Lecture 13: Forced Motion and Elastic Vehicle Mean-Axes

Textbook Sections 10.1 & 10.2

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Introduction

- Course perspective: elastic-body dynamics study effects of elastic deformation on flight dynamics of vehicle
  - Not purely structural/aeroelastic phenomena such as flutter or divergence

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  - Not purely structural/aeroelastic phenomena such as flutter or divergence
- Only lower frequency modes typically of interest for addition to rigid-body modes in 6-DOF EOMs
  - All real vehicles elastic & in many cases: vibration analysis only necessary to check if rigid-body assumption can be made for modeling flight vehicle
  - Larger flight vehicles typically require some elastic modeling as natural frequencies will be lower and more likely to interact with rigid-body or flight controller modes of vehicle

Introduction (continued)

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- Previous lectures: derived generalized coordinates to model vibration problems
  - Vibration modes orthogonal to rigid-body modes and demonstrated for some simple lumped-mass models to assist in visualizing construction of elastic-body dynamic models
- Types of coordinate frames satisfy mean-axis constraints: used for mean-axes derived in EOMs describing elastic-body flight dynamics in subsequent lectures
  - Before developing concept of mean-axes, first assume that navigation frame N inertial, i.e., “flat-Earth” model

Bi-Directional Beam Example

- Extend to multi-directional motion $\rightarrow$ more generalized vectors and matrices
  - 1 element of mode shape/eigenvector $\rightarrow$ 1 direction of motion

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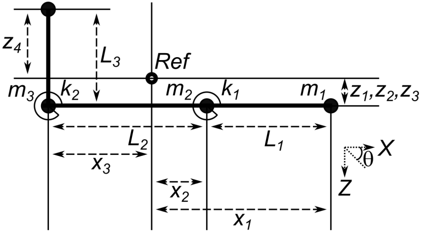
- Extend to multi-directional motion $\rightarrow$ more generalized vectors and matrices
  - 1 element of mode shape/eigenvector $\rightarrow$ 1 direction of motion
- Demonstration: consider bi-directional example of unforced 2D truss



- Directions of motion: X & Z
- Reference point, *Ref*: center of mass of truss

Energies

- Kinetic energy of truss:

$$
\begin{aligned}
T &= \frac{1}{2}\left[m_1(\dot{X}_1^2 + \dot{Z}_1^2) + m_2(\dot{X}_2^2 + \dot{Z}_2^2) + m_3(\dot{X}_3^2 + \dot{Z}_3^2) + m_4(\dot{X}_4^2 + \dot{Z}_4^2)\right] \\
&= \frac{1}{2}\begin{bmatrix}\dot{\vec{X}}^T & \dot{\vec{Z}}^T\end{bmatrix}\begin{bmatrix}M & 0 \\ 0 & M\end{bmatrix}\begin{bmatrix}\dot{\vec{X}} \\ \dot{\vec{Z}}\end{bmatrix}
\end{aligned}
\tag{1}
$$

Energies

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- Potential/strain energy of truss:

$$U = \frac{1}{2}k_1\theta_1^2 + \frac{1}{2}k_2\theta_2^2 = \frac{1}{2}\begin{bmatrix}\theta_1 & \theta_2\end{bmatrix}\begin{bmatrix}k_1 & 0 \\ 0 & k_2\end{bmatrix}\begin{bmatrix}\theta_1 \\ \theta_2\end{bmatrix} \quad (2)$$

  - 2 deflections, θ_1 & θ_2: relative angular displacements between rods 1-2 & 2-3

Geometric Constraints

- Geometry of truss and small angles $\rightarrow$ constraints for relative angular displacements

$$\begin{aligned}
\theta_1 &= \frac{Z_1 - Z_2}{x_1 - x_2} - \frac{Z_2 - Z_3}{x_2 + x_3} \\
&= \begin{bmatrix} \frac{1}{x_1 - x_2} & \left(\frac{-1}{x_1 - x_2} - \frac{1}{x_2 + x_3}\right) & \frac{1}{x_2 + x_3} & 0 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{bmatrix} \qquad (3) \\
&= C_1 \vec{Z}
\end{aligned}$$

Geometric Constraints (continued)

$$
\begin{aligned}
\theta_2 &= \frac{Z_3 - Z_2}{x_2 + x_3} - \frac{X_3 - X_4}{z_3 + z_4} \\
&= \begin{bmatrix} 0 & 0 & \frac{-1}{z_3+z_4} & \frac{1}{z_3+z_4} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} 0 & \left(\frac{-1}{x_2+x_3} - \frac{1}{x_2+x_3}\right) & \frac{1}{x_2+x_3} & 0 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{bmatrix} \\
&= \begin{bmatrix} C_2 & C_3 \end{bmatrix} \begin{bmatrix} \vec{X} \\ \vec{Z} \end{bmatrix}
\end{aligned} \tag{4}
$$

Equal Vibration Displacements

- Assume equal vibration displacements for colinear masses:

$$X_1 = X_2 = X_3 \qquad \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_3 \\ X_4 \end{bmatrix} \qquad \vec{X} = C_x' \vec{X}' \tag{5}$$

$$Z_3 = Z_4 \qquad \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \qquad \vec{Z} = C_z' \vec{Z}' \tag{6}$$

- Degrees of freedom of truss reduced using constraints $\rightarrow$ modified coordinate vectors: $\vec{X}'$, $\vec{Z}'$

Kinetic Energy of Truss

$$\begin{aligned} T &= \frac{1}{2}\begin{bmatrix}\dot{\vec{X}}'^T & \dot{\vec{Z}}'^T\end{bmatrix}\begin{bmatrix}C'_x & 0 \\ 0 & C'_z\end{bmatrix}^T\begin{bmatrix}M & 0 \\ 0 & M\end{bmatrix}\begin{bmatrix}C'_x & 0 \\ 0 & C'_z\end{bmatrix}\begin{bmatrix}\dot{\vec{X}}' \\ \dot{\vec{Z}}'\end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix}\dot{\vec{X}}'^T & \dot{\vec{Z}}'^T\end{bmatrix}[MM]'\begin{bmatrix}\dot{\vec{X}}' \\ \dot{\vec{Z}}'\end{bmatrix} \end{aligned} \tag{7}$$

Potential/Strain Energy of Truss

$$
\begin{aligned}
U &= \frac{1}{2}\begin{bmatrix}\vec{X}^T & \vec{Z}^T\end{bmatrix}\begin{bmatrix}0 & C_1\\ C_2 & C_3\end{bmatrix}^T\begin{bmatrix}k_1 & 0\\ 0 & k_2\end{bmatrix}\begin{bmatrix}0 & C_1\\ C_2 & C_3\end{bmatrix}\begin{bmatrix}\vec{X}\\ \vec{Z}\end{bmatrix}\\
&= \frac{1}{2}\begin{bmatrix}\vec{X}^T & \vec{Z}^T\end{bmatrix}K_c\begin{bmatrix}\vec{X}\\ \vec{Z}\end{bmatrix}\\
&= \frac{1}{2}\left[\vec{X}'^T \quad \vec{Z}'^T\begin{bmatrix}C'_x & 0\\ 0 & C'_z\end{bmatrix}^T\right]K_c\begin{bmatrix}C'_x & 0\\ 0 & C'_z\end{bmatrix}\begin{bmatrix}\vec{X}'\\ \vec{Z}'\end{bmatrix}\\
&= \frac{1}{2}\begin{bmatrix}\vec{X}'^T & \vec{Z}'^T\end{bmatrix}K'_c\begin{bmatrix}\vec{X}'\\ \vec{Z}'\end{bmatrix}
\end{aligned}
\tag{8}
$$

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- External forces on lumped-mass systems $\rightarrow$ incorporate aerodynamic forces into elastic-body flight dynamics

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- Consider Euler-Lagrange equation with external forces applied

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\vec{q}}}\right) - \frac{\partial T}{\partial \vec{q}} + \frac{\partial U}{\partial \vec{q}} = \vec{Q} \tag{9}$$

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- Q: generalized force

$$\vec{Q}^T = \frac{\partial \delta W}{\partial \delta \vec{q}} \tag{10}$$

  - $\delta \vec{q}$: virtual displacement of generalized coordinates $\vec{q}$

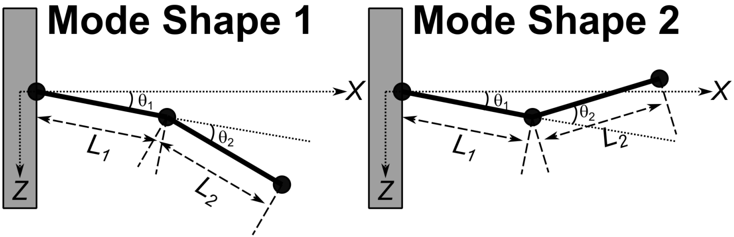
Virtual Work

- δW: **Virtual work**

$$\delta W = \sum_{i=1}^{m} \vec{F}_i \cdot \delta \vec{d}_i$$
$$\delta W = \begin{bmatrix} \vec{F}_1^T & \cdots & \vec{F}_m^T \end{bmatrix}^T \begin{bmatrix} \delta \vec{d}_1 \\ \vdots \\ \delta \vec{d}_m \end{bmatrix} \tag{11}$$

- $\delta \vec{d}_i$: virtual physical displacement of point of application of force F_i

2-Lumped-Mass Example

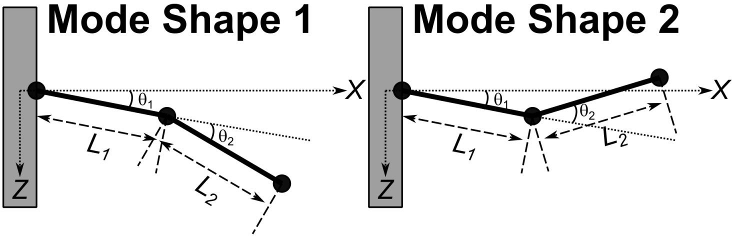


- Define vertical forces:

$$\vec{F}_1 = F_1\hat{k} \qquad \vec{F}_2 = F_1\hat{k} \tag{12}$$

  - $\hat{k}$: unit vector for Z direction

2-Lumped-Mass Example



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$$\vec{F}_1 = F_1\hat{k} \qquad \vec{F}_2 = F_1\hat{k} \tag{12}$$

  - $\hat{k}$: unit vector for Z direction
- Virtual physical displacements at points of application:

$$\begin{aligned} \delta\vec{d}_1 &= \delta Z_1\hat{k} \\ \delta\vec{d}_2 &= \delta Z_2\hat{k} \end{aligned} \tag{13}$$

2-Lumped-Mass Example (continued)

$$
\begin{aligned}
\delta W &= F_1 \delta Z_1 + F_2 \delta Z_2 \\
\delta W &= \begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} \delta Z_1 \\ \delta Z_2 \end{bmatrix}
\end{aligned}
\tag{14}
$$

2-Lumped-Mass Example (continued)

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$$
\begin{bmatrix} \delta Z_1 \\ \delta Z_2 \end{bmatrix} = \begin{bmatrix} L_1 & 0 \\ L_1 + L_2 & L_2 \end{bmatrix} \begin{bmatrix} \delta \theta_1 \\ \delta \theta_2 \end{bmatrix}
\tag{15}
$$

2-Lumped-Mass Example (continued)

$$\delta W = F_1 \delta Z_1 + F_2 \delta Z_2$$
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- Assuming unforced (i.e. "free") vibration problem solved in terms of θ_1 and θ_2
- Elements of free-vibration mode shapes correspond to angular displacements

2-Lumped-Mass Example (continued)

- Virtual displacements $\delta\theta_1$ and $\delta\theta_2$ expressed in terms of mode shapes and two vibration modal coordinates, η_1 and η_2:

$$\begin{aligned} \begin{bmatrix} \delta\theta_1 \\ \delta\theta_2 \end{bmatrix} &= \begin{bmatrix} \vec{\nu}_1 & \vec{\nu}_2 \end{bmatrix} \begin{bmatrix} \delta\eta_1 \\ \delta\eta_2 \end{bmatrix} \\ \delta\vec{\theta} &= \Psi\delta\vec{\eta} \end{aligned} \tag{16}$$

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- Substitution: virtual work

$$\begin{aligned} \delta W &= \begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} L_1 & 0 \\ L_1 + L_2 & L_2 \end{bmatrix} \Psi\delta\vec{\eta} \\ \delta W &= \vec{\mathcal{F}}^T\Psi\delta\vec{\eta} \end{aligned} \tag{17}$$

  - $\vec{\mathcal{F}}$: vector of applied forces relative to virtual displacements

2-Lumped-Mass Example (continued)

- Kinetic energy (modal coordinates):

$$T = \frac{1}{2}\dot{\vec{\theta}}^T M \dot{\vec{\theta}} = \frac{1}{2}\dot{\vec{\eta}}^T \Psi^T M \Psi \dot{\vec{\eta}} = \frac{1}{2}\dot{\vec{\eta}}^T \mathcal{M} \dot{\vec{\eta}} \tag{18}$$

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$$U = \frac{1}{2}\vec{\theta}^T K \vec{\theta} = \frac{1}{2}\vec{\eta}^T \Psi^T K \Psi \vec{\eta} = \frac{1}{2}\vec{\eta}^T \mathcal{K} \vec{\eta} \tag{19}$$

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- Lagrange's equation with external forces applied:

$$\mathcal{M}\ddot{\vec{\eta}} + \mathcal{K}\vec{\eta} = Q = \Psi^T \vec{\mathcal{F}} == \vec{\nu}_{vib}^T \vec{\mathcal{F}} \tag{20}$$

Unrestrained 3-Lumped-Mass Extension

- Similar expression in form:

$$
\begin{aligned}
M_{tot}\ddot{\vec{Z}}_{Ref} &= F_1 + F_2 + F_3 = \vec{1}^T \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \\
I_G\ddot{\theta}_{Ref} &= F_1x_1 + F_2x_2 - F_3x_3 = \vec{x}^T \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \\
\mathcal{M}_{vib}\ddot{\vec{\eta}}_{vib} + \mathcal{K}_{vib}\vec{\eta}_{vib} &= Q = \Psi_{vib}^T\vec{\mathcal{F}} = \vec{\nu}_{vib}^T\vec{\mathcal{F}}
\end{aligned}
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\end{gathered}
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$$

- For forced response of unrestrained beam's EOMs
  - Fundamental for elastic-body flight dynamics
- *Ref*: center of mass of lumped-mass system
  - Easily extended to *n*-lumped-mass systems

Kinetic Energy of Vehicle

$$T = \frac{1}{2}\int_{Vol} \vec{v}_N \cdot \vec{v}_N \rho_V dV \tag{22}$$

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- $\vec{v}_N$: inertial velocity of mass element of vehicle

$$\vec{v}_N = \vec{v}_{B/N} + \vec{v}_B + \omega_{B/N} \times \vec{x}_B \tag{23}$$

- $\vec{v}_B$: velocity of mass element in body-fixed frame
- $\vec{v}_{B/N}$: velocity of body-fixed frame relative to navigation frame
- $\vec{\omega}_{B/N}$: angular velocity of body-fixed frame relative to navigation frame
  - Note: velocities can be represented in body-fixed or navigation frames
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  - $\vec{x}_B$: position of mass element in body-fixed frame relative to center of mass

$$\vec{x}_N = \vec{x}_B + \vec{x}_{B/N} \tag{24}$$

  - $\vec{x}_{B/N}$: position of origin of body-fixed frame relative to navigation frame

Kinetic Energy of Vehicle (continued)

- By substitution:

$$
\begin{aligned}
T = \frac{1}{2}\int_{Vol} \Big[& \vec{v}_{B/N,N} \cdot \vec{v}_{B/N,N} \\
& + 2\vec{v}_{B/N,N} \cdot \vec{v}_{B} \\
& + 2\left(\vec{v}_{B/N,N} + \vec{v}_{B}\right) \cdot \left(\vec{\omega}_{B/N} \times \vec{x}_{B}\right) \\
& + \vec{v}_{B} \cdot \vec{v}_{B} \\
& + \left(\vec{\omega}_{B/N} \times \vec{x}_{B}\right) \cdot \left(\vec{\omega}_{B/N} \times \vec{x}_{B}\right) \Big] \rho_V dV
\end{aligned}
\tag{25}
$$

Potential Energy

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- Gravitational potential energy:

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\end{aligned}
\tag{26}
$$

  - $\vec{g}$: acceleration due to gravity
- Elastic strain energy: energy stored in elastic structure due to deformation resulting from applied force
  - Negative of work done on structure by applied force
  - Work: force acting over displacement $\vec{d}_e$

Potential Energy (continued)

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  - $\vec{x}_{r-b,B}$ invariant w.r.t. body-fixed frame

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- D'Alembert's principle:
  - Express force on mass element in terms of mass of element and acceleration

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- D'Alembert's principle:
  - Express force on mass element in terms of mass of element and acceleration
- Elastic strain energy:

$$U_e = -\frac{1}{2}\int_{Vol} \ddot{\vec{d}}_{e,B} \cdot \vec{d}_{e,B}\rho_V dV \tag{29}$$

Mean-Axes Body Frame

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  - $\rightarrow$ *additional* requirements for body-fixed frame to exhibit decoupled dynamic modes
- **Mean-axes constraints**: define coordinate axes about which relative translational and angular momenta (about center of mass) due to elastic vibrations are zero

$$\int_{Vol} \dot{\vec{d}}_{e,B} \rho_V dV = \int_{Vol} \vec{x}_B \times \dot{\vec{d}}_{e,B} \rho_V dV = \vec{0} \tag{30}$$

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- "Special" **mean-axes body-fixed frame**: always exist for elastic-body

Practical Mean-Axes

- By substitution in previous section:

$$\int_{Vol} \frac{d}{dt}\left(\vec{x}_{r-b} + \vec{d}_{e,B}\right) \rho_V dV = \vec{0} \tag{31}$$

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- Analogous to modal orthogonality constraints for lumped-mass systems

Mean-Axes Demonstration

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  - I.e. constrained eigenvectors and eigenvalues
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- Elastic vibrations:

$$\vec{d}_{e,B} = \sum_{i=1}^{n} \vec{\nu}_i(\vec{x})\eta_i(t) \tag{34}$$

  - $\eta_i(t)$: generalized coordinate associated with i-th vibration mode
  - $\vec{\nu}_i(\vec{x})$: vector with components defined in body-fixed frame
  - Each component function of $\vec{x}$ location on *undeformed* structure

Practical Mean-Axes Constraints

$$\int_{Vol} \dot{\vec{d}}_{e,B}\rho_V dV = \sum_{i=1}^{n} \dot{\eta}_i(t) \left(\int_{Vol} \vec{\nu}_i(\vec{x})\rho_V dV \right) = 0 \tag{35}$$

$$\int_{Vol} \vec{x}_{r-b} \times \dot{\vec{d}}_{e,B}\rho_V dV = \sum_{i=1}^{n} \left(\int_{Vol} \vec{x}_{r-b} \times \vec{\nu}_i(\vec{x})\rho_V dV \right) = 0 \tag{36}$$

- Satisfied as integrals inside parentheses correspond to momenta conservation requirements
- Selected vibration modes, by design, orthogonal to rigid-body body translation and rotation modes (w.r.t. mass distribution)

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$$\begin{aligned}\int_{Vol} \vec{v}_{B/N,N} \cdot \vec{v}_{B/N,N} \rho_V dV &= \vec{v}_{B/N,N} \cdot \vec{v}_{B/N,N} \int_{Vol} \rho_V dV \\ &= m \vec{v}_{B/N,N} \cdot \vec{v}_{B/N,N}\end{aligned} \tag{37}$$

  - m: total mass of vehicle

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  - m: total mass of vehicle
- 2nd term of kinetic energy $= 0$:

$$\int_{Vol} \vec{v}_{B/N,N} \cdot \dot{\vec{d}}_{e,B} \rho_V dV = \vec{v}_{B/N,N} \cdot \int_{Vol} \dot{\vec{d}}_{e,B} \rho_V dV = 0 \tag{38}$$

3rd Term of Kinetic Energy

- Recall: requirement for origin of mean-axis body-fixed frame $=$ center of mass:

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$$\begin{aligned} &\int_{Vol} \left(\vec{v}_{B/N,N} + \dot{\vec{d}}_{e,B}\right) \cdot \left(\vec{\omega}_{B/N} \times \vec{x}_B\right) \rho_V dV \\ &= \vec{v}_{B/N,N} \cdot \left(\vec{\omega}_{B/N} \times \int_{Vol} \vec{x}_B \rho_V dV\right) = 0 \end{aligned} \tag{41}$$

4th Term of Kinetic Energy

- Use mode shapes and generalized coordinate summations for displacement *velocity*:

$$
\begin{aligned}
\int_{Vol} \vec{v}_B \cdot \vec{v}_B \rho_V dV &= \frac{1}{2}\int_{Vol} \left(\sum_{i=1}^{n} \vec{\nu}_i(\vec{x})\dot{\eta}_i \cdot \sum_{i=1}^{n} \vec{\nu}_i(\vec{x})\dot{\eta}_i \right) \rho_V dV \\
&= \int_{Vol} \left(\sum_{i=1}^{n} \vec{\nu}_i(\vec{x}) \cdot \vec{\nu}_i(\vec{x})\dot{\eta}_i^2 \right) \rho_V dV \\
&= \sum_{i=1}^{n} \mathcal{M}_i \dot{\eta}_i^2
\end{aligned}
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- Simplification due to mutual orthogonality of vibration modes:

$$\int_{Vol} \vec{\nu}_i \cdot \vec{\nu}_j \rho_V dV = \begin{cases} 0 & i \neq j \\ \mathcal{M}_i & i = j \end{cases} \tag{43}$$

5th Term and Final of Kinetic Energy

- In terms of inertia matrix:

$$\int_{Vol} \left(\vec{\omega}_{B/N} \times \vec{x}_B\right) \cdot \left(\vec{\omega}_{B/N} \times \vec{x}_B\right) \rho_V dV = \vec{\omega}^T_{B/N} I_G \vec{\omega}_{B/N} \tag{44}$$

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- Kinetic energy:

$$T = \frac{1}{2} m \vec{v}^T_{B/N,N} \vec{v}_{B/N,N} + \frac{1}{2} \vec{\omega}^T_{B/N} I_G \vec{\omega}_{B/N} + \frac{1}{2} \sum_{i=1}^{n} \mathcal{M}_i \dot{\eta}_i^2 \tag{45}$$

Gravitional Potential Energy

- Applying *mean-axis constraints*:

$$
\begin{aligned}
U_g &= -\int_{Vol} \vec{g} \cdot \left(\vec{x}_B + \vec{x}_{B/N}\right) \rho_V dV \\
U_g &= -\vec{g} \cdot \int_{Vol} \vec{x}_B \rho_V dV - \vec{g} \cdot \vec{x}_{B/N} \int_{Vol} \rho_V dV \\
U_g &= -m\vec{g}^T \vec{x}_{B/N}
\end{aligned} \tag{46}
$$

Elastic Strain Energy

- Applying *mean-axis constraints*:

$$U_e = -\frac{1}{2}\int_{Vol}\sum_{i=1}^{n}\vec{\nu}_i(\vec{x})\ddot{\eta}_i(t)\cdot\sum_{i=1}^{n}\vec{\nu}_i(\vec{x})\eta_i(t)\rho_V dV \tag{47}$$

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- Recall sinusoidal solution for modal coordinates:

$$\eta_i(t) = A_i\cos(\omega_i t + \Gamma_i) \tag{49}$$

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  - Appropriate for rigid-body bodies: relative position of point on body represented with location of center of mass *and* attitude
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  - Not true for elastic vehicles
- B/N subscript dropped to easily compare elastic vehicle dynamics with rigid-body vehicle dynamics:selected generalized coordinates

$$\vec{x}_{B/N,N} = \begin{bmatrix} x_N & y_N & z_N \end{bmatrix}^T \tag{52}$$

  - Recall $z_N = -h$ for flat-Earth

Generalized Coordinates (continued)

- Velocity of body-fixed frame's origin (in navigation frame coordinates)

$$\vec{v}_{B/N,N} = \begin{bmatrix} \dot{x}_N & \dot{y}_N & \dot{z}_N \end{bmatrix}^T \tag{53}$$

Generalized Coordinates (continued)

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- Select angular velocity of body-fixed frame relative to navigation frame:

$$\vec{\omega}_{B/N} = \begin{bmatrix} p & q & r \end{bmatrix}^T \tag{55}$$

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- Select the 3-2-1 Euler angles ψ, θ, ϕ: attitude/orientation of mean-axes body-fixed frame w.r.t. navigation frame

Generalized Coordinates (continued)

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- Recall: Euler angles related to angular velocity

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & -\sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \tag{56}$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{57}$$

Generalized Coordinates (continued)

- Selected generalized coordinates:

$$\vec{q} = \begin{bmatrix} x_N & y_N & z_N & \phi & \theta & \psi & \eta_i, i = 1, 2, \dots \end{bmatrix}^T \tag{58}$$

Generalized Coordinates (continued)

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- Same for rigid-body vehicle dynamics *with* addition of vibration modal coordinates corresponding to mutually orthogonal vibration mode shapes

Rewritten Energies

- Kinetic energy of elastic vehicle:

$$T = \frac{1}{2} m \begin{bmatrix} \dot{x}_N & \dot{y}_N & \dot{z}_N \end{bmatrix} \begin{bmatrix} \dot{x}_N \\ \dot{y}_N \\ \dot{z}_N \end{bmatrix} + \frac{1}{2} \begin{bmatrix} p & q & r \end{bmatrix} I_G \begin{bmatrix} p \\ q \\ r \end{bmatrix} + \frac{1}{2} \sum_{i=1}^{n} \mathcal{M}_i \dot{\eta}_i^2 \tag{59}$$

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- Gravitational potential energy:

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- Elastic strain energy (same):

$$U_e = -\frac{1}{2}\sum_{i=1}^{n} \omega_i^2 \eta_i^2(t) \mathcal{M}_i$$

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  - Body-fixed frame origin: center of mass
  - Body-fixed frame axes: mean-axes
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