Lecture 10: Atmospheric and Mass Effects on Dynamics

Textbook Sections 9.5 & 9.6

Dr. Jordan D. Larson

Introduction

- Intro FDC assumes simple constant density, no wind atmospheric conditions:

- Variable mass, e.g. rocket engines, results in *additive* terms for nonlinear EOMs for flight vehicles
  - Including additional relevant states, easily added/neglected in flight dynamics modeling
  - Linearized dynamics: expanded dimension of LTI state-space model

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- Lecture: alter EOMs for atmospheric and variable mass effects

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- $\rightarrow$ include aerodynamic forces due to density variation

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\begin{aligned}
X_h &= \frac{Q_\infty S_w}{\rho_\infty m} \bar{C}_X \left(\frac{\partial \rho_\infty}{\partial h}\right) \\
Y_h &= \frac{Q_\infty S_w}{\rho_\infty m} \bar{C}_Y \left(\frac{\partial \rho_\infty}{\partial h}\right) \\
Z_h &= \frac{Q_\infty S_w}{\rho_\infty m} \bar{C}_Z \left(\frac{\partial \rho_\infty}{\partial h}\right)
\end{aligned} \tag{1}
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- Disregarding effect on moments as typically quite small
- Here density gradient $\partial \rho_\infty / \partial h$ obtained from atmospheric model

Aviation: International Standard Atmosphere

- Uses linear temperature vs. altitude model for different atmospheric layers

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Atmospheric Layer	**Lapse Rate ℓ** ($^\circ$R/ft)	**Lower Altitude h_0** (ft)	**Temperature T_0** ($^\circ$R)	**Pressure p_0** (psf)	**Density ρ_0** (sl/ft^3)
Troposphere	-3.5662×10^{-3}	0	518.69	2,116.2	2.3769×10^{-3}
Stratosphere I	0	36,089	389.99	472.68	7.0613×10^{-4}
Stratosphere II	5.4864×10^{-4}	65,617	389.99	114.35	1.7083×10^{-4}

Models for Atmospheric Layers

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$$\frac{dp}{dh} = -\rho g_0 \tag{3}$$

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- R: universal gas constant for air, 1716.5 $\mathrm{ft}^2/(\mathrm{s}^2 -^\circ R)$

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- Troposphere atmospheric model:

$$\begin{aligned} T &= 518.69 - (3.5662 \times 10^{-3})h \quad {}^\circ\text{R} \\ p &= (1.1376 \times 10^{-11})T^{5.256} \quad \text{psf} \\ \rho &= (6.6277 \times 10^{-15})T^{4.256} \quad \text{sl/ft}^3 \end{aligned} \tag{5}$$

Models for Atmospheric Layers (continued)

- Stratosphere I atmospheric model:

$$
\begin{aligned}
T &= 389.99 \quad {}^\circ\text{R} \\
p &= (2678.4)exp\left((-4.8063 \times 10^{-5})h\right) \quad \text{psf} \\
\rho &= (1.4939 \times 10^{-6})p \quad \text{sl/ft}^3
\end{aligned}
\tag{6}
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- Stratosphere II atmospheric model:

$$\begin{aligned} T &= 389.99 + (5.4864 \times 10^{-4})(h - 65617) \quad {}^\circ\text{R} \\ p &= (3.7930 \times 10^{90}) T^{-34.164} \quad \text{psf} \\ \rho &= (2.2099 \times 10^{87}) T^{-35.164} \quad \text{sl/ft}^3 \end{aligned} \tag{7}$$

Simple Approximation

$$\text{Troposphere: } \rho = (2.3769 \times 10^{-3}) \exp\left(\frac{-h}{29,730}\right) \quad \text{sl/ft}^3$$
$$\text{Stratosphere I: } \rho = (7.0613 \times 10^{-4}) \exp\left(\frac{h - 36,089}{20,806}\right) \quad \text{sl/ft}^3 \tag{8}$$
$$\text{Stratosphere II: } \rho = (1.7083 \times 10^{-4}) \exp\left(\frac{-(h - 65,617)}{29,730}\right) \quad \text{sl/ft}^3$$

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$$\vec{V}_{B/N} = \vec{V}_\infty + \vec{V}_{W/N} \tag{9}$$

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- In FDC: aerodynamic forces and moments typically function of airspeed (not ground speed)
- $\rightarrow$ useful to use airspeed velocity as part of state vector in EOMs

Rewriting EOMs with Airspeed

- Utilizing navigation and body-fixed frame coordinates

$$\vec{v}_{B/N,N} = C_{N \leftarrow B} \vec{v}_{\infty,B} + \vec{v}_{W/N,N} \tag{10}$$

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- Substituting and converting to body-fixed frame coordinates:

$$\vec{F}_{a,B} + \vec{F}_{p,B} + mC_{B\leftarrow N}\vec{g}_N = m\left(\dot{\vec{v}}_{\infty,B} + \left[\vec{\omega}_{B/N,B}\right]_{\times}\vec{v}_{\infty,B} + C_{B\leftarrow N}\dot{\vec{v}}_{W/N,N}\right) \tag{13}$$

Rewriting EOMs with Airspeed (continued)

- Via normalization and $\vec{v}_{\infty,B} = [u\ v\ w]^T$:

$$\begin{bmatrix} X - g\sin\theta \\ Y + g\cos\theta\sin\phi \\ Z + g\cos\theta\cos\phi \end{bmatrix} = \begin{bmatrix} \dot{u} + qw - rv \\ \dot{v} + ru - pw \\ \dot{w} + pv - qu \end{bmatrix} + C_{B\leftarrow N} \begin{bmatrix} \dot{u}_{W/N,N} \\ \dot{v}_{W/N,N} \\ \dot{w}_{W/N,N} \end{bmatrix} \tag{14}$$

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- Integrate velocity EOMs to find vehicle's inertial position

$$\vec{x}_{B/N,N} = \int \dot{\vec{x}}_{B/N,N} dt = \int C_{N\leftarrow B}(t)\vec{v}_{\infty,B}(t) + \vec{v}_{W/N,N} dt \tag{15}$$

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- Simply use airspeed components in EOMs $\rightarrow$ adds positional offset that grows linearly with time due to steady wind
  - Often preferred as airspeed vector affects aerodynamic forces and moments

Wind Shear

- **Wind shear**: variation of wind vector with respect to position, typically decoupled into vertical and horizontal coordinates
  - Vertical wind shear: wind vector varies with altitude
  - Horizontal wind shear describes the wind changing along some horizontal distance
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- Specific wind shear profiles caused by topography, frontal systems, thunderstorms obtained via numerical studies
- Basic logarithmic model for vertical wind shear profile based on point measurement 3 ft $< h <$ 1000 ft:

$$\bar{u}_{ws}(h) = \|\vec{V}_{w,20}\|_2 \frac{\log\left(\frac{h}{z_0}\right)}{\log\left(\frac{20}{z_0}\right)} \tag{16}$$

  - $\bar{u}_{ws}$: mean horizontal wind speed in navigation frame
  - $\vec{V}_{w,20}$: measured horizontal wind speed at height above ground level of 20 ft
  - $z_0 = 0.15$ ft: take-off, climb, approach, landing & $z_0 = 2$ ft: cruise

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- Augmented longitudinal state dynamics:

$$\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{\alpha} \\ \Delta\dot{q} \\ \Delta\dot{\theta} \\ \Delta\dot{h} \end{bmatrix} = \begin{bmatrix} X_u & X_\alpha & 0 & -g\cos\bar{\theta} & -X_u\left(\frac{du}{dh}\right) \\ \frac{Z_u}{\bar{u}-Z_{\dot{\alpha}}} & \frac{Z_\alpha}{\bar{u}-Z_{\dot{\alpha}}} & \frac{\bar{u}+Z_q}{\bar{u}-Z_{\dot{\alpha}}} & -\frac{g}{\bar{u}-Z_{\dot{\alpha}}}\sin\bar{\theta} & -\left(\frac{Z_u}{\bar{u}-Z_{\dot{\alpha}}}\right)\left(\frac{du}{dh}\right) \\ M_u + M_{\dot{\alpha}}\frac{Z_u}{\bar{u}-Z_{\dot{\alpha}}} & M_\alpha + M_{\dot{\alpha}}\frac{Z_\alpha}{\bar{u}-Z_{\dot{\alpha}}} & M_q + M_{\dot{\alpha}}\frac{\bar{u}+Z_q}{\bar{u}-Z_{\dot{\alpha}}} & -M_{\dot{\alpha}}\frac{g}{\bar{u}-Z_{\dot{\alpha}}}\sin\bar{\theta} & -\left(M_u + M_{\dot{\alpha}}\frac{Z_u}{\bar{u}-Z_{\dot{\alpha}}}\right)\left(\frac{du}{dh}\right) \\ 0 & 0 & 1 & 0 & \\ \sin\bar{\theta} & \bar{u}\cos\bar{\theta} & 0 & -\bar{u}\cos\bar{\theta} & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta\alpha \\ \Delta q \\ \Delta\theta \\ \Delta h \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & X_T \\ \frac{Z_{\delta_e}}{\bar{u}-Z_{\dot{\alpha}}} & \frac{Z_T}{\bar{u}-Z_{\dot{\alpha}}} \\ M_{\delta_e} + M_{\dot{\alpha}}\frac{Z_{\delta_e}}{\bar{u}-Z_{\dot{\alpha}}} & M_T + M_{\dot{\alpha}}\frac{Z_T}{\bar{u}-Z_{\dot{\alpha}}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\delta_e \\ \Delta T \end{bmatrix} \tag{17}$$

Unsteady Wind and Stochastic Gusts

- **Unsteady wind component** on velocity EOMs $\rightarrow$ unsteady wind gust as **stochastic gusts**, have stochastic properties of varying magnitude denoted as $\vec{v}_g$

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- u, v, w implicitly model velocity relative to hypothetical *still* air mass

Stochastic Gust Models

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- Dryden gust state equations in stability frame coordinates:

$$
\begin{aligned}
\begin{bmatrix} \dot{u}_g(t) \\ \dot{v}_g(t) \\ \dot{v}_{g_1}(t) \\ \dot{w}_g(t) \\ \dot{w}_{g_1}(t) \end{bmatrix} =& \begin{bmatrix} -\frac{\bar{u}}{L_u} & 0 & 0 & 0 & 0 \\ 0 & -\frac{\bar{u}}{L_u} & \sigma_v(1-\sqrt{3})\left(\frac{\bar{u}}{L_v}\right)^{3/2} & 0 & 0 \\ 0 & 0 & -\frac{\bar{u}}{L_v} & 0 & 0 \\ 0 & 0 & 0 & -\frac{\bar{u}}{L_w} & \sigma_w(1-\sqrt{3})\left(\frac{\bar{u}}{L_w}\right)^{3/2} \\ 0 & 0 & 0 & 0 & -\frac{\bar{u}a}{L_w} \end{bmatrix} \begin{bmatrix} u_g(t) \\ v_g(t) \\ v_{g_1}(t) \\ w_g(t) \\ w_{g_1}(t) \end{bmatrix} \\
&+ \begin{bmatrix} \sigma_u\left(\frac{2\bar{u}}{\pi L_u}\right)^{1/2} \\ \sigma_v\left(\frac{3\bar{u}}{L_v}\right)^{1/2} \\ 1 \\ \sigma_w\left(\frac{3\bar{u}}{L_w}\right)^{1/2} \\ 1 \end{bmatrix} n(t) \qquad (20) \\
\dot{\vec{x}}_g =& A_g \vec{x}_g + B_g n(t)
\end{aligned}
$$

Dryden Gust Model (continued)

- Driving function, $n(t)$: zero-mean, additive white Gaussian noise (AWGN) of unit intensity
  - Discrete approximation over short Δt: zero-mean Gaussian with variance $1/\Delta t$

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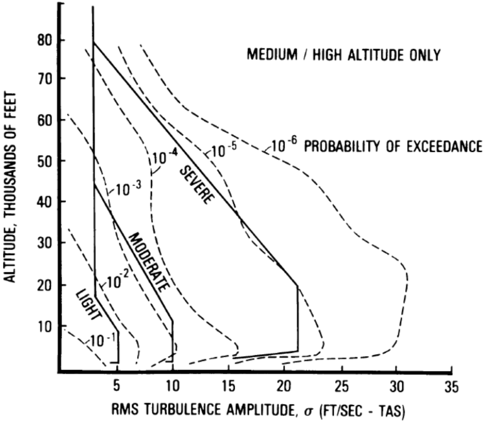
- σ_u, σ_v, σ_w: standard deviations of gusts, or **RMS gust intensities**
- Obtained from data, e.g. “MIL-F-8785C Military Specification: Flying Qualities of Piloted Airplanes”
  - Plot of three levels of RMS gust intensities for different altitudes: *light*, *moderate*, and *severe*

Dryden Gust Model (continued)



- **Probability of exceedance**: probability that RMS gust intensity exceeds value on curve at that altitude

Angle of Attack and Sideslip

- Wind frame Euler angles, i.e. angle of attack and sideslip angles, transform instantaneous velocity magnitude v_∞ to body frame coordinates:

$$\begin{bmatrix} v_\infty \cos\alpha_{tot} \cos\beta_{tot} \\ v_\infty \sin\beta_{tot} \\ v_\infty \sin\alpha_{tot} \cos\beta_{tot} \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} u_g \\ v_g \\ w_g \end{bmatrix} \tag{21}$$

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- 3rd row divided by first row:

$$\tan\alpha_{tot} = \frac{w + w_g}{u + u_g} \tag{23}$$

Angle of Attack and Sideslip (continued)

- 2nd row rewritten:

$$\sin \beta_{tot} = \frac{v + v_g}{V_\infty} \tag{24}$$

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  - Used instead of v_{tot}, w_{tot}, $\dot{w}_{tot}$

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- For computing stability derivative contributions w.r.t. u, v (or β), w (or α), $\dot{w}$ (or $\dot{\alpha}$) for X, Y, Z, L, M, and N

Augmented LTI State-Space

- For linearized flight dynamics: linear Dryden wind model above combined with nominal LTI state-space state equation

$$\Delta\dot{\vec{x}} = A\Delta\vec{x} + B\Delta\vec{u} \tag{28}$$

  - $\Delta\vec{x} = [\Delta u \;\; \Delta\beta \;\; \Delta\alpha \;\; \Delta p \;\; \Delta q \;\; \Delta r \;\; \Delta\phi \;\; \Delta\theta \;\; \Delta\psi]^T$

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- Form augmented LTI state equation

$$\begin{bmatrix} \Delta\dot{\vec{x}} \\ \dot{\vec{x}}_g \end{bmatrix} = \begin{bmatrix} A & G_g C_g \\ 0 & A_g \end{bmatrix} \begin{bmatrix} \Delta\vec{x} \\ \vec{x}_g \end{bmatrix} + \begin{bmatrix} B & G_g D_g \\ 0 & B_g \end{bmatrix} \begin{bmatrix} \Delta\vec{u} \\ n \end{bmatrix} \tag{29}$$

Augmented LTI State-Space (continued)

- Gust output matrix:

$$C_g = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\bar{u}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\bar{u}} & 0 \\ 0 & 0 & 0 & -\frac{1}{L_w} & \sigma_w(1-\sqrt{3})\left(\frac{\bar{u}}{L_w^3}\right)^{1/2} \end{bmatrix} \tag{30}$$

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- Outputs u_g, β_g, α_g, $\dot{\alpha}_g$ from gust state vector $\vec{x}_g = [u_g \;\; v_g \;\; v_{g_1} \;\; w_g \;\; w_{g_1}]^T$ and $n(t)$

Augmented LTI State-Space (continued)

- Multiplying gust outputs by stability derivatives for force/moment contributions

$$G_g = \begin{bmatrix} X_u + \frac{X_{\dot{\alpha}} Z_u}{\bar{u} - Z_{\dot{\alpha}}} & 0 & X_\alpha + \frac{X_{\dot{\alpha}} Z_\alpha}{\bar{u} - Z_{\dot{\alpha}}} & X_{\dot{\alpha}} + \frac{X_{\dot{\alpha}} Z_{\dot{\alpha}}}{\bar{u} - Z_{\dot{\alpha}}} \\ 0 & \frac{Y_\beta}{\bar{u}} & 0 & 0 \\ \frac{Z_u}{\bar{u} - Z_{\dot{\alpha}}} & 0 & \frac{Z_\alpha}{\bar{u} - Z_{\dot{\alpha}}} & \frac{Z_{\dot{\alpha}}}{\bar{u} - Z_{\dot{\alpha}}} \\ 0 & L_\beta^* & 0 & 0 \\ M_u + \frac{M_{\dot{\alpha}} Z_u}{\bar{u} - Z_{\dot{\alpha}}} & 0 & M_\alpha + \frac{M_{\dot{\alpha}} Z_\alpha}{\bar{u} - Z_{\dot{\alpha}}} & M_{\dot{\alpha}} + \frac{M_{\dot{\alpha}} Z_{\dot{\alpha}}}{\bar{u} - Z_{\dot{\alpha}}} \\ 0 & N_\beta^* & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{32}$$

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- Also possible to include linear Dryden gust model in nonlinear state equation

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- Rotation directly contributes to total angular momentum of vehicle
- Consider angular momentum of vehicle about its center of mass:

$$\begin{bmatrix} I_{xx}L \\ I_{yy}M \\ I_{zz}N \end{bmatrix} = \dot{\vec{H}}_N = \dot{\vec{H}}_B + \vec{\omega}_{B\leftarrow N} \times \vec{H}_B \tag{33}$$

Two Components of $\vec{H}_B$

- Rigid vehicle's angular momentum with mass of propeller "disk"
- With propeller's angular momentum, $\vec{H}_{prop,B}$:

$$\vec{H}_B = \vec{H}_{r-b,B} + \vec{H}_{prop,B} \tag{34}$$

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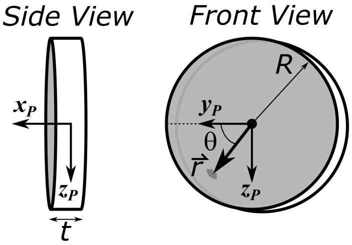
- $\vec{x}_B$: radial position of mass element $\rho_V dV$ w.r.t. center of mass

Idealized Propeller as Rotating Disk



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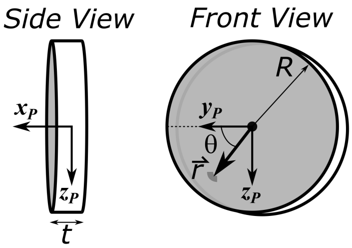

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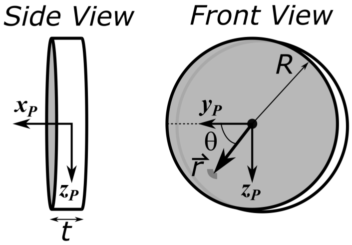

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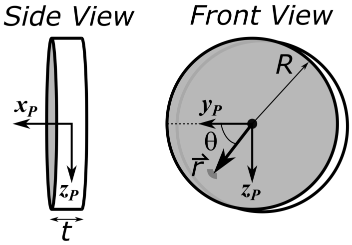

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- Disk radial mass distribution matches propeller
  - Disk density, ρ_{disk}, covers disk volume with equivalent mass

Mass Element Velocity

- Mass element velocity:

$$\dot{\vec{x}}_{B'} = \dot{\vec{x}}_P + \vec{\omega}_{B'/P} \times \vec{x} \tag{38}$$

- B' frame: body-fixed frame centered at propeller's center

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- Assuming propeller disk rigid:

$$\dot{\vec{x}}_{B'} = \omega_{B'/P} \times \vec{x} = \begin{bmatrix} \omega_{prop} \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ r\cos\theta \\ r\sin\theta \end{bmatrix} = \begin{bmatrix} 0 \\ \omega_{prop} r \sin\theta \\ -\omega_{prop} r \cos\theta \end{bmatrix} \tag{39}$$

Propeller Angular Momentum (B' Frame)

- Integrand of angular velocity

$$\vec{x}_{B'} \times \rho_V \dot{\vec{x}}_{B'} = \rho_V \begin{bmatrix} 0 \\ r\cos\theta \\ r\sin\theta \end{bmatrix} \times \begin{bmatrix} 0 \\ \omega_{prop} r\sin\theta \\ -\omega_{prop} r\cos\theta \end{bmatrix} = \begin{bmatrix} \omega_{prop} r^2 \\ 0 \\ 0 \end{bmatrix} \quad (40)$$

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$$\begin{aligned} \vec{H}_{prop,B'} &= \begin{bmatrix} \int_0^R \int_0^{2\pi} \omega_{prop} r^2 \rho_{disk} (rtd\theta dr) \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \omega_{prop} 2\pi t \int_0^R r^3 \rho_{disk} dr \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \omega_{prop} I_{prop} \\ 0 \\ 0 \end{bmatrix} \end{aligned} \tag{41}$$

- I_{prop}: moment of inertia of propeller about its center of mass

Propeller Angular Momentum (B Frame)

$$\vec{H}_{prop,B} = \begin{bmatrix} h_{x,prop} \\ h_{y,prop} \\ h_{z,prop} \end{bmatrix} = C_{B \leftarrow B'} \begin{bmatrix} \omega_{prop} I_{prop} \\ 0 \\ 0 \end{bmatrix} \tag{42}$$

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- $C_{B \leftarrow B'}$ depends on orientation of propeller relative to body frame
- Example: B' at some positive rotation angle, τ_P, about y_B axis

$$\vec{H}_{prop,B} = \begin{bmatrix} \omega_{prop} I_{prop} \cos \tau_P \\ 0 \\ -\omega_{prop} I_{prop} \sin \tau_P \end{bmatrix} \tag{43}$$

Total Angular Momentum

- By substitution, additive terms in differential equation:

$$\begin{bmatrix} I_{xx}L \\ I_{yy}M \\ I_{zz}N \end{bmatrix} = \left(\dot{\vec{H}}_{r-b,B} + \dot{\vec{H}}_{prop,B}\right) + \vec{\omega}_{B\leftarrow N} \times \left(\vec{H}_{r-b,B} + \vec{H}_{prop,B}\right) \tag{44}$$

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- Additive differential terms:

$$\begin{bmatrix} L \\ M \\ N \end{bmatrix} = \begin{bmatrix} \dot{p} + \frac{I_{zz}-I_{yy}}{I_{xx}}qr - \frac{I_{xz}}{I_{xx}}(\dot{r} + pq) \\ \dot{q} + \frac{I_{xx}-I_{zz}}{I_{yy}}pr - \frac{I_{xz}}{I_{yy}}(r^2 - p^2) \\ \dot{r} + \frac{I_{yy}-I_{xx}}{I_{zz}}pq - \frac{I_{xz}}{I_{zz}}(\dot{p} - qr) \end{bmatrix} + \begin{bmatrix} \frac{1}{I_{xx}}\left(\dot{h}_{x,prop} - rh_{y,prop} + qh_{z,prop}\right) \\ \frac{1}{I_{yy}}\left(\dot{h}_{y,prop} + rh_{x,prop} - ph_{z,prop}\right) \\ \frac{1}{I_{zz}}\left(\dot{h}_{z,prop} - qh_{x,prop} + ph_{y,prop}\right) \end{bmatrix} \tag{46}$$

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  - For even number, often each spun in opposite directions to counteract contribution

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  - For even number, often each spun in opposite directions to counteract contribution
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  - Function of additional states
  - Added dynamics for propeller/rotor angular momentum
- E.g. propeller angular rate $\rightarrow$ control input affects $\dot{p}$, $\dot{q}$ and $\dot{r}$ equations
  - In nonlinear state-space EOMs or LTI state-space EOMs
  - Changes in propeller angular velocity change $h_{x,prop}$, $h_{y,prop}$, $h_{z,prop}$
  - Propeller angular velocity controlled by speed controller for producing commanded thrust

Propulsion through Mass Expulsion

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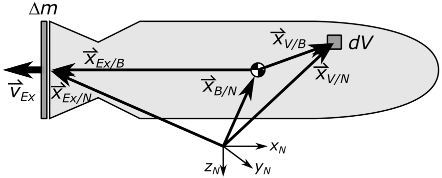
- Note: derivation consider position vectors as represented in inertial navigation frame N coordinates unless otherwise noted

Various Elements of Rocket



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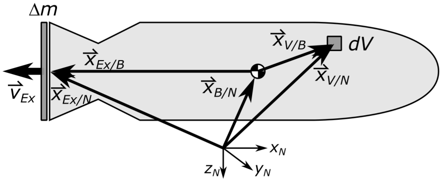

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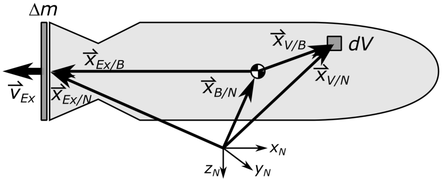

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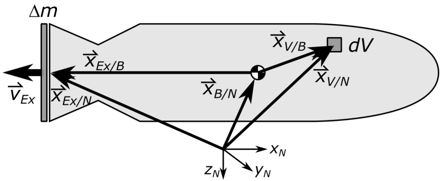

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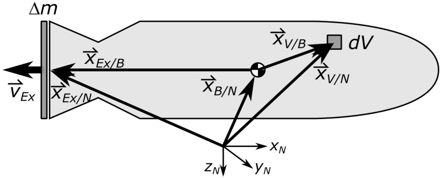

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- Expelled mass Δm
  - Position relative to body & navigation frames: $\vec{x}_{E/B}$ & $\vec{x}_{E/N}$
  - **Exit velocity** of Δm relative to vehicle: $\vec{v}_E$

$$\|\vec{v}_E\|_2^2 = I_{sp} g_0 \qquad (47)$$

- I_{sp}: specific impulse of rocket engine
- g_0: standard acceleration due to gravity

Change in Translational Momentum

- Vehicle linear momentum at time t:

$$\vec{P}_N(t) = \int_{Vol} \rho_V \dot{\vec{x}}_{V/N} dV \tag{48}$$

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$$\vec{P}_N(t+\Delta t) = \int_{Vol} \left(\dot{\vec{x}}_{V/N} + \Delta \dot{\vec{x}}_{V/N} \right) \rho_V dV + \Delta m \left(\dot{\vec{x}}_{E/N} + \Delta \dot{\vec{x}}_{E/N} \right) \tag{49}$$

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$$\dot{\vec{P}}_N = \int_{Vol} \frac{d}{dt} \left(\rho_V \dot{\vec{x}}_{V/N}\right) dV + \dot{m}\dot{\vec{x}}_{E/N} \tag{50}$$

Newton's Translational EOM

- Total rate of change in linear momentum:

$$\dot{\vec{P}}_N = \int_{Vol} \rho_V \vec{g}\, dV + \int_{Area} d\vec{F}_{ext} + \dot{m}\dot{\vec{x}}_{E/N} \tag{51}$$

  - dF_{ext}: external force acting at infinitesimal surface area, i.e. pressure forces

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- Inertial velocity of expelled mass *dm*:

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- As $\Delta t \to 0$:

$$\frac{d}{dt}\left(m\vec{x}_{B/N}\right) = \vec{P}_N(t) + \dot{m}\vec{x}_{E/N} \tag{56}$$

  - Note: $\dot{m} < 0$

Rewritting the Translational EOM

- Position of expelled mass *dm*

$$\vec{x}_{E/N} = \vec{x}_{B/N} + \vec{x}_{E/B} \tag{57}$$

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$$\vec{P}_N(t) = m\dot{\vec{x}}_{B/N} - \dot{m}\vec{x}_{E,B} \tag{59}$$

- Differentiating:

$$\dot{\vec{P}}_N = m\ddot{\vec{x}}_{B/N} + \dot{m}\left(\dot{\vec{x}}_{B/N} - \dot{\vec{x}}_{E/B}\right) - \ddot{m}\vec{x}_{E/B} \tag{60}$$

Combining Results

- Equating (53) & (60):

$$
\begin{aligned}
& m\ddot{\vec{x}}_{B/N} + \dot{m}\left(\dot{\vec{x}}_{B/N} - \dot{\vec{x}}_{E/B}\right) - \ddot{m}\vec{x}_{E/B} \\
& = \int_{Vol} \rho_V \vec{g}\, dV + \int_{Area} d\vec{F}_{ext} + \dot{m}\left(\vec{v}_{B/N} + \dot{\vec{x}}_{E/B} + \vec{v}_E\right)
\end{aligned}
\tag{61}
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- Gravity constant over volume

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- Propulsive force acts forward on vehicle:

$$\vec{F}_p = \dot{m}\vec{v}_{Ex} + \int_{Exit\ \ Area} d\vec{F}_{ext} \tag{63}$$

Variable Mass Translational EOM

$$m\dot{\vec{v}}_{B/N} = \vec{F}_{g,B} + \vec{F}_{p,B} + \vec{F}_{a,B} + 2\dot{m}\dot{\vec{x}}_{E/B} + \ddot{m}\vec{x}_{E} \tag{64}$$

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- Body frame coordinates

$$m\begin{bmatrix} u - qw + rv \\ v - ru + pw \\ w - pv + qu \end{bmatrix} = \vec{F}_{g,B} + \vec{F}_{p,B} + \vec{F}_{a,B} + 2\dot{m}\left(\dot{\vec{x}}_{Ex/B,B} + \vec{\omega}_{B/N} \times \vec{x}_{Ex/B,B}\right) + \ddot{m}\vec{x}_{Ex/B} \tag{65}$$

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    - $\vec{x}_{Ex/B}$ small magnitude relative to $\vec{v}_E$
    - $\ddot{m}$ small
- Note: $\dot{\vec{x}}_{Ex/B,B}$ changes with time
  - Expelled mass alters location of center of mass (body frame origin)

Angular Momentum

- Inertial rotational momentum at time t:

$$\vec{H}_N(t) = \int_{Vol} \vec{x}_{V/B} \times \rho_V \dot{\vec{x}}_{V/N} dV \tag{66}$$

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- Change in vehicle's rotational momentum due to change in mass
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Taking Limit and Euler's Rotation EOM

- As $\Delta t \to 0$ (neglecting higher order Δ terms):

$$\dot{\vec{H}}_N = \int_{Vol} \frac{d}{dt}\left(\vec{x}_{V/B} \times \rho_V \dot{\vec{x}}_{V/N}\right) dV + \vec{x}_{E/N} \times \dot{m}\dot{\vec{x}}_{E/N} \tag{68}$$

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- Comparing with Newton's 2nd law: total rate of change in rotational momentum

$$\dot{\vec{H}}_N = \int_{Vol} \vec{x}_{V/N} \times \rho_V \vec{g}\, dV + \int_{Area} \vec{x}_{V/N} \times d\vec{F}_{ext} + \vec{x}_{E/N}\dot{m}\dot{\vec{x}}_{E/N} \tag{69}$$

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- Note: $\vec{x}_{V/N} = \vec{x}_{B/N} + \vec{x}_{V/B}$

Differentiation

$$\begin{aligned}\vec{H}_N &= \vec{x}_{B/N} \times \int_{Vol} \rho_V \dot{\vec{x}}_{V/N} dV + \int_{Vol} \vec{x}_{V/B} \times \rho_V \dot{\vec{x}}_{V/B} \\ &= \vec{x}_{B/N} \times \int_{Vol} \rho_V \dot{\vec{x}}_{V/N} dV + \vec{H}_{B,N}\end{aligned} \tag{70}$$

- First term: angular momentum of body-fixed frame
- $\vec{H}_{B,N}$: angular momentum of vehicle in navigation frame coordinates

Differentiation

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- First term: angular momentum of body-fixed frame
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- Recall, by definition:

$$\vec{P}_N = \int_{Vol} \rho_V \dot{\vec{x}}_{V/N} dV \tag{71}$$

Differentiation

$$\vec{H}_N = \vec{x}_{B/N} \times \int_{Vol} \rho_V \dot{\vec{x}}_{V/N} dV + \int_{Vol} \vec{x}_{V/B} \times \rho_V \dot{\vec{x}}_{V/B}$$
$$= \vec{x}_{B/N} \times \int_{Vol} \rho_V \dot{\vec{x}}_{V/N} dV + \vec{H}_{B,N} \quad (70)$$

- First term: angular momentum of body-fixed frame
- $\vec{H}_{B,N}$: angular momentum of vehicle in navigation frame coordinates

- Recall, by definition:

$$\vec{P}_N = \int_{Vol} \rho_V \dot{\vec{x}}_{V/N} dV \quad (71)$$

- Differentiating w.r.t. navigation frame (in navigation frame coordinates):

$$\dot{\vec{H}}_N = \dot{\vec{x}}_{B/N} \times \vec{P}_N + \vec{x}_{B/N} \times \dot{\vec{P}}_N + \dot{\vec{H}}_{B,N} \quad (72)$$

Substitution and Combination

- Substituting for $\vec{P}$ and $\dot{\vec{P}}_N$:

$$\dot{\vec{H}}_N = -\dot{\vec{x}}_{B/N} \times \dot{m}\vec{x}_{E/B} + \vec{x}_{B/N} \times \left(\int_{Vol} \rho_V \vec{g}\, dV + \int_{Area} d\vec{F}_{ext} + \dot{m}\dot{\vec{x}}_{E/N} \right) + \dot{\vec{H}}_{B,N} \quad (73)$$

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- Equating (69) & (73):

$$\begin{aligned} &\dot{\vec{H}}_{B,N} - \dot{\vec{x}}_{B/N} \times \dot{m}\vec{x}_{E/B} + \vec{x}_{B/N} \times \left(\int_{Vol} \rho_V \vec{g}\, dV + \int_{Area} d\vec{F}_{ext} + \dot{m}\dot{\vec{x}}_{E/N} \right) \\ &= \int_{Vol} \vec{x}_{V/N} \times \rho_V \vec{g}\, dV + \int_{Area} \vec{x}_{V/N} \times d\vec{F}_{ext} + \vec{x}_{E/N}\dot{m}\dot{\vec{x}}_{E/N} \end{aligned} \quad (74)$$

Vector Relationships and Simplifications

- First mass moment about center of mass $=$ zero for no gravity gradients: first term eliminated

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\vec{x}_{E/N} &= \vec{x}_{B/N} + \vec{x}_{E/B} \\
\dot{\vec{x}}_{E/N} &= \dot{\vec{x}}_{B/N} + \dot{\vec{x}}_{E/B} + \vec{v}_{E}
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$$

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\end{aligned} \tag{75}
$$

- Rewrite derivative of angular momentum about its center of mass

$$
\dot{\vec{H}}_{B,N} = \int_{Area} \vec{x}_{V/B} \times d\vec{F}_{ext} + \vec{x}_{E/B} \times \dot{m} \left(\dot{\vec{x}}_{E/B} + \vec{v}_{E} \right) \tag{76}
$$

Final Derivative of Angular Momentum

- Separate external moments into aerodynamic and propulsive

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$$\dot{\vec{H}}_{B,N} = \vec{M}_{a,N} + \vec{M}_{p,N} + \vec{x}_{E/B} \times \dot{m}\left(\dot{\vec{x}}_{E/B,B} + \vec{\omega}_{B/N} \times \vec{x}_{E/B}\right) \tag{77}$$

  - $\vec{M}_{p,N} = \vec{x}_{E/B} \times \vec{F}_{p}$

Final Derivative of Angular Momentum

- Separate external moments into aerodynamic and propulsive

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- $\vec{M}_{p,N} = \vec{x}_{E/B} \times \vec{F}_p$

$$\dot{\vec{H}}_{B,N} = C_{N \leftarrow B}\begin{bmatrix} I_{xx}L \\ I_{yy}M \\ I_{zz}N \end{bmatrix} + \vec{x}_{E/B} \times \dot{m}\left(\dot{\vec{x}}_{E/B,B} + \vec{\omega}_{B/N} \times \vec{x}_{E/B}\right) \tag{78}$$

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- Additive triple product: **jet-damping effect**
  - Tends to act against angular motion, $\Delta m < 0$
  - Proportional to angular velocity, square of distance from nozzle exit to center of mass, & mass flow rate
  - More significant effect for maneuvering, long vehicles with large mass flow rates, e.g. missiles

Summary

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  - Density changes with altitude
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- Variable Mass:
  - Adds to apparent acceleration through mass rate
  - Adds to angular momentum change through jet-damping effect