Lecture 15: Elastic Airplane Dynamics and Structural-Mode Control

Textbook Sections 10.4 & 12.1

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Introduction

- Previous: background for developing elastic vibration EOMs alongside rigid body EOMs

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- Showed vibration modes have own EOMs, enter rigid body EOMs through elastic effects on forces and moments
- Airplanes: considered as **dynamic-elastic effects**
  - Elastic deformation effects on forces and moments in rigid body EOMs
  - Vibration dynamics
- Airplanes: considered as **static-elastic effects**:
  - Elastic deformation effects on forces and moments in rigid body EOMs
  - Do not include vibration dynamics

Elastic Flight Vehicle EOMs

$$\begin{bmatrix} \dot{u} + qw - rv \\ \dot{v} + ru - pw \\ \dot{w} + pv - qu \\ \dot{p} + \frac{I_{zz}-I_{yy}}{I_{xx}} qr - \frac{I_{xz}}{I_{xx}}(\dot{r} + pq) \\ \dot{q} + \frac{I_{xx}-I_{zz}}{I_{yy}} pr - \frac{I_{xz}}{I_{yy}}(r^2 - p^2) \\ \dot{r} + \frac{I_{yy}-I_{xx}}{I_{zz}} pq - \frac{I_{xz}}{I_{zz}}(\dot{p} - qr) \end{bmatrix} = \begin{bmatrix} X - g\sin\theta \\ Y + g\cos\theta\sin\phi \\ Z + g\cos\theta\cos\phi \\ L \\ M \\ N \end{bmatrix} \tag{1}$$

$$\ddot{\eta}_i + 2\zeta_i\omega_i\dot{\eta}_i + \omega_i^2\eta_i = \frac{Q_i}{\mathcal{M}_i}, \quad i = 1, \ldots, n$$

Alternative Wind Frame

- Use thrust, $\vec{T}$, and wind frame forces: lift L, side S, drag D

$$
\begin{aligned}
m\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} &= \vec{T} + C_{B\leftarrow W}(\alpha,\beta)\begin{bmatrix} -D \\ S \\ -L \end{bmatrix} \\
m\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} &= \vec{T} + \begin{bmatrix} \cos\alpha\cos\beta & -\cos\alpha\sin\beta & -\sin\alpha \\ \sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & -\sin\alpha\sin\beta & \cos\alpha \end{bmatrix}\begin{bmatrix} -D \\ S \\ -L \end{bmatrix}
\end{aligned} \tag{2}
$$

- Dynamic-elastic effects applied to L, S, D instead of X, Y, Z

Stability and Control Derivatives

- Recall: aerodynamic and propulsive forces using stability and control derivatives/coefficients ($\delta_t = T$)

Stability and Control Derivatives

- Recall: aerodynamic and propulsive forces using stability and control derivatives/coefficients ($\delta_t = T$)
- Add coefficients for modal coordinates and modal coordinate rates:

$$
\begin{aligned}
\begin{bmatrix} X \\ Z \\ M \end{bmatrix} &= \begin{bmatrix} X_0 \\ Z_0 \\ M_0 \end{bmatrix} + \begin{bmatrix} 0 & X_{\dot{\alpha}} & 0 \\ 0 & Z_{\dot{\alpha}} & 0 \\ 0 & M_{\dot{\alpha}} & 0 \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{\alpha} \\ \dot{q} \end{bmatrix} + \begin{bmatrix} X_u & X_\alpha & X_q \\ Z_u & Z_\alpha & Z_q \\ M_u & M_\alpha & M_q \end{bmatrix} \begin{bmatrix} u \\ \alpha \\ q \end{bmatrix} \\
&+ \begin{bmatrix} X_{\delta_e} & X_{\delta_t} \\ Z_{\delta_e} & Z_{\delta_t} \\ M_{\delta_e} & M_{\delta_t} \end{bmatrix} \begin{bmatrix} \delta_e \\ \delta_t \end{bmatrix} \\
&+ \begin{bmatrix} X_{\dot{\eta}_1} & \cdots & X_{\dot{\eta}_1} \\ Z_{\dot{\eta}_1} & \cdots & Z_{\dot{\eta}_1} \\ M_{\dot{\eta}_1} & \cdots & M_{\dot{\eta}_1} \end{bmatrix} \begin{bmatrix} \dot{\eta}_1 \\ \vdots \\ \dot{\eta}_n \end{bmatrix} + \begin{bmatrix} X_{\eta_1} & \cdots & X_{\eta_1} \\ Z_{\eta_1} & \cdots & Z_{\eta_1} \\ M_{\eta_1} & \cdots & M_{\eta_1} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix}
\end{aligned}
\tag{3}
$$

Stability and Control Derivatives (continued)

$$\begin{aligned}\begin{bmatrix} Y \\ L \\ N \end{bmatrix} &= \begin{bmatrix} Y_0 \\ L_0 \\ N_0 \end{bmatrix} + \begin{bmatrix} Y_\beta & Y_p & Y_r \\ L_\beta & L_p & L_r \\ N_\beta & N_p & N_r \end{bmatrix} \begin{bmatrix} \beta \\ p \\ r \end{bmatrix} + \begin{bmatrix} Y_{\delta_a} & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix} \\ &+ \begin{bmatrix} Y_{\dot{\eta}_1} & \cdots & Y_{\dot{\eta}_1} \\ L_{\dot{\eta}_1} & \cdots & L_{\dot{\eta}_1} \\ N_{\dot{\eta}_1} & \cdots & N_{\dot{\eta}_1} \end{bmatrix} \begin{bmatrix} \dot{\eta}_1 \\ \vdots \\ \dot{\eta}_n \end{bmatrix} + \begin{bmatrix} Y_{\eta_1} & \cdots & Y_{\eta_1} \\ L_{\eta_1} & \cdots & L_{\eta_1} \\ N_{\eta_1} & \cdots & N_{\eta_1} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix}\end{aligned} \tag{4}$$

Coefficient/Derivative Conversion

- Dynamic pressure: $Q_\infty = \frac{1}{2}\rho_\infty \bar{V}_\infty^2$
- Trimmed airspeed: $\bar{V}_\infty = \sqrt{u^2 + v^2 + w^2}$

$\bullet$	$X_\bullet$	$Z_\bullet$	$M_\bullet$
u	$\frac{Q_\infty S_W}{m\bar{V}_\infty} C_{X_u}$	$\frac{Q_\infty S_W}{m\bar{V}_\infty} C_{Z_u}$	$\frac{Q_\infty S_W \bar{c}_W}{I_{yy}\bar{V}_\infty} C_{m_u}$
α	$\frac{Q_\infty S_W}{m} C_{X_\alpha}$	$\frac{Q_\infty S_W}{m} C_{Z_\alpha}$	$\frac{Q_\infty S_W \bar{c}_W}{I_{yy}} C_{m_\alpha}$
q	$\frac{Q_\infty S_W \bar{c}_W}{2m\bar{V}_\infty} C_{X_q}$	$\frac{Q_\infty S_W \bar{c}_W}{2m\bar{V}_\infty} C_{Z_q}$	$\frac{Q_\infty S_W \bar{c}_W^2}{2I_{yy}\bar{V}_\infty} C_{m_q}$
$\dot{\alpha}$	$\frac{Q_\infty S_W \bar{c}_W}{2m\bar{V}_\infty} C_{X_{\dot{\alpha}}}$	$\frac{Q_\infty S_W \bar{c}_W}{2m\bar{V}_\infty} C_{Z_{\dot{\alpha}}}$	$\frac{Q_\infty S_W \bar{c}_W^2}{2I_{yy}\bar{V}_\infty} C_{m_{\dot{\alpha}}}$
δ_e	$\frac{Q_\infty S_W}{m} C_{X_{\delta_e}}$	$\frac{Q_\infty S_W}{m} C_{Z_{\delta_e}}$	$\frac{Q_\infty S_W \bar{c}_W}{I_{yy}} C_{m_{\delta_e}}$
δ_t	$\frac{Q_\infty S_W}{m} C_{X_{\delta_t}}$	$\frac{Q_\infty S_W}{m} C_{Z_{\delta_t}}$	$\frac{Q_\infty S_W \bar{c}_W}{I_{yy}} C_{m_{\delta_t}}$
η_i	$\frac{Q_\infty S_W}{m} C_{X_{\eta_i}}$	$\frac{Q_\infty S_W}{m} C_{Z_{\eta_i}}$	$\frac{Q_\infty S_W \bar{c}_W}{I_{yy}} C_{m_{\eta_i}}$
$\dot{\eta}_i$	$\frac{Q_\infty S_W}{m\bar{V}_\infty} C_{X_{\dot{\eta}_i}}$	$\frac{Q_\infty S_W}{m\bar{V}_\infty} C_{Z_{\dot{\eta}_i}}$	$\frac{Q_\infty S_W \bar{c}_W}{I_{yy}\bar{V}_\infty} C_{m_{\dot{\eta}_i}}$

Coefficient/Derivative Conversion (continued)

$\bullet$	$Y_\bullet$	$L_\bullet$	$N_\bullet$
β	$\frac{Q_\infty S_w}{m} C_{Y_\beta}$	$\frac{Q_\infty S_w b_w}{I_{xx}} C_{l_\beta}$	$\frac{Q_\infty S_w b_w}{I_{zz}} C_{n_\beta}$
p	$\frac{Q_\infty S_w b_w}{2m\bar{V}_\infty} C_{Y_p}$	$\frac{Q_\infty S_w b_w^2}{2I_{xx}\bar{V}_\infty} C_{l_p}$	$\frac{Q_\infty S_w b_w^2}{2I_{zz}\bar{V}_\infty} C_{n_p}$
r	$\frac{Q_\infty S_w b_w}{2m\bar{V}_\infty} C_{Y_r}$	$\frac{Q_\infty S_w b_w^2}{2I_{xx}\bar{V}_\infty} C_{l_r}$	$\frac{Q_\infty S_w b_w^2}{2I_{zz}\bar{V}_\infty} C_{n_r}$
δ_a	$\frac{Q_\infty S_w}{m} C_{Y_{\delta_a}}$	$\frac{Q_\infty S_w b_w}{I_{xx}} C_{l_{\delta_a}}$	$\frac{Q_\infty S_w b_w}{I_{zz}} C_{n_{\delta_a}}$
δ_r	$\frac{Q_\infty S_w}{m} C_{Y_{\delta_r}}$	$\frac{Q_\infty S_w b_w}{I_{xx}} C_{l_{\delta_r}}$	$\frac{Q_\infty S_w b_w}{I_{zz}} C_{n_{\delta_r}}$
η_i	$\frac{Q_\infty S_w}{m} C_{Y_{\eta_i}}$	$\frac{Q_\infty S_w b_w}{I_{xx}} C_{l_{\eta_i}}$	$\frac{Q_\infty S_w b_w}{I_{zz}} C_{n_{\eta_i}}$
$\dot{\eta}_i$	$\frac{Q_\infty S_w}{m\bar{V}_\infty} C_{Y_{\dot{\eta}_i}}$	$\frac{Q_\infty S_w b_w}{I_{xx}\bar{V}_\infty} C_{n_{\dot{\eta}_i}}$	$\frac{Q_\infty S_w b_w}{I_{zz}\bar{V}_\infty} C_{n_{\dot{\eta}_i}}$

n Vibration LTI ODEs

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$$\ddot{\eta}_i + 2\zeta_i\omega_i\dot{\eta}_i + \omega_i^2\eta_i = \frac{Q_i}{\mathcal{M}_i}, \quad i = 1, ..., n \tag{5}$$

Generalized Forces

$$
\begin{aligned}
Q_i = & Q_{i_0} + \begin{bmatrix} Q_{i_u} & Q_{i_\beta} & Q_{i_\alpha} & Q_{i_p} & Q_{i_q} & Q_{i_r} \end{bmatrix} \begin{bmatrix} u \\ \beta \\ \alpha \\ p \\ q \\ r \end{bmatrix} \\
& + \begin{bmatrix} Q_{i_{\delta_a}} & Q_{i_{\delta_e}} & Q_{i_{\delta_r}} & Q_{i_{\delta_t}} \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_e \\ \delta_r \\ \delta_t \end{bmatrix} \\
& + \begin{bmatrix} Q_{i_{\dot{\eta}_1}} & \cdots & Q_{i_{\dot{\eta}_n}} \end{bmatrix} \begin{bmatrix} \dot{\eta}_1 \\ \vdots \\ \dot{\eta}_n \end{bmatrix} + \begin{bmatrix} Q_{i_{\eta_1}} & \cdots & Q_{i_{\eta_n}} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix}
\end{aligned} \tag{6}
$$

Generalized Forces and Coefficients

$$Q_{i_\bullet} = Q_\infty S_W \bar{c}_W C_{Q_{i_\bullet}} \tag{7}$$

- $\bullet = \alpha,\ \beta,\ \delta_a,\ \delta_e,\ \delta_r,\ \delta_t,\ \eta_j$ for $j = 1, \ldots, n$

Generalized Forces and Coefficients

$$Q_{i_\bullet} = Q_\infty S_W \bar{c}_W C_{Q_{i_\bullet}} \tag{7}$$

- $\bullet = \alpha, \beta, \delta_a, \delta_e, \delta_r, \delta_t, \eta_j$ for $j = 1, ..., n$

$$Q_{i_\bullet} = \frac{Q_\infty S_W \bar{c}_W}{\bar{V}_\infty} C_{Q_{i_\bullet}} \tag{8}$$

- $\bullet = u, p, q, r, \dot{\eta}_j$ for $j = 1, ..., n$

Static-Elastic Effects

- Recall: elastic flight vehicle EOMs

$$
\begin{aligned}
\dot{\vec{x}}_{rig} &= f_{rig}(\vec{x}_{rig}, \phi, \theta) + \mathcal{A}_{rig\leftarrow rig}\vec{x}_{rig} + \begin{bmatrix} \mathcal{A}_{rig\leftarrow\eta} & \mathcal{A}_{rig\leftarrow\dot{\eta}} \end{bmatrix} \vec{x}_{vib} + \mathcal{B}_{rig}\vec{u} \\
\dot{\vec{x}}_{vib} &= \begin{bmatrix} 0_{n\times 6} \\ A_{vib\leftarrow rig} \end{bmatrix} \vec{x}_{rig} + \begin{bmatrix} 0_{n\times n} & I_{n\times n} \\ A_{vib\leftarrow\eta} & A_{vib\leftarrow\dot{\eta}} \end{bmatrix} \vec{x}_{vib} + \begin{bmatrix} 0_{n\times 4} \\ B_{vib} \end{bmatrix} \vec{u}
\end{aligned}
\tag{9}
$$

Static-Elastic Effects

- Recall: elastic flight vehicle EOMs

$$\begin{aligned}\dot{\vec{x}}_{rig} &= f_{rig}(\vec{x}_{rig}, \phi, \theta) + \mathcal{A}_{rig\leftarrow rig}\vec{x}_{rig} + \begin{bmatrix}\mathcal{A}_{rig\leftarrow\eta} & \mathcal{A}_{rig\leftarrow\dot{\eta}}\end{bmatrix}\vec{x}_{vib} + \mathcal{B}_{rig}\vec{u} \\ \dot{\vec{x}}_{vib} &= \begin{bmatrix}0_{n\times 6} \\ A_{vib\leftarrow rig}\end{bmatrix}\vec{x}_{rig} + \begin{bmatrix}0_{n\times n} & I_{n\times n} \\ A_{vib\leftarrow\eta} & A_{vib\leftarrow\dot{\eta}}\end{bmatrix}\vec{x}_{vib} + \begin{bmatrix}0_{n\times 4} \\ B_{vib}\end{bmatrix}\vec{u}\end{aligned} \tag{9}$$

- Deformation equilibrium, i.e. static-elastic effects: $\ddot{\eta}_i = \dot{\eta}_i = 0 \ \forall i$

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- Recall: elastic flight vehicle EOMs

$$\begin{aligned}\dot{\vec{x}}_{rig} &= f_{rig}(\vec{x}_{rig}, \phi, \theta) + \mathcal{A}_{rig\leftarrow rig}\vec{x}_{rig} + \begin{bmatrix}\mathcal{A}_{rig\leftarrow\eta} & \mathcal{A}_{rig\leftarrow\dot{\eta}}\end{bmatrix}\vec{x}_{vib} + \mathcal{B}_{rig}\vec{u} \\ \dot{\vec{x}}_{vib} &= \begin{bmatrix}0_{n\times 6} \\ A_{vib\leftarrow rig}\end{bmatrix}\vec{x}_{rig} + \begin{bmatrix}0_{n\times n} & I_{n\times n} \\ A_{vib\leftarrow\eta} & A_{vib\leftarrow\dot{\eta}}\end{bmatrix}\vec{x}_{vib} + \begin{bmatrix}0_{n\times 4} \\ B_{vib}\end{bmatrix}\vec{u}\end{aligned} \tag{9}$$

- Deformation equilibrium, i.e. static-elastic effects: $\ddot{\eta}_i = \dot{\eta}_i = 0 \ \forall i$
- Solve for **static-elastic modal coordinates**:

$$\bar{\eta} = \begin{bmatrix}\bar{\eta}_1 & \cdots & \bar{\eta}_n\end{bmatrix} \tag{10}$$

- In terms of rigid state and control inputs

Static-Elastic EOMs

- Vibration EOM:

$$\vec{0} = \begin{bmatrix} 0_{n\times 6} \\ A_{vib\leftarrow rig} \end{bmatrix} \vec{x}_{rig} + \begin{bmatrix} 0_{n\times n} & I_{n\times n} \\ A_{vib\leftarrow \eta} & A_{vib\leftarrow \dot{\eta}} \end{bmatrix} \begin{bmatrix} \bar{\eta} \\ \vec{0} \end{bmatrix} + \begin{bmatrix} 0_{n\times 4} \\ B_{vib} \end{bmatrix} \vec{u} \tag{11}$$

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- Non-trivial portion:

$$0 = A_{vib\leftarrow rig} \vec{x}_{rig} + A_{vib\leftarrow \eta} \bar{\eta} + B_{vib} \vec{u} \tag{12}$$

Static-Elastic EOMs

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- Non-trivial portion:

$$0 = A_{vib\leftarrow rig}\vec{x}_{rig} + A_{vib\leftarrow\eta}\bar{\eta} + B_{vib}\vec{u} \tag{12}$$

- **Static-elastic constraint**:

$$\bar{\eta} = A_{vib\leftarrow\eta}^{-1}\left(A_{vib\leftarrow rig}\vec{x}_{rig} + B_{vib}\vec{u}\right) \tag{13}$$

Static-Elastic Effects (continued)

- Rigid-body EOM:

$$\dot{\vec{x}}_{rig} = f_{rig}(\vec{x}_{rig}, \phi, \theta) + \mathcal{A}_{rig\leftarrow rig}\vec{x}_{rig} + \begin{bmatrix} \mathcal{A}_{rig\leftarrow\eta} & \mathcal{A}_{rig\leftarrow\dot{\eta}} \end{bmatrix} \begin{bmatrix} \bar{\eta} \\ \vec{0} \end{bmatrix} + \mathcal{B}_{rig}\vec{u} \tag{14}$$

Static-Elastic Effects (continued)

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- By back-substitution:

$$\begin{aligned}\dot{\vec{x}}_{rig} =& f_{rig}(\vec{x}_{rig}, \phi, \theta) + \mathcal{A}_{rig\leftarrow rig}\vec{x}_{rig} \\ &+ \mathcal{A}_{rig\leftarrow\eta}A_{vib\leftarrow\eta}^{-1}\left(A_{vib\leftarrow rig}\vec{x}_{rig} + B_{vib}\vec{u}\right) + B_{rig}\vec{u}\end{aligned} \tag{15}$$

Static-Elastic Effects (continued)

- Rigid-body EOM:

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- By back-substitution:

$$\begin{aligned} \dot{\vec{x}}_{rig} =& f_{rig}(\vec{x}_{rig}, \phi, \theta) + \mathcal{A}_{rig\leftarrow rig}\vec{x}_{rig} \\ &+ \mathcal{A}_{rig\leftarrow \eta} A^{-1}_{vib\leftarrow \eta} \left(A_{vib\leftarrow rig}\vec{x}_{rig} + B_{vib}\vec{u}\right) + B_{rig}\vec{u} \end{aligned} \tag{15}$$

- **Static-elastic rigid vehicle EOMs**

$$\begin{aligned} \dot{\vec{x}}_{rig} =& f_{rig}(\vec{x}_{rig}, \phi, \theta) + \left(\mathcal{A}_{rig\leftarrow rig} - \mathcal{A}_{rig\leftarrow \eta} A^{-1}_{vib\leftarrow \eta} A_{vib\leftarrow rig}\right)\vec{x}_{rig} \\ &+ \left(B_{rig} - \mathcal{A}_{rig\leftarrow \eta} \mathcal{A}^{-1}_{vib\leftarrow \eta} B_{vib}\right)\vec{u} \end{aligned} \tag{16}$$

Residualization

- Process called **residualization** of vibration degrees-of-freedom into new matrices of static-elastic stability and control derivatives/coefficients
  - Residualized static-elastic derivatives: elements of $\left(\mathcal{A}_{rig\leftarrow rig} - \mathcal{A}_{rig\leftarrow\eta} A^{-1}_{vib\leftarrow\eta} A_{vib\leftarrow rig}\right)$ & $\left(B_{rig} - \mathcal{A}_{rig\leftarrow\eta} A^{-1}_{vib\leftarrow\eta} B_{vib}\right)$

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- In general depend on flight conditions
  - Directly affect loads on vehicle's structure, especially dynamic pressure

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- In general depend on flight conditions
  - Directly affect loads on vehicle's structure, especially dynamic pressure
- If aerodynamic forces and moments not truly linear $\rightarrow$ numerical techniques to find static-elastic modal coordinates

Rigid Body Modeling

- Linearized EOMs for elastic airplanes typically use fuselage body frame (subscript F) instead of stability body frame (subscript S) for developing vibration and dynamic-elastic coefficients

Rigid Body Modeling

- Linearized EOMs for elastic airplanes typically use fuselage body frame (subscript F) instead of stability body frame (subscript S) for developing vibration and dynamic-elastic coefficients
- Linearized rigid airplane dynamics in stability frame $\rightarrow$ transform perturbed rigid body aerodynamic and propulsive forces and moments

$$\begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix}_F = \begin{bmatrix} \cos\bar{\alpha} & 0 & -\sin\bar{\alpha} \\ 0 & 1 & 0 \\ \sin\bar{\alpha} & 0 & \cos\bar{\alpha} \end{bmatrix} \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix}_S \tag{17}$$

$$\begin{bmatrix} \Delta L \\ \Delta M \\ \Delta N \end{bmatrix}_F = \begin{bmatrix} \cos\bar{\alpha} & 0 & -\sin\bar{\alpha} \\ 0 & 1 & 0 \\ \sin\bar{\alpha} & 0 & \cos\bar{\alpha} \end{bmatrix} \begin{bmatrix} \Delta L \\ \Delta M \\ \Delta N \end{bmatrix}_S \tag{18}$$

F Frame Linearized EOMs

- Note: $\bar{\alpha}$ may not equal 0

$$\begin{bmatrix}\Delta X\\ \Delta Y\\ \Delta Z\end{bmatrix} + g\begin{bmatrix}-\cos\bar{\theta} & 0\\ -\sin\bar{\theta}\sin\bar{\phi} & \cos\bar{\theta}\cos\bar{\phi}\\ \sin\bar{\theta}\cos\bar{\phi} & \cos\bar{\theta}\sin\bar{\phi}\end{bmatrix}\begin{bmatrix}\theta\\ \phi\end{bmatrix} = \begin{bmatrix}\Delta\dot{u}\\ \Delta\dot{v}\\ \Delta\dot{w}\end{bmatrix}$$
$$+\begin{bmatrix}0 & -\bar{r} & \bar{q}\\ \bar{r} & 0 & -\bar{p}\\ -\bar{q} & \bar{p} & 0\end{bmatrix}\begin{bmatrix}\Delta u\\ \Delta v\\ \Delta w\end{bmatrix} + \begin{bmatrix}0 & \bar{w} & -\bar{v}\\ -\bar{w} & 0 & \bar{u}\\ \bar{v} & -\bar{u} & 0\end{bmatrix}\begin{bmatrix}\Delta p\\ \Delta q\\ \Delta r\end{bmatrix} \tag{19}$$

F Frame Linearized EOMs (continued)

$$
\begin{bmatrix} \Delta L \\ \Delta M \\ \Delta N \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\frac{I_{xz}}{I_{xx}} \\ 0 & 1 & 0 \\ -\frac{I_{xz}}{I_{zz}} & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta \dot{p} \\ \Delta \dot{q} \\ \Delta \dot{r} \end{bmatrix}
+ \begin{bmatrix} -\frac{I_{xz}}{I_{xx}}\bar{q} & -\frac{I_{xz}}{I_{xx}}\bar{p} + \frac{I_{zz}-I_{yy}}{I_{xx}}\bar{r} & \frac{I_{zz}-I_{yy}}{I_{xx}}\bar{q} \\ \frac{I_{xx}-I_{zz}}{I_{yy}}\bar{r} + 2\frac{I_{xz}}{I_{yy}}\bar{p} & 1 & \frac{I_{xx}-I_{zz}}{I_{yy}}\bar{p} - 2\frac{I_{xz}}{I_{yy}}\bar{r} \\ \frac{I_{yy}-I_{xx}}{I_{zz}}\bar{q} & \frac{I_{xz}}{I_{zz}}\bar{r} + \frac{I_{yy}-I_{xx}}{I_{zz}}\bar{p} & \frac{I_{xz}}{I_{zz}}\bar{q} \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta q \\ \Delta r \end{bmatrix} \tag{20}
$$

Perturbed Forces/Moments

$$
\begin{aligned}
\begin{bmatrix} \Delta X \\ \Delta Z \\ \Delta M \end{bmatrix} &= \begin{bmatrix} 0 & X_{\dot{\alpha}} & 0 \\ 0 & Z_{\dot{\alpha}} & 0 \\ 0 & M_{\dot{\alpha}} & 0 \end{bmatrix} \begin{bmatrix} \Delta \dot{u} \\ \Delta \dot{\alpha} \\ \Delta \dot{q} \end{bmatrix} + \begin{bmatrix} X_u & X_\alpha & X_q \\ Z_u & Z_\alpha & Z_q \\ M_u & M_\alpha & M_q \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta \alpha \\ \Delta q \end{bmatrix} \\
&+ \begin{bmatrix} X_{\delta_e} & X_{\delta_t} \\ Z_{\delta_e} & Z_{\delta_t} \\ M_{\delta_e} & M_{\delta_t} \end{bmatrix} \begin{bmatrix} \Delta \delta_e \\ \Delta \delta_t \end{bmatrix} \\
&+ \begin{bmatrix} X_{\dot{\eta}_1} & \cdots & X_{\dot{\eta}_n} \\ Z_{\dot{\eta}_1} & \cdots & Z_{\dot{\eta}_n} \\ M_{\dot{\eta}_1} & \cdots & M_{\dot{\eta}_n} \end{bmatrix} \begin{bmatrix} \Delta \dot{\eta}_1 \\ \vdots \\ \Delta \dot{\eta}_n \end{bmatrix} + \begin{bmatrix} X_{\eta_1} & \cdots & X_{\eta_n} \\ Z_{\eta_1} & \cdots & Z_{\eta_n} \\ M_{\eta_1} & \cdots & M_{\eta_n} \end{bmatrix} \begin{bmatrix} \Delta \eta_1 \\ \vdots \\ \Delta \eta_n \end{bmatrix}
\end{aligned} \tag{21}
$$

Perturbed Forces/Moments (continued)

$$
\begin{aligned}
\begin{bmatrix} \Delta Y \\ \Delta L \\ \Delta N \end{bmatrix} &= \begin{bmatrix} Y_\beta & Y_p & Y_r \\ L_\beta & L_p & L_r \\ N_\beta & N_p & N_r \end{bmatrix} \begin{bmatrix} \Delta\beta \\ \Delta p \\ \Delta r \end{bmatrix} + \begin{bmatrix} Y_{\delta_a} & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \end{bmatrix} \begin{bmatrix} \Delta\delta_a \\ \Delta\delta_r \end{bmatrix} \\
&+ \begin{bmatrix} Y_{\dot{\eta}_1} & \cdots & Y_{\dot{\eta}_n} \\ L_{\dot{\eta}_1} & \cdots & L_{\dot{\eta}_n} \\ N_{\dot{\eta}_1} & \cdots & N_{\dot{\eta}_n} \end{bmatrix} \begin{bmatrix} \Delta\dot{\eta}_1 \\ \vdots \\ \Delta\dot{\eta}_n \end{bmatrix} + \begin{bmatrix} X_{\eta_1} & \cdots & X_{\eta_n} \\ Z_{\eta_1} & \cdots & Z_{\eta_n} \\ M_{\eta_1} & \cdots & M_{\eta_n} \end{bmatrix} \begin{bmatrix} \Delta\eta_1 \\ \vdots \\ \Delta\eta_n \end{bmatrix}
\end{aligned}
\tag{22}
$$

Alternative

- Angle of attack and sideslip angle substitutions:

$$\Delta w = \bar{u}\Delta\alpha \tag{23}$$

$$\Delta v = \bar{v}_{\infty}\Delta\beta \tag{24}$$

  - No-wind assumption
  - Small angle approximation

Alternative

- Angle of attack and sideslip angle substitutions:

$$\Delta w = \bar{u}\Delta\alpha \tag{23}$$

$$\Delta v = \bar{v}_\infty \Delta\beta \tag{24}$$

  - No-wind assumption
  - Small angle approximation
- Furthermore, if $\bar{\beta} = \bar{\phi} = \bar{p} = \bar{q} = \bar{r} = 0$, then one can decoupled the dynamics into the longitudinal and lateral-directional.

Linearized Vibrations

- Already linearly modeled (simply):

$$\Delta\ddot{\eta}_i + 2\zeta_i\omega_i\Delta\dot{\eta}_i + \omega_i^2\Delta\eta_i = \frac{\Delta Q_i}{\mathcal{M}_i}, \quad i = 1, \ldots, n \tag{25}$$

Generalized Forces

$$
\begin{aligned}
\Delta Q_i = {} & \begin{bmatrix} Q_{i_u} & Q_{i_\beta} & Q_{i_\alpha} & Q_{i_p} & Q_{i_q} & Q_{i_r} \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta \beta \\ \Delta \alpha \\ \Delta p \\ \Delta q \\ \Delta r \end{bmatrix} \\
& + \begin{bmatrix} Q_{i_{\delta_a}} & Q_{i_{\delta_e}} & Q_{i_{\delta_r}} & Q_{i_{\delta_t}} \end{bmatrix} \begin{bmatrix} \Delta \delta_a \\ \Delta \delta_e \\ \Delta \delta_r \\ \Delta \delta_t \end{bmatrix} \\
& + \begin{bmatrix} Q_{i_{\dot\eta_1}} & \cdots & Q_{i_{\dot\eta_n}} \end{bmatrix} \begin{bmatrix} \Delta \dot\eta_1 \\ \vdots \\ \Delta \dot\eta_n \end{bmatrix} + \begin{bmatrix} Q_{i_{\eta_1}} & \cdots & Q_{i_{\eta_n}} \end{bmatrix} \begin{bmatrix} \Delta \eta_1 \\ \vdots \\ \Delta \eta_n \end{bmatrix}
\end{aligned} \tag{26}
$$

Decoupled Longitudinal EOMs Example

- Straight-and-level flight provides :

$$\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{\alpha} \\ \Delta\dot{q} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u & X_\alpha & X_q & -g \\ \frac{Z_u}{\bar{u}} & \frac{Z_\alpha}{\bar{u}} & 1+\frac{Z_q}{\bar{u}} & 0 \\ M_u & M_\alpha & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta\alpha \\ \Delta q \\ \Delta\theta \end{bmatrix} + \begin{bmatrix} X_{\delta_e} & X_{\delta_t} \\ \frac{Z_{\delta_e}}{\bar{u}} & \frac{Z_{\delta_t}}{\bar{u}} \\ M_{\delta_e} & M_{\delta_t} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\delta_e \\ \Delta\delta_t \end{bmatrix} \tag{27}$$

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- Models portion of Equation **??**:

$$\begin{bmatrix} \Delta\dot{\vec{x}}_{rig} \\ \Delta\dot{\vec{x}}_{eul} \end{bmatrix} = \begin{bmatrix} A_{rig\leftarrow rig} & A_{rig\leftarrow eul} \\ A_{eul\leftarrow rig} & A_{eul\leftarrow eul} \end{bmatrix} \begin{bmatrix} \Delta\vec{x}_{rig} \\ \Delta\vec{x}_{eul} \end{bmatrix} + \begin{bmatrix} B_{rig} \\ 0 \end{bmatrix} \Delta\vec{u} \qquad (28)$$

  - $\Delta\alpha$ has been used in place of Δw

Decoupled Longitudinal EOMs Example (continued)

- Form other matrices:

$$A_{rig\leftarrow vib} = \begin{bmatrix} X_{\eta_1} & \cdots & X_{\eta n} & X_{\dot{\eta}_1} & \cdots & X_{\dot{\eta} n} \\ \frac{Z_{\eta_1}}{\bar{u}} & \cdots & \frac{Z_{\eta n}}{\bar{u}} & \frac{Z_{\dot{\eta}_1}}{\bar{u}} & \cdots & \frac{Z_{\dot{\eta} n}}{\bar{u}} \\ M_{\eta_1} & \cdots & M_{\eta n} & M_{\dot{\eta}_1} & \cdots & M_{\dot{\eta} n} \end{bmatrix} \tag{29}$$

Decoupled Longitudinal EOMs Example (continued)

- Form other matrices:

$$A_{rig \leftarrow vib} = \begin{bmatrix} X_{\eta_1} & \cdots & X_{\eta n} & X_{\dot{\eta}_1} & \cdots & X_{\dot{\eta} n} \\ \frac{Z_{\eta_1}}{\bar{u}} & \cdots & \frac{Z_{\eta n}}{\bar{u}} & \frac{Z_{\dot{\eta}_1}}{\bar{u}} & \cdots & \frac{Z_{\dot{\eta} n}}{\bar{u}} \\ M_{\eta_1} & \cdots & M_{\eta n} & M_{\dot{\eta}_1} & \cdots & M_{\dot{\eta} n} \end{bmatrix} \tag{29}$$

- Define **(aero)elastic stability and control derivative** for state or input •:

$$\Xi_{i_\bullet} = \frac{Q_{i_\bullet}}{\mathcal{M}_i} \tag{30}$$

Decoupled Longitudinal EOMs Example (continued)

- Vibration state and input sub-matrices:

$$A_{vib\leftarrow rig} = \begin{bmatrix} \Xi_{1_u} & \Xi_{1_\alpha} & \Xi_{1_q} \\ \vdots & \vdots & \vdots \\ \Xi_{n_u} & \Xi_{n_\alpha} & \Xi_{n_q} \end{bmatrix} \tag{31}$$

$$A_{vib\leftarrow\eta} = \begin{bmatrix} \Xi_{1_{\eta_1}} & \cdots & \Xi_{1_{\eta n}} \\ \vdots & \ddots & \vdots \\ \Xi_{n_{\eta_1}} & \cdots & \Xi_{n_{\eta n}} \end{bmatrix} - \Omega^2 \tag{32}$$

$$A_{vib\leftarrow\dot{\eta}} = \begin{bmatrix} \Xi_{1_{\dot{\eta}_1}} & \cdots & \Xi_{1_{\dot{\eta} n}} \\ \vdots & \ddots & \vdots \\ \Xi_{n_{\dot{\eta}_1}} & \cdots & \Xi_{n_{\dot{\eta} n}} \end{bmatrix} - 2\Omega_\zeta \tag{33}$$

$$B_{vib} = \begin{bmatrix} \Xi_{1_{\delta_e}} & \Xi_{1_{\delta_t}} \\ \vdots & \vdots \\ \Xi_{n_{\delta_e}} & \Xi_{n_{\delta_t}} \end{bmatrix} \tag{34}$$

Decoupled Longitudinal EOMs Example (continued)

$$
\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{\alpha} \\ \Delta\dot{q} \\ \Delta\dot{\theta} \\ \Delta\dot{\eta}_1 \\ \vdots \\ \Delta\dot{\eta}_n \\ \Delta\ddot{\eta}_1 \\ \vdots \\ \Delta\ddot{\eta}_n \end{bmatrix}
=
\begin{bmatrix}
X_u & X_\alpha & X_q & -g & X_{\eta_1} & \cdots & X_{\eta n} & X_{\dot{\eta}_1} & \cdots & X_{\dot{\eta} n} \\
\frac{Z_u}{\bar{u}} & \frac{Z_\alpha}{\bar{u}} & 1+\frac{Z_q}{\bar{u}} & 0 & \frac{Z_{\eta_1}}{\bar{u}} & \cdots & \frac{Z_{\eta n}}{\bar{u}} & \frac{Z_{\dot{\eta}_1}}{\bar{u}} & \cdots & \frac{Z_{\dot{\eta} n}}{\bar{u}} \\
M_u & M_\alpha & M_q & 0 & M_{\eta_1} & \cdots & M_{\eta n} & M_{\dot{\eta}_1} & \cdots & M_{\dot{\eta} n} \\
0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \\
\Xi_{1_u} & \Xi_{1_\alpha} & \Xi_{1_q} & 0 & \Xi_{1_{\eta_1}} - \omega_1^2 & \cdots & \Xi_{1_{\eta n}} & \Xi_{1_{\dot{\eta}_1}} - 2\zeta_1\omega_1 & \cdots & \Xi_{1_{\dot{\eta} n}} \\
\vdots & \vdots & \vdots & 0 & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\Xi_{n_u} & \Xi_{n_\alpha} & \Xi_{n_q} & 0 & \Xi_{n_{\eta_1}} & \cdots & \Xi_{n_{\eta n}} - \omega_n^2 & \Xi_{n_{\dot{\eta}_1}} & \cdots & \Xi_{n_{\dot{\eta} n}} - 2\zeta_1\omega_1
\end{bmatrix}
\begin{bmatrix} \Delta u \\ \Delta\alpha \\ \Delta q \\ \Delta\theta \\ \Delta\eta_1 \\ \vdots \\ \Delta\eta_n \\ \Delta\dot{\eta}_1 \\ \vdots \\ \Delta\dot{\eta}_n \end{bmatrix}
$$

$$
+ \begin{bmatrix}
X_{\delta_e} & X_{\delta_t} \\
\frac{Z_{\delta_e}}{\bar{u}} & \frac{Z_{\delta_t}}{\bar{u}} \\
M_{\delta_e} & M_{\delta_t} \\
\vec{0}_{n+1} & \vec{0}_{n+1} \\
\Xi_{1_{\delta_e}} & \Xi_{1_{\delta_t}} \\
\vdots & \vdots \\
\Xi_{n_{\delta_e}} & \Xi_{n_{\delta_t}}
\end{bmatrix}
\begin{bmatrix} \Delta\delta_e \\ \Delta\delta_t \end{bmatrix}
\tag{35}
$$

Notch Filtering

-

Notch Filtering

Co-Located Sensor and Actuator

Co-Located Sensor and Actuator

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Co-Located Sensor and Actuator

-

Summary

- Dynamic-Elastic Airplane Dynamics
  - Augment with linear terms for X, Y, Z, L, M, N derivatives based on η_i and $\dot{\eta}_i$ $\forall$ i
  - Model vibration dynamics for $\ddot{\eta}_i$ $\forall$ i

Summary

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- Static-Elastic Airplane Dynamics
  - Use $\bar{\eta}$ for $\dot{\eta}_i = \ddot{\eta}_i = 0$
  - Residualization of vibration into new stability and control derivatives

Summary

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  - Augment with linear terms for X, Y, Z, L, M, N derivatives based on η_i and $\dot{\eta}_i$ $\forall$ i
  - Model vibration dynamics for $\ddot{\eta}_i$ $\forall$ i
- Static-Elastic Airplane Dynamics
  - Use $\bar{\eta}$ for $\dot{\eta}_i = \ddot{\eta}_i = 0$
  - Residualization of vibration into new stability and control derivatives
- Linearized elastic airplane dynamics
  - Aeroelastic derivatives already linear model
  - Vibration dynamics already linear