AEM 668 Project 2

Nonlinear Performance Simulation of Subsonic Airplane

Learning Objective

This project is intended to introduce in MATLAB/Simulink the use of high-fidelity point-mass dynamics modeling for a subsonic airplane, including mass effects, atmospheric effects, gravity effects, and Earth frame effects.

Dynamical System

An important nonlinear simulation in FDC is a point-mass performance simulation which assesses a flight vehicle's response to following a commanded flight profile defined as a commanded groundspeed, rate of climb, and heading. Therefore, this type of simulation will only approximate the attitude dynamics where one can assume that an inner-loop attitude control system is being used to achieve the commanded attitude performance in this point-mass simulation of the translation equations of motion, specifically it is designed such that sideslip angle β and the side force S are both regulated quickly to zero, i.e., coordinated flight. This project will be developed for Simulink to solve the nonlinear equations through simulation.

For this simulation, let the translation equations of motion be expressed in a "wind frame" aligned with the inertial velocity with a spherical-Earth model as

$$m\left(\begin{bmatrix}\dot{v}_{B/N,W}\\0\\0\end{bmatrix}+\begin{bmatrix}p_{W/I,W}\\q_{W/I,W}\\r_{W/I,W}\end{bmatrix}_{\times}\begin{bmatrix}v_{W/N,W}\\0\\0\end{bmatrix}\right)=\vec{F}_{a,W}+\vec{F}_{p,W}+\vec{F}_{g,W}+\vec{F}_{Earth,W} \tag{1}$$

where $[p_{W/I,W}\ q_{W/I,W}\ r_{W/I,W}]^T$ is the inertial angular velocity of the wind frame, as opposed to the body-fixed frame, and $\vec{v}_{B/N,W}$ is the linear velocity of the vehicle's center of mass (the same in the body and wind frames) with respect to the navigation frame which is related to the airspeed vector and the wind vector via the wind triangle, i.e., in the navigation frame coodinates

$$\vec{v}_{\infty,N}=\begin{bmatrix}u_\infty\\v_\infty\\w_\infty\end{bmatrix}=\begin{bmatrix}v_{W/N,W}\cos\gamma\cos\sigma\\v_{W/N,W}\cos\gamma\sin\sigma\\v_{W/N,W}\sin\gamma\end{bmatrix}-\vec{v}_{Wind,N} \tag{2}$$

where the $3-2-1$ Euler angles for the wind-to-navigation frame rotation are the bank angle, μ, the flight path angle, γ, and the heading angle, σ, and it is assumed $\dot{\vec{v}}_{Wind,N}=\vec{0}$ for this simulation and

$$\|\vec{v}_\infty\|_2^2=u_\infty^2+v_\infty^2+w_\infty^2 \tag{3}$$

$\vec{F}_{a,W}$ is the aerodynamic force in the wind frame (with no side force)

$$\vec{F}_{a,W} = \begin{bmatrix} -D \\ 0 \\ -L \end{bmatrix} \tag{4}$$

$\vec{F}_{p,W}$ is the propulsion force assumed to be colinear with the x_B-axis, i.e.

$$\vec{F}_{p,W} = \begin{bmatrix} T\cos\alpha \\ 0 \\ -T\sin\alpha \end{bmatrix} \tag{5}$$

where thrust is related to the mass rate by

$$\dot{m} = -K_{\dot{f}} T \tag{6}$$

where $K_{\dot{f}}$ is the **thrust-specific fuel consumption** specific *constant* and depends on the fuel rate, $\dot{f}$.

$\vec{F}_{g,W}$ is the gravity force in the navigation frame given by the spherical-Earth model, i.e.

$$\vec{F}_{g,W} = \begin{bmatrix} -mg\sin\gamma \\ mg\sin\mu\cos\gamma \\ mg\cos\mu\cos\gamma \end{bmatrix} \tag{7}$$

where

$$g = g_0(\ell)\left(\frac{\bar{R}_E}{\bar{R}_E + h}\right)^2 \tag{8}$$

and ℓ is the (geodetic/geocentric for spherical-Earth) latitude and $\bar{R}_E$ is Earth's *mean* radius, 6,371,000 m.

$\vec{F}_{Earth,B}$ is the apparent force due to the rotating Earth frame effects expressed in body-fixed frame coordinates.

$$\vec{F}_{Earth,W} = m\left(\begin{bmatrix} 0 \\ -r_{E/I,W} v_{W/N,W} \\ q_{E/I,W} v_{W/N,W} \end{bmatrix} + C_{W\leftarrow N}(\mu,\gamma,\sigma)\begin{bmatrix} -\omega_E^2(\bar{R}_E + h)\cos\lambda\sin\lambda \\ 0 \\ -\omega_E^2(\bar{R}_E + h)\cos^2\lambda \end{bmatrix}\right) \tag{9}$$

where

$$\begin{bmatrix} p_{E/I,W} \\ q_{E/I,W} \\ r_{E/I,W} \end{bmatrix} = C_{W\leftarrow N}(\mu,\gamma,\sigma)\begin{bmatrix} \omega_E\cos\ell \\ 0 \\ -\omega_E\sin\ell \end{bmatrix} \tag{10}$$

$$C_{W\leftarrow N}(\mu,\gamma,\sigma) = \begin{bmatrix} \cos\gamma\cos\sigma & \cos\gamma\sin\sigma & -\sin\gamma \\ \sin\mu\sin\gamma\cos\sigma - \cos\mu\sin\sigma & \sin\mu\sin\gamma\sin\sigma + \cos\mu\cos\sigma & \sin\mu\cos\gamma \\ \cos\mu\sin\gamma\cos\sigma + \sin\mu\sin\sigma & \cos\mu\sin\gamma\sin\sigma - \sin\mu\cos\sigma & \cos\mu\cos\gamma \end{bmatrix} \tag{11}$$

or

$$
\vec{F}_{Earth,W} = m\left(\begin{bmatrix} 0 \\ -r_{E/I,W} v_{W/N,W} \\ q_{E/I,W} v_{W/N,W} \end{bmatrix} + \begin{bmatrix} -\omega_E^2 (\bar{R}_E + h) \cos\lambda \, (\sin\lambda \cos\gamma \cos\sigma - \cos\lambda \sin\gamma) \\ -\omega_E^2 (\bar{R}_E + h) \cos\lambda \, (\sin\lambda (\sin\mu \sin\gamma \cos\sigma - \cos\mu \sin\sigma) + \cos\lambda \sin\mu \cos\gamma) \\ -\omega_E^2 (\bar{R}_E + h) \cos\lambda \, (\sin\lambda (\cos\mu \sin\gamma \cos\sigma + \sin\mu \sin\sigma) + \cos\lambda \cos\mu \cos\gamma) \end{bmatrix} \right) \tag{12}
$$

and

$$
\begin{bmatrix} \dot{\ell} \\ \dot{\lambda} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} \frac{1}{\bar{R}_E + h} & 0 & 0 \\ 0 & \frac{1}{(\bar{R}_E + h)\cos\ell} & 0 \\ 0 & 0 & -1 \end{bmatrix} C_{N \leftarrow W} \begin{bmatrix} v_{W/N,W} \\ 0 \\ 0 \end{bmatrix} \tag{13}
$$

where

$$
C_{N \leftarrow W} = C_{W \leftarrow N}^T (\mu, \gamma, \sigma) \tag{14}
$$

or

$$
\begin{bmatrix} \dot{\ell} \\ \dot{\lambda} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} \frac{v_{W/N,W} \cos\gamma \cos\sigma}{\bar{R}_E + h} & 0 & 0 \\ 0 & \frac{v_{W/N,W} \cos\gamma \sin\sigma}{(\bar{R}_E + h)\cos\ell} & 0 \\ 0 & 0 & v_{W/N,W} \sin\gamma \end{bmatrix} \tag{15}
$$

Thus, one has

$$
\begin{bmatrix} \dot{v}_{W/N,W} \\ v_{W/N,W} r_{W/I,W} \\ -v_{W/N,W} q_{W/I,W} \end{bmatrix} = \begin{bmatrix} \frac{T}{m}\cos\alpha - \frac{D}{m} - g\sin\gamma \\ g \sin\mu \cos\gamma \\ \frac{L}{m} + \frac{T}{m}\sin\alpha - g\cos\mu\cos\gamma \end{bmatrix} + \frac{1}{m}\vec{F}_{Earth,W} \tag{16}
$$

Next, substituting for the wind frame inertial angular velocity components with the navigation-to-wind frame Euler angles, i.e.

$$
\begin{bmatrix} p_{W/I,W} \\ q_{W/I,W} \\ r_{W/I,W} \end{bmatrix} = \begin{bmatrix} p_{W/N,W} \\ q_{W/N,W} \\ r_{W/N,W} \end{bmatrix} + \begin{bmatrix} p_{N/I,W} \\ q_{N/I,W} \\ r_{N/I,W} \end{bmatrix} \tag{17}
$$

where

$$
\begin{bmatrix} p_{W/N,W} \\ q_{W/N,W} \\ r_{W/N,W} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\gamma \\ 0 & \cos\mu & \sin\mu\cos\gamma \\ 0 & -\sin\mu & \cos\mu\cos\gamma \end{bmatrix} \begin{bmatrix} \dot{\mu} \\ \dot{\gamma} \\ \dot{\sigma} \end{bmatrix} \tag{18}
$$

$$
\begin{bmatrix} p_{N/I,W} \\ q_{N/I,W} \\ r_{N/I,W} \end{bmatrix} = C_{W \leftarrow N}(\mu, \gamma, \sigma) \begin{bmatrix} \left(\omega_E + \dot{\lambda}\right)\cos\ell \\ -\dot{\ell} \\ -\left(\omega_E + \dot{\lambda}\right)\sin\ell \end{bmatrix} \tag{19}
$$

Thus, one has

$$
\begin{bmatrix} p_{W/I,W} \\ q_{W/I,W} \\ r_{W/I,W} \end{bmatrix} = \begin{bmatrix} \dot{\mu} - \dot{\sigma}\sin\gamma \\ \dot{\sigma}\cos\mu\cos\gamma - \dot{\gamma}\sin\mu \\ \dot{\gamma}\cos\mu + \dot{\sigma}\sin\mu\cos\gamma \end{bmatrix} + \begin{bmatrix} p_{N/I,W} \\ q_{N/I,W} \\ r_{N/I,W} \end{bmatrix} \tag{20}
$$

and by substitution and rearrangement

$$\begin{bmatrix} \dot{v}_{B/N,W} \\ v_{W/N,W}\,(\dot{\sigma}\cos\mu\cos\gamma - \dot{\gamma}\sin\mu) \\ -v_{W/N,W}(\dot{\gamma}\cos\mu + \dot{\sigma}\sin\mu\cos\gamma) \end{bmatrix} = \begin{bmatrix} \frac{T\cos\alpha - D}{m} - g\sin\gamma \\ g\sin\mu\cos\gamma \\ -\frac{L+T\sin\alpha}{m} + g\cos\mu\cos\gamma \end{bmatrix} + \begin{bmatrix} \tilde{F}_{app,1} \\ \tilde{F}_{app,2} \\ \tilde{F}_{app,3} \end{bmatrix} \tag{21}$$

where, for convenience, the apparent normalized forces are combined as three terms

$$\begin{aligned} \begin{bmatrix} \tilde{F}_{app,1} \\ \tilde{F}_{app,2} \\ \tilde{F}_{app,3} \end{bmatrix} &= \begin{bmatrix} 0 \\ -r_{N/I,W}v_{W/N,W} \\ q_{N/I,W}v_{W/N,W} \end{bmatrix} + \begin{bmatrix} 0 \\ -r_{E/I,W}v_{W/N,W} \\ q_{E/I,W}v_{W/N,W} \end{bmatrix} \\ &+ \begin{bmatrix} -\omega_E^2(\bar{R}_E + h)\cos\lambda\,(\sin\lambda\cos\gamma\cos\sigma - \cos\lambda\sin\gamma) \\ -\omega_E^2(\bar{R}_E + h)\cos\lambda\,(\sin\lambda(\sin\mu\sin\gamma\cos\sigma - \cos\mu\sin\sigma) + \cos\lambda\sin\mu\cos\gamma) \\ -\omega_E^2(\bar{R}_E + h)\cos\lambda\,(\sin\lambda(\cos\mu\sin\gamma\cos\sigma + \sin\mu\sin\sigma) + \cos\lambda\cos\mu\cos\gamma) \end{bmatrix} \end{aligned} \tag{22}$$

Multiplying the second term by $\sin\mu$ and the third term by $\cos\mu$ and adding, one has

$$\begin{aligned} &v_{W/N,W}\left(\dot{\sigma}\cos\mu\cos\gamma\sin\mu - \dot{\gamma}\sin^2\mu\right) - v_{W/N,W}(\dot{\gamma}\cos^2\mu + \dot{\sigma}\sin\mu\cos\gamma\cos\mu) \\ &\quad = g\sin^2\mu\cos\gamma + \tilde{F}_{app,2}\sin\mu - \frac{L + T\sin\alpha}{m}\cos\mu + g\cos^2\mu\cos\gamma + \tilde{F}_{app,3}\cos\mu \end{aligned} \tag{23}$$

or

$$v_{W/N,W}\dot{\gamma} = \frac{L + T\sin\alpha}{m}\cos\mu - g\cos\gamma - \left(\tilde{F}_{app,2}\sin\mu + \tilde{F}_{app,3}\cos\mu\right) \tag{24}$$

Similarly, multiplying the second term by $\cos\mu$ and the third term by $\sin\mu$ and subtracting, one has

$$\begin{aligned} &v_{W/N,W}\left(\dot{\sigma}\cos^2\mu\cos\gamma - \dot{\gamma}\sin\mu\cos\mu\right) + v_{W/N,W}(\dot{\gamma}\sin\mu\cos\mu + \dot{\sigma}\sin^2\mu\cos\gamma\cos\mu) \\ &\quad = g\sin\mu\cos\gamma\cos\mu + \tilde{F}_{app,2}\cos\mu + \frac{L + T\sin\alpha}{m}\sin\mu - g\cos\mu\cos\gamma\sin\mu \\ &\quad - \tilde{F}_{app,3}\sin\mu \end{aligned} \tag{25}$$

or

$$v_{W/N,W}\dot{\sigma}\cos\gamma = \frac{L + T\sin\alpha}{m}\sin\mu + \left(\tilde{F}_{app,2}\sin\mu - \tilde{F}_{app,3}\sin\mu\right) \tag{26}$$

Therefore, one has

$$\begin{bmatrix} \dot{v}_{W/N,W} \\ \dot{\gamma} \\ \dot{\sigma} \end{bmatrix} = \begin{bmatrix} \frac{T\cos\alpha - D}{m} - g\sin\gamma + \tilde{F}_{app,1} \\ \frac{1}{v_{W/N,W}}\left(\frac{L+T\sin\alpha}{m}\cos\mu - g\cos\gamma - \left(\tilde{F}_{app,2}\sin\mu + \tilde{F}_{app,3}\cos\mu\right)\right) \\ \frac{1}{v_{W/N,W}\cos\gamma}\left(\frac{L+T\sin\alpha}{m}\sin\mu + \left(\tilde{F}_{app,2}\sin\mu - \tilde{F}_{app,3}\sin\mu\right)\right) \end{bmatrix} \tag{27}$$

Here one can see how the thrust is used to control the velocity of the airplane, the magnitude of the lift force is used to control the flight path, and the direction of the lift force is used to control the heading, albeit with some coupling and Earth frame effects between these inputs. Here, the commanded lift magnitude, L_c, can be directly related to the commanded angle of attack, α and the commanded heading change, $\dot{\sigma}$, is related to the commanded bank angle, μ_c.

To capture the inner-loop attitude control dynamics, one can approximate their responses with first-order transfer functions, i.e.,

$$\frac{T(s)}{T_c(s)} = \frac{\omega_T}{s+\omega_T} \tag{28}$$

$$\frac{L(s)}{L_c(s)} = \frac{\omega_L}{s+\omega_L} \tag{29}$$

$$\frac{\mu(s)}{\mu_c(s)} = \frac{\omega_\mu}{s+\omega_\mu} \tag{30}$$

where ω_T, ω_L, and ω_μ are all frequencies selected to approximate the responses of the thrust, lift magnitude, and banking attitude. This simulation should also include limits to these responses as

$$0 \leq T \leq T_{max} \tag{31}$$

$$L \leq K_{L,max} v_{W/N,W}^2 \tag{32}$$

and

$$-\mu_{max} \leq \mu \leq \mu_{max} \tag{33}$$

whose values will generally be airplane specific.

For the aerodynamic lift and drag, one can use the following equations

$$L = Q_\infty S_w C_L \tag{34}$$

$$D = Q_\infty S_w C_D \tag{35}$$

where S_w is the wing reference area, Q_∞ is the free-stream dynamic pressure and can be modeled as

$$Q_\infty = \frac{1}{2}\rho_\infty v_\infty^2 \tag{36}$$

where ρ_∞ is the air density of the free-stream, C_L is the lift coefficient,

$$C_L = C_{L_\alpha}(\alpha - \alpha_0) \tag{37}$$

where C_{L_α} is the lift-curve slope, α_0 is the angle of attack, α_0 is the zero-lift angle of attack, C_D is the drag coefficient which is combination of parasitic drag and induced drag

$$C_D = C_{D,0} + \frac{1}{\pi AR_w e_{eff}} C_L^2 \tag{38}$$

$C_{D,0}$ is the parasitic drag, also known as the profile drag, AR_w is the aspect ratio of the wing, and e_{eff} is the effective efficiency of the aircraft.

Furthermore, for the 1962 standard atmosphere model, one has for the air density

$$\rho_\infty = \begin{cases} (6.6277\times10^{-15})\left(518.69 - (3.5662\times10^{-3})h\right)^{4.256} & \text{sl/ft}^3 & 0 < h < 36,089 \text{ ft} \\ (1.4939\times10^{-6})(2678.4)\exp\left((-4.8063\times10^{-5})h\right) & \text{sl/ft}^3 & 36,089 \leq h < 65,617 \text{ ft} \\ (2.2099\times10^{87})\left(389.99 + (5.4864\times10^{-4})(h - 65617)\right)^{-35.164} & \text{sl/ft}^3 & 65,617 \leq h < 104,990 \text{ ft} \end{cases} \tag{39}$$

However, in this simulation, if one wishes to use the lift as a control variable, these equations can be inverted to find the drag and angle of attack for a given lift by the equations

$$D = \frac{1}{2}\rho_\infty S_w C_{D,0} \|v_\infty\|_2^2 + \left(\frac{2}{\rho_\infty S_w \pi AR_w e_{eff}}\right) \frac{L^2}{\|v_\infty\|_2^2} \tag{40}$$

$$\alpha = \left(\frac{2}{\rho_\infty S_w C_{L_\alpha}}\right) \frac{L}{\|v_\infty\|_2^2} + \alpha_0 \tag{41}$$

For this project, assume the following subsonic airplane parameters are given:

$mg_0 = 157,000 - 327,000$ lbs	Airspeed range: 200-600 mph	$K_{\dot{f}} = 4 \times 10^{-6}$ sl/(lb-s)
$\omega_T = 2$ rad/s	$\omega_L = 2.5$ rad/s	$\omega_\mu = 1$ rad/s
$T_{max} = 72,000$ lb	$K_{L,max} = 2.6$ lb-s^2/ft^2	$\mu_{max} = 30°$
$C_{D,0} = 0.0183$	$C_{L_\alpha} = 0.0920/°$	$\alpha_0 = -0.05°$
$S_w = 1,745$ft^2	$AR_w = 10.1$	$e_{eff} = 0.613$

Project Assignment and Deliverables

For this project, determine in MATLAB/Simulink using the Simulink model provided:

a) Build the Simulink subsystem for solving the "Vehicle Dynamics" using Equations 2, 3, 6, 15, 27, 40, and 41 ***assuming***:

- $T \cos\alpha = T$, $T \sin\alpha = 0$, and $\vec{F}_{app} = \vec{0}$ in Equation 27; and
- the acceleration due to gravity and air density is constant.
- The inputs are the following:
    - T
    - L
    - μ
- Your "Vehicle Dynamics" subsystem is broken into the five further subsystems with the various required inputs and with outputs for
    - m;
    - $v_{W/N,W}$;
    - γ;
    - σ; and
    - ℓ, λ, h, and $\|\vec{v}_\infty\|_2$.

b) Build the Simulink subsystem for the "Guidance & Control System."

- The "Guidance & Control System" inputs are the following:
    - m;
    - $v_{B/N,W,c}$;
    - $v_{W/N,W}$;
    - γ_c;

* γ;
* $\|\vec{v}_\infty\|_2$;
* σ_c; and
* and σ;

– parallel PI guidance system for T_c, L_c, and μ_c with auxiliary commands for α_c and h_c:

$$\frac{T_c(s)}{v_{B/N,W,c}(s) - v_{W/N,W}(s)} = \frac{m\left(K_{T_p}s + K_{T_i}\right)}{s} \tag{42}$$

$$\frac{L_c(s)}{v_{B/N,W,c}(\sin\gamma_c(s) - \sin\gamma(s))} = \frac{m\left(K_{L_p}s + K_{L_i}\right)}{s} \tag{43}$$

$$\frac{\mu_c(s)}{(\sigma_c(s) - \sigma(s))} = K_{\mu,p}\frac{v_{B/N,W,c}}{g} \tag{44}$$

$$\alpha_c = \left(\frac{2}{\rho_\infty S_w C_{L_\alpha}}\right)\frac{L_c}{\|v_\infty\|_2^2} + \alpha_0 \tag{45}$$

$$h_c = \left(v_{B/N,W,c}\sin\gamma_c\right)t + h_0 \tag{46}$$

– where $K_{T_p} = 0.08$ s, $K_{T_i} = 0.002/\text{s}^2$, $K_{L_p} = 0.5/\text{s}$, $K_{L_i} = 0.01/\text{s}^2$, and $K_{\mu_p} = 0.075$ s

– the “Guidance & Control System” outputs the following by Equations 28, 29, and 30, respectively:
  * thrust;
  * lift; and
  * heading.

– include “Saturation” blocks in your Simulink model as limiters to the outputs as shown in Equations 31, 32, and 33

c) Simulate the Simulink model completed with parts a) and b) for 5 minutes

- for initial conditions
  * $v_{B/N,W,0} = 600$ ft/s
  * $\gamma_0 = 0°$
  * $\sigma_0 = 0°$
  * $h_0 = 20,000$ ft
  * $\ell_0 = 33.2098°$
  * $\lambda_0 = -87.5692°$
  * $\vec{v}_{wind,N} = [40\ 40\ 0]^T$ ft/s
  * $g = g_0 = 32.17$ ft/s^2 $\forall\ t$
  * $\rho = 2.3769 \times 10^{-3}\ \forall\ t$
  * $mg_0 = 200,000$ lbs
  * L_0, D_0, and T_0 should balance Equation 27 for these conditions.
- commanded trajectory
  * $v_{B/N,W,c} = 660$ ft/s
  * $\gamma_c = 5°$
  * $\sigma_c = 15°$
- and output time history plots of
  * $v_{B/N,W,c}$, $v_{W/N,W}$, and $\|\vec{v}_\infty\|_2$
  * γ_c and γ
  * σ_c and σ
  * α_c, α, and α_{max}
  * h_c and h

* λ vs. ℓ

* T, D, and T_{max}

* μ and μ_{max}

d) Build a second Simulink model with a higher fidelity subsystem for solving the "Vehicle Dynamics" using Equations 6, 15, 2, 3, 27, 40, and 41 ***including***:

   – all terms and the change in air density as a function of h.

e) Simulate the second Simulink model completed with parts b) and d) for the initial conditions in part c) and compare to the simulation output plots from part c).

Deliver: in the Blackboard assignment, all files to run your MATLAB script(s) and Simulink model(s). There is no need to zip your files.