$
\begin{aligned}
\begin{bmatrix} \Delta X \\ \Delta Z \\ \Delta M \end{bmatrix} &= \begin{bmatrix} 0 & X_{\dot\alpha} & 0 \\ 0 & Z_{\dot\alpha} & 0 \\ 0 & M_{\dot\alpha} & 0 \end{bmatrix} \begin{bmatrix} \Delta \dot u \\ \Delta \dot\alpha \\ \Delta \dot q \end{bmatrix} + \begin{bmatrix} X_u & X_\alpha & X_q \\ Z_u & Z_\alpha & Z_q \\ M_u & M_\alpha & M_q \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta \alpha \\ \Delta q \end{bmatrix} \\
&+ \begin{bmatrix} X_{\delta_e} & X_{\delta_t} \\ Z_{\delta_e} & Z_{\delta_t} \\ M_{\delta_e} & M_{\delta_t} \end{bmatrix} \begin{bmatrix} \Delta \delta_e \\ \Delta \delta_t \end{bmatrix} \\
&+ \begin{bmatrix} X_{\dot\eta_1} & \cdots & X_{\dot\eta_n} \\ Z_{\dot\eta_1} & \cdots & Z_{\dot\eta_n} \\ M_{\dot\eta_1} & \cdots & M_{\dot\eta_n} \end{bmatrix} \begin{bmatrix} \Delta \dot\eta_1 \\ \vdots \\ \Delta \dot\eta_n \end{bmatrix} + \begin{bmatrix} X_{\eta_1} & \cdots & X_{\eta_n} \\ Z_{\eta_1} & \cdots & Z_{\eta_n} \\ M_{\eta_1} & \cdots & M_{\eta_n} \end{bmatrix} \begin{bmatrix} \Delta \eta_1 \\ \vdots \\ \Delta \eta_n \end{bmatrix}
\end{aligned} \tag{13}
$$

Perturbed Forces/Moments (continued)

$$\begin{bmatrix} \Delta Y \\ \Delta L \\ \Delta N \end{bmatrix} = \begin{bmatrix} Y_\beta & Y_p & Y_r \\ L_\beta & L_p & L_r \\ N_\beta & N_p & N_r \end{bmatrix} \begin{bmatrix} \Delta\beta \\ \Delta p \\ \Delta r \end{bmatrix} + \begin{bmatrix} Y_{\delta_a} & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \end{bmatrix} \begin{bmatrix} \Delta\delta_a \\ \Delta\delta_r \end{bmatrix} + \begin{bmatrix} Y_{\dot\eta_1} & \cdots & Y_{\dot\eta_n} \\ L_{\dot\eta_1} & \cdots & L_{\dot\eta_n} \\ N_{\dot\eta_1} & \cdots & N_{\dot\eta_n} \end{bmatrix} \begin{bmatrix} \Delta\dot\eta_1 \\ \vdots \\ \Delta\dot\eta_n \end{bmatrix} + \begin{bmatrix} X_{\eta_1} & \cdots & X_{\eta_n} \\ Z_{\eta_1} & \cdots & Z_{\eta_n} \\ M_{\eta_1} & \cdots & M_{\eta_n} \end{bmatrix} \begin{bmatrix} \Delta\eta_1 \\ \vdots \\ \Delta\eta_n \end{bmatrix} \tag{14}$$

Alternative

- Angle of attack and sideslip angle substitutions:

$$\Delta w = \bar{u}\Delta\alpha \tag{15}$$

$$\Delta v = \bar{v}_\infty \Delta\beta \tag{16}$$

  - No-wind assumption
  - Small angle approximation

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$$\Delta w = \bar{u}\Delta\alpha \tag{15}$$

$$\Delta v = \bar{v}_\infty \Delta\beta \tag{16}$$

  - No-wind assumption
  - Small angle approximation
- Furthermore, if $\bar{\beta} = \bar{\phi} = \bar{p} = \bar{q} = \bar{r} = 0$, then one can decoupled the dynamics into the longitudinal and lateral-directional.

Linearized Vibrations

- Already linearly modeled (simply):

$$\Delta\ddot{\eta}_i + 2\zeta_i\omega_i\Delta\dot{\eta}_i + \omega_i^2\Delta\eta_i = \frac{\Delta Q_i}{\mathcal{M}_i}, \quad i = 1, ..., n \tag{17}$$

Generalized Forces

$$\Delta Q_i = \begin{bmatrix} Q_{i_u} & Q_{i_\beta} & Q_{i_\alpha} & Q_{i_p} & Q_{i_q} & Q_{i_r} \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta\beta \\ \Delta\alpha \\ \Delta p \\ \Delta q \\ \Delta r \end{bmatrix} + \begin{bmatrix} Q_{i_{\delta_a}} & Q_{i_{\delta_e}} & Q_{i_{\delta_r}} & Q_{i_{\delta_t}} \end{bmatrix} \begin{bmatrix} \Delta\delta_a \\ \Delta\delta_e \\ \Delta\delta_r \\ \Delta\delta_t \end{bmatrix} + \begin{bmatrix} Q_{i_{\dot\eta_1}} & \cdots & Q_{i_{\dot\eta_n}} \end{bmatrix} \begin{bmatrix} \Delta\dot\eta_1 \\ \vdots \\ \Delta\dot\eta_n \end{bmatrix} + \begin{bmatrix} Q_{i_{\eta_1}} & \cdots & Q_{i_{\eta_n}} \end{bmatrix} \begin{bmatrix} \Delta\eta_1 \\ \vdots \\ \Delta\eta_n \end{bmatrix} \tag{18}$$

Decoupled Longitudinal EOMs Example

- Straight-and-level flight provides :

$$\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{\alpha} \\ \Delta\dot{q} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u & X_\alpha & X_q & -g \\ \frac{Z_u}{\bar{u}} & \frac{Z_\alpha}{\bar{u}} & 1+\frac{Z_q}{\bar{u}} & 0 \\ M_u & M_\alpha & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta\alpha \\ \Delta q \\ \Delta\theta \end{bmatrix} + \begin{bmatrix} X_{\delta_e} & X_{\delta_t} \\ \frac{Z_{\delta_e}}{\bar{u}} & \frac{Z_{\delta_t}}{\bar{u}} \\ M_{\delta_e} & M_{\delta_t} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\delta_e \\ \Delta\delta_t \end{bmatrix} \tag{19}$$

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- Models portion of Equation **??**:

$$\begin{bmatrix} \Delta\dot{\vec{x}}_{rig} \\ \Delta\dot{\vec{x}}_{eul} \end{bmatrix} = \begin{bmatrix} A_{rig\leftarrow rig} & A_{rig\leftarrow eul} \\ A_{eul\leftarrow rig} & A_{eul\leftarrow eul} \end{bmatrix} \begin{bmatrix} \Delta\vec{x}_{rig} \\ \Delta\vec{x}_{eul} \end{bmatrix} + \begin{bmatrix} B_{rig} \\ 0 \end{bmatrix} \Delta\vec{u} \tag{20}$$

  - $\Delta\alpha$ has been used in place of Δw

Decoupled Longitudinal EOMs Example (continued)

- Form other matrices:

$$A_{rig \leftarrow vib} = \begin{bmatrix} X_{\eta_1} & \cdots & X_{\eta_n} & X_{\dot{\eta}_1} & \cdots & X_{\dot{\eta}_n} \\ \frac{Z_{\eta_1}}{\bar{u}} & \cdots & \frac{Z_{\eta_n}}{\bar{u}} & \frac{Z_{\dot{\eta}_1}}{\bar{u}} & \cdots & \frac{Z_{\dot{\eta}_n}}{\bar{u}} \\ M_{\eta_1} & \cdots & M_{\eta_n} & M_{\dot{\eta}_1} & \cdots & M_{\dot{\eta}_n} \end{bmatrix} \quad (21)$$

Decoupled Longitudinal EOMs Example (continued)

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- Define **(aero)elastic stability and control derivative** for state or input $\bullet$:

$$\Xi_{i_\bullet} = \frac{Q_{i_\bullet}}{\mathcal{M}_i} \tag{22}$$

Decoupled Longitudinal EOMs Example (continued)

- Vibration state and input sub-matrices:

$$A_{vib\leftarrow rig} = \begin{bmatrix} \Xi_{1_u} & \Xi_{1_\alpha} & \Xi_{1_q} \\ \vdots & \vdots & \vdots \\ \Xi_{n_u} & \Xi_{n_\alpha} & \Xi_{n_q} \end{bmatrix} \tag{23}$$

$$A_{vib\leftarrow \eta} = \begin{bmatrix} \Xi_{1_{\eta_1}} & \cdots & \Xi_{1_{\eta n}} \\ \vdots & \ddots & \vdots \\ \Xi_{n_{\eta_1}} & \cdots & \Xi_{n_{\eta n}} \end{bmatrix} - \Omega^2 \tag{24}$$

$$A_{vib\leftarrow \dot{\eta}} = \begin{bmatrix} \Xi_{1_{\dot{\eta}_1}} & \cdots & \Xi_{1_{\dot{\eta} n}} \\ \vdots & \ddots & \vdots \\ \Xi_{n_{\dot{\eta}_1}} & \cdots & \Xi_{n_{\dot{\eta} n}} \end{bmatrix} - 2\Omega_\zeta \tag{25}$$

$$B_{vib} = \begin{bmatrix} \Xi_{1_{\delta_e}} & \Xi_{1_{\delta_t}} \\ \vdots & \vdots \\ \Xi_{n_{\delta_e}} & \Xi_{n_{\delta_t}} \end{bmatrix} \tag{26}$$

Decoupled Longitudinal EOMs Example (continued)

$$
\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{\alpha} \\ \Delta\dot{q} \\ \Delta\dot{\theta} \\ \Delta\dot{\eta}_1 \\ \vdots \\ \Delta\dot{\eta}_n \\ \Delta\ddot{\eta}_1 \\ \vdots \\ \Delta\ddot{\eta}_n \end{bmatrix}
=
\begin{bmatrix}
X_u & X_\alpha & X_q & -g & X_{\eta_1} & \cdots & X_{\eta n} & X_{\dot{\eta}_1} & \cdots & X_{\dot{\eta} n} \\
\frac{Z_u}{\bar{u}} & \frac{Z_\alpha}{\bar{u}} & 1+\frac{Z_q}{\bar{u}} & 0 & \frac{Z_{\eta_1}}{\bar{u}} & \cdots & \frac{Z_{\eta n}}{\bar{u}} & \frac{Z_{\dot{\eta}_1}}{\bar{u}} & \cdots & \frac{Z_{\dot{\eta} n}}{\bar{u}} \\
M_u & M_\alpha & M_q & 0 & M_{\eta_1} & \cdots & M_{\eta n} & M_{\dot{\eta}_1} & \cdots & M_{\dot{\eta} n} \\
0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \\
\Xi_{1_u} & \Xi_{1_\alpha} & \Xi_{1_q} & 0 & \Xi_{1_{\eta_1}} - \omega_1^2 & \cdots & \Xi_{1_{\eta n}} & \Xi_{1_{\dot{\eta}_1}} - 2\zeta_1\omega_1 & \cdots & \Xi_{1_{\dot{\eta} n}} \\
\vdots & \vdots & \vdots & 0 & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\Xi_{n_u} & \Xi_{n_\alpha} & \Xi_{n_q} & 0 & \Xi_{n_{\eta_1}} & \cdots & \Xi_{n_{\eta n}} - \omega_n^2 & \Xi_{n_{\dot{\eta}_1}} & \cdots & \Xi_{n_{\dot{\eta} n}} - 2\zeta_1\omega_1
\end{bmatrix}
\begin{bmatrix} \Delta u \\ \Delta\alpha \\ \Delta q \\ \Delta\theta \\ \Delta\eta_1 \\ \vdots \\ \Delta\eta_n \\ \Delta\dot{\eta}_1 \\ \vdots \\ \Delta\dot{\eta}_n \end{bmatrix}
+
\begin{bmatrix}
X_{\delta_e} & X_{\delta_t} \\
\frac{Z_{\delta_e}}{\bar{u}} & \frac{Z_{\delta_t}}{\bar{u}} \\
M_{\delta_e} & M_{\delta_t} \\
\vec{0}_{n+1} & \vec{0}_{n+1} \\
\Xi_{1_{\delta_e}} & \Xi_{1_{\delta_t}} \\
\vdots & \vdots \\
\Xi_{n_{\delta_e}} & \Xi_{n_{\delta_t}}
\end{bmatrix}
\begin{bmatrix} \Delta\delta_e \\ \Delta\delta_t \end{bmatrix}
\tag{27}
$$

Structural-Mode Control

- **Structural-mode control (SMC)**: additional stages in flight control law to account for effects of vibrations or structural-modes at frequencies close to rigid-body dynamics that cannot simply be removed using low-pass filtering
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- Important aspect of feedback control: placement of inertial sensors, i.e., accelerometers and gyroscopes,
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- With sensor information, one can implement
    - Passive structural-mode control: simply filter measurements
    - Active structural-mode control: utilize specialized actuators with sensors

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- Provides method to mitigate structural-mode effects from disturbances and control inputs

Active SMC

- Direct way to mitigate structural-mode effects: **active SMC**: utilizes co-located actuators and sensors at proper locations on structure:
  - Measure modal acceleration and/or mode-slope rate via accelerometer and/or rate gyroscope
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- To avoid exciting rigid-body modes, typically use combination of low-pass and high-pass filters to target structural modes

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$$\begin{aligned} T_{x,c} &= -F_c \Delta \operatorname{sign}(\lambda_{n,x}) \\ T_{y,c} &= -F_c \Delta \operatorname{sign}(\lambda_{n,y}) \end{aligned} \tag{28}$$

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- Time delays and dead zones in system hardware typically vital to final design of TB-ANC systems

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- Linearized Euler equations:

$$\begin{aligned} H_x(t) &= I_{xx} p(t) = \frac{M_d}{\bar{r}\left(\frac{I_{zz}}{I_{xx}} - 1\right)} \sin(\lambda_n t) \\ H_y(t) &= I_{yy} q(t) = I_{xx}\omega_x(t) = \frac{M_d}{\bar{r}\left(\frac{I_{zz}}{I_{xx}} - 1\right)} (1 - \cos(\lambda_n t)) \\ \bar{r} &= \text{constant} \end{aligned} \tag{29}$$

- **Nutation frequency**:

$$\lambda_n = \bar{r}\frac{(I_{zz} - I_{xx})}{I_{xx}} = \bar{r}\left(\frac{I_{zz}}{I_{xx}} - 1\right) \tag{30}$$

Nutation Magnitude

- Total momentum in x_B-y_B plane perpendicular to z_B:

$$H_\perp(t) = \sqrt{H_x^2 + H_y^2} = \frac{\sqrt{2}M_d}{\bar{r}\left(\frac{I_{zz}}{I_{xx}} - 1\right)}\sqrt{1 - \cos\left(\lambda_n t\right)} = \frac{2M_d}{\bar{r}\left(\frac{I_{zz}}{I_{xx}} - 1\right)}\left|\sin\left(\frac{\lambda_n t}{2}\right)\right| \tag{31}$$

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- Recall relationship between nutation angle and perpendicular angular momentum:

$$\tan\theta_n = \frac{H_{\perp}}{H_z} = \frac{2M_d}{I_{zz}\bar{r}^2\left(\frac{I_{zz}}{I_{xx}} - 1\right)}\left|\sin\left(\frac{\lambda_n t}{2}\right)\right| \tag{32}$$

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- Magnitude of nutation inversely proportional to $\bar{r}^2$ which costs fuel to increase for TB-ANC systems

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- Flywheels typically connected rigidly to satellite, but wheels driven by electric motors
  - Rotate more or less independently, depending on type of gyro

Wheel-Based Nutation Control

- **Wheel-based nutation control** systems utilize single reaction wheel with some moment inertia, I_w, and angular velocity, ω_w, to add to total angular momentum of spacecraft and wheel system:

$$\vec{H}_G = I_G \vec{\omega}_{B/I} = \begin{bmatrix} I_{xx} p \\ (I_{yy} + I_w) q + I_w \omega_w \\ I_{zz} r \end{bmatrix} \tag{34}$$

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- Assuming $I_{xx} \gg I_w$, $I_{xx} = I_{yy}$:

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Wheel-Based Nutation Control (continued)

- Using Euler's equation:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \frac{d}{dt}\vec{H}_G = \begin{bmatrix} I_{xx}\dot{p} + (I_{zz} - I_{xx})qr - I_w r \omega_w \\ I_{xx}\dot{q} + I_w \dot{\omega}_w + (I_{xx} - I_{zz})pr \\ I_{zz}\dot{r} + I_w p \omega_w \end{bmatrix} \tag{36}$$

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- Assuming $I_{zz} \gg I_w$ and using nutation frequency, λ_n, and inertia ratio, $\epsilon_w = I_w/I_{xx}$:

$$\begin{bmatrix} \dot{p} + \lambda_n q - \epsilon_w \bar{r}\omega_w \\ \dot{q} + \epsilon_w\dot{\omega}_w - \lambda_n p \\ \dot{r} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{37}$$

  - $\dot{r} = 0$ infers constant $r = \bar{r}$

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- p and q dynamics:

$$\begin{bmatrix} \dot{p} + \lambda_n q - \epsilon_w \bar{r}\omega_w \\ \dot{q} + \epsilon_w \dot{\omega}_w - \lambda_n p \\ \dot{q} + \dot{\omega}_w + \frac{C_d}{I_w}\omega_w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{39}$$

Wheel-Based Passive Nutation Control (continued)

- In Laplace domain:

$$\begin{bmatrix} s & \lambda_n & -\epsilon_w \bar{r} \\ s & -\lambda_n & \epsilon_w s \\ 0 & s & s + \frac{C_d}{I_w} \end{bmatrix} \begin{bmatrix} p \\ q \\ \omega_w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{40}$$

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- Characteristic equation given by determinant of right matrix:

$$(1 - \epsilon_w)s^3 + \frac{C_d}{I_w}s^2 + (\lambda_n^2 - \lambda_n \epsilon_w \bar{r})s + \frac{C_d}{I_w}\lambda_n^2 = 0 \tag{41}$$

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- Rearranged:

$$1 + \frac{C_D}{I_w(1 - \epsilon_w)} \frac{s^2 + \lambda_n^2}{s(s^2 + \lambda_1^2)} \tag{42}$$

$$\lambda_1 = \lambda_n \sqrt{1 + \frac{I_{zz} I_w}{\;}} \tag{43}$$

Wheel-Based Passive Nutation Control (continued)

$$\begin{aligned} I_{xx} > Izz \quad &\rightarrow \quad \lambda_n > \lambda_1 \\ I_{xx} < Izz \quad &\rightarrow \quad \lambda_n < \lambda_1 \end{aligned} \tag{44}$$

- Analysis of roots of characteristic equation, spinning satellite with WB-PNC system nutationally stable if and only if $I_{xx} < I_{zz}$, i.e., spin axis must be major axis

Wheel-Based Passive Nutation Control (continued)

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- No external moment on wheel, equation for *total* angular velocity of wheel:

$$\dot{\omega}_w + \dot{q} = \frac{k_m}{R_m I_w} v_m - \frac{k_m k_{emf}}{R_m I_w} \omega_w \tag{45}$$

  - v_w: input voltage to wheel
  - R_m: **armature resistance**
  - I_w: wheel's moment of inertia
  - k_{emf}: **electromotive force (EMF) constant**
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- By substitution for $\dot{\omega}_w$, p and q dynamics:

$$\begin{bmatrix} \dot{p} + \lambda_n q - \epsilon_w \bar{r} \omega_w \\ \dot{q} + \epsilon_w \dot{\omega}_w - \lambda_n p \\ \dot{\omega}_m + \dot{q} + \frac{k_m k_{emf}}{R_m I_w} \omega_w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{k_m}{R_m I_w} \end{bmatrix} v_m \tag{46}$$

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- Simple proportional controller for voltage as function of one of spacecraft's perpendicular angular velocity components, e.g., p:

$$v_m = K_p p \tag{48}$$

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- From analysis of roots of closed-loop system, spinning satellite with WB-ANC system nutationally stable for both $I_{xx} < I_{zz}$ and $I_{xx} > I_{zz}$, i.e., spin axis can be major or minor axis

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  - Use $\bar{\eta}$ for $\dot{\eta}_i = \ddot{\eta}_i = 0$
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