Lecture 4: Aerospace Vehicle Rotations and Dynamics

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Textbook Sections 7.3, 7.7, & 8.1

Introduction

- Rigid body dynamics require attitude description of body relative orientation using reference frame rotations

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- Course: Euler angles for rotation

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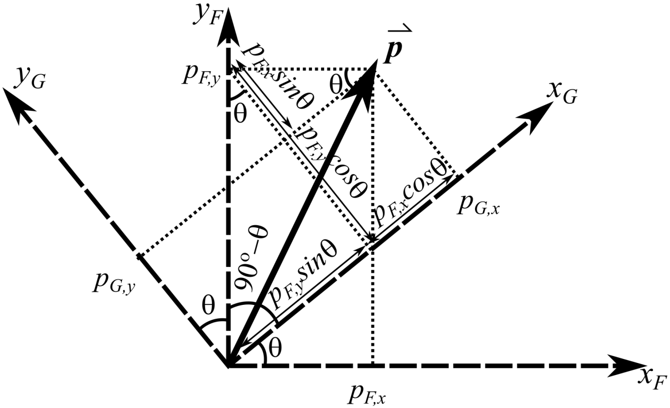
- Course: Euler angles for rotation
- Euler angles defined for rotation between two reference frames with *same* origin
- Flight vehicles operate in many different frames
  - Body-fixed frame describes rigid body model
  - Relate body to other frames through Euler angles and translations

Two-Dimensional Vectors

- Consider vector $\vec{p}$ in frames F and G:
  - $\vec{p}_F = [p_{F,x}\ p_{F,y}]^T$
  - $\vec{p}_G = [p_{G,x}\ p_{G,y}]^T$
  - With same origin



Rotation Matrix

$$\vec{p}_G = \begin{bmatrix} p_{G,x} \\ p_{G,y} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_{F,x} \\ p_{F,y} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \vec{p}_F \tag{1}$$

$$\vec{p}_G = C_{G\leftarrow F}(\theta)\vec{p}_F \tag{2}$$

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- $C_{G\leftarrow F}(\theta)$: **rotation matrix** from frame F to G by angle θ

- $C_{G\leftarrow F}$ a.k.a. **two-dimensional direction cosine matrix (DCM)**:

$$C_{G\leftarrow F} = \begin{bmatrix} \cos\theta & \cos(90^\circ - \theta) \\ \cos(90^\circ + \theta) & \cos\theta \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \tag{3}$$

- Identities: $\sin\theta = \cos(90 - \theta)$ and $-\sin\theta = \cos(90^\circ + \theta)$
- Each element, $C_{i,j}$, corresponds to cosine of angle between i-axis of G frame and j-axis of F frame
- Also seen by inspection of previous figure

Rotation Matrix Inverse Property

- Transform back: form inverse rotation matrix by using negative angle

$$\vec{p}_F = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix} \vec{p}_G = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \vec{p}_G \quad (4)$$

$$\vec{p}_F = C_{F \leftarrow G}(-\theta) \vec{p}_G \quad (5)$$

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- By inspection:

$$\vec{p}_F = C^T_{G\leftarrow F}(\theta)\vec{p}_G \tag{6}$$

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- Implies property for rotation matrices:

$$C^{-1}_{G \leftarrow F}(\theta) = C^T_{G \leftarrow F}(\theta) \tag{7}$$

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- By product rule:

$$\frac{d}{dt}\vec{p}_G(t) = C_{G\leftarrow F}\dot{\vec{p}}_F(t) + \dot{C}_{G\leftarrow F}\vec{p}_F = \dot{\vec{p}}_G(t) + \begin{bmatrix} -\sin\theta(t) & \cos\theta(t) \\ -\cos\theta(t) & -\sin\theta(t) \end{bmatrix}\dot{\theta}(t)\vec{p}_F(t) \tag{9}$$

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- $\dot{\vec{p}}$: time derivative of vector elements

$$\dot{\vec{p}}_F(t) = \frac{\partial}{\partial t}\vec{p}_F(t) \tag{10}$$

Velocity in Rotating Reference Frame

- Rearranging second term:

$$\frac{d}{dt}\vec{p}_G(t) = \dot{\vec{p}}_G(t) + \dot{\theta}(t)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \cos\theta(t) & \sin\theta(t) \\ -\sin\theta(t) & \cos\theta(t) \end{bmatrix}\vec{p}_F(t) \tag{11}$$

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- Defining $\omega(t) = \dot{\theta}(t)$: **2D angular velocity** & definition of rotation matrix:

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- **2D angular velocity tensor**:

$$[\omega(t)]_\times = \begin{bmatrix} 0 & \omega(t) \\ -\omega(t) & 0 \end{bmatrix} \tag{13}$$

$$\dot{C}_{G\leftarrow F} = [\omega(t)]_\times C_{G\leftarrow F} \tag{14}$$

Second Time Derivative of Rotation Matrix

- Define derivative of linear velocity:

$$\frac{d}{dt}\vec{v}_G(t) = \vec{a}_G(t) = \dot{\vec{v}}_G(t) + \begin{bmatrix} 0 & \omega(t) \\ -\omega(t) & 0 \end{bmatrix} \vec{v}_G(t) \tag{15}$$

  - $\vec{a}_G(t)$: **2D linear acceleration**
  - I.e. $\frac{d^2}{dt^2}\vec{p}_G(t)$

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  - $\vec{a}_G(t)$: **2D linear acceleration**
  - I.e. $\frac{d^2}{dt^2}\vec{p}_G(t)$
- Second time derivative in rotating frame G, product rule for both terms:

$$\frac{d^2}{dt^2}\vec{p}_G(t) = \frac{d}{dt}\vec{v}_G(t) = \vec{a}_G(t) \tag{16}$$

$$\frac{d^2}{dt^2}\vec{p}_G(t) = \frac{d}{dt}\left(\dot{\vec{p}}_G(t)\right) + \frac{d}{dt}\left(\begin{bmatrix} 0 & \omega(t) \\ -\omega(t) & 0 \end{bmatrix} \vec{p}_G(t)\right) \tag{17}$$

Second Time Derivative of Rotation Matrix (continued)

$$\begin{aligned}\frac{d^2}{dt^2}\vec{p}_G(t) =& \ddot{\vec{p}}_G(t) + \begin{bmatrix} 0 & \omega(t) \\ -\omega(t) & 0 \end{bmatrix} \dot{\vec{p}}_G(t) \\ &+ \begin{bmatrix} 0 & \dot{\omega}(t) \\ -\dot{\omega}(t) & 0 \end{bmatrix} \vec{p}_G(t) + \begin{bmatrix} 0 & \omega(t) \\ -\omega(t) & 0 \end{bmatrix} \frac{d}{dt}\vec{p}_G(t)\end{aligned} \tag{18}$$

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- By previous formula for $\frac{d}{dt}\vec{p}_G(t)$ & substitution $\alpha = \dot{\omega}$: **two-dimensional angular acceleration**

$$\begin{aligned}\frac{d^2}{dt^2}\vec{p}_G(t) =& \ddot{\vec{p}}_G(t) + 2\begin{bmatrix} 0 & \omega(t) \\ -\omega(t) & 0 \end{bmatrix} \dot{\vec{p}}_G(t) + \begin{bmatrix} 0 & \alpha(t) \\ -\alpha(t) & 0 \end{bmatrix} \vec{p}_G(t) \\ &+ \begin{bmatrix} 0 & \omega(t) \\ -\omega(t) & 0 \end{bmatrix} \begin{bmatrix} 0 & \omega(t) \\ -\omega(t) & 0 \end{bmatrix} \vec{p}_G(t)\end{aligned} \tag{19}$$

Three Additional Term Names

- **Coriolis acceleration**:

$$2\begin{bmatrix} 0 & \omega(t) \\ -\omega(t) & 0 \end{bmatrix} \dot{\vec{p}}_G(t) \tag{20}$$

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- Collectively: **fictitious accelerations** which produce **fictitious forces**

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1. **Axis-angle** representation:
   - Instantaneous axis of rotation, a.k.a. **Euler axis**: $\vec{e} = [\cos(e_x) \;\; \cos(e_y) \;\; \cos(e_z)]^T$, of unit length
     Rotation angle: θ about axis
     - 2D rotation angle: θ, Euler axis always normal to 2D plane, i.e. hypothetical "z-axis"
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     - 3D Euler axis arbitrary
   - Passive rotation vector between frames: **Euler vector**, $\vec{\theta} = \theta\vec{e}$, a.k.a. **passive rotation vector**, **orientation vector**, or **attitude vector**
     - **3D angular velocity**: $\vec{\omega}(t) = \dot{\vec{\theta}}(t) = \omega\vec{e}$
     - **3D angular acceleration**: $\vec{\alpha}(t) = \dot{\vec{\omega}}(t) = \ddot{\vec{\theta}}(t) = \alpha\vec{e}$
     - Magnitude of 3D angular velocity and acceleration consistent with 2D case

Direction Cosine Matrix

2 Extension from 2D rotation matrix: **direction cosine matrix (DCM)**, $C_{G\leftarrow F}$

$$\vec{v}_G = C_{G\leftarrow F}\vec{v}_F \tag{23}$$

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$$C_{G \leftarrow F} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \tag{24}$$

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- Each row corresponds to unit vector for x-, y-, and z-axis for G expressed in F frame coordinates
- Each column corresponds to unit vector for x-, y-, and z-axis for F expressed in G frame coordinates
- Each element, $C_{i,j}$, corresponds to cosine of angle between i-axis of G frame and j-axis of F frame

Basic Rotation Matrices and Euler Angles

3 3 **basic rotation matrices**

Basic Rotation Matrices and Euler Angles

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(1)

$$C_1(\theta_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{bmatrix} \tag{25}$$

(2)

$$C_2(\theta_y) = \begin{bmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{bmatrix} \tag{26}$$

Basic Rotation Matrices and Euler Angles (continued)

(3)

$$C_3(\theta_z) = \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{27}$$

- I.e. rotations about each primary coordinate axis

Basic Rotation Matrices and Euler Angles (continued)

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  - Three sequential angles for these basic rotation matrices: **Euler angles**

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- In FDC, typically Euler angles for $3-2-1$ rotation sequence:

$$\vec{v}_G = C_1(\theta_x)C_2(\theta_y)C_3(\theta_z)\vec{v}_F \tag{28}$$

  - G rotating w.r.t. F and θ_x, θ_y, θ_z: $3-2-1$ Euler angles
  - In FDC, occasionally Euler angles for $3-1-3$ rotation sequence, e.g., orbital elements and nutation analysis

Euler Symmetric Parameters

4 Mathematically efficient representation: **Euler symmetric parameters**

- A.k.a. **passive rotation quaternion**
- Can be defined several different ways
- 2 different conventions common for FDC: JPL and Hamilton

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  - 2 different conventions common for FDC: JPL and Hamilton
- Hamilton convention:

$$\vec{q} = \begin{bmatrix} q_w \\ q_x \\ q_y \\ q_z \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2)\cos(e_x) \\ \sin(\theta/2)\cos(e_y) \\ \sin(\theta/2)\cos(e_z) \end{bmatrix} \tag{29}$$

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- Note: $\vec{q}$ unit quaternion

$$\|\vec{q}\|_2 = \sqrt{q_w^2 + q_x^2 + q_y^2 + q_z^2} = 1 \tag{30}$$

Euler Symmetric Parameters to Axis-Angle Conversion

$$\theta = 2 \arccos \vec{q}_w \tag{31}$$

$$\vec{e} = \frac{\tilde{\vec{q}}}{\|\tilde{\vec{q}}\|_2} \tag{32}$$

- Where

$$\tilde{\vec{q}} = \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix} \tag{33}$$

Navigation-to-Wind 3-2-1 Euler Angles

- **Bank angle,** μ
- **Flight path angle,** γ
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- Vector from navigation frame coordinates, $\vec{v}_N$, $\to$ wind frame coordinates, $\vec{v}_W$:

$$\vec{v}_W = C_1(\mu)C_2(\gamma)C_3(\sigma)\vec{v}_N \tag{34}$$

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$$C_{W\leftarrow N} = \begin{bmatrix} \cos\gamma\cos\sigma & \cos\gamma\sin\sigma & -\sin\gamma \\ \sin\mu\sin\gamma\cos\sigma - \cos\mu\sin\sigma & \sin\mu\sin\gamma\sin\sigma + \cos\mu\cos\sigma & \sin\mu\cos\gamma \\ \cos\mu\sin\gamma\cos\sigma + \sin\mu\sin\sigma & \cos\mu\sin\gamma\sin\sigma - \sin\mu\cos\sigma & \cos\mu\cos\gamma \end{bmatrix} \tag{36}$$

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- σ arbitrarily set as subsequent σ rotations performed before other Euler angles, typically referenced to north

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- **Pitch angle,** θ
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$$C_{B\leftarrow N} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix} \tag{39}$$

Navigation-to-Body 3-2-1 Euler Angles

- **Roll angle**, ϕ
- **Pitch angle**, θ
- **Yaw angle**, ψ
- Vector from navigation frame coordinates, $\vec{v}_N$, $\rightarrow$ body frame coordinates, $\vec{v}_B$:

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- Note: Euler angles represent orientation or **attitude** of aircraft
- ψ arbitrarily set as subsequent ψ rotations performed before other angles

Wind-to-Body 3-2-1 Euler Angles

- **Angle of attack,** α
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$$\vec{v}_B = C_2(\alpha)C_3(-\beta)\vec{v}_W \tag{40}$$

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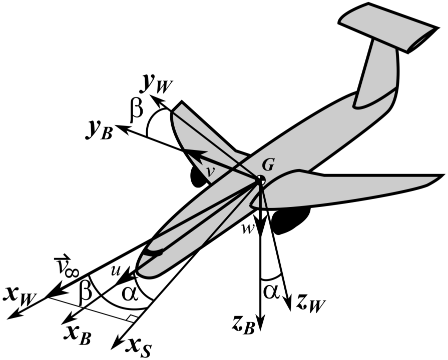
$$C_{B\leftarrow W} = \begin{bmatrix} \cos\alpha\cos\beta & -\cos\alpha\sin\beta & -\sin\alpha \\ \sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & -\sin\alpha\sin\beta & \cos\alpha \end{bmatrix} \tag{42}$$

Airspeed Vector Rotation

- Representation of airspeed vector $\vec{v}_\infty$ with magnitude v_∞ in body frame:

$$\vec{v}_{\infty,B} = [u\ v\ w]^T \tag{43}$$



Airspeed Vector Rotation (continued)

- From previous equations:

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\beta & -\cos\alpha\sin\beta & -\sin\alpha \\ \sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & -\sin\alpha\sin\beta & \cos\alpha \end{bmatrix} \begin{bmatrix} V_\infty \\ 0 \\ 0 \end{bmatrix} \tag{44}$$

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$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} V_\infty\cos\alpha\cos\beta \\ V_\infty\sin\beta \\ V_\infty\sin\alpha\cos\beta \end{bmatrix} \tag{45}$$

Airspeed Vector Rotation (continued)

- Dividing first row by third row:

$$\frac{u}{w} = \frac{V_\infty \cos\alpha \cos\beta}{V_\infty \sin\alpha \cos\beta} \tag{46}$$

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$$\frac{u}{w} = \frac{1}{\tan\alpha} \tag{48}$$

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- Relationship between angle of attack and components of airspeed vector:

$$\alpha = \tan^{-1} \frac{w}{u} \tag{49}$$

- Isolating second row:

$$v = V_\infty \sin \beta \tag{50}$$

- Provides relationship between sideslip angle and components of airspeed vector:

$$\beta = \sin^{-1} \frac{v}{V_\infty} \tag{51}$$

Point-Mass Gravity Modeling

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$$F_g = W = \frac{GMm}{r^2} = \frac{\mu m}{r^2} \tag{52}$$

  - G: gravitational constant, i.e. $6.674 \times 10^{-11}\ \mathrm{m^3 kg^{-1} s^{-2}}$
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- Celestial bodies' gravity field contains variation
  - Course: assumes gravitational field gradient does not produce any significant moments on aerospace vehicle
  - Often not neglected for spacecraft in low orbits

Gravity Modeling (continued)

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  - g: Earth's **gravitational acceleration**
  - g generally varies as function of latitude, altitude, and local topography and geology
- Module: **spherical-Earth gravity model** for g as function of altitude, h

$$g(h) = g_0 \left(\frac{R_E}{R_E + h} \right)^2 \tag{54}$$

  - R_E: **Earth's mean radius**, 6.3710088×10^6 m
  - g_0: **standard gravitational acceleration**, 9.80665 m/s^2
  - Higher fidelity modeling of gravitational force discussed in later modules

Propulsion Modeling

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- Rotating propellers/rotors create changing mass distribution due to rotation
  - Accommodated in rigid aerospace vehicle equations of motion in later modules

Aerodynamics

- Aerodynamic forces and moments due to air pressure distribution around aerospace vehicle
  - **Aerodynamic force** vector, $\vec{F}_a$, resist/assist translation through air mass
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- Determination of models for general aircraft studied as **aircraft system identification (SID)**
  - Typically employs **optimal parameter estimation** discussed in other courses

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3. **Deformable-body model** defines object by dynamic distribution of mass in space, experiences deformations caused by strain
   - Sub-type: **elastic-body model** assumes deformations caused by strain completely recoverable
   - Assumes that object's motion completely characterized by motion of nominal center of mass, rotation of nominal structure, *and* vibration of body about nominal structure
   - Vibrations: structural modes of object

Flight Dynamics Approximations

1. Rigid body model for aerospace vehicles
    - “Good enough” for majority of FDC analysis and design, structures designed so structural modes occur at much higher frequencies than aerodynamic or astrodynamic modes
    - Modern aircraft often incorporate flexible structures, necessitates elastic body model, i.e. aeroelastic dynamics
    - Later module: elastic aerospace vehicle model considered

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   - "Down" direction aligns with gravitational force, mg
   - Approximation typically "good enough" for FDC analysis and design
   - Although with hypersonic flying vehicle velocities and/or long distance flight analysis, flat-Earth approximation may not be suitable
   - Later module: additional effects of non-flat Earth modeling on flight dynamics

Flight Dynamics Approximations (continued)

3. **No-wind approximation** to introduce aircraft dynamics
   - Wind speeds significantly close to nominal aircraft airspeed and/or long distance flight analysis, no-wind approximation may not be suitable
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   - Rocket engines: mass rate crucial in deriving appropriate EOMs
   - Later module: variable and rotating mass effects

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- Vector-valued differential equation:

$$\sum \vec{F} = \frac{d}{dt}\vec{p} = \frac{d}{dt}(m\vec{v}) \tag{55}$$

  - $\sum \vec{F}$: total force on rigid body
  - $\vec{p}$: linear momentum
  - m: mass
  - $\vec{v}$: velocity of point-mass

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- First equation of Newton-Euler EOMs: **translation equation**, equivalent to point-mass equation
- Second equation: **rotation equation**

Integrated Mass Distribution

- Since rigid-body represented as continuous mass distribution: quantities computed as integrals over volume V by following equations:

$$\begin{aligned} \int_V \vec{f}\rho dV &= \frac{d}{dt}\int_V \vec{v}\rho dV \\ \int_V \vec{r}\times\vec{f}\rho dV &= \frac{d}{dt}\int_V \vec{r}\times\vec{v}\rho dV \end{aligned} \tag{57}$$

  - $\vec{f}$: forces acting on rigid body per unit mass
  - $dm = \rho dV$: infinitesimal mass element of body with density ρ
  - $\vec{v}$: velocity of mass element
  - $\vec{r}$: position vector of mass element w.r.t. origin of inertial reference frame

Inertia Matrix

- Alternative: represent static distribution of mass through **inertia matrix**, a.k.a. **moment of inertia tensor**
  - Composed of **moments of inertia** and **products of inertia** about body-fixed frame coordinates axes $x_B - y_B - z_B$

$$I_G = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}, \tag{58}$$

Body-Fixed Frame EOMs

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- Newton-Euler EOMs written in rotating reference frame:

$$\begin{aligned} \sum \vec{F}_B &= \frac{d}{dt}(m\vec{v}_B) = m\left(\dot{\vec{v}}_B + \omega_{B/N} \times \vec{v}_B\right) \\ \sum \vec{M}_B &= \frac{d}{dt}\left(I_G\omega_{B/N}\right) = I_G\dot{\vec{\omega}}_B + \omega_{B/N} \times I_G\omega_{B/N} \end{aligned} \tag{59}$$

  - $\sum \vec{F}_B$: total force acting on center of mass G in body-fixed frame
  - $\vec{v}_B$: linear velocity of center of mass
  - $\sum \vec{M}_B$: total moment acting about center of mass G in body-fixed frame
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- Additional cross product terms included due to body-fixed frame being non-inertial, i.e. coordinate axes rotating and origin accelerating

Body-Fixed Frame Representations

- For body-fixed frame linear and angular velocity components, typically use following representations

$$\vec{v}_B = \begin{bmatrix} u & v & w \end{bmatrix}^T \tag{60}$$

$$\omega_{B/N} = \begin{bmatrix} p & q & r \end{bmatrix}^T \tag{61}$$

Newton-Euler Equations of Motion

- For any rigid-body in rotating reference frame, Newton-Euler EOMs:

$$
\begin{aligned}
\sum \vec{F}_B &= m \begin{bmatrix} \dot{u} + qw - rv \\ \dot{v} + ru - pw \\ \dot{w} + pv - qu \end{bmatrix} \\
\sum \vec{M}_B &= \begin{bmatrix} I_{xx}\dot{p} + (I_{zz} - I_{yy})qr - I_{xy}(\dot{q} - pr) - I_{xz}(\dot{r} + pq) + I_{yz}(r^2 - q^2) \\ I_{yy}\dot{q} + (I_{xx} - I_{zz})pr - I_{xy}(\dot{p} + qr) + I_{xz}(p^2 - r^2) - I_{yz}(\dot{r} - pq) \\ I_{zz}\dot{r} + (I_{yy} - I_{xx})pq + I_{xy}(q^2 - p^2) - I_{xz}(\dot{p} - qr) - I_{yz}(\dot{q} + pr) \end{bmatrix}
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- Represent six coupled nonlinear ODEs with 6 free variables: u, v, w, p, q, r
  - Forms **six degree-of-freedom (6-DOF) equations of motion (EOM)**

Flight Vehicle Equations of Motion

- Assuming flat-earth model, write 6-DOF **rigid aerospace vehicle equations of motion** in body-fixed frame:

$$\begin{aligned}
\vec{F}_{a,B} + \vec{F}_{p,B} + C_{B\leftarrow N}\vec{F}_{g,N} &= m\begin{bmatrix} \dot{u} + qw - rv \\ \dot{v} + ru - pw \\ \dot{w} + pv - qu \end{bmatrix} \\
\vec{M}_{a,B} + \vec{M}_{p,B} &= \begin{bmatrix} I_{xx}\dot{p} + (I_{zz} - I_{yy})qr - I_{xy}(\dot{q} - pr) - I_{xz}(\dot{r} + pq) + I_{yz}(r^2 - q^2) \\ I_{yy}\dot{q} + (I_{xx} - I_{zz})pr - I_{xy}(\dot{p} + qr) + I_{xz}(p^2 - r^2) - I_{yz}(\dot{r} - pq) \\ I_{zz}\dot{r} + (I_{yy} - I_{xx})pq + I_{xy}(q^2 - p^2) - I_{xz}(\dot{p} - qr) - I_{yz}(\dot{q} + pr) \end{bmatrix}
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Rotation Matrix Representation

- $C_{B \leftarrow N}$ term depends on representation of attitude to be used, e.g. DCM, Euler angles, or Euler symmetric parameters,

  Changes as function of angular velocity

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  Changes as function of angular velocity
- Course: Euler angles

$$C_{B\leftarrow N}\vec{F}_{g,N} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} \tag{64}$$

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- $C_{B\leftarrow N}$ term depends on representation of attitude to be used, e.g. DCM, Euler angles, or Euler symmetric parameters,

  Changes as function of angular velocity
- Course: Euler angles

$$C_{B\leftarrow N}\vec{F}_{g,N} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} \tag{64}$$

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Aerospace Vehicle Equations of Motion (continued)

$$
\begin{aligned}
\vec{F}_{a,B} + \vec{F}_{p,B} + mg \begin{bmatrix} -\sin\theta \\ \sin\phi\cos\theta \\ \cos\phi\cos\theta \end{bmatrix} &= m \begin{bmatrix} \dot{u} + qw - rv \\ \dot{v} + ru - pw \\ \dot{w} + pv - qu \end{bmatrix} \\
\vec{M}_{a,B} + \vec{M}_{p,B} &= \begin{bmatrix} I_{xx}\dot{p} + (I_{zz} - I_{yy})qr - I_{xy}(\dot{q} - pr) - I_{xz}(\dot{r} + pq) + I_{yz}(r^2 - q^2) \\ I_{yy}\dot{q} + (I_{xx} - I_{zz})pr - I_{xy}(\dot{p} + qr) + I_{xz}(p^2 - r^2) - I_{yz}(\dot{r} - pq) \\ I_{zz}\dot{r} + (I_{yy} - I_{xx})pq + I_{xy}(q^2 - p^2) - I_{xz}(\dot{p} - qr) - I_{yz}(\dot{q} + pr) \end{bmatrix}
\end{aligned}
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\end{aligned}
\tag{66}
$$

- Only requires eight states to calculate derivatives at any instant: u, v, w, p, q, r, ϕ, θ
  - ψ: derived parameter

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$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{67}$$

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- Equations do not use rotation transformations: $[\dot{\phi}, \dot{\theta}, \dot{\psi}]^T$ not vector in 3D space

Position in Navigation Frame

- Let $[\dot{x}_N\ \dot{y}_N\ \dot{z}_N]^T$ represent velocity in navigation frame, quantities related to previous body-fixed frame by simple rotation:

$$\begin{bmatrix} \dot{x}_N \\ \dot{y}_N \\ \dot{z}_N \end{bmatrix} = C_{N\leftarrow B} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{69}$$

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  - Celestial “inertial” frame
  - Earth-centered frames
  - Reference ellipsoid
  - Local vehicle-centered frames

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- Rigid body equations written in body-fixed frame so that inertia tensor remains constant
  - Requires body-fixed frame Euler angles for transformation due to gravity force