Lecture 23: Introduction to $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Optimal Control

Textbook Sections 4.4 & 4.5

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Introduction

- LTI optimal control: minimize some objective
  - Example: minimize state error and control effort simultaneously

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  - Example: minimize state error and control effort simultaneously
- $\mathcal{H}_2$-norm: average of singular values averaged across all frequencies
  - I.e. for all frequencies, average gain of system
- $\mathcal{H}_2$ optimal control: minimize weighted average of control effort and error for all time, i.e. all frequencies

$\mathcal{H}_2$ Optimal Control Objective

- Find K that minimizes $\mathcal{H}_2$-norm of generalized feedback control system:

$$K^{opt} = \underset{K \text{ stabilizing}}{\operatorname{argmin}} \; \|F_L(G,K)\|_2 \tag{1}$$

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- $F_L(G, K)$ defines state-space model:

$$\begin{aligned}
\dot{\vec{x}}(t) &= A\vec{x}(t) + B_1\vec{d}(t) + B_2\vec{u}(t) \\
\vec{e}(t) &= C_1\vec{x}(t) + D_{12}\vec{u}(t) \\
\vec{y}(t) &= C_2\vec{x}(t) + D_{21}\vec{d}(t)
\end{aligned} \tag{2}$$

  - $D_{11} = 0$ for finite $\mathcal{H}_2$-norm and $D_{22} = 0$ without loss of generality for ease of notation
  - For solution to $\mathcal{H}_2$ OCP to exist, (A, B_2) must be stabilizable and (A, C_2) must be detectable

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- **Regular $\mathcal{H}_2$ OCP** additionally assumes:
  (1) For LTI plant state-space model (A, B_2, C_1, D_{12}): no zeros on $j\omega$ axis and $(D_{12}^T D_{12})^{-1}$ exists, i.e. injective
  (2) For LTI plant state-space model (A, B_1, C_2, D_{21}): no zeros on $j\omega$ axis and $(D_{21} D_{21}^T)^{-1}$ exists, i.e. surjective

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- For regular $\mathcal{H}_2$ OCPs: can find *unique* solution, whereas sub-optimal solution must be found for singular $\mathcal{H}_2$ OCPs using semidefinite programming

Alternative $\mathcal{H}_2$ Optimal Control

- By definition of $\mathcal{H}_2$-norm in time domain with $\vec{d}(t)$ constrained as impulse inputs, alternatively state $\mathcal{H}_2$ cost function as

$$\mathcal{J}(\vec{x}, \vec{u}) = \frac{1}{2} \int_0^\infty \vec{e}^{\,T} \vec{e} \, d\tau \tag{3}$$

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- By substitution:

$$\mathcal{J}(\vec{x}, \vec{u}) = \frac{1}{2} \int_0^\infty \left(C_1 \vec{x}(t) + D_{12} \vec{u}(t)\right)^T \left(C_1 \vec{x}(t) + D_{12} \vec{u}(t)\right) d\tau \tag{4}$$

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- More generalized form of infinite-horizon continuous-time LQ OCP with $Q = C_1^T C_1$, $S = C_1^T D_{12}$, $R = D_{12}^T D_{12}$

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- Module: focus on state feedback $\mathcal{H}_2$ OCP, simplifies to

$$\vec{y} = \vec{x} \tag{6}$$

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- Output feedback $\mathcal{H}_2$ OCPs may be regular or singular
- Observer feedback $\mathcal{H}_2$ OCP as regular type, allows one to apply separation principle to problem to obtain LQR and **linear-quadratic estimator (LQE)**
  - When $\vec{d}$ modeled as Gaussian noise a.k.a. **linear-quadratic-Gaussian (LQG) OCP**, fundamental OCP for stochastic dynamical systems, addressed in subsequent courses

State Feedback $\mathcal{H}_2$

- Recall solution to finite-horizon continuous-time LQR OCP used **Riccati differential equation**, i.e.

$$\dot{P} = -PA - A^T P + (PB + S) R^{-1} (B^T P + S^T) - Q \tag{8}$$

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- **Unconstrained infinite-horizon continuous-time LQ OCP** sets $t_f = \infty$ and **unconstrained infinite-horizon continuous-time LQR** uses steady-state solution of Riccati differential equation, i.e.

$$0 = PA + A^T P - (PB + S) R^{-1} (B^T P + S^T) + Q \tag{9}$$

  - A.k.a. **continuous algebraic Riccati equation (CARE)**, solved using standard algorithms

State Feedback $\mathcal{H}_2$ (continued)

- Optimal control:

$$\vec{u}(t) = D_K \vec{x}(t) = -R^{-1}(B^T P + S^T)\vec{x}(t) \tag{10}$$

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- Closed-loop dynamic:

$$\dot{\vec{x}}(t) = (A + BD_K)\,\vec{x}(t) \tag{11}$$

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- State matrix, A, stable and $\|G\|_2 < 1$ if and only if, there exists $W \in \mathbb{S}^{n_x}$ such that

$$\mathrm{Tr}\left[CWC^T\right] < 1 \tag{12}$$

and

$$AW + WA^T + BB^T < 0 \tag{13}$$

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- Consider perturbed expression for $\epsilon > 0$

$$W(\epsilon) = \int_0^\infty e^{At}(BB^T + \epsilon I)e^{At}dt \tag{15}$$

  - Note: $W(\epsilon) > 0$ for $\epsilon > 0$
  - $W(\epsilon)$ continuous function of ϵ and $W(\epsilon) = W_C$ for $\epsilon = 0 \rightarrow$ by continuity, $\mathrm{Tr}\left[CWC^T\right] < 1$ for some $\epsilon > 0$

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- Matrix satisfies matrix Lyapunov equation:

$$AW + WA^T + (BB^T + \epsilon I) = 0 \tag{16}$$

$$AW + WA^T + BB^T = -\epsilon I < 0 \tag{17}$$

$\mathcal{H}_2$ OCP as SDP: Preliminary (continued)

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- By definition:

$$\|G\|_2 < 1 \tag{23}$$

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- Statement: $D_K \in \mathbb{R}^{n_u \times n_y}$ exists s.t. A_L stable and satisfies $\|F_L(G, K)\|_2 < \gamma$ if and only if there exists $P > 0$ and D_K s.t.

$$A_L P + P A_L^T + B_L B_L^T < 0 \tag{24}$$

$$\mathrm{Tr}\left[C_L P C_L^T\right] < \gamma^2 \tag{25}$$

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- Substituting for closed-loop matrices:

$$(A + B_2 D_K)P + P(A + B_2 D_K)^T + B_1 B_1^T < 0 \tag{26}$$

$$\mathrm{Tr}\left[(C_1 + D_{12} D_K) P (C_1 + D_{12} D_K)^T\right] < \gamma^2 \tag{27}$$

  - Nonlinear matrix inequality in (P, D_K) due to bilinear terms

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- Define change of variable from D_K to

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- Not convex condition:

$$\text{Tr}\left[(C_1 + D_{12} Q P^{-1}) P (C_1 + D_{12} Q P^{-1})^T \right] < \gamma^2 \tag{30}$$

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- Convexify inequality by introducing **slack variable**, $R \in \mathbb{S}^{n_e}$, based on following linear algebra fact: if $M_1 \le M_2$, then $\mathrm{Tr}[M_1] \le M_2$

$\mathcal{H}_2$ OCP as SDP (continued)

- Write previous trace inequality as two separate inequalities

$$\begin{aligned} (C_1 P + D_{12} Q) P^{-1} (C_1 P + D_{12} Q)^T &< R \\ \mathrm{Tr}[R] &< \gamma^2 \end{aligned} \tag{32}$$

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- Subtract R from both sides and use Schur complement lemma:

$$\begin{aligned} \begin{bmatrix} R & C_1P + D_{12}Q \\ (C_1P + D_{12}Q)^T & P \end{bmatrix} &< 0 \\ \mathrm{Tr}[R] &< \gamma^2 \end{aligned} \tag{33}$$

$\mathcal{H}_2$ OCP as SDP (continued)

- $\mathcal{H}_2$ OCP as SDP in (P, Q, R, γ):

$$
\begin{aligned}
(P, Q, R, \gamma)^{opt} &= \underset{P\in\mathbb{S}^{nx}, Q\in\mathbb{R}^{nu\times nx}, R\in\mathbb{S}^{ne}, \gamma>0}{\text{argmin}} \quad \gamma \\
\underline{\text{subject to:}} \quad & \begin{bmatrix} A & B_2 \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} + \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} A^T \\ B_2^T \end{bmatrix} + B_1 B_1^T < 0 \\
& - \begin{bmatrix} R & C_1 P + D_{12} Q \\ (C_1 P + D_{12} Q)^T & P \end{bmatrix} < 0 \\
& \text{Tr}[R] - \gamma^2 < 0
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- $\mathcal{H}_2$ optimal controller, K^{opt}, as $\vec{u}(t) = D_K \vec{x}(t)$ with

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D_K = QP^{-1} \tag{35}
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$$Q = I \qquad R = \rho^2 I \tag{36}$$

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- Cost functional:

$$\mathcal{J} = \int_0^\infty \vec{x}^T \vec{x} + \rho^2 \vec{u}^T \vec{u} \tag{37}$$

  - Balances $\mathcal{L}_2$-norm of state and input through ρ
  - Varying ρ from $0 \to \infty$: sweep through different values of ρ to find satisfactory response

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  - Varying ρ from $0 \to \infty$: sweep through different values of ρ to find satisfactory response
  - **Optimal tradeoff curve**: sweeping ρ versus $\mathcal{J}$, identify lowest cost overall
  - Another analysis plot: root locus as function of ρ

Selection of Q and R (continued)

2. Second method uses relative cost, ρ, between output and input:

$$Q = C^T C \qquad R = \rho^2 I \tag{38}$$

- Incorporates output equation with no feedthrough term, $\vec{y} = C\vec{x}$, to directly use output in objective

Selection of Q and R (continued)

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- Balances L_2-norm of output and input through ρ
- Vary ρ from $0 \to \infty$: find satisfactory response from possible options

Selection of Q and R (continued)

3 Use individual diagonal costs in addition to relative cost, ρ:

$$Q = \begin{bmatrix} q_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & q_n \end{bmatrix} \qquad R = \rho^2 \begin{bmatrix} r_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_p \end{bmatrix} \tag{40}$$

- Each q_i and r_i selected to normalize state and input for "equal" levels of error or effort, respectively
- I.e. "weighted" $\mathcal{L}_2$-norm of state and input or weighted $\mathcal{L}_2$-norm of output and input using no feedthrough model, $\vec{y} = C\vec{x}$

Selection of Q and R (continued)

- Normalization typically based on units

Selection of Q and R (continued)

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  - Example: error of 5 m/s in x_1 *as poor as* error of 3° in x_2, set

$$q_1 = \left(\frac{1}{5}\right)^2 \tag{41}$$

$$q_2 = \left(\frac{1}{3}\right)^2 \tag{42}$$

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- Can vary ρ from $0 \to \infty$ to find satisfactory response
- Choosing diagonal costs may require additional trial and error if no simple method gives satisfactory controller
  - Primary task of control designer

Selection of Q and R (continued)

4 Use **Parseval's theorem** to convert scalar quadratic functions in time domain to frequency domain using Fourier transforms:

$$\begin{aligned} \mathcal{J} &= \int_0^\infty \vec{x}^T(t) Q \vec{x}(t) + \vec{u}^T(t) R \vec{u}(t) dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \vec{x}^T(-j\omega) Q \vec{x}(j\omega) + \vec{u}^T(-j\omega) R \vec{u}(j\omega) d\omega \end{aligned} \tag{43}$$

- $\vec{x}(j\omega)$: continuous-time Fourier transform of $\vec{x}(t)$
- $\vec{u}(j\omega)$: continuous-time Fourier transform of $\vec{u}(t)$
- Q and R formed as functions of frequency, ω
- Penalize state/output error or input over different regimes more than others: type of loop-shaping

Q, R, and Eigenvalues

- Q and R matrices of quadratic cost functional related to eigenvalues of closed-loop state dynamics for $\mathcal{H}_2$ controller:

$$\dot{\vec{x}} = (A + BD_K)\vec{x} = A_L\vec{x} \tag{44}$$

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- Consider Hamiltonian matrix, H, for continuous-time:

$$H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \tag{45}$$

  - Contains $2n_x$ eigenvalues
  - n_x eigenvalues of A_L with negative real parts
  - n_x have positive real parts, unstable, but stable “backward” in time

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  - n_x eigenvalues of A_L with negative real parts
  - n_x have positive real parts, unstable, but stable “backward” in time
- Recall characteristic equation of closed-loop: zeros are eigenvalues:

$$\phi_{cl} = \det\left[sI - A - BD_K\right] \tag{46}$$

$\mathcal{H}_\infty$ Optimal Control Objective

- Find stabilizing K that minimizes $\mathcal{H}_\infty$-norm of generalized feedback control system:

$$K^{opt} = \underset{K \text{ stabilizing}}{\operatorname{argmin}} \; \|F_L(G, K)\|_\infty \tag{47}$$

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$$\begin{aligned} \dot{\vec{x}}(t) &= A\vec{x}(t) + B_1\vec{d}(t) + B_2\vec{u}(t) \\ \vec{e}(t) &= C_1\vec{x}(t) + D_{11}\vec{d}(t) + D_{12}\vec{u}(t) \\ \vec{y}(t) &= C_2\vec{x}(t) + D_{21}\vec{d}(t) + D_{22}\vec{u}(t) \end{aligned} \tag{48}$$

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- Recall definition of $\mathcal{H}_\infty$-norm, recast OCP in min-max OCP framework:

$$K^{opt}, \vec{d}^{\,opt} = \underset{K \text{ stabilizing}}{\operatorname{argmin}} \; \underset{0 \neq \|\vec{d}\|_2 \leq \infty, \vec{x}(0)=0}{\operatorname{argmax}} \; \frac{\|\vec{e}\|_2^2}{\|\vec{d}\|_2^2} \tag{49}$$

  - Difficult problem to solve for general $F_L(G, K)$
  - Type of differential game

$\mathcal{H}_\infty$ Optimal Control Restatement

- Note: for all $\vec{d}$ with $\|\vec{d}\|_2 < \infty$

$$\|F_L(G,K)\|_\infty \geq \frac{\|\vec{e}\|_2}{\|\vec{d}\|_2} \tag{50}$$

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$$\|\vec{e}\|_2^2 - \gamma^2\|\vec{d}\|_2^2 \leq 0 \tag{53}$$

  - Equals zero for some worst case $\vec{d}$ and $\gamma = \gamma_{min} = \|F_L(G,K)\|_\infty$

$\mathcal{H}_\infty$ OCP Related Reformulation

- Solve for optimal controller *and* maximizing disturbance that solves following constrained min-max OCP

$$K^{opt}, \vec{d}^{\,opt} = \underset{K \text{ stabilizing}}{\text{argmin}} \quad \underset{0 \neq \|\vec{d}\|_2 \leq \infty, \vec{x}(0)=0}{\text{argmax}} \quad \mathcal{J}(K, \vec{d}) \tag{54}$$

$$\text{subject to : } \gamma \geq \|F_L(G, K)\|_\infty$$

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$$\text{subject to : } \gamma \geq \|F_L(G, K)\|_\infty$$

- Cost functional

$$\mathcal{J}(K, \vec{d}) = \frac{1}{2}\int_0^{t_f} \vec{e}^{\,T}\vec{e} - \gamma^2 \vec{d}^{\,T}\vec{d}\, d\tau \tag{55}$$

  - $\vec{x}_0 = 0$
  - t_f given, may be finite in
  - $\vec{x}(t_f)$ free to vary in optimization
  - Can include terminal cost on $\vec{x}(t_f)$ if desired

State Feedback $\mathcal{H}_\infty$ OCP Related Reformulation

- Assume $C_2 = I$, $D_{21} = D_{22} = 0$, $\vec{u}(t) = D_K(t)\vec{x}(t)$, cost functional:

$$\mathcal{J}(\vec{u}, \vec{d}) = \frac{1}{2}\int_0^{t_f} [C_1\vec{x} + D_{12}\vec{u} + D_{11}\vec{d}]^T[C_1\vec{x} + D_{12}\vec{u} + D_{11}\vec{d}] - \gamma^2\vec{d}^T\vec{d}\,d\tau \qquad (56)$$

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$$\begin{aligned}\mathcal{J}(\vec{u}, \vec{d}) = \frac{1}{2}\int_0^{t_f} &\vec{x}^T C_1^T C_1\vec{x} + 2\vec{x}^T\begin{bmatrix} C_1^T D_{12} & C_1^T D_{11}\end{bmatrix}\begin{bmatrix}\vec{u} \\ \vec{d}\end{bmatrix} \\ &+ \begin{bmatrix}\vec{u} \\ \vec{d}\end{bmatrix}^T \begin{bmatrix} D_{12}^T D_{12} & D_{12}^T D_{11} \\ D_{11}^T D_{12} & D_{11}^T D_{11} - \gamma^2 I\end{bmatrix}\begin{bmatrix}\vec{u} \\ \vec{d}\end{bmatrix} d\tau\end{aligned} \tag{57}$$

- Close to quadratic cost functional except for γ^2 term
- Minimized as part of optimization, thus not truly quadratic cost

LQR Sub-Problem

- Assume constant upper bound γ, sub-problem: find $\vec{d}$ and $\vec{u}$ which maximize and minimize cost functional to obtain $D_K(t)$

LQR Sub-Problem

- Assume constant upper bound γ, sub-problem: find $\vec{d}$ and $\vec{u}$ which maximize and minimize cost functional to obtain $D_K(t)$
- Define LQR sub-problem:

$$Q = C_1^T C_1 \tag{58}$$

$$S = \begin{bmatrix} C_1^T D_{12} & C_1^T D_{11} \end{bmatrix} \tag{59}$$

$$R = \begin{bmatrix} D_{11}^T D_{12} & D_{11}^T d_2 \\ D_{11}^T D_{12} & D_{11}^T D_{11} - \gamma^2 I \end{bmatrix} \tag{60}$$

$$\tilde{B} = \begin{bmatrix} B_2 & B_1 \end{bmatrix} \tag{61}$$

$$\tilde{u} = \begin{bmatrix} \vec{u} \\ \vec{d} \end{bmatrix} \tag{62}$$

LQR Sub-Problem (continued)

- LQR cost functional:

$$\mathcal{J}(\tilde{u}) = \frac{1}{2}\int_0^{t_f} \vec{x}^T Q\vec{x} + 2\vec{x}^T S\tilde{u} + \tilde{u}^T R\tilde{u} d\tau \tag{63}$$

LQR Sub-Problem (continued)

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- Solved by differential Riccati equation:

$$-\dot{P} = PA + A^T P + Q - (P\tilde{B} + S)R^{-1}(\tilde{B}^T P + S^T) \tag{64}$$

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$$\tilde{u} = -R^{-1}(\tilde{B}^T P(t) + S^T)\vec{x} \tag{65}$$

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- Optimal control sequence $\vec{u}$:

$$\begin{aligned}\vec{u} &= \begin{bmatrix} I_{n_u\times n_u} & 0\end{bmatrix}\tilde{u} \\ &= -\begin{bmatrix} I_{n_u\times n_u} & 0\end{bmatrix} R^{-1}(\tilde{B}^T P(t) + S^T)\vec{x} \\ &= D_K(t)\vec{x}\end{aligned} \tag{66}$$

LQR Sub-Problem (continued)

- For $t_f \to \infty$, D_K fixed-gain controller obtained by Riccati matrix $P \geq 0$ that solves CARE

$$0 = PA + A^T P + Q - (P\tilde{B} + S)R^{-1}(\tilde{B}^T P + S^T) \tag{67}$$

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- Choosing γ to solve $\mathcal{H}_\infty$ OCP still required for LQR solution method
  - Typically done through γ**-iteration**

Simple Iteration: Bi-Section Search

1. Initialize γ larger than the anticipated optimal γ for binary search
   - Form LQR cost matrices using γ
   - Solve continuous-time algebraic Riccati equation (CARE) for matrix P
   - If $P > 0$ and $(A - BD_K)$ Hurwitz:
     - Decrease γ by bisection (until convergence threshold)
   - Else:
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2. Convergence to γ_{min} to form D_K

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- Note: care must be taken as γ approaches γ_{min} as R typically becomes ill-conditioned
- Typically prudent to slightly increase γ from γ_{min} to reduce feedback gain magnitudes and improve accuracy of numerical solver for CARE

Bounded Real Lemma

- States: A_L Hurwitz, and $\|F_L(G,K)\|_\infty < \gamma^2$ if and only if there exists $P > 0$ satisfying strict ARI:

$$A_L^T P + PA_L + C_L^T C_L + \gamma^{-2} PB_L B_L^T P < 0 \tag{69}$$

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- Written as

$$\begin{bmatrix} (A + B_2 D_K)^T P + P(A + B_2 D_K) & PB_1 \\ B_1^T P & \gamma^{-2} I \end{bmatrix} + \begin{bmatrix} (C_1 + D_{12} D_K)^T \\ 0 \end{bmatrix} \begin{bmatrix} C_1 + D_{12} D_K & 0 \end{bmatrix} < 0 \tag{70}$$

  - Nonlinear matrix inequality in (P, D_K) due to bilinear and quadratic terms

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  - Nonlinear matrix inequality in (P, D_K) due to bilinear and quadratic terms
- Use **symmetric congruence transformation**

$$\begin{bmatrix} P^{-1} & 0 \\ 0 & I \end{bmatrix} \tag{71}$$

Bounded Real Lemma (continued)

- Left and right multiplied:

$$\begin{bmatrix} P^{-1}(A+B_2D_K)^T + (A+B_2D_K)P^{-1} & B_1 \\ B_1^T & \gamma^{-2}I \end{bmatrix} + \begin{bmatrix} P^{-1}(C_1+D_{12}D_K)^T \\ 0 \end{bmatrix} \begin{bmatrix} (C_1+D_{12}D_K)P^{-1} & 0 \end{bmatrix} < 0 \tag{72}$$

Bounded Real Lemma (continued)

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- Defining change of variables as $Q = P^{-1}$ and $R = D_K Q$, alternatively

$$\begin{bmatrix} QA^T + AQ + R^T B_2^T + B_2 R & B_1 \\ B_1^T & \gamma^{-2}I \end{bmatrix} + \begin{bmatrix} (C_1 Q + D_{12}R)^T \\ 0 \end{bmatrix} \begin{bmatrix} C_1 Q + D_{12}R & 0 \end{bmatrix} < 0 \tag{73}$$

  - Not LMI due to quadratic term, $(C_1 Q + D_{12}R)^T (C_1 Q + D_{12}R)$

Bounded Real Lemma (continued)

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- By Schur Complement Lemma:

$$\begin{bmatrix} QA^T+AQ+R^TB_2^T+B_2R & B_1 & (C_1Q+D_{12}R)^T \\ B_1^T & \gamma^{-2}I & 0 \\ C_1Q+D_{12}R & 0 & -I \end{bmatrix} < 0 \qquad (74)$$

$\mathcal{H}_\infty$ OCP as SDP

- State $\mathcal{H}_\infty$ OCP as SDP in (Q, R, γ):

$$
\begin{aligned}
(Q, R, \gamma)^{opt} = & \underset{Q \in \mathbb{S}^{n_x}, R \in \mathbb{R}^{n_u \times n_x}, \gamma > 0}{\operatorname{argmin}} \quad \gamma \\
\underline{\text{subject to}}: & \begin{bmatrix} QA^T + AQ + R^T B_2^T + B_2 R & B_1 & (C_1 Q + D_{12} R)^T \\ B_1^T & -\gamma^{-2} I & 0 \\ C_1 Q + D_{12} R & 0 & -I \end{bmatrix} < 0 \\
& -Q < 0
\end{aligned}
\tag{75}
$$

$\mathcal{H}_\infty$ OCP as SDP

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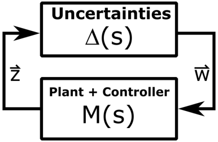
$$\underline{\text{subject to}}: \begin{bmatrix} QA^T + AQ + R^T B_2^T + B_2 R & B_1 & (C_1 Q + D_{12} R)^T \\ B_1^T & -\gamma^{-2} I & 0 \\ C_1 Q + D_{12} R & 0 & -I \end{bmatrix} < 0 \qquad (75)$$

$$-Q < 0$$

- State feedback $\mathcal{H}_\infty$ optimal controller, K^{opt}, as $\vec{u}(t) = D_K \vec{x}(t)$ reconstructed with

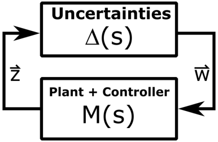
$$D_K = RQ^{-1} \qquad (76)$$

Generalized ΔM Framework



- Structured singular value (SSV), $\mu_{\boldsymbol{\Delta}}$: compute upper and lower bound for inverse of minimum possible $\bar{\sigma}\Delta$ which causes ΔM model to becomes unstable

Generalized ΔM Framework



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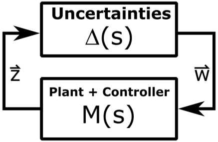

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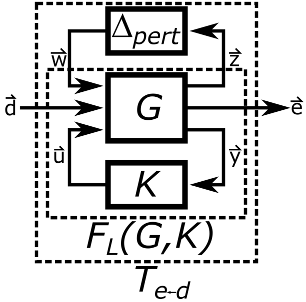
- Numerically approximated:

$$\max_{Q} \lambda(QM) \leq \mu_{\boldsymbol{\Delta}}(M) \leq \inf_{D} \bar{\sigma}(DMD^{-1}) \tag{78}$$

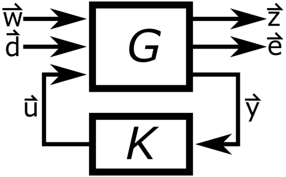
  - Frequency-dependent D matrices, commute with Δ (i.e. $D\Delta = \Delta D$): D **scalings**

ΔM and $\mathcal{H}_\infty$ Control Combination



- $\vec{d}$ and $\vec{e}$: generalized disturbance input and generalized error output vectors
- $\vec{w}$ and $\vec{z}$: perturbation input and output vectors to some plant perturbation uncertainty, Δ_{pert}
- $\vec{u}$ and $\vec{y}$: control input to plant and plant output to controller vectors
- Lower LFT, $F_L(G, K)$, equivalent to M in ΔM robust analysis model

Model $F_L(G, K)$ Setup



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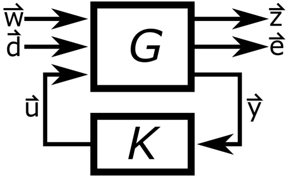

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- **μ-synthesis** OCP: minimize over all stabilizing controllers, K, peak value of μ_Δ of closed-loop transfer function

$$K^{opt} = \underset{K \text{ stabilizing}}{\text{argmin}} \ \max_{\omega} \mu_{\boldsymbol{\Delta}} \left(F_L(G, K)(j\omega)\right) \tag{80}$$

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$D - K$ Iteration (continued)

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- Approximation typically very close
  - $D-K$ iteration not guaranteed to converge to global or even local minimum: serious shortcoming of design approach
  - Shown to work well in many flight vehicle control problems: highly flexible airplanes, missile autopilots, modern fighter airplanes, space shuttle flight control system

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- $\mathcal{H}_2$-norm optimal control: design
  - Weigh Q and R relative to each other and internal elements if needed
  - Choose grid of relative weights versus ρ: optimal tradeoff and root locus
  - Check zeros of H_1