Lecture 12: Unforced Elastic-Body Dynamics

Textbook Section 10.1

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Structural Vibrations

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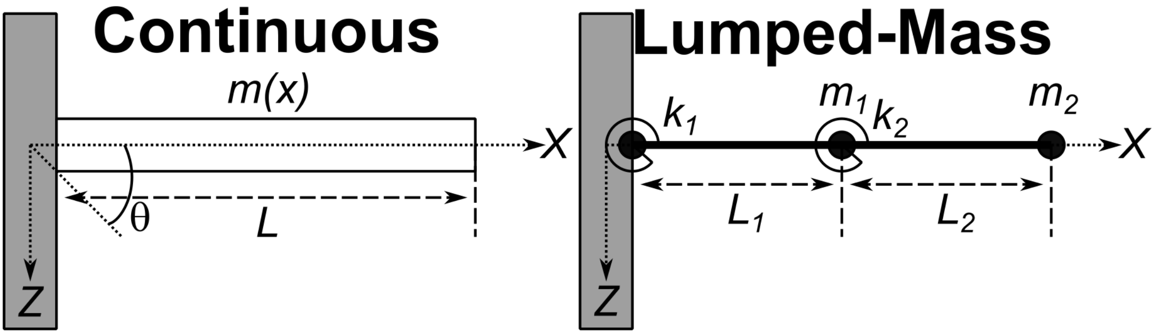
- In reality, not rigid, but some flexibility/elasticity in structures
  - Airborne and spaceborne vehicles
- Advanced course: additional modeling of structural vibrations in flight vehicles
  - Highlighting elastic airplane dynamics

Fixed-Beam Vibration Example



- Continuous deformable body with mass distribution, $m(x)$, as function of horizontal coordinate, x
- Finite-dimensional approximation: simplify to discrete mass model with $i = 1, ..., n$ particles, a.k.a. **lumped-mass model**

Continuous Model

- Partial differential equation (PDE) governing vertical deformation of beam, Z:

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 Z(x,t)}{\partial x^2}\right) + m(x)\frac{\partial^2 Z(x,t)}{\partial t^2} = 0 \tag{1}$$

  - E: elastic modulus of beam material
  - I: area moment of inertia of beam cross-section about neutral axis

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  - E: elastic modulus of beam material
  - I: area moment of inertia of beam cross-section about neutral axis
- Separation of variables: solution to PDE

$$Z(x,t) = \sum_{i=1}^{\infty} \nu_i(x)\eta_i(t) \tag{2}$$

  - Infinite sum of terms, each consisting of purely space-dependent function $\nu_i(x)$ and time-dependent function, $\eta_i(t)$
  - $\nu_i(x)$: **mode shapes**, a.k.a. **eigenfunctions**
  - $\eta_i(t)$: **modal coordinates**
  - Infinite sum solution: **infinite-dimensional problem**

Lumped-Mass Model

- Each mass particle, m_i, associated spring stiffness, k_i, connected by massless rigid rods of lengths L_i
  - Note: solution to vibration problem using finite-element analysis (FEA) results in lumped-mass model approximation
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  - Lumped-mass models always used for real, complex, FDC structures
- Fixed-beam vibration: approximated by two masses, two springs, two rods
  - Note: better approximation $\rightarrow$ add more lumped-masses

Euler-Lagrange Equation of Motion

- Lumped-mass systems: **Euler-Lagrange equation of motion**
  - Lagrangian: $L = T - U$
  - T: kinetic energy of system
  - U: potential (or strain) energy of system

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$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = 0 \tag{3}$$

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- **Generalized coordinates**: include both physical and nonphysical coordinates

Beam-Vibration Example EOMs

- Possible coordinates: transverse displacements of masses $Z_i(t)$, or angular displacements of masses, $\theta_i(t)$, where

$$\begin{bmatrix} \dot{Z}_1 \\ \dot{Z}_2 \end{bmatrix} = \begin{bmatrix} L_1 & 0 \\ L_1 + L_2 & L_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \tag{4}$$

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- Systems-of-particles dynamics kinetic energy of beam:

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- Potential energy of beam:

$$U = \frac{1}{2}\left(k_1 \theta_1^2 + k_2 \theta_2^2\right) = \frac{1}{2} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}^T \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \tag{6}$$

General Matrix-Vector EOMs

- Euler-Lagrange EOM:

$$\begin{bmatrix} m_1^2 L_1^2 + m_2(L_1 + L_2)^2 & m_1 L_2(L_1 + L_2) \\ m_1 L_2(L_1 + L_2) & m_2 L_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7}$$

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- General matrix-vector differential form for *all* lumped-mass vibration problems:

$$M\ddot{\vec{q}} + K\vec{q} = \vec{0} \tag{8}$$

  - M: **mass matrix**, always positive-definite, i.e. strictly positive eigenvalues
  - K: **stiffness matrix**
  - M, K real, symmetric matrices

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- Vibration problem completely described by generalized coordinates, initial conditions, and M, K

Eigenvalue Decomposition

- General lumped-mass vibration problem:

$$\ddot{\vec{q}} + D\vec{q} = \vec{0} \tag{9}$$

  - $D = M^{-1}K$: **dynamic matrix**
  - Always exists since M positive-definite

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  - $D = M^{-1}K$: **dynamic matrix**
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- Standard modal transformation:

$$\vec{q} = \Psi\vec{\eta} \tag{10}$$

  - Ψ: **modal matrix** of D consisting of n eigenvectors, $\vec{\nu}_i$ where $i = 1, ..., n$

$$\Psi^{-1}D\Psi = \Lambda \tag{11}$$

  - Λ diagonal matrix of n eigenvalues of D, λ_i where $i = 1, ..., n$

Eigenvalue Decomposition (continued)

- Eigenvectors satisfy

$$(\lambda_i I - D)\,\vec{\nu}_i = \vec{0} \tag{12}$$

$$\Psi = [\vec{\nu}_1 | \vec{\nu}_2 | \cdots | \vec{\nu}_n] \tag{13}$$

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- n differential equations now independent:

$$\ddot{\eta}_i + \lambda_i \eta_i = 0 \tag{15}$$

  - For $i = 1, \ldots, n$

Eigenvalue Solution

$$\eta_i(t) = A_i \cos(\sqrt{\lambda_i} t + \Gamma_i) \tag{16}$$

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- General lumped-mass vibration problems, find n natural modes each oscillating at natural frequencies $\omega_i = \sqrt{\lambda_i}$
- From definition of Ψ:

$$\vec{q}(t) = [\vec{\nu}_1 | \vec{\nu}_2 | \cdots | \vec{\nu}_n] \, \vec{\eta}(t) = \sum_{i=1}^{n} \vec{\nu}_i \eta_i(t) \tag{17}$$

- Each modal response η_i contributes to system response through eigenvectors or mode shapes
- Eigenvectors have arbitrary magnitude, typically normalized to
  - unit length
  - unity displacement of selected element
  - unity generalized mass

Generalization

- Definition of D and $\vec{\nu}_i$ provides

$$\left(\lambda_i I - M^{-1}K\right)\vec{\nu}_i = 0 \tag{18}$$

$$\lambda_i M\vec{\nu}_i = K\vec{\nu}_i \tag{19}$$

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- Same process:

$$\lambda_j \nu_i^T M \vec{\nu}_j = \vec{\nu}_i^T K \vec{\nu}_j \tag{21}$$

Generalization (continued)

- Noting M and K symmetric:

$$(\lambda_i - \lambda_j)\vec{\nu}_j^T M \vec{\nu}_i = 0 \tag{22}$$

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- Thus, if $\lambda_i \neq \lambda_j \; \forall i \neq j$, then **orthogonality property** holds for restrained lumped-mass modes:

$$\nu_j^T M \vec{\nu}_i = 0, i \neq j \tag{24}$$

$$\nu_j^T K \vec{\nu}_i = 0, i \neq j \tag{25}$$

Generalization (continued)

- If $i = j$, define **i-th generalized mass**

$$\mathcal{M}_i = \vec{\nu}_i^T M \vec{\nu}_i \tag{26}$$

- **i-th generalized stiffness**

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- Define $\mathcal{M} = \text{diag}[\mathcal{M}_1, ..., \mathcal{M}_n]$
- Define $\mathcal{K} = \text{diag}[\mathcal{K}_1, ..., \mathcal{K}_n]$

Lumped-Mass EOM Alternative

- Lumped-mass vibration equations of motion:

$$\Psi^{-1} M \Psi \ddot{\vec{\eta}} + \Psi^{-1} K \Psi \vec{\eta} = \vec{0} \tag{28}$$

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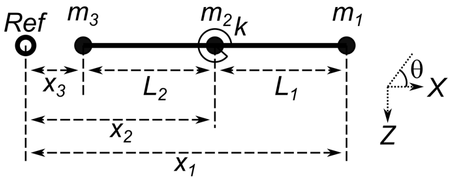
- Demonstrates:

$$\lambda_i = \frac{\mathcal{M}_i}{\mathcal{K}_i} \tag{31}$$

Unrestrained Beam

- Unrestrained three-lumped-mass system: free to translate and rotate



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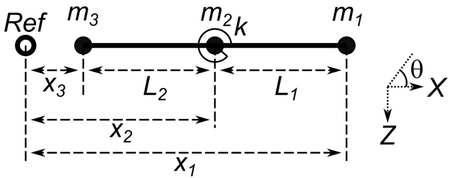

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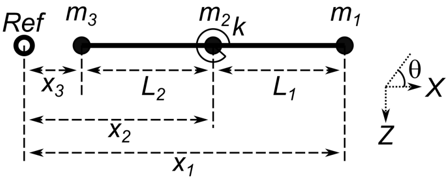

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- Note: bending displacement occurs by relative deflection angle θ between the lines for rods 1 and 2

$$\theta = \frac{Z_1 - Z_2}{x_1 - x_2} - \frac{Z_2 - Z_3}{x_2 - x_3} = \left[\frac{1}{x_1 - x_2} \quad \left(-\frac{1}{x_1 - x_2} - \frac{1}{x_2 - x_3} \right) \quad \frac{1}{x_2 - x_3} \right] \vec{Z} = C\vec{Z} \qquad (32)$$

  - C: constraint matrix relates beam-displacement coordinates

Restrained Beam Energies

- Kinetic energy of beam:

$$T = \frac{1}{2}\begin{bmatrix}\dot{Z}_1\\ \dot{Z}_2\\ \dot{Z}_3\end{bmatrix}^T \begin{bmatrix} m_1 & 0 & 0\\ 0 & m_2 & 0\\ 0 & 0 & m_3\end{bmatrix}\begin{bmatrix}\dot{Z}_1\\ \dot{Z}_2\\ \dot{Z}_3\end{bmatrix} = \frac{1}{2}\dot{\vec{Z}}^T M \dot{\vec{Z}} \tag{33}$$

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- Defining $D_c = M^{-1}K_c$ as **constrained dynamic matrix**:

$$\ddot{\vec{Z}} + D_c\vec{Z} = \vec{0} \tag{36}$$

Modal Analysis

- Continuing modal analysis:

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- Repeating previous equations as M and K_c still symmetric

$$(\lambda_i - \lambda_j) \vec{\nu}_j^T M \vec{\nu}_i = 0 \tag{40}$$

$$(\lambda_i - \lambda_j) \vec{\nu}_j^T K_c \vec{\nu}_i = 0 \tag{41}$$

Eigenvalues

- For unrestrained beam, two of eigenvalues of $D = 0$, i.e. equal
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- Further work required for entire elastic-body motion in terms of mutually orthogonal modes
- Linear algebra: matrix with repeated eigenvalues
  - *Any* linear combination of eigenvectors associated with repeated eigenvalues $=$ eigenvectors of given matrix $\rightarrow$ obtain mutually orthogonal modes for unrestrained bodies

Alternate Approach to Derivation

- Consider rigid-body degrees-of-freedom more directly with total (i.e. inertial) vertical position of masses referenced to reference axis

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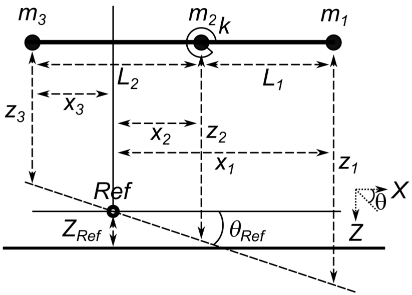
- Three-lumped-body example:



- *Ref*: arbitrary reference point of coordinates

Energies

- Total vertical displacements of lumped-masses (with small angle approximation):

$$\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = \begin{bmatrix} Z_{Ref} \\ Z_{Ref} \\ Z_{Ref} \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \\ -x_3 \end{bmatrix} \theta_{Ref} + \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \qquad \vec{Z} = \vec{1} Z_{Ref} + \vec{x} \theta_{Ref} + \vec{z} \tag{42}$$

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$$U = \frac{1}{2}\theta^T k\theta \tag{44}$$

Momenta Constraints

- Model angular displacement

$$\theta = \frac{z_1 - z_2}{x_1 - x_2} - \frac{z_2 - z_3}{x_2 + x_3} = \begin{bmatrix} \frac{1}{x_1 - x_2} & -\frac{1}{x_1 - x_2} - \frac{1}{x_2 + x_3} & \frac{1}{x_2 + x_3} \end{bmatrix} \vec{z} = C\vec{z} \tag{45}$$

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- Two constraints: absence of external forces and moments, constant translational and rotational momenta
- Taking arbitrary constant to 0, translational momenta:

$$m_1\dot{Z}_1 + m_2\dot{Z}_2 + m_3\dot{Z}_3 = \vec{1}^{\,T} M \dot{\vec{Z}} = 0 \tag{47}$$

Additional Modeling for Rigid Bodies

- Rotational momenta:

$$m_1 x_1 \dot{Z}_1 + m_2 x_2 \dot{Z}_2 - m_3 x_3 \dot{Z}_3 = \vec{x}^T M \dot{\vec{Z}} = 0 \tag{48}$$

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- Two shapes must be mutually orthogonal, with respect to M

$$m_1 x_1 + m_2 x_2 - m_3 x_3 = \vec{1}^T M \vec{x} = 0 \tag{49}$$

  - Reference point, *Ref*, at center of mass of beam

Derivation of Solution to Vibration Problem

- Total mass of beam:

$$M_{tot} = \vec{1}^T M \vec{1} \tag{50}$$

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- Invoke constraints and relative motion $\vec{z}_c$ in terms of mutually orthogonal modal responses $\rightarrow$ desired solution

Constraints

- Differentiating Equation 52 w.r.t. time and momenta constraints:

$$\vec{1}^T M \dot{\vec{Z}}_c = \vec{1}^T M \left[\vec{1} \dot{Z}_{Ref} + \vec{x} \dot{\theta}_{Ref} + \dot{\vec{z}}_c \right] = 0 \quad (53)$$

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- Total mass and moment of inertia equations and center of mass constraint:

$$\dot{Z}_{Ref} = -\frac{1}{M_{tot}} \vec{1}^{\,T} M \dot{\vec{z}}_c \tag{55}$$

$$\dot{\theta}_{Ref} = \frac{1}{I_G} \vec{x}^{\,T} M \dot{\vec{z}}_c \tag{56}$$

Constrained Energies

- Combining two constraints: constrained total velocities as functions of constrained relative velocities:

$$\dot{\vec{Z}}_c = \left[I_{3\times3} - \frac{1}{M_{tot}} \vec{1}\,\vec{1}^T M - \frac{1}{I_G} \vec{x}\,\vec{x}^T M \right] \dot{\vec{z}}_c = \Xi \dot{\vec{z}}_c \tag{57}$$

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Lagrange's Equation

- Utilizing $\vec{z}_c$ as generalized coordinates $\vec{q}$ in Lagrange's equation in vector form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\vec{q}}}\right) - \frac{\partial T}{\partial \vec{q}} + \frac{\partial U}{\partial \vec{q}} = \vec{0}^{T} \tag{60}$$

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- Constrained mass matrix M_c now singular and inverse does not exist
- For mode shapes and vibration frequencies $\rightarrow$ solve **generalized eigenvalue problem**, i.e.

$$(\lambda_i M_c - K_c)\vec{\nu}_i = 0 \tag{62}$$

Generalized Eigenvalue Problem

- Unrestrained beam modal analysis$\rightarrow$ generalized eigenvalue problem provide model for incorporating vibration modes into rigid-body dynamics to produce elastic-body dynamics

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- When $i \neq j$ and eigenvalues distinct $\rightarrow$ assures two associated eigenvectors are orthogonal with respect to M_c (not M) as required

Constrained Eigenvectors

- Definition of M_c provides

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- For unrestrained beam example: 3 mode shapes
  - 2 of transformed generalized eigenvectors $= \vec{0}$ due to two rigid-body modes
  - 1 satisfy three orthogonal constraints $\rightarrow$ single vibration mode shape $\vec{\nu}_{vib}$

General n-Lumped-Mass Beam

- Extend to relative displacement equation for $n-2$ vibration modes

$$\vec{z}_c(t) = \sum_{i=1}^{n-2} \vec{\nu}_{vib,i} A_i \cos(\omega_{vib,i} t + \Gamma_i) \tag{67}$$

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- To prove also orthogonal to rigid-body mode shapes: $\vec{1}$ and $\vec{x}$
- 1st consider relative motion $\vec{z}_c$ as function of original eigenvectors

$$\vec{z}_c(t) = \sum_{i=1}^{n} \vec{\nu}_i \eta_i(t) \tag{68}$$

Proof Constraints

- Recall: orthogonality constraints for $\vec{\dot{Z}}_c$ relative to $\vec{1}$ & $\vec{x}$

$$\vec{1}^{\,T} M \vec{\dot{Z}}_c = 0 \tag{69}$$

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- Thus

$$\vec{1}^T M \dot{\vec{Z}}_c = \vec{1}^T M \Xi \dot{\vec{z}}_c = \vec{1}^T M \Xi \sum_{i=1}^{n} \vec{\nu}_i \dot{\eta}_i(t) = 0 \tag{72}$$

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Proof by Inspection

- Requires

$$\vec{1}^T M \Xi \vec{\nu}_i = \vec{1}^T M \vec{\nu}_{c,i} = 0 \quad \forall i \tag{74}$$

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Key Idea for Orthogonality

- Physical responses of unrestrained beam expressed in terms of linear combination of *mutually orthogonal* modes

$$\sum_{i=1}^{n} \vec{\nu}_{c,i}\eta_i = \vec{1}\eta_n + \vec{x}\eta_{n-1} + \sum_{i=1}^{n-2} \vec{\nu}_{vib,i}\eta_i \tag{76}$$

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  - A.k.a. **Normal Modes**

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  - Rigid body + vibration modes
  - A.k.a. **Normal Modes**
- Recall: unrestrained three-lumped mass model for vertical displacements (with small angles)

$$\begin{aligned} \vec{Z}(t) &= \vec{1}Z_{Ref}(t) + \vec{x}\theta_{Ref}(t) + \vec{z}_{vib}(t) \\ \vec{Z}(t) &= \vec{1}Z_{Ref}(t) + \vec{x}\theta_{Ref}(t) + \sum_{i=1}^{n-2} \vec{\nu}_{vib,i}\eta_i(t) \end{aligned} \tag{77}$$

EOM for Z of Elastic Body

- From generalized coordinates for EOM for Z of elastic-body:

$$\vec{Z} = \begin{bmatrix} \vec{1} & \vec{x} & \vec{\nu}_{vib,1} & \cdots & \vec{\nu}_{vib,n-2} \end{bmatrix} \begin{bmatrix} Z_{Ref} \\ \theta_{Ref} \\ \eta_1 \\ \vdots \\ \eta_{n-2} \end{bmatrix} = \Psi \vec{q} \tag{78}$$

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- Kinetic energy of beam:

$$\begin{aligned} T &= \frac{1}{2} \dot{\vec{Z}} M \dot{\vec{Z}} = \frac{1}{2} \dot{\vec{q}} \Psi^T M \Psi \dot{\vec{q}} \\ &= \frac{1}{2} M_{tot} \dot{Z}_{Ref}^2 + \frac{1}{2} I_G \dot{\theta}_{Ref}^2 + \frac{1}{2} \dot{\vec{\nu}}_{vib}^T \mathcal{M}_{vib} \dot{\vec{\nu}}_{vib} \\ &= \frac{1}{2} \dot{\vec{q}} \mathcal{M} \dot{\vec{q}} \end{aligned} \tag{79}$$

EOM for Z of Elastic Body (continued)

- Rigid body kinetic energy and elastic kinetic energy linearly combine

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  - $\mathcal{K}_{vib}$: generalized stiffness matrix
  - Ψ_{vib}: vibration modal matrix

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- Free response of unrestrained beam EOMs
  - Fundamental for elastic-body flight dynamics

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- Solving Euler-Lagrange EOM using modified constrained mass matrix $[MM]'$ and modified constrained stiffness matrix K_c':

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- Elastic-bodies:
  - Continuous deformation: infinite dimensional problem
  - Lumped-mass approximation: finite dimensional problem
- Lumped-mass vibration dynamics: Euler-Lagrange EOM
  - Use generalized coordinates
  - Modal analysis of mass and stiffness matrices
- Unrestrained lumped-mass dynamics
  - Constrained mass matrix: singular
  - Requires generalized eigenvalues
- Solving Euler-Lagrange EOM using modified constrained mass matrix $[MM]'$ and modified constrained stiffness matrix K_c':
- Same modal analysis for mutually orthogonal modes and coordinates previously performed also possible for multi-directional motion
  - Caveat: “proper” mass and stiffness matrices used

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- Summary: generalized coordinates for vibration problems
  - Vibration modes orthogonal to rigid-body modes
  - Demonstrated for simple lumped-mass models for visualizing construction of elastic