Lecture 11: Earth Frame Effects on Dynamics

Textbook Sections 9.7

Dr. Jordan D. Larson

Introduction

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  - Important parameter for drastic changes in altitude, e.g., launch vehicles

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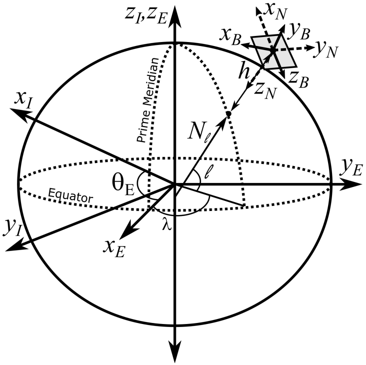
  - I.e. “flat-Earth” model
  - High supersonic velocities and/or long distances $\rightarrow$ may not be negligible effect
- Lecture: discuss effects of Earth frame as rotating **reference ellipsoid**
  - Adds to apparent forces and moments of vehicle
  - Does not discuss centripetal acceleration of Earth about Sun nor Sun about Milky Way
  - Earth truly non-analytical geoid and has very slight wobble in rotation axis
  - Develop effects as additions to rigid flight vehicle EOMs expressed in navigation frame
  - changes to EOMs for rigid flight vehicle with constant mass, no rotating mass, and no wind

Rotating, Ellipsoidal Earth Frames

- Frames:
  - ECI frame (subscript I)
  - ECEF frame (subscript E)
  - Geodetic coordinates (ℓ, λ, h)
  - Navigation frame (subscript N)
  - Body-fixed frame (subscript B)



Additional Parameters

- $\theta_E = \omega_E t$: rotation angle between ECI and ECEF frames
  - ω_E: 7.2921150×10^{-5} rad/s by WGS 84
  - t: reference time, e.g. Geocentric Celestial Reference Frame (GCRF) ECI frame referenced at 12:00 Terrestrial Time on January 1, 2000 with ITRF ECEF frame

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- N_ℓ: **prime-vertical radius of curvature** of reference ellipsoid, depends on latitude

$$N_\ell = \frac{R_e}{\sqrt{1 - e_E^2 \sin^2 \ell}} \tag{1}$$

  - R_e: equatorial radius of reference ellipsoid defined as $6,378,137.0$ m by WGS 84

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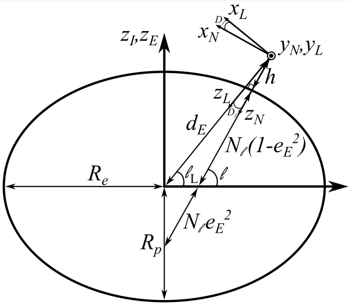
  - R_e: equatorial radius of reference ellipsoid defined as $6,378,137.0$ m by WGS 84
  - e_E: eccentricity of reference ellipsoid

$$e_E = \sqrt{f_E(2 - f_E)} = 0.081819190842622 \tag{2}$$

  - f_E: **flattening** of reference ellipsoid defined as $1/298.257223563$ by WGS 84
  - R_p: reference ellipsoid's polar radius

$$R_p = R_e(1 - f_E) = 6,356,752.3 \text{ m} \tag{3}$$

LVLH and Navigation Frames



- d_E: geocentric distance from Earth's center

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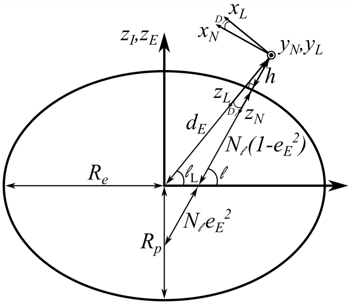

- d_E: geocentric distance from Earth's center
- N_ℓ split into two convenient sections
- D_ℓ: **deviation of normal**
  - Related to geodetic latitude, ℓ, & geocentric latitude, ℓ_L

$$\ell = \ell_L + D_\ell \quad (4)$$

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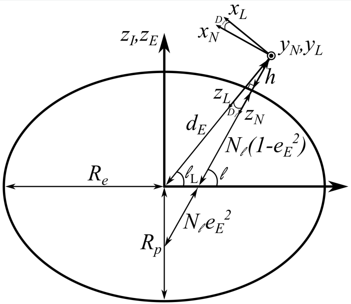

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  - Related to geodetic latitude, ℓ, & geocentric latitude, ℓ_L

$$\ell = \ell_L + D_\ell \tag{4}$$

  - Also defines rotation between local-vertical, local-horizontal (LVLH) frame (subscript L) and navigation frame: **spherical-Earth model**, $D_\ell = 0$ & LVLH = navigation frame

Relationships between ℓ and ℓ_L

$$n_\ell = \frac{e_E^2 N_\ell}{N_\ell + h} \tag{5}$$

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$$\tan D_\ell = \frac{n_\ell \sin\ell \cos\ell}{1 - n_\ell \sin^2\ell} = \frac{n_\ell \sin\ell_e \cos\ell_L}{1 - n_\ell \cos^2\ell_L} \tag{6}$$

$$\sin\ell_L = \frac{(1 - n_\ell)\sin\ell}{\sqrt{1 - n_\ell(2 - n_\ell)\sin^2\ell}} \tag{7}$$

$$\cos\ell_L = \frac{\cos\ell}{\sqrt{1 - n_\ell(2 - n_\ell)\sin^2\ell}} \tag{8}$$

$$\tan\ell_L = (1 - n_\ell)\tan\ell \tag{9}$$

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$$d_E = (N_\ell + h)\sqrt{1 - n(2 - n)\sin^2\ell} \tag{10}$$

Position of Vehicle's Center of Mass

- ECEF coordinates related to geocentric or geodetic coordinates:

$$\vec{x}_{B/E,E} = \begin{bmatrix} d_E \cos \ell_L \cos \lambda \\ d_E \cos \ell_L \sin \lambda \\ d_E \sin \ell_L \end{bmatrix} = \begin{bmatrix} (N_\ell + h) \cos \lambda \cos \ell \\ (N_\ell + h) \sin \lambda \cos \ell \\ (N_\ell(1 - e_E^2) + h) \sin \ell \end{bmatrix} \tag{11}$$

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- Change in coordinates as related to geocentric coordinates:

$$\dot{\vec{x}}_{B/E,E} = \begin{bmatrix} -\dot{\ell}_L d_E \sin\ell_L \cos\lambda - \dot{\lambda} d_E \cos\ell_L \sin\lambda + \dot{d}_E \cos\ell_L \cos\lambda \\ -\dot{\ell}_L d_E \sin\ell_L \sin\lambda + \dot{\lambda} d_E \cos\ell_L \cos\lambda + \dot{d}_E \cos\ell_L \sin\lambda \\ \dot{\ell}_L d_E \cos\ell_L + \dot{d}_E \sin\ell_L \end{bmatrix} \tag{12}$$

Change in Position of Vehicle's Center of Mass

- In LVLH frame coordinates:

$$\dot{\vec{x}}_{B/E,L} = C_{L\leftarrow E}\begin{bmatrix} -\dot{\ell}_L d_E \sin \ell_L \cos\lambda - \dot{\lambda} d_E \cos\ell_L \sin\lambda + \dot{d}_E \cos\ell_L \cos\lambda \\ -\dot{\ell}_L d_E \sin\ell_L \sin\lambda + \dot{\lambda} d_E \cos\ell_L \cos\lambda + \dot{d}_E \cos\ell_L \sin\lambda \\ \dot{\ell}_L d_E \cos\ell_L + \dot{d}_E \sin\ell_L \end{bmatrix} \tag{13}$$

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- Rotation matrix:

$$C_{L\leftarrow E} = C_2(-\ell_L - \pi/2)C_3(\lambda) = \begin{bmatrix} -\sin\ell_L\cos\lambda & -\sin\ell_L\sin\lambda & \cos\ell_L \\ -\sin\lambda & \cos\lambda & 0 \\ -\cos\ell_L\cos\lambda & -\cos\ell_L\sin\lambda & -\sin\ell_L \end{bmatrix} \tag{14}$$

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$$\dot{\vec{x}}_{B/E,L} = \begin{bmatrix} \dot{\ell}_L d_E \\ \dot{\lambda} d_E \cos\ell_L \\ -\dot{d}_E \end{bmatrix} \tag{15}$$

Geocentric Rates Supplementary Equation

- Define velocity of vehicle's center of mass in body-fixed frame coordinates:

$$\dot{\vec{x}}_{B/E,L} = C_{L \leftarrow B} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{16}$$

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$$\begin{bmatrix} \dot{\ell}_L \\ \dot{\lambda} \\ \dot{d}_E \end{bmatrix} = \begin{bmatrix} \frac{1}{d_E} & 0 & 0 \\ 0 & \frac{1}{d_E \cos \ell_L} & 0 \\ 0 & 0 & -1 \end{bmatrix} C_{L\leftarrow B} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{18}$$

Geodetic Coordinate Facts

- For geodetic coordinates:

$$\frac{d}{dt}\left((N_\ell + h)\cos\ell\right) = -(M_\ell + h)\sin\ell \tag{19}$$

$$\frac{d}{dt}\left((N_\ell(1 - e_{Earth}^2) + h)\sin\ell\right) = (M_\ell + h)\cos\ell \tag{20}$$

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- M_ℓ: **Earth's meridinal radius of curvature**:

$$M_\ell = \frac{R_e\left(1 - e_E^2\right)}{\left(1 - e_E^2\sin^2\ell\right)^{3/2}} \tag{21}$$

Change in Position of Vehicle's Center of Mass

- Using geodetic coordinates:

$$\frac{d}{dt}\vec{x}_{B/E,E} = \begin{bmatrix} -\dot{\ell}(M_\ell + h)\sin\ell\cos\lambda - \dot{\lambda}(N_\ell + h)\cos\ell\sin\lambda + \dot{h}\cos\ell\cos\lambda \\ -\dot{\ell}(M_\ell + h)\sin\ell\sin\lambda + \dot{\lambda}(N_\ell + h)\cos\ell\cos\lambda + \dot{h}\cos\ell\sin\lambda \\ \dot{\ell}(M_\ell + h)\cos\ell + \dot{h}\sin\ell \end{bmatrix} \tag{22}$$

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- In navigation frame coordinates:

$$\frac{d}{dt}\vec{x}_{B/E,N} = C_{N\leftarrow E}\begin{bmatrix} -\dot{\ell}(M_\ell + h)\sin\ell\cos\lambda - \dot{\lambda}(N_\ell + h)\cos\ell\sin\lambda + \dot{h}\cos\ell\cos\lambda \\ -\dot{\ell}(M_\ell + h)\sin\ell\sin\lambda + \dot{\lambda}(N_\ell + h)\cos\ell\cos\lambda + \dot{h}\cos\ell\sin\lambda \\ \dot{\ell}(M_\ell + h)\cos\ell + \dot{h}\sin\ell \end{bmatrix} \quad (23)$$

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$$\frac{d}{dt}\vec{x}_{B/E,N} = \begin{bmatrix} \dot{\ell}(M_\ell + h) \\ \dot{\lambda}(N_\ell + h)\cos\ell \\ -\dot{h} \end{bmatrix} \tag{25}$$

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- **Geodetic rates supplementary equation**

$$\begin{bmatrix} \dot{\ell} \\ \dot{\lambda} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} \frac{1}{M_\ell+h} & 0 & 0 \\ 0 & \frac{1}{(N_\ell+h)\cos\ell} & 0 \\ 0 & 0 & -1 \end{bmatrix} C_{N\leftarrow B} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{26}$$

Angular Velocity Effect of Rotating Earth

- Two additional angular velocity terms for ECI and ECEF frames:

$$\vec{\omega}_{B/I} = \vec{\omega}_{B/L} + \vec{\omega}_{L/E} + \vec{\omega}_{E/I} = \vec{\omega}_{B/N} + \vec{\omega}_{N/E} + \vec{\omega}_{E/I} \tag{27}$$

  - Flat-Earth model: $\vec{\omega}_{E/I} = \vec{\omega}_{L/E} = \vec{\omega}_{N/E} = 0$

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- For $\vec{\omega}_{E/I}$, by definition of ECI frame:

$$\vec{\omega}_{E/I,E} = \begin{bmatrix} 0 \\ 0 \\ \omega_E \end{bmatrix} \tag{28}$$

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- Rotation from ECI frame to ECEF frame:

$$C_{E\leftarrow I} = \begin{bmatrix} \cos\omega_E t & \sin\omega_E t & 0 \\ -\sin\omega_E t & \cos\omega_E t & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_E & \sin\theta_E & 0 \\ -\sin\theta_E & \cos\theta_E & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{29}$$

ECEF/ECI Angular Velocity

- LVLH frame coordinates:

$$\vec{\omega}_{E/I,L} = \begin{bmatrix} \omega_E \cos \ell_L & 0 & -\omega_E \sin \ell_L \end{bmatrix}^T \tag{30}$$

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- Navigation frame coordinates:

$$\vec{\omega}_{E/I,N} = \begin{bmatrix} \omega_E \cos \ell & 0 & -\omega_E \sin \ell \end{bmatrix}^T \tag{31}$$

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- Body-fixed frame coordinates:

$$\vec{\omega}_{E/I,B} = C_{B\leftarrow L}(\phi_L, \theta_L, \psi_L) \begin{bmatrix} \omega_E \cos \ell_L \\ 0 \\ -\omega_E \sin \ell_L \end{bmatrix} = C_{B\leftarrow N}(\phi, \theta, \psi) \begin{bmatrix} \omega_E \cos \ell \\ 0 \\ -\omega_E \sin \ell \end{bmatrix} \tag{32}$$

  - $(\phi_L, \theta_L, \psi_L)$: $3-2-1$ Euler angles from LVLH frame to body-fixed frame
  - (ϕ, θ, ψ): $3-2-1$ Euler angles from navigation frame to body-fixed frame
  - Flat-Earth and spherical-Earth models: equivalent

LVLH/ECEF Angular Velocity

- For $\vec{\omega}_{L/E}$, use definition of derivative of rotation matrix:

$$\dot{C}_{E\leftarrow L} = C_{E\leftarrow L}\left[\vec{\omega}_{L/E,L}\right]_{\times} \tag{33}$$

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- For $\vec{\omega}_{L/E}$, use definition of derivative of rotation matrix:

$$\dot{C}_{E\leftarrow L} = C_{E\leftarrow L}\left[\vec{\omega}_{L/E,L}\right]_{\times} \tag{33}$$

$$\begin{bmatrix} \dot{\lambda}\sin\ell_L\sin\lambda - \dot{\ell}_L\cos\ell_L\cos\lambda & -\dot{\lambda}\sin\ell_L\cos\lambda - \dot{\ell}_L\cos\ell_L\sin\lambda & -\dot{\ell}_L\sin\ell_L \\ -\dot{\lambda}\cos\lambda & -\dot{\lambda}\sin\lambda & 0 \\ \dot{\lambda}\cos\ell_L\sin\lambda + \dot{\ell}_L\sin\ell_L\cos\lambda & -\dot{\lambda}\cos\ell_L\cos\lambda + \dot{\ell}_L\sin\ell_L\sin\lambda & -\dot{\ell}_L\cos\ell_L \end{bmatrix}$$
$$= \begin{bmatrix} -\sin\ell_L\cos\lambda & -\sin\ell_L\sin\lambda & \cos\ell_L \\ -\sin\lambda & \cos\lambda & 0 \\ -\cos\ell_L\cos\lambda & -\cos\ell_L\sin\lambda & -\sin\ell_L \end{bmatrix}\left[\vec{\omega}_{L/E,L}\right]_{\times} \tag{34}$$

LVLH/ECEF Angular Velocity (continued)

- Can be shown:

$$\vec{\omega}_{L/E,L} = \begin{bmatrix} \dot{\lambda}\cos\ell_L & -\dot{\ell}_L & -\dot{\lambda}\sin\ell_L \end{bmatrix}^T \tag{35}$$

$$\vec{\omega}_{L/I,L} = \begin{bmatrix} \left(\omega_E + \dot{\lambda}\right)\cos\ell_L & -\dot{\ell}_L & -\left(\omega_E + \dot{\lambda}\right)\sin\ell_L \end{bmatrix}^T \tag{36}$$

LVLH/ECEF Angular Velocity (continued)

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- Similarly, for $\vec{\omega}_{N/E}$:

$$\vec{\omega}_{N/E,N} = \begin{bmatrix} \dot{\lambda}\cos\ell & -\dot{\ell} & -\dot{\lambda}\sin\ell \end{bmatrix}^T \tag{37}$$

$$\vec{\omega}_{N/I,N} = \begin{bmatrix} \left(\omega_E + \dot{\lambda}\right)\cos\ell & -\dot{\ell} & -\left(\omega_E + \dot{\lambda}\right)\sin\ell \end{bmatrix}^T \tag{38}$$

Navigation Frame Rotation (continued)

- Flat-Earth angular velocity definition:

$$\vec{\omega}_{B/L,B} = \begin{bmatrix} p_{B/L,B} \\ q_{B/L,B} \\ r_{B/L,B} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta_L \\ 0 & \cos\phi_L & -\sin\phi_L\cos\theta_L \\ 0 & -\sin\phi_L & \cos\phi_L\cos\theta_L \end{bmatrix} \begin{bmatrix} \dot{\phi}_L \\ \dot{\theta}_L \\ \dot{\psi}_L \end{bmatrix} \tag{39}$$

$$\vec{\omega}_{B/N,B} = \begin{bmatrix} p_{B/N,B} \\ q_{B/N,B} \\ r_{B/N,B} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & -\sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \tag{40}$$

- Equivalent for spherical-Earth model and “inertial” angular velocity for flat-Earth model

Rotation of Earth Effect

- Redefine "new" inertial angular velocity coordinates

$$\vec{\omega}_{B/I,B} = \begin{bmatrix} 1 & 0 & -\sin\theta_L \\ 0 & \cos\phi_L & -\sin\phi_L\cos\theta_L \\ 0 & -\sin\phi_L & \cos\phi_L\cos\theta_L \end{bmatrix} \begin{bmatrix} \dot{\phi}_L \\ \dot{\theta}_L \\ \dot{\psi}_L \end{bmatrix} + C_{B\leftarrow L}(\phi_L, \theta_L, \psi_L) \begin{bmatrix} \left(\omega_E + \dot{\lambda}\right)\cos\ell_L \\ -\dot{\ell}_L \\ -\left(\omega_E + \dot{\lambda}\right)\sin\ell_L \end{bmatrix} \quad (41)$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} p_{B/L,B} \\ q_{B/L,B} \\ r_{B/L,B} \end{bmatrix} + \begin{bmatrix} p_{L/I,B} \\ q_{L/I,B} \\ r_{L/I,B} \end{bmatrix}$$

Rotation of Earth Effect (continued)

- Redefine "new" inertial angular velocity coordinates

$$\vec{\omega}_{B/I,B} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & -\sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} + C_{B\leftarrow N}(\phi,\theta,\psi) \begin{bmatrix} \left(\omega_E + \dot{\lambda}\right)\cos\ell \\ -\dot{\ell} \\ -\left(\omega_E + \dot{\lambda}\right)\sin\ell \end{bmatrix}$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} p_{B/N,B} \\ q_{B/N,B} \\ r_{B/N,B} \end{bmatrix} + \begin{bmatrix} p_{N/I,B} \\ q_{N/I,B} \\ r_{N/I,B} \end{bmatrix} \tag{42}$$

Rotation Equation of Motion

- With "new" inertial angular acceleration, *rotation equation of motion* same as before:

$$\begin{bmatrix} L \\ M \\ N \end{bmatrix} = \begin{bmatrix} \dot{p} + \frac{I_{zz}-I_{yy}}{I_{xx}} qr - \frac{I_{xz}}{I_{xx}}(\dot{r} + pq) \\ \dot{q} + \frac{I_{xx}-I_{zz}}{I_{yy}} pr - \frac{I_{xz}}{I_{yy}}(r^2 - p^2) \\ \dot{r} + \frac{I_{yy}-I_{xx}}{I_{zz}} pq - \frac{I_{xz}}{I_{zz}}(\dot{p} - qr) \end{bmatrix} \tag{43}$$

Euler Rates Supplemental Equations

- LVLH-to-body-fixed frame Euler angle rates:

$$\begin{bmatrix}\dot{\phi}_L\\ \dot{\theta}_L\\ \dot{\psi}_L\end{bmatrix} = \begin{bmatrix}1 & \sin\phi_L\tan\theta_L & \cos\phi_L\tan\theta_L\\ 0 & \cos\phi_L & -\sin\phi_L\\ 0 & \sin\phi_L\sec\theta_L & \cos\phi_L\sec\theta_L\end{bmatrix}\left(\begin{bmatrix}p\\ q\\ r\end{bmatrix} - \begin{bmatrix}p_{L/I,B}\\ q_{L/I,B}\\ r_{L/I,B}\end{bmatrix}\right) \tag{44}$$

- Navigation-to-body-fixed frame Euler angle rates:

$$\begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix} = \begin{bmatrix}1 & \sin\phi\tan\theta & \cos\phi\tan\theta\\ 0 & \cos\phi & -\sin\phi\\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta\end{bmatrix}\left(\begin{bmatrix}p\\ q\\ r\end{bmatrix} - \begin{bmatrix}p_{N/I,B}\\ q_{N/I,B}\\ r_{N/I,B}\end{bmatrix}\right) \tag{45}$$

Vehicle's Center of Mass

- Recall position of vehicle's center of mass w.r.t. center of Earth, i.e. ECEF/ECI frame origin:

$$\vec{x}_{B/E,E} = \begin{bmatrix} d_E \cos \ell_L \cos \lambda \\ d_E \cos \ell_L \sin \lambda \\ d_E \sin \ell_L \end{bmatrix} = \begin{bmatrix} (N_\ell + h) \cos \lambda \cos \ell \\ (N_\ell + h) \sin \lambda \cos \ell \\ (N_\ell (1 - e_E^2) + h) \sin \ell \end{bmatrix} \tag{46}$$

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- In ECI frame coordinates:

$$\vec{x}_{B/I,I} = C_{I \leftarrow E} \vec{x}_{B/E,E} \tag{47}$$

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- In ECI frame coordinates:

$$\vec{x}_{B/I,I} = C_{I \leftarrow E} \vec{x}_{B/E,E} \quad (47)$$

- Differentiating and relating this to body-fixed frame coordinates:

$$\dot{\vec{x}}_{B/I,I} = C_{I \leftarrow E} \dot{\vec{x}}_{B/E,E} + \dot{C}_{I \leftarrow E} \vec{x}_{B/E,E} \quad (48)$$

Body-Fixed Frame Velocity Components

- By definition:

$$\dot{\vec{x}}_{B/E,E} = C_{E \leftarrow B} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{49}$$

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- Substituting:

$$\dot{\vec{x}}_{B/I,I} = C_{I \leftarrow B} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + C_{I \leftarrow E} \left[\vec{\omega}_{E/I,E}\right]_{\times} \vec{x}_{B/E,E} \tag{50}$$

Body-Fixed Frame Velocity Components

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$$\dot{\vec{x}}_{B/I,I} = C_{I\leftarrow B}\begin{bmatrix} u \\ v \\ w \end{bmatrix} + C_{I\leftarrow E}\left[\vec{\omega}_{E/I,E}\right]_{\times}\vec{x}_{B/E,E} \tag{50}$$

- Taking derivative for inertial acceleration and assuming $\dot{\omega}_{E/I,E} = 0$:

$$\begin{aligned}\ddot{\vec{x}}_{B/I,I} =& C_{I\leftarrow B}\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} + C_{I\leftarrow B}\left[\vec{\omega}_{B/I,B}\right]_{\times}\begin{bmatrix} u \\ v \\ w \end{bmatrix} + C_{I\leftarrow E}\left[\vec{\omega}_{E/I,E}\right]_{\times}\dot{\vec{x}}_{B/E,E} \\ &+ C_{I\leftarrow E}\left[\vec{\omega}_{E/I,E}\right]_{\times}\left[\vec{\omega}_{E/I,E}\right]_{\times}\vec{x}_{B/E,E}\end{aligned} \tag{51}$$

ECEF Frame Transformation

- In ECEF frame coordinates:

$$\ddot{\vec{x}}_{B/I,E} = C_{E\leftarrow B}\begin{bmatrix}\dot{u}\\ \dot{v}\\ \dot{w}\end{bmatrix} + C_{E\leftarrow B}\left[\vec{\omega}_{B/I,B}\right]_{\times}\begin{bmatrix}u\\ v\\ w\end{bmatrix} + \left[\vec{\omega}_{E/I,E}\right]_{\times} C_{E\leftarrow B}\begin{bmatrix}u\\ v\\ w\end{bmatrix} + \begin{bmatrix}0 & \omega_E & 0\\ -\omega_E & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & \omega_E & 0\\ -\omega_E & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\vec{x}_{B/E,E} \tag{52}$$

Body-Fixed Frame Transformation

- In body-fixed frame coordinates:

$$\ddot{\vec{x}}_{B/I,B} = \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} + \begin{bmatrix} p \\ q \\ r \end{bmatrix}_\times \begin{bmatrix} u \\ v \\ w \end{bmatrix} + [\vec{\omega}_{E/I,B}]_\times \begin{bmatrix} u \\ v \\ w \end{bmatrix} + C_{B\leftarrow E} \begin{bmatrix} -\omega_E^2 & 0 & 0 \\ 0 & -\omega_E^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{x}_{B/E,E} \tag{53}$$

$$\vec{\omega}_{E/I,B} = \begin{bmatrix} p_{E/I,B} & q_{E/I,B} & r_{E/I,B} \end{bmatrix}^T \tag{54}$$

  - Additive terms: Coriolis and centrifugal accelerations of Earth

Translation Equation of Motion

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \vec{g}_B - \begin{bmatrix} q_{E/I,B} w - r_{E/I,B} v \\ r_{E/I,B} u - p_{E/I,B} w \\ p_{E/I,B} v - q_{E/I,B} u \end{bmatrix} - C_{B\leftarrow E} \begin{bmatrix} -\omega_E^2 & 0 & 0 \\ 0 & -\omega_E^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{x}_{B/E,E} = \begin{bmatrix} \dot{u} + qw - rv \\ \dot{v} + ru - pw \\ \dot{w} + pv - qu \end{bmatrix} \tag{55}$$

$$\begin{bmatrix} p_{E/I,B} \\ q_{E/I,B} \\ r_{E/I,B} \end{bmatrix} = C_{B\leftarrow L}(\phi_L, \theta_L, \psi_L) \begin{bmatrix} \omega_E \cos \ell_L \\ 0 \\ -\omega_E \sin \ell_L \end{bmatrix} = C_{B\leftarrow N}(\phi, \theta, \psi) \begin{bmatrix} \omega_E \cos \ell \\ 0 \\ -\omega_E \sin \ell \end{bmatrix} \tag{56}$$

$$\vec{x}_{B/E,E} = \begin{bmatrix} d_E \cos \ell_L \cos \lambda \\ d_E \cos \ell_L \sin \lambda \\ d_E \sin \ell_L \end{bmatrix} = \begin{bmatrix} (N_\ell + h) \cos \lambda \cos \ell \\ (N_\ell + h) \sin \lambda \cos \ell \\ (N_\ell(1 - e_E^2) + h) \sin \ell \end{bmatrix} \tag{57}$$

Spherical-Earth Model

- $\ell_L = \ell$ and $d_E = \bar{R}_E + h$
  - $\bar{R}_E$: **mean radius of Earth** defined as $6,366,707.0195$ m by WGS 84

$$\begin{bmatrix} \dot{\ell} \\ \dot{\lambda} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} \frac{1}{\bar{R}_E+h} & 0 & 0 \\ 0 & \frac{1}{(\bar{R}_E+h)\cos\ell} & 0 \\ 0 & 0 & -1 \end{bmatrix} C_{L\leftarrow B} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{58}$$

$$\begin{bmatrix} X - g(\ell,h)\sin\theta \\ Y + g(\ell,h)\cos\theta\sin\phi \\ Z + g(\ell,h)\cos\theta\cos\phi \end{bmatrix} - \begin{bmatrix} q_{E/I,B}w - r_{E/I,B}v \\ r_{E/I,B}u - p_{E/I,B}w \\ p_{E/I,B}v - q_{E/I,B}u \end{bmatrix} - C_{B\leftarrow E} \begin{bmatrix} \omega_E^2(\bar{R}_E + h)\cos\ell\sin\lambda \\ 0 \\ \omega_E^2(\bar{R}_E + h)\sin\ell \end{bmatrix} = \begin{bmatrix} \dot{u} + qw - rv \\ \dot{v} + ru - pw \\ \dot{w} + pv - qu \end{bmatrix} \tag{59}$$

J_2 Gravitation Model

- Higher fidelity gravitation model: expand Newton's point-mass Law of Gravitation in ECEF frame:

$$\vec{G}_E = -\frac{GM_E}{\|\vec{x}_E\|_2^3} \begin{bmatrix} \left(1 + \frac{3}{2}\left(\frac{R_e}{\|\vec{x}_E\|_2}\right)^2 J_2(1 - 5\sin^2\ell)\right) x_E \\ \left(1 + \frac{3}{2}\left(\frac{R_e}{\|\vec{x}_E\|_2}\right)^2 J_2(1 - 5\sin^2\ell)\right) y_E \\ \left(1 + \frac{3}{2}\left(\frac{R_e}{\|\vec{x}_E\|_2}\right)^2 J_2(3 - 5\sin^2\ell)\right) z_E \end{bmatrix} \tag{60}$$

  - $\vec{x}_E = [x_E\ y_E\ z_E]^T$: position of flight vehicle in ECEF frame
  - R_e: Earth's equatorial radius, 6378137.0 m

$$GM_E = 3986004.418 \times 10^8 m^3/s^2 \tag{61}$$

  - Second-order term of Earth's gravitation field, **WGS 84 J_2 parameter**:

$$J_2 = -\sqrt{5}\bar{C}_{2,0} = 0.001082626684 \tag{62}$$

Ellipsoidal Gravity Model

- Gravitational attraction minus centripetal acceleration of rotating Earth contributes to weight force:

$$\vec{F}_{g,E} = m\vec{g}_E = m\left(\vec{G}_E - [\vec{\omega}_{E/I}]_\times [\vec{\omega}_{E/I}]_\times \vec{x}_E\right) \tag{63}$$

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- Often assume gravity vector acts straight downward in navigation frame:

$$\vec{F}_{g,N} = m\vec{g}_N \approx \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix}^T \tag{64}$$

  - Generally be function of latitude ℓ and altitude h

Altitude Correction of Gravity

- As function of altitude *above* MSL **free air correction (FAC)**:

$$g(h) = g_0 \left(\frac{\bar{R}_E}{\bar{R}_E + h} \right)^2 \tag{65}$$

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$$g(h) = g_0 \left(\frac{\bar{R}_E}{\bar{R}_E + h} \right)^2 \tag{65}$$

  - $\bar{R}_E$: Earth's *mean* radius, 6,371,000 m
- Linear approximation for $h \ll \bar{R}_E$:

$$g(h) \approx g_0 - 3.086 \times 10^{-6} h \tag{66}$$

Latitude Correction of Gravity

- **Latitude correction** at MSL: **WGS 84 Ellipsoidal Gravity Formula**

$$g_0(\ell) = g_e \left(\frac{1 + k \sin^2 \ell}{\sqrt{1 - e_E^2 \sin^2 \ell}} \right) \tag{67}$$

  - $e_E^2 = 1 - (R_p/R_e)^2$: Earth's eccentricity $= 0.00669437999013$
  - $k = \frac{R_p g_p - R_e g_e}{R_e g_e}$: formula constant $= 0.00193185138639$
  - R_e: Earth's equatorial radius $= 6378137.0$ m
  - R_p: Earth's polar radius $= 6356752.3$ m
  - g_e: acceleration due to gravity at equator $= 9.7803253359$ m/s^2
  - g_p: acceleration due to gravity at poles $= 9.8321849378$ m/s^2

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  - g_e: acceleration due to gravity at equator $= 9.7803253359$ m/s^2
  - g_p: acceleration due to gravity at poles $= 9.8321849378$ m/s^2
- Combining two corrections:

$$g(\ell, h) = g_0(\ell) \left(\frac{\bar{R}_E}{\bar{R}_E + h} \right)^2 \tag{68}$$

Summary

- Rotating, ellipsoidal Earth effects:
  - Requires geocentric/geodetic rates supplemental equations
  - Redefined inertial angular velocity components
  - Requires Euler angle rate supplemental equations
  - Additive apparent forces due to Coriolis and centrifugal accelerations of Earth's rotation, ω_E

Summary

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    - J_2 term and Earth's centrifugal acceleration
  - Spherical gravity model
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