Lecture 21: Introduction to Energy-, Time-, and Fuel-Optimal Control

With Spacecraft Attitude-Maneuver Example
Textbook Sections 4.2 & 12.5

Dr. Jordan D. Larson

Introduction: Classical OCP

$$
\begin{aligned}
\vec{u}_{opt}(t) = \underset{u(t)\ \forall t \in [t_0,\ t_f]}{\text{argmin}} \quad & \mathcal{J} = \mathcal{E}\left(\vec{x}(t_f), t_f\right) + \int_{t_0}^{t_f} \mathcal{L}(\vec{x}(t), \vec{u}(t), t)dt \\
\underline{\text{subject to:}} \quad & \\
\text{continuous-time dynamics:} \quad & \dot{\vec{x}}(t) = f(\vec{x}(t), \vec{u}(t), t) \\
\text{initial conditions:} \quad & \vec{x}(t_0) - \vec{x}_0 = 0 \\
\text{control constraints:} \quad & |\vec{u}(t)| < u_{max}
\end{aligned} \tag{1}
$$

- t_0 fixed and t_f free
- $\mathcal{E}$: endpoint cost
- $\mathcal{L}$: running cost or Lagrangian
- $\mathcal{H} = \mathcal{L} + \vec{\lambda}^T f(\vec{x}(t), \vec{u}(t), t)$: Hamiltonian

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- Lecture: sub-types of OCPs applicable to flight vehicles for guidance and control
  - I.e. energy, fuel, time optimal control problems

Minimum-Energy OCP

- **Minimum-energy OCP**:

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  - Integral term: $\mathcal{L}_2$-norm of control input signal

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  - Integral term: $\mathcal{L}_2$-norm of control input signal
- Special case of LQR OCP with $Q = 0$ and $R = I$
  - $\vec{x}_f$: origin
  - Quadratic terminal cost: $\mathcal{E}\left(\vec{x}(t_f), t_f\right) = \vec{x}^T(t_f)E\vec{x}(t_f)$
  - Linear dynamics: $f(\vec{x}(t), \vec{u}(t), t) = A(t)\vec{x} + B(t)\vec{u}$

Minimum-Energy LQR OCP

- Assume (A, B) controllable $\forall t \in [0,\ t_f]$: solution by differential Riccati equation

$$\dot{P}(t) = -P(t)A(t) - A^T(t)P(t) + P(t)B(t)B^T(t)P(t) \tag{3}$$

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$$P(t_f) = E \tag{4}$$

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- Optimal control:

$$u_{opt} = -B^T(t)P(t)\vec{x}(t) \tag{5}$$

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- Closed-loop state-space dynamics:

$$\dot{\vec{x}}(t) = \left(A(t) - B(t)B^T(t)P(t)\right)\vec{x}(t) \tag{6}$$

Notes on Minimum-Energy LQR OCP

- If require system to converge to some fixed $x(t_f) = x_f$

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- If require system to converge to some fixed $x(t_f) = x_f$
- Define

$$E = diag(\vec{E}) \tag{7}$$

- Take limit of optimal control as cost weight goes to ∞:

$$\lim_{\vec{E}\to\infty} u_{opt}(E, t) \tag{8}$$

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- If $\vec{u}$ constrained, e.g. $|\vec{u}_i| < u_{c,i} \ \forall \ i = 1, ..., n_u$
  - Constrained optimization problem solved using Pontryagin's principle for admissible control set

Minimum-Time OCP

- **Minimum-time OCP**:

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\text{final conditions:} \quad & \vec{x}(t_f) - \vec{x}_c = 0 \\
\text{control constraints:} \quad & \vec{u}(t) \in \mathcal{U}
\end{aligned}
\tag{9}
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- Hamiltonian:

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\mathcal{H}(\vec{x}(t), \vec{u}(t), \vec{\lambda}(t), t) = 1 + \vec{\lambda}^T f(\vec{x}(t), \vec{u}(t), t) \tag{10}
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- Generally requires use of Pontryagin's principle to solve as well as methods for determining admissible controls, $\vec{u}(t) \in \mathcal{U}$

Introductory Minimum-Time Problem

- Consider linear dynamics, i.e. $f(\vec{x}(t), \vec{u}(t), t) = A(t)\vec{x} + B(t)\vec{u}$:

$$\mathcal{H}(\vec{x}(t), \vec{u}(t), \vec{\lambda}(t), t) = \vec{\lambda}^T A\vec{x} + \vec{\lambda}^T B\vec{u} \tag{11}$$

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- n_u-dimensional hypercube for independent control actuators as admissible control set:

$$\mathcal{U} = \{\vec{u} \in \mathbb{R}^{n_u} \ : \ u_i \in [u_{i,min}, u_{i,max}], \ i = 1, ..., n_u\} \tag{12}$$

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- Pontryagin's principle:

$$\mathcal{H}(\vec{x}_{opt}(t), \vec{u}_{opt}(t), \vec{\lambda}_{opt}(t)) \leq \mathcal{H}(\vec{x}(t), \vec{u}(t), \vec{\lambda}(t)) \quad \forall \ t \in [0, t_f], \quad \vec{u} \in \mathcal{U} \tag{13}$$

Introductory Minimum-Time Problem (continued)

- Simplified:

$$\vec{\lambda}_{opt}^{T} B \vec{u}_{opt} = \min_{\vec{u} \in \mathcal{U}} \vec{\lambda}^{T} B \vec{u} \quad \forall\, t \in [0, t_f] \tag{14}$$

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$$\sum_{i=1}^{n_u} \vec{\lambda}_{opt}^T b_i u_{i,opt} = \min_{u_{i,min} \leq |u_i| \leq u_{i,max}} \sum_{i=1}^{n_u} \vec{\lambda}^T \vec{b}_i u_i \quad \forall\, t \in [0, t_f] \tag{15}$$

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- Optimal control cases:

$$u_{i,opt}(t) = \begin{cases} u_{i,max} & \text{if } \vec{\lambda}_{opt}^T \vec{b}_i > 0 \\ ? & \text{if } \vec{\lambda}_{opt}^T \vec{b}_i = 0 \\ u_{i,min} & \text{if } \vec{\lambda}_{opt}^T \vec{b}_i < 0 \end{cases} \tag{16}$$

Introductory Minimum-Time Problem (continued)

- To solve for ? element, recall costate equation:

$$\dot{\vec{\lambda}}^T_{opt} = -\frac{\partial \mathcal{H}}{\partial \vec{x}} = -A^T \vec{\lambda}_{opt} \tag{17}$$

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$$\vec{\lambda}_{opt}(t) = \exp\left(A^T (t_{opt} - t)\right) \vec{\lambda}_{opt}(t_{opt}) \tag{18}$$

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- $\vec{\lambda}^T_{opt} \vec{b}_i = \vec{\lambda}^T_{opt}(t_{opt}) \exp\left(A^T(t_{opt} - t)\right) \vec{b}_i$ only $= 0$ over some time *interval* if zero for all t and its derivatives:

$$\vec{\lambda}^T_{opt}(t_{opt}) \vec{b}_i = \vec{\lambda}^T_{opt}(t_{opt}) A \vec{b}_i = \cdots = \vec{\lambda}^T_{opt}(t_{opt}) A^{n_x - 1} \vec{b}_i \tag{19}$$

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- While $\vec{\lambda}_{opt}(t_{opt})$ cannot be 0 for $\vec{x}(t_0) \neq \vec{x}_0$, require each pair $(A, \vec{b}_i)$ to be controllable, a.k.a. **normal LTI system**

Bang-Bang Property for LTI Systems

- Conclusion: **bang-bang property**
  - $\vec{u}_{opt}$ only takes values at vertices of hypercube $\mathcal{U}$
  - Finite number of discontinuities, i.e. *switches*, between these values
  - Unique

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- Assumption of normality for bang-bang property of minimum time optimal control for hypercube can be relaxed to modified bang-bang principle for linear systems
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- Assumption of normality for bang-bang property of minimum time optimal control for hypercube can be relaxed to modified bang-bang principle for linear systems
  - Not *every* minimum-time optimal control is bang-bang, but one solution is bang-bang if $\mathcal{U}$ convex polyhedron
- Note: can demonstrate bang-bang property for normal control-affine systems

Bang-Bang Property for LTI Systems (continued)

- Bang-bang property for LTI systems established without Pontrayagin's principle by recalling general solution to LTI system:

$$\vec{x}(t) = \exp\left(A(t - t)\right) \vec{x}_0 + \int_{t_0}^{t_f} \exp\left(A(t - \tau)\right) B\vec{u}(\tau) d\tau \tag{20}$$

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- For $t \geq t_0$, *reachable set* from $\vec{x}(t_0) = \vec{x}_0$ at time t:

$$\mathcal{R}^t(\vec{x}_0) = \left\{\exp\left(A(t-t)\right)\vec{x}_0 + \textstyle\int_{t_0}^{t_f} \exp\left(A(t-\tau)\right)B\vec{u}(\tau)d\tau \ : \ \vec{u} \in \mathcal{U},\ t_0 \leq \tau \leq t\right\} \tag{21}$$

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$$t_{opt} = \operatorname*{argmin}_{t} \vec{x}_{opt} \in \mathcal{R}^t(\vec{x}_0) \tag{22}$$

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- $\vec{x}_{opt}$ must occur on boundary of $\mathcal{R}^{t_{opt}}(\vec{x}_0)$
  - If in interior, could reach sooner

Bang-Bang Property for LTI Systems (continued)

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- $\vec{x}_{opt}$ must occur on boundary of $\mathcal{R}^{t_{opt}}(\vec{x}_0)$
  - If in interior, could reach sooner
- Note: $\mathcal{R}^{t_{opt}}(\vec{x}_0)$ compact and convex $\rightarrow$ exists hyperplane that passes through $\vec{x}_{opt}$ and contains $\mathcal{R}^T(\vec{x}_0)$ on one side

Bang-Bang Property for LTI Systems (continued)

- Chosing normal vector to hyperplane as $\vec{\lambda}_{opt}(t_{opt})$:

$$\vec{\lambda}^{T}_{opt}(t_{opt})\vec{x}_{opt} \geq \vec{\lambda}^{T}_{opt}(t_{opt})\vec{x} \quad \forall\ \vec{x} \in \mathcal{R}^{t_{opt}}(\vec{x}_0) \tag{23}$$

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- By definition

$$\int_{t_0}^{t_{opt}} \vec{\lambda}^{T}_{opt}(t_{opt}) \exp\left(A(t-\tau)\right) B\vec{u}_{opt}(\tau)d\tau \geq \int_{t_0}^{t_{opt}} \vec{\lambda}^{T}_{opt}(t_{opt}) \exp\left(A(t-\tau)\right) B\vec{u}(\tau)d\tau \quad \forall \vec{u} \in \mathcal{U} \text{ from } [t_0, t_{opt}] \tag{24}$$

Bang-Bang Property for LTI Systems (continued)

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- By definition

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- Note: $\vec{\lambda}(\tau) = \exp\left(A^T(t_{opt}-\tau)\right)\vec{\lambda}_{opt}(t_{opt})$ &

$$\int_{t_0}^{t_{opt}} \vec{\lambda}^T_{opt}(\tau)B\vec{u}_{opt}(\tau)d\tau \geq \int_{t_0}^{t_{opt}} \vec{\lambda}^T_{opt}(\tau)B\vec{u}(\tau)d\tau \quad \forall \vec{u} \in \mathcal{U} \text{ from } [t_0, t_{opt}] \tag{25}$$

  - Can show optimal control cases as previously

Minimum-Fuel OCP

$$
\begin{aligned}
\vec{u}_{opt}(t) = \underset{u(t)\ \forall t\in[0,\ t_f]}{\text{argmin}} \quad & \mathcal{J} = \int_0^{t_f} \sum_{i=1}^{n_u} |u_i(t)| dt \\
\underline{\text{subject to:}} \quad & \dot{\vec{x}}(t) = f(\vec{x}(t), \vec{u}(t), t) \\
\text{initial conditions:} \quad & \vec{x}(t_0) - \vec{x}_0 = 0 \\
\text{final conditions:} \quad & \vec{x}(t_f) - \vec{x}_c = 0 \\
\text{control constraints:} \quad & \vec{u}(t) \in \mathcal{U}
\end{aligned} \tag{26}
$$

Minimum-Fuel OCP

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$$\mathcal{H}(\vec{x}(t), \vec{u}(t), \vec{\lambda}(t), t) = \sum_{i=1}^{n_u} |u_i(t)| + \vec{\lambda}^T f(\vec{x}(t), \vec{u}(t), t) \tag{27}$$

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- Integral term: $\mathcal{L}_1$-norm of control input
  - Minimum-fuel optimal control a.k.a. $\mathcal{L}_1$-optimal control
- Generally requires use of Pontryagin's principle to solve as well as methods for determining admissible controls, $\vec{u}(t) \in \mathcal{U}$

Introductory Minimum-Fuel Problem

- Consider linear dynamics, $f(\vec{x}(t), \vec{u}(t), t) = A(t)\vec{x} + B(t)\vec{u}$:

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- n_u-dimensional hypercube for independent control actuators as admissible control set:

$$\begin{aligned} \mathcal{U} = \{\vec{u} \in \mathcal{R}^{n_u} \ : \ u_i \in [u_{i,min}, u_{i,max}],\ i = 1, ..., n_u\} \\ u_{i,min} < 0 \\ u_{i,min} > 0 \end{aligned} \tag{29}$$

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- Pontryagin's principle:

$$\mathcal{H}(\vec{x}_{opt}(t), \vec{u}_{opt}(t), \vec{\lambda}_{opt}(t)) \leq \mathcal{H}(\vec{x}(t), \vec{u}(t), \vec{\lambda}(t)) \quad \forall\ t \in [0, t_f], \quad \vec{u} \in \mathcal{U} \tag{30}$$

Introductory Minimum-Fuel Problem (continued)

- Simplified:

$$\sum_{i=1}^{n_u} |u_{i,opt}| + \vec{\lambda}_{opt}^T B \vec{u}_{opt} = \min_{\vec{u} \in \mathcal{U}} \sum_{i=1}^{n_u} |u_i| + \vec{\lambda}^T B \vec{u} \quad \forall \ t \in [0, t_f] \tag{31}$$

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- Optimal control cases:

$$u_{i,opt}(t) = \begin{cases} u_{i,max} & \text{if } \vec{\lambda}_{opt}^T \vec{b}_i \leq -1 \\ 0 & \text{if } -1 \leq \vec{\lambda}_{opt}^T \vec{b}_i \leq 1 \\ u_{i,min} & \text{if } \vec{\lambda}_{opt}^T \vec{b}_i \geq 1 \end{cases} \tag{33}$$

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- Similar nature of minimum-time and minimum-fuel: two conditions linearly combinable into minimum-time-fuel OCP

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- For vehicles, Euclidean shortest-path problem a.k.a. **Dubins path problem** solved in 2D and 3D using Pontryagin's principle with constraints on control

2D Dubins Path Problem

- 2D **Dubins vehicle model**:

$$\dot{\vec{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \end{bmatrix} = f(\vec{x}, \vec{u}) = \begin{bmatrix} V\cos\psi \\ V\sin\psi \\ u \end{bmatrix} \tag{34}$$

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  - (x, y): position of vehicle
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- Objective for Dubins path starting at $t = 0$ at some state (x_0, y_0, ψ_0): achieve some other state, (x_f, y_f, ψ_f), at time $t = t_f$ in shortest amount of time

2D Dubins Path OCP

$$\vec{u}^{opt}(t) = \underset{u(t)\ \forall t \in [0,\ t_f]}{\text{argmin}} \quad J = \int_0^{t_f} dt = t_f$$

$$\begin{aligned}
&\underline{\text{subject to:}} \\
&\text{dynamics} \quad \begin{bmatrix} \dot{\vec{x}} \\ \dot{t} \end{bmatrix} = \begin{bmatrix} f(\vec{x}, \vec{u}) \\ 1 \end{bmatrix} \\
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- p, q, e constant on $[0, t_f]$
- Define $r = \sqrt{p^2 + q^2} > 0$; $p = r\cos\phi$ & $q = r\sin\phi$

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- Optimal path: concatenation of arcs of circles of radius R and line segments all parallel to some fixed direction ϕ

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- Consider spinning satellite with some initial $\vec{H}_{G,0}$ about spin axis, z_B
  - Thruster pair impulsively fires creating some $\Delta H_{G,1}$ normal to spin axis
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- After precessing π radians, thruster pair impulsively fires creating some $\Delta H_{G,2}$ in same direction relative to satellite as $\Delta H_{G,1}$
  - $\|\Delta\vec{H}_{G,2}\|_2 = \|\Delta\vec{H}_{G,1}\|_2$
  - Stabilizing spin vector in commanded reorientation, θ

Coning Maneuvers (continued)

- Total ΔH for single maneuver:

$$\Delta H_{tot} = \|\Delta \vec{H}_{G,1}\|_2 + \|\Delta \vec{H}_{G,2}\|_2 = 2\|\vec{H}_{G,0}\|_2 \tan\left(\frac{\theta_c}{2}\right) \tag{43}$$

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- For N small coning maneuvers:

$$\Delta H_{tot} = 2N\|\vec{H}_{G,0}\|_2 \tan\left(\frac{\theta_c}{2N}\right) \tag{45}$$

Coning Maneuvers (continued)

- For large N by small angle approximation:

$$\Delta H_{tot} \approx 2N \|\vec{H}_{G,0}\|_2 \left(\frac{\theta_c}{2N}\right) \approx \|\vec{H}_{G,0}\|_2 \theta_c \tag{46}$$

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- Ratio form by small angle approximation:

$$\frac{t_N}{t_1} = N\frac{\cos\left(\frac{\theta_c}{2N}\right)}{\cos\left(\frac{\theta_c}{2}\right)} \approx \frac{N}{\cos\left(\frac{\theta_c}{2}\right)} \tag{48}$$

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