Lecture 20: Introduction to Optimal Control

with Planar Intercept Guidance Example
Textbook Sections 4.1 & 12.6

Dr. Jordan D. Larson

Optimal Control Theory

- **Decision theory**: determination of optimal decision given constraints and assumptions on decision problem

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  - Chosen objective involves actions of other, separate decision maker(s): extension of optimal control theory called **differential game theory**
- Dynamic optimization, i.e. optimization over time: critical to optimal control theory
  - Formulated for continuous-time and discrete-time dynamical systems
  - Course: continuous-time dynamical system optimal control
  - Discrete-time dynamical systems optimal control addressed in additional sections of textbook

Optimal Control Solution Methods

- To solve for optimal control policy: methods from **mathematical optimization**, a.k.a. **mathematical programming**
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- Sets of possible elements: continuous, discrete, and/or constrained
- Objective in optimal control can be imposed superficially by control designer or dictated by real dynamic process
- Optimization procedure provides general **optimal control problem (OCP)**: typically stated as minimization

Optimal Control Problem for Continuous-Time

$$
\begin{aligned}
\vec{u}^{opt}(t) = \underset{u(t)\ \forall t\in[t_0,\ t_f]}{\text{argmin}} \quad & \mathcal{J}\left(\vec{x}, \vec{u}, t, t_0, t_f\right) \\
\underline{\text{subject to:}} \quad & \\
\text{continuous-time dynamics:} \quad & \dot{\vec{x}}(t) = f(\vec{x}(t), \vec{u}(t), t) \\
\text{boundary conditions:} \quad & e\left(\vec{x}(t_0), t_0, \vec{x}(t_f), t_f\right) = 0 \\
\text{path constraints:} \quad & c\left(\vec{x}(t), \vec{u}(t), t\right) \leq 0
\end{aligned}
\tag{1}
$$

- $\vec{u}^{\text{opt}}(t)$: **optimal control function**
- argmin stands for “argument which minimizes” following expression
- t_0: start time
- t_f: final time
- $t_f - t_0$: **time horizon**
- $\mathcal{J}\left(\vec{x}, \vec{u}, t\right)$: **cost functional**, a.k.a. **objective functional**, a.k.a. **performance index**
  - Generally depends on state, input, & time
  - Term **functional**: mathematical definition for “function of function”

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   - May be multiple solutions to OCP

- Often not necessarily interested in finding truly optimal solution to OCP, but efficient near-optimal solution for complex OCPs

Types of Optimal Control Problems

1. Optimize for finite- or infinite-time horizons typically shortened to finite- or infinite-horizon OCPs, i.e. $t_f \neq \infty$ or $t_f = \infty$
   - Often infinite-horizon OCPs easier to solve finite-horizon OCPs: optimal control policy not depend on specific time, but only state, becoming *fixed-gain* control policy
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   - Effects of constraints on solving method typically revolves around whether state constraints, input constraints, or both
4. Cost functional: linear, quadratic, convex, or non-convex, each requiring more complex solvers
   - For finite minimum cost to exist, simplest model: linear cost for constrained OCPs and quadratic cost for unconstrained OCPs
   - Distinction between convex/non-convex costs: convexity implies local minimum $=$ global minimum, simplifies many efficient numerical search methods

Classical Optimal Control Problem

- Consider classical optimal control problem:

$$\begin{aligned} \vec{u}^{opt}(t) = \underset{u(t)\ \forall t \in [t_0,\ t_f]}{\operatorname{argmin}} \quad & \mathcal{E}\left(\vec{x}(t_f), t_f\right) + \int_{t_0}^{t_f} \mathcal{L}(\vec{x}(t), \vec{u}(t), t)dt \\ \underline{\text{subject to}}: \quad & \\ \text{continuous-time dynamics:} \quad & \dot{\vec{x}}(t) = f(\vec{x}(t), \vec{u}(t), t) \\ \text{initial conditions:} \quad & \vec{x}(t_0) - \vec{x}_0 = 0 \end{aligned} \tag{2}$$

  - t_f and t_0 fixed
  - $\mathcal{E}$: **Mayer term**, a.k.a. **endpoint cost** or **terminal cost**
  - $\mathcal{L}$: **Lagrangian term**, a.k.a. **running cost** or **instant cost**, which must be nonnegative

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- To solve classical OCP: use **generalized calculus of variations**

Generalized Calculus of Variations

- Augment Lagrangian with **costate vector**, a.k.a. **adjoint vector**, $\vec{\lambda}(t)$
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$$\bar{\mathcal{J}} = \mathcal{E}\left(\vec{x}(t_f), t_f\right) + \int_{t_0}^{t_f} \left(\mathcal{L}(\vec{x}(t), \vec{u}(t), t) + \vec{\lambda}^T(f(\vec{x}(t), \vec{u}(t), t) - \dot{\vec{x}}(t))dt\right) \tag{3}$$

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- Note: $\vec{\lambda}(t)$ can be *any* vector: state dynamics require that

$$f(\vec{x}(t), \vec{u}(t), t) - \dot{\vec{x}}(t) = 0 \tag{4}$$

  holds for all time
  - $\vec{\lambda}(t)$ being multiplied by zero

Generalized Calculus of Variations (continued)

- Define **Hamiltonian**:

$$\mathcal{H}(\vec{x}(t), \vec{u}(t), \vec{\lambda}(t), t) = \mathcal{L}(\vec{x}(t), \vec{u}(t), t) + \vec{\lambda}^T f(\vec{x}(t), \vec{u}(t), t) \tag{5}$$

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- Variation of $\bar{\mathcal{J}}$

$$\delta\bar{\mathcal{J}} = \frac{\partial \mathcal{E}(t_f)}{\partial \vec{x}} \partial \vec{x}(t_f) + \int_{t_0}^{t_f} \frac{\partial \mathcal{H}}{\partial \vec{x}} \partial \vec{x} + \frac{\partial \mathcal{H}}{\partial \vec{u}} \partial \vec{u} - \vec{\lambda}^T \partial \dot{x} dt \tag{7}$$

Generalized Calculus of Variations (continued)

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- Expanding last term using integration by parts:

$$-\int_{t_0}^{t_f} \vec{\lambda}^T \partial \dot{x} dt = -\vec{\lambda}^T(t_f)\partial x(t_f) + \vec{\lambda}^T(t_0)\partial \vec{x}(t_0) + \int_{t_0}^{t_f} \dot{\vec{\lambda}}^T \partial \vec{x} dt \quad (8)$$

Generalized Calculus of Variations (continued)

- Substituting and rearranging, separate $\delta\bar{\mathcal{J}}$ into four different components:

$$\delta\bar{\mathcal{J}} = \left(\frac{\partial\mathcal{E}(t_f)}{\partial\vec{x}} - \vec{\lambda}^T(t_f)\right)\partial\vec{x}(t_f) + \vec{\lambda}^T(t_0)\partial\vec{x}(0) + \int_0^{t_f}\left(\frac{\partial\mathcal{H}}{\partial\vec{x}} + \dot{\vec{\lambda}}^T\right)\partial\vec{x} + \frac{\partial\mathcal{H}}{\partial\vec{u}}\partial\vec{u}\,dt \quad (9)$$

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- Adjoint vectors arbitrary, select to make coefficients of $\partial\vec{x}(t)$ and $\partial\vec{x}(t_f)$ equal 0:

$$\begin{aligned}\dot{\vec{\lambda}}^T &= -\frac{\partial\mathcal{H}}{\partial\vec{x}} \\ \vec{\lambda}(t_f) &= \frac{\partial\mathcal{E}(t_f)}{\partial\vec{x}}\end{aligned} \quad (10)$$

  - First equation: **costate equation** a.k.a. **adjoint equation** for dynamics of $\vec{\lambda}$
  - Second equation: final condition for $\vec{\lambda}$

Control Variation Requirement

- With choice:

$$\delta\bar{\mathcal{J}} = \int_0^{t_f} \frac{\partial \mathcal{H}}{\partial \vec{u}} \partial \vec{u}\, dt \tag{11}$$

  - For $\mathcal{J}$ to be minimized requires $\delta\bar{\mathcal{J}} = 0$

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- If $\vec{u}$ constrained in set $\mathcal{U}$: use **Pontryagin's principle**: replaces previous equation with requirement

$$\mathcal{H}(\vec{x}^{opt}(t), \vec{u}^{opt}(t), \vec{\lambda}^{opt}(t), t) \leq \mathcal{H}(\vec{x}(t), \vec{u}(t), \vec{\lambda}(t), t) \quad \forall\, t \in [t_0, t_F], \quad \vec{u} \in \mathcal{U} \tag{13}$$

Pontryagin's Principle Example

- If bound control simply by

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- For feasibility of variation of $\vec{u}$ at limits: must only be in direction allowed
- Requires:

$$\begin{cases} u(t) = \vec{u}_{min} & \text{for} \frac{\partial \mathcal{H}}{\partial \vec{u}} \geq 0 \\ \vec{u}_{min} < u(t) < \vec{u}_{max} & \text{for} \frac{\partial \mathcal{H}}{\partial \vec{u}} = 0 \\ u(t) = \vec{u}_{max} & \text{for} \frac{\partial \mathcal{H}}{\partial \vec{u}} \leq 0 \end{cases} \tag{15}$$

Hamilton-Jacobi-Bellman Equation

- Pontryagin's principle: necessary condition, but only sufficient if $\mathcal{L}(\vec{x}(t), \vec{u}(t), t)$ and $f(\vec{x}(t), \vec{u}(t), t)$ both convex in $\vec{x}(t)$ and $\vec{u}(t)$

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- Necessary and sufficient condition as alternative to Pontryagin's principle: **Hamilton-Jacobi-Bellman (HJB) equation**

$$\begin{aligned}\frac{\partial \mathcal{V}^{opt}(\vec{x}, t)}{\partial t} &= -\frac{\partial \mathcal{H}}{\partial \vec{x}} \\ \mathcal{V}^{opt}(\vec{x}(t_f), t_f) &= \mathcal{E}\left(\vec{x}(t_f), t_f\right)\end{aligned} \tag{16}$$

  - $\mathcal{V}(\vec{x}, t)$: continuous-time **cost-to-go**, a.k.a. **value function**

Hamilton-Jacobi-Bellman Equation

- **Principle of optimality**: regardless of initial state and initial decision, remaining decisions must constitute optimal policy with regard to state resulting from first decision, i.e. for any $t > t_0$

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- Discrete-time version: Bellman equation
  - Basis of dynamic programming: additional sections of textbook

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- Cost functional contains **cost/weight matrices**
-

$$J = \frac{1}{2}x^T(t_f)Ex(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\left(x^T(t)Q(t)x(t) + u^T(t)R(t)u(t) + 2x^T(t)S(t)u(t)dt\right) \tag{19}$$

  - E: **endpoint cost/weight matrix** or **terminal cost/weight matrix**
  - Q: **state cost/weight matrix**
  - R: **input cost/weight matrix**
  - S: **cross-cost/weight matrix**

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- Controller which solves LQ OCP: **linear-quadratic regulator (LQR)**, i.e. $\vec{u}^{\text{opt}}(t)$
  - Term **regulator** denotes controller steers system state to 0
  - Recall robust servomechanism control: with servomechanism to augment dynamics $\rightarrow$ **linear-quadratic tracker (LQT)**

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- Note: A/F, B/G, Q, R, S can all vary with t

Unconstrained Finite-Horizon Continuous-Time LQ OCP

- **Unconstrained finite-horizon continuous-time LQ OCP**:

$$\begin{aligned} \vec{u}^{\text{opt}}(t) = \underset{u(t)\ \forall t \in [0,\ t_f]}{\text{argmin}} \quad & J = \vec{x}^T(t_f) E \vec{x}(t_f) + \int_0^{t_f} \vec{x}^T(t) Q \vec{x}(t) + \vec{u}^T(t) R \vec{u}(t) + 2\vec{x}^T(t) S \vec{u}(t) dt \\ \text{subject to:} \quad & \dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t) \\ \text{initial condition:} \quad & \vec{x}(0) \end{aligned} \tag{20}$$

  - Unconstrained finite-horizon continuous-time LQR: optimal control function, $\vec{u}^{\text{opt}}(t)$, which minimizes quadratic cost functional, J

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- Generalized calculus of variations: assign Hamiltonian

$$\mathcal{H}(\vec{x}(t), \vec{u}(t), \vec{\lambda}(t), t) = \vec{x}^T(t) Q \vec{x}(t) + \vec{u}^T(t) R \vec{u}(t) + 2\vec{x}^T(t) S \vec{u}(t) + \vec{\lambda}^T \left(A\vec{x}(t) + B\vec{u}(t) \right) \tag{21}$$

Unconstrained F-H C-T LQR (continued)

- LQ OCP dropping explicit t:

$$\begin{cases} \dot{\vec{\lambda}} & = -Q\vec{x} - S\vec{u} - A^T\vec{\lambda} \\ 0 & = \vec{u}^T R + \vec{x}^T S + \vec{\lambda}^T B \\ \vec{\lambda}(t_f) & = \vec{x}^T(t_F)E \end{cases} \tag{22}$$

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$$\begin{cases} \dot{\vec{\lambda}} & = -Q\vec{x} - S\vec{u} - A^T\vec{\lambda} \\ 0 & = \vec{u}^T R + \vec{x}^T S + \vec{\lambda}^T B \\ \vec{\lambda}(t_f) & = \vec{x}^T(t_F)E \end{cases} \tag{22}$$

- To solve this OCP, assume costate has linear form:

$$\vec{\lambda}(t) = P(t)\vec{x}(t) \tag{23}$$

  - $P(t)$: symmetric matrix

Unconstrained F-H C-T LQR (continued)

- Substituting into costate equations and including state dynamics equation:

$$\begin{cases} \dot{\vec{x}} & = A\vec{x} + B\vec{u} \\ \frac{d}{dt}(P\vec{x}) = \dot{P}\vec{x} + P\dot{\vec{x}} & = -Q\vec{x} - S\vec{u} - A^T P\vec{x} \\ 0 & = \vec{u}^T R + \vec{x}^T S + \vec{x}^T PB \\ P(t_f)\vec{x}(t_f) & = E\vec{x}(t_f) \end{cases} \tag{24}$$

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- Substituting first equation into second equation:

$$\dot{P}\vec{x} + PA\vec{x} + PB\vec{u} = -Q\vec{x} - S\vec{u} - A^T P\vec{x} \tag{25}$$

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- Rewriting third equation:

$$\vec{u} = -R^{-1}(B^T P + S^T)\vec{x} \tag{26}$$

Unconstrained F-H C-T LQR (continued)

- By substitution for $\vec{u}$ into newly derived equation:

$$\dot{P}\vec{x} + PA\vec{x} - PBR^{-1}(B^TP + S^T)\vec{x} = -Q\vec{x} + SR^{-1}(B^TP + S^T)\vec{x} - A^TP\vec{x} \tag{27}$$

Unconstrained F-H C-T LQR (continued)

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- Removing common $\vec{x}$ term and rearranging: **Riccati differential equation**

$$\dot{P} = -PA - A^T P + (PB + S)\,R^{-1}(B^T P + S^T) - Q \tag{28}$$

  - Describes dynamics of $P(t)$
  - Solved using boundary condition on costate:

$$P(t_f) = E \tag{29}$$

  - Lagrangian multiplier problem: matrix-valued ODE solved in reverse time from end condition

Unconstrained F-H C-T LQR (continued)

- **Unconstrained finite-horizon continuous-time LQR**:

$$\vec{u}^{\text{opt}}(t) = -K(t)\vec{x}(t) \tag{30}$$

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- Closed-loop state-space dynamics:

$$\dot{\vec{x}}(t) = (A - BK(t))\,\vec{x}(t) \tag{32}$$

Alternative Infinite-Horizon OCP for $F_L(G, K)$

- Generalized feedback control system setting: design stabilizing LTI controller $K(s)$ to minimize input-to-output gain from $\vec{d} \to \vec{e}$ by setting cost functional $J(\vec{u})$ for LTI systems as some system norm $\|F_L(G, K)\|$, i.e.

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$$\begin{aligned} \vec{u}^{opt}(s) = K^{opt}(s)\vec{y}(s) = \underset{u(s)}{\text{argmin}} \quad & \|F_L(G, K)\| \\ \underline{\text{subject to:}} \quad & \\ \text{continuous dynamics:} \quad & \dot{\vec{x}}(t) = f(\vec{x}(t), \vec{u}(t), t) \\ \text{constraints:} \quad & K \text{ stabilizing} \end{aligned} \tag{33}$$

Alternative Infinite-Horizon OCP for $F_L(G, K)$ (continued)

- $F_L(G, K)$ defines state-space:

$$\begin{aligned}
\dot{\vec{x}}(t) &= A\vec{x}(t) + B_1\vec{d}(t) + B_2\vec{u}(t) \\
\vec{e}(t) &= C_1\vec{x}(t) + D_{12}\vec{u}(t) \\
\vec{y}(t) &= C_2\vec{x}(t) + D_{21}\vec{d}(t)
\end{aligned} \tag{34}$$

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- Rewritten:

$$K^{opt}(s) = \underset{K \text{ stabilizing}}{\text{argmin}} \; \|F_L(G, K)\| \tag{35}$$

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- Module: $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms for this type of OCP

Introduction

- Flight vehicles operate in 3D, many, if not most, guidance laws devise and implement planar guidance laws in each of two maneuver planes
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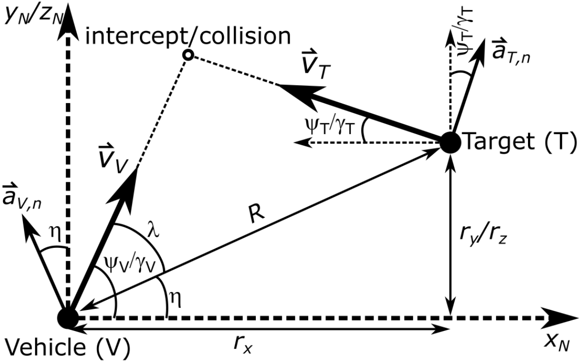
  - E.g. longitudinal and lateral-directional
- Includes **intercept guidance problem**, a.k.a. **pursuit-evasion guidance problem**



Introduction (continued)

- ψ_V/γ_V: heading/flight-path angle of vehicle
- ψ_T/γ_T: heading/flight-path angle of target
- $\vec{v}_V$: velocity of vehicle
- $\vec{v}_T$: velocity of target
- r_x: x_N relative position
- r_y/r_z: relative y_N/z_N position
- R: range-to-target
- η: **line-of-sight (LOS) angle**
- $\vec{a}_{V,n}$: acceleration of vehicle normal to LOS
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- λ: **lead angle** between LOS vector and vehicle velocity vector
  - $\lambda > 0$ and $\dot{R} < 0$: intercept exists

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  - If vehicle actively reduces and holds r_y/r_z to zero, then r_x continues to decrease until collision occurs

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  - Determining lead angle key purpose of guidance law
  - If vehicle actively reduces and holds r_y/r_z to zero, then r_x continues to decrease until collision occurs
  - Regulating r_y/r_z: key analysis required for planar intercept

Intercept Kinematics

- In Cartesian form, intercept kinematics as target-vehicle relative position, $\vec{r}$:

$$\vec{r} = \vec{r}_T - \vec{r}_V = \begin{bmatrix} r_x \\ r_y \end{bmatrix} = \begin{bmatrix} R\cos\eta \\ R\sin\eta \end{bmatrix} \tag{36}$$

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- Relative velocity:

$$\vec{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} -\|\vec{v}_T\|_2 \cos\gamma_T - \|\vec{v}_V\|_2 \cos(\lambda+\eta) \\ \|\vec{v}_T\|_2 \sin\gamma_T - \|\vec{v}_V\|_2 \sin(\lambda+\eta) \end{bmatrix} \tag{37}$$

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- Relative acceleration:

$$\vec{a} = \begin{bmatrix} a_x \\ a_y \end{bmatrix} = \begin{bmatrix} \|\vec{a}_{T,n}\|_2\sin\gamma_T + \|\vec{a}_{V,n}\|_2\sin\eta \\ \|\vec{a}_{T,n}\|_2\cos\gamma_T - \|\vec{a}_{V,n}\|_2\cos\eta \end{bmatrix} \tag{38}$$

  - Clearly nonlinear equations

Linearized Intercept Kinematics

- Small angle approximations for γ_T and η, i.e. near-collision course conditions:

$$r_y \approx R\eta \tag{39}$$

$$a_y \approx \|\vec{a}_{T,n}\|_2 - \|\vec{a}_{V,n}\|_2 \tag{40}$$

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- Kinematics:

$$\dot{v}_y = \|\vec{a}_{T,n}\|_2 - \|\vec{a}_{V,n}\|_2 \tag{41}$$

$$\dot{r}_y = \ddot{v}_y = \|\vec{a}_{T,n}\|_2 - \|\vec{a}_{V,n}\|_2 \tag{42}$$

LTI State-Space Model

- Augmented LTI state-space model:

$$\begin{aligned} \begin{bmatrix} \dot{r}_y \\ \dot{v}_y \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r_y \\ v_y \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \|\vec{a}_{V,n}\|_2 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \|\vec{a}_{T,n}\|_2 \\ \vec{y} &= \begin{bmatrix} r_y \\ v_y \end{bmatrix} \end{aligned} \tag{43}$$

  - Form different linear-quadratic guidance laws depending on additional assumptions

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  - Form different linear-quadratic guidance laws depending on additional assumptions
- Note: if r_y and v_y not both available for feedback, employ separation principle and use parallel guidance filter to estimate r_y and v_y

LQ Minimum-Energy OCP 1

- Assume:
  - Vehicle and target speeds, $\|\vec{v}_V\|_2$ and $\|\vec{v}_T\|_2$, constant
  - Target non-manuevering, i.e. $\|\vec{a}_{T,n}\|_2 = 0$
  - Vehicle responds instantaneously to acceleration command, $a_{V,c} = \|\vec{a}_{V,n}\|_2$

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\begin{aligned}
\vec{u}^{\text{opt}}(t) = \underset{u(t)\ \forall t \in [0,\ t_f]}{\text{argmin}} \quad & \mathcal{J} = \vec{x}^T(t_f) \begin{bmatrix} E_r & 0 \\ 0 & E_v \end{bmatrix} \vec{x}(t_f) + \int_0^{t_f} \vec{u}^T(t)\vec{u}(t)dt \\
\text{subject to:} \quad & \dot{\vec{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \vec{u}(t) \\
\text{initial condition:} \quad & \vec{x}(0)
\end{aligned}
\tag{44}
$$

- $\vec{x} = [r_y \ v_y]^T$
- $u = \|\vec{a}_{V,n}\|_2$
- E_r: cost on relative range at time t_f
- E_v: cost on relative velocity at time t_f
  - $E_r \to \infty$ and $E_v = 0$: **intercept problem**
  - $E_r \to \infty$ and $E_v \to \infty$: **rendezvous problem**

OCP 1 Solution

- Differential Riccati equation:

$$\dot{P} = -PA - A^T P + PBB^T P \tag{45}$$

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$$u^{opt}(t) = -B^T P(t)\vec{x} \tag{47}$$

$$u^{opt}(t) = \frac{3}{t_{go}^2}\left[\frac{\left(1+\frac{1}{2}E_v t_{go}\right) r_y(t) + \left(1+\frac{1}{3}E_v t_{go} + \frac{E_v}{E_r t_{go}^2}\right) v_y(t) t_{go}}{1 + \frac{3}{E_r t_{go}^3}(1 + E_v t_{go}) + \frac{E_v t_{go}}{4}}\right] \tag{48}$$

- $t_{go} = t_f - t$

OCP 1 Solution (continued)

- Expressible as:

$$u^{opt}(t) = \frac{\tilde{N}(E_r, E_v, t_{go})}{t_{go}^2} Z(r_y, v_y, E_r, E_v, t_{go}) \tag{49}$$

- **Effective navigation ratio**

$$\tilde{N}(E_r, E_v, t_{go}) = \frac{3}{1 + \frac{3}{E_r t_{go}^3}(1 + E_v t_{go}) + \frac{E_v t_{go}}{4}} \tag{50}$$

$$Z(r_y, v_y, E_r, E_v, t_{go}) = \left(1 + \frac{1}{2} E_v t_{go}\right) r_y(t) + \left(1 + \frac{1}{3} E_v t_{go} + \frac{E_v}{E_r t_{go}^2}\right) v_y(t) t_{go} \tag{51}$$

Intercept and Rendezvous Solutions

- **Proportional guidance (PN)** law occurs for intercept problem, i.e. if $E_v = 0$ and as $E_r \to \infty$:

$$u_{PN}(t) = \frac{3}{t_{go}^2}\left[r_y(t) + v_y(t)t_{go}\right] \tag{52}$$

  - $\tilde{N}_{PN} = 3$
  - $Z = ZEM_{PN} = r_y(t) + v_y(t)t_{go}$: referred to as **zero-effort-miss**, i.e. current miss distance that would result if vehicle and target did not maneuver over time period $[t, t_f]$
  - In practice, ZEM estimated and fed to controller by guidance filter

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  - In practice, ZEM estimated and fed to controller by guidance filter
- **Rendezvous (REN) guidance** law occurs for rendezvous problem, i.e. as $E_v \to \infty, E_r \to \infty$:

$$u_{REN}(t) = \frac{6}{t_{go}^2}\left[r_y(t) + \frac{2}{3}v_y(t)t_{go}\right] \tag{53}$$

Proportional Navigation Aside

- Recall:

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- $R = -\dot{R}t_{go}$

Proportional Navigation Aside

- Recall:

$$r_y \approx R\eta \tag{54}$$

$$v_y = \dot{R}\eta + R\dot{\eta} \tag{55}$$

- $R = -\dot{R}t_{go}$
- Substituting for η and R:

$$v_y = \dot{R}\frac{r_y}{-\dot{R}t_{go}} - \dot{R}t_{go}\dot{\eta} \tag{56}$$

Proportional Guidance Aside (continued)

- Rearranging:

$$\dot{R}\dot{\eta} = \frac{r_y(t) + v_y(t)t_{go}}{t_{go}^2} \tag{57}$$

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$$u_{PN}(t) = -\tilde{N}\dot{R}\dot{\eta} \tag{58}$$

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$$u_{PN}(t) = -\tilde{N}\dot{R}\dot{\eta} \tag{58}$$

- $\tilde{N} = 3$: “energy-optimal” PN gain as shown

LQ Minimum-Energy OCP 2

- Assume:
  - Vehicle and target speeds, $\|\vec{v}_V\|_2$ and $\|\vec{v}_T\|_2$, constant
  - Target undergoing constant acceleration, i.e. $\|\vec{a}_{T,n}\|_2 = a_{T,n,y}$
    - Added to state and estimated
  - Vehicle responds instantaneously to acceleration command, $a_{V,c} = \|\vec{a}_{V,n}\|_2$

LQ Minimum-Energy OCP 2

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$$\vec{u}^{\text{opt}}(t) = \underset{u(t)\ \forall t \in [0,\ t_f]}{\text{argmin}} \ \mathcal{J} = \vec{x}^T(t_f) \begin{bmatrix} E_r & 0 & 0 \\ 0 & E_v & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{x}(t_f) + \int_0^{t_f} \vec{u}^T(t)\vec{u}(t)dt$$

$$\text{subject to: } \dot{\vec{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \vec{u}(t) \quad (59)$$

$$\text{initial condition: } \vec{x}(0)$$

- $\vec{x} = [r_y\ v_y\ a_{T,n,y}]^T$
- $u = \|\vec{a}_{V,n}\|_2$

OCP 2 Solution

- General solution

$$u^{opt}(t) = \frac{3}{t_{go}^2}\left[\frac{\left(1+\frac{1}{2}E_v t_{go}\right) r_y(t) + \left(1+\frac{1}{3}E_v t_{go} + \frac{E_v}{E_r t_{go}^2}\right) v_y(t) t_{go} + \frac{1}{2}\left(1+\frac{1}{6}E_v t_{go} + \frac{2E_v}{E_r t_{go}^2}\right) a_{T,n,y}(t) t_{go}^2}{1+\frac{3}{E_r t_{go}^3}\left(1+E_v t_{go}\right)+\frac{E_v t_{go}}{4}}\right] \tag{60}$$

- Only difference: additional term in numerator
  - Time-varying gain multiplying target acceleration state

Intercept and Rendezvous Solutions

- **Augmented proportional guidance (APN)** law occurs for intercept problem, i.e. if $E_v = 0$ and as $E_r \to \infty$:

$$u_{APN}(t) = \frac{3}{t_{go}^2}\left[r_y(t) + v_y(t)t_{go} + \frac{1}{2}a_{T,n,y}t_{go}^2\right] \tag{61}$$

  - $\tilde{N}_{APN} = 3$
  - $Z = ZEM_{APN} = r_y(t) + v_y(t)t_{go} + \frac{1}{2}a_{T,n,y}t_{go}^2$
  - Only change in APN from PN: change in ZEM estimate
  - APN guidance law will only perform better than PN if vehicle able to sufficiently estimate target normal acceleration

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$$u_{APN}(t) = \frac{3}{t_{go}^2}\left[r_y(t) + v_y(t)t_{go} + \frac{1}{2}a_{T,n,y}t_{go}^2\right] \tag{61}$$

  - $\tilde{N}_{APN} = 3$
  - $Z = ZEM_{APN} = r_y(t) + v_y(t)t_{go} + \frac{1}{2}a_{T,n,y}t_{go}^2$
  - Only change in APN from PN: change in ZEM estimate
  - APN guidance law will only perform better than PN if vehicle able to sufficiently estimate target normal acceleration
- **Augmented rendezvous (AREN) guidance** law occurs for rendezvous problem, i.e. as $E_v \to \infty$, $E_r \to \infty$:

$$u_{AREN}(t) = \frac{6}{t_{go}^2}\left[r_y(t) + \frac{2}{3}v_y(t)t_{go}\right] + a_{T,n,y} \tag{62}$$

  - Same as REN, but adds direct cancellation of target maneuver in acceleration command

Minimum-Energy OCP 3

- Assume:
  - Vehicle and target speeds, $\|\vec{v}_V\|_2$ and $\|\vec{v}_T\|_2$, constant
  - Target undergoing constant acceleration, i.e. $\|\vec{a}_{T,n}\|_2 = a_{T,n,y}$
    - Added to state and estimated
  - Vehicle experiences acceleration command lag modeled as first-order transfer function:

$$\frac{\|\vec{a}_{V,n}\|_2}{a_{V,c}} = \frac{\omega}{s+\omega} \tag{63}$$

Minimum-Energy OCP 3 (continued)

$$\vec{u}^{\text{opt}}(t) = \underset{u(t)\ \forall t\in[0,\ t_f]}{\text{argmin}}\ \mathcal{J} = \vec{x}^T(t_f)\begin{bmatrix} E_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\vec{x}(t_f) + \int_0^{t_f} \vec{u}^T(t)\vec{u}(t)dt$$

$$\text{subject to: } \dot{\vec{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega \end{bmatrix}\vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \omega \end{bmatrix}\vec{u}(t) \tag{64}$$

$$\text{initial condition: } \vec{x}(0)$$

- $\vec{x} = [r_y\ v_y\ a_{T,n,y}\ \|\vec{a}_{V,n}\|_2]^T$
- $u = a_{V,c}$

OCP 3 Solution

- General solution:

$$u^{opt}(t) = \frac{6\omega^2 t_{go}^2 \left(\omega t_{go} + e^{-\omega t_{go}} - 1\right)}{t_{go}^2} \left[\frac{r_y(t) + v_y(t) t_{go} + \frac{1}{2} a_{T,n,y}(t) t_{go}^2 - \frac{1}{\omega^2}\left(\omega t_{go} + e^{-\omega t_{go}} - 1\right) \|\vec{a}_{V,n}\|_2}{\frac{6\omega^3}{E_r} + 3 + 6\omega t_{go} - 6\omega^2 t_{go}^2 + 2\omega^3 t_{go}^3 - 12\omega t_{go} e^{-\omega t_{go}} - 3e^{-2\omega t_{go}}} \right] \tag{65}$$

  - Considerably more complex than non-ideal vehicle response to acceleration command

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  - Considerably more complex than non-ideal vehicle response to acceleration command
- **Optimal guidance law (OGL)** occurs for intercept problem, i.e. as $E_v \to \infty$:

$$u_{OGL}(t) = \frac{6\omega^2 t_{go}^2 \left(\omega t_{go} + e^{-\omega t_{go}} - 1\right)}{t_{go}^2} \left[\frac{r_y(t) + v_y(t) t_{go} + \frac{1}{2} a_{T,n,y} t_{go}^2 - \frac{1}{\omega^2}\left(\omega t_{go} + e^{-\omega t_{go}} - 1\right) \|\vec{a}_{V,n}\|_2}{3 + 6\omega t_{go} - 6\omega^2 t_{go}^2 + 2\omega^3 t_{go}^3 - 12\omega t_{go} e^{-\omega t_{go}} - 3e^{-2\omega t_{go}}} \right] \tag{66}$$

OCP 3 Solution (continued)

- Time-varying effective navigation ratio:

$$\tilde{N}_{OGL} = \frac{6\omega^2 t_{go}^2 \left(\omega t_{go} + e^{-\omega t_{go}} - 1\right)}{3 + 6\omega t_{go} - 6\omega^2 t_{go}^2 + 2\omega^3 t_{go}^3 - 12\omega t_{go} e^{-\omega t_{go}} - 3e^{-2\omega t_{go}}} \tag{67}$$

OCP 3 Solution (continued)

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- Zero-effort-miss:

$$ZEM_{OGL} = r_y(t) + v_y(t) t_{go} + \frac{1}{2} a_{T,n,y} t_{go}^2 - \frac{1}{\omega^2}\left(\omega t_{go} + e^{-\omega t_{go}} - 1\right) \|\vec{a}_{V,n}\|_2 \tag{68}$$

  - Adds another term to ZEM_{APN} based on response lag

Minimum-Energy OCP 4

- Assume:
  - Vehicle and target speeds, $\|\vec{v}_V\|_2$ and $\|\vec{v}_T\|_2$, constant
  - Target undergoing constant jerk, $j_{T,n,y}$
    - Added to state and estimated along with target acceleration
  - Vehicle responds instantaneously to acceleration command, $a_{V,c} = \|\vec{a}_{V,n}\|_2$

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$$\text{initial condition:} \quad \vec{x}(0)$$

- $\vec{x} = [r_y\ v_y\ a_{T,n,y}\ j_{T,n,y}]^T \qquad u = \|\vec{a}_{V,n}\|_2$

OCP 4 Solution

- General solution:

$$u^{opt}(t) = \frac{3}{t_{go}^2}\left[\frac{\left(1+\frac{1}{2}E_v t_{go}\right) r_y(t) + \left(1+\frac{1}{3}E_v t_{go}+\frac{E_v}{E_r t_{go}^2}\right) v_y(t) t_{go} + \frac{1}{2}\left(1+\frac{1}{6}E_v t_{go}+\frac{2E_v}{E_r t_{go}^2}\right) a_{T,n,y}(t) t_{go}^2 + \frac{1}{6}\left(1+\frac{3E_v}{E_r t_{go}^2}\right) j_{T,n,y}(t) t_{go}^3}{1+\frac{3}{E_r t_{go}^3}(1+E_v t_{go})+\frac{E_v t_{go}}{4}}\right] \tag{70}$$

- Additional term in numerator, time-varying gain multiplying target jerk state

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  - Additional term in numerator, time-varying gain multiplying target jerk state
- **Extended proportional guidance (EPN)** occurs for intercept problem, i.e. if $E_v = 0$ and as $E_r \to \infty$:

$$u_{EPN}(t) = \frac{3}{t_{go}^2}\left[r_y(t) + v_y(t) t_{go} + \frac{1}{2} a_{T,n,y} t_{go}^2 + \frac{1}{6} j_{T,n,y}(t) t_{go}^3\right] \tag{71}$$

  - $\tilde{N}_{EPN} = 3 \qquad Z = ZEM_{EPN} = r_y(t) + v_y(t) t_{go} + \frac{1}{2} a_{T,n,y} t_{go}^2 + \frac{1}{6} j_{T,n,y}(t) t_{go}^3$

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  - Only change in EPN from APN and PN: change in ZEM estimate
    $\rightarrow$ EPN guidance law will only perform better if vehicle able to sufficiently estimate target normal acceleration *and* jerk

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  - Extension of decision theory for dynamical systems
  - Wide application
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