Lecture 16: Advanced Concepts from Linear Algebra and LTI Systems

Textbook Sections A.2 & 3.1

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Introduction

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  - Theory applies concepts from linear algebra

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- Introductory concepts:
  - Vector differential equations
  - Matrix eigenvalue decomposition
- Advanced concepts:
  - Norms: vectors, signals, systems, matrices
  - Other matrix decompositions
  - Matrix inequalities

Important Matrix Definitions

- Define $A_{i,j}$ as element of A in i^{th} row and j^{th} column, then
  - **Matrix transpose** of A: $A^T = B$ which assigns $B_{i,j} = A_{j,i}$.
  - **Conjugate matrix transpose** or **Hermitian transpose** of A: $A^H = A^* = B$ which assigns $B_{i,j} = A^*_{j,i}$
  - **matrix inverse** of A: solution to $A^{-1}A = I$.
  - A: **orthogonal matrix** if $A^{-1} = A^T$.
  - A: **unitary matrix** if $A^{-1} = A^*$.
  - A: **symmetric matrix** if $A = A^T$.
  - A: **Hermitian matrix** if $A = A^*$.
  - A: **diagonal matrix** if for $i \neq j, \ A_{i,j} = 0$.
  - A: **upper triangular matrix** if for $i < j, \ A_{i,j} = 0$.
  - A: **lower triangular matrix** if for $i > j, \ A_{i,j} = 0$.
  - A: **square matrix** if equal number of rows and columns.

Matrix Rank

- **Rank** of matrix M, denoted by $\text{rank}(M)$: measure of "non-degenerateness" of system of linear equations encoded by M
  - More formally, expressed mathematically as maximal number of linearly independent rows/columns of M or as **dimension of vector space** spanned by rows/columns of M

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  - Fundamental theorem of linear algebra: column rank equal to row rank

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  - **Row rank**: dimension of row space, i.e. linearly independent rows, of M
  - Fundamental theorem of linear algebra: column rank equal to row rank
- M has **full rank** if rank equals lesser of number of rows and columns, i.e., $\mathrm{rank}(M) = \min(m, n))$
  - M **rank deficient** if not have full rank

Vector Norm

- **Vector norm**, $\|\vec{x}\|$, of any vector $\vec{x} \in \mathbb{R}^{n_x}$: real valued function from $\mathbb{R}^{n_x} \to \mathbb{R}$ with following properties
  1. $\|\vec{x}\| \geq 0$;
  2. $\|\vec{x}\| = 0$ if and only if $\vec{x}$ is zero vector in $\mathbb{R}^{n_x}$;
  3. for any $\lambda \in \mathbb{R}$, $\|\lambda\vec{x}\| = |\lambda| \, \|\vec{x}\|$; and
  4. for any $\vec{y} \in \mathbb{R}^{n_x}$, **triangle inequality** holds

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\| \tag{1}$$

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- Important class of vector norms: L_p**-norms**, a.k.a. p**-norms**, defined as

$$\|\vec{x}\|_p = \left(\sum_{i=1}^{n_x} |x_i|^p \right)^{\frac{1}{p}}, \qquad 1 \leq p \leq \infty \tag{2}$$

  - Subscript p dropped and any norm operation implicitly L^p-norm

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- L_∞**-norm**, a.k.a. **vector maximum norm**:

$$\|\vec{x}\|_\infty = \max |x_i| \tag{5}$$

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- **Matrix norm**, $\|A\|$, of matrix $A \in \mathbb{R}^{m\times n}$: real valued function from $\mathbb{R}^{m\times n} \to \mathbb{R}$ with following properties
  1. $\|A\| \geq 0$;
  2. $\|A\| = 0$ if and only if A zero matrix in $\mathbb{R}^{m\times n}$;
  3. for any $\lambda \in \mathbb{R}$, $\|\lambda A\| = |\lambda|\ \|A\|$; and
  4. for any $B \in \mathbb{R}^{m\times n}$, **triangle inequality** holds

$$\|A + B\| \leq \|A\| + \|B\| \tag{6}$$

First Important Class of Matrix Norms

- Important class of matrix norms: **induced matrix norms**, defined for matrix A and some specified norm $\|\vec{x}\|$ as

$$\|A\| = \sup_{\vec{x} \neq 0} \frac{\|A\vec{x}\|}{\|\vec{x}\|} \tag{7}$$

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- Typically one uses L_p-norms for induced matrix norms:

$$\|A\|_p = \sup_{\vec{x} \neq 0} \frac{\|A\vec{x}\|_p}{\|\vec{x}\|_p} \tag{8}$$

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- $L_{\infty,\infty}$**-norm** a.k.a. **matrix maximum norm**:

$$\|A\|_{\max} = \max_{i,j} |a_{i,j}| \tag{11}$$

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- For nontrivial solutions, i.e., $\vec{v} \neq 0$, solve for

$$\det(\lambda I - A) = 0 \tag{15}$$

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- Based on values of λ, define following:
  - A: **positive definite**, denoted by $A > 0$, if real parts of all $\lambda(A)$ are > 0
  - A: **positive semi-definite**, denoted by $A \geq 0$, if real parts of all $\lambda(A) \geq 0$
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- Negative definite A matrix a.k.a. **stable matrix** or **Hurwitz matrix**

Diagonalizable Matrices

- Square matrix A **diagonalizable**: if eigenvalue decomposition performed as

$$A = V \Lambda V^{-1} \tag{16}$$

- n (left) eigenvectors of A makeup V as

$$V = [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n] \tag{17}$$

- n corresponding *non-repeated* eigenvalues of A makeup diagonal Λ:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \tag{18}$$

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$$J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_k \end{bmatrix} \tag{19}$$

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  - 0's: zero-valued matrices
- k **Jordan blocks**, J_k: specified by dimension r and eigenvalue λ_r, i.e.,

$$J_k(r, \lambda_r) = \begin{bmatrix} \lambda_r & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_r & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_r & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_r & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_r \end{bmatrix} \tag{20}$$

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- Continuous-time LTI state-space system rewritten as

$$\begin{aligned} V\dot{\vec{z}}(t) &= AV\vec{z}(t) + B\vec{u}(t) \\ \vec{y}(t) &= CV\vec{z}(t) + D\vec{u}(t) \end{aligned} \tag{22}$$

Jordan Canonical Form (continued)

$$\begin{aligned} \dot{\vec{z}}(t) &= V^{-1}AV\vec{z}(t) + V^{-1}B\vec{u}(t) \\ \vec{y}(t) &= CV\vec{z}(t) + D\vec{u}(t) \end{aligned} \tag{23}$$

$$\begin{aligned} \dot{\vec{z}}(t) &= \Lambda\vec{z}(t) + \bar{B}\vec{u}(t) \\ \vec{y}(t) &= \bar{C}\vec{z}(t) + D\vec{u}(t) \end{aligned} \tag{24}$$

- Λ: Jordan matrix, i.e., diagonal or nearly diagonal
- $\bar{B}$ new input matrix
- $\bar{C}$ new output matrix

Generalized Eigenvectors

- To obtain V, solve for **generalized eigenvectors** for each Jordan block, $J_k(r, \lambda_r)$, which solve

$$\begin{aligned}(A - \lambda_r I)\vec{v}_1 &= 0 \\ (A - \lambda_r I)\vec{v}_2 &= \vec{v}_1 \\ &\vdots \\ (A - \lambda_r I)\vec{v}_r &= \vec{v}_{r-1}\end{aligned} \tag{25}$$

Singular Value Decomposition

- When considering LTI system behavior from input to output: **singular value decomposition (SVD)** defined for any $m \times n$ real-valued matrix M:

$$M = U\Sigma V^{-1} \tag{26}$$

  - U: $m \times m$ orthogonal matrix
  - V: $n \times n$ orthogonal matrix

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- Σ diagonal $m \times n$ matrix with *non-negative* real numbers on diagonal:

$$\Sigma = \begin{bmatrix} \Sigma_1 & \tilde{0} \\ \tilde{0} & \tilde{0} \end{bmatrix}, \qquad \Sigma_1 = \mathrm{diag}(\sigma_1, \cdots, \sigma_k) \tag{27}$$

Singular Value Decomposition (continued)

- Diagonal entries of Σ: **singular values** of M and ordered in size with $\bar{\sigma} = \sigma_1 \geq \cdots \geq \sigma_k = \underline{\sigma}$ with $k = \min(m, n)$ singular values because of non-square nature

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- U & V orthogonal, i.e., $U^{-1} = U^T$ & $V^{-1} = V^T$, thus SVD:

$$M = U \Sigma V^T \tag{28}$$

  - Much simpler to compute

Singular Values

- Singular values, σ_i, for M: non-negative real numbers for which exists unit-length real-valued vectors $\vec{u}$ and $\vec{v}$:

$$M\vec{v}_i = \sigma_i\vec{u} \tag{29}$$

$$\vec{u}_i^T M = \sigma_i\vec{v}^T \quad \text{or} \quad M^T\vec{u}_i = \sigma_i\vec{v} \tag{30}$$

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$$V = \begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_n\end{bmatrix} \tag{31}$$

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$$\sigma_i(M) = \sqrt{\lambda_i(M^* M)} = \sqrt{\lambda_i(MM^*)} > 0 \tag{33}$$

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- All σ_i^2: eigenvalues of MM^* and M^*M
- All $\vec{v}_i$: eigenvectors of M^*M
- All $\vec{u}_i$: eigenvectors of MM^*

Maximum and Minimum Singular Values

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- **Condition number** for A:

$$\kappa(A) = \frac{\bar{\sigma}(A)}{\underline{\sigma}(A)} \tag{37}$$

  - Determines how "well" A can be inverted

Other Matrix Decompositions

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2. **Cholesky decomposition** for Hermitian positive definite matrices:

$$A = LL^* \tag{39}$$

   - L: upper triangular matrix with real and positive entries along its main diagonal
   - If A real-valued, then reduces to $A = LL^T$
   - Hermitian defined next

Nyquist/Bode Plots

- Frequency response of SISO LTI system: Fourier transform of system, i.e. $G(j\omega)$
  - Fourier inversion theorem: $G(j\omega)$ provides system response to *all* harmonic sinusoids within any “well-behaved” input signal
  - From which one can exactly replicate output signal

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- Frequency response provides magnitude and phase values for all *steady-state* step and sinusoidal responses
- Nyquist and Bode plots used to analyze frequency response of SISO LTI systems

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- Recall definition of singular value decomposition (SVD) $\rightarrow$ model transfer function matrix at particular frequency as

$$[G(j\omega)] = U(j\omega)\Sigma(j\omega)V^{-1}(j\omega) \tag{40}$$

  - Each matrix depends on frequency
  - Values of unitary matrices, U and V, may change
  - Gain of “directions” from input to output captured by singular values within Σ

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  - Albeit potential coupling between all inputs and outputs via transfer function matrix
- Recall definition of singular value decomposition (SVD) $\rightarrow$ model transfer function matrix at particular frequency as

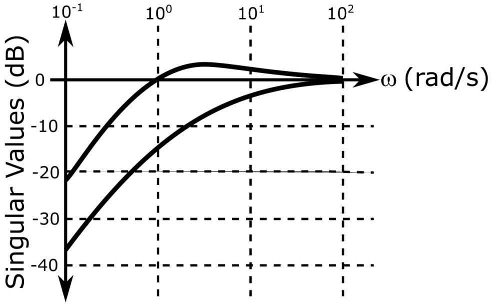
$$[G(j\omega)] = U(j\omega)\Sigma(j\omega)V^{-1}(j\omega) \tag{40}$$

  - Each matrix depends on frequency
  - Values of unitary matrices, U and V, may change
  - Gain of "directions" from input to output captured by singular values within Σ
- Frequency response of MIMO LTI systems analyzed by plotting singular values as function of ω: **singular value plot**
  - A.k.a. **Sigma plot** or σ**-plot**

Sigma Plot Example

- Consider

$$[G(s)] = \begin{bmatrix} \frac{s}{s+3} & \frac{-6s}{s^2+6s+9} \\ 0 & \frac{s}{s+3} \end{bmatrix} \tag{41}$$



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  - $u_i(t)$: i^{th} element of $\vec{u}(t)$, may or may not be finite
- Particular interest: **signal 2-norm**, i.e.

$$\|\vec{u}(t)\|_2 = \left(\int_0^\infty \sum_{i=1}^{n_u} u_i(t)^2 dt \right)^{\frac{1}{2}} \tag{43}$$

Signal p-norm (continued)

- Redefined as

$$\|\vec{u}(t)\|_2 = \left(\int_0^\infty \mathrm{Tr}\left[\vec{u}(t)^T \vec{u}(t) \right] dt \right)^{\frac{1}{2}} \tag{44}$$

Signal p-norm (continued)

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- By Parseval's theorem:

$$\|\vec{u}(t)\|_2 = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Tr}\left[\vec{u}(j\omega)^* \vec{u}(j\omega)\right] d\omega\right)^{\frac{1}{2}} \tag{45}$$

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$$\|G\|_{p \leftarrow q} = \max_{0 \neq \|\vec{u}\|_q \leq \infty, \vec{x}(0)=0} \frac{\|\vec{y}\|_p}{\|\vec{u}\|_q} \tag{46}$$

  - Also stated: smallest constant, c, such that $\|\vec{y}\|_p \leq c\|\vec{u}\|_q$
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- Particular interest: $\|G\|_{2\leftarrow 2}$ norms, a.k.a. $\mathcal{H}_\infty$-norm of G for *stable* G:

$$\|G\|_\infty = \|G\|_{2\leftarrow 2} \tag{47}$$

  - Finite if and only if G strictly proper: $D = 0$ in LTI state-space, with no poles on $j\omega$ axis

$\bar{\sigma}$ and $\mathcal{H}_\infty$-Norm

- By definition of LTI system as $\vec{y}(j\omega) = G(j\omega)\vec{u}(j\omega)$, and definition of singular values of transfer function matrix, $G(j\omega)$:

$$\bar{\sigma}(G(j\omega)) = \max_{0 \neq \|\vec{u}(j\omega)\|_2} \frac{\|G(j\omega)\vec{u}(j\omega)\|_2}{\|\vec{u}(j\omega)\|_2} \tag{48}$$

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  - $\bar{\sigma}(G(j\omega))$: maximum singular value of transfer function matrix $G(j\omega)$ evaluated at ω
- Over entire frequency spectrum:

$$\|G\|_\infty = \max_{\omega} \bar{\sigma}(G(j\omega)) \tag{49}$$

  - SISO systems: maximum magnitude of Bode plot
  - MIMO systems, $\mathcal{H}_\infty$-norm: maximum singular value across all frequencies, i.e. peak on σ-plot

$\mathcal{H}_2$-Norm

- $\mathcal{H}_2$-norm for stable LTI system, G:

$$\|G\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \|G(j\omega)\|_F^2 d\omega \right)^{\frac{1}{2}} \tag{50}$$

  - $\|G(j\omega)\|_F$: Frobenius or entry-wise $L_{2,2}$-norm of transfer function matrix evaluated at $j\omega$

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- For SISO systems:

$$\|G\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega)|^2 d\omega \right)^{\frac{1}{2}} \tag{52}$$

  - Measure of area under Bode plot of system

$\mathcal{H}_2$-Norm (continued)

- Using SVD of $G(j\omega) = U(j\omega)\Sigma(j\omega)V(j\omega)^*$:

$$\|G\|_2 = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Tr}\left[V(j\omega)\Sigma(j\omega)^* U(j\omega)^* U(j\omega)\Sigma(j\omega)V(j\omega)^*\right] d\omega\right)^{\frac{1}{2}} \tag{53}$$

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- Important interpretation of $\mathcal{H}_2$-norm occurs by considering $\vec{u}(t)$ as white noise random process with unit variance, i.e.

$$\mathbb{E}\left[\vec{u}(t_1)\vec{u}(t_2)^T\right] = I_{n_u}\delta(t_1 - t_2) \tag{56}$$

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$$\|G\|_2^2 = \mathbb{E}\left[\lim_{t_f \to \infty} \frac{1}{t_f}\int_0^{t_f} \mathrm{Tr}\left[\vec{y}(t)\vec{y}(t)^T\right] dt\right] \tag{57}$$

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- Provides useful motivation for using $\mathcal{H}_2$ OCPs when *unobservable* random processes significant inputs to plant

Calculating $\mathcal{H}_2$-Norm

- By Parseval's theorem: transfer function matrix as solution to impulse response of MIMO system with $D = 0$, i.e.

$$\|G\|_2 = \left(\int_0^\infty \mathrm{Tr}\left[C e^{At} B B^T e^{A^T t} C^T \right] dt \right)^{\frac{1}{2}} \tag{58}$$

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Calculating $\mathcal{H}_2$-Norm (continued)

- W_C found by noting

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$$-BB^T = AW_C + W_C A^t \tag{64}$$

- W_C unique solution to

$$AW_C + W_C A^T + BB^T = 0 \tag{65}$$

  - Exists if and only if A stable

Calculating $\mathcal{H}_2$-Norm (continued)

- Alternatively:

$$\|G\|_2 = \left(\int_0^{\infty} \mathrm{Tr}\left[B^T e^{A^T t} C^T C e^{At} B \right] dt \right)^{\frac{1}{2}} \tag{66}$$

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Matrix Lyapunov Equations

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- **Matrix Lyapunov equation**: type of LME of form

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- Unique solution

$$X = \int_0^\infty e^{A^T t} Q e^{At} dt \tag{71}$$

  - Well-defined as A assumed stable
  - X typically obtained numerically using certain eigenvalue decompositions

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  - Certain results crucial to control theory and design
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