Lecture 18: MIMO LTI Feedback Control Systems

Textbook Sections 3.1 & 3.2

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Introduction

- MIMO LTI control systems: may use combination of
  - Feedforward terms, i.e. control gains dependent on $\vec{r}$
  - Tracking error terms, i.e. control gains dependent on $\vec{e}$
  - Output terms, i.e. control gains dependent on $\vec{y}$
  - Other combinations, e.g. integrated error terms
  - Coupling between all of these

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  - Tracking error terms, i.e. control gains dependent on $\vec{e}$
  - Output terms, i.e. control gains dependent on $\vec{y}$
  - Other combinations, e.g. integrated error terms
  - Coupling between all of these
- For studying MIMO LTI systems, useful to define generalized framework
  - Generalize all possible interconnections of feedback systems: utilize linear fractional transformations (LFTs)
  - System properties including stability, controllability, observability, and robustness

Linear Fractional Transformations

- **Linear fractional transformations (LFT)** models feedback connection of two matrices
  - Can be used to represent MIMO LTI systems via transfer function matrix

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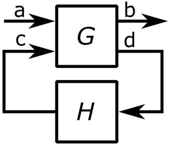
- **Lower LFT**: $F_L(G, H)$



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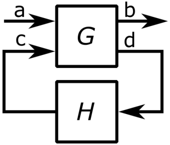
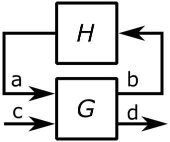


- **Upper LFT**: $F_U(G, H)$



Well-Posed Lower LFT Solution

- Lower LFT:

$$\begin{bmatrix} \vec{b} \\ \vec{d} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} \vec{a} \\ \vec{c} \end{bmatrix} \qquad (1)$$

$$\vec{c} = H\vec{d}$$

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- Lower LFT:

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$$\vec{c} = H\vec{d} \tag{1}$$

- Substituting third equation into second equation:

$$\vec{d} = G_{21}\vec{a} + G_{22}H\vec{d} \tag{2}$$

Well-Posed Lower LFT Solution

- If $I_{n_d} - G_{22}H$ non-singular, i.e. invertible, then lower LFT well-posed and equation solved as:

$$\vec{d} = \left(I_{n_d} - G_{22}H\right)^{-1} G_{21}\,\vec{a} \tag{3}$$

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- Substituted into first equation with $\vec{c} = H\vec{d}$:

$$\begin{aligned} \vec{b} &= \left(G_{11} + G_{12}H\left(I_{n_d} - G_{22}H\right)^{-1} G_{21}\right)\vec{a} \\ \vec{b} &= F_L(G, H)\,\vec{a} \end{aligned} \tag{4}$$

Well-Posed Upper LFT Solution

- Upper LFT:

$$\begin{bmatrix} \vec{b} \\ \vec{d} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} \vec{a} \\ \vec{c} \end{bmatrix} \tag{5}$$

$$\vec{a} = H\vec{b}$$

Well-Posed Upper LFT Solution

- Upper LFT:

$$\begin{bmatrix} \vec{b} \\ \vec{d} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} \vec{a} \\ \vec{c} \end{bmatrix} \qquad (5)$$

$$\vec{a} = H\vec{b}$$

- Substituting third equation into first equation:

$$\vec{b} = G_{11} H \vec{b} + G_{12} \vec{c} \qquad (6)$$

Well-Posed Upper LFT Solution

- If $I_{n_b} - G_{11}H$ non-singular, i.e. invertible, then upper LFT well-posed and equation solved as

$$\vec{b} = \left(I_{n_b} - G_{11}H\right)^{-1} G_{12}\vec{c} \tag{7}$$

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- Substituted into second equation with $\vec{a} = H\vec{b}$:

$$\begin{aligned} \vec{b} &= \left(G_{22} + G_{21}H\left(I_{n_b} - G_{11}H\right)^{-1} G_{12}\right)\vec{a} \\ \vec{d} &= F_U(G, H)\vec{c} \end{aligned} \tag{8}$$

Serial/Parallel Interconnection Example: 2 Upper LFTs

$$\begin{bmatrix} \vec{b}_1 \\ \vec{d}_1 \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} \vec{a}_1 \\ \vec{c}_1 \end{bmatrix} \tag{9}$$

$$\vec{a}_1 = H_1 \vec{b}_1$$

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$$\begin{bmatrix} \vec{b}_2 \\ \vec{d}_2 \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} \vec{a}_2 \\ \vec{c}_2 \end{bmatrix} \qquad (10)$$

$$\vec{a}_2 = H_2 \vec{b}_2$$

Serial/Parallel Interconnection Example: 2 Upper LFTs

$$\begin{bmatrix} \vec{b}_1 \\ \vec{d}_1 \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} \vec{a}_1 \\ \vec{c}_1 \end{bmatrix} \qquad (9)$$
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- Connected in series from $F_U(L, H_1)$ to $F_U(N, H_2)$:

$$\vec{c}_2 = \vec{d}_1 \qquad (11)$$

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- Connected in series from $F_U(L, H_1)$ to $F_U(N, H_2)$:

$$\vec{c}_2 = \vec{d}_1 \tag{11}$$

- Connected in parallel:

$$\vec{c} = \vec{c}_1 = \vec{c}_2 \quad \& \quad \vec{d} = \vec{d}_1 + \vec{d}_2 \tag{12}$$

Serial/Parallel Interconnection Example (continued)

- Series: upper LFT

$$\begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{d}_2 \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & L_{12} \\ N_{12}L_{21} & N_{11} & N_{21}L_{22} \\ N_{22}L_{21} & N_{21} & N_{22}L_{22} \end{bmatrix} \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{c}_1 \end{bmatrix}$$
$$\begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \end{bmatrix} = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \end{bmatrix} \tag{13}$$

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$$\begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \end{bmatrix} = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \end{bmatrix}$$

- Parallel: upper LFT

$$\begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{d} \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & L_{12} \\ 0 & N_{11} & N_{12} \\ L_{21} & N_{21} & L_{22} + N_{22} \end{bmatrix} \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{c} \end{bmatrix} \qquad (14)$$

$$\begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \end{bmatrix} = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \end{bmatrix}$$

Serial/Parallel Interconnection Example (continued)

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$$\begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \end{bmatrix} = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \end{bmatrix}$$

- For both LFTs, H_1 and H_2 matrices form “structured” upper LFT, i.e. zero matrices exist on off-diagonal block elements

Negative Feedback Interconnection

- Example of feedback interconnection, consider upper LFT:

$$\begin{bmatrix} \vec{b} \\ \vec{d} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} \vec{a} \\ \vec{c} \end{bmatrix}$$
$$\vec{a} = H\vec{b} \tag{15}$$

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- Negative feedback interconnection for $\vec{c}$:

$$\vec{c} = \vec{r} - \vec{d} \tag{16}$$

  - $\vec{r}$: some reference command input to feedback loop

Negative Feedback Interconnection (continued)

- From second equation, $\vec{d} = G_{21}\,\vec{a} + G_{22}\,\vec{c}$
  Assuming $I_{n_d} + G_{22}$ invertible, by substituting for $\vec{d}$:

$$\vec{c} = \left(I_{n_d} + G_{22}\right)^{-1}\left(\vec{r} - G_{21}\,\vec{a}\right) \tag{17}$$

Negative Feedback Interconnection (continued)

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- Substituted for $\vec{c}$ to obtain upper LFT:

$$\begin{bmatrix} \vec{b} \\ \vec{d} \end{bmatrix} = \begin{bmatrix} G_{11} - G_{12}\left(I_{n_d} + G_{22}\right)^{-1} G_{21} & G_{12}\left(I_{n_d} + G_{22}\right)^{-1} \\ \left(I_{n_d} + G_{22}\right)^{-1} G_{21} & G_{22}\left(I_{n_d} + G_{22}\right)^{-1} \end{bmatrix} \begin{bmatrix} \vec{a} \\ \vec{r} \end{bmatrix} \tag{18}$$

$$\vec{a} = H\,\vec{b}$$

- Well-posed if $I_{n_b} - \left(G_{11} - G_{12}\left(I_{n_d} + G_{22}\right)^{-1} G_{21}\right) H$ invertible

Lower LFT Inverse

- Consider lower LFT:

$$\begin{bmatrix} \vec{b} \\ \vec{d} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} \vec{a} \\ \vec{c} \end{bmatrix}$$
$$\vec{c} = H\vec{d} \tag{19}$$

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$$\vec{c} = H\vec{d}$$

LFT inverse obtained if G_{11} invertible via rewritten first equation:

$$\vec{a} = G_{11}^{-1}\vec{b} - G_{11}^{-1}G_{12}\vec{c} \tag{20}$$

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$$\vec{a} = G_{11}^{-1}\vec{b} - G_{11}^{-1}G_{12}\vec{c} \qquad (20)$$

- Substituted into second equation:

$$\begin{bmatrix} \vec{a} \\ \vec{d} \end{bmatrix} = \begin{bmatrix} G_{11}^{-1} & -G_{11}^{-1}G_{12} \\ G_{21}G_{11}^{-1} & G_{22} - G_{21}G_{11}^{-1}G_{12} \end{bmatrix} \begin{bmatrix} \vec{b} \\ \vec{c} \end{bmatrix} \qquad (21)$$

$$\vec{c} = H\vec{d}$$

  - Well-posed if $I_{n_d} - \left(G_{22} - G_{21}G_{11}^{-1}G_{12}\right)H$ invertible

Upper LFT Inverse

- Consider upper LFT

$$\begin{aligned} \begin{bmatrix} \vec{b} \\ \vec{d} \end{bmatrix} &= \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} \vec{a} \\ \vec{c} \end{bmatrix} \\ \vec{a} &= H\vec{b} \end{aligned} \tag{22}$$

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$$\vec{a} = H\vec{b}$$

- **LFT inverse** obtained if G_{22} invertible via rewritten second equation:

$$\vec{c} = -G_{22}^{-1} G_{21}\,\vec{a} + G_{22}^{-1}\,\vec{d} \tag{23}$$

Upper LFT Inverse

- Consider upper LFT

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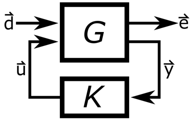
- Substituted into first equation:

$$\begin{bmatrix} \vec{b} \\ \vec{c} \end{bmatrix} = \begin{bmatrix} G_{11} - G_{12}G_{22}^{-1}G_{21} & G_{12}G_{22}^{-1} \\ -G_{22}^{-1}G_{21} & G_{22}^{-1} \end{bmatrix} \begin{bmatrix} \vec{a} \\ \vec{d} \end{bmatrix} \qquad (24)$$

$$\vec{a} = H\vec{b}$$

  - Well-posed if $I_{n_b} - \left(G_{11} - G_{12}G_{22}^{-1}G_{21}\right) H$ invertible

Generalized LTI Feedback Control System



- Linear fractional transformation (LFT): $F_L(G, K)$
- $\vec{d} \in \mathbb{R}^{n_d}$: **generalized disturbance**
  - Includes: reference commands, $\vec{r}$, process noise, $\vec{w}$, measurement noise, $\vec{v}$
- $\vec{e} \in \mathbb{R}^{n_e}$: **generalized error**
  - Includes: at least weighted tracking error and weighted control effort
- $\vec{u} \in \mathbb{R}^{n_u}$: **generalized control input**
- $\vec{y} \in \mathbb{R}^{n_y}$: **generalized output**

Generalized LTI Feedback Control System (continued)

- G: **generalized plant**:

$$\begin{aligned}\dot{\vec{x}}(t) &= A\vec{x}(t) + \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} \vec{d}(t) \\ \vec{u}(t) \end{bmatrix} \\ \begin{bmatrix} \vec{e}(t) \\ \vec{y}(t) \end{bmatrix} &= \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} \vec{d}(t) \\ \vec{u}(t) \end{bmatrix}\end{aligned} \tag{25}$$

  - Includes: actuators, sensors, original plant dynamics, weighting filters, any routing and operations on disturbance, tracking error, state, inputs, output signals
  - Note: $\vec{y}$ may be tracking error instead of “true” plant state or output, may contain reference command and plant output

Generalized LTI Feedback Control System (continued)

- G: **generalized plant**:

$$
\begin{aligned}
\dot{\vec{x}}(t) &= A\vec{x}(t) + \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} \vec{d}(t) \\ \vec{u}(t) \end{bmatrix} \\
\begin{bmatrix} \vec{e}(t) \\ \vec{y}(t) \end{bmatrix} &= \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} \vec{d}(t) \\ \vec{u}(t) \end{bmatrix}
\end{aligned}
\tag{25}
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  - Note: $\vec{y}$ may be tracking error instead of "true" plant state or output, may contain reference command and plant output
- K: **generalized LTI feedback controller**

$$
\begin{aligned}
\dot{\vec{x}}_K &= A_K \vec{x}_K + B_K \vec{y} \\
\vec{u} &= C_K \vec{x}_K + D_K \vec{y}
\end{aligned}
\tag{26}
$$

Generalized LTI Feedback Control System (continued)

- For input and output vectors:

$$\begin{bmatrix} I & -D_K \\ D_{22} & I \end{bmatrix} \begin{bmatrix} \vec{u}(t) \\ \vec{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} \begin{bmatrix} \vec{x}(t) \\ \vec{x}_K(t) \end{bmatrix} + \begin{bmatrix} 0 \\ D_{21} \end{bmatrix} \begin{bmatrix} \vec{d}(t) \end{bmatrix} \tag{27}$$

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- Interconnection of G and K well-posed if and only if $(I - D_{22}D_K)^{-1}$ exists
  - i.e. $\vec{u}$, $\vec{y}$ can be substituted into state equation

$$\begin{bmatrix} \dot{\vec{x}}(t) \\ \dot{\vec{x}}_K(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} \begin{bmatrix} \vec{x}(t) \\ \vec{x}_K(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \vec{d}(t) + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} \vec{u}(t) \\ \vec{y}(t) \end{bmatrix} \tag{28}$$

  -][]¡2-¿ & unique solution can be found

Generalized LTI Feedback Control Loop-Shifting

- In control synthesis, D_{22} set to 0 without loss of generality as one can alternatively use **loop-shifting** to form $F_L(\tilde{G}, \tilde{K})$ from $\tilde{G}$ and $\tilde{K}$ and undo loop-shifting to get original K

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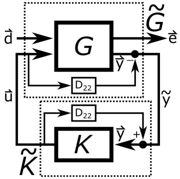
- Visualized as following alteration to $F_L(G, K)$:



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- In control synthesis, D_{22} set to 0 without loss of generality as one can alternatively use **loop-shifting** to form $F_L(\tilde{G}, \tilde{K})$ from $\tilde{G}$ and $\tilde{K}$ and undo loop-shifting to get original K
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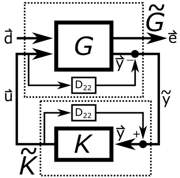

- Typically assume generalized LTI plant, G, specified as

$$\begin{aligned}\dot{\vec{x}}(t) &= A\vec{x}(t) + \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} \vec{d}(t) \\ \vec{u}(t) \end{bmatrix} \\ \begin{bmatrix} \vec{e}(t) \\ \vec{y}(t) \end{bmatrix} &= \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & 0 \end{bmatrix} \begin{bmatrix} \vec{d}(t) \\ \vec{u}(t) \end{bmatrix}\end{aligned} \tag{29}$$

Generalized LTI Feedback Closed-Loop Model

- State equation for closed-loop system rewritten as

$$\begin{bmatrix} \dot{\vec{x}}(t) \\ \dot{\vec{x}}_K(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} \begin{bmatrix} \vec{x}(t) \\ \vec{x}_K(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \vec{d}(t) + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} \vec{u}(t) \\ \vec{y}(t) \end{bmatrix} \quad (30)$$

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- Following relationship for $\vec{u}(t)$ and $\vec{y}(t)$

$$\begin{bmatrix} I & -D_K \\ 0 & I \end{bmatrix} \begin{bmatrix} \vec{u}(t) \\ \vec{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} \begin{bmatrix} \vec{x}(t) \\ \vec{x}_K(t) \end{bmatrix} + \begin{bmatrix} 0 \\ D_{21} \end{bmatrix} \begin{bmatrix} \vec{d}(t) \end{bmatrix} \tag{31}$$

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Generalized LTI Feedback Closed-Loop Model (continued)

- By substitution, obtain closed-loop LTI state-space system model:

$$\begin{aligned}\begin{bmatrix}\dot{\vec{x}}(t)\\ \dot{\vec{x}}_K(t)\end{bmatrix} &= \begin{bmatrix}A+B_2D_KC_2 & B_2C_K\\ B_KC_2 & A_K\end{bmatrix}\begin{bmatrix}\vec{x}(t)\\ \vec{x}_K(t)\end{bmatrix}+\begin{bmatrix}B_1+B_2D_KD_{21}\\ B_KD_{21}\end{bmatrix}\vec{d}(t)\\ \vec{e}(t) &= \begin{bmatrix}C_1+D_{12}D_KC_2 & D_{12}C_K\end{bmatrix}\begin{bmatrix}\vec{x}(t)\\ \vec{x}_K(t)\end{bmatrix}+\left(D_{11}+D_{12}D_KD_{21}\right)\vec{d}(t)\end{aligned} \tag{34}$$

Generalized LTI Feedback Closed-Loop Model (continued)

- Closed-loop state matrix, A_L:

$$A_L = \begin{bmatrix} A + B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_K \end{bmatrix} \tag{35}$$

- Closed-loop input matrix, B_L:

$$B_L = \begin{bmatrix} B_1 + B_2 D_K D_{21} \\ B_K D_{21} \end{bmatrix} \tag{36}$$

- Closed-loop output matrix, C_L:

$$C_L = \begin{bmatrix} C_1 + D_{12} D_K C_2 & D_{12} C_K \end{bmatrix} \tag{37}$$

- Closed-loop feedthrough matrix, D_L:

$$D_L = D_{11} + D_{12} D_K D_{21} \tag{38}$$

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  - Stability of generalized feedback control system: A_L stable

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- Closed-loop system:

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Output Feedback Control (continued)

- Closed-loop state matrix, A_L:

$$A_L = A + B_2 D_K C_2 \tag{41}$$

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State Feedback Control

- **State feedback control**, i.e. $C_2 = I$, $D_{21} = 0$
- State feedback controller:

$$\vec{u}(t) = D_K \vec{x}(t) \tag{45}$$

$$A_L = A + B_2 D_K \tag{46}$$

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Observer Feedback Control

- General case of dynamic-controller feedback control a.k.a. **observer feedback control**: define controller state as **state estimate**, $\vec{\hat{x}}$, i.e.

$$\vec{x}_K = \vec{\hat{x}} \tag{50}$$

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- Observer designed to form $\hat{\vec{x}}$ and use this to form control input:

$$\vec{u}(t) = -K\hat{\vec{x}}(t) \tag{51}$$

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- **Open-loop observer** formed based on linear state-space model for continuous-time, assuming disturbances $\vec{d}$ unknown:

$$\dot{\vec{\hat{x}}} = A\vec{\hat{x}} + B_2\vec{u} \tag{52}$$

Closed-Loop Observer

- Feedback control receives output signal from output equation $\rightarrow$ form **closed-loop observer** for continuous-time:

$$\begin{aligned}\dot{\hat{\vec{x}}}(t) &= A\hat{\vec{x}}(t) + B_2\vec{u}(t) + L\left(\vec{y}(t) - \hat{\vec{y}}(t)\right) \\ \hat{\vec{y}}(t) &= C_2\hat{\vec{x}}(t)\end{aligned} \tag{53}$$

  - $\hat{\vec{y}}$: output estimate based on output equation model
  - L: **Luenberger observer matrix**

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  - $\hat{\vec{y}}$: output estimate based on output equation model
  - L: **Luenberger observer matrix**
- Via substitution, form continuous-time **observer feedback control system** as generalized LTI feedback controller:

$$\begin{aligned} \dot{\hat{\vec{x}}}(t) &= (A - B_2K - LC_2)\,\hat{\vec{x}}(t) + L\vec{y}(t) \\ \vec{u}(t) &= -K\hat{\vec{x}}(t) \end{aligned} \tag{54}$$

Observer Feedback Control System

$$A_K = A - B_2 K - LC_2 \tag{55}$$

$$B_K = L \tag{56}$$

$$C_K = -K \tag{57}$$

$$D_K = 0 \tag{58}$$

- Closed-loop dynamics

$$\begin{aligned}\begin{bmatrix}\dot{\vec{x}}(t)\\ \dot{\hat{\vec{x}}}(t)\end{bmatrix} &= \begin{bmatrix}A & -B_2K\\ LC_2 & A - B_2K - LC_2\end{bmatrix}\begin{bmatrix}\vec{x}(t)\\ \hat{\vec{x}}(t)\end{bmatrix} + \begin{bmatrix}B_1\\ LD_{21}\end{bmatrix}\vec{d}(t)\\ \vec{e}(t) &= \begin{bmatrix}C_1 & -D_{12}K\end{bmatrix}\begin{bmatrix}\vec{x}(t)\\ \hat{\vec{x}}(t)\end{bmatrix} + D_{11}\vec{d}(t)\end{aligned} \tag{59}$$

State Error Dynamics

- To assess stability of closed-loop system, consider **state error**, $\vec{e}_x(t)$:

$$\vec{e}_x(t) = \vec{x}(t) - \hat{\vec{x}}(t) \tag{60}$$

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$$\begin{aligned}
\begin{bmatrix} \dot{\vec{x}}(t) \\ \dot{\vec{x}}(t) - \dot{\vec{e}}_x(t) \end{bmatrix} &= \begin{bmatrix} A & -B_2K \\ LC_2 & A - B_2K - LC_2 \end{bmatrix} \begin{bmatrix} \vec{x}(t) \\ \vec{x}(t) - \vec{e}_x(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ LD_{21} \end{bmatrix} \vec{d}(t) \\
\vec{e}(t) &= \begin{bmatrix} C_1 & -D_{12}K \end{bmatrix} \begin{bmatrix} \vec{x}(t) \\ \vec{x}(t) - \vec{e}_x(t) \end{bmatrix} + D_{11}\vec{d}(t)
\end{aligned} \tag{61}$$

State Error Dynamics Closed-Loop Alternative

$$\begin{bmatrix} \dot{\vec{x}}(t) \\ \dot{\vec{x}}(t) - \dot{\vec{e}}_x(t) \end{bmatrix} = \begin{bmatrix} (A - B_2 K) & B_2 K \\ (A - B_2 K) & -A + B_2 K + L C_2 \end{bmatrix} \begin{bmatrix} \vec{x}(t) \\ \vec{e}_x(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ L D_{21} \end{bmatrix} \vec{d}(t)$$
$$\vec{e}(t) = \begin{bmatrix} (C_1 - D_{12} K) & D_{12} K \end{bmatrix} \begin{bmatrix} \vec{x}(t) \\ \vec{e}_x(t) \end{bmatrix} + D_{11} \vec{d}(t) \tag{62}$$

State Error Dynamics Closed-Loop Alternative

$$\begin{aligned}\begin{bmatrix}\dot{\vec{x}}(t)\\ \dot{\vec{x}}(t)-\dot{\vec{e}}_x(t)\end{bmatrix} &= \begin{bmatrix}(A-B_2K) & B_2K\\ (A-B_2K) & -A+B_2K+LC_2\end{bmatrix}\begin{bmatrix}\vec{x}(t)\\ \vec{e}_x(t)\end{bmatrix}+\begin{bmatrix}B_1\\ LD_{21}\end{bmatrix}\vec{d}(t)\\ \vec{e}(t) &= \begin{bmatrix}(C_1-D_{12}K) & D_{12}K\end{bmatrix}\begin{bmatrix}\vec{x}(t)\\ \vec{e}_x(t)\end{bmatrix}+D_{11}\vec{d}(t)\end{aligned} \tag{62}$$

- Subtracting second row of state equation from first row:

$$\begin{aligned}\begin{bmatrix}\dot{\vec{x}}(t)\\ \dot{\vec{e}}_x(t)\end{bmatrix} &= \begin{bmatrix}(A-B_2K) & B_2K\\ 0 & A-LC_2\end{bmatrix}\begin{bmatrix}\vec{x}(t)\\ \vec{e}_x(t)\end{bmatrix}+\begin{bmatrix}B_1\\ B_1-LD_{21}\end{bmatrix}\vec{d}(t)\\ \vec{e}(t) &= \begin{bmatrix}(C_1-D_{12}K) & D_{12}K\end{bmatrix}\begin{bmatrix}\vec{x}(t)\\ \vec{e}_x(t)\end{bmatrix}+D_{11}\vec{d}(t)\end{aligned} \tag{63}$$

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- Eigenvalues of upper triangular matrix dependent only on block diagonal terms
- Stability of LTI observer feedback control system: $A - B_2K$ <u>and</u> $A - LC_2$ stable

Separation Principle

- **Separation principle**: linear observer feedback control systems have independent criteria for design of L and K
  - Foundational result in control and estimation theory

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  - Foundational result in control and estimation theory
- Justifies separate design of estimation and control algorithms in dynamical systems applications
- Course: focus on state feedback control design
  - Other courses: optimal state estimation and application to flight vehicles' state, i.e. navigation and target tracking

Eigenvalue/Pole Placement Control Design

- **Eigenvalue placement** a.k.a. **pole placement** design: “place” eigenvalues of linear feedback control system by choosing feedback gain matrices
  - Set modal characteristics for system response
  - Generalization of root locus technique for selecting single gain parameter in SISO LTI control system $\rightarrow$ selecting two gain matrices independently in MIMO LTI control systems and effect on system’s root/poles/eigenvalues for control and state estimation
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  - Computer algorithms solve eigenvalue placement problem, e.g. `place` in MATLAB, derivations beyond scope of course
- For MIMO LTI systems: may not be able to set eigenvalues arbitrarily, eigenvalue equations depend on generalized plant state, input, and potentially output matrices
  - Ability to place eigenvalues: concepts of controllability and observability for LTI systems
  - To place poles of state feedback: system must be state controllable
  - To place poles of output feedback, system must be output controllable
  - To place poles of observer feedback, system must be state controllable to place $A - B_2K$ and observable to place $A - LC_2$

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  - Conditions for state reachability implies state controllability for LTI systems

1 Any $\vec{x}(t)$ can be reached and pair (A, B) is controllable if and only if **controllability matrix**, $\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$, invertible, i.e.

$$\operatorname{rank}\left(\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}\right) = n \tag{64}$$

PBH Controllability Test

2 **Popov-Belevitch-Hautus (PBH) controllability test**: (A, B) controllable if and only if for all eigenvalues $\lambda \in \mathbb{C}$ of A:

$$\operatorname{rank}\left(\begin{bmatrix}\lambda I - A & B\end{bmatrix}\right) = n \tag{65}$$

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- Weaker notion than controllability: **stabilizability**
  - Stabilizable system: if all *uncontrollable* state variables can be controlled
  - (A, B) stabilizable if and only if for all eigenvalues $\lambda \in \mathbb{C}$ of A with $Real(\lambda) \geq 0$:

$$\operatorname{rank}\left(\begin{bmatrix} \lambda I - A & B \end{bmatrix}\right) = n \tag{66}$$

Controllability Gramian Test

3 Use **controllability Gramian** for continuous-time:

$$W_C(t) = \int_0^t e^{A\tau} BB^T e^{A\tau} d\tau \tag{67}$$

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- If and only if $W_C(t) > 0$ for *any* $t > 0$, then system controllable
- Equation reducible to solving

$$AW_C + W_C A^T = -BB^T \tag{68}$$

Check if $W_C > 0$

- Eigenvalues of $n \times n$ Gramian, W_C, characterize relative degree of controllability

Other Types of Controllability

- Two other common notions of controllability

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- **Output controllability**: ability of input to move *output* from any initial condition to any final condition in finite time
- Naturally involves output matrix in addition to input matrix, e.g. continuous-time **output controllability matrix**:

$$\begin{bmatrix} C & CAB & CA^2B & \cdots & CA^{n_x-1}B & D \end{bmatrix} \tag{69}$$

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  - Note: state and output controllability not equivalent nor does one imply other
- **Controllability under constraints** which may be imposed upon practical systems modeled as LTI system
  - Constraints may be inherent to system, e.g. saturating actuator, or imposed by control designer, e.g. due to safety-related concerns
  - Effect of constraints to systems: vast larger topic in control and mentioned later in course

Observability Test

- **Observability**: ability to observe system's *past* initial state, $\vec{x}(0)$
  - I.e., if, for some finite time interval, $[0, t_f]$, inputs $\vec{u}(t)$, and outputs $\vec{y}(t)$, $\rightarrow$ initial state $\vec{x}(0)$ can be determined

Observability Test

- **Observability**: ability to observe system's *past* initial state, $\vec{x}(0)$
  - I.e., if, for some finite time interval, $[0, t_f]$, inputs $\vec{u}(t)$, and outputs $\vec{y}(t)$, $\rightarrow$ initial state $\vec{x}(0)$ can be determined
- Consider $n-1$ continuous output derivatives which necessitate measurements of $y(t)$ over time interval, $[0, t_f]$

$$\begin{aligned}
\vec{y}(0) &= C\vec{x}_0 \\
\dot{\vec{y}}(0) &= C\dot{\vec{x}}(0) = C\left(A\vec{x}(0) + B\vec{u}(0)\right) \\
\ddot{\vec{y}}(0) &= C\ddot{\vec{x}}(0) = C\frac{d}{dt}\left(A\vec{x}(0) + B\vec{u}(0)\right) \\
\ddot{\vec{y}}(0) &= CA\left(A\vec{x}(0) + B\vec{u}(0)\right) + CB\dot{\vec{u}}(0) \\
\vdots &= \vdots \\
\vec{y}^{[n-1]}(0) &= CA^{n-1}\vec{x}(0) + CA^{n-2}B\vec{u}^{[n-2]}(0) + \cdots CB\dot{\vec{u}}(0)
\end{aligned} \tag{70}$$

Observability Matrix Test

- $\vec{y}(0)$ and $\vec{u}(0)$ and derivatives all known
  $\rightarrow$ any $\vec{x}(0)$ can be estimated and pair (A, C) observable if and only if:

1. **Observability matrix,** $\begin{bmatrix} C \\ CA \\ \dots \\ CA^{n-1} \end{bmatrix}$, invertible, i.e.

$$\text{rank}\left(\begin{bmatrix} C \\ CA \\ \dots \\ CA^{n-1} \end{bmatrix}\right) = n \tag{71}$$

PBH Observability Test

2 **Popov-Belevitch-Hautus (PBH) observability test**: (A, C) observable if and only if for all eigenvalues $\lambda \in \mathbb{C}$ of A:

$$\operatorname{rank}\left(\begin{bmatrix} \lambda I - A & C \end{bmatrix}\right) = n \tag{72}$$

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- Weaker notion than observability: **detectability**
  - Detectable system if all *uncontrollable* state variables can be observed
  - (A, C) detectable if and only if for all eigenvalues $\lambda \in \mathbb{C}$ of A with $Real(\lambda) \geq 0$:

$$\operatorname{rank}\left(\begin{bmatrix} \lambda I - A & C \end{bmatrix}\right) = n \tag{73}$$

Observability Grammian Test

3 **Observability Gramian** for continuous-time:

$$W_O(t) = \int_0^t e^{A\tau} C C^T e^{A\tau} d\tau \tag{74}$$

- If and only if $W_O(t) > 0$ for *any* $t > 0$, then system observable

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- Equation reducible to solving

$$A W_O + W_O A^T = -C C^T \tag{75}$$

Check if $W_O > 0$

- Eigenvalues of $n \times n$ Gramian, W_O, characterize relative degree of observability

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- Generalized LTI Feedback Control System
  - Distinguishes state, output, observer feedback controllers
- Separation principle: separate state feedback control and state estimator design
  - Requires state controllability for state feedback control
  - Requires observability for state estimator
  - Stabilizability, detectability may provide sufficient solutions
  - Course: focus on MIMO LTI state feedback design