Lecture 18: Robust Servomechanism and Stability of LTI Systems

Textbook Sections 3.3 & 3.4

Dr. Jordan D. Larson

Introduction

- Stability robustness to model uncertainties: crucial design criterion for feedback control systems of flight vehicles

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- Lecture:
  - General types of uncertainties
  - Generalized framework for studying uncertain LTI systems
  - Primary robust stability results using framework

Internal Model Principle: SISO Systems

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- Control system design: design control system to regulate tracking error of reference command, $r(t)$, to zero in presence of unknown disturbances to system, $w(t)$
- Classical control theory for SISO systems: integral control action necessary to achieve this zero steady-state tracking error
- **Internal model principle**: number of integrators in open-loop transfer function, i.e. **system type**, must be $\geq$ reference and disturbance order
  - If $r(t) = w(t) = 0$, 0 integrators, type 0 control system, a.k.a. **regulator**
  - If $\dot{r}(t) = \dot{w}(t) = 0$, 1 integrator, type 1 control system
  - If $\ddot{r}(t) = \ddot{w}(t) = 0$, 2 integrators, type 2 control system

Internal Model Principle: MIMO Systems

- For MIMO systems, state feedback controllers act as type 0 control system, i.e. regulate system state to zero
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- Other types of reference commands and disturbances require additional feedback control considerations for zero-error tracking of array of reference commands in presence of array disturbances
  - Standard design approach: robust servomechanism control system
  - 2 components: servomechanism and state feedback
  - "Robust" term: reach zero steady-state tracking error in presence of specified classes of *disturbances*

Robust Servomechanism Problem Setup

- Consider continuous-time LTI state-space system:

$$\begin{aligned} \dot{\vec{x}} &= A\vec{x} + B\vec{u} + M\vec{w} \\ \vec{y} &= C\vec{x} + D\vec{u} \end{aligned} \tag{1}$$

  - Additive unknown bounded disturbance: $\vec{w} \in \mathbb{R}^{n_w}$
  - Assume controllable & observable

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  - Additive unknown bounded disturbance: $\vec{w} \in \mathbb{R}^{n_w}$
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- Assume reference command as some commanded output, $\vec{r}(t) \in \mathbb{R}^{n_y}$: p^{th} order ODE

$$\vec{r}^{[p]} = \sum_{i=1}^{p} a_i \vec{r}^{[p-i]} \tag{2}$$

  - a_i: scalar coefficients known & superscript $[j]$ denotes j^{th} derivative
  - Constant command: ODE form of $\dot{\vec{r}} = 0$ with $p = 1$, $a_1 = 0$
  - Ramp command: ODE form of $\ddot{\vec{r}} = 0$ with $p = 2$, $a_2 = a_1 = 0$
  - Sinusoidal command at frequency ω_0: ODE form of $\ddot{\vec{r}} = -\omega_0^2\vec{r}$ with $p = 2$, $a_2 = -\omega_0^2$, $a_1 = 0$

Robust Servomechanism: Disturbances & Tracking Error

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$$\vec{e}^{[p]} - \sum_{i=1}^{p} a_i \vec{e}^{[p-i]} = \left(\vec{y}^{[p]} - \sum_{i=1}^{p} a_i \vec{y}^{[p-i]} \right) - \left(\vec{r}^{[p]} - \sum_{i=1}^{p} a_i \vec{r}^{[p-i]} \right) \tag{5}$$

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  - By definition, second term on right side will be zero
- First term rewritten using output equation and its derivatives:

$$\vec{e}^{[p]} - \sum_{i=1}^{p} a_i \vec{e}^{[p-i]} = C\left(\vec{x}^{[p]} - \sum_{i=1}^{p} a_i \vec{x}^{[p-i]}\right) + D\left(\vec{u}^{[p]} - \sum_{i=1}^{p} a_i \vec{u}^{[p-i]}\right) \tag{6}$$

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$$\vec{\eta} = \vec{x}^{[p]} - \sum_{i=1}^{p} a_i \vec{x}^{[p-i]} \tag{7}$$

$$\vec{\mu} = \vec{u}^{[p]} - \sum_{i=1}^{p} a_i \vec{u}^{[p-i]} \tag{8}$$

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Robust Servomechanism: Eliminating Disturbance

- Differentiating:

$$\dot{\vec{\eta}} = \vec{x}^{[p+1]} - \sum_{i=1}^{p} a_i \vec{x}^{[p-i+1]} \tag{10}$$

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$$\dot{\vec{\eta}} = A\vec{\eta} + B\vec{\mu} \tag{12}$$

  - Original system without additive disturbance term

Servomechanism State-Space Model

- **Servomechanism state-space model**:

$$\dot{\vec{z}} = \tilde{A}\vec{z} + \tilde{B}\vec{\mu} \tag{13}$$

- $\vec{z}$: augmented $n_x + p \times n_y$ state vector

$$\vec{z} = \begin{bmatrix} \vec{e} \\ \dot{\vec{e}} \\ \vdots \\ \vec{e}^{[p-1]} \\ \vec{\eta} \end{bmatrix} \tag{14}$$

Servomechanism State-Space Model

- Augmented state matrix:

$$\tilde{A} = \begin{bmatrix} 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \\ a_p I & a_{p-1} I & \cdots & a_1 I & C \\ 0 & 0 & \cdots & 0 & A \end{bmatrix} \tag{15}$$

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- Check for controllability of servomechanism state-space model:
  - (A, B) be controllable
  - $n_u \geq n_y$
  - (A, B, C, D) must not have any zeros in common with ODE for $\vec{y}$

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- **Parametric uncertainty**: specified as unknown parameters, members of some set of values called **uncertainty set**
  - **Complex parametric uncertainty** or **real parametric uncertainty**
  - Scalar, i.e. some $\alpha \in \mathbb{C}$ or some $\beta \in \mathbb{R}$, or matrix-defined sets

Analysis Approaches

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  - Model dependencies between parameters
- *Worst-case analysis approach*: equally likely values within some finite uncertainty set
- Probability theory beyond scope of course, robustness analysis of LTI systems under parametric uncertainty in course: worst-case analysis approach

Important Set for Complex Parameters

- Uncertainty disk:

$$\alpha \in Disk\left(\frac{1-m}{1+m}, \frac{1+m}{1-m}\right) = \left\{\alpha = \frac{1}{1-m\delta} \in \mathbb{C} \ : \ \delta \in \mathbb{C}, \ |\delta| \leq 1\right\} \tag{18}$$

  - $m \in [0, 1)$

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- Center of disk:

$$\frac{1-m}{1+m} + \frac{2m}{(1+m)(1-m)} = \frac{1-2m+m^2+2m}{(1+m)(1-m)} = m/2 \quad (20)$$

  - Center defines *Disk* instead of m
  - As $m \to 1$, disk converges to right half of complex plane (RHP)

Important Set for Real Parameters

- Closed interval:

$$\beta \in [\underline{\beta}, \bar{\beta}] = \left\{ \beta = \frac{\bar{\beta} + \underline{\beta}}{2}\delta_{\beta}^{2} + \frac{\bar{\beta} - \underline{\beta}}{2}\delta_{\beta} + \beta_0 \ : \ \delta_{\beta} \in [-1, 1] \right\} \tag{21}$$

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  - δ: normalized uncertainty
- $\delta_\beta = 0$ corresponds to nominal β_0
- $\delta_\beta = -1$ corresponds to $\underline{\beta}$
- $\delta_\beta = 1$ corresponds to $\bar{\beta}$

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- Can relate LTI dynamic uncertainty to complex parametric uncertainty (later)
- Course: focus on LTI dynamic uncertainty for robustness analysis of LTI systems
  - Robustness analysis for nonlinear and/or time-varying dynamic uncertainty requires more general **integral quadratic constraints** (not covered)

Important Set for LTI Dynamic Uncertainty

- Perturbed model:

$$G(s) \in \mathcal{M}_W = \{G(s) = G_0(s)(I + W(s)\Delta(s)) \ : \ \Delta(s) \ LTI, \ \|\Delta\|_\infty \leq 1\} \tag{22}$$

  - $G_0(s)$: nominal LTI model
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  - $W(j\omega)$: bound on relative error at all frequencies from $G_0(j\omega)$
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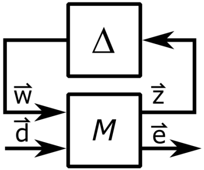
- Δ assumed stable: restricts $G(s)$ only has RHP poles at the same locations as $G_0(s)$, often used in robustness analysis

Multiplicative Uncertain LTI Model



- Denoted by $F_U(M, \Delta)$: LFT

$$\begin{aligned} \begin{bmatrix} \vec{z} \\ \vec{d} \end{bmatrix} &= \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \vec{w} \\ \vec{e} \end{bmatrix} \\ \vec{w} &= \Delta \vec{z} \end{aligned} \tag{24}$$

Multiplicative Uncertain LTI Model (continued)

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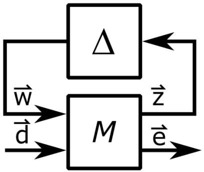
Additive Uncertain LTI Model



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Additive Uncertain LTI Model (continued)

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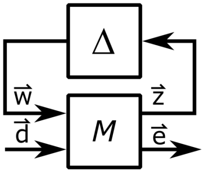
Generalized Uncertain LTI Model



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- LFT well-posed if $I_{n_z} - M_{11}(\infty)\Delta(\infty)$ invertible
- Δ may have "structure" depending on types of uncertainties
  - E.g. multiple scalar uncertainties and/or real parametric uncertainties
  - If $\Delta \in \mathbb{C}^{n_w \times n_z}$ or single LTI dynamic uncertainty, then Δ **unstructured uncertainty set**

Generalized Uncertain LTI State-Space

- LTI state-space model for nominal model, M:

$$\begin{aligned}
\dot{\vec{x}}(t) &= A\vec{x}(t) + \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} \vec{w}(t) \\ \vec{d}(t) \end{bmatrix} \\
\begin{bmatrix} \vec{z}(t) \\ \vec{e}(t) \end{bmatrix} &= \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} \vec{w}(t) \\ \vec{d}(t) \end{bmatrix}
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- LTI state-space model for generalized uncertainty set, Δ:

$$\begin{aligned} \dot{\vec{x}}_\Delta &= A_\Delta \vec{x}_\Delta + B_\Delta \vec{z} \\ \vec{w} &= C_\Delta \vec{x}_\Delta + D_\Delta \vec{z} \end{aligned} \tag{28}$$

  - If Δ contains only real or complex uncertainty, then $A_\Delta = B_\Delta = C_\Delta = 0$ with no $\vec{x}_\Delta$

Generalized Uncertain LTI State-Space (continued)

- Combining for uncertainty input and output vectors

$$\begin{bmatrix} I & -D_\Delta \\ -D_{11} & I \end{bmatrix} \begin{bmatrix} \vec{w}(t) \\ \vec{z}(t) \end{bmatrix} = \begin{bmatrix} 0 & C_\Delta \\ C_1 & 0 \end{bmatrix} \begin{bmatrix} \vec{x}(t) \\ \vec{x}_\Delta(t) \end{bmatrix} + \begin{bmatrix} 0 \\ D_{12} \end{bmatrix} \begin{bmatrix} \vec{d}(t) \end{bmatrix} \tag{29}$$

  - Interconnection of M and Δ well-posed if and only if $I_{n_z} - D_{11}D_\Delta$ invertible

Generalized Uncertain LTI State-Space (continued)

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- Similar to generalized LFT feedback control system using loop-shifting, form closed-loop LTI state-space system model as

$$\begin{aligned} \begin{bmatrix} \dot{\vec{x}}(t) \\ \dot{\vec{x}}_\Delta(t) \end{bmatrix} &= \begin{bmatrix} A + B_1 D_\Delta C_1 & B_1 C_\Delta \\ B_\Delta C_1 & A_\Delta \end{bmatrix} \begin{bmatrix} \vec{x}(t) \\ \vec{x}_\Delta(t) \end{bmatrix} + \begin{bmatrix} B_2 + B_1 D_\Delta D_{21} \\ B_\Delta D_{12} \end{bmatrix} \vec{d}(t) \\ \vec{e}(t) &= \begin{bmatrix} C_2 + D_{21} D_\Delta C_1 & D_{21} C_\Delta \end{bmatrix} \begin{bmatrix} \vec{x}(t) \\ \vec{x}_\Delta(t) \end{bmatrix} + (D_{22} + D_{21} D_\Delta D_{12}) \vec{d}(t) \end{aligned} \tag{30}$$

Generalized Uncertain LTI State-Space (continued)

- Closed-loop state matrix, A_L:

$$A_L = \begin{bmatrix} A + B_1 D_\Delta C_1 & B_1 C_\Delta \\ B_\Delta C_2 & A_\Delta \end{bmatrix} \tag{31}$$

- Closed-loop input matrix, B_L:

$$B_L = \begin{bmatrix} B_2 + B_1 D_\Delta D_{12} \\ B_\Delta D_{12} \end{bmatrix} \tag{32}$$

- Closed-loop output matrix, C_L:

$$C_L = \begin{bmatrix} C_2 + D_{21} D_\Delta C_1 & D_{21} C_\Delta \end{bmatrix} \tag{33}$$

- Closed-loop feedthrough matrix, D_L:

$$D_L = D_{22} + D_{21} D_\Delta D_{12} \tag{34}$$

Robust Stability Analysis

- Assume M and Δ stable and $F_U(M, \Delta)$ well-posed, then $F_U(M, \Delta)$ has pole at s_p if and only if $\det(s_p I - A_L) = 0$
  - Can be shown through determinant formulas and M and Δ stability

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- Only occurs if and only if $\det(I - M_{11}(s_p)\Delta(s_p)) = 0$
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- Also seen from LFT from d to e:

$$F_U(M, \Delta) = M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1} M_{12} \tag{35}$$

  - Δ, M_{11}, M_{12}, M_{21}, M_{22} all stable by assumption
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  - $F_U(M, \Delta)$ stable if and only if $(I - M_{11}\Delta)^{-1}$: unstable
- Allows definition: generalized LTI uncertainty system, $F_U(M, \Delta)$, **robustly stable** if and only if $I - M_{11}\Delta$ invertible for all $\Delta \in \boldsymbol{\Delta}$
  - $\boldsymbol{\Delta}$: modeled uncertainty set, structured or unstructured

Robust Stability Margin

- Robust stability margin: find $\Delta \in \boldsymbol{\Delta}$ with smallest $\|\Delta\|_\infty$ such that $\det(I - M_{11}(s_p)\Delta(s_p)) = 0$ for some s_p in closed RHP

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- If $m > 1$, then $F_U(M, \Delta)$ robustly stable
- Worst-case uncertainty, $\Delta_{w-c} \in \mathbf{\Delta}$: destabilizes $F_U(M, \Delta)$ and has $\|\Delta\|_\infty = m$
  - Note: Δ_{w-c} causes $F_U(M, \Delta)$ to have system poles on imaginary axis at some $\pm j\omega_{w-c}$

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  - Sufficient condition: if $\bar{\sigma}(M) \leq \frac{1}{m}$, then

$$\bar{\sigma}(M_{11}\Delta) \leq \bar{\sigma}(M)\bar{\sigma}(\Delta) < 1 \quad \forall \, \Delta \text{ with } \bar{\sigma}(\Delta) < m \tag{36}$$

$\rightarrow \|M\Delta\vec{v}\| < \|\vec{v}\|$ for any non-zero $\vec{v} \in \mathbb{C}^{n_v}$ or $(I - M\Delta)\vec{v} \neq 0$, i.e. $\det(I - M\Delta) \neq 0$ for all $\bar{\sigma}(\Delta) < m$

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  - Necessary condition by contradiction: if $\bar{\sigma}(M) > \frac{1}{m}$, then SVD of M:

$$\bar{\sigma}(M)\vec{v} = M\vec{z} \ \text{ and } \ \|\vec{v}\|_2 = \|\vec{z}\|_2 = 1 \tag{37}$$

    Selecting $\Delta_0 = \frac{1}{\bar{\sigma}(M)}\vec{z}\vec{v}^* \in \mathbb{C}^{n_z \times n_w} \rightarrow \bar{\sigma}(\Delta_0) = \frac{1}{\bar{\sigma}(M)} < m \rightarrow (I - M\Delta_0)\vec{v} = 0$ & $\det(I - M\Delta_0) = 0$

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- Note: equivalence between complex and dynamic uncertainty breaks down at $\omega_0 = 0$ and ∞ as LTI system with real-valued matrices has real frequency response at $\omega_0 = 0$ and ∞, i.e. $\Delta(0), \Delta(\infty) \in \mathbb{R}^{n_z \times n_w}$

Robust Stability for Unstructured LTI Dynamic Uncertainty

- Consider generalized LTI uncertainty system, $F_U(M, \Delta)$ where Δ unstructured, stable LTI system and M stable

  $F_U(M, \Delta)$ well-posed and stable for all $\|\Delta\|_\infty < m$ if and only if $\|M_{11}\|_\infty \leq \frac{1}{m}$, i.e.

$$\max_{\omega \in \mathbb{R} \cup \{\infty\}} \bar{\sigma}(M_{11}(j\omega)) \leq \frac{1}{m} \tag{38}$$

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- Note: if M_{11} achieves peak gain at $\omega_0 = 0$ or ∞, then interpolation performed at some arbitrarily small or large finite frequency

SGT as Optimization

- Consequence of SGT: $\det(I - M_{11}\Delta) \neq 0$ for any $\Delta \in \mathbb{C}^{n_w \times n_z}$ with $\bar{\sigma}(\Delta) < \frac{1}{\bar{\sigma}(M_{11})}$

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$$
\begin{aligned}
\frac{1}{\bar{\sigma}(M_{11})} &= \min_{\Delta \in \mathbb{C}^{n_w \times n_z}} \bar{\sigma}(\Delta) \\
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- For some structured $\Delta \in \boldsymbol{\Delta}$, define **structured singular value (SSV)**for some $M_{11} \in \mathbb{C}^{n_w \times n_z}$ as

$$\mu_{\boldsymbol{\Delta}}(M_{11}) = \begin{cases} \left[\min\limits_{\Delta \in \boldsymbol{\Delta}} \left(\bar{\sigma}(\Delta) \text{ s.t. } \det(I - \Delta M_{11}) = 0 \right) \right]^{-1} \\ 0, \text{ if no } \Delta \in \boldsymbol{\Delta} \text{ causes } \det(I - \Delta M_{11}) = 0 \end{cases} \tag{40}$$

  - Second case *may* occur if $\boldsymbol{\Delta}$ consists of only real parametric uncertainties

SSV Generalization of SGT

- Almost directly from SSV definition: if $\Delta(s) \in \boldsymbol{\Delta} \subset \mathbb{C}^{n_w \times n_z}$ and $M_{11} \in \mathbb{C}^{n_z \times n_w}$, then $\det(I - M_{11}\Delta) \neq 0$ for all $\Delta \in \boldsymbol{\Delta}$ with $\bar{\sigma}(\Delta) < m$ if and only if $\mu_{\boldsymbol{\Delta}}(M_{11}) \leq \frac{1}{m}$

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- If $\mathbf{\Delta} \in \mathbb{C}^{n_w \times n_z}$ full complex matrix, then SGT-based maximum singular value optimization provides SSV as

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  - Implementations of SSV-based robustness bounds use upper and lower bounds on $\mu_{\mathbf{\Delta}}(M_{11})$
- Generally $\mathbf{\Delta} \subset \mathbb{C}^{n_w \times n_z}$: simple upper bound

$$\mu_{\mathbf{\Delta}}(M_{11}) \leq \bar{\sigma}(M_{11}) \tag{42}$$

  - Extended via tighter bounds through D-scalings which account for non-uniqueness in LFT representation of M

Simple SSV Lower Bound

- Simple lower bound for $\mu_{\mathbf{\Delta}}(M_{11})$ computed if $\Delta(s)$ diagonal matrix with single complex scalar $\delta \in \mathbb{C}$:

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- For simplest structure:

$$\mu_{\boldsymbol{\Delta}}(M_{11}) = \bar{\rho}(M_{11}) = \max_i |\lambda(M_{11})| \tag{45}$$

  - $\bar{\rho}(M_{11})$: **spectral radius** of matrix, M_{11}, i.e. largest absolute value of eigenvalues of M_{11}

SSV Bounds

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- Numerical methods use versions of power iteration and D-scalings on general M_{11} to numerically compute SSV bounds via

$$\max_{Q} \lambda(QM_{11}) \leq \mu_{\boldsymbol{\Delta}}(M_{11}) \leq \inf_{D} \bar{\sigma}(DM_{11}D^{-1}) \tag{47}$$

μ_{Δ}-Analysis

- Assuming interpolation to replace LTI dynamic uncertainty blocks with corresponding complex parametric uncertainty of same size

  State exact, necessary, and sufficient condition for robust stability margin of generalized LTI uncertainty system with structured real and complex parametric uncertainty, and stable, LTI dynamic uncertainty, $\Delta \in \mathbf{\Delta}$

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  - Provides structured robust stability and margin based on structured singular value, μ_Δ
- Structured singular value bounds performed numerically
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