Lecture 8: Rigid Satellite Dynamics & Stability

Textbook Section 8.5 & 8.6

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Introduction

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- Airplane attitude:
  - Navigation-to-Body $3-2-1$ Euler angles: roll, pitch, yaw

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- Requires body-fixed frame to inertial frame transformation due to gravity force
- Airplane attitude:
  - Navigation-to-Body $3-2-1$ Euler angles: roll, pitch, yaw
- Satellite attitude:
  - Inertial-to-body $3-1-3$ Euler angles: precession
  - LVLH-to-body $3-2-1$ Euler angles: roll, pitch, yaw

Satellite Euler Angles

- $3-1-3$ Euler angles for body-fixed frame relative to inertial frame:
  - **Precession angle**,: ϕ_p
  - **Nutation angle**: θ_n
  - **Spin angle**: ψ_s
  - Subscripts added to differentiate Euler angles from $3-2-1$ LVLH-to-body frame Euler angles: roll, pitch, yaw

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  - Subscripts added to differentiate Euler angles from $3-2-1$ LVLH-to-body frame Euler angles: roll, pitch, yaw
- Vector expressed in inertial frame coordinates, $\vec{v}_I$, in body-fixed frame coordinates, $\vec{v}_B$:

$$\vec{v}_B = C_3(\psi_s)C_1(\theta_n)C_3(\phi_p)\vec{v}_I \tag{1}$$

$$\vec{v}_B = C_{B\leftarrow I}\vec{v}_I \tag{2}$$

More Definitions

$$C_{B\leftarrow I} = \begin{bmatrix} \cos\phi_p\cos\psi_s - \cos\theta_n\sin\phi_p\sin\psi_s & \sin\phi_p\cos\psi_s - \cos\theta_n\cos\phi_p\sin\psi_s & \sin\theta_n\sin\psi_s \\ -\cos\phi_p\sin\psi_s - \cos\theta_n\sin\phi_p\cos\psi_s & -\sin\phi_p\sin\psi_s - \cos\theta_n\cos\phi_p\cos\psi_s & \sin\theta_n\cos\psi_s \\ \sin\theta_n\sin\psi_s & -\sin\theta_n\cos\psi_s & \cos\theta_n \end{bmatrix} \quad (3)$$

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- **Precession rate**: $\omega_p = \dot{\phi}_p$
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- **Precession rate**: $\omega_p = \dot{\phi}_p$
  Nutation rate: $\omega_n = \dot{\theta}_n$
  Spin rate: $\omega_s = \dot{\psi}_s$
- Angular velocity of body-fixed frame relative to inertial frame expressed in body-fixed frame coordinates: $\vec{\omega}_{B/I,B} = [p\ q\ r]^T$

$$\begin{bmatrix} \omega_p \\ \omega_n \\ \omega_s \end{bmatrix} = \begin{bmatrix} \dot{\phi}_p \\ \dot{\theta}_n \\ \dot{\psi}_s \end{bmatrix} = \begin{bmatrix} \sin\psi_s \csc\theta_n & \cos\psi_s \csc\theta_n & 0 \\ \cos\psi_s & -\sin\psi_s & 0 \\ -\sin\psi_s \cot\theta_n & -\cos\psi_s \cot\theta_n & 1 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{4}$$

Principal Axes

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  - Inertia tensor varies depending on choosing body-fixed frame axes
- **Principal body-fixed frame**: axes for which inertia tensor diagonal

$$I_G = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \tag{5}$$

  - I_1, I_2, I_3: **principal moments of inertia**
  - Largest-to-smallest: major, intermediate, minor

Principal Axes (continued)

- Another body-fixed frame with inertia tensor, I'_G: eigenvalue decomposition of *symmetric* matrix I'_G to form I_G

$$I_G = V I_G V^T \tag{6}$$

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$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} I'_G \begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \\ \vec{v}_3^T \end{bmatrix} \tag{7}$$

  - I_1, I_2, I_3: eigenvalues of I'_G
  - $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$: corresponding right eigenvectors

Rigid Satellite Equations of Motion

- For angular momentum of satellite in principal body-fixed frame coordinates:

$$\vec{H}_{G,B} = I_G \vec{\omega}_{B/I,B} = \begin{bmatrix} I_1 p \\ I_2 q \\ I_3 r \end{bmatrix} \tag{8}$$

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- Principal body-fixed frame for Newton-Euler equations of motion: **rigid satellite equations of motion**

$$\begin{aligned} \vec{F}_{a,B} + \vec{F}_{p,B} + \vec{F}_{g,B} &= m \begin{bmatrix} \dot{u} + qw - rv \\ \dot{v} + ru - pw \\ \dot{w} + pv - qu \end{bmatrix} \\ \vec{M}_{a,B} + \vec{M}_{p,B} + \vec{M}_{g,B} &= \begin{bmatrix} I_1 \dot{p} + (I_3 - I_2) qr \\ I_2 \dot{q} + (I_1 - I_3) pr \\ I_3 \dot{r} + (I_2 - I_2) pq \end{bmatrix} \end{aligned} \tag{9}$$

Torque-Free Motion

- Neglecting secondary atmospheric drag forces and other celestial bodies, gravity from orbited body only force acting on satellite during **coast**:
  - I.e. no propulsive forces or moments applied
  - Unless satellite unusually large, gravitational force concentrated at center of mass G

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- Net moment about the center of mass is zero and the satellite undergoing **torque-free motion**

$$\sum \vec{M}_B = \frac{d}{dt}\vec{H}_G = \begin{bmatrix} I_1\dot{p} + (I_3 - I_2)qr \\ I_2\dot{q} + (I_1 - I_3)pr \\ I_3\dot{r} + (I_2 - I_1)pq \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{10}$$

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  - Without loss of generality, assume $\vec{H}_G$ defines z_I-axis as orbital plane assumed static
- Translation governed by orbital mechanics in inertial frame
  - Independent of attitude
  - Can transform to body-fixed frame velocity components using orbital elements and angular velocities from attitude dynamics

Nutation Angle Analysis

- By definition of $3-1-3$ Euler angles, nutation angle, θ_n, defines angle between z_B-axis and $\vec{H}_G$

$$\cos\theta_n = \frac{\vec{H}_G}{\|\vec{H}_G\|_2} \cdot \vec{k} \tag{11}$$

  - $\vec{k}$: unit vector of z_B-axis

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$$\|\vec{H}_G\|_2 = \frac{I_3 r}{\cos\theta_n} \tag{14}$$

Nutation Angle Analysis (continued)

- Taking derivative w.r.t. time and assuming torque-free motion:

$$\frac{d\cos\theta_n}{dt} = \frac{1}{\|\vec{H}_G\|_2}\begin{bmatrix} I_1 p \\ I_2 q \\ I_3 r \end{bmatrix} \cdot \left([\vec{\omega}_{B/I,B}]_\times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \tag{15}$$

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$$\dot{\theta}_n = \omega_n = -\frac{(I_1 - I_2)pq}{\|\vec{H}_G\|_2 \sin\theta_n} \tag{17}$$

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- For torque-free motion: nutation rate, ω_n, vanishes only if $I_1 = I_2$
  - I.e. z_B-axis: axis of “rotational symmetry” as x_B and y_B can be switched for principal body-fixed frame
  - Placement of x_B- and y_B-axes arbitrary: **axisymmetric** about z_B

Introductory Torque-Free Motion EOMS

- Assume $I_1 = I_2$:

$$\sum \vec{M}_B = \frac{d}{dt}\vec{H}_G = \begin{bmatrix} I_1\dot{p} + (I_3 - I_1)qr \\ I_1\dot{q} + (I_1 - I_3)pr \\ I_3\dot{r} \end{bmatrix} = 0 \tag{18}$$

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- Spin rate constant, i.e. $r = \bar{r}$

$$\|\vec{H}_G\|_2 = \frac{I_3\bar{r}}{\cos\theta_n} \tag{19}$$

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- Spin rate constant, i.e. $r = \bar{r}$

$$\|\vec{H}_G\|_2 = \frac{I_3\bar{r}}{\cos\theta_n} \tag{19}$$

- Define *temporarily* angular rate, ω_*:

$$\omega_* = \frac{I_1 - I_3}{I_1}\bar{r} \tag{20}$$

  - Related to another angular rate

Torque-Free Motion Characteristic Polynomial

- Rewrite torque-free motion dynamics as two coupled differential equations:

$$\begin{aligned} \dot{p} - \omega_* q &= 0 \\ \dot{q} + \omega_* p &= 0 \end{aligned} \tag{21}$$

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- Taking Laplace transform and separating variables: eigenvalue equation

$$\begin{bmatrix} s & -\omega_* \\ \omega_* & s \end{bmatrix} \begin{bmatrix} p(s) \\ q(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{22}$$

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- Determinant, i.e. characteristic polynomial:

$$s^2 + \omega_*^2 \tag{23}$$

  - Two purely imaginary roots/poles

Introductory Torque-Free Motion EOMS (continued)

- Well-known solution:

$$\begin{aligned} p &= \Omega \sin \omega_* t \\ q &= \Omega \cos \omega_* t \end{aligned} \tag{24}$$

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$$\vec{\omega} = \begin{bmatrix} \Omega \sin \omega_* t \\ \Omega \cos \omega_* t \\ \bar{r} \end{bmatrix} \tag{25}$$

  - Describes sweeping out of **space cone** at constant nutation angle, θ_n, about $\vec{H}_G$
  - Constant height, $\bar{r}$, above $x_B - y_B$ plane with circular base of radius Ω about z_B-axis

$3-1-3$ Euler Angle Description

- Substituting these expressions into $3-1-3$ Euler angle rate equations:

$$\begin{bmatrix}\omega_p\\ \omega_n\\ \omega_s\end{bmatrix} = \begin{bmatrix}\sin\psi_s\csc\theta_n & \cos\psi_s\csc\theta_n & 0\\ \cos\psi_s & -\sin\psi_s & 0\\ -\sin\psi_s\cot\theta_n & -\cos\psi_s\cot\theta_n & 1\end{bmatrix}\begin{bmatrix}\Omega\sin\omega_* t\\ \Omega\cos\omega_* t\\ \bar{r}\end{bmatrix} \tag{26}$$

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$$\begin{bmatrix}\omega_p\\ \omega_n\\ \omega_s\end{bmatrix}=\begin{bmatrix}\Omega\sin\omega_* t\sin\psi_s\csc\theta_n+\Omega\cos\omega_* t\cos\psi_s\csc\theta_n\\ \Omega\sin\omega_* t\cos\psi_s-\Omega\cos\omega_* t\sin\psi_s\\ -\Omega\sin\omega_* t\sin\psi_s\cot\theta_n-\Omega\cos\omega_* t\cos\psi_s\cot\theta_n+\bar{r}\end{bmatrix} \tag{27}$$

3 – 1 – 3 Euler Angle Description (continued)

- By trigonometric identities:

$$\begin{bmatrix}\omega_p \\ \omega_n \\ \omega_s\end{bmatrix} = \begin{bmatrix}\Omega \csc\theta_n \cos(\omega_* t - \psi_s) \\ \Omega (\sin\omega_* t - \psi_s) \\ \bar{r} - \Omega \cot\theta_n \cos(\omega_* t - \psi_s)\end{bmatrix} \tag{28}$$

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- As $\omega_n = 0$ for $I_1 = I_2$:
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$3-1-3$ Euler Angle Description (continued)

- Requires:

$$\omega_s = \dot{\psi}_s = \omega_* = \frac{I_1 - I_3}{I_1}\bar{r} \tag{30}$$

$$\vec{H}_{G,B} = \begin{bmatrix} I_1 \Omega \sin \omega_s t \\ I_1 \Omega \cos \omega_s t \\ I_3 \bar{r} \end{bmatrix} \tag{31}$$

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- By back-substitution into third component:

$$\Omega = \frac{I_3}{I_1}\bar{r} \tan \theta_n \tag{32}$$

$3-1-3$ Euler Angle Description (continued)

- Substituting for Ω into first component of angular velocity:

$$\bar{r} = \frac{I_1}{I_3}\omega_p \cos\theta_n \tag{33}$$

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- Substituting into angular momentum magnitude:

$$\|\vec{H}_G\|_2 = I_1\omega_p \tag{34}$$

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- Lastly, substituting for $\bar{r}$ in equation for ω_s:

$$\omega_s = \frac{I_1 - I_3}{I_3}\cos\theta_n\omega_p \tag{35}$$

$3-1-3$ Euler Angle Description (continued)

- For **oblate body**, i.e. $I_1 < I_3$: ω_p has opposite sign as ω_s
  - **Retrograde precession**

$3-1-3$ Euler Angle Description (continued)

- For **oblate body**, i.e. $I_1 < I_3$: ω_p has opposite sign as ω_s
  - **Retrograde precession**
- For **prolate body**, i.e. $I_1 > I_3$: ω_p has same sign as ω_s
  - **Prograde procession**

Wobble Angle

- **Wobble angle**, γ: angle between $\vec{w}_{B/I,B}$ and z_B-axis:

$$\cos\gamma = \frac{r}{\|\vec{\omega}_{B/I,B}} = \frac{\bar{r}}{\sqrt{\Omega^2\sin^2(\omega_* t) + \Omega^2\cos^2(\omega_* t) + \bar{r}^2}} = \frac{\bar{r}}{\sqrt{\Omega^2 + \bar{r}^2}} \tag{36}$$

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- Substituting for Ω:

$$\cos\gamma = \frac{\omega_0}{\sqrt{\left(\frac{I_3}{I_1}\bar{r}\tan\theta_n\right)^2}} = \frac{I_1}{\sqrt{I_1^2 + I_3^2\tan^2\theta_n}} \tag{37}$$

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$$\cos\gamma = \frac{\omega_0}{\sqrt{\left(\frac{I_3}{I_1}\bar{r}\tan\theta_n\right)^2}} = \frac{I_1}{\sqrt{I_1^2 + I_3^2\tan^2\theta_n}} \tag{37}$$

- Using trigonometric identities:

$$\cos\gamma = \frac{\cos\theta_n}{\sqrt{\frac{I_3^2}{I_1^2} + \left(1 - \frac{I_3^2}{I_1^2}\right)\cos^2\theta_n}} \tag{38}$$

  - $\gamma > \theta$: prograde precession
  - $\gamma < \theta$: retrograde precession

Introductory Stability of Torque-Free Motion

- Consider equilibrium condition: angular velocity about only one axis
  - Without loss of generality, assume z_B-axis as spin axis
  - I.e. $r(t) = \bar{r}$, $\bar{p} = 0^\circ$, $\bar{q} = 0^\circ$, for all $t > 0$
  - Trivially solves torque-free EOMs

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- Using perturbation notation, satellite torque-free motion:

$$\begin{bmatrix} I_1 \Delta \dot{p} + (I_3 - I_2)(\bar{r} + \Delta r)\Delta q \\ I_2 \Delta \dot{q} + (I_1 - I_3)(\bar{r} + \Delta r)\Delta p \\ I_3 \Delta \dot{r} + (I_2 - I_1)\Delta p \Delta q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{39}$$

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- Retaining first-order terms:

$$\begin{bmatrix} I_1 \Delta\dot{p} + (I_3 - I_2)\bar{r}\Delta q \\ I_2 \Delta\dot{q} + (I_1 - I_3)\bar{r}\Delta p \\ I_3 \Delta\dot{r} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{40}$$

  - Implies $\Delta\dot{r} = \Delta\dot{r}_0$ decoupled from other perturbed velocities

Linearized Model Characteristic Polynomial

$$\begin{bmatrix} \Delta\dot{p} + \frac{(I_3 - I_2)}{I_1}\bar{r}\Delta q \\ \Delta\dot{q} + \frac{(I_1 - I_3)}{I_2}\bar{r}\Delta p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{41}$$

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- Taking Laplace transform and separating variables: eigenvalue equation

$$\begin{bmatrix} s & \frac{(I_3 - I_2)}{I_1}\bar{r} \\ \frac{(I_1 - I_3)}{I_2}\bar{r} & s \end{bmatrix} \begin{bmatrix} \Delta p(s) \\ \Delta q(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{42}$$

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- Spin axis: major axis, i.e. $I_3 > I_1$ & $I_3 > I_2$ OR minor axis, i.e. $I_3 < I_1$ & $I_3 < I_2$
  Then, poles purely imaginary, sinusoidal motion, marginally stable w/ no damping
  - Spin axis: intermediate axis, i.e. $I_1 > I_3 > I_2$ or $I_2 > I_3 > I_1$,
    Then, motion unstable due to RHP system pole

Flexible Structure Analysis

- In reality, all satellites experience some degree of flexibility
  - Additional stability analysis of spin axis as major or minor axis due to energy dissipation as structural vibrations

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- Differentiating w.r.t. time:

$$\dot{T}_R = \frac{1}{2} I_1 \frac{d\omega_\perp^2}{dt} + I_3 r \dot{r} \tag{46}$$

  - $\dot{T}_R < 0$ for all satellites due to energy dissipation

Angular Momentum Relationship

- Recall angular momentum of satellite with $I_1 = I_2$:

$$\vec{H}_{G,B} = \begin{bmatrix} I_1 p \\ I_1 q \\ I_3 r \end{bmatrix} \tag{47}$$

$$\|\vec{H}_G\|_2^2 = I_1^2 \left(p^2 + q^2\right) + I_3^2 r^2 = I_1^2 \omega_\perp^2 + I_3^2 r^2 \tag{48}$$

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$$\dot{r} = -\frac{I_1^2}{2 I_3^2 r} \frac{d\omega_\perp^2}{dt} \tag{50}$$

Angular Velocity and Energy Dissipation

- Substituting expression into energy dissipation equation:

$$\dot{T}_R = \frac{1}{2} I_1 \frac{d\omega_\perp^2}{dt} + I_3 r \left(-\frac{I_1^2}{2 I_3^2 r} \frac{d\omega_\perp^2}{dt} \right) \tag{51}$$

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$$\dot{T}_R = \left(\frac{I_1 I_3}{2 I_3} - \frac{I_1^2}{2 I_3} \right) \frac{d\omega_\perp^2}{dt} \tag{52}$$

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$$\dot{T}_R = \left(\frac{I_1 I_3}{2 I_3} - \frac{I_1^2}{2 I_3} \right) \frac{d\omega_\perp^2}{dt} \tag{52}$$

- Rearranging:

$$\frac{d\omega_\perp^2}{dt} = \frac{2 I_3}{I_1 (I_3 - I_1)} \dot{T}_R \tag{53}$$

  - Oblate satellite, i.e. $I_3 > I_1$, then $\frac{d\omega_\perp^2}{dt} < 0$: asymptotically stable spin
    - Any perturbation in p or q will asymptotically decay to 0
  - Prolate satellite, i.e. $I_3 < I_1$, then $\frac{d\omega_\perp^2}{dt} > 0$: asymptotically unstable spin

Recall: Attitude Equation of Motion

- Satellite attitude equation of motion:

$$\vec{M}_B = \begin{bmatrix} I_{xx}\dot{p} + (I_{zz} - I_{yy})qr \\ I_{yy}\dot{q} + (I_{xx} - I_{zz})pr \\ I_{zz}\dot{r} + (I_{yy} - I_{xx})pq \end{bmatrix} \tag{54}$$

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- Define following for inertial angular velocity:

$$\vec{\omega}_{B/P,B} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \vec{\omega}_{B/L,B} + \vec{\omega}_{L/P,B} \tag{55}$$

Orbital Angular Velocity

- Orbital angular velocity $\omega_O = -\frac{\mu}{a^3}$ for circular orbit enters dynamics via

$$\vec{\omega}_{L/P,L} = \begin{bmatrix} 0 \\ -\omega_O \\ 0 \end{bmatrix} \tag{56}$$

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$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \vec{\omega}_{B/L,B} + C_{B\leftarrow L} \begin{bmatrix} 0 \\ -\omega_O \\ 0 \end{bmatrix} \tag{57}$$

Linearized Inertial Angular Velocity

- In terms of $3-2-1$ Euler angles of body-fixed frame relative to LVLH frame and corresponding Euler angle rates:

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\phi \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix} + \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ -\omega_O \\ 0 \end{bmatrix} \tag{58}$$

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- Angular velocity linearized for small Euler angles *and* small Euler angle rates using small angle approximation:

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} \approx \begin{bmatrix} \dot\phi - \phi\dot\psi \\ \dot\theta + \sin\phi\dot\psi \\ \dot\psi - \phi\dot\theta \end{bmatrix} + \begin{bmatrix} 1 & \psi & -\theta \\ -\psi & 1 & \phi \\ \theta & -\phi & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -\omega_O \\ 0 \end{bmatrix} \tag{59}$$

Linearized Inertial Angular Velocity

- Discarding higher-order terms, one has

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} \approx \begin{bmatrix} \dot{\phi} - \psi\omega_O \\ \dot{\theta} - \omega_O \\ \dot{\psi} + \phi\omega_O \end{bmatrix} \tag{60}$$

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- By differentiation:

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} \approx \begin{bmatrix} \ddot{\phi} - \omega_O\dot{\psi} \\ \ddot{\theta} \\ \ddot{\psi} + \omega_O\dot{\phi} \end{bmatrix} \tag{61}$$

Linearized Attitude Dynamics

- By substitution:

$$\vec{M}_B = \begin{bmatrix} I_{xx}\ddot{\phi} - I_{xx}\omega_O\dot{\psi} + (I_{zz} - I_{yy})(\dot{\theta} - \omega_O)(\dot{\psi} + \phi\omega_O) \\ I_{yy}\ddot{\theta} + (I_{xx} - I_{zz})(\dot{\phi} - \psi\omega_O)(\dot{\psi} + \phi\omega_O) \\ I_{zz}\ddot{\psi} - I_{zz}\omega_O\dot{\phi} + (I_{yy} - I_{xx})(\dot{\phi} - \psi\omega_O)(\dot{\theta} - \omega_O) \end{bmatrix} \tag{62}$$

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- Note: linearized pitch dynamics decoupled from linearized roll and yaw dynamics

Gravity-Gradient Definition

- For non-spinning satellites, typically consider effects of gravity-gradient moment, $\vec{M}_g$, separate from other moment disturbances, $\vec{M}_d$, and control inputs, $\vec{M}_c$

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- For rigid-body:

$$\vec{M}_{g,B} = \int [\vec{x}_B]_\times d\vec{F}_G = \int_V [\vec{x}_B]_\times \left(-\frac{\mu \rho dV}{\|\vec{r}_m\|_2^3} \vec{r}_m \right) \tag{64}$$

  - $dm = \rho dV$: infinitesimal mass element of body with density ρ
  - $\vec{r}_m = \vec{r}_{P,L} + \vec{x}_B$: position vector of mass element w.r.t. origin of perifocal frame
  - $\vec{r}_{P,B}$: position of center of mass of rigid-body w.r.t. origin of perifocal frame expressed in body-fixed frame coordinates

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- For circular orbit

$$\vec{r}_{P,B} = C_{B\leftarrow L} \begin{bmatrix} 0 \\ 0 \\ -a \end{bmatrix} = \begin{bmatrix} a \sin\theta \\ -a \sin\phi \cos\theta \\ -a \cos\phi \cos\theta \end{bmatrix} \tag{65}$$

Gravity-Gradient Approximation

- With assumption $\vec{x}_B \lll \vec{r}_P$, approximate radial distance cubed by truncated Taylor series:

$$\frac{1}{\|\vec{r}_m\|_2^3} \approx \frac{1}{a^3}\left(1 - \frac{3\vec{r}_P \cdot \vec{x}_B}{a^2}\right) \tag{66}$$

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$$\vec{M}_{g,B} = \frac{3\mu}{a^5}\int_V (\vec{r}_P \cdot \vec{x}_B)([\vec{x}_B] \times \vec{r}_P)\rho dV \tag{67}$$

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- Use of principal axes, all cross-products of inertia's zero:

$$\vec{M}_{g,B} \approx \frac{3\mu}{a^5}\begin{bmatrix} r_{P,y}r_{P,z}\left(\int y_B^2\rho dV - \int z_B^2\rho dV\right) \\ r_{P,x}r_{P,z}\left(\int z_B^2\rho dV - \int x_B^2\rho dV\right) \\ r_{P,x}r_{P,y}\left(\int x_B^2\rho dV - \int y_B^2\rho dV\right) \end{bmatrix} \tag{68}$$

Gravity-Gradient Approximation (continued)

- Substituting for moment of inertia integrals and components of $\vec{r}_P$:

$$\vec{M}_{g,B} \approx \frac{3\mu}{a^5} \begin{bmatrix} (-a\sin\phi\cos\theta)(-a\cos\phi\cos\theta)(I_{zz} - I_{yy}) \\ (a\sin\theta)(-a\cos\phi\cos\theta)(I_{xx} - I_{zz}) \\ (a\sin\theta)(-a\sin\phi\cos\theta)(I_{yy} - I_{xx}) \end{bmatrix} \tag{69}$$

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- Simplifies to:

$$\vec{M}_{g,B} \approx \frac{3\mu}{2a^3} \begin{bmatrix} (I_{zz} - I_{yy})\sin(2\phi)\cos^2(\theta) \\ (I_{zz} - I_{xx})\sin(2\theta)\cos\phi \\ (I_{xx} - I_{yy})\sin(2\theta)\sin\phi \end{bmatrix} \tag{70}$$

Gravity-Gradient Linearization

- Using small-angle approximation:

$$\vec{M}_{g,B} = \frac{3\omega_O^2}{2} \begin{bmatrix} (I_{zz} - I_{yy})2\phi \\ (I_{zz} - I_{xx})2\theta \\ (I_{xx} - I_{yy})(2\theta)\phi \end{bmatrix} \tag{71}$$

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- Discarding higher-order terms, **linearized gravity-gradient moment** vector:

$$\vec{M}_{g,B} = \begin{bmatrix}3\omega_O^2(I_{zz} - I_{yy})\phi \\ \omega_O^2(I_{zz} - I_{xx})\theta \\ 0\end{bmatrix} \tag{72}$$

Linearized Gravity-Gravity Dynamics

- By substitution, **linearized gravity-gradient attitude dynamics**:

$$\begin{bmatrix} M_{d,x} + M_{c,x} \\ M_{d,y} + M_{c,y} \\ M_{d,y} + M_{c,z} \end{bmatrix} + \begin{bmatrix} 3\omega_O^2(I_{zz} - I_{yy})\phi \\ \omega_O^2(I_{zz} - I_{xx})\theta \\ 0 \end{bmatrix} = \begin{bmatrix} I_{xx}\ddot{\phi} - \omega_O(I_{xx} + I_{zz} - I_{yy})\dot{\psi} + \omega_O^2(I_{zz} - I_{yy})\phi \\ I_{yy}\ddot{\theta} \\ I_{zz}\ddot{\psi} + \omega_O(I_{xx} + I_{zz} - I_{yy})\dot{\phi} + \omega_O^2(I_{yy} - I_{zz})\psi \end{bmatrix} \tag{73}$$

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$$\begin{bmatrix} M_{d,x}+M_{c,x} \\ M_{d,y}+M_{c,y} \\ M_{d,y}+M_{c,z} \end{bmatrix} + \begin{bmatrix} 3\omega_O^2(I_{zz}-I_{yy})\phi \\ \omega_O^2(I_{zz}-I_{xx})\theta \\ 0 \end{bmatrix} = \begin{bmatrix} I_{xx}\ddot{\phi}-\omega_O(I_{xx}+I_{zz}-I_{yy})\dot{\psi}+\omega_O^2(I_{zz}-I_{yy})\phi \\ I_{yy}\ddot{\theta} \\ I_{zz}\ddot{\psi}+\omega_O(I_{xx}+I_{zz}-I_{yy})\dot{\phi}+\omega_O^2(I_{yy}-I_{zz})\psi \end{bmatrix} \tag{73}$$

$$\begin{bmatrix} M_{d,x}+M_{c,x} \\ M_{d,y}+M_{c,y} \\ M_{d,y}+M_{c,z} \end{bmatrix} = \begin{bmatrix} I_{xx}\ddot{\phi}+4\omega_O^2(I_{yy}-I_{zz})\phi-\omega_O(I_{xx}+I_{zz}-I_{yy})\dot{\psi} \\ I_{yy}\ddot{\theta}+3\omega_O^2(I_{xx}-I_{zz})\theta \\ I_{zz}\ddot{\psi}+\omega_O^2(I_{yy}-I_{xx})\psi+\omega_O(I_{xx}+I_{zz}-I_{yy})\dot{\phi} \end{bmatrix} \tag{74}$$

Stability Analysis: Gravity-Gradient

- Define moments of inertia ratios:

$$\sigma_{xx} = \frac{I_{yy} - I_{zz}}{I_{xx}} = \frac{\int (x^2 + z^2) dm - \int (x^2 + y^2) dm}{\int (y^2 + z^2) dm} = \frac{\int (z^2 - y^2) dm}{\int (z^2 + y^2) dm} \tag{75}$$

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- Note:

$$|\sigma_{xx}| < 1 \;\; \& \;\; |\sigma_{yy}| < 1 \;\; \& \;\; |\sigma_{zz}| < 1 \tag{78}$$

Linearized Pitch State-Space Dynamics

- Ignoring disturbances, linearized pitch state-space dynamics:

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3\omega_O^2 \sigma_{yy} & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ I_{yy}^{-1} \end{bmatrix} M_{c,y} \tag{79}$$

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  - Either purely imaginary or one positive real and one negative real
- Gravity-gradient pitch dynamics: marginally stable if and only if $\sigma_{yy} > 0$ or $I_{xx} > I_{zz}$, otherwise unstable

Alternative Pitch Stability Criterion

- Recalling $I_{yy} < I_{xx} + I_{zz}$ must also hold and multiplying by $I_{xx} - I_{zz} > 0$ for pitch dynamics stability:

$$I_{yy} I_{xx} - I_{yy} I_{zz} < I_{xx}^2 - I_{zz}^2 \tag{82}$$

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- By definition, alternative criterion for pitch dynamics stability

$$\sigma_{zz} < \sigma_{xx} \tag{86}$$

Linearized Roll-Yaw State-Space Dynamics

- Ignoring disturbances, linearized roll-yaw state-space dynamics:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\psi} \\ \ddot{\phi} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4\omega_O^2\sigma_{xx} & 0 & 0 & \omega_O(1-\sigma_{xx}) \\ 0 & -\omega_O^2\sigma_{zz} & -\omega_O(1-\sigma_{zz}) & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \psi \\ \dot{\phi} \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I_{xx}^{-1} & 0 \\ 0 & I_{zz}^{-1} \end{bmatrix} \begin{bmatrix} M_{c,x} \\ M_{c,z} \end{bmatrix} \tag{87}$$

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- Characteristic equation

$$\lambda^4 + (3\sigma_{xx} + \sigma_{xx}\sigma_{zz} + 1)\omega_O^2\lambda^2 + 4\sigma_{xx}\sigma_{zz}\omega_O^4 = 0 \tag{88}$$

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- Four roots:

$$\lambda_{1,2,3,4} = \pm\omega_O\sqrt{\frac{1}{2}\left(-(3\sigma_{xx} + \sigma_{xx}\sigma_{zz} + 1) \pm \sqrt{(3\sigma_{xx} + \sigma_{xx}\sigma_{zz} + 1)^2 - 16\sigma_{xx}\sigma_{zz}}\right)} \tag{89}$$

  - $\lambda_1 = -\lambda_2$ and $\lambda_3 = -\lambda_4$

Roll-Yaw Stability Criteria

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as purely real negative number for both terms

- Requires following three criteria

$$\begin{aligned} -(3\sigma_{xx}+\sigma_{xx}\sigma_{zz}+1) &< 0 \\ (3\sigma_{xx}+\sigma_{xx}\sigma_{zz}+1)^2-16\sigma_{xx}\sigma_{zz} &> 0 \\ -(3\sigma_{xx}+\sigma_{xx}\sigma_{zz}+1) &< \sqrt{(3\sigma_{xx}+\sigma_{xx}\sigma_{zz}+1)^2-16\sigma_{xx}\sigma_{zz}} \end{aligned} \tag{91}$$

Roll-Yaw Stability Criteria (continued)

$$\begin{aligned}
3\sigma_{xx} + \sigma_{xx}\sigma_{zz} + 1 &> 0 \\
(3\sigma_{xx} + \sigma_{xx}\sigma_{zz} + 1)^2 &> 16\sigma_{xx}\sigma_{zz} \\
(3\sigma_{xx} + \sigma_{xx}\sigma_{zz} + 1)^2 &> (3\sigma_{xx} + \sigma_{xx}\sigma_{zz} + 1)^2 - 16\sigma_{xx}\sigma_{zz}
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$$

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\begin{aligned}
3\sigma_{xx} + \sigma_{xx}\sigma_{zz} + 1 &> 0 \\
3\sigma_{xx} + \sigma_{xx}\sigma_{zz} + 1 &> 4\sqrt{\sigma_{xx}\sigma_{zz}} \\
(3\sigma_{xx} + \sigma_{xx}\sigma_{zz} + 1)^2 &> (3\sigma_{xx} + \sigma_{xx}\sigma_{zz} + 1)^2 - 16\sigma_{xx}\sigma_{zz}
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Roll-Yaw Stability Criteria (continued)

$$\begin{aligned} 3\sigma_{xx} + \sigma_{xx}\sigma_{zz} + 1 &> 0 \\ 3\sigma_{xx} + \sigma_{xx}\sigma_{zz} + 1 &> 4\sqrt{\sigma_{xx}\sigma_{zz}} \\ \sigma_{xx}\sigma_{zz} &> 0 \end{aligned} \tag{94}$$

- By inspection, third criterion demonstrates satisfying second criterion naturally satisfies first criterion

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- By inspection, third criterion demonstrates satisfying second criterion naturally satisfies first criterion
- Squaring second criterion on both sides:

$$
9\sigma_{xx}^2 + \sigma_{xx}^2\sigma_{zz}^2 + 1 + 6\sigma_{xx}^2\sigma_{zz} + 2\sigma_{xx}\sigma_{zz} + 6\sigma_{xx} > 16\sigma_{xx}\sigma_{zz} \tag{95}
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- Gravity-gradient roll-yaw dynamics marginally stability if and only if:

$$
\begin{aligned}
(\sigma_{zz}^2 + 6\sigma_{zz} + 9)\sigma_{xx}^2 + (-14\sigma_{zz} + 6)\sigma_{xx} + 1 &> 0 \\
\sigma_{xx}\sigma_{zz} &> 0
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- First:

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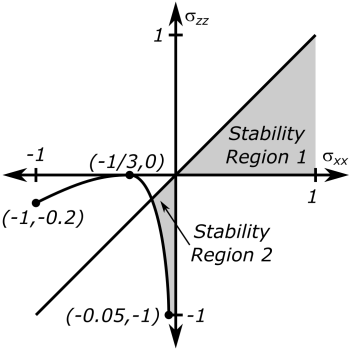
- Second:

$$\sigma_{xx} < 0, \quad \sigma_{zz} < 0, \quad \sigma_{zz} < \sigma_{xx}, \quad (\sigma_{zz}^2 + 6\sigma_{zz} + 9)\sigma_{xx}^2 + (-14\sigma_{zz} + 6)\sigma_{xx} + 1 > 0 \tag{98}$$

Gravity-Gradient Stability Regions Plot



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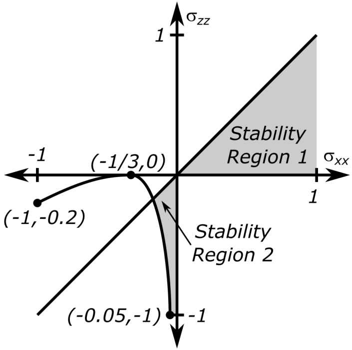

- Stability region 2: very small region and seldom used owing to practical structural difficulties
- Note: linearized dynamics only marginally stable, passive and/or active damping typically necessary for gravity-gradient-stabilized satellites

Moments of Inertia Stability Criteria

- By definition of moment of inertia ratios, stability region 1 equivalently requires

$$I_{yy} - I_{zz} > 0, \quad I_{yy} - I_{xx} > 0, \quad I_{xx} > I_{zz} \tag{99}$$

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- Gravity-gradient (marginal) stability criteria/regions
  1. Common: $I_{xx} + I_{zz} > I_{yy} > I_{xx} > I_{zz}$
  2. Very Rare: $I_{yy} + I_{zz} > I_{xx} > I_{zz} > I_{yy}, \quad I_{yy} - I_{zz} \ll I_{xx}$
  - Gravity-gradient-stabilized satellites: damping required