Lecture 14: Elastic Aerospace Vehicle Dynamics

Textbook Section 10.3

Dr. Jordan D. Larson

Recall Previous Definitions

- Generalized coordinates are

$$\vec{q} = \begin{bmatrix} x_N & y_N & z_N & \phi & \theta & \psi & \eta_i, i = 1, 2, ... \end{bmatrix}^T \tag{1}$$

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- Kinetic energy of elastic vehicle:

$$T = \frac{1}{2} m \begin{bmatrix} \dot{x}_N & \dot{y}_N & \dot{z}_N \end{bmatrix} \begin{bmatrix} \dot{x}_N \\ \dot{y}_N \\ \dot{z}_N \end{bmatrix} + \frac{1}{2} \begin{bmatrix} p & q & r \end{bmatrix} I_G \begin{bmatrix} p \\ q \\ r \end{bmatrix} + \frac{1}{2} \sum_{i=1}^{n} \mathcal{M}_i \dot{\eta}_i^2 \tag{2}$$

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- Gravitational potential energy of elastic vehicle:

$$U_g = -mgz_N = mgh \tag{3}$$

Recall Previous Definitions (continued)

- Elastic strain energy of elastic vehicle:

$$U_e = -\frac{1}{2}\sum_{i=1}^{n}\omega_i^2\eta_i^2(t)\mathcal{M}_i \tag{4}$$

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- Euler-Lagrange equation (vector form):

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\vec{q}}}\right) - \frac{\partial T}{\partial \vec{q}} + \frac{\partial U}{\partial \vec{q}} = \vec{Q}^T = \frac{\partial \delta W}{\partial \delta q} \tag{5}$$

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- Derive equations of motion for elastic flight vehicles in three coupled EOMs:
  - Rigid body translation
  - Rigid body rotation
  - Elastic vibration

Constant-Mass Flight Vehicle

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$$T_{tran} = \frac{1}{2} m \begin{bmatrix} \dot{x}_N & \dot{y}_N & \dot{z}_N \end{bmatrix} \begin{bmatrix} \dot{x}_N \\ \dot{y}_N \\ \dot{z}_N \end{bmatrix} \tag{6}$$

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$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\vec{x}}_N}\right) - \frac{\partial T}{\partial \vec{x}_N} = m \begin{bmatrix} \ddot{x}_N \\ \ddot{y}_N \\ \ddot{z}_N \end{bmatrix} = m\ddot{\vec{x}}_N \tag{7}$$

Gravity and Forces

- Gravitational potential energy:

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$$\ddot{\vec{x}}_N = \begin{bmatrix} \ddot{x}_N \\ \ddot{y}_N \\ \ddot{z}_N \end{bmatrix} = \begin{bmatrix} \frac{Q_{x,N}}{m} \\ \frac{Q_{y,N}}{m} \\ \frac{Q_{z,N}}{m} + g \end{bmatrix} \tag{9}$$

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- Conversion of velocity:

$$\ddot{\vec{x}}_N = \ddot{\vec{x}}_B + \vec{\omega}_{B/N} \times \dot{\vec{x}}_B \tag{10}$$

FDC Rewrite

- Body frame linear and angular velocity components:

$$
\begin{aligned}
\ddot{\vec{x}}_N &= \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} + \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} u \\ v \\ w \end{bmatrix} \\
&= \begin{bmatrix} \dot{u} - rv + qw \\ \dot{v} + ru - wp \\ \dot{w} - qu + pv \end{bmatrix}
\end{aligned} \tag{11}
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- Generalized forces for flight vehicle: net force (besides gravitational) $=$ net aerodynamics and propulsion (in body-fixed frame):

$$\vec{F}_{a,B} + \vec{F}_{p,B} = \begin{bmatrix} mX & mY & mZ \end{bmatrix}^T \tag{12}$$

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- In generalized coordinates use DCM from $N \to B$ (function of Euler angles, also generalized coordinates):

$$\delta W_F = \begin{bmatrix} mX & mY & mZ \end{bmatrix} C_{B \leftarrow N}(\phi, \theta, \psi) \begin{bmatrix} \delta x_N \\ \delta y_N \\ \delta z_N \end{bmatrix} \tag{14}$$

Generalized Forces

$$\begin{aligned}\begin{bmatrix} Q_{x,N} \\ Q_{y,N} \\ Q_{z,N} \end{bmatrix} &= \frac{\partial \delta W_F}{\partial \delta \vec{x}_N} = \begin{bmatrix} mX & mY & mZ \end{bmatrix} C_{B \leftarrow N}(\phi, \theta, \psi) \\ &= C^T_{B \leftarrow N}(\phi, \theta, \psi) \begin{bmatrix} mX \\ mY \\ mZ \end{bmatrix}\end{aligned} \tag{15}$$

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- Rewritten in body-fixed frame coordinates:

$$\begin{bmatrix} Q_{x,B} \\ Q_{y,B} \\ Q_{z,B} \end{bmatrix} = C_{B \leftarrow N}(\phi, \theta, \psi) C^T_{B \leftarrow N}(\phi, \theta, \psi) \begin{bmatrix} mX \\ mY \\ mZ \end{bmatrix} = \begin{bmatrix} mX \\ mY \\ mZ \end{bmatrix} \tag{16}$$

Elastic Vehicle Translation EOMs

- For gravitational force (in body-fixed frame coordinates):

$$\begin{bmatrix} -mg\sin\theta \\ mg\cos\theta\sin\phi \\ mg\cos\theta\cos\phi \end{bmatrix} = C_{B\leftarrow N}(\phi,\theta,\psi)\begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} \tag{17}$$

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- Same rigid vehicle translation EOMs for elastic translation:

$$\begin{bmatrix} \dot{u} - rv + qw \\ \dot{v} + ru - wp \\ \dot{w} - qu + pv \end{bmatrix} = \begin{bmatrix} X - g\sin\theta \\ Y + g\cos\theta\sin\phi \\ Z + g\cos\theta\cos\phi \end{bmatrix} \tag{18}$$

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- Euler angles represent three sequential rotations $\rightarrow$ relationship between body-fixed frame angular velocity (appears in kinetic energy) and Euler angles:

$$\vec{\omega}_{B/N} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = C_{\omega}(\vec{q}_{\angle})\dot{\vec{q}}_{\angle} = \begin{bmatrix} 1 & 0 & -\sin\phi \\ 0 & \cos\phi & \cos\theta\sin\phi \\ 0 & -\sin\phi & \cos\theta\cos\phi \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \tag{19}$$

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$$\frac{\partial \vec{\omega}_{B/N}}{\partial \dot{\vec{q}}_\angle} = C_\omega \quad (20)$$

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$$\begin{aligned}\frac{\partial \omega_{B/N}}{\partial \vec{q}_\angle} &= \begin{bmatrix} 0 & -\dot{\psi}\cos\theta & 0 \\ \dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi & -\dot{\psi}\sin\theta\sin\phi & 0 \\ -\dot{\psi}\cos\theta\sin\phi - \dot{\theta}\cos\phi & -\dot{\psi}\sin\theta\cos\phi & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\dot{\psi}\cos\theta & 0 \\ r & -\dot{\psi}\sin\theta\sin\phi & 0 \\ -q & -\dot{\psi}\sin\theta\cos\phi & 0 \end{bmatrix}\end{aligned} \tag{23}$$

Lagrange's Equation

- Euler-Lagrange equation for kinetic energy:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\vec{q}}_\angle} - \frac{\partial T}{\partial \vec{q}_\angle} = \frac{d}{dt}\left(\frac{\partial T}{\partial \omega_{B/N}}\frac{\partial \omega_{B/N}}{\partial \dot{\vec{q}}_\angle}\right) - \frac{\partial T}{\partial \omega_{B/N}}\frac{\partial \vec{\omega}}{\partial \vec{q}_\angle} \quad (24)$$

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- Matrix multiplications and (some) algebra:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\vec{q}}_\angle} - \frac{\partial T}{\partial \vec{q}_\angle} = C_\omega^T\left(I_G\dot{\omega}_{B/N} + \vec{\omega}_{B/N} \times I_G\vec{\omega}_{B/N}\right) = \vec{Q}_\angle \tag{25}$$

Virtual Work by Moments

- Virtual work associated with net moment on flight vehicle in body-fixed frame related to virtual angular displacements:

$$\delta W_M = \begin{bmatrix} I_{xx}L & I_{yy}M & I_{zz}N \end{bmatrix} \begin{bmatrix} \delta\phi \\ \delta\theta \\ \delta\psi \end{bmatrix} \tag{26}$$

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$$\vec{Q}_\angle^T = \frac{\partial \delta W_M}{\partial \vec{q}_\angle} = \begin{bmatrix} I_{xx}L & I_{yy}M & I_{zz}N \end{bmatrix} C_\omega \tag{28}$$

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- For $I_{xy} = I_{yz} = 0$:

$$\begin{bmatrix} \dot{p} + \frac{I_{zz}-I_{yy}}{I_{xx}} qr - \frac{I_{xz}}{I_{xx}}(\dot{r} + pq) \\ \dot{q} + \frac{I_{xx}-I_{zz}}{I_{yy}} pr - \frac{I_{xz}}{I_{yy}}(r^2 - p^2) \\ \dot{r} + \frac{I_{yy}-I_{xx}}{I_{zz}} pq - \frac{I_{xz}}{I_{zz}}(\dot{p} - qr) \end{bmatrix} = \begin{bmatrix} L \\ M \\ N \end{bmatrix} \tag{31}$$

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- Applying Euler-Lagrange equation for vibration kinetic energy and elastic strain energy for *each* individual modal coordinate:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\eta}_i}\right) - \frac{\partial T}{\partial \eta_i} + \frac{\partial U_e}{\partial \eta_i} = Q_i = \frac{\partial \delta W}{\partial \delta \eta_i} \tag{32}$$

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- $\mathcal{M}_i$: i-th generalized mass
- Q_i: i-th generalized force

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$$d\delta W_P = \vec{P}(\vec{x}_B) \cdot \sum_{i=1}^{n} \vec{\nu}_i(\vec{x}_B)\delta\eta_i(t)dS \tag{35}$$

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- dS: infinitesimal surface area for pressure

Virtual Work by Pressure

- Total virtual work:

$$\begin{aligned} \delta W_P &= \int_{Area} \vec{P}(\vec{x}_B) \cdot \sum_{i=1}^{n} \vec{\nu}_i(\vec{x}_B) \delta \nu_i(t) dS \\ &= \sum_{i=1}^{n} \int_{Area} \vec{P}(\vec{x}_B) \cdot \vec{\nu}_i(\vec{x}_B) dS \delta \eta_i(t) \end{aligned} \tag{36}$$

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- n vibration EOM:

$$\ddot{\eta}_i + \omega_i^2\eta_i = \frac{1}{\mathcal{M}_i}\int_{Area} \vec{P}(\vec{x}_B) \cdot \vec{\nu}_i(\vec{x}_B)dS \qquad i = 1, \cdots, n \tag{37}$$

Virtual Work by Pressure

- Total virtual work:

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- Connects aerodynamic pressure with structural deformations

Motion of Any Point on Vehicle

- 3 sets of EOMs $\rightarrow$ able to describe motion of *any* point on elastic flight vehicle

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$$\vec{x}_N = \vec{x}_{B/N} + \vec{x}_{r-b} + d_e(\vec{x}_B, t) \tag{38}$$

Motion of Any Point on Vehicle (continued)

$$\vec{x}_N = \vec{x}_{B/N} + \vec{x}_{r-b} + \sum_{i=1}^{n} \vec{\nu}_i(\vec{x}_B)\eta_i(t) \tag{39}$$

Motion of Any Point on Vehicle (continued)

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- Each term determined by elastic vehicle EOMs (assuming solution for free-vibration mode shapes)

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- Note: position of instantaneous center of mass $\leftarrow$ rigid body translation EOM
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- Each term determined by elastic vehicle EOMs (assuming solution for free-vibration mode shapes)
- Note: position of instantaneous center of mass $\leftarrow$ rigid body translation EOM
  - Typically uses body-fixed frame coordinates for velocity of body-fixed frame
- Recall:

$$\dot{\vec{x}}_{B/N} = \begin{bmatrix} \dot{x}_N \\ \dot{y}_N \\ \dot{z}_N \end{bmatrix} = C_{N\leftarrow B} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{40}$$

- Based on final EOMs given

Vector Definitions

- Linear and angular velocity of body-fixed frame as rigid state vector:

$$\vec{x}_{rig} = \begin{bmatrix} u & v & w & p & q & r \end{bmatrix}^T \tag{41}$$

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$$\vec{x}_{vib} = \begin{bmatrix} \eta_1 & \cdots & \eta_n & \dot{\eta}_1 & \cdots & \dot{\eta}_n \end{bmatrix}^T \tag{42}$$

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$$\vec{x}_{vib} = \begin{bmatrix} \eta_1 & \cdots & \eta_n & \dot{\eta}_1 & \cdots & \dot{\eta}_n \end{bmatrix}^T \tag{42}$$

- Control surface deflections and thrust input as control input vector example:

$$\vec{u} = \begin{bmatrix} \delta_a \\ \delta_e \\ \delta_r \\ \delta_t \end{bmatrix} \tag{43}$$

Nonlinear State-Space Aeroelastic EOMs

$$
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{I_{xz}}{I_{zz}} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{I_{xz}}{I_{zz}} & 0 & 1 \end{bmatrix} \dot{\vec{x}}_{rig} = \begin{bmatrix} rv - qw - g\sin\theta \\ pw - ru + g\cos\theta\sin\phi \\ qu - pv + g\cos\theta\cos\phi \\ \frac{I_{yy}-I_{zz}}{I_{xx}} qr + \frac{I_{xz}}{I_{xx}} pq \\ \frac{I_{zz}-I_{xx}}{I_{yy}} pr + \frac{I_{xz}}{I_{yy}} (r^2 - p^2) \\ \frac{I_{xx}-I_{yy}}{I_{zz}} pq - \frac{I_{xz}}{I_{zz}} qr \end{bmatrix} + \begin{bmatrix} X \\ Y \\ Z \\ L \\ M \\ N \end{bmatrix} \tag{44}
$$

$$
\dot{\vec{x}}_{vib} = \begin{bmatrix} 0_{n\times n} & I_{n\times n} \\ -\Omega^2 & -2\Omega_\zeta \end{bmatrix} \vec{x}_{vib} + \begin{bmatrix} \vec{0}_n \\ \frac{Q_1}{\mathcal{M}_1} \\ \vdots \\ \frac{Q_n}{\mathcal{M}_n} \end{bmatrix}
$$

More Definitions

$$\Omega^2 = \begin{bmatrix} \omega_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \omega_n^2 \end{bmatrix} \tag{45}$$

$$\Omega_\zeta = \begin{bmatrix} \zeta_1\omega_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \zeta_n\omega_n \end{bmatrix} \tag{46}$$

- ζ added for potential damping

More Definitions (continued)

- Aerodynamic forces and moments & generalized forces modeled as linear functions of rigid states, vibration states, control inputs:

More Definitions (continued)

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$$\mathcal{I} = \begin{bmatrix} 1 & 0 & -X_{\dot{w}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 - Z_{\dot{w}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{I_{xz}}{I_{zz}} \\ 0 & 0 & -M_{\dot{w}} & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{I_{xz}}{I_{zz}} & 0 & 1 \end{bmatrix} \tag{47}$$

More Definitions (continued)

- Aerodynamic forces and moments & generalized forces modeled as linear functions of rigid states, vibration states, control inputs:

$$\mathcal{I} = \begin{bmatrix} 1 & 0 & -X_{\dot{w}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 - Z_{\dot{w}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{I_{xz}}{I_{zz}} \\ 0 & 0 & -M_{\dot{w}} & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{I_{xz}}{I_{zz}} & 0 & 1 \end{bmatrix} \tag{47}$$

- If $I_{xz} = 0$ and ignore $\dot{w}$ effects, $\mathcal{I} = I_{6\times 6}$

More Definitions (continued)

$$\mathcal{M} = \begin{bmatrix} \mathcal{M}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathcal{M}_n \end{bmatrix} \tag{48}$$

More Definitions (continued)

$$\mathcal{M} = \begin{bmatrix} \mathcal{M}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathcal{M}_n \end{bmatrix} \tag{48}$$

$$f_{rig}(\vec{x}_{rig}, \theta, \phi) = \mathcal{I}^{-1} \begin{bmatrix} rv - qw - g\sin\theta + X_0 \\ pw - ru + g\cos\theta\sin\phi + Y_0 \\ qu - pv + g\cos\theta\cos\phi + Z_0 \\ \frac{I_{xx}-I_{zz}}{I_{xx}} qr + \frac{I_{xz}}{I_{xx}} pq + L_0 \\ \frac{I_{zz}-I_{xx}}{I_{yy}} pr + \frac{I_{xz}}{I_{yy}} (r^2 - p^2) + M_0 \\ \frac{I_{xx}-I_{yy}}{I_{zz}} pq - \frac{I_{xz}}{I_{zz}} qr + N_0 \end{bmatrix} \tag{49}$$

More Definitions (continued)

$$\mathcal{A}_{rig\leftarrow rig} = \mathcal{I}^{-1} \begin{bmatrix} X_u & 0 & X_w & 0 & X_q & 0 \\ 0 & Y_v & 0 & Y_p & 0 & Y_r \\ Z_u & 0 & Z_w & 0 & Z_q & 0 \\ 0 & L_v & 0 & L_p & 0 & L_r \\ M_u & 0 & M_w & 0 & M_q & 0 \\ 0 & N_v & 0 & N_p & 0 & N_r \end{bmatrix} \tag{50}$$

More Definitions (continued)

$$\mathcal{A}_{rig\leftarrow rig} = \mathcal{I}^{-1} \begin{bmatrix} X_u & 0 & X_w & 0 & X_q & 0 \\ 0 & Y_v & 0 & Y_p & 0 & Y_r \\ Z_u & 0 & Z_w & 0 & Z_q & 0 \\ 0 & L_v & 0 & L_p & 0 & L_r \\ M_u & 0 & M_w & 0 & M_q & 0 \\ 0 & N_v & 0 & N_p & 0 & N_r \end{bmatrix} \tag{50}$$

$$\mathcal{A}_{rig\leftarrow \eta} = \mathcal{I}^{-1} \begin{bmatrix} X_{\eta_1} & \cdots & X_{\eta_n} \\ Y_{\eta_1} & \cdots & Y_{\eta_n} \\ Z_{\eta_1} & \cdots & Z_{\eta_n} \\ L_{\eta_1} & \cdots & L_{\eta_n} \\ M_{\eta_1} & \cdots & M_{\eta_n} \\ N_{\eta_1} & \cdots & N_{\eta_n} \end{bmatrix} \tag{51}$$

More Definitions (continued)

$$\mathcal{A}_{rig\leftarrow\dot{\eta}} = \mathcal{I}^{-1} \begin{bmatrix} X_{\dot{\eta}_1} & \cdots & X_{\dot{\eta}_n} \\ Y_{\dot{\eta}_1} & \cdots & Y_{\dot{\eta}_n} \\ Z_{\dot{\eta}_1} & \cdots & Z_{\dot{\eta}_n} \\ L_{\dot{\eta}_1} & \cdots & L_{\dot{\eta}_n} \\ M_{\dot{\eta}_1} & \cdots & M_{\dot{\eta}_n} \\ N_{\dot{\eta}_1} & \cdots & N_{\dot{\eta}_n} \end{bmatrix} \tag{52}$$

More Definitions (continued)

$$\mathcal{A}_{rig\leftarrow\dot{\eta}} = \mathcal{I}^{-1}\begin{bmatrix} X_{\dot{\eta}_1} & \cdots & X_{\dot{\eta}_n} \\ Y_{\dot{\eta}_1} & \cdots & Y_{\dot{\eta}_n} \\ Z_{\dot{\eta}_1} & \cdots & Z_{\dot{\eta}_n} \\ L_{\dot{\eta}_1} & \cdots & L_{\dot{\eta}_n} \\ M_{\dot{\eta}_1} & \cdots & M_{\dot{\eta}_n} \\ N_{\dot{\eta}_1} & \cdots & N_{\dot{\eta}_n} \end{bmatrix} \tag{52}$$

$$\mathcal{B}_{rig} = \mathcal{I}^{-1}\begin{bmatrix} 0 & X_{\delta_e} & 0 & X_{\delta_t} \\ 0 & 0 & Y_{\delta_r} & 0 \\ 0 & Z_{\delta_e} & 0 & Z_{\delta_t} \\ L_{\delta_a} & 0 & L_{\delta_r} & 0 \\ 0 & M_{\delta_e} & 0 & M_{\delta_t} \\ N_{\delta_a} & 0 & N_{\delta_r} & 0 \end{bmatrix} \tag{53}$$

More Definitions (continued)

$$A_{vib\leftarrow rig} = \mathcal{M}^{-1} \begin{bmatrix} Q_{1_u} & Q_{1_v} & Q_{1_w} & Q_{1_p} & Q_{1_q} & Q_{1_r} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ Q_{n_u} & Q_{n_v} & Q_{n_w} & Q_{n_p} & Q_{n_q} & Q_{n_r} \end{bmatrix} \tag{54}$$

More Definitions (continued)

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$$A_{vib\leftarrow\eta} = \mathcal{M}^{-1} \begin{bmatrix} Q_{1_{\eta_1}} & \cdots & Q_{1_{\eta n}} \\ \vdots & \ddots & \vdots \\ Q_{n_{\eta_1}} & \cdots & Q_{n_{\eta n}} \end{bmatrix} - \Omega^2 \tag{55}$$

More Definitions (continued)

$$A_{vib \leftarrow \dot{\eta}} = \mathcal{M}^{-1} \begin{bmatrix} Q_{1_{\dot{\eta}_1}} & \cdots & Q_{1_{\dot{\eta}_n}} \\ \vdots & \ddots & \vdots \\ Q_{n_{\dot{\eta}_1}} & \cdots & Q_{n_{\dot{\eta}_n}} \end{bmatrix} - 2\Omega_{\zeta} \tag{56}$$

More Definitions (continued)

$$A_{vib\leftarrow\dot{\eta}} = \mathcal{M}^{-1}\begin{bmatrix} Q_{1_{\dot{\eta}_1}} & \cdots & Q_{1_{\dot{\eta}_n}} \\ \vdots & \ddots & \vdots \\ Q_{n_{\dot{\eta}_1}} & \cdots & Q_{n_{\dot{\eta}_n}} \end{bmatrix} - 2\Omega_\zeta \tag{56}$$

$$B_{vib} = \mathcal{M}^{-1}\begin{bmatrix} Q_{1_{\delta_a}} & Q_{1_{\delta_e}} & Q_{1_{\delta_r}} & Q_{1_{\delta_t}} \\ \vdots & \vdots & \vdots & \vdots \\ Q_{n_{\delta_a}} & Q_{n_{\delta_e}} & Q_{n_{\delta_r}} & Q_{n_{\delta_t}} \end{bmatrix} \tag{57}$$

Rewritten Elastic State-Space EOMs

$$
\begin{aligned}
\dot{\vec{x}}_{rig} &= f_{rig}(\vec{x}_{rig}, \phi, \theta) + \mathcal{A}_{rig\leftarrow rig}\vec{x}_{rig} + \begin{bmatrix}\mathcal{A}_{rig\leftarrow\eta} & \mathcal{A}_{rig\leftarrow\dot{\eta}}\end{bmatrix}\vec{x}_{vib} + \mathcal{B}_{rig}\vec{u} \\
\dot{\vec{x}}_{vib} &= \begin{bmatrix}0_{n\times 6} \\ A_{vib\leftarrow rig}\end{bmatrix}\vec{x}_{rig} + \begin{bmatrix}0_{n\times n} & I_{n\times n} \\ A_{vib\leftarrow\eta} & A_{vib\leftarrow\dot{\eta}}\end{bmatrix}\vec{x}_{vib} + \begin{bmatrix}0_{n\times 4} \\ B_{vib}\end{bmatrix}\vec{u}
\end{aligned}
\tag{58}
$$

- Note: v, w, $\dot{w}$ used in place of β, α, $\dot{\alpha}$
- Could be replaced by linear approximations and coefficient conversions

Rewritten Elastic State-Space EOMs

$$\begin{aligned}\dot{\vec{x}}_{rig} &= f_{rig}(\vec{x}_{rig}, \phi, \theta) + \mathcal{A}_{rig\leftarrow rig}\vec{x}_{rig} + \begin{bmatrix}\mathcal{A}_{rig\leftarrow\eta} & \mathcal{A}_{rig\leftarrow\dot{\eta}}\end{bmatrix}\vec{x}_{vib} + \mathcal{B}_{rig}\vec{u} \\ \dot{\vec{x}}_{vib} &= \begin{bmatrix}0_{n\times 6} \\ A_{vib\leftarrow rig}\end{bmatrix}\vec{x}_{rig} + \begin{bmatrix}0_{n\times n} & I_{n\times n} \\ A_{vib\leftarrow\eta} & A_{vib\leftarrow\dot{\eta}}\end{bmatrix}\vec{x}_{vib} + \begin{bmatrix}0_{n\times 4} \\ B_{vib}\end{bmatrix}\vec{u}\end{aligned} \tag{58}$$

- Note: v, w, $\dot{w}$ used in place of β, α, $\dot{\alpha}$
- Could be replaced by linear approximations and coefficient conversions

- Also require supplemental Euler angle equation: relate p, q, r to $\dot{\phi}$, $\dot{\theta}$ to complete state-space

$$\begin{bmatrix}\dot{\phi} \\ \dot{\theta} \\ \dot{\psi}\end{bmatrix} = \begin{bmatrix}1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta\end{bmatrix}\begin{bmatrix}p \\ q \\ r\end{bmatrix} \tag{59}$$

Linearized EOMs

- Linearization of EOMs for easier analysis, simulation, and design

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- $f_{rig}(\vec{x}_{rig}, \theta, \phi)$ and Euler angle equation $\rightarrow$ *only* nonlinear term *AND* rigid vehicle specific
- $\rightarrow$ rigid flight vehicle linearization method to form LTI state-space system:

$$\begin{bmatrix} \Delta \dot{\vec{x}}_{rig} \\ \Delta \dot{\vec{x}}_{eul} \\ \Delta \dot{\vec{x}}_{vib} \end{bmatrix} = \begin{bmatrix} A_{rig \leftarrow rig} & A_{rig \leftarrow eul} & A_{rig \leftarrow vib} \\ A_{eul \leftarrow rig} & A_{eul \leftarrow eul} & 0_{3 \times 2n} \\ A_{vib \leftarrow rig} & 0_{2n \times 3} & A_{vib \leftarrow vib} \end{bmatrix} \begin{bmatrix} \Delta \vec{x}_{rig} \\ \Delta \vec{x}_{eul} \\ \Delta \vec{x}_{vib} \end{bmatrix} + \begin{bmatrix} B_{rig} \\ 0_{3 \times 4} \\ B_{vib} \end{bmatrix} \Delta \vec{u} \tag{60}$$

Definitions

$$\Delta \vec{x}_{eul} = \begin{bmatrix} \Delta\phi \\ \Delta\theta \\ \Delta\psi \end{bmatrix} \tag{61}$$

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Definitions

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- $A_{rig\leftarrow rig}$ from $\mathcal{A}_{rig\leftarrow rig}$ due to linearization of $f_{rig}(\vec{x}_{rig}, \theta, \phi)$ w.r.t. $\vec{x}_{rig}$
- $A_{rig\leftarrow eul}$ from linearization of $f_{rig}(\vec{x}_{rig}, \theta, \phi)$ w.r.t. $\vec{x}_{eul}$
- $A_{eul\leftarrow rig}$ and $A_{eul\leftarrow eul}$ from linearization of supplemental Euler angle equation

Summary

- 3-DOF Translation
  - Use mean-axis constraints
  - Coordinates: center of mass velocity
- 3-DOF Rotation:
  - Use mean-axis constraints
  - Coordinates: mean-axis body-fixed frame Euler angles
- *n*-DOF Vibration:
  - Orthogonal mode shapes
  - Additional modal coordinates and rates of vibrations
  - Number dependent on Finite-Element Analysis “lumped-mass” model