Lecture 23: Command Tracking Control System Design

Textbook Section 3.5

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Introduction

- Chapter: methods for MIMO LTI feedback control systems to be analyzed:
  - Stability
  - Controllability & observability
  - Robust stability

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- Specific design of control systems varies widely
- Lecture: introduces two additional control systems that augment the LTI or nonlinear plant dynamics to provide an architecture to design optimal MIMO LTI controllers alongside these augmentations:
  - Servomechanism augmentation $\rightarrow$ robust servomechanism control
  - Feedback linearization augmentation $\rightarrow$ dynamic inversion control

Gain-Scheduling & Dynamic Inversion

- Flight vehicles not inherently LTI systems $\rightarrow$ adaptive technique: gain-scheduling
  - Linearize nonlinear, time-invariant dynamics over grid of flight conditions design MIMO LTI feedback controllers at those conditions
  - Interpolate gains between points
  - Necessitates smooth transitions between flight conditions, linearized model good approximation to dynamics, robustness to neglected higher-order terms (HOT) in dynamics about flight conditions

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- During some flight maneuvers, may not sufficiently neglect nonlinear dynamics for flight vehicles
  - Alternative: **dynamic inversion (DI) control**
  - Transforms nonlinear, time-invariant dynamical system into LTI system via inversion: can then be controlled using MIMO LTI controller

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  - $\vec{y}(t)$ a.k.a. **controlled variable**
- Tracking error, $\vec{e}$:

$$\vec{e}(t) = \vec{r}(t) - \vec{y}(t) \tag{2}$$

Input-Output Feedback Linearization

- This type of dynamic inversion differentiates controlled variable, $\vec{y}(t)$, until control, $\vec{u}(t)$, appears in expression for derivative
  - A.k.a. **input-output feedback linearization control**
  - First presented for linear, time-invariant systems as introduction to concept
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- Note: method relies heavily on differentiation of dynamical system: may not be suitably robust to model uncertainties
  - Nonlinear dynamic inversion often used alongside adaptive control methods for linear-in-control systems to account for **matched uncertainties** in dynamics model, i.e. those that can be canceled by choosing some $\vec{u}(t)$, as opposed to **unmatched uncertainties**, i.e. those that can't

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- Introduce basic control designs
  - Can be extended to be robust to unmatched uncertainties
  - Exhaustive discussion including stability and robustness beyond scope of course
  - Typically addressed in nonlinear systems and/or adaptive control courses

LTI Dynamic Inversion Control

- Introduction to dynamic inversion control, consider LTI system:

$$\begin{aligned} \dot{\vec{x}} &= A\vec{x} + B\vec{u} \\ \vec{y} &= C\vec{x} \end{aligned} \tag{3}$$

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- Define inner **feedback linearization loop**:

$$\vec{u} = (CB)^{-1}\left(-CA\vec{x} + \dot{\vec{r}} + \vec{v}\right) \tag{5}$$

  - $\vec{v}$: **virtual control input**

Error Dynamics

- Substituting this control law for $\dot{\vec{y}}$:

$$\dot{\vec{y}} = CA\vec{x} + CB\,(CB)^{-1}\left(-CA\vec{x} + \dot{\vec{r}} + \vec{v}\right) \tag{6}$$

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$$\dot{\vec{e}} = -\vec{v} \tag{9}$$

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- Selecting this $\vec{u}$ canceled $CA\vec{x}$ term and relates tracking error without CB term, making system from $\vec{v}$ to $\vec{y}$ appear like linear system with poles at origin

Outer Tracking Loop

- Design **outer tracking loop** as state feedback control law, K, using MIMO LTI control on tracking error, $\vec{e}$:

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- Overall **LTI dynamic inversion (LDI) controller**:

$$\vec{u} = (CB)^{-1}\left(-CA\vec{x} + \dot{\vec{r}} + K\vec{e}\right) \tag{12}$$

  - Sums feedback linearization loop output, LTI control gain on tracking error, and

Closed-Loop System Dynamics

- Substituting LDI controller into state equation, closed-loop system dynamics:

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\dot{\vec{x}} &= A\vec{x} + B\left(CB\right)^{-1}\left(-CA\vec{x} + \dot{\vec{r}} + \vec{v}\right) \\
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- System stability: error dynamics & **zero dynamics**, i.e. $\vec{y}(t) = 0$ or $\vec{v} = -\dot{\vec{r}}$:

$$\dot{\vec{x}} = \left(I - B\,(CB)^{-1}\right)A\vec{x} = A_z\vec{x} \tag{14}$$

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- Dimension of $\vec{e}$: $n_y < n_x$
- Remaining $n_x - n_y$ system poles must be LHP stable for stable DI controller
  - Unobservable by selecting controlled variable $\vec{y} = C\vec{x}$
  - $I - B\left(CB\right)^{-1}$: projection on null space of C along range of B
  - $\left(I - B\left(CB\right)^{-1}\right)A = A_z$: dynamics in null space of C & range perpendicular of B
  - Designing C, i.e. controlled variables of state: stable LDI

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$$\vec{y}^{[n]} = CA^{n+1}\vec{x} + C\begin{bmatrix} A^nB\cdots & AB & B \end{bmatrix}\begin{bmatrix} \vec{u} \\ \vdots \\ \vec{u}^{[n-1]} \\ \vec{u}^{[n]} \end{bmatrix} \tag{15}$$

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- Form dynamic inversion controller:

$$\vec{u} = \left(CA^{n-1}B\right)^{-1}\left(-CA^n\vec{x} + \vec{r}^{[n]} + \vec{v}\right) \tag{17}$$

Special Cases: (continued)

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- If CB non-zero and singular, but system controllable: take $n = n_x$ and form controllability matrix:

$$\mathcal{C} = \begin{bmatrix} A^{n_x}B \cdots & AB & B \end{bmatrix} \tag{19}$$

  - Rank n_x
  - C matrix has rank n_u

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- Define **pseudoinverse** of $C\mathcal{C}$:

$$(C\mathcal{C})^{+} = (C\mathcal{C})^{T} \left((C\mathcal{C}) (C\mathcal{C})^{T} \right)^{-1} \tag{20}$$

Special Cases: (continued)

- Form dynamic inversion controller:

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- Error dynamics:

$$\dot{\vec{e}} + K_{n-1}\,\vec{e}^{[n-1]} + \cdots + K_0\,\vec{e} = 0 \tag{24}$$

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- For CB as non-zero and singular: use dynamical system to extract $\vec{u}(t)$ from its derivative vector $\begin{bmatrix}\vec{u} & \cdots & \vec{u}^{[n-1]} & \vec{u}^{[n]}\end{bmatrix}^T$

Linear-In-Control, Time-Variant System

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$$\dot{\vec{y}} = \frac{\partial h}{\partial \vec{x}}\dot{\vec{x}} = \frac{\partial h}{\partial \vec{x}} f(\vec{x}) + \frac{\partial h}{\partial \vec{x}} g(\vec{x})\vec{u} \tag{26}$$

  - $\frac{\partial h}{\partial \vec{x}} f(\vec{x})$: **Lie derivative** of $h(\vec{x})$ along $f(\vec{x})$

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  - $\frac{\partial h}{\partial \vec{x}} f(\vec{x})$: **Lie derivative** of $h(\vec{x})$ along $f(\vec{x})$
- Inner **feedback linearization loop**:

$$\vec{u} = \left(\frac{\partial h}{\partial \vec{x}} g(\vec{x})\right)^{-1}\left(-\frac{\partial h}{\partial \vec{x}} f(\vec{x}) + \dot{\vec{r}} + \vec{v}\right) \tag{27}$$

  - **Virtual control input**: $\vec{v}(t)$

Error Dynamics

- Substituting this control law for $\dot{\vec{y}}$: **error dynamics**

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Nonlinear Dynamic Inversion Controller

- Overall **nonlinear dynamic inversion (NDI) controller**:

$$\vec{u} = \left(\frac{\partial h}{\partial \vec{x}} g(\vec{x})\right)^{-1} \left(-\frac{\partial h}{\partial \vec{x}} f(\vec{x}) + \dot{\vec{r}} + K\vec{e}\right) \tag{31}$$

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- Note: $\vec{u}$ requires knowing nonlinear dynamics or available lookup tables to compute $\frac{\partial h}{\partial \vec{x}} f(\vec{x})$ and $\frac{\partial h}{\partial \vec{x}} g(\vec{x})$
- Similar to CB singular or zero for LDI: if $\frac{\partial h}{\partial \vec{x}} g(\vec{x})$ singular, must use successive Lie derivatives to form suitable feedback linearization controllers

Closed-Loop Dynamics

- Substituting NDI controller into state equation, closed-loop system dynamics:

$$
\begin{aligned}
\dot{\vec{x}} &= f(\vec{x}) + g(\vec{x}) \left(\frac{\partial h}{\partial \vec{x}} g(\vec{x}) \right)^{-1} \left(-\frac{\partial h}{\partial \vec{x}} f(\vec{x}) + \dot{\vec{r}} + K\vec{e} \right) \\
\dot{\vec{x}} &= (I - g(\vec{x})) \left(\frac{\partial h}{\partial \vec{x}} g(\vec{x}) \right)^{-1} \left(-\frac{\partial h}{\partial \vec{x}} \right) f(\vec{x}) + g(\vec{x}) \left(\frac{\partial h}{\partial \vec{x}} g(\vec{x}) \right)^{-1} \left(\dot{\vec{r}} + \vec{v} \right)
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- **Zero dynamics**, i.e. $\vec{y}(t) = 0$ or $\vec{v} = -\dot{\vec{r}}$:

$$\dot{\vec{x}} = (I - g(\vec{x}))\left(\frac{\partial h}{\partial \vec{x}} g(\vec{x})\right)^{-1}\left(-\frac{\partial h}{\partial \vec{x}}\right) f(\vec{x}) \tag{33}$$

  - Linearized at specific flight conditions to check suitability of controlled variable, $\vec{y} = h(\vec{x})$
  - OR simulated at different initial conditions and checked

Beyond Standard Nonlinear Dynamic Inversion

- To improve robustness to model uncertainty in NDI control strategy, alternatively design **incremental nonlinear dynamic inversion (INDI) controller** instead of full NDI
  - Calculates required *change* to control input as opposed to full control input
  - Full system model not required, only local part of model: INDI more robust

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- Another control strategy to improve performance of NDI to model uncertainty: **NDI-Based Model Reference Adaptive Control (MRAC)**
  - Reference signal, $\vec{r}$, and derivative, $\dot{\vec{r}}$, defined via specified dynamical system called **reference model**
  - Reference model tracks plant state, i.e. regulate error $\vec{e} = \vec{r} - \vec{x}$ to zero
  - Control system incorporates additional adaptive feedback loop to account for matched uncertainties in dynamics

NDI-Based MRAC Example

- Consider linear-in-control dynamical system with additive uncertainty $\Delta(\vec{x})$:

$$\dot{\vec{x}} = f(\vec{x}) + g(\vec{x})\vec{u} + \Delta(\vec{x}, \vec{u}) \tag{34}$$

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$$\vec{u} = g(\vec{x})^{-1}\left(-f(\vec{x}) + \dot{\vec{r}} + K\vec{e} - \hat{\Delta}(\vec{x}, \vec{u}, \vec{\theta})\right) \tag{35}$$

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$$\dot{\vec{e}} = \dot{\vec{r}} - \dot{\vec{x}} \tag{36}$$

$$\dot{\vec{e}} = \dot{\vec{r}} - f(\vec{x}) - g(\vec{x})\vec{u} - \Delta(\vec{x}, \vec{u}) \tag{37}$$

NDI-Based MRAC Example (continued)

$$\dot{\vec{e}} = \dot{\vec{r}} - f(\vec{x}) - \left(-f(\vec{x}) + \dot{\vec{r}} + K\vec{e} - \hat{\Delta}(\vec{x}, \vec{u}, \vec{\theta})\right) - \Delta(\vec{x}) \tag{38}$$

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  - Often $\hat{\Delta}(\vec{x}, \vec{u}, \vec{\theta})$ may not even converge to true $\Delta(\vec{x}, \vec{u})$, but system still stable

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  - Often $\hat{\Delta}(\vec{x}, \vec{u}, \vec{\theta})$ may not even converge to true $\Delta(\vec{x}, \vec{u})$, but system still stable
- Adaptive neural networks have been used to perform the uncertainty cancellation term $\hat{\Delta}(\vec{x}, \vec{u})$ & MRAC can be performed with and without NDI
  - Such control system design and analysis beyond scope of course, simply mentioned to highlight importance of linear feedback control design element present even in many nonlinear control systems

Single Input, Single Output (SISO) Plant

- Dynamics:

$$\dot{x} = ax + b(u + f(x)) \tag{40}$$

  - x: system state
  - u: control input
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  - Sign of b known & controllable
- System dynamics depend on unknown function $f(x)$ defined as linear combination of N known basic functions $\phi_i(x)$ with N unknown constants, $\vec{\theta}_i$:

$$f(x) = \sum_{i=1}^{N} \vec{\theta}_i \phi_i(x) = \vec{\theta}^T \Phi(x) \tag{41}$$

  - $\Phi(x) = [\phi_1 \ldots \phi_N]^T \in \mathbb{R}^n$: known **regressor vector**
  - Components $\phi_i(x)$ assumed Lipschitz-continuous in x

Stable Reference Model

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- Stable reference model dynamics described by first-order differential equation:

$$\dot{x}_{ref} = a_{ref} x_{ref} + b_{ref} r(t) \tag{43}$$

  - $a_{ref} < 0$ & b_{ref}: desired constants chosen to represent desired response due to bounded commands
  - E.g. $b_{ref} = -a_{ref}$ for unity DC gain
  - E.g. select a_{ref} such that reference time constant as small as desired and indirectly indicative of control effort possible

Matching Conditions

- Control objective of interest in SISO case: asymptotically track state x_{ref} of reference model driven by any bounded command $r(t)$

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  - In presence of $N + 2$ unknown constant parameters $\{a, b, \theta_1, ..., \theta_N\}$
- Define ideal feedback and feedforward control law as if unknown parameters known:

$$u_{ideal} = k_x x + k_r r - \theta^T \Phi(x) \tag{44}$$

  - k_x & k_r: ideal feedback & feedforward gains

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  - A.k.a. **matching conditions**,
- Clear for SISO plants: unknown ideal gains, k_x and k_r, always exist
  - Not true for MIMO dynamics

Tracking Control Law

- Tracking control law:

$$u = \hat{k}_x x + \hat{k}_r r - \hat{\theta}^T \Phi(x) \tag{47}$$

  - Adaptive feedback gain: $\hat{k}_x$
  - Adaptive feedforward gain: $\hat{k}_r$
  - Vector of estimated parameters: $\vec{\theta}$
  - $\rightarrow$ achieve global uniform asymptotic tracking of reference model trajectories

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- To show $\rightarrow$ substitute into system dynamics:

$$\dot{x} = (a + b\hat{k}_x)x + b(\hat{k}_r r - (\hat{\theta} - \vec{\theta})^T \Phi(x)) \tag{48}$$

- Rewrite using matching conditions:

$$\dot{x} = a_{ref}x + bk_r r + b(\hat{k}_x - k_x)x + b(\hat{k}_r - k_r)r - b(\hat{\theta} - \vec{\theta})^T \tag{49}$$

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$$\Delta k_x = \hat{k}_x - k_x \tag{50}$$

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- Parameter estimation error vector:

$$\Delta \vec{\theta} = \hat{\theta} - \vec{\theta} \tag{52}$$

Tracking Error Dynamics

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  - Choose adaptive gains $\hat{k}_x, \hat{k}_r, \vec{\hat{\theta}}$ to enforce global uniform asymptotic stability of origin
- Accomplished: inverse Lyapunov design approach
  - Choose Lyapunov function candidate
  - Select adaptive laws s.t. Lyapunov function time derivative evaluated along trajectories of error dynamics becomes nonpositive
  - By Lyapunov stability theory: tracking error asymptotically converges to origin
  - System state asymptotically tracks state of reference model

Lyapunov Function Candidate and Power

- Quadratic Lyapunov function candidate:

$$V(e, \Delta k_x, \Delta k_r, \Delta \vec{\theta}) = e^2 + |b|(\gamma_x^{-1} \Delta k_x^2 + \gamma_r^{-1} \Delta k_r^2 + \Delta \vec{\theta}^T \Gamma_\theta^{-1} \Delta \vec{\theta}) \quad (55)$$

  - $\gamma_x > 0$, $\gamma_r > 0$, $\Gamma_\theta \in \mathbb{R}^{n \times n}$: **rates of adaptation**
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  - Tunable by control system designer
- Time derivative of V along trajectories of current SISO MRAC problem:

$$\begin{aligned}
\dot{V}(e, \Delta k_x, \Delta k_r, \Delta\vec{\theta}) &= 2e\dot{e} + 2|b|(\gamma_x^{-1}\Delta k_x\dot{\hat{k}}_x + \gamma_r^{-1}\Delta k_r\dot{\hat{k}} + \Delta\vec{\theta}^T\Gamma_\theta\dot{\hat{\theta}}) \\
\dot{V}(e, \Delta k_x, \Delta k_r, \Delta\vec{\theta}) &= 2e(a_{ref}e + b(\Delta k_x x + \Delta k_r r - \Delta\vec{\theta}^T\Phi(x))) \\
&\quad + 2|b|(\gamma_x^{-1}\Delta k_x\dot{\hat{x}} + \gamma_r^{-1}\Delta k_r\dot{\hat{k}}_r + \Delta\vec{\theta}^T\Gamma_\theta^{-1}\dot{\hat{\theta}}) \\
\dot{V}(e, \Delta k_x, \Delta k_r, \Delta\vec{\theta}) &= 2a_{ref}e^2 + 2|b|(\Delta k_x(xe\,\mathrm{sign}(b) + \gamma_x^{-1}\dot{\hat{k}})) \\
&\quad + 2|b|(\Delta k_r(re\,\mathrm{sign}(b) + \gamma_r^{-1}\dot{\hat{k}}_r)) \\
&\quad + 2|b|\Delta\vec{\theta}^T(-\Phi(x)e\,\mathrm{sign}(b) + \Gamma_\theta^{-1}\dot{\hat{\theta}})
\end{aligned} \tag{56}$$

Lyapunov Stability Theory

- Sufficient to choose adaptive laws s.t. $\dot{V}(e, \Delta k_x, \Delta k_r, \Delta \vec{\theta}) \leq 0$ occur if

$$
\begin{aligned}
\dot{\hat{k}}_x &= -\gamma_x x e \operatorname{sign}(b) \\
\dot{\hat{k}}_r &= -\gamma_r r e \operatorname{sign}(b) \\
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- Then,

$$\dot{V}(e, \Delta k_x, \Delta k_r, \Delta \vec{\theta}) = 2 a_{ref} e(t)^2 \leq 0 \tag{58}$$

  - $a_{ref} < 0$ as specified

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  - $\rightarrow$ $\dot{x}$ and $\dot{x}_{ref}$ bounded
- Differentiating:

$$\ddot{V}(e, \Delta k_x, \Delta k_r, \Delta \vec{\theta}) = 4a_{ref}e(t)\dot{e}(t) \tag{59}$$

- Bounded $\rightarrow$ $\dot{V}$ continuous function of time

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- Bounded $\rightarrow$ $\dot{V}$continuous function of time
- As V lower bounded and $\dot{V} \leq 0$, then V has finite limit and Barbalot's lemma:

$$\lim_{t \rightarrow \infty} \dot{V}(t) = 0 \tag{60}$$

- Thus, $e(t) \rightarrow 0$ as $t \rightarrow \infty$

Additional Notes

- As Lyapunov function radially unbounded and does not depend explicitly on time $\rightarrow$ stability property global and uniform
  - I.e. closed-loop tracking error dynamics globally uniformly asymptotically stable
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  - Command tracking problem solved
- Note: estimated parameters $\hat{\theta}$ not guaranteed to converge to ideal parameters $\vec{\theta}$, but uniformly bounded
  - Persistency of excitation provides sufficient conditions for estimates to converge
- Note: MRAC "tuning knobs" are rates of adaptation: γ_x, γ_r, Γ_θ
  - Larger rates $\rightarrow$ faster adaptive laws evolve $\rightarrow$ fast tracking
  - Can lead to undesirable oscillations during transient times as system output forced closer to command
  - Trade-off between fast tracking and smooth transients: design-dependent challenge

Direct MRAC for MIMO Nonlinear Systems

- Consider nonlinear plant with dynamics of form:

$$\dot{\vec{x}} = A\vec{x} + B\Lambda(\vec{u} + f(\vec{x})) \tag{61}$$

  - $\vec{x} \in \mathbb{R}^n$: system state
  - $\vec{u} \in \mathbb{R}^p$: control input
  - $B \in \mathbb{R}^{n \times p}$: known control matrix
  - $A \in \mathbb{R}^{n \times n}$: unknown state matrix
  - $\Lambda \in \mathbb{R}^{p \times p}$: unknown diagonal matrix of control uncertainties with diagonal elements $\lambda_i > 0$

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- Assume pair $(A, B\Lambda)$ controllable
- Unknown nonlinear vector function $f(\vec{x}) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ represents system matched uncertainty

$$f(\vec{x}) = \vec{\theta}^T \Phi(\vec{x}) \tag{62}$$

Direct MIMO State Feedback

- Require: $\vec{x}$ globally uniformly asymptotically track $\vec{x}_{ref}$
- **Reference Model**

$$\dot{\vec{x}}_{ref} = A_{ref}\vec{x}_{ref} + B_{ref}\vec{r}(t) \tag{63}$$

  - $A_{ref} \in \mathbb{R}^n$ chosen such that all eigenvalues in LHP
  - $B_{ref} \in \mathbb{R}^{n \times m}$: reference input matrix
  - $\vec{r}(t) \in \mathbb{R}^p$: external, bounded reference command

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- Requires: all signals in system remain uniformly bounded
- Note: reference command has same dimension as plant control input
- Given any bounded $\vec{r}(t)$, control input $\vec{u}(t)$ chosen such that state tracking error, $\vec{e}(t) = \vec{x}(t) - \vec{x}_{ref}(t)$, globally uniformly asymptotically tends to zero

$$\lim_{t\to\infty} \|\vec{x}(t) - \vec{x}_{ref}(t)\| = 0 \tag{64}$$

Ideal Fixed-Gain Control

- If matrices A and Λ known $\rightarrow$ ideal fixed-gain control law:

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$$\vec{u} = K_x^T \vec{x} + K_r^T \vec{r} - \vec{\theta}^T \Phi(\vec{x}) \tag{65}$$

- Obtain closed-loop system:

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- Comparing with desired reference dynamics: existence of controller of ideal fixed-gain form $\rightarrow$ ideal unknown control gains, K_x and K_r, must satisfy **matching conditions**

$$\begin{aligned} A + B\Lambda K_x^T &= A_{ref} \\ B\Lambda K_r^T &= B_{ref} \end{aligned} \tag{67}$$

Adaptive Control Law

- Assuming these hold, then ideal fixed-gain control law results in closed-loop system exactly same as reference model

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- In practice, structure of A known and A_{ref} and B_{ref} chosen so system has at least one ideal solution for K_x and K_r
- Assuming K_x and K_r exist $\rightarrow$ adaptive control law (based on ideal fixed-gain control law)

$$\vec{u} = \hat{K}_x^T \vec{x} + \hat{K}_r^T \vec{r} - \hat{\theta}^T \Phi(\vec{x}) \tag{68}$$

- $\hat{K}_x \in \mathbb{R}^{n \times p}$, $\hat{K}_r \in \mathbb{R}^{p \times p}$, $\hat{\theta} \in \mathbb{R}^{N \times n}$: estimates of ideal unknown matrices K_x, K_r, $\vec{\theta}$

Closed-Loop Error Dynamics

- Substitute into plant dynamics $\rightarrow$ closed-loop system dynamics:

$$\dot{\vec{x}} = (A + B\Lambda\hat{K}_x^T)\vec{x} + B\Lambda\left(\hat{K}_r^T\vec{r} - (\hat{\theta} - \vec{\theta})^T\Phi(\vec{x})\right) \tag{69}$$

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- Subtracting reference model from closed-loop system dynamics → closed-loop tracking error dynamics:

$$\begin{aligned}\dot{\vec{e}} =& (A + B\Lambda\hat{K}_x^T)\vec{x} + B\Lambda\left(\hat{K}_r^T\vec{r} - (\hat{\theta} - \vec{\theta})^T\Phi(\vec{x})\right) \\ & - A_{ref}\vec{x}_{ref} - B_{ref}\vec{r}\end{aligned} \tag{70}$$

Closed-Loop Error Dynamics

- Including matching conditions:

$$
\begin{aligned}
\dot{\vec{e}} &= \left(A_{ref} + B\Lambda(\hat{K}_x - K_x)\right)\vec{x} - A_{ref}\vec{x}_{ref} \\
&\quad + B\Lambda(\hat{K}_r - K_r)\vec{r} - B\Lambda(\hat{\theta} - \vec{\theta})^T\Phi(\vec{x}) \\
\dot{\vec{e}} &= A_{ref}\vec{e} + B\Lambda\left((\hat{K}_x - K_x)^T\vec{x} + (\hat{K}_r - K_r)^T\vec{r} - (\hat{\theta} - \vec{\theta})^T\Phi(\vec{x})\right)
\end{aligned} \tag{71}
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Closed-Loop Error Dynamics

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- Defining $\Delta K_x = \hat{K}_x$, $\Delta K_r = \hat{K}_r - K_r$, & $\Delta\vec{\theta} = \hat{\theta} - \vec{\theta}$: parameter estimation errors

$$\dot{\vec{e}} = A_{ref}\vec{e} + B\Lambda\left(\Delta K_x^T\vec{x} + \Delta K_r^T\vec{r} - \Delta\vec{\theta}^T\Phi(\vec{x})\right) \tag{72}$$

Lyapunov Function

- Define rates of adaptation: $\Gamma_x = \Gamma_x^T > 0$, $\Gamma_r = \Gamma_r^T > 0$, $\Gamma_\theta = \Gamma_\theta^T > 0$

Lyapunov Function

- Define rates of adaptation: $\Gamma_x = \Gamma_x^T > 0$, $\Gamma_r = \Gamma_r^T > 0$, $\Gamma_\theta = \Gamma_\theta^T > 0$
- Consider globally radially unbounded quadratic Lyapunov function candidate:

$$
\begin{aligned}
&V(\vec{e}, \Delta K_x, \Delta K_r, \Delta\vec{\theta}) = \vec{e}^{\,T} P \vec{e} \\
&+ \operatorname{Tr}\left(\left(\Delta K_x^T \Gamma_x^{-1} \Delta K_x + \Delta K_r^T \Gamma_r^{-1} \Delta K_r + \Delta\vec{\theta}^{\,T} \Gamma_\theta^{-1} \Delta\vec{\theta}\right) \Lambda\right)
\end{aligned} \tag{73}
$$

- $P = P^T > 0$ satisfies **algebraic Lyapunov equation**:

$$
P A_{ref} + A_{ref}^T P = -Q \tag{74}
$$

  - For some $Q = Q^T > 0$

Time Derivative of V Along Trajectories

$$
\begin{aligned}
\dot{V} &= \dot{\vec{e}}^{\,T} P \vec{e} + \vec{e}^{\,T} P \dot{\vec{e}} \\
&+ 2\mathrm{Tr}\left(\left(\Delta K_x^T \Gamma_x^{-1} \dot{\hat{K}}_x + \Delta K_r^T \Gamma_r^{-1} \dot{\hat{K}}_r + \Delta \vec{\theta}^{\,T} \Gamma_\theta^{-1} \dot{\hat{\theta}}\right) \Lambda\right)
\end{aligned} \tag{75}
$$

Time Derivative of V Along Trajectories

$$\begin{aligned}\dot{V} &= \dot{\vec{e}}^T P \vec{e} + \vec{e}^T P \dot{\vec{e}} \\ &+ 2\mathrm{Tr}\left(\left(\Delta K_x^T \Gamma_x^{-1} \dot{\hat{K}}_x + \Delta K_r^T \Gamma_r^{-1} \dot{\hat{K}}_r + \Delta \vec{\theta}^T \Gamma_\theta^{-1} \dot{\hat{\theta}}\right) \Lambda\right)\end{aligned} \tag{75}$$

$$\begin{aligned}\dot{V} = &\left(A_{ref} \vec{e} + B\Lambda(\Delta K_x^T \vec{x} + \Delta K_r^T \vec{r} - \Delta \vec{\theta}^T \Phi(\vec{x}))\right)^T P \vec{e} \\ &+ \vec{e}^T P \left(A_{ref} \vec{e} + B\Lambda(\Delta K_x^T \vec{x} + \Delta K_r^T \vec{r} - \Delta \vec{\theta}^T \Phi(\vec{x})\right) \\ &+ 2\mathrm{Tr}\left(\left(\Delta K_x^T \Gamma_x^{-1} \dot{\hat{K}}_x + \Delta K_r^T \Gamma_r^{-1} \dot{\hat{K}}_r + \Delta \vec{\theta}^T \Gamma_\theta^{-1} \dot{\hat{\theta}}\right) \Lambda\right)\end{aligned} \tag{76}$$

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$$\begin{aligned}\dot{V} &= \vec{e}^T (A_{ref} P + P A_{ref}) \vec{e} + 2 \vec{e}^T P B \Lambda(\Delta K_x^T \vec{x} + \Delta K_r \vec{r} - \Delta \vec{\theta}^T \Phi(\vec{x})) \\ &+ 2\mathrm{Tr}\left(\left(\Delta K_x^T \Gamma_x^{-1} \dot{\hat{K}}_x + \Delta K_r^T \Gamma_r^{-1} \dot{\hat{K}}_r + \Delta \vec{\theta}^T \Gamma_\theta^{-1} \dot{\hat{\theta}}\right) \Lambda\right)\end{aligned} \tag{77}$$

Time Derivative of V

$$
\begin{aligned}
\dot{V} = & -\vec{e}^{\,T} Q \vec{e} + \left(2\vec{e}^{\,T} PB\Lambda\Delta K_x^T \vec{x} + 2\mathrm{Tr}\left(\Delta K_x^T \Gamma_x^{-1} \dot{\hat{K}}_x \Lambda\right)\right) \\
& + \left(2\vec{e}^{\,T} PB\Lambda\Delta K_r \vec{r} + 2\mathrm{Tr}\left(\Delta K_r^T \Gamma_r^{-1} \dot{\hat{K}}_r \Lambda\right)\right) \\
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\end{aligned}
\tag{78}
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- Use vector trace identity $\vec{a}^{\,T}\vec{b} = \mathrm{Tr}(\vec{b}\,\vec{a}^{\,T})$:

$$
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\dot{V} = & -\vec{e}^{\,T} Q \vec{e} + 2\mathrm{Tr}\left(\Delta K_x^T \left(\Gamma_x^{-1} \dot{\hat{K}}_x + \vec{x}\,\vec{e}^{\,T} PB\right)\Lambda\right) \\
& + 2\mathrm{Tr}\left(\Delta K_r^T \left(\Gamma_r^{-1} \dot{\hat{K}}_r + \vec{r}\,\vec{e}^{\,T} PB\right)\Lambda\right) \\
& + 2\mathrm{Tr}\left(\Delta\vec{\theta}^{\,T} \left(\Gamma_x^{-1} \dot{\hat{\theta}} - \Phi(\vec{x})\,\vec{e}^{\,T} PB\right)\Lambda\right)
\end{aligned} \tag{79}
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Direct MIMO MRAC Adaptive Laws

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\begin{aligned}
\dot{\hat{K}}_x &= -\Gamma_x \vec{x}\,\vec{e}^{\,T} PB \\
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\end{aligned} \tag{80}
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- Closed-loop error dynamics uniformly stable
  $\rightarrow$ $\vec{e}(t)$ and $\Delta K_x(t)$, $\Delta K_r(t)$, $\Delta\theta(t)$ uniformly bounded
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- Since $\vec{r}(t)$ bounded and A_{ref} has all LHP eigenvalues
  - $\rightarrow \vec{x}_{ref}$ and $\dot{\vec{x}}_{ref}$ bounded
  - $\rightarrow \vec{x}(t)$ uniformly bounded, $\vec{u}(t)$ bounded, $\dot{\vec{x}}(t)$ bounded
  - $\rightarrow \dot{\vec{e}}(t)$ bounded

MRAC System Results

- Second time derivative of $V(t)$ bounded:

$$\ddot{V} = -2\vec{e}^{\,T} Q \dot{\vec{e}} \tag{82}$$

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- Tuning knobs in MIMO case: Γ_x, Γ_r, Γ_θ, also Q for algebraic Lyapunov equation
  - All must be symmetric, positive definite matrices

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- Model reference adaptive control: track specified dynamical system reference under uncertainties
  - Model reference adaptive control (MRAC): often used with NDI