# Lecture 5: Introductory Airplane Dynamics

## Textbook Sections 7.4 & 8.1

Dr. Jordan D. Larson

# Intro

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  - Identify system model to be controlled

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  - Identify system model to be controlled
- Flight dynamics & control (FDC) plant: flight vehicle
- First principles modeling implements physics-derived mathematical model for flight vehicle dynamics
- Lecture: develop rigid-body flight vehicle dynamics model

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- For rigid flight vehicle dynamics, fixed-wing aircraft generate aerodynamic forces and moments from several different static lifting surfaces
- For inclusion into the rigid flight vehicle dynamics, one must define the aerodynamic forces and moments for entire aircraft in body frame
- Similar to finite-wing theory, model fixed-wing aircraft's total aerodynamic force vector, $\vec{F}_a$, in wind frame:

$$\vec{F}_{a,W} = \begin{bmatrix} -D \\ S \\ -L \end{bmatrix} \tag{1}$$

  - $D$: **drag force** for entire fixed-wing aircraft
  - $S$: **side force** for entire fixed-wing aircraft
  - $L$: **lift force** for entire fixed-wing aircraft

# Lift, Side, and Drag Coefficients

- Typically models each by vehicle coefficients:

$$L = Q_\infty S_w C_L \tag{2}$$

$$S = Q_\infty S_w C_S \tag{3}$$

$$D = Q_\infty S_w C_D \tag{4}$$

  - $Q_\infty = 0.5\rho v_\infty^2$: dynamic pressure
  - $S_w$: wing surface area

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  - $Q_\infty = 0.5\rho v_\infty^2$: dynamic pressure
  - $S_w$: wing surface area
- For subsonic fixed-wing aircraft:

$$C_D = C_{D_0} + \frac{C_L^2}{\pi e_{eff} AR_w} \tag{5}$$

$e_{eff}$: **effective Oswald's efficiency**

# Body-Fixed Frame Aerodynamic Forces

- Wind frame aerodynamic forces rotated to body-fixed frame:

$$\vec{F}_{a,B} = C_{B\leftarrow W} \begin{bmatrix} -D \\ S \\ -L \end{bmatrix} \tag{6}$$

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$$\vec{F}_{a,B} = \begin{bmatrix} \cos\alpha\cos\beta & -\cos\alpha\sin\beta & -\sin\alpha \\ \sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & -\sin\alpha\sin\beta & \cos\alpha \end{bmatrix} \begin{bmatrix} -D \\ S \\ -L \end{bmatrix} \tag{7}$$

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$$\vec{F}_{a,B} = \begin{bmatrix} -D\cos\alpha\cos\beta - S\cos\alpha\sin\beta + L\sin\alpha \\ S\cos\beta - D\sin\beta \\ -D\sin\alpha\cos\beta - S\sin\alpha\sin\beta - L\cos\alpha \end{bmatrix} \tag{8}$$

  - $S$: without subscript should not be confused with surface area of lifting surface, always has specifying subscript with it

# Airplanes

- For airplanes, typically model propulsive force:

$$\vec{F}_{p,B} = \begin{bmatrix} T\cos\theta_T \\ 0 \\ T\sin\theta_T \end{bmatrix} \tag{9}$$

  - $T$: **thrust force**
  - $\theta_T$: potential offset angle w.r.t. $x_B$-axis of body frame

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- Propulsive moment may also be present nominally about $y_B$-axis:

$$\vec{M}_{p,B} = \begin{bmatrix} 0 \\ T(z_T\cos\theta_T - x_T\sin\theta_T) \\ 0 \end{bmatrix} \tag{10}$$

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  - $(x_T, z_T)$ denotes location of thrust force in $x_B - z_B$ plane
  - Often $\theta_T \approx 0^\circ$
  - $T$ generally function of airspeed, altitude, throttle setting

# Normalized Body-Fixed Forces

- Convention: aerodynamic and propulsive forces along $x_B$-, $y_B$-, and $z_B$-axes combined and normalized by mass of airplane, $m$: denoted by $X$, $Y$, and $Z$

$$\vec{F}_{p,B} + \vec{F}_{a,B} = \begin{bmatrix} mX \\ mY \\ mZ \end{bmatrix} \tag{11}$$

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$$\vec{F}_{p,B} + \vec{F}_{a,B} = \begin{bmatrix} mX \\ mY \\ mZ \end{bmatrix} \tag{11}$$

$$\frac{1}{m}\left(\vec{F}_{p,B} + \vec{F}_{a,B}\right) = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{12}$$

# Normalized Body-Fixed Forces and Wind Frame

$$\frac{1}{m}\left(\begin{bmatrix} T\cos\theta_T \\ 0 \\ T\sin\theta_T \end{bmatrix} + \begin{bmatrix} -D\cos\alpha\cos\beta - S\cos\alpha\sin\beta + L\sin\alpha \\ S\cos\beta - D\sin\beta \\ -D\sin\alpha\cos\beta - S\sin\alpha\sin\beta - L\sin\alpha \end{bmatrix}\right) = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{13}$$

- Used for $X$, $Y$, $Z$ in conventional translation equation for rigid airplane dynamics

# Normalized Body-Fixed Moments

- Convention: aerodynamic and propulsive moments about $x_B$-, $y_B$-, $z_B$-axes normalized by moments of inertia, $I_{xx}$, $I_{yy}$, $I_{zz}$, about $x_B$, $y_B$, and $z_B$: denoted by $L$, $M$, $N$

$$\vec{M}_{p,B} + \vec{M}_{a,B} = \begin{bmatrix} I_{xx} L_{roll} \\ I_{yy} M \\ I_{zz} N \end{bmatrix} \tag{14}$$

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$$\begin{bmatrix} I_{xx}^{-1} & 0 & 0 \\ 0 & I_{yy}^{-1} & 0 \\ 0 & 0 & I_{zz}^{-1} \end{bmatrix} \left( \begin{bmatrix} 0 \\ T(z_T \cos\theta_T - x_T \sin\theta_T) \\ 0 \end{bmatrix} + \vec{M}_{a,B} \right) = \begin{bmatrix} L_{roll} \\ M \\ N \end{bmatrix} \tag{16}$$

# Body-Fixed Force Coefficients

- Similarly for $L$, $S$, and $D$: $X$, $Y$, $Z$, written with aerodynamic coefficients:

$$X = \frac{Q_\infty S_w}{m} C_X \tag{17}$$

$$Y = \frac{Q_\infty S_w}{m} C_Y \tag{18}$$

$$Z = \frac{Q_\infty S_w}{m} C_Z \tag{19}$$

# Body-Fixed Moment Coefficients

- $L_{roll}$, $M$, and $N$ written with aerodynamic coefficients:

$$L_{roll} = \frac{Q_\infty S_w b_w}{I_{xx}} C_l \tag{20}$$

$$M = \frac{Q_\infty S_w \bar{c}_w}{I_{yy}} C_m \tag{21}$$

$$N = \frac{Q_\infty S_w b_w}{I_{zz}} C_n \tag{22}$$

  - Lowercase letters for $L$, $M$, $N$ coefficients: $C_L$ already lift coefficient

# Final Thoughts

- All aerodynamic coefficients in lecture modeled using aircraft system indentification (SID) via:
  - Analytical equations, e.g. **build-up component model** for conventional airplanes presented in appendix A
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- Coefficients functions of:
  - Local airspeed, angle of attack, sideslip angle at each lifting surface
  - Geometric layout
  - Control inputs to elevator, rudder, and ailerons
  - Linear velocity
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  - Angular velocity
- One typically linearizes rigid airplane dynamics about equilibrium flight conditions to analyze response characteristics and design suitable control systems

# Point-Mass Model in Wind Frame

- Use wind frame for translation equation:

$$\sum \vec{F}_W = \frac{d}{dt}(m\vec{v}_W) = m\left(\dot{\vec{v}}_W + \vec{\omega}_{W/N} \times \vec{v}_W\right) \tag{23}$$

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$$\vec{\omega}_{W/N} = \begin{bmatrix} p_W & q_W & r_W \end{bmatrix}^T \tag{24}$$

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$$\vec{F}_{a,W} + \vec{F}_{p,W} + \vec{F}_{g,W} = \begin{bmatrix} m\dot{v}_\infty \\ 0 \\ 0 \end{bmatrix} + m \begin{bmatrix} p_W \\ q_W \\ r_W \end{bmatrix} \times \begin{bmatrix} v_\infty \\ 0 \\ 0 \end{bmatrix} \tag{26}$$

# Force Models

- Gravitational force:

$$\vec{F}_{g,W} = C_{W \leftarrow N}(\mu, \gamma, \sigma) \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} \tag{27}$$

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- Aerodynamic force:

$$\vec{F}_{a,W} = \begin{bmatrix} -D \\ S \\ -L \end{bmatrix} \tag{29}$$

# Equation of Motion

$$\begin{bmatrix} -D \\ S \\ -L \end{bmatrix} + \begin{bmatrix} T\cos\alpha\cos\beta \\ -T\cos\alpha\sin\beta \\ -T\sin\alpha \end{bmatrix} + \begin{bmatrix} -mg\sin\gamma \\ mg\sin\mu\cos\gamma \\ mg\cos\mu\cos\gamma \end{bmatrix} = \begin{bmatrix} \dot{v}_\infty \\ mv_\infty r_W \\ -mv_\infty q_W \end{bmatrix} \tag{30}$$

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- Substituting for wind frame angular velocity components with navigation-to-wind frame Euler angles:

$$\begin{bmatrix} p_W \\ q_W \\ r_W \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\gamma \\ 0 & \cos\mu & \sin\mu\cos\gamma \\ 0 & -\sin\mu & \cos\mu\cos\gamma \end{bmatrix} \begin{bmatrix} \dot{\mu} \\ \dot{\gamma} \\ \dot{\sigma} \end{bmatrix} \tag{31}$$

# Equation of Motion (continued)

- Rearranging:

$$\begin{bmatrix} -D + T\cos\alpha\cos\beta - mg\sin\gamma \\ S - T\cos\alpha\sin\beta + mg\sin\mu\cos\gamma \\ L + T\sin\alpha - mg\cos\mu\cos\gamma \end{bmatrix} = \begin{bmatrix} m\dot{v}_\infty \\ mv_\infty\left(\dot{\sigma}\cos\mu\cos\gamma - \dot{\gamma}\sin\mu\right) \\ v_\infty\left(\dot{\gamma}\cos\mu + \dot{\sigma}\sin\mu\cos\gamma\right) \end{bmatrix} \tag{32}$$

  - Can add control inputs to model through thrust $T$ and lift $L$

## Equation of Motion (continued)

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  - Can add control inputs to model through thrust $T$ and lift $L$
- Include navigation frame velocity components:

$$\begin{bmatrix} \dot{x}_N \\ \dot{y}_N \\ \dot{h} \end{bmatrix} = \begin{bmatrix} V_\infty\cos\gamma\cos\sigma \\ V_\infty\cos\gamma\sin\sigma \\ V_\infty\sin\gamma \end{bmatrix} \tag{33}$$

  - $-\dot{h} = \dot{z}_N$: altitude rate instead of down rate

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- Forces and trim states of airplane related as

$$\begin{bmatrix} -\bar{D} + \bar{T}\cos\bar{\alpha}\cos\bar{\beta} - mg\sin\bar{\gamma} \\ \bar{S} - T\cos\bar{\alpha}\sin\bar{\beta} + mg\sin\bar{\mu}\cos\bar{\gamma} \\ -\bar{T}\sin\bar{\alpha} - \bar{L} + mg\cos\bar{\mu}\cos\bar{\gamma} \end{bmatrix} = \begin{bmatrix} 0 \\ m\bar{v}_\infty\dot{\sigma}\cos\bar{\mu}\cos\bar{\gamma} \\ m\bar{v}_\infty\dot{\sigma}\sin\bar{\mu}\cos\bar{\gamma} \end{bmatrix} \tag{34}$$

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- By substitution:

$$\begin{bmatrix} -\bar{D} + \bar{T}\cos\bar{\alpha} - mg\sin\bar{\gamma} \\ mg\sin\bar{\mu}\cos\bar{\gamma} \\ -\bar{T}\sin\bar{\alpha} - \bar{L} + mg\cos\bar{\mu}\cos\bar{\gamma} \end{bmatrix} = \begin{bmatrix} 0 \\ m\bar{v}_\infty\dot{\sigma}\cos\bar{\mu}\cos\bar{\gamma} \\ m\bar{v}_\infty\dot{\sigma}\sin\bar{\mu}\cos\bar{\gamma} \end{bmatrix} \tag{35}$$

# Performance Steady-Flight Equations

- Defining **instantaneous radius of curvature** in navigation frame, $R_c$:

$$R_c = \frac{\bar{V}_\infty \cos \bar{\gamma}}{\dot{\sigma}} \tag{36}$$

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$$\begin{bmatrix} \bar{T}\cos\bar{\alpha} - \bar{D} - mg\sin\bar{\gamma} \\ mg\sin\bar{\mu}\cos\bar{\gamma} \\ -\bar{T}\sin\bar{\alpha} - \bar{L} + mg\cos\bar{\mu}\cos\bar{\gamma} \end{bmatrix} = \begin{bmatrix} 0 \\ m\frac{(\bar{V}_\infty \cos\bar{\gamma})^2}{R_c}\cos\bar{\mu} \\ m\frac{(\bar{V}_\infty \cos\gamma)^2}{R_c}\sin\bar{\mu} \end{bmatrix} \tag{37}$$

# Notes on Performance Steady-Flight Equations

- From second equation:

$$mg \sin \bar{\mu} \cos \bar{\gamma} = m \frac{(\bar{v}_\infty \cos \bar{\gamma})^2}{R} \cos \bar{\mu} \tag{38}$$

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- For any non-zero bank angle

$$\dot{\sigma} = \frac{g \tan \bar{\mu}}{\bar{v}_\infty} \tag{39}$$

$$R_c = \frac{\bar{v}_\infty^2 \cos \bar{\gamma}}{g \tan \bar{\mu}} \tag{40}$$

# Performance Steady-Flight Equations (continued)

- Most general maneuver described by point-mass steady-flight equations: steady climbing or descending coordinated turn, primarily controlled by altering lift, thrust, bank angle of airplane
  - Trajectory of airplane during maneuver: helix about $z_N$-axis and circular projection on $x_N - y_N$ plane

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- Three special cases of steady-flight maneuver: straight climbs/descents, level turns, straight-and-level
  - **Straight flight**: $\dot{\sigma} = \bar{\mu} = 0°$
  - **Level flight**: $\bar{\gamma} = 0°$

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  - For given airplane's aerodynamic and mass properties

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  - For given airplane's aerodynamic and mass properties
- Considering airplane as rigid body: additional moment equations must be balanced to ensure airplane remains at prescribed steady-flight conditions
  - Additional moment equations altered by control inputs to ailerons, rudder, elevator: moment balance as **trimming airplane** considered in subsequent chapter

# Airplane Propulsion and Aerodynamics

- Vast majority of airplanes designed as symmetric in $x_B - y_B$ and $y_B - z_B$ planes: inertia matrix simplified

$$I_G = \begin{bmatrix} I_{xx} & 0 & -I_{xz} \\ 0 & I_{yy} & 0 \\ -I_{xz} & 0 & I_{zz} \end{bmatrix} \tag{41}$$

  - Note: often $I_{xz}$ also neglected due to relatively small magnitude

# Rigid Airplane Equations of Motion

$$\begin{bmatrix} X - g\sin\theta \\ Y + g\sin\phi\cos\theta \\ Z - g\cos\phi\cos\theta \end{bmatrix} = \begin{bmatrix} \dot{u} + qw - rv \\ \dot{v} + ru - pw \\ \dot{w} + pv - qu \end{bmatrix}$$
$$\begin{bmatrix} L \\ M \\ N \end{bmatrix} = \begin{bmatrix} \dot{p} + \frac{I_{zz}-I_{yy}}{I_{xx}} qr - \frac{I_{xz}}{I_{xx}}(\dot{r} + pq) \\ \dot{q} + \frac{I_{xx}-I_{zz}}{I_{yy}} pr + \frac{I_{xz}}{I_{yy}}(p^2 - r^2) \\ \dot{r} + \frac{I_{yy}-I_{xx}}{I_{zz}} pq - \frac{I_{xz}}{I_{zz}}(\dot{p} - qr) \end{bmatrix} \tag{42}$$

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- Supplemental equations

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{43}$$

# Alternative Rigid Airplane EOMs

- Assuming no wind, substitute for lateral and vertical velocity terms:

$$v = u \tan\beta \tag{44}$$

$$w = u \sin\alpha \tag{45}$$

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- Derivatives:

$$\dot{v} = \dot{u} \tan \beta + \dot{\beta} u \sec^2 \beta \tag{46}$$

$$\dot{w} = \dot{u} \sin \alpha + \dot{\alpha} u \cos \alpha \tag{47}$$

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# Rigid Airplane Steady-Flight Conditions

- For airplanes: rigid body **equilibrium flight conditions** a.k.a. rigid body **steady-flight conditions** by definition, occur when state variables in rigid airplane EOMs constant

$$\dot{u} = \dot{\alpha} = \dot{\beta} = \dot{p} = \dot{q} = \dot{r} = \dot{\phi} = \dot{\theta} = 0 \tag{49}$$

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- Implies steady-flight conditions solve rigid body **steady-flight equations** in body frame:

$$\begin{bmatrix} \bar{X} - g\sin\bar{\theta} \\ \bar{Y} + g\sin\bar{\phi}\cos\bar{\theta} \\ \bar{Z} - g\cos\bar{\phi}\cos\bar{\theta} \\ \bar{L}_{roll} \\ \bar{M} \\ \bar{N} \end{bmatrix} = \begin{bmatrix} \bar{q}\bar{u}\sin\bar{\alpha} - \bar{r}\bar{u}\tan\bar{\beta} \\ \bar{r}\bar{u} - \bar{p}\bar{u}\sin\bar{\alpha} \\ \bar{p}\bar{u}\tan\bar{\beta} - \bar{q}\bar{u} \\ \frac{I_{zz}-I_{yy}}{I_{xx}}\bar{q}\bar{r} - \frac{I_{xz}}{I_{xx}}\bar{p}\bar{q} \\ \frac{I_{xx}-I_{zz}}{I_{yy}}\bar{p}\bar{r} + \frac{I_{xz}}{I_{yy}}(\bar{p}^2 - \bar{r}^2) \\ \frac{I_{yy}-I_{xx}}{I_{zz}}\bar{p}\bar{q} + \frac{I_{xz}}{I_{zz}}\bar{q}\bar{r} \end{bmatrix} \tag{50}$$

# Angular Velocity and Yaw Rate Relationship

$$\begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \bar{p} & \bar{q}\sin\bar{\phi}\tan\bar{\theta} & \bar{r}\cos\bar{\phi}\tan\bar{\theta} \\ 0 & \bar{q}\cos\bar{\phi} & -\bar{r}\sin\bar{\phi} \\ 0 & \bar{q}\sin\bar{\phi}\sec\bar{\theta} & \bar{r}\cos\bar{\phi}\sec\bar{\theta} \end{bmatrix} \tag{51}$$

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$$\begin{bmatrix} \bar{p} \\ \bar{q} \\ \bar{r} \end{bmatrix} = \begin{bmatrix} -\dot{\psi}\sin\bar{\theta} \\ -\dot{\psi}\sin\bar{\phi}\cos\bar{\theta} \\ \dot{\psi}\cos\bar{\phi}\cos\bar{\theta} \end{bmatrix} \tag{52}$$

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- $\bar{u}$, $\bar{\beta}$, $\bar{\alpha}$, $\bar{p}$, $\bar{q}$, $\bar{r}$, $\bar{\phi}$, $\bar{\theta}$, $\dot{\psi}$: steady-flight conditions

# Notes on General Steady-Flight Equations

- $\bar{X}$, $\bar{Y}$, $\bar{Z}$, $\bar{L}$, $\bar{M}$, $\bar{N}$: aerodynamic and propulsive forces and moments at steady-flight
  - Functions of steady-flight conditions
  - Alternatively use thrust, lift, side, drag forces at steady-flight: $\bar{T}$, $\bar{L}$, $\bar{S}$, and $\bar{D}$, instead of $\bar{X}$, $\bar{Y}$, and $\bar{Z}$

# Notes on General Steady-Flight Equations

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  - Functions of steady-flight conditions
  - Alternatively use thrust, lift, side, drag forces at steady-flight: $\bar{T}$, $\bar{L}$, $\bar{S}$, and $\bar{D}$, instead of $\bar{X}$, $\bar{Y}$, and $\bar{Z}$
- Gravitational acceleration, $g$, & air density, $\rho$: vary as function of altitude, $h$
  - Requires strict steady-flight condition: constant altitude
  - Variations occur slowly: assume constant for analyzing different "steady-flight" maneuvers

# Stability Frame

- Recall previous modeling defined aerodynamic and propulsive forces in body-fixed frame affixed to fuselage, a.k.a. **fuselage-fixed frame**

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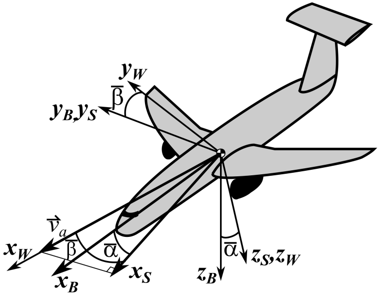
- For linearized rigid airplane dynamics: use alternative body-fixed frame known as **stability frame** (subscript $S$)
- Stability frame related to fuselage-fixed frame through rotation of $\bar{\alpha}$ about $y_B$-axis as shown:



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- Note: $\theta = \gamma$ if $\theta$ describes pitch angle from *N* to *S*

# Transformations to Stability Frame

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- Related to fuselage-fixed body frame $B$ using $C_2$ basic rotation matrix about $\bar{\alpha}$

$$\vec{v}_B = C_2(\bar{\alpha})\vec{v}_S \tag{53}$$

  - $\vec{v}$: some vector

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$$\vec{v}_B = C_2(\bar{\alpha})\vec{v}_S \tag{53}$$

  - $\vec{v}$: some vector
- **Inertia matrix in stability frame**, $I_S$, related to inertia matrix in fuselage-fixed frame, $I_B$, by transformation

$$I_B = C_2(\bar{\alpha}) I_S C_2^T(\bar{\alpha}) \tag{54}$$

# Straight Flight in Stability Frame

- For straight flight, i.e. $\dot{\psi} = 0°/s$, in stability frame: $\bar{\alpha} = 0°$ &

$$\begin{bmatrix} \bar{p} \\ \bar{q} \\ \bar{r} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{55}$$

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- Substitution $\bar{\theta} = \bar{\gamma}$ for stability frame:

$$\begin{bmatrix} \bar{X} - g \sin\bar{\gamma} \\ \bar{Y} + g \sin\bar{\phi} \cos\bar{\gamma} \\ \bar{Z} - g \cos\bar{\phi} \cos\bar{\gamma} \\ \bar{L}_{roll} \\ \bar{M} \\ \bar{N} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{56}$$

# Force Substitution

- Substituting for propulsive and aerodynamic forces separately in stability frame:

$$\begin{bmatrix} -\bar{D}\cos\bar{\beta} - \bar{S}\sin\bar{\beta} + \bar{T}\cos(\theta_T + \bar{\alpha}) \\ \bar{S}\cos\bar{\beta} - \bar{D}\sin\bar{\beta} \\ \bar{L} + \bar{T}\sin(\theta_T + \bar{\alpha}) \\ \bar{L}_{a,S} \\ \bar{M}_{a,S} + \bar{T}(z_T\cos\theta_T - x_T\sin\theta_T) \\ \bar{N}_{a,S} + \bar{N}_{p,S} \end{bmatrix} = \begin{bmatrix} mg\sin\bar{\gamma} \\ -mg\sin\bar{\phi}\cos\bar{\gamma} \\ mg\cos\bar{\phi}\cos\bar{\gamma} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{57}$$

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- $\bar{N}_{p,S} \neq 0$ included in case of **engine-out flight condition**: occurs for multiple-engine airplanes
- $(x_T, z_T)$ defined in fuselage frame coordinates

# Linear Models for Coefficients

- For simplified analysis: assume linear relationships for aerodynamic coefficients w.r.t. $\bar{\alpha}$ and $\bar{\delta}_e$

$$\bar{C}_D = C_{D_0} + C_{D_\alpha}\bar{\alpha} + C_{D_{\delta_e}}\bar{\delta}_e \tag{58}$$

$$\bar{C}_L = C_{L_0} + C_{L_\alpha}\bar{\alpha} + C_{L_{\delta_e}}\bar{\delta}_e \tag{59}$$

$$\bar{C}_m = C_{m_0} + C_{m_\alpha}\bar{\alpha} + C_{m_{\delta_e}}\bar{\delta}_e \tag{60}$$

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- By substitution:

$$\begin{bmatrix} -Q_\infty S_w\left(C_{D_0} + C_{D_\alpha}\bar{\alpha} + C_{D_{\delta_e}}\bar{\delta}_e\right) + \bar{T}\cos(\theta_T + \bar{\alpha}) \\ Q_\infty S_w\left(C_{L_0} + C_{L_\alpha}\bar{\alpha} + C_{L_{\delta_e}}\bar{\delta}_e\right) + \bar{T}\sin(\theta_T + \bar{\alpha}) \\ Q_\infty S_w \bar{c}_w\left(C_{m_0} + C_{m_\alpha}\bar{\alpha} + C_{m_{\delta_e}}\bar{\delta}_e\right) + \bar{T}\frac{z_T\cos\theta_T - x_T\sin\theta_T}{Q_\infty S_w \bar{c}_w} \end{bmatrix} = \begin{bmatrix} mg\sin\bar{\gamma} \\ mg\cos\bar{\gamma} \\ 0 \end{bmatrix} \tag{61}$$

# Approximation Methods for Determination

- For given $Q_\infty$, i.e. given airspeed and altitude, & $\bar{\gamma}$: equations determine three unknowns for trimming airplane
  - I.e. $\bar{T}$, $\bar{\beta}$, $\bar{\alpha}$, $\bar{\delta}_a$, $\bar{\delta}_e$, $\bar{\delta}_r$
  - Modern solution: solved using numerical search methods

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  - Modern solution: solved using numerical search methods

1 Approximate values: assume $\sin(\theta_T + \bar{\alpha}) = 0$ & $\cos(\theta_T + \bar{\alpha}) = 1$

$$\begin{bmatrix} C_{D_\alpha} & C_{D_{\delta_e}} & \frac{1}{Q_\infty S_w} \\ C_{L_\alpha} & C_{L_{\delta_e}} & 0 \\ C_{m_\alpha} & C_{m_{\delta_e}} & \frac{z_T \cos\theta_T - x_T \sin\theta_T}{Q_\infty S_w \bar{c}_w} \end{bmatrix} \begin{bmatrix} \bar{\alpha} \\ \bar{\delta}_e \\ \bar{T} \end{bmatrix} = \begin{bmatrix} \frac{mg}{Q_\infty S_w} \sin\bar{\gamma} + C_{D_0} \\ \frac{mg}{Q_\infty S_w} \cos\bar{\gamma} - C_{L_0} \\ -C_{m_0} \end{bmatrix} \tag{62}$$

  - Solved by multiplying both sides by inverse matrix on left side

# Approximation Methods for Determination (continued)

2 Approximate values: assume $z_T \cos\theta_T - x_T \sin\theta_T = 0$ and $L \gg T \sin(\theta_T + \bar{\alpha})$

$$\begin{bmatrix} \bar{C}_{D_\alpha} & C_{D_{\delta_e}} & \frac{\cos(\theta_T + \bar{\alpha})}{Q_\infty S_w} \\ C_{L_\alpha} & C_{L_{\delta_e}} & 0 \\ C_{m_\alpha} & C_{m_{\delta_e}} & 0 \end{bmatrix} \begin{bmatrix} \bar{\alpha} \\ \bar{\delta}_e \\ \bar{T} \end{bmatrix} = \begin{bmatrix} \frac{mg}{Q_\infty S_w} \sin\bar{\gamma} + C_{D_0} \\ \frac{mg}{Q_\infty S_w} \cos\bar{\gamma} - C_{L_0} \\ -C_{m_0} \end{bmatrix} \tag{63}$$

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- Decouple $\bar{\alpha}$ & $\bar{\delta}_e$ from $\bar{T}$:

$$\begin{aligned} \bar{\alpha} &\approx \frac{\left(\frac{mg}{Q_\infty S_w}\cos\bar{\gamma} - C_{L_0}\right) C_{m_{\delta_e}} + C_{m_0} C_{L_{\delta_e}}}{C_{L_\alpha} C_{m_{\delta_e}} - C_{m_\alpha} C_{L_{\delta_e}}} \\ \bar{\delta}_e &\approx -\frac{\left(\frac{mg}{Q_\infty S_w}\cos\bar{\gamma} - C_{L_0}\right) C_{m_\alpha} + C_{m_0} C_{L_\alpha}}{C_{L_\alpha} C_{m_{\delta_e}} - C_{m_\alpha} C_{L_{\delta_e}}} \end{aligned} \tag{64}$$

  - Only function of lift and $M$-moment coefficients

# Approximation Methods for Determination (continued)

- Determine trim thrust via original equation:

$$\bar{T} = \frac{mg \sin \bar{\gamma} + \left( C_{D_0} + C_{D_\alpha} \bar{\alpha} + C_{D_{\delta_e}} \bar{\delta}_e \right) Q_\infty S_w}{\cos(\theta_T + \bar{\alpha})} \tag{65}$$

  - Note: function of drag coefficients and trim angle of attack and elevator deflection

# Approximation Methods for Determination (continued)

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  - Note: function of drag coefficients and trim angle of attack and elevator deflection
- Recall:

$$\bar{L} = Q_\infty S_w \bar{C}_L \quad (66)$$

$$\bar{L} = Q_\infty S_w \left( C_{L_0} + C_{L_\alpha} \bar{\alpha} + C_{L_{\delta_e}} \bar{\delta}_e \right) \quad (67)$$

# Longitudinal Trim Analysis Notes

- 3 trim values coupled:
  - Maximum limit on thrust, $T_{max}$: limiting factor of achievable lift, $\bar{L}$, via maximizing dynamic pressure, $Q_{\infty,max}$
  - Maximum limit on elevator deflection, $\delta_{e,max}$, OR angle of attack, $\alpha_{max}$: limiting factor of achievable lift, $\bar{L}$, via maximizing lift coefficient, $C_{L,max}$

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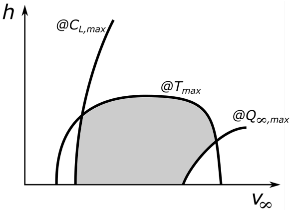
  - Maximum limit on elevator deflection, $\delta_{e,max}$, OR angle of attack, $\alpha_{max}$: limiting factor of achievable lift, $\bar{L}$, via maximizing lift coefficient, $C_{L,max}$
- Limits for straight-and-level flight, i.e. $\bar{\gamma} = 0$, define vehicle's **flight envelope** plot
  - Depicts three curves of $T_{max}$, $C_{L,max}$, and $Q_{\infty,max}$ for airspeed, $v_\infty$ versus altitude, $h$
  - Generally accounts for nonlinear coefficients and numerical solutions
  - Center region: possible steady-flight conditions for airplane



# Lateral-Directional Trim Analysis

- Express aerodynamic forces in terms of coefficients, $C_S$, $C_l$, $C_n$:

$$\begin{bmatrix} Q_\infty S_w \bar{C}_S \cos\bar{\beta} - Q_\infty S_w \bar{C}_D \sin\bar{\beta} \\ Q_\infty S_w \bar{C}_l \\ Q_\infty S_w \bar{C}_n + \bar{N}_{p,S} \end{bmatrix} = \begin{bmatrix} -mg \sin\bar{\phi}\cos\bar{\gamma} \\ 0 \\ 0 \end{bmatrix} \tag{68}$$

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- For simplified analysis: assume linear relationships for aerodynamic coefficients w.r.t. trim drag coefficient, $\bar{C}_D$, trim flight-path angle, $\bar{\gamma}$, trim sideslip angle, $\bar{\beta}$, trim aileron deflection $\bar{\delta}_a$, & trim rudder deflection, $\bar{\delta}_r$

$$\bar{C}_S = C_{S_\beta}\bar{\beta} + C_{S_{\delta_a}}\bar{\delta}_a + C_{S_{\delta_a}}\bar{\delta}_r \tag{69}$$

$$\bar{C}_l = C_{l_\beta}\bar{\beta} + C_{l_{\delta_a}}\bar{\delta}_a + C_{l_{\delta_a}}\bar{\delta}_r \tag{70}$$

$$\bar{C}_n = C_{n_\beta}\bar{\beta} + C_{n_{\delta_a}}\bar{\delta}_a + C_{n_{\delta_a}}\bar{\delta}_r \tag{71}$$

# Simplified Lateral-Directional Trim Analysis

- By substitution:

$$\begin{bmatrix} Q_\infty S_w \left( C_{S_\beta}\bar{\beta} + C_{S_{\delta_a}}\bar{\delta}_a + C_{S_{\delta_a}}\bar{\delta}_r \right) \cos\bar{\beta} - Q_\infty S_w \bar{C}_D \sin\bar{\beta} \\ Q_\infty S_w b_w \left( C_{l_\beta}\bar{\beta} + C_{l_{\delta_a}}\bar{\delta}_a + C_{l_{\delta_a}}\bar{\delta}_r \right) \\ Q_\infty S_w b_w \left( C_{n_\beta}\bar{\beta} + C_{n_{\delta_a}}\bar{\delta}_a + C_{n_{\delta_a}}\bar{\delta}_r \right) + \bar{N}_{p,S} \end{bmatrix} = \begin{bmatrix} -mg\sin\bar{\phi}\cos\bar{\gamma} \\ 0 \\ 0 \end{bmatrix} \tag{72}$$

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$$\begin{bmatrix} Q_\infty S_w \left( C_{S_\beta}\bar{\beta} + C_{S_{\delta_a}}\bar{\delta}_a + C_{S_{\delta_a}}\bar{\delta}_r \right) \cos\bar{\beta} - Q_\infty S_w \bar{C}_D \sin\bar{\beta} \\ Q_\infty S_w b_w \left( C_{l_\beta}\bar{\beta} + C_{l_{\delta_a}}\bar{\delta}_a + C_{l_{\delta_a}}\bar{\delta}_r \right) \\ Q_\infty S_w b_w \left( C_{n_\beta}\bar{\beta} + C_{n_{\delta_a}}\bar{\delta}_a + C_{n_{\delta_a}}\bar{\delta}_r \right) + \bar{N}_{p,S} \end{bmatrix} = \begin{bmatrix} -mg\sin\bar{\phi}\cos\bar{\gamma} \\ 0 \\ 0 \end{bmatrix} \quad (72)$$

- Assuming $\bar{\beta}$ small, i.e. $\sin\bar{\beta} \approx \bar{\beta}$ and $\cos\bar{\beta} \approx 1$:

$$\begin{bmatrix} \bar{C}_{S_\beta} - \bar{C}_D & C_{S_{\delta_a}} & C_{S_{\delta_a}} \\ C_{l_\beta} & C_{l_{\delta_a}} & C_{l_{\delta_r}} \\ C_{n_\beta} & C_{n_{\delta_a}} & C_{n_{\delta_r}} \end{bmatrix} \begin{bmatrix} \bar{\beta} \\ \bar{\delta}_a \\ \bar{\delta}_r \end{bmatrix} = \begin{bmatrix} \frac{mg}{Q_\infty S_w}\cos\bar{\gamma}\sin\bar{\phi} \\ 0 \\ -\frac{\bar{N}_{p,S}}{Q_\infty S_w b_w} \end{bmatrix} \quad (73)$$

  - Solved by multiplying both sides by inverse matrix on left side
  - Note: maximum allowable roll angle, $\phi_{max}$, plays role in possible trim states for airplane

# Engine-Out Balance

- If required to balance engine-out moment, $\bar{N}_{p,S}$ only using rudder:

$$\bar{\delta}_r = -\frac{\bar{N}_{p,S}}{Q_\infty S_w b_w C_{n_{\delta_r}}} \tag{74}$$

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- By setting rudder deflection to limit, $\bar{\delta}_r = \delta_{r,max}$: **minimum control airspeed**, $v_{mc}$

$$v_{mc} = \sqrt{-\frac{\bar{N}_{p,S}}{\frac{1}{2}\rho S_w b_w C_{n_{\delta_r}}}\delta_{r,max}} \tag{75}$$

  - Typically used in sizing vertical tail and rudder

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  - A.k.a. steady-flight
- Airplane trim analysis simplified for straight flight
  - I.e. $\dot{\psi} = 0°/\text{s}$
  - Use stability frame based on $\bar{\alpha}$