Lecture 22: Convex Optimization in Control With Powered Descent Guidance Example

Textbook Sections 4.3 & 12.7

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Introduction

- Optimal control problems: can be solved using generalized calculus of variations
  - Pontryagin's principle
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  - Special convex optimization: semidefinite programming
  - Non-convex constraints may make solution more difficult
- Lecture: introduce concepts behind standard semidefinite programming solvers for optimal control

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- Line segment $L(\vec{v}_1, \vec{v}_2)$ between them contains set of points:

$$L(\vec{v}_1, \vec{v}_2) = \{\vec{v} \in \mathcal{V} : \vec{v} = \mu\vec{v}_1 + (1-\mu)\vec{v}_2 \text{ for some } \mu \in [0, 1]\} \tag{1}$$

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- Let $\mathcal{Q} \subset \mathcal{V}$ be nonempty, then $\mathcal{Q}$ **convex set** of $\mathcal{V}$ if for any $\vec{v}_1, \vec{v}_2 \in \mathcal{Q}$, $L(\vec{v}_1, \vec{v}_2) \subset \mathcal{Q}$

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- Consider expression $\vec{v} = \mu\vec{v}_1 + (1-\mu)\vec{v}_2$ as weighed average, i.e. if $\mu_1, \mu_2 \in [0, 1]$ with $\mu_1 + \mu_2 = 1$, then $\vec{v} = \mu_1\vec{v}_1 + \mu_2\vec{v}_2$

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- Generalized to average of n points as $\vec{v} = \mu_1\vec{v}_1 + \cdots + \mu_n\vec{v}_n$ with $\mu_1, ..., \mu_n \in [0, 1]$ and $\mu_1 + \cdots + \mu_n = 1$

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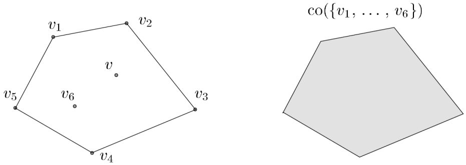

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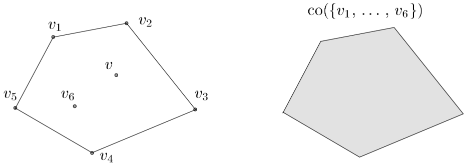
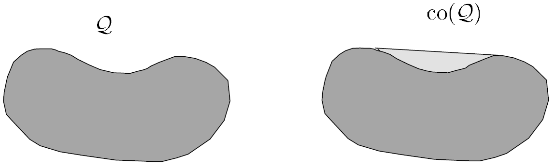


- More generally, given set $\mathcal{Q}$,
  Convex hull, $co(\mathcal{Q})$, by set$\{\vec{v} \in \mathcal{V}$: there exists n and $\vec{v}_1, \ldots, \vec{v}_n \in \mathcal{Q}$ such that $\vec{v} \in co(\{\vec{v}_1, \ldots, \vec{v}_n\})\}$
  - I.e. convex hull of $\mathcal{Q}$: collection of all possible weighted averages of points in $\mathcal{Q}$

2D Example of Convex Hull

- Set $\mathcal{Q}$



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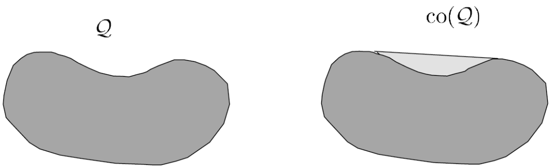

- Can show:
  - Subset $\mathcal{Q} \subset co(\mathcal{Q})$ satisfied
  - Convex hull $co(\mathcal{Q})$ convex
  - $co(\mathcal{Q}) = co(co(\mathcal{Q}))$
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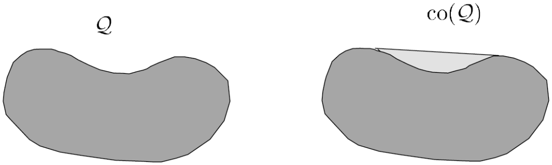

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- Note: by definition, intersection of convex sets always convex

Convex Cones

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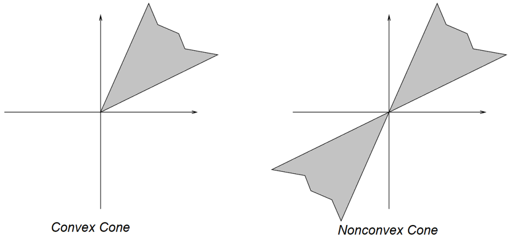
- Two-dimensional example of convex and nonconvex cone



Linear Programs (LP)

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Quadratic Programs (QP)

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Second-Order Cone Programs (SOCP)

-

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- Many generalized OCPs formulated as **semidefinite programs (SDP)**: type of convex optimization
  - Use LMI constraints on solution space $\mathcal{X}$, e.g. ARIs
- General form of SDP formulation:

$$\begin{aligned} X^{opt} = \underset{X \in \mathcal{X}}{\text{argmin}} \quad & c(X) \\ \underline{\text{subject to}}: \quad & F(X) \leq Q \\ & X \in \mathcal{X} \end{aligned} \tag{3}$$

  - $c(X)$: *linear* functional on vector space $\mathcal{X}$
  - A.k.a. type of **linear objective problem**

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- Feasibility question can be made part of SDP and focus of solving SDPs on solving linear objective problems with LMI constraints

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- Using linearity of $F()$:

$$F(X_3) = \mu F(X_1) + (1-\mu)F(X_2) < \mu Q + (1-\mu)Q = Q \tag{5}$$

  - Inequality follows from fact that positive definite matrices: convex cones
  - Therefore: $X_3 \in \mathcal{C}$

Linear Objective Problem Example

- For $X = [x_1 \ x_2]^T \in \mathbb{R}^2$ with

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- Represent feasibility set $\mathcal{C} = \{X \in \mathbb{R}^2 : F(X) < Q\}$ for linear objective problem:
  - X_n: current guess
  - X_{min}: element of $\mathcal{C}$ with most negative projection in direction of Y and solution to problem

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  - Namely, for every other point exists “descent” direction
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- Function $f(\mu) = c(\mu X_1 + (1 - \mu)X_0)$ linear in $\mu \in [0, \epsilon)$ and $f(0) \leq f(\mu)$ for $\mu \in [0, \epsilon)$ by hypothesis
  - $f(\mu)$ non-decreasing and $f(0) \leq f(1)$, or equivalently, $c(X_0) \leq c(X_1)$

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- Can progressively shrink feasibility region to zero, provided able to successively generate “good” feasible point, e.g. X_{n+1}

Ellipsoid Algorithm

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    - Alternates between “cutting” and bounding resulting set by ellipsoid
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  - X_{n+1} would be center of such ellipsoid
- More efficient methods for SDPs: based on **barrier functions** to impose feasibility constraint.
  - Idea: minimize function

$$c(X) + \alpha\phi(X) \tag{8}$$

  - Where $\alpha > 0$ and barrier function $\phi(X)$ convex and approaches infinity on boundary of feasible set
  - E.g. for set $\mathcal{C}$

$$\phi(X) = -\log(\det[Q - F(X)]) \tag{9}$$

  Serve as barrier function

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- Additional details on convex optimization algorithms beyond scope of textbook and left to reader

Algebraic Riccati Equation

- *Will be shown*: solutions to infinite-horizon OCPs found by setting up algebraic Riccati equations (ARE) of general form:

$$A^T P + PA + Q + PRP = 0 \tag{10}$$

  - Typically, required to find particular solution $P = P^T \in \mathbb{R}^{n_x \times n_x}$ such that $A + RP$ stable, i.e. all its eigenvalues have strict negative real part

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- For such problems, define Hamiltonian matrix of ARE of (A, Q, R):

$$H = \begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix} \tag{11}$$

Algebraic Riccati Equation (continued)

- Alternate ARE:

$$\begin{bmatrix} P & -I \end{bmatrix} H \begin{bmatrix} I \\ P \end{bmatrix} = 0 \tag{12}$$

  - Eigenvalue decomposition of H: key to solving ARE

Algebraic Riccati Equation (continued)

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  - Eigenvalue decomposition of H: key to solving ARE
- In this case, state if:
  - A, $Q = Q^T$ and $R = R^T$ given
  - H has no purely imaginary axis eigenvalues
  - $R \geq 0$ or $R \leq 0$
  - (A, R) stabilizable

  Then ARE of (A, Q, R) has unique solution $P = P^T$ such that $A + RP$ stable

Algebraic Riccati Inequality

- For more advanced control synthesis, instead use **algebraic Riccati inequality (ARI)**:

$$A^T P + PA + Q + PRP < 0 \tag{13}$$

  - Type of linear matrix inequality discussed in following section.

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  - Type of linear matrix inequality discussed in following section.
- For example, consider inequality

$$A^T P + PA + C^T C + \gamma^{-2} PBB^T P < 0 \tag{14}$$

  - Quadratic in $P \in \mathbb{S}^n > 0$

Algebraic Riccati Inequality

- Rewriting this as

$$\begin{bmatrix} A^T P + PA + C^T C + \gamma^{-2} PBB^T P & 0 \\ 0 & -\gamma^{-2} I \end{bmatrix} < 0 \tag{15}$$

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- By Schur Complement Lemma, form equivalent LMI for P

$$\begin{bmatrix} A^T P + PA + C^T C & PB \\ B^T P & -\gamma^2 I \end{bmatrix} < 0 \tag{16}$$

Bounded Real Lemma

$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & \gamma^{-2} I \end{bmatrix} < \begin{bmatrix} -C^T C & 0 \\ 0 & 0 \end{bmatrix} \tag{17}$$

- LMI in P as left side can be assigned as $F(P) : \mathbb{S}^n \to \mathbb{S}^n$
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- One of important lemmas between AREs/ARIs: known as **Bounded Real Lemma**, a.k.a. **Kalman-Yacubovich-Popov (KYP) lemma** stated as follows

Bounded Real Lemma

$$\begin{bmatrix} A^TP + PA & PB \\ B^TP & \gamma^{-2}I \end{bmatrix} < \begin{bmatrix} -C^TC & 0 \\ 0 & 0 \end{bmatrix} \tag{17}$$

- LMI in P as left side can be assigned as $F(P) : \mathbb{S}^n \to \mathbb{S}^n$
- Right side assigned as $Q \in \mathbb{S}$

- In this case, relate solutions to AREs and ARIs for OCPs
- One of important lemmas between AREs/ARIs: known as **Bounded Real Lemma**, a.k.a. **Kalman-Yacubovich-Popov (KYP) lemma** stated as follows
- Consider following LTI system, $F_L(G, K)$:

$$\begin{aligned} \dot{\vec{x}}(t) &= A_L\vec{x}(t) + B_1\vec{d}(t) \\ \vec{e}(t) &= C_1\vec{x}(t) \end{aligned} \tag{18}$$

- Let $\gamma > 0$ be given

Bounded Real Lemma (continued)

- Following three statements equivalent
  1. $F_L(G, K)$ stable, i.e. A_L stable, and $\|F_L(G, K)\|_\infty < \gamma^2$
  2. Exists unique $P_1 \geq 0$ such that $A_L + \gamma^{-2} P_1 B_1 B_1^T P_1$ stable and satisfies ARE:

$$A^T P_1 + P_1 A + C_1^T C_1 + \gamma^{-2} P_1 B_1 B_1^T P_1 = 0 \tag{19}$$

  3. Exists $P_2 > 0$ satisfying strict ARI:

$$A^T P_2 + P_2 A + C_1^T C_1 + \gamma^{-2} P_2 B_1 B_1^T P_2 < 0 \tag{20}$$

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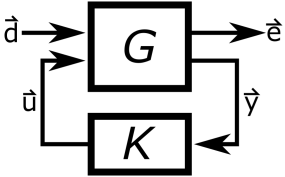
- To solve ARIs for P_2 requires **semidefinite programming (SDP)**
  - Class of convex optimization
  - Discussed in later subsection and then applied to both $\mathcal{H}_2$ and $\mathcal{H}_\infty$ OCPs in later lectures

Generalized Feedback Control System



- Generalized plant, $G(s)$:

$$\begin{aligned} \dot{\vec{x}}(t) &= A\vec{x}(t) + B_1\vec{d}(t) + B_2\vec{u}(t) \\ \vec{e}(t) &= C_1\vec{x}(t) + D_{11}\vec{d}(t) + D_{12}\vec{u}(t) \end{aligned} \tag{21}$$

- State feedback control policy for $K(s)$:

$$\vec{u}(t) = D_K\vec{x}(t) \tag{22}$$

Generalized Feedback Control System (continued)

- Results in closed-loop system:

$$\begin{aligned} \dot{\vec{x}}(t) &= (A + B_2 D_K)\vec{x}(t) + B_1 \vec{d}(t) \\ \vec{e}(t) &= (C_1 + D_{12} D_K)\vec{x}(t) + D_{11} \vec{d}(t) \end{aligned} \tag{23}$$

  - Stable if and only if $A_L = A + B_2 D_K$ stable

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- LMI for stabilizing D_K, enforce stable condition by matrix Lyapunov inequality:

$$A_L P + P A_L^T < 0 \tag{24}$$

  - Solution $P > 0$

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  - Solution $P > 0$
- Substituting for A_L:

$$(A + B_2 D_K)P + P(A + B_2 D_K)^T < 0 \tag{25}$$

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- Expanded:

$$AP + PA^T + B_2(D_K P) + (PD_K^T)B_2^T < 0 \tag{26}$$

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$$\begin{bmatrix} A & B_2 \end{bmatrix} \begin{bmatrix} P \\ Y \end{bmatrix} - \begin{bmatrix} P & Y^T \end{bmatrix} \begin{bmatrix} A^T \\ B_2^T \end{bmatrix} < 0 \tag{27}$$

  - Solved for P and Y and obtain stabilizing feedback controller as $D_K = YP^{-1}$

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- Note: process can be expanded to general LMI characterizations for non-state feedback controllers D_K

Summary

- Linear optimal control problems:
  - Finite-horizon, quadratic cost: Riccati differential equation
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- Simple LMI for stabilizing state feedback control