Lecture 9: Advanced Airplane Trim and Relative Orbital Dynamics

Textbook Sections 9.1 & 9.2

Dr. Jordan D. Larson

Introduction

- Airplane 6-DOF EOMs
  - Complicated

Introduction

- Airplane 6-DOF EOMs
  - Complicated
- Stability analysis and control design
  - Lyapunov indirect method: linearize about trimmed steady-flight
  - LTI controls theory

Introduction

- Airplane 6-DOF EOMs
  - Complicated
- Stability analysis and control design
  - Lyapunov indirect method: linearize about trimmed steady-flight
  - LTI controls theory
- Introductory course:
  - Straight, coordinated, wings-level flight
  - Decoupled longitudinal and lateral-directional trim analysis, linearization, and control design

Introduction

- Airplane 6-DOF EOMs
  - Complicated
- Stability analysis and control design
  - Lyapunov indirect method: linearize about trimmed steady-flight
  - LTI controls theory
- Introductory course:
  - Straight, coordinated, wings-level flight
  - Decoupled longitudinal and lateral-directional trim analysis, linearization, and control design
- Lecture: advanced, i.e. turning, steady-flight

Steady-Flight Conditions

- Recall: rigid-body **steady-flight conditions**, by definition, occur when state variables in rigid airplane EOMs constant:

$$\dot{u} = \dot{\alpha} = \dot{\beta} = \dot{p} = \dot{q} = \dot{r} = \dot{\phi} = \dot{\theta} = 0 \tag{1}$$

Steady-Flight Conditions

- Recall: rigid-body **steady-flight conditions**, by definition, occur when state variables in rigid airplane EOMs constant:

$$\dot{u} = \dot{\alpha} = \dot{\beta} = \dot{p} = \dot{q} = \dot{r} = \dot{\phi} = \dot{\theta} = 0 \tag{1}$$

- Steady-flight conditions solve **rigid-body steady-flight equations** in stability frame, i.e., $\bar{\alpha} = 0$:

$$\begin{bmatrix} \bar{X} - g\sin\bar{\gamma} \\ \bar{Y} + g\sin\bar{\phi}\cos\bar{\gamma} \\ \bar{Z} - g\cos\bar{\phi}\cos\bar{\gamma} \\ \bar{L}_{roll} \\ \bar{M} \\ \bar{N} \end{bmatrix} = \begin{bmatrix} -\bar{r}\bar{u}\tan\bar{\beta} \\ \bar{r}\bar{u} \\ \bar{p}\bar{u}\tan\bar{\beta} - \bar{q}\bar{u} \\ \frac{I_{zz}-I_{yy}}{I_{xx}}\bar{q}\bar{r} - \frac{I_{xz}}{I_{xx}}\bar{p}\bar{q} \\ \frac{I_{xx}-I_{zz}}{I_{yy}}\bar{p}\bar{r} + \frac{I_{xz}}{I_{yy}}(\bar{p}^2 - \bar{r}^2) \\ \frac{I_{yy}-I_{xx}}{I_{zz}}\bar{p}\bar{q} + \frac{I_{xz}}{I_{zz}}\bar{q}\bar{r} \end{bmatrix} \tag{2}$$

Steady-Flight Conditions (continued)

- Supplemental kinematic equations given as

$$\begin{bmatrix} \bar{p} \\ \bar{q} \\ \bar{r} \end{bmatrix} = \begin{bmatrix} -\dot{\psi} \sin \bar{\theta} \\ \dot{\psi} \sin \bar{\phi} \cos \bar{\theta} \\ \dot{\psi} \cos \bar{\phi} \cos \bar{\theta} \end{bmatrix} \tag{3}$$

Steady-Flight Conditions (continued)

- Supplemental kinematic equations given as

$$\begin{bmatrix} \bar{p} \\ \bar{q} \\ \bar{r} \end{bmatrix} = \begin{bmatrix} -\dot{\psi}\sin\bar{\theta} \\ \dot{\psi}\sin\bar{\phi}\cos\bar{\theta} \\ \dot{\psi}\cos\bar{\phi}\cos\bar{\theta} \end{bmatrix} \tag{3}$$

- $\dot{\psi}$ constant
  - $\dot{\psi} = 0$: straight steady-flight
  - $\dot{\psi} \neq 0$: turning steady-flight

Level Steady-Flight Conditions

- Simplified analysis: assume coordinated **level** turns, i.e. $\bar{\gamma} = 0$

Level Steady-Flight Conditions

- Simplified analysis: assume coordinated **level** turns, i.e. $\bar{\gamma} = 0$
- Supplemental kinematic equations:

$$\begin{bmatrix} \bar{p} \\ \bar{q} \\ \bar{r} \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{\psi} \sin \bar{\phi} \\ \dot{\psi} \cos \bar{\phi} \end{bmatrix} \tag{4}$$

  - Steady-state pitch rate, $\bar{q} \neq 0$, due to non-zero roll angle of airplane required to turn

Level Steady-Flight Conditions

- Simplified analysis: assume coordinated **level** turns, i.e. $\bar{\gamma} = 0$
- Supplemental kinematic equations:

$$\begin{bmatrix} \bar{p} \\ \bar{q} \\ \bar{r} \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{\psi} \sin \bar{\phi} \\ \dot{\psi} \cos \bar{\phi} \end{bmatrix} \tag{4}$$

  - Steady-state pitch rate, $\bar{q} \neq 0$, due to non-zero roll angle of airplane required to turn
- Substituting for aerodynamic and propulsive forces and moments:

$$\begin{bmatrix} -\bar{D}\cos\bar{\beta} - \bar{S}\sin\bar{\beta} + \bar{T}\cos(\theta_T + \bar{\alpha}) \\ \bar{S}\cos\bar{\beta} - \bar{D}\sin\bar{\beta} + mg\sin\bar{\phi} \\ -\bar{L} - \bar{T}\sin(\theta_T + \bar{\alpha}) + mg\cos\bar{\phi} \\ \bar{L}_{a,S} \\ \bar{M}_{a,S} + \bar{T}(z_T\cos\theta_T - x_T\sin\theta_T) \\ \bar{N}_{a,S} \end{bmatrix} = \begin{bmatrix} -m\bar{r}\bar{u}\tan\bar{\beta} \\ m\bar{r}\bar{u} \\ -m\bar{q}\bar{u} \\ (I_{zz} - I_{yy})\,\bar{q}\bar{r} \\ -I_{xz}\bar{r}^2 \\ I_{xz}\bar{q}\bar{r} \end{bmatrix} \tag{5}$$

  - Assumes thrust force symmetric w.r.t. $x_B - z_B$ plane of vehicle

Level Steady-Flight Conditions (continued)

- Assume zero lateral aerodynamic force:

$$\bar{S}\cos\bar{\beta} - \bar{D}\sin\bar{\beta} = 0 \tag{6}$$

Level Steady-Flight Conditions (continued)

- Assume zero lateral aerodynamic force:

$$\bar{S}\cos\bar{\beta} - \bar{D}\sin\bar{\beta} = 0 \tag{6}$$

- Second and third equations for lateral and vertical forces:

$$\begin{bmatrix} mg\sin\bar{\phi} \\ \bar{L} + \bar{T}\sin(\theta_T + \bar{\alpha}) - mg\cos\bar{\phi} \end{bmatrix} = \begin{bmatrix} m\bar{r}\bar{u} \\ m\bar{q}\bar{u} \end{bmatrix} \tag{7}$$

Level Steady-Flight Conditions (continued)

- Assume zero lateral aerodynamic force:

$$\bar{S}\cos\bar{\beta} - \bar{D}\sin\bar{\beta} = 0 \tag{6}$$

- Second and third equations for lateral and vertical forces:

$$\begin{bmatrix} mg\sin\bar{\phi} \\ \bar{L} + \bar{T}\sin(\theta_T + \bar{\alpha}) - mg\cos\bar{\phi} \end{bmatrix} = \begin{bmatrix} m\bar{r}\bar{u} \\ m\bar{q}\bar{u} \end{bmatrix} \tag{7}$$

- Substituting in terms of $\dot{\psi}$:

$$\begin{bmatrix} mg\sin\bar{\phi} \\ \bar{L} + \bar{T}\sin(\theta_T + \bar{\alpha}) - mg\cos\bar{\phi} \end{bmatrix} = \begin{bmatrix} m\bar{u}\left(\dot{\psi}\cos\bar{\phi}\right) \\ m\bar{u}\left(\dot{\psi}\sin\bar{\phi}\right) \end{bmatrix} \tag{8}$$

Level Steady-Flight Conditions (continued)

- Assume zero lateral aerodynamic force:

$$\bar{S}\cos\bar{\beta} - \bar{D}\sin\bar{\beta} = 0 \tag{6}$$

- Second and third equations for lateral and vertical forces:

$$\begin{bmatrix} mg\sin\bar{\phi} \\ \bar{L} + \bar{T}\sin(\theta_T + \bar{\alpha}) - mg\cos\bar{\phi} \end{bmatrix} = \begin{bmatrix} m\bar{r}\bar{u} \\ m\bar{q}\bar{u} \end{bmatrix} \tag{7}$$

- Substituting in terms of $\dot{\psi}$:

$$\begin{bmatrix} mg\sin\bar{\phi} \\ \bar{L} + \bar{T}\sin(\theta_T + \bar{\alpha}) - mg\cos\bar{\phi} \end{bmatrix} = \begin{bmatrix} m\bar{u}\left(\dot{\psi}\cos\bar{\phi}\right) \\ m\bar{u}\left(\dot{\psi}\sin\bar{\phi}\right) \end{bmatrix} \tag{8}$$

- First equation rewritten:

$$mg\tan\bar{\phi} = m\bar{u}\dot{\psi} \tag{9}$$

Level Steady-Flight Conditions (continued)

- Note:

$$\dot{\psi} = \frac{g}{\bar{u}} \tan \bar{\phi} \tag{10}$$

Level Steady-Flight Conditions (continued)

- Note:

$$\dot{\psi} = \frac{g}{\bar{u}} \tan \bar{\phi} \tag{10}$$

- Substituting into second equation:

$$\bar{L} + \bar{T} \sin(\theta_T + \bar{\alpha}) = mg \left(\cos \bar{\phi} + \tan \bar{\phi} \sin \bar{\phi} \right) \tag{11}$$

Level Steady-Flight Conditions (continued)

- Note:

$$\dot{\psi} = \frac{g}{\bar{u}} \tan \bar{\phi} \tag{10}$$

- Substituting into second equation:

$$\bar{L} + \bar{T} \sin(\theta_T + \bar{\alpha}) = mg \left(\cos \bar{\phi} + \tan \bar{\phi} \sin \bar{\phi} \right) \tag{11}$$

- Multiplying by $\cos \bar{\phi}$:

$$\left(\bar{L} + \bar{T} \sin(\theta_T + \bar{\alpha}) \right) \cos \bar{\phi} = mg \left(\cos^2 \bar{\phi} + \sin^2 \bar{\phi} \right) \tag{12}$$

$$\left(\bar{L} + \bar{T} \sin(\theta_T + \bar{\alpha}) \right) \cos \bar{\phi} = mg \tag{13}$$

Normal Load Factor

- Define dimensionless **normal load factor**, n:

$$n(mg) = L + T\sin(\theta_T + \alpha) \tag{14}$$

  - Referred to in terms of “g’s”

$$\bar{n}(mg) = \frac{1}{\cos\bar{\phi}} \tag{15}$$

Normal Load Factor

- Define dimensionless **normal load factor**, n:

$$n(mg) = L + T\sin(\theta_T + \alpha) \tag{14}$$

  - Referred to in terms of "g's"

$$\bar{n}(mg) = \frac{1}{\cos\bar{\phi}} \tag{15}$$

- Often specified instead of $\bar{\phi}$ for steady turns
  - Requires direction of turn to be specified
  - Wings-level flight, i.e., $\bar{\phi} = 0$: $\bar{n} = 1$ g

Normal Load Factor (continued)

- Denote pitch and yaw rates:

$$\begin{bmatrix} \bar{q} \\ \bar{r} \end{bmatrix} = \begin{bmatrix} \frac{g}{\bar{u}} \tan \bar{\phi} \sin \bar{\phi} \\ \frac{g}{\bar{u}} \sin \bar{\phi} \end{bmatrix} \tag{16}$$

$$\begin{bmatrix} \bar{q} \\ \bar{r} \end{bmatrix} = \begin{bmatrix} \frac{g}{\bar{u}} \left(\bar{n} - \frac{1}{\bar{n}}\right) \\ \pm \frac{g}{\bar{u}\bar{n}} \sqrt{\bar{n}^2 - 1} \end{bmatrix} \tag{17}$$

Lateral-Directional Trim Conditions

- Derivations assume:

$$\bar{S}\cos\bar{\beta} - \bar{D}\sin\bar{\beta} = 0 \tag{18}$$

Lateral-Directional Trim Conditions

- Derivations assume:

$$\bar{S}\cos\bar{\beta} - \bar{D}\sin\bar{\beta} = 0 \tag{18}$$

- Expressing lateral-directional aerodynamic forces and moments in terms of coefficients, C_S, C_l, and C_n, lateral-directional trim conditions for level turns:

$$\begin{bmatrix} Q_\infty S_w \bar{C}_S \cos\bar{\beta} - Q_\infty S_w \bar{C}_D \sin\bar{\beta} \\ Q_\infty S_w \bar{C}_l \\ Q_\infty S_w \bar{C}_n \end{bmatrix} = \begin{bmatrix} 0 \\ \left(I_{zz} - I_{yy}\right)\bar{q}\bar{r} \\ I_{xz}\bar{q}\bar{r} \end{bmatrix} \tag{19}$$

Lateral-Directional Trim Conditions

- Derivations assume:

$$\bar{S}\cos\bar{\beta} - \bar{D}\sin\bar{\beta} = 0 \tag{18}$$

- Expressing lateral-directional aerodynamic forces and moments in terms of coefficients, C_S, C_l, and C_n, lateral-directional trim conditions for level turns:

$$\begin{bmatrix} Q_\infty S_w \bar{C}_S \cos\bar{\beta} - Q_\infty S_w \bar{C}_D \sin\bar{\beta} \\ Q_\infty S_w \bar{C}_l \\ Q_\infty S_w \bar{C}_n \end{bmatrix} = \begin{bmatrix} 0 \\ \left(I_{zz} - I_{yy}\right)\bar{q}\bar{r} \\ I_{xz}\bar{q}\bar{r} \end{bmatrix} \tag{19}$$

- Substituting for $\bar{r}$ and $\bar{q}$:

$$\begin{bmatrix} Q_\infty S_w \bar{C}_S \cos\bar{\beta} - Q_\infty S_w \bar{C}_D \sin\bar{\beta} \\ Q_\infty S_w \bar{C}_l \\ Q_\infty S_w \bar{C}_n \end{bmatrix} = \begin{bmatrix} 0 \\ \left(I_{zz} - I_{yy}\right)\left(\frac{g}{\bar{u}}\right)^2 \tan\bar{\phi}\sin^2\bar{\phi} \\ I_{xz}\left(\frac{g}{\bar{u}}\right)^2 \tan\bar{\phi}\sin^2\bar{\phi} \end{bmatrix} \tag{20}$$

Simplified Analysis: Linear Coefficients

- Assume linear relationships for aerodynamic coefficients w.r.t. trim drag coefficient, $\bar{C}_D$, trim side coefficient, $\bar{C}_S$, trim sideslip angle, $\bar{\beta}$, trim yaw rate, $\bar{r}$, trim aileron deflection $\bar{\delta}_a$, and trim rudder deflection, $\bar{\delta}_r$:

$$\bar{C}_S = C_{S_\beta}\bar{\beta} + C_{S_r}\bar{r} + C_{S_{\delta_a}}\bar{\delta}_a + C_{S_{\delta_a}}\bar{\delta}_r \tag{21}$$

$$\bar{C}_l = C_{l_\beta}\bar{\beta} + C_{l_r}\bar{r} + C_{l_{\delta_a}}\bar{\delta}_a + C_{l_{\delta_a}}\bar{\delta}_r \tag{22}$$

$$\bar{C}_n = C_{n_\beta}\bar{\beta} + C_{n_r}\bar{r} + C_{n_{\delta_a}}\bar{\delta}_a + C_{n_{\delta_a}}\bar{\delta}_r \tag{23}$$

Lateral-Directional Trim Computation

- By substitution for coefficients and $\bar{r}$ in terms of $\bar{\phi}$:

$$\begin{bmatrix} Q_\infty S_w \left(C_{S_\beta}\bar{\beta} + C_{S_r}\frac{g}{\bar{u}}\sin\bar{\phi} + C_{S_{\delta_a}}\bar{\delta}a + C_{S_{\delta_a}}\bar{\delta}r \right)\cos\bar{\beta} - Q_\infty S_w \bar{C}_D \sin\bar{\beta} \\ Q_\infty S_w b_w \left(C_{l_\beta}\bar{\beta} + C_{l_r}\frac{g}{\bar{u}}\sin\bar{\phi} + C_{l_{\delta_a}}\bar{\delta}a + C_{l_{\delta_a}}\bar{\delta}r \right) \\ Q_\infty S_w b_w \left(C_{n_\beta}\bar{\beta} + C_{n_r}\frac{g}{\bar{u}}\sin\bar{\phi} + C_{n_{\delta_a}}\bar{\delta}a + C_{n_{\delta_a}}\bar{\delta}r \right) \end{bmatrix} = \begin{bmatrix} 0 \\ (I_{zz} - I_{yy})\left(\frac{g}{\bar{u}}\right)^2 \tan\bar{\phi}\sin^2\bar{\phi} \\ I_{xz}\left(\frac{g}{\bar{u}}\right)^2 \tan\bar{\phi}\sin^2\bar{\phi} \end{bmatrix} \quad (24)$$

Lateral-Directional Trim Computation

- By substitution for coefficients and $\bar{r}$ in terms of $\bar{\phi}$:

$$\begin{bmatrix} Q_\infty S_w \left(C_{S_\beta} \bar{\beta} + C_{S_r} \frac{g}{\bar{u}} \sin \bar{\phi} + C_{S_{\delta_a}} \bar{\delta}_a + C_{S_{\delta_a}} \bar{\delta}_r \right) \cos \bar{\beta} - Q_\infty S_w \bar{C}_D \sin \bar{\beta} \\ Q_\infty S_w b_w \left(C_{l_\beta} \bar{\beta} + C_{l_r} \frac{g}{\bar{u}} \sin \bar{\phi} + C_{l_{\delta_a}} \bar{\delta}_a + C_{l_{\delta_a}} \bar{\delta}_r \right) \\ Q_\infty S_w b_w \left(C_{n_\beta} \bar{\beta} + C_{n_r} \frac{g}{\bar{u}} \sin \bar{\phi} + C_{n_{\delta_a}} \bar{\delta}_a + C_{n_{\delta_a}} \bar{\delta}_r \right) \end{bmatrix} = \begin{bmatrix} 0 \\ (I_{zz} - I_{yy}) \left(\frac{g}{\bar{u}} \right)^2 \tan \bar{\phi} \sin^2 \bar{\phi} \\ I_{xz} \left(\frac{g}{\bar{u}} \right)^2 \tan \bar{\phi} \sin^2 \bar{\phi} \end{bmatrix} \quad (24)$$

- Given Q_∞, i.e. given airspeed $\bar{u}$ and altitude, and roll angle, $\bar{\phi}$ or normal load factor $\bar{n}$:
  - Equations determine 3 unknowns for lateral-directional trim conditions: $\bar{\beta}$, $\bar{\delta}_a$, $\bar{\delta}_r$
  - Can be solved using numerical methods

Lateral-Directional Trim Computation

- By substitution for coefficients and $\bar{r}$ in terms of $\bar{\phi}$:

$$\begin{bmatrix} Q_\infty S_w \left(C_{S_\beta}\bar{\beta} + C_{S_r}\frac{g}{\bar{u}}\sin\bar{\phi} + C_{S_{\delta_a}}\bar{\delta}_a + C_{S_{\delta_a}}\bar{\delta}_r \right)\cos\bar{\beta} - Q_\infty S_w \bar{C}_D \sin\bar{\beta} \\ Q_\infty S_w b_w \left(C_{l_\beta}\bar{\beta} + C_{l_r}\frac{g}{\bar{u}}\sin\bar{\phi} + C_{l_{\delta_a}}\bar{\delta}_a + C_{l_{\delta_a}}\bar{\delta}_r \right) \\ Q_\infty S_w b_w \left(C_{n_\beta}\bar{\beta} + C_{n_r}\frac{g}{\bar{u}}\sin\bar{\phi} + C_{n_{\delta_a}}\bar{\delta}_a + C_{n_{\delta_a}}\bar{\delta}_r \right) \end{bmatrix} = \begin{bmatrix} 0 \\ (I_{zz} - I_{yy})\left(\frac{g}{\bar{u}}\right)^2 \tan\bar{\phi}\sin^2\bar{\phi} \\ I_{xz}\left(\frac{g}{\bar{u}}\right)^2 \tan\bar{\phi}\sin^2\bar{\phi} \end{bmatrix} \tag{24}$$

- Given Q_∞, i.e. given airspeed $\bar{u}$ and altitude, and roll angle, $\bar{\phi}$ or normal load factor $\bar{n}$:
  - Equations determine 3 unknowns for lateral-directional trim conditions: $\bar{\beta}$, $\bar{\delta}_a$, $\bar{\delta}_r$
  - Can be solved using numerical methods
- Requires $\bar{C}_D$ already been obtained from longitudinal trim analysis
  - Longitudinal analysis depends on trim sideslip angle, $\bar{\beta}$
  - Iterate between trim computations
  - Initial attempt typically assumes $\bar{C}_D \approx 0$

Analytical Approach

- Assume $\bar{\beta}$ small, i.e. $\sin\bar{\beta} \approx \bar{\beta}$ and $\cos\bar{\beta} \approx 1$:

$$\begin{bmatrix} \bar{C}_{S_\beta} - \bar{C}_D & C_{S_{\delta_a}} & C_{S_{\delta_r}} \\ C_{l_\beta} & C_{l_{\delta_a}} & C_{l_{\delta_r}} \\ C_{n_\beta} & C_{n_{\delta_a}} & C_{n_{\delta_r}} \end{bmatrix} \begin{bmatrix} \bar{\beta} \\ \bar{\delta}_a \\ \bar{\delta}_r \end{bmatrix} = \begin{bmatrix} -C_{S_r} \frac{g}{\bar{u}} \sin\bar{\phi} \\ \left(\frac{I_{zz} - I_{yy}}{Q_\infty S_w b_w}\right) \left(\frac{g}{\bar{u}}\right)^2 \tan\bar{\phi} \sin^2\bar{\phi} - C_{l_r} \frac{g}{\bar{u}} \sin\bar{\phi} \\ \frac{I_{xz}}{Q_\infty S_w b_w} \left(\frac{g}{\bar{u}}\right)^2 \tan\bar{\phi} \sin^2\bar{\phi} - C_{n_r} \frac{g}{\bar{u}} \sin\bar{\phi} \end{bmatrix} \tag{25}$$

Analytical Approach

- Assume $\bar{\beta}$ small, i.e. $\sin\bar{\beta} \approx \bar{\beta}$ and $\cos\bar{\beta} \approx 1$:

$$\begin{bmatrix} \bar{C}_{S_\beta} - \bar{C}_D & C_{S_{\delta_a}} & C_{S_{\delta_r}} \\ C_{l_\beta} & C_{l_{\delta_a}} & C_{l_{\delta_r}} \\ C_{n_\beta} & C_{n_{\delta_a}} & C_{n_{\delta_r}} \end{bmatrix} \begin{bmatrix} \bar{\beta} \\ \bar{\delta}_a \\ \bar{\delta}_r \end{bmatrix} = \begin{bmatrix} -C_{S_r} \frac{g}{\bar{u}} \sin\bar{\phi} \\ \left(\frac{I_{zz}-I_{yy}}{Q_\infty S_w b_w}\right) \left(\frac{g}{\bar{u}}\right)^2 \tan\bar{\phi}\sin^2\bar{\phi} - C_{l_r}\frac{g}{\bar{u}}\sin\bar{\phi} \\ \frac{I_{xz}}{Q_\infty S_w b_w}\left(\frac{g}{\bar{u}}\right)^2 \tan\bar{\phi}\sin^2\bar{\phi} - C_{n_r}\frac{g}{\bar{u}}\sin\bar{\phi} \end{bmatrix} \tag{25}$$

- With trim normal load factor:

$$\begin{bmatrix} \bar{C}_{S_\beta} - \bar{C}_D & C_{S_{\delta_a}} & C_{S_{\delta_r}} \\ C_{l_\beta} & C_{l_{\delta_a}} & C_{l_{\delta_r}} \\ C_{n_\beta} & C_{n_{\delta_a}} & C_{n_{\delta_r}} \end{bmatrix} \begin{bmatrix} \bar{\beta} \\ \bar{\delta}_a \\ \bar{\delta}_r \end{bmatrix} = \begin{bmatrix} \mp C_{S_r} \pm \frac{g}{\bar{u}\bar{n}}\sqrt{\bar{n}^2-1} \\ \pm\left(\frac{I_{zz}-I_{yy}}{Q_\infty S_w b_w}\right)\left(\frac{g}{\bar{u}}\right)^2\left(1-\frac{1}{\bar{n}^2}\right)\sqrt{\bar{n}^2-1} \mp C_{l_r}\frac{g}{\bar{u}\bar{n}}\sqrt{\bar{n}^2-1} \\ \pm\frac{I_{xz}}{Q_\infty S_w b_w}\left(\frac{g}{\bar{u}}\right)^2\left(1-\frac{1}{\bar{n}^2}\right)\sqrt{\bar{n}^2-1} \mp C_{n_r}\frac{g}{\bar{u}\bar{n}}\sqrt{\bar{n}^2-1} \end{bmatrix} \tag{26}$$

  - Solved by multiplying both sides by inverse matrix on left side

Longitudinal Trim Conditions

- Solved for trim sideslip angle, $\bar{\beta}$, trim side force coefficient, $\bar{C}_S$, trim aileron deflection, $\bar{\delta}_a$, trim rudder deflection $\bar{\delta}_r$, longitudinal trim conditions for level turns:

$$\begin{bmatrix} -\bar{D}\cos\bar{\beta} - \bar{S}\sin\bar{\beta} + \bar{T}\cos(\theta_T + \bar{\alpha}) \\ -\bar{L} - \bar{T}\sin(\theta_T + \bar{\alpha}) + mg\cos\bar{\phi} \\ \bar{M}_a + \bar{T}(z_T\cos\theta_T - x_T\sin\theta_T) \end{bmatrix} = \begin{bmatrix} -m\bar{r}\bar{u}\tan\bar{\beta} \\ -m\bar{q}\bar{u} \\ -I_{xz}\bar{r}^2 \end{bmatrix} \tag{27}$$

Longitudinal Trim Conditions

- Solved for trim sideslip angle, $\bar{\beta}$, trim side force coefficient, $\bar{C}_S$, trim aileron deflection, $\bar{\delta}_a$, trim rudder deflection $\bar{\delta}_r$, longitudinal trim conditions for level turns:

$$\begin{bmatrix} -\bar{D}\cos\bar{\beta} - \bar{S}\sin\bar{\beta} + \bar{T}\cos(\theta_T + \bar{\alpha}) \\ -\bar{L} - \bar{T}\sin(\theta_T + \bar{\alpha}) + mg\cos\bar{\phi} \\ \bar{M}_a + \bar{T}(z_T\cos\theta_T - x_T\sin\theta_T) \end{bmatrix} = \begin{bmatrix} -m\bar{r}\bar{u}\tan\bar{\beta} \\ -m\bar{q}\bar{u} \\ -I_{xz}\bar{r}^2 \end{bmatrix} \tag{27}$$

- Express longitudinal aerodynamic forces and moments in terms of coefficients, C_L, C_S, C_D, C_m:

$$\begin{bmatrix} -Q_\infty S_w \bar{C}_D\cos\bar{\beta} - Q_\infty S_w \bar{C}_S\sin\bar{\beta} + \bar{T}\cos(\theta_T + \bar{\alpha}) \\ Q_\infty S_w \bar{C}_L + \bar{T}\sin(\theta_T + \bar{\alpha}) \\ Q_\infty S_w \bar{c}_w \bar{C}_m + \bar{T}(z_T\cos\theta_T - x_T\sin\theta_T) \end{bmatrix} = \begin{bmatrix} -m\bar{r}\bar{u}\tan\bar{\beta} \\ -m\bar{q}\bar{u} \\ -I_{xz}\bar{r}^2 \end{bmatrix} \tag{28}$$

Simplified Analysis: Linear Coefficients

- Assume linear relationships for C_L, C_D, C_m w.r.t. trim angle of attack, $\bar{\alpha}$, trim pitch rate, $\bar{q}$, trim elevator deflection, $\bar{\delta}_e$, trim aileron deflection, $\bar{\delta}_a$, trim rudder deflection $\bar{\delta}_r$:

$$\bar{C}_D = C_{D_0} + C_{D_\alpha}\bar{\alpha} + C_{D_q}\bar{q} + C_{D_{\delta_e}}\bar{\delta}_e + C_{D_{\delta_a}}\bar{\delta}_a + C_{D_{\delta_r}}\bar{\delta}_r \tag{29}$$

$$\bar{C}_L = C_{L_0} + C_{L_\alpha}\bar{\alpha} + C_{L_q}\bar{q} + C_{L_{\delta_e}}\bar{\delta}_e \tag{30}$$

$$\bar{C}_m = C_{m_0} + C_{m_\alpha}\bar{\alpha} + C_{m_q}\bar{q} + C_{m_{\delta_e}}\bar{\delta}_e \tag{31}$$

Longitudinal Trim Computation

- By substitution:

$$\begin{bmatrix} -Q_\infty S_w \left(C_{D_0} + C_{D_\alpha}\bar{\alpha} + C_{D_q}\bar{q} + C_{D_{\delta_e}}\bar{\delta}_e + C_{D_{\delta_a}}\bar{\delta}_a + C_{D_{\delta_r}}\bar{\delta}_r\right)\cos\bar{\beta} - Q_\infty S_w \bar{C}_S \sin\bar{\beta} + \bar{T}\cos(\theta_T + \bar{\alpha}) \\ Q_\infty S_w \left(C_{L_0} + C_{L_\alpha}\bar{\alpha} + C_{L_q}\bar{q} + C_{L_{\delta_e}}\bar{\delta}_e\right) + \bar{T}\sin(\theta_T + \bar{\alpha}) \\ Q_\infty S_w \bar{c}_w \left(C_{m_0} + C_{m_q}\bar{q} + C_{m_\alpha}\bar{\alpha} + C_{m_{\delta_e}}\bar{\delta}_e\right) + \bar{T}\frac{z_T\cos\theta_T - x_T\sin\theta_T}{Q_\infty S_w \bar{c}_w} \end{bmatrix} = \begin{bmatrix} -m\bar{r}\bar{u}\tan\bar{\beta} \\ -m\bar{q}\bar{u} \\ -I_{xz}\bar{r}^2 \end{bmatrix} \tag{32}$$

Longitudinal Trim Computation

- By substitution:

$$\begin{bmatrix} -Q_\infty S_w \left(C_{D_0} + C_{D_\alpha}\bar{\alpha} + C_{D_q}\bar{q} + C_{D_{\delta_e}}\bar{\delta}_e + C_{D_{\delta_a}}\bar{\delta}_a + C_{D_{\delta_r}}\bar{\delta}_r\right)\cos\bar{\beta} - Q_\infty S_w \bar{C}_S \sin\bar{\beta} + \bar{T}\cos(\theta_T + \bar{\alpha}) \\ Q_\infty S_w \left(C_{L_0} + C_{L_\alpha}\bar{\alpha} + C_{L_q}\bar{q} + C_{L_{\delta_e}}\bar{\delta}_e\right) + \bar{T}\sin(\theta_T + \bar{\alpha}) \\ Q_\infty S_w \bar{c}_w \left(C_{m_0} + C_{m_q}\bar{q} + C_{m_\alpha}\bar{\alpha} + C_{m_{\delta_e}}\bar{\delta}_e\right) + \bar{T}\frac{z_T\cos\theta_T - x_T\sin\theta_T}{Q_\infty S_w \bar{c}_w} \end{bmatrix} = \begin{bmatrix} -m\bar{r}\bar{u}\tan\bar{\beta} \\ -m\bar{q}\bar{u} \\ -I_{xz}\bar{r}^2 \end{bmatrix} \quad (32)$$

- Given Q_∞, i.e. given airspeed $\bar{u}$ and altitude, and roll angle, $\bar{\phi}$ or normal load factor $\bar{n}$:
  - Equations determine three unknowns for longitudinal trim conditions, i.e. $\bar{\alpha}$, $\bar{\delta}_e$, $\bar{T}$
  - Can be solved using numerical methods

Analytical Approach 1

- Assume $\bar{\beta}$ small, i.e. $\sin\bar{\beta} \approx \bar{\beta}$ and $\cos\bar{\beta} \approx 1$, $\cos(\theta_T + \bar{\alpha}) = 1$, and $\sin(\theta_T + \bar{\alpha}) = 0$:

$$
\begin{bmatrix} -C_{D_\alpha} & -C_{D_{\delta_e}} & \frac{1}{Q_\infty S_w} \\ C_{L_\alpha} & C_{L_{\delta_e}} & 0 \\ C_{m_\alpha} & C_{m_{\delta_e}} & \frac{z_T \cos\theta_T - x_T \sin\theta_T}{Q_\infty S_w \bar{c}_w} \end{bmatrix} \begin{bmatrix} \bar{\alpha} \\ \bar{\delta}_e \\ \bar{T} \end{bmatrix} = \begin{bmatrix} \left(\bar{C}_S - \frac{mg}{Q_\infty S_w}\sin\bar{\phi}\right)\bar{\beta} + C_{D_0} + C_{D_q}\frac{g}{\bar{u}}\tan\bar{\phi}\sin\bar{\phi} + C_{D_{\delta_a}}\bar{\delta}_a + C_{D_{\delta_r}}\bar{\delta}_r \\ \frac{mg}{Q_\infty S_w \cos\bar{\phi}} - C_{L_0} - C_{L_q}\frac{g}{\bar{u}}\tan\bar{\phi}\sin\bar{\phi} \\ -I_{xz}\left(\frac{g}{\bar{u}}\right)^2 \sin^2\bar{\phi} - C_{m_0} - C_{m_q}\frac{g}{\bar{u}}\tan\bar{\phi}\sin\bar{\phi} \end{bmatrix} \tag{33}
$$

Analytical Approach 1

- Assume $\bar{\beta}$ small, i.e. $\sin\bar{\beta} \approx \bar{\beta}$ and $\cos\bar{\beta} \approx 1$, $\cos(\theta_T + \bar{\alpha}) = 1$, and $\sin(\theta_T + \bar{\alpha}) = 0$:

$$\begin{bmatrix} -C_{D_\alpha} & -C_{D_{\delta_e}} & \frac{1}{Q_\infty S_w} \\ C_{L_\alpha} & C_{L_{\delta_e}} & 0 \\ C_{m_\alpha} & C_{m_{\delta_e}} & \frac{z_T \cos\theta_T - x_T \sin\theta_T}{Q_\infty S_w \bar{c}_w} \end{bmatrix} \begin{bmatrix} \bar{\alpha} \\ \bar{\delta}_e \\ \bar{T} \end{bmatrix} = \begin{bmatrix} \left(\bar{C}_S - \frac{mg}{Q_\infty S_w}\sin\bar{\phi}\right)\bar{\beta} + C_{D_0} + C_{D_q}\frac{g}{\bar{u}}\tan\bar{\phi}\sin\bar{\phi} + C_{D_{\delta_a}}\bar{\delta}_a + C_{D_{\delta_r}}\bar{\delta}_r \\ \frac{mg}{Q_\infty S_w \cos\bar{\phi}} - C_{L_0} - C_{L_q}\frac{g}{\bar{u}}\tan\bar{\phi}\sin\bar{\phi} \\ -I_{xz}\left(\frac{g}{\bar{u}}\right)^2\sin^2\bar{\phi} - C_{m_0} - C_{m_q}\frac{g}{\bar{u}}\tan\bar{\phi}\sin\bar{\phi} \end{bmatrix} \tag{33}$$

- With trim normal load factor:

$$\begin{bmatrix} -C_{D_\alpha} & -C_{D_{\delta_e}} & \frac{1}{Q_\infty S_w} \\ C_{L_\alpha} & C_{L_{\delta_e}} & 0 \\ C_{m_\alpha} & C_{m_{\delta_e}} & \frac{z_T \cos\theta_T - x_T \sin\theta_T}{Q_\infty S_w \bar{c}_w} \end{bmatrix} \begin{bmatrix} \bar{\alpha} \\ \bar{\delta}_e \\ \bar{T} \end{bmatrix} = \begin{bmatrix} \left(\bar{C}_S - \frac{mg}{Q_\infty S_w}\sqrt{1 - \frac{1}{\bar{n}^2}}\right)\bar{\beta} + C_{D_0} + C_{D_q}\frac{g}{\bar{u}}\tan\bar{\phi}\sin\bar{\phi} + C_{D_{\delta_a}}\bar{\delta}_a + C_{D_{\delta_r}}\bar{\delta}_r \\ \frac{mg}{Q_\infty S_w} - C_{L_0} - C_{L_q}\frac{g}{\bar{u}}\left(\bar{n} - \frac{1}{\bar{n}}\right) \\ -I_{xz}\left(\frac{g}{\bar{u}}\right)^2\left(1 - \frac{1}{\bar{n}^2}\right) - C_{m_0} - C_{m_q}\frac{g}{\bar{u}}\left(\bar{n} - \frac{1}{\bar{n}}\right) \end{bmatrix} \tag{34}$$

  - Solved by multiplying both sides by inverse matrix on left side

Analytical Approach 2

- Alternatively assume $z_T \cos\theta_T - x_T \sin\theta_T \approx 0$:

$$\begin{bmatrix} -C_{D_\alpha} & -C_{D_{\delta_e}} & \frac{\cos(\theta_T+\bar{\alpha})}{Q_\infty S_w} \\ C_{L_\alpha} & C_{L_{\delta_e}} & 0 \\ C_{m_\alpha} & C_{m_{\delta_e}} & 0 \end{bmatrix} \begin{bmatrix} \bar{\alpha} \\ \bar{\delta}_e \\ \bar{T} \end{bmatrix} = \begin{bmatrix} \left(\bar{C}_S - \frac{mg}{Q_\infty S_w}\sqrt{1-\frac{1}{\bar{n}^2}}\right)\bar{\beta} + C_{D_0} + C_{D_q}\frac{g}{\bar{u}}\tan\bar{\phi}\sin\bar{\phi} + C_{D_{\delta_a}}\bar{\delta}_a + C_{D_{\delta_r}}\bar{\delta}_r \\ \bar{n}\frac{mg}{Q_\infty S_w} - C_{L_0} - C_{L_q}\frac{g}{\bar{u}}\left(\bar{n}-\frac{1}{\bar{n}}\right) \\ -I_{xz}\left(\frac{g}{\bar{u}}\right)^2\left(1-\frac{1}{\bar{n}^2}\right) - C_{m_0} - C_{m_q}\frac{g}{\bar{u}}\left(\bar{n}-\frac{1}{\bar{n}}\right) \end{bmatrix} \tag{35}$$

  - Decouple $\bar{\alpha}$ and $\bar{\delta}_e$ from $\bar{T}$

Analytical Approach 2

- Alternatively assume $z_T \cos\theta_T - x_T \sin\theta_T \approx 0$:

$$\begin{bmatrix} -C_{D_\alpha} & -C_{D_{\delta_e}} & \frac{\cos(\theta_T+\bar{\alpha})}{Q_\infty S_w} \\ C_{L_\alpha} & C_{L_{\delta_e}} & 0 \\ C_{m_\alpha} & C_{m_{\delta_e}} & 0 \end{bmatrix} \begin{bmatrix} \bar{\alpha} \\ \bar{\delta}_e \\ \bar{T} \end{bmatrix} = \begin{bmatrix} \left(\bar{C}_S - \frac{mg}{Q_\infty S_w}\sqrt{1-\frac{1}{\bar{n}^2}}\right)\bar{\beta} + C_{D_0} + C_{D_q}\frac{g}{\bar{u}}\tan\bar{\phi}\sin\bar{\phi} + C_{D_{\delta_a}}\bar{\delta}_a + C_{D_{\delta_r}}\bar{\delta}_r \\ \bar{n}\frac{mg}{Q_\infty S_w} - C_{L_0} - C_{L_q}\frac{g}{\bar{u}}\left(\bar{n}-\frac{1}{\bar{n}}\right) \\ -I_{xz}\left(\frac{g}{\bar{u}}\right)^2\left(1-\frac{1}{\bar{n}^2}\right) - C_{m_0} - C_{m_q}\frac{g}{\bar{u}}\left(\bar{n}-\frac{1}{\bar{n}}\right) \end{bmatrix} \tag{35}$$

  - Decouple $\bar{\alpha}$ and $\bar{\delta}_e$ from $\bar{T}$

$$\begin{aligned} \bar{\alpha} &\approx \frac{C_1(n)C_{m_{\delta_e}} - C_2(n)C_{L_{\delta_e}}}{C_{L_\alpha}C_{m_{\delta_e}} - C_{m_\alpha}C_{L_{\delta_e}}} \\ \bar{\delta}_e &\approx -\frac{C_1(n)C_{m_\alpha} - C_2(n)C_{L_\alpha}}{C_{L_\alpha}C_{m_{\delta_e}} - C_{m_\alpha}C_{L_{\delta_e}}} \\ C_1(n) &= \bar{n}\frac{mg}{Q_\infty S_w} - C_{L_0} - C_{L_q}\frac{g}{\bar{u}}\left(\bar{n}-\frac{1}{\bar{n}}\right) \\ C_2(n) &= -I_{xz}\left(\frac{g}{\bar{u}}\right)^2\left(1-\frac{1}{\bar{n}^2}\right) - C_{m_0} - C_{m_q}\frac{g}{\bar{u}}\left(\bar{n}-\frac{1}{\bar{n}}\right) \end{aligned} \tag{36}$$

Analytical Approach 2 (continued)

- Only function of normal load factor and lift and M-moment coefficients

Analytical Approach 2 (continued)

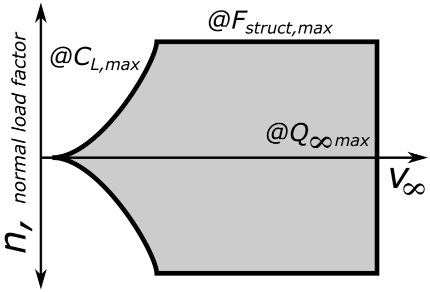
- Only function of normal load factor and lift and *M*-moment coefficients
- Determine trim thrust:

$$\bar{T} = \frac{-mg\sqrt{1-\frac{1}{\bar{n}^2}}\tan\bar{\beta} + Q_\infty S_w \bar{C}_D \cos\bar{\beta} + Q_\infty S_w \bar{C}_S \sin\bar{\beta}}{\cos(\theta_T + \bar{\alpha})} \tag{37}$$

  - Note: function of drag coefficients and trim angle of attack and elevator deflection

V-n Analysis Plot

- Lift coefficient, $C_{L,max}$, dynamic pressure, $Q_{\infty,max}$, structural limits, $F_{struct,max}$, for level, turns define **v-n plot**: airspeed, v_∞, vs. normal load factor, n
  - $C_{L,max}$ results from either maximum angle of attack or maximum elevator deflection
  - $Q_{\infty,max}$ often due to flutter limit
  - Center region defines possible steady-flight conditions for airplane



Coupled Airplane Linearized Dynamics

- For non-coordinated, non-wings-level steady-flight conditions, including turning flight: linearized longitudinal and lateral-directional state-space models coupled
  - Often coupling weak and ignored in control design
  - Under certain trim conditions, e.g., high angles of attack or sideslip, fully coupled equations necessary assess coupling effects of control inputs on *all* airplane states

Coupled Airplane Linearized Dynamics

- For non-coordinated, non-wings-level steady-flight conditions, including turning flight: linearized longitudinal and lateral-directional state-space models coupled
  - Often coupling weak and ignored in control design
  - Under certain trim conditions, e.g., high angles of attack or sideslip, fully coupled equations necessary assess coupling effects of control inputs on *all* airplane states
- Coupling uses coupled stability and control derivatives, e.g. X_β, Y_α, L^*_α, N^*_α, and X_{δ_r}

Coupled Airplane Linearized Dynamics

- For non-coordinated, non-wings-level steady-flight conditions, including turning flight: linearized longitudinal and lateral-directional state-space models coupled
  - Often coupling weak and ignored in control design
  - Under certain trim conditions, e.g., high angles of attack or sideslip, fully coupled equations necessary assess coupling effects of control inputs on *all* airplane states
- Coupling uses coupled stability and control derivatives, e.g. X_β, Y_α, L^*_α, N^*_α, and X_{δ_r}
- Lecture: coupled airplane linearized dynamics using **polynomial-matrix model** for more clarity in expressions:

$$P(s)\vec{y}(s) = Q(s)\vec{u}(s) \tag{38}$$

- System transfer function matrix:

$$[G(s)] = P^{-1}(s)Q(s) \tag{39}$$

Coupled Polynomial-Matrix, Inputs, Outputs

$$\begin{bmatrix} P_{long}(s) & P_{long-lat}(s) \\ P_{lat-long}(s) & P_{lat}(s) \end{bmatrix} \begin{bmatrix} \vec{y}_{long}(s) \\ \vec{y}_{lat}(s) \end{bmatrix} = \begin{bmatrix} Q_{long}(s) & Q_{long-lat}(s) \\ Q_{lat-long}(s) & Q_{lat}(s) \end{bmatrix} \begin{bmatrix} \vec{u}_{long}(s) \\ \vec{u}_{lat}(s) \end{bmatrix} \tag{40}$$

$$\vec{y}_{long}(s) = \begin{bmatrix} \Delta u(s) & \Delta\alpha(s) & \Delta q(s) & \Delta\theta(s) \end{bmatrix}^T \tag{41}$$

$$\vec{y}_{lat}(s) = \begin{bmatrix} \Delta\beta(s) & \Delta p(s) & \Delta r(s) & \Delta\phi(s) & \Delta\psi(s) \end{bmatrix}^T \tag{42}$$

$$\vec{u}_{long}(s) = \begin{bmatrix} \Delta\delta_e(s) & \Delta\delta_T(s) \end{bmatrix}^T \tag{43}$$

$$\vec{u}_{lat}(s) = \begin{bmatrix} \Delta\delta_a(s) & \Delta\delta_r(s) \end{bmatrix}^T \tag{44}$$

Diagonal Polynomial-Matrices

$$P_{long}(s) = \begin{bmatrix} s - X_u & -X_\alpha + \bar{q}\bar{u} & -X_q & g\cos\bar{\theta} \\ -Z_u - \bar{q} & (\bar{u} - Z_{\dot{\alpha}})s - Z_\alpha & -Z_q - \bar{u} & g\sin\bar{\theta}\cos\bar{\phi} \\ -M_u & -M_{\dot{\alpha}}s - M_\alpha & s - M_q & 0 \\ 0 & 0 & -\cos\bar{\phi} & s \end{bmatrix} \tag{45}$$

$$Q_{long}(s) = \begin{bmatrix} X_{\delta_e} & X_{\delta_T} \\ Z_{\delta_e} & Z_{\delta_T} \\ M_{\delta_e} & M_{\delta_T} \\ 0 & 0 \end{bmatrix} \tag{46}$$

Diagonal Polynomial-Matrices (continued)

$$P_{lat}(s) = \begin{bmatrix} \bar{u}s - Y_\beta & -Y_p & \bar{u} - Y_r & -g\cos\bar{\theta}\cos\bar{\phi} & 0 \\ -L^*_\beta & s - L^*_p - C_1\bar{q} & -L^*_r - C_2\bar{q} & 0 & 0 \\ -N^*_\beta & -N^*_p - C_3\bar{q} & s - N^*_r + C_1\bar{q} & 0 & 0 \\ 0 & 1 & \tan\bar{\theta}\cos\bar{\phi} & -s + \left(\bar{q}\cos\bar{\phi} - \bar{r}\sin\bar{\phi}\right)\tan\bar{\theta} & 0 \\ 0 & 0 & \cos\bar{\phi} & \bar{q}\cos\bar{\phi} - \bar{r}\sin\bar{\phi} & -s\cos\bar{\phi} \end{bmatrix} \tag{47}$$

$$L^*_\bullet = \left(L_\bullet + N_\bullet \frac{I_{xz}}{I_{zz}}\right) D_{xz} \tag{48}$$

$$N^*_\bullet = \left(N_\bullet + L_\bullet \frac{I_{xz}}{I_{zz}}\right) D_{xz} \tag{49}$$

$$D_{xz} = \left(1 - \frac{I^2_{xz}}{I_{xx} I_{zz}}\right)^{-1} \tag{50}$$

For $\bullet = \beta$, p, r, δ_a, and δ_r due to coupling of L and N for non-zero I_{xz}

Diagonal Polynomial-Matrices (continued)

$$C_1 = \left(I_{xx} - I_{yy} + I_{zz}\right) I_{xz} I_{xx}^{-1} I_{zz}^{-1} D_{xz} \tag{51}$$

$$C_2 = \left(I_{yy} - I_{zz} + \frac{I_{xz}^2}{I_{zz}}\right) I_{xx}^{-1} D_{xz} \tag{52}$$

$$C_3 = \left(I_{xx} - I_{yy} + \frac{I_{xz}^2}{I_{xx}}\right) I_{zz}^{-1} D_{xz} \tag{53}$$

$$Q_{lat}(s) = \begin{bmatrix} Y_{\delta_a} & Y_{\delta_r} \\ L^*_{\delta_a} & L^*_{\delta_r} \\ N^*_{\delta_a} & N^*_{\delta_r} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{54}$$

Off-Diagonal Polynomial-Matrices

$$P_{long-lat}(s) = \begin{bmatrix} -\bar{r}\bar{u} - X_\beta & 0 & -\bar{\beta}\bar{u} & 0 & 0 \\ \bar{p}\bar{u} & \bar{\beta}\bar{u} & 0 & g\cos\bar{\theta}\sin\bar{\phi} & 0 \\ 0 & (I_{xx} - I_{zz})\, I_{yy}^{-1}\bar{r} + 2I_{xz}I_{yy}^{-1}\bar{p} & (I_{xx} - I_{zz})\, I_{yy}^{-1}\bar{p} - 2I_{xz}I_{yy}^{-1}\bar{r} & 0 & 0 \\ 0 & 0 & \sin\bar{\phi} & \dot{\psi}\cos\bar{\theta} & 0 \end{bmatrix} \tag{55}$$

$$Q_{long-lat}(s) = \begin{bmatrix} 0 & X_{\delta_r} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{56}$$

Off-Diagonal Polynomial-Matrices (continued)

$$P_{lat-long}(s) = \begin{bmatrix} \bar{r} & \bar{p}\bar{u} - Y_\alpha & 0 & g \sin\bar{\theta} \sin\bar{\phi} \\ 0 & -L_\alpha^* & I_{zz}^{-1} D_{xz} C_4 + I_{xz} I_{xx}^{-1} I_{zz}^{-1} D_{xz} C_5 & 0 \\ 0 & -N_\alpha^* & I_{xx}^{-1} D_{xz} C_5 + I_{xz} I_{xx}^{-1} I_{zz}^{-1} D_{xz} C_4 & 0 \\ 0 & 0 & \tan\bar{\theta} \sin\bar{\phi} & \bar{q} \sin\bar{\phi} + \bar{r} \cos\bar{\phi} + \dot{\psi} \sin\bar{\theta} \tan\bar{\theta} \\ 0 & 0 & \sin\bar{\phi} & \dot{\psi} \sin\bar{\theta} \end{bmatrix} \tag{57}$$

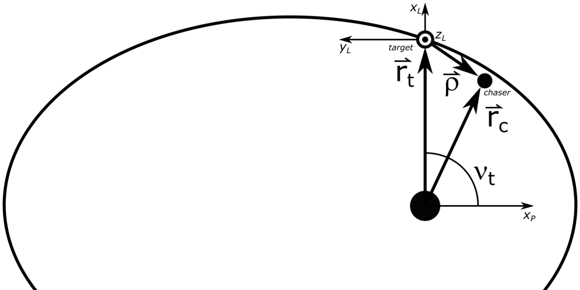
$$C_4 = \left(I_{zz} - I_{yy}\right) \bar{r} - I_{xz}\bar{p} \tag{58}$$

$$C_5 = \left(I_{yy} - I_{xx}\right) \bar{p} + I_{xz}\bar{r} \tag{59}$$

$$Q_{lat-long}(s) = 0_{5\times 2} \tag{60}$$

More

- In many instances for spacecraft dynamics and control, one is interested in modeling the dynamics between multiple spacecraft, e.g., a **rendezvous and proximity operation (RPO).** To that end, consider the following simplified model of two satellites operating in proximity, i.e. a simplified **three-body problem**, involving a **chaser spacecraft** and a **target spacecraft** (subscript t) on an elliptical orbit. One can represent this relative motion in **Hill's frame (HF)** for the target spacecraft as shown in the following figure.



More

- Thus, one can represent the target position relative to the celestial body in the target's HF axes as

$$\vec{r}_t = \begin{bmatrix} r_t & 0 & 0 \end{bmatrix}^T \tag{61}$$

and the relative position of the chaser in the target's HF axes as

$$\vec{\rho} = \begin{bmatrix} x & y & z \end{bmatrix}^T \tag{62}$$

Then, noting

$$\vec{r}_c = \vec{r}_t + \vec{\rho} \tag{63}$$

and using Newton's Law of Gravitation for the relative motion of the chaser spacecraft relative to the target spacecraft, one has the equations of motion

$$\ddot{\vec{\rho}} + 2\vec{\omega}_{t,P/I} \times \dot{\vec{\rho}} + \vec{\alpha} \times \vec{\rho} + \vec{\omega}_{t,P/I} \times [\vec{\omega}_{t,P/I} \times \vec{\rho}] = \frac{\mu m_c}{\|\vec{r}_t + \vec{\rho}\|_2^3} (\vec{r}_t + \vec{\rho}) - \frac{\mu m_t}{r_t^3} \vec{r}_t \tag{64}$$

which is a nonlinear due to the gravity dependence on $\|r_c\|_2^{-3}$ and time-varying due to the $\vec{r}_t$ dependence.

More

- If one assumes a circular orbit for the target spacecraft and by linearizing about the resulting constant $\vec{r}_t$,

More

- Equations

More

- Equations

More

- LTI **Clohessy-Wiltshire (CW) equations** for artificial satellites and the **Hill equations** for natural satellites, in the target's HF:

$$\begin{aligned} \ddot{x} &= 2n_t\dot{y} + 3n_t^2 x + u_x \\ \ddot{y} &= -2n_t\dot{x} + u_y \\ \ddot{z} &= -n_t^2 z + u_z \end{aligned} \tag{65}$$

  - where $n = \sqrt{\mu/r_t^3}$ and $\vec{u} = [u_x \; u_y \; u_z]^T$ is the acceleration input, e.g., the mass-normalized thrust force, for the chaser satellite.

More

- LTI **Clohessy-Wiltshire (CW) equations** for artificial satellites and the **Hill equations** for natural satellites, in the target's HF:

$$\begin{aligned} \ddot{x} &= 2n_t\dot{y} + 3n_t^2 x + u_x \\ \ddot{y} &= -2n_t\dot{x} + u_y \\ \ddot{z} &= -n_t^2 z + u_z \end{aligned} \tag{65}$$

  - where $n = \sqrt{\mu/r_t^3}$ and $\vec{u} = [u_x\ u_y\ u_z]^T$ is the acceleration input, e.g., the mass-normalized thrust force, for the chaser satellite.
- A.k.a. Clohessy-Wiltshire-Hill (CWH) and Hill-Clohessy-Wiltshire (HCW) equations

More

- Then, one has the continuous-time LTI dynamics equation

$$\dot{\vec{x}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3n_t^2 & 0 & 0 & 0 & 2n_t & 0 \\ 0 & 0 & 0 & -2n_t & 0 & 0 \\ 0 & 0 & -n_t^2 & 0 & 0 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{u} \tag{66}$$

where

$$\vec{x} = \begin{bmatrix} \vec{\rho} \\ \dot{\vec{\rho}} \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \tag{67}$$

More

- This can also be discretized at Δt using a zero-order hold as the discrete-time LTI dynamics equation

$$\vec{x}[k+1] = F\vec{x}[k] + G\vec{u}[k]$$

$$= \begin{bmatrix} 4-3\cos(n_t\Delta t) & 0 & 0 & n_t^{-1}\sin(n_t\Delta t) & 2n_t^{-1}(1 - \\ 6(\sin(n_t\Delta t)-n_t\Delta t) & 1 & 0 & -2n_t^{-1}(1-\cos(n_t\Delta t)) & n_t^{-1}(4\sin(\\ 0 & 0 & \cos(n_t\Delta t) & 0 & \\ 3n_t\sin(n_t\Delta t) & 0 & 0 & \cos(n_t\Delta t) & 2\sin \\ -6n_t(1-\cos(n_t\Delta t)) & 0 & 0 & -2\sin(n_t\Delta t) & 4\cos(\\ 0 & 0 & -n_t\sin(n_t\Delta t) & 0 & \end{bmatrix}$$

$$+ \begin{bmatrix} n_t^{-2}(1-\cos(n_t\Delta t)) & 2n_t^{-2}(n_t\Delta t-\sin(n_t\Delta t)) & 0 \\ -2n_t^{-2}(n_t\Delta t-\sin(n_t\Delta t)) & 4n_t^{-2}(1-\cos(n_t\Delta t))-\frac{3}{2}n_t\Delta t^2 & 0 \\ 0 & 0 & n_t^{-2}(1-\cos(\\ n_t^{-1}\sin(n_t\Delta t) & 2n_t^{-1}(1-\cos(n_t\Delta t)) & 0 \\ -2n_t^{-1}(1-\cos(n_t\Delta t)) & 4n_t^{-1}\sin(n_t\Delta t)-3\Delta t & 0 \end{bmatrix}$$

Summary

- Airplane performance/stability analysis and control design
  - Steady-flight and linearized modeling

Summary

- Airplane performance/stability analysis and control design
  - Steady-flight and linearized modeling
- Two important steady-flight maneuvers
  - Straight flight
  - Turning flight
  - Both with or without climbing/descending

Summary

- Airplane performance/stability analysis and control design
  - Steady-flight and linearized modeling
- Two important steady-flight maneuvers
  - Straight flight
  - Turning flight
  - Both with or without climbing/descending
- Level turns: coupled longitudinal and lateral-directional trim analysis

Summary

- Airplane performance/stability analysis and control design
  - Steady-flight and linearized modeling
- Two important steady-flight maneuvers
  - Straight flight
  - Turning flight
  - Both with or without climbing/descending
- Level turns: coupled longitudinal and lateral-directional trim analysis
- Coupled linearized dynamics important for
  - Significant $\bar{p}$, $\bar{q}$, $\bar{r}$, i.e. turning
  - Significant $\bar{\beta}$ and $\bar{\alpha}$
  - High maneuverability airplanes