Lecture 1: Dynamical Systems, Equilibrium, and Stability

Textbook Sections 1.1, 1.2, & 1.3

Dr. Jordan D. Larson

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- Model-based design uses coverall approach to standard embedded systems development $\rightarrow$ time to port between modeling software and embedded systems can outweigh temporal value for alternative lab-based design

Model-Based Design Software

- Uses graphical modeling tools: generic and unified graphical modeling environment
  - Reduces complexity of model designs by breaking into hierarchies of individual design blocks
  - Helps design engineers to conceptualize entire system, typically with **graphical user interface (GUI)**
  - Connection between signals and systems in these GUIs typically depicted using **block diagrams**, for which each “block” represents system and each “arrow” represents signal

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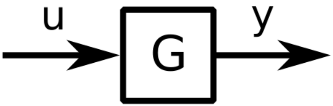
- Example of basic block diagram:
  - System: G
  - Input signal: u
  - Output signal: y



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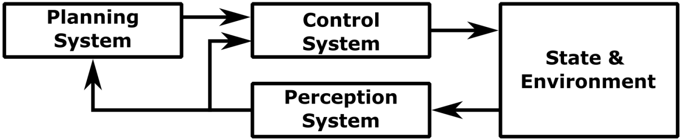

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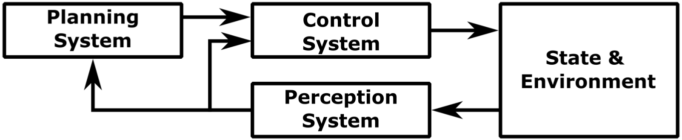
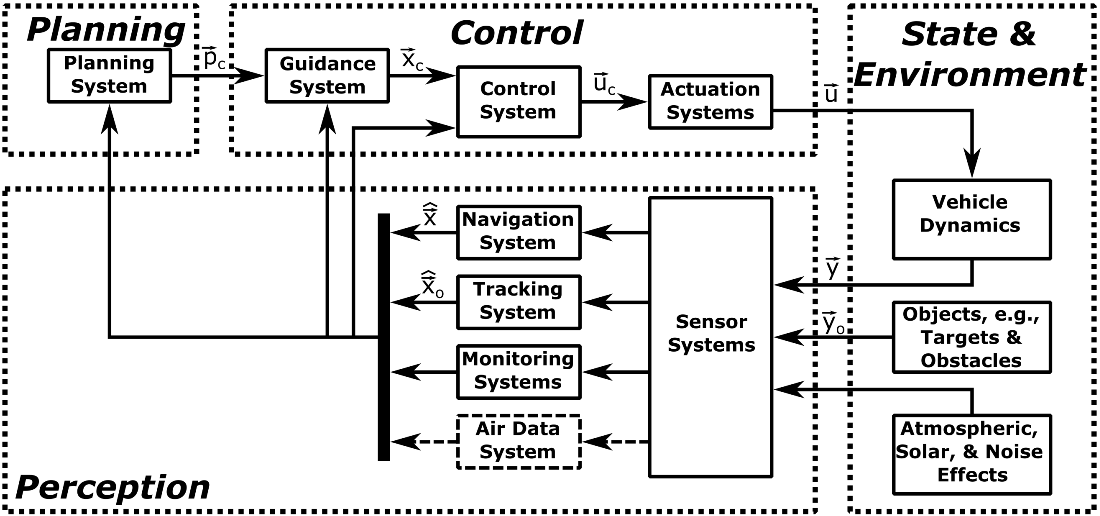


- Planning system: determines what mission/path/trajectory, $\vec{p}_c$, aerospace vehicle should fly
- Control system: executes path/trajectory for aerospace vehicle by actuating control inputs, $\vec{u}$
- Perception system(s): senses and estimates state of aerospace vehicle, $\hat{\vec{x}}$, state of object(s) in environment, $\hat{\vec{x}}_o$, and state of environment
  - For aerospace vehicles, perception systems provide feedback to planning system & control system

Aerospace Planning, Control, and Perception System



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- 1950s and 1960s: space race generated interest in embedded control and perception systems
  - Engineers constructed control and perception systems as part of end product, e.g. engine control units and flight simulators
- 1970s, computer-based control and perception systems introduced and became standard
  - Drastic shift in control and perception system design
  - Led to modern use of model-based design for control and perception systems

4 Steps of Model-Based Design

1. **Plant modeling**: consists of identifying system model to be controlled, i.e. **plant** which comes from “powerplant” in early days of control and perception systems
    - Plant modeling can be based on first principles or machine learning
    - First principles modeling implements physics-derived mathematical model for plant dynamics, e.g. Newton-Euler EOMs
    - Machine learning modeling processes raw data from real-world system and uses learning algorithms to identify data-driven model for plant dynamics

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2. **Control and perception design**: consists of **control and perception analysis** and **control and perception synthesis**
   - Mathematical plant model developed in first step used to identify suitable **design requirements** for controlled plant
   - Based on these identified requirements, control and perception strategies chosen
   - Control system and perception system synthesized: explicit **control law** and **observer law**, i.e. two mathematical algorithms for computing plant inputs automatically based on external commands and sensor data

4 Steps of Model-Based Design (continued)

3. **System simulation**: performed as multiple simulations with lower/higher fidelity plant models as well as offline/real-time simulations
   - Allow specification, requirements, and modeling errors to be found immediately, rather than later in overall system design effort
   - Real-time simulation: test control system, perception system, and plant on real-time modeling computer, a.k.a. **software-in-the-loop (SIL)** simulation
   - Control and perception systems implemented on embedded system hardware and simulated with real-time plant and sensor models, a.k.a. **hardware-in-the-loop (HIL)** simulation

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4. **Validation and verification (V&V)** of control and perception system
   - Control and perception systems implemented on embedded system hardware and tested with actual plant, e.g. aerospace vehicle, and sensors
   - Control and perception systems highly unlikely to work exactly same on actual plant and sensors as in simulation $\rightarrow$ iterative debugging process by analyzing test results and updating control and perception system designs untildesign requirements met in actuality

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- **System**: process that produces output signals in response to input signals
  - Output signal from system, a.k.a. **system response**
  - System characterized by its different properties

Dynamical Systems

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- Dynamical systems theory encompasses three broad topics:
  - Simulation
  - Control
  - **System identification (SID)**
  - Course: advanced simulation and control for aerospace vehicles

Dynamical System Characterization

- Characterize dynamical systems: signal types
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  - **Time-varying system**: mathematical rule that depends *explicitly* on time
  - **Deterministic system**: always produces same output signal for given input signal
  - **Stochastic system**: never produces same output signal for given input signal

Dynamical Systems Theory Scope

- Characterize system: number of input and output signals
  - **Single Input, Single Output (SISO) system**
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- Unfortunately, no dynamical system perfectly linear, time-invariant, or deterministic
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- Assumptions simplify mathematical theory of dynamical systems & provide valuable insight into FDC

State-Space Representation

- Continuous-time MIMO dynamical systems: **continuous-time state-space model**

$$\begin{aligned} \dot{\vec{x}}(t) &= f(t, \vec{x}, \vec{u}) \\ \vec{y}(t) &= h(t, \vec{x}, \vec{u}) \end{aligned} \tag{1}$$

  - $\vec{u}(t) \in \mathbb{R}^{n_u}$: **input vector** of n_u input signals
  - $\vec{y}(t) \in \mathbb{R}^{n_y}$: **output vector**of n_y output signals
  - $\vec{x}(t) \in \mathbb{R}^{n_x}$: **state vector** of n_x^{th}-order dynamical system

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- First vector-valued differential equation: **dynamics equation** for state-space systems, a.k.a. **state equation**
- Second vector-valued algebraic equation: **output equation**
  - Relates state and input vectors to output vector

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- First vector-valued differential equation: **dynamics equation** for state-space systems, a.k.a. **state equation**
- Second vector-valued algebraic equation: **output equation**
  - Relates state and input vectors to output vector
- System state $\vec{x}$ dynamically *controlled* by input $\vec{u}$ and *observed* through output $\vec{y}$

State-Space by Component

- Vector-valued state-space system rewritten as n_x first order ODEs for state equation and n_y algebraic equations for output equation:

$$
\begin{aligned}
\dot{x}_1 &= f_1(t, x_1, \cdots, x_{n_x}, u_1, \cdots, u_{n_u}) \\
\vdots &= \vdots \\
\dot{x}_{n_x} &= f_{n_x}(t, x_1, \cdots, x_{n_x}, u_1, \cdots, u_{n_u}) \\
y_1 &= h_1(t, x_1, \cdots, x_{n_x}, u_1, \cdots, u_{n_u}) \\
\vdots &= \vdots \\
y_{n_y} &= h_{n_y}(t, x_1, \cdots, x_{n_x}, u_1, \cdots, u_{n_u})
\end{aligned}
\tag{2}
$$

  - x_i denotes i^{th} element of $\vec{x}$
  - u_j denotes j^{th} element of $\vec{u}$
  - y_k denotes k^{th} element of $\vec{y}$

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  - Let output $y_1(t)$ be solution to linear ODE with input $u_1(t)$ and zero initial conditions
  - Scaling property: for any real number c, solution of linear ODE with input $u_s(t) = cu_1(t)$ and zero initial conditions is $y_s(t) = cy_1(t)$

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  - Let $y_2(t)$ be solution with $u_2(t)$ and zero initial conditions
  - Additivity property: for solution of linear ODE with input $u_a(t) = u_1(t) + u_2(t)$ and zero initial conditions is $y_a(t) = y_1(t) + y_2(t)$

Continuous-Time Linear System Models

- **Continuous-time linear state-space model** general form:

$$\begin{aligned} \dot{\vec{x}}(t) &= A(t)\vec{x}(t) + B(t)\vec{u}(t) \\ \vec{y}(t) &= C(t)\vec{x}(t) + D(t)\vec{u}(t) \end{aligned} \tag{3}$$

  - $A(t) \in \mathbb{R}^{n_x \times n_x}$: **state matrix**
  - $B(t) \in \mathbb{R}^{n_x \times n_u}$: **input matrix**
  - $C(t) \in \mathbb{R}^{n_y \times n_x}$: **output matrix**
  - $D(t) \in \mathbb{R}^{n_y \times n_u}$: **feedthrough matrix**

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- **Continuous-time linear time-invariant state-space models** general form

$$\begin{aligned} \dot{\vec{x}}(t) &= A\vec{x}(t) + B\vec{u}(t) \\ \vec{y}(t) &= C\vec{x}(t) + D\vec{u}(t) \end{aligned} \tag{5}$$

  - LTI systems theory borrows heavily from mathematical methods in differential equations and linear algebra, utilized throughout course

Notes on LTI Systems

- Particular LTI state-space model can be denoted by the quadruple (A, B, C, D)
  - Different choices for (A, B, C, D) that represent same dynamical system in terms of input-to-output relationship
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- To set $\vec{y} = \vec{x}$: set output matrix to **identity matrix**

$$C = I = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \vdots & 1 \end{bmatrix} \quad (6)$$

& set feedthrough matrix to **zero matrix**

$$D = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \quad (7)$$

Transfer Function

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- Numerator and denominator: polynomials $\rightarrow$ have real and/or complex roots
  - **Zero** of transfer function z: root of numerator
    - $|G(s)| \rightarrow 0$ for s equal to zero
  - **Pole** of transfer function p: root of denominator
    - $|G(s)| \rightarrow \infty$ for s equal to pole
    - Plotting $|G(s)|$ over complex plane, ∞ looks like tent pole in 3D

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    - Plotting $|G(s)|$ over complex plane, ∞ looks like tent pole in 3D
- Can always factor transfer function as

$$G(s) = \frac{K(s - z_1)\cdots(s - z_m)}{(s - p_1)\cdots(s - p_n)} \tag{10}$$

Notes on Transfer Functions

- Can always factor transfer function as **partial-fraction decomposition**, i.e.

$$G(s) = \frac{r_1}{(s - p_1)} + \cdots + \frac{r_n}{(s - p_n)} \tag{11}$$

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- **Final Value Theorem (FVT)**: if every pole of transfer function $F(s)$ not purely imaginary *except*, at most, single pole at origin, then:

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) \tag{12}$$

  - $s \to 0$ denotes s approaching through positive numbers

Transfer Functions in Multivariate Domain

- For continuous-time MIMO LTI systems as state-space model, i.e.

$$\begin{aligned}\dot{\vec{x}}(t) &= A\vec{x}(t) + B\vec{u}(t)\\ \vec{y}(t) &= C\vec{x}(t) + D\vec{u}(t)\end{aligned} \tag{13}$$

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$$\begin{aligned} \dot{\vec{x}}(t) &= A\vec{x}(t) + B\vec{u}(t) \\ \vec{y}(t) &= C\vec{x}(t) + D\vec{u}(t) \end{aligned} \tag{13}$$

- Transform to Laplace domain with zero initial conditions as

$$\begin{aligned} s\vec{x}(s) &= A\vec{x}(s) + B\vec{u}(s) \\ \vec{y}(s) &= C\vec{x}(s) + D\vec{u}(s) \end{aligned} \tag{14}$$

Transfer Function Matrix

- To find **transfer function matrix**, $[G(s)]$, maps each input to each output through matrix of transfer functions, i.e.

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  - $(sI - A)^{-1}$: **resolvent** of A

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- By substitution

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- By definition of transfer function matrix, **standard transfer function matrix**:

$$[G(s)] = C\left(sI - A\right)^{-1} B + D \tag{21}$$

- $n \times m$ matrix of transfer function elements, $G_{ij}(s)$, i.e.

$$[G(s)] = \begin{bmatrix} G_{11}(s) & \cdots & G_{1m}(s) \\ \vdots & \ddots & \vdots \\ G_{n1} & \cdots & G_{nm}(s) \end{bmatrix} \tag{22}$$

Equilibrium Points for State-Space Models

- State-input pair $(\vec{x}, \vec{u})$ **equilibrium point**: if $\dot{\vec{x}}$ zero for all $t > 0$, i.e. if

$$f(t, \vec{x}, \vec{u}) = 0 \tag{23}$$

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  - I.e. $\vec{x}$ "steady"
- Also may be multiple, even infinite, solutions for particular **trim state**, $\bar{\vec{x}}$
  - Fewer equations than free variables for vector-valued state equation
  - I.e. n_x elements and $n_x + n_u$ free variables

Lyapunov Stability Definitions

- Important characterization of system equilibrium points: stability
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- LTI FDC considers Lyapunov's first method for time-invariant systems
  - Provide methods for assessing flight vehicle's stability and designing control systems
  - Nonlinear and time-varying control: discusses Lyapunov's second method

Multivariate Linearization

- **Taylor Series** for multivariate function $f(\vec{x}, \vec{u})$ about vector pair $(\bar{\vec{x}}, \bar{\vec{u}})$:

$$\dot{\vec{x}}(t) = f(\vec{x}, \vec{u}) = f(\bar{\vec{x}}, \bar{\vec{u}}) + \left[\frac{\partial f}{\partial \vec{x}}(\bar{\vec{x}}, \bar{\vec{u}})\right](\vec{x} - \bar{\vec{x}}) + \left[\frac{\partial f}{\partial \vec{u}}(\bar{\vec{x}}, \bar{\vec{u}})\right](\vec{u} - \bar{\vec{u}}) + \text{HOT} \quad (24)$$

&

$$\vec{y}(t) = h(\vec{x}, \vec{u}) = h(\bar{\vec{x}}, \bar{\vec{u}}) + \left[\frac{\partial h}{\partial \vec{x}}(\bar{\vec{x}}, \bar{\vec{u}})\right](\vec{x} - \bar{\vec{x}}) + \left[\frac{\partial h}{\partial \vec{u}}(\bar{\vec{x}}, \bar{\vec{u}})\right](\vec{u} - \bar{\vec{u}}) + \text{HOT} \quad (25)$$

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- $\left[\frac{\partial f}{\partial \vec{x}}(\bar{\vec{x}}, \bar{\vec{u}})\right]$: **Jacobian** of $f()$
- Multivariate linearization a.k.a. **Jacobian linearization**

Perturbation Vectors

- State **perturbation vector** about constant $\bar{\vec{x}}$:

$$\Delta \vec{x}(t) = \vec{x}(t) - \bar{\vec{x}} \tag{26}$$

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- Recognize for trim, $f(\bar{\vec{x}}, \bar{\vec{u}}) = 0$ and $h(\bar{\vec{x}}, \bar{\vec{u}}) = \bar{\vec{y}}$:

$$\Delta\dot{\vec{x}}(t) = \dot{\vec{x}}(t) = f(\vec{x}, \vec{u}) = f(\bar{\vec{x}}, \bar{\vec{u}}) + \left[\frac{\partial f}{\partial \vec{x}}(\bar{\vec{x}}, \bar{\vec{u}})\right]\Delta\vec{x}(t) + \left[\frac{\partial f}{\partial \vec{u}}(\bar{\vec{x}}, \bar{\vec{u}})\right]\Delta\vec{u}(t) + \text{HOT} \tag{29}$$

Time-Invariant State-Space Model

- Recall general form for time-invariant state-space model:

$$\begin{aligned} \dot{\vec{x}} &= f(\vec{x}, \vec{u}) \\ \vec{y} &= h(\vec{x}, \vec{u}) \end{aligned} \tag{31}$$

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- Set

$$A = \left[\frac{\partial f}{\partial \vec{x}}(\bar{\vec{x}}, \bar{\vec{u}})\right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\bar{\vec{x}}, \bar{\vec{u}}) & \cdots & \frac{\partial f_1}{\partial x_{n_x}}(\bar{\vec{x}}, \bar{\vec{u}}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{n_x}}{\partial x_1}(\bar{\vec{x}}, \bar{\vec{u}}) & \cdots & \frac{\partial f_{n_x}}{\partial x_{n_x}}(\bar{\vec{x}}, \bar{\vec{u}}) \end{bmatrix} \tag{32}$$

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Time-Invariant State-Space Model (continued)

$$C = \left[\frac{\partial h}{\partial \vec{x}}(\bar{\vec{x}}, \bar{\vec{u}})\right] = \begin{bmatrix} \frac{\partial h_1}{\partial x_1}(\bar{\vec{x}}, \bar{\vec{u}}) & \cdots & \frac{\partial h_1}{\partial x_{n_x}}(\bar{\vec{x}}, \bar{\vec{u}}) \\ \vdots & \ddots & \vdots \\ \frac{\partial h_{n_y}}{\partial x_1}(\bar{\vec{x}}, \bar{\vec{u}}) & \cdots & \frac{\partial h_{n_y}}{\partial x_{n_x}}(\bar{\vec{x}}, \bar{\vec{u}}) \end{bmatrix} \tag{34}$$

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- Yields approximate LTI state-space model about $\bar{\vec{x}}$ and $\bar{\vec{u}}$:

$$\begin{aligned} \Delta\dot{\vec{x}}(t) &\approx A\Delta\vec{x}(t) + B\Delta\vec{u}(t) \\ \Delta\vec{y}(t) &\approx C\Delta\vec{x}(t) + D\Delta\vec{u}(t) \end{aligned} \tag{36}$$

General Solution for LTI Systems

- General LTI state-space form

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  - Boundary conditions: $\vec{x}(t_0)$

- General solution for vector-valued first-order state equation:

$$\begin{aligned}\vec{x}(t) &= e^{A(t-t_0)}\vec{x}(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}B\vec{u}(\tau)d\tau \\ \vec{y}(t) &= C\left[e^{A(t-t_0)}\vec{x}(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}B\vec{u}(\tau)d\tau\right] + D\vec{u}(t)\end{aligned} \tag{38}$$

State-Space Characteristic Equation

- **State-space characteristic equation** for general LTI systems:

$$\det\left[\lambda I - A\right] = 0 \tag{39}$$

  - Solutions to equation: n_x roots of state-space characteristic polynomial, may be repeated and will either be real numbers or complex-conjugate pairs

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- Roots of characteristic polynomials a.k.a. **system poles**
  - If no pole-zero cancellations, then poles of any $G_{ij}(s)$ also system poles
- Each distinct real pole and each distinct complex-conjugate pair: **system mode**
  - Poles in LHP: stable

Stability

- Real part of each individual mode: characterize stability of each mode
  - **Stable** mode: real part of $\lambda < 0$
  - **Marginally stable** mode: real part of $\lambda = 0$
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- Stable LTI system: *all* modes stable
  - If *any* mode unstable $\rightarrow$ unstable LTI system
- Eigenvalues may be complex-valued $\rightarrow$ system poles located in left half of complex plane, i.e. **left half plane (LHP)**, correspond to stable modes

Modal Characteristics

- Each mode corresponds to unique **modal time constant**
  - Describes exponential decay/growth rate

$$\tau = \frac{1}{\text{Real}(\lambda)} \tag{40}$$

  - Real(λ) represents the real part of λ, i.e. if $\tau < 0$, then stable mode
  - Modes with faster decay on output $=$ modes with higher eigenvalues

Transfer Function Modal Analysis

- For complex-conjugate pairs, $\lambda = \frac{1}{\tau} \pm j\omega_d$, define **modal damped frequency**: ω_d
  - Related to **modal undamped natural frequency**: ω_n
  - **Modal damping ratio**, $0 < \zeta_i < 1$

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- Note: one can compare:

$$(\lambda - \frac{1}{\tau} + j\omega_d)(\lambda - \frac{1}{\tau} - j\omega_d) \tag{43}$$

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 \tag{44}$$

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