Lecture 24: Time-Varying Systems Theory and Control

Textbook Section 6.1 & 6.2

Dr. Jordan D. Larson

Introduction

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  - Utilize linearization dependence on non-state flight conditions, i.e. parameters
  - Requires additional dynamical systems theory
- Introduces concepts used in nonlinear systems as well

Time-Varying Dynamical System

- **Time-varying dynamical system**:

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- **Initial value problem (IVP)**: system at time $t_0 \geq 0$ as some $\vec{x}(t_0) = \vec{x}_0 \in \mathbb{R}^n$
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  - May have many solutions, one unique solution, or no existing solution
- Existence and uniqueness of solutions to IVPs for non-LTI dynamical systems not always guaranteed

Existence and Uniqueness of Solutions

- **Cauchy-Peano Theorem**: sufficient conditions for IVP to admit solution, may not be unique:
  If, for some $T > 0$, some $\epsilon > 0$, and $f(t, \vec{x}) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ continuous in closed region, i.e.

$$B = \{(t, \vec{x}) : |t - t_0| \leq T, \|\vec{x} - \vec{x}_0\| \leq \epsilon\} \subseteq \mathbb{R} \times \mathbb{R}^n \tag{3}$$

  Then, exists $t_0 < t_1 << T$ s.t. IVP has at least one continuously differentiable solution $\vec{x}(t)$ on interval $[t_0, T]$
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  - Does not guarantee uniqueness of solution
- Key constraint that yields uniqueness: **Lipschitz condition** whereby $f(t, \vec{x})$ satisfies inequality

$$\|f(t, \vec{x}) - f(t, \vec{y})\| \leq L\|\vec{x} - \vec{y}\| \tag{4}$$

  for all $(t, \vec{x})$ and $(t, \vec{y})$ in some "neighborhood" of $(t_0, \vec{x}_0)$ with finite constant $L > 0$

Existence and Uniqueness of Solutions (continued)

- Sufficient conditions for IVP to admit *local* existence and uniqueness of solution: If, for some $\epsilon > 0$, $f(t, \vec{x}) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ piece-wise continuous in t and satisfies Lipschitz condition, i.e.

$$\forall \vec{x}, \vec{y} \in B = \{\vec{x} \in \mathbb{R}^n : \|\vec{x} - \vec{x}_0\| \leq \epsilon\},\ \forall t \in [t_0, t_1] \tag{5}$$

then, exists some $\delta > 0$ such that IVP for state equation $\dot{\vec{x}} = f(t, \vec{x})$ with $\vec{x}(t_0) = \vec{x}_0$ has unique solution over $[t_0, t_0 + \delta]$.

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$$\|f(t, \vec{x}) - f(t, \vec{y})\| \leq L\|\vec{x} - \vec{y}\|,\ \ \forall\ \vec{x}, \vec{y} \in \mathbb{R}^n,\ \forall\ t \in [t_0, t_1] \tag{6}$$

  then, IVP has unique solution over $[t_0, t_1]$ where final time t_1 may be arbitrarily large

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- Define: new time as $\tau = t - t_0$ and new state as $\vec{z}(\tau) = \vec{x}(\tau + t_0) - \bar{\vec{x}}$

$$\frac{d\vec{z}(\tau)}{d\tau} = \frac{\vec{x}(\tau + t_0)}{dt} = f(\tau + t_0, \vec{z}(\tau) + \bar{\vec{x}}) = g(\tau, \vec{z}(\tau)) \tag{9}$$

  - With $g(0, \vec{0}) = f(t_0, \bar{\vec{x}}) = 0$
  - Can shift equilibrium point to origin and initial time to zero

Zero and Origin as Arbitrary Equilibrium

- Suppose one has state trajectory $\vec{x}(t)$ that starts at $t = t_0$, i.e.

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  - With $g(0, \vec{0}) = \vec{0}$
- Analyzing these new dynamics around origin as equilibrium point while starting at t_0: determine original system behavior around original nonzero equilibrium $\vec{\bar{x}}$
  - I.e. can assess system relative dynamics w.r.t. any time-dependent trajectory $\vec{\bar{x}}(t)$, starting at arbitrary initial time $t_0 \geq 0$
  - Key for stability analysis

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- System stability can be interpreted as continuity of system trajectories, w.r.t. initial conditions, over *infinite* time interval
- Infinite time interval highlights primary notion of stability as continuity property of Lipschitz-continuous differential equations holding infinitely in time
  - Let $\vec{x}(t, \vec{x}_0)$ define unique solution of $\dot{\vec{x}} = f(t, \vec{x})$ with initial condition $\vec{x}(t_0) = \vec{x}_0$ which exists on finite, possibly open-ended interval $[t_0, T)$
- Continuity property of $\vec{x}(t, \vec{x}_0)$ due to changes in $\vec{x}_0$:
  - Given any constant $\epsilon > 0$, there must exist sufficiently small constant $\delta > 0$ such that for all perturbed initial conditions $\vec{x}_0 + \Delta\vec{x}_0$ with $\|\Delta\vec{x}_0\| \leq \delta$
    Corresponding perturbed solution $\vec{x}(t, \vec{x}_0 + \Delta\vec{x}_0)$ deviates from original $\vec{x}_0$ by no more than ϵ
    - I.e. $\|\vec{x}(t, \vec{x}_0 + \Delta\vec{x}_0) - \vec{x}(t, \vec{x}_0)\| \leq \epsilon$, for all $t_0 \leq t < T$

Lyapunov Stability of Equilibrium Points

- **Lyapunov stability of equilibrium point**, $\vec{x} = \vec{0}$, i.e. origin, for time-varying unforced dynamics:
  - *Stable*: if for any $\epsilon > 0$ and $t \geq 0$ there exists some $\delta(\epsilon, t_0) > 0$ such that for all initial conditions $\|\vec{x}_0\| < \delta$ and for all $t \geq t_0 \geq 0$, the corresponding system trajectories bounded, i.e. $\|\vec{x}(t)\| < \epsilon$
  - Otherwise, *unstable*
  - Given outer "hyper-sphere" $B_\epsilon = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| \leq \epsilon\}$, can find inner "hyper-sphere" $B_\delta = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| \leq \delta\}$, such that any trajectory that starts inside B_δ will evolve inside B_ϵ for *all* future times

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- Nonlinear dynamical systems may display completely different behavior in various domains of state: distinguish between local and global stability
  - Equilibrium point, $\vec{x} = \vec{0}$, i.e. origin, has **global stability** if stable and $\lim_{\epsilon \to \infty} \delta(\epsilon, t_0) = \infty$
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- System trajectories of time-varying dynamical systems depend on initial time t_0
  - Stability of equilibrium point for time-varying systems may depend on t_0
  - Equilibrium point has **uniform stability** if stable and δ does not depend on t_0

Stricter Lyapunov Stability of Equilibrium Points

- Equilibrium point, $\vec{x} = \vec{0}$, i.e. origin, has **asymptotic stability**:
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  If uniformly asymptotically stable and $\lim_{\epsilon \to \infty} \delta(\epsilon) = \infty$
- Uniform asymptotic stability typically highly desirably property of control system design: able to maintain their closed-loop performance in presence of state perturbations and disturbances

Lyapunov Indirect and Direct Methods

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- **Lyapunov's direct method** requires concepts:
  - Positive and negative definite functions
  - Time derivative of scalar function along state trajectories of differential equation, i.e. possible solutions

Positive and Negative Definite Functions

- $V(\vec{x}) : \mathbb{R}^n \to \mathbb{R}$ of vector argument $\vec{x} \in \mathbb{R}^n$: **locally positive definite**
  If $V(\vec{0}) = 0$ and there exists constant ϵ such that $V > 0$ for all $\vec{x} \in \mathbb{R}^n$ in neighborhood of origin, i.e. $B_\epsilon = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| \leq \epsilon\}$
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  If $V(\vec{0}) = 0$ and there exists constant ϵ such that $V \leq 0$ for all $\vec{x} \in \mathbb{R}^n$ in neighborhood of origin, i.e. $B_\epsilon = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| \leq \epsilon\}$
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  - $\nabla V(\vec{x}) = [\frac{\partial V}{\partial x_1}, ..., \frac{\partial V}{\partial x_n}]$: row vector gradient of $V(\vec{x})$ w.r.t. $\vec{x}$
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- Lyapunov direct method: let $\bar{\vec{x}} = \vec{0} \in \mathbb{R}^n$ as equilibrium point for time-varying dynamics, initial conditions drawn from domain $D \subset \mathbb{R}^n$ with $\bar{\vec{x}} \in D$ and $t_0 = 0$

Lyapunov Direct Method

- If on domain D, there exists continuously differentiable locally positive definite function $V(\vec{x}) : D \to \mathbb{R}$, whose time derivative along system trajectories locally negative semi-definite, i.e.

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for all $t \geq 0$ and for all $\vec{x} \in D$, then $\vec{\bar{x}} = \vec{0}$ locally uniformly stable

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    - If $\dot{V}(\vec{x}) < 0$ for all $t \geq 0$, i.e. time derivative along the system trajectories locally negative definite, then $\vec{\bar{x}} = \vec{0}$ locally uniformly asymptotically stable
- Any locally positive definite $V(\vec{x})$: **Lyapunov function candidate**, if satisfies time derivative condition: **Lyapunov function**
    - Existence of Lyapunov function sufficient to claim uniform stability for equilibrium point, if one cannot be found, nothing can be stated about stability of equilibrium point
    - Lyapunov functions not unique

Lyapunov Functions and Definitions

- Lyapunov function viewable as "energy-like" function for testing system stability
  - If values of V do not increase along system trajectories, then origin uniformly stable
  - If V strictly decreases, then, in addition, system trajectories will approach origin asymptotically
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- Define $\Omega_c = \{\vec{x} \in \mathbb{R}^n : V(\vec{x}) \leq c\}$ as union of interior set of V_c and V_c itself

Lyapunov Functions (continued)

- Consider converging sequence $\lim_{k\to\infty} \vec{x}_k = \vec{a}$ with all $\vec{x}$ from Ω_c, then limit point $\vec{a}$ must also be in Ω_c

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  - Ω_c: **closed set**
    - Since $V(\vec{x})$ continuous on $\mathbb{R}^n$ and $V(\vec{x}) \leq c$ for all $k = 1, 2, \ldots \to c \geq \lim_{k\to\infty} V(\vec{x}_k) = V(\vec{a})$, and consequently $\vec{a} \in \Omega_c$: every converging sequence has limit point in same set

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- **Krasovskii-LaSalle theorem**:
  If $\vec{x} = \vec{0}$: equilibrium point of $\dot{\vec{x}} = f(t, \vec{x})$ and $V(\vec{x}) : \mathbb{R}^n \to \mathbb{R}$: radially unbounded Lyapunov function
  Then $\vec{x}$ globally uniformly asymptotically stable equilibrium point
  - Simple example of radially unbounded Lyapunov function candidates include quadratic form $V(\vec{x}) = \vec{x}^T P \vec{x}$ where P symmetric positive definite matrix, i.e. $P = P^T > 0$

Linear, Time-Varying Dynamical System

- **Linear, time-varying (LTV) systems**:

$$\begin{aligned}\dot{\vec{x}}(t) &= A(t)\vec{x}(t) + B(t)\vec{u}(t)\\ \vec{y}(t) &= C(t)\vec{x}(t) + D(t)\vec{u}(t)\end{aligned} \tag{14}$$

- Similar to LTI, general solution of LTV/LPV system

$$\vec{x}(t) = \Phi(t, t_0)\vec{x}(t_0) + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)\vec{u}(\tau)d\tau \tag{15}$$

  - $\Phi(t, t_0)$: state transition matrix of LTV/LPV system

$$\Phi(t, t_0) = I + \int_{\tau}^{t} A(\sigma_1)d\sigma_1 + \int_{\tau}^{t} A(\sigma_1)\int_{\tau}^{\sigma_1} A(\sigma_2)d\sigma_2 d\sigma_1 + \cdots \tag{16}$$

  - $\Phi(t, t_0)$ converges uniformly and absolutely to solution that exists and unique

Lyapunov Stability of LTV Systems

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$$V(\vec{x}) = \vec{x}^T P \vec{x} \tag{17}$$

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  for all t
- Stable LTV systems: globally uniformly asymptotically stable

Controllability and Observability

- **Linear, time-varying controllability Gramian** using state-transition matrix:

$$W_C(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, t)B(t)B(t)^T \Phi(t_0, t)^T dt \tag{21}$$

  - $W_C(t_0, t_1)$: symmetric, positive semi-definite, satisfies

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Linear, Parameter-Varying Systems

- Special class of LTV systems: **linear, parameter-varying (LPV) systems**:

$$\begin{aligned}\dot{\vec{x}}(t) &= A(\vec{\beta}(t))\vec{x}(t) + B(\vec{\beta}(t))\vec{u}(t) \\ \vec{y}(t) &= C(\vec{\beta}(t))\vec{x}(t) + D(\vec{\beta}(t))\vec{u}(t)\end{aligned} \tag{25}$$

  - $\vec{\beta}(t)$: **parameter vector** must be continuously differentiable function of time
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  - Typically arise when linearizing nonlinear system about equilibrium specified by $\vec{\beta}$
- $\vec{\beta}(t)$ typically also has restricted admissible trajectories for any particular realization
  - Typically at least restricted by some upper and lower bounds, e.g. $\vec{\beta}_U$ and $\vec{\beta}_L$
  - May also be rate restricted by some upper and lower bounds, e.g. $\dot{\vec{\beta}}_U$ and $\dot{\vec{\beta}}_L$
  - $\mathcal{B}$: set of admissible trajectories for $\vec{\beta}$

$$\begin{aligned}\mathcal{B} = \Big\{ &\vec{\beta}(t) : \mathbb{R}_+ \to \mathbb{R}^{n_\beta} \text{ such that: } \vec{\beta}(t) \text{ continuously differentiable,} \\ &\vec{\beta}_L \leq \vec{\beta}(t) \leq \vec{\beta}_U \; \forall\, t \geq 0, \;\; \dot{\vec{\beta}}_L \leq \dot{\vec{\beta}}(t) \leq \dot{\vec{\beta}}_U \; \forall\, t \geq 0 \Big\}\end{aligned} \tag{26}$$

LPV Control Design Approaches

- Common LPV control design approach: **gain scheduling**
  - Compute array of optimal LTI controllers at variety of *different* equilibrium points across ranges of admissible $\vec{\beta}$, e.g. obtain *different* optimal LTI feedback gain matrices
  - Blend various LTI controllers to obtain single, integrated control design, e.g. interpolate between different optimal LTI feedback gain matrices if operating away from equilibrium points
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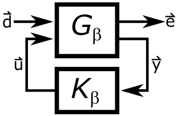
- More advanced approaches for LPV systems proposed based on specific structure of state-space matrices, e.g. linear or rational dependence on $\vec{\beta}$
- For arbitrary dependence, specify performance of LPV system, G_β in terms of maximum possible induced $\mathcal{L}_{2\leftarrow 2}$ gain from input $\vec{u}$ to output $\vec{y}$, i.e.

$$\|G_\beta\|_{2\leftarrow 2} = \max_{\vec{\beta}\in\mathcal{B},0\neq\|\vec{u}\|_2\leq\infty,\vec{x}(0)=0} \frac{\|\vec{y}\|_2}{\|\vec{u}\|_2} \tag{27}$$

State Feedback Linear, Parameter-Varying OCP



- LPV plant:

$$\begin{aligned} \dot{\vec{x}}(t) &= A(\vec{\beta})\vec{x}(t) + \begin{bmatrix} B_1(\vec{\beta}) & B_2(\vec{\beta}) \end{bmatrix} \begin{bmatrix} \vec{d}(t) \\ \vec{u}(t) \end{bmatrix} \\ \begin{bmatrix} \vec{e}(t) \\ \vec{y}(t) \end{bmatrix} &= \begin{bmatrix} C_1(\vec{\beta}) \\ I \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 & D_{12}(\vec{\beta}) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{d}(t) \\ \vec{u}(t) \end{bmatrix} \end{aligned} \tag{28}$$

- State feedback LPV controller:

$$\vec{u} = D_K(\vec{\beta})\vec{x} \tag{29}$$

  - Could also use output feedback or observer feedback

Generalized Bounded Real Lemma

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  - **Generalized bounded real lemma**: derive sufficient conditions for upper bound on $\|G_\beta\|_{2\leftarrow 2}$
- Rate-unbounded LPV system globally uniformly asymptotically stable and $\|G_\beta\|_{2\leftarrow 2} \leq \gamma$:
  If there exists $P > 0$ such that $\forall \vec{\beta} \in [\vec{\beta}_L, \vec{\beta}_U]$:

$$\begin{bmatrix} A^T(\vec{\beta})P + PA(\vec{\beta}) & PB(\vec{\beta}) \\ B^T(\vec{\beta})P & -I \end{bmatrix} + \frac{1}{\gamma^2} \begin{bmatrix} C^T(\vec{\beta}) \\ D^T(\vec{\beta}) \end{bmatrix} \begin{bmatrix} C(\vec{\beta}) & D(\vec{\beta}) \end{bmatrix} < 0 \tag{30}$$

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- To show $\|G_\beta\|_{2\leftarrow 2} \leq \gamma$: left and right multiply by $[\vec{x}^T \ \vec{u}^T]^T$:

$$\dot{\vec{x}}^T P\vec{x} + \vec{x}^T P\dot{\vec{x}} + \frac{1}{\gamma^2}\vec{y}^T\vec{y} - \vec{u}^T\vec{u} \leq 0 \tag{31}$$

$$\dot{V} + \frac{1}{\gamma^2}\vec{y}^T\vec{y} - \vec{u}^T\vec{u} \leq 0 \tag{32}$$

Generalized Bounded Real Lemma (continued)

- Integrate from $t = 0$ to $t = t_f$:

$$V(\vec{x}(t_f)) - V(\vec{x}(0)) + \frac{1}{\gamma^2}\int_0^{t_f} \vec{y}^T\vec{y}\,dt - \int_0^{t_f} \vec{u}^T\vec{u}\,dt \leq 0 \tag{33}$$

Generalized Bounded Real Lemma (continued)

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- As $t_f \to \infty$:

$$\|\vec{y}\|_2^2 \leq \gamma^2 \|\vec{u}\|_2^2 \tag{35}$$

$$\|G_\beta\|_{2\leftarrow 2} \leq \gamma \tag{36}$$

Generalized Bounded Real Lemma (continued)

- Matrix inequality as parameterized LMI condition: one LMI for each value of $\vec{\beta} \in \mathcal{B}$
  - In practice: infinite LMI conditions approximated by enforcing only on finite grid of points
  - Finite dimensional LMI conditions directly obtained to bound $\|G_\beta\|_{2\leftarrow 2}$ without approximation if state matrices have rational dependence on $\vec{\beta}$

Generalized Bounded Real Lemma (continued)

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  - Finite dimensional LMI conditions directly obtained to bound $\|G_\beta\|_{2\leftarrow 2}$ without approximation if state matrices have rational dependence on $\vec{\beta}$
- Rate-bounded LPV system: alter matrix inequality with variation on Lyapunov theory
  - Check 2^{n_β} LMI conditions evaluated at endpoints defined by hypercube of elements of $\dot{\vec{\beta}}_U$ and $\dot{\vec{\beta}}_L$
  - Also need to search over infinitely dimensional space of functions, $P(\beta)$, must be restricted to finite dimensional subspace
  - In practice: specify collection of scalar basis functions, $g_i(\vec{\beta})$, and use linear combination

$$P(\vec{\beta}) = \sum_{i=1}^{N} g_i(\vec{\beta}) P_i \tag{37}$$

  - P_i: symmetric matrices that form finite collection of decision variables, often polynomials chosen as basis functions

State Feedback LPV Optimal Control

- Closed-loop LPV system:

$$\begin{aligned}\dot{\vec{x}}(t) &= \left(A(\vec{\beta}) + B_2(\vec{\beta})D_K(\vec{\beta})\right)\vec{x}(t) + B_1(\vec{\beta})\vec{d}(t)\\ \vec{e}(t) &= \left(C_1(\vec{\beta}) + D_{12}(\vec{\beta})D_K(\vec{\beta})\right)\vec{x}(t)\end{aligned} \tag{38}$$

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- By generalized bounded real lemma, closed-loop LPV system internally stable and $\|G_\beta\|_{2\leftarrow 2} \leq \gamma$ if there exists $P > 0$ such that $\forall \vec{\beta} \in [\vec{\beta}_L, \vec{\beta}_U]$:

$$\begin{bmatrix} A_L^T(\vec{\beta})P + PA_L(\vec{\beta}) & PB_L(\vec{\beta}) \\ B_L^T(\vec{\beta})P & -I \end{bmatrix} + \frac{1}{\gamma^2}\begin{bmatrix} C_L^T(\vec{\beta}) \\ D_L^T(\vec{\beta}) \end{bmatrix}\begin{bmatrix} C_L(\vec{\beta}) & D_L(\vec{\beta}) \end{bmatrix} < 0 \tag{39}$$

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- Use change of variables:

$$Q = P^{-1} \tag{40}$$

$$R(\vec{\beta}) = D_K(\vec{\beta})Q \tag{41}$$

State Feedback LPV Optimal Control (continued)

- By Schur Complement Lemma:

$$\begin{bmatrix} QA^T(\vec{\beta}) + A(\vec{\beta})Q + R^T(\vec{\beta})B_2^T(\vec{\beta}) + B_2(\vec{\beta})R(\vec{\beta}) & B_1(\vec{\beta}) & (C_1(\vec{\beta})Q + D_{12}(\vec{\beta})R(\vec{\beta}))^T \\ B_1^T(\vec{\beta}) & \gamma^{-2}I & 0 \\ C_1(\vec{\beta})Q + D_{12}(\vec{\beta})R(\vec{\beta}) & 0 & -I \end{bmatrix} < 0 \tag{42}$$

State Feedback LPV Optimal Control (continued)

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- State feedback LPV OCP as SDP in (Q, R, γ):

$$(Q, R(\vec{\beta}), \gamma)^{opt} = \underset{Q \in \mathbb{S}^{n_x}, R \in \mathbb{R}^{n_u \times n_x}, \gamma > 0}{\text{argmin}} \quad \gamma$$

$$\underline{\text{subject to:}} \quad \begin{bmatrix} QA^T(\vec{\beta}) + A(\vec{\beta})Q + R^T(\vec{\beta})B_2^T(\vec{\beta}) + B_2(\vec{\beta})R(\vec{\beta}) & B_1(\vec{\beta}) & (C_1(\vec{\beta})Q + D_{12}(\vec{\beta})R(\vec{\beta}))^T \\ B_1^T(\vec{\beta}) & -\gamma^{-2}I & 0 \\ C_1(\vec{\beta})Q + D_{12}(\vec{\beta})R(\vec{\beta}) & 0 & -I \end{bmatrix} < 0 \quad (43)$$

$$Q > 0$$

State Feedback LPV Optimal Control (continued)

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- State feedback LPV optimal controller as $\vec{u}(t) = D_K(\vec{\beta})\vec{x}(t)$ reconstructed with

$$D_K(\vec{\beta}) = R(\vec{\beta})Q^{-1} \tag{44}$$

  - Solved approximately by gridding over $[\vec{\beta}_L, \vec{\beta}_U]$

Gain-Scheduling

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Summary

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  - Need Lyapunov direct stability method for various definitions of stability for equilibrium points

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- Non-LTI systems requires additional dynamical systems theory
    - Need to show existence and uniqueness: Lipschitz sufficient
    - Need Lyapunov direct stability method for various definitions of stability for equilibrium points
- Linear, Time-Varying (LTV) systems exhibit excellent stability properties using Lyapunov theory
    - General solution, controllability, and observability analysis similar to LTI systems using state-transition matrix