Lecture 6: Linearized Airplane Dynamics and Stability

Textbook Sections 8.3, 8.4, & 11.3

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Introduction

- Rigid-body aerospace vehicle model: 6-DOF differential equations
    - Complicated
    - Module focus: airplane

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- Second analysis: linearization of simplified conventional airplane dynamics
  - Stability frame
  - Coordinated flight: $\bar{\beta} = 0$
  - Wings-level flight: $\bar{\phi} = 0$

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  - Define trim states and inputs for airplane for steady-flight
- Second analysis: linearization of simplified conventional airplane dynamics
  - Stability frame
  - Coordinated flight: $\bar{\beta} = 0$
  - Wings-level flight: $\bar{\phi} = 0$
- Airplane: linearized, time-invariant state equation:
  - States: u, α, β, p, q, r, ϕ, θ
  - Inputs: δ_a, δ_e, δ_r, T
  - Perturbed form with leading Δ's
  - Output equation not addressed as depends on use of dynamics

6-DOF Airplane EOMs

- 6-DOF airplane EOMs:

$$\begin{bmatrix} X - g\sin\theta \\ Y + g\cos\theta\sin\phi \\ Z + g\cos\theta\cos\phi \\ L \\ M \\ N \end{bmatrix} = \begin{bmatrix} \dot{u} + qu\sin\alpha - ru\tan\beta \\ \dot{u}\tan\beta + \dot{\beta}u\sec^2\beta + ru - pu\sin\alpha \\ \dot{u}\sin\alpha + \dot{\alpha}u\cos\alpha + pu\tan\beta - qu \\ \dot{p} + \frac{I_{zz}-I_{yy}}{I_{xx}}qr - \frac{I_{xz}}{I_{xx}}(\dot{r} + pq) \\ \dot{q} + \frac{I_{xx}-I_{zz}}{I_{yy}}pr - \frac{I_{xz}}{I_{yy}}(r^2 - p^2) \\ \dot{r} + \frac{I_{yy}-I_{xx}}{I_{zz}}pq - \frac{I_{xz}}{I_{zz}}(\dot{p} - qr) \end{bmatrix} \tag{1}$$

Decouple for Straight, Wings-Level, Coordinated Flight

- For $\dot{\psi} = \bar{\phi} = \bar{\beta} = 0$, : decouple into linearized longitudinal and lateral-directional EOMs using trim and perturbed states, forces, moments

$$\begin{bmatrix} \bar{X} + \Delta X - g \sin\left(\bar{\theta} + \Delta\theta\right) \\ \bar{Z} + \Delta Z + g \cos\left(\bar{\theta} + \Delta\theta\right) \cos\Delta\phi \\ \bar{M} + \Delta M \end{bmatrix} = \begin{bmatrix} \Delta\dot{u} + \Delta q \left(\bar{u} + \Delta u\right) \sin\Delta\alpha \\ \Delta\dot{u} \sin\Delta\alpha + \Delta\dot{\alpha}\left(\bar{u} + \Delta u\right) \cos\Delta\alpha - \Delta q \left(\bar{u} + \Delta u\right) \\ \Delta\dot{q} \end{bmatrix} \quad (2)$$

$$\begin{bmatrix} \bar{Y} + \Delta Y + g \cos\bar{\theta} \sin\Delta\phi \\ \bar{L} + \Delta L \\ \bar{N} + \Delta N \end{bmatrix} = \begin{bmatrix} \Delta\dot{\beta}\bar{u} \sec^2\Delta\beta + \bar{u}\left(\Delta r\right) \\ \Delta\dot{p} - \frac{I_{xz}}{I_{xx}}\Delta\dot{r} \\ \Delta\dot{r} - \frac{I_{xz}}{I_{zz}}\Delta\dot{p} \end{bmatrix} \quad (3)$$

Linearized Trigonometric Funcations

- Using trigonometric addition formulas and small angle approximation:

$$\sin(\bar{a} + \Delta a) = \sin\bar{a}\cos\Delta a + \cos\bar{a}\sin\Delta a = \sin\bar{a} + \cos\bar{a}\Delta a \tag{4}$$

$$\cos(\bar{a} + \Delta a) = \cos\bar{a}\cos\Delta a - \sin\bar{a}\sin\Delta a = \cos\bar{a} - \sin\bar{a}\Delta a \tag{5}$$

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  - For θ, ϕ, α, β, $\bar{\phi}=\bar{\alpha}=\bar{\beta}=0$ already assumed

$$\begin{bmatrix}\bar{X}+\Delta X-g\sin\bar{\theta}-g\cos\bar{\theta}\Delta\theta\\ \bar{Z}+\Delta Z-g\sin\bar{\theta}\Delta\theta\\ \bar{M}+\Delta M\end{bmatrix}=\begin{bmatrix}\Delta\dot{u}+\Delta q\left(\bar{u}+\Delta u\right)\Delta\alpha\\ \Delta\dot{u}\Delta\alpha+\Delta\dot{\alpha}\left(\bar{u}+\Delta u\right)-\Delta q\left(\bar{u}+\Delta u\right)\\ \Delta\dot{q}\end{bmatrix} \tag{6}$$

$$\begin{bmatrix}\bar{Y}+\Delta Y-g\cos\bar{\theta}\Delta\phi\\ \bar{L}+\Delta L\\ \bar{N}+\Delta N\end{bmatrix}=\begin{bmatrix}\Delta\dot{\beta}\bar{u}+\Delta r\bar{u}\\ \Delta\dot{p}-\frac{I_{xz}}{I_{xx}}\Delta\dot{r}\\ \Delta\dot{r}-\frac{I_{xz}}{I_{zz}}\Delta\dot{p}\end{bmatrix} \tag{7}$$

Further Linearization

- Eliminate higher-order terms of perturbations and separate out perturbation terms:

$$\begin{bmatrix} \bar{X} - g\sin\bar{\theta} \\ \bar{Z} \\ \bar{M} \end{bmatrix} + \begin{bmatrix} \Delta X \\ \Delta Z \\ \Delta M \end{bmatrix} + \begin{bmatrix} -g\cos\bar{\theta} \\ -g\sin\bar{\theta} \\ 0 \end{bmatrix} \Delta\theta = \begin{bmatrix} \Delta\dot{u} \\ \bar{u}\Delta\dot{\alpha} - \bar{u}\Delta q \\ \Delta\dot{q} \end{bmatrix} \tag{8}$$

$$\begin{bmatrix} \bar{Y} \\ \bar{L} \\ \bar{N} \end{bmatrix} + \begin{bmatrix} \Delta Y \\ \Delta L \\ \Delta N \end{bmatrix} + \begin{bmatrix} -g\cos\bar{\theta} \\ 0 \\ 0 \end{bmatrix} \Delta\phi = \begin{bmatrix} \Delta\dot{\beta}\bar{u} + \Delta r\bar{u} \\ \Delta\dot{p} - \frac{I_{xz}}{I_{xx}}\Delta\dot{r} \\ \Delta\dot{r} - \frac{I_{xz}}{I_{zz}}\Delta\dot{p} \end{bmatrix} \tag{9}$$

Straight Steady-Flight Conditions

$$
\begin{aligned}
\bar{X} - g\sin\bar{\theta} &= 0 \\
\bar{Y} &= 0 \\
\bar{Z} + g\cos\bar{\theta} &= 0 \\
\bar{L} &= 0 \\
\bar{M} &= 0 \\
\bar{N} &= 0
\end{aligned}
\tag{10}
$$

- Due to $\vec{\omega}_{S/N} = 0$

Linearized Models

- Using matrices and states:

$$\begin{bmatrix} \Delta X \\ \Delta Z \\ \Delta M \end{bmatrix} + \begin{bmatrix} -g\cos\bar{\theta} \\ -g\sin\bar{\theta} \\ 0 \end{bmatrix} \Delta\theta + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{u} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta\alpha \\ \Delta q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \bar{u} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{\alpha} \\ \Delta\dot{q} \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} \Delta Y \\ \Delta L \\ \Delta N \end{bmatrix} + \begin{bmatrix} -g\cos\bar{\theta} \\ 0 \\ 0 \end{bmatrix} \Delta\phi - \begin{bmatrix} 0 & 0 & \bar{u} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\beta \\ \Delta p \\ \Delta r \end{bmatrix} = \begin{bmatrix} \bar{u} & 0 & 0 \\ 0 & 1 & -\frac{I_{xz}}{I_{xx}} \\ 0 & -\frac{I_{xz}}{I_{xx}} & 1 \end{bmatrix} \begin{bmatrix} \Delta\dot{\beta} \\ \Delta\dot{p} \\ \Delta\dot{r} \end{bmatrix} \tag{12}$$

General Airplane Modeling Principle

- Perturbed aerodynamic and propulsive forces and moments modeled as two sets

$$\begin{bmatrix} \Delta X \\ \Delta Z \\ \Delta M \end{bmatrix} = \begin{bmatrix} X_{\dot{u}} & X_{\dot{\alpha}} & X_{\dot{q}} \\ Z_{\dot{u}} & Z_{\dot{\alpha}} & Z_{\dot{q}} \\ M_{\dot{u}} & M_{\dot{\alpha}} & M_{\dot{q}} \end{bmatrix} \begin{bmatrix} \Delta \dot{u} \\ \Delta \dot{\alpha} \\ \Delta \dot{q} \end{bmatrix} + \begin{bmatrix} X_u & X_\alpha & X_q \\ Z_u & Z_\alpha & Z_q \\ M_u & M_\alpha & M_q \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta \alpha \\ \Delta q \end{bmatrix} + \begin{bmatrix} X_{\delta_e} & X_T \\ Z_{\delta_e} & Z_T \\ M_{\delta_e} & M_T \end{bmatrix} \begin{bmatrix} \Delta \delta_e \\ \Delta T \end{bmatrix} \tag{13}$$

$$\begin{bmatrix} \Delta Y \\ \Delta L \\ \Delta N \end{bmatrix} = \begin{bmatrix} Y_{\dot{\beta}} & Y_{\dot{p}} & Y_{\dot{r}} \\ L_{\dot{\beta}} & L_{\dot{p}} & L_{\dot{r}} \\ N_{\dot{\beta}} & N_{\dot{p}} & N_{\dot{r}} \end{bmatrix} \begin{bmatrix} \Delta \dot{\beta} \\ \Delta \dot{p} \\ \Delta \dot{r} \end{bmatrix} + \begin{bmatrix} Y_\beta & Y_p & Y_r \\ L_\beta & L_p & L_r \\ N_\beta & N_p & N_r \end{bmatrix} \begin{bmatrix} \Delta \beta \\ \Delta p \\ \Delta r \end{bmatrix} + \begin{bmatrix} Y_{\delta_a} & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \end{bmatrix} \begin{bmatrix} \Delta \delta_a \\ \Delta \delta_r \end{bmatrix} \tag{14}$$

- Coefficients of perturbed states and inputs inside matrices: **stability and control derivatives**
- Correspond to Jacobian partial derivative terms about trimmed steady-flight

Stability and Control Derivative Assumption

- Derivatives generally change with airplane's trim conditions, typically calculate tables of derivatives at many steady-flight conditions
  - Use wind tunnels tests, flight tests, and/or computational fluid dynamics (CFD)

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- Course: linearized dynamics derivation for airplanes assume following stability and control derivatives dominate perturbed forces and moments:

$$\begin{bmatrix} \Delta X \\ \Delta Z \\ \Delta M \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & Z_{\dot{\alpha}} & 0 \\ 0 & M_{\dot{\alpha}} & 0 \end{bmatrix} \begin{bmatrix} \Delta \dot{u} \\ \Delta \dot{\alpha} \\ \Delta \dot{q} \end{bmatrix} + \begin{bmatrix} X_u & X_\alpha & 0 \\ Z_u & Z_\alpha & Z_q \\ M_u & M_\alpha & M_q \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta \alpha \\ \Delta q \end{bmatrix} + \begin{bmatrix} 0 & X_T \\ Z_{\delta_e} & Z_T \\ M_{\delta_e} & M_T \end{bmatrix} \begin{bmatrix} \Delta \delta_e \\ \Delta T \end{bmatrix} \tag{15}$$

$$\begin{bmatrix} \Delta Y \\ \Delta L \\ \Delta N \end{bmatrix} = \begin{bmatrix} Y_\beta & Y_p & Y_r \\ L_\beta & L_p & L_r \\ N_\beta & N_p & N_r \end{bmatrix} \begin{bmatrix} \Delta \beta \\ \Delta p \\ \Delta r \end{bmatrix} + \begin{bmatrix} 0 & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \end{bmatrix} \begin{bmatrix} \Delta \delta_a \\ \Delta \delta_r \end{bmatrix} \tag{16}$$

Perturbed Forces and Moments Substitution

- Substitute for assumed perturbed aerodynamic and propulsive forces & moments:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & Z_{\dot\alpha} & 0 \\ 0 & M_{\dot\alpha} & 0 \end{bmatrix} \begin{bmatrix} \Delta\dot u \\ \Delta\dot\alpha \\ \Delta\dot q \end{bmatrix} + \begin{bmatrix} X_u & X_\alpha & 0 \\ Z_u & Z_\alpha & Z_q \\ M_u & M_\alpha & M_q \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta\alpha \\ \Delta q \end{bmatrix} + \begin{bmatrix} 0 & X_T \\ Z_{\delta_e} & Z_T \\ M_{\delta_e} & M_T \end{bmatrix} \begin{bmatrix} \Delta\delta_e \\ \Delta T \end{bmatrix}$$
$$+ \begin{bmatrix} -g\cos\bar\theta \\ -g\sin\bar\theta \\ 0 \end{bmatrix} \Delta\theta + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar u \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta\alpha \\ \Delta q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \bar u & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta\dot u \\ \Delta\dot\alpha \\ \Delta\dot q \end{bmatrix} \tag{17}$$

$$\begin{bmatrix} Y_\beta & Y_p & Y_r \\ L_\beta & L_p & L_r \\ N_\beta & N_p & N_r \end{bmatrix} \begin{bmatrix} \Delta\beta \\ \Delta p \\ \Delta r \end{bmatrix} + \begin{bmatrix} 0 & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \end{bmatrix} \begin{bmatrix} \Delta\delta_a \\ \Delta\delta_r \end{bmatrix}$$
$$+ \begin{bmatrix} -g\cos\bar\theta \\ 0 \\ 0 \end{bmatrix} \Delta\phi - \begin{bmatrix} 0 & 0 & \bar u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\beta \\ \Delta p \\ \Delta r \end{bmatrix} = \begin{bmatrix} \bar u & 0 & 0 \\ 0 & 1 & -\frac{I_{xz}}{I_{xx}} \\ 0 & -\frac{I_{xz}}{I_{xx}} & 1 \end{bmatrix} \begin{bmatrix} \Delta\dot\beta \\ \Delta\dot p \\ \Delta\dot r \end{bmatrix} \tag{18}$$

Combining Matrices and Rearranging

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \bar{u} - Z_{\dot{\alpha}} & 0 \\ 0 & -M_{\dot{\alpha}} & 1 \end{bmatrix} \begin{bmatrix} \Delta \dot{u} \\ \Delta \dot{\alpha} \\ \Delta \dot{q} \end{bmatrix} = \begin{bmatrix} X_u & X_\alpha & 0 & -g\cos\bar{\theta} \\ Z_u & Z_\alpha & \bar{u} + Z_q & -g\sin\bar{\theta} \\ M_u & M_\alpha & M_q & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta \alpha \\ \Delta q \\ \Delta \theta \end{bmatrix} + \begin{bmatrix} 0 & X_T \\ Z_{\delta_e} & Z_T \\ M_{\delta_e} & M_T \end{bmatrix} \begin{bmatrix} \Delta \delta_e \\ \Delta T \end{bmatrix} \tag{19}$$

$$\begin{bmatrix} \bar{u} & 0 & 0 \\ 0 & 1 & -\frac{I_{xz}}{I_{xx}} \\ 0 & -\frac{I_{xz}}{I_{xx}} & 1 \end{bmatrix} \begin{bmatrix} \Delta \dot{\beta} \\ \Delta \dot{p} \\ \Delta \dot{r} \end{bmatrix} = \begin{bmatrix} Y_\beta & Y_p & Y_r - \bar{u} & -g\cos\bar{\theta} \\ L_\beta & L_p & L_r & 0 \\ N_\beta & N_p & N_r & 0 \end{bmatrix} \begin{bmatrix} \Delta \beta \\ \Delta p \\ \Delta r \\ \Delta \phi \end{bmatrix} + \begin{bmatrix} 0 & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \end{bmatrix} \begin{bmatrix} \Delta \delta_a \\ \Delta \delta_r \end{bmatrix} \tag{20}$$

Using Small Angles

- Recall for small angles:

$$\Delta\dot{\theta} = \Delta q \tag{21}$$

$$\Delta\dot{\phi} = \Delta p + \tan\bar{\theta}\Delta r \tag{22}$$

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$$\begin{bmatrix} \bar{u} & 0 & 0 & 0 \\ 0 & 1 & -\frac{I_{xz}}{I_{xx}} & 0 \\ 0 & -\frac{I_{xz}}{I_{xx}} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta\dot{\beta} \\ \Delta\dot{p} \\ \Delta\dot{r} \\ \Delta\dot{\phi} \end{bmatrix} = \begin{bmatrix} Y_\beta & Y_p & Y_r - \bar{u} & -g\cos\bar{\theta} \\ L_\beta & L_p & L_r & 0 \\ N_\beta & N_p & N_r & 0 \\ 0 & 1 & \tan\bar{\theta} & 0 \end{bmatrix} \begin{bmatrix} \Delta\beta \\ \Delta p \\ \Delta r \\ \Delta\phi \end{bmatrix} + \begin{bmatrix} 0 & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\delta_a \\ \Delta\delta_r \end{bmatrix} \tag{24}$$

Linearized Longitudinal Dynamics

- Inverse matrices on left side: **linearized longitudinal rigid airplane dynamics**

$$
\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{\alpha} \\ \Delta\dot{q} \\ \Delta\dot{\theta} \end{bmatrix} =
\begin{bmatrix}
X_u & X_\alpha & 0 & -g\cos\bar{\theta} \\
\frac{Z_u}{\bar{u}-Z_{\dot{\alpha}}} & \frac{Z_\alpha}{\bar{u}-Z_{\dot{\alpha}}} & \frac{\bar{u}+Z_q}{\bar{u}-Z_{\dot{\alpha}}} & -\frac{g}{\bar{u}-Z_{\dot{\alpha}}}\sin\bar{\theta} \\
M_u + M_{\dot{\alpha}}\frac{Z_u}{\bar{u}-Z_{\dot{\alpha}}} & M_\alpha + M_{\dot{\alpha}}\frac{Z_\alpha}{\bar{u}-Z_{\dot{\alpha}}} & M_q + M_{\dot{\alpha}}\frac{\bar{u}+Z_q}{\bar{u}-Z_{\dot{\alpha}}} & -M_{\dot{\alpha}}\frac{g}{\bar{u}-Z_{\dot{\alpha}}}\sin\bar{\theta} \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix} \Delta u \\ \Delta\alpha \\ \Delta q \\ \Delta\theta \end{bmatrix}
+ \begin{bmatrix}
0 & X_T \\
\frac{Z_{\delta_e}}{\bar{u}-Z_{\dot{\alpha}}} & \frac{Z_T}{\bar{u}-Z_{\dot{\alpha}}} \\
M_{\delta_e} + M_{\dot{\alpha}}\frac{Z_{\delta_e}}{\bar{u}-Z_{\dot{\alpha}}} & M_T + M_{\dot{\alpha}}\frac{Z_T}{\bar{u}-Z_{\dot{\alpha}}} \\
0 & 0
\end{bmatrix}
\begin{bmatrix} \Delta\delta_e \\ \Delta T \end{bmatrix}
\tag{25}
$$

Linearized Lateral-Directional Dynamics

- **Linearized lateral-directional rigid airplane dynamics**:

$$\begin{bmatrix} \Delta\dot{\beta} \\ \Delta\dot{p} \\ \Delta\dot{r} \\ \Delta\dot{\phi} \end{bmatrix} = \begin{bmatrix} \frac{Y_\beta}{\bar{u}} & \frac{Y_p}{\bar{u}} & \frac{Y_r}{\bar{u}} - 1 & \frac{g}{\bar{u}}\cos\bar{\theta} \\ L_\beta^* & L_p^* & L_r^* & 0 \\ N_\beta^* & N_p^* & N_r^* & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\beta \\ \Delta p \\ \Delta r \\ \Delta\phi \end{bmatrix} + \begin{bmatrix} 0 & \frac{Y_{\delta_r}}{\bar{u}} \\ L_{\delta_a}^* & L_{\delta_r}^* \\ N_{\delta_a}^* & N_{\delta_r}^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\delta_a \\ \Delta\delta_r \end{bmatrix} \tag{26}$$

Linearized Lateral-Directional Dynamics (continued)

$$L_\bullet^* = \frac{L_\bullet + N_\bullet \frac{I_{xz}}{I_{zz}}}{1 - \frac{I_{xz}^2}{I_{xx} I_{zz}}} \tag{27}$$

$$N_\bullet^* = \frac{N_\bullet + L_\bullet \frac{I_{xz}}{I_{zz}}}{1 - \frac{I_{xz}^2}{I_{xx} I_{zz}}} \tag{28}$$

- For $\bullet = \beta, p, r, \delta_a, \delta_r$
- Due to coupling of L and N for non-zero I_{xz}
- If $I_{xz} = 0$, then $L_\bullet^* = L_\bullet$ and $N_\bullet^* = N_\bullet$

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  - Perturbed altitude, Δh: perturbed longitudinal state
  - Perturbed yaw, $\Delta\psi$: perturbed lateral-directional state

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- Extended to fifth-order systems by including:
  - Perturbed altitude, Δh: perturbed longitudinal state
  - Perturbed yaw, $\Delta\psi$: perturbed lateral-directional state
- For wings-level, coordinated steady-flight:

$$\Delta\dot{h} = \sin\bar{\theta}\Delta u + \bar{u}\cos\bar{\theta}\,(\Delta\theta - \Delta\alpha) \tag{29}$$

$$\Delta\dot{\psi} = \sec\bar{\theta}\Delta r \tag{30}$$

  - No direct control derivatives

Acceleration at Some Position on Vehicle

- Important output sometimes used in output equation
  Acceleration vector at some position along rigid airplane:
  $\vec{p}_S = [x_p \; y_p \; z_p]$ in stability frame coordinates

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  Acceleration vector at some position along rigid airplane:
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- In general:

$$\vec{a}_p = \dot{\vec{v}}_S + [\vec{\omega}_{S/N}]_\times \vec{v}_S + [\vec{\alpha}_{S/N}]_\times \vec{p}_S + [\vec{\omega}_{S/N}]_\times [\vec{\omega}_{S/N}]_\times \vec{p}_S \qquad (31)$$

Acceleration at Some Position on Vehicle

- Important output sometimes used in output equation
  Acceleration vector at some position along rigid airplane:
  $\vec{p}_S = [x_p \; y_p \; z_p]$ in stability frame coordinates
- In general:

$$\vec{a}_p = \dot{\vec{v}}_S + [\vec{\omega}_{S/N}]_\times \vec{v}_S + [\vec{\alpha}_{S/N}]_\times \vec{p}_S + [\vec{\omega}_{S/N}]_\times [\vec{\omega}_{S/N}]_\times \vec{p}_S \tag{31}$$

- Linearizing, $\bar{\beta} = 0$, & $\bar{\phi} = 0$:

$$\begin{bmatrix} \Delta a_{p,x} \\ \Delta a_{p,y} \\ \Delta a_{p,z} \end{bmatrix} = \begin{bmatrix} \Delta \dot{u} + \bar{w}\Delta q + z_p \Delta \dot{q} - y_p \Delta \dot{r} \\ \Delta \dot{v} + \bar{u}\Delta r - \bar{w}\Delta p + x_p \Delta \dot{r} - z_p \Delta \dot{p} \\ \Delta \dot{w} - \bar{u}\Delta q + y_p \Delta \dot{p} - x_p \Delta \dot{q} \end{bmatrix} \tag{32}$$

  - Used for output equation for accelerometer placed at $\vec{p}$

Linearized Longitudinal Model

- Fourth-order linearized longitudinal dynamics for rigid airplanes:

$$\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{\alpha} \\ \Delta\dot{q} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u & X_\alpha & 0 & -g\cos\bar{\theta} \\ \frac{Z_u}{\bar{u}-Z_{\dot{\alpha}}} & \frac{Z_\alpha}{\bar{u}-Z_{\dot{\alpha}}} & \frac{\bar{u}+Z_q}{\bar{u}-Z_{\dot{\alpha}}} & -\frac{g}{\bar{u}-Z_{\dot{\alpha}}}\sin\bar{\theta} \\ M_u + M_{\dot{\alpha}}\frac{Z_u}{\bar{u}-Z_{\dot{\alpha}}} & M_\alpha + M_{\dot{\alpha}}\frac{Z_\alpha}{\bar{u}-Z_{\dot{\alpha}}} & M_q + M_{\dot{\alpha}}\frac{\bar{u}+Z_q}{\bar{u}-Z_{\dot{\alpha}}} & -M_{\dot{\alpha}}\frac{g}{\bar{u}-Z_{\dot{\alpha}}}\sin\bar{\theta} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta\alpha \\ \Delta q \\ \Delta\theta \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & X_{\delta_T} \\ \frac{Z_{\delta_e}}{\bar{u}-Z_{\dot{\alpha}}} & \frac{Z_{\delta_T}}{\bar{u}-Z_{\dot{\alpha}}} \\ M_{\delta_e} + M_{\dot{\alpha}}\frac{Z_{\delta_e}}{\bar{u}-Z_{\dot{\alpha}}} & M_{\delta_T} + M_{\dot{\alpha}}\frac{Z_{\delta_T}}{\bar{u}-Z_{\dot{\alpha}}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\delta_e \\ \Delta\delta_T \end{bmatrix} \tag{33}$$

Linearized Longitudinal Model

- Fourth-order linearized longitudinal dynamics for rigid airplanes:

$$\begin{bmatrix}\Delta\dot{u}\\ \Delta\dot{\alpha}\\ \Delta\dot{q}\\ \Delta\dot{\theta}\end{bmatrix} = \begin{bmatrix} X_u & X_\alpha & 0 & -g\cos\bar{\theta} \\ \frac{Z_u}{\bar{u}-Z_{\dot{\alpha}}} & \frac{Z_\alpha}{\bar{u}-Z_{\dot{\alpha}}} & \frac{\bar{u}+Z_q}{\bar{u}-Z_{\dot{\alpha}}} & -\frac{g}{\bar{u}-Z_{\dot{\alpha}}}\sin\bar{\theta} \\ M_u + M_{\dot{\alpha}}\frac{Z_u}{\bar{u}-Z_{\dot{\alpha}}} & M_\alpha + M_{\dot{\alpha}}\frac{Z_\alpha}{\bar{u}-Z_{\dot{\alpha}}} & M_q + M_{\dot{\alpha}}\frac{\bar{u}+Z_q}{\bar{u}-Z_{\dot{\alpha}}} & -M_{\dot{\alpha}}\frac{g}{\bar{u}-Z_{\dot{\alpha}}}\sin\bar{\theta} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix}\Delta u\\ \Delta\alpha\\ \Delta q\\ \Delta\theta\end{bmatrix}$$

$$+ \begin{bmatrix} 0 & X_{\delta_T} \\ \frac{Z_{\delta_e}}{\bar{u}-Z_{\dot{\alpha}}} & \frac{Z_{\delta_T}}{\bar{u}-Z_{\dot{\alpha}}} \\ M_{\delta_e} + M_{\dot{\alpha}}\frac{Z_{\delta_e}}{\bar{u}-Z_{\dot{\alpha}}} & M_{\delta_T} + M_{\dot{\alpha}}\frac{Z_{\delta_T}}{\bar{u}-Z_{\dot{\alpha}}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix}\Delta\delta_e\\ \Delta\delta_T\end{bmatrix} \tag{33}$$

- Note: $\Delta w \approx \bar{u}\Delta\alpha$ may be substituted:

$$M_\alpha = \bar{u}M_w, \qquad Z_\alpha = \bar{u}Z_w, \qquad M_{\dot{\alpha}} = \bar{u}M_{\dot{w}} \tag{34}$$

Longitudinal Modes

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  - **Short-period mode**
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- Phugoid mode considered as gradual interchange of kinetic & potential energy through varying velocity & altitude

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- Fourth-order linearized lateral-directional dynamics for rigid airplanes:

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\begin{bmatrix} \Delta\dot{\beta} \\ \Delta\dot{p} \\ \Delta\dot{r} \\ \Delta\dot{\phi} \end{bmatrix} = \begin{bmatrix} \frac{Y_\beta}{\bar{u}} & \frac{Y_p}{\bar{u}} & \frac{Y_r}{\bar{u}} - 1 & \frac{g}{\bar{u}}\cos\bar{\theta} \\ L_\beta^* & L_p^* & L_r^* & 0 \\ N_\beta^* & N_p^* & N_r^* & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\beta \\ \Delta p \\ \Delta r \\ \Delta\phi \end{bmatrix} + \begin{bmatrix} 0 & \frac{Y_{\delta_r}}{\bar{u}} \\ L_{\delta_a}^* & L_{\delta_r}^* \\ N_{\delta_a}^* & N_{\delta_r}^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\delta_a \\ \Delta\delta_r \end{bmatrix} \tag{37}
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- Typically mode decays rapidly compared to other lateral-directional modes, albeit sometimes too quickly
  - Reducing roll mode time constant done through SAS called **roll damper**

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- Airplane design characteristics affect spiral and dutch-roll modes in opposite ways
  - Increasing dihedral effect makes dutch-roll mode less stable and spiral mode more stable
  - Increasing directional stability makes dutch-roll mode more stable and spiral mode less stable
  - Often use roll/yaw SAS to sufficiently stabilize both spiral and dutch roll modes

Flying/Handling Qualities

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- As pilots generally charged with performing various tasks or **missions**: flying qualities generally specified by airplane pilots according to following three subjective levels:
  - **Level 1** (Good): Flying qualities clearly adequate for mission flight phase
  - **Level 2** (Acceptable): Flying qualities adequate to accomplish mission flight phase, but some increase in pilot workload and/or degradation in mission effectiveness or both
  - **Level 3** (Poor): Flying qualities such that airplane controlled safely, but pilot workload excessive and/or mission effectiveness inadequate or both

Flight Phase Categories

- Depend on three generalized flight phase categories:
  - **Category A**: nonterminal flight phases requiring rapid maneuvering, precision tracking, or highly accurate flight-path control
  - **Category B**: nonterminal flight phases normally accomplished using gradual maneuvers and without precision tracking, although accurate flight-path control may be required
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- For Level 3 flying qualities, Category A flight phases can be terminated safely and Category B and C flight phases can be completed

Airplane Classes

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- Airplanes generally classified:
  - **Class I**: Small, light airplanes
  - **Class II**: Medium-weight, low-to-medium maneuverability airplanes
  - **Class III**: Large, heavy, low-to-medium maneuverability airplanes
  - **Class IV**: High-maneuverability airplanes

Longitudinal Modal Characteristics

- Different flying quality levels and flight phase categories for different classes of airplanes
  - Phugoid mode requirements: period T for potentially *unstable* mode

Mode	Level	Category	Class	Characteristic(s)
Phugoid	1	All	All	$\zeta > 0.04$
	2	All	All	$\zeta > 0$
	3	All	All	$T > 55s$
Short-Period	1	A and C	All	$0.35 \leq \zeta \leq 1.3$
		B	All	$0.3 \leq \zeta \leq 2.0$
	2	A and C	All	$0.25 \leq \zeta \leq 2.0$
		B	All	$0.2 \leq \zeta \leq 2.0$
	3	All	All	$0.15 \leq \zeta$

Lateral-Directional Modal Characteristics

- Different flying quality levels and flight phase categories for different classes of airplanes
  - Spiral mode requirements: doubling of amplitude for potentially *unstable* mode
  - C and L denote carrier- or land-based airplanes

Mode	Level	Category	Class	Characteristic(s)
Roll	1	A and C	I, IV	$\tau \leq 1.0$ sec
			II, III	$\tau \leq 1.4$ sec
		B	All	$\tau \leq 1.4$ sec
	2	A and C	I, IV	$\tau \leq 1.4$ sec
			II, III	$\tau \leq 3.0$ sec
		B	All	$\tau \leq 3.0$ sec
	3	All	All	$\tau \leq 10$ sec
Spiral	1	A	I, IV	Doubling amplitude $\geq$ 12 sec
			II, III	Doubling amplitude $\geq$ 20 sec
		B and C	All	Doubling amplitude $\geq$ 20 sec
	2	All	All	Doubling amplitude $\geq$ 12 sec
	3	All	All	Doubling amplitude $\geq$ 4 sec
Dutch-Roll	1	A	I, IV	$\zeta\omega_n \geq 0.35$ rad/s, $\zeta \geq 0.19$, $\omega_n > 1.0$ rad/s
			II, III	$\zeta\omega_n \geq 0.35$ rad/s, $\zeta \geq 0.19$, $\omega_n > 0.4$ rad/s
		B	All	$\zeta\omega_n \geq 0.15$ rad/s, $\zeta \geq 0.08$, $\omega_n > 0.4$ rad/s
		C	I, II-C, IV	$\zeta\omega_n \geq 0.15$ rad/s, $\zeta \geq 0.08$, $\omega_n > 1.0$ rad/s
			II-L, III	$\zeta\omega_n \geq 0.15$ rad/s, $\zeta \geq 0.08$, $\omega_n > 0.4$ rad/s
	2	All	All	$\zeta\omega_n \geq 0.05$ rad/s, $\zeta \geq 0.02$, $\omega_n > 0.4$ rad/s
	3	All	All	$\zeta \geq 0.02$, $\omega_n \geq 0.4$ rad/s

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- CHRS scale goes from 1 to 10 with lower numbers corresponding to better flying qualities

Pilot Rating	Aircraft Characteristic	Demand of Pilot	Overall Assessment
1	Excellent, highly desirable	Pilot compensation not a factor for desired performance	Good
2	Good, negligible deficiencies	Pilot compensation not a factor for desired performance	Good
3	Fair, some mildly unpleasant deficiencies	Minimal pilot compensation required for desired performance	Good
4	Minor, but annoying deficiencies	Desired performance requires moderate pilot compensation	Acceptable
5	Moderately objectionable deficiencies	Adequate performance requires considerable pilot compensation	Acceptable
6	Very objectionable, but tolerable deficiencies	Adequate performance requires extensive pilot compensation	Acceptable
7	Major deficiencies	Adequate performance not attainable with maximum tolerable compensation	Poor
8	Major deficiencies	Considerable pilot compensation is required for control	Poor
9	Major deficiencies	Intense pilot compensation is required for control	Poor
10	Major deficiencies	Control will be lost during some portion of required operation	Unacceptable

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- SAS generally single-output feedback systems that use single gain term, $K_\bullet$, which changes location of system poles

Root Locus Design Technique

- To choose value of $K_\bullet$: use **root locus** plot
  - I.e. plot of roots of closed-loop characteristic polynomial in complex plane as function of $K_\bullet$

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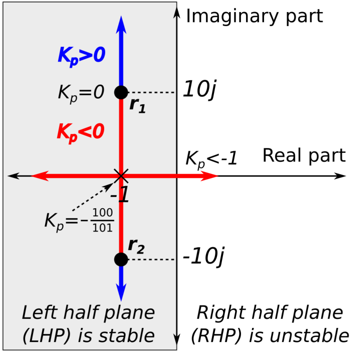
- Example root locus:



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- **Pitch damper**, $K_q(s)$: reduce damping ratio of short-period mode of airplane
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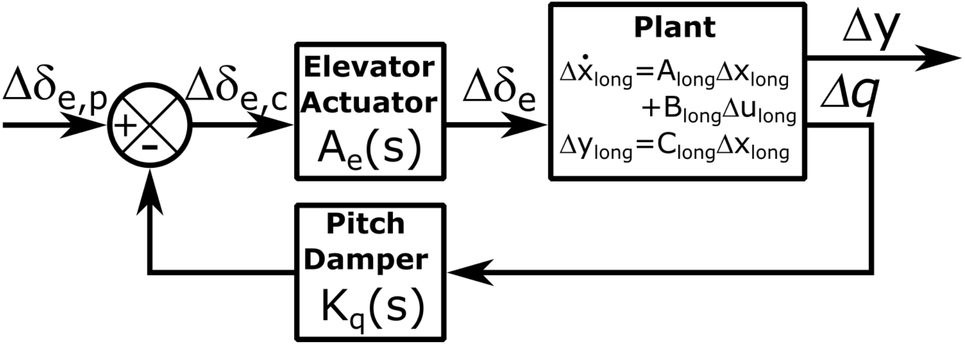

AoA SAS

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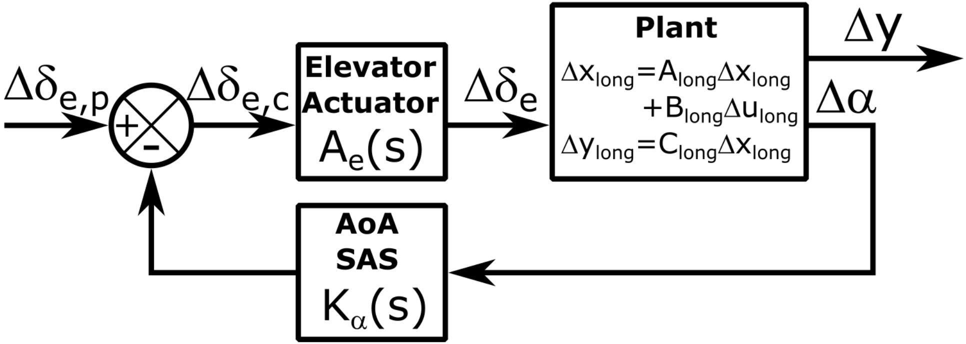

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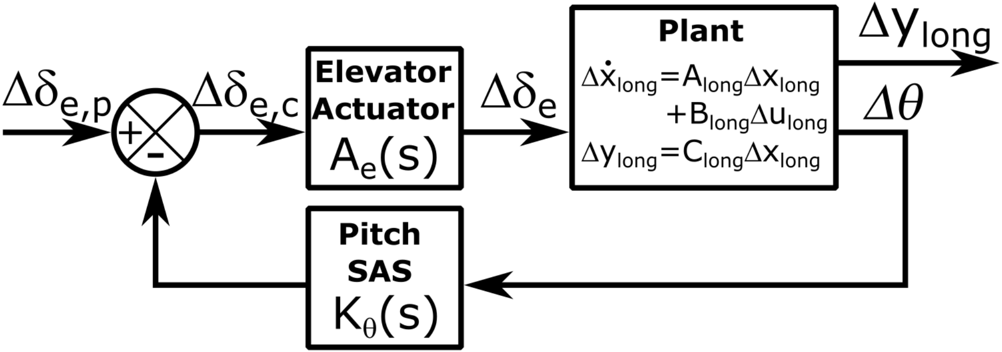

Yaw Damper

- **Yaw damper**, $K_r(s)$: increase damping ratio of dutch-roll mode of airplane

$$K_r(s) = \frac{K_r s}{s + \omega_w} \tag{42}$$

  - K_r: yaw damper gain
  - ω_w: washout frequency
    - Turns off yaw damper for sustained turns: yaw rate not zero
    - Typically selected well below the dutch-roll natural frequency
    - Effect of first-order high-pass filter “washes out” yaw damper for $\omega < \omega_w$
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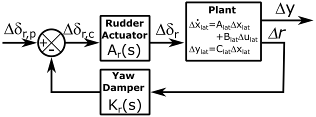

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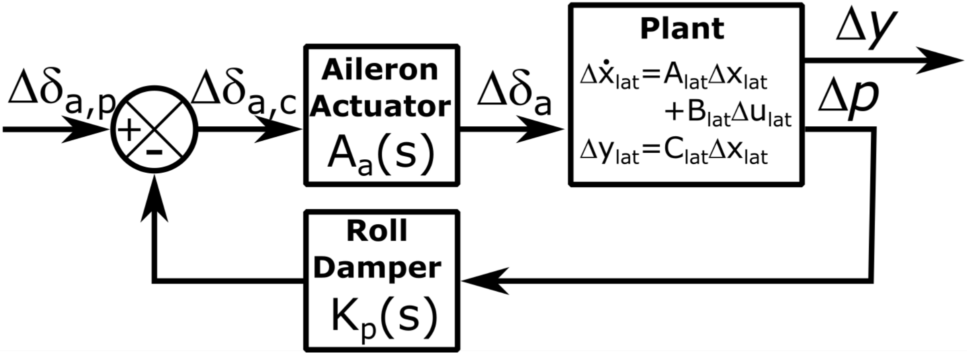

Aileron-Rudder Interconnect

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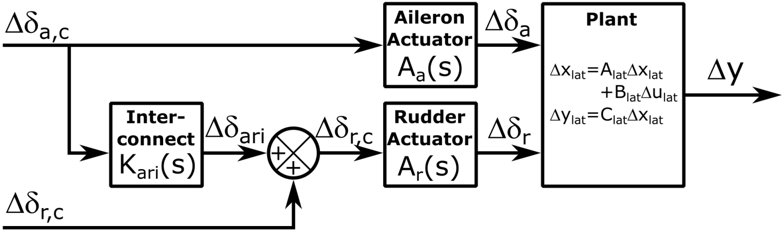
  - E.g. aileron deflections often cause undesirable excitation of dutch roll mode and/or adverse yawing moment, N_{δ_a}
- Many airplanes typically use some sort of **aileron-rudder interconnect (ARI)** to reduce effects



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$$K_{ari} = -\frac{N_{\delta_a}}{N_{\delta_r}} \tag{45}$$

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    - Spiral
- Modal characteristics:
    - Quantify design criteria and flying qualities
    - Can be adjusted with SAS